Investigation and development of alternating-direction-implicit finite-difference time-domain method

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Investigation and Development of Alternating- Direction-Implicit Finite-Difference Time- Domain Method

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School of Electrical & Electronic Engineering

A thesis submitted to the Nanyang Technological University
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Summary

The conventional Finite-Difference Time-Domain (FDTD) method is widely used for solving various problems related to electromagnetism. However, the Courant-Friedrich-Levy (CFL) condition must be satisfied when this method is applied so as to ensure the algorithm to be stable. To eliminate the limit on time step, the Alternating-Direction-Implicit FDTD (ADI-FDTD) method was proposed, which was proven to be unconditionally stable and free from the CFL condition so that the iteration number of time step may be decreased considerably. Hence it can be found that the ADI-FDTD method is a promising and useful method for electromagnetic computation due to its unconditional stability. My thesis work is mainly dedicated to the investigation and development of ADI-FDTD method.

Although the stability and numerical dispersion have been analyzed for the ADI-FDTD method, most analyses are considering the case of lossless media. This thesis performs the stability and dispersion analyses for the ADI-FDTD method in lossy media. This will be meaningful for the evaluation and further development of the ADI-FDTD method in lossy media.

In order to achieve improved numerical dispersion performance, a series of three-dimensional (3-D) higher order ADI-FDTD methods are presented, as well as a parameter optimized ADI-FDTD method based on the (2,4) stencil (denoted as (2,4) PO-ADI-FDTD method). The stability and numerical dispersion of these methods are also studied.

Summary

The thesis also proposes the ADI-FDTD method including passive lumped elements, which is proven to be unconditionally stable. So the stability of these proposed schemes is neither related to the mesh size, nor related to the values of the elements, which is therefore a distinguished advantage over the previous extended FDTD. The general linear lumped network is also incorporated into the ADI-FDTD method to extend the previous work, and the higher efficiency achieved has been demonstrated by the numerical experiment as well.

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Chapter 1

Introduction

1.1 Motivation

The Finite-Difference Time-Domain (FDTD) method [1] is simple, flexible and very useful numerical technique for solving various problems related to electromagnetism [2]. However, the FDTD method is based on an explicit finite-difference algorithm. Therefore the Courant-Friedrich-Levy (CFL) condition, which ensures no exponential growth during the updating procedure so as to the stability of the FDTD method [2], must be satisfied when FDTD method is used. Due to the restriction from the CFL condition, a maximum time-step size is limited by the minimum cell size in a computational domain. The minimum cell size is related to not only the smallest wavelength of interest, but also the finest scale dimension of the object to be analyzed. Thus sometimes the object to be analyzed, which has much finer scale dimensions compared with the wavelength, will lead to a very small time step and a significant increase in computing time.

To eliminate the limit on time step, an unconditional stable FDTD method, the Alternating-Direction-Implicit FDTD (ADI-FDTD) method [3], was proposed by Namiki in 1999. This method was proven to be unconditionally stable and free from

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the CFL condition, and thus the iteration number of time step may be reduced considerably. Although there are still other unconditionally stable FDTD methods, which include the one based on the weighted Laguerre polynomials [4], Split-Step method [5], [6] and Crank-Nicolson method [7]-[9], the ADI-FDTD method should be the most popular one among them, since compared with other unconditionally stable methods much more research works have been carried out for the investigation and development of the ADI-FDTD method according to the literature survey in Chapter 2. My thesis work is mainly dedicated to the investigation and development of the ADI-FDTD method, which involves the stability and dispersion analyses of the ADI-FDTD method, development of ADI-FDTD methods with improved dispersion performance, and incorporation of lumped elements to the ADI-FDTD method.

Since the unconditional stability ensures that the time step is free from the CFL condition and the numerical dispersion is often used to assess the accuracy of the FDTD method, it is important to study the stability and numerical dispersion of the ADI-FDTD method. Although the stability and numerical dispersion have been analyzed for the ADI-FDTD method [3], [10]-[19], most analyses are dedicated to the case of lossless media. So it is necessary to study the individual stability and numerical dispersion of the ADI-FDTD method in lossy media.

In order to reduce the numerical dispersion error, the ADI-FDTD methods with improved dispersion performance also need to be developed.

There have been many efforts made for the extended FDTD to incorporate the

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passive and active lumped elements into the explicit FDTD method so as to integrate electromagnetic analysis and circuit simulation seamlessly, e.g. [20]-[34]. According to the investigation of the stability for the schemes including passive elements like resistors, capacitors and inductors [35]-[38], one can find that the stability of the previous schemes, which were developed from the conventional explicit FDTD method, is related to either the mesh size, or both the mesh size and the values of the elements. This thesis will incorporate the lumped elements into the ADI-FDTD to achieve higher computation efficiency based on its unconditional stability.

1.2 Objectives

According to the motivation presented in the preceding section, the main objectives of my thesis work include:

- 1) Perform the stability and numerical dispersion analyses for the ADI-FDTD method in lossy media so as to prove its unconditional stability and reveal its numerical loss and dispersion characteristics.
- 2) Propose ADI-FDTD method with improved dispersion performance and perform the corresponding stability and dispersion analyses.
- 3) Incorporate lumped element into ADI-FDTD method with stability analysis and numerical demonstration experiments.

1.3 Major Contributions of This Thesis

This thesis achieves several original contributions and some of them have already been accepted for publication. The major contributions are listed below:

The stability and dispersion analyses for the ADI-FDTD method in lossy media are presented. The stability analysis theoretically proves the unconditional stability of this ADI-FDTD method in lossy media. The dispersion analysis reveals the numerical loss and dispersion characteristics of this scheme. This will be meaningful for the evaluation and further development of the ADI-FDTD method in lossy media.

