Monotone optimal control for a class of discrete-event stochastic systems and its applications in revenue management

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**MONOTONE OPTIMAL CONTROL FOR A CLASS OF
DISCRETE-EVENT STOCHASTIC SYSTEMS AND ITS
APPLICATIONS IN REVENUE MANAGEMENT**

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2010**

Monotone Optimal Control for a Class of Discrete-Event Stochastic Systems and its Applications in Revenue Management

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Abstract

This dissertation studies monotone optimal control for a class of discrete-event stochastic systems and its applications in revenue management. It first provides a unified treatment for a class of discrete-event stochastic systems with applications in many different fields. Then it explores applications in three important revenue management problems: capacity control for multiple medical diagnostic facilities, dynamic pricing with two revenue streams, and risk-sensitive dynamic pricing.

The first essay investigates a class of two-dimensional discrete-event stochastic systems that can be modeled by controlled Markov chains with each possible transition bounded by the unit vector. It provides a unified treatment to study monotone optimal control for this class of systems through D -multimodularity under the framework of Markov decision process. It demonstrates that each system of this class can be classified as either a substitutable-type or a complementary-type system according to the possible transition sets and provides a generic approach to prove the structural properties. It identifies D -multimodularity as the general sufficient condition for monotone optimal control for different systems. It also illustrates this integrated approach by classifying representative models in the literature of various fields and solves a two-flight joint dynamic pricing and inventory control problem with two customer segments.

The second essay studies capacity management for multiple medical diagnostic facilities with stochastic demands from three patient segments, which involves two interrelated tasks, namely, outpatient appointment scheduling and dynamic resource allocation. It fully characterizes the optimal allocation policy as the monotone switching-curve policy for any given appointment schedule and proposes two tractable scheduling rules, namely vertical scheduling and horizontal scheduling. The multiple-facility model is extremely relevant and useful in improving utilization of diagnostic facilities, enhancing overall revenue and facilitating resource coordination.

The third essay studies a generic dynamic pricing problem with two revenue streams, primary revenue and ancillary revenue, which plays an important part in the total revenue. It investigates multi-segment dynamic pricing of a perishable product in the presence of another revenue stream. It first studies the multi-segment dynamic pricing problem for a casino operation where customers are segmented through their gaming profile, and the casino dynamically prices its limited hotel rooms as a screening mechanism. It identifies a stochastic order on customer's gaming revenue profile which leads to a monotone inventory price of rooms. It identifies a threshold-type pricing policy for a special customer segmentation scheme in which the customer's winning profile can be ranked in terms of the failure rate order. It then studies the multi-segment dynamic pricing problem of a perishable inventory by integrating ancillary revenues. It derives comparative statics results on the average ancillary revenues across segments and offers new managerial insights. It shows that integrating ancillary revenues into multi-segment dynamic pricing will achieve higher expected total revenue than taking no account of ancillary revenue.

The fourth essay studies risk-sensitive dynamic pricing for a single perishable product. It shows that the intuitively appealing monotone structures under risk-neutrality are preserved under risk-sensitivity with the additive general utility and atemporal exponential utility functions. It also shows that the optimal price is decreasing over the degree of risk-sensitivity under the exponential class of both additive and atemporal utility functions. The risk-sensitive decision maker will set a lower price to hedge against the risk originating from uncertainty about a customer's willingness-to-pay.

This dissertation aims to make several contributions to the literature. In terms of methodology, it provides a unified and systematic framework to study monotone optimal control for a class of two-dimensional discrete-event stochastic systems. In terms of applications, it will enrich the revenue management literature and offer new managerial insights to the current revenue management practices.

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1 Introduction

1.1 Motivation and Contributions

Unlike continuous-state and time-driven physical systems in which state changes as time changes, discrete-event stochastic systems are discrete-state, event-driven and stochastic in system behavior (Cassandras and Lafortune, 2008). The state transition of discrete-event stochastic system is triggered by an instantaneous event, for instance, a customer-arrival in queue or a service completion. Discrete-event stochastic systems are common in various fields such as queueing, manufacturing and production, revenue management, tele-communication networks, call center management, healthcare management, traffic flow control, among others.

The capabilities of discrete-event stochastic systems are extremely powerful while the complexity is overwhelming. The vast body of continuous mathematical tools and techniques used to analyze and control physical systems is inadequate or simply inappropriate for discrete-event stochastic systems. New techniques are needed to study these highly complex systems. Markov decision processes (MDP) are usually used to model and identify structural properties of discrete-event stochastic systems. However, characterizing the optimal control policy under MDP approach is quite challenging. Without any special structure, identifying the optimal policy for MDP is computationally infeasible in general (Glasserman and Yao, 1994a).

Examining the structure of optimal control for smaller systems can lead to useful insights for developing heuristics for larger systems and dramatically simplify computational algorithms. Therefore the first goal of this dissertation is to provide a unified and systematic treatment to study structural properties and monotone optimal control for a class of two-dimensional discrete-event stochastic systems with a wide range of applications. The key innovation is to use D -multimodularity to study this class of discrete-event stochastic systems. In particular, it sheds light on some key questions related to the two-dimensional discrete-event stochastic systems: (1)

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what are the underlying common features of different systems? (2) what are the structural properties that are sufficient for monotone optimal control policies? (3) how can structural properties be propagated through different types of control operators? (4) is there a unified approach to deal with structural properties and monotone optimal control policies? (5) can this unified approach be applied to solve new managerial problems?

I firmly believe that the technical heart of OR/MS (operations research / management science) is modeling and methodology, both of which serve as tools to solve operational problems and gain managerial insights. In fact, this dissertation is dedicated to the goal of studying monotone optimal control for a class of discrete-event stochastic systems and its applications in revenue management. Revenue management, as one of the most successful application areas of Operations Research, was pioneered in the airline industry in the 1970s. Revenue management has now gradually taken hold in a growing spectrum of service and manufacturing industries, profit and nonprofit sectors (Talluri and van Ryzin 2004b; Phillips 2005; Ng 2008; etc.).

The concept of revenue management has been applied successfully in many industries and it can be expanded much further. Some segments of healthcare appear to be logical candidates to apply and benefit from revenue management (Chapman and Carmel, 1992; Gupta and Wang, 2008; etc.). As a result of escalating healthcare costs, government has enforced regulation on expensive equipment such as Magnetic Resonance Imaging (MRI) for cost containment and efficient utilization of limited resources, which calls for an effective capacity management for expensive diagnostic facilities. Capacity management for medical diagnostic facilities, in essence, is demand management or revenue management in healthcare (Chapman and Carmel, 1992).

However, there are some complications in healthcare that make the direct implementation of traditional revenue management a nontrivial exercise. The second goal of this dissertation is to enhance the understanding of revenue management in healthcare and to investigate the capacity management problem for multiple medical diagnostic facilities with stochastic demand from three patient segments. The multiple-facility model is extremely relevant and useful in improving

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healthcare access and utilization of diagnostic facilities, enhancing overall revenue and facilitating resource coordination. The generic methodology developed for certain classes of discrete-event stochastic systems in the first essay is then extended to solve this complicated healthcare problem. It demonstrates its strong efficiency and makes exploring structural properties and monotone optimal control policies for the multiple-facility problem tractable.

Even though revenue management has become a successful main strategy of revenue enhancement for many industries, it is far from fully explored and leveraged. One shortcoming is that traditional revenue management only takes account of *one* revenue stream from selling the product and falls short of taking full account of customer value and overall revenue (primary revenue and ancillary revenue). This is a generic problem faced by many industries such as airlines, hotels (standard or casino), events, third-party logistics and retailers with the presence of an ancillary revenue stream. To enhance the overall revenue, companies must go beyond one revenue stream and examine all revenue sources. Surprisingly to the best of our knowledge, there is no existing literature addressing such an important issue as revenue management with multiple revenue streams analytically. The third goal of this dissertation is to make the first attempt along this direction and study multi-segment dynamic pricing of a perishable product in the presence of another revenue stream. It will shed new light on customer valuation and market segmentation in the revenue management literature.

Another shortcoming in traditional revenue management is that it focuses on risk-neutral decision makers and falls short of taking account of revenue volatility, the financial risk to the firm. Risk-neutrality is often appropriate for repeated business activities where a single poor realization has no severe impact on the financial condition of the firm. However, this may not hold for event organizers (Levin et al. 2008), small airlines/hotels/casinos with cash-flow problems (Barz, 2007), healthcare industry (Secomandi et al., 2002), perishable high-profit products (Schweitzer and Cachon, 2000), firms with short-term revenue targets (Feng and Xiao, 2008), etc. These firms are sensitive to revenue variations and are willing to trade off lower

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expected revenue for downside protection against possible underperformances. Moreover, most product managers in charge of dynamic pricing policies present some degree of risk-aversion (Bitran and Caldentey, 2003). All these findings call for the development of dynamic pricing strategies for risk-sensitive decision makers. The final goal of this dissertation is dedicated to risk-sensitive dynamic pricing for a single perishable product.

This dissertation aims to make several contributions to the literature. In terms of methodology, it provides a unified framework to study monotone optimal control for a class of discrete-event stochastic systems. In terms of revenue management applications, it will enrich the revenue management literatures and offer new managerial insights to the current revenue management practices.

1.2 Dissertation Outline

This dissertation studies monotone optimal control for a class of discrete-event stochastic systems and its application in revenue management. To pursue the goals step by step, the dissertation first presents a methodology essay on a unified and systematic treatment to study monotone optimal control for a class of two-dimensional discrete-event stochastic systems with vast applications. Then the dissertation explores its applications in three important revenue management problems: optimal capacity control for multiple medical diagnostic facilities, dynamic pricing with two revenue streams and risk-sensitive dynamic pricing for a single perishable product.

The first application in healthcare belongs to the *discrete-action control*^{*} for two-dimensional systems, and we fully characterize the monotone optimal switching-curve policy. The two applications in dynamic pricing fall into the *continuous-action control* for one-dimensional systems and we focus on studying the monotone optimal pricing policy. Note that D -

^{*} Please refer to Section 2.1 on discrete-action control and continuous-action control.

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multimodularity in two-dimensions reduces to convexity in one-dimension and the results of the first methodology essay still apply for one-dimensional systems.

This dissertation is organized as follows. Chapter 1 provides motivation, contributions and an outline of this dissertation. Chapter 2 is a relevant literature review. Each chapter from Chapter 3 to Chapter 6 addresses a motivated research problem and a summary of the main results of each chapter is presented below. Chapter 7 is conclusions.

Chapter 3 investigates a class of two-dimensional discrete-event stochastic systems that can be modeled by controlled Markov chains with each possible transition bounded by the unit vector. This class of systems has vast applications in queueing systems, inventory control, manufacturing and production, revenue management, healthcare management, among others. We provide a unified treatment to study structural properties and monotone optimal control for this class of systems through D -multimodularity under the framework of Markov decision processes. We demonstrate that each system of this class can be classified into either a substitutable-type or a complementary-type system according to the possible transition sets and provide a generic approach to prove the structural properties. We identify D -multimodularity as the general condition that guarantees the existence of monotone optimal control for different systems. We also illustrate this integrated approach by classifying representative models in the literature of various fields and solve a two-flight joint dynamic pricing and inventory control problem with two customer segments. This unification leads to a systematic approach to study monotone optimal control for this particular class of two-dimensional discrete-event stochastic systems, which will shed new light on the general two-dimensional systems.

Escalating healthcare cost, limited resources, burgeoning demand, and trends toward integration and coordination critically call for an effective and efficient capacity management for expensive medical diagnostic facilities. Chapter 4 studies capacity management for multiple diagnostic facilities with stochastic demands from three patient segments which involves two interrelated tasks, namely, outpatient appointment scheduling and dynamic resource allocation.

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We have fully characterized the optimal allocation policy as the monotone switching-curve policy for the stochastic dynamic multiple-facility allocation problem for any given appointment scheduling. We propose two tractable scheduling rules, namely vertical scheduling and horizontal scheduling, and evaluate their relative performances through numerical studies under different cost and revenue structures. Our multiple-facility model is extremely relevant and useful in improving utilization of diagnostic facilities, enhancing overall revenue and facilitating resource coordinations.

Chapter 5 studies a generic dynamic pricing problem with two revenue streams, primary revenue and ancillary revenue, which plays an important part in the total revenue. We investigate multi-segment dynamic pricing of a perishable product in the presence of another revenue stream. We first study the multi-segment dynamic pricing problem for a casino operation where customers are segmented through their gaming profile and the casino dynamically prices its limited hotel rooms as a screening mechanism. We identify a stochastic order on customer's gaming revenue profile which leads to a monotone inventory price of rooms. We show a threshold-type pricing policy for a special customer segmentation scheme in which the customer's winning profile can be ranked in terms of the failure rate order. We then study the multi-segment dynamic pricing problem of a perishable inventory by integrating ancillary revenues. We derive comparative statics results on the average ancillary revenues across segments and offer new managerial insights. We show that integrating ancillary revenues into multi-segment dynamic pricing will achieve higher expected total revenue than that taking no account of ancillary revenue. Our models and results will shed new light on the issue of customer valuation and market segmentation.

Chapter 6 studies risk-sensitive dynamic pricing for a single perishable product. The intuitively appealing monotone structures of dynamic pricing for single perishable product under risk-neutrality are shown to be preserved under risk-sensitivity with additive general utility functions and atemporal exponential utility functions. In particular, it shows the monotonicity of

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the optimal price on the degree of risk-sensitivity under the exponential class of both additive and atemporal utility functions. The risk-sensitive decision maker will set a lower price to hedge against the risk originating from uncertainty about a customer's willingness-to-pay.

2 Literature Review

Before moving on to the main development of the dissertation, brief overviews of monotone optimal control for discrete-event stochastic systems and the state-of-the-art of revenue management are presented. These overviews provide the necessary background information needed to evaluate the contributions of this dissertation and serve as the motivation for a unified treatment for a class of discrete-event stochastic systems.

2.1 Monotone Optimal Control for Discrete-Event Stochastic Systems

Discrete-event stochastic systems are extremely useful in modeling real-life problems, but characterizing the optimal control policy is quite challenging. Without any special structure, it is generally infeasible to compute the optimal policies. The lattice programming approach by Topkis (1978), the optimal control approach by Glasserman and Yao (1994a, b) and Altman et al. (2003) have been developed to study the monotone structure for discrete-event stochastic systems.

When actions take values in a discrete set (discrete-action control), structural forms of the optimal control policy usually studied are either the *threshold-type* policy for one-dimensional systems or the *switching-curve* policy for two-dimensional systems (Glasserman and Yao, 1994a; Altman et al., 2003). When actions take values in a continuous set (continuous-action control), studying the *monotone* optimal control policy (a generalization of the *threshold-type* policy and the *switching-curve* policy) becomes the focus of the study, where the monotonicity is with respect to the components of the state or the occurrence of other events (Glasserman and Yao, 1994a). Multi-dimensional systems still remain a challenging open problem.

There is extensive literature in various fields studying structural optimal control policies for particular one- and two-dimensional discrete-event stochastic systems. Table 2.1 lists some

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examples of literature which are classified in terms of the problem, action and structural form of the optimal policy.

Table 2.1 Examples of literature on the structural policies of discrete-event stochastic systems

Revenue Management				
System	Literature	Problem	Action	Optimal Policy
One-dim	Lee & Hersh (1993) Lautenbacher & Stidham (1999)	Admission control for a single flight	Discrete	Threshold policy
	Gallego & van Ryzin (1994) Bitran & Mondschein (1997) Zhao & Zheng (2000)	Dynamic pricing for a single perishable product	Continuous	Monotone
Two-dim	Savin et al. (2005)	Admission control for car rentals with two customer segments	Discrete	Switching-curve
	You (1999)	Dynamic pricing for the two-flight-leg	Continuous	Monotone
Manufacturing and Production				
Two-dim	Veatch & Wein (1994) Ha (1997a,b,c,2000) Carr & Duenyas (2000) Zhao et al. (2008)	Two-station production scheduling MTS production scheduling MTS/MTO production MTS production with transshipment	Discrete	Switching-curve
Queueing System				
Two-dim	Ghoneim & Stidham (1985)	Admission control for two series queues	Discrete	Switching-curve
Multi-dim	Weber & Stidham (1987)	Service rate control for a cycle-queue	Continuous	Monotone
Healthcare Management				
One-dim	Gupta & Wang (2008)	Admission control for one physician	Discrete	Threshold
Two-dim	Green et al. (2006)	Admission control for one-facility with three patient segments	Discrete	Switching-curve

There are many general studies in the literature on structural properties and monotone optimal control under Markov or semi-Markov decision processes. In the context of network queues, Stidham and Weber (1993) categorize MDP models in terms of control types: transition rate, service rate, admission, routing and scheduling. Stidham and Weber (1993), Veatch and Wein (1992) have independently shown that submodularity in the direction of transitions of a cycle/tandem queue, which is equivalent to multimodularity in the state space, is sufficient for transition/service rate monotone. In line with Hajek (1984), Stidham and Weber (1993) also show that multimodularity implies admission monotone and routing monotone. Glasserman and Yao

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(1994a) establish the linkage between monotone optimal controls and structural conditions for generalized semi-Markov decision processes (GSMP). They consider Markovian GSMP in which event rates are subject to control and transform the GSMP on its state space to an equivalent controlled MDP for the counting process on the score space (score is event counts and the actual state is a function of score). They show that submodularity or supermodularity in the score space (which is called $\mathcal{S}$ -modularity) is sufficient for monotone optimal controls on event rates. Although elegant, the score space has more dimensions than the state space and limits the models that can be handled (Koole, 2006). Smith and McCardle (2002) study the closed convex cone properties which include monotonicity, convexity and supermodularity of the value functions and derive comparative statics results for MDP models. Koole (2006) studies monotonicity in Markov reward/decision chains and defines environment operators, arrival operators, departure operators, and transfer operators. He summarizes the propagation results for these operators and characterizes the monotone control policy for some operators. Morton (2006) explores structural properties of network revenue management models from an economic perspective and finds an interesting parallelism between revenue earning systems and cost incurring systems: cost substitutability (complementarity) is associated with supermodularity (submodularity) and strong convexity; while revenue substitutability (complementarity) is associated with submodularity (supermodularity) and strong concavity. Other excellent synthesis and surveys in this area include Koole (1998), Altman and Koole (1998), Stidham (2002), Koole (2004), among others.

However, most of these articles are queueing-oriented and the multimodularity concept they discussed is defined over specific multimodular base under the queueing transfer framework. Multimodularity has not been systematically utilized to study structural properties and monotone optimal controls for discrete-event stochastic systems in general. The type of control operators and the propagation of structural properties have not been fully explored. The concepts of substitutability and complementarity have appeared in the literature without realizing that they

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can be used as a classification mechanism. Motivated from all above, Chapter 3 aims to fill this gap.

2.2 Revenue Management

Revenue management (RM), pioneered in the airline industry in the 1970s, has quickly spread to a wide range of businesses and has proven to be an effective and successful competitive device. To name a few, the transportation industry (railway, air cargo, freight transport), hospitality, service industry (hotel, casino, car rental, cruise ships and ferry lines, restaurant, tour operator), entertainment industry (theaters and sporting events), tele-communication network, internet, media and broadcasting (radio and TV advertising), energy industry (natural-gas, electricity), retail industry, manufacturing, healthcare sectors, among others.

This dissertation studies monotone optimal control for a class of discrete-event stochastic systems and its applications in revenue management. Three application essays aim to address three key issues in revenue management: a RM problem in healthcare (Chapter 4), dynamic pricing with two revenue streams (Chapter 5), risk-sensitive dynamic pricing (Chapter 6). To gain a full picture of the state-of-the-art of revenue management, we will have a brief overview in terms of the key elements of RM, types of RM problems and RM strategies, model developments of RM.

Revenue management is the integration of market segmentation, pricing decisions and inventory control strategies to maximize the expected revenue through matching fluctuating demand with constrained capacity. In general, there are six key characteristics that prove to be conducive to the effective application of revenue management (Kimes 1989): (1) relatively fixed capacity; (2) perishable product; (3) fluctuating random demand; (4) ability to segment the market; (5) ability to sell the product in advance; and (6) low marginal service cost but high marginal capacity cost. The key elements of revenue management are illustrated in Figure 2.1.

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Figure 2.1 Key elements of revenue management

Customer heterogeneity enables firms to exploit varying willingness-to-pay and segment the market. Through market segmentation, revenue management seeks to execute two main RM strategies, namely, quantity-based inventory control and price-based dynamic pricing. For quantity-based control, the problem for the firm is to allocate capacity to different segments and control admissions; for price-based control, firms use dynamic pricing over the sales season according to the remaining capacity and selling horizon. Which strategy to use in any given industry depends on the context, and it may vary even across firms within a given industry. For instance, most legacy airlines commit to fixed prices and dynamically allocate capacity among fare classes, whereas low cost airlines tend to use dynamic pricing (Talluri and van Ryzin, 2004b).

However, pricing and inventory control strategies are closely interrelated to each other, since price decisions influence demand forecasts and inventory control, which in turn affect pricing due to inventory availability. Thus there is a growing concern for integrating pricing and inventory control. Nevertheless, the literature on joint pricing and inventory control is limited

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mainly to the newsvendor problems with pricing decisions for the single-period problems or static control for the multi-period problems.

Single-resource problems have been extensively studied in line with the Markov decision process framework and the optimal control policy for most single-resource models can be fully characterized. For the multiple-resource problems, structural results for optimal policies are very limited except some specific two-dimensional systems, thus approximation methods are usually adopted. Therefore structural properties of single-resource models are useful as building blocks in developing heuristics for the multiple-resource problems.

To help gain an outline of revenue management, a partial classification of traditional revenue management literature in terms of RM problems and RM strategies is summarized in Table 2.2. We will have an overview on the model development in revenue management next.

Table 2.2 Traditional RM literature in terms of RM problems and RM strategies

RM strategy RM problem	Capacity control (quantity-based)	Dynamic pricing (priced-based)	Joint pricing and inventory control
Single-resource	Littlewood (1972), Belobaba (1987,1989), Curry (1990), Wollmer (1992), Brumelle & McGill (1993), Robinson (1995), Li & Oum (2002), Lee & Hersh (1993), Liang (1999), Lautenbacher & Stidham (1999) Feng & Xiao (2000a) Brumelle & Walczak (2003), etc.	Gallego & van Ryzin (1994), Bitran & Mondschein (1997), Zhao & Zheng (2000), Feng & Xiao (2000b), Lin (2004), etc.	Lau & Lau (1988), Petruzzi & Dana (1999), etc.
Multiple-resource	Williamson (1992), Talluri & van Ryzin (1998, 1999), Bertsimas & Popescue (2003), Bertsimas & de Boer (2005b), Cooper & Homem-de-Mello (2007) van Ryzin & Vulcano (2008a), etc.	Gallego & van Ryzin (1997), You (1999), Maglaras & Meissner (2006), etc.	Bertsimas & de Boer (2005a), Kuyumcu & Popescue (2006), etc.

Littlewood's model (Littlewood, 1972) is the seminal model in revenue management, which establishes the foundations for RM models subsequently developed. Revenue management

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models have developed from static to dynamic, exogenous independent demand to customer choice behavior, myopic customer to strategic customer behavior, monopoly to competition, risk-neutrality to risk-sensitivity, restriction-based segmentation to value-based segmentation. A partial classification in terms of model development in revenue management is summarized in Table 2.3.

Table 2.3 Model development of revenue management

Static Model (sequential demand arrival) Littlewood (1972), Belobaba (1987, 1989), Curry (1990), Wollmer (1992), Brummell & McGill (1993), Robinson (1995), Li & Oum (2002), etc.	Dynamic Model (random demand arrival) Lee & Hersh (1993), Liang (1999), Lautenbacher et al. (1999), Feng & Xiao (2000a, b), Brummelle & Walczak (2003), etc.
Exogenous independent demand Traditional RM literature	Customer choice behavior Brummelle et al. (1990), Belobaba & Weatherford (1996), Mahajan & van Ryzin (2001), Gallego & Phillips (2004), Talluri & van Ryzin (2004a), Tang et al. (2004), Zhang & Cooper (2005), Liu & van Ryzin (2008b), Dong et al. (2008), van Ryzin & Vulcano (2008b), Zhang & Cooper (2009), Akcay et al. (2009), Shumsky & Zhang (2009), Meissner & Strauss (2009), Suh and Aydin (2009), etc.
Myopic customer Traditional RM literature	Strategic customer behavior Popescu & Wu (2006), Marcus & Anderson (2006), Ahn et al. (2007), Su (2007), Levin et al. (2007), Aviv and Pazgal (2008), Liu & van Ryzin (2008a), Levina et al. (2009), Levin et al. (2009a, b), etc.
Monopoly Traditional RM literature	Competition Netessine and Shumsky (2005), Gallego et al. (2006), Xu & Hopp (2006), Li et al. (2007), Li et al. (2008), Levin et al. (2009a), etc.
Risk-neutrality Traditional RM literature	Risk-sensitivity Feng & Xiao (1999), Lancaster (2003), Weatherford (2004), Barz (2007), Feng & Xiao (2008), etc.; Lim & Shanthikumar (2007), Levin et al. (2008), etc.
Restriction-based segmentation Traditional RM literature	Value-based segmentation Esse (2003); Noone et al. (2003); Hendler & Hendler (2004); Metters et al. (2007); etc.

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Most static revenue management models assume *sequential* arriving order, that is, independent demands for different classes arrive in non-overlapping intervals in a strict low-to-high order. The decision period corresponds to the open fare class and decisions are made at the beginning of each period similar to the periodic review of inventory management. Dynamic revenue management models assume *Markovian* arriving order of independent demands. Dynamic models monitor the state of booking processes over time and make decisions on a particular booking request when it arrives based on the state at that time point. Therefore, dynamic models account for the stochastic and dynamic nature of revenue management problems better than static models.

However, the exogenous *independent* demand assumption has its inherent flaw as it is not affected by the availability of other products and it fails to account for customer choice behavior observed in the real practice of revenue management. Flawed models produce suboptimal controls which can get even worse over time. Therefore, a spiral-down effect (Cooper et al., 2006) can occur and the existing revenue management suffers from suboptimality.

Dynamic pricing models have partially addressed customer choice behavior in which demands depend on the current price. However, dynamic pricing models assume one price at any time point and customers face a binary choice: to buy or not to buy, which fails to capture the real practice in which multiple fares are offered simultaneously and customers choose among them based on prices, availabilities, and preferences. Therefore, modeling demand by incorporating discrete customer *choice* behavior has emerged as an appealing alternative to the independent demand model.

Traditional dynamic pricing models formulate *Markovian* demand as a function of *current* price only, which is independent of price path *history* as well as *future* price. This is appropriate for modeling *myopic* customers who make purchase decisions immediately if the price is below their reservation, without taking reference effect of history prices into account or anticipating future prices. Yet, under the new business environment where customers are empowered with

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plenty of information and choices, it is arguably more realistic to assume that customers will behave strategically when customer faces different prices over time. Taking account of *strategic* purchasing behavior of customers makes dynamic pricing more effective and challenging.

Traditional revenue management is under the framework of *monopoly* assumption and ignores competitive interactions. However, to study a major product of deregulation such as revenue management, in isolation from the main driver of deregulation, that is competition, is just like studying fish out of water (Zhao, 2003). There is a strong need and growing concern on *competitive* revenue management.

While in reality decision-makers are normally risk-averse, most problems in revenue management are analyzed in the *risk-neutral* setting. Even though most dynamic pricing managers present some degree of risk-aversion (Bitran and Caldentey, 2003), literature on *risk-averse* revenue management is very limited and quite recent.

There is another important development in terms of customer segmentation schemes. Traditional revenue management uses fences and restrictions to segment the market and exploit heterogeneous willingness to pay. However, this transaction-orientated revenue management falls short of taking full account of customer value which exhibits deficiencies and risks in today's highly competitive business environment. Integrating customer value with revenue management is becoming increasingly imperative and value-based segmentation schemes have been proposed recently (Esse, 2003; Noone et al. 2003; Hendler and Hendler, 2004; Metters et al., 2008; etc.). Effective inventory control or customized pricing for the limited resources among customer segments based on the overall value can ensure that the most valued or profitable segment get resources with appropriate prices.

For comprehensive reviews on revenue management readers can refer to Belobaba (1989), Weatherford and Bodily (1992), McGill and van Ryzin (1999), Elmaghraby and Keskinocak (2003), Bitran and Caldentey (2003) and the books by Talluri and van Ryzin (2004), Phillips (2005), and Ng (2008), etc.

3 Monotone Optimal Control for a Class of Discrete-Event Stochastic Systems

3.1 Introduction

This chapter studies monotone optimal control for a class of two-dimensional discrete-event stochastic systems that can be modeled by controlled Markov chains with each possible transition bounded by the unit vector. This class of systems has vast applications in queueing systems, inventory control, manufacturing and production systems, revenue management, healthcare management, among others. It makes the following contributions.

First, it provides a unified framework to study structural properties and monotone optimal control for a class of two-dimensional discrete-event stochastic systems through D -multimodularity under the framework of MDP.

Second, we demonstrate that each system of this class can be classified into either a substitutable-type or a complementary-type according to the possible transition set and illustrate this unified approach by classifying representative models in the literature in various fields.

Third, we identify D -multimodularity as the general condition that guarantee the existence of monotone optimal control for different systems. A necessary and sufficient condition for two-dimensional D -multimodularity is given for the two multimodular bases which correspond to the substitutable-type and complementary-type system respectively. It is shown that the corresponding D -multimodularity will be propagated for full and partial control operators on each atom of the corresponding multimodular triangulation.

Finally, this unified approach is applied to solve a joint dynamic pricing and inventory control problem for airlines with two parallel flights, which is new to the literature.

The rest of the chapter is organized as follows. In Section 3.2 we study two-dimensional D -multimodularity. Section 3.3 introduces the classification mechanism for different systems, which

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will be applied to classify representative models in the literature. Section 3.4 studies the propagation of D -multimodularity over control operators. Monotone control policies through D -multimodularity will be characterized in Section 3.5. In Section 3.6, we study a new application problem - joint dynamic pricing and inventory control problem for two parallel flights, with the new method. Section 3.7 concludes. Proofs are provided in appendices in Section 3.8.

3.2 D -Multimodularity

Convex functions are tractable in continuous optimization mainly because local optimality guarantees global optimality. However many optimization problems from real life applications are discrete instead of continuous, for example, units of inventory, customers in queue, airline seats and hotel rooms. Discrete-event stochastic systems are defined over the discrete state space. Therefore analyzing discrete-event stochastic systems requires a good understanding of the discrete objective function first.

Originally introduced by Hajek (1985), the discrete counterpart of *convex* functions is called *multimodular* functions, which implies that, for a multimodular function, a local minimum is a global minimum. Altman et al. (2003) extend the multimodularity concept which is defined over particular multimodular base under the queueing transfer framework to the so-called D -multimodularity which is defined over all possible multimodular bases. D -multimodularity may be called as *simplex convexity* (Altman et al., 2003) and a local optimality for a D -multimodular function assures a global optimality.

One-dimensional control problems for discrete-event stochastic systems can be solved with a threshold-type optimal policy fully identified. Two-dimensional control problems for discrete-event stochastic systems are common in the literature and a generalization of convexity to two-dimensional lattice lies at the heart of monotone optimal controls. The general multi-dimensional control problem still remains open because regularity properties that are used to characterize

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optimal policies in one-dimensional or two-dimensional problems can not be generalized to multi-dimensional problems except for very special cases (Altman et al., 2003). In this chapter, we focus on studying a particular class of two-dimensional discrete-event stochastic systems which have broad applications in various fields.

3.2.1 Preliminaries

In the literature, there are various concepts that are closely related to the multimodularity concept. To standardize the presentation, we first present some preliminary definitions and results before introducing D -multimodularity. Throughout this dissertation, the symbol $\uparrow$ and $\downarrow$ represent increasing and decreasing in the weak sense respectively.

Definition 3.1 Let $f: \mathbb{Z}^n \rightarrow \mathbb{R}$ be any real-valued function. Denote $\Delta_i f(x) = f(x) - f(x - e_i)$

for $\forall x \in \mathbb{Z}^n$ and e_i the unit vector in dimension i . For all $i \neq j$, f is

(1) *supermodular* if $\Delta_i \Delta_j f(x) \geq 0$, *submodular* if $\Delta_i \Delta_j f(x) \leq 0$;

(2) *componentwise convex* if $\Delta_i \Delta_i f(x) \geq 0$;

(3) *integer convex* if $f(x + d) - f(x) \geq f(x) - f(x - d)$ for $d \in \mathbb{Z}^n$;

(see Altman et al., 2003)

(4) *superconvex* if $\Delta_i f(x) \geq \Delta_i f(x - e_i + e_j)$, *subconvex* if $\Delta_i f(x) \geq \Delta_i f(x - e_i - e_j)$;

(see Koole, 1998; Morton, 2006; etc.)

(5) *directionally convex* if f is supermodular and componentwise convex;

antidirectionally convex if f is submodular and componentwise convex;

(see Chang and Yao, 1993; Chang et al., 1994; Morton, 2006; etc.)

(6) *directionally supermodular* if f is supermodular and superconvex;

antidirectionally submodular if f is submodular and subconvex.

(See directional modularity properties in Morton, 2006).

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The dual properties in the concavity context are summarized in Definition 3.2. For a better reference, we still keep property (1) of supermodular and submodular in Definition 3.2.

Definition 3.2 Let $f: \mathbb{Z}^n \rightarrow \mathbb{R}$ be any real-valued function. Denote $\Delta_i f(x) = f(x) - f(x - e_i)$

for $\forall x \in \mathbb{Z}^n$ and e_i the unit vector in dimension i . For all $i \neq j$, f is

- (1) *supermodular* if $\Delta_i \Delta_j f(x) \geq 0$, *submodular* if $\Delta_i \Delta_j f(x) \leq 0$;
- (2) *componentwise concave* if $\Delta_i \Delta_i f(x) \leq 0$;
- (3) *integer concave* if $f(x + d) - f(x) \leq f(x) - f(x - d)$ for $d \in \mathbb{Z}^n$;
- (4) *subconcave* if $\Delta_i f(x) \leq \Delta_i f(x - e_i + e_j)$, *superconcave* if $\Delta_i f(x) \leq \Delta_i f(x - e_i - e_j)$;

(See Morton, 2006)

- (5) *directionally concave* if f is submodular and componentwise concave;
antidirectionally concave if f is supermodular and componentwise concave;
- (6) *directionally submodular* if f is submodular and subconcave;
antidirectionally supermodular if f is supermodular and superconcave.

(See directional modularity properties in Morton, 2006).

Next we will introduce concepts relevant to D -multimodularity. The relationships among these important concepts and properties will be discussed in the Section 3.2.2.

Definition 3.3 (Altman et al., 2003) Any $(n+1) \times n$ matrix D of rank n such that the $n+1$ row-vectors $(s_0, s_1, \dots, s_n)$ of matrix D satisfies $s_0 + \dots + s_n = 0$ is called a *multimodular matrix*.

Note that because D has rank n , any n row-vectors of a multimodular matrix D are linearly independent while $n+1$ row-vectors are linearly dependent. For any given multimodular matrix D , the array of $n+1$ row-vectors $(s_0, s_1, \dots, s_n)$ constitutes a *multimodular base*, which is denoted by $\mathcal{F} = \{s_0, s_1, \dots, s_n\}$.

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Definition 3.4 (Definition 17, Altman et al., 2003) The *mesh* M_D associated with the multimodular matrix D is the set of all points $\{\sum_{i=0}^n \alpha_i s_i, \alpha_i \in \mathbb{Z}\}$.

Definition 3.5 (Schrijver, 1986) A matrix is *totally unimodular* if every square submatrix has determinant 0, 1 or -1.

Note that every entry of a totally unimodular matrix is 0, 1 or -1. If the multimodular matrix D is totally unimodular, then its mesh M_D is $\mathbb{Z}^n$. Totally unimodular matrices typify transportation network problems and can lead to integer solutions to continuous problems[†].

Definition 3.6 (Definition 18, Altman et al., 2003) A *D-atom* is a simplex $\mathcal{S}\{x_0, \dots, x_n\}$ in $\mathbb{R}^n$, made of the following $n+1$ points,

$$\begin{aligned} x_0 &= a \\ x_1 &= x_0 + s_{\tau(0)}, \\ x_2 &= x_1 + s_{\tau(1)}, \\ &\vdots \\ x_n &= x_{n-1} + s_{\tau(n-1)}, \end{aligned}$$

where $a \in M_D$ is the root and τ is a permutation of $(0, \dots, n)$.

It is evident that $x_0 = x_n + s_{\tau(n)}$ and a *D-atom* is a simplex since D is of rank n . We denote a *D-atom* by $\mathcal{A}(a, D, \tau)$ with root a , multimodular matrix D and permutation τ .

Definition 3.7 (Definition 19, Altman et al., 2003) A collection of simplexes is a *triangulation* of $E \subseteq \mathbb{R}^n$ if E is the union of all the simplexes and the intersection of any two simplexes is either the empty set or a common face.

Lemma 3.1 (Theorem 70, Altman et al., 2003) The set of all the *D-atoms* forms a triangulation of $\mathbb{R}^n$, which is called a *multimodular triangulation*.

[†] Maximum flow and minimum cost flow problems yield a constraint matrix that is totally unimodular. Thus such network flow problems with bounded integer capacities have an integral optimal solution. Refer to Theorem 19.1 of Schrijver, 1986.

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We now present the definition of D -multimodularity.

Definition 3.8 (Definition 20, Altman et al., 2003) A function $f : M_D \rightarrow \mathbb{R}$ is D -multimodular if and only if for all $x \in M_D$ and for all $0 \leq i < j \leq n$,

$$f(x + s_i) + f(x + s_j) \geq f(x) + f(x + s_i + s_j). \quad (3.1)$$

Let function $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ be the linear interpolation of f over the atoms defined by D . The following lemma connects D -multimodularity with convexity.

Lemma 3.2 (Theorem 71, Altman et al., 2003) f is D -multimodular if and only if $\tilde{f}$ is convex.

To have a full exploration on monotone optimal control for a class of two-dimensional discrete-event stochastic systems, we need to study multimodularity in mesh $\mathbb{Z}^2$ next.

