Blind deconvolution algorithms for medical ultrasound imaging

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Blind Deconvolution Algorithms for Medical Ultrasound Imaging

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A thesis submitted to the Nanyang Technological University
in fulfilment of the requirement for the degree of
Doctor of Philosophy

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To my family.

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Summary

Medical ultrasound images are degraded representations of the true acoustical reflectors in the imaged anatomical tissue. The images are modelled mathematically as a spatio-temporal convolution between the space-variant ultrasonic system impulse response or point-spread function (PSF) and the biological tissue, with the addition of observation noise introduced in the image formation process. The dominant degradation effects are blurring and speckle, both can significantly degrade image quality and increase the difficulty in discriminating fine details in images during diagnostic examinations. Under this formulation, the process of recovering the biological images from the observations is known as the *blind deconvolution problem*. This thesis focuses on the development of novel ultrasound blind image deconvolution algorithms.

Blind deconvolution aims at extracting original images from blurry observations. The restoration process consists of two main procedures. First, the two-dimensional (2-D) PSF is estimated using a noise robust 2-D phase unwrapping algorithm and noise-robust procedure in the cepstral domain. With the incorporation of the 2-D PSF, the process of image restoration becomes a deconvolution problem.

Two different deconvolution approaches are adopted in our research studies. The first approach uses the constrained least squares (LS) optimization to deal with difficulties associated with inverse filter, while retaining the simplicity of using a single linear space-invariant filter for restoring the images. An iterative filtering scheme

that can further improve the solution in the frequency domain is then proposed.

Wavelets are scaled waveforms that measure signal variations. Wavelet multiresolution analysis has excellent property of space and scale localization such that objects in the ultrasound images are observed to occur at different scales. Features that might go undetected at one resolution may be easy spot at another. The alternate approach proposes two multiscale wavelet deconvolution algorithms to combat the limitations of the Fourier transform.

The first wavelet deconvolution algorithm utilizes the discrete nonorthogonal wavelet transform which offers the flexibility of having the wavelet follows the shape of the estimated PSF. We design the wavelet to adaptively estimate the ultrasonic pulse-wavelet. The second wavelet deconvolution algorithm differs from the first in that it does not need the wavelet to resemble the wave shape of the ultrasonic pulse-wavelet. Instead, the key component is the design of a new symmetrical mirror wavelet basis that approximates the Karhunen-Love transform. Estimation of tissue images is then accomplished in the wavelet subbands by making effective use of higher-order statistics under the Bayesian framework. This approach has an additional advantage in computational efficiency. The former algorithm segments the ultrasound image into lattices of homogenous regions. Wavelet deconvolutions are performed separately in each of these regions and the outcomes are concatenated to form the final deconvolved image. The second algorithm goes beyond this restriction where the entire ultrasound image is processed by a pseudo-inverse filter and then denoised in the symmetrical mirror wavelet basis.

These algorithms are evaluated based on computer simulation and real ultrasound radio frequency (RF) data. The deconvolved ultrasound images can show significant improvement in the speckle reduction and they are visually more appealing. The restored images are de-blurred, which implies that they are relatively independent of the properties of the imaging system. Embedded objects in the biological tissues

are more clearly seen in the deconvolved ultrasound images.

In summary, this thesis is a contribution to the development of novel blind deconvolution algorithms for medical ultrasound imaging. Our proposed new ultrasound tissue model is in the center of the development. Under this model, this thesis provides an extensive study of the proposed blind deconvolution algorithms and presents them in a unified framework to give a new paradigm of the deconvolution process.

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Notations and Abbreviations

The mathematical variables or symbols used within limited contexts are not listed here. They are described within their context. Symbols listed may stand for other entities or variables in different applications. Only the common associations in this thesis are listed for ready reference.

List of Symbols

Symbol	Description
$d(m, n)$	2-D tissue noise
$d(n)$	1-D tissue noise
$D(\omega_m, \omega_n)$	FT of $d(m, n)$
$D(\omega_n)$	FT of $d(n)$
$E\{\cdot\}$	statistical average
F	FT operator
F^{-1}	IFT operator
$f(m, n)$	2-D noiseless tissue
$f(n)$	1-D noiseless tissue
$F(\omega_m, \omega_n)$	FT of $f(m, n)$
$F(\omega_n)$	FT of $f(n)$

$\check{f}(m, n)$	estimate noiseless tissue
$h_{pe}(\vec{r}_1, \vec{r}_s, t)$	modified pulse-echo spatial impulse response
$\vec{r}_1$	position of scatterer
$\vec{r}_s$	position of ultrasound transducer
$s(m, n)$	2-D noisy tissue
$s(n)$	1-D noisy tissue
$S(\omega_m, \omega_n)$	FT of $s(m, n)$
$S(\omega_n)$	FT of $s(n)$
$s(\vec{r}_1)$	inhomogeneities of the tissue
t	continuous time variable
m	horizontal discrete sample coordinate
n	vertical discrete sample coordinate
$\mathcal{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$U(n)$	Unit step function
$v(m, n)$	2-D PSF
$v(n)$	1-D PSF
$V(\omega_m, \omega_n)$	FT of $v(m, n)$
$V(\omega_n)$	FT of $v(n)$
$v_{pe}(t)$	ultrasonic pulse-echo wavelet
$w(m, n)$	2-D inevitable noise in the measured signal
$w(n)$	1-D inevitable noise in the measured signal
$W(\omega_m, \omega_n)$	FT of $w(m, n)$
$W(\omega_n)$	FT of $w(n)$
$y(m, n)$	2-D ultrasonic RF observed signal
$y(n)$	1-D ultrasonic RF observed signal
$Y(\omega_m, \omega_n)$	FT of $y(m, n)$
$Y(\omega_n)$	FT of $y(n)$
Z	ZT operator

Z^{-1}	IZT operator
$\delta(n)$	Kronecker delta (discrete Dirac impulse in n)
$*$	2-D linear convolution operator
$*_{\tau}$	spatial convolution operator
$*_t$	temporal convolution operator
$ \cdot $	magnitude operator
$\ \cdot\ $	L^2 norm
∞	infinity

List of Abbreviations

1-D	one-dimensional
2-D	two-dimensional
3-D	three-dimensional
ACF	Autocovariance Function
CLS	Constrained Least Square
CR	Contrast Ratio
CWT	Continuous Wavelet Transform
DFT	Discrete Fourier Transform
DNWT	Discrete Nonorthogonal Wavelet Transform
DWT	Discrete Wavelet Transform
FT	Fourier Transform
GGD	Generalized Gaussian Distribution
HOS	Higher-Order Statistics
IDFT	Inverse Discrete Fourier Transform
IDNWT	Inverse Discrete Nonorthogonal Wavelet Transform
IDWT	Inverse Discrete Wavelet Transform
IFT	Inverse Fourier Transform

IZT	Inverse Z-Transform
LS	Least Square
MAD	Median Absolute Deviation
MAP	Maximum A Posteriori
MSE	Mean Square Error
MMSE	Minimum Mean Square Error
PSD	Power Spectrum Density
pdf	probability density function
PSF	Point-Spread Function
RF	Radio Frequency
RG	Resolution Gain
ROI	Region Of Interest
SNR	Signal-to-Noise Ratio
dB	deciBel
WSS	White Sense Stationary
ZT	Z-Transform

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Chapter 1

Introduction

Ultrasound imaging has been an important advancement in medical diagnosis for many reasons. The risk to patients of diagnostic ultrasound at the intensities currently used is minimal with no adverse effect being report [1]. For this reason, ultrasonography is an indispensable modality in obstetrics and management of a large number of diseases. As an example in fetal imaging, the outlines of the head and face, as well as the spinal column are clearly visible in the image. These enable the measurement of the size of the head and head-to-sacrum length of the fetus. These measurements facilitate the detection of abnormalities in the fetus development, especially distension of the stomach, hydrocephalus, and complications due to maternal (or gestational) diabetes such as the lack of development of the sacrum. Figure 1.1 shows an ultrasonic image of a 35-week fetal profile. The image demonstrated a superior soft tissue and frontal bone definition of the fetal profile.

Ultrasound is low-cost and portable. It seems certain that, in the future, many physicians will carry ultrasonic scanners along with their other paraphernalia such as stethoscopes and ophthalmoscopes. Physicians will have to learn how to interpret the images. This shift in practice will greatly benefit the patients. Figure 1.2 shows the photograph of the modern portable ultrasound system. It is complete with all

ultrasound functionality in a small and hand-carried package.

Figure 1.1: Superior soft tissue and frontal bone definition in an ultrasonic image of a 35-week fetal profile. Image courtesy of GE Healthcare.

Ultrasound is increasingly important in surgery. The real-time capability of ultrasonography makes it suitable as an intraoperative imaging modality that can cope with the problems that arise from changes in the anatomy during surgery. Furthermore, the potential of ultrasound has expanded from generating real time 2-D and repetitive three-dimensional (3-D), to also real-time 3-D images. It is a logical evolution of ultrasound technology and complements other parallel developments underway at this time, which is expected to dominate in next generation ultrasound systems. Viewing the 3-D anatomy in the computer enables the physician to visualize the relevant structures with a perspective similar to that found during surgery, facilitating the surgical plan, particularly in cases of congenital pathology. The 3-D imaging capability of ultrasound represents a valuable supplement or alternative to interventional magnetic resonance imaging and computer tomography because of lower investment and running costs as well as more convenient and portable equipment.

Figure 1.2: Portable ultrasound system has the size of a laptop computer. Image courtesy of GE Healthcare.

This thesis is a part of the ongoing research interests on biomedical signal processing at the Biomedical Instrumentation Laboratory, in the Division of Control & Instrumentation of Nanyang Technological University, Singapore. The research activity has concentrated on ultrasound image processing using blind deconvolution techniques on the received ultrasound RF signal. The central theme to this approach is based on linear systems theory where the images are interpreted as a spatio-temporal convolution between the ultrasound pulse-echo wavelet or ultrasonic 2-D PSF and the biological tissue, contaminated by an addition of white Gaussian noise. We aim at enhancing the ultrasound image and reduce distractions in image interpretation.

1.1 Motivation

The non-invasive nature, low cost, portability, and real-time image formation make ultrasound imaging a valuable tool for medical diagnosis. It is now the second most widely used diagnostic imaging modality (next to X-ray). Ultrasound imaging has found numerous applications in today's clinics [2]–[4]. Some of these applications are the diagnosis of valvular insufficiency (flail leaflet), myocardial dyskinesia, septal defects and congenital malformations in cardiology and it has become widely known

and commonly used in obstetrics for fetal monitoring and congenital malformations examination. Ultrasound is also generally far better than plain X-ray at distinguishing the subtle variations of soft, fluid-filled tissue structures, and can be used in any of several modes, depending on the need at hand. This makes it particularly useful in providing diagnostic images of the abdomen where identification of tumors and cysts in both liver and kidneys are performed.

Many of these diagnoses are based on relatively gross anatomical information that is available from the ultrasound image. This information is of critical value to the clinician for diagnoses such as diffuse disease processes, the identification of tumors, quality of myocardial dynamics and so forth. It is for these reasons that the signal integrity and retention of maximum dynamic range is of key value in the image formation process.

The full potential of ultrasound imaging in diagnostic and monitoring capabilities, as well as its clinical and biological applications has not yet been realized. It still suffers from a number of limitations. The ultrasound images exhibit speckle artifacts which mask the presence of low contrast lesions and hence reduce the ability of human observer to discriminate fine details in diagnostic examination. Speckle is caused by the constructive and destructive interference of backscattered echoes originating from unresolved tissue inhomogeneities, is responsible for a characteristic granularity, or texture, of the image which bears little resemblance to the actual acoustical tissue microstructures, but it is instead highly dependent on the performance characteristics of the imaging system.

Over the years, a number of image processing techniques such as weighted median filtering [5], [6], adaptive filtering [7], [8], wavelet shrinkage [9]–[11], and anisotropic diffusion [12], [13] have been explored on the ultrasound envelope image to reduce speckle. A brief survey of enhancement methods other than blind deconvolution that have been tried before on ultrasound images are given in Appendix A.

Despite being simple and fast, these techniques have achieved only a moderate degree of success because of inherent limitations in processing the demodulated envelope image. The ultrasound envelope images have a lower resolution and they are also severely blurred in both the axial and lateral directions by the scanner probe. Moreover, factors originating from imaged tissues such as acoustic phase aberrations and velocity variations due to tissue inhomogeneity are also of important causes. The ultrasonic pulse wavelet extends its shape in its axial and lateral directions, thereby changes its shape and frequency content as it propagates through the tissues. The observed ultrasound image is actually a distorted version of the actual anatomical image.

An ultrasound image is a representation of the reality. It can do no more than convey information to the observer concerning structure and function as represented by the ultrasonic characteristics of the tissue. Knowing that image formation actually occurs in the RF domain, recent techniques modelled the ultrasound image formation process in the RF domain as a spatio-temporal convolution between the ultrasonic 2-D PSF and tissue response, most of them described and referred to in [14]–[30]. This paradigm opens a new era spring from the remarkable advances in modern signal processing techniques for the restoration of the imaged tissue structure.

1.2 Objectives

The overall goal of this study is to develop blind image deconvolution techniques for ultrasound in medical diagnosis.

In our studies, we are mainly using simulated ultrasound images. This has the advantage that the test data are eliminated from effects highly susceptible by a variety of other ultrasound research topics such as phase aberrations and attenuation. This approach of investigation had also been widely adopted. Examples of which

are [12], [17], [22], [25], [31]–[40]. The use of simulated ultrasound images offers a means of studying the developed blind deconvolution algorithms. Specially, it can reveal the quality of the estimated PSF of which has been recognized the importance to have an accurately estimated PSF in the recovery process, most of them described and referred to in [14]–[30].

We want to develop new and simpler methods for more accurate deconvolution of medical ultrasound images. Under the convolution expression of the ultrasonic RF echo, a known degradation function is used in the recovery of the mathematical tissue structure. The quality of the deconvolved image depends on the fidelity in the mathematical modeling of the tissue structure. Therefore, we also want to improve our algorithms through the use of a better ultrasound tissue model. The incorporation of the new tissue model will help in discriminating the tissue structures from the random fluctuation in the diffuse scatterers. We emphasize noise suppression in both the biological tissue response and that inevitably presents during the RF echo measurement and, hence, improve visualization of the medical ultrasound images.

An iterative algorithm is powerful when prior knowledge about the underlying signal or image is available in the form of constraints on the allowable restorations, when the blurring function is only approximately known or when it is nonstationary. We are interested in deriving constrained LS filtering algorithm, which is based on the specific nature of our prior knowledge available about the imaged tissue structure and the degradation function. An iterative scheme is desirable for updating the estimate to enhance the deconvolution results.

Finally, since ultrasound imaging relies on the launching of an acoustic wavelet into the body under investigation and the ultrasound image is generated from the received echo wavelets, this mechanism generally motivates the use of wavelet analysis as a means for ultrasound image deblurring while preserving the tissue microstruc-

tures in a nature way. Thus, we expect that wavelets would be able to concentrate both the energy of the tissue structures and diffuse scatterers into few coefficients. We want to propose wavelet deconvolution algorithms as a natural selection for ultrasound image deconvolution.

1.3 Major Contribution of the Thesis

Modern ultrasound image deconvolution algorithms are developed based on simple convolution model. As the *first* contribution of this thesis, we propose an ultrasound tissue model that goes beyond this restriction to give a new paradigm of the convolution process. It incorporates random fluctuations of the tissue response and offers more realistic interpretation of the received pulse-echo ultrasound signal. Using the new model, we formulate two different approaches of ultrasound image deconvolution.

The *second* contribution of the thesis is the development of a modification from the classical regularization technique which combines the Wiener filter and constrained LS algorithm for restoration of the ultrasound image. The criterion function is chosen based on prior knowledge of the imaging system for a trade-off between errors caused by regularization and noise magnification. An iterative filtering scheme is then proposed to further improve the solution of the frequency domain deconvolution.

The *third* contribution of our thesis is to propose two multiscale wavelet deconvolution algorithms to overcome the limitations of the Fourier transform. Wavelet transforms offer the promise of compact representation and efficient detection of image components that match the wave shape of the chosen wavelet. The first wavelet deconvolution algorithm utilizes the discrete nonorthogonal wavelet transform which offers the flexibility of having the wavelet follow the shape of the estimated ultrasonic

PSF. We design the wavelet to adaptively estimate the ultrasound tissue response using the linear Bayesian estimator.

Inverting the degradation is often numerically unstable and noise is amplified considerably. The key component of the second algorithm is the design of a new symmetrical mirror wavelet basis that approximates the Karhunen-Love transform. Higher-order statistics can capture image properties that elude the second-order statistic in the linear Bayesian estimator. We utilize Bayesian estimator that exploits these higher-order statistics. In the wavelet domain, the tissue subband coefficients are modelled as the generalized Gaussian distribution (GGD) whereas the noise subband coefficients follow that of the white Gaussian distribution. Under the Bayesian framework, a threshold estimator is numerically computed for the restoration of the ultrasound tissue subband coefficients.

1.4 Organization of the Thesis

The thesis is divided into seven chapters in which we address the multiple connections between the essential principles of ultrasound imaging, the mathematical formulation of the ultrasonic RF convolution model, and the blind image deconvolution algorithms. It is organized as follows.

In Chapter 2, we first review the background knowledge on the mathematical formulation of the research problem. We present a concise and self-contained review emphasizing the most relevant aspects for the topic of this thesis. These include physical principles of medical ultrasound imaging and an illustration of the pulse-echo technique for medical ultrasound imaging. Readers familiar with physics of ultrasound can skip Section 2.1 and 2.2. In Section 2.3, we present a thoroughly mathematical derivation leading to the formulation of the received ultrasonic pressure field as a convolution expression. This section is rather lengthy; readers who

are not interested in the derivations may accept the final outcome and move on to Subsection 2.3.5. At this point, we systematically describe different statistical modelling of ultrasonic backscattering and introduced our new ultrasound tissue model. Then, in Section 2.4 we propose our model of the ultrasound image degradation and restoration process. Through Section 2.5 we link them to our later developments of ultrasound image blind deconvolution algorithms.

Chapter 3 begins with introduction to the concept taken in the estimation of the degradation function. We explain in detail the most successfully methods so far in this field. They are based on the assumption that the PSF and the tissue signal reside in different bands of the cepstral domain. Particularly, we provide motivations and thoroughly illustrations to the complex cepstrum analysis and the required 2-D phase unwrapping algorithm as it promises a good estimation for the 2-D PSF. This method of 2-D PSF estimation is used in the algorithms described in the following chapters.

Chapter 4 concerns with removing the linear blur from the ultrasound image through constrained LS filtering algorithm and an iterative deblurring procedure. The chapter first reviews image restoration with unconstrained and constrained LS optimization techniques. Readers familiar with these techniques are invited to skip these portions of the chapter. With these knowledge, we present a deconvolution algorithm according to the new ultrasound tissue model.

Chapter 5 and Chapter 6 present our main contributions to ultrasound image wavelet deconvolution. We propose two algorithms. The first proposal is recorded in Chapter 5, it is derived from a nonorthogonal wavelet transform which is able to provide flexible in the wavelet design. This algorithm can adaptively estimate the ultrasound tissue response using the linear Bayesian estimator. Our second algorithm that will be presented in Chapter 6 incorporates our newly designed symmetrical mirror wavelet basis which can approximate the Karhunen-Loève transform. We

statistically model the tissue subband coefficients with GGD prior and we propose the estimation of the tissue subband coefficients in the Bayesian framework. The results of both algorithms demonstrate the advantage of the new wavelet deconvolution techniques both in terms of quantitative and visual aspects of image quality. Finally, Chapter 7 is devoted to conclusions and our recommendations for further research.

Chapter 2

Background

This chapter begins with a review of the physics of ultrasound imaging. We briefly illustrate the physical properties of an ultrasonic wave propagating in biological tissues. We will also highlight the essential concepts for ultrasound imaging. They are reflection, refraction, scattering, attenuation and diffraction. Then, in Section 2.2 we describe the pulse-echo technique of medical ultrasound which leads to the formation of the 2-D ultrasound images. These two sections are meant for readers who are new to ultrasound imaging.

In Section 2.3, we present the modelling of the ultrasound image formation process in the RF domain as a 2-D spatio-temporal convolution expression from the wave equation. This section derives the wave equation and describes different linear assumptions made in Subsection 2.3.1. We calculate the scattered field and introduce the transducer model for calculating the incident field in Subsection 2.3.2 and Subsection 2.3.3 respectively. Subsection 2.3.4 combines the wave equation solution and transducer model to give the final expression for the received pressure field.

Furthermore, in Subsection 2.3.5, we show that the scattering term which is a function of density and propagation velocity perturbation can be represented as the

combination of the mean scattering field and a random fluctuating field. We introduce a new ultrasound tissue model into the convolution model so as to offer a more realistic interpretation of the received pulse-echo ultrasound signal. With this knowledge, in Section 2.4 we propose a model for the ultrasound image degradation and restoration process. Finally in the last section of the chapter, we briefly introduce the two-step ultrasound image blind deconvolution approach in our research studies.

2.1 Physics of Ultrasound Imaging

Sound waves with frequencies outside the human hearing range may reach human ear, but we are not generally aware of them. Frequencies above 20 KHz are called ultrasound. This means that ultrasonic waves have a much shorter wavelength which allows them to reflect off smaller surfaces. It is this property that makes ultrasound useful for nondestructive examination of materials.

In medicine, ultrasound imaging is widely used in diagnosis. It is non-invasive in nature, low cost, portability, and provides real-time image formation. The diagnostic ultrasound images are generated by making use of pulse-echo technique with frequencies typically between 1 and 20 MHz. A short ultrasonic wavelet is directed into the human body, and the images are formed via backscattering of mechanical energy from boundaries between tissues or interfaces between organs, lesions in the body, and from small structures within tissue. Typically, ultrasound pulses of about $1 \mu\text{s}$ duration at a repetition rate of about 1 Kpps (pulses per second) are applied, and the resulting echoes are used for locating tissue interfaces and imaging.

Higher frequencies mean shorter wavelengths, and thus in principle visualization of finer details. However, attenuation of ultrasonic waves increases with frequencies in soft tissues. The higher the frequency, the more the wave energy is absorbed

by the body, and the more the reflections from deeper inside the body are lost. Since the frequency and the wavelength are inversely related and lower frequencies achieve deeper penetration, lower frequencies of 1–5 MHz are used for studies of deep-lying structures, such as cardiology, abdominal, obstetrics and liver whereas higher frequencies are for those applications which require shallow penetration but high resolution, such as ophthalmology, peripheral vascular and testicular. Some essential physical properties of ultrasound for imaging can be found in [41]–[47]:

Reflection An ultrasonic wave bounced off a non-absorbing interface. It occurs at interfaces between large regions (much larger than a wavelength) of media with different acoustic properties such as density or compressibility. A single incident beam resulting in a single reflected beam is termed specular reflection. When the ultrasound beam is reflected, the incident angle is equal to the reflective angle. This suggests that a properly shaped surface can be used to focus or diffract ultrasound beam.

Refraction When an ultrasonic wave is incident upon a boundary between two different media of different characteristic acoustic impedances, some of the waves are reflected and some passes to the second medium. Refraction is most troublesome at a bone and soft tissue interface, where large angular deviations of up to 20° in the direction of the transmitted wave can occur.

Scattering Reflection and refraction of the ultrasonic beam occurs at boundaries where the dimensions of the surfaces are large compared to the wavelength of the ultrasonic wave. If the beam encounters tissue surface inhomogeneities or particles within tissue that are comparable to or smaller than the ultrasound wavelength, then the wave is scattered in many directions. The angular dependence and the

Figure 2.1: The effect of a boundary between two tissues having different acoustic impedances on the propagation of an ultrasonic wave. (Left) The incident wave is travelling at an angle of 90° with respect to the boundary, (center) the wave travels at an arbitrary angle with respect to the boundary, and (right) wave encounters surface irregularities give rise to scattering.

magnitude of the scattering are dependent upon the shape, the size, and the physical and acoustic properties of the scatterer. Scattering from objects much smaller than a wavelength typically increases with frequency, while that from an object comparable to a wavelength is constant with frequency. The effect of ultrasonic beams that are scattered to a direction that is significantly different from the original path will likely be perceived as background noise.

Attenuation Attenuation of the ultrasonic beam as it propagates through the tissue is the sum of the scattering and the absorption process where it does not include the amount of energy reflected from interfaces. Attenuation depends on the distance travelled and the frequency of the ultrasonic wave. It is characterized by an exponential decrease in both the pressure and the intensity of the ultrasonic beam as a function of its propagation distance. Since both the process of absorption and scattering are strongly dependent on frequency, attenuation is specified in terms of dB/cm/MHz. In inhomogeneous media such as soft tissues, the values of attenuation reported lie around 0.5 dB/cm/MHz.

Diffraction This is a bending of the direction of propagation that occurs when a wave impinges on an object of different density embedded within, and surrounded by, the incident media. Depending on the size of the object and the wavelength of

the incident ultrasonic wave, a shadow zone may be created on the distal side of the object which can distort the direction information of the signal.

2.2 Pulse-echo Technique for Medical Imaging

The basic principle of ultrasound imaging is based on pulse-echo technique works as follows. A short pulse of ultrasound is transmitted into the body using as ultrasonic transducer, which is a device that transforms an electrical pulse into ultrasonic pressure field. The transducer is focused to produce a narrow ultrasonic beam, which propagates as a pressure wave through tissue at a speed of approximately 1540 m/s. When the ultrasonic wavelet encounters tissue surfaces, boundaries between tissues, or structures within organs, a part of the energy of the pulse is scattered in all directions, with a certain fraction of the energy being scattered along the original transmission path and returning to the same transducer, while most of the energy continue on. The strength of a received pulse-echo wavelet depends mainly on the difference in density of the two materials on either side of the interface. It also depends on the ultrasound velocity in each material, but this normally has little effect since in most tissue the velocity is within a few percent of the average 1540 m/s.

The detection of the reflected pulse-echo wavelets is done by the same transducer, which converts the backscattered pressure fields into electrical wavelets. After time-gain compensation, rectified and filtering, the signals are digitized. The final signals are then displayed on a display monitor. As an example, consider an ultrasonic pulse-wavelet passing through the abdomen, as shown in Figure 2.2. At various interfaces, part of the transmitted pulse is reflected. Using the measured time elapse from pulse transmission to echo reception, and the propagation velocity of 1540 m/s, one can estimated the depth of the feature. If the total time taken for an

2.2 Pulse-echo Technique for Medical Imaging

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Figure 2.2: (Top) Ultrasound beam passes through abdomen, reflecting from surfaces in its path. (Middle) Reflected pulses plotted as a function of time when received by transducer. The vertical dashed lines point out which reflected pulse goes with which surface, and (Bottom) B-mode scan display for the same echo where the brightness of each dot is related to signal strength.

ultrasonic wavelet from the time of its transmission to the time it backscattered by the vertebra and then received by the transducer be $320\ \mu\text{s}$, the pulse would have travelled a round-trip distance of 0.5 m and one can estimate the distance from transducer to the vertebra is approximately 0.25 m.

A trace such as that shown in Figure 2.2 can be displayed directly on a monitor. Such a display is called an amplitude (A)-mode scan. After all the echoes have been received from the first beam trajectory, the direction of the beam is steered to acquire a second line of data adjacent to the first. Brightness (B)-mode scanning produces a 2-D image as shown in the Figure 2.3. Each line in the image consists of an A-mode

Figure 2.3: (Top) Twelve traces are made across the abdomen by moving the transducer, or by using an array of transducers. (Bottom) The echoes are plotted to produce the image. More closely spaced traces would give a more detailed image.

scan with the brightness of the signal being proportional to the amplitude of the backscattered echo. This mode of scanning can be used to study both stationary and moving structures, such as the heart, because complete images of the structures can be acquired rapidly. For example, in the case of ultrasound imaging with a depth-of-view of 10 cm, the total time taken for the round-trip distance travelled of 20 cm is $130 \mu\text{s}$. If the image is formed with 120 lines, then the total time taken to acquire one frame is 15.6 ms (i.e., $120 \times 130 \mu\text{s}$) and the frame rate is 64 Hz (i.e., $\frac{1}{15.6 \text{ ms}}$).

2.3 Modelling of Ultrasound Images

Diagnostic ultrasound is used in almost all medical fields and it is recognized with great success in the diagnosis of abnormalities in soft tissue structures in the human body. In Chapter 1, we have highlighted its increasingly importance in a variety of clinical situations. Unfortunately, several factors affect the image quality. Limited resolution of the ultrasound imaging system is the major factor degrading the image resolution. The deficiency of the axial and lateral resolutions is due to the finite bandwidth of the ultrasonic transducers and the non-negligible width of the transmitted ultrasonic beams respectively. Besides, other factors which are originating from the nature of the tissue such as phase aberrations and velocity variations will also affect the image contrast. Therefore, the observed ultrasound image can be seen as a distorted version of the tissue image.

Subsection 2.3.1 to Subsection 2.3.4 present the mathematical formulation of the physical ultrasound RF images as a convolution expression. This expression laid the groundwork for using deconvolution methods as a means to improve the quality of the ultrasound images. The soul of this section is an introduction of the new tissue model into the convolution expression in Subsection 2.3.5. In doing that, we identify the desirable signals and unwanted signals in the ultrasound image formation process.

2.3.1 Wave Representation of Mechanical Energy

In this subsection, we present the derivation of the wave equation by Chernov [53]. This formulation is also adopted in [31] where the scattering term for the wave equation is a function of the density and propagation velocity perturbations. Before we proceed on, we want to highlight that there is another equivalent formulation of the wave equation that will lead to the same interpretation of the ultrasound

images as a convolution expression. It is presented in a number of papers where the scattering term is considered as a function of the adiabatic compressibility and density in the wave equation [48]–[52].

In order to arrive at a solvable wave equation, several assumptions and approximations are required. The first assumption states that the instantaneous acoustic pressure and density can be written as:

$$P_{ins}(\vec{r}, t) = P + p_1(\vec{r}, t); \quad \rho_{ins}(\vec{r}, t) = \rho(\vec{r}) + \rho_1(\vec{r}, t), \quad (2.1)$$

where P is the mean pressure of the medium, ρ is the density of the undisturbed medium and t is the continuous time. Both the pressure variation $p_1(\vec{r}, t)$ and density change $\rho_1(\vec{r}, t)$ are small quantities of first order caused by the ultrasonic wave. They are of small values compared to P and ρ respectively.

The second assumption is that no heat conduction or conversion of ultrasound to thermal energy takes place. Thus, the pressure and density satisfy the adiabatic equation:

$$P_{ins}(\vec{r}, t) = c^2 \frac{\partial \rho_{ins}(\vec{r}, t)}{\partial t}, \quad (2.2)$$

where c is the adiabatic ultrasound velocity at the given point of the medium. The pressure, density and particle velocity $\vec{v}$, where $d\vec{v} = \frac{d\vec{r}}{dt}$, must also satisfy the *hydrodynamical equations*:

$$\rho_{ins} \frac{\partial \vec{v}}{\partial t} = -\nabla P_{ins}, \quad (2.3)$$

$$\frac{\partial \rho_{ins}}{\partial t} = -\nabla \cdot (\rho_{ins} \vec{v}), \quad (2.4)$$

which are correspondingly the *linear inviscid force equation* and the *equation of continuity*. Equation (2.2) contains total rather than partial time derivatives since the relation must be satisfied for a given particle of the medium and for a given

point of space. Using (2.1) to substitute for the pressure and the density in (2.2) and using the assumption $\rho_1(\vec{r}, t) \ll \rho(\vec{r})$, we obtain:

$$\frac{1}{c^2} \frac{\partial p_1(\vec{r}, t)}{\partial t} = \frac{\partial \rho_1(\vec{r}, t)}{\partial t} + \vec{v}(\vec{r}, t) \cdot \nabla \rho(\vec{r}). \quad (2.5)$$

In the case of a homogeneous medium, the convective term $\vec{v}(\vec{r}, t) \cdot \nabla \rho(\vec{r})$ in the last equation vanishes, and (2.2) can be written for $p_1(\vec{r}, t)$ and $\rho_1(\vec{r}, t)$ using either the total or partial time derivatives as in (2.2) and (2.5). Likewise, using the same assumption and upon substituting (2.1) into (2.3) and (2.4), we obtain:

$$\rho(\vec{r}) \frac{\partial \vec{v}(\vec{r}, t)}{\partial t} = -\nabla p_1(\vec{r}), \quad (2.6)$$

$$\frac{\partial \rho_1(\vec{r}, t)}{\partial t} = -\nabla \cdot (\rho(\vec{r}) \vec{v}(\vec{r}, t)). \quad (2.7)$$

Differentiating (2.7) with respect to t and inserting (2.6) gives:

$$\frac{\partial^2 \rho_1(\vec{r}, t)}{\partial t^2} = \nabla^2 p_1(\vec{r}, t). \quad (2.8)$$

Differentiating (2.5) with respect to t yields:

$$\frac{1}{c^2} \frac{\partial^2 p_1(\vec{r}, t)}{\partial t^2} = \frac{\partial^2 \rho_1(\vec{r}, t)}{\partial t^2} + \frac{\partial \vec{v}(\vec{r}, t)}{\partial t} \cdot \nabla \rho(\vec{r}). \quad (2.9)$$

Substituting (2.6) and (2.8) into (2.9) leads to:

$$\nabla^2 p_1(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 p_1(\vec{r}, t)}{\partial t^2} = \frac{1}{\rho(\vec{r})} \nabla \rho(\vec{r}) \cdot \nabla p_1(\vec{r}, t). \quad (2.10)$$

We now assuming that the propagation velocity and density only vary slightly from their mean values, so that:

$$\rho(\vec{r}) = \rho_0 + \Delta \rho(\vec{r}); \quad c(\vec{r}) = c_0 + \Delta c(\vec{r}), \quad (2.11)$$

where $\rho_0 \gg \Delta\rho(\vec{r})$ and $c_0 \gg \Delta c(\vec{r})$. Then, using the approximation $\frac{1}{1+x} \approx 1 - x$ and $\rho_0 \gg \Delta\rho(\vec{r})$, (2.11) becomes:

$$\nabla^2 p_1(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 p_1(\vec{r}, t)}{\partial t^2} = -\frac{2\Delta c(\vec{r})}{c_0^3} \frac{\partial^2 p_1(\vec{r}, t)}{\partial t^2} + \frac{1}{\rho_0} \nabla \left(\Delta\rho(\vec{r}) \cdot \nabla p_1(\vec{r}, t) \right). \quad (2.12)$$

This is the homogeneous wave equation in inhomogeneous media. The last two terms on the right side of the equation are the scattering terms that vanish for a homogeneous medium.

2.3.2 Calculation of the Scattered Field

Having derived the wave equation, it is used in calculating the scattered field from a small inhomogeneity identified by $\vec{r}_1$ embedded in a homogeneous surrounding and is enclosed in the volume V' as shown in Figure 2.4.

Figure 2.4: Coordinate system for calculating the scattered field.

The scattered field is calculated at the point indicated by $\vec{r}_1$ by integrating all the spherical waves emanating from the scattering region V' using the time dependent

Green's function for unbounded space. The scattered field is found to be [31], [48], [49]:

$$p_s(\vec{r}_2, t) = \int_{V'} \int_T \left[\frac{1}{\rho_0} \nabla(\Delta \rho(\vec{r}_1)) \cdot \nabla p_1(\vec{r}_1, t_1) - \frac{2\Delta c(\vec{r}_1)}{c_0^3} \frac{\partial^2 p_1(\vec{r}_1, t_1)}{\partial t^2} \right] \times G(\vec{r}_1, t_1 | \vec{r}_2, t) dt_1 d^3 \vec{r}_1, \quad (2.13)$$

where the free space Green's function is:

$$G(\vec{r}_1, t_1 | \vec{r}_2, t) = \frac{\delta(t - t_1 - \frac{|\vec{r}_2 - \vec{r}_1|}{c_0})}{4\pi |\vec{r}_2 - \vec{r}_1|}, \quad (2.14)$$

and $|\cdot|$ denotes the magnitude. The pressure field inside the scattering region is given as:

$$p_1(\vec{r}_2, t) = p_i(\vec{r}_2, t) + p_s(\vec{r}_2, t), \quad (2.15)$$

where $p_i(\vec{r}_2, t)$ is the incident pressure field. As can be seen, the integral in (2.13) cannot be solved directly. Fortunately, it is solved by applying *Born-Neumann expansion*:

$$p_s(\vec{r}_2, t) = p_{s1}(\vec{r}_2, t) + p_{s2}(\vec{r}_2, t) + p_{s3}(\vec{r}_2, t) + \dots, \quad (2.16)$$

where the scattering fields are given by:

$$p_{sn}(\vec{r}_2, t) = \begin{cases} \int_{V'} \int_T \left[\frac{1}{\rho_0} \nabla(\Delta \rho(\vec{r}_1)) \cdot \nabla p_i(\vec{r}_1, t_1) - \frac{2\Delta c(\vec{r}_1)}{c_0^3} \frac{\partial^2 p_i(\vec{r}_1, t_1)}{\partial t^2} \right] G(\vec{r}_1, t_1 | \vec{r}_2, t) dt_1 d^3 \vec{r}_1 & , \text{if } n = 1 \\ \int_{V'} \int_T \left[\frac{1}{\rho_0} \nabla(\Delta \rho(\vec{r}_1)) \cdot \nabla p_{s(n-1)}(\vec{r}_1, t_1) - \frac{2\Delta c(\vec{r}_1)}{c_0^3} \frac{\partial^2 p_{s(n-1)}(\vec{r}_1, t_1)}{\partial t^2} \right] G(\vec{r}_1, t_1 | \vec{r}_2, t) dt_1 d^3 \vec{r}_1 & , \text{if } n > 1 \end{cases} \quad (2.17)$$

For weak scattering throughout V' , i.e., $p_{s1}(\vec{r}_2, t) \ll p_i(\vec{r}_2, t)$, the first-order Born approximation will give a satisfactory representation of the behavior of the scattered ultrasound. The scattered field is found to be [31], [48], [49]:

$$\begin{aligned}
p_s(\vec{r}_2, t) &\approx p_{s1}(\vec{r}_2, t) \\
&\approx \int_{V'} \int_T \left[\frac{1}{\rho_0} \nabla(\Delta \rho(\vec{r}_1)) \cdot \nabla p_i(\vec{r}_1, t_1) - \frac{2\Delta c(\vec{r}_1)}{c_0^3} \frac{\partial^2 p_i(\vec{r}_1, t_1)}{\partial t^2} \right] \\
&\quad \times G(\vec{r}_1, t_1 | \vec{r}_2, t) dt_1 d^3 \vec{r}_1.
\end{aligned} \tag{2.18}$$

Now, we have an expression of the scattered field expressed in terms of the incident field. So, the incident field has to be calculated in order to calculate the scattered field.

2.3.3 Calculation of the Incident Field

The incident field is generated by the transducer, assuming no other sources exist in the tissue. The method presented in this section adopts the convolution method developed in [54]–[56]. The incident field is found by solving the wave equation for the homogeneous case:

$$\nabla^2 p_i(\vec{r}, t) - \frac{1}{c_0^2} \frac{\partial^2 p_i(\vec{r}, t)}{\partial t^2} = 0. \tag{2.19}$$

The field is conveniently calculated by employing the velocity potential $\Psi(\vec{r}, t)$, and enforcing appropriate boundary conditions. From *Eular's equation* of particle velocity, $\rho(\vec{r}) \frac{\partial \vec{v}(\vec{r}, t)}{\partial t} = -\nabla p_i(\vec{r}, t)$, and since $\nabla \times \nabla p_i(\vec{r}, t) = 0$ which implies $\nabla \times \vec{v}(\vec{r}, t) = 0$. Hence, the particle velocity has a scalar potential function:

$$\vec{v}(\vec{r}, t) = \nabla \Psi(\vec{r}, t). \tag{2.20}$$

Substituting (2.20) into the Euler's equation, the acoustic pressure wave can be expressed as:

$$p_i(\vec{r}, t) = \rho_0 \frac{\partial \Psi(\vec{r}, t)}{\partial t}. \tag{2.21}$$

2.3 Modelling of Ultrasound Images

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Using the coordinate system shown in Figure 2.5 in the calculation, the velocity potential satisfies the wave equation for the homogeneous medium:

$$\Psi(\vec{r}_1, \vec{r}_3, t) = \int_S \int_T v(\vec{r}_3 + \vec{r}_4, t_3) g(\vec{r}_1, t | \vec{r}_3 + \vec{r}_4, t_3) dt_3 d^2 \vec{r}_4, \quad (2.22)$$

where the transducer is mounted in a rigid infinite planar baffle, $v(\vec{r}_3 + \vec{r}_4, t_3)$ denotes the particle velocity normal to the transducer surface. Here, S denotes the transducer surface and the Green's function for a bounded medium is:

$$g(\vec{r}_1, t | \vec{r}_3 + \vec{r}_4, t_3) = \frac{\delta(t - t_3 - \frac{|\vec{r}_1 - \vec{r}_3 + \vec{r}_4|}{c_0})}{2\pi |\vec{r}_1 - (\vec{r}_3 + \vec{r}_4)|}, \quad (2.23)$$

where $\vec{r}_3$ identifies the position of the transducer and $\vec{r}_4$ a point on the transducer.

Figure 2.5: Coordinate system for calculating the incident field.

Assume that the surface vibration can be split into a spatial component $a(\vec{r}_3 + \vec{r}_4)$ and a temporal component $v(t_3)$ [95]:

$$\Psi(\vec{r}_1, \vec{r}_3, t) = \int_S \int_T v(t_3) a(\vec{r}_3 + \vec{r}_4) g(\vec{r}_1, t | \vec{r}_3 + \vec{r}_4, t_3) dt_3 d^2 \vec{r}_4, \quad (2.24)$$

where $a(\vec{r})$ denotes the spatial source velocity distribution [57]. Now, if we assume that the particle velocity is uniform over the surface of the transducer, (2.24) is

reduced to:

$$\begin{aligned}\Psi(\vec{r}_1, \vec{r}_3, t) &= \int_T v(t_3) \int_S a(\vec{r}_3 + \vec{r}_4) g(\vec{r}_1, t | \vec{r}_3 + \vec{r}_4, t_3) d^2 \vec{r}_4 dt_3 \\ &= v(t) *_i h(\vec{r}_1, \vec{r}_3, t),\end{aligned}\quad (2.25)$$

where $*_i$ denotes temporal convolution. Therefore, upon substituting (2.25) into (2.21), we arrived at:

$$p_i(\vec{r}_1, \vec{r}_3, t) = \rho_0 \frac{\partial v(t)}{\partial t} *_i h(\vec{r}_1, \vec{r}_3, t). \quad (2.26)$$

Noted, the separation between the excitation and the transducer geometry and that $h(\vec{r}_1, \vec{r}_3, t)$ is spatially varying depends on the difference between $\vec{r}_1$ and $\vec{r}_3$.

2.3.4 Calculation of the Received Signal

Figure 2.6: Coordinate system for calculating the received signal.