A series of 3-D higher order ADI-FDTD methods are discussed, where their stability and dispersion analyses are presented. The higher order scheme is one usual way to achieve better numerical dispersion performance, and the study on stability and dispersion will be useful for the selection and evaluation of various higher order ADI methods.

A parameter optimized (PO) ADI-FDTD method based on the (2,4) stencil ((2,4) PO-ADI-FDTD) to achieve better dispersion performance is also developed. The (2,4) PO-ADI-FDTD method has been proven to be unconditionally stable for any real parameters. By setting different optimization objectives, the method can be applied to satisfy different requirements. The variation of the performance of the (2,4) PO-ADI-FDTD method for different frequencies has also been studied. One can find that the (2,4) PO-ADI-FDTD method can basically preserve the desired

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performance within a band below the frequency where the optimization of parameters is performed. Compared with the existing (2,2) PO-ADI-FDTD method, this (2,4) PO-ADI-FDTD method possesses the following advantages: 1) The (2,4) PO-ADI-FDTD method is unconditionally stable for any real parameters, however, the (2,2) PO-ADI-FDTD requires the parameters to satisfy some conditions to maintain the unconditional stability; 2) The (2,4) PO-ADI-FDTD method achieves much better performance than the (2,2) PO-ADI-FDTD method for the same requirements; 3) The (2,4) PO-ADI-FDTD method basically preserves the desired performance within a band below certain frequency, while the (2,2) PO-ADI-FDTD method only maintains the performance within a narrow band.

The ADI-FDTD method including passive lumped elements is proposed, which is proven to be unconditionally stable. So the stability of the proposed schemes is neither related to the mesh size, nor related to the values of the elements, which is therefore a distinguished advantage over the previous extended FDTD, whose stability is related to either the mesh size, or both the mesh size and the values of the elements. Some numerical experiments have been performed for the demonstration of unconditional stability and higher efficiency, as well as validation of the proposed schemes. This thesis also incorporates the linear lumped network to the ADI-FDTD method as an extension of the preceding work, and the higher efficiency achieved has also been demonstrated by the numerical experiment.

1.4 Organization of This Thesis

In the next chapter, a few previous representative works in this area of ADI-FDTD

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are surveyed and some basic concepts of this method will be introduced. My own thesis work will be presented in Chapters 3 to 6. Particularly, the stability and numerical dispersion analyses of the ADI-FDTD in lossy media are presented in Chapter 3. Chapter 4 discusses a series of 3-D higher order ADI-FDTD methods and performs the stability and dispersion analyses. A parameter optimized ADI-FDTD method based on the (2,4) stencil to achieve better dispersion performance is developed in Chapter 5, where the stability and dispersion analyses are also provided. Chapter 6 incorporates the lumped elements into the ADI-FDTD method, whose unconditional stability is proved. The ADI-FDTD method including general linear lumped network is also presented in this chapter. Finally, the conclusion is provided in Chapter 7 and some recommendations for future research are highlighted.

Chapter 2

Literature Survey of ADI-FDTD Method

The ADI-FDTD method has been proven to be unconditionally stable and free from the CFL condition so that the limit on time step due to the CFL condition can be eliminated and the iteration number of time step may be decreased considerably. There are usually some electrically very small parts in the geometries to be analyzed, so very fine spatial grids have to be adopted to resolve such geometries and the time step of the conventional explicit FDTD subsequently has to be very small because of the CFL condition. This will lead to much more computation iterations so that the problems may be difficult to be solved. The ADI-FDTD method is especially effective for such problems owing to its unconditional stability. Hence it can be found that the ADI-FDTD method is a promising and useful method for electromagnetic computation because of its unconditional stability. In fact, a number of researchers have recognized this potential and many research works have been carried out in this area during recent years. Some representative ones among them will be listed below as a literature survey, which also demonstrates the popularity of this method.

Chapter2: Literature Survey of ADI-FDTD Method

Considering the stability and numerical dispersion characteristics of the ADI-FDTD method, a few works have been carried out for the investigation of these characteristics [3], [10]-[19]. The perfectly matched layer (PML) absorbing boundary condition (ABC) [39]-[42] is also implemented for the truncation of ADI method [43]-[53], which facilitates the application of this method to practical problems. The ADI-FDTD methods including various dispersive media have been developed [54]-[56], which are justified to be more efficient than the conventional explicit FDTD including dispersion media. So far the ADI method have been applied to solve many practical problems [57]-[66] so as to utilize its advantage of unconditional stability. All the works surveyed above illustrate the potential of the ADI-FDTD to be a promising solver for electromagnetic problems.

My thesis work is mainly dedicated to the investigation and development of ADI-FDTD method. This chapter will focus on the introduction to some basic concepts of the ADI-FDTD method, such as the formulas, stability and numerical dispersion analyses, so as to facilitate the further discussion. The later chapters will present my own thesis work based on these introductions.