3.2.2 Two-Dimensional D -Multimodularity

Note that a function f on $\mathbb{Z}^n$ is multimodular with respect to $\mathcal{F}$ if and only if f is multimodular with respect to $\bar{\mathcal{F}} = \{-u : u \in \mathcal{F}\}$ (Altman et al., 2003). Therefore there are only two possible multimodular bases for mesh $\mathbb{Z}^2$, namely, $\mathcal{F}^s = \{-e_1, e_1 - e_2, e_2\}$ and $\mathcal{F}^c = \{e_1, -e_1 - e_2, e_2\}$ with the totally unimodular multimodular matrix

$$D^s = \begin{pmatrix} -e_1 \\ e_1 - e_2 \\ e_2 \end{pmatrix}, \quad D^c = \begin{pmatrix} e_1 \\ -e_1 - e_2 \\ e_2 \end{pmatrix}$$

respectively, where $e_1 \equiv (1, 0)$ and $e_2 \equiv (0, 1)$. The meanings of the superscripts will become clear later. There are 6 ($=3!$) permutations of $(0, 1, 2)$, which are denoted by τ^i , $i = 1, \dots, 6$ and each permutation corresponds to a D -atom. The D -atom $\mathcal{A}(a, D, \tau^i)$ is a two-simplex in $\mathbb{Z}^2$ and there are six atoms for each multimodular triangulation. For ease of presentation, we denote the multimodular triangulation with respect to multimodular base $\mathcal{F}^s$ and $\mathcal{F}^c$ by $\mathcal{MT}(\mathcal{F}^s)$ and

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$\mathcal{MT}(\mathcal{F}^c)$ respectively and denote the D -atom on $\mathcal{MT}(\mathcal{F}^s)$ and $\mathcal{MT}(\mathcal{F}^c)$ by $\mathcal{A}(a, D^s, \tau^i)$ and $\mathcal{A}(a, D^c, \tau^i)$, $i=1, \dots, 6$ respectively. Multimodular triangulations and D -atoms in $\mathbb{Z}^2$ are illustrated in Figure 3.1.

$$\tau^1 = (0, 1, 2), \tau^2 = (0, 2, 1), \tau^3 = (2, 0, 1), \tau^4 = (2, 1, 0), \tau^5 = (1, 2, 0), \tau^6 = (1, 0, 2)$$

Figure 3.1 D -atoms $\mathcal{A}(a, D, \tau^i)$, $i=1, \dots, 6$ of multimodular triangulations in $\mathbb{Z}^2$

The following two propositions identify the necessary and sufficient conditions for D -multimodularity with respect to the multimodular matrix D^s and D^c respectively. It is worthwhile to highlight the main results first since the seemingly new concept of D -multimodularity is actually equivalent to some well-known properties. We show that D -multimodularity with respect to D^s is equivalent to supermodularity and superconvexity; D -multimodularity with respect to D^c is equivalent to submodularity and subconvexity.

Proposition 3.1 Let $\mathcal{F}^s = \{-e_1, e_1 - e_2, e_2\}$ be the multimodular base of mesh $\mathbb{Z}^2$ with the multimodular matrix D^s . A real-valued function $f: \mathbb{Z}^2 \rightarrow \mathbb{R}$ is D -multimodular with respect to

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D^s if and only if $f(x) \in \mathcal{V}^s$, where $\mathcal{V}^s$ is the class of real-valued functions with the following properties: for $i, j=1, 2$ and $i \neq j$,

$$\Delta_i \Delta_j f(x) \geq 0, \quad (3.2)$$

$$\Delta_i f(x) \geq \Delta_i f(x - e_i + e_j), \quad (3.3)$$

where $\Delta_i f(x) = f(x) - f(x - e_i)$, $x \in \mathbb{Z}^2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Proof. The multimodular matrix D^s is given by

$$D^s = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix}$$

with $s_0 + s_1 + s_2 = (0, 0)$. From Definition 3.8 it can be shown that for all $0 \leq i < j \leq 2$, (3.1) is equivalent to (3.2) and (3.3) as follows:

Case $(i, j) = (0, 1)$: $f(x_1 - 1, x_2) + f(x_1 + 1, x_2 - 1) \geq f(x_1, x_2) + f(x_1, x_2 - 1)$,

$$\Leftrightarrow \Delta_1 f(x_1, x_2) \geq \Delta_1 f(x_1 - 1, x_2 + 1),$$

Case $(i, j) = (1, 2)$: $f(x_1 + 1, x_2 - 1) + f(x_1, x_2 + 1) \geq f(x_1, x_2) + f(x_1 + 1, x_2)$,

$$\Leftrightarrow \Delta_2 f(x_1, x_2) \geq \Delta_2 f(x_1 + 1, x_2 - 1),$$

Case $(i, j) = (0, 2)$: $f(x_1 - 1, x_2) + f(x_1, x_2 + 1) \geq f(x_1, x_2) + f(x_1 - 1, x_2 + 1)$,

$$\Leftrightarrow \Delta_1 \Delta_2 f(x) \geq 0.$$

Thus (3.2) and (3.3) are necessary and sufficient conditions for D -multimodularity with respect to D^s . ■

By Proposition 3.1, we know that *supermodularity* and *superconvexity* are necessary and sufficient conditions for D -multimodularity with respect to D^s . Hence *submodularity* and

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subconcavity are necessary and sufficient conditions for D -multimodularity with respect to D^s in the concavity context, which is summarized in the following corollary.

Corollary 3.1 A real-valued function $f: \mathbb{Z}^2 \rightarrow \mathbb{R}$ is D -multimodular with respect to D^s in the concavity context if and only if $f(x) \in \mathcal{U}^s$, where $\mathcal{U}^s$ is the class of real-valued functions with the following properties: for $i, j = 1, 2$ and $i \neq j$,

$$\Delta_i \Delta_j f(x) \leq 0,$$

$$\Delta_i f(x) \leq \Delta_i f(x - e_i + e_j),$$

where $\Delta_i f(x) = f(x) - f(x - e_i)$, $x \in \mathbb{Z}^2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Note that property (3.3) is equivalent to

$$\Delta_i \Delta_i f(x) \geq \Delta_i \Delta_j f(x) \tag{3.4}$$

which is called as “diagonal dominance” in Ha (1997a). Lemma 2(c) of Altman et al. (2003) shows that (3.2) and (3.4) are sufficient conditions for two-dimensional multimodularity. Proposition 3.1 has extended their results and shows that (3.2) and (3.4) are both necessary and sufficient conditions for D -multimodularity with respect to D^s .

The multimodular base $\mathcal{F}^s = \{-e_1, e_1 - e_2, e_2\}$ has been extensively studied in the literature, especially in the context of queueing transfer framework where e_1 represents the event of a customer entering into queue-1, $-e_1 + e_2$ the event of transferring a customer from queue-1 to queue-2, and $-e_2$ the event of a customer leaving from queue-2. However, the multimodular base $\mathcal{F}^c = \{e_1, -e_1 - e_2, e_2\}$ has not been much explored in the literature. The following proposition identifies the necessary and sufficient condition for D -multimodularity with respect to D^c .

Proposition 3.2 Let $\mathcal{F}^c = \{e_1, -e_1 - e_2, e_2\}$ be the multimodular base of mesh $\mathbb{Z}^2$ with the multimodular matrix D^c . A real-valued function $f: \mathbb{Z}^2 \rightarrow \mathbb{R}$ is D -multimodular with respect to

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D^c if and only if $f(x) \in \mathcal{V}^c$, where $\mathcal{V}^c$ is the class of real-valued functions with the following properties: for $i, j = 1, 2$ and $i \neq j$,

$$\Delta_i \Delta_j f(x) \leq 0, \quad (3.5)$$

$$\Delta_i f(x) \geq \Delta_i f(x - e_i - e_j). \quad (3.6)$$

Proof. The multimodular matrix D^c is given by

$$D^c = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix}$$

with $s_0 + s_1 + s_2 = (0, 0)$. By Definition 3.8 it can be shown that for all $0 \leq i < j \leq 2$, (3.1) is equivalent to (3.5) and (3.6) as follows:

$$\text{Case } (i, j) = (0, 1): f(x_1 + 1, x_2) + f(x_1 - 1, x_2 - 1) \geq f(x_1, x_2) + f(x_1, x_2 - 1),$$

$$\Leftrightarrow \Delta_1 f(x_1, x_2) \geq \Delta_1 f(x_1 - 1, x_2 - 1),$$

$$\text{Case } (i, j) = (1, 2): f(x_1 - 1, x_2 - 1) + f(x_1, x_2 + 1) \geq f(x_1, x_2) + f(x_1 - 1, x_2),$$

$$\Leftrightarrow \Delta_2 f(x_1, x_2) \geq \Delta_2 f(x_1 - 1, x_2 - 1),$$

$$\text{Case } (i, j) = (0, 2): f(x_1 + 1, x_2) + f(x_1, x_2 + 1) \geq f(x_1, x_2) + f(x_1 + 1, x_2 + 1),$$

$$\Leftrightarrow \Delta_1 \Delta_2 f(x) \leq 0.$$

Thus (3.5) and (3.6) are necessary and sufficient conditions for D -multimodularity with respect to D^c . ■

By Proposition 3.2, we know that *submodularity* and *subconvexity* are necessary and sufficient conditions for D -multimodularity with respect to D^c . Hence *supermodularity* and *superconcavity* are necessary and sufficient conditions for D -multimodularity with respect to D^c in the concavity context, which is summarized in the following corollary.

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Corollary 3.2 A real-valued function $f: \mathbb{Z}^2 \rightarrow \mathbb{R}$ is D -multimodular with respect to D^c in the concavity context if and only if $f(x) \in \mathcal{U}^c$, where $\mathcal{U}^c$ is the class of real-valued functions with the following properties: for $i, j=1, 2$ and $i \neq j$,

$$\Delta_i \Delta_j f(x) \geq 0,$$

$$\Delta_i f(x) \leq \Delta_i f(x - e_i - e_j),$$

where $\Delta_i f(x) = f(x) - f(x - e_i)$, $x \in \mathbb{Z}^2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Note that properties (3.5) and (3.6) have appeared in the literature, for example, Ha (1997b), You (1999), Carr and Duenyas (2000), Morton (2006), without making the connection with the D -multimodularity concept. Table 3.1 summarizes necessary and sufficient conditions for two-dimensional D -multimodularity with respect to D^s and D^c in the convexity and concavity context respectively.

Table 3.1 Necessary and sufficient conditions for two-dimensional D -multimodularity with respect to D^s and D^c in the convexity and concavity context

D -multimodularity w.r.t D^s and D^c in the convexity context	D -multimodularity w.r.t D^s and D^c in the concavity context
$f(x) \in \mathcal{V}^s$ Supermodular: $\Delta_i \Delta_j f(x) \geq 0$ Superconvex: $\Delta_i f(x) \geq \Delta_i f(x - e_i + e_j)$	$f(x) \in \mathcal{U}^s$ Submodular: $\Delta_i \Delta_j f(x) \leq 0$ Subconcave: $\Delta_i f(x) \leq \Delta_i f(x - e_i + e_j)$
$f(x) \in \mathcal{V}^c$ Submodular: $\Delta_i \Delta_j f(x) \leq 0$ Subconvex: $\Delta_i f(x) \geq \Delta_i f(x - e_i - e_j)$	$f(x) \in \mathcal{U}^c$ Supermodular: $\Delta_i \Delta_j f(x) \geq 0$ Superconcave: $\Delta_i f(x) \leq \Delta_i f(x - e_i - e_j)$

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Figure 3.2 Necessary and sufficient conditions for two-dimensional D -multimodularity

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Let us have a further graphical investigation of two-dimensional D -multimodularity on atoms of the multimodular triangulation which is summarized in Figure 3.2. It is evident that D -multimodularity requires supermodularity (submodularity) on the square and superconvexity (subconvexity) on the parallelogram.

For ease of presentation and visualization, let $\widehat{V_i V_j}$ represent the difference of the value function at vertex V_i and V_j , i.e., $\widehat{V_i V_j} = f(V_i) - f(V_j)$. It is evident from Figure 3.2(a) that $\widehat{V_4 V_0} \geq \widehat{V_3 V_2}$ (superconvexity) and $\widehat{V_3 V_2} \geq \widehat{V_0 V_1}$ (supermodularity) imply $\widehat{V_4 V_0} \geq \widehat{V_0 V_1}$ (componentwise convexity), which is the convexity over the horizontal and vertical direction. Similarly, $\widehat{V_5 V_0} \geq \widehat{V_6 V_1}$ (superconvexity) and $\widehat{V_6 V_1} \geq \widehat{V_0 V_2}$ (superconvexity) imply $\widehat{V_5 V_0} \geq \widehat{V_0 V_2}$ (integer convexity), which is the convexity over the diagonal direction. Same results can be obtained from Figure 3.2(b) for D -multimodularity with respect to D^c . These properties are illustrated in Figure 3.3 and summarized in the subsequent corollary.

Figure 3.3 D -Multimodularity implies componentwise convexity and integer convexity

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Corollary 3.3 If the real-valued function $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is D -multimodular then f is componentwise convex and integer convex.

However the converse of the above corollary is not true, that is, componentwise convex and integer convex are insufficient for D -multimodularity. The relationships among these important concepts are illustrated in Figure 3.4.

Figure 3.4 Relationships among important concepts

An interesting finding is that directional supermodularity and antidirectional submodularity in two-dimension defined by Morton (2006) coincide with D -multimodularity with respect to D^s and D^c respectively; directional submodularity and antidirectional supermodularity coincide with D -multimodularity with respect to D^s and D^c in the concavity context respectively. We have unified those concepts under D -multimodularity. However for higher dimensions, it can be shown that directional modularity in Morton (2006) is not equivalent to D -multimodularity.

Note that D -multimodularity with respect to D^s and D^c implies directional convexity and antidirectional convexity respectively (Chang and Yao, 1993; Chang et al., 1994). However, the weaker condition of directional (antidirectional) convexity is insufficient for monotone optimal control for the full control, which will become clear in Section 3.4 and Section 3.5.

Having discussed the necessary and sufficient conditions for two-dimensional D -multimodularity and its associated properties, we now turn our attention to structural properties of

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different systems and how D -multimodularity can be utilized to characterize monotone optimal control policies for discrete-event stochastic systems. Careful readers will notice a clear pattern from the multimodular bases $\mathcal{F}^s = \{-e_1, e_1 - e_2, e_2\}$ and $\mathcal{F}^c = \{e_1, -e_1 - e_2, e_2\}$. Each vector in the multimodular base represents a particular event or control, and the second vector determines the essential property of the multimodular base. Note that in the queueing transfer context, $-e_1 + e_2$ (or $e_1 - e_2$) corresponds to the event of shifting a customer from one queue to another and it indicates “*either-or*” nature in the sense that we can put a customer in either queue 1 or queue 2; $e_1 + e_2$ (or $-e_1 - e_2$) implies the event of requiring both resources and it indicates “*both-and*” nature. This intrinsic feature will enable us to classify a system according to the possible transition set associated with the multimodular base.

The next section will give a general MDP formulation and propose a useful mechanism to classify different systems.

3.3 A General MDP Formulation and System Classification

Mechanism

We consider a particular class of discrete-event stochastic systems that can be modeled by controlled Markov chains. This class of systems has broad applications in many fields. We first state the general MDP formulation and then propose a system classification mechanism. Through the mechanism, a system of this class can be classified into either a substitutable-type or a complementary-type according to the underlying transition set, which is associated with atoms of multimodular triangulations $\mathcal{MT}(\mathcal{F}^s)$ and $\mathcal{MT}(\mathcal{F}^c)$ respectively.

For a discrete-event stochastic system, the state space is discrete and the state transition is event-driven, and it can be modeled in either continuous time or discrete time. We may limit our treatment to discrete-time MDP formulation in the minimization context. For a comprehensive

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treatment of discrete-time MDP, please refer to Puterman (2005). Note that techniques for deriving the average-cost problem from the discount-cost problem are well developed (Stidham and Weber, 1993), and there exists a stationary optimal policy for discounted MDP under reasonable assumptions (Puterman, 2005). We limit our discussion to the discounted-cost criteria and stationary optimal policies.

A general Markov decision process can be represented by $\{\mathcal{T}, \mathcal{S}, \mathcal{A}, c_t(x, u), p_t(\cdot | x, u)\}$, where $\mathcal{T}$ is the set of decision epochs, $\mathcal{S}$ is the state space and $\mathcal{A}$ is the action space that can be either a discrete set or a continuous set, which corresponds to discrete-action control and continuous-action control respectively. At a specific point of time t , the decision maker observes the state of the system $x \in \mathcal{S}$ and chooses an action $u \in \mathcal{A}$. As a result, a cost $c_t(x, u)$ incurs and the system evolves to a new state according to a transition probability $p_t(\cdot | x, u)$.

For an n -period MDP problem, $\pi = \{u_0^*, u_1^*, \dots, u_{n-1}^*\}$ is chosen to minimize the total expected discounted cost

$$V_0^\pi(x_0) = E^\pi \left[\sum_{t=0}^{n-1} \alpha^t \cdot c(X_t, u_t) + \alpha^n \cdot c(X_n) \mid X_0 = x_0 \right] \quad (3.7)$$

and for an infinite-horizon MDP problem, $\pi = \{u_0^*, u_1^*, \dots\}$ is chosen to minimize

$$V^\pi(x_0) = E^\pi \left[\sum_{t=0}^{\infty} \alpha^t \cdot c(X_t, u_t) \mid X_0 = x_0 \right] \quad (3.8)$$

The corresponding optimality equations for (3.9) and (3.10) are given by

$$V_t(x_t) = \min_{u \in \mathcal{A}} \{c(x_t, u) + \alpha \sum_{x_{t+1} \in \mathcal{S}} p(x_{t+1} | x_t, u) V_{t+1}(x_{t+1})\} \quad (3.9)$$

and

$$V(x) = \min_{u \in \mathcal{A}} \{c(x, u) + \alpha \sum_{\hat{x} \in \mathcal{S}} p(\hat{x} | x, u) V(\hat{x})\} \quad (3.10)$$

respectively.

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We now study a class of two-dimensional discrete-event stochastic systems that can be modeled by controlled Markov chains with each possible transition bounded by the unit vector. Note that a system can be controlled through transition probabilities in order to influence the state evolution of the underlying Markov chain and to achieve the optimal system performance. Define the controlled state transition operator by $T: X \rightarrow \mathbb{Z}^2$ where $X \subseteq \mathbb{Z}^2$. Let $Tx = x + \delta$ with $\delta = (\delta_1, \delta_2) \in \mathbb{Z}^2$ such that $|\delta_i| \leq 1$ for $i = 1$ and 2 . This operator can be used to model many different event controls. For example, $\delta = (0,0)$ represents rejection or stay-put control; $\delta = (1,0)$ and $\delta = (0,1)$ represent arrival/admission control on two resources; $\delta = (-1,0)$ and $\delta = (0,-1)$ represent departure/service control; $\delta = (-1,1)$ and $\delta = (1,-1)$ represent routing (rationing / transferring / switching) control; $\delta = (1,1)$ and $\delta = (-1,-1)$ represent simultaneous arrival/through control. Therefore this class of two-dimensional discrete-event stochastic systems has vast applications in queueing, manufacturing and production, revenue management, among others.

According to the possible transition set, we propose a mechanism to classify a system of this class into either a substitutable-type or a complementary-type. Substitutable-type systems indicate “either-or” nature of the possible transition control while complementary-type systems indicate “both-and” nature of the possible transition control. Denote the set of possible transitions for multimodular triangulation $\mathcal{MT}(\mathcal{F}^s)$ and $\mathcal{MT}(\mathcal{F}^c)$ by

$$\begin{aligned} \mathcal{T}^s &= \left\{ \{(0,0),(0,-1),(-1,0)\}, \{(0,0),(-1,0),(-1,1)\}, \{(0,0),(-1,1),(0,1)\}, \right. \\ &\quad \left. \{(0,0),(0,1),(1,0)\}, \{(0,0),(1,0),(1,-1)\}, \{(0,0),(1,-1),(0,-1)\} \right\} \\ &\equiv \{\mathcal{T}_1^s, \mathcal{T}_2^s, \mathcal{T}_3^s, \mathcal{T}_4^s, \mathcal{T}_5^s, \mathcal{T}_6^s\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}^c &= \left\{ \{(0,0),(0,-1),(1,0)\}, \{(0,0),(1,0),(1,1)\}, \{(0,0),(1,1),(0,1)\}, \right. \\ &\quad \left. \{(0,0),(0,1),(-1,0)\}, \{(0,0),(-1,0),(-1,-1)\}, \{(0,0),(-1,-1),(0,-1)\}, \right\} \\ &\equiv \{\mathcal{T}_1^c, \mathcal{T}_2^c, \mathcal{T}_3^c, \mathcal{T}_4^c, \mathcal{T}_5^c, \mathcal{T}_6^c\}, \end{aligned}$$

respectively as illustrated by Figure 3.5.

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Figure 3.5 Possible transitions on atoms of multimodular triangulation $\mathcal{MT}(\mathcal{F}^s)$ and $\mathcal{MT}(\mathcal{F}^c)$

Note that each element $\mathcal{T}_i^s$ of $\mathcal{T}^s$ is a possible transition set on the atom $\mathcal{A}(0, D^s, \tau^i)$, $i = 1, \dots, 6$ of $\mathcal{MT}(\mathcal{F}^s)$; similarly each element $\mathcal{T}_i^c$ of $\mathcal{T}^c$ is a possible transition set on the atom $\mathcal{A}(0, D^c, \tau^i)$, $i = 1, \dots, 6$ of $\mathcal{MT}(\mathcal{F}^c)$. We now give the formal definition of our classification mechanism for the class of systems under consideration as follows.

Definition 3.9 Given the possible transition set $\mathcal{T} = \{\mathcal{T}^1, \dots, \mathcal{T}^k\}$ of a system with k types of control, where element $\mathcal{T}^j$ is a possible transition set for j^{th} control, $j = 1, \dots, k$. It is called a *substitutable-type* system if for $\forall 1 \leq j \leq k$ there exists $\mathcal{T}_i^s$, $i = 1, \dots, 6$ such that $\mathcal{T}^j \subseteq \mathcal{T}_i^s$; it is called a *complementary-type* system if for $\forall 1 \leq j \leq k$ there exists $\mathcal{T}_i^c$, $i = 1, \dots, 6$ such that $\mathcal{T}^j \subseteq \mathcal{T}_i^c$.

According to the above definition, a system of this class can be classified into either a substitutable-type or a complementary-type through the possible transition set of its underlying MDP. Note that Morton (2006) has also defined a possible transition set for the substitutable case

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and the complementary case by using seven points similar to our nodes in the corresponding multimodular triangulation. However, the possible transition set defined by Morton (2006) is insufficient for system classification. To illustrate the usefulness of this new classification mechanism, we have reviewed a small set of representative papers in the literature. The findings are reported in Table 3.2.

Table 3.2 Classification examples of a class of two-dimensional discrete-event stochastic systems

Literature	System	Decision	Action	Possible Transition Set	System Classification
Hajek (IEEE,1984)	Queueing sys.	Admission, service, transfer	Discrete	$\mathcal{T}^1 = \{(1,0), (0,1)\} \subseteq \mathcal{T}_4^s$ $\mathcal{T}^2 = \{(-1,0), (0,-1)\} \subseteq \mathcal{T}_1^s$ $\mathcal{T}^3 = \{(0,0), (-1,1)\} \subseteq \mathcal{T}_2^s$ $\mathcal{T}^4 = \{(0,0), (1,-1)\} \subseteq \mathcal{T}_6^s$	Substitutable -Type
Veatch & Wein (OR,1994)	Two-station tandem production/inv. sys.	Production	Discrete	$\mathcal{T}^1 = \{(0,0), (1,0)\} \subseteq \mathcal{T}_4^s$ $\mathcal{T}^2 = \{(0,0), (-1,1)\} \subseteq \mathcal{T}_2^s$	Substitutable -Type
Ha (OR,1997a)	Make-to-stock production system	Production ($\mu_1 = \mu_2$)	Discrete	$\mathcal{T} = \{(0,0), (1,0), (0,1)\} \subseteq \mathcal{T}_4^s$	Substitutable -Type
Zhao et al. (OR, 2008)	Make-to-stock Production system	Production and inventory transshipment	Discrete	$\mathcal{T}^1 = \{(0,0), (1,0), (0,1)\} \subseteq \mathcal{T}_4^s$ $\mathcal{T}^2 = \{(0,0), (0,1), (1,0)\} \subseteq \mathcal{T}_4^s$ $\mathcal{T}^3 = \{(-1,0), (0,-1)\} \subseteq \mathcal{T}_1^s$ $\mathcal{T}^4 = \{(0,-1), (-1,0)\} \subseteq \mathcal{T}_1^s$	Substitutable -Type
Green et al. (OR,2006)	Queueing sys. with same service rate	Capacity allocation	Discrete	$\mathcal{T} = \{(-1,0), (0,-1)\} \subseteq \mathcal{T}_1^s$	Substitutable -Type
Ha (NRL,1997b)	Make-to-stock production system	Production, Stock rationing	Discrete	$\mathcal{T}^1 = \{(0,0), (1,0), (0,-1)\} \subseteq \mathcal{T}_1^c$ $\mathcal{T}^2 = \{(0,0), (1,0)\} \subseteq \mathcal{T}_1^c$ $\mathcal{T}^3 = \{(-1,0), (0,1)\} \subseteq \mathcal{T}_4^c$	Complementary -Type
You (TS,1999)	Revenue management sys.	Pricing	Continuous	$\mathcal{T}^1 = \{(0,0), (-1,0)\} \subseteq \mathcal{T}_4^c$ $\mathcal{T}^2 = \{(0,0), (0,-1)\} \subseteq \mathcal{T}_1^c$ $\mathcal{T}^3 = \{(0,0), (-1,-1)\} \subseteq \mathcal{T}_6^c$	Complementary -Type
Carr & Duenyas (OR,2000)	Make-to-stock Make-to-order production system	Admission Production	Discrete	$\mathcal{T}^1 = \{(0,0), (0,1)\} \subseteq \mathcal{T}_1^c$ $\mathcal{T}^2 = \{(1,0), (0,-1)\} \subseteq \mathcal{T}_1^c$	Complementary -Type

Substitutable-type systems are generally related to resource sharing or queueing transferring. Hajek (1984) studies two interacting transferable service station with two independent arrivals and one common stream of arrivals controlled through routing. The common

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stream of arrivals can be routed to either server and the queueing transfer indicates “either-or” nature of the system. Veatch and Wein (1994) study optimal production control of two-station tandem queue and production at station 2 will transfer one inventory from station 1 to station 2, a typical queueing transfer system. Ha (1997a) studies the dynamic production scheduling of a single-server make-to-stock queue (M/M/1) with two products, a typical resource sharing system where production capacity is shared by two products. Green et al. (2006) study dynamic capacity allocation problem for one medical facility among three customer segments with one absolute priority class. Zhao et al. (2008) consider a distribution network of two-location, make-to-stock production systems with backorders and transshipment. By checking the possible transition sets of the underlying controlled Markov chains we conclude that these systems are of substitutable-type.

Complementary-type systems are generally related to resource complementary. Ha (1997b) consider the stock and production rationing in a make-to-order production system (M/M/1) with one product and two-class customers. Demands from class 1 customers can be backordered only when there is no inventory on-hand; demands from class 2 customers will be rationed and can be backorder if there is inventory on-hand. The production rationing decisions are: (a) idle, (b) produce one unit to be added to inventory, and (c) to produce one unit to satisfy the backorder of class 2 demand. You (1999) studies dynamic pricing for a multiple flight leg problem and characterizes structural properties for monotone prices for two-leg problem which faces demand for leg 1, demand for leg 2, demand for both legs. Morton (2006) reexamines the same two-leg problem in You (1999) under admission control scheme and shows its structural properties. Carr and Duenyas (2000) study the admission control and production scheduling for a make-to-stock and make-to-order production system (M/M/1) with one product and two-class demands. Class 1 demand is by on-contract which can not be denied and is make-to-stock; class 2 demand is from spot market and is make-to-order. The admission control is to ration the class 2 demand and the production scheduling decisions are whether to produce one unit to be added to inventory or to

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produce one unit to satisfy class 2 demand (decrease one unit of product in process of class 2).

These problems can be classified as complementary-type according to the possible transition sets.

3.4 Propagation of D -Multimodularity of the Control Operators on Atoms

The standard approach for stochastic dynamic control problems is to identify structural properties that produce structural forms of optimal policies and show that the structural properties propagate through dynamic programming recursions. The intrinsic differences in “either-or” and “both-and” nature of systems enable us to classify a system of the class into either a substitutable-type or a complementary-type according to the possible transition set which is associated with the multimodular triangulation $\mathcal{MT}(\mathcal{F}^s)$ and $\mathcal{MT}(\mathcal{F}^c)$ respectively. The next task is to show that the corresponding D -multimodularity is preserved over the control operators on the corresponding atoms. We first give the following definition of full control and partial control to classify the control operators.

Definition 3.10 A control over the full possible transition set on an atom is called a *full control*; a control over a subset of the full possible transition set on an atom is called a *partial control*.

Denote the full control operator on the atoms $\mathcal{A}(x, D^s, \tau^i)$ and $\mathcal{A}(x, D^c, \tau^i)$ by $T_i^s f(x_1, x_2)$ and $T_i^c f(x_1, x_2)$, $i = 1, \dots, 6$ respectively, where $x = (x_1, x_2)$ is the root of the atom.

For example,

$$T_1^s f(x_1, x_2) = \min \{f(x_1, x_2), f(x_1 - 1, x_2), f(x_1, x_2 - 1)\},$$

$$T_4^s f(x_1, x_2) = \min \{f(x_1, x_2), f(x_1 + 1, x_2), f(x_1, x_2 + 1)\},$$

$$T_1^c f(x_1, x_2) = \min \{f(x_1, x_2), f(x_1 + 1, x_2), f(x_1, x_2 - 1)\},$$

$$T_4^c f(x_1, x_2) = \min \{f(x_1, x_2), f(x_1 - 1, x_2), f(x_1, x_2 + 1)\}.$$

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Denote the partial control operator on the atom $\mathcal{A}(x, D^s, \tau^i)$ and $\mathcal{A}(x, D^c, \tau^i)$ by $\tilde{T}_{i,j}^s f(x_1, x_2)$ and $\tilde{T}_{i,j}^c f(x_1, x_2)$, $i = 1, \dots, 6$; $j = 1, 2, 3$ respectively. There are three partial control operators on each atom. For consistence, let $j = 1$ represent the control over two horizontal nodes; $j = 2$ represent the control over two vertical nodes and $j = 3$ represent the control over two diagonal nodes. For example, three partial control operators on the atom $\mathcal{A}(x, D^s, \tau^1)$ are given by

$$\begin{aligned}\tilde{T}_{1,1}^s f(x_1, x_2) &= \min\{f(x_1, x_2), f(x_1 - 1, x_2)\}, \\ \tilde{T}_{1,2}^s f(x_1, x_2) &= \min\{f(x_1, x_2), f(x_1, x_2 - 1)\}, \\ \tilde{T}_{1,3}^s f(x_1, x_2) &= \min\{f(x_1 - 1, x_2), f(x_1, x_2 - 1)\},\end{aligned}$$

respectively; and three partial control operators on the atom $\mathcal{A}(x, D^c, \tau^1)$ are given by

$$\begin{aligned}\tilde{T}_{1,1}^c f(x_1, x_2) &= \min\{f(x_1, x_2), f(x_1 + 1, x_2)\}, \\ \tilde{T}_{1,2}^c f(x_1, x_2) &= \min\{f(x_1, x_2), f(x_1, x_2 - 1)\}, \\ \tilde{T}_{1,3}^c f(x_1, x_2) &= \min\{f(x_1 + 1, x_2), f(x_1, x_2 - 1)\},\end{aligned}$$

respectively.

There are two well-known approaches in the literature to show the structural properties of the value function. One approach is to follow Topkis's theory of ordered optimal solutions (Topkis (1978), Theorem 4.3), which has been applied in many circumstances, for instance, Glasserman and Yao (1994a, Lemma 4.2), Ha (1997a&b, Lemma 2), Carr and Duenyas (2000, Theorem 1), and Zhao et al. (2008, Theorem 1). Another approach is to apply case-by-case analysis, for example, Hajek (1984, Lemma 3.2) and a recent application from Green et al. (2006, Proposition 1). The following lemma paves the way for us to develop an efficient alternative method.

Lemma 3.3 Let $\{f_i, g_i, \tilde{f}_i, \tilde{g}_i : i = 1, \dots, m\}$ be four arbitrary real sequences. Assume that, for any i and j , there exist i_j and $j_j \in \{1, \dots, m\}$ such that

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$$f_i - g_{i_i} \geq \tilde{f}_{j_j} - \tilde{g}_j \quad (3.11)$$

Then the following inequality holds,

$$\min\{f_1, \dots, f_m\} - \min\{g_1, \dots, g_m\} \geq \min\{\tilde{f}_1, \dots, \tilde{f}_m\} - \min\{\tilde{g}_1, \dots, \tilde{g}_m\} \quad (3.12)$$

Proof. Given any $i = 1, \dots, m$, note that by (3.11),

$$f_i - \min\{g_1, \dots, g_m\} \geq f_i - g_{i_i} \geq \tilde{f}_{j_j} - \tilde{g}_j \geq \min\{\tilde{f}_1, \dots, \tilde{f}_m\} - \tilde{g}_j, \forall j.$$

Hence it follows that

$$f_i - \min\{g_1, \dots, g_m\} \geq \min\{\tilde{f}_1, \dots, \tilde{f}_m\} - \min\{\tilde{g}_1, \dots, \tilde{g}_m\}, \text{ for all } i,$$

which immediately leads to the inequality (3.12). ■

It is important to highlight that examining structural properties of certain operator by using Lemma 3.3 will be much more efficient than the method of checking all possible combinations especially when m is large because the method in Lemma 3.3 requires m^2 combinations while the normal approach requires m^4 combinations. In addition, checking (3.11) is also easier than the method of ordered optimal solutions.

In the following we will use Lemma 3.3 to prove structural properties for the control operators. For ease of presentation, we have suppressed the cost/reward collected from each action taken. It is important to note that all the structural results remain valid when incorporating the corresponding one-period cost/reward. To facilitate the exploration of the propagations of structural properties of control operators on different atoms, we introduce the following lemma.

Lemma 3.4 $f(x_1, x_2) \in \mathcal{V}^s$ if and only if $f(-x_1, x_2) \in \mathcal{V}^c$ or $f(x_1, -x_2) \in \mathcal{V}^c$.

Lemma 3.4 follows immediately from Proposition 3.2 of Morton (2006) since $\mathcal{V}^s$ and $\mathcal{V}^c$ is equivalent to directional supermodularity and antidirectional submodularity in two-dimension respectively. It says that property $\mathcal{V}^s$ is changed to $\mathcal{V}^c$ through reflection transformation around x_1 or x_2 . Next we show that D -multimodularity with respect to D^s is preserved over the full control operator $T_1^s f(x_1, x_2)$ and the partial control operator $\tilde{T}_{1,1}^s f(x_1, x_2)$, $\tilde{T}_{1,2}^s f(x_1, x_2)$,

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$\tilde{T}_{1,3}^s f(x_1, x_2)$ in the following lemmas. The proofs of Lemma 3.5, 3.7, 3.8 and 3.9 are presented in appendix.

Lemma 3.5 If $f(x_1, x_2) \in \mathcal{V}^s$, then $T_1^s f(x_1, x_2) \in \mathcal{V}^s$.

Lemma 3.6 If $f(x_1, x_2) \in \mathcal{V}^s$, then $T_4^s f(x_1, x_2) \in \mathcal{V}^s$.

Proof. Let $f(x_1, x_2) = g(-x_1, -x_2)$ and note that $f(x_1, x_2) \in \mathcal{V}^s$ if and only if $g(x_1, x_2) \in \mathcal{V}^s$ by Lemma 3.4. Since $g(x_1, x_2) \in \mathcal{V}^s$ implies $T_1^s g(x_1, x_2) \in \mathcal{V}^s$ and $T_1^s g(-x_1, -x_2) \in \mathcal{V}^s$ by Lemma 3.5 and Lemma 3.4, and $T_4^s f(x_1, x_2) = T_1^s g(-x_1, -x_2)$, thus we have $T_4^s f(x_1, x_2) \in \mathcal{V}^s$. The result follows. ■

Lemma 3.7 If $f(x_1, x_2) \in \mathcal{V}^s$, then $\tilde{T}_{1,1}^s f(x_1, x_2) \in \mathcal{V}^s$ and $\tilde{T}_{1,2}^s f(x_1, x_2) \in \mathcal{V}^s$.

Lemma 3.8 If $f(x_1, x_2) \in \mathcal{V}^s$, then $\tilde{T}_{1,3}^s f(x_1, x_2) \in \mathcal{V}^s$.

By Lemma 3.5 to Lemma 3.8, it follows that D -multimodularity with respect to D^s will be propagated for both full control operators $T_i^s f(x_1, x_2)$ and partial control operators $\tilde{T}_{i,j}^s f(x_1, x_2)$, $i=1, \dots, 6$; $j=1, 2, 3$ on each atom of multimodular triangulation $\mathcal{MT}(\mathcal{F}^s)$ through shifting transformations. The following proposition summarizes these results.

Proposition 3.3 If $f(x_1, x_2) \in \mathcal{V}^s$, then $T_i^s f(x_1, x_2) \in \mathcal{V}^s$ and $\tilde{T}_{i,j}^s f(x_1, x_2) \in \mathcal{V}^s$, $i=1, \dots, 6$; $j=1, 2, 3$.

Proof. It is evident that

$$\begin{aligned} T_2^s f(x_1, x_2) &= T_4^s f(x_1 - 1, x_2), \quad T_6^s f(x_1, x_2) = T_4^s f(x_1, x_2 - 1), \\ T_5^s f(x_1, x_2) &= T_1^s f(x_1 + 1, x_2), \quad T_3^s f(x_1, x_2) = T_1^s f(x_1, x_2 + 1). \end{aligned}$$

Thus $T_i^s f(x_1, x_2) \in \mathcal{V}^s$, $i=1, \dots, 6$ by Lemma 3.5 and Lemma 3.6. Similarly,

$$\begin{aligned} \tilde{T}_{2,1}^s f(x_1, x_2) &= \tilde{T}_{1,1}^s f(x_1, x_2), \quad \tilde{T}_{3,1}^s f(x_1, x_2) = \tilde{T}_{1,1}^s f(x_1, x_2 + 1), \\ \tilde{T}_{4,1}^s f(x_1, x_2) &= \tilde{T}_{5,1}^s f(x_1, x_2) = \tilde{T}_{1,1}^s f(x_1 + 1, x_2), \quad \tilde{T}_{6,1}^s f(x_1, x_2) = \tilde{T}_{1,1}^s f(x_1 + 1, x_2 - 1); \end{aligned}$$

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$$\begin{aligned}\tilde{T}_{2,2}^s f(x_1, x_2) &= \tilde{T}_{1,2}^s f(x_1 - 1, x_2 + 1), \quad \tilde{T}_{3,2}^s f(x_1, x_2) = \tilde{T}_{4,2}^s f(x_1, x_2) = \tilde{T}_{1,2}^s f(x_1, x_2 + 1), \\ \tilde{T}_{5,2}^s f(x_1, x_2) &= \tilde{T}_{1,2}^s f(x_1 + 1, x_2), \quad \tilde{T}_{6,2}^s f(x_1, x_2) = \tilde{T}_{1,2}^s f(x_1, x_2); \\ \tilde{T}_{2,3}^s f(x_1, x_2) &= \tilde{T}_{3,3}^s f(x_1, x_2) = \tilde{T}_{1,3}^s f(x_1, x_2 + 1), \quad \tilde{T}_{4,3}^s f(x_1, x_2) = \tilde{T}_{1,3}^s f(x_1 + 1, x_2 + 1), \\ \tilde{T}_{5,3}^s f(x_1, x_2) &= \tilde{T}_{6,3}^s f(x_1, x_2) = \tilde{T}_{1,3}^s f(x_1 + 1, x_2).\end{aligned}$$

Thus $\tilde{T}_{i,j}^s f(x_1, x_2) \in \mathcal{V}^s$, $i = 1, \dots, 6; j = 1, 2, 3$ by Lemma 3.7 and Lemma 3.8. ■

Through reflection transformation of $f(x)$ around x_2 , we can show that D -multimodularity with respect to D^c is propagated for both full control operators $T_i^c f(x_1, x_2)$ and partial control operators $\tilde{T}_{i,j}^c f(x_1, x_2), i = 1, \dots, 6; j = 1, 2, 3$ on each atom of the multimodular triangulation $\mathcal{MT}(\mathcal{F}^c)$, which is summarized below.