The coordinate system for the calculation of the received signal is shown in Figure 2.6. The received signal is the scattered pressure field of the inhomogeneity integrated over the transducer surface, convolved with the electromechanical impulse response $E_m(t)$ of the transducer:

$$p_r(\vec{r}_5, t) = E_m(t) *_i \int_S p_s(\vec{r}_5 + \vec{r}_6, t) d^2 \vec{r}_6. \quad (2.27)$$

Substituting the scattered field derived in (2.18) and that of the incident field in (2.26) into (2.27) gives:

$$\begin{aligned}
 p_r(\vec{r}_5, t) &= E_m(t) *_{\vec{t}} \frac{1}{2} \int_S \int_{V'} \int_T \left[\frac{1}{\rho_0} \nabla(\Delta \rho(\vec{r}_1) \cdot \nabla p_i(\vec{r}_1, t)) \right. \\
 &\quad \left. - \frac{2\Delta c(\vec{r}_1)}{c_0^3} \frac{\partial^2 p_i(\vec{r}_1, t)}{\partial t^2} \right] 2 G(\vec{r}_1, t_1 | \vec{r}_5 + \vec{r}_6, t) dt_1 d^3 \vec{r}_1 d^3 \vec{r}_6 \\
 &= E_m(t) *_{\vec{t}} \frac{1}{2} \int_{V'} \left[\frac{1}{\rho_0} \nabla(\Delta \rho(\vec{r}_1) \cdot \nabla [\rho_0 \frac{\partial v(t)}{\partial t} *_{\vec{t}} h(\vec{r}_1, \vec{r}_3, t)] \right. \\
 &\quad \left. - \frac{2\Delta c(\vec{r}_1)}{c_0^3} \frac{\partial^2 [\rho_0 \frac{\partial v(t)}{\partial t} *_{\vec{t}} h(\vec{r}_1, \vec{r}_3, t)]}{\partial t^2} \right] *_{\vec{t}} h(\vec{r}_5, \vec{r}_1, t) d^3 \vec{r}_1.
 \end{aligned} \tag{2.28}$$

If the position of the transmitting and the receiving transducer are the same, i.e., $\vec{r}_3 = \vec{r}_5$, we have:

$$\begin{aligned}
 p_r(\vec{r}_5, t) &= \frac{\rho_0}{2} E_m(t) *_{\vec{t}} \frac{\partial v(t)}{\partial t} *_{\vec{t}} \int_{V'} \left[\frac{1}{\rho_0} \nabla(\Delta \rho(\vec{r}_1) \cdot \nabla H_{pe}(\vec{r}_1, \vec{r}_5, t)) \right. \\
 &\quad \left. - \frac{2\Delta c(\vec{r}_1)}{c_0^3} \frac{\partial^2 H_{pe}(\vec{r}_1, \vec{r}_5, t)}{\partial t^2} \right] d^3 \vec{r}_1.
 \end{aligned} \tag{2.29}$$

Here, the pulse-echo spatial impulse response is:

$$H_{pe}(\vec{r}_1, \vec{r}_5, t) = h(\vec{r}_1, \vec{r}_5, t) *_{\vec{t}} h(\vec{r}_5, \vec{r}_1, t). \tag{2.30}$$

Assume that $H_{pe}(\vec{r}_1, \vec{r}_5, t)$ is slowly varying so that the spatial frequency content is constant over a finite volume, we have:

$$\nabla(\Delta \rho(\vec{r}_1) \cdot \nabla H_{pe}(\vec{r}_1, \vec{r}_5, t)) = \Delta \rho(\vec{r}_1) \nabla^2 H_{pe}(\vec{r}_1, \vec{r}_5, t). \tag{2.31}$$

Since the time taken for the propagation of the pressure field between the transducer and the scatterer is given by:

$$t = \frac{|\vec{r}_1 - \vec{r}_5|}{c_0}, \tag{2.32}$$

and assuming only small deviations from the mean propagation velocity, the Laplace operator, through the use of chain rule, corresponds to the second derivative with

respect to time. Thus,

$$\nabla^2 H_{pe}(\vec{r}_1, \vec{r}_s, t) = \frac{1}{c_0^2} \frac{\partial^2 H_{pe}(\vec{r}_1, \vec{r}_s, t)}{\partial t^2}. \quad (2.33)$$

Using the approximations in (2.31) and (2.33), (2.29) can be rewritten as:

$$p_r(\vec{r}_s, t) = \frac{\rho_0}{2} E_m(t) \underset{i}{*} \frac{\partial v(t)}{\partial t} \underset{i}{*} \int_{V'} \left[\frac{\Delta \rho(\vec{r}_1)}{\rho_0} - \frac{2\Delta c(\vec{r}_1)}{c_0} \right] \frac{1}{c_0^2} \frac{\partial^2 H_{pe}(\vec{r}_1, \vec{r}_s, t)}{\partial t^2} d^3 \vec{r}_1. \quad (2.34)$$

Symbolically, this is written as:

$$p_r(\vec{r}_s, t) = v_{pe}(t) \underset{i}{*} s(\vec{r}_1) \underset{\tau}{*} h_{pe}(\vec{r}_1, \vec{r}_s, t). \quad (2.35)$$

Here, $\underset{\tau}{*}$ denotes spatial convolution, $v_{pe}(t)$ is the pulse-echo wavelet that accounts for the transducer excitation and the electromechanical impulse response during emission and reception of the pulse, and $h_{pe}(\vec{r}_1, \vec{r}_s, t)$ is the modified pulse-echo spatial impulse response that relates the transducer geometry to the spatial extent of the scattered field. The term $s(\vec{r}_1) = \frac{\Delta \rho(\vec{r}_1)}{\rho_0} - \frac{2\Delta c(\vec{r}_1)}{c_0}$ accounts for the inhomogeneities in the tissue due to density and propagation velocity perturbations that give rise to the scattered signal.

2.3.5 Proposed Ultrasound Tissue Signal Model

In medical ultrasound imaging, a 3-D pulsed pressure field is emitted into the body and is scattered and reflected by small density and propagation velocity perturbations. The backscattered field is then received by the transducer for image formation. Under the assumptions of linear propagation and weak scattering, an expression for the received pressure field is derived using the first-order *Born approximation* and presented as a convolution model in (2.35). In the model, the term $s(\vec{r}_1)$ accounts for the inhomogeneities in the tissue due to density and propagation velocity per-

turbations that give rise to the scattered signal. Therefore, in the study of tissue characteristics, it is often interpreted as a collection of point scatterers embedded in a uniform nonscattering medium. The biological variability associated with tissues, the spatial distribution, and the scattering strength associated with the scatterers, are often described in statistical terms. Shankar *et al.* [33], [37], [58] studied the statistics of the ultrasound echo by interpreting the ultrasonic backscattering from tissue as point scatterers. The strength of scatterers is modelled as Gamma distribution [33], [35], [37], [58], [59] and the locations of them are random and followed a uniform distribution [33], [35], [37], [58]. Thus, the structures within tissue that are responsible for the backscattered ultrasonic field may be expressed as follows:

$$s(\vec{r}_1) = \sum_{k=1}^K \alpha_k \delta(\vec{r}_1 - \vec{r}_k), \quad (2.36)$$

where α_k is the amplitude of the k th scatterer of Gamma distribution which accounts for the variation in the scattering amplitudes within the tissue, $\delta(\cdot)$ is the Kronecker delta and K is the number of scatterers in the range cell.

Insana *et al.* [51] represent the scattered field as a sum of average field and fluctuating field. The average field is called the coherent field and the fluctuating field is called the incoherent field which has the ensemble average of zero. The scattering region may also contain subresolvable periodic alignment of scatterers in addition to a collection of randomly located scatterers. In [59], Abeyratne *et al.* attempt to characterize the tissue based on a three-component point scatter model: a) the diffused component, b) resolvable periodic component, and c) a mixed non-Gaussian component. Georgia *et al.* in [34], [35], however, decompose the RF echo into a coherent component and a diffused component. The spacing of the regular structure are considered to be a Gaussian-distributed [37] or Gamma-distributed [59] random variable.

Figure 2.7 shows three separate scattering conditions. For a small number of scatterers will lead to K-distributed envelope while a large number of randomly located scatterers in the resolution cell will lead to fully developed speckle. Contribution from both the periodic resolvable and unresolvable diffuse scatterers gives Rician or generalized Rician distributed envelope depends on the spacing between the regular structures.

Figure 2.7: The three separate scattering conditions. (Left) Small number of scatterers, (Middle) completely random alignment of a large number of scatterers, and (right) randomly located scatterers along with regularly spaced scatterers.

We can deduce that the scatterers forming an alignment have a similar spatial distribution. Hence, we introduce a new tissue response model to describe the tissue response:

$$s(\vec{r}_1) = f(\vec{r}_1) + d(\vec{r}_1), \quad (2.37)$$

where $f(\vec{r}_1)$ represents the mean scattering amplitude, the subresolvable or the resolvable periodic alignment of scatterers and $d(\vec{r}_1)$ modelled the randomness in the variation of $s(\vec{r}_1)$. More specifically, $f(\vec{r}_1)$ is composed of the coherent scatterers and the mean strength of the diffuse scatterers whereas the fluctuation in the diffuse scattering is described by $d(\vec{r}_1)$. Coherent scatterers exist in organs such as liver tissue (the lobular structure and the portal triads) and the statistics of their inter-scatterer spacing distribution are commonly used as tissue signatures [34], [59]. As mentioned earlier on, there also exist fluctuation in the diffuse scattering which is a

consequence of fluctuations in the scattering cross section in the sample volume and they depend on the physical and biological changes in the tissue [33]. These fluctuations in scattering cross section will also be random due to a number of different factors such as changes in blood flow, and local attenuating attenuation velocity from the presence of growth in the tissue. In other words, the mean strength of the backscattered echo will also be a random variable.

Thus, the response $s(\vec{r}_1)$ is considered as noisy tissue with the noiseless component $f(\vec{r}_1)$. The noiseless tissue can be interpreted as a random variable, whereas $d(\vec{r}_1)$, as an additive noise, has Gamma distribution in the amplitude and its location is uniformly distributed across the sampled domain [33], [35], [37], [59].

In the ultrasound image formation, the ultrasound images are blurred in both the axial and the lateral directions due to the finite resolution of the imaging system, and the received pulse-echo pressure field modelled in (2.35) is presented in the following form [17]:

$$p_r(\vec{r}_5, t) = v_{pe}(t) *_{\vec{r}_1} s(\vec{r}_1) *_{\vec{r}_5} h_{pe}(\vec{r}_1, \vec{r}_5, t) + w(\vec{r}_5, t). \quad (2.38)$$

Clearly, $f(\vec{r}_1)$ is the desired signal to be displayed. However, what we can measure is its smoothed version which has been corrupted by additive noise obscuring the finer details in the image. The smoothing process consists of a convolution in time with a fixed wavelet $v_{pe}(t)$ and a spatial convolution with a spatially varying $h_{pe}(\vec{r}_1, \vec{r}_5, t)$. They are known as the axial PSF and lateral PSF or, collectively, called the 2-D PSF in the thesis. The noise term $w(\vec{r}_5, t)$ is added to describe the inevitable noise in the measured signal, accounting for electrical noise from amplifiers and for physical effects not explained by the convolution model.

2.4 Proposed Model of the Image Degradation and Restoration Process

Ultrasound image degradation can affect human visual interpretation and will have significant impact on the diagnostic quality. The compensation of these imaging distortions is a crucial issue to address in order to improve the quality of ultrasonic diagnostic examination. Knowing the fact that the ultrasound image formation process actually occurs in the RF domain, signal processing techniques applied to the RF data to obtain high quality image becomes very attractive.

In Section 2.3, we presented a physical modelling of the ultrasound image formation process in the RF domain and showed that the received pulse-echo pressure field in medical ultrasound B-mode images is described by a convolution between scattered field and the emitted pressure field. We further elaborated that the scattered field is a combination of the mean scattering field and a random fluctuating field. The incorporation of the random fluctuations of the tissue response into the convolution model offers a more realistic interpretation of the received pulse-echo ultrasound signal. For simplicity, the vector representation in (2.37) and (2.38) are rewritten in the following form:

$$y(\mathbf{r}, t) = v(\mathbf{r}, t) * s(\mathbf{r}, t) + w(\mathbf{r}, t), \quad (2.39)$$

$$s(\mathbf{r}, t) = f(\mathbf{r}, t) + d(\mathbf{r}, t), \quad (2.40)$$

where $y(\mathbf{r}, t)$ represents the ultrasonic received pulse-echo pressure field, $v(\mathbf{r}, t)$ is the 2-D PSF, $s(\mathbf{r}, t)$ is the noisy tissue image, both $w(\mathbf{r}, t)$ and $d(\mathbf{r}, t)$ are additive noise, and $*$ denotes the 2-D linear convolution operator with respect to the variables $\mathbf{r}$ and t representing the horizontal and vertical continuous variables. Equation (2.40) describes the representation of the tissue signal with two components, where $f(\mathbf{r}, t)$

represents the mean scattering amplitude, the subresolvable or the resolvable periodic alignment of scatterers and $d(\mathbf{r}, t)$ is the random fluctuation of $s(\mathbf{r}, t)$.

Figure 2.8: Model for ultrasound image degradation and restoration process.

Figure 2.8 shows the model we shall use in the development of the discrete spatial restoration techniques. The *top* row indicates the desired but impossible situation where the anatomical tissue structures $f(\mathbf{r}, t)$ is discretized directly by an ideal digitizer. This produces a discrete version of the undegraded continuous signal in the sampled domain where the variables m and n representing the horizontal and vertical discrete sample coordinates respectively.

The *second* row of the figure models what actually happens when an ultrasound RF data is digitalized and recorded. The biological tissue image $f(\mathbf{r}, t)$ is contaminated by the randomness of the fluctuating field and then blurred by the linear ultrasonic system response $v(\mathbf{r}, t)$. Blurring occurs in both the axial and lateral directions due to the finite resolution of the imaging system. Noise inevitably presents in any imaging system, and is customarily modelled as a white Gaussian noise $w(\mathbf{r}, t)$ at the output when the dominant source is the random thermal motion of electrons. The noise term could also explain the unwanted signal in the measurement that

cannot be explained by the convolution model.

An ideal digitizer forms the 2-D discrete RF data which is then subjected to a restoration operation that produces estimate $\check{f}(m, n)$, which is an approximation to the desired result $f(m, n)$. As we have explained earlier, $f(m, n)$ is the tissue image to be identified and displayed. Unfortunately, we can only measure its smoothed version, which is also corrupted by the additive noise, obscuring the finer details in the image. The smoothing process is the resultant of a convolution in time with a pulsed-echo wavelet and a spatial convolution with a spatially varying impulse response of the transducer aperture. Therefore, while the blur is linear, it may not be spatially invariant.

The *third* row of the figure shows the model for the restoration process. In this case, all the continuous signals and operations are converted into the digital form. After discretization, the convolution model (2.39) has the following expression in the sampled domain [14]–[30]:

$$y(m, n) = v(m, n) * s(m, n) + w(m, n). \quad (2.41)$$

Likewise, the proposed new tissue model in (2.40) becomes:

$$s(m, n) = f(m, n) + d(m, n), \quad (2.42)$$

where $y(m, n)$ is the ultrasound RF echo, $v(m, n)$ is the 2-D PSF, $s(m, n)$ is the noisy tissue image, both $w(m, n)$ and $d(m, n)$ are additive white noise. Now, we can create very impressive simulations with this model. We are able to design the degradation process and implement it with precision. As before, the restoration algorithm produces the estimate $\check{f}(m, n)$.

2.5 The Blind Deconvolution Problem

Subsection 2.3.4 established the convolution expression for the ultrasound image formation process. In Subsection 2.3.5, we introduced a new tissue model into the convolution expression, and then we developed the concept of the ultrasound image degradation and restoration model and its discrete approximation in Section 2.4. In this section, we explain our approach chosen to restore the tissue image.

In ultrasound image acquisition, the captured image is the result of the biological tissue structures being convolved with the 2-D PSF. Given the convolution expression in (2.41), if one of the functions $v(m, n)$ or $s(m, n)$ is known, restoration methods such as Wiener filtering and iterative restoration can recover the other function. The problem of separation of the two signals becomes difficult if neither of the functions is known. The problem is now termed *blind image deconvolution*.

The general blind image deconvolution problem refers to the task of separating the two convolved signals, $v(m, n)$ and $s(m, n)$, when both the signals are either unknown or partially known. This problem has been the subject of a vast literature spanning the numerous special fields of science [60]–[73]. The existing literature differs fundamentally on the notion of a solution, which in turn leads to different mathematical approaches explored to solve the problem. This recovery process is critical to many image processing applications such as astronomical imaging [64]–[66], remote sensing [60], and medical ultrasound [13]–[30]. In addition to image restoration, there are many other important applications such as seismic data analysis, blind channels equalization in communication [73], transmission monitoring, and echo cancellation in wireless telephony.

Deconvolution is necessary in order to undo the effects of the blur. Based on our proposed ultrasound image degradation model, our deblurring challenge is therefore a blind image deconvolution problem, where we seek to deconvolve our observation

$y(m, n)$ to obtain a close approximation to the unknown tissue structures $f(m, n)$ without precise knowledge of the degradation $v(m, n)$. With or without the noise, (2.41) admits an infinite number of solutions. Fortunately, we have prior knowledge on the 2-D PSF. We know that its axial PSF is an oscillating wavelet about the transducer center frequency and its lateral PSF is approximately Gaussian function. Given the difficulty of the deconvolution problem, it may seem outrageous to suggest simultaneous extraction of both the imaged tissue structures and the 2-D PSF. In the following chapters of the thesis, we have chosen a two-step approach for the ultrasound image blind deconvolution problem. We first estimate the 2-D PSF using complex analysis and then use it in the recovering of the tissue structures. In doing so, we have transformed the blind image deconvolution problem to an image deconvolution problem. This approach is illustrated in Figure 2.9.

Figure 2.9: Proposed two-step ultrasound image blind deconvolution approach.

Chapter 3

Estimating the Degradation Function

Blurring in ultrasound images arise from practical limitations of the image acquisition systems and the intrinsic properties of the imaged tissues. The aperture and geometry of the ultrasonic transducer probe will contribute to the lateral spread in the ultrasonic beam while the nonnegligible transmission pulse width of the ultrasonic pulse-wavelet propagating into the nonhomogeneous tissues will result in the axial PSF deviations from the ideal impulse response. Moreover, as the ultrasonic pulse-wavelet propagates in the intervening biological tissues, the intrinsic properties of the tissue such as frequency-dependent attenuation, dispersive, and phase aberrations will cause the pulse-wavelet amplitude and shape to change considerably along the depth of interrogation. These collective effects lead to spatially varying 2-D PSF in our studies.

This chapter concerns the estimation of the degradation function in ultrasound imaging. We have shown in Chapter 2 that the ultrasound images can be modelled as a 2-D convolution of the tissue structures and the spatially varying 2-D PSF. As we have mentioned earlier, this 2-D PSF is governed by the physical characteristics

of the imaging ultrasonic transducer probe such as its geometry, its beam width and also the biological tissue properties. It is thus the source of degradation influencing the ultrasound image quality. If it can be accurately estimated, the tissue structures would have restored via a reverse convolution process known as *deconvolution*. This perspective has been confirmed in numerous studies and it attracts continuous efforts of estimating the PSF directly from the acquired RF data. In this chapter, we will present the most successful techniques used in ultrasound research community in estimating the PSF.

In Section 3.1, we present the concept of the PSF estimation. It is based on the existence of the distinctive smoothness properties between the PSF and the tissue scatterings that makes the estimation possible. Section 3.2 covers the estimation by utilizing the higher-order statistics (HOS). This technique has received quite a lot of attention because of its ability to identify nonminimum-phase signal and the elimination of the Gaussian processes. Unfortunately, it is a one-dimensional (1-D) PSF estimation approach. In order to obtain best possible results, we resort to the complex cepstrum analysis which is capable of accomplishing 2-D PSF estimation while not being restricted to minimum-phase signals. Consequently, in Section 3.3, we exploit the potential of the cepstrum analysis in estimating the 2-D PSF. We will also explain in this section how the 2-D phase wrapping operation is incorporated into the computation of the complex cepstrum. Because 2-D phase wrapping operation is a crucial operation, we have adopted a robust algorithm known as the 2-D weighted LS phase wrapping. This algorithm and its robustness will be illustrated through an example in Section 3.4.

3.1 Fundamentals of the Approach

In medical ultrasound imaging, a 3-D acoustic pulsed pressure field is emitted into the human body and is scattered and reflected by small density and propagation velocity perturbations. The backscattered field is then a 3-D function that describes the strength of the acoustic tissues and scatterers. In Chapter 2, we introduced the 2-D convolution model to describe the formation of the ultrasound images in the RF domain as a function of spatial coordinates. In this model, we assume that the tissue properties are approximately uniform in the direction perpendicular to the scanning plane such that it is reasonable to reduce from 3-D physical phenomena to a 2-D model:

$$y(m, n) = v(m, n) * s(m, n) + w(m, n). \quad (3.1)$$

In reality, (3.1) may not be true throughout the entire RF images (or data) since it is derived under the assumption of weak scattering. The frequency-dependent attenuation and dispersive nature of biological tissues cause the interrogating ultrasonic pulse-wavelet continuously varying its amplitude and shape. Fortunately, the RF images which are divided into segments, if short enough, are quasi-stationary. Under this condition, each segment can be modelled by (3.1). Without loss of generality, this thesis focuses on an arbitrary RF segment where (3.1) can be used in modelling the ultrasound images. The same condition holds when the deconvolution is done based on a sliding window across the entire RF images.

For the deconvolution to be feasible, either of the function $v(m, n)$ or $s(m, n)$ has to be known so that they can be separated. In the context of ultrasound imaging, $s(m, n)$ is the noisy tissue structure. It contains the information of the imaged objects and it is our interests to restore a noiseless version of it for diagnostic examination. Ideally, we should not presume how it looks like. On the other hand, $v(m, n)$ is governed by the characteristics of the imaging system and its characteris-

tics generally follows that of the specific ultrasonic transducer probe used. It is the existence of these distinct characteristics among the two convolution components that make deconvolution feasible. This is usually seen with an alternative expression in the frequency domain by taking Fourier transform (FT) on both sides of the convolution model in (3.1). Now, the expression becomes:

$$Y(\omega_m, \omega_n) = V(\omega_m, \omega_n) S(\omega_m, \omega_n) + W(\omega_m, \omega_n), \quad (3.2)$$

where $Y(\omega_m, \omega_n)$, $V(\omega_m, \omega_n)$, $S(\omega_m, \omega_n)$, and $W(\omega_m, \omega_n)$ are the 2-D FT of $y(m, n)$, $v(m, n)$, $s(m, n)$, and $w(m, n)$ respectively. In this case, we can use $V(\omega_m, \omega_n)$ as a division operator on $Y(\omega_m, \omega_n)$ to serve as a preliminary step in the restoration of $S(\omega_m, \omega_n)$. In the following, we show that there exists different smoothness features in $V(\omega_m, \omega_n)$ and $S(\omega_m, \omega_n)$ that make them discriminable. For clarity, the noise term $W(\omega_m, \omega_n)$ will be ignored for the time being and the concept of the PSF is elaborated as a 1-D problem:

$$Y(\omega_n) = V(\omega_n) S(\omega_n). \quad (3.3)$$

In (3.3), $Y(\omega_n)$, $V(\omega_n)$, and $S(\omega_n)$ are the RF line, axial PSF, and tissue structures all in the axial direction respectively. Although we are illustrating the distinctly different smoothness features of $V(\omega_m, \omega_n)$ and $S(\omega_m, \omega_n)$ in only the axial direction, the results hold as well as that in the lateral direction. Tissue structures behave in the same manner in all directions while the lateral PSF is generally a lowpass response which is much smoother than that of the axial PSF [12], [18]–[25], [74].

The PSF $V(\omega_n)$ is an ultrasonic pulse-wavelet as depicted in Figure 3.1. The pulse-wavelet echo sequence is measured from a thick planar reflector in a water tank using an unfocused Panametrics V382-SU, 3.5-MHz transducer which has a diameter of 0.5 in, and the sequence is sampled at 50 MHz. The right plot is the estimate of

its power spectrum density (PSD). As shown, the spectrum of the ultrasonic PSF is mostly a smooth function, with a maximum in the vicinity of the nominal frequency of the transducer [18]–[25], [74], [75].

Figure 3.1: Characteristics of the pulse-wavelet echo sequence measured underwater. (a) Pulse-wavelet echo sequence measured from a thick planar reflector in a water-tank, and (b) periodogram PSD estimate of the pulse-wavelet echo.

The features of biological tissue structures have totally different characteristics. As we have mentioned in Chapter 2, there exist inhomogeneities in the tissue due to density and ultrasound propagation velocity perturbations that give rise to the scattered signal. It is often interpreted as a collection of point scatterers embedded in a uniform nonscattering medium. The biological variability associated with tissues, the spatial distribution, and the scattering strength associated with these numerous, independent small scatterers, are often described in statistical terms. That is, tissues are normally modelled as statistical random process. Therefore, we expect that the tissues to experience broadband, very irregular and spiky spectrum. Since the true tissues are not known, its spectrum cannot be visualized directly. Fortunately, we can observe its characteristics via the spectrum of the RF line. From (3.3), the spectrum of the RF line $Y(\omega_n)$ is the multiplication of the two spectra, i.e., $V(\omega_n)$ and $S(\omega_n)$. PSD estimate of $V(\omega_n)$ shows that the axial PSF is a smooth function,

we can attribute the deviation in the PSD estimate of $Y(\omega_n)$ away from smoothness being contributed by $S(\omega_n)$. If the spectrum of $S(\omega_n)$ is also a smooth function, we will also obtain a smooth function from that of $Y(\omega_n)$. However, if $S(\omega_n)$ does not have a smooth spectrum, the spectrum of $Y(\omega_n)$ will not be smooth and we may deduce from here that the various ruptures and dips are caused by $S(\omega_n)$. Figure 3.2 shows RF line measured from a commercially available tissue-mimicking phantom, ATS model 532 contrast resolution phantom. Comparing the two spectra in Figure 3.1 and 3.2 reveal that both spectra generally follow the same shape except that the one in Figure 3.2 is spikier. It appears as if there is a realization of a random process multiplied with the smooth function of Figure 3.1. Thus, we can conclude that the tissues have a distinctive feature different from the ultrasonic PSF. It is this feature that allows estimation of the PSF to be feasible.

Figure 3.2: Characteristics of the RF line measured from tissue-mimicking phantom. (a) A measured RF line measured from tissue-mimicking phantom, and (b) periodogram PSD estimate of it.

3.2 Estimation by Higher-order Statistics

This section introduces a powerful, noise robust and mixed phase technique used to estimate the ultrasonic PSF based on HOS. The reconstruction of the ultrasonic PSF is achieved by estimating its complex cepstrum through the higher-order spectra of the RF lines. Since estimations are based on HOS of the RF data, the estimations are immune to additive Gaussian observation noise and also produce the correct phase. This method takes advantage of the fact that Gaussian process, which has zero cumulants of order greater than two, will automatically be eliminated in the HOS. Hence, it is expected that this method is applicable only for non-Gaussian tissues with white Gaussian observation noise [76]. This technique is originally proposed by Abeyratne *et al.* [21], [23]. With this technique, the ultrasonic PSF in both the axial and lateral directions are obtained separately using higher-order spectral methods. Therefore, it is not a true 2-D approach.

For brevity, we slightly abuse the use of notation. The notation $v(n)$ is used to represent the discrete-time impulse response of either the axial or the lateral PSF without loss of generality. It is a deterministic, causal, finite-impulse response (FIR) filter of length M . As before, $s(n)$ is the tissue response function, which is assumed to be stationary, white, zero-mean and non-Gaussian, and $w(n)$ is a zero-mean Gaussian noise that is independent of $s(n)$. The third-order cumulant of the zero-mean observation $y(n)$ is defined as [77], [78]:

$$\begin{aligned} c_y(\rho, \tau) &= E\{y(n)y(n+\rho)y(n+\tau)\} \\ &= \gamma_s \frac{1}{M} \sum_{k=0}^{M-1} v(k)v(k+\rho)v(k+\tau) + c_w(\rho, \tau), \end{aligned} \quad (3.4)$$

where $\gamma_s = E\{s^3(n)\}$ is a constant equal to the third-order cumulant of the zero-mean signal $s(n)$, which is assumed to be nonzero. The expression $E\{\cdot\}$ represents statistical average. The bispectrum of $y(n)$ is defined to be the Z-transform (ZT) of

the third-order cumulant:

$$\begin{aligned} C_y(z_m, z_n) &= \sum_p \sum_q c_y(p, q) z_m^{-p} z_n^{-q} \\ &= \gamma_s V(z_m) V(z_n) V(z_m^{-1}, z_n^{-1}) + C_w(z_m, z_n), \end{aligned} \quad (3.5)$$

where $V(z_m)$, $V(z_n)$, $V(z_m, z_n)$ are the spectrum of $v(m)$, $v(n)$ and $v(m, n)$ respectively, and $C_w(z_m, z_n)$ is the noise spectrum. If the additive noise $w(n)$ is zero-mean Gaussian, then $C_w(z_m, z_n) = 0$, and (3.5) becomes [77], [78]:

$$C_y(z_m, z_n) = \gamma_s V(z_m) V(z_n) V(z_m^{-1}, z_n^{-1}). \quad (3.6)$$

The bicepstrum is defined as the cepstrum of the bispectrum, which is defined as the inverse Z-transform (IZT) of the logarithm of the bispectrum:

$$b_y(m, n) = Z^{-1} \left[\log_e (C_y(z_m, z_n)) \right]. \quad (3.7)$$

The following expression can be used for the computation of the bicepstrum in order to avoid the required bispectrum estimation procedures as well as phase unwrapping algorithm associated with (3.5) and (3.7):

$$b_y(m, n) = \frac{1}{m} F^{-1} \left[\frac{F[m c_y(m, n)]}{F[c_y(m, n)]} \right], \quad (3.8)$$

where F and F^{-1} represent FT and its inverse, respectively. To relate the bicepstrum to the PSF, substitute (3.6) into the logarithm of the bispectrum, we have:

$$\begin{aligned} \log_e (C_y(z_m, z_n)) &= \log_e (\gamma_s) + \log_e (V(z_m)) \\ &\quad + \log_e (V(z_n)) + \log_e (V(z_m^{-1}, z_n^{-1})). \end{aligned} \quad (3.9)$$

Denoting the cepstrum of $v(n)$ as $c_v(n)$ and combining (3.7) and (3.9), we have:

$$\begin{aligned}
 Z[b_y(m, n)] &= \sum_p \sum_q b_y(p, q) z_m^{-p} z_n^{-q} \\
 &= \log_e(\gamma_s) + \sum_p c_v(p) z_m^{-p} + \sum_q c_v(q) z_n^{-q} + \sum_{p, q=p} c_v(p) z_m^p z_n^q \\
 &= \sum_p \sum_q \left[\log_e(\gamma_s) \delta(p) \delta(q) + c_v(p) \delta(p) \right. \\
 &\quad \left. + c_v(q) \delta(q) + c_v(-p) \delta(p - q) \right] z_m^{-p} z_n^{-q}.
 \end{aligned} \tag{3.10}$$

Thus, $c_v(q)$ can be obtained for all $q \neq 0$ by evaluating the bicepstrum along the diagonal $p = q$:

$$c_v(n) = Z^{-1} \left[\log_e(V(z_n)) \right] = b_y(-n, -n) \quad , n \neq 0 \tag{3.11}$$

As shown in (3.9), if $s(n)$ is stationary independent identically distributed, γ_s is flat. One is expected to see that its bicepstrum will be an impulse located at the origin and this causes the value of $c_v(0)$ to be undetermined as γ_s is not known. Luckily, its exact value is not needed because different $c_v(0)$ only results in a different overall scaling for the pulse $v(n)$. Therefore, using bicepstral values along the diagonal except at the origin we can construct a scaled and shifted version of $v(n)$:

$$v(n) = Z^{-1} \left[e^{Z[c_v(n)]} \right], \tag{3.12}$$

where

$$c_v(n) = \begin{cases} b_y(-n, -n) & , n \neq 0 \\ 0 & , n = 0 \end{cases} \tag{3.13}$$

The demonstration of the HOS method in estimating the PSF is carried out through a B-scan image of an ATS model 532 contrast resolution phantom acquired in a water tank. The transducer used is an unfocused immersion transducer from Panametrics model V382-SU with a diameter of 0.5 in and a nominal center frequency of

3.5 MHz. A stepper motor with a precision of 0.01 mm per step controlled the position of the transducer under the guidance of a personal computer. The transducer is moved across the scanning plane in steps of 1.00 mm, and RF echoes are sampled at 12.0482 MHz. Data acquisition is done through a 12-bit ADC Acquiris DC440 interfaced to the Panametrics model 5077PR Ultrasonic Pulser/Receiver. The procedure to estimate the third-order cumulants from the RF data in both the axial and lateral directions is as follows:

- (S1) The RF data $y(n)$ is segmented into K records, each record $y_j(n)$ has a length of N samples, and the mean value of each record is removed.
- (S2) The third-order cumulants $\hat{c}_{y_j}(\rho, \tau)$ of each record $y_j(n)$, $j = 1, 2, \dots, K$ are estimated according to:

$$\hat{c}_{y_j}(\rho, \tau) = \frac{1}{N} \sum_{k=0}^{N-1} y_j(k) y_j(k + \tau) y_j(k + \rho) \quad , |\rho| \leq M, |\tau| \leq M \quad (3.14)$$

where M is also the length of correlation lags considered in the computation.

- (S3) The cumulant estimates $\hat{c}_y(\rho, \tau)$ are computed as:

$$\hat{c}_y(\rho, \tau) = \frac{1}{K} \sum_{j=0}^{K-1} \hat{c}_{y_j}(\rho, \tau) \quad , -M \leq \rho, \tau \leq M \quad (3.15)$$

- (S4) The bispectrum $\hat{b}_y(m, n)$ is estimated via (3.8) where $c_y(m, n)$ is replaced with $\hat{c}_y(m, n)$ given in (3.15). Likewise, the estimated PSF $v(n)$ is obtained via (3.12) by replacing $b_y(m, n)$ with $\hat{b}_y(m, n)$ in (3.13).

Figure 3.3 depicts the estimates obtained for different depths in the phantom. In the estimation, parameters chosen are $M = 32$, $N = 78$ and, K is 17 and 10 for the axial and lateral PSF estimates respectively. It is seen from the figure that the dispersive attenuation of the phantom only causes slight smearing of the pulses.

This observation is inline to the characteristics of the phantom designated to be homogeneous. Moreover, the waveforms of the estimates deviate from that shown in Figure 3.1 which is also measured underwater. In reality, the true PSF in the clinical images are not known and cannot be measured. The reason is that the actual tissues introduce effects like phase aberrations, nonlinearities and dispersive attenuation. Thus, measurements done underwater are not valid in clinical images and we expect them to be different from the estimates.

(a)

(b)

(c)

(d)

Figure 3.3: Estimation of the axial and lateral PSF at various depths using HOS. Ultrasonic axial PSF estimates (Right) and lateral PSF estimates (Left) computed for different depths in the tissue-mimicking phantom. (a), (b) PSF estimates at reference depth, (c), (d) PSF estimates at 2 cm, and (e), (f) PSF estimates at 4 cm below reference depth respectively.

Further examination of the quality of the PSF estimation is carried out through the use of simulated ultrasound RF data which are generated according to the procedure described in Appendix B. To give an indication of the variations in the estimated PSF, 100 different simulated ultrasound RF data are used and the result is shown in Figure 3.4. As seen from the figure, the estimated axial PSF is quite close approximation to the true axial PSF whereas the estimated lateral PSF seen to deviate away from the true lateral PSF when it comes near to the center of the profile. The quality of the PSF estimation quantified based on the mean square error (MSE) between the estimation and that simulated, and the MSE for both the axial and lateral estimations are given by -18.7453 ± 2.07474 dB and -21.4736 ± 0.395107 dB respectively.

Figure 3.4: HOS-based axial PSF and lateral PSF estimates.

HOS-based ultrasonic PSF estimate (:) compared with the simulated PSF (-). Axial PSF (Top) and lateral PSF (Bottom) estimates computed from 100 different simulated ultrasound RF data.

3.3 Estimation by Cepstrum Analysis

The term *complex cepstrum* is used by Oppenheim *et al.* [79], [80] and Childers *et al.* [81] to describe the IZT of the complex logarithm of the ZT of a signal. The use of the word *complex* implies that the complex logarithm is used in the definition. This terminology is somewhat confusing because the complex cepstrum of a real signal is actually real. Consequently, it is also widely accepted to use the term *cepstrum* interchangeably with it for the same expression:

$$\hat{y}(n) = Z^{-1} \left[\log_e (Z[y(n)]) \right], \quad (3.16)$$

where $\hat{y}(n)$ is the cepstrum and $y(n)$ is the ultrasonic RF signal. As discussed in Oppenheim and Schaffer [80], the following function must be analytic for the cepstrum to exist:

$$\begin{aligned} \hat{Y}(z_n) &= \log_e (Z[y(n)]) \\ &= \log_e \left(\sum_n y(n) z_n^{-1} \right). \end{aligned} \quad (3.17)$$

For a real and stable signal $y(n)$, its ZT, $Y(z_n)$, is analytic in the region of convergence that includes the unit circle so that its FT, $Y(\omega_n)$, exists and whose cepstra are also stable. This implies that:

$$0 < |Y(\omega_n)| < \infty, \quad -\pi < \omega_n \leq \pi \quad (3.18)$$

The requirement that $\hat{Y}(z_n)$ be analytic means that the phase of $y(n)$, that is the imaginary part of $\hat{Y}(z_n)$, must be continuous and periodic. Both are ensured by the proper definition of the complex logarithm and the adjustment of the signal origin by removing any linear phase components. If we have the convolution of two sequences as in (3.3):

$$y(n) = v(n) * s(n). \quad (3.19)$$

Then, according to the definition of (3.16) and (3.17), we will have the following correspondingly:

$$\hat{Y}(z_n) = \hat{V}(z_n) + \hat{S}(z_n), \quad (3.20)$$

$$\hat{y}(n) = \hat{v}(n) + \hat{s}(n). \quad (3.21)$$

Further, if $\hat{v}(n)$ and $\hat{s}(n)$ occupy different cepstrums, then the complex cepstrum can be liftered (filtered) to remove one or the other of the convolved sequences through the use of lifter (frequency-invariant linear filter). Since the phase information is retained, the complex cepstrum is invertible. In the context of PSF estimation, the cepstra of $\hat{v}(n)$ is to be retained. Thus, we may then apply a reverse operation of (3.16) to recover the sequence $v(n)$. This process consists of ZT, exponentiate, and followed by an IZT. Further elaboration of this process is covered in Subsection 3.3.1. In the following, we present the analyticity of the complex cepstrum.

In ultrasound imaging, the observed ultrasonic RF line is the combinations of 1-D damped sinusoids, and may be modelled as a 1-D complex exponential sequence [20], [82], [83]. The ZT of a stable exponential sequence is a rational function of z_n :

$$Y(z_n) = \frac{A z_n^r \prod_{k=1}^{M_i} (1 - a_k z_n^{-1}) \prod_{k=1}^{M_o} (1 - b_k z_n)}{\prod_{k=1}^{N_i} (1 - c_k z_n^{-1}) \prod_{k=1}^{N_o} (1 - d_k z_n)}, \quad (3.22)$$

where $|a_k|$, $|b_k|$, $|c_k|$, and $|d_k|$ are all less than unity, so that $z_n = a_k$ and $z_n = c_k$ correspond to the zero and pole inside the unit circle whereas $z_n = 1/b_k$ and $z_n = 1/d_k$ correspond to the zero and pole outside the unit circle. Our interest is the case where the sequence $y(n)$ has a finite length of $M_o + M_i + 1$, there are no poles in (3.22) and its denominator is unity. More generally, computing using (3.17) will

transform the product of terms in (3.22) to the sum of logarithmic terms:

$$\begin{aligned}\hat{Y}(z_n) = & \log_e(A) + r \log_e(z_n) + \sum_{k=1}^{M_i} (1 - a_k z_n^{-1}) \\ & + \sum_{k=1}^{M_o} (1 - b_k z_n) - \sum_{k=1}^{N_i} (1 - c_k z_n^{-1}) - \sum_{k=1}^{N_o} (1 - d_k z_n) .\end{aligned}\quad (3.23)$$

At this stage, it is worth noting that $\hat{y}(n)$ is not computed directly by substituting (3.23) into (3.16). In fact, several treatments are adopted before (3.16) is readily implemented digitally with the inverse discrete FT (IDFT) [80]. They are elaborated as follows.

For real sequences, A is real. But if A is negative, it is more difficult to determine its contribution to the complex cepstrum. Consequently, in practice, its polarity is made positive by multiply the sequence with -1 if $Y(\omega_n)|_{\omega_n=0} < 0$. Now, the term $\log_e(A)$ in (3.23) simply contributes only to $\hat{y}(0)$. In addition to this manipulation, the terms z_n^r will correspond to either a delay or advance of the sequence $y(n)$. This induces a linear term with a slope r in the unwrapped phase function $\phi(\omega_n)$ as can be seen from (3.23). This is undesirable as it will force discontinuities in $\phi(\omega_n)$ at $\omega_n = \pm\pi$ and causes $\hat{Y}(z_n)$ to be no longer analytic on the unit circle. There are several proposals in the removal of the linear phase existing in $y(n)$. Some of which are presented in [20], [84]–[86]. After manipulating the polarity of A and the removal of the linear phase term through altering the input, (3.22) has the form:

$$Y(z_n) = \frac{|A| \prod_{k=1}^{M_i} (1 - a_k z_n^{-1}) \prod_{k=1}^{M_o} (1 - b_k z_n)}{\prod_{k=1}^{N_i} (1 - c_k z_n^{-1}) \prod_{k=1}^{N_o} (1 - d_k z_n)} . \quad (3.24)$$

Again, computing $\hat{Y}(z_n)$ using (3.17) will transform the product of terms in (3.24)

to the sum of logarithmic terms:

$$\begin{aligned}\hat{Y}(z_n) = & \log_e(|A|) + \sum_{k=1}^{M_i} (1 - a_k z_n^{-1}) \\ & + \sum_{k=1}^{M_o} (1 - b_k z_n) - \sum_{k=1}^{N_i} (1 - c_k z_n^{-1}) - \sum_{k=1}^{N_o} (1 - d_k z_n) .\end{aligned}\quad (3.25)$$

Using the power series expansion $\log_e(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} x^n}{n}$ for $|x| < 1$, applying IZT on $\hat{Y}(z_n)$ according to (3.16) gives:

$$\hat{y}(n) = \begin{cases} \log_e(|A|) & , n = 0 \\ -\sum_{k=1}^{M_i} \frac{a_k^n}{n} + \sum_{k=1}^{N_i} \frac{c_k^n}{n} & , n > 0 \\ -\sum_{k=1}^{\infty} \frac{\beta^{3k}}{k} z_n^{-3 n_0 k} + \sum_{k=1}^{\infty} \frac{\beta^k}{k} z_n^{-n_0 k} & , n < 0 \end{cases} \quad (3.26)$$

This expression shows that the equivalent complex cepstrum of (3.22) is two-sided. It is worth noting that the minimum-phase components a_k and the maximum-phase components b_k reside separately in the right and left sides of the complex cepstrum respectively. As phase information is retained, complex cepstrum is capable of handling all types of signal regardless of whether they are minimum-phase, maximum-phase or mixed-phase.

3.3.1 PSF Recovery via Complex Cepstrum

In this subsection, we will demonstrate the capability of the complex cepstrum in estimating the 2-D PSF of ultrasound images. For clarity, we first illustrate the methodology used to extract the PSF for both cases of maximum-phase and minimum-phase sequences. Their combinations lead to the more general case where all signals are handled as mixed-phase sequences. Next, we use a representative example to illustrate this methodology based on analytical computation and its

practical implementation using discrete FT (DFT).

Under a simplified scenario, if we have $s(n) = \delta(n) + a\delta(n - n_0)$. According to the convolution expression in (3.19), we will have obtained $y(n) = v(n) + a v(n - n_0)$. Taking the ZT and evaluating it on the unit circle will arrive at:

$$Y(\omega_n) = V(\omega_n) (1 + a e^{-j\omega_n n_0}). \quad (3.27)$$

For the case of $y(n)$ being a maximum-phase sequence (due to $a \geq 1$), $Y(\omega_n)$ is rearranged as $Y(\omega_n) = a e^{-j\omega_n n_0} V(\omega_n)(1 + \frac{j\omega_n n_0}{a})$ and its cepstrum is given by (3.16) making use of the power series expansion with both ZT and IZT replaced by FT and IFT respectively:

$$\hat{y}(n) = \hat{v}(n) + \log_e(a)\delta(n) + \frac{1}{a}\delta(n - n_0) - \frac{1}{2a^2}\delta(n - 2n_0) + \dots. \quad (3.28)$$

Using a longpass lifter with a time response as in Figure 3.5(a) will extract the cepstrum of the PSF:

$$\hat{v}(n) = \hat{l}(n)\hat{y}(n). \quad (3.29)$$

Likewise, for the case when $y(n)$ is a minimum-phase sequence (due to $a < 1$), we will have:

$$\hat{y}(n) = \hat{v}(n) + a\delta(n - n_0) - \frac{a^2}{2}\delta(n - 2n_0) + \dots. \quad (3.30)$$

Similarly, a shortpass lifter showed in Figure 3.5(b) will extract the cepstrum of the PSF based on the expression (3.29). Using the same rationale, one is expected to see a lifter with a time response that is a combination of both longpass and shortpass lifters to be used for the case of mixed-phase sequence.