2.1 Formulas of ADI-FDTD Method

Let us consider Maxwell's curl equations in a source-free, lossless, linear diagonal anisotropy and non-dispersive medium:

$$\vec{\varepsilon} \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{H} \quad (2-1a)$$

$$\vec{\mu} \frac{\partial \vec{H}}{\partial t} = -\nabla \times \vec{E} \quad (2-1b)$$

where $\vec{\varepsilon}$ and $\vec{\mu}$ are the tensors of the electric permittivity and magnetic permeability respectively:

Chapter2: Literature Survey of ADI-FDTD Method

$$\vec{\varepsilon} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}, \vec{\mu} = \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \quad (2-2)$$

The vector equation (2-1) can be expressed in a 3-D rectangular coordinates system as:

$$\frac{\partial}{\partial t} \begin{bmatrix} \varepsilon_x \vec{E}_x \\ \varepsilon_y \vec{E}_y \\ \varepsilon_z \vec{E}_z \\ \mu_x \vec{H}_x \\ \mu_y \vec{H}_y \\ \mu_z \vec{H}_z \end{bmatrix} = \begin{bmatrix} \frac{\partial \vec{H}_z}{\partial y} \\ \frac{\partial \vec{H}_x}{\partial z} \\ \frac{\partial \vec{H}_y}{\partial x} \\ \frac{\partial \vec{E}_y}{\partial x} \\ \frac{\partial \vec{E}_z}{\partial x} \\ \frac{\partial \vec{E}_x}{\partial y} \end{bmatrix} + \begin{bmatrix} -\frac{\partial \vec{H}_y}{\partial z} \\ -\frac{\partial \vec{H}_z}{\partial x} \\ -\frac{\partial \vec{H}_x}{\partial y} \\ -\frac{\partial \vec{E}_z}{\partial y} \\ -\frac{\partial \vec{E}_x}{\partial z} \\ -\frac{\partial \vec{E}_y}{\partial x} \end{bmatrix} \quad (2-3)$$

If the Maxwell's curl equations are discretized by the standard Crank-Nicolson finite difference scheme, it can be represented in a matrix form as

$$\mathbf{D}_d \frac{\vec{u}^{n+1} - \vec{u}^n}{\Delta t} = \mathbf{A} \frac{\vec{u}^{n+1} + \vec{u}^n}{2} + \mathbf{B} \frac{\vec{u}^{n+1} - \vec{u}^n}{2} + O(\Delta t^2) \quad (2-4)$$

where

$$\vec{u}^n = \begin{bmatrix} \vec{E}_x^n & \vec{E}_y^n & \vec{E}_z^n & \vec{H}_x^n & \vec{H}_y^n & \vec{H}_z^n \end{bmatrix}^T \quad (2-5a)$$

$$\mathbf{D}_d = \begin{bmatrix} \mathbf{D}_{\varepsilon x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{D}_{\varepsilon y} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{D}_{\varepsilon z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{D}_{\mu x} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{D}_{\mu y} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{D}_{\mu z} \end{bmatrix} \quad (2-5b)$$

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$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{D}_{Ay} \\ 0 & 0 & 0 & \mathbf{D}_{Az} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{D}_{Ax} & 0 \\ 0 & -\mathbf{D}_{Az}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{D}_{Ax}^T & 0 & 0 & 0 \\ -\mathbf{D}_{Ay}^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2-5c)$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{D}_{Bz} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{D}_{Bx} \\ 0 & 0 & 0 & \mathbf{D}_{By} & 0 & 0 \\ 0 & 0 & -\mathbf{D}_{By}^T & 0 & 0 & 0 \\ -\mathbf{D}_{Bz}^T & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathbf{D}_{Bx}^T & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2-5d)$$

Δt is the time step size and n is the time index. Vector $\vec{u}$ comprises the electric field vectors $\vec{E}_x$, $\vec{E}_y$, $\vec{E}_z$, and magnetic field vectors $\vec{H}_x$, $\vec{H}_y$, $\vec{H}_z$. The matrix $\mathbf{D}_d$ consists of $\mathbf{D}_{\varepsilon x}$, $\mathbf{D}_{\varepsilon y}$, $\mathbf{D}_{\varepsilon z}$, $\mathbf{D}_{\mu x}$, $\mathbf{D}_{\mu y}$, $\mathbf{D}_{\mu z}$, which are composed of the permittivity ε and permeability μ at corresponding positions. The sub-matrices $\mathbf{D}_{Ax}$, $\mathbf{D}_{Ay}$, $\mathbf{D}_{Az}$, $\mathbf{D}_{Bx}$, $\mathbf{D}_{By}$, $\mathbf{D}_{Bz}$, forming $\mathbf{A}$ and $\mathbf{B}$, are derived from the discretization of spatial derivative. By rearranging (2-4), one can obtain

$$\begin{aligned} & \left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{A} \right) \left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{B} \right) \vec{u}^{n+1} \\ &= \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{A} \right) \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{B} \right) \vec{u}^n - \frac{\Delta t^2}{4} \mathbf{D}_d^{-1} \mathbf{A} \mathbf{D}_d^{-1} \mathbf{B} \left(\vec{u}^{n+1} - \vec{u}^n \right) + O(\Delta t^3) \end{aligned} \quad (2-6)$$

Ignoring the perturbation terms in (2-6) and splitting it into two sub-procedures, the update equations for ADI-FDTD can be presented in matrix form as:

$$\left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{A} \right) \vec{u}^{n+1/2} = \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{B} \right) \vec{u}^n \quad (2-7a)$$

$$\left(\mathbf{I} - \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{B} \right) \vec{u}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{A} \right) \vec{u}^{n+1/2} \quad (2-7b)$$

(2-7a) and (2-7b) are the equations updating the fields components from n^{th} to

$(n+1/2)^{\text{th}}$ time step and from $(n+1/2)^{\text{th}}$ to $(n+1)^{\text{th}}$ time step respectively. These

derivations can be found in [6]. If these equations are specified, they can be written

as:

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$$E_x \Big|_{I+1/2,J,K}^{n+1/2} = E_x \Big|_{I+1/2,J,K}^n + b \left[\begin{array}{c} \frac{H_z \Big|_{I+1/2,J+1/2,K}^{n+1/2} - H_z \Big|_{I+1/2,J-1/2,K}^{n+1/2}}{\Delta y} \\ - \frac{H_y \Big|_{I+1/2,J,K+1/2}^n - H_y \Big|_{I+1/2,J,K-1/2}^n}{\Delta z} \end{array} \right] \quad (2-8a)$$

$$E_y \Big|_{I,J+1/2,K}^{n+1/2} = E_y \Big|_{I,J+1/2,K}^n + b \left[\begin{array}{c} \frac{H_x \Big|_{I,J+1/2,K+1/2}^{n+1/2} - H_x \Big|_{I,J+1/2,K-1/2}^{n+1/2}}{\Delta z} \\ - \frac{H_z \Big|_{I+1/2,J+1/2,K}^n - H_z \Big|_{I-1/2,J+1/2,K}^n}{\Delta x} \end{array} \right] \quad (2-8b)$$

$$E_z \Big|_{I,J,K+1/2}^{n+1/2} = E_z \Big|_{I,J,K+1/2}^n + b \left[\begin{array}{c} \frac{H_y \Big|_{I+1/2,J,K+1/2}^{n+1/2} - H_y \Big|_{I-1/2,J,K+1/2}^{n+1/2}}{\Delta x} \\ - \frac{H_x \Big|_{I,J+1/2,K+1/2}^n - H_x \Big|_{I,J-1/2,K+1/2}^n}{\Delta y} \end{array} \right] \quad (2-8c)$$

$$H_x \Big|_{I,J+1/2,K+1/2}^{n+1/2} = H_x \Big|_{I,J+1/2,K+1/2}^n + d \left[\begin{array}{c} \frac{E_y \Big|_{I,J+1/2,K+1}^{n+1/2} - E_y \Big|_{I,J+1/2,K}^{n+1/2}}{\Delta z} \\ - \frac{E_z \Big|_{I,J+1,K+1/2}^n - E_z \Big|_{I,J,K+1/2}^n}{\Delta y} \end{array} \right] \quad (2-8d)$$

$$H_y \Big|_{I+1/2,J,K+1/2}^{n+1/2} = H_y \Big|_{I+1/2,J,K+1/2}^n + d \left[\begin{array}{c} \frac{E_z \Big|_{I+1,J,K+1/2}^{n+1/2} - E_z \Big|_{I,J,K+1/2}^{n+1/2}}{\Delta x} \\ - \frac{E_x \Big|_{I+1/2,J,K+1}^n - E_x \Big|_{I+1/2,J,K}^n}{\Delta z} \end{array} \right] \quad (2-8e)$$

$$H_z \Big|_{I+1/2,J+1/2,K}^{n+1/2} = H_z \Big|_{I+1/2,J+1/2,K}^n + d \left[\begin{array}{c} \frac{E_x \Big|_{I+1/2,J+1,K}^{n+1/2} - E_x \Big|_{I+1/2,J,K}^{n+1/2}}{\Delta y} \\ - \frac{E_y \Big|_{I+1,J+1/2,K}^n - E_y \Big|_{I,J+1/2,K}^n}{\Delta x} \end{array} \right] \quad (2-8f)$$

$$E_x \Big|_{I+1/2,J,K}^{n+1} = E_x \Big|_{I+1/2,J,K}^{n+1/2} + b \left[\begin{array}{c} \frac{H_z \Big|_{I+1/2,J+1/2,K}^{n+1/2} - H_z \Big|_{I+1/2,J-1/2,K}^{n+1/2}}{\Delta y} \\ - \frac{H_y \Big|_{I+1/2,J,K+1/2}^{n+1} - H_y \Big|_{I+1/2,J,K-1/2}^{n+1}}{\Delta z} \end{array} \right] \quad (2-9a)$$

$$E_y \Big|_{I,J+1/2,K}^{n+1} = E_y \Big|_{I,J+1/2,K}^{n+1/2} + b \left[\begin{array}{c} \frac{H_x \Big|_{I,J+1/2,K+1/2}^{n+1/2} - H_x \Big|_{I,J+1/2,K-1/2}^{n+1/2}}{\Delta z} \\ - \frac{H_z \Big|_{I+1/2,J+1/2,K}^{n+1} - H_z \Big|_{I-1/2,J+1/2,K}^{n+1}}{\Delta x} \end{array} \right] \quad (2-9b)$$

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$$E_z|_{I,J,K+1/2}^{n+1} = E_z|_{I,J,K+1/2}^{n+1/2} + b \left[\frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} - H_y|_{I-1/2,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{H_x|_{I,J+1/2,K+1/2}^{n+1} - H_x|_{I,J-1/2,K+1/2}^{n+1}}{\Delta y} \right] \quad (2-9c)$$

$$H_x|_{I,J+1/2,K+1/2}^{n+1} = H_x|_{I,J+1/2,K+1/2}^{n+1/2} + d \left[\frac{E_y|_{I,J+1/2,K+1}^{n+1/2} - E_y|_{I,J+1/2,K}^{n+1/2}}{\Delta z} - \frac{E_z|_{I,J+1,K+1/2}^{n+1} - E_z|_{I,J,K+1/2}^{n+1}}{\Delta y} \right] \quad (2-9d)$$