Proposition 3.4 If $f(x_1, x_2) \in \mathcal{V}^c$, then $T_i^c f(x_1, x_2) \in \mathcal{V}^c$ and $\tilde{T}_{i,j}^c f(x_1, x_2) \in \mathcal{V}^c$, $i = 1, \dots, 6; j = 1, 2, 3$.

Proof. Let $f(x_1, x_2) = h(-x_1, x_2)$ and note that $f(x_1, x_2) \in \mathcal{V}^c$ if and only if $h(x_1, x_2) \in \mathcal{V}^s$ by Lemma 3.4. Since $h(x_1, x_2) \in \mathcal{V}^s$ implies $T_i^s h(x_1, x_2) \in \mathcal{V}^s$ and $T_i^s h(-x_1, x_2) \in \mathcal{V}^c$ by Proposition 3.2 and Lemma 3.4, and $T_i^c f(x_1, x_2) = T_i^s h(-x_1, x_2)$, thus we have $T_i^c f(x_1, x_2) \in \mathcal{V}^c$. The same argument goes with $\tilde{T}_{i,j}^c f(x_1, x_2)$ since $\tilde{T}_{i,j}^c f(x_1, x_2) = \tilde{T}_{i,j}^s h(-x_1, x_2)$. Hence the results follow. ■

It is evident from Figure 3.5 that there are five nodes that are common to both $\mathcal{MT}(\mathcal{F}^s)$ and $\mathcal{MT}(\mathcal{F}^c)$. We name these four partial controls with possible transition sets $\{(0,0), (1,0)\}$, $\{(0,0), (0,1)\}$, $\{(0,0), (-1,0)\}$, $\{(0,0), (0,-1)\}$ as a *neutral-type*, which can have D -multimodularity with respect to either D^s or D^c and the structural properties depend on the cost function and boundary conditions.

In summary we have shown that D -multimodularity with respect to D^s (D^c) is propagated through all full and partial control operators on each atom of multimodular triangulation $\mathcal{MT}(\mathcal{F}^s)$ ($\mathcal{MT}(\mathcal{F}^c)$) which is associated with substitutable-type

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(complementary-type) systems. Readers are reminded that each result in the convexity context has a parallel version in the concavity context.

Koole (2006) has defined the control operators in terms of environment operator, arrival operator, departure operator, and transfer operator and summarized structural properties that the operators can propagate. These operators in Koole (2006) that have been shown to propagate multimodularity under two-dimensional case correspond to our partial control operators, which are summarized in Table 3.3. It is interesting to highlight that there are no operators in Koole (2006) that correspond to our full control operators in the two-dimensional case.

Note that we have shown structural properties for optimal controls when the state space has no boundaries. Problems with boundaries can be handled by modifying the analysis. One way is following Glasserman and Yao (1994) to introduce penalties at the boundary through assigning infinite cost to infeasible states. Thus an optimal control for the unconstrained problem provides an optimal control for the original constrained problem. Another way is that a system with boundaries may be viewed as a partial control of the one with no boundaries as the transitions turned off at boundaries.

In this section, we have provided a unified and systematic way to identify structural properties for different systems and their propagation on control operators. These new results will save future efforts on ad hoc proofs and can be applied directly to analyze this class of systems. Therefore the research effort should be directed at characterizing optimal policies and developing easy-to-implement heuristic control policies.

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Table 3.3 Link between Koole's operators and partial control operator in our framework

Koole's Operator (2006)	Operator in our framework
<p>Arrival Operator</p> <p>1. controlled arrival $T_{CA(1)}f(x) = \min\{f(x) + c, f(x_1 + 1, x_2) + c'\}$ $T_{CA(2)}f(x) = \min\{f(x) + c, f(x_1, x_2 + 1) + c'\}$</p> <p>2. controlled simultaneous arrivals: $m = 2$ $T_{CAF}f(x) = \min\{f(x_1, x_2) + c, f(x_1 + 1, x_2 + 1)\}$</p> <p>3. routing: $m=2$ $T_Rf(x) = \min\{f(x_1 + 1, x_2), f(x_1, x_2 + 1)\}$</p>	<p>$\tilde{T}_{4,1}^s f(x_1, x_2), \tilde{T}_{5,1}^s f(x_1, x_2)$ $\tilde{T}_{3,2}^s f(x_1, x_2), \tilde{T}_{4,2}^s f(x_1, x_2)$ $\tilde{T}_{2,3}^c f(x_1, x_2), \tilde{T}_{3,3}^c f(x_1, x_2)$ $\tilde{T}_{4,3}^s f(x_1, x_2)$</p>
<p>Departure Operator</p> <p>1. controlled departure: $x_1 > 0, x_2 > 0$ $T_{CD(1)}f(x) = \min\{c(1) + f(x_1 - 1, x_2), c(0) + f(x_1, x_2)\}$ $T_{CD(2)}f(x) = \min\{c(1) + f(x_1, x_2 - 1), c(0) + f(x_1, x_2)\}$</p> <p>2. movable server: $x_i > 0, m = 2, \mu_1 = \mu_2 = \mu$ $T_{MS}f(x) = \min\{f(x_1 - 1, x_2), f(x_1, x_2 - 1)\} + (1 - \mu)f(x_1, x_2)$</p>	<p>$\tilde{T}_{1,1}^s f(x_1, x_2), \tilde{T}_{2,1}^s f(x_1, x_2)$ $\tilde{T}_{1,2}^s f(x_1, x_2), \tilde{T}_{6,2}^s f(x_1, x_2)$ $\tilde{T}_{1,3}^s f(x_1, x_2)$</p>
<p>Transfer Operator</p> <p>1. controlled transfer: $x_i > 0, m = 2$ $T_{CTD(1)}f(x) = \min\{c(0) + f(x), c(1) + f(x - e_1 + e_2)\},$ $T_{CTD(2)}f(x) = \min\{c(0) + f(x), c(1) + f(x + e_1 - e_2)\}$</p>	<p>$\tilde{T}_{2,3}^s f(x_1, x_2), \tilde{T}_{3,3}^s f(x_1, x_2)$ $\tilde{T}_{5,3}^s f(x_1, x_2), \tilde{T}_{6,3}^s f(x_1, x_2)$</p>

3.5 D -Multimodularity and Monotone Optimal Control Policies

One of the main issues in MDP approach is to establish the structural optimal policies. The optimality of structural policies is not only appealing to decision makers but also easy in implementation and efficient for computation (Puterman, 2005).

It is evident from Figure 3.2 that D -multimodularity requires that the difference of the value function over each direction of the basic atom be monotone on x_1 and x_2 , which is sufficient for the existence of monotone switching-curve policies. Next we will identify the general conditions to assure the existence of monotone optimal control policies (under discrete-action control or continuous-action control) for the class of two-dimensional discrete-event stochastic systems with each possible transition bounded by the unit vector.

For $d = s$ and c , let us first study the discrete-action case and demonstrate that D -multimodularity with respect to D^d is sufficient for monotone optimal switching-curve policies for the full control operator $T_1^d f(x_1, x_2)$.

Given $f(x_1, x_2) \in \mathcal{V}^s$, define the following switching-curves for the control operator $T_1^s f(x_1, x_2) = \min\{f(x_1, x_2), f(x_1 - 1, x_2), f(x_1, x_2 - 1)\}$:

$$S_1(x_1) = \max\{x_2 : \Delta_1 f(x_1, x_2) \leq 0, \Delta_2 f(x_1, x_2) \leq 0\},$$

$$S_2(x_1) = \max\{x_2 : f(x_1, x_2 - 1) - f(x_1 - 1, x_2) \geq 0\}.$$

It is evident that $S_1(x_1)$ is well-defined because $\Delta_2 f(x_1, x_2) \uparrow x_2$ and $\Delta_1 f(x_1, x_2) \uparrow x_2$; $S_2(x_1)$ is well-defined as $f(x_1, x_2 - 1) - f(x_1 - 1, x_2) \downarrow x_2$. Since $\Delta_1 f(x_1, x_2) \uparrow x_1$ and $\Delta_2 f(x_1, x_2) \uparrow x_1$, it follows that

$$\Delta_1 f(x_1 - 1, S_1(x_1)) \leq \Delta_1 f(x_1, S_1(x_1)) \leq 0, \quad \Delta_2 f(x_1 - 1, S_1(x_1)) \leq \Delta_2 f(x_1, S_1(x_1)) \leq 0.$$

By the definition of $S_1(x_1)$ we get that $S_1(x_1) \leq S_1(x_1 - 1)$, i.e., $S_1(x_1)$ is decreasing in x_1 .

Similarly, it can be shown that $S_2(x_1)$ is increasing in x_1 as $f(x_1, x_2 - 1) - f(x_1 - 1, x_2) \uparrow x_1$.

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Therefore the state space is separated into three different decision regions as illustrated in Figure 3.6(a).

Given $f(x_1, x_2) \in \mathcal{V}^c$, define the following switching-curves for the control operator $T_1^c f(x_1, x_2) = \min\{f(x_1, x_2), f(x_1 + 1, x_2), f(x_1, x_2 - 1)\}$:

$$S_1(x_1) = \max\{x_2 : \Delta_1 f(x_1 + 1, x_2) \geq 0, \Delta_2 f(x_1, x_2) \leq 0\},$$

$$S_2(x_2) = \max\{x_2 : f(x_1 + 1, x_2) - f(x_1, x_2 - 1) \leq 0\}.$$

$S_1(x_1)$ is well-defined because $\Delta_1 f(x_1 + 1, x_2) \downarrow x_2$ and $\Delta_2 f(x_1, x_2) \uparrow x_2$; and $S_2(x_1)$ is well-defined as $f(x_1 + 1, x_2) - f(x_1, x_2 - 1) \uparrow x_2$. It can be shown that $S_1(x_1)$ is increasing in x_1 because $\Delta_1 f(x_1 + 1, x_2) \uparrow x_1$ and $\Delta_2 f(x_1, x_2) \downarrow x_1$; similarly, $S_2(x_1)$ is decreasing in x_1 because $f(x_1 + 1, x_2) - f(x_1, x_2 - 1) \uparrow x_1$. Therefore the state space is separated into three different decision regions as illustrated in Figure 3.6(b).

Next we take the partial control operator $\tilde{T}_{1,1}^s f(x_1, x_2)$ and $\tilde{T}_{1,3}^s f(x_1, x_2)$ as examples to illustrate that D -multimodularity with respect to D^s is sufficient for monotone optimal switching-curve policies. Given $f(x_1, x_2) \in \mathcal{V}^s$, define the switching-curve for control operator

$$\tilde{T}_{1,1}^s f(x_1, x_2) = \min\{f(x_1, x_2), f(x_1 - 1, x_2)\} :$$

$$S_1(x_1) = \max\{x_2 : \Delta_1 f(x_1, x_2) \leq 0\}.$$

$S_1(x_1)$ is well-defined due to $\Delta_1 f(x_1, x_2) \uparrow x_2$ and $S_1(x_1)$ is decreasing in x_1 because $\Delta_1 f(x_1, x_2) \uparrow x_1$, thus the state space is separated into two decision regions. Given $f(x_1, x_2) \in \mathcal{V}^s$, define the switching-curve for control operator $\tilde{T}_{1,3}^s f(x_1, x_2) = \min\{f(x_1 - 1, x_2), f(x_1, x_2 - 1)\}$:

$$S_1(x_1) = \max\{x_2 : f(x_1 - 1, x_2) - f(x_1, x_2 - 1) \leq 0\}.$$

$S_1(x_1)$ is well-defined and is increasing in x_1 because $f(x_1 - 1, x_2) - f(x_1, x_2 - 1) \downarrow x_1 \uparrow x_2$. The monotone switching-curves of the partial control operator $\tilde{T}_{1,1}^s f(x_1, x_2)$ and $\tilde{T}_{1,1}^c f(x_1, x_2)$ are

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illustrated in Figure 3.6(c)&(d). Similarly, we can show the monotone switching-curve policy for other control operators.

Figure 3.6 An illustration of optimal monotone switching-curve policies for full control and partial control operators on the state space

From the above discussion, we can conclude that directional (antidirectional) convexity is sufficient for monotone optimal control for the partial control operator $\tilde{T}_{1,1}^s, \tilde{T}_{1,2}^s (\tilde{T}_{1,1}^c, \tilde{T}_{1,2}^c)$ but is insufficient for the partial control operator $\tilde{T}_{1,3}^s (\tilde{T}_{1,3}^c)$ and the full control operator.

To summarize the results, we denote the control operator by $H^d f(x_1, x_2)$ for $d = s$ and c which can be either full control operator $T_i^d f(x_1, x_2)$ or partial control operator $\tilde{T}_{i,j}^d f(x_1, x_2)$. Let

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$|\cdot|$ be the number of elements in a set, thus $|T_i^d f(x_1, x_2)| = 3$ and $|\tilde{T}_{i,j}^d f(x_1, x_2)| = 2$. Now we are ready to present the main results on discrete-action controls in the following proposition.

Proposition 3.5 For the class of two-dimensional discrete-event stochastic systems that can be modeled by controlled Markov chains with each possible transition bounded by the unit vector, D -multimodularity with respect to D^d (for $d = s$ and c) of the value function is sufficient for monotone optimal switching-curve policies for the corresponding type of systems, which can be characterized as follows:

- (a) There exists a series of switching-curves $S_i(x_k)$, $i = 1, \dots, |H^d f(x_1, x_2)| - 1$; $k = 1, 2$, separating the state space into $|H^d f(x_1, x_2)|$ regions, each of which corresponds to an action which leads to a possible transition of the control operator.
- (b) The switching-curve $S_i(x_k)$, $k = 1, 2$ is monotone on x_k .

Note that sufficient conditions in Proposition 3.5 only require the one-period cost function and the boundary function belong to the class of $\mathcal{V}^s$ or $\mathcal{V}^c$. Proposition 3.5 summarizes the monotone optimal switching-curve policies for the control operator $H^s f(x_1, x_2)$ and $H^c f(x_1, x_2)$ with the discrete-action control. For the continuous-action control, we have the following general conditions for monotone optimal control policies.

Proposition 3.6 Consider the class of two-dimensional discrete-event stochastic system that can be modeled by controlled Markov chains with each possible transition bounded by the unit vector. If the value function is submodular or supermodular with respect to the control variable u that triggers the state transition Tx and the marginal transition cost $\Delta_T f(x) = f(x) - f(Tx)$, then D -multimodularity with respect to D^d (for $d = s$ and c) of the value function is sufficient for the monotone optimal control u^* on the state variables for the corresponding type of systems.

Proposition 3.6 follows immediately from the following two facts: (a) submodularity or supermodularity of the value function on u and $\Delta_T f(x)$ guarantees that the optimal control u^* is

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monotone on $\Delta_T f(x)$; and (b) D -multimodularity is sufficient for the monotonicity of $\Delta_T f(x)$ over the state variables.

Veatch and Wein (1992) and Stidham and Weber (1993) have obtained similar results that submodularity or supermodularity in the direction of transition (which is equivalent to multimodularity on the state space, Koole (2006)) imply transition monotone or service rate monotone in the queue network which belongs to a substitutable-type system in our framework. In revenue management literature, Morton (2006) studies structural properties for the same two-flight-leg problem as that of You (1999), a complementary-type system, while through admission control (discrete-action control) instead of dynamic pricing (continuous-action control). As stated in Morton (2006), “results we prove here apply only to admission control... However, we can see no reason why the qualitative insights, does not apply with equal force to dynamic pricing models”. Our unified approach can address not only the monotone switching-curve policy but also general monotone optimal control policy, for both substitutable-type and complementary-type two-dimensional systems of this class.

In summary, under our unified framework, we can classify a system of this class according to the possible transition set of the underlying MDP model. We identify the general conditions for the existence of the monotone optimal control policy for different type of systems. Under our unified framework, structural properties and monotone optimal control policies follow readily. In the next section, we will demonstrate that this unified approach can be used to solve a new and fairly complex control problem at ease.

3.6 Joint Dynamic Pricing and Inventory Control for Two

Parallel Flights – An Application

Previous studies on dynamic control of discrete-event stochastic systems generally fall into two streams, either the discrete-action control with a transition operator $Tx = x + \delta$ such that $\delta = (\delta_1,$

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δ_2): $|\delta_i| \leq 1$ for $i = 1, 2$ (see examples in Table 3.2) or the continuous-action control which includes production/service rate control (Stidham and Weber, 1993), event rate control (Glasserman and Yao, 1994a), dynamic pricing control (You, 1999), among others. To the best of our knowledge, there is no literature studying joint types of control for two-dimensional discrete-event stochastic systems. In this section, we will solve a new application problem in revenue management.

Examining the literature in revenue management shows that inventory control and dynamic pricing are usually studied separately. Inventory control for single-resource problems has been extensively studied with structural policies identified (Lee and Hersh, 1993; Liang, 1999; Lautenbacher and Stidham, 1999; Brumelle and Walczak, 2003; etc.). Dynamic control for multiple resources (network revenue management) has been investigated through approximation (Talluri and van Ryzin, 1998, 1999; Bertsimas and Popescue, 2003; etc.). Dynamic pricing for the single-resource problem investigates the structure of price monotonicity (Gallego and van Ryzin, 1994; Bitran and Mondschein, 1997; Zhao and Zheng, 2000; etc.). Dynamic pricing for multiple resources is studied through heuristics (Gallego and van Ryzin, 1997; etc.) and structural properties are characterized only for some two-resource problems (You, 1999; Aydin and Ziya, 2008; etc.). For comprehensive reviews in revenue management, readers can refer to McGill and van Ryzin (1999), Elmaghraby and Keskinocak (2003), Bitran and Caldentey (2003), Talluri and van Ryzin (2004).

Integrating joint pricing and inventory control has been studied recently. However, the existing literatures of joint pricing and inventory control either study static control model (e.g., Bertsimas and de Boer, 2005a), deterministic model (e.g., Kuyumcu and Popescue, 2006), or joint dynamic pricing and inventory control for one common pool of resource (e.g., Feng and Xiao, 2006; Gans and Savin, 2007; Lin et. al., 2008). One-resource inventory system cannot demonstrate interactions among resources. However there is no literature addressing dynamic joint pricing and inventory control of two sources with monotone optimal control policies

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identified. Next we will demonstrate how our unified approach can be used to solve this new and challenging problem of joint dynamic pricing and inventory control for two-resource systems.

Consider that an airline operates two flights (e.g., a peak-time flight and an off-peak time flight on the same day) on the same origin-destination market with two general customer segments. Segment 1 consists of membership customers and segment 2 consists of public customers. There are different subclasses of membership customers, such as tour operators, cooperate clients, government clients, clients associated with a partner airline and internal travelers (Li, 2006). Special fares of different classes of membership customers are committed prior to the reservations and the availability of the seats for such clientele should be controlled. The airline adopts dynamic pricing for public customers. Together, this becomes a joint dynamic pricing and inventory control problem with two resources.

The booking horizon is separated into small intervals during which no more than one booking request arrives. Denote the arrival rate at time t of class- i membership customers by λ_t^i ($i = 1, \dots, m$) and that of public customers for flight- j by $\tilde{\lambda}_t^j$ ($j = 1, 2$). Based on the inventory level of two flights, the airline makes inventory control decision when facing a request from the class- i membership customer: reject, accept to flight-1, accept to flight-2; and makes pricing decision when facing a request from the public customer. Note that penalty cost and switch cost of membership customers can be adjusted through the fare rate (e.g. Savin et. al. 2005) so we simply denote the revenue of class- i membership customer for flight- j by R_{ij} which is negotiated. Assume that the public customers have a willingness-to-pay distribution with tail probability $\bar{F}_j(p)$ for flight- j . We adopt traditional dynamic pricing approach that the demand of the public customer only depends on the price of the flight requested. If the price of the flight the public customer quoted is above his/her reservation, the customer either is lost or joins the arrival process for the other flight. Admittedly a demand model with customer choice that depends on prices of both flights would be more reasonable; we decide not to consider customer choice here

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since it is already a very complicated model with two flights, two customer segments with multiple fare classes, and two types of controls. Nevertheless, incorporating customer choice under our setting will be an interesting question for further research.

Let $V_t(x_1, x_2)$ be the optimal revenue-to-go at time t with x_1 and x_2 seats booked in flight-1 and flight-2 respectively. The underlying MDP formulation is given by

$$V_t(x_1, x_2) = \sum_{i=1}^m \lambda_t^i G_i V_{t-1}(x_1, x_2) + \sum_{j=1}^2 \tilde{\lambda}_t^j \tilde{G}_j V_{t-1}(x_1, x_2) + (1 - \sum_{i=1}^m \lambda_t^i - \sum_{j=1}^2 \tilde{\lambda}_t^j) V_{t-1}(x_1, x_2),$$

where

$$G_i V_t(x_1, x_2) = \max \{V_t(x_1, x_2), V_t(x_1 + 1, x_2) + R_{i1}, V_t(x_1, x_2 + 1) + R_{i2}\}, i = 1, \dots, m,$$

$$\tilde{G}_j V_t(x_1, x_2) = \max_{r_j} \{\bar{F}_j(r_j)(V_t(x + e_j) + r_j) + F_j(r_j)V_t(x)\}, j = 1, 2,$$

with boundary condition

$$V_0(x_1, x_2) = -c_o [(x_1 - K_1)I(x_1 > K_1) + (x_2 - K_2)I(x_2 > K_2)],$$

where c_o is the overbooking cost adopted from Lautenbacher and Stidham (1999) and K_i is the capacity of flight- j for $j = 1$ and 2 .

The possible transition set of this problem is given by $\mathcal{T} = \{\mathcal{T}_1, \dots, \mathcal{T}_m, \tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2\}$ with

$$\begin{aligned} \mathcal{T}_1 &= \dots = \mathcal{T}_m = \{(0, 0), (1, 0), (0, 1)\} \subseteq \mathcal{T}_4^s, \\ \tilde{\mathcal{T}}_1 &= \{(0, 0), (1, 0)\} \subseteq \mathcal{T}_4^s, \\ \tilde{\mathcal{T}}_2 &= \{(0, 0), (0, 1)\} \subseteq \mathcal{T}_4^s. \end{aligned}$$

The system we are studying is a substitutable-type by Definition 3.9 which is intuitive in the sense that two parallel flights are substitutable resources. According to Proposition 3.3 and Proposition 3.5, structural properties of $\mathcal{U}^s$ are relevant to characterize the monotone optimal control policies, which are summarized in the following two propositions.

Proposition 3.7 $V_t(x_1, x_2) \in \mathcal{U}^s$.

Proof. See appendix. ■

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Proposition 3.8 The optimal monotone control policies are characterized as follows:

(a) For the class- i ($i=1, \dots, m$) membership customers, there exist optimal switching-curve policies

$$S_t^{i1}(x_1) = \min \{x_2 : \Delta_1 V_{t-1}(x_1 + 1, x_2) \leq -R_{i1}, \Delta_2 V_{t-1}(x_1, x_2 + 1) \leq -R_{i2},$$

$$S_t^{i2}(x_1) = \min \{x_2 : V_{t-1}(x_1, x_2 + 1) - V_{t-1}(x_1 + 1, x_2) \leq R_{i1} - R_{i2}\}, \quad i = 1, \dots, m,$$

and $S_t^{i1}(x_1) \downarrow x_1, S_t^{i2}(x_1) \uparrow x_1$, such that

- reject class- i customer if $x_2 > S_t^{i1}(x_1)$;
- accept $(i,1)$ customer to flight-1 if $S_t^{i2}(x_1)(x_1) \leq x_2 \leq S_t^{i1}(x_1)$;
- accept $(i,1)$ customer to flight-2 if $x_2 \leq S_t^{i1}(x_1)$ and $x_2 \leq S_t^{i2}(x_1)$.

(b) For the public customers, the optimal price is monotone over the seats booked in each flight:

$$r_1^*(x_1, x_2) \uparrow x_1 \uparrow x_2; r_2^*(x_1, x_2) \uparrow x_1 \uparrow x_2.$$

Proof. (a). It follows directly by the parallel results of Proposition 3.3 and Proposition 3.5.

Note that

$$\tilde{G}_j V_t(x_1, x_2) = V_t(x) + \max_{r_j} \{\bar{F}_j(r_j)(r_j + \Delta_j V_t(x + e_j))\}, j = 1, 2 \quad (3.13)$$

and

$$\frac{\partial^2}{\partial r_j \partial \Delta_j} [\bar{F}_j(r_j)(r_j + \Delta_j)] = -f_j(r_j) < 0,$$

i.e., r_j and Δ_j are submodular, thus $r_j^*(\Delta_j) \downarrow \Delta_j$. Therefore the results follows since

$$\Delta_1 V_t(x_1, x_2) \downarrow x_1 \downarrow x_2 \text{ and } \Delta_2 V_t(x_1, x_2) \downarrow x_1 \downarrow x_2. \blacksquare$$

We will use a simple numerical example to illustrate the above analytical results. The total booking horizon is $T=30$ and basic parameters are listed in Table 3.4.a. We adopt a Weibull distribution for willingness-to-pay of public customers (e.g. Bitran and Mondschein, 1997). Monotone optimal switching-curve policies for inventory control of membership customers and monotone optimal pricing control for public customers are illustrated by Table 3.4(b) and (c), where 0 stands for rejection, 1 stands for acceptance to flight-1, 2 for acceptance to flight-2.

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Membership customers will be either rationed between two flights or rejected depending on the opportunity cost $|\Delta_1 V_t(x_1, x_2)|$ and $|\Delta_2 V_t(x_1, x_2)|$. By the first-order-condition (Weibull distribution is sufficient for the quasiconcavity of the value function) of (3.13), we obtain

$$r_j^*(x_1, x_2) = \frac{\bar{F}_j(r_j^*)}{f_j(r_j^*)} + |\Delta_j V_t(x_1, x_2)|,$$

which says that the optimal price equals the current value $\bar{F}_j(r_j^*)/f_j(r_j^*)$ plus the future value $|\Delta_j V_t(x_1, x_2)|$. From Table 3.4(c), we observe that even though the optimal price for public customers is much higher than the revenue from membership customers, the expected revenue from public customers is not much higher since airline faces buy or no-buy from customers when offering a price whereas the revenue from membership customers is certain. The optimal control depends on the opportunity cost which is interrelated with decisions of both membership customers and public customers. Joint dynamic pricing and inventory control for two flights take advantage of the flexibility of allocating membership customers and pricing public customers. Monotone optimal control policies are extremely useful for developing heuristics for large-scale problem in the real life.

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Table 3.4 Monotone optimal control policies for membership and public customers

$$R_{11}=200, R_{12}=150, R_{21}=120, R_{22}=80, c_0=500, K_1=K_2=10, \\ F_1 \sim \text{weibull}(500,2), F_2 \sim \text{weibull}(400,2), T=30$$

a. Booking Request Probabilities

Probability	30 - 26	25 - 19	18 - 12	11 - 5	4 - 1
λ_1	0.08	0.06	0.10	0.14	0.15
λ_2	0.08	0.06	0.10	0.14	0.15
$\hat{\lambda}_1$	0.14	0.14	0.10	0.16	0
$\hat{\lambda}_2$	0.14	0.14	0.10	0.16	0

b. Optimal monotone switching-curve policies for membership customers ($t=20$)

Optimal control for class-1 membership customers

x_2	0	1	2	3	4	5	6	7	8	9
9	1	1	1	1	1	1	1	1	0	0
8	1	1	1	1	1	1	1	1	1	0
7	1	1	1	1	1	1	1	1	2	2
6	1	1	1	1	1	1	1	2	2	2
5	1	1	1	1	1	1	1	2	2	2
4	1	1	1	1	1	1	1	2	2	2
3	1	1	1	1	1	1	1	2	2	2
2	1	1	1	1	1	1	1	2	2	2
1	1	1	1	1	1	1	1	2	2	2
0	1	1	1	1	1	1	1	2	2	2

Optimal control for class-2 membership customers

x_2	0	1	2	3	4	5	6	7	8	9
9	1	1	1	1	1	1	0	0	0	0
8	1	1	1	1	1	1	0	0	0	0
7	1	1	1	1	1	1	1	0	0	0
6	1	1	1	1	1	1	2	2	0	0
5	1	1	1	1	1	1	2	2	2	2
4	1	1	1	1	1	1	2	2	2	2
3	1	1	1	1	1	1	2	2	2	2
2	1	1	1	1	1	1	2	2	2	2
1	1	1	1	1	1	1	2	2	2	2
0	1	1	1	1	1	1	2	2	2	2

c. Optimal monotone pricing control for public customers ($t=20$)

Optimal price of flight-1 for public customers

x_2	0	1	2	3	4	5	6	7	8	9
9	356	360	367	379	395	411	426	447	470	525
8	355	358	364	374	389	405	420	439	463	523
7	355	357	361	369	381	396	411	427	453	520
6	355	356	360	366	375	387	400	417	445	516
5	354	356	359	364	371	380	391	406	438	513
4	354	356	359	363	369	376	384	397	431	509
3	354	356	359	363	368	374	380	392	425	504
2	354	356	359	363	368	373	378	388	421	501
1	354	356	359	363	368	373	378	387	419	499
0	354	356	359	363	368	373	377	386	418	498

Optimal price of flight-2 for public customers

x_2	0	1	2	3	4	5	6	7	8	9
9	402	403	405	408	412	416	419	422	425	427
8	326	327	329	332	338	345	351	357	366	372
7	296	296	298	300	306	313	322	331	343	352
6	286	286	287	288	292	298	307	318	327	335
5	283	283	284	284	286	290	296	306	316	322
4	283	283	283	283	284	286	289	296	304	311
3	283	283	283	283	283	284	285	289	295	300
2	283	283	283	283	283	283	284	285	288	292
1	283	283	283	283	283	283	283	284	285	287
0	283	283	283	283	283	283	283	283	284	284

3.7 Conclusions

In this chapter, we have presented a unified treatment to study structural properties and monotone optimal control policies through D -multimodularity for a class of two-dimensional discrete-event stochastic systems that can be modeled by controlled Markov chains with each possible transition bounded by the unit vector. We demonstrate that each system of this class can be classified into either a substitutable-type or a complementary-type according to the possible transition set of the underlying MDP. We identify D -multimodularity as the general sufficient conditions for monotone optimal control for different type of systems under this unified framework. This integrated approach has been used to classify representative models in the literature. We also demonstrate that this new approach can be applied to solve new application problems with the monotone optimal control policies fully characterized.

The unification leads to a systematic approach to study monotone optimal control for this particular class of two-dimensional discrete-event stochastic systems, which will shed new light on the general two-dimensional systems. It is worthwhile to note that the limitation on two-dimensional systems is not a limitation of the approach or theory developed in this paper. It is rooted in the limitation of the switching-curve type of control itself, which by its very nature suffers from combinatorial explosion.

There are many interesting avenues for future research. First, this framework can be extended to study more general two-dimensional discrete-event stochastic systems which we are undergoing. For instance, multiple-resource allocation, batch-size demand in inventory control or manufacturing and production, all belong to a class of two-dimensional discrete-event stochastic systems with each possible transition beyond the unit vector. The methodology developed in this paper can be applied in studying monotone optimal control for this new class of two-dimensional systems with great efficiency. Therefore, it is worthwhile to explore structural properties and monotone optimal control for the general two-dimensional discrete-event stochastic systems.

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Second, it will be interesting to explore possible applications of all control operators on atoms to formulate appropriate models and solve new and more complex two-dimensional problems. Finally, it is quite challenging to investigate multi-dimensional discrete-event stochastic systems. Even though directional submodularity and antidirectional supermodularity in two-dimension are equivalent to D -multimodularity with respect to D^s and D^c respectively, Morton (2006) has shown that antidirectional supermodularity is false for a three-leg problem. We conjecture that structural properties of D -multimodularity will be preserved for the three-dimensional system and remain to be sufficient for monotone optimal control.

3.8 Appendix – Proofs

Proof of Lemma 3.5, Lemma 3.7, Lemma 3.8:

As superconvexity/subconvexity with respect to i and j are pairwise symmetry, it suffices to prove $i=1$.

Proof of Lemma 3.5:

Supermodularity (3.2): $T_1^s f(x_1, x_2) - T_1^s f(x_1, x_2 - 1) \geq T_1^s f(x_1 - 1, x_2) - T_1^s f(x_1 - 1, x_2 - 1)$,

i.e.,

$$\min\{f(x_1, x_2), f(x_1 - 1, x_2), f(x_1, x_2 - 1)\} - \min\{f(x_1, x_2 - 1), f(x_1 - 1, x_2 - 1), f(x_1, x_2 - 2)\} \\ \geq \min\{f(x_1 - 1, x_2), f(x_1 - 2, x_2), f(x_1 - 1, x_2 - 1)\} - \min\{f(x_1 - 1, x_2 - 1), f(x_1 - 2, x_2 - 1), f(x_1 - 1, x_2 - 2)\}.$$

Since

$$\begin{aligned} f(x_1, x_2) - f(x_1, x_2 - 1) &\geq f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1), \\ f(x_1, x_2) - f(x_1, x_2 - 1) &\geq f(x_1 - 2, x_2) - f(x_1 - 2, x_2 - 1), \\ f(x_1, x_2) - f(x_1, x_2 - 1) &\geq f(x_1 - 1, x_2 - 1) - f(x_1 - 1, x_2 - 2), \end{aligned}$$

$$\begin{aligned} f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1) &\geq (=) f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1), \\ f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1) &\geq f(x_1 - 2, x_2) - f(x_1 - 2, x_2 - 1), \\ f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1) &\geq f(x_1 - 1, x_2 - 1) - f(x_1 - 1, x_2 - 2), \end{aligned}$$

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$$\begin{aligned} f(x_1, x_2 - 1) - f(x_1, x_2 - 1) &\geq (=) f(x_1 - 1, x_2 - 1) - f(x_1 - 1, x_2 - 1), \\ f(x_1, x_2 - 1) - f(x_1 - 1, x_2 - 1) &\geq f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1), \\ f(x_1, x_2 - 1) - f(x_1, x_2 - 2) &\geq f(x_1 - 1, x_2 - 1) - f(x_1 - 1, x_2 - 2), \end{aligned}$$

where the first, second, fifth and ninth inequality follows from (3.2); the sixth and eighth inequality follows from (3.7); the third inequality follows from (3.3) and (3.2) through

$$\Delta_2 f(x_1, x_2) \geq \Delta_2 f(x_1 + 1, x_2 - 1) \geq \Delta_2 f(x_1 - 1, x_2 - 1).$$

Therefore supermodularity (3.2) of $T_1^s f(x_1, x_2)$ is proved by Lemma 3.3.

Superconvexity (3.3): $T_1^s f(x_1, x_2 - 1) - T_1^s f(x_1 - 1, x_2 - 1) \geq T_1^s f(x_1 - 1, x_2) - T_1^s f(x_1 - 2, x_2)$,

i.e.

$$\begin{aligned} &\min\{f(x_1, x_2 - 1), f(x_1 - 1, x_2 - 1), f(x_1, x_2 - 2)\} - \min\{f(x_1 - 1, x_2 - 1), f(x_1 - 2, x_2 - 1), f(x_1 - 1, x_2 - 2)\} \\ &\geq \min\{f(x_1 - 1, x_2), f(x_1 - 2, x_2), f(x_1 - 1, x_2 - 1)\} - \min\{f(x_1 - 2, x_2), f(x_1 - 3, x_2), f(x_1 - 2, x_2 - 1)\}. \end{aligned}$$

Since

$$\begin{aligned} f(x_1, x_2 - 1) - f(x_1 - 1, x_2 - 1) &\geq f(x_1 - 1, x_2) - f(x_1 - 2, x_2), \\ f(x_1, x_2 - 1) - f(x_1 - 1, x_2 - 1) &\geq f(x_1 - 2, x_2) - f(x_1 - 3, x_2), \\ f(x_1, x_2 - 1) - f(x_1 - 1, x_2 - 1) &\geq f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1), \\ f(x_1 - 1, x_2 - 1) - f(x_1 - 1, x_2 - 1) &\geq (=) f(x_1 - 2, x_2) - f(x_1 - 2, x_2), \\ f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1) &\geq f(x_1 - 2, x_2) - f(x_1 - 3, x_2), \\ f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1) &\geq (=) f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1), \\ f(x_1, x_2 - 2) - f(x_1 - 1, x_2 - 1) &\geq f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2), \\ f(x_1, x_2 - 2) - f(x_1 - 1, x_2 - 2) &\geq f(x_1 - 2, x_2) - f(x_1 - 3, x_2), \\ f(x_1, x_2 - 2) - f(x_1 - 1, x_2 - 2) &\geq f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1), \end{aligned}$$

where the first, fifth and eighth inequality follows from (3.3); the third and ninth inequality follows from (3.2); the seventh inequality follows from (3.8); the second inequality follows from (3.3) and (3.2) through

$$\Delta_1 f(x_1, x_2 - 1) \geq \Delta_1 f(x_1 - 1, x_2) \geq \Delta_1 f(x_1 - 2, x_2).$$

Therefore superconvexity (3.3) of $T_1^s f(x_1, x_2)$ is proved by Lemma 3.3. ■

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Proof of Lemma 3.7:

Since $\tilde{T}_{1,1}^s f(x_1, x_2)$ and $\tilde{T}_{1,2}^s f(x_1, x_2)$ are pairwise symmetry, so it suffices to prove the preservation of $\mathcal{V}^s$ for $\tilde{T}_{1,1}^s f(x_1, x_2)$ and $\tilde{T}_{1,2}^s f(x_1, x_2)$ can be proved analogously which is omitted.

Supermodularity (3.2): $\tilde{T}_{1,1}^s f(x_1, x_2) - \tilde{T}_{1,1}^s f(x_1, x_2 - 1) \geq \tilde{T}_{1,1}^s f(x_1 - 1, x_2) - \tilde{T}_{1,1}^s f(x_1 - 1, x_2 - 1)$,

i.e.,

$$\begin{aligned} & \min\{f(x_1, x_2), f(x_1 - 1, x_2)\} - \min\{f(x_1, x_2 - 1), f(x_1 - 1, x_2 - 1)\} \\ & \geq \min\{f(x_1 - 1, x_2), f(x_1 - 2, x_2)\} - \min\{f(x_1 - 1, x_2 - 1), f(x_1 - 2, x_2 - 1)\}. \end{aligned}$$

Since

$$\begin{aligned} & f(x_1, x_2) - f(x_1, x_2 - 1) \geq f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1), \\ & f(x_1, x_2) - f(x_1, x_2 - 1) \geq f(x_1 - 2, x_2) - f(x_1 - 2, x_2 - 1), \\ & f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1) \geq (=) f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1), \\ & f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1) \geq f(x_1 - 2, x_2) - f(x_1 - 2, x_2 - 1), \end{aligned}$$

where the three inequalities follow from (3.2), therefore supermodularity (3.2) is true.