With the abovementioned lifters emplaced, the PSF is separated from the tissue scattering function $s(n)$ in the cepstrum domain. Now, we further elaborate this methodology with a sequence in which complex cepstrum can be computed analyti-

Figure 3.5: Time response of frequency-invariant linear filters.
(a) Longpass lifter, and (b) shortpass lifter.

cally so that comparison with its practical implementation is possible. The sequence of the PSF is shown in Figure 3.6(a) and has the following expression:

$$v(n) = b_0 w(n) + b_1 w(n-1), \quad (3.31)$$

where $w(n) = r^n [\cos(\theta n) + \cot(\theta) \sin(\theta n)] U(n)$, $\theta \neq 0, \pi$, and $U(n)$ is the unit step function. Then, the ZT of $v(n)$ is given by:

$$V(z_n) = \frac{b_1 z_n^{-1} \left(1 + \frac{b_0}{b_1} z_n\right)}{(1 - r e^{j\theta} z_n^{-1})(1 - r e^{-j\theta} z_n^{-1})}. \quad (3.32)$$

As we had outlined earlier on, the factor z_n^{-1} contributes a linear component to the unwrapped phase that will force a discontinuity at $\omega_n = \pm\pi$ in $\hat{V}(\omega_n)$, so $\hat{V}(z_n)$ will not be analytic on the unit circle. This is avoided by altering $v(n)$ with one-sample time shift so that the evaluation is carried out on $v(n+1)$ and consequently also on $y(n+1)$. With $v(n)$ replaced by $v(n+1)$ and the corresponding $V(z_n)$ replaced by $z_n V(z_n)$, we consider $V(z_n)$ to have the following form:

$$V(z_n) = \frac{b_1 \left(1 + \frac{b_0}{b_1} z_n\right)}{(1 - r e^{j\theta} z_n^{-1})(1 - r e^{-j\theta} z_n^{-1})}. \quad (3.33)$$

With this expression, its complex cepstrum showed in Figure 3.6(b) is governed by:

$$\hat{v}(n) = \begin{cases} \log_e(b_1) & , n = 0 \\ \frac{1}{n} [(r e^{j\theta})^n + (r e^{-j\theta})^n] & , n > 0 \\ \frac{1}{n} \left(\frac{-b_0}{b_1} \right)^{-n} & , n < 0 \end{cases} \quad (3.34)$$

Now, if we consider $s(n)$ to be an impulse train of the form depicted in Figure 3.6(c):

$$s(n) = \delta(n) + \beta \delta(n - n_0) + \beta^2 \delta(n - 2n_0) + \beta^3 \delta(n - 3n_0). \quad (3.35)$$

The corresponding $S(z_n)$, $\hat{S}(z_n)$ and $\hat{s}(n)$ are written as:

$$S(z_n) = 1 + \beta z_n^{-n_0} + \beta^2 z_n^{-2n_0} + \beta^3 z_n^{-3n_0} = \frac{1 - \beta^4 z_n^{-4n_0}}{1 - \beta z_n^{-n_0}}, \quad (3.36)$$

$$\begin{aligned} \hat{S}(z_n) &= \log_e(1 - \beta^4 z_n^{-4n_0}) - \log_e(1 - \beta z_n^{-n_0}) \\ &= -\sum_{k=1}^{\infty} \frac{\beta^{4k}}{k} z_n^{-4n_0 k} + \sum_{k=1}^{\infty} \frac{\beta^k}{k} z_n^{-n_0 k}, \quad |\beta| < 1 \end{aligned} \quad (3.37)$$

$$\hat{s}(n) = -\sum_{k=1}^{\infty} \frac{\beta^{4k}}{k} \delta(n - 4n_0 k) + \sum_{k=1}^{\infty} \frac{\beta^k}{k} \delta(n - n_0 k). \quad (3.38)$$

Figure 3.6(d) shows the complex cepstrum $\hat{s}(n)$. Since convolution in the time domain results in an addition in the cepstral domain, the complex cepstrum of the RF sequence $\hat{y}(n)$ is an addition of both $\hat{v}(n)$ and $\hat{s}(n)$ as illustrated in Fig. 3.3.2(f). As shown, $\hat{v}(n)$ and $\hat{s}(n)$ occupy different band in the cepstral domain, the cepstrum of $\hat{v}(n)$ can be extracted by using a lowpass lifter.

3.3 Estimation by Cepstrum Analysis

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(a)

(b)

(c)

(d)

(e)

(f)

Figure 3.6: Demonstration of the PSF estimation via the complex cepstrum.
 (a) PSF $v(n)$ with $r=0.9$, $\theta = \pi/6$, (b) $\hat{v}(n)$, (c) $s(n)$ with $\beta = 0.9$, $n_0 = 15$, (d) $\hat{s}(n)$, (e) $y(n)$, and (f) extraction of $\hat{v}(n)$ from $\hat{y}(n)$ using a lowpass lifter.

In reality, we do not attempt to compute the cepstrum analytically. Instead, the DFT and phase unwrapping algorithm are used. The overall implementation strategy of the PSF estimation methodology is incorporated in the block diagram of Figure 3.7. The whole process involves three major steps. The forward transformation to the cepstral domain, the separation of the cepstrum of the PSF from other signals using a lowpass lifter, and the inverse transformation of the estimated PSF back to the sample domain. The computation of the forward complex cepstrum is much more complicated compared to the inverse complex cepstrum as the phase function has to be unwrapped and linear phase component needs to be removed. As for the inverse complex cepstrum, no special treatment is required and so it is computed directly according to the inverse operations of (3.16).

Figure 3.7: Overall PSF estimation system using the complex cepstrum method.

Following the flow of the block diagram in Figure 3.7, the computation of the forward complex cepstrum $\hat{y}(n)$ begins by applying DFT on the signal sequence $y(n)$ to give its discrete spectrum $Y(\omega_n)$. Since $y(n)$ is a real signal, $Y(\omega_n)$ will be a complex quantity. The natural logarithm of $Y(\omega_n)$ is achieved by recognizing that

$\log_e (Y(\omega_n))$ has the following relationship:

$$\begin{aligned}\log_e (Y(\omega_n)) &= \log_e (|Y(\omega_n)| e^{j\phi(\omega_n)}) \\ &= \log_e |Y(\omega_n)| + j\phi(\omega_n),\end{aligned}\tag{3.39}$$

where $|Y(\omega_n)|$ is the magnitude of $Y(\omega_n)$ whereas $\phi(\omega_n)$ is either wrapped (principal phase values) or unwrapped phase of $Y(\omega_n)$. Unfortunately, the computation of $\hat{y}(n)$ requires an unwrapped phase whereas the arctangent subroutine in the computer provides the principal phase values. This means that the phase curve in (3.39) has to be computed explicitly using an independent phase unwrapping algorithm. By definition, the phase unwrapping algorithm is to solve for:

$$\phi(\omega_n) = \angle Y(\omega_n) + 2\pi r(\omega_n),\tag{3.40}$$

where $\angle Y(\omega_n)$ denotes the principal phase values falling in the range $-\pi < \angle Y(\omega_n) \leq \pi$ and $r(\omega_n)$ takes on appropriate integer values derived by the phase unwrapping algorithm to make $\phi(\omega_n)$ continuous. In our example, $\log_e |Y(\omega_n)|$, $\angle Y(\omega_n)$ and $\phi(\omega_n)$ are shown in Figure 3.8(a), (b), and (c) respectively. We observe that in Figure 3.8(c) there is a linear phase component within $\phi(\omega_n)$ that makes $\phi(\omega_n)$ discontinuous at the boundary. Since a real signal will have a FT that its real part is even symmetry and its imaginary part has an odd symmetry. This linear phase component has to be removed to satisfy the continuity requirement of the IDFT after the logarithm operation to give a valid cepstrum:

$$\hat{y}(n) = F^{-1} [\log_e (Y(\omega_n))].\tag{3.41}$$

The unwrapped phase curve in our example with the linear phase component removed is depicted in Figure 3.8(d). The phase unwrapping process is a rather complicate operation. There are various algorithms available, e.g., [90], and the

detail of our treatment will be presented in Section 3.4.

From the discussion detailed earlier, the complex cepstrum is computed according to (3.41) using IDFT after the removal of the linear phase component and the result is visually no different from that in Figure 3.6(f). As stated earlier, a lowpass lifter enters into this stage of the PSF estimation process to filter off all the unwanted signals only to retain $\hat{v}(n)$. From Section 3.1, the scattering function $s(n)$ is random in nature. It will most likely have a cepstrum that occupies a wideband of the cepstral domain. Hence, it is generally impossible to extract $\hat{v}(n)$ from $\hat{y}(n)$ completely without portion of $\hat{s}(n)$ being included. Therefore, what we obtained will be an estimation of $\hat{v}(n)$. Fortunately, the quality of the estimation can readily be improved by averaging from the various estimations of $\hat{v}(n)$ using sample sequence segments in the neighborhood such that $v(n)$ is spatially invariant.

Unlike the computation of the complex cepstrum which requires unwrapping of the phase function explicitly, the computation of the inverse complex cepstrum is straightforward:

$$\begin{aligned} v(n) &= F^{-1} [V(\omega_n)] \\ &= F^{-1} [\text{Re}\{V(\omega_n)\} + j \text{Im}\{V(\omega_n)\}], \end{aligned} \quad (3.42)$$

where the real and imagery parts of $V(\omega_n)$ are computed, respectively, according to:

$$\text{Re}\{V(\omega_n)\} = e^{\text{Re}\{\hat{V}(\omega_n)\}} \cos\left(\text{Im}\{\hat{V}(\omega_n)\}\right), \quad (3.43)$$

$$\text{Im}\{V(\omega_n)\} = e^{\text{Re}\{\hat{V}(\omega_n)\}} \sin\left(\text{Im}\{\hat{V}(\omega_n)\}\right), \quad (3.44)$$

and

$$\begin{aligned} \hat{V}(\omega_n) &= F [\hat{v}(n)] \\ &= \text{Re}\{\hat{V}(\omega_n)\} + j \text{Im}\{\hat{V}(\omega_n)\}. \end{aligned} \quad (3.45)$$

It is worth mentioning that the linear phase reinsertion in Figure 3.7 is not essen-

Figure 3.8: Computation of the complex cepstrum using 1024-point DFT.
 (a) Log magnitude $\log_e |Y(\omega_n)|$, (b) principle value of the phase, (c) unwrapped phase, and (d) unwrapped phase with linear phase component removed.

tial in the PSF estimation as it simply gives spatially shifted estimation. In the practical implementation of the PSF estimation algorithm, it will not be reinserted. Its inclusion in Figure 3.7 solely serves as completeness in the sense that a signal experiencing a forward followed by the reverse transformations should be the same.

3.3.2 Minimum-phase/All-pass Decomposition

Any nonminimum-phase causal sequence can be represented as a convolution expression of a minimum-phase and an all-pass sequences as follows [80], [88]:

$$y(n) = y_{min}(n) * y_{ap}(n), \quad (3.46)$$

where $y_{min}(n)$ and $y_{ap}(n)$ is a minimum-phase sequence and an all-pass sequence respectively such that:

$$|Y_{min}(\omega_n)| = |Y(\omega_n)|, \quad (3.47)$$

$$|Y_{ap}(\omega_n)| = 1. \quad (3.48)$$

Following the block diagram in Figure 3.9, the cepstrum of the minimum-phase component can be obtained:

$$\hat{y}_{min}(n) = \hat{c}_y(n) \hat{l}_{min}(n), \quad (3.49)$$

where $\hat{c}_y(n)$ and $\hat{l}_{min}(n)$ are given as:

$$\hat{c}_y(n) = F^{-1} [\log_e (|Y(\omega_n)|)], \quad (3.50)$$

$$\hat{l}_{min}(n) = 2U(n) - \delta(n). \quad (3.51)$$

The basic strategy incorporated in the block diagram relies on the fact that:

$$\hat{y}_{ap}(n) = F^{-1} [\log_e (|Y_{ap}(\omega_n)|)], \quad (3.52)$$

$$Y_{ap}(\omega_n) = \frac{Y(\omega_n)}{Y_{min}(\omega_n)}, \quad (3.53)$$

$$\phi_{ap}(\omega_n) = \phi(\omega_n) - \phi_{min}(\omega_n). \quad (3.54)$$

Therefore, the complex cepstrum computed based on minimum-phase/all-pass decomposition is given as:

$$\hat{y}(n) = \hat{y}_{min}(n) + \hat{y}_{ap}(n). \quad (3.55)$$

As convolution of two signals results in an addition of their FT phases in the frequency domain and since the minimum-phase component has phase function of continuity fall within the $\pm\pi$, the phase unwrapping operation is only required for the all-pass component. Compared to the FT phase of $y(n)$, the FT phase of $y_{ap}(n)$ is supposed to contain less wrapped phases such that the errors introduced in the phase unwrapping operation can now be further alleviated. This is the motivation underlying the exploration of using the additional efforts to decompose the signal into these two components. Unfortunately, this decomposition is not used in the lateral direction because the lateral PSF is noncausal and can not have a minimum-phase component.

Figure 3.9: Computation of the complex cepstrum using minimum-phase/all-pass decomposition.

3.3.3 Results of 2-D PSF Estimation

In this section, we review the quality of the 2-D PSF estimation in the complex cepstrum. Two different approaches of computing the complex cepstrum are explored.

They are the direct approach written in (3.16) and that utilizing the fact that a nonminimum-phase causal sequence can have minimum-phase/all-pass decomposition. Both methods require the use of 2-D phase unwrapping algorithm to unwrap their phases. In our studies, the weighted LS phase unwrapping algorithm proposed in [89], [90] is used to obtain an unwrapped solution by iteratively minimizing the differences between the discrete partial derivatives of the wrapped phase data and the discrete partial derivatives of the unwrapped solution. This algorithm is proven robust to phase noise and data inconsistencies. Unlike path-following unwrapping algorithms, degradations are automatically accommodated by the LS formulation without the need of operator intervention [90]. Unfortunately, the LS algorithm does not produce an unwrapped surface that is congruent to the wrapped phase. Hence, a postprocessing operation is required [90]. In order to maintain the flow of the presentation, we postpone the theoretical formulation of the weighted LS phase unwrapping algorithm to Section 3.4.

In our implementation, the low frequency content is centered before phase unwrapping. Besides, $y(m, n)$ being the ultrasound RF signal is real-valued. Thus the properties of the FT mandate that the resulting complex cepstrum is entirely real. This is accomplished by forcing the logarithm of the magnitude function even and phase function odd with respect to frequency. These treatments show improved results.

In a similar manner to the 1-D formulation in (3.16), the 2-D formulation of the complex cepstrum follows exactly the same procedure. Omitting the noise $w(m, n)$ in (3.1), if $v(m, n)$ and $s(m, n)$ have valid complex cepstra $\hat{v}(m, n)$ and $\hat{s}(m, n)$, the complex cepstrum has a property that the process of convolution of two signals in the time domain is reduced to mere addition in the cepstral domain. Thus we have:

$$\hat{y}(m, n) = \hat{v}(m, n) + \hat{s}(m, n). \quad (3.56)$$

An important implication of this additive property is that signals combined by convolution can now be easily separated with a lifter in the complex cepstrum. As in the 1-D case, the complex cepstrum is invertible because its phase information is retained. The previous sections pointed out the fact that the scattering of tissues have a distinctive smoothness feature different from the ultrasonic PSF which causes both signals $\hat{v}(m, n)$ and $\hat{s}(m, n)$ occupying different band in the cepstral domain [26], [29], [80], [81]. This phenomenon furthers simplified the extraction of $\hat{v}(m, n)$ to the use of a simple lowpass lifter with appropriate cutoffs:

$$\hat{v}(m, n) = \hat{l}(m, n) \hat{y}(m, n), \quad (3.57)$$

where the cutoffs m_o and n_o are usually expressed as a fraction of the whole domain:

$$\hat{l}(m, n) = \begin{cases} 1 & , \text{if } |m| \leq m_o \text{ and } |n| \leq n_o \\ 0 & , \text{otherwise} \end{cases} \quad (3.58)$$

Furthermore, it has been widely known that direct truncation from the frequency response of a signal will result in Gibbs ringing phenomenon. This artifact is said to be eliminated by replacing the lowpass lifter with a 2-D Butterworth lowpass filter (or lifter) [26], [29]. In the review, we are also interested in comparing results obtained with both lifters. The 2-D Butterworth lowpass lifter has an order of R , and with cutoff at a distance L_o from the origin, in the complex cepstrum. Thus, the estimated 2-D PSF is given by:

$$\check{v}(m, n) = F^{-1} \left[e^{F \left[\frac{\hat{y}(m, n)}{1 + [L(m, n)/L_o]^{2R}} \right]} \right], \quad (3.59)$$

where $L(m, n)$ is the distance from the point (m, n) to the origin [91].

Both simulated and real ultrasound RF data are used in our studies. Using simulated ultrasound data have an added advantage that investigation on the quality of the

2-D PSF can be carried out. We will be able to compare the estimated 2-D PSF with that of simulated both visually and quantitatively. This comparison is not possible with the real ultrasound data.

From the prior knowledge of the smoothness properties of the 2-D PSF, one is expected to see that there will be some forms of smoothness in the lateral PSF and the oscillatory nature of the axial PSF about the center frequency of the transmitting ultrasound. In reality, the use of the real ultrasound RF data in the PSF estimation simply serves this purpose. In fact, there is not much to deduce from the estimated 2-D PSF with the real data. We can assess its quality based only on the resultant deconvolved ultrasound images. Moreover, all the existing deconvolution algorithms are derived based on the convolution model in (3.1). Therefore, all algorithms developed should be expected to give improved results on the simulated data which are created based on this model. Because of these reasons, it is of course more logical to focus the performance evaluations based on the simulated ultrasound RF data rather than the real data whereas the latter will solely serve as a final validation. As a matter of convenience, all settings used in the study of the PSF estimation with the simulated ultrasound RF data are used later with the real ultrasound data.

The simulated ultrasound RF data are generated according to the procedure described in Appendix B. Each of the simulated ultrasound RF data has the size of 796 by 360. All the dimensions are specified in sample numbers, unless specifically stated otherwise. In order to infer the quality of the PSF estimation algorithms, three hundreds simulated ultrasound RF data are generated and they are divided into three sets of one hundred each. For each of the simulated ultrasound RF data, it is segmented into smaller overlapping RF segments with a size of 64 by 32. This setting is arrived either empirically or through underwater measurement such that it should have a spatial extent larger than that measured so as to account for the inhomogeneity of the tissues. The overlapping of the segment is chosen to be 70 %

of its size so that there are sufficient segments to be used for the evaluation. As the final estimation is an averaging of the numerous estimates, we investigate the required number for averaging necessary to ensure a good estimation. In our studies, the numbers of averaging involved are 10, 20, and 30. Each of the set of hundred simulated ultrasound RF data is used in each of the studies to avoid biasing to a certain data set. The selection of the segments is based on their envelope SNR that are closer to 1.91 which ensures that the segments involved have fully developed speckle.

The two types of lowpass lifters involved in our investigations are illustrated in Figure 3.10. Both lifters are set to equivalent cutoffs of 0.5 and 0.7 in the axial and lateral directions respectively, and an additional parameter which is the lifter order R of 5 is needed for the Butterworth lowpass lifter. These parameters are set empirically and they are also used for the real ultrasound RF data.

Figure 3.10: Response of the lowpass lifters in the complex cepstrum.
 (a) Lowpass lifter with $m_0 = 0.7$ and $n_0 = 0.5$, and (b) Butterworth lowpass lifter with an equivalent cutoffs $L(m, n)$ and order $R = 5$.

Table 3.1 and 3.2 present the estimation results. Like the HOS-based estimates, the quality of the PSF estimation is quantified based on the MSE between the estimation and that simulated. As shown, the results are quite close among each other with

that utilizing Butterworth lowpass lifter being slightly better. Besides, increasing the number of averaging improves the variance of the estimation. Surprisingly, the results show that making use of minimum-phase/all-pass decomposition of the signal did not give improvement.

Table 3.1: Quality of the estimated 2-D PSF based on direct computation of the complex cepstrum (in dB). These values are tabulated in terms of mean $\pm$ one standard deviation.

Number of averages	Direct computation of complex cepstrum	
	Lowpass lifter	Butterworth lowpass lifter
10	-39.2116 ± 1.2133	-39.3051 ± 1.1823
20	-39.9803 ± 1.1442	-39.9859 ± 1.1891
30	-39.9644 ± 0.9575	-40.0203 ± 0.9173

Table 3.2: Quality of the estimated 2-D PSF utilizing minimum-phase/all-pass decomposition (in dB). These values are tabulated in terms of mean $\pm$ one standard deviation.

Number of averages	Computation of complex cepstrum based on minimum-phase/all-pass Decomposition	
	Lowpass lifter	Butterworth lowpass lifter
10	-39.2860 ± 1.2295	-39.3061 ± 1.1903
20	-39.8010 ± 0.9965	-39.8304 ± 1.1428
30	-39.9194 ± 0.8556	-39.9044 ± 0.8865

In the following, we illustrate the operations involved in the 2-D PSF estimation. The weighted LS phase unwrapping algorithm provides the feasibility that we can assign weights to the quality of the phase values. Since the lateral PSF generally has a lowpass response, we can deduce that most portions of the phase beyond a certain lowpass cutoff in the lateral direction are dominated by noise. This means that, if one can exclude these portions of the phase in the unwrapping process, it will greatly improve the overall estimation quality. For simplicity, a quality map assigned with weights of either 0 or 1 is adopted. The active area of the quality map where weights assigned as 1 in the axial and lateral directions is set to 100 % and 30 % of their corresponding total area at the center coordinate respectively. For this reason, the weighted wrapped phase curve for the phase unwrapping has the appearance showed in Figure 3.11(a). From the discussion detailed earlier, one can observe that a linear phase component exists in the unwrapped phase in Figure 3.11(b). When it is removed, the unwrapped phase will be an odd function and continuous at the boundaries as depicted in Figure 3.11(c).

(a)

(b)

(c)

Figure 3.11: Surface plot of the 2-D phase unwrapping process.
(a) Wrapped phase function, (b) unwrapped phase curve, and (c) unwrapped phase curve with linear phase component removed.

From the results tabulated in Table 3.1 and 3.2, the MSE computed for the various methods are very small. Consequently, they are visually of little difference. To be convinced of the results, we superimpose both the estimated axial and lateral PSF onto that of the corresponding simulated PSF to give a close comparison of their profiles. These profiles are extracted through slicing at the center of the 2-D PSF in both the axial and lateral directions. Figure 3.12 presents the surface plot of the 2-D PSF of the simulated ultrasound data and its estimation based on both the lowpass and Butterworth lowpass lifters. Their comparisons in the axial and lateral PSF are shown in Figure 3.13. Their corresponding comparisons with the complex cepstrum that is computed using the minimum-phase/all-pass decomposition is also presented in Figure 3.14. As mentioned before, they are hardly discriminable by visual examination.

(b)

(c)

Figure 3.12: Surface plot of the 2-D PSF estimated from the simulated ultrasound RF data. (a) Simulated 2-D PSF, (b) estimated 2-D PSF using lowpass lifter, and (c) that using Butterworth lowpass lifter.

(a)

(b)

Figure 3.13: Comparison between the estimated axial and lateral PSF obtained using different lifters with that of the corresponding simulated PSF. (a) Using lowpass lifter, and (b) using Butterworth lowpass lifter.

Figure 3.14: Comparison of the estimated PSF under minimum-phase/all-pass decomposition obtained from the simulated ultrasound RF data. Without (-) and with (o) minimum-phase/all-pass decomposition using both lowpass lifter and Butterworth lowpass lifter under the same settings. (a) Using lowpass lifter, and (b) using Butterworth lowpass lifter.

The description of the acquisition of the real ultrasound RF data is described in Appendix C. Figure 3.15 through Figure 3.17 present the estimation results of the 2-D PSF using both the lowpass and Butterworth lowpass lifters under the same parameters setting as that used in the simulated ultrasound RF data. The same observations are noted with the two different complex cepstrum computations and lowpass lifters on the real ultrasound RF data.

Figure 3.15: Surface plot of the estimated 2-D PSF obtained from the real ultrasound RF data. (a) Using lowpass lifter, and (b) using Butterworth lowpass lifter. Minimum-phase/all-pass decomposition is adopted in computing the complex cepstrum and the 2-D PSF is extracted by using (c) lowpass lifter, and (d) Butterworth lowpass lifter.

(a)

(b)

Figure 3.16: Comparison of the estimated PSF obtained from the real ultrasound RF data under minimum-phase/all-pass decomposition. Without (-) and with (o) minimum-phase/all-pass decomposition using both lowpass lifter and Butterworth lowpass lifter under the same settings. (a) Using lowpass lifter, and (b) using Butterworth lowpass lifter.

Figure 3.17: Comparison of the estimated PSF obtained from the real ultrasound RF data using lowpass lifter (-) and Butterworth lowpass lifter (o).

3.4 2-D Weighted Least Square Phase Unwrapping Algorithm

When the angle of a complex number is computed, the arctangent subroutine of the computer outputs an angle in the form of its principal value. This means that if the phase angle of complex function $Y(\omega_n)$ is computed by the computer system, what we get will be the principal value of its phase given as:

$$-\pi < \angle Y(\omega_n) \leq \pi. \quad (3.60)$$

Any other angles will also lead to the same complex function $Y(\omega_n)$ if they are summing of its principal values in (3.60) with multiple of 2π of an integer number:

$$\phi(\omega_n) = \angle Y(\omega_n) + 2\pi r(\omega_n), \quad (3.61)$$

where $r(\omega_n)$ takes an appropriate integer value. Although $r(\omega_n)$ can be any integer function, we are only interested in the determination of $r(\omega_n)$ by the phase unwrapping algorithm to make $\phi(\omega_n)$ continuous. An illustration of the wrapped and unwrapped phase function is shown in Figure 3.8(b) and (c) respectively.

In the following, we will present the 2-D phase unwrapping algorithm which was originated from Ghiglia and Romero [89]. It is used explicitly to unwrap the FT-phase in the computation of the forward complex cepstrum. This method is robust to phase noise and data inconsistencies for unwrapping 2-D principal phase values, in the LS sense, iteratively for solving the weighted unwrapping problem with the incorporation of the fast cosine transform.

For the ease of explanation, we simplify the notations used in the frequency coordinates ω_m and ω_n of $\phi(\omega_m, \omega_n)$ to i and j respectively. Under this condition, we may interpret $\phi(i, j)$, generally, as an unwrapped phase surface. 2-D phase unwrapping

is the process of attempting to recover unambiguous phase values from a 2-D array of phase values known only modulo 2π . Recently, it has received a great deal of attention. A detailed description of the 2-D phase unwrapping theory is given by Ghiglia and Pritt in [90]. The weighted LS phase unwrapping algorithm in [89] yields an unwrapped solution by minimizing the differences between the discrete partial derivatives of the wrapped phase data and the discrete partial derivatives of the unwrapped solution with respect to the unwrapped phase data expressed as follows iteratively:

$$\begin{aligned} \xi^2 = & \sum_{i,j} U(i,j) [\phi(i+1,j) - \phi(i,j) - \Delta^x(i,j)]^2 \\ & + \sum_{i,j} V(i,j) [\phi(i,j+1) - \phi(i,j) - \Delta^y(i,j)]^2, \end{aligned} \quad (3.62)$$

$\phi(i,j)$ is the unknown unwrapped phase value, $\Delta^x(i,j) = W_p\{\psi(i+1,j) - \psi(i,j)\}$, and $\Delta^y(i,j) = W_p\{\psi(i,j+1) - \psi(i,j)\}$ with W_p being the wrapping operation. $\psi(i,j)$ is the known wrapped phase value, $U(i,j)$ and $V(i,j)$ are collectively known as the quality map contains of the appropriate gradient weights assigned to indicate the quality of the phases. The weights are selected to isolate the regions of low signal-to-noise ratio or to impose preferences on the desired solution. As the ultrasound echo is generally over-sampled, there will be large frequency domains without signal and only noise due to band-limitation. Besides, magnitude zeros are the generators of residues which means that the phase is not uniquely unwrappable by any method. These are the frequency regions assigned zero weights so that they did not contribute to the phase unwrapping while the rest of the active regions are set to one to include their contributions.

The LS solution $\phi(i,j)$ to this problem is obtained using the discrete variational

approach and is given by the equation:

$$\begin{aligned} U(i, j) [\phi(i+1, j) - \phi(i, j)] - U(i-1, j) [\phi(i, j) - \phi(i-1, j)] \\ + V(i, j) [\phi(i, j+1) - \phi(i, j)] - V(i, j-1) [\phi(i, j) - \phi(i, j-1)] \\ = c(i, j), \end{aligned} \quad (3.63)$$

where $c(i, j)$ is the *weighted phase Laplacian* defined by:

$$\begin{aligned} c(i, j) = U(i, j) \Delta^x(i, j) - U(i-1, j) \Delta^x(i-1, j) \\ + V(i, j) \Delta^y(i, j) - V(i, j-1) \Delta^y(i, j-1). \end{aligned} \quad (3.64)$$

The heart of the algorithm is the unweighted LS phase unwrapping algorithm which is being applied iteratively to minimize the cost function in (3.62). The weighted and unweighted LS phase unwrapping algorithms are equivalent when all the weights are set to one. It is shown in [89], [90] that in this case it is the *discretized Poisson's equation* on a rectangular M by N grid, i.e.,

$$\frac{\partial^2 \phi(i, j)}{\partial i^2} + \frac{\partial^2 \phi(i, j)}{\partial j^2} = \rho(i, j). \quad (3.65)$$

In this case, (3.63) and (3.64) are of the following discrete equivalent forms:

$$[\phi(i+1, j) - 2\phi(i, j) + \phi(i-1, j)] + [\phi(i, j+1) - 2\phi(i, j) + \phi(i, j-1)] = \rho(i, j), \quad (3.66)$$

$$\rho(i, j) = [\Delta^x(i, j) - \Delta^x(i-1, j)] + [\Delta^y(i, j) - \Delta^y(i, j-1)], \quad (3.67)$$

and they are computed under the following *Neumann boundary conditions* on *Poisson's equation*:

$$\begin{aligned} \Delta^x(-1, j) &= 0, \\ \Delta^x(M-1, j) &= 0, \quad j = 0, \dots, N-1 \\ \Delta^y(i, -1) &= 0, \\ \Delta^y(i, N-1) &= 0, \quad i = 0, \dots, M-1 \end{aligned} \quad (3.68)$$

The phases are unwrapped by using the fast cosine transform as follows:

$$\hat{\rho}(m, n) = \begin{cases} \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} 4 \rho(i, j) \cos \left[\frac{(2i+1)m\pi}{2M} \right] \cos \left[\frac{(2j+1)n\pi}{2N} \right] \\ , 0 \leq m \leq M-1, 0 \leq n \leq N-1 \\ 0 , \text{otherwise} \end{cases} \quad (3.69)$$

$$\hat{\phi}(m, n) = \frac{\hat{\rho}(m, n)}{2 \left[\cos \left(\frac{i\pi}{M} \right) + \cos \left(\frac{j\pi}{N} \right) \right]}, \quad (3.70)$$

$$\phi(i, j) = \begin{cases} \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} w_1(m) w_2(n) \hat{\phi}(m, n) \times \\ \cos \left[\frac{(2i+1)m\pi}{2M} \right] \cos \left[\frac{(2j+1)n\pi}{2N} \right] \\ , 0 \leq i \leq M-1, 0 \leq j \leq N-1 \\ 0 , \text{otherwise} \end{cases} \quad (3.71)$$

$$\begin{aligned} w_1(m) &= \frac{1}{2} , m = 0 \\ w_1(m) &= 1 , 0 \leq m \leq M-1 \\ w_2(n) &= \frac{1}{2} , n = 0 \\ w_2(n) &= 1 , 0 \leq n \leq N-1 \end{aligned} \quad (3.72)$$

Solving $\phi(i, j)$ based on (3.66) to (3.72) is equivalent to finding the solution of the over-determined set of linear equations $\mathbf{A} \boldsymbol{\phi} = \mathbf{b}$ where it is, of course, the solution to the normal equation:

$$\mathbf{P} \boldsymbol{\phi} = \boldsymbol{\rho}, \quad (3.73)$$

where $\mathbf{T}$ refers to matrix transposition, $\mathbf{P} = \mathbf{A}^T \mathbf{A}$ is a matrix that performs the *discrete Laplacian operation* on the vector as presented in (3.66) and $\boldsymbol{\rho} = \mathbf{A}^T \mathbf{b}$ is a vector containing the *discrete Laplacian operation* on the wrapped phase differences as in (3.67).

Likewise, the weighted LS phase unwrapping problem stated in (3.62) to (3.64) can be posed as a weighted LS problem:

$$\mathbf{W} \mathbf{A} \boldsymbol{\phi} = \mathbf{W} \mathbf{b}, \quad (3.74)$$

where $\mathbf{W}$ is a matrix of weighting values corresponding to $U(i, j)$ and $V(i, j)$ in (3.62). The LS solution of (3.74) is also the solution of the normal equation of $\mathbf{A}^T \mathbf{W}^T \mathbf{W} \mathbf{A} \boldsymbol{\phi} = \mathbf{A}^T \mathbf{W}^T \mathbf{W} \mathbf{b}$ and is given as:

$$\mathbf{Q} \boldsymbol{\phi} = \mathbf{c}, \quad (3.75)$$

where $\mathbf{Q} = \mathbf{A}^T \mathbf{W}^T \mathbf{W} \mathbf{A}$ and $\mathbf{c} = \mathbf{A}^T \mathbf{W}^T \mathbf{W} \mathbf{b}$ is nothing more than a vector formed from the *modified discrete Laplacian* of the weighted wrapped phase differences. Hence, (3.75) is of the same form as that of (3.73) except that the wrapped phase differences are weighted as in (3.63) and (3.64).

The *classical Gauss-Seidel relaxation* method can be invoked to solve (3.75) iteratively. However, the rate of convergence using *classical Gauss-Seidel relaxation* method is too slow to be of practical use. The weighted LS phase unwrapping algorithm adopts preconditioned conjugate gradient (PCG) algorithm to give rapid and robust convergence properties. Figure 3.18 shows the flow diagram of the algorithm where ϵ is a small value for the determination of the convergence of the iterative procedure whereas K_{Max} is the predefined maximum number of iterations. They are set to default values of 0.01 and 20 respectively. The beauty of this algorithm is that it allows the incorporation of the fast cosine transform elaborated in (3.68) to (3.72) to speed up the unwrapping process within the robust structure of using the conjugate gradient in order to solve the weighted LS problem iteratively.

3.4 2-D Weighted Least Square Phase Unwrapping Algorithm

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Figure 3.18: Flow diagram of the PCG for 2-D weighted LS phase unwrapping algorithm.

In general, the FT-phase of an image is a very complicated function. The FT-magnitude will likely have many zeros throughout the transform domain. Besides, the LS algorithms do not produce an unwrapped surface that is congruent to the wrapped phase as the LS formulation minimizes the squares of the differences between the gradients. Therefore, a postprocessing operation is required. The simplest way is to subtract the unwrapped surface from the wrapped phase, wrap the result, and then added it to the unwrapped surface.

We demonstrate the robustness of the 2-D weighted LS phase unwrapping algorithm through an example described as follows. Figure 3.19(a) shows the wrapped values of a 512 by 512 pixel 2-D phase-plane image. The phase image depicting wrapped phases in the range $(-\pi, \pi]$ are scaled between black and white in the range of 0 to 255 for display. Likewise, the unwrapped phase images that will present are also scaled to within this range for this reason.

Figure 3.19: Test phase data used for the demonstration of the 2-D LS phase unwrapping algorithm. (a) Noisy wrapped 2-D phase values, (b) wrapped 2-D phase values with a small square region corrupted by randomly distributed noise.

It can be seen in Figure 3.19(a) that the phase surface increases linearly from the top-left corner to the bottom-right corner across the image. In Figure 3.19(b), a small square region contaminated with randomly distributed noise is added into Figure 3.19(a) to model the scenario whereby good phase values are destroyed locally because of noise exists locally within the region.

Figure 3.20: Demonstration of the robustness of the 2-D unweighted LS phase unwrapping algorithm to data inconsistencies. (a) Unwrapped 2-D phase values, the smoothness of the degraded locally at the region where the phase values are contaminated. (b) Rewrapping of (a) shows that inconsistent data influences the unwrapped results locally, with diminishing effects further away.

As shown in Figure 3.20(a), the unwrapped phase image by the unweighted LS algorithm shows some influence at the noisy region. Because of the dynamic range of the unwrapped result is large, visual display is difficult to validate the correctness in unwrapping the phase data. Fortunately, if the unwrapped phase image is rewrapped, it can permit a direct comparison as subtle deviations can now be seen clearly. This methodology is adopted and the rewrapped phase image of Figure 3.20(a) is shown in Figure 3.20(b). Figure 3.20(b) confirms that the phase values are inconsistent in the square region and around its boundaries where the phase data are contaminated

by the phase noise in Figure 3.19(b). It is worth noting that these inconsistencies are accommodated by the 2-D unweighted LS phase unwrapping algorithm so that they influence the unwrapping result only locally.

Assuming that we have additional information on the quality of the phase data, we can prevent noisy phase from influencing the unwrapping result by assigning zero weight to the noisy phase data. The set of additional weights is the weights, $U(i, j)$ and $V(i, j)$, in (3.62) or W in (3.74), and it is implemented as the quality map [90]. Figure 3.21(a) depicts the corresponding weightings accompanying the wrapped phase image of Figure 3.19(b). The black square region corresponds to zero weight and the remainder of the region is set to unity weight and it is shown as white in the figure. The dark boundaries are explicitly superimposed just to highlight where the boundaries of the quality map are in the figure.

Figure 3.21(b) illustrates the unwrapped phase image based on the 2-D weighted LS phase unwrapping algorithm where the weights are assigned via the quality map. It is interesting to note that the unwrapped phase image appears to be continuous even across the noisy region where the weights are assigned zeros. The rewrapped result in Figure 3.21(c) confirms that the 2-D weighted LS phase unwrapping algorithm produces a correct solution. Through this example, we show the great potential of the 2-D weighted LS phase algorithm. The incorporation of the weightings elegantly eliminated corrupted regions. Moreover, it is efficiently implemented via the fast 2-D discrete cosine transform as the basis for LS unwrapping.

Figure 3.21: Demonstration of the robustness of the 2-D weighted LS phase unwrapping algorithm to data inconsistencies. (a) Quality map to accompany the wrapped phase data depicted in Figure 3.19(b). Black corresponds to zero-valued weights and white corresponds to unity weight. The dark boundaries are explicitly added to show the boundaries solely for display. (b) Unwrapped phase using the weighted LS algorithm, and (c) the rewrapping of (b) shows that it produces a correct solution.

Chapter 4

Constrained Least Squares Filter

This chapter consists of six sections. The first section gives a complete derivation of the constrained least squares (CLS) filtering algorithm based on the classical signal distortion model. This presentation is included mainly for researchers working in the field of ultrasound research who are not familiar with image restoration using CLS filter. The section consists of two parts. First, the classical signal distortion model is introduced. Using this model, we present the use of the optimization techniques leading to both the unconstrained restoration and constrained restoration techniques. Readers who are familiar with the optimization techniques may skip this section to go to Section 4.2.

In Section 4.2, we propose a deconvolution algorithm which is derived based on our proposed signal degradation and restoration model in Chapter 2.4. Moreover, the developed CLS filter is extended to an iterative filtering scheme in Section 4.3. This iterative CLS (ICLS) filter can show progressively improvement in the deconvolved ultrasound images. In Section 4.4 and Section 4.5, we present both the simulation and experimental results of the proposed algorithms respectively. Section 4.6 concludes the chapter summarizing its main results.

4.1 Introduction

Image restoration attempts to recover the original image that has been degraded by using prior knowledge of the degradation phenomenon. It has a notable feature as compared to the statistical filtering approach and produces optimal estimation in the average sense that its result is optimal for each image to which it is applied. This approach involves formulating a criterion of goodness that will yield a desirable estimate. However, it is important to keep in mind that the choice of one criterion function over the other will almost be determined by the perceived visual quality of the resulting restored images.

Figure 4.1: A model of the classical image degradation/restoration process.

Figure 4.1 shows the model of the classical image degradation/restoration process. The original image $f(m, n)$ is blurred by the degradation phenomenon which is modelled as a degradation function $v(m, n)$. The resultant image is then contaminated by an additive noise $w(m, n)$ to form the degraded observation $y(m, n)$. Given that $v(m, n)$ is a linear space-invariant process, then the degraded image is given in the spatial domain by:

$$y(m, n) = v(m, n) * f(m, n) + w(m, n). \quad (4.1)$$

Since convolution in the spatial domain is equivalent to multiplication in the frequency domain, we can write the model in (4.1) in an equivalent frequency domain

representation:

$$Y(\omega_m, \omega_n) = V(\omega_m, \omega_n) F(\omega_m, \omega_n) + W(\omega_m, \omega_n). \quad (4.2)$$

Given $Y(\omega_m, \omega_n)$ and some knowledge about the degradation process $V(\omega_m, \omega_n)$ and the additive noise $W(\omega_m, \omega_n)$, our objective is to produce an estimate as close as possible to the original image.

In the following two subsections, we will introduce the derivations of the optimization techniques leading to the unconstrained and constrained LS filters.

4.1.1 Unconstrained Restoration

If the noise source $w(m, n)$ is not present or we have no information of it, the restoration is set up as a LS minimization problem as follows. Let $E(\omega_m, \omega_n)$ be the residual errors that result from using $\check{F}(\omega_m, \omega_n)$ as an approximation of $F(\omega_m, \omega_n)$:

$$E(\omega_m, \omega_n) = Y(\omega_m, \omega_n) - V(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n). \quad (4.3)$$

Now, we are going to minimize the cost function:

$$\begin{aligned} W(\check{F}(\omega_m, \omega_n)) &= ||E(\omega_m, \omega_n)||^2 \\ &= ||Y(\omega_m, \omega_n) - V(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n)||^2 \\ &= [Y(\omega_m, \omega_n) - V(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n)]^* \times \\ &\quad [Y(\omega_m, \omega_n) - V(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n)] \end{aligned} \quad (4.4)$$

where $*$ denotes complex conjugate, and $||\cdot||$ denotes the L^2 norm, that is, the sum of the squares of its elements, and $\check{F}(\omega_m, \omega_n)$ is the estimate of the undegraded image. This means that we are selecting $\check{F}(\omega_m, \omega_n)$ so that if it is blurred by $V(\omega_m, \omega_n)$, the result will differ from the observation by as little as possible in the mean square

sense. Here, we seek to minimize the difference between the blurred original and a similarly blurred estimate of the original.

Setting to zero the derivative of $W(\check{F}(\omega_m, \omega_n))$ with respect to $\check{F}(\omega_m, \omega_n)$, we obtain:

$$\begin{aligned} \frac{\partial W(\check{F}(\omega_m, \omega_n))}{\partial \check{F}(\omega_m, \omega_n)} &= -2 V(\omega_m, \omega_n)^* [Y(\omega_m, \omega_n) - V(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n)] \\ &= 0, \end{aligned} \quad (4.5)$$

and solving for $\check{F}(\omega_m, \omega_n)$ yields:

$$\check{F}(\omega_m, \omega_n) = \frac{Y(\omega_m, \omega_n)}{V(\omega_m, \omega_n)}. \quad (4.6)$$

We notice that $Y(\omega_m, \omega_n)$ is essentially $F(\omega_m, \omega_n)$ blurred by $V(\omega_m, \omega_n)$, this approach is satisfying. Since $F(\omega_m, \omega_n)$ and $\check{F}(\omega_m, \omega_n)$ are both blurred by $V(\omega_m, \omega_n)$, the LS minimization may result in $\check{F}(\omega_m, \omega_n)$ that is a good approximation of $F(\omega_m, \omega_n)$.