$$H_y|_{I+1/2,J,K+1/2}^{n+1} = H_y|_{I+1/2,J,K+1/2}^{n+1/2} + d \left[\frac{E_z|_{I+1,J,K+1/2}^{n+1/2} - E_z|_{I,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{E_x|_{I+1/2,J,K+1}^{n+1} - E_x|_{I+1/2,J,K}^{n+1}}{\Delta z} \right] \quad (2-9e)$$

$$H_z|_{I+1/2,J+1/2,K}^{n+1} = H_z|_{I+1/2,J+1/2,K}^{n+1/2} + d \left[\frac{E_x|_{I+1/2,J+1,K}^{n+1/2} - E_x|_{I+1/2,J,K}^{n+1/2}}{\Delta y} - \frac{E_y|_{I+1,J+1/2,K}^{n+1} - E_y|_{I,J+1/2,K}^{n+1}}{\Delta x} \right] \quad (2-9f)$$

where I, J , and K are the spatial indexes for Yee cell and

$$b = \frac{\Delta t}{2\epsilon} \quad (2-10a)$$

$$d = \frac{\Delta t}{2\mu} \quad (2-10b)$$

First let us consider the computation of electric field components updating from n^{th}

to $(n+1/2)^{\text{th}}$ time step. For (2-8a), there are unknown fields components on its both

sides, so expression of $H_z|_{I+1/2,J+1/2,K}^{n+1/2}$ in (2-8f) should be substituted into (2-8a) and

after some arrangement one can obtain

$$\begin{aligned} & -\left(\frac{\Delta t^2}{4\mu\epsilon\Delta y^2}\right)E_x|_{I+1/2,J-1,K}^{n+1/2} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta y^2}\right)E_x|_{I+1/2,J,K}^{n+1/2} - \left(\frac{\Delta t^2}{4\mu\epsilon\Delta y^2}\right)E_x|_{I+1/2,J+1,K}^{n+1/2} \\ & = E_x|_{I+1/2,J,K}^n + \frac{\Delta t}{2\epsilon\Delta y} \left(H_z|_{I+1/2,J+1/2,K}^n - H_z|_{I+1/2,J-1/2,K}^n \right) \\ & \quad - \frac{\Delta t}{2\epsilon\Delta z} \left(H_y|_{I+1/2,J,K+1/2}^n - H_y|_{I+1/2,J,K-1/2}^n \right) \\ & \quad - \frac{\Delta t^2}{4\mu\epsilon\Delta y\Delta x} \left(E_y|_{I+1,J+1/2,K}^n - E_y|_{I,J+1/2,K}^n - E_y|_{I+1,J-1/2,K}^n + E_y|_{I,J-1/2,K}^n \right) \end{aligned} \quad (2-11)$$

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If these update equations involving E_x at $(n+1/2)^{\text{th}}$ combined with proper boundary conditions, such as the PEC boundary condition, one can find the coefficient matrix on the left hand side is a tri-diagonal matrix and thus the electric component E_x at $(n+1/2)^{\text{th}}$ time step can be solved efficiently. Other electric field components at $(n+1/2)^{\text{th}}$ time step can also be achieved by the similar procedures, which read

$$\begin{aligned}
& -\left(\frac{\Delta t^2}{4\mu\epsilon\Delta z^2}\right)E_y\bigg|_{I,J+1/2,K-1}^{n+1/2} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta z^2}\right)E_y\bigg|_{I,J+1/2,K}^{n+1/2} - \left(\frac{\Delta t^2}{4\mu\epsilon\Delta z^2}\right)E_y\bigg|_{I,J+1/2,K+1}^{n+1/2} \\
& = E_y\big|_{I,J+1/2,K}^n + \frac{\Delta t}{2\epsilon\Delta z}\left(H_x\big|_{I,J+1/2,K+1/2}^n - H_x\big|_{I,J+1/2,K-1/2}^n\right) \\
& \quad - \frac{\Delta t}{2\epsilon\Delta x}\left(H_z\big|_{I+1/2,J+1/2,K}^n - H_z\big|_{I-1/2,J+1/2,K}^n\right) \\
& \quad - \frac{\Delta t^2}{4\mu\epsilon\Delta z\Delta y}\left(E_z\big|_{I,J+1,K+1/2}^n - E_z\big|_{I,J,K+1/2}^n - E_z\big|_{I,J+1,K-1/2}^n + E_z\big|_{I,J,K-1/2}^n\right)
\end{aligned} \tag{2-12}$$

$$\begin{aligned}
& -\left(\frac{\Delta t^2}{4\mu\epsilon\Delta x^2}\right)E_z\bigg|_{I-1,J,K+1/2}^{n+1/2} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta x^2}\right)E_z\bigg|_{I,J,K+1/2}^{n+1/2} - \left(\frac{\Delta t^2}{4\mu\epsilon\Delta x^2}\right)E_z\bigg|_{I+1,J,K+1/2}^{n+1/2} \\
& = E_z\big|_{I,J,K+1/2}^n + \frac{\Delta t}{2\epsilon\Delta x}\left(H_y\big|_{I+1/2,J,K+1/2}^n - H_y\big|_{I-1/2,J,K+1/2}^n\right) \\
& \quad - \frac{\Delta t}{2\epsilon\Delta y}\left(H_x\big|_{I,J+1/2,K+1/2}^n - H_x\big|_{I,J-1/2,K+1/2}^n\right) \\
& \quad - \frac{\Delta t^2}{4\mu\epsilon\Delta x\Delta z}\left(E_x\big|_{I+1/2,J,K+1}^n - E_x\big|_{I+1/2,J,K}^n - E_x\big|_{I-1/2,J,K+1}^n + E_x\big|_{I-1/2,J,K}^n\right)
\end{aligned} \tag{2-13}$$