Superconvexity (3.3): $\tilde{T}_{1,1}^s f(x_1, x_2 - 1) - \tilde{T}_{1,1}^s f(x_1 - 1, x_2 - 1) \geq \tilde{T}_{1,1}^s f(x_1 - 1, x_2) - \tilde{T}_{1,1}^s f(x_1 - 2, x_2)$,

i.e.,

$$\begin{aligned} & \min\{f(x_1, x_2 - 1), f(x_1 - 1, x_2 - 1)\} - \min\{f(x_1 - 1, x_2 - 1), f(x_1 - 2, x_2 - 1)\} \\ & \geq \min\{f(x_1 - 1, x_2), f(x_1 - 2, x_2)\} - \min\{f(x_1 - 2, x_2), f(x_1 - 3, x_2)\}. \end{aligned}$$

Since

$$\begin{aligned} & f(x_1, x_2 - 1) - f(x_1 - 1, x_2 - 1) \geq f(x_1 - 1, x_2) - f(x_1 - 2, x_2), \\ & f(x_1, x_2 - 1) - f(x_1 - 1, x_2 - 1) \geq f(x_1 - 2, x_2) - f(x_1 - 3, x_2), \\ & f(x_1 - 1, x_2 - 1) - f(x_1 - 1, x_2 - 1) \geq (=) f(x_1 - 2, x_2) - f(x_1 - 2, x_2), \\ & f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1) \geq f(x_1 - 2, x_2) - f(x_1 - 3, x_2), \end{aligned}$$

where the first and third inequality follows from (3.3) and the second inequality follows from (3.3)

and (3.7) through

$$\Delta_1 f(x_1, x_2 - 1) \geq \Delta_1 f(x_1 - 1, x_2) \geq \Delta_1 f(x_1 - 2, x_2),$$

therefore superconvexity (3.3) is true. ■

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Proof of Lemma 3.8:

Supermodularity (3.2): $\tilde{T}_{1,3}^s f(x_1, x_2) - \tilde{T}_{1,3}^s f(x_1, x_2 - 1) \geq \tilde{T}_{1,3}^s f(x_1 - 1, x_2) - \tilde{T}_{1,3}^s f(x_1 - 1, x_2 - 1)$,

i.e.,

$$\begin{aligned} & \min\{f(x_1 - 1, x_2), f(x_1, x_2 - 1)\} - \min\{f(x_1 - 1, x_2 - 1), f(x_1, x_2 - 2)\} \\ & \geq \min\{f(x_1 - 2, x_2), f(x_1 - 1, x_2 - 1)\} - \min\{f(x_1 - 2, x_2 - 1), f(x_1 - 1, x_2 - 2)\}. \end{aligned}$$

Since

$$\begin{aligned} f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1) & \geq f(x_1 - 2, x_2) - f(x_1 - 2, x_2 - 1), \\ f(x_1 - 1, x_2) - f(x_1 - 1, x_2 - 1) & \geq f(x_1 - 1, x_2 - 1) - f(x_1 - 1, x_2 - 2), \\ f(x_1, x_2 - 1) - f(x_1 - 1, x_2 - 1) & \geq f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1), \\ f(x_1, x_2 - 1) - f(x_1, x_2 - 2) & \geq f(x_1 - 1, x_2 - 1) - f(x_1 - 1, x_2 - 2), \end{aligned}$$

where the first and forth inequality follows from (3.2); the second and third inequality follows from (3.7), therefore supermodularity (3.2) is true.

Superconvexity (3.3): $\tilde{T}_{1,3}^s f(x_1, x_2 - 1) - \tilde{T}_{1,3}^s f(x_1 - 1, x_2 - 1) \geq \tilde{T}_{1,3}^s f(x_1 - 1, x_2) - \tilde{T}_{1,3}^s f(x_1 - 2, x_2)$,

i.e.,

$$\begin{aligned} & \min\{f(x_1 - 1, x_2 - 1), f(x_1, x_2 - 2)\} - \min\{f(x_1 - 2, x_2 - 1), f(x_1 - 1, x_2 - 2)\} \\ & \geq \min\{f(x_1 - 2, x_2), f(x_1 - 1, x_2 - 1)\} - \min\{f(x_1 - 3, x_2), f(x_1 - 2, x_2 - 1)\}. \end{aligned}$$

Since

$$\begin{aligned} f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1) & \geq f(x_1 - 2, x_2) - f(x_1 - 3, x_2), \\ f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1) & \geq (=) f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1), \\ f(x_1, x_2 - 2) - f(x_1 - 1, x_2 - 2) & \geq f(x_1 - 2, x_2) - f(x_1 - 3, x_2), \\ f(x_1, x_2 - 2) - f(x_1 - 1, x_2 - 2) & \geq f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2 - 1), \end{aligned}$$

where all the inequalities follow from (3.3), therefore superconvexity (3.3) is true. ■

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Proof of Proposition 3.7 :

Since $V_0(x_1, x_2)$ is linear and separable in x_1 and x_2 so $V_0(x_1, x_2) \in \mathcal{U}^s$. By induction, assume $V_{t-1}(x_1, x_2) \in \mathcal{U}^s$ and we need to show $V_t(x_1, x_2) \in \mathcal{U}^s$. It follows directly from the parallel results of Proposition 3.3 in the concavity context that $G_i V_{t-1}(x_1, x_2) = T_4^s \in \mathcal{U}^s, i=1, \dots, m$ and we need to show $\tilde{G}_j V_{t-1}(x_1, x_2) \in \mathcal{U}^s, j=1, 2$. Rewrite

$$\begin{aligned}\tilde{G}_1 V_{t-1}(x_1, x_2) &= V_{t-1}(x_1, x_2) + \max_{r_1} \{\bar{F}_1(r_1)(r_1 + \Delta_1 V_{t-1}(x_1 + 1, x_2))\}, \\ \tilde{G}_2 V_t(x_1, x_2) &= V_{t-1}(x_1, x_2) + \max_{r_2} \{\bar{F}_2(r_2)(r_2 + \Delta_2 V_{t-1}(x_1, x_2 + 1))\}.\end{aligned}$$

Denote $K_i(\Delta) = \bar{F}_i(r)(r + \Delta), i=1, 2$. Similar parts of (c.1) and (f.2) of Lemma 2.1 (You, 1999), we can easily show the following results:

$$K_i(\Delta) \uparrow \Delta; \quad (\text{A.1})$$

$$K_i(\Delta^1) - K_i(\Delta^2) \leq \bar{F}_i(r_i^*(\Delta^1))(\Delta^1 - \Delta^2); \quad (\text{A.2})$$

If $\Delta^2 \leq \Delta^1 \wedge \Delta^4, \Delta^1 \vee \Delta^4 \leq \Delta^3$ then

$$K_i(\Delta^1) - K_i(\Delta^2) - K_i(\Delta^3) + K_i(\Delta^4) \leq \bar{F}_i(r_i^*(\Delta^4))(\Delta^1 - \Delta^2 - \Delta^3 + \Delta^4) \quad (\text{A.3})$$

Proof of Submodularity of $\tilde{G}_1 V_{t-1}(x_1, x_2)$: $\Delta_1 \tilde{G}_1 V_{t-1}(x_1, x_2) \leq \Delta_1 \tilde{G}_1 V_{t-1}(x_1, x_2 - 1)$

$$\begin{aligned}& \tilde{G}_1 V_{t-1}(x_1, x_2) - \tilde{G}_1 V_{t-1}(x_1 - 1, x_2) - \tilde{G}_1 V_{t-1}(x_1, x_2 - 1) + \tilde{G}_1 V_{t-1}(x_1 - 1, x_2 - 1) \\&= \Delta_1 V_{t-1}(x_1, x_2) - \Delta_1 V_{t-1}(x_1, x_2 - 1) + \left[\begin{aligned} & K(\Delta_1 V_{t-1}(x_1 + 1, x_2)) - K(\Delta_1 V_{t-1}(x_1, x_2)) \\ & - K(\Delta_1 V_{t-1}(x_1 + 1, x_2 - 1)) + K(\Delta_1 V_{t-1}(x_1, x_2 - 1)) \end{aligned} \right] \\&\leq \Delta_1 V_{t-1}(x_1, x_2) - \Delta_1 V_{t-1}(x_1, x_2 - 1) + K(\Delta_1 V_{t-1}(x_1, x_2 - 1)) - K(\Delta_1 V_{t-1}(x_1, x_2)) \\&\leq (1 - \bar{F}_1(r_1^*(x_1, x_2 - 1)))(\Delta_1 V_{t-1}(x_1, x_2) - \Delta_1 V_{t-1}(x_1, x_2 - 1)) \leq 0,\end{aligned}$$

where the first inequality follows from submodularity and (A.1); the second follows from (A.2) and the third follows from submodularity.

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Proof of Subconcavity of $\tilde{G}_1 V_{t-1}(x_1, x_2)$: $\Delta_1 \tilde{G}_1 V_{t-1}(x_1, x_2) \leq \Delta_1 \tilde{G}_1 V_{t-1}(x_1 - 1, x_2 + 1)$

$$\begin{aligned} & \tilde{G}_1 V_{t-1}(x_1, x_2) - \tilde{G}_1 V_{t-1}(x_1 - 1, x_2) - \tilde{G}_1 V_{t-1}(x_1 - 1, x_2 + 1) - \tilde{G}_1 V_{t-1}(x_1 - 2, x_2 + 1) \\ &= \Delta_1 V_{t-1}(x_1, x_2) - \Delta_1 V_{t-1}(x_1 - 1, x_2 + 1) + \left[\begin{aligned} & K(\Delta_1 V_{t-1}(x_1 + 1, x_2)) - K(\Delta_1 V_{t-1}(x_1, x_2)) \\ & - K(\Delta_1 V_{t-1}(x_1, x_2 + 1)) + K(\Delta_1 V_{t-1}(x_1 - 1, x_2 + 1)) \end{aligned} \right] \\ &\leq \Delta_1 V_{t-1}(x_1, x_2) - \Delta_1 V_{t-1}(x_1 - 1, x_2 + 1) + K(\Delta_1 V_{t-1}(x_1 - 1, x_2 + 1)) - K(\Delta_1 V_{t-1}(x_1, x_2)) \\ &\leq (1 - \bar{F}_1(r_1^*(x_1 - 1, x_2 + 1)))(\Delta_1 V_{t-1}(x_1, x_2) - \Delta_1 V_{t-1}(x_1 - 1, x_2 + 1)) \leq 0, \end{aligned}$$

where the first inequality follows from subconcavity and (A.1); the second follows from (A.2) and the third follows from subconcavity.

Proof of Subconcavity of $\tilde{G}_1 V_{t-1}(x_1, x_2)$: $\Delta_2 \tilde{G}_1 V_{t-1}(x_1, x_2) \leq \Delta_2 \tilde{G}_1 V_{t-1}(x_1 + 1, x_2 - 1)$

$$\begin{aligned} & \tilde{G}_1 V_{t-1}(x_1, x_2) - \tilde{G}_1 V_{t-1}(x_1, x_2 - 1) - \tilde{G}_1 V_{t-1}(x_1 + 1, x_2 - 1) - \tilde{G}_1 V_{t-1}(x_1 + 1, x_2 - 2) \\ &= \Delta_2 V_{t-1}(x_1, x_2) - \Delta_2 V_{t-1}(x_1 + 1, x_2 - 1) + \left[\begin{aligned} & K(\Delta_1 V_{t-1}(x_1 + 1, x_2)) - K(\Delta_1 V_{t-1}(x_1 + 2, x_2 - 1)) \\ & - K(\Delta_1 V_{t-1}(x_1 + 1, x_2 - 1)) + K(\Delta_1 V_{t-1}(x_1 + 2, x_2 - 2)) \end{aligned} \right] \\ &\leq \Delta_2 V_{t-1}(x_1, x_2) - \Delta_2 V_{t-1}(x_1 + 1, x_2 - 1) + \\ & \quad \bar{F}_1(r_1^*(x_1 + 2, x_2 - 2)) \left[\begin{aligned} & \Delta_1 V_{t-1}(x_1 + 1, x_2) - \Delta_1 V_{t-1}(x_1 + 2, x_2 - 1) \\ & - \Delta_1 V_{t-1}(x_1 + 1, x_2 - 1) + \Delta_1 V_{t-1}(x_1 + 2, x_2 - 2) \end{aligned} \right] \\ &= \Delta_2 V_{t-1}(x_1, x_2) - \Delta_2 V_{t-1}(x_1 + 1, x_2 - 1) + \\ & \quad \bar{F}_1(r_1^*(x_1 + 2, x_2 - 2)) \left[\begin{aligned} & \Delta_2 V_{t-1}(x_1 + 1, x_2) - \Delta_2 V_{t-1}(x_1, x_2) \\ & - \Delta_2 V_{t-1}(x_1 + 2, x_2 - 1) + \Delta_2 V_{t-1}(x_1 + 1, x_2 - 1) \end{aligned} \right] \\ &\leq (1 - \bar{F}_1(r_1^*(x_1 + 2, x_2 - 2)))(\Delta_2 V_{t-1}(x_1, x_2) - \Delta_2 V_{t-1}(x_1 + 1, x_2 - 1)) \leq 0, \end{aligned}$$

where the first inequality follows from submodularity, subconcavity and (A.3); the second and third follows from subconcavity. Thus $\tilde{G}_1 V_{t-1}(x_1, x_2) \in \mathcal{U}^s$. Similarly $\tilde{G}_2 V_{t-1}(x_1, x_2) \in \mathcal{U}^s$ as follows.

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Proof of Submodularity of $\tilde{G}_2 V_{t-1}(x_1, x_2)$: $\Delta_1 \tilde{G}_2 V_{t-1}(x_1, x_2) \leq \Delta_1 \tilde{G}_2 V_{t-1}(x_1, x_2 - 1)$

$$\begin{aligned}
 & \tilde{G}_2 V_{t-1}(x_1, x_2) - \tilde{G}_2 V_{t-1}(x_1 - 1, x_2) - \tilde{G}_2 V_{t-1}(x_1, x_2 - 1) + \tilde{G}_2 V_{t-1}(x_1 - 1, x_2 - 1) \\
 &= \Delta_2 V_{t-1}(x_1, x_2) - \Delta_2 V_{t-1}(x_1 - 1, x_2) + \left[\begin{aligned} & K(\Delta_2 V_{t-1}(x_1, x_2 + 1)) - K(\Delta_2 V_{t-1}(x_1 - 1, x_2 + 1)) \\ & - K(\Delta_2 V_{t-1}(x_1, x_2)) + K(\Delta_2 V_{t-1}(x_1 - 1, x_2)) \end{aligned} \right] \\
 &\leq \Delta_2 V_{t-1}(x_1, x_2) - \Delta_2 V_{t-1}(x_1, x_2 - 1) + K(\Delta_2 V_{t-1}(x_1 - 1, x_2)) - K(\Delta_2 V_{t-1}(x_1, x_2)) \\
 &\leq (1 - \bar{F}_2(r_2^*(x_1 - 1, x_2))) (\Delta_2 V_{t-1}(x_1, x_2) - \Delta_2 V_{t-1}(x_1, x_2 - 1)) \leq 0.
 \end{aligned}$$

Proof of Subconcavity of $\tilde{G}_2 V_{t-1}(x_1, x_2)$: $\Delta_1 \tilde{G}_2 V_{t-1}(x_1, x_2) \leq \Delta_1 \tilde{G}_2 V_{t-1}(x_1 - 1, x_2 + 1)$

$$\begin{aligned}
 & \tilde{G}_2 V_{t-1}(x_1, x_2) - \tilde{G}_2 V_{t-1}(x_1 - 1, x_2) - \tilde{G}_2 V_{t-1}(x_1 - 1, x_2 + 1) - \tilde{G}_2 V_{t-1}(x_1 - 2, x_2 + 1) \\
 &= \Delta_1 V_{t-1}(x_1, x_2) - \Delta_1 V_{t-1}(x_1 - 1, x_2 + 1) + \left[\begin{aligned} & K(\Delta_2 V_{t-1}(x_1, x_2 + 1)) - K(\Delta_2 V_{t-1}(x_1 - 1, x_2 + 2)) \\ & - K(\Delta_2 V_{t-1}(x_1 - 1, x_2 + 1)) + K(\Delta_2 V_{t-1}(x_1 - 2, x_2 + 2)) \end{aligned} \right] \\
 &\leq \Delta_1 V_{t-1}(x_1, x_2) - \Delta_1 V_{t-1}(x_1 - 1, x_2 + 1) \\
 &\quad + \bar{F}_2(r_2^*(x_1 - 2, x_2 + 2)) \left[\begin{aligned} & \Delta_2 V_{t-1}(x_1, x_2 + 1) - \Delta_2 V_{t-1}(x_1 - 1, x_2 + 2) \\ & - \Delta_2 V_{t-1}(x_1 - 1, x_2 + 1) + \Delta_2 V_{t-1}(x_1 - 2, x_2 + 2) \end{aligned} \right] \\
 &= \Delta_1 V_{t-1}(x_1, x_2) - \Delta_1 V_{t-1}(x_1 - 1, x_2 + 1) + \\
 &\quad \bar{F}_2(r_2^*(x_1 - 2, x_2 + 2)) \left[\begin{aligned} & \Delta_1 V_{t-1}(x_1, x_2 + 1) - \Delta_1 V_{t-1}(x_1 - 1, x_2 + 2) \\ & - \Delta_1 V_{t-1}(x_1, x_2) + \Delta_1 V_{t-1}(x_1 - 1, x_2 + 1) \end{aligned} \right] \\
 &\leq (1 - \bar{F}_2(r_2^*(x_1 - 2, x_2 + 2))) (\Delta_1 V_{t-1}(x_1, x_2) - \Delta_1 V_{t-1}(x_1 - 1, x_2 + 1)) \leq 0.
 \end{aligned}$$

Proof of Subconcavity of $\tilde{G}_2 V_{t-1}(x_1, x_2)$: $\Delta_2 \tilde{G}_2 V_{t-1}(x_1, x_2) \leq \Delta_2 \tilde{G}_2 V_{t-1}(x_1 + 1, x_2 - 1)$

$$\begin{aligned}
 & \tilde{G}_2 V_{t-1}(x_1, x_2) - \tilde{G}_2 V_{t-1}(x_1, x_2 - 1) - \tilde{G}_2 V_{t-1}(x_1 + 1, x_2 - 1) - \tilde{G}_2 V_{t-1}(x_1 + 1, x_2 - 2) \\
 &= \Delta_2 V_{t-1}(x_1, x_2) - \Delta_2 V_{t-1}(x_1 + 1, x_2 - 1) + \left[\begin{aligned} & K(\Delta_2 V_{t-1}(x_1, x_2 + 1)) - K(\Delta_2 V_{t-1}(x_1, x_2)) \\ & - K(\Delta_2 V_{t-1}(x_1 + 1, x_2)) + K(\Delta_2 V_{t-1}(x_1 + 1, x_2 - 1)) \end{aligned} \right] \\
 &\leq \Delta_2 V_{t-1}(x_1, x_2) - \Delta_2 V_{t-1}(x_1 + 1, x_2 - 1) + K(\Delta_2 V_{t-1}(x_1 + 1, x_2 - 1)) - K(\Delta_2 V_{t-1}(x_1, x_2)) \\
 &\leq (1 - \bar{F}_1(r_1^*(x_1 + 1, x_2 - 1))) (\Delta_2 V_{t-1}(x_1, x_2) - \Delta_2 V_{t-1}(x_1 + 1, x_2 - 1)) \leq 0.
 \end{aligned}$$

Thus $\tilde{G}_2 V_{t-1}(x_1, x_2) \in \mathcal{U}^s$, and $V_t(x_1, x_2) \in \mathcal{U}^s$. ■

4 Optimal Capacity Control for Multiple Medical Diagnostic Facilities

4.1 Introduction

Healthcare expenditure in the US has increased rapidly from 5% of the Gross Domestic Product (GDP) in 1965 (Smith-Daniels et al., 1988) to 17% of GDP in 2008 and is projected to reach 20% of GDP by 2017[‡]. With the increasing costs, limited resources, and burgeoning demand, proper capacity management is crucial to improve the access, efficiency and quality of healthcare services.

Capacity management in healthcare services primarily concerns minimizing investment and operating cost, maximizing facility utilization, minimizing patient waiting time and inappropriate service care (Smith-Daniels et al., 1988). As a result of escalating healthcare costs, government has enforced regulation on purchasing expensive diagnostic facilities such as Magnetic Resonance Imaging (MRI), computed tomography (CT), for cost containment and efficient utilization of limited resources. Therefore, effective capacity management for medical diagnostic facilities is one of the primary concerns in healthcare management.

Effective capacity management for medical diagnostic facilities can help to achieve the goals: (1) matching demand with capacity so that resources are better utilized; (2) improving healthcare access; (3) minimizing patient waiting time; (4) enhancing revenue streams and cost-effectiveness of hospital; (5) achieving regional-level resource coordination among hospitals; etc. However, the complexity is overwhelming.

Demand for medical diagnostic facilities arises from multiple sources – inpatients, outpatients, and emergency patients, from both inside and outside the hospital. The three segments of patients have different arrival patterns and service priorities: outpatients are reserved

[‡] According to the Organization for Economic Cooperation and Development.

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through an appointment schedule, inpatients and emergency patients arrive randomly while emergency patients have the priority over the other two types. Inpatients and outpatients have class-specific revenue, waiting cost, and penalty cost from untreatments.

Capacity management for diagnostic facilities involves two interrelated tasks: outpatient appointment scheduling and dynamic resource allocation among segments of patients (Green et al., 2006). Access of outpatients to diagnostic facilities is controlled through appointment scheduling. Effective appointment scheduling can improve the efficiency and timely access to health services. It is complicated due to “cancellation” or “no-show” which will waste resources, impact accessibility and increase cost if not treated appropriately. It is further complicated with random arrivals of inpatients and emergency patients. Dynamic resource allocation provides real-time control of access to facilities by potentially competing patient segments. It also dynamically adjusts the congestion in queue for different patient segments by taking account of waiting time (or cost). Appointment scheduling affects the dynamic allocation policy, which in turn further affects the impact of the scheduling policy on the overall performance of the service system.

In this chapter we investigate capacity management problem for *multiple* medical diagnostic facilities with stochastic demands from three patient segments. This is one of the imperative problems that hospitals are confronting. For instance, the department of diagnostic radiology of Singapore General Hospital (SGH), the largest radiology department in Singapore, is equipped with two MRIs and four CTs. The department provides radiology services for emergency patients, inpatients and outpatients referred from within SGH as well as from other hospitals locally and overseas. How should the radiology department manage access via the appointment scheduling for outpatients? How should the limited facilities be allocated dynamically and efficiently to serve different patient segments, by taking account of different cost and revenue structure with the objective to maximize the expected revenue adjusted by social costs?

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We model the problem as a sequential allocation-scheduling problem for multiple facilities, namely, given any scheduling policy we first solve dynamic resource allocation problem through Markov decision process (MDP) and then solve outpatient scheduling problem. We have fully characterized the optimal dynamic allocation policy given any appointment scheduling for outpatients. It is extremely difficult to identify the optimal scheduling policy for outpatient appointment scheduling. We propose two numerically tractable scheduling rules that are commonly used in hospital, namely vertical scheduling and horizontal scheduling for the multiple-facility problem. We evaluate their relative performances through numerical examples under different cost and revenue structure, and gain managerial insights.

The appointment scheduling and dynamic allocation of medical diagnostic facilities in the presence of multiple patient segments have received very limited and recent attention. The work of Green et al. (2006) and Kolisch and Sickinger (2008) are closely related to ours. Green et al. (2006) make the first step to study managing the one-facility problem with three patient segments. They identify the optimal allocation policy under an arbitrary outpatient appointment schedule and develop heuristics for appointment scheduling. They pose the multiple-facility problem as the future research. Kolisch and Sickinger (2008) investigate the two-facility problem with three patient segments. They develop heuristics for both the capacity allocation problem and the appointment scheduling problem. They pose analyzing structural properties and the optimal allocation policy as the future research.

This chapter has solved the research problems raised in both literatures above and extended them in many significant ways. First, we go beyond a single facility problem. This provides the opportunities for region-level coordination among hospitals. Second, we introduce more general arrival processes for emergency patients and inpatients, which can be stationary or nonstationary. In particular, the arrival process of emergency patients can be any distribution bounded by the number of facilities; while the arrival process of inpatients can be any bounded probability distribution. Our model has incorporated one-facility model of Green et al. (2006) and two-

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facility model of Kolisch and Sickinger (2008) as a special case. Third, the technical innovation of this paper is to introduce the *multimodularity* concept and *complete rationing* to fully characterize the optimal allocation policy for the multiple-facility problem. This technical innovation has made the multiple-facility problem tractable and demonstrated strong efficiency in establishing structural properties and optimal control policies.

4.1.1 Related Literature Review

This work is related to two streams of literature – appointment scheduling and revenue management. Abundant literatures in healthcare management are addressing appointment scheduling and comprehensive reviews include Magerlein and Martin (1978), Cayirli and Veral (2003), Mondschein and Weintraub (2003), Gupta and Denton (2008). Appointment scheduling system is classified into *advance* scheduling (schedule patients in advance of the service date) and *allocation* scheduling (schedule available patients on the day of service) (Magerlein and Martin, 1978), which is related to two types of access delay respectively, *indirect* waiting time (time different between request and appointment) and *direct* waiting time (time difference between appointment and service) (Gupta and Denton, 2008). Performance measure of the scheduling system usually used includes time-based measure such as minimizing patient waiting time, and cost-based (or profit-based) measure etc. Gerchak et al. (1996) investigate *advance* scheduling of two-class of surgeries (elective and emergency) to maximize the expected daily profit and characterize the nature of the optimal policy which is not necessarily a control-limit type. Green and Savin (2008) study reducing delays for patient *advance* scheduling with cancellation or no-show through queueing approach and derive system performance measures. Patrick et al. (2008) examine dynamic *advance* scheduling through MDP for *one* diagnostic facility with *two* sources of demand – inpatients and outpatients, to minimize system cost and solve the equivalent linear program through approximate dynamic programming. In contrast, studies in *allocation*

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scheduling are quite small and only recently researchers have recognized the importance of dynamic allocation resources on a daily basis (Magerlein and Martin, 1978). In this paper, we study *allocation* scheduling in which *direct* waiting time (cost) is relevant, for *multiple* diagnostic facilities with stochastic demands from *three* patient segments.

Capacity management for medical diagnostic facilities, in essence, is *demand management* or *revenue management* in healthcare systems (Chapman and Carmel, 1992) and it meets the necessary conditions (Kimes, 1989) for effective revenue management applications. For instance, expensive medical diagnostic facilities such as MRI have relatively fixed capacity with perishable service slots. The market demand for such facilities has at least three clearly defined segments: inpatient, outpatient and emergency. And the demand in each segment is clearly uncertain. The service slots for each facility can be booked in advance. The marginal costs of performing a standard procedure is relatively low; while adding additional capacity in the short-notice is usually infeasible. However there are some complications that make the direct implementation of traditional revenue management a nontrivial exercise. First, among three distinctive patient segments in healthcare services, emergency patients have the priority for service over the other two types. On one hand, emergency patients are similar to full-fare customers in airline in the sense that they have the relative priority over the other patients. But they also differ from the full-fare customers in the sense that they enjoy the absolute priority over the other patients, which mean they cannot be denied. Furthermore, even though emergency fees are usually the highest among the three segments, but hospitals must treat all emergency patients regardless of the patients' ability to pay. Therefore, the expected revenue from emergency patients may not necessarily be the highest among three segments. Another distinctive element is that both outpatients and inpatients can be bumped and put on queue for the next available service slot. Reservations by outpatients provide some degree of assurance on demand, but the reserved time slot can not be guaranteed because of the absolute priority given to emergency patients as well as the dynamic adjustment of congestion in queue for different patient segments. In contrast to the

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prevailing bumping practice (denied service) in airlines and hotels, bumping a scheduled patient is involuntary without direct compensation and the associated costs are the *direct* waiting costs for the case of bumping any patient and the additional hospitalization costs for the case of delaying the treatment to an inpatient.

Patrick et al. (2008), Gupta and Denton (2008) have discussed some other connections and differences between capacity allocation of medical facilities and revenue management. As characterizing structural properties of the optimal policies for stochastic dynamic control problems is quite challenging, generally there are no structural properties for multi-resource revenue management problem except for some specific cases such as two-resource problems. For instance, You (1999) derives the monotone structure of the optimal dynamic pricing policies for *two-flight-leg* problem. Gupta and Wang (2008) study revenue management for a primary-care clinic in the presence of patient choice and examine the appointment scheduling through MDP. They have derived a threshold policy for a *single* physician and *partially* characterized the structure of the optimal policy for *multi*-physician clinic. Note that the *multi*-physician problem of Gupta and Wang (2008) and our *multiple*-facility problem are akin to *multiple*-parallel-flight problem. In this chapter we have *fully* characterized the optimal allocation policy as the monotone switching-curve policy for the *multiple*-facility problem. This may shed new light on the multi-resource revenue management problem.

The rest of the chapter is organized as follows. In Section 4.2, we set up the multiple-facility model through MDP. In Section 4.3 we characterize the structure of the optimal dynamic allocation policy for the multiple-facility model and numerical examples are used to illustrate the optimal control policies. In Section 4.4, we propose two scheduling rules for outpatient and evaluate their relative performance through numerical experiments. Section 4.5 summarizes our findings and suggests directions for further research. A model comparison with Chen (2004) is made in Appendix of Section 4.6.

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4.2 Multiple-Facility Model

Consider a given work day with N equal-length service slots of m diagnostic facilities serving three types of patients, namely, emergency patient, inpatient and outpatient. Outpatients are reserved through an appointment schedule. Inpatients and emergency patients arrive randomly while emergency patients have the priority over the other two types. Let us start with the model setting in sequence of revenue and cost structure, arrival processes, scheduling and rationing.

Revenues collected for serving an inpatient and an outpatient are given by r_n and r_s respectively with $r_n < r_s$ since the hospital receives a “fee-for-service” from outpatients while inpatient care is typically reimbursed. The waiting costs for an inpatient and an outpatient per each service-slot duration are w_n and w_s with $w_n < w_s$ because the inpatient is in the hospital. However the end-of-day penalty costs for each unserved inpatient and outpatient are π_n and π_s with $\pi_n > \pi_s$ since an untreatment of an inpatient at the end of day typically results in an extra day of hospitalization which incurs much higher cost than an untreatment of an outpatient (Green et al., 2006). Cost and revenue of emergency do not need to be taken into account as it has to be served immediately with the available capacity.

Emergency cases are rare and we assume that at most one emergency case arrives at each facility in each service slot. Let Z_1^j be the total emergency arrivals in j^{th} slot. Then Z_1^j is a random variable bounded from above by m , the number of facilities. For example, if we use Bernoulli trial to model the individual arrival of an emergency patient at each facility at j^{th} slot with a “success probability” p_e^j , then Z_1^j follows a binomial distribution $B(m, p_e^j)$. It is worthwhile highlighting that the upper bound m for Z_1^j assures the Markovian property for the service system since if the emergency arrivals in one slot exceed the number of facilities then the future state depends on the current state as well as the past history, violating Markovian property. Let Z_2^j be the total number of inpatient arrivals in j^{th} slot such that $\sum_{k=0}^M P(Z_2^j = k) = 1$ for some

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finite $M > 0$, which can be interpreted as the maximum possible number of inpatient arrivals during one service slot across m facilities.

For outpatients, the key information is the appointment schedule represented by an $m \times N$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mN} \end{pmatrix},$$

where $a_{ij} = 1$ if i^{th} facility's j^{th} slot is scheduled and 0 otherwise. Thus, the i^{th} row of A , denoted A_i , is the schedule for N slots of i^{th} facility; the j^{th} column of A , denoted A^j , is the j^{th} slot schedule for m facilities. Define $m_j \equiv \sum_{i=1}^m a_{ij}$, representing the number of scheduled outpatients at j^{th} slot across m facilities. Let Z_3^j be the number of outpatient show-ups for the j^{th} service slot. If outpatients show up independently with the same probability p_s^j , then Z_3^j follows a binomial distribution $B(m_j, p_s^j)$.

Given the appointment schedule A , the state space of the j^{th} service slot for the multiple-facility model is given by

$$S_j^m = \{(n, s) \mid 0 \leq n \leq M \times (j-1), 0 \leq s \leq \sum_{k=1}^{j-1} m_k\}, \quad j = 1, \dots, N,$$

which is characterized by the number of inpatients (n) and outpatients (s) waiting in queue at the beginning of the j^{th} service slot *after* decision. This means that at most $M(j-1)$ inpatients and at most $\sum_{k=1}^{j-1} m_k$ outpatients have been displaced and put in waiting because of giving absolute priority to emergency patients or through discretionary rationing between outpatients and inpatients. The system evolutions are illustrated in Figure 4.1.

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Figure 4.1 System evolutions of the multiple-facility model

When all facilities are subject to rationing, we call it a *complete rationing*, and all other cases are *partial rationings*. Let $V_j^A(n, s)$ be the optimal expected adjusted revenue-to-go from j^{th} slot onward after decision with n inpatients and s outpatients waiting in queue during j^{th} service slot given any outpatient appointment schedule A . The underlying complete rationing operator is defined as follow,

$$\tilde{T}_m V_j^A(n, s) = \max_{x+y=m} \{V_j^A(n-x, s-y) + x \cdot r_n + y \cdot r_s\}, n \geq m, s \geq m. \quad (4.1)$$

which can be reduced to a complete rationing. For instance, when $n \geq m$ and $s < m$, the rationing on m facilities is a partial rationing and the complete rationing is made only on s facilities because $(m-s)$ facilities can always be made available to inpatients. In this case, the partial rationing on m facilities is reduced to a complete rationing $\tilde{T}_s V_j^A(n-(m-s), s) + (m-s) \cdot r_n$.

Figure 4.2 below illustrates the rationing region in the state space S_j^m . When $n \geq m$ and $s \geq m$, the best action is to allocate m facilities among inpatients and outpatients. When $n \geq m$ and $1 \leq s \leq m-1$, $(m-s)$ inpatients will always be guaranteed for service and the rationing decision is made on the remaining s facilities. Similarly, when $1 \leq n \leq m-1$ and $s \geq m$, $(m-n)$ outpatients will always be guaranteed and rationing decision is made on the remaining n facilities.

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When $1 \leq n, s \leq m-1$ and $n+s \geq m+1$, $(m-s)$ inpatients and $(m-n)$ outpatients will always be served and the rationing decision is made on the remaining $(n+s-m)$ facilities. The remaining area is subject to no rationing.

Figure 4.2 Rationing region in the state space $S_j^m = \{(n, s) | 0 \leq n \leq M(j-1), 0 \leq s \leq \sum_{k=1}^j m_k\}$

We now present the dynamic capacity control problem for m facilities for any given appointment schedule A through Markov dynamic programming. For $(n, s) \in S_j^m$, $j = 1, \dots, N$,

$$V_j^A(n, s) = -nw_n - sw_s + \mathbf{E}_{Z_1^j, Z_2^j, Z_3^{j+1}} [H_{j+1}^{m-Z_1^j, A}(n + Z_2^j, s + Z_3^{j+1})] \quad (4.2)$$

The optimal allocation operator for the above problem is given by,

$$H_j^{m, A}(n, s) = \begin{cases} \tilde{T}_m V_j^A(n, s), & \text{if } n \geq m, s \geq m, \\ \tilde{T}_s V_j^A(n + s - m, s) + (m - s)r_n, & \text{if } n \geq m, 1 \leq s \leq m - 1, \\ \tilde{T}_n V_j^A(n, n + s - m) + (m - n)r_s, & \text{if } 1 \leq n \leq m - 1, s \geq m, \\ \tilde{T}_{n+s-m} V_j^A(n + s - m, n + s - m) + (m - s)r_n + (m - n)r_s, & \text{if } 1 \leq n, s \leq m - 1, n + s \geq m + 1, \\ V_j^A((n - m)^+, (s - m)^+) + \min(n, m) \cdot r_n + \min(s, m) \cdot r_s, & \text{otherwise,} \end{cases} \quad (4.3)$$

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for $m \geq 1$ and $H_j^{m,A}(n,s) = V_j^A(n,s)$ for $m = 0$, together with following boundary conditions:

$$H_{N+1}^{m-Z_1^j,A}(n,s) = f(n,s), Z_1^j = 0, \dots, m, j = 1, \dots, N, \quad (4.4)$$

where $f(n,s)$ is the *penalty function* defined over S_{N+1}^m .

The optimal allocation decision among m facilities is fully captured by $H_j^{m,A}(n,s)$ in (4.3).

$E[H_{j+1}^{m-Z_1^j,A}(n+Z_2^j, s+Z_3^{j+1})]$ represents the expected revenue-to-go from $(j+1)^{th}$ slot onward under optimal rationings. If Z_1^j emergency patients and Z_2^j inpatients arrive, in addition Z_3^{j+1} outpatients scheduled for $(j+1)^{th}$ slot appointment show up during service slot j , the rationing decision at the beginning of $(j+1)^{th}$ slot is made by allocating $m - Z_1^j$ facilities among $n + Z_2^j$ inpatients and $s + Z_3^{j+1}$ outpatients. Note that Z_1^j , Z_2^j and Z_3^{j+1} are independent random variables under our setting.

The optimal appointment scheduling problem can then be formulated as follows:

$$V^* = \max_A \{V_1^A(0,0)\}, \quad (4.5)$$

where V^* denotes the optimal expected profits simply assuming that the system is empty after the first decision each day.

Note that the one-facility model of Green et al. (2006) and two-facility model of Kolisch and Sickinger (2008) are a special case of our above multiple-facility model corresponding to $m = 1$, Z_1^j , Z_2^j and Z_3^{j+1} are all 0-1 Bernoulli-type random variables and $m = 2$, Z_1^j , Z_2^j are 0-1 and a_{ij} is any nonnegative number, respectively.

4.3 Optimal Allocation Policy of the Multiple-Facility Model

In this section, we investigate structural properties of the multiple-facility model and characterize the optimal allocation policy for the multiple-facility problem under any appointment schedule.

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4.3.1 Multimodularity Property of the Value Function

Note that the possible transitions of the underlying MDP of the multiple-facility model (4.2) lie on the line of $x + y = m$. Even though the multiple-facility model does not fall into the class of discrete-event stochastic systems with each possible transition bounded by the unit vector discussed in Chapter 3, it is evidently a two-dimensional resource-sharing problem with revenue maximization. We will show that multimodularity property $\mathcal{U}^s$ in the concavity context is preserved under the optimal rationing operator of the multiple-facility model and is sufficient for the monotone optimal switching-curve allocation policy for the multiple-facility problem.

Recall the property $\mathcal{U}^s$ defined in Table 3.1, a real-valued function $f: \mathbb{Z}^2 \rightarrow \mathbb{R}$, $f(x) \in \mathcal{U}^s$ if for $i = 1, 2$ and $j = 1, 2$

$$\Delta_i \Delta_j f(x) \leq 0, \quad (4.6)$$

$$\Delta_i f(x) \leq \Delta_i f(x - e_i + e_j). \quad (4.7)$$

Furthermore, if $f(x) \in \mathcal{U}^s$, then similarly by Corollary 3.3 we have

$$\Delta_i \Delta_i f(x) \leq 0, \quad (4.8)$$

$$f(x_1 - 2, x_2) - f(x_1 - 1, x_2 - 1) \leq f(x_1 - 1, x_2 - 1) - f(x_1, x_2 - 2). \quad (4.9)$$

In the multiple-facility context, (x_1, x_2) are the numbers of inpatients and outpatients waiting in queue, $\Delta_1 f(x_1, x_2)$ and $\Delta_2 f(x_1, x_2)$ are the opportunity costs for serving one inpatient and one outpatient respectively. The opportunity cost of serving one patient of a certain type is lower when the queue of its own is larger (componentwise concavity: $\Delta_j f(x + e_i) \leq \Delta_j f(x)$), and is also lower when the queue of the other type is larger (submodularity: $\Delta_j f(x + e_j) \leq \Delta_j f(x)$) because the problem exhibits partial substitutability among different types of patients due to sharing common resources. However, the opportunity cost of serving one patient of a certain type when the queue of its own is larger dominates that when the queue of the other type is larger (subconcavity:

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$\Delta_j f(x+e_i) \leq \Delta_j f(x+e_j)$). Properties of $\mathcal{U}^s$ highlight the importance of reduction in waiting or delaying as the congestion in queue grows.