4.1.2 Constrained Least Square Restoration

The LS solution in (4.6) produces $F(\omega_m, \omega_n)$ in the absence of noise. Unfortunately, central to this method is the issue of sensitivity of $V(\omega_m, \omega_n)$ to the noise. If $V(\omega_m, \omega_n)$ has zeros, then the solution does not exist as there will be large amplification of the noise in the neighborhood of frequencies where $V(\omega_m, \omega_n)$ vanishes. One way to alleviate the noise sensitivity problem is to reinforce smoothness on the solution of a LS error problem by the introduction of some desired linear operation $\Omega(\omega_m, \omega_n)$ on $\check{F}(\omega_m, \omega_n)$. To be meaningful, the restoration is constrained by the parameters of problems at hand. Then, what is desired is to find the minimum of a

criterion function, $C(\omega_m, \omega_n)$, defined as:

$$C(\omega_m, \omega_n) = \Omega(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n), \quad (4.7)$$

subject to the constraint that the norms of each side of (4.3) be the same; that is:

$$\|Y(\omega_m, \omega_n) - V(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n)\|^2 = \|E(\omega_m, \omega_n)\|^2. \quad (4.8)$$

Now we can set up the restoration as the problem of minimization of the following performance function:

$$\begin{aligned} W(\check{F}(\omega_m, \omega_n)) &= \|C(\omega_m, \omega_n)\|^2 \\ &+ \lambda \left[\|Y(\omega_m, \omega_n) - V(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n)\|^2 - \|E(\omega_m, \omega_n)\|^2 \right], \end{aligned} \quad (4.9)$$

where λ is a constant called a *Lagrange multiplier*. The ability to specify $\Omega(\omega_m, \omega_n)$ gives us flexibility in setting the goal of the restoration. The solution of the constrained optimization problem is obtained by taking the derivative of $W(\check{F}(\omega_m, \omega_n))$ with respect to $\check{F}(\omega_m, \omega_n)$ and is given by:

$$\begin{aligned} \frac{\partial W(\check{F}(\omega_m, \omega_n))}{\partial \check{F}(\omega_m, \omega_n)} &= 2 \Omega(\omega_m, \omega_n)^* \Omega(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n) \\ &- 2 \lambda V(\omega_m, \omega_n)^* [Y(\omega_m, \omega_n) - V(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n)]. \end{aligned} \quad (4.10)$$

Setting to zero the derivative and solving for $\check{F}(\omega_m, \omega_n)$ yields:

$$\check{F}(\omega_m, \omega_n) = \frac{V(\omega_m, \omega_n)^* Y(\omega_m, \omega_n)}{|V(\omega_m, \omega_n)|^2 + \gamma |\Omega(\omega_m, \omega_n)|^2}, \quad (4.11)$$

where $\gamma = 1/\lambda$ is the *reciprocal Lagrange multiplier* chosen so that the constraint is satisfied.

4.2 Derivation of the Proposed Algorithm

In Chapter 2, we presented a new ultrasound tissue model, which incorporates random fluctuations of the tissue signal within the received ultrasound RF echo signal. The inclusion of the model in the received pulse-echo pressure field reveals that there are two different natures of noise sources existing in the received ultrasonic RF echo. One is inherent in the tissue response and the other is the measurement noise. A desirable ultrasound image can be obtained only when effects of both kinds of noise are suppressed.

In this section, we will be presenting two ultrasound image deconvolution algorithms for improving the quality of the medical ultrasound images. Under the new model, we propose to use a modified version of classical regularization techniques which combine optimal Wiener filter and CLS filtering algorithms for the estimation of the tissue image. We then present an iterative filter scheme that can further improve the solution of the frequency domain deconvolution. The performance of the algorithms is evaluated based on both the simulated phantom images and real ultrasound RF data. The results show that the algorithms can provide improved ultrasound imaging performance in terms of the resolution gain. In the deconvolved images, the speckle is reduced and the presentation is visually more appealing.

A naive deconvolution would be to inverse the smoothing process $v(m, n)$ directly to obtain $s(m, n)$. However, such inverse filtering [91] can perform poorly in the presence of noise because the filter design ignores the noise process. To resolve this issue, Taxt *et al.* [26], [29] resort to the use of the classical regularization techniques [91] in order to reduce the measurement noise effect of cancelling the inverse filter singularities, so as to restore the original noiseless image. Their approach is originated from the signal distortion model in Figure 4.1 which neglects the random fluctuations nature inherent in the tissue image.

Our proposed algorithm arises from the new model showed in the earlier chapter in Figure 2.8 and for convenience it is presented again in Figure 4.2. As indicated, there exists two undesirable terms $d(m, n)$ and $w(m, n)$ in the model which are to be eliminated in restoring the tissue image. Our goal is to eliminate them. One effective method is to remove one of the noise sources with an optimal filter and then estimate the tissue image using constrained optimization techniques. Constrained LS filtering is an approach for overcoming difficulties associated with inverse filter, while retaining the simplicity of using a single linear space-invariant filter for restoring the image.

Figure 4.2: Block diagram of the proposed ultrasound image formation process.

If all the noise sources are omitted momentarily, we have $y_{fn}(m, n) = y(m, n)$. A good restoration should give a blurred estimate approximately given by:

$$v(m, n) * \check{f}(m, n) = y_{fn}(m, n), \quad (4.12)$$

where $\check{f}(m, n)$ denotes an estimate solution of $f(m, n)$. When an inverse filter is used, this approximation is made exact which causes problem when there are randomness in $y_{fn}(m, n)$, i.e., $v(m, n) * d(m, n)$, because the inverse filter tries to get an exact fit to the noisy data. Another interpretation is to consider the signal $v(m, n) * d(m, n)$ as an uncertainty on $v(m, n)$. A reasonable expectation for the

restoration which accounts for the noise term is that it satisfies the relation:

$$\|y_{f_n}(m, n) - v(m, n) * \check{f}(m, n)\| = \|v(m, n) * d(m, n)\|. \quad (4.13)$$

This means that we are selecting $\check{f}(m, n)$ so that if it is blurred by $v(m, n)$ the result will differ from $y_{f_n}(m, n)$ by as little as possible in the mean square sense.

Now, we want to set up a constraint minimization problem where we select a criterion function $\Omega(m, n)$ to define some desired linear operations on $\check{f}(m, n)$. The observation noise $w(m, n)$ is a white noise occupies that the whole spectrum. Prior knowledge of PSF tells us that its axial and lateral frequency spectrums are bandpass and lowpass respectively. Restoration with an inverse filter will have the tendency of emphasizing the out-of-band noise. Thus, we are looking for a criterion function which emphasizes the restoration of passband signals while deemphasizes those out-of-band signals. With this idea in mind, we have considered $\Omega(m, n) = q(m, n) * \check{f}(m, n)$, where $q(m, n) = (-1)^{m+n} v(m, n)$, for a trade-off between errors caused by regularization and noise magnification [92]. This operator is essentially the quadrature mirror filter of $v(m, n)$. Such an operator shows that the out-of-band signals which are essentially noise are suppressed. Another useful operator of our interest is the highpass convolution kernel which is the discrete approximation to the *Laplacian* operation:

$$q(m, n) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (4.14)$$

Because $y_{f_n}(m, n)$ and $d(m, n)$ in (4.13) are not available, in the following we will elaborate a procedure leading to an expression satisfying the constraint.

Let $z(m, n)$ account all undesirable signals, $d(m, n)$ and $w(m, n)$, written as:

$$\begin{aligned} z(m, n) &= y(m, n) - v(m, n) * f(m, n) \\ &= v(m, n) * d(m, n) + w(m, n). \end{aligned} \quad (4.15)$$

This representation has the form of the signal distortion model of Figure 4.1. We have explained earlier that $d(m, n)$ is an unwanted stationary signal. Hence, we can minimize the mean square error $E\{[d(m, n) - \check{d}(m, n)]^2\}$ to obtain an approximation of it given by $\check{d}(m, n) = h(m, n) * z(m, n)$ where $h(m, n)$ is the Wiener filter:

$$h(m, n) = F^{-1} \left[\frac{V^*(\omega_m, \omega_n)}{|V(\omega_m, \omega_n)|^2 + \beta_0} \right], \quad (4.16)$$

$V(\omega_m, \omega_n)$ is the 2-D FT of $v(m, n)$, i.e., $V(\omega_m, \omega_n) = F[v(m, n)]$, $*$ denotes complex conjugate, and β_0 is the power spectrum ratio of $w(m, n)$ against that of $d(m, n)$. In practice, β_0 can be estimated as $\max\left(0, \frac{\sigma_w^2}{\sigma_y^2 - \sigma_w^2}\right)$ from a homogeneous region where σ_w^2 is the measurement noise variance obtained via calibration and σ_y^2 is the variance of $y(m, n)$. For the case of imaging in a homogeneous region, it may be assigned as a constant value.

The Wiener filter makes use of MSE criterion which weights all errors equally, regardless of their locations in the image, while the eye is considerably more tolerant of errors in dark areas and high-gradient areas than elsewhere. In minimizing the MSE, it smoothes the image more than the eye would prefer. Hence, the MSE criterion of optimality is not particularly good for restoration of image for the human eye. Thus, the generalization of the Wiener filter may be adopted [91]:

$$H(\omega_m, \omega_n) = \left[\frac{V^*(\omega_m, \omega_n)}{|V(\omega_m, \omega_n)|^2} \right]^\alpha \left[\frac{V^*(\omega_m, \omega_n)}{|V(\omega_m, \omega_n)|^2 + \beta_0 \beta_1} \right]^{1-\alpha}, \quad (4.17)$$

where $H(\omega_m, \omega_n) = F[h(m, n)]$, α and β_1 are positive real constants. The transfer function in (4.17) is parameterized by these two parameters. When $\alpha = 0$ this

filter reduces to the parametric Wiener filter, which further turns into the standard Wiener filter when $\beta_1 = 1$. With $\alpha = 1$ this filter reduces to the inverse filter. Thus, when α falls between 0 and 1, the behaviour of the filter deviates in between these two filters. Also, when $\alpha = 1/2$ and $\beta_1 = 1$, the filter is commonly referred to as the spectrum equalization filter. The motivation for generalizing the filter lies in the belief that, in 2-D signal processing, the best filter is not necessarily a single filter.

With $\check{d}(m, n)$ in place of $d(m, n)$, an approximation of $y_{f_n}(m, n)$ is obtained as follows:

$$\check{y}_{f_n}(m, n) = v(m, n) * [f(m, n) + \check{d}(m, n)]. \quad (4.18)$$

Thus, by introducing into the minimization the constraint that the norms of each side of (4.18) be the same; we arrived at an approximation of (4.13):

$$\|\check{y}_{f_n}(m, n) - v(m, n) * \check{f}(m, n)\| = \|v(m, n) * \check{d}(m, n)\|. \quad (4.19)$$

The removal of $\check{d}(m, n)$ is accomplished by introducing the Lagrange multiplier, λ ; which is an approach used in an attempt to enforce prior knowledge on the solution of a LS error problem. The constrained LS solution is such that the restoration $\check{f}(m, n)$ is chosen to minimize the criterion function $\|\Omega(m, n)\|^2$ subject to the condition that $\|\check{y}_{f_n}(m, n) - v(m, n) * \check{f}(m, n)\|^2 = \|v(m, n) * \check{d}(m, n)\|^2$ [91], [92]. Then, using the method of Lagrange multipliers gives the estimated noiseless tissue image as:

$$\check{f}(m, n) = \arg \min_{\check{f}} \left\{ \|\Omega(m, n)\|^2 + \lambda \left(\|\check{y}_{f_n}(m, n) - v(m, n) * \check{f}(m, n)\|^2 - \|v(m, n) * \check{d}(m, n)\|^2 \right) \right\}. \quad (4.20)$$

Using $\check{d}(m, n) = h(m, n) * z(m, n)$ and substituting (4.15) into (4.18), we have:

$$\begin{aligned} \check{f}_{f_n}(m, n) - v(m, n) * \check{f}(m, n) &= v(m, n) * h(m, n) * \\ &[y(m, n) - v(m, n) * f(m, n)]. \end{aligned} \quad (4.21)$$

From (4.21), we observe that (4.20) has the following representation:

$$\begin{aligned} \check{f}(m, n) &= \arg \min_{\check{f}} \left\{ \|\Omega(m, n)\|^2 + \lambda \left(\|v(m, n) * h(m, n) * \right. \right. \\ &\quad \left. \left. [y(m, n) - v(m, n) * \check{f}(m, n)]\|^2 - \|v(m, n) * \check{d}(m, n)\|^2 \right) \right\}. \end{aligned} \quad (4.22)$$

For simplicity, the derivations from this point onwards are carried out in the frequency domain. Using the spectrum convention established earlier and $Q(\omega_m, \omega_n) = F[q(m, n)]$, it follows that $V(\omega_m, \omega_n)$, $\check{D}(\omega_m, \omega_n)$, and $F[\Omega(m, n)] = Q(\omega_m, \omega_n) \times \check{F}(\omega_m, \omega_n)$ are the spectrum of $v(m, n)$, $\check{d}(m, n)$, and $\Omega(m, n)$ respectively. Now, we let $Y'(\omega_m, \omega_n) = V(\omega_m, \omega_n) H(\omega_m, \omega_n) Y(\omega_m, \omega_n)$ and $V'(\omega_m, \omega_n) = V(\omega_m, \omega_n) \times H(\omega_m, \omega_n) V(\omega_m, \omega_n)$, (4.22) is equivalent to the minimization of the following cost function:

$$\begin{aligned} W(\check{F}(\omega_m, \omega_n)) &= \|Q(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n)\|^2 + \lambda \left(\|Y'(\omega_m, \omega_n) - \right. \\ &\quad \left. V'(\omega_m, \omega_n) \check{F}(\omega_m, \omega_n)\|^2 - \|V(\omega_m, \omega_n) \check{D}(\omega_m, \omega_n)\|^2 \right). \end{aligned} \quad (4.23)$$

To make the equation easier to read, we will discard all the associated discrete spatial frequency variables in the notations of (4.23) and expanded it as follows:

$$\begin{aligned} W(\check{F}) &= (Q\check{F})^* (Q\check{F}) + \\ &\lambda \left[(Y' - V'\check{F})^* (Y' - V'\check{F}) - (V\check{D})^* (V\check{D}) \right]. \end{aligned} \quad (4.24)$$

Minimizing the cost function with respect to $\check{F}$ is accomplished by taking the deriva-

tive of $W(\check{F})$ with respect to $\check{F}$:

$$\frac{\partial W(\check{F})}{\partial \check{F}} = 2Q^*Q\check{F} - 2\lambda V'^* (Y' - V'\check{F}). \quad (4.25)$$

Setting to zero the derivative and solving for $\check{F}$ yields:

$$\check{F}(\omega_m, \omega_n) = \frac{V(\omega_m, \omega_n)^* Y(\omega_m, \omega_n)}{|V(\omega_m, \omega_n)|^2 + \gamma \left| \frac{Q(\omega_m, \omega_n)}{V(\omega_m, \omega_n) H(\omega_m, \omega_n)} \right|^2}, \quad (4.26)$$

where $\check{F}(\omega_m, \omega_n) = F[\check{f}(m, n)]$, $\gamma = 1/\lambda$ is the reciprocal Lagrange multiplier chosen so that the constraint $\|\check{y}_{f_n}(m, n) - v(m, n) * \check{f}(m, n)\|^2 = \|v(m, n) * \check{d}(m, n)\|^2$ is satisfied [92]. Equation (4.26), if rearranged, can show to arrive at the following expression:

$$\check{F}(\omega_m, \omega_n) = \frac{V(\omega_m, \omega_n)^* Y(\omega_m, \omega_n)}{|V(\omega_m, \omega_n)|^2 + \gamma \left| Q(\omega_m, \omega_n) + \beta_0 \frac{Q(\omega_m, \omega_n)}{|V(\omega_m, \omega_n)|^2} \right|^2}. \quad (4.27)$$

Equation (4.27) is in a form of the classical constrained LS filter for the classical signal distortion model in Figure 4.1 with $\beta_0 = 0$. The introduction of the term $\beta_0 Q(\omega_m, \omega_n)/|V(\omega_m, \omega_n)|^2$ can be seen as a mechanism to alleviate the presence of undesirable signal $d(m, n)$. Sekko *et al.* has arrived at a same expression with slight different formulation and interpretation in [93].

For the proposed algorithm in (4.26), it is possible to obtain an estimation of $y_{f_n}(m, n)$ with the Wiener filter of the form $F^{-1}[|V(\omega_m, \omega_n)|^2 / (|V(\omega_m, \omega_n)|^2 + \beta_2)]$ applied on $y(m, n)$, where β_2 is the power spectrum ratio of $w(m, n)$ against the sum of that of $f(m, n)$ and $d(m, n)$. Unfortunately, as this does not omit the need to search for γ which meets the constraint; and for simplicity, one may desire to find a value for γ which results in the restoration to be appealing to a human from

the perceptual viewpoint. Besides, the fact is that the variance of the noisy tissue varying across the whole image, it is therefore not possible for a single value of γ producing optimal results across the whole image. Alternatively, one may look for a value of γ best satisfying the constraint through searching within a preset range. These two ways of fine tuning γ are tedious and computationally expensive. In the following section, we propose to use an iterative filtering algorithm to overcome this problem.

4.3 An Iterative Filtering Scheme

In the above section, fine tuning of γ is necessary to arrive at the optimal result. Particularly, we want to make the algorithm less reliant on the chosen γ . In the actual implementation, the ultrasound RF data is prolonged in the axial direction and the whole piece of RF data will be segmented into various overlapping segments. Since the physical properties of tissue may be different for different regions, the desired γ will be different for the different segments. These procedures are tedious and computationally expensive. In this section, we propose to use an iterative filtering algorithm to overcome this problem. This algorithm can be shown to progressively lead to better result.

Let the estimated tissue image during the i th iteration be $\check{\check{F}}_i(\omega_m, \omega_n) = F(\omega_m, \omega_n) - \Delta F_i(\omega_m, \omega_n)$. Applying the deconvolution algorithm derived in (4.26) iteratively, the error at the i th iteration being the difference between the observation and the re-blurred i th estimate and is given by:

$$\begin{aligned} \epsilon_i(\omega_m, \omega_n) &= Y(\omega_m, \omega_n) - V(\omega_m, \omega_n) \check{\check{F}}_i(\omega_m, \omega_n) \\ &= V(\omega_m, \omega_n) [\Delta F_i(\omega_m, \omega_n) + D(\omega_m, \omega_n)] + W(\omega_m, \omega_n). \end{aligned} \quad (4.28)$$

As shown, (4.28) is of the same form as our proposed new model in Figure 4.2. with

$F(\omega_m, \omega_n)$ replaced by $\Delta F_i(\omega_m, \omega_n)$. Therefore, we may deduce an estimation of $\Delta F_i(\omega_m, \omega_n)$ following the form of (4.26) and is given as:

$$\Delta \check{F}_i(\omega_m, \omega_n) = \frac{V(\omega_m, \omega_n)^* \epsilon_i(\omega_m, \omega_n)}{|V(\omega_m, \omega_n)|^2 + \gamma \left| \frac{Q(\omega_m, \omega_n)}{V(\omega_m, \omega_n) H(\omega_m, \omega_n)} \right|^2}. \quad (4.29)$$

Then, we can update our estimation of $F(\omega_m, \omega_n)$ as follows:

$$\check{F}_{i+1}(\omega_m, \omega_n) = \check{F}_i(\omega_m, \omega_n) + \Delta \check{F}_i(\omega_m, \omega_n). \quad (4.30)$$

Because $\epsilon_{i+1}(\omega_m, \omega_n) = Y(\omega_m, \omega_n) - V(\omega_m, \omega_n) \check{F}_{i+1}(\omega_m, \omega_n)$, by simple substitution of (4.30) into it, we obtain the following relation between the errors at i th and $(i+1)$ th iterations:

$$\begin{aligned} \epsilon_{i+1}(\omega_m, \omega_n) &= \frac{\gamma \left| \frac{Q(\omega_m, \omega_n)}{V(\omega_m, \omega_n) H(\omega_m, \omega_n)} \right|^2 \epsilon_i(\omega_m, \omega_n)}{|V(\omega_m, \omega_n)|^2 + \gamma \left| \frac{Q(\omega_m, \omega_n)}{V(\omega_m, \omega_n) H(\omega_m, \omega_n)} \right|^2} \\ &= \left[\frac{\gamma \left| \frac{Q(\omega_m, \omega_n)}{V(\omega_m, \omega_n) H(\omega_m, \omega_n)} \right|^2}{|V(\omega_m, \omega_n)|^2 + \gamma \left| \frac{Q(\omega_m, \omega_n)}{V(\omega_m, \omega_n) H(\omega_m, \omega_n)} \right|^2} \right]^i \epsilon_1(\omega_m, \omega_n). \end{aligned} \quad (4.31)$$

Since $|V(\omega_m, \omega_n)|^2 \geq 0$, $|Q(\omega_m, \omega_n)|^2 \geq 0$ and $\gamma|H(\omega_m, \omega_n)|^2 \geq 0$, we know that $\|\epsilon_{i+1}(\omega_m, \omega_n)\|^2 \leq \|\epsilon_i(\omega_m, \omega_n)\|^2$ and hence the error reduces with an increases in the number of iterations. Ideally, one should stop the iteration when $\|\epsilon_{i+1}(\omega_m, \omega_n)\|^2 \approx \|V(\omega_m, \omega_n) D(\omega_m, \omega_n)\|^2 + \|W(\omega_m, \omega_n)\|^2$. These features have a subtle effect in obtaining an optimal value for γ and the iterative process. Since error reduces along with iteration, we can therefore set γ to a large value so that the initial error is large and the iteration is terminated once the stopping condition is reached which will result in an optimal result.

In practice, the true PSF, $V(\omega_m, \omega_n)$, is not known. Under this circumstance, its estimation $\check{V}(\omega_m, \omega_n)$ is used in equations (4.26) and (4.29). This substitution imposes an unavoidable error to the estimation which can only be alleviated by the use of a better PSF estimation algorithm. In the following, we elaborate the accuracy of the iterative algorithm when it is equipped with the estimated PSF.

Let the estimated PSF deviate from the true PSF by $\Delta V(\omega_m, \omega_n)$ such that $\check{V}(\omega_m, \omega_n) = V(\omega_m, \omega_n) - \Delta V(\omega_m, \omega_n)$. Thus, in the proposed iterative deconvolution algorithm, the error at the i th iteration is given by:

$$\begin{aligned}\epsilon_i(\omega_m, \omega_n) &= Y(\omega_m, \omega_n) - \check{V}(\omega_m, \omega_n) \check{F}_i(\omega_m, \omega_n) \\ &= \check{V}(\omega_m, \omega_n) [\Delta F_i(\omega_m, \omega_n) + D(\omega_m, \omega_n)] + \\ &\quad \Delta V(\omega_m, \omega_n) [F(\omega_m, \omega_n) + D(\omega_m, \omega_n)] + W(\omega_m, \omega_n).\end{aligned}\quad (4.32)$$

In this case, if we let $\zeta_i(\omega_m, \omega_n) = \epsilon_i(\omega_m, \omega_n) - \Delta V(\omega_m, \omega_n) [F(\omega_m, \omega_n) + D(\omega_m, \omega_n)]$, we will have:

$$\zeta_i(\omega_m, \omega_n) = \check{V}(\omega_m, \omega_n) [\Delta F_i(\omega_m, \omega_n) + D(\omega_m, \omega_n)] + W(\omega_m, \omega_n). \quad (4.33)$$

Equation (4.33) is again of the same form as our proposed new model in Figure 4.2 with $V(\omega_m, \omega_n)$ and $F(\omega_m, \omega_n)$ replaced by $\check{V}(\omega_m, \omega_n)$ and $\Delta F_i(\omega_m, \omega_n)$ respectively. In this spirit, we may deduce an estimation of $\Delta F_i(\omega_m, \omega_n)$ following the form of (4.29) with $\zeta_i(\omega_m, \omega_n)$ in place of $\epsilon_i(\omega_m, \omega_n)$. Thus, the estimated tissue signal is updated as follows:

$$\begin{aligned}
\check{\check{F}}_{i+1}(\omega_m, \omega_n) &= \check{\check{F}}_i(\omega_m, \omega_n) + \Delta \check{\check{F}}_i(\omega_m, \omega_n) \\
&= \check{\check{F}}_i(\omega_m, \omega_n) + \\
&\quad \frac{\check{V}(\omega_m, \omega_n)^* \xi_i(\omega_m, \omega_n)}{|\check{V}(\omega_m, \omega_n)|^2 + \gamma \left| \frac{Q(\omega_m, \omega_n)}{\check{V}(\omega_m, \omega_n) H(\omega_m, \omega_n)} \right|^2} \\
&= \check{\check{F}}_1(\omega_m, \omega_n) + \\
&\quad \frac{\check{V}(\omega_m, \omega_n)^* \xi_i(\omega_m, \omega_n)}{|\check{V}(\omega_m, \omega_n)|^2 + \gamma \left| \frac{Q(\omega_m, \omega_n)}{\check{V}(\omega_m, \omega_n) H(\omega_m, \omega_n)} \right|^2},
\end{aligned} \tag{4.34}$$

where $\check{\check{F}}_1(\omega_m, \omega_n) = \check{F}(\omega_m, \omega_n)$ and $\xi_i(\omega_m, \omega_n)$ is given as follows:

$$\xi_i(\omega_m, \omega_n) = \sum_{j=1}^i \epsilon_j(\omega_m, \omega_n) - i \times \Delta V(\omega_m, \omega_n) [F(\omega_m, \omega_n) + D(\omega_m, \omega_n)]. \tag{4.35}$$

When there is no error in the 2-D PSF estimation, we will have $\Delta V(\omega_m, \omega_n) = 0$. Thus, equation (4.35) reveals that an unavoidable error occurred in the iterative filtering algorithm due to an inaccuracy in the estimation of the 2-D PSF. This error appears as terms on the right hand side of (4.35) associated with $\Delta V(\omega_m, \omega_n)$ and it increases along with the number of iterations.

If we can have an estimated 2-D PSF which is incorporated into the iterative algorithm as in (4.29) such that $\Delta V(\omega_m, \omega_n)$ is small, the iterative algorithm will perform according to (4.31). Under this condition, it will have the characteristic to produce ultrasound images that improve in quality progressively in time. Hence, in the application of ultrasound diagnosis; whenever close examination of the patient is desired, the iterative scheme is triggered once the ultrasound transducer is positioned stationary.

4.4 Simulation Results

Computer simulation is carried out to study the performance of the proposed algorithms. We design a simple ultrasound tissue phantom which has an object embedded in it and it is insonified under a severe scattering condition. Using simulated ultrasound data has an additional advantage that it facilitates the comparison between algorithms. The evaluation can be repeated many times with little efforts and the results can be easily compiled. The generation of the simulated ultrasound RF data is described in Appendix B. It consists of a lower scattering object embedded in a homogeneous background region.

The sampled RF data of the receiving ultrasound echo signal $y(m, n)$ has a dimension of 796 x 360. All the dimensions are specified in sample numbers, unless specifically stated otherwise. In the following, we describe the parameter settings for the 2-D PSF estimation and the proposed CLS filter. The same settings are used in the iterative algorithm (ICLS filter). These settings are arrived at empirically and they are also used later for the real ultrasound RF data.

Each 2-D PSF estimate is based on sampled RF data from a block of 32 x 64. This setting is obtained based on a prior knowledge of the ultrasonic system settings which is large enough to accommodate the spatial extent of the 2-D PSF throughout different imaging depth. The RF images are then segmented into various rectangular blocks of 70 % overlapping. Among them, thirty blocks that have envelope SNR closer to 1.91 are undergone the 2-D PSF estimation.

The estimation of the 2-D PSF is accomplished through the cepstral domain using a 2-D Butterworth lowpass lifter as described in Chapter 3. The parameters of the Butterworth lifter are set to an order of 5 and lowpass cutoffs of 0.5 and 0.7 from the origin of the axial and lateral frequency axes respectively. The quality map in the weighted LS phase unwrapping algorithm is set to 100 % and 30 % corresponding to

the axial and lateral axes. The estimated PSF are averaged to improve the robustness of the estimation. In the studies, we adopt the complex cepstrum method that does not decompose signal into all-pass and minimum-phase components when computing the complex cepstrum as we did not observe improvement in the estimation during our investigations in Chapter 3.

Two regions of interests (ROI) are identified on the simulated ultrasound image of 3 cm by 3 cm as shown in Figure 4.3(a). These two ROIs define two homogeneous regions of 100 by 80 in the simulated tissue phantom. ROI 1 is located at the center of the cylindrical disc whereas ROI 2 is situated in the scattering background region. To assess the performance of the proposed algorithm, we studied the performance metric used by many authors, e.g., Abeyratne *et al.* [21], [23], Taxt *et al.* [20], [29], and Suiren Wan *et al.* [28]. We have adopted the speckle signal-to-noise ratio (SNR), and contrast ratio (CR), and resolution gain (RG) in our studies.

Speckle SNR is computed by taking the mean value of the image against its standard deviation whereas the CR is defined as $|20 \log_{10}(\mu_1/\mu_2)|$ in decibel where μ_1 and μ_2 are the mean value of ROI 1 and ROI 2 respectively. Since the ability of an object to be detected depends of the mean brightness difference between the object and its surrounding background, and also the noise level, it is straightforward to see that detectability improves with higher CR and higher speckle SNR. Hence, the desirable improvement in the quality of the deconvolved ultrasound image is a high CR accompanied with a high SNR.

Following the reasoning given by Abeyratne *et al.* in [21], a reduction in the size of speckle suggesting a gain in the resolution. In order to assess the apparent increase in the resolution, the -5 dB and -10 dB correlation lengths of the RF data corresponding to the images are computed before and after deconvolution. To facilitate the computation, the 2-D autocovariance function (ACF) is computed based on a size of 100 by 80 located at the center of the data. The RG is defined as

the ratio of the correlation lengths of the RF data before and after deconvolution. Defining the resolution at two different dB levels allow us to have a better idea of the shape of the 2-D ACF. These computations are performed on both the axial slice, A_{00} , and lateral slice, L_{00} , where they are extracted through the peak of the 2-D ACF. Similar measures have been used previously by other researchers [21], [26], [28], [29].

Although these metrics are commonly used by researchers [21], [23], [24], [26], [28], [29], we observe that they cannot supercede the visual evaluation by human observers. Based on our experience, the reduction of speckle size, which improves the discrimination of fine details in the ultrasound image during diagnostic examination, can hardly be quantified. While there is no metric to best quantify this improvement, visual inspection is necessary.

One hundred simulated ultrasound RF data are generated and the evaluation results are compiled in Table 4.1 and Table 4.2. These are results of the one hundred simulated data reported in the form of the statistical mean value with standard deviation of one, i.e., mean value $\pm$ one standard deviation. Both tables compare the performances of the CLS filter and the reference algorithm (Wiener filter). In the evaluations, the noise-to-signal ratio of the Wiener filter is set to 10^{-2} which is chosen interactively to yield the best visual results, and then a range of values of γ is examined centered about this value used together with the default settings of all other parameters, i.e., $\alpha = 0$, $\beta_1 = 1$, and γ of 10^{-3} , 10^{-2} , and 10^{-1} . Table 4.1 reveals that there is hardly any improvement in speckle SNR with both filters. This is also observed and explained in an early work [29]. The explanation given for the speckle SNR being unchanged by the deconvolution is that the speckle SNR is based on the variance of the envelope values within a uniform region. It is not related to the prevailing spatial frequencies of the speckle pattern, which is changed by deconvolution [29].

Moreover, the results show that the CR of the CLS filtered images is approximately the same as the original images. Preservation of the CR is essentially to ensure the visibility of the embedded tissue structures. In comparison, the resulting CR of the Wiener solution has deteriorated.

Table 4.1: Performance of the proposed CLS filter with the various settings of γ based on the one hundred simulated RF data. These values are tabulated in terms of mean $\pm$ one standard deviation.

Description	γ	Speckle SNR		CR (dB)
		ROI 1	ROI 2	
Original	-	1.3240 ± 0.0500	1.3221 ± 0.0509	11.9194 ± 0.8648
Wiener	-	1.3185 ± 0.0326	1.3215 ± 0.0232	10.5192 ± 0.6151
CLS filter	10^{-3}	1.3292 ± 0.0423	1.3385 ± 0.0811	11.5742 ± 0.9426
	10^{-2}	1.3297 ± 0.0400	1.3357 ± 0.0965	11.6554 ± 0.9434
	10^{-1}	1.3261 ± 0.0388	1.3495 ± 0.1365	11.5744 ± 1.1737

Table 4.2 tabulates the results in RG. The results of both filters show improvement and their readings are comparable. The logarithmic compression of the original ultrasound envelope and the deconvolved envelopes are presented in Figure 4.3. The deconvolved images are scaled using logarithmic pixel transformation for displaying because the physiological response of the eye to light intensity is itself logarithmic.

This practice has also been adopted by Taxt *et al.* [20], [29], Suiren Wan *et al.* [28], and Abeyratne *et al.* [21], [23].

Table 4.2: Resolution gains obtained due to deconvolution using the CLS filter computed in the RF domain based on the one hundred simulated RF data. These values are tabulated in terms of mean $\pm$ one standard deviation.

Description	γ	Axial slice, A_{00}		Lateral slice, L_{00}	
		RG _{-5dB}	RG _{-10dB}	RG _{-5dB}	RG _{-10dB}
Wiener	-	2.9692 $\pm$	1.3796 $\pm$	1.3327 $\pm$	1.3236
		0.5562	0.6570	0.1166	0.1589
CLS filter	10^{-3}	2.2878 $\pm$	1.7875 $\pm$	1.3335 $\pm$	1.3601 $\pm$
		0.2956	0.8532	0.1141	0.1534
	10^{-2}	2.1095 $\pm$	1.8812 $\pm$	1.2997 $\pm$	1.3262 $\pm$
		0.1843	0.6879	0.1048	0.1440
	10^{-1}	1.8917 $\pm$	1.8895 $\pm$	1.2589 $\pm$	1.2874 $\pm$
		0.1825	0.5072	0.0905	0.1266

As shown, the size of the granular patterns in both Figure 4.3(c) and Figure 4.3(d) has reduced. Furthermore, the embedded object is better defined with the CLS filter than in both the original image and Wiener solution. The borders of the cylindrical disc of both the original image and Wiener solution are smeared and more difficult to detect. In particular, the Wiener solution is also more noisy. These subjective observations in the Wiener solution seem to indicate that factors of its improvement in RG are because that it is being more noisy and that RG is also related to the degree of decorrelation.

Figure 4.3: Deconvolution of the simulated ultrasound image using CLS filter. (a) Description of the image, (b) Simulated ultrasound image, (c) Wiener solution, and (d) CLS filtered image, $\gamma = 10^{-2}$.

Next, the performance of the proposed ICLS filter is studied. The filtering algorithm requires the use of (4.17), (4.26), (4.28), (4.29), and (4.30). Since the 2-D PSF is obtained through estimation, the 2-D PSF $V(\omega_m, \omega_n)$ in (4.17), (4.26), (4.28), and (4.29) are substituted by its estimation $\check{V}(\omega_m, \omega_n)$.

The maximum number of iteration is set to 30 and the results at iteration 10, 20, and 30 are collected and used for investigations. The results of both the speckle SNR and CR are tabulated in Table 4.3 whereas the resulting RG is presented in Table 4.4. The behavior of the algorithm did not work according to our expectation. We expect that a better performance is possible with a larger value of γ to work inline with the formulation of the iterative algorithm since a larger setting of γ allows estimation of the tissue error to be computed under a finer step size of increment. The estimations are expected to improve along with the iterations for an initial error larger than $\|V(\omega_m, \omega_n) D(\omega_m, \omega_n)\|^2 + \|W(\omega_m, \omega_n)\|^2$ and the result should converge to an optimal deconvolved value. Besides, rather than delivering a progressively improved result, the results tabulated in Table 4.3 and Table 4.4 for the various settings of γ are comparable. We believe that these deviations from the designated behaviors are caused by the use of the estimated 2-D PSF. This behavior is, in fact, in accordance with our analysis in Section 4.3 where we have seen in (4.34) and (4.35) that an introduction of the estimated PSF will incur an unavoidable error which can only be reduced with better PSF estimation methods. This is not surprising as inaccuracy of the estimation always gives error in the final result.

Table 4.3: Performance of the proposed ICLS filter with the various settings of γ based on the one hundred simulated RF data. These values are tabulated in terms of mean $\pm$ one standard deviation.

γ	Iteration number	Speckle SNR		CR (dB)
		ROI 1	ROI 2	
10^{-3}	10	1.3290 ± 0.0384	1.3247 ± 0.0353	11.4962 ± 0.5407
	20	1.3275 ± 0.0371	1.3213 ± 0.0342	11.4471 ± 0.5554
	30	1.3262 ± 0.0363	1.3200 ± 0.0336	11.4053 ± 0.5701
10^{-2}	10	1.3312 ± 0.0420	1.3302 ± 0.0387	11.6602 ± 0.4817
	20	1.3312 ± 0.0395	1.3292 ± 0.0368	11.6144 ± 0.4821
	30	1.3306 ± 0.0392	1.3283 ± 0.0361	11.5863 ± 0.4926
10^{-1}	10	1.3320 ± 0.0397	1.3214 ± 0.0370	11.7634 ± 0.5232
	20	1.3340 ± 0.0407	1.3229 ± 0.0377	11.7483 ± 0.5083
	30	1.3343 ± 0.0410	1.3245 ± 0.0380	11.7289 ± 0.5010

Table 4.4: Resolution gains obtained due to deconvolution with the proposed ICLS filter computed in the RF domain based on the one hundred simulated RF data. These values are tabulated in terms of mean $\pm$ one standard deviation.

γ	Iteration number	Axial slice, A_{00}		Lateral slice, L_{00}	
		RG _{-5dB}	RG _{-10dB}	RG _{-5dB}	RG _{-10dB}
10^{-3}	10	2.5774 $\pm$	1.5637 $\pm$	1.3746 $\pm$	1.4033 $\pm$
		0.3697	0.8938	0.1110	0.1483
	20	2.6691 $\pm$	1.4657 $\pm$	1.3809 $\pm$	1.4080 $\pm$
		0.3370	0.8692	0.1123	0.1508
	30	2.6746 $\pm$	1.4525 $\pm$	1.3842 $\pm$	1.4104 $\pm$
		0.4286	0.8914	0.1128	0.1517
10^{-2}	10	2.3671 $\pm$	1.7456 $\pm$	1.3477 $\pm$	1.3771 $\pm$
		0.2688	0.8938	0.1151	0.1536
	20	2.4355 $\pm$	1.7473 $\pm$	1.3601 $\pm$	1.3906 $\pm$
		0.2792	0.9389	0.1119	0.1491
	30	2.4756 $\pm$	1.7074 $\pm$	1.3650 $\pm$	1.3954 $\pm$
		0.2883	0.9019	0.1115	0.1484
10^{-1}	10	2.1713 $\pm$	1.9014 $\pm$	1.3154 $\pm$	1.3439 $\pm$
		0.1786	0.7399	0.1081	0.1473
	20	2.2363 $\pm$	1.7960 $\pm$	1.3262 $\pm$	1.3549 $\pm$
		0.1814	0.7715	0.1101	0.1487
	30	2.2410 $\pm$	1.8076 $\pm$	1.3325 $\pm$	1.3614 $\pm$
		0.2901	0.8236	0.1111	0.1493

4.5 Experimental Results

In this section, we present the visual improvement in the deconvolved images of the real ultrasound RF data using the proposed algorithms. A total of eight different real ultrasound RF data are investigated in this study. Their acquisitions are depicted in Appendix C.

In the simulation, we simulate a spatially invariant 2-D PSF so that the quality of the PSF estimation can be examined as we have already elaborated in Chapter 3. However, this is not the case for the real ultrasound RF data. The 2-D PSF is expected to vary to some extent. In order to speed up the computation of the algorithm, there exists a distinct treatment in the real ultrasound data. In this case, the whole piece of ultrasound RF data is segmented into various overlapping segments and each of these segments are considered to be the convolved result of a spatially invariant 2-D PSF and the noisy tissue image. Since the sample RF data has a size of 1024 by 301, which is prolonged in the axial direction, the segmentation is only performed in the axial direction. Each of these segments is then deconvolved by the respective estimated 2-D PSF and their results are then concatenated to form the final deconvolved result. Furthermore, the Laplacian operator in (4.14) is also used in the experimental data.

Likewise, the evaluations of the experimental results are also summarized in terms of mean value $\pm$ one standard deviation. In the experimental data, CR is not computed mainly because of the difficulties in identifying two homogeneous regions of similar characteristics in every data set.

The speckle SNR computed over the entire region of the original ultrasound envelopes is 1.3600 ± 0.0630 whereas that of the deconvolved data using the CLS filter, ICLS filter with 10, 20, and 30 iterations, are computed to be 1.4598 ± 0.0851 , 1.4914 ± 0.1083 , 1.5022 ± 0.1203 , and 1.5095 ± 0.1246 respectively. Therefore, we

conclude that there is a progressively slight improvement in the speckle SNR with the number of iterations.

For the experimental data, the 2-D ACF is computed based on a smaller size of 64 by 64 located at the center of the data. The surface plot of a 2-D ACF is depicted in Figure 4.4. The computations of the RG on both the A_{00} and L_{00} where they are extracted through the peak of the 2-D ACF are illustrated in Figure 4.5.

Figure 4.4: Surface plot of the 2-D autocovariance function. (a) Original, and (b) deconvolved 2-D ACF.

Table 4.5 presents the evaluation results of the axial and lateral resolution gains. The results indicate an improvement in the resolutions of the deconvolved data. This is a possible indication of decorrelation of the deconvolved data which means that deblurring has probably occurred. Since the deconvolved data is deblurred, the resultant images will have higher resolution. Moreover, we observe that there is also a progressively higher resolution gains as the number of iterations increased. In this light, both the CLS and ICLS filters show promising results.

Table 4.5: Resolution gains obtained due to deconvolution using the proposed CLS and ICLS filters computed in the RF domain. These values are tabulated in terms of mean $\pm$ one standard deviation.

Description	Axial slice, A_{00}		Lateral slice, L_{00}	
	RG _{-5dB}	RG _{-10dB}	RG _{-5dB}	RG _{-10dB}
CLS filter	1.7214 $\pm$ 0.2266	1.9341 $\pm$ 0.5954	1.3151 $\pm$ 0.1654	1.3671 $\pm$ 0.2036
ICLS filter, 10 iterations	1.8330 $\pm$ 0.2372	2.1635 $\pm$ 0.4732	1.4461 $\pm$ 0.2555	1.5091 $\pm$ 0.3069
ICLS filter, 20 iterations	1.8565 $\pm$ 0.2548	2.1923 $\pm$ 0.4925	1.4784 $\pm$ 0.2857	1.5465 $\pm$ 0.3421
ICLS filter, 30 iterations	1.8679 $\pm$ 0.2651	2.2047 $\pm$ 0.5040	1.4999 $\pm$ 0.3068	1.5710 $\pm$ 0.3675

As shown in Figure 4.6(a), there exists a granular pattern masked on the ultrasound image. The Wiener solution in Figure 4.6(b) produces an image with higher resolution. Unfortunately, the ringing artefacts are quite obvious as can be seen in the top portion of the image. Figure 4.6(c) and (d) show the deconvolved images using the proposed algorithms. The blurring which is more noticeable in the far field of the original images is corrected.

Besides, the Wiener solution appears to be more noisy whereas there is a better definition of the tissue structures and blood vessels in Figure 4.6(c) and (d). As shown, there is no ringing artefact in the images and the fine features of the tissue structures are now more visible. Moreover, under close visual examination, we can notice that at the center of the deconvolved images, Figure 4.6(d) appears to be

clearer than that of Figure 4.6(c). All these contribute to an improvement in the visibility of the images.

Figure 4.5: Computation of the resolution gain based on the axial and lateral slices of the ACF. (a) The axial slice, A_{00} , and (b) the lateral slice, L_{00} , of the 2-D ACF corresponding to the original image ($\cdots$) and the deconvolved image ($-$). (c) and (d) are the envelopes of (a) and (b) respectively, with the RG_{5dB} and RG_{10dB} lines.

In general, it is difficult to understand what quality means to the observer and there is currently, to the best knowledge of the author, no evidence that the existing performance metrics are closely related to the dynamic of visual perception. Consequently, the choice of one algorithm will almost always be determined by the perceived visual quality of the resulting images. In this spirit, all metrics used should solely serve as a general guideline indicating the possibility of an improvement. Most importantly, we performed visual examination on the images to validate the quality of the deconvolved images. The same observations that the deconvolved images of the proposed algorithms are visually more appealing can also be appreciated in another six set of real ultrasound images presented in Figure 4.7. A medical expert is invited to grade the visual quality of the deconvolved images of the Wiener and CLS filters. Seven out of the eight deconvolved images were credited to the CLS filter (Six of the eight set of deconvolved images involved in the assessment are shown in Figure 4.7).

Very consistent observations are made from the experimental results. All the structures in the standard images are blurred and the speckle in the images is pronounced. The figures show that the size of the speckle is reduced in the deconvolved images. The shrinkage in the granular pattern makes small details sharper. The speckle is more distinct and the improvement in the spatial resolution is also evident from the reduction in its spatial correlation axially and laterally. This is demonstrated by the relatively narrower 2-D ACF compared to the corresponding widths of the standard image leading to an increase of resolution gain in these two directions. The medical expert has confirmed that images with better perceived visual quality are credited to those processed by the CLS filter.

Figure 4.6: Deconvolution results of the real ultrasound image using the CLS and ICLS filters. (a) Original, (b) Wiener solution (constant noise-to-signal power ratio of 10^{-2}), (c) CLS filter ($\alpha = 0, \beta_1 = 1, \gamma = 10^{-2}$), and (d) ICLS filter under 10 iterations.