The magnetic field components will be solved directly from (2-8d)-(2-8f) by substituting the electric field components at $(n+1/2)^{\text{th}}$ time step obtained from (2-11)-(2-13). All the electric field components at $(n+1)^{\text{th}}$ time step can be attained by the similar approach, the corresponding update equations read:

$$\begin{aligned}
& -\left(\frac{\Delta t^2}{4\mu\epsilon\Delta z^2}\right)E_x\bigg|_{I+1/2,J,K-1}^{n+1} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta z^2}\right)E_x\bigg|_{I+1/2,J,K}^{n+1} - \left(\frac{\Delta t^2}{4\mu\epsilon\Delta z^2}\right)E_x\bigg|_{I+1/2,J,K+1}^{n+1} \\
& = E_x\big|_{I+1/2,J,K}^{n+1/2} + \frac{\Delta t}{2\epsilon\Delta y}\left(H_z\big|_{I+1/2,J+1/2,K}^{n+1/2} - H_z\big|_{I+1/2,J-1/2,K}^{n+1/2}\right) \\
& \quad - \frac{\Delta t}{2\epsilon\Delta z}\left(H_y\big|_{I+1/2,J,K+1/2}^{n+1/2} - H_y\big|_{I+1/2,J,K-1/2}^{n+1/2}\right) \\
& \quad - \frac{\Delta t^2}{4\mu\epsilon\Delta z\Delta x}\left(E_z\big|_{I+1,J,K+1/2}^{n+1/2} - E_z\big|_{I,J,K+1/2}^{n+1/2} - E_z\big|_{I+1,J,K-1/2}^{n+1/2} + E_z\big|_{I,J,K-1/2}^{n+1/2}\right)
\end{aligned} \tag{2-14}$$

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$$\begin{aligned}
& -\left(\frac{\Delta t^2}{4\mu\epsilon\Delta x^2}\right)E_y\Big|_{I-1,J+1/2,K}^{n+1} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta x^2}\right)E_y\Big|_{I,J+1/2,K}^{n+1} - \left(\frac{\Delta t^2}{4\mu\epsilon\Delta x^2}\right)E_y\Big|_{I+1,J+1/2,K}^{n+1} \\
& = E_y\Big|_{I,J+1/2,K}^{n+1/2} + \frac{\Delta t}{2\epsilon\Delta z}\left(H_x\Big|_{I,J+1/2,K+1/2}^{n+1/2} - H_x\Big|_{I,J+1/2,K-1/2}^{n+1/2}\right) \\
& \quad - \frac{\Delta t}{2\epsilon\Delta x}\left(H_z\Big|_{I+1/2,J+1/2,K}^{n+1/2} - H_z\Big|_{I-1/2,J+1/2,K}^{n+1/2}\right) \\
& \quad - \frac{\Delta t^2}{4\mu\epsilon\Delta x\Delta y}\left(E_x\Big|_{I+1/2,J+1,K}^{n+1/2} - E_x\Big|_{I+1/2,J,K}^{n+1/2} - E_x\Big|_{I-1/2,J+1,K}^{n+1/2} + E_x\Big|_{I-1/2,J,K}^{n+1/2}\right) \\
& \quad - \left(\frac{\Delta t^2}{4\mu\epsilon\Delta y^2}\right)E_z\Big|_{I-1,J,K+1/2}^{n+1} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta y^2}\right)E_z\Big|_{I,J,K+1/2}^{n+1} - \left(\frac{\Delta t^2}{4\mu\epsilon\Delta y^2}\right)E_z\Big|_{I+1,J,K+1/2}^{n+1} \\
& = E_z\Big|_{I,J,K+1/2}^{n+1/2} + \frac{\Delta t}{2\epsilon\Delta x}\left(H_y\Big|_{I+1/2,J,K+1/2}^{n+1/2} - H_y\Big|_{I-1/2,J,K+1/2}^{n+1/2}\right) \\
& \quad - \frac{\Delta t}{2\epsilon\Delta y}\left(H_x\Big|_{I,J+1/2,K+1/2}^{n+1/2} - H_x\Big|_{I,J-1/2,K+1/2}^{n+1/2}\right) \\
& \quad - \frac{\Delta t^2}{4\mu\epsilon\Delta y\Delta z}\left(E_y\Big|_{I,J+1/2,K+1}^{n+1/2} - E_y\Big|_{I,J+1/2,K}^{n+1/2} - E_y\Big|_{I,J-1/2,K+1}^{n+1/2} + E_y\Big|_{I,J-1/2,K}^{n+1/2}\right)
\end{aligned} \tag{2-15}$$

$$\begin{aligned}
& -\left(\frac{\Delta t^2}{4\mu\epsilon\Delta y^2}\right)E_z\Big|_{I-1,J,K+1/2}^{n+1} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta y^2}\right)E_z\Big|_{I,J,K+1/2}^{n+1} - \left(\frac{\Delta t^2}{4\mu\epsilon\Delta y^2}\right)E_z\Big|_{I+1,J,K+1/2}^{n+1} \\
& = E_z\Big|_{I,J,K+1/2}^{n+1/2} + \frac{\Delta t}{2\epsilon\Delta x}\left(H_y\Big|_{I+1/2,J,K+1/2}^{n+1/2} - H_y\Big|_{I-1/2,J,K+1/2}^{n+1/2}\right) \\
& \quad - \frac{\Delta t}{2\epsilon\Delta y}\left(H_x\Big|_{I,J+1/2,K+1/2}^{n+1/2} - H_x\Big|_{I,J-1/2,K+1/2}^{n+1/2}\right) \\
& \quad - \frac{\Delta t^2}{4\mu\epsilon\Delta y\Delta z}\left(E_y\Big|_{I,J+1/2,K+1}^{n+1/2} - E_y\Big|_{I,J+1/2,K}^{n+1/2} - E_y\Big|_{I,J-1/2,K+1}^{n+1/2} + E_y\Big|_{I,J-1/2,K}^{n+1/2}\right)
\end{aligned} \tag{2-16}$$

Then the magnetic field components at $(n+1)^{\text{th}}$ time step will be solved directly from (2-9d)-(2-9f).