Next we will apply Lemma 3.3, which makes the complete rationing operator $\tilde{T}_m V_j^A(n, s)$ tractable, to show structural properties of the multiple-facility model.

Proposition 4.1 Given any penalty function $f(n, s) \in \mathcal{U}^s$, let the value function of the multiple-facility model defined by (4.2) – (4.4). Then $V_j^A(n, s) \in \mathcal{U}^s$.

Proof. We prove by induction on j . For $j = N$, the property that $V_N^A(n, s) \in \mathcal{U}^s$ follows from the boundary condition (4.4). Suppose $V_{j+1}^A(n, s) \in \mathcal{U}^s$, we need to show that $V_j^A(n, s) \in \mathcal{U}^s$, which is equivalent to proving $H_{j+1}^{m,A}(n, s) \in \mathcal{U}^s$ for any $m \geq 1$ since taking expectation preserves those properties. It is evident from (4.3) that the key is to show $\tilde{T}_m V_{j+1}^A(n, s) \in \mathcal{U}^s$ for any $m \geq 1$. Note that as the revenues collected do not affect the properties of the operator, we can drop them for ease of presentation. Redefine an operator

$$g_m(n, s) \equiv T_m V_j^A(n, s) = \max_{x+y=m} V_j^A(n-x, s-y).$$

So the remaining task is to show $T_m V_j^A(n, s) \in \mathcal{U}^s$ if $V_j^A(n, s) \in \mathcal{U}^s$, that is, $g_m(n, s) \in \mathcal{U}^s$.

Proof of submodularity (4.6): $g_m(n, s) - g_m(n-1, s) \leq g_m(n, s-1) - g_m(n-1, s-1)$

Note that the property (4.6) can be stated explicitly as follows:

$$\begin{aligned} & \max_{x+y=m} V_j^A(n-x, s-y) - \max_{x+y=m} V_j^A(n-1-x, s-y) \\ & \leq \max_{x+y=m} V_j^A(n-x, s-1-y) - \max_{x+y=m} V_j^A(n-1-x, s-1-y), \end{aligned}$$

or alternatively,

$$\begin{aligned} & \max_{0 \leq x \leq m} V_j^A(n-x, s-m+x) - \max_{0 \leq x \leq m} V_j^A(n-1-x, s-m+x) \\ & \leq \max_{0 \leq x \leq m} V_j^A(n-x, s-m-1+x) - \max_{0 \leq x \leq m} V_j^A(n-1-x, s-m-1+x). \end{aligned}$$

For $x = 1, \dots, m$ and $\xi = 0, \dots, x-1$, there exist $i_x = x-1$ and $j_\xi = \xi+1$ such that

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$$\begin{aligned} & V_j^A(n-x, s-m+x) - V_j^A(n-1-(x-1), s-m+(x-1)) \\ & \leq V_j^A(n-(\xi+1), s-m-1+(\xi+1)) - V_j^A(n-1-\xi, s-m-1+\xi), \end{aligned}$$

which is equivalent to

$$\Delta_2 V_j^A(n-x, s-m+x) \leq \Delta_2 V_j^A(n-1-\xi, s-m+\xi).$$

The above inequality is derived from subconcavity (4.7) and componentwise concavity (4.8) of

$V_j^A(n, s)$ as follows:

$$\begin{aligned} & \Delta_2 V_j^A(n-x, s-m+x) \leq \Delta_2 V_j^A(n-x+(x-1-\xi), s-m+x-(x-1-\xi)) \\ & = \Delta_2 V_j^A(n-1-\xi, s-m+\xi+1) \leq \Delta_2 V_j^A(n-1-\xi, s-m+\xi). \end{aligned}$$

For $x = 0, \dots, m$ and $\xi = x, \dots, m$, there exist $i_x = x$ and $j_\xi = \xi$ such that

$$\begin{aligned} & V_j^A(n-x, s-m+x) - V_j^A(n-1-x, s-m+x) \\ & \leq V_j^A(n-\xi, s-m-1+\xi) - V_j^A(n-1-\xi, s-m-1+\xi), \end{aligned}$$

which is equivalent to

$$\Delta_1 V_j^A(n-x, s-m+x) \leq \Delta_1 V_j^A(n-\xi, s-m-1+\xi).$$

The above inequality follows from subconcavity (4.7) and submodularity (4.6) of $V_j^A(n, s)$:

$$\begin{aligned} & \Delta_1 V_j^A(n-x, s-m+x) \leq \Delta_1 V_j^A(n-x-(\xi-x), s-m+x+(\xi-x)) \\ & = \Delta_1 V_j^A(n-\xi, s-m+\xi) \leq \Delta_1 V_j^A(n-\xi, s-m-1+\xi). \end{aligned}$$

Therefore submodularity (4.6) is proved by Lemma 3.3.

Proof of Subconcavity (4.7): $g_m(n, s-1) - g_m(n-1, s-1) \leq g_m(n-1, s) - g_m(n-2, s)$

Note that subconcavity property (4.7) can be explicitly stated as follows:

$$\begin{aligned} & \max_{0 \leq x \leq m} V_j^A(n-x, s-m-1+x) - \max_{0 \leq x \leq m} V_j^A(n-1-x, s-m-1+x) \\ & \leq \max_{0 \leq x \leq m} V_j^A(n-1-x, s-m+x) - \max_{0 \leq x \leq m} V_j^A(n-2-x, s-m+x). \end{aligned}$$

For $x = 1, \dots, m$ and $\xi = 0, \dots, m-1$, there exist $i_\xi = \xi+1$ and $j_x = x-1$ such that

$$\begin{aligned} & V_j^A(n-x, s-m-1+x) - V_j^A(n-1-(\xi+1), s-m-1+(\xi+1)) \\ & (\leq) = V_j^A(n-1-(x-1), s-m+(x-1)) - V_j^A(n-2-\xi, s-m+\xi). \end{aligned}$$

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For $x = 1, \dots, m$, $\xi = m$ and $x = 0$, $\xi = 0, \dots, m$, there exist $i_x = x$ and $j_\xi = \xi$ such that

$$\begin{aligned} & V_j^A(n-x, s-m-1+x) - V_j^A(n-1-x, s-m-1+x) \\ & \leq V_j^A(n-1-\xi, s-m+\xi) - V_j^A(n-2-\xi, s-m+\xi), \end{aligned}$$

which is equivalent to

$$\Delta_1 V_j^A(n-x, s-m-1+x) \leq \Delta_1 V_j^A(n-1-\xi, s-m+\xi).$$

The above inequality follows from subconcavity (4.7) of $V_j^A(n, s)$:

$$\begin{aligned} \Delta_1 V_j^A(n-x, s-m-1+x) & \leq \Delta_1 V_j^A(n-x-(\xi+1-x), s-m-1+x+(\xi+1-x)) \\ & = \Delta_1 V_j^A(n-1-\xi, s-m+\xi). \end{aligned}$$

Therefore subconcavity (4.7) is proved by Lemma 3.3. ■

The essence of Proposition 4.1 is that the complete rationing operator

$$\tilde{T}_m V(n, s) = \max_{x+y=m} \{V(n-x, s-y) + x \cdot r_n + y \cdot r_s\}, n \geq m, s \geq m$$

preserves the multimodularity property of the value function $V(n, s)$. For the one-dimensional cases, Stidham (1978) has shown that the operator $\max_{\{0 \leq x \leq m\}} \{V(n-x) + x \cdot r\}$ preserves concavity of $V(n)$ in studying the optimal admission control for $GI/M/1$ queues. This has been adopted by Lautenbacher and Stidham (1999) and Talluri and van Ryzin (2004b) to prove the concavity of the value function for the single resource problem in revenue management. For the two-dimensional cases, the operators most studied have the form of either $\max_{\{x+y \leq 1\}} \{V(n-x, s-y)\}$ or $\min_{\{x+y \leq 1\}} V(n+x, s+y\}$ which are associated with 0-1 decision control, for instance, Ha (1997a,b), Carr and Duenyas (2000), Altman et al. (2001), Örmeci et al. (2001), Savin et al. (2005), and Green et al. (2006). However, these literatures usually study properties (4.6) – (4.9), which are not connected with multimodularity and make some properties redundant. In addition, they usually adopt the two approaches discussed in Section 3.4 to show the structural properties. Lemma 3.3 has demonstrated its strong efficiency in proving structural properties of $\tilde{T}_m V_j^A(n, s)$ beyond the class of control operators studied in Chapter 3. The parallel

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operator to ours in the continuous state space has been studied by Chen (2004) who considers one-machine two-product flexible manufacturing systems under periodic review. We will have a comparison between the multiple-facility model and the model of Chen (2004) in Section 4.6.

4.3.2 Structural Properties of the Optimal Allocation Policy

After establishing the multimodularity property of the value function, we now demonstrate that the optimal control policy for the allocation operator $H_j^{m,A}(n,s)$ can be fully characterized by a series of monotone switching curves.

With one facility, Green et al. (2006) have shown that there is a switching curve $n_{j,1}^0(s)$ that partitions the state space into two regions: above the switching curve accept an inpatient and below it accept an outpatient. Intuitively, it says that it is optimal to accept one type of patients in the state region where the system is subject to high loads of this type of patients in order to balance the congestion in queue.

With two facilities, for allocation operator $H_j^{2,A}(n,s)$, there are three possible admission decisions: (a) two inpatients, (b) one inpatient and one outpatient, and (c) two outpatients. We can imagine that there are two switching curves with a higher curve $n_{j,2}^0(s)$ separating the state space into two regions. Above $n_{j,2}^0(s)$ we allocate both facilities to two inpatients, below $n_{j,2}^0(s)$ we can re-separate it by a lower curve $n_{j,2}^1(s)$ into two regions, that is, between the curves of $n_{j,2}^0(s)$ and $n_{j,2}^1(s)$ we accept one inpatient and one outpatient, below $n_{j,2}^1(s)$ we accept two outpatients.

The same picture goes to $H_j^{m,A}(n,s)$ when m facilities are available and there should be m switching curves to separate the state space into $m+1$ regions which correspond to $m+1$ rationing decisions.

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The rest of this section is to validate the above conjecture and provide concrete description of the switching curves. To help understanding the analysis, we present the main result first in Proposition 4.2 which is further illustrated in Figure 4.3.

Proposition 4.2 For arbitrary appointment schedule A , the optimal capacity allocation policies if m facilities available for rationing are:

- (a). Accept m inpatients if and only if $n \geq n_{j,m}^0(s)$;
- (b). Accept $m-k$ inpatients and k outpatients if and only if $n_{j,m}^k(s) \leq n < n_{j,m}^{k-1}(s)$ for $k=1, \dots, m-1$;
- (c). Accept m outpatients if and only if $n < n_{j,m}^{m-1}(s)$;
- (d). $n_{j,m}^k(s) \uparrow s \downarrow k, \forall k=0, \dots, m-1$.

The monotone optimal switching-curve policy if m facilities available is illustrated in Figure 4.3 below.

Figure 4.3 Monotone optimal switching-curve policies if m facilities available

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We will start with showing the optimal switching-curve policy for the complete rationing operator $\tilde{T}_m V_j^A(n, s)$ and then for the allocation operator $H_j^{m,A}(n, s)$ in the following.

Proposition 4.3 For any appointment schedule A , the optimal allocation policy for the complete rationing operator $\tilde{T}_m V_j^A(n, s)$ is characterized by a series of monotone switching curves $\{\tilde{n}_{j,m}^k(s), k = 0, \dots, m-1\}$ with the following properties:

- (a). $\tilde{n}_{j,m}^k(s)$ is nondecreasing in s ;
- (b). $\tilde{n}_{j,m}^k(s)$ is nonincreasing in k .

Proof. (a). From the definition of $\tilde{T}_m V_j^A(n, s)$, we can rewrite it as follows,

$$\begin{aligned} \tilde{T}_m V_j^A(n, s) &= \max_{x+y=m} \{V_j^A(n-x, s-y) + xr_n + yr_s\} \\ &= \max_{k=0, \dots, m} \{V_j^A(n-m+k, s-k) + (m-k)r_n + kr_s\}, \quad n \geq m, s \geq m. \end{aligned}$$

For $k = 0, \dots, m-1$ and $i = k+1, \dots, m$, we need to show that

$$V_j^A(n-m+k, s-k) - V_j^A(n-m+i, s-i) \uparrow n \quad (4.10)$$

so that we can define the following series:

$$\tilde{n}_{j,m}^{k,i}(s) = \min \{n \mid V_j^A(n-m+k, s-k) - V_j^A(n-m+i, s-i) \geq (i-k)(r_s - r_n)\} \quad (4.11)$$

From subconcavity (4.7), we have

$$\begin{aligned} \Delta_1 V_j^A(n-m+i+1, s-i) &\leq \Delta_1 V_j^A(n-m+i+1-(i-k), s-i+(i-k)) \\ &= \Delta_1 V_j^A(n-m+k+1, s-k), \end{aligned}$$

i.e.,

$$V_j^A(n-m+k, s-k) - V_j^A(n-m+i, s-i) \leq V_j^A(n-m+k+1, s-k) - V_j^A(n-m+i+1, s-i).$$

Therefore (4.10) is true; hence (4.11) is well defined. Note that $\tilde{n}_{j,m}^{k,i}(s)$ is the minimum of n at decision point of service slot j when m facilities are available for rationing such that

$$V_j^A(n-m+k, s-k) + (m-k)r_n + kr_s \geq V_j^A(n-m+i, s-i) + (m-i)r_n + ir_s.$$

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We denote the switching-curve for the complete rationing operator $\tilde{T}_m V_j^A(n, s)$ as $\tilde{n}_{j,m}^k(s)$: above this curve we accept $(m-k)$ inpatients and k outpatients into service. Then $\tilde{n}_{j,m}^k(s)$ is the highest curve among the series $\tilde{n}_{j,m}^{k,i}(s)$, that is,

$$\tilde{n}_{j,m}^k(s) = \max_{i=k+1, \dots, m} \{\tilde{n}_{j,m}^{k,i}(s)\} \quad (4.12)$$

Next we show that

$$V_j^A(n-m+k, s-k) - V_j^A(n-m+i, s-i) \downarrow s \quad (4.13)$$

for $k=0, \dots, m-1$ and $i=k+1, \dots, m$ so that $\tilde{n}_{j,m}^k(s)$ is nondecreasing in s .

From subconcavity (4.7), we have

$$\Delta_2 V_j^A(n-m+k, s-k) \leq \Delta_2 V_j^A(n-m+k+(i-k), s-k-(i-k)) = \Delta_2 V_j^A(n-m+i, s-i),$$

i.e.,

$$V_j^A(n-m+k, s-k) - V_j^A(n-m+i, s-i) \leq V_j^A(n-m+k, s-k-1) - V_j^A(n-m+i, s-i-1).$$

From above inequality, we have

$$\begin{aligned} & V_j^A(\tilde{n}_{j,m}^{k,i}(s) - m + k, s - k - 1) - V_j^A(\tilde{n}_{j,m}^{k,i}(s) - m + i, s - i - 1) \\ & \geq V_j^A(\tilde{n}_{j,m}^{k,i}(s) - m + k, s - k) - V_j^A(\tilde{n}_{j,m}^{k,i}(s) - m + i, s - i) \geq (i - k) \cdot (r_s - r_n) \end{aligned}$$

Consequently, by definition of (4.11), the following inequality holds,

$$\tilde{n}_{j,m}^{k,i}(s-1) \leq \tilde{n}_{j,m}^{k,i}(s).$$

Therefore it follows from (4.12) that $\tilde{n}_{j,m}^k(s)$ is nondecreasing in s .

(b) After having established the monotonicity of $\tilde{n}_{j,m}^k(s)$ in s , we now prove the monotonicity of

$\tilde{n}_{j,m}^k(s)$ in k . For $k=0, \dots, m-1$, from (4.12) we have

$$\tilde{n}_{j,m}^k(s) = \max_{i=k+1, \dots, m} \{\tilde{n}_{j,m}^{k,i}(s)\} = \max_{i=k, \dots, m-1} \{\tilde{n}_{j,m}^{k,i+1}(s)\} \quad (4.14)$$

$$\tilde{n}_{j,m}^{k-1}(s) = \max_{i=k, \dots, m} \{\tilde{n}_{j,m}^{k-1,i}(s)\} = \max_{i=k, \dots, m-1} \{\max_{j=m} \{\tilde{n}_{j,m}^{k-1,i}(s)\}, \tilde{n}_{j,m}^{k-1,m}(s)\} \} \quad (4.15)$$

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Let us now compare $\max_{i=k, \dots, m-1} \{\tilde{n}_{j,m}^{k,i+1}(s)\}$ with $\max_{i=k, \dots, m-1} \{\tilde{n}_{j,m}^{k-1,i}(s)\}$. For $k=1, \dots, m-1$ and

$i=k, \dots, m-1$, by (4.11) we have

$$\tilde{n}_{j,m}^{k,i+1}(s) = \min \{n \mid V_j^A(n-m+k, s-k) - V_j^A(n-m+i+1, s-i-1) \geq (i+1-k)(r_s - r_n)\} \quad (4.16)$$

$$\tilde{n}_{j,m}^{k-1,i}(s) = \min \{n \mid V_j^A(n-m+k-1, s-k+1) - V_j^A(n-m+i, s-i) \geq (i+1-k)(r_s - r_n)\} \quad (4.17)$$

For $k=1, \dots, m-1$ and $i=k, \dots, m-1$, the following inequality holds

$$\begin{aligned} & V_j^A(n-m+k, s-k) - V_j^A(n-m+i+1, s-i-1) \\ & \geq V_j^A(n-m+k-1, s-k+1) - V_j^A(n-m+i, s-i), \end{aligned} \quad (4.18)$$

i.e.,

$$\begin{aligned} & V_j^A(n-m+k, s-k) - V_j^A(n-m+k-1, s-k+1) \\ & \geq V_j^A(n-m+i+1, s-i-1) - V_j^A(n-m+i, s-i). \end{aligned}$$

The above inequality is established through (4.9), integer concavity of $V_j^A(n, s)$ in (n, s) , as follows:

$$\begin{aligned} V_j^A(n-m+1, s-1) - V_j^A(n-m, s) & \geq V_j^A(n-m+2, s-2) - V_j^A(n-m+1, s-1) \\ & \geq V_j^A(n-m+3, s-3) - V_j^A(n-m+2, s-2) \\ & \geq \dots \\ & \geq V_j^A(n-1, s-m+1) - V_j^A(n-2, s-m+2) \\ & \geq V_j^A(n, s-m) - V_j^A(n-1, s-m+1). \end{aligned}$$

From inequality (4.18) and definitions (4.16) – (4.17), we can establish that

$$\begin{aligned} & V_j^A(\tilde{n}_{j,m}^{k-1,i}(s) - m + k, s - k) - V_j^A(\tilde{n}_{j,m}^{k-1,i}(s) - m + i + 1, s - i - 1) \\ & \geq V_j^A(\tilde{n}_{j,m}^{k-1,i}(s) - m + k - 1, s - k + 1) - V_j^A(\tilde{n}_{j,m}^{k-1,i}(s) - m + i, s - i) \geq (i+1-k) \cdot (r_s - r_n). \end{aligned}$$

Thus

$$\tilde{n}_{j,m}^{k-1,i}(s) \geq \tilde{n}_{j,m}^{k,i+1}(s), \text{ for } k=1, \dots, m-1 \text{ and } i=k, \dots, m-1,$$

i.e.,

$$\max_{i=k, \dots, m-1} \{\tilde{n}_{j,m}^{k-1,i}(s)\} \geq \max_{i=k, \dots, m-1} \{\tilde{n}_{j,m}^{k,i+1}(s)\} \text{ for } k=1, \dots, m-1.$$

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It can be easily followed from (4.16) and (4.17) that

$$\tilde{n}_{j,m}^{k-1}(s) \geq \tilde{n}_{j,m}^k(s) \text{ for } k = 1, \dots, m-1,$$

that is, $\tilde{n}_{j,m}^k(s)$ is nonincreasing in k . This proves the proposition. ■

Note that if there is no $n \in S_j^m$ such that (4.11) is true, we can artificially set $\tilde{n}_{j,m}^{k,i}(s) = n_j^{\max} + 1$ with $n_j^{\max} = M(j-1)$. If there exists some $k_1 \in [0, m-1)$ such that $\tilde{n}_{j,m}^{k_1}(s) = m$, then we set $\tilde{n}_{j,m}^{k_1+1}(s) = \dots = \tilde{n}_{j,m}^{m-1}(s) = m$ because $\tilde{n}_{j,m}^k(s)$ is nonincreasing in k . By adopting a similar approach as the proof of monotonicity of $\tilde{n}_{j,m}^k(s)$ in s , we can prove that $\tilde{n}_{j,m}^k(s)$ is also nondecreasing in m . However it is very complicated to prove the monotonicity of $\tilde{n}_{j,m}^k(s)$ in j for m -facility model because of the general operator $H_j^{m,A}(n, s)$. We conjecture that $\tilde{n}_{j,m}^k(s)$ is nonincreasing in j when $\pi_n > \pi_s$ and the penalty function $f(n, s)$ is nondecreasing in n and s . This is quite intuitive since the end-of-day penalty cost of inpatients are much higher than outpatients, the switching curves will be lower as nearing the end of the day. Our numerical examples will validate this property.

Let us now characterize the optimal switching-curve policy for the allocation operator $H_j^{m,A}(n, s)$. By separating rationing and no-rationing, the allocation operator can be rewritten as

$$H_j^{m,A}(n, s) = \begin{cases} \tilde{H}_j^{m,A}(n, s), & \text{if rationing,} \\ \bar{H}_j^{m,A}(n, s), & \text{if no-rationing,} \end{cases} \quad (4.19)$$

where

$$\begin{aligned} \tilde{H}_j^{m,A}(n, s) &= \max \{V_j^A(n-x, s-y) + xr_n + yr_s\} \\ \text{s. t. } &x + y = m, \\ &n + s > m, \\ &n \cdot s \neq 0, \\ &x \leq n, y \leq s, \end{aligned} \quad (4.20)$$

and $\bar{H}_j^{m,A}(n, s) = V_j^A((n-m)^+, (s-m)^+) + \min(n, m) \cdot r_n + \min(s, m) \cdot r_s$.

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The rationing operator $\tilde{H}_j^{m,A}(n,s)$ consists of four parts $\tilde{T}_s, \tilde{T}_{n+s-m}, \tilde{T}_m, \tilde{T}_n$ as illustrated in Figure 4.2. There exist a series of switching-curves respectively from Proposition 4.3 defined as follow:

$$\tilde{T}_s : \{\tilde{n}_{j,s}^k(s), k = 0, \dots, s-1; s = 1, \dots, m-1\} \text{ with } \tilde{n}_{j,s}^k(s) = \max_{i=k+1, \dots, s} \{\tilde{n}_{j,s}^{k,i}(s)\};$$

$$\tilde{T}_{n+s-m} : \{\tilde{n}_{j,n+s-m}^k(s), k = m-n, \dots, s-1; s = 1, \dots, m-1\} \text{ with } \tilde{n}_{j,n+s-m}^k(s) = \max_{i=k+1, \dots, s} \{\tilde{n}_{j,n+s-m}^{k,i}(s)\};$$

$$\tilde{T}_m : \{\tilde{n}_{j,m}^k(s), k = 0, \dots, m-1; s \geq m\} \text{ with } \tilde{n}_{j,m}^k(s) = \max_{i=k+1, \dots, m} \{\tilde{n}_{j,m}^{k,i}(s)\};$$

$$\tilde{T}_n : \{\tilde{n}_{j,n}^k(s), k = m-n, \dots, m-1; s \geq m\} \text{ with } \tilde{n}_{j,n}^k(s) = \max_{i=k+1, \dots, m} \{\tilde{n}_{j,n}^{k,i}(s)\}.$$

For all $k = m-n, \dots, s-1$ such that $\tilde{n}_{j,s}^k(s) \geq m$, we can artificially set $\tilde{n}_{j,n+s-m}^k(s) = \tilde{n}_{j,s}^k(s)$. For all $k = m-n, \dots, m-1$ such that $\tilde{n}_{j,m}^k(s) \geq m$, we set $\tilde{n}_{j,n}^k(s) = \tilde{n}_{j,m}^k(s)$. If there exists $k_1 = 0, \dots, s-2$ such that $\tilde{n}_{j,s}^{k_1}(s) = m$, then we set $\tilde{n}_{j,s}^{k_1+1}(s) = \dots = \tilde{n}_{j,s}^{s-1}(s) = m$. Similarly, if there exists $k_2 = 0, \dots, m-2$ such that $\tilde{n}_{j,m}^{k_2}(s) = m$, we set $\tilde{n}_{j,m}^{k_2+1}(s) = \dots = \tilde{n}_{j,m}^{m-1}(s) = m$.

Now we can define a series of switching-curves $\{n_{j,m}^k(s), k = 0, \dots, m-1\}$ for $H_j^{m,A}(n,s)$.

For $s = 1, \dots, m-1$,

$$n_{j,m}^k(s) = \tilde{n}_{j,s}^k(s) \text{ if } \tilde{n}_{j,s}^k(s) \geq m \text{ for all } k = 0, \dots, s-1 \quad (4.21)$$

otherwise,

$$n_{j,m}^k(s) = \begin{cases} \tilde{n}_{j,s}^k(s), & \text{for } k = 0, \dots, k_1, \\ \tilde{n}_{j,n+s-m}^k(s), & \text{for } k = k_1 + 1, \dots, s-1. \end{cases} \quad (4.22)$$

For $s \geq m$,

$$n_{j,m}^k(s) = \tilde{n}_{j,m}^k(s) \text{ if } \tilde{n}_{j,m}^k(s) \geq m, \text{ for all } k = 0, \dots, m-1 \quad (4.23)$$

otherwise,

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$$n_{j,m}^k(s) = \begin{cases} \tilde{n}_{j,m}^k(s), & \text{for } k = 0, \dots, k_2, \\ \tilde{n}_{j,n}^k(s), & \text{for } k = k_2 + 1, \dots, m-1. \end{cases} \quad (4.24)$$

Proposition 4.4 For arbitrary appointment schedule A , the optimal control policies if m facilities available are characterized by a series of monotone switching-curves $\{n_{j,m}^k(s), k = 0, \dots, m-1\}$ with the following properties:

- (a). $n_{j,m}^k(s)$ is nondecreasing in s ;
- (b). $n_{j,m}^k(s)$ is nonincreasing in k .

Proof. (a) From Proposition 4.3, we know that $\tilde{n}_{j,s}^k(s)$, $\tilde{n}_{j,n+s-m}^k(s)$, $\tilde{n}_{j,m}^k(s)$ and $\tilde{n}_{j,n}^k(s)$ are all nondecreasing in s . To prove that $n_{j,m}^k(s)$ is nondecreasing in s , we first show that

$$\tilde{n}_{j,s}^k(s)|_{s=m-1} \leq \tilde{n}_{j,m}^k(s)|_{s=m} \text{ for } k = 0, \dots, s-1 \text{ and}$$

$$\tilde{n}_{j,n+s-m}^k(s)|_{s=m-1} \leq \tilde{n}_{j,n}^k(s)|_{s=m} \text{ for } k = m-n, \dots, s-1.$$

For $k = 0, \dots, s-1$ and $i = k+1, \dots, s$ it can be shown that $\tilde{n}_{j,m}^{k,i}(m-1) \leq \tilde{n}_{j,m}^{k,i}(m)$ by adopting the approach of Proposition 4.3. Note that

$$\tilde{n}_{j,m-1}^k(m-1) = \max_{i=k+1, \dots, m-1} \tilde{n}_{j,m}^{k,i}(m-1) \leq \max_{i=k+1, \dots, m-1} \tilde{n}_{j,m}^{k,i}(m) \leq \max_{i=k+1, \dots, m} \tilde{n}_{j,m}^{k,i}(m) = \tilde{n}_{j,m}^k(m).$$

Consequently $\tilde{n}_{j,s}^k(s) \leq \tilde{n}_{j,m}^k(s)$ and $k_1 \leq k_2$ where k_1 and k_2 are defined by $\tilde{n}_{j,s}^{k_1}(s) = m$ and

$\tilde{n}_{j,m}^{k_2}(s) = m$ respectively when they do exist. Similarly we have $\tilde{n}_{j,n+(m-1)-m}^k(m-1) \leq \tilde{n}_{j,n}^k(m)$.

Consequently it follows that $\tilde{n}_{j,n+s-m}^k(s) \leq \tilde{n}_{j,n}^k(s)$ for $k = m-n, \dots, m-1$.

If (4.21) is true, then (4.23) is true and $n_{j,m}^k(s)$ is nondecreasing in s since $\tilde{n}_{j,s}^k(s) \leq \tilde{n}_{j,m}^k(s)$.

If (4.22) and (4.23) are true, then $\tilde{n}_{j,s}^k(s) \leq \tilde{n}_{j,m}^k(s)$ for $k = 0, \dots, k_1$ and $\tilde{n}_{j,n+s-m}^k(s) < m \leq \tilde{n}_{j,m}^k(s)$

for $k = k_1 + 1, \dots, s-1$. Hence $n_{j,m}^k(s)$ is nondecreasing in s . If (4.22) and (4.24) are true, then

$$\tilde{n}_{j,s}^k(s) \leq \tilde{n}_{j,m}^k(s) \text{ for } k = 0, \dots, k_1,$$

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$$\tilde{n}_{j,n+s-m}^k(s) < m \leq \tilde{n}_{j,m}^k(s) \text{ for } k = k_1 + 1, \dots, k_2,$$

$$\tilde{n}_{j,n+s-m}^k(s) \leq \tilde{n}_{j,n}^k(s) \text{ for } k = k_2 + 1, \dots, s - 1.$$

For all these cases there is one more switching curve $n_{j,m}^{m-1}(s)$ if $s \geq m$, which is nondecreasing in s . Summarizing the above discussions, it follows that $n_{j,m}^k(s)$ is nondecreasing in s .

(b) The monotonicity of $n_{j,m}^k(s)$ in k follows from the fact that $\tilde{n}_{j,m}^k(s)$ is nonincreasing in k . ■

Combining the results of Proposition 4.3 and Proposition 4.4, we can clearly describe the decision-making associated with the optimal capacity allocation policy, which is summarized in Proposition 4.2 with no need for a proof. In this section we have fully characterized the optimal resource allocation policy if m facilities available given any outpatient scheduling as the monotone switching-curve policy.

4.3.3 Numerical Examples

In this subsection, we perform a numerical experiment to illustrate the monotone optimal switching-curve policy for a three-facility model given certain appointment scheduling A .

The arrival distributions of emergencies and inpatients as well as the show-up probability of outpatients are the inputs for our multiple-facility model. They can be estimated from empirical data. For simplicity, we assume stationary arrival distributions of inpatients and emergencies in this numerical example. Readers are reminded that the structure of optimal control policies preserves under nonstationary arrival distributions as well. The cost and revenue data are taken from Green et al. (2006) and all parameters used in this numerical example on optimal control for 3-facility are summarized in Table 4.1.

Let the penalty function be a linear function $f(n, s) = -n\pi_n - s\pi_s$ and the appointment scheduling for outpatients be given by

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$$A = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}_{3 \times 20} \quad (4.25)$$

where the first 10 columns are ones and the last 10 columns are all zeros.

Figure 4.4 illustrates the optimal capacity allocation policy as the monotone switching-curve policy at slot $j = 15$ and 19 for our three-facility example. When three emergency patients arrive, there is no rationing decision since all three facilities are used to serve emergency patients. If two emergency patients arrive, the rationing decision is made on one facility and there is only one switching curve $\{n_{j,1}^0(s)\}$ separating the state space into two regions: above the curve serving the inpatient and under the curve serving the outpatient. If there is only one emergency arrival, there will be two switching curves $\{n_{j,2}^0(s), n_{j,2}^1(s)\}$ separating the state space into three regions corresponding to three rationing decisions as demonstrated in the graph. If no emergency arrives, the allocation decision is to ration three facilities among inpatients and outpatients and there exist three switching curves $\{n_{j,3}^0, n_{j,3}^1, n_{j,3}^2\}$. It is evident that the optimal switching curves are nondecreasing in s and nonincreasing in k . In addition they are changing over real time and nonincreasing in the decision time slot j .

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Table 4.1 Parameters of the dynamic resource allocation for the three-facility problem

Parameter	Value / Distribution / Function
Inpatient revenue, r_n	\$200
Outpatient revenue, r_s	\$1000
Inpatient waiting cost, w_n	\$0
Outpatient waiting cost, w_s	\$15
Inpatient end-of-day penalty, π_n	\$1000
Outpatient end-of-day penalty, π_s	\$100
Service slots in a workday, N	20
Arrival Distribution of Emergencies, $q_e(k)$	$q_e(0) = 0.72, q_e(1) = 0.2, q_e(2) = 0.06, q_e(3) = 0.02$
Arrival Distribution of Inpatients, $q_n(k)$	$q_n(0) = 0.1, q_n(1) = 0.2, q_n(2) = 0.4,$
	$q_n(3) = 0.25, q_n(4) = 0.04, q_n(5) = 0.01$
Show-up Probability of Outpatients, p_s	$p_s = 0.84$
Outpatient Scheduling, A	$a_{ij} = 1, i=1, \dots, 3; j=1, \dots, 10; a_{ij} = 0$ otherwise
Penalty Function, $f(n,s)$	$f(n,s) = -n\pi_n - s\pi_s$

Notes: $q_e(k) \equiv P(Z_1^{j-1} = k)$; $q_n(k) \equiv P(Z_2^{j-1} = k)$; $Z_3^j \sim B(m_j, p_s)$, $m_j = \sum_{i=1}^m a_{ij}$

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Figure 4.4 Optimal switching-curve policy for the three-facility problem

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4.4 Two Scheduling Policies and Relative Performances

We have fully characterized the structure of the optimal allocation policy for the multiple-facility problem under any given outpatient schedule A . Unfortunately, the subsequent optimal scheduling problem (4.5) is extremely “hard to solve”. In this section, we propose two numerically tractable scheduling rules, namely, vertical scheduling and horizontal scheduling, and evaluate their relative performances through numerical examples. Let us start with the following formal definition.

Definition 4.1 Given any appointment schedule represented by the matrix $A = (a_{ij})_{m \times N}$, then

- (a). it is called a *vertical scheduling* (VS) if there exists a pair of integer $(i_v, j_v): 0 \leq i_v \leq m$ and $1 \leq j_v \leq N$ such that

$$a_{ij} = \begin{cases} 1, & \text{for } i = 1, \dots, m; j = 1, \dots, j_v - 1, \\ 1, & \text{for } i = 1, \dots, i_v; j = j_v, \\ 0, & \text{otherwise.} \end{cases}$$

- (b). it is called a *horizontal scheduling* (HS) if there exists a pair of integers $(i_h, j_h): 1 \leq i_h \leq m$ and $0 \leq j_h \leq N$ such that

$$a_{ij} = \begin{cases} 1, & \text{for } i = 1, \dots, i_h - 1; j = 1, \dots, N, \\ 1, & \text{for } i = i_h; j = 1, \dots, j_h, \\ 0, & \text{otherwise.} \end{cases}$$

The two parameters for the vertical scheduling in above definition have the following interpretations: j_v is the last time slot open for outpatient appointment and i_v is the number of facilities available for outpatient booking on the last available time slot. The total number of bookings to be accepted under the vertical scheduling (i_v, j_v) , named as *panel size* (Green and Savin, 2008), is given by

$$S_v = (j_v - 1)m + i_v.$$

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For the horizontal scheduling (i_h, j_h) , the parameter i_h represents the number of facilities available for outpatient booking and among $i_h - 1$ of them can be fully booked for outpatient; and the parameter j_h represents the last time slot on i_h -th facility available for outpatient booking. The total number of bookings to be accepted under the horizontal schedule (i_h, j_h) is given by

$$S_h = (i_h - 1)N + j_h.$$

Given a total number of outpatient appointments $S = x$, we can easily derive the corresponding vertical scheduling and horizontal scheduling respectively as follows,

$$j_v = \left\lceil \frac{x}{m} \right\rceil + 1, i_v = x - \left\lceil \frac{x}{m} \right\rceil m$$

and

$$i_h = \left\lceil \frac{x}{N} \right\rceil + 1, j_h = x - \left\lceil \frac{x}{N} \right\rceil N,$$

where $[a]$ represents the integer component of a . Therefore we can characterize these two scheduling with only one parameter S_v or S_h as well.

The schedule matrix represented by (4.25) is a vertical scheduling with $(i_v, j_v) = (3, 10)$. An example of a horizontal scheduling with $(i_h, j_h) = (2, 4)$ is given as follows,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{3 \times 20} \quad (4.26)$$

Note that for any horizontal scheduling, what really matters is the number of facilities is available for outpatient bookings at each service slot instead of which specific facilities are available. For example, the following schedule leads to the same solution as schedule (4.26),

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 1 \end{pmatrix}_{3 \times 20}.$$

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This flexibility allows the decision maker to allocate the schedules among different facilities associated with different demands.

The two scheduling rules proposed in this section, vertical scheduling and horizontal scheduling, belong to threshold appointment schedules commonly used in medical diagnostic facilities (Green et al., 2006). The panel size S_v and S_h serve to balance the expected demand of three patient segments and supply in the first step. Then dynamic capacity allocation makes real-time control based on the congestion in queue for different patient segments.

Both vertical scheduling and horizontal scheduling can be easily implemented. For vertical scheduling, all facilities are scheduled for outpatients before j_v -th slot and are open for outpatient booking up to i_v facilities at j_v -th service slot. For horizontal scheduling, i_h facilities are scheduled for outpatients before (at) j_h -th slot and $(i_h - 1)$ facilities are scheduled after that. Furthermore, the proposed two scheduling heuristics are computationally tractable. The original optimal scheduling problem (4.5) requires checking all 2^{mN} possible schedules under two-dimensional dynamic programming, which is a NP-hard problem. However, two scheduling proposed in this section reduces the scheduling problem to search the panel size S_v or S_h over the integer range of $[0, mN]$.

We now use numerical examples to illustrate and compare two scheduling heuristics for a 3-facility problem with 20 service slots under different cost and revenue structures. We assess their relative performances and hope to gain managerial insights by comparing the difference of the optimal profits under the two heuristics. We use the same cost and revenue parameters from Green et al. (2006), which have studied three threshold-type scheduling heuristics for one facility by considering a broad range of scenarios. Two parameters are fixed, the revenue of outpatient r_s and the waiting cost of inpatient w_n . For other cost and revenue parameters, for simplicity, we just consider two values: low or high. Both stationary and nonstationary arrival processes for emergency patients will be examined. For the nonstationary case, our example assumes that the

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9th to 11th slots are associated with high frequency of emergency arrivals. Parameter values are summarized in Table 4.2.