"WIENER"

(a)

"CLSF"

(b)

"WIENER"

(c)

"CLSF"

(d)

"WIENER"

(e)

"CLSF"

(f)

"WIENER"

(g)

"CLSF"

(h)

Figure 4.7: Deconvolution results of another six different real ultrasound images using the CLS filter. (Right) Wiener solution (constant noise-to-signal power ratio of 3×10^{-2}), and (Left) CLS filter ($\alpha = 0, \beta_1 = 1, \gamma = 10^{-2}$).

4.6 Conclusion

The results in this chapter are based on the proposed model of image degradation and restoration process in Chapter 2.4. Our approach of ultrasound blind image deconvolution follows the two-step procedure in Chapter 2.5 where the 2-D PSF is first estimated using the complex cepstrum method described in Chapter 3.3. Then, the estimated 2-D PSF is incorporated into the formulation of the deconvolution problem to obtain the tissue image.

In this chapter, we have proposed two deconvolution algorithms. This is the CLS filter in Section 4.2 and the ICLS filter in Section 4.3. The constrained optimization expression to derive the CLS filter is presented in (4.22) and the resultant filter equation is given in (4.26). The ICLS filter is an extension of the CLS filter with an intention to further improve the image quality of the deconvolved ultrasound images. The associated equations require performing the iterative filtering operation are given in (4.26), (4.28), (4.29), and (4.30).

Performance metrics used to evaluate the image quality are the speckle SNR, CR, and RG. They are described in Section 4.4. Besides, the Wiener filter is used as a reference algorithm. Simulation results show no significant improvement on speckle SNR and no improvement on CR. These observations are also made by other researchers and are discussed in Section 4.4. The results of RG show improvement and they are tabulated in Table 4.2 and Table 4.4 correspondingly to the CLS filter and ICLS filter. While the CLS filter did not show better lateral deconvolution results over that of the Wiener filter, its axial deconvolution results are much better. This can be seen from its larger RG computed at -10 dB which indicates a consistently better axial deconvolution.

The computed RG of the ICLS filter is generally better than that of the CLF filter. However, the tabulated results in Table 4.4 for iteration number beyond 10 deter-

orated. This behavior may be attributed to the use of the estimated 2-D PSF as discussed in Section 4.4. As for the experimental results tabulated in Table 4.5, the ICLS filter shows consistently better RG over that of the CLS filter. The deconvolved images using the CLS filter show in Figure 4.6 and Figure 4.7 have better perceived visual quality.

Chapter 5

Deconvolution via Nonorthogonal Wavelet Transform

Over the years, researchers in applied mathematics and signal processing have developed powerful wavelet methods for multiscale signal representation and analysis [96]–[99]. These methods differ from the traditional Fourier techniques by the way information is localized in the time-frequency plane. Unlike Fourier transform whose basis functions are sinusoids that provide excellent frequency resolution, but fail to deliver good time resolution as they do not have compact support, the wavelet transforms are based on small waves, called *wavelets*, of varying time and frequency resolution. This feature allows them to tradeoff between time and frequency resolutions, which makes them suitable for analysis of nonstationary signals. In image processing, this salient features make wavelets particularly attractive because many important image features, such as edges, are highly localized in the spatial position.

Recently, there has been considerable interest in the use of the orthogonal discrete wavelet transform (DWT) as an effective tool for recovering the ultrasound envelope images from the noisy observation [9]–[11]. The intuition of using the DWT is that wavelet transform is good at energy compaction. The wavelet transform

decorrelates a signal with structures while leaving the uncorrelated noise uncorrelated. Noise is then spread out evenly over all wavelet coefficients, and important signal singularities are still distinguishable from the noisy coefficients. Thus, small wavelet coefficients are more likely due to noise and large wavelet coefficients are more likely due to important signal features. This allows the development of simple denoising algorithms to effectively suppressing the noise through the wavelet coefficients. Unfortunately, the existing methods process on the ultrasound envelope results in a denoised envelope which is of lower spatial resolution mainly due to that their treatments did not consider the blurring effect introduced by the ultrasound scanner probe.

In this Chapter, we propose the use of the nonorthogonal wavelet deconvolution algorithm as a means to recover the tissue image from the ultrasound RF signal. This algorithm is developed based on the proposed ultrasound tissue signal model in Subsection 2.3.5. The adaptation of the nonorthogonal wavelet transform offers the flexibility of having the wavelet follow the shape of the estimated ultrasonic PSF and can effectively suppressing the tissue noise. When the noise is signal-independent, the minimum mean square error (MMSE) estimator in the undecimated wavelet domain can optimally rescale the wavelet coefficients according to the local statistics. In Section 5.1, we define and briefly introduce the continuous wavelet transform and its discretization. Section 5.2 contains an abbreviated derivation of the algorithm à trous followed by the undecimated implementation of the nonorthogonal wavelet transform. Undecimated wavelet coefficients are preferred for denoising and yields superior visual results (in terms of noise suppression and sharpness preservation) because of its translation-invariant property whereas the classical dyadic wavelet decompositions are characterized by iterated filtering and downsampling, their estimation of both the nonstationary signal and noise variances are critical due to the missing translation invariance. Section 5.3 formulates the discrete inverse of the nonorthogonal wavelet transform. In Section 5.4 we present our contribution

of this work in the application of Shensa's guidelines [101], [102] in choosing the nonorthogonal wavelet filters for ultrasound image deconvolution. We further adaptively adjusted the wavelet filters to take advantage of the energy compaction it offered. Moreover, we concisely present the proposed nonorthogonal wavelet deconvolution algorithm. This includes the description of the theoretical framework for the blind deconvolution, a review of the MMSE filtering approach in the spatial domain, and its formulation as a rescaling of wavelet coefficients depending on the SNR of the coefficients themselves. Experimental results carried out on both the simulated and real ultrasound RF data are reported and discussed in Section 5.5 and Section 5.6 respectively. The chapter is concluded in Section 5.7.

5.1 Introduction

Nonorthogonal wavelet transform offers greater flexibility in the choice of wavelet than their orthogonal counterparts. Our design of the discrete nonorthogonal wavelet transform (DNWT) follows the proposal of Shensa which is inspired by the algorithm à trous originally developed for music synthesis [100]. In the following, we present DNWT that coincides with the continuous wavelet transform (CWT) by a wavelet $\psi(t)$ whose samples $\psi(n)$ form the highpass filter, $g(n) = \psi(-n)$, when the lowpass filter satisfies the à trous condition $h(2n) = \delta(n)/\sqrt{2}$. The standard inversion procedure for DNWT is a finite expansion in terms of the analyzing wavelet based on the frame theory [97], [103].

The CWT of a signal $s(t)$ transforms the continuous function $s(t)$ into a highly redundant function of two continuous variables, time translation b and scale a ,

defined as:

$$\begin{aligned} W(a, b) &= \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} s(t) \psi^* \left(\frac{t-b}{a} \right) dt \\ &= \int_{-\infty}^{\infty} s(t) \psi_{a,b}^*(t) dt, \end{aligned} \quad (5.1)$$

where $\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi \left(\frac{t-b}{a} \right)$, $\psi(t)$ is called the *mother wavelet*, and the factor $1/\sqrt{a}$ is for energy normalization across the different scales. This equation shows how a function $s(t)$ is decomposed into a set of wavelet basis functions $\psi_{a,b}(t)$. The mother wavelet is assumed to lie in $L^2(\mathbb{C})$, the space of square integrable complex functions, and satisfies the *admissibility condition*:

$$C_\psi \triangleq \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty, \quad (5.2)$$

where $\Psi(\omega)$ is the FT of $\psi(t)$. In most cases, this simply means that $\Psi(0) = 0$ and $\Psi(\omega)$ tends to zero as ω tends to infinity fast enough to make $C_\psi < \infty$. Since the admissibility condition implies that $\Psi(\omega)$ vanishes at both the zero and high frequencies, the wavelets must have a bandpass like spectrum. The mother wavelet is the generator of both families of decimated wavelets $\psi_{j,n}(t)$ and undecimated wavelets $\psi_n^j(t)$ defined as follows:

$$\psi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \psi \left(\frac{t}{2^j} - n \right), \quad (5.3)$$

$$\psi_n^j(t) = \frac{1}{\sqrt{2^j}} \psi \left(\frac{t-n}{2^j} \right). \quad (5.4)$$

Equation (5.1) reveals that the basis functions have time resolution that decreases with a and frequency resolution which increases with a . This property offers great advantage in ultrasound RF signal analysis since transmitted ultrasounds are pulse wavelet and the high frequency signal characteristics (such as edges) are generally highly localized in time whereas slowly varying signals have lower frequency reso-

lution. The discretization of (5.1) is done by discretize the integral, and often at dyadic scales and shifts, that is $a = 2^j$ and b to be a multiple of a . Hence, the discrete wavelet transform of the sampled function is:

$$w(2^j, 2^j n) \triangleq \frac{1}{\sqrt{2^j}} \sum_k s(k) \psi^* \left(\frac{k}{2^j} - n \right). \quad (5.5)$$

For generality, the sample rate has been set to equal one. Comparing (5.2) and (5.5) shows that (5.5) represents a decimated wavelet transform which means that octave j is only output every 2^j samples and so the resulting transform is not translation invariant. This simplifies our explanation as it readily lends itself to the physical interpretation, its similarity with the *Mallat's algorithm* [96], and the derivation of the undecimated à trous algorithm. In this spirit, if we let $g(n)$ be the discrete filter acquired by truncating the sampled wavelet function, i.e., $g(n) = \psi^*(-n)$, we shall arrive at an expression of the filter band implementation of the wavelet decomposition:

$$s_{j+1}(n) = \sum_k s_j(k) h(2n - k), \quad (5.6)$$

$$w(2^{j+1}, 2^{j+1}n) = \sum_k s_{j+1}(k) g(n - k), \quad (5.7)$$

where $s_0(n)$ corresponds to $s(n)$ of (5.5). The filter band structure is shown in Figure 5.1. The mystery in this implementation is the determination of the discrete filter $h(n)$ in (5.6) which will be unraveled in the derivation of the à trous algorithm in the following section.

Figure 5.1: An implementation of the DWT using filter bank structure. The down-arrow indicates downsampling by 2. The output of the transform is a family of decimated wavelet coefficients.

5.2 The À Trous Algorithm

A discrete approximation of the CWT in (5.1) at dyadic scales and shifts is presented in (5.5). For dealing with real data such as that of the received pulse-echo ultrasound RF signal, the complex conjugation in the approximation can be discarded and it is rewritten as:

$$w(2^j, 2^j n) = \frac{1}{\sqrt{2^j}} \sum_k s(k) \psi\left(\frac{k}{2^j} - n\right). \quad (5.8)$$

This equation shows that the rate of sampling of $\psi(t)$ for its computation varies with the scale. For a coarser scale, more number of samples per unit time is required. Conversely, a finer scale will require lesser number of samples of $\psi(t)$ per unit time. As an example, $\psi(n)$ is samples of $\psi(t)$ at integer samples of t whereas $\psi(n/2)$ is samples of $\psi(t)$ at values of t that are integer multiples of $1/2$. This means that the computation of the CWT requires storage of the samples of $\psi(t)$ obtained at multiple sampling rates. This is rather cumbersome and necessitates huge memory for the storage of samples of $\psi(t)$ at multiple sampling rates required during real-time computation. Instead, we might choose to start with samples at one rate and then fill in approximate values for the higher sampling rates. This filling is accomplished through an interpolating or à trous (literally, with holes) filter $h(n)$.

The interpolation scheme is elaborated as follows. Given $\psi(n)$, an interpolation filter $h(n)$ is used for the approximation of the next higher scale $\tilde{\psi}(n/2)$:

$$\tilde{\psi}\left(\frac{n}{2}\right) = \sum_k \psi(k) h(n - 2k). \quad (5.9)$$

From (5.9), in order to satisfy the approximation $\tilde{\psi}(n/2) = \psi(n/2)$ for even n , we require that $h(2n) = \delta(n)/\sqrt{2}$ where $\sqrt{2}$ is simply a convenience means of including the normalization factor of (5.8) in the filter. Following this methodology, we can deduce that:

$$\tilde{\psi}\left(\frac{n}{2^{j+1}}\right) = \sum_k \tilde{\psi}\left(\frac{k}{2^j}\right) h(n - 2k). \quad (5.10)$$

The limiting form of this operation as j tends to infinity leads to a function $\tilde{\psi}(t)$ which is an approximation of $\psi(t)$. Therefore, the strategy of the à trous algorithm is to compute the CWT in (5.8) using $\tilde{\psi}(t)$ rather than $\psi(t)$. As an example of $j = 1$, we have:

$$\begin{aligned} \tilde{w}(2, 2n) &= \sum_k s(k) \tilde{\psi}\left(\frac{k}{2} - n\right) \\ &= \sum_k s(k) \tilde{\psi}\left(\frac{k - 2n}{2}\right). \end{aligned} \quad (5.11)$$

Substituting (5.9) into (5.11) gives:

$$\begin{aligned} \tilde{w}(2, 2n) &= \sum_k s(k) \sum_p \psi(p) h(k - 2n - 2p) \\ &= \sum_q \psi(q - n) \sum_k s(k) h(k - 2q). \end{aligned} \quad (5.12)$$

Now, we can let $c_1(q) = \sum_k s(k) h(k - 2q)$ and (5.12) becomes:

$$\tilde{w}(2, 2n) = \sum_q \psi(q - n) c_1(q). \quad (5.13)$$

Hence, the sequence $c_1(n)$ can be expressed as a result of filtering the sequence $s(n)$

by a discrete filter of impulse response $h(-n)$ followed by downsampling by a factor of two. Furthermore, the approximate wavelet transform $\tilde{w}(2, 2n)$ is now in the form as the filtering of the sequence $c_1(n)$ by a filter of impulse response $\psi(-n)$. Likewise for scale $j = 2$,

$$\begin{aligned}\tilde{w}(4, 4n) &= \sum_k s(k) \tilde{\psi}\left(\frac{k}{4} - n\right) \\ &= \sum_k s(k) \tilde{\psi}\left(\frac{k - 4n}{4}\right).\end{aligned}\quad (5.14)$$

From (5.10), we know that $\tilde{\psi}(n/4) = \sum_k \tilde{\psi}(n/2) h(n - 2k)$ and upon substituting (5.9) into it, we arrive at:

$$\tilde{\psi}\left(\frac{n}{4}\right) = \sum_k h(n - 2k) \sum_p \psi(p) h(k - 2p). \quad (5.15)$$

Upon substituting (5.15) into (5.14):

$$\begin{aligned}\tilde{w}(4, 4n) &= \sum_k s(k) \sum_q h(k - 4n - 2q) \sum_p \psi(p) h(q - 2p) \\ &= \sum_p \psi(p) \sum_q h(q - 2p) \sum_k s(k) h(k - 4n - 2q).\end{aligned}\quad (5.16)$$

To simplify the expression, we let $r = 2n + q$ and (5.16) becomes:

$$\tilde{w}(4, 4n) = \sum_p \psi(p) \sum_r h(r - 2(n + p)) c_1(r). \quad (5.17)$$

Now if we let $q = n + p$, we observe that (5.17) leads to a form similar to (5.13):

$$\begin{aligned}\tilde{w}(4, 4n) &= \sum_q \psi(q - n) \sum_r c_1(r) h(r - 2q) \\ &= \sum_q \psi(q - n) c_2(r).\end{aligned}\quad (5.18)$$

This process is called the *à trous algorithm* and can be generalized to other scales,

as depicted below:

$$c_{j+1}(n) = \sum_k c_j(k) h(k - 2n), \quad c_0(n) = s(n) \quad (5.19)$$

$$\tilde{w}(2^j, 2^j n) = \sum_k c_j(k) \psi(k - n). \quad (5.20)$$

A glance at these two equations, (5.19) and (5.20), suffice to reveal that they are of the filter bank structure depicted in Figure 5.1 with $c_j(n) = s_j(n)$ and $\tilde{w}(2^j, 2^j n) = w(2^j, 2^j n)$. The à trous structure is the same as that of the Mallat's algorithm with the difference that there is no downsampling following the filtering with the filter $g(n)$. As we are not addressing the accuracy of the approximation, we will assume that $\tilde{w}(2^j, 2^j n)$ can be taken as $w(2^j, 2^j n)$ in the following context.

The decimated à trous algorithm is a time-varying algorithm in that any time shift in the original signal causes the wavelet to change its shape. For a time-invariant approximation, the à trous algorithm is modified to compute the CWT at uniform shifts by using the noble identities, which allows the reversing of the order of downsampling and filtering as shown in Figure 5.2. The shifting of the downsampling operators in the decimated à trous structure leads to the undecimated à trous algorithm as shown in Figure 5.3.

Figure 5.2: Illustration of the noble identities. It describes the reversing of the order of upsampling/downsampling and filtering.

From the illustration in Figure 5.3, the conversion from the decimated to the undecimated à trous algorithm is simply accomplished by the utilization of the noble identities. One is expected to see that the same interpolation filter $h(n)$ is used in both cases. The interpretation of the filter in the undecimated à trous algorithm can

Figure 5.3: The undecimated à trous algorithm.

be visualized via Figure 5.4. Given $\psi(n)$, if there exists an interpolation filter that leaves the even points fixed and interpolates only to get an approximated sample of $\psi(n/2)$ at the odd points. This aim is again achieved by using the à trous filter $h(2n) = \delta(n)/\sqrt{2}$. It has been proven in [101], [102] that the Lagrange à trous filters are the square of the Daubechies filters $h_{db}(n)$ given as:

$$h(n) = \frac{h_{db}(-n) * h_{db}(n)}{\sqrt{2}}. \quad (5.21)$$

Figure 5.4: Diagram illustrating the dilation and interpolation of the wavelet function $\psi(t)$.

5.3 Discrete Inverse

The undecimated DNWT in Figure 5.3 is implemented by cascading the single-stage filter bank structure shown in Figure 5.5(a). In the figure, $\check{s}_j^{(d)}(n)$ corresponds to $w(2^j, n)$ and D^j is the operator that inserts $2^j - 1$ zeros between the elements of a filter. As an example, if $h(n)$ is the filter with coefficients $(-0.1294, 0.2241, 0.8365, 0.4830)$, then $D^1[h(n)]$ is $(-0.1294, 0, 0.2241, 0, 0.8365, 0, 0.4830)$.

Figure 5.5: Illustration of a single stage of (a) the DNWT, and (b) the IDNWT. The same filter bank structure is used in undecimated DWT (a) and its inverse, IDWT (b).

The goal of this section is to present the filter bank structure and the associated discrete filters required for the realization of the undecimated inverse discrete nonorthogonal wavelet transform (IDNWT). An explicit representation of the output of the IDNWT is, as follows, from Figure 5.5(b):

$$\check{s}(n) \cong \sum_{j=0}^{J-1} \left[\prod_{i=0}^{j-1} \left(D^i [\tilde{h}(n)] * \right) \right] \left(D^j [\tilde{g}(n)] * \right) \check{s}_j^{(d)}(n) + \left[\prod_{i=0}^{J-1} \left(D^i [\tilde{h}(n)] * \right) \right] \check{s}_J^{(a)}(n), \quad (5.22)$$

where $\tilde{h}(n)$ is the discrete lowpass synthesis filter, $\tilde{g}(n)$ is the discrete highpass synthesis filter, $\check{s}_j^{(d)}(n)$ and $\check{s}_j^{(a)}(n)$ are the detailed and approximation wavelet coef-

ficients, respectively, and J is the level of decomposition. Since it is feasible to invert a single stage of the DNWT, thereby inverting the entire DNWT, this approach is adopted to simplify the determination of the discrete synthesis filters. For the undecimated DNWT, it suffices to find the filters that satisfy the transfer function relationship:

$$\tilde{h}(n) * h(n) + \tilde{g}(n) * g(n) = \delta(n). \quad (5.23)$$

As the operator D^i has the property that $D^i[\tilde{h}(n) * h(n)] = D^i[\tilde{h}(n)] * D^i[h(n)]$, $D^i[\tilde{g}(n) * g(n)] = D^i[\tilde{g}(n)] * D^i[g(n)]$, and $D^i[\delta(n)] = \delta(n)$, it can be easily seen that (5.23) is equivalent to:

$$D^i[\tilde{h}(n)] * D^i[h(n)] + D^i[\tilde{g}(n)] * D^i[g(n)] = \delta(n). \quad (5.24)$$

Moreover, it is worth noting that the same filter bank structure is used in the undecimated DWT and its inverse. The differences are in the determination of the discrete filters.

Shensa in [101], [102] derived several alternatives for the determination of synthesis filters and delineated their relationships with the CWT. Among them are the adjoint DNWT, discrete inverse analogous to the single integration approximation, and discrete inverse analogous to the double integration approximation. For a full exposition, the reader is referred to [101] for the derivation of the adjoint DNWT and [102] for both the discrete inverse analogous to the single and double integration approximations. All these methods show satisfactory approximation. In our studies, we have adopted the adjoint DNWT. Its synthesis filters are given as:

$$\tilde{h}(n) = \frac{h(-n)}{2}, \quad (5.25)$$

$$\tilde{g}(n) = c_E g(-n), \quad (5.26)$$

where $1/2$ and c_E are normalization factors included to ensure that the energy of the signal is conserved. The requirement of the existence of $c_E > 0$ is chosen to satisfy $c_E \|g(n)\|^2 + \frac{1}{2} \|h(n)\|^2 = 1$.

5.4 Derivation of the Proposed Algorithm

In this section, we propose the use of the nonorthogonal wavelet deconvolution algorithm on the received ultrasound RF echoes to improve the quality of the ultrasound images. To address the issue of suppressing noise in the tissue response, the proposed algorithm applies equal emphasis on the PSF estimation and noise reduction. A key operation in the noise suppression is the application of wavelet decomposition to the estimated noisy tissue image for noise rejection. An important rule for selecting a suitable wavelet is to choose one that has a shape similar to that of the processed signal so that the decomposition of the signal is highly efficient with fewer coefficients. This is highly desirable because the denoising algorithm in the wavelet domain is more effective for signal with sparse or near-sparse representations. In [28], the authors compared the wave shape of the estimated PSF with that of the available wavelets in the library to decide a suitable DWT for the decomposition of the signal. However, this is not always possible, so is not a practically feasible approach. In our proposal, we adopt the DNWT which provides flexibility for the wavelet such that it will always have a wave shape similar to that of the processed signal. As a result, the decomposition of the signal is highly efficient with fewer nonzero coefficients. This feature is particularly valuable when the underlying PSF is spatially-variant. The wavelet adapts to the varying PSF thereby always assuring that there will be an efficient decomposition of the received ultrasound RF echo signal.

5.4.1 Framework for the Blind Deconvolution

The proposed ultrasound blind image deconvolution algorithm is illustrated in a block diagram showed in Figure 5.6. The entire procedure consists of two stages. In the first stage, the received ultrasonic RF echo $y(m, n)$ is processed to estimate the 2-D PSF by applying a frequency-invariant linear filter in the cepstral domain. The second stage is known as wavelet deconvolution which consists of an inverse filtering process cascaded with a wavelet denoising algorithm. In the wavelet deconvolution algorithm, the estimated 2-D PSF $\check{v}(m, n)$ is incorporated in the regularized Wiener inverse filter to obtain the estimated noisy tissue $\check{s}(m, n)$ from the measurements $y(m, n)$. The estimation of the noiseless tissue $f(m, n)$ is then accomplished by decomposing $\check{s}(m, n)$ with a separable 2-D DWT. The 2-D DWT consists of the 1-D DNWT in the axial direction and a conventional 1-D DWT in the lateral direction. We assume a separable 2-D DWT, such that $s(m, n) = s(m)s(n)$ is satisfied. Since both wavelet transforms have the same wavelet equations except that their wavelet filters differed, we will only be concentrated on the design of the DNWT for our application while the wavelet filters for the conventional DWT are easily available, e.g., in [96], and it will only be mentioned whenever necessary.

Without loss of generality, only the description of the wavelet decomposition in the axial direction is presented. Under the condition $\check{s}(m, n) = \check{s}(m)\check{s}(n)$, the DNWT decomposition of $\check{s}(n)$ is the set of wavelet coefficients $\{\check{s}_1^{(d)}(n), \check{s}_2^{(d)}(n), \dots, \check{s}_J^{(d)}(n), \check{s}_J^{(a)}(n)\}$.

In a region of a large number of scatterers and without correlation between scatterers, we consider that the condition of central limit theorem is met and the statistics will be Gaussian [37]. Hence, it asserts that utilizing a linear Bayesian estimator, known as local linear minimum mean square error (LLMMSE) estimator, on the wavelet coefficients can give an optimal estimation of the noiseless tissue coefficients

Figure 5.6: Block diagram of the proposed blind nonorthogonal wavelet deconvolution algorithm. The two stages of the algorithm are the 2-D PSF estimation followed by the wavelet deconvolution. The wavelet deconvolution consists of an inverse filtering operation, and the estimation of the noiseless coefficients in the wavelet domain by the LLMMSE estimator. The 2-D DWT consists of the 1-D NDWT and 1-D DWT in the axial and lateral directions respectively. Likewise, the reconstruction is accomplished by the use of the 1-D IDNWT and 1-D IDWT in the corresponding directions.

$\check{f}_{LLMMSE}(n)$. To define the equations, we let $\check{f}_{j,LLMMSE}^{(d)}(n)$ be the LLMMSE estimator of $\check{s}_j^{(d)}(n)$ and $\check{f}_{j,LLMMSE}^{(a)}(n)$ be the corresponding LLMMSE estimator based on $\check{s}_j^{(a)}(n)$ where $1 \leq j \leq J$. The 2-D equivalent representation of these signals is illustrated in Figure 5.6.

While the first stage of the algorithm has a known result in the literature, e.g., [26], [29], and the Wiener solution in Chapter 4, the second stage in dealing with the tissue response model and tissue noise suppression is novel and has not been studied before. The details of the proposed algorithm are elaborated in the following.

5.4.2 Wavelet Deconvolution

A naive deconvolution would be to inverse the smoothing process $\check{v}(m,n)$ directly to obtain $\check{s}(m,n)$. Unfortunately, such inverse filtering [91] can perform poorly in the presence of noise $w(m,n)$ because the filter design ignores the noise process. Improvement of the restored quality is possible with the regularized Wiener inverse

filter technique [26], [91] which performs as an inverse filter at passband frequencies, and as a smooth roll-off filter at stopband frequencies of $\check{v}(m, n)$. The regularized Wiener inverse filter implemented in the Fourier domain often distorts the localized features leading to ringing effects. This problem is alleviated by wavelets which capture the localized phenomena.

We then apply an LLMMSE estimator to the wavelet coefficients which removes the additive noise in the tissue by exploiting the local statistics. This methodology is suitable for ultrasound imaging due to the fact that distance imaged objects are independent. Since both the tissue signal and tissue noise energies are concentrated into fewer wavelet coefficients, their magnitudes in the wavelet domain are their genuine representations which will facilitate in discriminating between them by the Bayesian estimator. The estimation of noiseless tissue signal processing in the wavelet domain can be thought of as suitable manipulations of the wavelet coefficients.

5.4.2.1 Regularized Wiener Inverse Filter

The procedure for deconvolution with wavelets consists of two steps. The received ultrasound RF echo signal is first processed by the regularized Wiener inverse filter. The estimated noisy tissue image is given by:

$$\check{s}(m, n) = F^{-1} \left[\frac{\check{V}^*(\omega_m, \omega_n) Y(\omega_m, \omega_n)}{|\check{V}(\omega_m, \omega_n)|^2 + \gamma} \right], \quad (5.27)$$

where γ is the regularization parameter, $\check{V}(\omega_m, \omega_n)$ is the 2-D FT of $\check{v}(m, n)$, and γ is a small value determined empirically. In situation where high quality PSF estimation is attainable and also the observation noise level is low, γ can further be reduced so as to preserve the fine features of the tissue image. In practice, its value has to be chosen interactively to yield the desirable visual results.

5.4.2.2 Discrete Nonorthogonal Wavelet Transform

The ultrasound RF echo signal after the inverse filtering operation is decomposed by the undecimated NDWT. The rationale for working in the undecimated wavelet domain is that classical dyadic wavelet decompositions, characterized by iterated filtering and downsampling, make estimation of nonstationary signal and noise variances critical, due to the missing translation invariance. Thus, wavelet-based estimation of the tissue signal is superior in the undecimated wavelet domain. The same rationale holds for the lateral wavelet transform.

The undecimated DNWT used in our algorithm is derived from its continuous counterpart elaborated in previous sections. In practice, both the DNWT and its inverse, so as the DWT and its inverse in the lateral direction, are implemented efficiently in a filter bank framework using a cascade algorithm. Following the notations used earlier, the undecimated DNWT has the following forward transform :

$$\begin{aligned} \check{s}_j^{(d)}(n) &= D^j[g(n)] * \left[\prod_{i=0}^{j-1} \left(D^i[h(n)] * \right) \right] \check{s}(n), \\ \check{s}_j^{(a)}(n) &= \left[\prod_{i=0}^{J-1} \left(D^i[h(n)] * \right) \right] \check{s}(n), \end{aligned} \quad (5.28)$$

and the IDNWT is given by:

$$\begin{aligned} \check{s}(n) \approx & \sum_{j=0}^{J-1} \left[\prod_{i=0}^{j-1} \left(D^i[\check{h}(n)] * \right) \right] \left(D^j[\check{g}(n)] * \right) \check{s}_j^{(d)}(n) \\ & + \left[\prod_{i=0}^{J-1} \left(D^i[\check{h}(n)] * \right) \right] \check{s}_J^{(a)}(n). \end{aligned} \quad (5.29)$$

With respect to (5.28) and (5.29), we see that the structural scheme of a single stage DNWT is of the form illustrated in Figure 5.5. As mentioned earlier, both the DNWT (for axial decomposition) and DWT (for lateral decomposition) share the same wavelet equations, (5.28) and (5.29), and also the same filter structure showed

in Figure 5.5. They differ only in the determination of the wavelet filters $h(n)$, $\tilde{h}(n)$, $g(n)$ and $\tilde{g}(n)$.

It is presented in Section 5.2 that if the lowpass filter satisfies à trous condition $h(2n) = \delta(n)/\sqrt{2}$, then the DWT coincides with a CWT by a wavelet $\psi(t)$ whose samples form the highpass filter $g(n)$, i.e., $g(n) = \psi(-n)$. Equation (5.21) reveals that the square of the Daubechies filters satisfies this condition. Hence, in our studies, the interpolation filter $h(n)$ is chosen such that $h_{db4}(-n) * h_{db4}(n)/\sqrt{2}$ where $h_{db4}(n)$ is the Daubechies-4 basis filter. The synthesis filters are designed to be $\tilde{h}(n) = h(-n)/2$ and $\tilde{g}(n) = c_E g(-n)$ with $c_E > 0$. They are required to satisfy the relation $\tilde{h}(n) * h(n) + \tilde{g}(n) * g(n) = \delta(n)$.

The transmitted ultrasound pressure field into the body is a pulse-wavelet. The received RF echo signal is thus also a pulse-echo wavelet in the axial direction. To some extent, the lateral PSF may not be of oscillating nature [15], [17], [21], [26], [28], [29] depending on the ultrasound imaging system. In the case if it does oscillate, it may not meet the admissibility condition required to make a wavelet. For this reason, the adoption of the DNWT is only applicable in the axial direction. Our incorporation of the lateral decomposition using the DWT follows a heuristic approach proposed in [28] where the estimated lateral PSF is compared with the scaling function of the wavelet library in choosing a suitable DWT.

The following describes the determination of the highpass analysis filter $g(n)$. The highpass filter is derived immediately from the estimated axial PSF, i.e., $g(n) = \check{v}(-n)$. However, as the sampling rate F_S adopted in ultrasound imaging is typically around 4 to 6 times of the transmitted ultrasound frequency, this means that the designed DNWT will not have adequate frequency coverage within the frequency spectrum from $F_S/4$ to $F_S/2$. This is not a problem as signal within this frequency band is mostly noise. On the other hand, in theory, the wavelet transform should be designed to cover the complete spectrum. Taking this factor into consideration, the

estimated axial PSF is resampled to twice of its frequency to cover the high frequency band and the highpass analysis filter coefficients are then given by $g(n) = \check{v}(-2n)$. To assure a good wavelet, the resampled wavelet is fitted to the Morlet wavelet $\psi_m(n) = e^{-\beta^2 n^2 / 2} e^{2\pi f_0 n}$ before assigning as the highpass filter. Both the resampled PSF and the Morlet wavelet are tested under the admissibility condition of wavelet, and only that which has a value closer to zero is used as the discrete highpass filter. This treatment is used for fine tuning the resampled PSF to a good wavelet.

Therefore, instead of choosing a wavelet from the existing wavelet library which has no guarantee that it will always be possible to find a wave shape resembling that of the estimated axial PSF, the wavelet designed under this methodology resolves this issue so that the decomposition of signal under the DNWT can give fewer nonzero wavelet coefficients.

5.4.2.3 Estimation of Noiseless Tissue Wavelet Coefficients

After applying the DNWT to $\check{s}(n)$, we obtain a set of decomposed signals which represent projections of $f(n)$ and $d(n)$ in the wavelet domain. For this reason, the projections of their estimations are expressed as follows:

$$\begin{aligned}
 \check{s}_j^{(d)}(n) &= D^j[g(n)] * \left[\prod_{i=0}^{j-1} \left(D^i[h(n)] * \right) \right] (\check{f}(n) + \check{d}(n)) \\
 &= \check{f}_j^{(d)}(n) + \check{d}_j^{(d)}(n), \\
 \check{s}_j^{(a)}(n) &= \left[\prod_{i=0}^{J-1} \left(D^i[h(n)] * \right) \right] (\check{f}(n) + \check{d}(n)) \\
 &= \check{f}_j^{(a)}(n) + \check{d}_j^{(a)}(n).
 \end{aligned} \tag{5.30}$$

For simplicity, the superscripts $^{(a)}$ and $^{(d)}$, unless specified, are omitted in the following elaborations. In the biological tissue, the scattered signal is the sum of the tissue signal and a random fluctuating field of which its location is random and char-

acterized by a uniform distribution. The difficulty in discriminating between them so as to remove the random fluctuating field is dealt with by using a form of the linear Bayesian estimator which exploits the local statistics elaborated as follows. To impose a global MSE minimization over the whole image within the constraint of a linear solution, the LMMSE estimation [104], [105] of the noiseless signal is used:

$$\check{f}_{j,MMSE}(n) = E\{\check{f}_j(n)\} + C_{\check{f}\check{s},j} C_{\check{s},j}^{-1} [\check{s}_j(n) - E\{\check{s}_j(n)\}], \quad (5.31)$$

where $C_{\check{s},j}$ is the autocovariance matrix of $\check{s}_j(n)$, and $C_{\check{f}\check{s},j}$ is the crosscovariance matrix of $\check{f}_j(n)$ and $\check{s}_j(n)$. The solution for (5.31) requires the computations of $E\{\check{f}_j^{(d)}(n)\}$, $E\{\check{f}_j^{(a)}(n)\}$, $E\{\check{s}_j^{(d)}(n)\}$, $E\{\check{s}_j^{(a)}(n)\}$, $E\{\check{d}_j^{(d)}(n)\}$, and $E\{\check{d}_j^{(a)}(n)\}$. For simplicity, let $\check{x}_j(n)$ be an arbitrary signal representing either $\check{f}_j(n)$, $\check{s}_j(n)$ or $\check{d}_j(n)$, the expectation of $\check{x}_j^{(d)}(n)$ and $\check{x}_j^{(a)}(n)$ are given by:

$$\begin{aligned} E\{\check{x}_j^{(d)}(n)\} &= E\left\{ D^j[g(n)] * \left[\prod_{i=0}^{j-1} (D^i[h(n)] *) \right] \check{x}(n) \right\} \\ &= D^j[g(n)] * \left[\prod_{i=0}^{j-1} (D^i[h(n)] *) \right] E\{\check{x}(n)\}, \end{aligned} \quad (5.32)$$

$$\begin{aligned} E\{\check{x}_j^{(a)}(n)\} &= E\left\{ \left[\prod_{i=0}^{J-1} (D^i[h(n)] *) \right] \check{x}(n) \right\} \\ &= \left[\prod_{i=0}^{J-1} (D^i[h(n)] *) \right] E\{\check{x}(n)\}. \end{aligned} \quad (5.33)$$

Let $\sigma_{\check{f},j}^2(n)$ and $\sigma_{\check{d},j}^2(n)$ denote the ensemble variance of $\check{f}_j(n)$ and $\check{d}_j(n)$ at the n th sample position with signal length of N . As $E\{\check{d}_j(n)\} = 0$ and assuming that $\check{f}_j(n)$ is spatially uncorrelated, the covariance matrices $C_{\check{s},j}$ and $C_{\check{f}\check{s},j}$ in (5.31) become diagonal:

$$\begin{aligned} C_{\check{s},j} &= C_{\check{f},j} + C_{\check{d},j} \\ &= \text{diag} [\sigma_{\check{f},j}^2(0) + \sigma_{\check{d},j}^2(0), \dots, \sigma_{\check{f},j}^2(N-1) + \sigma_{\check{d},j}^2(N-1)], \end{aligned} \quad (5.34)$$

$$\begin{aligned}
C_{\check{f}_{s,j}} &= C_{s,j} \\
&= \text{diag} \left[\sigma_{\check{f}_{s,j}}^2(0), \dots, \sigma_{\check{f}_{s,j}}^2(N-1) \right].
\end{aligned} \tag{5.35}$$

Substituting (5.34) and (5.35) into (5.31) and noting that $E\{\check{s}_j(n)\} = E\{\check{f}_j(n)\}$, the LLMMSE of $\check{f}_j(n)$ is obtained as [105], [106]:

$$\begin{aligned}
\check{f}_{j,LLMMSE}(n) &= E\{\check{f}_j(n)\} + \frac{\sigma_{\check{f}_{s,j}}^2(n)}{\sigma_{\check{f}_{s,j}}^2(n) + \sigma_{\check{d}_{s,j}}^2(n)} \left[\check{s}_j(n) - E\{\check{s}_j(n)\} \right] \\
&= E\{\check{f}_j(n)\} + \max \left\{ 0, \frac{\sigma_{\check{s}_{s,j}}^2(n) - \sigma_{\check{d}_{s,j}}^2(n)}{\sigma_{\check{s}_{s,j}}^2(n)} \right\} \left[\check{s}_j(n) - E\{\check{s}_j(n)\} \right].
\end{aligned} \tag{5.36}$$

The function $\max\{\cdot\}$ in (5.36) is introduced to prevent the estimated signal variance, $\sigma_{\check{s}_{s,j}}^2(n) - \sigma_{\check{d}_{s,j}}^2(n)$, from being negative. Estimation of $\check{f}_j(n)$ with (5.36) requires substitution of the second-order statistics, $\sigma_{\check{s}_{s,j}}^2(n)$ and $\sigma_{\check{d}_{s,j}}^2(n)$, into the equation. The values $\sigma_{\check{s}_{s,j}}^2(n)$ are defined as follows:

$$\begin{aligned}
\sigma_{\check{s}_{s,j}}^2(n) &= E \left\{ \left[\check{s}_j(n) - E\{\check{s}_j(n)\} \right]^2 \right\} \\
&= E\{\check{s}_j^2(n)\} - E\{\check{s}_j(n)\}^2.
\end{aligned} \tag{5.37}$$

The calculation requires estimation of $E\{\check{s}_j^2(n)\}$ and $E\{\check{s}_j(n)\}$. If we assume that the noisy tissue signals are locally stationary and ergodic, we may substitute ensemble expectations with sample averages. Thus, they are obtained as averages of the wavelet coefficients of the observed noisy signal samples and of its squares, respectively. Their extensions to the 2-D representation are straight forward and are given as:

$$\begin{aligned}
E\{\check{s}_j(m, n)\} &\approx \overline{\check{s}}(m, n) \triangleq \frac{1}{(2P+1)(2Q+1)} \sum_{p=-P}^P \sum_{q=-Q}^Q \check{s}_j(m+p, n+q), \\
E\{\check{s}_j^2(m, n)\} &\approx \overline{\check{s}^2}(m, n) \triangleq \frac{1}{(2P+1)(2Q+1)} \sum_{p=-P}^P \sum_{q=-Q}^Q \check{s}_j^2(m+p, n+q),
\end{aligned} \tag{5.38}$$

where $2P+1$ and $2Q+1$ is the size of the local window in lateral and axial directions respectively, in which the coefficients are used. Because the tissue noise has a uniform spatial distribution, we may assume that the estimation of $\sigma_{d,j}^2(m, n) \cong E\{\sigma_{s,j}^2(m, n)\}$, we thus obtain the expression for the estimation of the projection of $\check{f}(m, n)$ in the wavelet domain:

$$\begin{aligned} \check{f}_{j,LLMMSE}^{(d)}(n) &= \overline{\check{s}}_j^{(d)}(m, n) + \max \left\{ 0, \frac{[\sigma_{s,j}^{(d)}(m, n)]^2 - \sigma_{d,j}^{(d)}(m, n)]^2}{\sigma_{s,j}^{(d)}(m, n)]^2} \right\} \\ &\times [\check{s}_j^{(d)}(m, n) - \overline{\check{s}}_j^{(d)}(m, n)], \end{aligned} \quad (5.39)$$

$$\begin{aligned} \check{f}_{J,LLMMSE}^{(a)}(n) &= \overline{\check{s}}_J^{(a)}(m, n) + \max \left\{ 0, \frac{[\sigma_{s,J}^{(a)}(m, n)]^2 - \sigma_{d,J}^{(a)}(m, n)]^2}{\sigma_{s,J}^{(a)}(m, n)]^2} \right\} \\ &\times [\check{s}_J^{(a)}(m, n) - \overline{\check{s}}_J^{(a)}(m, n)]. \end{aligned} \quad (5.40)$$

The above equations show that the estimated tissue image depends on its local statistics which are determined by the wavelet coefficients of the neighbouring samples. This method of tissue image estimation may be regarded as a spatially adaptive wavelet threshold operation which can yield optimal estimation of the tissue image.

5.5 Simulation Results

This section reports evaluation of our proposed deconvolution algorithm by computer simulation and real ultrasound RF data.

The sampled RF data of the receiving ultrasound echo signal has a size of 796 by 360. All the dimensions are specified in sample numbers, unless specifically stated otherwise. In the following, we describe the parameter settings for the 2-D PSF estimation, Wiener filter and the proposed algorithm. The same settings are used later in the real ultrasound RF data.

The sampled simulated RF data has an equivalent 2-D PSF with a spatial extent of 17 by 60. For convenience, the 2-D PSF is estimated based on a rectangular block where its axial and lateral dimensions are set to be the nearest numbers to the power of 2, i.e., 32 by 64. If there is no prior knowledge of the actual PSF, they can be roughly determined by examining the variance of the estimated PSFs. Nevertheless, the axial excitation and aperture geometry can always reveal information on the spatial extent of the PSF. The simulated RF images are then segmented into various 70 % overlapping rectangular blocks of this size. Among them, thirty blocks that have envelope SNR closer to 1.91 are used for the 2-D PSF estimation. These settings are chosen empirically.

Estimations of the 2-D PSF are performed in the cepstral domain using a 2-D Butterworth lowpass lifter described in Chapter 3. The parameters of the Butterworth lifter are set to an order of 5, D_0 of 0.5 and 0.7 for the axial and lateral axes respectively. The active map in the weighted LS phase unwrapping algorithm is set to 100 % and 30 % in the axial and lateral axes respectively. These values are set based on preliminary experiments in which the spatial extent of the 2-D PSF in the complex cepstrum is determined. Theoretically, all these settings (D_0 and active map) are kept small so as to eliminate noise. Recall that the complex cepstrum method of estimation assumes that the 2-D PSF occupied in a different cepstrum from all other signals. This is not always true as some tissue signals and noise will also cross-over into it. Since averaging noise level will decrease with an increase in the number of PSF estimates, thirty estimated 2-D PSF obtained from the thirty rectangular blocks are averaged in the spatial domain to improve the robustness of the estimation. Figure 5.7 shows the surface plot of the simulated 2-D PSF and its estimation. In Figure 5.8, we compare the axial and lateral slices of the estimated 2-D PSF with the simulated 2-D PSF. One hundred simulated ultrasound images are generated and the MSE between the simulated and estimated 2-D PSF is -39.9644 ± 0.957504 dB (mean value $\pm$ one standard deviation). It is worth mentioning that

this sort of quantitative comparison is not possible with the real ultrasound RF data. This comparison reveals the quality of our estimated 2-D PSF.

Figure 5.7: Surface plot of the 2-D PSF. (Left) Simulated, and (Right) estimated.

Figure 5.8: Comparison between the simulated (-) and estimated (o) PSF. (Top) Axial, and (Bottom) lateral PSF.