2.2 Stability Analysis of ADI-FDTD Method

The unconditional stability is the advantage of the ADI-FDTD method over the conventional explicit FDTD. This section will present the theoretical proof of the unconditional stability. There are a few analysis methods able to study the stability of the finite difference methods, such as the von Neumann method [67], spectral analysis method [68], energy method [69], and von Neumann method combined with the Routh-Hurwitz criteria [70]. The von Neumann method is usually used because of its simplicity. Here the stability analysis is performed by this method.

Assuming k_x , k_y and k_z are the spatial frequency along x , y , and z directions, the plane, monochromatic traveling-wave trial solution with angular frequency ω reads

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$$E_x \Big|_{I+1/2,J,K}^n = E_x^n e^{-j(k_x(I+1/2)\Delta x + k_y J \Delta y + k_z K \Delta z)} \quad (2-17a)$$

$$E_y \Big|_{I,J+1/2,K}^n = E_y^n e^{-j(k_x I \Delta x + k_y (J+1/2)\Delta y + k_z K \Delta z)} \quad (2-17b)$$

$$E_z \Big|_{I,J,K+1/2}^n = E_z^n e^{-j(k_x I \Delta x + k_y J \Delta y + k_z (K+1/2)\Delta z)} \quad (2-17c)$$

$$H_x \Big|_{I,J+1/2,K+1/2}^n = H_x^n e^{-j(k_x I \Delta x + k_y (J+1/2)\Delta y + k_z (K+1/2)\Delta z)} \quad (2-17d)$$

$$H_y \Big|_{I+1/2,J,K+1/2}^n = H_y^n e^{-j(k_x (I+1/2)\Delta x + k_y J \Delta y + k_z (K+1/2)\Delta z)} \quad (2-17e)$$

$$H_z \Big|_{I+1/2,J+1/2,K}^n = H_z^n e^{-j(k_x (I+1/2)\Delta x + k_y (J+1/2)\Delta y + k_z K \Delta z)} \quad (2-17f)$$

where

$$\psi^n = \psi_0 e^{j\omega n \Delta t} \quad (2-18)$$

Here ψ can be any electric field component or magnetic field component in (2-17),

i.e. E_x, E_y, E_z, H_x, H_y , or H_z . ψ_0 is related to its initial value.

If (2-17) is substituted into (2-8) and (2-9), the time marching relations in these two equations can be written in a matrix form [10] as

$$\vec{F}^{n+1/2} = \mathbf{M}_1 \vec{F}^n \quad (2-19a)$$

$$\vec{F}^{n+1} = \mathbf{M}_2 \vec{F}^{n+1/2} \quad (2-19b)$$

where

$$\vec{F} = \begin{bmatrix} E_x & E_y & E_z & H_x & H_y & H_z \end{bmatrix}^T \quad (2-20a)$$

$$\mathbf{M}_1 = \begin{bmatrix} \frac{1}{Q_y} & \frac{W_x W_y}{\mu \epsilon Q_y} & 0 & 0 & j \frac{W_z}{\epsilon Q_y} & -j \frac{W_y}{\epsilon Q_y} \\ 0 & \frac{1}{Q_z} & \frac{W_y W_z}{\mu \epsilon Q_z} & -j \frac{W_z}{\epsilon Q_z} & 0 & j \frac{W_x}{\epsilon Q_z} \\ \frac{W_x W_z}{\mu \epsilon Q_x} & 0 & \frac{1}{Q_x} & j \frac{W_y}{\epsilon Q_x} & -j \frac{W_x}{\epsilon Q_x} & 0 \\ 0 & -j \frac{W_z}{\mu Q_z} & j \frac{W_z}{\mu Q_z} & \frac{1}{Q_z} & 0 & \frac{W_x W_z}{\mu \epsilon Q_z} \\ j \frac{W_z}{\mu Q_x} & 0 & -j \frac{W_x}{\mu Q_x} & \frac{W_x W_y}{\mu \epsilon Q_x} & \frac{1}{Q_x} & 0 \\ -j \frac{W_y}{\mu Q_y} & j \frac{W_x}{\mu Q_y} & 0 & 0 & \frac{W_y W_z}{\mu \epsilon Q_y} & \frac{1}{Q_y} \end{bmatrix} \quad (2-20b)$$