Table 4.2 Parameters for scheduling of a three-facility example

Parameter	Value / Distribution / Function
Inpatient revenue, r_n	\$200, \$800
Outpatient revenue, r_s	\$1000
Inpatient waiting cost, w_n	\$0
Outpatient waiting cost, w_s	\$10, \$20
Inpatient end-of-day penalty, π_n	\$500, \$2000
Outpatient end-of-day penalty, π_s	\$100, \$300
Service slots in a workday, N	20
Stationary Distribution of Emergencies, $q_e(k)$	$q_e(0) = 0.72, q_e(1) = 0.2, q_e(2) = 0.06, q_e(3) = 0.02$
Nonstationary Distribution of Emergencies, $q_e^j(k)$	$q_e^j(0) = 0.72, q_e^j(1) = 0.2, q_e^j(2) = 0.06, q_e^j(3) = 0.02$
	$j=1, \dots, 8, 12, \dots, 20$
	$q_e^j(0) = 0.5, q_e^j(1) = 0.3, q_e^j(2) = 0.12, q_e^j(3) = 0.08$
	$j = 9, 10, 11$
Arrival Distribution of Inpatients, $q_n(k)$	$q_n(0) = 0.1, q_n(1) = 0.2, q_n(2) = 0.4,$ $q_n(3) = 0.25, q_n(4) = 0.04, q_n(5) = 0.01$
Show-up Probability of Outpatients, p_s	$p_s = 0.84$
Outpatient Scheduling, A	Vertical Scheduling, Horizontal Scheduling
Penalty Function, $f(n, s)$	$f(n, s) = -n\pi_n - s\pi_s$

Notes: $q_e(k) \equiv P(Z_1^j = k)$ is the stationary distribution of emergencies

$q_e^j(k) \equiv P(Z_1^j = k)$ is the nonstationary distribution of emergencies

Table 4.3 Optimal Profits for Vertical Scheduling (V_v^*) and Horizontal Scheduling (V_h^*)

		Stationary Emergency Arrival				Nonstationary Emergency Arrival			
$w_s = 10$		$\pi_s = 100$		$\pi_s = 300$		$\pi_s = 100$		$\pi_s = 300$	
r_n	π_n	V_v^*	V_h^*	V_v^*	V_h^*	V_v^*	V_h^*	V_v^*	V_h^*
200	500	28447	28447	28172	28172	27282	27282	26967	26964
200	2000	12393	12701	11689	11454	11079	11302	10375	9913
800	500	38903	39084	38786	38262	37650	37775	37552	36886
800	2000	34201	34516	33491	33254	32844	33069	32142	31672
		Stationary Emergency Arrival				Nonstationary Emergency Arrival			
$w_s = 20$		$\pi_s = 100$		$\pi_s = 300$		$\pi_s = 100$		$\pi_s = 300$	
r_n	π_n	V_v^*	V_h^*	V_v^*	V_h^*	V_v^*	V_h^*	V_v^*	V_h^*
200	500	28214	28214	27948	27947	26976	26976	26666	26668
200	2000	11515	12148	11052	10990	10178	10665	9721	9457
800	500	38740	38800	38740	38255	37508	37459	37508	36878
800	2000	33231	33912	32732	32743	31863	32375	31372	31211

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Table 4.3 presents the derived profits for vertical scheduling and horizontal scheduling heuristics under stationary and nonstationary emergency arrival distribution. Examining results from Table 4.3 shows that under both stationary and nonstationary arrival distribution of emergency patients, a horizontal scheduling generally achieves higher profit when the end-of-day penalty cost of outpatient π_s is relatively low; while a vertical scheduling outperforms a horizontal scheduling when π_s is relatively high.

Intuitively, horizontal scheduling should perform better because at each service slot it maintains some spare capacity protected for emergency and inpatient arrivals. However, if the penalty cost of outpatient π_s is high, vertical scheduling outperforms horizontal scheduling. The rationale behind is that scheduling outpatients vertically can avoid heavy penalty cost at the end of the day. Furthermore, the difference in realized profits is particularly significant when the revenue of inpatient (r_n) is high and penalty cost of inpatient (π_n) is low: $r_n = \$800$ and $\pi_n = \$500$. When the penalty cost of outpatient (π_s) is low and π_n is high, the profit improvement of a horizontal scheduling over a vertical scheduling is very significant and is increasing in w_s . This implies that the performance of a vertical scheduling or a horizontal scheduling is most sensitive to the penalty costs. This observation is consistent with the findings in Green et al. (2006), which say that the performance of threshold-type scheduling heuristics for one facility is most sensitive to the penalty costs.

Our scheduling heuristics are captured by the underlying panel size for outpatient appointments, namely S_v^* and S_h^* respectively, as summarized in Table 4.4. It is evident from Table 4.4 that both S_v^* and S_h^* are nonincreasing in π_s , w_s , r_n and π_n respectively. This is quite intuitive since higher waiting cost or penalty cost of outpatients will lead to less bookings. And similarly higher revenue or penalty cost of inpatients will reduce the number of service slots available for outpatient appointments because of high opportunity costs of displacing inpatients

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through rationing. We also notice that the total number of slots for outpatients under a horizontal scheduling is generally not lower than that of a vertical scheduling. Table 4.5 summarizes these qualitative observations from our numerical examples on relative performance of two heuristics.

Table 4.4 Optimal panel size for vertical scheduling and horizontal scheduling

		Stationary Emergency Arrival				Nonstationary Emergency Arrival			
$w_s = 10$		$\pi_s = 100$		$\pi_s = 300$		$\pi_s = 100$		$\pi_s = 300$	
r_n	π_n	S_v^*	S_h^*	S_v^*	S_h^*	S_v^*	S_h^*	S_v^*	S_h^*
200	500	60	60	59	59	59	59	58	58
200	2000	23	25	21	24	22	25	20	23
800	500	24	26	23	25	22	25	22	24
800	2000	23	25	21	24	22	24	19	23
		Stationary Emergency Arrival				Nonstationary Emergency Arrival			
$w_s = 20$		$\pi_s = 100$		$\pi_s = 300$		$\pi_s = 100$		$\pi_s = 300$	
r_n	π_n	S_v^*	S_h^*	S_v^*	S_h^*	S_v^*	S_h^*	S_v^*	S_h^*
200	500	60	60	58	58	59	59	58	58
200	2000	21	24	20	23	20	24	18	16
800	500	23	25	23	25	22	25	22	24
800	2000	21	24	19	23	20	24	18	15

Table 4.5 Summary on relative performance of VS and HS

π_s	w_s	π_n	r_n	Recommended Scheduling	Comments on Profit improvement
H				VS	
H		L	H	VS	most significant
L				HS	
L	H	H	H	HS	most significant, nondecreasing in w_s

Heuristic schedule for outpatient

$$S_v^*: \downarrow \pi_n, r_n, w_s, \pi_s, \sum_{j=1}^N E[Z_1^j]; S_h^*: \downarrow \pi_n, r_n, w_s, \pi_s, \sum_{j=1}^N E[Z_1^j]$$

$$S_v^* \leq S_h^*$$

Notes: H $\equiv$ high, L $\equiv$ low, $\downarrow \equiv$ nonincreasing

In this section, two tractable scheduling, vertical scheduling and horizontal scheduling, are proposed and their relative performances are evaluated. We recommend scheduling rules for different scenarios and gain managerial insights. Based on the two scheduling rules, we can further develop simple heuristics to determine the panel size.

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4.5 Conclusions

Escalating healthcare costs, fast-growing demands, and trends toward integration and coordination critically call for an effective and efficient capacity management for expensive medical facilities. In this chapter, we study capacity management problem for multiple medical diagnostic facilities with stochastic demands from three patient segments. We first solve dynamic capacity allocation problem and fully characterize the optimal dynamic capacity allocation policy given any outpatient appointment schedule. Then we solve scheduling problem through two numerically tractable scheduling rules and evaluate their relative performances.

In terms of modeling, our approach has extended the existing literatures to multiple-facility models and allowed more general arrival processes that can be either stationary or nonstationary. The key methodological innovation is to introduce the multimodularity and complete rationing concept together with a novel approach to study structural properties of the underlying stochastic dynamic control problem. The main contribution of this chapter is the result that the optimal allocation policy for the multiple-facility stochastic capacity management problem can be fully characterized by a series of monotone switching curves. In our view, the multiple-facility model developed in this chapter should be relevant to practice because key inputs to our model can be routinely estimated and validated from empirical data, as illustrated in our three-facility numerical example.

Given the optimal control policies, the underlying optimal scheduling problem is unfortunately intractable. In this chapter, we propose and evaluate two scheduling rules for outpatients – vertical scheduling and horizontal scheduling. Through numerical examples we find that the relative performance of the two scheduling is most sensitive to the end-of-day penalty costs. If the penalty cost for outpatient is not high and the penalty cost for inpatient is high, horizontal scheduling achieves higher profits whereas if the penalty cost of outpatient is high, vertical scheduling outperforms horizontal scheduling. This numerical finding on the relative

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performance of vertical scheduling and horizontal scheduling under different cost and revenue structures should provide some managerial insights in designing appropriate scheduling rules for multiple-facility management systems.

There are a few directions for further research. First, our model assumes a fixed service rate for each time slot at each facility. Hence, a challenging question is to study the multi-facility capacity management problem under stochastic service rates. Second, our model is a finite-horizon stochastic dynamic formulation with the objective to maximize expected daily revenue adjusted by social cost. It will be worthwhile to study the infinite-horizon problem, under which we must deal with the complicated issue related to carry-overs of inpatients and outpatients. Third, with multiple facilities, it seems to be natural to apply the network-based revenue management tools, which should be a fruitful area for further research. Last, but not the least, the optimal outpatient scheduling problem remains to be a challenging open question.

4.6 Appendix – Model Comparisons

In this appendix, we will make a comparison between our multiple-facility model (4.2) – (4.3) in healthcare and the model of Chen (2004) in manufacturing. Even though the two models are in different scenarios, they have some interesting linkages and essential differences between them. One of the key differences between these two models lies in that the multiple-facility model is defined over the discrete state space while the model of Chen (2004) is defined over the continuous state space. Through this comparison we can gain a deeper understanding of the structural property and monotone optimal control for different systems.

Chen (2004) studies one flexible production system with two products under periodic review and determines the production decision to minimize the total expected discounted cost over the planning horizon. The system evolutions are illustrated in Figure 4.5(a) below. Given the inventory level before decision x and y at the beginning of each period, any independently and

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identically distributed demands over period D_1 and D_2 , production decisions q_1 and q_2 , production cost c_1 and c_2 , production rate μ_1 and μ_2 for product 1 and product 2 respectively, the optimality equation is given by

$$f_n(x, y) = \min_{\mu_1^{-1} q_1 + \mu_2^{-1} q_2 \leq 1} \{L_1(X) + L_2(Y) + c_1 q_1 + c_2 q_2 + \alpha E[f_{n-1}(X - D_1, Y - D_2)]\}, \quad (\text{A-1})$$

where L_1 and L_2 are period inventory cost function, X and Y are inventory level after decision respectively.

Figure 4.5 Model comparisons between the multiple-facility model and the model of Chen (2004)

The model of Chen (2004) involves two streams of demands with the total capacity normalized as one, and the unmet demands can be backordered. Our multiple-facility model

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involves three streams of demands with the total service capacity of m facilities, and the emergency demand must be satisfied and the unmet demands from the other two streams can be put in queue or “backordered”. If interpreted in the model setting of Chen (2004), our multiple-facility model may be viewed as two streams of demand with *random* capacity due to the random emergency demand that must be satisfied. Another linkage is that the *negative* inventory levels of the model of Chen (2004) may be viewed as the number of patients in queue in our multiple-facility model.

Now let us turn our attention to the structural properties of the two models. Chen (2004) shows structural properties of the so-called μ -differential monotone for the value function of (A-1) in the continuous state space, which is in essence D -multimodularity with respect to D^s in the discrete state space. Chen (2004) proves the propagation of μ -differential monotone (see Theorem 1 of Chen (2004)) through discussing all the relative positions as illustrated in Figure 4.6(b). In contrast, we show the structural properties in Proposition 4.1 through a simple algebraic method directly.

Note that due to the assumption on the continuous state space, Chen (2004) shows the preservation of μ -differential monotone under different production rates, which may not hold for discrete state space (please refer to a similar paper of Ha (1997a) on the operator $\min_{\{x+y \leq 1\}} V(n+x, s+y)$ with $\mu_1 = \mu_2$). The optimal allocation operator (4.3) with *certain* capacity corresponds to that of Chen (2004) in the discrete state space with $\mu_1 = \mu_2$, which is natural and reasonable under our setting.

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Figure 4.6 μ -differential monotone and D -multimodularity with respect to D^s

With the structural properties, the optimal policies of the two models are characterized in the following Figure 4.7. Chen (2004) characterizes the hedging point policy which separate the state space corresponding to three decisions, namely, do not produce, produce product 1 only, produce product 2 only. Note that the negative inventory level of Chen (2004) corresponds to the number of patients in queue for our multiple-facility model, the region that does not produce corresponds to the third quadrant of the state space of our multiple-facility model. As there is no negative queue, therefore we do not have the corresponding region that does not serve any

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patients. The region that produces product 1 only and produces product 2 only in Figure 4.7(a) correspond to the highest and the lowest region in Figure 4.7(b) that serve inpatients only and serve outpatients only respectively. We have a full characterization of the optimal control policy on the state space between the two extreme cases.

Figure 1. Hedging policy regions.

(a) The model of Chen (2004)

(b) Multiple-facility model

Figure 4.7 Comparisons of the optimal control policies

To summarize, our multiple-facility model differs from the model of Chen (2004) in many significant ways. First, the multiple-facility model deals with *random* service capacity. Second, the multiple-facility model is defined over more realistic *discrete* state space. Finally, we identify *multimodularity* which fully characterizes the optimal monotone switching-curve policy. We show the propagation of structural properties through a simple and novel approach.

However, the two pieces of work present the essence of the different systems through different approaches, which will be helpful in enhancing the understanding of monotone optimal control for discrete-event stochastic systems.

5 Dynamic Pricing with Two Revenue Streams

5.1 Introduction

Traditional revenue management (RM) only takes account of *one* revenue stream from selling a perishable product with limited capacity. In many businesses such as airlines, hotels (standard or casino), events, freight companies (third-party logistics) and retailers, there is another significant stream of revenue associated with selling a product or service, namely, the presence of an *ancillary revenue* stream. Ancillary revenues for airlines, the so-called non-ticket revenues, include a la carte features, commission-based services (hotel, car rental, insurance, local tours, etc.) and frequent flier activities. Ancillary revenues for hotels, the so-called non-room revenues, include food and beverage, SPA, golf and function space, among others. Third-party logistics such as cargo and trucking companies can generate ancillary revenue from optional services related to time, handling and security of package delivery etc. Retailers can also generate ancillary revenue from extended warranties and additional services.

Total revenue consists of *primary revenue* and *ancillary revenue*. Airline revenue includes primary revenue from tickets and ancillary revenue from the related services. The ancillary revenue of Ryanair, the largest LCC in Europe, accounts for nearly 20% of Ryanair's total revenue. The ancillary revenue of American airlines, the second largest U.S. airline, has increased 60% to \$2.1 billion in 2008 from \$1.3 billion in 2002. Ancillary revenue has created a new opportunity for both full service carriers (FSC) and low cost carriers (LCC) to expand airline's overall revenue potential beyond traditional revenue management levers. Casino revenue includes gaming revenue, room revenue and other revenues. Gaming revenue is the main source of casino revenue and room revenue is the ancillary revenue while casino hotel serves to screen customers and attract these who generate higher gaming revenues (Talluri and van Ryzin, 2004).

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It is evident that ancillary revenue has played an important role in enhancing a firm's bottom line. To enhance the overall revenue companies must go beyond one revenue stream and examine all revenue sources. Harrah, the biggest gaming company in the world, has the goal for the Harrah's Cherokee to have a full occupancy for its hotel with an average room rate of \$0/night (Metters et al., 2008). Through customized pricing Harrah uses its hotel room inventory effectively to enhance its total revenue. Customers who generate high gaming revenues are offered a free room together with many complementary services while customers who generate little gaming revenues are requested to pay a very high price for a hotel room, which effectively discourages them to visit the casino. Harrah's 'killer' application of revenue management has increased revenue per room across the hotel chain by 15% compared to an average 3% to 7% in other industries. The unorthodox business model envisioned by Michael O'Leary, the CEO of Ryanair, predicts that customers may someday "fly for free". When the price of iPhone dropped from \$399 to \$199, AT&T was confident that the increased ancillary revenues from AT&T data plans and other services would pay off in the long-run. Many firms in various industries are now shifting the focus from one revenue stream to the total revenue that a customer can contribute.

However given the rich literature in revenue management and dynamic pricing, there is few literature addressing multiple revenue streams. This chapter studies a generic problem of dynamic pricing with two revenue streams, primary revenue and ancillary revenue. It differs from bundling pricing[§] such as cross selling of Netessine et al. (2006) and upselling of Aydin and Ziya (2008) in a significant way. We control one revenue stream of either primary or ancillary revenue in order to induce another revenue stream which is random. We are not aware of any papers that have analyzed this important issue of dynamic pricing with two revenue streams.

In this chapter, we first propose a multi-segment dynamic pricing model for casinos where customers are segmented through customer's primary gaming revenue profile and casinos have

[§] Cross selling or upselling is a special form of bundling (Aydin and Ziya, 2008) in which the customer is given the option to purchase the package only after purchasing a product.

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the direct pricing control on ancillary revenue from the room inventory. Customer's valuation for the whole experience in casino is dependent on the room price and the customer's winning profile. We identify a stochastic order on customer's winning profile which leads to a monotone room price. We show a threshold-type pricing policy for a special customer segmentation scheme that the customer's winning profile can be ranked in terms of the failure rate order.

We then study multi-segment dynamic pricing with ancillary revenue where customers are segmented through customer's utility for the primary product itself and firms have direct control on primary revenue through integrating ancillary revenue. Two business settings are further analyzed through the comparative statics analysis on the average ancillary revenue. When segments differ in average ancillary revenues, higher ancillary revenue of one segment will lead to a lower inventory price for this particular segment while a higher inventory price for the other segments. When all segments have the same average ancillary revenue, higher ancillary revenue always lead to lower inventory prices across all segments. Finally we also show that integrating ancillary revenue into multi-segment dynamic pricing will achieve higher expected total revenue than that takes no account of ancillary revenue.

There are three interesting new findings. First, in the casino operations, average gaming revenue that is usually used for market segmentation (see Metters et al., 2008) is insufficient to segment the market. Second, the optimal price can be negative, which is new in the literature even though it is common in practice. Third, in airline and standard hotel operations with ancillary revenue, one would expect that there should be no difference between incorporating and not incorporating average ancillary revenue into dynamic pricing as it is a constant. However, we show that taking account of average ancillary revenue can do better in revenue improvement. Our models and results may shed new light on customer valuation and market segmentation in revenue management.

The rest of this chapter is organized as follows. In the remainder of this section we survey related literature. In Section 5.2, we study multi-segment dynamic pricing for casinos. In section

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5.3 we study multi-segment dynamic pricing with ancillary revenue. In Section 5.4 numerical studies are used to illustrate the major analytical results. Section 5.5 gives the concluding remarks and future research directions. Proofs are provided in appendices in Section 5.6.

5.1.1 Literature Review

Dynamic pricing of a single perishable product has been studied by Gallego and van Ryzin (1994), Bitran and Mondschein (1997), Zhao and Zheng (2000), etc. This line of research focuses on dynamic pricing of limited inventories over the finite selling horizon with the Markovian demand from the homogeneous customer population. Dynamic pricing for multiple products has been studied by Gallego and van Ryzin (1997), You (1999), Maglaras and Meissner (2006), etc. For comprehensive reviews of the early work on dynamic pricing, we refer the reader to Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003), Talluri and van Ryzin (2004).

In response to more realistic demand assumptions, recent work has introduced non-Markovian demand, strategic customer behavior and customer choice behavior into dynamic pricing. Non-Markovian demand model studies dynamic pricing when demand in each period is dependent on history prices over multiple periods (Popescu and Wu, 2006; Ahn et al., 2007). Dynamic pricing with strategic customer behavior assumes that customers act strategically when facing different prices over time (Elmaghraby et al., 2006; Su, 2007; Levin et al., 2007; Aviv and Pazgal, 2008; Levina et al., 2009; Levin et al., 2009). Dynamic pricing with substitutable products explicitly incorporates customer choice behavior into pricing (Dong et al., 2008; Akcay et al., 2009; Zhang and Cooper, 2009; Suh and Aydin, 2009).

There is another growing interest in utilizing customer information for customized pricing in the context of dynamic pricing (Elmaghraby and Keskinocak, 2003). Advance information technology enables customer tracking systems to record detailed customer information including demographic profile, transaction history and customer purchase behavior etc. Customer

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information can be exploited for customer recognition and segmentation in order to implement multi-segment pricing and offer customized-services.

Netessine et al. (2006) study dynamic cross selling in e-commerce retailing which involves dynamic package selection and dynamic pricing of packages based on the customer information. Aydin and Ziya (2008) study the pricing of promotional products under upselling based on the information that the customer has just bought a regular product at the advertised price. Aydin and Ziya (2009) investigate personalized dynamic pricing of limited inventories by using the signal provided by the arriving customer to infer the customer's reservation. Kuo et al. (2009) examine dynamic pricing of limited inventories when a customer is allowed to negotiate the price and to make an offer.

Our work of multi-segment dynamic pricing for casinos where customers are segmented through the customer's gambling profile is closely related to personalized dynamic pricing model that leverages on customer signals (Aydin and Ziya, 2009). However, Aydin and Ziya (2009) only consider one revenue stream from the inventory. The signal information revealed by the customer is used to estimate the customer's reservation price for the product and thus engage in personalized pricing. This personalization may give customers an incentive to act strategically, which has not been addressed in their paper. In our paper, we consider two revenue streams associated with the inventory. Using the customer's gaming profile as the basis for market segmentation, we determine the optimal price for hotel room for each segment to maximize the expected total revenue. Customers self-segment themselves with their own gambling behavior and this behavior-based pricing scheme is consistent with the revelation principle; hence is incentive compatible (Fudenberg and Villas-Boas, 2006).

In summary, this chapter aims to make three main contributions. First, we make the first attempt to study dynamic pricing with two revenue streams, primary revenue and ancillary revenue. Second, we propose a novel model to study multi-segment dynamic pricing for casinos and study its structural properties and optimal pricing policies. Third, we study multi-segment

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dynamic pricing with ancillary revenue and gain managerial insights through the comparative statics analysis.

5.2 Multi-segment Dynamic Pricing for Casinos

Casinos' primary revenue is gaming revenue and ancillary revenue is from rooms and other services. Two popular edicts in the industry are that gamers play where they stay, and gamers play more when they stay (Talluri and van Ryzin, 2004). The average daily gaming revenue from different type of games can vary from \$20 to \$20,000 (Talluri and van Ryzin, 2004). Hence casino hotels are functioned as a screening mechanism to attract the most profitable customers to play in casino (Hendler and Hendler, 2004; Metters et al., 2008). Casinos use advanced customer tracking systems of reward program to record customer's information and gambling profile. Customers have strong incentives to use reward program when booking rooms and playing games (Metters et al., 2008). Thus casino is able to segment customers according to customer's profile, through which casino determines a room rate for each customer segment. In this section, we study multi-segment dynamic pricing the casino hotel rooms based on customer's gaming profile with the objective to maximize the expected total revenue. We first propose the basic model and explore its structure properties, then study the threshold-type pricing policy for a special segmentation scheme.

Before describing the model, we first give some definitions needed in this chapter.

Definition 5.1 $F(\cdot)$ is an increasing failure rate (IFR) distribution if $h(x) = \frac{f(x)}{\bar{F}(x)}$ is increasing for x such that $F(x) < 1$.

Increasing failure rate implies increasing generalized failure rate which is equivalent to increasing price elasticity of demand (Ziya et al. 2004). Increasing failure rate captures most common distributions such as uniform, normal, exponential (IFR in the weak sense, i.e., constant

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failure rate) and logistic. Distributions such as the Weibull, Beta, and Gamma are IFR distributions if their parameters fall into certain ranges (Bagnoli and Bergstrom, 2005).

Definition 5.2 For two cumulative distribution functions F_1 and F_2 (with the corresponding probability density function f_1 and f_2), F_1 is said to be less than F_2 in the failure rate order if

$$\frac{f_1(x)}{\bar{F}_1(x)} \geq \frac{f_2(x)}{\bar{F}_2(x)}, \forall x \text{ and we write as } F_1 \leq_{fr} F_2.$$

Note that failure rate order implies stochastic order, that is, if $F_1 \leq_{fr} F_2$ then F_1 is first-order stochastically dominant by F_2 , namely, $F_1 \leq_{FSD} F_2$ and $E[X_1] \leq E[X_2]$.

5.2.1 Model and Basic Structures

Consider a casino hotel that has N rooms to sell over the finite horizon T . The casino partitions customers into m segments according to the customer's gambling profile of winning. Denote the winning of customers from segment- i by a random variable W_i with the cumulative distribution F_i on the real-valued support and $E[W_i] = w_i$. Gaming revenue from segment- i for casino is given by $G_i = -W_i$ with $E[G_i] = g_i$. Note that either G_i or W_i can be negative and generally $g_i > 0$ (i.e., $w_i < 0$) which reflects that casino games are to the advantage of the casino house in the long run.

The valuation of the customer from segment- i is described by $V_i = W_i - p_i$. The customer is willing to accept the room rate p_i as long as $V_i > 0$. Therefore, the customer's willingness-to-pay can be modelled by $P(W_i > p_i)$, i.e., $\bar{F}_i(p_i)$, the probability that the customer believes to win more than the room rate. Casinos have sufficient information to estimate this distribution from empirical customer profiles. The selling horizon is divided into small subintervals within which at most one customer arrives. For $t = 1, \dots, T$, let λ_t^i be the probability of one customer from

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segment- i arrives during period t , $i = 1, \dots, m$ and λ_t^0 be the probability of no arrival during period t with $\sum_{i=0}^m \lambda_t^i = 1$. Note that the casino industry is highly competitive and rooms are yielded as a daily base (Hendler and Hendler, 2004).

Let $J_t(x)$ be the optimal expected total-revenue-to-go from period t onward given room inventory level x . The optimality equation of multi-segment dynamic pricing in casino is given by

$$J_t(x) = \max_{p_1, \dots, p_m} \left\{ \sum_{i=1}^m \lambda_t^i \bar{F}_i(p_i) (p_i - w_i + J_{t-1}(x-1)) + (1 - \sum_{i=1}^m \lambda_t^i \bar{F}_i(p_i)) J_{t-1}(x) \right\} \quad (5.1)$$

That is, casino needs to determine m prices for all customer segments given any time t and room inventory level x . The first term in (5.1) says that if the customer from segment- i accepts the room rate offered by the casino hotel, a demand is realized with the room revenue p_i and the average gaming revenue $-w_i$; the second term represents no selling. With some algebraic manipulations, it is observed that (5.1) is separable in p_i . The optimality equation can be rewritten by

$$\begin{aligned} J_t(x) &= J_{t-1}(x) + \sum_{i=1}^m \lambda_t^i \max_{p_i} \{ \bar{F}_i(p_i) (p_i - w_i - \Delta J_{t-1}(x)) \}, \\ J_0(x) &= 0, \quad x = 1, \dots, N; \quad J_t(0) = 0, \quad t = 1, \dots, T. \end{aligned} \quad (5.2)$$

Denote the revenue rate function from segment- i by

$$R_i(p_i, w_i, \Delta J_{t-1}(x)) = \bar{F}_i(p_i) (p_i - w_i - \Delta J_{t-1}(x)) \quad (5.3)$$

and the optimal price for segment- i is given by

$$p_i^*(w_i, x, t) = \arg \max_{p_i} R_i(p_i, w_i, \Delta J_{t-1}(x)). \quad (5.4)$$

Note that the opportunity cost $\Delta J_{t-1}(x)$ is implicitly dependent on $\vec{w} = (w_1, \dots, w_m)$, each element of which is further dependent on the winning distribution of each segment $F_1, \dots, F_m$ correspondingly, and the opportunity cost in turn will affect the optimal price for each segment.

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The sufficient conditions for the regularity of the revenue rate function (5.3) are identified below (also see Proposition 1 in Bitran and Mondschein (1997) for the case of homogenous customers).

Lemma 5.1 If $F_i(\cdot)$ is an increasing failure rate (IFR) distribution, then $R_i(p_i, w_i, \Delta J_{t-1}(x))$ is quasiconcave on p_i .

Proof. See Appendix. ■

By Lemma 5.1 there exists a unique optimal solution to the optimality equation (5.2) which is determined by the FOC of (5.3), i.e.,

$$1 - (p_i - w_i - \Delta J_{t-1}(x)) \cdot \frac{f_i(p_i)}{\bar{F}_i(p_i)} = 0 \quad (5.5)$$

Thus the optimal price for segment- i is given by

$$p_i^*(w_i, x, t) = \frac{\bar{F}_i(p_i^*(w_i, x, t))}{f_i(p_i^*(w_i, x, t))} + \Delta J_{t-1}(x) + w_i \quad (5.6)$$

which equals the current value plus the future value and plus the average winning (or minus the average gaming revenue).

For simplicity we use $\downarrow$ to indicate decreasing and $\uparrow$ to indicate increasing in the weak sense throughout this paper. Structural properties of the problem (5.2) are characterized in the following lemma.

Lemma 5.2 The multi-segment dynamic pricing problem (5.2) for casinos has the following structural properties:

- (a) $\Delta J_t(x) \downarrow x \uparrow t$;
- (b) $J_t(x) - J_{t-1}(x) \leq J_{t-1}(x) - J_{t-2}(x)$;
- (c) $p_i^*(w_i, x, t) \downarrow x \uparrow t$.

Proof. See Appendix. ■

Lemma 5.2 says that the intuitively monotone structures of the opportunity cost and the optimal price with respect to the inventory level and time still hold for multi-segment dynamic

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pricing for casinos based on the customer's gambling profile. In addition one would intuitively expect that higher average gaming revenue should be associated with a lower room rate, which is interestingly not true in general. The following proposition identifies the sufficient condition on the gaming profile for the price monotonicity.

Proposition 5.1 For the multi-segment dynamic pricing problem (5.2) for casinos, if $F_i \leq_{fr} F_j$ then $p_i^*(w_i, x, t) \leq p_j^*(w_j, x, t)$.

Proof. Note that when $F_i \leq_{fr} F_j$, we have $w_i \leq w_j$ and $\frac{f_i(x)}{\bar{F}_i(x)} \geq \frac{f_j(x)}{\bar{F}_j(x)}$. Hence, it follows that

$$\begin{aligned} \left. \frac{\partial R_i(p_i, w_i, \Delta J_{t-1}(x))}{\partial p_i} \right|_{p_i = p_j^*(w_j, x, t)} &= \bar{F}_i(p_j^*) \left[1 - (p_j^* - w_i - \Delta J_{t-1}(x)) \frac{f_i(p_j^*)}{\bar{F}_i(p_j^*)} \right] \\ &\leq \bar{F}_i(p_j^*) \left[1 - (p_j^* - w_j - \Delta J_{t-1}(x)) \frac{f_j(p_j^*)}{\bar{F}_j(p_j^*)} \right] \\ &= 0. \end{aligned}$$

Therefore, $p_i^*(w_i, x, t) \leq p_j^*(w_j, x, t)$ by the quasiconcavity of $R_i(p_i, w_i, \Delta J_{t-1}(x))$. ■

Proposition 5.1 says that merely higher average gaming revenue is insufficient to guarantee a lower room rate and a stronger condition on stochastic order is needed. It is shown that the failure rate order of customer's winning profile leads to monotone optimal room prices for different segments.

5.2.2 Threshold-type Optimal Pricing Policy for a Special Segmentation

Scheme

Now assume that casino partitions customers according to the failure rate order of customer's winning profile. Casino ranks m customer segments in terms of the failure rate order as $F_1 \leq_{fr} F_2 \leq_{fr} \dots \leq_{fr} F_m$ and we can gain sharper structures on the optimal pricing policy. With this segmentation, we have $w_1 \leq w_2 \leq \dots \leq w_m$, or equivally $g_1 \geq g_2 \geq \dots \geq g_m$. We call segment-1 the

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highest segment in ranking or the most valued customers and segment- m the lowest segment in ranking.

Under the regularity assumption of increasing failure distribution of $F_i(\cdot)$, the right-hand-side (RHS) function of (5.6) is decreasing in p_i , i.e.,

$$\frac{\bar{F}_i(p_i)}{f_i(p_i)} + \Delta J_{t-1}(x) + w_i \downarrow p_i,$$

and the left-hand-side (LHS) function is increasing in p_i . Thus there exists a unique cross point of the LHS and RHS function which determines the unique optimal solution $p_i^*(w_i, x, t)$. In addition, the RHS function of (5.6) is increasing in the segment i due to the segment ranking in terms of failure rate order, that is

$$\frac{\bar{F}_i(p_i)}{f_i(p_i)} + \Delta J_{t-1}(x) + w_i \uparrow i.$$

Hence the optimal segmented price $p_i^*(w_i, x, t)$ is monotone increasing in i . As the ranking of the segment becomes higher, the RHS function of (5.6) will drop down which will lead to a possible negative cross point or $p_i^*(w_i, x, t)$. We can define a threshold value $i_0(x, t)$ on segment as follows,

$$i_0(x, t) = \min \{i \mid p_i^*(w_i, x, t) \geq 0\}, \quad (5.7)$$

which separates the nonnegative optimal prices from negative optimal prices. Figure 5.1 illustrates the optimal segmented prices and the threshold value on segment.

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Figure 5.1 Optimal segmented prices and the threshold value on segment

Structures of the optimal pricing policy under the failure rate order segmentation scheme are characterized in the following proposition.

Proposition 5.2 The multi-segment dynamic pricing problem (5.2) for casinos with the segmentation scheme of $F_1 \leq_{fr} F_2 \leq_{fr} \dots \leq_{fr} F_m$ has the optimal pricing policy with the following structure:

(a) $p_i^*(w_i, x, t) \uparrow i$, i.e., $p_1^*(w_1, x, t) \leq p_2^*(w_2, x, t) \leq \dots \leq p_m^*(w_m, x, t)$;

(b) There exists a threshold value $i_0(x, t)$ such that

- if $i \geq i_0(x, t)$ then $p_i^*(w_i, x, t) \geq 0$;
- if $i < i_0(x, t)$ then $p_i^*(w_i, x, t) < 0$;

(c) $i_0(x, t) \uparrow x \downarrow t$.

Proof. (a) & (b) follow immediately from Proposition 1 and the argument above.

(c) According to (5.6), $i_0(x, t)$ is also determined by

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$$i_0(x, t) = \min \left\{ i \mid \frac{\bar{F}(p_i^*(w_i, x, t))}{f(p_i^*(w_i, x, t))} + \Delta J_{t-1}(x) + w_i \geq 0 \right\}.$$

Since $\Delta J_t(x) \downarrow x$ it follows that

$$\frac{\bar{F}(p_{i_0}^*(w_{i_0}, x, t))}{f(p_{i_0}^*(w_{i_0}, x, t))} + \Delta J_{t-1}(x-1) + w_{i_0} \geq \frac{\bar{F}(p_{i_0}^*(w_{i_0}, x, t))}{f(p_{i_0}^*(w_{i_0}, x, t))} + \Delta J_{t-1}(x) + w_{i_0} \geq 0.$$

By the definition of $i_0(x, t)$, we have $i_0(x, t) \geq i_0(x-1, t)$. Similarly we can show that $i_0(x, t) \leq i_0(x, t-1)$ due to that $\Delta J_t(x) \uparrow t$. ■

Proposition 5.2 shows a threshold-type pricing policy for a special customer segmentation scheme that the customer's winning profile can be ranked in terms of the failure rate order. Under such segmentation scheme, the optimal room price is monotone increasing in the ranking of the segments. Furthermore there exists a threshold value on the ranking of the segments such that if the segment ranking is lower than the threshold value, the optimal prices are nonnegative; if the ranking is higher than the threshold value, the optimal prices are negative, which, to our knowledge, is an interesting property that is new in the dynamic pricing literature even though it does exist in practice.

Negative price in casinos can be interpreted as the free room plus other concessions. As the gaming revenue outweighs the need to purely optimize room revenue, the purpose of a hotel room is to secure a more profitable revenue stream from another product (Pinchuk, 2002). This multi-segment dynamic pricing through customer's gambling profile of winning is to give away room revenue to attract high-valued customers with the objective to optimize the total revenue in casinos.

In this section, we propose a model studying multi-segment dynamic pricing for casinos where customers are segmented through customer's primary revenue profile. Casinos have direct control on ancillary revenue from rooms in order to induce primary revenue of gaming and leverage the expected total revenue. We identify a stochastic order on customer's primary

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revenue profile which leads to a monotone inventory price of rooms. We show a threshold-type pricing policy for a special customer segmentation scheme that the customer's winning profile can be ranked in terms of the failure rate order. The model and results will shed new light on customer valuation and market segmentation.

5.3 Multi-segment Dynamic Pricing with Ancillary Revenue

In airlines or standard hotels, the primary revenue is from ticket or room revenue; the ancillary revenue is from non-ticket or non-room revenue associated with the primary product. Different segments such as transient customers or corporate customers have different utility for the primary product as well as different spending behavior which generates different ancillary revenue. In this section, we study multi-segment dynamic pricing the primary product by integrating ancillary revenue with the objective to maximize the expected total revenue. We first propose the model and conduct a comparative statics analysis on the average ancillary revenue under two business settings. Then we compare the optimal pricing strategies and the expected total revenue of multi-segment dynamic pricing which integrates ancillary revenue and multi-segment dynamic pricing which takes no account of ancillary revenue.

5.3.1 Model and Comparative Statics Analysis

Assume that there are m customer segments for the primary product. The maximum willingness-to-pay of the product for customers from segment- i is U_i with the cumulative distribution F_i on the non-negative support. The customer from segment- i will purchase the product as long as the surplus $V_i = U_i - p_i$ is positive. Therefore the willingness-to-pay for the customer from segment- i is given by the tail probability of $P(U_i > p_i) = \bar{F}_i(p_i)$, which is effectively the demand function from segment- i . The average ancillary revenue from segment- i is g_i .

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Let $J_t(x)$ be the optimal expected total-revenue-to-go from period t onward given inventory level x . The optimality equation of multi-segment dynamic pricing with ancillary revenue is given by

$$J_t(x) = \max_{p_1, \dots, p_m} \left\{ \sum_{i=1}^m \lambda_i^t \bar{F}_i(p_i) (p_i + g_i + J_{t-1}(x-1)) + (1 - \sum_{i=1}^m \lambda_i^t \bar{F}_i(p_i)) J_{t-1}(x) \right\} \quad (5.8)$$

which can be rewritten as

$$\begin{aligned} J_t(x) &= J_{t-1}(x) + \sum_{i=1}^m \lambda_i^t \max_{p_i} \{ \bar{F}_i(p_i) (p_i + g_i - \Delta J_{t-1}(x)) \}, \\ J_0(x) &= 0, \quad x = 1, \dots, N; \quad J_t(0) = 0, \quad t = 1, \dots, T. \end{aligned} \quad (5.9)$$

The key difference between the model (5.2) and the model (5.9) lies in the customer's willingness-to-pay model. In the casino model (5.2), demand is mainly driven by the whole experience and customers' willingness-to-pay is dependent with the primary revenue which is stochastic. In the model (5.9) of an airline or a hotel with substantial ancillary revenue from different segments, the demand is mainly driven by the primary product itself and customers' willingness-to-pay is independent of ancillary revenue, which is considered to be incidental.