In our proposed algorithm, the estimated PSF is used in the regularized Wiener inverse filter, with γ set to be 10^{-2} , to inverse the convolution process. Our proposed tissue signal model suggests that the restored image is not yet the tissue image. Indeed, it is the noisy tissue image. To reduce the noise fluctuations in the tissue; we resort to the use of our proposed 2-D DWT denoising algorithm for the reduction of the tissue noise via the wavelet coefficients. To facilitate our discussions, we postpone our elaborations of the 2-D DWT design to Section 5.6 when we come to the real ultrasound RF data. Three levels of wavelet decomposition are adopted. In the wavelet domain, we make use of an LMMSE estimator to estimate the noiseless tissue under a local window of 3 by 3. Noise is estimated at only the highest level of the detail diagonal subband and it is used for other subbands. For fully developed speckle, the noisy tissue has a Gaussian distribution and LLMMSE estimator will give optimal performance.

Two ROI are identified on the simulated ultrasound image of 3 cm by 3 cm. These two ROIs define two homogeneous regions of 64 by 64 in the simulated tissue phantom. ROI 1 is located at the centre of the cylindrical disc while ROI 2 is situated in the scattering background region. The computations of the axial and lateral resolution gains, at both the -5 dB and -10 dB, based on the axial slice A_{00} and the lateral slice L_{00} of the 2-D ACF respectively, are illustrated in Figure 5.9. For the one hundred simulated ultrasound RF data, the evaluation results of the proposed algorithm are compiled in Table 5.1 and Table 5.2. Results in Table 5.1 show no improvement in speckle SNR and CR whereas results in Table 5.2 clearly indicate an improvement in resolution. This can also be verified visually at a glance by looking at the surface plot of the 2-D ACF of the original and deconvolved images in Figure 5.10. As shown, the 2-D ACF of the deconvolved image has a smaller 2-D correlation lag as compared to that of the original image which indicates an improvement in the resolution.

Figure 5.9: Computation of the resolution gain based on axial and lateral slices of the ACF. (a) The axial slice, A_{00} , and (b) the lateral slice, L_{00} , of the 2-D ACF corresponding to the original image ($\cdots$) and the deconvolved image ($-$). (c) and (d) are the envelopes of (a) and (b) respectively, with the RG_{-5dB} and RG_{-10dB} lines.

(a)

(b)

Figure 5.10: Surface plot of the 2-D autocovariance function. (a) Original, and (b) deconvolved 2-D ACF.

Table 5.1: The effect of deconvolution on the speckle SNR and contrast ratio (in dB). These values are tabulated in terms of mean $\pm$ one standard deviation.

Description	Speckle SNR	CR (dB)
Original	1.3237 ± 0.0498	11.9080 ± 0.8681
DNWT Solution	1.3173 ± 0.0419	11.3644 ± 0.8829

Table 5.2: Resolution gains obtained due to deconvolution using DNWT algorithm. These values are tabulated in terms of mean $\pm$ one standard deviation.

Axial slice, A_{00}		Lateral slice, L_{00}	
RG _{-5dB}	RG _{-10dB}	RG _{-5dB}	RG _{-10dB}
$1.3778 \pm$ 0.3286	$1.1650 \pm$ 0.3633	$1.5885 \pm$ 0.3444	$1.5010 \pm$ 0.2676

A set of the ultrasound images is presented in Figure 5.11. As shown, the deconvolved image in both Figure 5.11(b) and Figure 5.11(c) are deblurred whereas the proposed algorithm in Figure 5.11(c) appears to be much shaper.

Figure 5.11: Deconvolution of the simulated ultrasound image using blind nonorthogonal wavelet transform. (a) Logarithmically compressed envelope of simulated image, (b) Wiener solution, and (c) Wavelet solution.

5.6 Experimental Results

In this section, we present the visual improvement in the deconvolved images of the real ultrasound RF data using the proposed algorithm. The acquisition of the real ultrasound RF data is depicted in Appendix C. As mentioned earlier, the same parameter settings of the algorithm used in the simulated ultrasound RF data are used here for the real ultrasound RF data. In spite of that, there exists a distinct treatment between the simulated and real ultrasound RF data. In the simulation, we simulate a spatially invariant 2-D PSF so that the quality of the PSF estimation can be examined. This is not the case in the real ultrasound RF data. The 2-D PSF is expected to vary to some extent. Because of this reason, the whole piece of ultrasound RF data is segmented into various overlapping segments and each of these segments are considered to be the convolved result of a spatially invariant 2-D PSF and the noisy tissue image. Since the sample RF data has a size of 1024 by 301, which is prolonged in the axial direction, the segmentation is only performed in the axial direction.

The estimated 2-D PSF from a RF segment of the real ultrasound RF data is showed in Figure 5.12. Its axial and lateral PSF are then extracted to design the DNWT and DWT. Figure 5.13(a) shows the estimated axial PSF used in shaping the design of the wavelet. After it is resampled to twice its frequency, its nominal frequency and bandwidth are used to determine a Morlet wavelet which has similar characteristics. Both the Morlet wavelet and the resampled PSF are then tested under the admissibility criteria of wavelet and only the one which is closer to the criteria is chosen to be the highpass filters of the DNWT. The spectrum of the resampled PSF and the Morlet wavelet is showed in Figure 5.13(b). As mentioned before in Section 5.4.2.2, this procedure serves to regulate the estimated PSF for being a good wavelet. Since the designated wavelet is adapted to the estimated PSF, it is a highly efficient decomposition whereby the noisy tissue signals are represented

by few nonzero coefficients. In addition, the transfer function of the DNWT filter bank (i.e., equation (5.23)) and its frequency responses are also presented in Figure 5.13(c) and Figure 5.13(d) respectively.

Figure 5.12: Estimation of the 2-D PSF from the real ultrasound RF data.

Figure 5.13: Characteristics of the filterbank of the nonorthogonal discrete wavelet transform. (a) Morlet wavelet fitted to the estimated axial PSF, (b) frequency responses of the Morlet wavelet, (c) transfer function, and (d) frequency responses of the DNWT filter bank.

As for the decomposition with DWT in the lateral direction, the estimated lateral PSF is compared to the scaling function in the wavelet library to find a close match. This methodology is recommended in [28] for the lateral decomposition. We show in Figure 5.14 that the ‘Symlets 8’ wavelet has a scaling function reasonably close to the estimated lateral PSF. Hence, its filters are used in DWT.

Figure 5.14: Estimated lateral PSF (-) is compared to scaling function ‘Symlets 8’ of DWT (o).

Figure 5.15 visually compares the result obtained by the Wiener filter alone and that subsequently denoised by the proposed 2-D DWT algorithm. The results can show a significant improvement in the visual quality. Figure 5.15(a) is the original ultrasound image where the size of speckle is large. This granular pattern is reduced in both Figure 5.15(b) and Figure 5.15(c). Particularly, the speckle which is more obvious in the center of Figure 5.15(b) is also suppressed in Figure 5.15(c). More significantly, we notice that the ringing artifacts in the Wiener solution at the top-right corner of the image in Figure 5.15(b) are alleviated in that of the proposed algorithm depicted in Figure 5.15(c).

Figure 5.15: Deconvolution of the real ultrasound image. (a) Original, (b) the Wiener solution, and (c) the Wavelet solution.

"NONORTH"

(a)

"NONORTH"

(b)

"NONORTH"

(c)

"NONORTH"

(d)

Figure 5.16: Deconvolution results of another seven different real ultrasound images using the Nonorthogonal Wavelet solution.

5.7 Conclusion

This chapter concerned the use of the DNWT that offers the flexibility of having the wavelet follows the shape of the estimated ultrasonic PSF. The wavelet is designed to adaptively estimate the ultrasonic pulse-wavelet. This feature enables the wavelet to concentrate the signal energy and facilitates tissue signal restoration through the wavelet domain.

Section 5.1 provided a brief introduction of the basic concepts of wavelet analysis, and the continuous wavelet transform. We showed that the discrete implementation of the wavelet transform is feasible. This idea is expanded further and we derived the à trous algorithm in Section 5.2. Then, we established the relationship between the decimated transform and the undecimated transform and it is demonstrated via Figure 5.3. The required filters are obtained from the sampled ultrasonic wavelet and square of the Daubechies filters.

Then in Section 5.3, we illustrated the filter bank implementation of the DNWT through Figure 5.5 and we showed its reconstruction algorithm in (5.22). In Section 5.4 we presented our proposed framework of blind deconvolution. It is worthwhile to note that this framework is also used in Chapter 6. In this section, we further elaborated the filter bank implementation of the DNWT and its corresponding IDNWT through (5.28) and (5.29) respectively. The LMMSE estimator used provides an optimal estimation of the tissue wavelet coefficients when the noise is signal-independent. It accomplished the objective by rescaling the wavelet coefficients based on the local statistics. Finally, Section 5.5 and Section 5.6 are dedicated correspondingly to the simulation and experimental results of our proposed algorithm.

Chapter 6

Deconvolution in Symmetrical Mirror Wavelet Basis

Recall in Chapter 2, the ultrasound image formation process in the RF domain is interpreted as a spatio-temporal convolution between the tissue response and the ultrasonic system response, and the observations are contaminated by white noise. The convolution model has the following expression in the space domain:

$$y(m, n) = v(m, n) * s(m, n) + w(m, n), \quad (6.1)$$

$$s(m, n) = f(m, n) + d(m, n), \quad (6.2)$$

where $y(m, n)$ is the ultrasound RF echo, $v(m, n)$ is the 2-D PSF, $s(m, n)$ is the noisy tissue response, both $w(m, n)$ and $d(m, n)$ are additive noise, and $*$ denotes the discrete 2-D linear convolution operator with respect to the variables m and n representing the horizontal and vertical discrete sample coordinates. In other words, the distortion introduced by the ultrasound system is modelled as a convolution of the desired signal with the ultrasonic system impulse response. Inverting the effects of the distortions corresponds to a deconvolution procedure. A direct inversion is

not feasible because the observed ultrasound RF echo is also contaminated by the environmental noise which makes the deconvolution problem more difficult.

In Chapter 5, we presented the ultrasound blind image deconvolution algorithm where the wavelets are designated to follow the shape of the PSF in order to attain sparse signal representation. In this chapter, we propose a symmetrical mirror wavelet deconvolution algorithm in the place of the nonorthogonal wavelet deconvolution algorithm under the same framework. This algorithm offers the advantage that the wavelets are not to be derived during real-time computation thereby speeding up the computation. It is accomplished via proper segmentation of the frequency plane to concentrate the signal energy. This chapter is organized as follows. In Section 6.1, we give an introduction of wavelet deconvolution based on the pseudo-inverse filter. Section 6.2 discusses the desired wavelet basis for denoising. In Section 6.3, we derive and illustrate our contribution of this work in the designing of a symmetrical mirror wavelet basis and address the quality of the wavelet denoising algorithms. Next in Section 6.4, we present the experimental results of the wavelet deconvolution algorithm. Finally, Section 6.5 contains the conclusions.

6.1 Introduction

The 2-D PSF is primarily determined by the characteristics of the imaging system such that it has a lowpass and a bandpass response in the lateral and axial directions respectively. For the sake of simplicity, we present our proposed algorithm by separating the 2-D model in (6.1) and (6.2) into 1-D models with signal size N in these two directions. Unless otherwise specified, the convolution model in the axial

direction is used for discussion:

$$y(n) = v(n) * s(n) + w(n), \quad (6.3)$$

$$s(n) = f(n) + d(n). \quad (6.4)$$

The analysis assumes periodic boundary condition, which means that all the convolutions in this chapter are circular. The use of periodic boundary condition is not problem specific; it solely serves to simplify the border issue. The noise $w(n)$ is circular stationary and its covariance is therefore diagonalized by the discrete Fourier basis. We suppose that the axial PSF is obtained using contemporary PSF estimation techniques described in Chapter 3 and the noise is known through calibration.

Let $V(\omega_n)$ denote the DFT of $v(n)$, i.e., $V(\omega_n) = \mathcal{F}[v(n)]$. Direct inversion of (6.3) with $V(\omega_n)$ necessarily amplifies noise components at frequencies where $V(\omega_n)$ is small. We employ a conditional inversion to moderate the noise amplification. It is the pseudo-inverse filter $v^{-1}(n)$ whose FT $V^{-1}(\omega_n)$ is:

$$V^{-1}(\omega_n) = \begin{cases} \frac{1}{V(\omega_n)} & , |V(\omega_n)| > \tau \\ 0 & , |V(\omega_n)| \leq \tau \end{cases} \quad (6.5)$$

where τ is a threshold controlling the tradeoff between the amount of noise suppression and the amount of signal distortion. The range of τ varies from 0 to the maximum value of $|V(\omega_n)|$. Setting $\tau = 0$ gives an unbiased but noisy estimate whereas setting it to the maximum value completely suppresses the noise but also completely distort the signal. A small τ is desired as it can lead to huge reduction in the degree of noise amplification only at the expenses of slight increase in signal distortion. In our studies, τ is set to 1 % the magnitude of $|V(\omega_n)|$ which is equivalent to having the effect of setting the frequency segments above critical frequencies to small values. More elaborations on τ are presented in the subsequent section.

The use of the pseudo-inverse filter avoids amplification of the deconvolved noise in frequencies where there is no observed signal. Applying $v^{-1}(n)$ to $y(n)$ rewrite the task of restoration of tissue signals into a standard denoising problem:

$$\begin{aligned} x(n) &= v^{-1}(n) * y(n) \\ &= c(n) * s(n) + v^{-1}(n) * w(n) \\ &= c(n) * f(n) + z(n), \end{aligned} \tag{6.6}$$

where we have assigned $z(n) = c(n) * d(n) + v^{-1}(n) * w(n)$ to account for all the undesirable signals. Estimating $f(n)$ from $y(n)$ is equivalent to estimate it from $x(n)$ whereas $c(n) = v^{-1}(n) * v(n)$ projects $s(n)$ over frequencies where $V(\omega_n)$ does not vanish. Hence, $C(\omega_n) = F[c(n)]$ has the following spectrum:

$$C(\omega_n) = \begin{cases} 1 & , |V(\omega_n)| > \tau \\ 0 & , |V(\omega_n)| \leq \tau \end{cases} \tag{6.7}$$

Since both $d(n)$ and $w(n)$ are circular stationary, the deconvolved noise $z(n)$ is also circular stationary with its power spectrum $P_Z(\omega_n)$ related to that of $d(n)$ and $w(n)$ as follows:

$$P_Z(\omega_n) = C(\omega_n) P_D(\omega_n) + P_W(\omega_n) |V^{-1}(\omega_n)|^2, \tag{6.8}$$

where $P_D(\omega_n)$ and $P_W(\omega_n)$ are the power spectrum of $d(n)$ and $w(n)$ respectively. Since $V(\omega_n)$ has a bandpass frequency response, its frequency response decays at the low and high frequencies. In the neighborhood of frequencies where it vanishes, $V^{-1}(\omega_n)$ becomes large so that the deconvolved noise spectrum $P_W(\omega_n) |V^{-1}(\omega_n)|^2$ is amplified considerably.

In the following context, we propose a wavelet basis that is designated for ultrasound deconvolution. It is a modification of the orthonormal mirror wavelet basis proposed by Mallat in [99], [107] which nearly diagonalizes the covariance of the noise.

6.2 Choice of Basis for Deconvolution

As we have mentioned at the beginning of the chapter, our objective is to derive a basis that is well adapted to concentrate both the signal and noise energies. From the *Karhunen-Love energy compaction theorem* [99], the basis that diagonalizes the covariance of a random process is the basis that best concentrates the expected energy of the process over any specified number of its basis vectors. Since $z(n)$ is circular stationary, the Karhunen-Love basis that diagonalizes its covariance is the discrete Fourier basis which means that the discrete Fourier basis is the optimal basis for the deconvolved noise. Unfortunately, the discrete Fourier basis is not optimal for signal because image contains sharp transitions. It is therefore of our interests to find a different basis, other than the discrete Fourier basis, that can nearly diagonalize the covariance of the deconvolved noise while still being able to provide sparse signal representation.

If there exists is a basis $\mathcal{B} = \{g_m\}_{1 \leq m \leq N}$ that is able to concentrate the noise energy. Then, the variances of $z(n)$ in $\mathcal{B}$ will remain of the same order as the power spectrum values and they are related as follows:

$$\sigma_m^2 = \frac{1}{N} \sum_{\omega_n \in S_m} |g_m(\omega_n)|^2 P_Z(\omega_n), \quad (6.9)$$

where S_m is the frequency support of $g_m(\omega_n)$. As $N^{-1} \sum_{\omega_n \in S_m} |g_m(\omega_n)|^2 = \|g_m\|^2 = 1$, we may deduce from (6.9) the following:

$$\min_{\omega_n \in S_m} P_Z(\omega_n) \leq \sigma_m^2 \leq \max_{\omega_n \in S_m} P_Z(\omega_n). \quad (6.10)$$

To ensure the validity of the equalities (6.10) under a small frequency support, we choose $g_m(\omega_n)$ which has fast decay or low shape factor so that the sum in (6.9) is dominated by the values of $P_Z(\omega_n)$ within the passband of $g_m(\omega_n)$ to give (6.10). If

there exists a constant λ such that:

$$\lambda^2 \min_{\omega_n \in \mathcal{S}_m} P_Z(\omega_n) \geq \max_{\omega_n \in \mathcal{S}_m} P_Z(\omega_n), \quad (6.11)$$

then (6.10) implies that σ_m^2 is of the same order as $P_Z(\omega_n)$ for $\omega_n \in \mathcal{S}_m$, up to the factor λ^2 . Because both $d(n)$ and $w(n)$ are white noises, we can let $P_D(\omega_n) = \sigma_d^2$ and $P_W(\omega_n) = \sigma_w^2$, the condition (6.11) becomes:

$$\lambda^2 \min_{\omega_n \in \mathcal{S}_m} C(\omega_n) \sigma_d^2 + \sigma_w^2 |V^{-1}(\omega_n)|^2 \geq \max_{\omega_n \in \mathcal{S}_m} C(\omega_n) \sigma_d^2 + \sigma_w^2 |V^{-1}(\omega_n)|^2. \quad (6.12)$$

Since the spectrum of both $d(n)$ and $w(n)$ are white, they remain approximately constant within the frequency support $\mathcal{S}_m$. Hence, (6.12) is equivalent to:

$$\lambda \min_{\omega_n \in \mathcal{S}_m} |V^{-1}(\omega_n)| \geq \max_{\omega_n \in \mathcal{S}_m} |V^{-1}(\omega_n)|. \quad (6.13)$$

Now, we have arrived at a condition for ensuring the noise energy will concentrate in the basis $\mathcal{B}$. It will be used to examine the chosen basis for the signal if it can concentrate the noise energy.

Wavelet bases are well-known for their efficiency in concentrating the signal energy and have been widely used to enhance images which have been corrupted by noise. We will explore the feasibility of utilizing the wavelet basis through verifying whether they can also meet the condition of concentrating the noise energy via (6.13). An N periodic discrete orthonormal dyadic scale wavelet basis is written as:

$$\psi_{j,q}(n) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{n - N 2^j q}{2^j}\right) = \psi_j(n - N 2^j q) = \psi_j(n - 2^{j-L} q), \quad (6.14)$$

where $0 \leq q < 2^{-j}$ and the frequency support of $\psi_j(n)$ is proportional to 2^j such that $\Psi_j(\omega_n) = \mathcal{F}[\psi_j(n)]$ has an energy essentially concentrated in the frequency band

$|\omega_n| \in [2^{-j-1}, 2^{-j}]$. The wavelet family $\{\psi_{j,q}\}_{L+1 \leq j \leq 1, 0 \leq q < 2^{-j}}$ is an orthonormal basis of $\mathcal{C}^N$, with $L = -\log_2 N$ and the largest scale wavelet is constant, i.e., $\psi_{1,0}(n) = 1$. The modification of the condition specified in (6.13) for the wavelet basis is as follows:

$$\lambda_j \min_{\omega_n \in [2^{-j-1}, 2^{-j}]} |V^{-1}(\omega_n)| \geq \max_{\omega_n \in [2^{-j-1}, 2^{-j}]} |V^{-1}(\omega_n)|. \quad (6.15)$$

To simplify notations, we use $\alpha(\omega_n) \sim \beta(\omega_n)$ to represent the following relationship for two constant A and B of the order of 1 such that:

$$A \alpha(\omega_n) \leq \beta(\omega_n) \leq B \alpha(\omega_n). \quad (6.16)$$

For $|V^{-1}(\omega_n)|$ having a rational growth, it can be modelled as $|V^{-1}(\omega_n)| \sim |\omega_n|^\gamma$ whereby γ is a real exponential. In this case, it can be shown that (6.15) is satisfied for $\lambda_j \sim 2^{|\gamma|}$ which implies that the wavelet basis concentrates efficiently the noise energy [115].

6.3 Mirror Wavelet Basis for Ultrasound Deconvolution

For ultrasound imaging, the 2-D PSF has a lowpass response in the lateral direction and a bandpass response in the axial direction. Therefore, the 2-D PSF can be modelled by transfer function separable in the lateral and axial directions with zeros of order $p_1, p_2, p_3 \geq 1$, as displayed in Figure 6.1, written as follows:

$$|V(\omega_m)| \sim \left| \frac{2\omega_m}{N} - 1 \right|^{p_2}, \quad (6.17)$$

$$|V(\omega_n)| \sim |\omega_n|^{p_1} \left| \frac{2\omega_n}{N} - 1 \right|^{p_3}. \quad (6.18)$$

Therefore, the deconvolved noise has the following spectrum in the lateral and axial directions respectively:

$$P_Z(\omega_m) = C(\omega_m) P_D(\omega_m) + P_W(\omega_m) |V^{-1}(\omega_m)|^2 \sim \sigma_w^2 \left| \frac{2\omega_m}{N} - 1 \right|^{-2p_2}, \quad (6.19)$$

$$P_Z(\omega_n) = C(\omega_n) P_D(\omega_n) + P_W(\omega_n) |V^{-1}(\omega_n)|^2 \sim \sigma_w^2 |\omega_n|^{-p_1} \left| \frac{2\omega_n}{N} - 1 \right|^{-2p_3}. \quad (6.20)$$

Substituting (6.19) and (6.20) into (6.11) and letting $\mathcal{S}_m \in [2^{-j-1}, 2^{-j}]$ indicate that we are choosing the N periodic discrete orthonormal dyadic scale wavelet basis presented in (6.14). It can be shown that we will arrive at (6.15) under the following conditions:

$$\lambda_j \min_{\omega_m \in [2^{-j-1}, 2^{-j}]} \left| \frac{2\omega_m}{N} - 1 \right|^{-p_2} \geq \max_{\omega_m \in [2^{-j-1}, 2^{-j}]} \left| \frac{2\omega_m}{N} - 1 \right|^{-p_2}, \quad (6.21)$$

$$\lambda_j \min_{\omega_n \in [2^{-j-1}, 2^{-j}]} |\omega_n|^{-p_1} \left| \frac{2\omega_n}{N} - 1 \right|^{-p_3} \geq \max_{\omega_n \in [2^{-j-1}, 2^{-j}]} |\omega_n|^{-p_1} \left| \frac{2\omega_n}{N} - 1 \right|^{-p_3}. \quad (6.22)$$

For the lowpass response $|V(\omega_m)|$ of (6.17), (6.21) is valid (i.e., λ_j should remain of the order of 1) for $j > L + 1$. However, at the finest scale $j = L + 1$, $|V^{-1}(\omega_m)|$ varies by a huge factor within the interval $[N/4, N/2 - 1]$ with $\lambda_j \sim (N/4)^{p_2}$. This is an indication that the finest wavelet does not have enough frequency resolution to concentrate the noise energy. As for the bandpass response $|V(\omega_n)|$ of (6.18), there is large amplification of noise in both the low and the high frequencies. Due to the dyadic scale of wavelet basis which results having a higher frequency resolution in the larger scale of $j > L + 1$, (6.22) is then satisfied for $j > L + 1$ with $\lambda_j \sim 2^{p_1}$ [107]. Likewise, at the finest scale $j > L + 1$, (6.22) is not valid with $\lambda_j \sim (N/4)^{p_3}$.

We will now modify the wavelet basis so that it can concentrate both signal and noise energies. We know that the wavelet basis of scale $j > L + 1$ is able to adapt well

to the amplified noise with $\lambda_j \sim 2^{p_1}$ while the signal is still well approximated by relative small proportion of nonzero wavelet coefficients. For this reason, we are now of interest in constructing a new wavelet basis such that it is a mirror reflection of the original wavelet basis about the frequency $N/4$. Hence, the newly built wavelet basis is known as the *mirror wavelet basis*. The frequency support of the mirror wavelet $\tilde{\psi}_{j,q}(n)$ is given by $[N/2 - 2^{-j}, N/2 - 2^{-j-1}]$. One does not want to further reduce the size of these frequency intervals because it would increase the spatial support of the basis vectors which reduce the efficiency of the signal representation.

The designated composite wavelet basis composes of the original wavelet basis and the mirror wavelet basis, both are used with scales $j > L + 1$. This is the composite wavelet basis we propose for ultrasound image deconvolution. It has satisfied $\lambda_j \sim 2^{p_1}$ in the wavelet basis and $\lambda_j \sim 2^{p_3}$ in the mirror wavelet basis while still being well adapted to approximate signals efficiently. In the next section, we will illustrate its realization using cascading of filter bank.

Figure 6.1: Modelling of the PSF. (a) Axial PSF, and (b) lateral PSF.

6.3.1 Proposed Symmetrical Mirror Wavelet Basis

In this section, we propose an orthonormal basis that will diagonalize both the covariance of signal and noise. It is a modification of the mirror wavelet basis to satisfy the scenario of the ultrasound deconvolution.

The mirror wavelet basis is realized by combining two wavelet basis. A discrete and periodic orthogonal wavelet basis is given as:

$$\mathcal{B} = \{ \psi_{j,q}(n) \}_{L+1 \leq j \leq 1; 0 \leq q < 2^{-j}}, \quad (6.23)$$

where the scaling function $\phi_{0,0}(n)$ is represented as the wavelet $\psi_{1,0}(n)$, and $\psi_{j,q}(n) = 2^{-j/2} \psi(2^{-j}n - Nq) = \psi_j(n - N2^j q) = \psi_j(n - 2^{j-L} q)$. The finest wavelet $\psi_{L+1,q}(n)$ is replaced by wavelets whose discrete Fourier support decrease exponentially at high frequencies known as the mirror wavelet:

$$\left\{ \tilde{\psi}_{j,q}(n) = (-1)^{n-1} \psi_{j,q}(1-n) \right\}_{L+2 \leq j \leq 1; 0 \leq q < 2^{-j}}, \quad (6.24)$$

whereby $\Psi_{j,q}(\omega_n) = F[\psi_{j,q}(n)]$, $\tilde{\Psi}_{j,q}(\omega_n) = F[\tilde{\psi}_{j,q}(n)]$ and $|\tilde{\Psi}_{j,q}(\omega_n)| = |\Psi_{j,q}(N/2 - \omega_n)|$. The result of combining the two wavelet basis forms an orthonormal basis of $\mathcal{C}^N$, called the mirror wavelet basis. In the realization of the multiresolution discrete wavelet transform, two finite impulse response conjugate mirror filters, $h(n)$ and $g(n)$, are cascaded with circular convolution. These two filters are related to the discrete orthonormal wavelets $\psi_j(n)$ in the discrete Fourier domain as shown below:

$$\Psi_j(\omega_n) = G(2^{j-L-1} \omega_n) \prod_{p=0}^{j-L-2} H(2^p \omega_n), \quad (6.25)$$

$$\Phi_j(\omega_n) = \prod_{p=0}^{j-L-1} H(2^p \omega_n), \quad (6.26)$$

where $H(\omega_n) = F[h(n)]$, $G(\omega_n) = F[g(n)]$, $\Phi_j(\omega_n) = F[\phi_j(n)]$, $\phi_{j,q}(n) = \phi_j(n -$

$2^{j-L}q) = 2^{-j/2}\phi(2^{-j}n - Nq)$, and $\int_{-\infty}^{\infty} \phi(t) dt = 1$. For a periodic signal $s(n)$ and slight abuse of notation, the discrete wavelet coefficients, $d_j(q) = \langle s(n), \psi_{j,q}(n) \rangle$ (do not confuse it with the tissue noise $d(n)$) and $a_j(q) = \langle s(n), \phi_{j,q}(n) \rangle$, are computed using a fast filter bank algorithm for $L \leq j < 0$:

$$\begin{aligned} a_{j+1}(q) &= a_j(q) * \bar{h}(2q), \\ d_{j+1}(q) &= a_j(q) * \bar{g}(2q), \end{aligned} \quad (6.27)$$

where $\bar{h}(n) = h(-n)$, $\bar{g}(n) = g(-n)$, $d_1(0) = a_0(0)$, $a_L(n) = s(n)$, and the coefficient $a_j(n)$ provides an approximation of $s(n)$ at the scale 2^j . The cascade of filtering and subsampling with conjugate filters produces the set of detail wavelet coefficients $\{\tilde{d}_j\}_{L-1 \leq j \leq 1}$. The finest scale wavelet coefficients $d_{L+1}(q)$ carry the highest subband frequencies of $s(n)$ essentially concentrated in the frequency interval $[N/4, N/2]$. These are the coefficients that are decomposed by the mirror wavelet transform to give the mirror wavelet coefficients $\tilde{d}_j(q)$. This means that for $\tilde{a}_{L+1}(q) = d_{L+1}(q)$ and $\tilde{d}_1(0) = \tilde{a}_0(0)$, we obtain a family of discrete wavelet coefficients $\{\tilde{d}_j\}_{L+2 \leq j \leq 1}$ with the following expression applied recursively:

$$\begin{aligned} \tilde{a}_{j+1}(q) &= \tilde{a}_j(q) * h(2q), \\ \tilde{d}_{j+1}(q) &= \tilde{a}_j(q) * g(2q). \end{aligned} \quad (6.28)$$

Since $g(n) = (-1)^{1-n}h(1-n)$, one can derive from (6.27), (6.28), and (6.24) that the resulting coefficients are the mirror wavelet coefficients of $s(n)$ as $\tilde{d}_j(q) = \langle s(n), \tilde{\psi}_{j,q}(n) \rangle$. Figure 6.2 shows the proposed combined orthonormal wavelet basis.

The inverse wavelet transform is computed by reversing the operation. Let us denote:

$$l(n) = \begin{cases} l\left(\frac{n}{2}\right) & , n \text{ is even} \\ 0 & , n \text{ is odd} \end{cases} \quad (6.29)$$

For $L+1 \leq j < 0$, the inverse wavelet transform of the mirror wavelet coefficients produces the finest wavelet coefficients $d_{L+1}(q)$ through the following recursive equations:

$$\tilde{a}_i(q) = \check{\tilde{a}}_{i+1}(q) * \overline{g}(q) + \check{\tilde{d}}_{i+1}(q) * \overline{h}(q). \quad (6.30)$$

Following the decomposition algorithm, by setting $d_{L+1}(q) = \tilde{a}_{L+1}(q)$, the signal $s(n) = a_0(n)$ is reconstructed with the fast inverse wavelet transform algorithm for $L < j < 0$:

$$a_i(q) = \check{a}_{i+1}(q) * h(q) + \check{d}_{i+1}(q) * g(q). \quad (6.31)$$

Figure 6.2: Positive frequency decomposition induced by the mirror wavelet basis. There exists a cut-off frequency at both low and high frequency, shaded in the figure, above which the variance of the deconvolved noise is too large and the estimator sets all coefficients to zero. Setting the threshold τ to 1 % the magnitude of the PSF automatically determined these cut-off frequencies and prevents the wavelet coefficients from over amplification.

At low frequencies $(\omega_m, \omega_n) \in [0, N/4] \times [0, N/4]$, a 2-D separable wavelet basis is constructed with separable products of discrete periodic scaling functions $\phi_j(n)$ and the corresponding periodic wavelet $\psi_j(n)$. If we let $\psi_{1,0}(m) = \phi_{0,0}(m)$ and $\psi_{1,0}(n) = \phi_{0,0}(n)$, we have:

$$\mathcal{B}_0 = \{ \psi_{j_1,p}(m) \psi_{j_2,q}(n) \}_{L+2 \leq j_1, j_2 \leq 1; 0 \leq p < 2^{-j_1}, 0 \leq q < 2^{-j_2}}. \quad (6.32)$$

The remaining frequency annulus is covered by the 2-D symmetrical mirror wavelet basis that is formed by the separable products of mirror wavelets and the periodic orthogonal wavelets:

$$\mathcal{B}_1 = \{ \psi_{j_1,p}(m) \tilde{\psi}_{j_2,q}(n), \tilde{\psi}_{j_1,p}(m) \psi_{j_2,q}(n), \tilde{\psi}_{j_1,p}(m) \tilde{\psi}_{j_2,q}(n) \}_{\substack{L+2 \leq j_1, j_2 \leq 1 \\ 0 \leq p < 2^{-j_1}, 0 \leq q < 2^{-j_2}}}. \quad (6.33)$$

The union of these two set of bases $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$ defines an orthonormal symmetrical mirror wavelets basis of image of N^2 pixels. The segmentation of the 2-D frequency plane by the composite wavelet basis is shown in Figure 6.3.

The decomposition of images in a separable symmetrical mirror wavelet basis is computed by the filter bank algorithm illustrated in Figure 6.4. Each branch is a separable convolution with a separable product of two filters where the image rows and columns are convolved respectively with the first and the second filter. As an example, the branch labelled “ hg ” denotes a convolution first with $h(n)$ along the rows and then followed by a convolution with $g(n)$ along the columns. The outputs of the convolution with $\delta(n)$ are not subsampled while all other convolutions are followed by a subsampling by a factor of 2 along both the rows and the columns. Following the notations established earlier, the filter $\bar{h}(n)$, $\bar{g}(n)$, $h(n)$, and $g(n)$ represent the analysis lowpass, analysis highpass, synthesis lowpass, and synthesis highpass filters respectively.

Figure 6.3: Proposed symmetrical mirror wavelet basis. The grey squares correspond to critical frequencies beyond which the estimator sets all coefficients to zeros. Setting the threshold τ to 1 % the magnitude of the PSF automatically determined these critical frequencies and prevents the wavelet coefficients from over amplification.

Figure 6.4: Decomposition in a separable symmetrical mirror wavelet basis with a filter bank. Due to space constraint, the discrete value 'n' is discarded in the figure, e.g., $\delta(n)$ is shown as δ .

6.3.2 Estimation in Mirror Wavelet Basis

In the wavelet context, when the noise is independent and remains white in all bases, it does not influence the choice of bases. Its variance will be the same for all scales and can, as is conventional, be estimated from the median of the absolute values of the coefficients at the finest scale. As we have mentioned in the earlier section, the inversion of the PSF colors the white corrupting noise. The noise variances are not the same in each scale. Furthermore, it is typical that wavelet coefficients of a signal will be relatively sparser at the fine resolution scales than at the coarse scales. To use this prior information, it is therefore desirable to develop an estimator that adapts its parameters scale by scale. This section describes the employment of the scale-dependent estimator in the mirror wavelet basis. Following the general definition of an estimator, the deconvolved signal $x(n) = v^{-1}(n) * y(n)$ is decomposed by the proposed composite wavelet basis and a decision operator ρ_m is used to estimate the noiseless tissue wavelet coefficients:

$$\begin{aligned} \check{f}(n) &= D[x(n)] \\ &= \sum_{j=L+2}^1 \sum_{q=0}^{2^{-j}-1} \rho_m(\langle x(n), \psi_{j,q}(n) \rangle) \psi_{j,m} \\ &\quad + \sum_{j=L+2}^1 \sum_{q=0}^{2^{-j}-1} \rho_{\tilde{m}}(\langle x(n), \tilde{\psi}_{j,q}(n) \rangle) \tilde{\psi}_{j,m}. \end{aligned} \quad (6.34)$$

6.3.3 Restoration with Bayes Estimator

To attenuate the deconvolved noise while preserving the signal, we adopt a representation that discriminates the signal from the noise. The representation is optimized by utilizing Bayes theory which takes advantage of prior information we have on the probability distribution of the tissue and noise. The Bayes estimation of $f(n)$ is calculated by decomposing $x(n)$ on an orthonormal basis $\mathcal{B} = \{g_m\}_{1 \leq m \leq N}$ and then estimate its coefficients with the Bayes decision operator ρ_m . The resulting

estimator $\check{f}(n)$ in $\mathcal{B}$ is defined as:

$$\begin{aligned}\check{f}(n) &= D[x(n)] \\ &= \sum_{m=1}^N \rho_m (\langle x(n), g_m \rangle) g_m.\end{aligned}\quad (6.35)$$

Our goal is to minimize the error of the estimation which is measured by the loss function. The loss function allows appropriate weighting of the outcomes to achieve desirable objective or subjective properties. It can be chosen to associate a high cost with outcomes that are undesirable or disastrous. For ultrasound imaging, the loss should measure the visual degradation which is currently not available. Hence, we adopt the quadratic loss function which is mathematically simple and sufficiently precise in most applications. The risk of the estimation is computed as follows [99]:

$$\begin{aligned}r(D, f(n)) &= E\{\|f(n) - D[x(n)]\|^2\} \\ &= \sum_{m=1}^N E\{|\langle f(n), g_m \rangle - \rho_m(\langle x(n), g_m \rangle)|^2\}.\end{aligned}\quad (6.36)$$

In order to devise a simple means to estimate all the coefficients independently without losing performance, diagonal estimation is highly desirable. For example, given an observed data vector $\mathbf{x}$ modelled as the Bayesian general linear model [104]:

$$\mathbf{x} = \mathbf{H}\mathbf{f} + \mathbf{z}, \quad (6.37)$$

where $\mathbf{x}$ is an $N \times 1$ data vector, $\mathbf{H}$ is a known $N \times p$ matrix, $\mathbf{f}$ is a $p \times 1$ random vector with prior probability density function (pdf) $\mathcal{N}(\mathbf{0}, \mathbf{C}_f)$, and $\mathbf{z}$ is an $N \times 1$ noise vector with pdf $\mathcal{N}(\mathbf{0}, \mathbf{C}_z)$ and independent of $\mathbf{f}$, then the MMSE estimation of $\mathbf{f}$ is given by the mean of the posterior pdf:

$$\check{\mathbf{f}} = E\{\mathbf{f}/\mathbf{x}\} = \mathbf{C}_f \mathbf{H}^T (\mathbf{H}\mathbf{C}_f \mathbf{H}^T + \mathbf{C}_z)^{-1} \mathbf{x}. \quad (6.38)$$

Thus, if C_f and C_z are the diagonal covariance matrices of $\mathbf{f}$ and $\mathbf{z}$ respectively will greatly simplified the computation of the estimation of $\mathbf{f}$ from matrix computation to simply scalar multiplication. This example brought out the advantage of diagonalizes the coefficients in the wavelet basis. We now can estimate the noiseless coefficient according to a simple scaling function derived from Bayesian philosophy.

If the basis $\mathcal{B}$ defines a sparse signal representation, then the nonlinear diagonal estimators are nearly optimal among all nonlinear estimators [99]. The task becomes that of finding the basis that will nearly diagonalize the covariance of the noise and provides sparse signal representation so that energy of the signal compacts into just a few large wavelet coefficients while the noise remains disbursed over a large number of small coefficients. This disparity is exploited to distinguish signal from noise. As the wavelet coefficients are independent Gaussian random processes, the linear estimator for the estimation of $f(n)$ which is suboptimal becomes optimal and it is an optimal estimation among all linear and nonlinear estimators.

The use of Bayes estimator is highly desirable. Bayesian philosophy combines the evidence contained in the signal with prior knowledge of the probability distribution of the process. As such, many of the most popular denoising algorithms are simplification of it [99]. It has been shown that the risk of estimation using Bayes estimator with oracle is less than or equal to that of the projection with oracle. Therefore, the risk of estimation with Bayesian estimator is the lowest bound of all estimators inherited from it. Mallat in [99] proved that there exists a Bayes decision operator and a corresponding Bayes estimator that achieves the minimum Bayes risk. Even more, he has proven that if the covariance of the signal and noise are diagonal in the same Karhunen-Loève basis $\mathcal{B} = \{g_m\}_{1 \leq m \leq N}$, then the optimal linear estimator is diagonal in this basis and is given as follows:

$$\begin{aligned}
\check{f}(n) &= D[x(n)] \\
&= \sum_{m=1}^N \frac{\beta_m^2}{\beta_m^2 + \sigma_m^2} x_B(m) g_m \\
&= \sum_{m=1}^N \frac{\beta_m^2 / \sigma_m^2}{\beta_m^2 / \sigma_m^2 + 1} x_B(m) g_m,
\end{aligned} \tag{6.39}$$

where $x_B(m) = \langle x(n), g_m \rangle$, $f_B(m) = \langle f(n), g_m \rangle$, $z_B(m) = \langle z(n), g_m \rangle$, $\beta_m^2 = E\{|f_B(m)|^2\}$, and $\sigma_m^2 = E\{|z_B(m)|^2\}$. As shown in (6.39), the Bayes estimator is implemented with a diagonal attenuation of each coefficient $x_B(m)$ by a factor depending on the SNR, β_m^2 / σ_m^2 , in the direction of g_m . The smaller the SNR, the larger is the attenuation. The minimum linear Bayes risk is given by:

$$\begin{aligned}
r_l(D, f(n)) &= \sum_{m=1}^N \frac{\beta_m^2 \sigma_m^2}{\beta_m^2 + \sigma_m^2} \\
&= \begin{cases} 0 & , \frac{\beta_m^2}{\sigma_m^2} \gg 1 \\ \sum_{m=1}^N \beta_m^2 & , \frac{\beta_m^2}{\sigma_m^2} \ll 1 \end{cases} \tag{6.40}
\end{aligned}$$

Recently, a number of authors observed that sparse signal representation of the wavelet transform gives rise to wavelet coefficients of non-Gaussian statistics [108], [96], [109]. The intuitive reason for this behavior is that images typically contain smooth regions which lead to small filter responses that generate sharp peak at zero, and the localized features produce large-amplitude responses that generate the extensive tails. Histograms for subbands of the proposed symmetrical mirror wavelet decompositions of the ultrasound signal are plotted in Appendix F. These densities are accurately modelled with a two-parameter density function known as the generalized Gaussian distribution or GGD of the form:

$$p_{f_B}(f_B(m)) = \frac{\beta \eta(\beta)}{2 \Gamma\left(\frac{1}{\beta}\right) \sigma_{f_B}} e^{-\left(\frac{\eta(\beta)|f_B(m)|}{\sigma_{f_B}}\right)^\beta}, \tag{6.41}$$

where $f_B(m)$ is the wavelet coefficient, σ_{fB} be the standard deviation, β is the shape parameter, and $\eta(\beta)/\sigma_{fB} = 1/\alpha$. The distribution is zero-mean and symmetric. The description of the distribution is presented in Appendix D. In order to keep the presentation concise, the estimation of the parameters, $\{\alpha, \beta\}$, using the method of moments is postponed to Appendix E. Let κ be the kurtosis which is the fourth moment divided by squared of the variance, the parameters α and β of GGD are directly related to the second moment and fourth moments of GGD as follows:

$$\alpha = \sqrt{\frac{\Gamma\left(\frac{3}{\beta}\right)}{\sigma_{fB}^2 \Gamma\left(\frac{1}{\beta}\right)}} \quad \text{and} \quad \kappa = \frac{\Gamma\left(\frac{1}{\beta}\right) \Gamma\left(\frac{5}{\beta}\right)}{\Gamma\left(\frac{3}{\beta}\right)^2}. \quad (6.42)$$

The distribution is Laplacian with β of 1.0 and it becomes Gaussian when β is 2.0. This implies that random variables following GGD with small β are highly impulsive which is an evident of the presence of resolvable structures in the underlying data. Figure 6.5 shows the density function of GGD with various β . It shows that smaller β gives a sharper peak at zero with broad tails indicating a larger kurtosis which is typically well above that of the Gaussian distributed data.

Figure 6.5: Tail behaviour of GGD for $\beta = 0.5$, $\beta = 1.0$ (Laplacian), and $\beta = 2.0$ (Gaussian).

As mentioned earlier, the orthonormality of the wavelet transform for an observation corrupted by a white Gaussian noise will produce wavelet coefficient that is also Gaussian and white in the wavelet domain. This implies that each of the wavelet coefficient in the wavelet expansion of the noisy measurement in the wavelet basis, $\mathcal{B}$, has the form written as $x_B(m) = f_B(m) + z_B(m)$. Unfortunately, this is not for the case of an inverse problem where the noise is now colored. In the previous section, modification of the wavelet basis to the symmetrical mirror wavelet basis is proposed so as to alleviate this problem. The resultant subband wavelet coefficient is the combination of the signal coefficient which is of GGD distribution and observation noise coefficient that is nearly white. In the following, we compare two formulations of Bayesian estimator for the estimation of the noiseless coefficients in the wavelet domain. They are known as the maximum a posterior (MAP) and the minimum mean square error (MMSE) estimators [110], [111].