Let $g_{-i} = (g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_m)$ and the average ancillary revenue vector from m segments be

$$\vec{g} = (g_1, \dots, g_m) \equiv (g_i, g_{-i}), \quad i = 1, \dots, m.$$

$\vec{g}$ is a parameter vector that will affect the opportunity cost and finally influence the optimal segment-based price. Denote the revenue rate function from segment- i by

$$R_i(p_i, g_i, \Delta J_{t-1}(x, \vec{g})) = \max_{p_i} \{ \bar{F}_i(p_i) (p_i + g_i - \Delta J_{t-1}(x, \vec{g})) \} \quad (5.10)$$

and the optimal price for segment- i by

$$p_i^*(\vec{g}, x, t) = \arg \max_{p_i} R_i(p_i, g_i, \Delta J_{t-1}(x, \vec{g})) \quad (5.11)$$

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Note that the opportunity cost $\Delta J_{t-1}(x)$ is implicitly dependent on the ancillary revenues $\bar{g} = (g_1, \dots, g_m)$ and the distribution functions $F_1, \dots, F_m$, even though $g_1, \dots, g_m$ are independent with $F_1, \dots, F_m$. The opportunity cost in turn affects the optimal price for each segment.

Increasing failure rate distribution $F_i(\cdot)$ is sufficient for the quasiconcavity of the revenue rate function (similar to the proof of Lemma 5.1). Therefore the first-order-condition determines the unique optimal price $p_i^*(\bar{g}, x, t)$, i.e.,

$$p_i^*(\bar{g}, x, t) = \frac{\bar{F}_i(p_i^*(\bar{g}, x, t))}{f_i(p_i^*(\bar{g}, x, t))} + \Delta J_{t-1}(x, \bar{g}) - g_i \quad (5.12)$$

The basic monotone structures of the opportunity cost and the optimal price with respect to the inventory level and time still hold for multi-segment dynamic pricing with ancillary revenue. That is, the results of Lemma 5.2 for the problem (5.2) are valid for the problem (5.9). For the sake of completeness, we summarize these properties in the following lemma without a proof.

Lemma 5.3 The multi-segment dynamic pricing problem (5.9) with ancillary revenue has the following structural properties:

- (a) $\Delta J_t(x) \downarrow x \uparrow t$;
- (b) $J_t(x) - J_{t-1}(x) \leq J_{t-1}(x) - J_{t-2}(x)$;
- (c) $p_i^*(\bar{g}, x, t) \downarrow x \uparrow t$.

Next we will investigate on the effect of the ancillary revenues, through which we hope to gain deeper insights on its impact on the opportunity cost and the optimal segmented prices. We consider two business settings: different and same average ancillary revenue. We first consider the general case when different segment has different average ancillary revenue and the results on the comparative statics analysis are summarized below.

Proposition 5.3 The multi-segment dynamic pricing problem (5.9) with different average ancillary revenue has the following comparative statics results on the average ancillary revenue:

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- (a) $g_i - \Delta J_t(x, g_i, g_{-i}) \uparrow g_i$, i.e., $g_i - \Delta J_t(x, g_i, g_{-i}) \geq \tilde{g}_i - \Delta J_t(x, \tilde{g}_i, g_{-i})$ whenever $g_i > \tilde{g}_i$;
- (b) $\Delta J_t(x, g_i, g_{-i}) \uparrow g_i$, i.e., $\Delta J_t(x, g_i, g_{-i}) \geq \Delta J_t(x, \tilde{g}_i, g_{-i})$ whenever $g_i > \tilde{g}_i$;
- (c) $p_i^*(g_i, g_{-i}, x, t) \downarrow g_i$ for $\forall i$;
- (d) $p_j^*(g_i, g_{-i}, x, t) \uparrow g_i$ for $\forall j \neq i$.

Proof. See Appendix. ■

Proposition 5.3 studies the marginal effect of ancillary revenue on the opportunity cost and the optimal prices. It says that when all other factors hold constant, higher average ancillary revenue from one segment implies a higher opportunity cost and the marginal difference for the opportunity cost is bounded by the marginal difference of the average ancillary revenue. Higher average ancillary revenue from one segment will lead to a lower inventory price for this particular segment while a higher inventory price for other segments. For instance, when customers from certain segment change the reservation channel (such as Global Distribution Systems or Internet Distribution Services) which is associated with different ancillary revenue through that channel, higher average ancillary revenue of this segment will lead to a lower inventory price while increase the inventory price for the other segments.

We then consider the special case when different segment has the same average ancillary revenue, for instance, in sport stadium there is not much difference of the ancillary revenue among customers. Assume that all customer segments generate the same average ancillary revenue, e.g., $g_1 = \dots = g_m = g$. The optimality equation of multi-segment dynamic pricing with the same average ancillary revenue is given by

$$\begin{aligned} J_t(x) &= J_{t-1}(x) + \sum_{i=1}^m \lambda_t^i \max_{p_i} \left\{ \bar{F}_i(p_i)(p_i + g - \Delta J_{t-1}(x)) \right\}, \\ J_0(x) &= 0, \quad x = 1, \dots, N; \quad J_t(0) = 0, \quad t = 1, \dots, T. \end{aligned} \quad (5.13)$$

We show that the optimal price for each segment is monotone on the average ancillary revenue in the following proposition.

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Proposition 5.4 For any $g > \tilde{g}$, the multi-segment dynamic pricing problem (5.13) with the same average ancillary revenue, has the following results on the comparative statics:

- (a) $g - \Delta J_i(x, g) \uparrow g$, i.e., $g - \Delta J_i(x, g) \geq \tilde{g} - \Delta J_i(x, \tilde{g})$ whenever $g_i > \tilde{g}_i$;
- (b) $\Delta J_i(x, g) \uparrow g$, i.e., $\Delta J_i(x, g) \geq \Delta J_i(x, \tilde{g})$ whenever $g_i > \tilde{g}_i$;
- (c) $p_i^*(g, x, t) \downarrow g$, for $\forall i$.

Proof. See Appendix. ■

Proposition 5.4 says when the all customer segments generate the same average ancillary revenue, higher average ancillary revenue will lead to a lower inventory price across all segments. For instance, a firm has two same hotels in different locations (e.g., one in the downtown, the other is in the resort) with the same customer segments. Assume that each segment has the same ancillary revenue. However, different location is associated with different ancillary revenue such as restaurant revenue. The downtown hotel has lower ancillary revenue from restaurant due to the plenty of choices and competitors while the resort hotel has higher ancillary revenue. Proposition 5.4 gives a guideline on how to price hotel rooms for the two hotels.

5.3.2 Comparison with Traditional Dynamic Pricing

Multi-segment dynamic pricing with ancillary revenue has integrated average ancillary revenue into pricing decision. Traditional dynamic pricing, for instance the multi-segment dynamic pricing model in Maglaras and Meissner (2006), does not take account of ancillary revenue into pricing. In this subsection we will compare the optimal pricing strategies and the expected total revenue of multi-segment dynamic pricing which integrates ancillary revenue with traditional multi-segment dynamic pricing which takes no account of ancillary revenue.

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Let $p_i^*(0, x, t)$ be the optimal price of multi-segment dynamic pricing which takes no account of ancillary revenue. When all customer segments generate the same average ancillary revenue, we have

$$p_i^*(g, x, t) \leq p_i^*(0, x, t) \text{ for } \forall i,$$

by the result of Proposition 5.4(c). That is, multi-segment dynamic pricing with same ancillary revenue will set a lower inventory price for each segment than multi-segment dynamic pricing which takes no account of ancillary revenue. However when the average ancillary revenue is different among customer segments it is not necessarily true for

$$p_i^*(\bar{g}, x, t) \leq p_i^*(\bar{0}, x, t) \text{ for } \forall i$$

and the trade-off among segments are much more complicated. Next we discuss the impact of different pricing strategies on the expected total revenue.

Let $I_t(x)$ be the optimal expected inventory-revenue-to-go from t period onward. The optimality equation of multi-segment dynamic pricing which takes no account of ancillary revenue is given by

$$\begin{aligned} I_t(x) &= I_{t-1}(x) + \sum_{i=1}^m \lambda_t^i \max_{p_i} \{ \bar{F}_i(p_i)(p_i - \Delta I_{t-1}(x)) \}, \\ I_0(x) &= 0, \quad x = 1, \dots, N; \quad I_t(0) = 0, \quad t = 1, \dots, T. \end{aligned} \tag{5.14}$$

Denote the optimal inventory-revenue-maximization price and pricing policy by $p_i^*(0, x, t)$ and π_i^* respectively. So the expected inventory revenue starting with N seats and T period of time under the optimal pricing policy π_i^* is denoted by $I_T(N) \equiv I^*$.

For multi-segment dynamic pricing which takes no account of ancillary revenue, the expected ancillary revenue is contingent on the optimal inventory-revenue-maximization pricing policy π_i^* . Let $U_t(x)$ be the expected secondary-revenue-to-go under policy π_i^* , we have the following recursive relationship

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$$U_t(x) = \sum_{i=1}^m \lambda_i^i \bar{F}(p_i^*(0, x, t))(g_i + U_{t-1}(x-1)) + (1 - \sum_{i=1}^m \lambda_i^i \bar{F}(p_i^*(0, x, t)))U_t(x),$$

which is rewritten as

$$\begin{aligned} U_t(x) &= U_{t-1}(x) + \sum_{i=1}^m \lambda_i^i \bar{F}(p_i^*(0, x, t))(g_i - \Delta U_{t-1}(x)), \\ U_0(x) &= 0, \quad x = 1, \dots, N; \quad U_t(0) = 0, \quad t = 1, \dots, T. \end{aligned} \quad (5.15)$$

In addition, we have the following recursive relationship for the expected total-revenue-to-go

$\hat{J}_t(x)$ under π_t^* ,

$$\begin{aligned} &I_t(x) + U_t(x) \\ &= I_{t-1}(x) + U_{t-1}(x) + \sum_{i=1}^m \lambda_i^i \bar{F}(p_i^*(0, x, t))(p_i^*(0, x, t) - \Delta I_{t-1}(x)) + \sum_{i=1}^m \lambda_i^i \bar{F}(p_i^*(0, x, t))(g_i - \Delta U_{t-1}(x)), \end{aligned}$$

i.e.,

$$\hat{J}_T(N) = I^* + U_T(N) \quad (5.16)$$

where $\hat{J}_t(x) = I_t(x) + U_t(x)$ and $\Delta \hat{J}_t(x) = \Delta I_t(x) + \Delta U_t(x)$.

The expected total revenue under multi-segment dynamic pricing which takes no account of ancillary revenue is given by

$$\hat{J}_T(N) = I^* + U_T(N) \quad (5.17)$$

The impact of different pricing strategies on the expected total revenue is given in the following proposition.

Proposition 5.5 Multi-segment dynamic pricing which integrates ancillary revenue achieves higher expected total revenue than that of multi-segment dynamic pricing which takes no account of ancillary revenue, that is, $J_T(N) \geq \hat{J}_T(N)$.

Proof. It follows that (5.16) satisfies the recursive relationship of the optimality equation (5.9). ■

Proposition 5.5 says that integrating ancillary revenue into dynamic pricing decisions will achieve higher expected total revenue than that of taking no account of ancillary revenue into pricing decisions. In particular, for multi-segment dynamic pricing with the same average

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ancillary revenue, we summarize the results on different pricing strategies in the following corollary.

Corollary 5.1 For multi-segment dynamic pricing with the same average ancillary revenue g , it will set a lower inventory price while achieve higher expected total revenue than multi-segment dynamic pricing which takes no account of ancillary revenue. That is,

$$(a) \ p_i^*(g, x, t) \leq p_i^*(0, x, t);$$

$$(b) \ J_T(N) \geq \hat{J}_T(N).$$

Corollary 5.1 says that traditional dynamic pricing taking no account of ancillary revenue underestimates the revenue. A higher inventory price of traditional dynamic pricing has two marginal impacts, reducing the likelihood of sale and further affecting the ancillary revenue.

In summary, this section studies multi-segment dynamic pricing with ancillary revenue where customers are segmented according to their demand for the primary product. When different segments have different average ancillary revenue, it is shown that increasing the average ancillary revenue of one segment will lead to a lower inventory price of this particular segment while a higher inventory price of the other segments. However, when all segments have the same average ancillary revenue, higher ancillary revenue always leads to lower inventory prices across all segments. It is also shown that multi-segment dynamic pricing which integrates ancillary revenue will achieve higher expected total revenue than multi-segment dynamic pricing which takes no account of ancillary revenue.

5.4 Numerical Examples

In this section, we conduct a series of numerical studies to validate the major analytical results discussed in the previous sections. We first give an example of multi-segment dynamic pricing for casinos and then an example of multi-segment dynamic pricing with ancillary revenue.

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Example 1 Multi-segment dynamic pricing for the casino

Assume that the casino hotel has the room inventory of $N=10$ and a selling horizon of $T=15$. There are three segments with normally distributed winning profile $W_i \sim N(\mu_i, \sigma_i)$, i.e., $W_1 \sim N(-1000, 1000)$, $W_2 \sim N(-500, 250)$, $W_3 \sim N(-250, 250)$. Note that a normal distribution allows either positive or negative winning. For normal distribution it can be show that $\mu_1 < \mu_2, \sigma_1 = \sigma_2$ implies $F_1 \leq_{fr} F_2$; while $\mu_1 = \mu_2, \sigma_1 < \sigma_2$ may not necessarily imply $F_1 \leq_{fr} F_2$. The arrival probabilities of the three segments are listed in Table 5.1. Figure 5.1 illustrates the optimal room prices $p_i^*(w_i, x, t)$.

Table 5.1 Arrival probabilities

t	1-5	6-10	11-15
λ_t^1	0.4	0.3	0.1
λ_t^2	0.3	0.3	0.3
λ_t^3	0.1	0.2	0.5

Figure 5.2 Optimal room prices $p_i^*(w_i, x, t)$ for multi-segment dynamic pricing in casino

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From Figure 5.2, we have the following observations:

- a. The optimal price $p_i^*(w_i, x, t)$ is decreasing in the inventory level x and increasing in time t .
- b. As the winning profile of segment-2 is less than that of segment-3 in the failure rate order, i.e., $W_2 \leq_{fr} W_3$, by Proposition 5.1 we have $p_2^*(w_2, x, t) \leq p_3^*(w_3, x, t)$ which is evident in the graph.
- c. Even though the average gaming revenue from segment-1 is higher than that from segment-2, i.e., $E[G_1] \geq E[G_2]$, we do not observe $p_1^*(w_1, x, t) \leq p_2^*(w_2, x, t)$, but to the contrary. It can be easily verified that $f_1(x)/\bar{F}_1(x) \geq f_2(x)/\bar{F}_2(x)$ does not hold for all x , thus it does not satisfy the sufficient condition in Proposition 5.1. It is intuitive that although the average gaming revenue of customers in segment-1 is higher, its variation and the probability to win is also higher, thus its room rate should be higher. Therefore, merely higher average gaming revenue can not guarantee a lower room rate and a stronger condition of failure rate order is needed.
- d. The optimal price can be negative, which can be interpreted as free room plus other amenities in casino (also see Table 1 in Metters et al., 2008). As the gaming revenue outweighs the need to purely optimize room revenue, the purpose of casino's hotel room is to secure a more profitable revenue stream from another product to be in the package (Pinchuk, 2002). This multi-segment dynamic pricing model for casinos allows giving away rooms to attract high-valued gamblers with the objective to optimize the expected total revenue of casinos. It will shed new light on customer valuation and market segmentation.

Example 2 Multi-segment dynamic pricing with ancillary revenue

Assume that a firm has the perishable inventory of $N = 10$ such as standard hotel rooms or airline seats etc., and a selling horizon of $T = 20$. There are two segments, for instance, segment-1 is corporate customers and segment-2 is transient customers. It is estimated that segment-1 has a willingness-to-pay distribution of Weibull(200, 2) and segment-2 has a willingness-to-pay distribution of Weibull(100, 2). Note that Weibull distribution is an increasing failure rate

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distribution and it is flexible with a variety of behavior of reservation prices, such as normal, exponential, a heavier left tail or a heavier right tail, etc. Thus Weibull distribution is frequently used to model reservation distribution (see Bitran and Mondschein, 1997). The arrival probabilities of the two segments are listed in Table 2, where there is a clear pattern that corporate customers are more likely to arrive when closer to the date of arrival while transient customers book earlier. The optimal pricing policies $p_i^*(\vec{g}, x, t)$ for multi-segment dynamic pricing with ancillary revenue are demonstrated in Figure 5.3.

Figure 5.3(a) illustrates the comparative statics of the ancillary revenue. For instance, consider two scenarios of $\vec{g} = (0, 50)$ and $\vec{g}' = (150, 50)$, i.e., with the segment-2 customers fixed at an average ancillary revenue of 50. When segment-1 customers increase ancillary revenue from no spending to an average of 150, the optimal inventory price for segment-1 is decreasing while the optimal inventory price for segment-2 is increasing, which is consistent with Proposition 5.3. However, when both segments increase ancillary revenue from no spending to an average of 150 and 50 respectively, it is not necessarily true that $p_i^*(\vec{g}, x, t) \leq p_i^*(\vec{0}, x, t)$ with $\vec{g} = [150 \ 50]$. It is observed from Figure 5.3(b) that the optimal price of segment-1 is decreasing while the optimal price of segment-2 can either increase or decrease. That is, integrating ancillary revenue into multi-segment dynamic pricing does not necessarily lead to a lower inventory price for each segment except when each segment has same ancillary revenue. Nevertheless, the expected total revenue of integrating ancillary revenue $\vec{g}$ into multi-segment dynamic pricing is 1809 while the expected total revenue of traditional multi-segment dynamic pricing which takes no account of ancillary revenue is 1711, with a 5.4% of improvement on the expected total revenue. This finding validates Proposition 5.5.

Table 5.2 Arrival probabilities

t	1-10	11-20
λ_t^1	0.6	0.1
λ_t^2	0.1	0.6

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(a)

(b)

Figure 5.3 Multi-segment dynamic pricing with ancillary revenue

5.5 Conclusions

In order to expand the overall revenue potential and enhance long-term profitability, it is critically important for the firms to look at the whole revenue picture and take into account of the full customer value into revenue management. For this purpose, this paper studies a generic revenue management problem through dynamic pricing with two revenue streams, primary revenue and ancillary revenue. The two revenue streams have played an integral part in total revenue of a firm and constitute a full valuation of a customer's total contribution.

In this paper, we have mainly considered two different but related models which can be applied in many different business settings. In the first model, the primary revenue is stochastic and the customer's primary revenue profile is used as the basis for market segmentation. Firms make dynamic control on the inventory price of the ancillary product (which generates ancillary revenue) in order to induce more profitable primary revenue. This model is developed in the context of casino operations and it can be applied to other situations with similar characteristics. In the second model, the ancillary revenue is stochastic and incidental, and the demand is mainly driven by the primary product itself with different utilities or segments. Firms make dynamic pricing decision for the inventory of the primary product (which generates primary revenue) by integrating ancillary revenues. This model can be applied to airlines, standard hotels and among others which share similar characteristics on the revenue sources. Structural properties and comparative statics results of these two models should be useful in implementation as well as gaining managerial insights.

We believe that incorporating two or multiple revenue streams is of great importance to both RM practices and RM research. This paper makes a first attempt along this direction and studies multi-segment dynamic pricing of a perishable product in the presence of another revenue stream. There are many directions for future research. First, the second revenue stream, either primary or ancillary, is stochastic and can not be collected until the consumption day or the end of

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the service. How a firm makes a trade-off between the first assured stream of inventory revenue with the second stochastic stream of revenue is quite interesting and challenging to be explored because the stochastic stream of revenue is intrinsically riskier than the inventory revenue. Second, in this paper we have not explicitly considered the capacity constraint of the other product. For instance, when the casino has the capacity constraint, or a large hotel which serves both casino and resort customers, how will the pricing decision affected by the capacity constraint of another product? How will the allocation or pricing decision be affected by the stochastic primary revenue from casino customers and resort customers? This will lead to a two-dimensional Markov decision process, which will surely be very challenging.

5.6 Appendix – Proofs

Proof of Lemma 5.1:

Proof. The First-Order-Condition (FOC) of (5.3) is given by

$$\frac{\partial R_i(p_i, w_i, \Delta J_{t-1}(x))}{\partial p_i} = \bar{F}_i(p_i) \left[1 - (p_i - w_i - \Delta J_{t-1}(x)) \cdot \frac{f_i(p)}{\bar{F}_i(p)} \right] = 0 \quad (\text{A-1})$$

The second partial derivative of $R_i(p_i, w_i, \Delta J_{t-1})$ evaluated at the FOC point p_i^* is given by

$$\left. \frac{\partial^2 R_i(p_i, w_i, \Delta J_{t-1}(x))}{\partial p_i^2} \right|_{p=p_i^*} = \bar{F}_i(p_i) \left[-\frac{f_i(p_i)}{\bar{F}_i(p_i)} - (p_i - w_i - \Delta J_{t-1}(x)) \frac{\partial}{\partial p_i} \left(\frac{f_i(p_i)}{\bar{F}_i(p_i)} \right) \right] < 0,$$

where the inequality follows from the fact that $p_i - w_i - \Delta J_{t-1}(x) \geq 0$ by FOC (A.1) and IFR of

$F(\cdot)$. Therefore $R_i(p_i, w_i, \Delta J_{t-1}(x))$ is quasiconcave on p_i . ■

To facilitate the proof on structure properties, we need the following Lemma A-1, which follows from $K_i(\Delta) \downarrow \Delta$ and $K_i(\Delta) + \Delta \uparrow \Delta$ immediately.

Lemma A-1 Denote $K_i(\Delta) = \max_{p_i} \{ \bar{F}_i(p_i)(p_i - w_i - \Delta) \}$. If $\Delta^1 \leq \Delta^2$ then

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- (a) $K(\Delta^1) \geq K(\Delta^2)$;
 (b) $K(\Delta^1) - K(\Delta^2) \leq \Delta^2 - \Delta^1$.

Proof of Lemma 5.2:

Proof.

(a) We prove $\Delta J_t(x) \downarrow x$ by induction on t . It is trivially true when $t = 0$. Assume that

$\Delta J_{t-1}(x) \leq \Delta J_{t-1}(x-1)$, we need to show that $\Delta J_t(x) \leq \Delta J_t(x-1)$, and it is true since

$$\begin{aligned} \Delta J_t(x) - \Delta J_t(x-1) &= \Delta J_{t-1}(x) - \Delta J_{t-1}(x-1) + \sum_{i=1}^m \lambda_t^i \left[K_i(\Delta J_{t-1}(x)) - K_i(\Delta J_{t-1}(x-1)) \right. \\ &\quad \left. - K_i(\Delta J_{t-1}(x-1)) + K_i(\Delta J_{t-1}(x-2)) \right] \\ &\leq \Delta J_{t-1}(x) - \Delta J_{t-1}(x-1) + \sum_{i=1}^m \lambda_t^i [K_i(\Delta J_{t-1}(x)) - K_i(\Delta J_{t-1}(x-1))] \\ &\leq (1 - \sum_{i=1}^m \lambda_t^i) [\Delta J_{t-1}(x) - \Delta J_{t-1}(x-1)] \\ &\leq 0, \end{aligned}$$

where the inequalities follows by Lemma 5.2(a), Lemma 5.2(b) and the induction assumption of

$$\Delta J_{t-1}(x) \leq \Delta J_{t-1}(x-1).$$

The result of $\Delta J_t(x) \downarrow t$ follows immediately from

$$\Delta J_t(x) = \Delta J_{t-1}(x) + \sum_{i=1}^m \lambda_t^i [K_i(\Delta J_{t-1}(x)) - K_i(\Delta J_{t-1}(x-1))] \geq \Delta J_{t-1}(x).$$

$$(b) \quad J_t(x) - J_{t-1}(x) = \sum_{i=1}^m \lambda_t^i K_i(\Delta J_{t-1}(x)) \leq \sum_{i=1}^m \lambda_t^i K_i(\Delta J_{t-2}(x)) = J_{t-1}(x) - J_{t-2}(x).$$

(c) Note that $p_i^*(w_i, x, t)$ is increasing in $\Delta J_{t-1}(x)$ since $R_i(p_i, w_i, \Delta J_{t-1})$ is supermodular in p_i

and $\Delta J_{t-1}(x)$, i.e., $\frac{\partial^2 R_i(p_i, w_i, \Delta J_{t-1})}{\partial p_i \partial \Delta J_{t-1}} = f_i(p_i) > 0$. Hence the results follow by Lemma 5.2(a). ■

5 Dynamic Pricing with Two Revenue Streams

To facilitate the study of the comparative statics we need the following Lemma A-2 and readers are referred to You (1999) for some similar results.

Lemma A-2 Let $K_i(v) = \max_{p_i} \{ \bar{F}_i(p_i)(p_i + v) \}$ and $K_i(v)$ has the following properties:

- (a) $K_i(v) \uparrow v$
- (b) $K(v) - v \downarrow v$, i.e., $v_1 \leq v_2 \Rightarrow K(v_1) - K(v_2) \geq v_1 - v_2$;
- (c) If $v_1 \leq v_2 \wedge v_3, v_2 \vee v_3 \leq v_4$ then

$$K_i(v_1) - K_i(v_2) - K_i(v_3) + K_i(v_4) \geq \bar{F}(p_i^*(v_2))(v_1 - v_2 - v_3 + v_4).$$

Proof. (a)&(b) are obvious thus omitted.

(c) It is evident that $p^*(v) \downarrow v$ and $\bar{F}_i(p_i^*(v)) \uparrow v$ because of submodularity of the function $\bar{F}_i(p_i)(p_i + v)$ on p and v . We prove (c) by considering two cases:

if $v_2 \leq v_3$ then

$$\begin{aligned} K_i(v_1) - K_i(v_2) - K_i(v_3) + K_i(v_4) &\geq \bar{F}(p^*(v_2))(v_1 - v_2) + \bar{F}(p^*(v_3))(v_4 - v_3) \\ &\geq \bar{F}(p^*(v_2))(v_1 - v_2 - v_3 + v_4); \end{aligned}$$

if $v_2 \leq v_3$ then

$$\begin{aligned} K_i(v_1) - K_i(v_2) - K_i(v_3) + K_i(v_4) &\geq \bar{F}(p^*(v_3))(v_1 - v_3) + \bar{F}(p^*(v_2))(v_4 - v_2) \\ &\geq \bar{F}(p^*(v_2))(v_1 - v_2 - v_3 + v_4). \end{aligned}$$

Thus the result follows. ■

Proof of Proposition 5.3:

Proof. We prove (a) and (b) by induction together. It is trivially true for $t = 0$. Assume (a) and (b) is true for $t - 1$ and we need to show that they are true for t .

$$\begin{aligned} &[g_i - \Delta J_t(x, g_i, g_{-i})] - [\tilde{g}_i - \Delta J_t(x, \tilde{g}_i, g_{-i})] \\ &= [g_i - \Delta J_{t-1}(x, g_i, g_{-i})] - [\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})] + \end{aligned}$$

5 Dynamic Pricing with Two Revenue Streams

$$\begin{aligned}
& \lambda_t^i \left[K_i(\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) - K_i(\tilde{g}_i - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i})) \right] - \\
& \lambda_t^i \left[K_i(g_i - \Delta J_{t-1}(x, g_i, g_{-i})) - K_i(g_i - \Delta J_{t-1}(x-1, g_i, g_{-i})) \right] + \\
& \sum_{j \neq i} \lambda_t^j \left[K_j(g_j - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) - K_j(g_j - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i})) \right] - \\
& \sum_{j \neq i} \lambda_t^j \left[K_j(g_j - \Delta J_{t-1}(x, g_i, g_{-i})) - K_j(g_j - \Delta J_{t-1}(x-1, g_i, g_{-i})) \right] \\
& \geq [g_i - \Delta J_{t-1}(x, g_i, g_{-i})] - [\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})] + \\
& \lambda_t^i \left[K_i(\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) - K_i(g_i - \Delta J_{t-1}(x, g_i, g_{-i})) \right] + \\
& \sum_{j \neq i} \lambda_t^j \left[K_j(g_j - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) - K_j(g_j - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i})) \right] - \\
& \sum_{j \neq i} \lambda_t^j \left[K_j(g_j - \Delta J_{t-1}(x, g_i, g_{-i})) - K_j(g_j - \Delta J_{t-1}(x-1, g_i, g_{-i})) \right] \\
& \geq [g_i - \Delta J_{t-1}(x, g_i, g_{-i})] - [\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})] + \\
& \lambda_t^i \left[K_i(\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) - K_i(g_i - \Delta J_{t-1}(x, g_i, g_{-i})) \right] + \\
& \sum_{j \neq i} \lambda_t^j \bar{F}(p_j^*(g_j - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i}))) \left[\begin{array}{l} -\Delta J_{t-1}(x-1, g_i, g_{-i}) + \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i}) \\ +\Delta J_{t-1}(x, g_i, g_{-i}) - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i}) \end{array} \right] \\
& \geq [g_i - \Delta J_{t-1}(x, g_i, g_{-i})] - [\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})] - \\
& \lambda_t^i \left[(g_i - \Delta J_{t-1}(x, g_i, g_{-i})) - (\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) \right] + \\
& \sum_{j \neq i} \lambda_t^j \bar{F}(p_j^*(g_j - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i}))) \left[\begin{array}{l} (g_i - \Delta J_{t-1}(x-1, g_i, g_{-i})) - (\tilde{g}_i - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i})) \\ - (g_i - \Delta J_{t-1}(x, g_i, g_{-i})) + (\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) \end{array} \right] \\
& \geq (1 - \lambda_t^i) \left[(g_i - \Delta J_{t-1}(x, g_i, g_{-i})) - (\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) \right] - \\
& \sum_{j \neq i} \lambda_t^j \bar{F}(p_j^*(g_j - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i}))) \left[(g_i - \Delta J_{t-1}(x, g_i, g_{-i})) - (\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) \right] \\
& = (1 - \lambda_t^i - \sum_{j \neq i} \lambda_t^j \bar{F}(p_j^*)) \left[(g_i - \Delta J_{t-1}(x, g_i, g_{-i})) - (\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) \right] \\
& \geq 0,
\end{aligned}$$

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where the first inequality follows by assumption (a) for $t-1$ and Lemma A-2(a); the second inequality follows by Lemma A-2(c) with the sequence of v_4, v_2, v_3, v_1 ; the third inequality follows by Lemma A-2(b); the forth and the final inequality follow by assumption (a) for $t-1$.

Thus it follows that $g_i - \Delta J_t(x, g_i, g_{-i}) \geq \tilde{g}_i - \Delta J_t(x, \tilde{g}_i, g_{-i})$.

Next we show that $\Delta J_t(x, g_i, g_{-i}) \geq \Delta J_t(x, \tilde{g}_i, g_{-i})$.

$$\begin{aligned}
 & \Delta J_t(x, g_i, g_{-i}) - \Delta J_t(x, \tilde{g}_i, g_{-i}) \\
 &= \Delta J_{t-1}(x, g_i, g_{-i}) - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i}) + \\
 & \quad \lambda_t^i \left[K_i(g_i - \Delta J_{t-1}(x, g_i, g_{-i})) - K_i(g_i - \Delta J_{t-1}(x-1, g_i, g_{-i})) \right] - \\
 & \quad \lambda_t^i \left[K_i(\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) - K_i(\tilde{g}_i - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i})) \right] + \\
 & \quad \sum_{j \neq i} \lambda_t^j \left[K_j(g_j - \Delta J_{t-1}(x, g_i, g_{-i})) - K_j(g_j - \Delta J_{t-1}(x-1, g_i, g_{-i})) \right] - \\
 & \quad \sum_{j \neq i} \lambda_t^j \left[K_j(g_j - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) - K_j(g_j - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i})) \right] \\
 & \geq \Delta J_{t-1}(x, g_i, g_{-i}) - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i}) + \\
 & \quad \lambda_t^i \left[(\tilde{g}_i - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i})) - (g_i - \Delta J_{t-1}(x-1, g_i, g_{-i})) \right] + \\
 & \quad \left[-(\tilde{g}_i - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) + (g_i - \Delta J_{t-1}(x, g_i, g_{-i})) \right] \\
 & \quad \sum_{j \neq i} \lambda_t^j \left[K_j(g_j - \Delta J_{t-1}(x, g_i, g_{-i})) - K_j(g_j - \Delta J_{t-1}(x-1, g_i, g_{-i})) \right] - \\
 & \quad \sum_{j \neq i} \lambda_t^j \left[K_j(g_j - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) - K_j(g_j - \Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i})) \right] \\
 & \geq \Delta J_{t-1}(x, g_i, g_{-i}) - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i}) + \\
 & \quad \lambda_t^i \left[-\Delta J_{t-1}(x-1, \tilde{g}_i, g_{-i}) + \Delta J_{t-1}(x-1, g_i, g_{-i}) \right] + \\
 & \quad \left[+\Delta J_{t-1}(x, \tilde{g}_i, g_{-i}) - \Delta J_{t-1}(x, g_i, g_{-i}) \right] \\
 & \quad \sum_{j \neq i} \lambda_t^j \left[K_j(g_j - \Delta J_{t-1}(x, g_i, g_{-i})) - K_j(g_j - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) \right] \\
 & \geq (1 - \lambda_t^i) [\Delta J_{t-1}(x, g_i, g_{-i}) - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})] +
 \end{aligned}$$

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$$\begin{aligned}
 & \sum_{j \neq i} \lambda_t^j \left[K_j(g_j - \Delta J_{t-1}(x, g_i, g_{-i})) - K_j(g_j - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})) \right] \\
 & \geq (1 - \lambda_t^i - \sum_{j \neq i} \lambda_t^j) [\Delta J_{t-1}(x, g_i, g_{-i}) - \Delta J_{t-1}(x, \tilde{g}_i, g_{-i})] \\
 & \geq 0,
 \end{aligned}$$

where the first inequality follows by Lemma A-2(c) with the sequence of v_4, v_2, v_3, v_1 ; the second inequality follows by assumption (b) for $t-1$ and Lemma A-2(a); the third follows by assumption (b) for $t-1$; the forth follows by assumption (b) for $t-1$ and Lemma A-2(b).

(c) By the result of (a) and (b), we have

$$0 \leq \frac{\partial \Delta J_{t-1}(x, g_i, g_{-i})}{\partial g_i} = \frac{\Delta J_t(x, g_i, g_{-i}) - \Delta J_t(x, \tilde{g}_i, g_{-i})}{g_i - \tilde{g}_i} \leq 1.$$

Note that

$$\frac{\partial^2 R_i(p_i, g_i, \Delta J_{t-1}(x, g_i, g_{-i}))}{\partial p_i \partial g_i} = -f_i(p_i) \left(1 - \frac{\partial \Delta J_{t-1}(x, g_i, g_{-i})}{\partial g_i} \right) < 0,$$

Therefore, it follows that $p_i(g_i, g_{-i}, x, t) \downarrow g_i$ due to the submodularity of p_i and g_i .

(d) By the FOC of (5.12) we have

$$p_j^*(\tilde{g}, x, t) = \frac{\bar{F}_j(p_j^*(\tilde{g}, x, t))}{f_j(p_j^*(\tilde{g}, x, t))} + \Delta J_{t-1}(x, g_i, g_{-i}) - g_j. \quad (\text{A-2})$$

Therefore $p_j^*(g, x, t) \uparrow g_i$ follows by $\Delta J_{t-1}(x, g_i, g_{-i}) \uparrow g_i$. ■

Proof of Proposition 5.4:

Proof. We prove (a) and (b) by induction together. It is trivially true for $t = 0$. Assume (a) and (b) is true for $t-1$ and we need to show that they are true for t .

$$\begin{aligned}
 & [g - \Delta J_t(x, g)] - [\tilde{g} - \Delta J_t(x, \tilde{g})] = [g - \Delta J_{t-1}(x, g)] - [\tilde{g} - \Delta J_{t-1}(x, \tilde{g})] + \\
 & \sum_{i=1}^m \lambda_t^i \left[K_i(\tilde{g} - \Delta J_{t-1}(x, \tilde{g})) - K_i(\tilde{g} - \Delta J_{t-1}(x-1, \tilde{g})) - K_i(g - \Delta J_{t-1}(x, g)) + K_i(g - \Delta J_{t-1}(x-1, g)) \right]
 \end{aligned}$$

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$$\begin{aligned}
 &\geq [g - \Delta J_{t-1}(x, g)] - [\tilde{g} - \Delta J_{t-1}(x, \tilde{g})] + \sum_{i=1}^m \lambda_t^i [K_i(\tilde{g} - \Delta J_{t-1}(x, \tilde{g})) - K_i(g - \Delta J_{t-1}(x, g))] \\
 &\geq (1 - \sum_{i=1}^m \lambda_t^i) [(g - \Delta J_{t-1}(x, g)) - (\tilde{g} - \Delta J_{t-1}(x, \tilde{g}))] \\
 &\geq 0,
 \end{aligned}$$

where the first inequality follows by assumption (a) for $t-1$ and Lemma A-2(a); the second inequality follows by Lemma A-2(b). Thus the result of (a) is proved.

$$\begin{aligned}
 \Delta J_t(x, g) - \Delta J_t(x, \tilde{g}) &= \Delta J_{t-1}(x, g) - \Delta J_{t-1}(x, \tilde{g}) + \\
 &\quad \sum_{i=1}^m \lambda_t^i \left[K_i(g - \Delta J_{t-1}(x, g)) - K_i(g - \Delta J_{t-1}(x-1, g)) \right. \\
 &\quad \left. - K_i(\tilde{g} - \Delta J_{t-1}(x, \tilde{g})) + K_i(\tilde{g} - \Delta J_{t-1}(x-1, \tilde{g})) \right] \\
 &\geq \Delta J_{t-1}(x, g) - \Delta J_{t-1}(x, \tilde{g}) + \sum_{i=1}^m \lambda_t^i \left[(\tilde{g} - \Delta J_{t-1}(x-1, \tilde{g})) - (g - \Delta J_{t-1}(x-1, g)) \right. \\
 &\quad \left. - (\tilde{g} - \Delta J_{t-1}(x, \tilde{g})) + (g - \Delta J_{t-1}(x, g)) \right] \\
 &\geq (1 - \sum_{i=1}^m \lambda_t^i) [\Delta J_{t-1}(x, g) - \Delta J_{t-1}(x, \tilde{g})] \\
 &\geq 0,
 \end{aligned}$$

where the first inequality follows by Lemma A-2(c) with the sequence of v_4, v_2, v_3, v_1 ; the second and third inequality follow by assumption (b) for $t-1$. Thus the result of (b) is proved.