6.3.3.1 The MAP Estimator

The MAP estimate is obtained as the coefficient that maximizes the posterior pdf $p_{f_B/x_B}(f_B(m)/x_B(m))$ given the noisy observation $x_B(m)$:

$$\begin{aligned} \check{f}_B(m) &= \arg \max_{f_B} p_{f_B/x_B}(f_B(m)/x_B(m)) \\ &= \arg \max_{f_B} p_{x_B/f_B}(x_B(m)/f_B(m)) p_{f_B}(f_B(m)) \\ &= \arg \max_{f_B} p_{z_B}(x_B(m) - f_B(m)) p_{f_B}(f_B(m)), \end{aligned} \quad (6.43)$$

where $p_{z_B}(z_B(m))$ is the pdf of the deconvolved noise coefficient in the symmetrical wavelet basis that is modelled as a white Gaussian noise. The Bayes' rule allows the posterior pdf to be written in terms of the probability density of the noise $p_{z_B}(z_B(m))$ and the prior density of the signal coefficient $p_{f_B}(f_B(m))$.

Figure 6.6 shows a set of numerically computed MAP estimators for the model

equation in (6.43) with different values of the shape parameter assuming a Gaussian noise density. As shown in the figure, when $\beta = 0.5$, the estimator resembles a hard thresholding decision operator. When $\beta = 1.0$, the estimator acts as if it is a soft thresholding decision operator whereas for $\beta = 2.0$ the estimator is in the linear form resembles that of a Wiener estimator.

Figure 6.6: MAP estimators for the model given in equation (6.43), with three different shape parameters. The noise is additive and Gaussian, with variance one third that of the signal. Dash line indicates the identity function.

6.3.3.2 The MMSE Estimator

Now, we will present the MMSE estimator. The MMSE estimate is obtained as the coefficient that minimizes a mean square error cost function defined in (6.36). The mean of the posterior distribution provides an optimal estimate of the noiseless coefficient, $f_B(m)$, given a measurement of $x_B(m)$:

$$\begin{aligned}
 \check{f}_B(m) &= \arg \min_{D[x_B(m)]} r(D, f_B(m)) \\
 &= \int f_B(m) p_{f_B/x_B}(f_B(m)/x_B(m)) df_B \\
 &= \frac{\int f_B(m) p_{x_B}(x_B(m) - f_B(m)) p_{f_B}(f_B(m)) df_B}{\int p_{x_B}(x_B(m) - f_B(m)) p_{f_B}(f_B(m)) df_B},
 \end{aligned} \tag{6.44}$$

where for a given set of observation data, the pdf $p_{x_B}(x_B(m))$ is a constant and has only a normalizing effect. From (6.44), the posterior pdf is proportional to the product of the likelihood $p_{x_B/f_B}(x_B(m)/f_B(m))$ that the observation $x_B(m)$ is contributed by signal coefficient, $f_B(m)$, and the prior pdf $p_{f_B}(f_B(m))$.

Figure 6.7 shows the set of numerically computed MMSE estimators with three different values of the shape parameters. Like the MAP estimators, smaller values of β produce a nonlinear shrinkage operator, somewhat smoother than that of the MAP estimators. For $\beta = 0.5$, the estimator preserves large coefficients and suppress the small coefficients. This is intuitively sensible because small value of β corresponds to sparse wavelet representation. As the value of β increases toward 2.0, this indicates that the statistic of the underlying data tends toward more Gaussian. Therefore, the MMSE estimator eventually attained the linear form when $\beta = 2.0$. Besides, it is worth mentioning that for posterior pdf that is of Gaussian distribution (i.e., $\beta = 2$), the MAP and MMSE estimators are identical.

Figure 6.7: MMSE estimators for the model given in equation (6.44), with three different shape parameters. The noise is additive and Gaussian, with variance one third that of the signal. Dash line indicates the identity function.

6.3.3.3 Quality of the Estimators

The model parameter β reflects the presence of resolvable structures in the underlying data. An efficient estimator will adapt itself when β varies. Hence, we examine the quality of the estimators as a function of β . Figure 6.8 shows how the error variance of both the MAP and MMSE estimators change with β . The risks of the estimations are computed numerically according to the following formulation:

$$\begin{aligned} r(D, f_B(n)) &= E\{ \|f_B(n) - D[x_B(n)]\|^2 \} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f_B(n) - D[x_B(n)]]^2 \times \\ &\quad p_{x_B/f_B}(x_B(n) - f_B(n)) p_{f_B}(f_B(n)) df_B dx_B . \end{aligned} \quad (6.45)$$

As the figure reveals that the error variance of the MAP estimator is higher than that of the MMSE estimator, we will expect a better performance using the later. Besides, both estimators changes significantly for values of β less than one. These are the indications of the estimators adapts to the shape of the signal statistical distribution.

Figure 6.9 shows the estimation risk for both estimators plotted against the signal standard deviation. We observe a consistent in the behavior of both estimators in Figure 6.9 and that displayed in Figure 6.8. Two different behaviors are observed on the two estimators which react differently with β . Firstly, the MMSE estimator has lower risk for smaller β whereas the MAP estimator has higher risk for smaller β . Moreover, the estimation risk of the MMSE estimator is also lower than that of the MAP estimators.

From these results, the MMSE estimator is preferred over the MAP estimator. Figure 6.7 shows that the MMSE estimator behaves more like a soft thresholding deci-

Figure 6.8: Error variance of the estimators relative to the noise variance plotted as a function of the model parameter β . (a) MAP estimator, and (b) MMSE estimator. The noise is additive and Gaussian, with variance one third that of the signal.

sion operator, we can expect to derive an expression of it in a form resemble the soft thresholding decision function. Then, we may substitute the numerical computation with expression (6.44) using the thresholding shrinkage and the computation load can relieve significantly.

We note that in [109] an approximation of it is proposed based on numerical calculation as shown in Figure 6.10. This method is known as BayesShrink proposed in [109] to use in the actual implementation. It employs a soft thresholding scheme with the threshold chosen to be $T_B = \frac{\sigma_m^2}{\beta_B}$, and the soft thresholding decision function is governed by:

$$\rho(x_B(m)) = \begin{cases} x_B(m) - T_B & , \text{ if } x_B(m) \geq T_B \\ x_B(m) + T_B & , \text{ if } x_B(m) \leq -T_B \\ 0 & , \text{ if } |x_B(m)| \leq T_B \end{cases} \quad (6.46)$$

Figure 6.9: Error variance of the estimators against the standard deviation of the model signal. a) MAP estimator, and (b) MMSE estimator. $\beta = 0.5$ (—), $\beta = 1$ ($\cdots$), and $\beta = 2$ (-.-). The noise is additive and Gaussian, with variance of one.

6.3.4 Noise Estimation

Our modelling in Chapter 2 reveals that the tissue image is contaminated by an approximated white Gaussian noise in the symmetrical mirror wavelet basis. In Appendix D, we present the derivation of the GGD statistic. Furthermore, we illustrate through Appendix F that it serves as a good model in describing the statistics of the signal wavelet coefficients. The GGD is modelled by a set of parameters $\{\alpha, \beta\}$. Both the MAP and MMSE estimators incorporate these model parameters into their estimations of the noiseless signal wavelet coefficients. Therefore, one must in practice estimate these parameters from the noisy wavelet coefficients. Appendix E estimates $\{\alpha, \beta\}$ using the method of moments with knowledge of the noise variance $\sigma_{z_B}^2$. In the following, we investigate the estimation of $\sigma_{z_B}^2$. To facilitate our studies, the notations of the subband wavelet coefficients defined in Figure F.1 of Appendix F are adopted.

Our studies of the noise variance estimation in the symmetrical mirror basis utilized two commonly used methods in the wavelet literature. These are the noise standard deviations computed using the statistical averaging under on Gaussian pdf and the robust noise estimator based on the median of absolute deviation (MAD) used by

Figure 6.10: Compare the Error variance of the BayesShrink (—) with the MMSE estimator for $\beta = 0.5$, $\beta = 1$, and $\beta = 2$ (···). The noise is additive and Gaussian, with variance of one.

Donoho and Johnstone [112]:

$$\hat{\sigma}_{z_B} = \frac{\text{median}(|x_B - \text{median}(x_B)|)}{0.6745}. \quad (6.47)$$

This expression is often written with $\text{median}(x_B) = 0$ because the noiseless wavelet coefficients are thought to be mostly zeros [114]. Moreover, in a typical wavelet denoising application, an estimate of the noise level based only on the empirical wavelet coefficients at the highest level of resolution is accepted for denoising all the subbands. The reason for computing only at the highest level coefficients is that these tend to consist mostly of noise. This reasoning is also valid in our case but modification is necessary because our subbands suffered from different noise amplification. More precisely, we have to estimate the noise level at each of the subbands and the noise estimated in the specific subband is used for denoising the subband accordingly.

Following this methodology means that we need to estimate the noise at each and every subbands. But before it can be adopted, we have to examine if the noise variances are accurately estimated in all the subbands. The studies are carried out

on five homogeneous images that are contaminated by zero-mean white Gaussian noise with standard deviation of 5. These test images are then decomposed in the symmetrical mirror wavelet basis using the cascading of filter bank algorithm described earlier. In the investigation, Daubechies wavelets with 8 vanishing moments and 3 levels of decomposition are used. We then utilized the two noise estimation methods on the wavelet coefficients to examine their performances. From the results in Table 6.1 and Table 6.2, we may deduce that the most accurate estimation occurred at the diagonal segmented frequency subbands. Thus, at each scale, the noise variance estimated at the diagonal subband is a reasonable estimate for that of its neighboring horizontal and vertical subbands. As an example, the noise variance estimated in $d_{L+3}^{LL,d}$ can also be used for that in $d_{L+3}^{LL,h1}$, $d_{L+3}^{LL,h2}$, $d_{L+3}^{LL,v1}$, and $d_{L+3}^{LL,v2}$. Because noise is considerably amplified after inverse filtering, we need to also examine the noise estimation at the diagonal subbands after the inverse operation.

Table 6.1: Estimation of noise standard deviation in the wavelet subbands based on statistical averaging. These values are tabulated in terms of mean $\pm$ one standard deviation.

Scale	Index	LL-plane	LH-plane	HL-plane	HH-plane
L+2	d	5.0657 $\pm$	4.9879 $\pm$	5.0021 $\pm$	4.9721 $\pm$
		0.0622	0.0441	0.0781	0.0125
	h1	4.0315 $\pm$	4.1235 $\pm$	3.9333 $\pm$	3.9463 $\pm$
		0.0785	0.0760	0.1022	0.0656
	v1	3.9762 $\pm$	3.9365 $\pm$	3.9742 $\pm$	3.9785 $\pm$
		0.0603	0.0673	0.1047	0.0752
	h2	3.9523 $\pm$	4.0050 $\pm$	4.0308 $\pm$	4.0666 $\pm$
		0.1137	0.0990	0.1636	0.1250

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6.3 Mirror Wavelet Basis for Ultrasound Deconvolution

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Scale	Index	LL-plane	LH-plane	HL-plane	HH-plane
L+2	v2	4.0285 $\pm$	3.9778 $\pm$	4.0099 $\pm$	4.0162 $\pm$
		0.0792	0.0820	0.1381	0.1256
	h3	3.9523 $\pm$	4.0050 $\pm$	4.0308 $\pm$	4.0667 $\pm$
		0.1137	0.0990	0.1636	0.1250
	v3	4.0285 $\pm$	3.9785 $\pm$	4.0099 $\pm$	4.0176 $\pm$
		0.0792	0.0822	0.1381	0.1265
L+3	d	5.1542 $\pm$	5.1321 $\pm$	5.0960 $\pm$	5.0110 $\pm$
		0.0519	0.2398	0.2207	0.1726
	h1	3.5290 $\pm$	3.5973 $\pm$	4.9097 $\pm$	4.9274 $\pm$
		0.0921	0.1479	0.2069	0.1572
	v1	4.0458 $\pm$	3.9700 $\pm$	4.0031 $\pm$	3.9943 $\pm$
		0.1032	0.0732	0.1900	0.1414
	h2	3.5258 $\pm$	3.5941 $\pm$	4.9061 $\pm$	4.9171 $\pm$
		0.0897	0.1461	0.2081	0.1489
L+4	v2	4.0458 $\pm$	3.9624 $\pm$	4.0031 $\pm$	3.9904 $\pm$
		0.1032	0.0766	0.1901	0.1462
	d	4.8186 $\pm$	4.8524 $\pm$	4.6961 $\pm$	4.8227 $\pm$
		0.1614	0.3090	0.3912	0.1769
	h1	2.5527 $\pm$	2.7071 $\pm$	4.8011 $\pm$	4.8299 $\pm$
		0.1719	0.1395	0.2388	0.1436
L+5	v1	4.0288 $\pm$	3.9221 $\pm$	4.0118 $\pm$	4.0934 $\pm$
		0.0617	0.1591	0.2097	0.1424
	d	4.8063 $\pm$	5.0249 $\pm$	5.0771 $\pm$	4.9917 $\pm$
		0.2738	0.3455	0.2611	0.2286

Table 6.2: Estimation of noise standard deviation in the wavelet subbands based on robust estimator. These values are tabulated in terms of mean $\pm$ one standard deviation.

Scale	Index	LL-plane	LH-plane	HL-plane	HH-plane
L+2	d	5.0613 $\pm$	5.0113 $\pm$	5.0412 $\pm$	5.0078 $\pm$
		0.0799	0.0855	0.0965	0.1086
	h1	2.6696 $\pm$	2.6553 $\pm$	2.0466 $\pm$	2.1232 $\pm$
		0.0628	0.1008	0.0977	0.0711
	v1	2.3868 $\pm$	2.2627 $\pm$	2.2248 $\pm$	2.2798 $\pm$
		0.0986	0.0380	0.1209	0.0892
	h2	2.2925 $\pm$	2.2664 $\pm$	2.1629 $\pm$	2.2342 $\pm$
		0.0507	0.1339	0.1335	0.0688
	v2	2.3913 $\pm$	2.1979 $\pm$	2.1666 $\pm$	2.1869 $\pm$
		0.0532	0.1220	0.1091	0.1311
	h3	2.2925 $\pm$	2.2664 $\pm$	2.1629 $\pm$	2.2342 $\pm$
		0.0507	0.1339	0.1335	0.0688
L+3	d	2.3913 $\pm$	2.2008 $\pm$	2.1666 $\pm$	2.1921 $\pm$
		0.0532	0.1226	0.1091	0.1324
	h1	5.0806 $\pm$	5.1459 $\pm$	5.1096 $\pm$	4.9607 $\pm$
		0.1555	0.2730	0.3048	0.2290
	v1	1.0481 $\pm$	1.0145 $\pm$	4.6801 $\pm$	4.7298 $\pm$
		0.0920	0.1328	0.3083	0.1997
		2.3919 $\pm$	2.2011 $\pm$	2.1593 $\pm$	2.1496 $\pm$
		0.0674	0.0854	0.2112	0.0863
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Scale	Index	LL-plane	LH-plane	HL-plane	HH-plane
L+3	h2	1.0346 ± 0.0959	1.0036 ± 0.1309	4.6699 ± 0.3119	4.7099 ± 0.1953
		2.3919 ± 0.0674	2.1972 ± 0.0862	2.1593 ± 0.2112	2.1477 ± 0.0855
	v2	4.6321 ± 0.2356	4.8131 ± 0.2221	4.6944 ± 0.3907	4.6929 ± 0.2535
		0.1415 ± 0.0220	0.1499 ± 0.0148	4.4969 ± 0.3350	4.5240 ± 0.1421
L+4	h1	2.3956 ± 0.0680	2.2339 ± 0.1356	2.2158 ± 0.2113	2.2444 ± 0.1417
		4.7752 ± 0.2931	4.9111 ± 0.2826	5.1349 ± 0.3790	4.7929 ± 0.1093
	v1				
L+5	d				

In the following, we demonstrate the quality of the noise estimated at the diagonal subbands of the inverse filtered ultrasound RF image with threshold, τ in (6.5), set to 1 % of $|V(\omega_m, \omega_m)|$. A blood vessel image is assumed as the scattering ultrasound tissue image and it is circular convolved with the PSF, and then contaminated by the five different noise data that have been used in Table 6.1 and Table 6.2 to generate five test RF data. The blood vessel image is shown in Figure 6.11(a) and the PSF is shown in Figure 6.11(b). All the tests are repeated twice; first with only the measurement noise and then together with the ultrasound scattering signal circular convolved with the PSF and then contaminated by the additive measurement noise.

We realize that the most accurate results are obtained when the estimation is performed on a predetermined homogeneous region and the results for these tests are tabulated in Table 6.3 and Table 6.4. We made three observations on the estimated

Figure 6.11: Blood vessel image used in the study of noise estimation. (a) Blood vessel image, and (b) the 2-D PSF used.

noise variances. Firstly, the variances at the HL and HH frequency planes are relatively small. This is caused by the pseudo-inverse filter which attenuates wavelet subband coefficients occurred in spectrum where noise experiences huge amplification. Secondly, due to normalization of the PSF incorporated in the pseudo-inverse filter, the noise variances estimated in scale $L+2$ where the spectrum of the pseudo-inverse filter will be relative flat may be slightly smaller than that in Table 6.1 and Table 6.2. Lastly, the estimated noise variances increase along with larger scale is in accordance to the profile of the pseudo-inverse filter spectrum. From these results, we conclude that estimations are relatively accurate except at the subband for the approximation coefficients; that is LL-plane level $L+5$. This phenomenon is within our expectation as the spectrum within this subband is less flat than the others. This problem can be alleviated by increasing the number of levels of wavelet decomposition such that the frequency plane is segmented into more subbands of smaller segments situated further away from the center frequency coordinate. Moreover, it is worth mentioning that the estimations at $L+4$ are close to the actual noise variances at $L+5$. Thus, we may make use of the estimations at $L+4$ as the ap-

proximations of that at L+5 for LL-plane. Besides, as subband coefficients of all the four corners frequency segments are likely to set to zeros by the pseudo-inverse filter, this problem may no longer exists.

Table 6.3: Estimation of noise standard deviation in the diagonal subbands based on statistical averaging on a homogeneous region of the blood vessel image. These values are tabulated in terms of mean $\pm$ one standard deviation.

Scale	Type	LL-plane	LH-plane	HL-plane	HH-plane
L+2	true	4.9878 $\pm$ 0.7478	6.2936 $\pm$ 0.9194	0.8366 $\pm$ 0.1267	1.0394 $\pm$ 0.1531
	estimated	5.6054 $\pm$ 0.5799	6.3208 $\pm$ 0.8545	0.9516 $\pm$ 0.0976	1.0466 $\pm$ 0.1572
L+3	true	21.2727 $\pm$ 3.2600	18.7110 $\pm$ 5.0960	1.1164 $\pm$ 0.1017	1.4795 $\pm$ 0.7199
	estimated	21.9484 $\pm$ 3.4888	18.8045 $\pm$ 4.8704	1.4439 $\pm$ 0.1074	1.6136 $\pm$ 0.7019
L+4	true	23.1735 $\pm$ 5.8872	20.0121 $\pm$ 1.0791	1.0423 $\pm$ 0.4425	1.1290 $\pm$ 0.4314
	estimated	30.9773 $\pm$ 6.0484	20.2312 $\pm$ 1.3632	1.3379 $\pm$ 0.5334	1.1626 $\pm$ 0.4307
L+5	true	28.9188 $\pm$ 7.4096	21.2637 $\pm$ 5.0933	1.3408 $\pm$ 0.5003	1.1046 $\pm$ 0.3517
	estimated	99.7386 $\pm$ 16.152	21.4004 $\pm$ 5.2276	4.0234 $\pm$ 0.7383	1.1185 $\pm$ 0.3861

Table 6.4: Estimation of noise standard deviation in the diagonal subbands based on robust estimator on a homogeneous region of the blood vessel image. These values are tabulated in terms of mean $\pm$ one standard deviation.

Scale	Type	LL-plane	LH-plane	HL-plane	HH-plane
L+2	true	3.5966 $\pm$ 0.4854	4.8264 $\pm$ 1.1542	0.3862 $\pm$ 0.0967	0.4699 $\pm$ 0.0857
	estimated	3.8716 $\pm$ 0.4898	4.8918 $\pm$ 1.1764	0.3935 $\pm$ 0.0464	0.4435 $\pm$ 0.0444
L+3	true	21.2108 $\pm$ 3.2865	18.0622 $\pm$ 5.7671	0.7004 $\pm$ 0.1243	0.9840 $\pm$ 0.5904
	estimated	22.7559 $\pm$ 3.6040	18.1226 $\pm$ 5.7460	0.8873 $\pm$ 0.1695	1.0341 $\pm$ 0.5654
L+4	true	20.8278 $\pm$ 5.6425	20.1138 $\pm$ 1.8999	0.4521 $\pm$ 0.1796	0.5697 $\pm$ 0.2400
	estimated	28.8796 $\pm$ 6.1778	20.4157 $\pm$ 1.5470	0.6205 $\pm$ 0.2416	0.5504 $\pm$ 0.1989
L+5	true	29.5406 $\pm$ 6.5443	20.9913 $\pm$ 5.3932	1.0178 $\pm$ 0.3067	0.8538 $\pm$ 0.3678
	estimated	123.5519 $\pm$ 21.4403	21.3274 $\pm$ 5.3382	3.8314 $\pm$ 0.7714	0.8628 $\pm$ 0.3984

6.4 Experimental Results

The description of the real ultrasound RF data used in the experiments is depicted in Appendix C. At this stage, we summarize our algorithm and provide some modifications to our treatments. These modifications are devised empirically to give results that are more appealing to our visual perception. We recall that given an observation $y(m, n)$, we attempt to restore the frequency content of $s(m, n)$ by multiplying $y(m, n)$ with a function that is the inverse of PSF. This operation involves large amplification factors of frequencies at locations where the PSF approaches zero, the deviation of the estimated PSF from the true response will create large errors and the noise located at these frequencies are greatly amplified. We presented in (6.5) a pseudo-inverse filter to set to zero the frequency amplification beyond certain frequency defined by a threshold τ so as to reduce the effect of the noise. In our implementation, we have replaced the pseudo-inverse filter with a regularized inverse filter:

$$V^{-1}(\omega_m, \omega_n) = \frac{V^*(\omega_m, \omega_n)}{|V(\omega_m, \omega_n)|^2 + \gamma}, \quad (6.48)$$

where γ is a regularization factor. Although this filter has an expression of the form of a regularized Wiener filter, strictly speaking, they have different implication. In the regularized Wiener filter, γ is supposed to vary according to the SNR of the underlying data in order to achieve optimality in filtering. As for the regularized inverse filter, it is kept constant throughout. There are advantages in substituting the pseudo-inverse filter with the regularized inverse filter. Firstly, the action in setting the frequency amplification to zero beyond a certain frequency will create artifacts commonly known as ringing, over- and under-shooting. These artifacts can be explained in terms of Gibbs phenomenon where in this case we truncate in the frequency domain causes oscillations to be generated in the spatial domain. With regularized inverse filter, the regularization factor can avoid these artifacts through providing a gentler cutoff in the frequency domain. Moreover, since the number

of vanishing moments of the quadrature mirror filters used to generate the chosen wavelet needs to be greater than that of the zeros of the PSF such that the noise covariance will be diagonalized in the mirror wavelet basis, the regularization factor helps loosen this requirement through moderating the noise magnification. Our experiments show that wavelets of 2, 4, 6, and 8 vanishing moments give equally good results.

In the following experiments, we record our results using maximally asymmetric wavelet of Daubechies with 4 vanishing moments and 3 level of wavelet decomposition. We compare the results of our algorithm under different configurations, and we are also keen to know the performance of the wavelet deconvolution if it is equipped with BayesShrink. To get an appealing image, the estimated noise standard deviations are multiplied by a factor of 5. Increasing the estimated noise standard deviation has the effect of increasing the thresholding by a multiplication factor. This is a treatment quite commonly used in wavelet denoising, and the multiplication factor is called the *threshold multiplier* [115].

Table 6.5 tabulates the resolution gains of the proposed algorithm under different configurations such as that with and without cycle-spinning technique, and also studies the situation if hard threshold decision operator is employed. Besides, we are also interested to know the deviation in the performance if the regularized-inverse filter is used to replace the pseudo-inverse filter. As shown in the tabulated results, their performances are comparable and these are also confirmed through visual examination of their deconvolved images. Furthermore, the speckle SNR is not tabulated because the improvement is not significant.

Table 6.5: Resolution gains obtained due to deconvolution using the proposed mirror wavelet solution computed in the RF domain with the different configurations: pseudo-inverse filter (Pinv), regularized-inverse filter (Rinv), cycle-spinning technique (Cspin), and Hard Threshold (HardThres). These values are tabulated in terms of mean $\pm$ one standard deviation.

Description	Axial slice, A_{00}		Lateral slice, L_{00}	
	RG _{-5dB}	RG _{-10dB}	RG _{-5dB}	RG _{-10dB}
Pinv	1.9014 $\pm$ 0.2651	2.0685 $\pm$ 0.5304	1.4464 $\pm$ 0.2043	1.5284 $\pm$ 0.2678
Pinv + HardThres	2.2110 $\pm$ 0.3047	2.3380 $\pm$ 0.6507	1.4714 $\pm$ 0.1969	1.5394 $\pm$ 0.2610
Pinv + Cspin	1.9009 $\pm$ 0.2612	2.0712 $\pm$ 0.5275	1.4474 $\pm$ 0.2050	1.5293 $\pm$ 0.2689
Pinv + Cspin + HardThres	2.1756 $\pm$ 0.2662	2.3351 $\pm$ 0.6217	1.4783 $\pm$ 0.2005	1.5429 $\pm$ 0.2602
Rinv	1.8111 $\pm$ 0.2559	1.9096 $\pm$ 0.5913	1.5645 $\pm$ 0.1881	1.6530 $\pm$ 0.2651
Rinv + HardThres	2.1331 $\pm$ 0.2416	2.2196 $\pm$ 0.5078	1.5761 $\pm$ 0.2009	1.6404 $\pm$ 0.2617
Rinv + Cspin	1.8099 $\pm$ 0.2534	1.9080 $\pm$ 0.5895	1.5630 $\pm$ 0.1898	1.6540 $\pm$ 0.2656
Rinv + Cspin + HardThres	2.1220 $\pm$ 0.2480	2.2346 $\pm$ 0.5071	1.5459 $\pm$ 0.2071	1.6096 $\pm$ 0.2713

We also observe that the resultant images are improved slightly if they are again denoising in the symmetrical mirror wavelet basis using hard threshold decision operator with mild threshold multiplier.

Figure 6.12 shows the deconvolved ultrasound images of an ultrasound RF data under different configurations. We observe that the result obtained using BayesShrink is visually much poorer and cycle-spinning technique is required for it to produce result with moderate image quality. For this reason, its resolution gains are not computed in Table 6.5 for its apparently poorer image quality. We also notice the case when the MMSE estimator is incorporated in the proposed wavelet basis, the observed deconvolved images' quality are hardly discriminable whether cycle-spinning technique is deployed. In addition, the resultant images are improved slightly if they are again denoising in the symmetrical mirror wavelet basis using hard threshold decision operator with mild threshold multiplier. Since the hard threshold procedure creates discontinuities at threshold levels, the recovered images can have small artifacts if the hard threshold level, which is determined empirically, is not properly chosen. For simplicity, the algorithm is used without the hard threshold operation and Figure 6.13 presents the results for the rest of the real ultrasound data.

Figure 6.12: Deconvolution of the real ultrasound image using blind mirror wavelet transform. (a) Original, (b) BayesShrink with cycle-spin technique, (c) Mirror Wavelet solution, and (d) Mirror Wavelet solution with hard threshold.

"MIRROR:l3,db4,noiseMult 5,cspin 4"

(a)

"MIRROR:l3,db4,noiseMult 5,cspin 4"

(b)

"MIRROR:l3,db4,noiseMult 5,cspin 4"

(c)

"MIRROR:l3,db4,noiseMult 5,cspin 4"

(d)

Figure 6.13: Deconvolution results of another seven different real ultrasound images using Mirror Wavelet solution.

6.5 Conclusion

In this chapter we have considered the restoration of ultrasound images as a wavelet deconvolution problem. This chapter began with a formulation of removing the noisy components as a wavelet denoising problem in (6.6) which facilitates the use of simple thresholding operation across the wavelet subbands' coefficients. This methodology is efficient in computations because it only requires the wavelet coefficients to be scaled according to the thresholding decision function.

Section 6.2 reviewed the desired wavelet basis for deconvolution that can nearly diagonalize the covariance of the deconvolved noise while still being able to provide sparse signal representation. Next in Section 6.3 we proposed a symmetrical mirror wavelet basis which is derived based on the key concept elaborated in Section 6.2 and is tailored for ultrasound image deconvolution. The designated wavelet basis is a union of the basis given in (6.32) and (6.33) and the 2-D frequency plane of it is shown in Figure 6.3. Moreover, we modeled the tissue image wavelet coefficients as GGD function where its shape parameter is an indication of its signal content. In this section, we also reviewed the two most well-known Bayes estimators. Based on the lower error variance of the MMSE estimator demonstrated in Figure 6.8 and Figure 6.9, we have chosen the MMSE estimator as the thresholding decision operator.

Finally in Section 6.4, we examined the various configurations of wavelet denoising for the algorithms. These included an additional hard thresholding operation and applying cycle-spinning technique. These configurations are widely practiced by the wavelet community. Their results are tabulated in Table 6.5. The results showed that configuration with an additional hard thresholding operator gave most satisfactory results.

Chapter 7

Conclusion and Recommendations

7.1 Conclusion

In this thesis, we proposed three different ultrasound image deconvolution algorithms. The details of their formulations are presented in Chapter 4, Chapter 5, and Chapter 6.

The thesis starts with a proposal of a new ultrasound tissue model for ultrasound image deconvolution in Chapter 2.3.5. It is based on careful studies of the works presented by researchers working in the field of tissue characterization. The incorporation of this model into the convolution expression of the ultrasonic RF echo explains the needs to further processing of the ultrasound image after inverse filtering, which has not been carried out in the existing results in the area of ultrasound image deconvolution. Our signal model points out the fact that even if there is no contribution by the external white Gaussian noise, there is still a random fluctuations of the tissue signal existing within the received ultrasound RF echo frequency spectrum.

In our work, we addressed this issue and proposed algorithms to alleviate both

the noise in the tissue and that due to external interferences. The proposed algorithms are a two-step ultrasound image blind deconvolution approach elaborated in Chapter 2.5. Under this approach, we first estimated the 2-D PSF via the 2-D Butterworth lowpass filter in the cepstral domain detailed in Chapter 3.3. The complex cepstrum method requires the phase values to be unwrapped explicitly and we have illustrated the performance of a robust 2-D phase unwrapping algorithm in Chapter 3.4.

With the estimated 2-D PSF, the problem of ultrasound image blind deconvolution is converted to a deconvolution problem. We proposed three different deconvolution algorithms for the final stage. They are the constrained least square filter in Chapter 4, the nonorthogonal wavelet solution in Chapter 5, and the symmetrical mirror wavelet solution in Chapter 6.

In Chapter 4, we have proposed the CLS and ICLS filtering algorithms. They are derived in Chapter 4.2 and Chapter 4.3 respectively. The ICLS filter is an extension of the CLS filter to further improve the image quality of the deconvolved ultrasound images. In this chapter, we also described the performance metrics for the evaluation of the deconvolution results of Chapter 4, Chapter 5, and Chapter 6 in Chapter 4.4. These metrics are the speckle SNR, CR, and RG. Simulation results show no significant improvement on speckle SNR and no improvement on CR. These observations are also made by other researchers and are discussed in Chapter 4.4. On the real ultrasound RF data, the performance of the ICLS filter is generally better than that of the CLF filter.

In Chapter 5, we presented the ultrasound blind image deconvolution algorithm where the wavelets are designated to follow the shape of the PSF in order to attain sparse signal representation. Then in the wavelet domain, we employed a LMMSE estimator to estimate the noiseless tissue wavelet coefficient. This framework is also adopted in Chapter 6 where we have proposed a symmetrical mirror wavelet decon-

volution algorithm in place of the nonorthogonal wavelet deconvolution algorithm. This algorithm offers the advantage that the wavelets are not to be derived during real-time computation thereby speeding up the computation. It is accomplished via proper segmentation of the frequency plane to concentrate the signal energy. Furthermore, we modelled the wavelet tissue coefficients as GGD function. The estimation of the tissue image is then accomplished via thresholding the wavelet coefficients using the MMSE estimator. The results showed superior performance over other techniques.

The results of these three algorithms on both the real and simulated ultrasound RF data can show superior visual improvement as compared to the existing Wiener solution. We can see that the embedded object in the simulated tissue phantoms and the tissue structures of the real ultrasound image are now more easily observed. In terms of solution quality, Mirror Wavelet solution seems to be better than the rest, followed by CLS/ICLS filters, and then the Nonorthogonal Wavelet solution.

In summary, this thesis is a contribution to the development of novel blind deconvolution algorithms for medical ultrasound imaging. Our proposed new ultrasound tissue model is in the center of the development. Under this model, this thesis provides an extensive study of the proposed blind deconvolution algorithms and presents them in a unified framework to give a new paradigm of the deconvolution process.

7.2 Recommendations for Further Research

The complex cepstrum representation of a 2-D signal is a straightforward extension of the 1-D case. This extension is accepted quite intuitively based on analogy from the extension of 1-D Fourier transform to 2-D Fourier transform. The fact is that, to the best knowledge of the author, there is no much works in this study because

of the absence of the ability to factorize a general 2-D polynomial. Moreover, phase unwrapping on noisy data is known to be the main source of errors in this method. Since cepstrum has great potential in the separation of mixed-phase signals combined by convolution, we would like to further perform the theoretical treatise of its 2-D extension. Preferably, we want to avoid the 2-D phase unwrapping operation and develop fast algorithms.

7.2.1 PSF Estimation via Separate Filtering on Logarithm Magnitude and Phase Responses

The distinctive smoothness features of the ultrasonic 2-D PSF different from the tissue scatterings are the underlying principle that allow it to be estimated. Since tissue scatterings are random in nature, consequently, there will be some cross-over into the cepstrum occupied by the 2-D PSF which introduce estimation error. This problem is usually resolved by averaging several independent estimates in the neighborhood. Although this methodology produces satisfactory results, it burdens the computational load. One feasible approach to improve the efficiency of the computation without sacrificing the performance is to apply smoothing operator onto both the logarithm magnitude and phase function in the frequency domain instead of utilizing lowpass filter in the cepstral domain. In Chapter 3, we have showed that the logarithm magnitude of the 2-D PSF $\log_e |V(\omega_m, \omega_n)|$ and that of the scattering tissue $\log_e |S(\omega_m, \omega_n)|$ have considerably different smoothness properties. In the following, we show that similar distinctive smoothness properties are presence in their phase responses.

The phase response of a signal can be deduced from the behavior of the zeros of its corresponding ZT. Unfortunately, analysis of 2-D polynomial is not available. The following attempt is a rather superficial investigation mainly to stimulate future

research studies. The analysis is carried out on a simplest phase response which is an irreducible factor of the 1-D phase response. Subsequently, it is assumed that the same results hold to the 2-D signal as well. The ZT of the signal $y(n)$ of length N is a polynomial in z_n^{-1} of degree $N-1$, which may be factorized into irreducible factors of $Y(z_n)$:

$$Y(z_n) = \sum_{n=0}^{N-1} y(n) z_n^{-1} = \prod_{k=0}^{N-1} (1 - z_k z_n^{-1}) , \quad (7.1)$$

where it is assumed without loss of generality that $y(N-1) = 1$ and each irreducible factor $1 - z_k z_n^{-1}$ will contribute a zero $z_k = r_k e^{j\theta_k}$ to $Y(z_n)$. It follows that the phase response of $Y(\omega_n)$ is given by the sum of all the contributions from each of the irreducible factors, namely:

$$\phi_Y(\omega_n) = \sum_{k=0}^{N-1} \tan^{-1} \left(\frac{r_k \sin(\omega_n - \theta_k)}{1 - r_k \cos(\omega_n - \theta_k)} \right) . \quad (7.2)$$

Therefore, the properties of $\phi_Y(\omega_n)$ are uniquely defined by the distribution of the complex zeros z_k where its distribution depends on the statistical properties of $y(n)$. For simplicity, the overall phase response of $\phi_Y(\omega_n)$ can be examined more easily through the analysis of the behavior of a single contribution in (7.2) which is the phase response of an irreducible factor of $Y(\omega_n)$. Figure 7.1 shows the phase response of an irreducible factor of $Y(\omega_n)$ when the zero varies from within the unit circle to outside of the unit circle. As illustrated, the phase function is smooth so long that it is always from the unit circle. As the zero moves closer to the unit circle, its smoothness degraded and eventually becomes discontinuous when it is located at the vicinity of the unit circle.

For the received ultrasound RF signal, $y(n)$ is the convolution of the PSF and tissue scattering function $s(n)$. The expression in (7.2) is therefore the sum of both the phase responses of $v(n)$ and $s(n)$:

$$\begin{aligned}
\phi_Y(\omega_n) &= \phi_V(\omega_n) + \phi_S(\omega_n) \\
&= \prod_{k \in \Gamma_V} \tan^{-1} \left(\frac{r_k \sin(\omega_n - \theta_k)}{1 - r_k \cos(\omega_n - \theta_k)} \right) \\
&\quad + \prod_{k \in \Gamma_S} \tan^{-1} \left(\frac{r_k \sin(\omega_n - \theta_k)}{1 - r_k \cos(\omega_n - \theta_k)} \right),
\end{aligned} \tag{7.3}$$

where Γ_V and Γ_S are the set of indexes such that $\{r_k e^{j\theta_k}\}_{\Gamma_V}$ and $\{r_k e^{j\theta_k}\}_{\Gamma_S}$ are the zeros of $V(\omega_n)$ and $S(\omega_n)$ respectively. The studies in [75] show that $\phi_V(\omega_n)$ typically have a smooth response due to the contributions of its individual terms are zero further away from the unit circle. On the contrary, the tissue scattering function $s(n)$ usually described as a random process will be random polynomial which will have zeros random and uniformly distributed in an annulus pattern about the unit circle. Consequently, $\phi_S(\omega_n)$ will be irregular and abrupt. This considerably different smoothness property between $\phi_V(\omega_n)$ and $\phi_S(\omega_n)$ reveals the feasibility of estimating $\phi_V(\omega_n)$ from $\phi_Y(\omega_n)$.

From the discussion above it stems that the estimation of the PSF may be done by separately estimated its logarithm magnitude and phase response. More precisely, the current approach attempts to estimate $\log_e [V(\omega_m, \omega_n)]$ from $\log_e [Y(\omega_m, \omega_n)]$ based on the relationship:

$$\log_e [Y(\omega_m, \omega_n)] = \log_e [V(\omega_m, \omega_n)] + \log_e [S(\omega_m, \omega_n)], \tag{7.4}$$

whereas the new proposal suggests that since:

$$\begin{aligned}
\log_e [Y(\omega_m, \omega_n)] &= \log_e [|Y(\omega_m, \omega_n)| e^{j\phi_Y(\omega_m, \omega_n)}] \\
&= \log_e |Y(\omega_m, \omega_n)| + j\phi_Y(\omega_m, \omega_n),
\end{aligned} \tag{7.5}$$

Figure 7.1: (Upper) Phase responses of zeros with $r = 0.75$ ($- \cdot$), 0.85 ($\cdots$), 0.95 ($--$), 0.9999 ($—$) and $\theta = \pi$. (Lower) Phase responses of zeros with $r = 1.25$ ($- \cdot$), 1.15 ($\cdots$), 1.05 ($--$), 1.0001 ($—$) and $\theta = \pi$.

and

$$\log_e |Y(\omega_m, \omega_n)| = \log_e |V(\omega_m, \omega_n)| + \log_e |S(\omega_m, \omega_n)|, \quad (7.6)$$

$$\phi_Y(\omega_m, \omega_n) = \phi_V(\omega_m, \omega_n) + \phi_S(\omega_m, \omega_n), \quad (7.7)$$

Therefore, we may estimate $\log_e |V(\omega_m, \omega_n)|$ and $\phi_V(\omega_m, \omega_n)$ from $\log_e |Y(\omega_m, \omega_n)|$ and $\phi_Y(\omega_m, \omega_n)$ respectively.

7.2.2 Computing the Complex Cepstrum bypassing the 2-D Phase Unwrapping Operation

The main limitation of the complex cepstrum is the ambiguity in its phase relationship and the necessary for a complicated 2-D phase unwrapping algorithm.

The original wrapped phase is made up of both the irrotational part and the rotational part. The *Helmholtz decomposition theorem* describes this partitioning [90]. The rotational part of the phase contains all the residues. In general, the phase of the Fourier transform of an image is a very complicated function. The Fourier transform magnitude will likely have many zeros throughout the transform domain. These magnitude zeros are the generators of residues and if any phase function contains residues, it is not uniquely unwrappable by any algorithm. The severity of the unwrapping problem may dictate the degree of success or failure with cepstral processing. We will have to determine by experiment and analysis which algorithm (if any) is most appropriate and gives the most useful results. Therefore, it is to the most of our interests to explore an equivalent spatial domain method of computation so that the computation of the cepstrum can be more readily carried out.

7.2.3 Recovering of Ultrasound Image via the Transform Domain

Existing procedures of blind deconvolution algorithms are typically a two consecutive separate independent-step process. First, the 2-D PSF is estimated. Then, the estimated 2-D PSF is incorporated in the algorithms which are designed to accomplish both the deblurring and noise removal operations. In these approaches, both steps are performed in different domain. The existing methods obtain the estimated 2-D PSF in the cepstral domain followed by deconvolution in either the spatial or frequency domain. These approaches are derived from the fact that the 2-D PSF has significantly different characteristics from that of the tissues in the transformed domain. In this light, we can theoretically discard the signature of the 2-D PSF, so as to retain the tissue image. Figure 7.2, through a 1-D blind deconvolution problem, illustrates the difference between the existing approach and that we have proposed for our future exploitation. For generality, the notation $T_1[\cdot]$, $T_1^{-1}[\cdot]$, and k are

used to represent the forward transform, inverse transform, and domain used for PSF estimation respectively. And let the notation $T_2[\cdot]$, and $T_2^{-1}[\cdot]$ denote the corresponding forward and inverse transform to the second domain for performing tissue image estimation.

(a)

(b)

Figure 7.2: Comparison between the existing and proposed future frameworks for ultrasound image blind deconvolution. (a) Existing two-step approach, and (b) proposed future approach.

Figure 7.2 reveals a potential in cutting down the processing time thereby reduces the computation time. In our future studies, we will explore the feasibility of these approaches. We hope that this will help to improve the computational efficiency and promote the real-time processing using ultrasound image deconvolution as a means of improving the ultrasound image quality.

Author's Publications

- [1] W. S. Yeoh, and C. Zhang, "Constrained least squares filtering algorithm for ultrasound image deconvolution," submitted to *IEEE Trans. Biomed. Eng.*, 2006, accepted.
- [2] W. S. Yeoh, and C. Zhang, "Blind deconvolution of medical ultrasound images using new tissue model and wavelets," submitted to *IEEE Trans. Ultrason. Ferroelec. Freq. Contr.*, 2006, under review.
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Appendix A

Brief Survey of Enhancement Methods other than Blind Deconvolution

This Appendix is a brief survey of the enhancement methods other than blind deconvolution that have been tried before on ultrasound images. These techniques work on the ultrasound envelope and are grouped under the categories of adaptive filtering, wavelet shrinkages and anisotropic diffusion. They are derived based on the case of fully formed speckle-noise models.

The two speckle-noise models available are elaborated as follows. The limited dynamic range of commercial display monitors require the echo signal to be compressed to fit into the display range in the ultrasound imaging systems. Since compression changes the characteristics of the signal pdf, at the output of the beamformer on the ultrasound imaging system and prior to the logarithmic compression stage for the envelope signal, speckle noise is approximated as a multiplicative noise such that [121]:

$$y(m, n) = s(m, n)w_m(m, n) + w_a(m, n). \quad (\text{A.1})$$

Here, $s(m, n)$ is an unknown piecewise constant 2-D function representing the noise-free original image, $y(m, n)$ is the noisy observation of $s(m, n)$, $w_m(m, n)$ and $w_a(m, n)$ are multiplicative and additive noise respectively. m and n are variables of spatial locations that belong to 2-D space of all real numbers, $(m, n) \in \mathbb{R}^2$. The additive noise, such as sensor noise, is considerably small compared with that of multiplicative noise due to coherent interference, $\|w_a(m, n)\|^2 \ll \|w_m(m, n)\|^2$, the expression is

rewritten as [121]:

$$y(m, n) \approx s(m, n)w_m(m, n). \quad (\text{A.2})$$

The above is the multiplicative noise model. The logarithmic amplification transforms this model into the classical signal in additive noise form known as the additive speckle-noise model [9], [11], [13]:

$$\log(y(m, n)) = \log(s(m, n)) + \log(w_m(m, n)), \quad (\text{A.3})$$

or

$$y^l(m, n) = s^l(m, n) + w_m^l(m, n). \quad (\text{A.4})$$

The assumption considering that the speckle nature, initially Rayleigh scattered after envelope detection, after logarithmic compression can be modelled as an additive white Gaussian noise, $w_m^l(m, n)$.

The second speckle-noise model in this appendix is arrived empirically by Loupas *et al.* [5] and subsequently used by Karaman *et al.* [6], and Hao *et al.* [10]. It was obtained from the observation that the envelope-detected echo signal with fully developed speckle has a Rayleigh distribution whose mean is proportional to the standard deviation. This implies that the speckle could be modelled as multiplicative noise [5]. However the signal processing stages inside the scanner, such as logarithmic compression, low-pass filtering, and interpolation, modify the statistics of the original signal. By taking into account these factors Loupas *et al.* have arrived at the graph shown in Figure A.1.

The solid curve models the envelope signal after the signal processing stages whereas the points were the experimental measurements obtained from scans of objects which have acoustic properties similar to those of soft tissue (tissue mimicking phantoms).

From this figure it is evident that the speckle no longer multiplicative in the sense that the mean is proportional to the variance rather than the standard deviation. Therefore, if $s(m, n)$ denotes the true signal, $w(m, n)$ is a noise term which is independent of $s(m, n)$ and has mean = 0 and variance = 1, while $y(m, n)$ is the observed signal, the following signal-dependence noise model can be used for signal processing.

$$y(m, n) = s(m, n) + s(m, n)^{1/2}w(m, n). \quad (\text{A.5})$$

Assuming that a uniform area is scanned, i.e., $s(m, n) = K$ is constant; the variance of the observed signal is $\sigma^2 = K\sigma_w^2$ where σ_w^2 is the noise variance.

Figure A.1: Standard deviation versus the mean of ultrasonic speckle noise.

A.1 Adaptive Filtering

A.1.1 Adaptive Weighted Median Filter

Adaptive weighted median filters (AWMF) are designed to combine edge-preserving properties of the weighted median (WM) with the space-varying implementation based on the local image characteristics to reduce significantly the speckle with negligible loss of genuine image detail. One of the first papers on AWMF is from Loupas *et al.*. The main idea behind this method is to choose a family of weights which decrease as they move away from the center of the window and the rate of decrease is controlled by the local image content.

Let K and σ^2 to be the local mean and the local variance inside the $2k+1$ by $2k+1$ window respectively. AWMF takes advantage of the fact that the ratio σ^2/K can characterize the local image content by performing space-varying weighted median filtering with the weight coefficients adjusted according to the local statistics of the image. This methodology is illustrated in Figure A.2.

In Figure A.2, the variable slope $\alpha = c\sigma^2/K$ is derived based on the observation that the logarithmic compression stage inside the ultrasound scanner has modified the envelope-detected signal such that it arrived with the graph shown in Figure A.1

Figure A.2: A family of linear weight coefficients with variable slope α .

by using the formula:

$$L(i, j) = \left[L(k+1, k+1) - cd \frac{\sigma^2}{K} \right], \quad (\text{A.6})$$

where c is a scaling constant chosen to make $c \frac{\sigma^2}{K}$ low in uniform areas, d is the distance of the point (i, j) from the centre of the window $(k+1, k+1)$, and $[x]$ denotes the nearest integer to x if x is positive, or zero if x is negative.

A.1.2 Unsharp Masking Filter

Most of the introduced adaptive techniques for speckle reduction in the literature use local statistics [5], [6], [7], [8], [116], [117], [118], [119], [120]. This section describes studies on adaptive filtering with log-compressed images. Their working principle may be described by a weighted average calculation using sub-region statistics to estimate statistical measures over pixel windows (typically 3x3, 5x5, and 7x7 sliding pixel windows). The algorithms in this class may be traced back to the following equation:

$$f(m, n) = \overline{y(m, n)} + L(m, n) \left[y(m, n) - \overline{y(m, n)} \right], \quad (\text{A.7})$$

where $f(m, n)$ is the estimated image pixel, $y(m, n)$ is the pixel in the middle of a moving window, $\overline{y(m, n)}$ is the local mean value, $L(m, n)$ is a weighting factor with $L(m, n) \in [0 \dots 1]$, and (m, n) is the absolute pixel coordinate. The factor $L(m, n)$ is a function of the local statistics in a moving window.

If the logarithmic compressor has parameter D and linear gain G , i.e., $f(m, n) = D \log_e(R) + G$ where R is the envelope image, Dutt *et al.* [7] worked out the statistics of the log-compressed envelope based on K-distributed data. The mean and variance for the log compressed amplitude are written as:

$$\overline{y(m, n)} = G + \frac{D}{2} \left[\log_e(2\sigma^2) - \left(\gamma + \frac{1}{2\alpha} \right) \right], \quad (\text{A.8})$$

$$\sigma^2 = \frac{\pi^2}{24} D^2 [1 + \Gamma(\alpha)], \quad (\text{A.9})$$

where $\gamma \approx 0.5772$, is the Euler's constant, α is the scatterer density in the K-distribution. If $\Gamma''(\cdot)$ is the second derivative of the Gamma function, $\Gamma(\alpha)$ is defined as:

$$\Gamma(\alpha) \approx \frac{0.608}{\alpha} + \frac{0.231}{\alpha^2}. \quad (\text{A.10})$$

Figure A.3: Variance of log-compressed K-distributed data with K parameter α .

The variance for the log-compressed amplitude, with $D = 1$, is shown in Figure A.3

and the normalized variance is defined as:

$$\begin{aligned} V_N &= \frac{24\sigma^2}{\pi^2 \hat{D}^2} \\ &= 1 + \Gamma(\alpha). \end{aligned} \quad (\text{A.11})$$

Equation (A.11) is used by the local statistic $L(m, n)$ in (A.7) formulated as:

$$\begin{aligned} L(m, n) &= 1 - \frac{1}{V_N} \\ &\approx 1 - \frac{\pi^2 \hat{D}^2}{24\sigma^2}, \end{aligned} \quad (\text{A.12})$$

where $\hat{D}$ is an estimate of the logarithmic compression parameter and $\hat{\sigma}^2$ is the estimated local variance of the image. In [7], the filter statistic $L(m, n)$ is limited to $[0, 1]$. Ghofrani *et al.* [8], [120] took a similar approach. Instead of using K-distribution, they developed their algorithm based on Nakagami distribution [37]. The statistics of the log-compressed envelope for Nakagami distributed data is derived and the mean and variance values are given as:

$$\overline{y(m, n)} = G - \frac{D}{2} \log_e \left(\frac{b}{\Omega} \right) + \frac{D \Gamma'(b)}{2 \Gamma(b)}, \quad (\text{A.13})$$

$$\sigma^2 = \frac{D^2}{4} \zeta(b), \quad (\text{A.14})$$

where $\Gamma'(b)$ is the derivative of Gamma and $\zeta(b)$ is defined as:

$$\zeta(b) = \frac{1}{b^2} + \frac{1}{(b+1)^2} + \frac{1}{(b+2)^2} + \dots, \quad (\text{A.15})$$

where Ω and b are parameter of Nakagami distribution defined as follow:

$$\Omega = E\{R^2\}, \quad (\text{A.16})$$

and

$$b = \frac{\Omega^2}{E\{R^4\} - \Omega^2}. \quad (\text{A.17})$$

Similar treatments as that in (A.11) and (A.12) are then carried out to obtain the required local statistic for adaptive filtering.

A.2 Wavelet Shrinkage Technique

The main reason for the choice of multiscale bases of decompositions is that the statistics of many natural signals, when decomposed in such bases, are significantly simplified [9], [10], [11], [121]. Methods based on multiscale decompositions consist of three main steps: First, the raw data are analyzed by means of the wavelet transform, then the empirical wavelet coefficients are shrunk, and finally, the denoised signal is synthesized from the processed wavelet coefficients through the inverse discrete wavelet transform (IDWT)

A.2.1 Soft Thresholding and Nonlinear Feature Emphasis

Zong *et al.* [9] used the additive speckle noise model in (A.4) under a framework of multiscale wavelet analysis for speckle reduction and feature enhancement. A discrete dyadic wavelet transform (DWT) is a linear transform, so we have:

$$W[y^l(m, n)] = W[s^l(m, n)] + W[w_a^l(m, n)], \quad (\text{A.18})$$

where

$$W[s^l(m, n)] = \left\{ (W_j^d[s^l(m, n)])_{d=1,2;1 \leq j \leq J}, S_J[s^l(m, n)] \right\}, \quad (\text{A.19})$$

$$W[w_a^l(m, n)] = \left\{ (W_j^d[w_a^l(m, n)])_{d=1,2;1 \leq j \leq J}, S_J[w_a^l(m, n)] \right\}. \quad (\text{A.20})$$

Speckle which lies in high spatial frequency will be reduced to near zero after a finite number of repeated smoothing operations, so $S_J[w_a^l(m, n)] \rightarrow 0$. Feature enhancement is achieved through nonlinear stretching via feature energy gain on signal details $W_j^d[s^l(m, n)]$. The DWT coefficients are modified for noise reduction by:

$$W_j^{d,*}[y^l(m, n)] = T(W_j^d[y^l(m, n)], t_j^d), \quad (\text{A.21})$$

where $d = 1, 2; j = 1, \dots, k; k \leq J$, and t_j^d is a threshold related to the noise level, orientation and scale. It is computed through a linearly decreasing function:

$$t_j^d = \begin{cases} (T_{max} - \alpha(j-1)) \sigma_j^d & , \text{if } T_{max} - \alpha(j-1) > T_{min} \\ T_{min} \sigma_j^d & , \text{otherwise} \end{cases} \quad (\text{A.22})$$

where σ_j^d is the standard deviation, α is a decreasing factor between two consecutive levels, T_{max} and T_{min} are the maximum and minimum factor for $\sigma_j^d, 1 \leq j \leq J$ and $d \in \{1, 2\}$. The soft threshold operation is represented as:

$$T(v, t_j^d) = \text{sign}(v) (|v| - t_j^d)_+, \quad (\text{A.23})$$

$$(|v| - t_j^d)_+ = \begin{cases} |v| - t_j^d & , \text{if } |v| > t_j^d \\ 0 & , \text{otherwise} \end{cases} \quad (\text{A.24})$$

The feature is emphasized by the generalized adaptive gain (GAG) which is a non-linear operator defined as:

$$E_{GAG}(v) = \begin{cases} 0 & , |v| < T_1 \\ \text{sign}(v) T_2 + a(T_3 - T_2) \times \\ \quad \left[\text{sigm}(c(u - b)) - \text{sigm}(-c(u + b)) \right] & , T_2 \leq |v| \leq T_3 \\ v & , \text{otherwise} \end{cases} \quad (\text{A.25})$$

where $v \in [-1, 1]$, $u = \text{sign}(v)(|v| - T_2)/(T_3 - T_2)$, $b \in \{0, 1\}$, $0 \leq T_1 \leq T_2 \leq T_3 \leq 1$, c is a gain factor and a can be computed as:

$$a = \frac{1}{\text{sigm}(c(1 - b)) - \text{sigm}(-c(1 + b))}, \quad (\text{A.26})$$

$$\text{sigm}(v) = \frac{1}{1 + e^{-v}}, \quad (\text{A.27})$$

where T_1 , T_2 and T_3 are selected parameters. The interval $[T_2, T_3]$ serves as a sliding window for feature selectivity.

A.2.2 Bayesian Estimation

Achim *et al.* in [11] designed a Bayesian estimator to recover the signal component of the wavelet coefficients in ultrasound images by using a symmetric alpha-stable ($S\alpha S$) signal prior distribution. The processor consists of two major modules: 1) a subband representation function that utilizes the wavelet transform; and 2) a Bayesian denoising algorithm based on $S\alpha S$ prior for the signal. The processing stages of this method are illustrated in Figure A.4.

First, the original image is logarithmically transformed to change multiplicative

Figure A.4: Block diagram of the multiscale homomorphic Bayesian-based algorithm.

speckle to additive white noise. The transformed image is then analyzed in the multiscale wavelet domain where the signal and noise components in various scales are modelled as $S\alpha S$ and Gaussian process, respectively. The parameters of the distributions are estimated from the measurements by means of a least squares (LS) fitting in the characteristic function domain. At the final step, a Bayesian processor based on an $S\alpha S$ signal prior is built at each scale for statistically optimal signal feature extraction and speckle suppression. Since this algorithm makes use of standard wavelet transform, the following will only elaborate the Bayesian processor which is designed for speckle removal.

The wavelet transform is a linear operation. Consequently, after applying the DWT to the additive speckle-noise model in (A.4) the results are, in each of the three directions, sets of noisy wavelet coefficients written as the sum of the transformations of the signal and of the noise:

$$d_{j,k}^i = s_{j,k}^i + \zeta_{j,k}^i, \quad (\text{A.28})$$

where $k = 0, \dots, 2^{J+j} - 1$ and $-1 < j < -J$ refer to the decomposition level or scale and $i = 1, 2, 3$ refers to the three spatial orientations.

In a Bayesian framework, referring to the above expression, $d_{j,k}$, $s_{j,k}$ and $\zeta_{j,k}$ are considered as samples of the random variables d , s , and ζ , respectively. The signal component s is modelled according to a $S\alpha S$ distribution with zero location parameter, while the noise component ζ is modelled as a zero-mean Gaussian random variable. The Bayes risk estimator $\hat{s}$ given the set of wavelet coefficients, d , is computed numerically:

$$\hat{s}(d) = \frac{\int P_{\zeta}(\zeta) P_s(s) s \, ds}{\int P_{\zeta}(\zeta) P_s(s) \, ds}, \quad (\text{A.29})$$

where $P_{\zeta}(\zeta)$ is the pdf of the wavelet coefficients corresponding to the noise.

To construct the Bayesian processor, first the parameters of the prior distributions of the signal s and noise ζ components of the wavelet coefficients d are estimated. Then, these parameters are used to build the two prior pdfs $P_\zeta(\zeta)$ and $P_s(s)$ and the nonlinear (in general) input-output relationship $\hat{s}(d)$. Equation (A.29) requires a parameterized model for the two pdfs that provides a good fit to the statistics of the ultrasound images.

Considering the signal and noise components to be independent and because of the lack of closed-form expressions for the general $S\alpha S$ pdf, a method based on characteristic functions is used. In particular, since the pdf of the measured coefficients is the convolution between the pdfs of the signal and noise components, the associated characteristic function of the measurements is given by the product of the characteristic functions of the signal and noise:

$$\Phi_d(\omega) = \Phi_s(\omega) \Phi_\zeta(\omega), \quad (\text{A.30})$$

where

$$\Phi_s(\omega) = e^{-\gamma_s |\omega|^{\alpha_s}} \quad ; \quad 1 < \alpha_s \leq 2 \quad (\text{A.31})$$

and

$$\Phi_\zeta(\omega) = e^{-\frac{\sigma^2}{2} |\omega|^2}. \quad (\text{A.32})$$

In the above equations, α_s and ζ_s are the $S\alpha S$ parameters, and σ is the noise standard deviation. Using the above expressions, estimation of the parameters α_s , γ_s , and σ are achieved by fitting the Fourier transform of the empirical pdf of the measured coefficients with function $\Phi_d(\omega)$ in the LS sense:

$$\{\hat{\alpha}_s, \hat{\gamma}_s, \hat{\sigma}\} = \arg \min_{\alpha_s, \gamma_s, \sigma} \sum_i^n [\Phi_d(\omega_i) - \Phi_{de}(\omega_i)]^2, \quad (\text{A.33})$$

where $\Phi_{de}(\omega_i)$ denotes the empirical characteristic function. In practice, the level of noise σ is estimated, and then the above equation is optimized with respect to the $S\alpha S$ parameters α_s and γ_s . A robust estimate of the noise standard deviation, σ , is obtained in the finest decomposition scale by the measured wavelet coefficients as:

$$\hat{\sigma} = \frac{1}{0.6745} \text{MAD}(\{d_{J,k}, 0 \leq k < 2^J\}), \quad (\text{A.34})$$

where the operator MAD signifies the median absolute deviation and J denotes the highest level of wavelet decomposition.

A.3 Nonlinear Anisotropic Diffusion

Diffusion algorithms remove noise from an image by modifying the image via a partial differential equation (PDE). Consider applying the isotropic diffusion (the heat equation) given by $\partial I(m, n; t) / \partial t = \text{div}(c \nabla I)$, using the original (degraded) image $I(m, n; 0)$ as the initial condition, where $I(m, n; 0) : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ is an image in the continuous domain, (m, n) specifies spatial position, t is an artificial parameter, c is the diffusion constant, where ∇I is the image gradient. Modifying the image according to this linear isotropic diffusion equation is equivalent to filtering the image with a Gaussian filter. Perona and Malik [122] replaced the classical isotropic diffusion with:

$$\frac{\partial I(m, n; t)}{\partial t} = \text{div} [g(\nabla I) \nabla I]. \quad (\text{A.35})$$

Here $g(\nabla I)$ is the diffusivity function or the edge-stopping function to adaptively control the smoothing. This function is chosen to satisfy $g(x) \rightarrow 0$ as $x \rightarrow \infty$ and is monotonically decreasing so the gradient strength increases and the diffusion is stopped across edges. The two diffusion functions by Perona and Malik are:

$$g_1(x) = \frac{1}{1 + \left(\frac{x}{k}\right)^2}, \quad (\text{A.36})$$

and

$$g_2(x) = e^{-\left(\frac{x}{k}\right)^2}, \quad (\text{A.37})$$

where k is the edge magnitude parameter. Diffusion encourages intra-region smoothing with preference to smoothing across the boundaries. The basis of this method is to suppress smoothing at the boundaries by selecting locally adaptive diffusion strengths. Flux $f(\nabla I) = g(\nabla I) \nabla I$ reveals the relation between the parameter k and the image gradient ∇I . Jin *et al.* [123] showed that the maximum flux $g_2(\nabla I) \nabla I$ is generated in regions with the gradient equal to $k/\sqrt{2}$, as shown in Figure A.5.

Below $k/\sqrt{2}$, the flux reduces to zero which represents the case of a homogeneous region. Above $k/\sqrt{2}$, the flux also reduces to zero, stopping diffusion across high gradients. The parameter k plays an important role in diffusion. If the k value is set too high, the filter will act as a smoothing filter, diffusing across the edge boundary, while if k is too low, small dilation will result in many iterations. At some k values, an edge will be introduced between the regions of high and low intensity.

Figure A.5: Section view of the flux shows that it has a peak at $k/\sqrt{2}$, $k=1$.

A.3.1 Speckle Reducing Anisotropic Diffusion

The selection of the threshold is vital to the filtering process. This section presents a method called speckle reducing anisotropic diffusion (SRAD) [12]. SRAD is adaptive and does not utilize hard thresholds to alter performance in homogeneous regions or in regions near edges and small features. It exploits the instantaneous coefficient of variation, which is shown to be a function of the local gradient magnitude and Laplacian operators. The instantaneous coefficient of variation is defined as:

$$q(m, n; t) = \sqrt{\frac{\frac{1}{2} \left(\frac{\nabla I}{I} \right)^2 - \frac{1}{4^2} \left(\frac{\nabla^2 I}{I} \right)^2}{\left[1 + \frac{1}{4} \left(\frac{\nabla^2 I}{I} \right) \right]^2}}. \quad (\text{A.38})$$

In this equation, $q(m, n; t)$ combines a normalized gradient magnitude operator and a normalized Laplacian operator to act like an edge detector for speckle imagery. High relative gradient magnitude and low relative Laplacian tend to indicate an edge. At the center of an edge, the Laplacian term undergoes zero crossing and the gradient term dominates, resulting in $q_{m,n} \rightarrow \nabla I_{m,n} / I_{m,n}$. $q_{m,n}$ allows for edge detection in bright regions as well as in dark regions. The diffusion function adopted has an expression resembles that of (A.36) and is given as:

$$g(q) = \frac{1}{1 + \frac{q^2(m, n; t) - q_0^2(t)}{q_0^2(t)(1 + q_0^2(t))}}, \quad (\text{A.39})$$

where $q_0(t)$ is the speckle scale function which effectively controls the amount of smoothing applied to the image by SRAD. It is estimated using:

$$q_0(t) \approx e^{-\frac{t}{\epsilon}}, \quad (\text{A.40})$$

A three-stage approach is used to calculate the SRAD PDE. In the first stage, the derivative approximations and the Laplacian approximation are calculated:

$$\begin{aligned} \nabla_R I_{m,n}^t &= [I_{m+1,n}^t - I_{m,n}^t, I_{m,n+1}^t - I_{m,n}^t], \\ \nabla_L I_{m,n}^t &= [I_{m,n}^t - I_{m-1,n}^t, I_{m,n}^t - I_{m,n-1}^t], \\ \nabla^2 I_{m,n}^t &= [I_{m+1,n}^t + I_{m-1,n}^t + I_{m,n+1}^t + I_{m,n-1}^t + 4I_{m,n}^t]. \end{aligned} \quad (\text{A.41})$$

In the second stage, the diffusion coefficient is calculated according to:

$$g_{m,n}^t = g \left[q \left(\frac{1}{I_{m,n}^t} \sqrt{|\nabla_R I_{m,n}^t|^2 + |\nabla_L I_{m,n}^t|^2}, \frac{1}{I_{m,n}^t} \nabla^2 I_{m,n}^t \right) \right]. \quad (\text{A.42})$$

Within the third stage, the divergence of $g(\cdot) \nabla I$ is calculated:

$$\begin{aligned} d_{m,n}^t &= g_{m+1,n}^t (I_{m+1,n}^t - I_{m,n}^t) + g_{m,n}^t (I_{m-1,n}^t - I_{m,n}^t) + \\ &\quad g_{m,n+1}^t (I_{m,n+1}^t - I_{m,n}^t) + g_{m,n}^t (I_{m,n-1}^t - I_{m,n}^t). \end{aligned} \quad (\text{A.43})$$

The numerical approximation to the differential equation is given by:

$$I_{m,n}^{t+1} = I_{m,n}^t + \frac{1}{4} d_{m,n}^t. \quad (\text{A.44})$$

A.3.2 Coherence Enhancing Diffusion

This section presents a technique called nonlinear coherent diffusion (NCD) [13]. This technique combines three different models according to speckle extent and image anisotropy. It changes progressively from (i) isotropic diffusion through (ii) anisotropic coherent diffusion to (iii) mean curvature motion (MCM). This structure maximally low-pass filters those parts of the image that correspond to fully developed speckle, while substantially preserving information associated with resolved-object structures. NCD uses a diffusivity matrix, D , to measure local coherence of structures such that the diffusion process becomes more directional in both the

gradient and the contour directions. The coherent diffusion model takes the form:

$$\frac{\partial I(\mathbf{m}, \mathbf{n}; t)}{\partial t} = \text{div} [\mathbf{D} \nabla I] . \quad (\text{A.45})$$

The design of $\mathbf{D}$ requires estimating the local coherence, which can be represented by the local contour and gradient principal directions and their relative contrast. To detect the local coherence, structure tensor is used:

$$\begin{pmatrix} I_m^2 & I_m I_n \\ I_m I_n & I_n^2 \end{pmatrix} . \quad (\text{A.46})$$

The multiscale structure matrix takes the form:

$$J_\rho(\nabla I) = k_\rho * (\nabla I \nabla I^T) \quad ; \rho \geq 0 \quad (\text{A.47})$$

or equivalently,

$$J_\rho(I) = \begin{pmatrix} k_\rho * I_m^2 & k_\rho * (I_m I_n) \\ k_\rho * (I_m I_n) & k_\rho * I_n^2 \end{pmatrix} . \quad (\text{A.48})$$

The symbol $*$ stands for convolution where the convolution kernel, $k_\sigma(\mathbf{m}, \mathbf{n}) = \frac{1}{2\pi\sigma^2} e^{-\frac{\mathbf{m}^2 + \mathbf{n}^2}{2\sigma^2}}$. The above convolution is done component-wise mainly to average a feature over a known neighborhood (scale) where σ is the integration scale (the window size) over which the orientation information is averaged. Using eigenvalue decomposition, we have:

$$J(I) = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{pmatrix} . \quad (\text{A.49})$$

Here, the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ and the eigenvalues u_1, u_2 correspond to the directions of maximum and minimum variations and the strength of these variations, respectively. The NCD method chose the diffusion tensor, $\mathbf{D}$, with the same eigenvectors of the structure matrix and the eigenvalues that represent the strength of diffusion in each principal direction. That is:

$$D(I) = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{pmatrix} . \quad (\text{A.50})$$

and

$$\lambda_1 = \begin{cases} \alpha \left(1 - \frac{(u_1 - u_2)^2}{s^2} \right) & , \text{if } (u_1 - u_2)^2 \leq s^2 \\ 0 & , \text{otherwise} \end{cases} \quad (\text{A.51})$$

$$\lambda_2 = \alpha. \quad (\text{A.52})$$

The main objective of this formulation is to produce a filter that is not only edge preserving but also coherence enhancing. $(u_1 - u_2)$ serves to measure the local coherence of the region whereas λ_1 is related to the anisotropy of the image $(u_1 - u_2)$ through a monotonically decreasing function. The stopping level s^2 is determined experimentally corresponding to the fully structured region resulting from the imaging system and it is manually set by the user.

A.4 Summary

In this appendix, we have given a brief survey of enhancement methods other than blind deconvolution proposed in recent publications. We have grouped them under the categories of adaptive filtering, wavelet shrinkage technique and nonlinear anisotropic diffusion. All of these algorithms work on ultrasound backscattered echo envelope instead of RF echo signal. The ultrasound image is interpreted as an independent piece of 2-D image.

AWMF processes the envelope image based on the assumed speckle-noise model in (A.5). The other two adaptive algorithms in Appendix A.1.2 assumed that the underlying echo envelope is K-distribution and Nakagami distribution respectively. Based on this assumption, they derived their statistics for the log-compressed image and used the local statistic in the adaptation parameter for adaptive filtering. In Appendix A.2, the wavelet shrinkage techniques adopted the additive noise model in (A.4). Being a linear transformation, it enables different operation on the signal and the noise components. This operation can be a nonlinear threshold operation for noise removal or an operation to enhance features. Appendix A.3 presents the anisotropic diffusions. The algorithms attempt to overcome the ambiguity of using SNR alone and to further enhance the image structure. Generally speaking, the diffusion process is difficult to control. The number of iterations is determined through experiment and, very often, the noise and features are difficult to differentiate which lead to confusing results.

Appendix B

Simulation of the Ultrasound Images

To test our restoration algorithms on ultrasound images, we model an observed ultrasound image of 3 cm by 3 cm. Let $y(\mathbf{r}, t)$ represent the 2-D ultrasonic RF echo data set, where $\mathbf{r}$ and t denote the lateral coordinate and propagation of the ultrasound in the axial direction, respectively. It is assumed that the imaging system has a linear, spatially invariant point spread function (PSF). We model the structures within tissue that are responsible for backscattered ultrasound field by point-scatterers and allowing the population of scatterers of the imaging region to be represented by a scattering function, $s(\mathbf{r}, t)$, the ultrasound RF echo is governed by:

$$y(\mathbf{r}, t) = v(\mathbf{r}, t) * s(\mathbf{r}, t), \quad (\text{B.1})$$

where $v(\mathbf{r}, t)$ is the ultrasonic 2-D PSF and $*$ denotes linear convolution operator. We assume that $v(\mathbf{r}, t)$ is separable [12], [49], i.e., $v(\mathbf{r}, t) = h_{pe}(\mathbf{r})v_{pe}(t)$ where $h_{pe}(\mathbf{r})$ is the spatial response for the transmitting and receiving aperture determined by:

$$h_{pe}(\mathbf{r}) = e^{-\frac{\mathbf{r}^2}{2\sigma_r^2}}, \quad (\text{B.2})$$

where $\sigma_r = 0.95$ mm representing the beam-width of the scanner probe. The second term $v_{pe}(t)$ is the transmitted pulse which is modelled as a sine modulated Gaussian function [33], [37], [58] given by:

$$v_{pe}(t) = -te^{-4\beta^2 t^2} \sin(2\pi f_0 t), \quad (\text{B.3})$$

where f_0 is the center frequency (3.5 MHz) and β is the bandwidth (1.72 MHz). In the simulation, the ultrasonic system impulse response is normalized with respect to the total energy such that $\sum_{\mathbf{r},t} v^2(\mathbf{r},t) = 1$.

The function $s(\mathbf{r},t)$ representing underlying tissue structure and is written as:

$$s(\mathbf{r},t) = \sum_{p=1}^M \sum_{q=1}^N \alpha_{p,q} \delta \left(\mathbf{r} - \mathbf{r}_p, t - 2\frac{z_q}{c} \right), \quad (\text{B.4})$$

where $\alpha_{p,q}$ is the backscattering scattering strength of the (p,q) th scatterer has a Gamma distribution [33], [35], [37], [58], [59] and the scatterers are randomly distributed in the tissue with a uniform distribution [33], [35], [37], [58]. The randomness in the variation of scatterers is expressed as the signal-to-noise ratio of the scattering amplitude which is set to 20 and the densities of randomly located scatterers are set to 5 scatterers/mm and they are uniformly distributed. MN is the number of scatterers in the range cell defined as $c\Delta\tau/2$ where τ is the pulse width (set via β), z_q is the location of the scatterer from the origin, and c is the velocity of sound (1540 m/s).

The simulated tissue is designated to give fully developed speckle and it has a cylindrical disc located at its central. The mean scattering strengths for the background and the cylindrical regions are assigned to be 25 and 99 respectively. Therefore, the received ultrasound echo signal, $y(\mathbf{r},t)$, is a superposition of echoes from all the scatterers present in the scattering volume:

$$y(\mathbf{r},t) = \sum_{p=1}^M \sum_{q=1}^N \alpha_{p,q} v \left(\mathbf{r} - \mathbf{r}_p, t - 2\frac{z_q}{c} \right). \quad (\text{B.5})$$

To model the inevitable noise in the measured signal, $y(\mathbf{r},t)$ is then corrupted by adding zero mean white Gaussian noise to give root-mean-square signal-to-noise ratio of 20 which is worst than that of the practical value [17].

In each simulation, the received RF backscattered signal is sampled at 20 MHz. The step size of transducer along the lateral direction is set to 0.1 mm. The resulting sampled ultrasound RF data has a size of 796 by 360. The ultrasound RF data if demodulated at 3.5 MHz is an envelope image such as that shown in Figure B.1. Due to the nature of sampling, the number of sample points along the axial (temporal) direction is much more than that of the lateral and the resultant image is actually elongated in the axial direction. In order to approximately maintain the aspect

ratio of the original target, it is down sampled by 3 in the axial direction for display. Besides, it is worth noted that the original scatter map cannot be shown in the printout as the scatterers are too small dots scattering over the scatter map. If printed out it will be a completely dark piece of image because of the domination of scatter map by the empty cells, i.e., as darker dots correspond to those scatters of lower scattering amplitude.

STANDARD

Figure B.1: Simulated ultrasound envelope image.

Appendix C

Description of the Real Ultrasound RF Data

A set of eight different unfiltered RF data is recorded from the abdomen of adult volunteers with a VIVID3 (GE, Medical ultrasound, Inc.) commercial ultrasound scanner, equipped with a special data-transfer board. These data are obtained by a linear array probe, with a central frequency of about 3.5 MHz. These images are more desired as they are simpler in visualizations which do not require a scan conversion stage. The images are acquired with a single transmission focal point, localized approximately at the center of the field of view. All the images are composed, on average, of 200 RF-line, each of 1024 sample points in length. The sampling rate and resolution are 20 MHz and 16-bit, respectively.

The ultrasound RF data if demodulated at 3.5 MHz is an envelope image such as that shown in Figure C.1.

STANDARD

(a)

STANDARD

(b)

STANDARD

(c)

STANDARD

(d)

Figure C.1: Real array test ultrasound envelope images.

Appendix D

Description of the Generalized Gaussian Density Function

This appendix gives the associated parameters of the generalized Gaussian density (GGD) function. Experimentally, we have found that detailed image histograms can be modelled with the family of histograms [96], [108]:

$$h(x) = K e^{-\left(\frac{|x|}{\alpha}\right)^\beta}. \quad (\text{D.1})$$

The histogram is zero-mean and symmetric, and the parameters α and β are directly related to the second and fourth moments. The parameter β modifies the decreasing rate of the peak and α models the variance. The constant K is adjusted so that $\int_{-\infty}^{\infty} h(x) dx = N$ where N is the total number of pixels of the given detail image. Thus, K is found as follows:

$$K = \frac{N}{\int_{-\infty}^{\infty} e^{-\left(\frac{|x|}{\alpha}\right)^\beta} dx}. \quad (\text{D.2})$$

By changing the variables in the integral, $u = (|x|/\alpha)^\beta$, $dx = u^{1/\beta-1} \alpha/\beta du$, we have:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\left(\frac{|x|}{\alpha}\right)^\beta} dx &= \frac{2\alpha}{\beta} \int_0^{\infty} e^{-u} u^{1/\beta-1} du \\ &= \frac{2\alpha}{\beta} \Gamma\left(\frac{1}{\beta}\right). \end{aligned} \quad (\text{D.3})$$

Therefore, when (D.3) is substituted into (D.2), we obtain:

$$K = \frac{N \beta}{2 \alpha \Gamma\left(\frac{1}{\beta}\right)}, \quad (\text{D.4})$$

where $\Gamma(t) = \int_0^\infty e^{-u} u^{t-1} du$. The coefficients α and β of the histogram model can be computed by measuring the first and second moment of the detail image histogram:

$$\begin{aligned} m_1 &= \int_{-\infty}^{\infty} |x| h(x) dx, \\ m_2 &= \int_{-\infty}^{\infty} x^2 h(x) dx. \end{aligned} \quad (\text{D.5})$$

Following the same procedure, by inserting the histogram model in (D.1) into (D.5) and changing of variables in these two integrals, it can easily show that:

$$\begin{aligned} m_1 &= \frac{2 K \alpha^2}{\beta} \Gamma\left(\frac{2}{\beta}\right), \\ m_2 &= \frac{2 K \alpha^3}{\beta} \Gamma\left(\frac{3}{\beta}\right). \end{aligned} \quad (\text{D.6})$$

In practice, both m_1 and m_2 can be computed from the given samples. Therefore, it is of our interest to derive an expression for both α and β in terms of m_1 and m_2 . From the expressions (D.6), we observe that:

$$\frac{m_1^2}{m_2 N} = \frac{\Gamma\left(\frac{2}{\beta}\right)^2}{\Gamma\left(\frac{1}{\beta}\right) \Gamma\left(\frac{3}{\beta}\right)}. \quad (\text{D.7})$$

Therefore, β can be obtained through defining a function:

$$\beta = F^{-1}\left(\frac{m_1^2}{m_2 N}\right), \quad (\text{D.8})$$

where

$$F(u) = \frac{\Gamma\left(\frac{2}{u}\right)^2}{\Gamma\left(\frac{1}{u}\right) \Gamma\left(\frac{3}{u}\right)}. \quad (\text{D.9})$$

The function $F^{-1}(u)$ is shown in the Figure ??:

Figure D.1: Graph of the function $F^{-1}(u)$ against u characterized by (D.8).

Moreover, it can be shown from (D.4) and (D.6) that $m_2 = N \alpha^2 \Gamma(3\beta^{-1})/\Gamma(\beta^{-1})$. Noted that $m_2 = \sigma_x^2$ and, hence, we have:

$$\alpha = \sigma_x \sqrt{\frac{\Gamma\left(\frac{1}{\beta}\right)}{N \Gamma\left(\frac{3}{\beta}\right)}} \Rightarrow \frac{1}{\alpha} = \frac{\sqrt{\frac{N \Gamma(3\beta^{-1})}{\Gamma(\beta^{-1})}}}{\sigma_x} = \frac{\eta(\beta)}{\sigma_x}. \quad (\text{D.10})$$

Equation (??) shows that α is a function of β . If we let $\eta(\beta)/\sigma_x = 1/\alpha$ for $\sigma_x = \sqrt{m_2}$, the histogram model in (D.1), upon normalization such that N equal to 1, can now be rewritten in a form resemble the Gaussian density function called the *generalized Gaussian density* or GGD:

$$p_x(x) = \frac{\beta \eta(\beta)}{2 \Gamma\left(\frac{1}{\beta}\right) \sigma_x} e^{-\left(\frac{\eta(\beta)|x|}{\sigma_x}\right)^\beta}. \quad (\text{D.11})$$

An exponent of $\beta = 1$ corresponds to the Laplacian density, and $\beta = 2$ corresponds to a Gaussian density. In general, smaller values of β lead to a density that is both more concentrated at zero and has more expansive tails.

Appendix E

Parameters Estimation of the Generalized Gaussian Density Function

This appendix gives the second and fourth moments of the noisy observations whose signals are of generalized Gaussian distribution and contaminated by independent and identically distributed additive white Gaussian noise. If the noisy observations are given by:

$$x = f + z, \quad (\text{E.1})$$

where f is the zero-mean signal following the generalized Gaussian distribution and z is zero-mean white Gaussian noise. The second and fourth moments of x are calculated:

$$\begin{aligned} m_2 &= \int_{-\infty}^{\infty} x^2 p_x(x) dx, \\ m_4 &= \int_{-\infty}^{\infty} x^4 p_x(x) dx, \end{aligned} \quad (\text{E.2})$$

where $p_x(x)$ is the probability density function (pdf) of x . From the binominal expansion of $(f + z)^2 = f^2 + 2fz + z^2$, we deduce that:

$$\begin{aligned} m_2 &= \int_{-\infty}^{\infty} f^2 p_f(f) df + \int_{-\infty}^{\infty} z^2 p_z(z) dz \\ &= \sigma_f^2 + \sigma_z^2, \end{aligned} \quad (\text{E.3})$$

where $p_x(z)$ is the pdf of z , σ_f^2 and σ_x^2 are the signal and noise variances respectively. For the noiseless case given in (D.6) and since $\int_{-\infty}^{\infty} p_x(x) dx = N = 1$, the expression of σ_f^2 is as follows:

$$\begin{aligned}\sigma_f^2 &= \frac{2K\alpha^3}{\beta} \Gamma\left(\frac{3}{\beta}\right) \\ &= \frac{\alpha^3 \Gamma\left(\frac{3}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}.\end{aligned}\tag{E.4}$$

Likewise, from the binomial expansion of $(f+z)^4 = f^4 + 4f^3z + 6f^2z^2 + 4fz^3 + z^4$, we deduce that:

$$\begin{aligned}m_4 &= \int_{-\infty}^{\infty} f^4 p_f(f) df + 6\sigma_f^2\sigma_x^2 + \int_{-\infty}^{\infty} z^4 p_x(z) dz \\ &= \frac{\alpha^4 \Gamma\left(\frac{5}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)} + \frac{6\sigma_x^2\alpha^3 \Gamma\left(\frac{3}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)} + 3\sigma_x^4.\end{aligned}\tag{E.5}$$

Assuming that σ_x^2 is known, the measurements of the second and fourth moments of the noisy data, thus provide an estimation of the model parameters of the generalized Gaussian distribution via (E.3) and (E.5). An estimation of parameters is found as:

$$\begin{aligned}\hat{\beta} &= \arg \min_{\beta} \left| \frac{\Gamma\left(\frac{1}{\beta}\right) \Gamma\left(\frac{5}{\beta}\right)}{\Gamma\left(\frac{3}{\beta}\right)^2} - \frac{m_4 - 3\sigma_x^4 - 6\sigma_x^2(m_2 - \sigma_x^2)}{(m_2 - \sigma_x^2)^2} \right|, \\ \hat{\alpha} &= \sqrt{\frac{\sigma_f^2 \Gamma\left(\frac{1}{\hat{\beta}}\right)}{\Gamma\left(\frac{3}{\hat{\beta}}\right)}}.\end{aligned}\tag{E.6}$$

The computation of (E.6) is sped up with the help of lookup tables. If we let $u = \frac{m_4 - 3\sigma_x^4 - 6\sigma_x^2(m_2 - \sigma_x^2)}{(m_2 - \sigma_x^2)^2}$ and $F_{\beta}(\beta) = \Gamma\left(\frac{1}{\beta}\right) \Gamma\left(\frac{5}{\beta}\right) / \Gamma\left(\frac{3}{\beta}\right)^2$, we can pre-compute $F_{\beta}(\beta)$ and deduce an estimation of β being $\hat{\beta} = F_{\beta}^{-1}(u)$ as shown in Figure E.1. In practice, $F_{\beta}^{-1}(u)$ is stored as discrete values in the computer. Therefore, the finding of the value of u that is close to that stored in the lookup table, $F_{\beta}^{-1}(u)$, which is equivalent to solving (E.6). Likewise, we may pre-compute $F_{\alpha}(\hat{\beta})$ for the estimation of α using $\hat{\alpha} = \sigma_f \sqrt{\Gamma\left(\frac{1}{\hat{\beta}}\right) / \Gamma\left(\frac{3}{\hat{\beta}}\right)} = \sigma_f F_{\alpha}(\hat{\beta})$. Since $F_{\alpha}(\hat{\beta})$ is a monotonically

increasing, Figure E.2 shows that α models the signal standard deviation σ_f by a scaling function $F_\alpha(\hat{\beta})$ which is dependent on β .

Figure E.1: Graph of the function $F_\beta^{-1}(u)$ against u .

Figure E.2: Graph of the function $F_\alpha(\beta)$ against β .

Appendix F

Histogram of the Symmetrical Mirror Wavelet Coefficients

This appendix uses a noiseless scattering image as an example to demonstrate the general distribution of the typical histograms of the noiseless subband wavelet coefficients under the symmetrical mirror wavelet basis. More importantly, we show that the generalized Gaussian distribution described in Appendix D is a good model of the noiseless subband wavelet coefficients. The noiseless scattering image is generated according to Appendix B but with a constant scattering strength. Each of the histograms of the subband wavelet coefficients of the noiseless scattering image is plotted in the following figures with a dashed curve corresponding to the best fitting instance of this density function, with its parameters $\{\alpha, \beta\}$ estimated by method of moments of the data under the model. The density model fits the histograms remarkably well. We have observed that values of the exponent β typically lie in the range $[0.5, 2]$. The factor α varies monotonically with the scale of the basis functions, with correspondingly higher variance for coarser-scale components.

To facilitate our studies, we label the subband wavelet coefficients at the specific segmented frequency plane of the symmetrical mirror wavelet basis as depicted in Figure F.1. The symbolic labeling of the of the subband coefficients is referenced to the mirror origin of the frequency plane; that is $(N/4, N/4)$. Using these notations and for simplicity, only the histograms of the diagonal subband wavelet coefficients are presented in Figure F.2 to Figure F.5.

Figure F.1: Symbolic representation of the subband wavelet coefficients defined by the symmetrical mirror wavelet basis illustrated at the frequency plane.

Figure F.2: Histogram of the mirror wavelet coefficients. (a) $d_{L+2}^{LL,d}$, (b) $d_{L+3}^{LL,d}$, (c) $d_{L+4}^{LL,d}$, and (d) $d_{L+5}^{LL,d}$.

Figure F.3: Histogram of the mirror wavelet coefficients. (a) $\tilde{d}_{L+2}^{LH,d}$, (b) $\tilde{d}_{L+3}^{LH,d}$, (c) $\tilde{d}_{L+4}^{LH,d}$, and (d) $\tilde{d}_{L+5}^{LH,d}$.

Figure F.4: Histogram of the mirror wavelet coefficients. (a) $\tilde{d}_{L+2}^{HL,d}$, (b) $\tilde{d}_{L+3}^{HL,d}$, (c) $\tilde{d}_{L+4}^{HL,d}$, and (d) $\tilde{d}_{L+5}^{HL,d}$.

Figure F.5: Histogram of the mirror wavelet coefficients. (a) $\tilde{d}_{L+2}^{HH,d}$, (b) $\tilde{d}_{L+3}^{HH,d}$, (c) $\tilde{d}_{L+4}^{HH,d}$, and (d) $\tilde{d}_{L+5}^{HH,d}$.