(c) By the result of (a) and (b), we have

$$0 \leq \frac{\partial \Delta J_{t-1}(x, g)}{\partial g} = \frac{\Delta J_t(x, g) - \Delta J_t(x, \tilde{g})}{g - \tilde{g}} \leq 1.$$

Note that

$$\frac{\partial^2 R_i(p_i, g, \Delta J_{t-1}(x, g))}{\partial p_i \partial g} = -f_i(p_i) \left(1 - \frac{\partial \Delta J_{t-1}(x, g)}{\partial g} \right) < 0,$$

Therefore, it follows that $p_i(g) \downarrow g$ due to submodularity of p_i and g . ■

6 Risk-Sensitive Dynamic Pricing for a Single Perishable Product

6.1 Introduction

Traditional research on dynamic pricing considers risk-neutral decision makers to maximize the expected revenue. This approach falls short of taking account of revenue volatility and risk-sensitivity. Although risk-neutrality is often appropriate for repeated business activities where a single poor outcome has no severe impact on the financial condition of the firm, it may not hold for event organizers (Levin et al., 2008), firms with cash-flow problems (Barz, 2007), firms with short-term revenue targets (Feng and Xiao, 2008), among others. These firms are sensitive to revenue variations and are willing to trade off lower expected revenue for downside protection against possible underperformances. Moreover, most managers in charge of dynamic pricing policies present some degree of risk aversion (Bitran and Caldentey, 2003). These findings call for the development of dynamic pricing strategies for risk-sensitive decision makers.

The literature on dynamic pricing strategies that incorporate risk-sensitivity is very limited and quite recent. Lim and Shanthikumar (2007) establish an interesting equivalence between dynamic pricing for a single product with an atemporal exponential utility and robust dynamic pricing when studying uncertainty originated from forecast errors in revenue management. However, no structural properties of the optimal policy have been characterized. Levin et al. (2008) study the risk-adjusted dynamic pricing and incorporate a risk factor to control the probability that total revenue falls below a minimum acceptable level. They show that the risk-adjusted optimal policy coincides with the risk-neutral optimal policy (Gallego and van Ryzin, 1994) after the required revenue level has been reached, and the risk-adjusted optimal price may decrease whereas the risk-neutral optimal price always increases immediately following a sale.

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Capacity control and dynamic pricing are two main strategies in revenue management to balance constrained capacity and fluctuating demand. The literature on risk-sensitive capacity control is also very limited. Our work is closely related to the generic single-resource capacity control problem under risk-sensitivity studied in Barz (2007) and Feng and Xiao (2008). Barz (2007) studies structural properties of the static and dynamic model under the additive general utility and atemporal exponential utility function, and raises a monotonicity conjecture that the optimal protection level is decreasing over the degree of risk-sensitivity for the decision maker with an atemporal exponential utility function. Feng and Xiao (2008) study risk-sensitive capacity control for single perishable product with an atemporal exponential utility function. They show that the optimal control policy preserves the attractive properties under risk-neutrality such as nested active price set, monotonicity over time and inventory, and threshold-type control. They also show that the optimal price set will grow larger as the degree of risk-sensitivity goes up, indicating that the firm is lowering the threshold price. This is consistent with Barz's monotonicity conjecture.

In contrast to Barz (2007) and Feng and Xiao (2008) that fall into the capacity control framework with prices given, this work studies generic dynamic pricing strategies for a single perishable product under risk-sensitivity. Two main findings are made. First, we show that the intuitively appealing monotone structures with respect to inventory level and time under risk-neutrality, originally established in Gallego and van Ryzin (1994), are preserved under risk-sensitivity with the additive general utility and atemporal exponential utility functions. Second, we prove that the optimal price is decreasing over the degree of risk-sensitivity under the exponential class of both additive and atemporal utility functions.

In accordance with the monotonicity conjecture in Barz (2007) and results from Feng and Xiao (2008), we show that a risk-sensitive decision maker will set a lower price to hedge against risks originated from uncertainties about consumer's willingness-to-pay. In view of findings from Feng and Xiao (2008), we conclude that a risk-sensitive decision maker will adopt a more

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conservative control policy under both capacity control and dynamic pricing to trade off lower expected revenue against possible unfavorable losses.

The rest of the chapter is organized as follows. Section 6.2 presents the models and preliminaries. Section 6.3 and Section 6.4 study structural properties of risk-sensitive dynamic pricing under the additive general utility and atemporal exponential utility functions respectively. In Section 6.5, numerical studies are used to illustrate the analytical results. Section 6.7 Concludes.

6.2 The Models

Consider a firm that has C units of a perishable product to sell over a finite horizon. The objective is to develop a dynamic pricing strategy to maximize the expected revenue under risk-neutrality or to maximize the expected utility under risk-sensitivity. Since the selling horizon for the perishable product is relatively short, discounting is generally not considered.

Let $\{\mathcal{T}, \mathcal{S}, \mathcal{A}, q_t(\cdot|x, p), r_t(x, p)\}$ denote the underlying Markov decision process (MDP) for the dynamic pricing problem. The set of decision epochs is given by $\mathcal{T} = \{T, T-1, \dots, 1, 0\}$ with time going backward. There is at most one customer arrival with probability λ within each period t . The state space is $\mathcal{S} = \{x \in \mathbb{Z} \mid 0 \leq x \leq C\}$ where x is the inventory level. Action space $\mathcal{A} = P \cup p_\infty$ is the set of allowable prices, where P is a compact set within $(0, +\infty)$ and p_∞ is a null price used to model out-of-stock situation (see Gallego and van Ryzin, 1994). Consumer's willingness-to-pay for the product has a tail probability $\bar{F}(p)$ such that $\bar{F}(p_\infty) = 0$ and $\bar{F}(p) > 0$ for all $p \in P$. The dynamic pricing decision rule is a mapping $p_t : \mathcal{S} \rightarrow \mathcal{A}$. The transition law is given by $q_t(x-1|x, p) = \lambda \bar{F}(p)$ and $q_t(x|x, p) = 1 - \lambda \bar{F}(p)$. The one-period reward function is defined by $r_t : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$. A T -stage

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Markovian pricing policy is given by $\pi = (p_T, \dots, p_1) \in \Pi \equiv \mathcal{A} \times \dots \times \mathcal{A}$. The risk-neutral decision maker determines an optimal pricing policy π^* that maximizes expected revenue:

$$J^{\pi^*}(C) = \sup_{\pi \in \Pi} J_T(C) = \max_{\pi \in \Pi} \mathbb{E}^{\pi} \left[\sum_{t=1}^T r_t(X_t, p_t(X_t)) + J_0(X_0) \middle| X_T = C \right] \quad (6.1)$$

Due to tractability two classes of utility functions are commonly used to study sequential decision problems under risk (Barz, 2007; Li and Zhuang, 2009), namely, additive utility function and atemporal utility function. Given any wealth profile $\omega = (\omega_0, \omega_1, \dots, \omega_T)$ over $T + 1$ periods, a utility function u is said to be additive if

$$u(\omega) = \sum_{t=0}^T u_t(\omega_t),$$

and it is called atemporal if

$$u(\omega) = u\left(\sum_{t=0}^T \omega_t\right).$$

The key difference between an additive utility and an atemporal utility lies in the decision maker's time preference over reward streams. An additive utility deals with utility dynamically over time; while an atemporal utility is insensitive to the time aspect and evaluates the total rewards within the planning horizon. Technically, an additive utility is separable in time, hence simple to work analytically; while an atemporal utility is generally non-separable in time, thus more challenging analytically.

An additive utility maximizer determines an optimal pricing policy π_{add}^* that maximizes the following expected utility:

$$J^{\pi_{add}^*}(C) = \sup_{\pi \in \Pi} J_T^{add}(C) = \max_{\pi \in \Pi} \mathbb{E}^{\pi} \left[\sum_{t=1}^T u_t(r_t(X_t, p_t(X_t))) + u_0(J_0^{add}(X_0)) \middle| X_T = C \right] \quad (6.2)$$

An atemporal utility maximizer determines an optimal pricing policy π_{atmp}^* that maximizes the following expected utility:

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$$J^{\pi^*}_{amp}(C) = \sup_{\pi \in \Pi} J^{\pi}_{amp}(C) = \max_{\pi \in \Pi} E^{\pi} \left[u \left(\sum_{t=1}^T r_t(X_t, p_t(X_t)) + J^{\pi}_{amp}_0(X_0) \right) \middle| X_T = C \right] \quad (6.3)$$

Next we will explore structural properties and comparative statics for risk-sensitive dynamic pricing. The following lemma derived from the envelop theorem (see Silberberg and Suen, 2001) is used to study the comparative statics.

Lemma 6.1 Let $f : X \times \Theta \rightarrow \mathbb{R}$.

(a) If $x^*(\theta) = \arg \max_x f(x, \theta)$, then $f_{x\theta}(x^*(\theta), \theta) \cdot \frac{\partial x^*(\theta)}{\partial \theta} > 0$.

(b) If $x^*(\theta) = \arg \min_x f(x, \theta)$, then $f_{x\theta}(x^*(\theta), \theta) \cdot \frac{\partial x^*(\theta)}{\partial \theta} < 0$.

6.3 Additive General Utility

Additive utility is frequently used in MDP (e.g. Barz, 2007; Chen et al., 2007) because of its time dynamics and analytical simplicity. For instance, firms with certain financial obligations to fulfill at certain time may have time-varying risk preferences which can be modeled by additive utility functions.

The Bellman equation of (6.2), dynamic pricing under the additive general utility, is given by

$$\begin{aligned} J_t^{add}(x) &= J_{t-1}^{add}(x) + \lambda \cdot \max_{p \in P} \left\{ \bar{F}(p) (u_t(p) - \Delta J_{t-1}^{add}(x)) \right\}, \quad \forall x = 1, \dots, C; \quad \forall t = 1, \dots, T, \\ J_0^{add}(x) &= 0, \quad \forall x = 1, \dots, C \text{ and } J_t^{add}(0) = 0, \quad \forall t = 1, \dots, T, \end{aligned} \quad (6.4)$$

where $u_t(\cdot)$ is a utility function with $u'_t(\cdot) \geq 0$, $u''_t(\cdot) \leq 0$ and $\Delta J_t^{add}(x) = J_t^{add}(x) - J_t^{add}(x-1)$ is the marginal opportunity cost in terms of utility. Let

$$R^{add}(p, \Delta J_{t-1}^{add}(x)) = \bar{F}(p) (u_t(p) - \Delta J_{t-1}^{add}(x)) \quad (6.5)$$

The optimal price is given by

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$$p_{add}^*(\Delta J_{t-1}^{add}(x)) = \arg \max_{p \in P} R^{add}(p, \Delta J_{t-1}^{add}(x)).$$

The sufficient condition for the regularity of $R^{add}(p, \Delta J_{t-1}^{add}(x))$ is characterized in Lemma 6.2.

Lemma 6.2 If $F(\cdot)$ is an increasing failure rate distribution, then $R^{add}(p, \Delta J_{t-1}^{add}(x))$ is quasiconcave on p .

Proof. The first-order condition (FOC) for (6.5) is given by

$$\frac{\partial R^{add}(p, \Delta J_{t-1}^{add}(x))}{\partial p} = \bar{F}(p) \left[u_t'(p) - (u_t(p) - \Delta J_{t-1}^{add}(x)) \cdot \frac{f(p)}{\bar{F}(p)} \right] = 0 \quad (6.6)$$

The second partial derivative of $R^{add}(p, \Delta J_{t-1}^{add}(x))$ evaluated at the FOC point is

$$\begin{aligned} & \left. \frac{\partial^2 R^{add}(p, \Delta J_{t-1}^{add}(x))}{\partial p^2} \right|_{p=p_{add}^*(\Delta J_{t-1}^{add}(x))} \\ &= \bar{F}(p) \left[u_t''(p) - u_t'(p) \frac{f(p)}{\bar{F}(p)} - (u_t(p) - \Delta J_{t-1}^{add}(x)) \frac{\partial}{\partial p} \left(\frac{f(p)}{\bar{F}(p)} \right) \right] < 0, \end{aligned}$$

where the third term is nonnegative since $u_t(p_{add}^*(\Delta J_{t-1}^{add}(x))) - \Delta J_{t-1}^{add}(x) \geq 0$ from (6.6) and

$\frac{\partial}{\partial p} \left(\frac{f(p)}{\bar{F}(p)} \right) > 0$ due to IFR of $F(\cdot)$. Therefore, $R^{add}(p, \Delta J_{t-1}^{add}(x))$ is quasiconcave on p and (6.6)

determines the unique optimal solution $p_{add}^*(\Delta J_{t-1}^{add}(x))$. ■

Note that increasing failure rate (IFR) implies increasing generalized failure rate which is equivalent to increasing price elasticity (see Ziya et al., 2004). IFR captures most common distributions such as uniform, normal, exponential and logistic. Distributions such as the Weibull, Beta, and Gamma are IFR distributions if their parameters fall into certain ranges.

To facilitate the characterization of structural properties, we let

$$K(\Delta) = \max_p \{ \bar{F}(p)(u(p) - \Delta) \}$$

and present a lemma summarizing properties of $K(\Delta)$. For simplicity of exposition, we use $\uparrow$ to indicate increasing and $\downarrow$ to indicate decreasing in the weak sense.

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Lemma 6.3 $K(\Delta)$ has the following properties:

- (a) $K(\Delta) \downarrow \Delta$;
- (b) If $\Delta_1 \leq \Delta_2$ then $K(\Delta_1) - K(\Delta_2) \leq \Delta_2 - \Delta_1$.

Lemma 6.3 follows immediately as $\bar{F}(p)(u(p) - \Delta)$ is decreasing in Δ while $\bar{F}(p)(u(p) - \Delta) + \Delta$ is increasing in Δ for any p . Structural properties are summarized below.

Proposition 6.1 The problem (6.4), namely risk-averse dynamic pricing under the additive general utility, has the following structural properties:

- (a) $\Delta J_t^{add}(x) \downarrow x \uparrow t$;
- (b) $p_{add}^*(\Delta J_{t-1}^{add}(x)) \uparrow \Delta J_{t-1}^{add}(x) \downarrow x \uparrow t$.

Proof. (a) We prove by induction on t . It is trivial when $t = 0$. Assume $\Delta J_{t-1}^{add}(x) \leq \Delta J_{t-1}^{add}(x-1)$.

We need to show $\Delta J_t^{add}(x) \leq \Delta J_t^{add}(x-1)$, which is true because

$$\begin{aligned} \Delta J_t^{add}(x) - \Delta J_t^{add}(x-1) &= \Delta J_{t-1}^{add}(x) - \Delta J_{t-1}^{add}(x-1) + \\ &\quad \lambda \left[K(\Delta J_{t-1}^{add}(x)) - K(\Delta J_{t-1}^{add}(x-1)) - K(\Delta J_{t-1}^{add}(x-1)) + K(\Delta J_{t-1}^{add}(x-2)) \right] \\ &\leq \Delta J_{t-1}^{add}(x) - \Delta J_{t-1}^{add}(x-1) + \lambda \left[K(\Delta J_{t-1}^{add}(x)) - K(\Delta J_{t-1}^{add}(x-1)) \right] \\ &\leq (1 - \lambda) (\Delta J_{t-1}^{add}(x) - \Delta J_{t-1}^{add}(x-1)) \\ &\leq 0, \end{aligned}$$

where the three inequalities follow by Lemma 6.3(a), Lemma 6.3 (b) and $\Delta J_{t-1}^{add}(x) \leq \Delta J_{t-1}^{add}(x-1)$ respectively.

For the time monotonicity, it follows because

$$\Delta J_t^{add}(x) = \Delta J_{t-1}^{add}(x) + \lambda \left[K(\Delta J_{t-1}^{add}(x)) - K(\Delta J_{t-1}^{add}(x-1)) \right] \geq \Delta J_{t-1}^{add}(x).$$

- (b) The property that $p_{add}^*(\Delta J_{t-1}^{add}(x))$ is increasing in $\Delta J_{t-1}^{add}(x)$ follows from the supermodularity of the function R in p and $\Delta J_{t-1}^{add}(x)$, i.e.,

$$\frac{\partial^2 R(p, \Delta J_{t-1}^{add}(x))}{\partial p \partial \Delta J_{t-1}^{add}(x)} = f(p) > 0.$$

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As $\Delta J_t^{add}(x) \downarrow x \uparrow t$, it follows that $p_{add}^*(\Delta J_{t-1}^{add}(x)) \downarrow x \uparrow t$. ■

Proposition 6.1 shows the monotone structures with respect to inventory level and remaining time for risk-sensitive dynamic pricing with the additive general utility, which includes risk-neutrality as a special case with $u_t(p) = p$. It is worth highlighting that the proof of Proposition 6.1 is different from that of Theorem 1 in Gallego and van Ryzin (1994) that uses a continuous-time MDP formulation. It is also different from the proof usually adopted in dynamic pricing literature through discrete-time formulation, which follows the approach of Proposition 4.3 in Ross (1983) that is an induction proof on $x + t$ (e.g. Lemma 5 in Aydin and Ziya, 2008). We prove the concavity of $\Delta \hat{J}_t(x)$ in x by induction on t and other properties follow immediately. This approach is much simpler.

6.3.1 Additive Exponential Utility

An exponential utility function has many appealing properties and is the most widely used risk-sensitive utility function (see Barz, 2007). Furthermore, a carefully selected exponential utility function can be a good approximation for a general utility function in most cases.

Consider the exponential utility function $u_t(\omega) = 1 - e^{-\gamma_t \omega}$. Under this specification, the degree of the decision maker's risk-sensitivity to the reward variation is completely captured by one parameter, namely the absolute risk-aversion coefficient,

$$\gamma_u(\omega) = -\frac{u_t''(\omega)}{u_t'(\omega)} = \gamma_t,$$

where $\gamma_t > 0$ corresponds to risk-aversion. The Bellman equation of risk-sensitive dynamic pricing under the additive exponential utility function is given by

$$J_t^{add}(x) = J_{t-1}^{add}(x) + \lambda \max_{p \in P} \left\{ \bar{F}(p) (1 - e^{-\gamma_t p} - \Delta J_{t-1}^{add}(x)) \right\} \quad (6.7)$$

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The monotonicity property of the optimal price in the degree of risk-sensitivity is stated in the following result.

Proposition 6.2 The optimal price $p_{add}^*(\gamma_t, \Delta J_{t-1}^{add}(x))$ of problem (6.7), risk-sensitive dynamic pricing under the additive exponential utility function, is decreasing in γ_t .

Proof. Let $R^{add}(p, \gamma_t, \Delta J_{t-1}^{add}(x)) = \bar{F}(p)(1 - e^{-\gamma_t p} - \Delta J_{t-1}^{add}(x))$, and

$$p_{add}^*(\gamma_t, \Delta J_{t-1}^{add}(x)) = \arg \max_{p \in P} R^{add}(p, \gamma_t, \Delta J_{t-1}^{add}(x))$$

By Proposition 6.1(b), we have

$$p_{add}^*(\gamma_t, \Delta J_{t-1}^{add}(x)) \geq p_{add}^*(\gamma_t, 0),$$

and

$$\left. \frac{\partial R^{add}(p, \gamma_t, 0)}{\partial p} \right|_{p=p_{add}^*(\gamma_t, \Delta J_{t-1}^{add}(x))} = \left[-f(p)(1 - e^{-\gamma_t p}) + \bar{F}(p)\gamma_t e^{-\gamma_t p} \right]_{p=p_{add}^*(\gamma_t, \Delta J_{t-1}^{add}(x))} \leq 0 \quad (6.8)$$

Thus it is true for

$$\begin{aligned} \left. \frac{\partial^2 R^{add}(p, \gamma_t, \Delta J_{t-1}^{add}(x))}{\partial p \partial \gamma_t} \right|_{p=p_{add}^*(\gamma_t, \Delta J_{t-1}^{add}(x))} &= \left[(1 - \gamma_t p)\bar{F}(p) - f(p)p \right] e^{-\gamma_t p} \\ &\leq \frac{1 - e^{-\gamma_t p} - \gamma_t p}{1 - e^{-\gamma_t p}} \bar{F}(p) e^{-\gamma_t p} < 0, \end{aligned}$$

where the first inequality follows from (6.8) and the second inequality follows from the fact that

$1 - e^{-\gamma_t p} - \gamma_t p < 0$ for $\forall \gamma_t > 0$. By Lemma 1 we have $\frac{\partial p_{add}^*(\gamma_t, \Delta J_{t-1}^{add}(x))}{\partial \gamma_t} < 0$. Therefore

$p_{add}^*(\gamma_t, \Delta J_{t-1}^{add}(x))$ is decreasing in γ_t . ■

Barz (2007) shows that the optimal protection level is monotone on the degree of risk-aversion with an additive utility function.

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6.4 Atemporal Exponential Utility

When the planning horizon is short and the time elapsing between two decisions is negligible, it is realistic to assume an atemporal utility if the decision maker is concerned on the total revenue gained with no time preference. However, using an atemporal utility function in the context of MDP is more challenging analytically than an additive utility function as it requires an enlarged state space to track the wealth level accumulated up to the decision period, which is computationally intractable. An atemporal exponential utility nevertheless has the unique advantage of not increasing the dimension of the state space for the underlying MDP; hence it is commonly used in the literature (see Barz, 2007; Feng and Xiao, 2008).

Risk-sensitive dynamic pricing problem (6.3) under the atemporal exponential utility function $u(\omega) = 1 - e^{-\gamma\omega}$ can be rewritten as follows,

$$\max_{\pi \in \Pi} E^{\pi} [u(\sum_{t=0}^T r_t(X_t, p_t(X_t)) | X_T = C)] = 1 - \min_{\pi \in \Pi} E^{\pi} [e^{-\gamma \sum_{t=0}^T r_t(X_t, p_t(X_t))} | X_T = C] \quad (6.9)$$

Let

$$J_T^{atmp}(C) = \min_{\pi \in \Pi} E^{\pi} [e^{-\gamma \sum_{t=0}^T r_t(X_t, p_t(X_t))} | X_T = C] \quad (6.10)$$

and the Bellman equation of (6.10) is given by

$$J_t^{atmp}(x) = \min_p \left\{ \lambda \bar{F}(p) \cdot e^{-\gamma p} J_{t-1}^{atmp}(x-1) + (1 - \lambda \bar{F}(p)) J_{t-1}^{atmp}(x) \right\} \quad (6.11)$$

Note that a similar derivation of (6.11) in the discrete-time formulation can be found in Howard and Matheson (1972) and for the continuous-time formulation in Feng and Xiao (2008) and Lim and Shanthikumar (2007). The certainty equivalent of $J_t^{atmp}(x)$ is given by

$$\tilde{J}_t^{atmp}(x) = -\frac{1}{\gamma} \ln J_t^{atmp}(x) \quad (6.12)$$

We can rewrite (6.11) in terms of $\tilde{J}_t^{atmp}(x)$:

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$$\tilde{J}_t^{atmp}(x) = \tilde{J}_{t-1}^{atmp}(x) - \frac{1}{\gamma} \ln \left[1 + \min_p \{ \lambda \bar{F}(p) \cdot (e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} - 1) \} \right] \quad (6.13)$$

where $\Delta \tilde{J}_t^{atmp}(x) = \tilde{J}_t^{atmp}(x) - \tilde{J}_t^{atmp}(x-1)$ is the opportunity cost in terms of certainty equivalent.

Let

$$R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x)) = \bar{F}(p) \cdot (e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} - 1) \quad (6.14)$$

The optimal price is given by

$$p_{atmp}^*(\gamma, \Delta \tilde{J}_{t-1}^{atmp}(x)) = \arg \min_{p \in P} R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x)).$$

The sufficient condition for the regularity of $R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))$ is characterized below.

Lemma 6.4 If $F(\cdot)$ is an increasing failure rate distribution, then $R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))$ is quasiconvex on p .

Proof. The FOC for (6.14) is given by

$$\frac{\partial R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))}{\partial p} = \bar{F}(p) \left[-\gamma e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} - (e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} - 1) \frac{f(p)}{\bar{F}(p)} \right] = 0 \quad (6.15)$$

The second partial derivative of $R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))$ evaluated at the FOC point is

$$\begin{aligned} & \frac{\partial^2 R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))}{\partial p^2} \bigg|_{\frac{\partial R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))}{\partial p} = 0} \\ &= \bar{F}(p) \left[\gamma^2 e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} - \left(e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} - 1 \right) \frac{\partial}{\partial p} \left(\frac{f(p)}{\bar{F}(p)} \right) + \gamma e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} \frac{f(p)}{\bar{F}(p)} \right] > 0, \end{aligned}$$

where the second term is negative since $e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} - 1 \leq 0$ from (6.15) and IFR of $F(\cdot)$.

Therefore, quasiconvexity of $R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))$ in p , together with (6.15), leads to the unique optimal solution $p_{atmp}^*(\gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))$. ■

To facilitate the proofs of structural properties, define

$$\tilde{K}(\Delta) = \min_p \{ \bar{F}(p) (e^{-\gamma(p - \Delta)} - 1) \} \text{ and } G(\Delta) = \ln [1 + \lambda \tilde{K}(\Delta)],$$

and their properties are summarized in the following lemma.

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Lemma 6.5 $\tilde{K}(\Delta)$ and $G(\Delta)$ have the following properties:

- (a) $\tilde{K}(\Delta) \uparrow \Delta$;
- (b) $G(\Delta) \uparrow \Delta$;
- (c) If $\Delta_1 \leq \Delta_2$ then $G(\Delta_2) - G(\Delta_1) \leq \gamma(\Delta_2 - \Delta_1)$.

Proof. It is evident that parts (a) and (b) are trivial.

- (c) Let $\tilde{G}(\Delta) = G(\Delta) - \gamma\Delta = \ln \left[1 + \min_p \{ \lambda \bar{F}(p) \cdot (e^{-\gamma(p-\Delta)} - 1) \} \right] - \gamma\Delta$. Note that

$$\frac{\partial \tilde{G}(\Delta)}{\partial \Delta} = \frac{(\lambda \bar{F}(p^*) - 1)\gamma}{1 + \lambda \bar{F}(p^*)(e^{-\gamma(p^*-\Delta)} - 1)} < 0,$$

since $p^* \geq \Delta$ by the first-order condition of $\tilde{K}(\Delta)$. Hence if $\Delta_1 \leq \Delta_2$ then $\tilde{G}(\Delta_1) \geq \tilde{G}(\Delta_2)$. ■

Structural properties are summarized in the following proposition.

Proposition 6.3 The problem (6.13), which is an equivalent problem of (6.9) on risk-sensitive dynamic pricing under the atemporal exponential utility $u(\omega) = 1 - e^{-\gamma\omega}$, has the following structural properties:

- (a) $\Delta \tilde{J}_t^{atmp}(x) \downarrow x \uparrow t$
- (b) $p_{atmp}^*(\gamma, \Delta \tilde{J}_{t-1}^{atmp}(x)) \uparrow \Delta \tilde{J}_{t-1}^{atmp}(x) \downarrow x \uparrow t \downarrow \gamma$

Proof. (a) We prove by induction on t . It is trivially true when $t = 0$.

Assume that $\Delta \tilde{J}_{t-1}^{atmp}(x) \leq \Delta \tilde{J}_{t-1}^{atmp}(x-1)$. We need to show that $\Delta \tilde{J}_t^{atmp}(x) \leq \Delta \tilde{J}_t^{atmp}(x-1)$ and it is true because

$$\begin{aligned} \Delta \tilde{J}_t^{atmp}(x) - \Delta \tilde{J}_t^{atmp}(x-1) &= \Delta \tilde{J}_{t-1}^{atmp}(x) - \Delta \tilde{J}_{t-1}^{atmp}(x-1) - \\ &\quad \frac{1}{\gamma} \left[G(\Delta \tilde{J}_{t-1}^{atmp}(x)) - G(\Delta \tilde{J}_{t-1}^{atmp}(x-1)) - G(\Delta \tilde{J}_{t-1}^{atmp}(x-1)) + G(\Delta \tilde{J}_{t-1}^{atmp}(x-2)) \right] \\ &\leq \Delta \tilde{J}_{t-1}^{atmp}(x) - \Delta \tilde{J}_{t-1}^{atmp}(x-1) + \frac{1}{\gamma} \left[G(\Delta \tilde{J}_{t-1}^{atmp}(x-1)) - G(\Delta \tilde{J}_{t-1}^{atmp}(x)) \right] \\ &\leq 0, \end{aligned}$$

where two inequalities follow by Lemma 6.5(b), Lemma 6.5(c) and $\Delta \tilde{J}_{t-1}^{atmp}(x) \leq \Delta \tilde{J}_{t-1}^{atmp}(x-1)$

respectively. Then it follows readily that

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$$\Delta \tilde{J}_t^{atmp}(x) = \Delta \tilde{J}_t^{atmp}(x-1) - \frac{1}{\gamma} \left[G(\Delta \tilde{J}_{t-1}^{atmp}(x)) - G(\Delta \tilde{J}_{t-1}^{atmp}(x-1)) \right] \geq \Delta \tilde{J}_t^{atmp}(x-1).$$

(b) Note that

$$\frac{\partial^2 R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))}{\partial p \partial \Delta \tilde{J}_{t-1}^{atmp}(x)} = \left[-f(p)\gamma - \bar{F}(p)\gamma^2 \right] e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} < 0,$$

thus $p_{atmp}^*(\gamma, \Delta \tilde{J}_{t-1}^{atmp}(x)) \uparrow \Delta \tilde{J}_{t-1}^{atmp}(x)$ by Lemma 6.1 and $p_{atmp}^*(\gamma, \Delta \tilde{J}_{t-1}^{atmp}(x)) \downarrow x \uparrow t$ follows immediately from (a).

Also note that

$$\begin{aligned} & \left. \frac{\partial^2 R^{atmp}(p, \gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))}{\partial p \partial \gamma} \right|_{p=p_{atmp}^*(\gamma, \Delta \tilde{J}_{t-1}^{atmp}(x))} \\ &= \frac{\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x)) + e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} - 1}{1 - e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))}} \cdot \bar{F}(p) e^{-\gamma(p - \Delta \tilde{J}_{t-1}^{atmp}(x))} > 0, \end{aligned}$$

because $p_{atmp}^*(\gamma, \Delta \tilde{J}_{t-1}^{atmp}(x)) \geq \Delta \tilde{J}_{t-1}^{atmp}(x)$ by the FOC (6.15) and $f(x) = x + e^{-x} - 1 > 0, \forall x > 0$. By Lemma 6.1 we must have $p_{atmp}^*(\gamma, \Delta \tilde{J}_{t-1}^{atmp}(x)) \downarrow \gamma$. ■

Proposition 6.3 says that the monotone structures are preserved under risk-sensitive dynamic pricing with an atemporal exponential utility and the optimal price is monotone on the degree of risk-sensitivity. Our results are consistent with the monotonicity conjecture of Barz (2007) and the results from Feng and Xiao (2008). In view of the results from Feng and Xiao (2008), it can be concluded that a risk-sensitive decision maker will, not surprisingly, adopt a more conservative control policy under both capacity control and dynamic pricing.

6.5 Numerical Examples

In this section, we use a simple numerical example to illustrate the analytical results developed in this study. Given an initial inventory level $C = 10$ and a selling time horizon $T = 15$. Assume that the probability of a customer arrival during each period of time is $\lambda = 0.78$. We consider additive

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exponential utility functions $u_t(\omega) = 1 - e^{-\gamma\omega}$, $t = 0, \dots, T$ and atemporal exponential utility function $u(\omega) = 1 - e^{-\gamma\omega}$ respectively. Consumer's willingness-to-pay is assumed to have a Weibull distribution with scale 100 and shape 2, which is an increasing failure rate distribution. Note that a Weibull distribution is flexible with a variety of behavior of reservation prices, such as normal, exponential, a heavier left tail, or a heavier right tail, etc, thus it is frequently used to model the reservation distribution (e.g. Aydin and Ziya, 2008).

From Figure 6.1 we observe that under both utility functions, the optimal price is decreasing in inventory level x , increasing in time t and decreasing in the degree of risk-sensitivity γ . The risk-sensitive decision maker will set a lower price to hedge against the risks originated from the uncertainty about consumer's willingness-to-pay.

6.6 Concluding Remarks

In this chapter, we study risk-sensitive dynamic pricing for single perishable product and explore its structural properties. We have shown that all the intuitively appealing monotone structural properties under risk-neutrality are preserved under risk-sensitivity with additive utility and atemporal exponential utility functions. Moreover, we investigate how the decision maker's attitude toward risk affects the optimal price. We have shown the monotonicity of the optimal price in the degree of risk-sensitivity for additive and atemporal exponential utility functions. The risk-sensitive decision maker will set a lower price to hedge against the risks originated from the uncertainty about customer's willingness-to-pay.

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a. Monotonicity of price on inventory level x and remaining time t

b. Monotonicity of price on the degree of risk-sensitivity γ

Figure 6.1 Monotonicity of the optimal price under additive and atemporal exponential utility

7 Conclusions and Future Directions

This dissertation studies monotone optimal control for a class of discrete-event stochastic systems and its applications in revenue management. The contribution of this dissertation is two-fold, namely, methodology and applications. Major contributions of this dissertation are highlighted, limitations of this work and future directions are indicated hereby.

From the methodology point of view, Chapter 3 has made the following significant contributions.

- It provides a unified and systematic framework to study a typical class of two-dimensional discrete-event stochastic systems that can be modeled by controlled Markov chains with each possible transition bounded by the unit vector.
- The system classification mechanism is extremely useful in analyzing a system and the generic proofs of structural properties will save the future effort on ad hoc proofs and can be applied directly to analyze this class of systems
- It identifies D -multimodularity as the general condition that guarantees the existence of monotone optimal control (switching-curve policy for the discrete-action control and monotone control policy for the continuous-action control) for different systems.

Note that there is a growing need for joint dynamic pricing and inventory control and some recent literatures have addressed the problem for one common pool of resource (Feng and Xiao, 2006; Gans and Savin, 2007; Feng et al. 2008; Lin et. al., 2008; etc.). Joint dynamic pricing and inventory control for two parallel flights with two distinct customer segments discussed in this dissertation is a new application example of two-resource problem. There are many other potential applications to be explored in the future. For instance, for joint production control (scheduling) and demand rate control for manufacturing and production systems of two machines with two products, the results obtained in Chapter 3 will lay important foundations and gain

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useful insights for this managerial problem. This work is ongoing and there is much work in other areas to be explored in the future.

The limitation of Chapter 3 is that it only discusses a particular class of two-dimensional systems. Even though this class of systems has vast applications in many fields, there are many other systems that are not covered. Therefore, an interesting and challenging future direction is to investigate monotone optimal control for the general two-dimensional systems.

From the revenue management application point of view, this dissertation explores application in healthcare; investigates dynamic pricing with two revenue streams and risk-sensitive dynamic pricing. For the capacity management problem in healthcare, Chapter 4 has made the following contributions.

- The multiple-facility model has gone beyond one-facility model and allowed more general arrival processes of three patient segments.
- For this highly complicated multiple-facility model which involves two related tasks, namely, outpatient appointment scheduling and dynamic resource allocation, it has fully characterized the optimal resource allocation policy for any given appointment scheduling.
- The multiple-facility model is relevant to practice because key inputs of our model can be routinely estimated and validated from empirical data and the multiple-facility model is extremely useful in improving utilization of diagnostic facilities, enhancing overall revenue and facilitating resource coordination.
- The technical innovation is to apply D -multimodularity. The generic approach developed in Chapter 3 can be also used to dramatically simplify the tedious proofs of one-facility model (Green et al. 2006) which occupies over 40 pages in a companion file.

Note that the multiple-facility system (which belongs to substitutable system due to resource sharing) does not fall into the particular class of system studied in Chapter 3, the methodology still applies and the generic approach to prove structural properties has demonstrated its strong efficiency for the multiple-facility system. This demonstrates the great usefulness of the unified

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framework developed in the dissertation to study monotone optimal control for discrete-event stochastic systems. It also shows the strong need to investigate the unified framework for the general systems.

The key contribution of the multiple-facility model is a full characterization of optimal allocation policy for any given appointment scheduling and the limitation is the intractability of outpatient scheduling. However the multiple-facility model is highly complex with many complicated factors: interrelated appointment scheduling and dynamic resource allocation, random arrival processes of three patient segments with outpatient no-shows, and multiple facilities. Typical streams of healthcare management literature usually focus on solving one issue at a time: patient scheduling with no-shows, or outpatient (advance) scheduling with the random arrivals of walk-ins. It is usually difficult to get analytical solutions for scheduling problem even with one added perturbation factor. Our multiple-facility model is among the very few models (Green et al., 2006; Kolisch and Sickinger, 2008) addressing both outpatient (allocation) scheduling and resource allocation, and thus it focuses on the dynamic resource allocation policy. It is worthwhile to purely model the scheduling problem for the same problem setting considering advance scheduling or allocation scheduling, or both scheduling, in the future.

Chapter 5 investigates a generic dynamic pricing problem with two revenue streams, primary revenue and ancillary revenue, which has not gained the attention in the literature yet. Chapter 5 makes the first attempt to investigate multi-segment dynamic pricing of a perishable product in the presence of another revenue stream. It has obtained the following major results.

- We first study the multi-segment dynamic pricing problem for a casino operation where customers are segmented through customer's gaming profile and the casino dynamically prices its limited hotel rooms that are served as a screening mechanism. We identify a stochastic order on customer's gaming revenue profile which leads to a monotone inventory price of rooms. We show a threshold-type pricing policy for a special customer segmentation scheme that the customer's winning profile can be ranked in terms of the failure rate order.

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- We then study the multi-segment dynamic pricing problem of a perishable inventory by integrating ancillary revenues. We derive the comparative statics results on the average ancillary revenues across segments and gain new managerial insights. We show that integrating ancillary revenues into multi-segment dynamic pricing will achieve higher expected total revenue than that takes no account of ancillary revenue.
- There are some interesting new and counter-intuitive findings. This study will shed new light on the issue of customer valuation and market segmentation in revenue management.

However, the second revenue stream, either primary or ancillary, is stochastic and can not be collected until the consumption day or the end of the service and it is stochastic. How a firm makes a trade-off between the first assured stream of inventory revenue with the second stochastic stream of revenue is quite interesting and challenging to be explored because the stochastic stream of revenue is intrinsically riskier than the inventory revenue. In this dissertation, we have not explicitly considered the capacity constraint of the other product. For instance, when the casino has the capacity constraint, or a large hotel which serves both casino and resort customers, how will the pricing decision affected by the capacity constraint of another product? How will the allocation or pricing decision be affected by the stochastic primary revenue from casino customers and resort customers? This will lead to a two-dimensional MDP problem, which will surely be very challenging for future research.

Chapter 6 has shown the following main results on risk-sensitive dynamic pricing:

- For dynamic pricing with single perishable product, risk sensitive decision maker will set a lower price to hedge against the risk originated from the uncertainty about customer's willingness-to-pay, and the optimal price is monotone on the degree of risk-aversion under both additive and atemporal exponential utility.
- Risk attitude has different degree of impact on the optimal capacity allocation for advance market when facing the same market risk; risk-neutral and risk-averse newsvendor behave fundamentally different when facing the same increase in market risk exposure.

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- Ignoring risk sensitivity of the decision maker may lead to improper control and undesired financial consequences.

The key contribution of Chapter 6 is the analytical structural results for risk-sensitive dynamic pricing under the additive and atemporal exponential utility, which will have important applications such as robust controls and differential games, given their connections with risk-sensitive control. Risk-sensitive dynamic pricing for multiple products remain to be future research.

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