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$$\mathbf{M}_2 = \begin{bmatrix} \frac{1}{Q_z} & 0 & \frac{W_x W_z}{\mu \epsilon Q_z} & 0 & j \frac{W_z}{\epsilon Q_z} & -j \frac{W_y}{\epsilon Q_z} \\ \frac{W_x W_y}{\mu \epsilon Q_x} & \frac{1}{Q_x} & 0 & -j \frac{W_z}{\epsilon Q_x} & 0 & j \frac{W_x}{\epsilon Q_x} \\ 0 & \frac{W_y W_z}{\mu \epsilon Q_y} & \frac{1}{Q_y} & j \frac{W_y}{\epsilon Q_y} & -j \frac{W_x}{\epsilon Q_y} & 0 \\ 0 & -j \frac{W_z}{\mu Q_y} & j \frac{W_y}{\mu Q_y} & \frac{1}{Q_y} & \frac{W_x W_z}{\mu \epsilon Q_z} & 0 \\ j \frac{W_z}{\mu Q_z} & 0 & -j \frac{W_x}{\mu Q_z} & 0 & \frac{1}{Q_z} & \frac{W_z W_y}{\mu \epsilon Q_z} \\ -j \frac{W_y}{\mu Q_x} & j \frac{W_x}{\mu Q_x} & 0 & 0 & \frac{W_x W_z}{\mu \epsilon Q_x} & \frac{1}{Q_x} \end{bmatrix} \quad (2-20c)$$

$$W_\xi = \frac{\Delta t}{\Delta \xi} \sin\left(\frac{k_\xi \Delta \xi}{2}\right) \quad (2-20d)$$

$$Q_\xi = 1 + \frac{W_\xi^2}{\mu \epsilon} \quad (2-20e)$$

Here ξ can be x, y or z. From (2-19), it can be deduced that

$$\vec{F}^{n+1} = \mathbf{M}_{21} \vec{F}^n = \mathbf{M}_2 \mathbf{M}_1 \vec{F}^n \quad (2-21)$$

The eigenvalues of the update matrix M_{21} can be solved as

$$\lambda_1 = \lambda_2 = 1 \quad (2-22a)$$

$$\lambda_3 = \lambda_4 = \frac{\sqrt{R^2 - S^2} + jS}{R} \quad (2-22b)$$

$$\lambda_5 = \lambda_6 = \frac{\sqrt{R^2 - S^2} - jS}{R} \quad (2-22c)$$

where

$$R = (\mu \epsilon + W_x^2)(\mu \epsilon + W_y^2)(\mu \epsilon + W_z^2) \quad (2-23a)$$

$$S = \sqrt{4\mu \epsilon (\mu^3 \epsilon^3 + W_x^2 W_y^2 W_z^2)} \left[\frac{\mu \epsilon (W_x^2 + W_y^2 + W_z^2)}{+W_x^2 W_y^2 + W_x^2 W_z^2 + W_y^2 W_z^2} \right] \quad (2-23b)$$

It can be seen that all of the magnitudes of the six eigenvalues are unity, so the ADI-

FDTD method is concluded to be unconditionally stable.

2.3 Numerical Dispersion Analysis for ADI-FDTD Method

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Considering (2-17), (2-18), and (2-21), one can obtain

$$\left(e^{j\omega\Delta t}\mathbf{U}_6 - \mathbf{M}_{21}\right)\vec{F}^n = 0 \quad (2-24)$$

Here $\mathbf{U}_6$ is a 6×6 identity matrix. $\vec{F}^n$ is related to the initial field vector $\vec{F}_0$:

$$\vec{F}^n = \vec{F}_0 e^{j\omega n\Delta t} \quad (2-25)$$

For a nontrivial solution of (2-24), the determinant of the coefficient matrix should be zero, i.e.

$$\det\left(e^{j\omega\Delta t}\mathbf{U}_6 - \mathbf{M}_{21}\right) = 0 \quad (2-26)$$

The dispersion relation can be achieved by simplifying (2-26), which is

$$\sin^2(\omega\Delta t) = \frac{4\mu\epsilon\left(\mu^3\epsilon^3 + W_x^2W_y^2W_z^2\right)\left[\mu\epsilon\left(W_x^2 + W_y^2 + W_z^2\right) + W_x^2W_y^2 + W_x^2W_z^2 + W_y^2W_z^2\right]}{\left(\mu\epsilon + W_x^2\right)^2\left(\mu\epsilon + W_y^2\right)^2\left(\mu\epsilon + W_z^2\right)^2} \quad (2-27)$$

The numerical dispersion characteristics of the ADI-FDTD method can be achieved by the investigation on the dispersion relation in (2-27).

This chapter has provided some basic introduction to the ADI-FDTD method, as well as the literature survey. Starting from the next chapter, my own thesis work dedicated to the investigation and development of ADI-FDTD method will be presented.

Chapter 3

Stability and Dispersion Analyses for ADI-FDTD Method in Lossy Media

3.1 Introduction

According to the literature survey in Chapter 2, there have been a few works carried out for the stability and numerical dispersion analyses for the ADI-FDTD method. However, the preceding stability and dispersion analyses performed are usually based on the assumption of lossless media, e.g. [3], [10]-[19]. Although the ADI-FDTD method in lossy media has been presented and applied [71]-[72], its stability and dispersion have not been studied theoretically according to the best of my knowledge. This chapter will analyze the stability and numerical dispersion characteristics of the ADI-FDTD method in lossy media.

The von Neumann method [67] is often applied to the stability analysis of the ADI-FDTD method in lossless media for its simplicity. However, the form of the eigenvalues becomes much more complex for the lossy media case so that it is difficult to analyze their magnitudes. Therefore the stability analysis of the ADI-FDTD method in lossy medium is performed by another stability analysis method

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based on the energy concept, i.e. the energy method [69]. This analysis method has already been used for the stability analysis for the explicit FDTD method [36], [37] and the ADI-FDTD method with uniform mesh in homogeneous lossless medium [11]. Resorting to this method, the unconditional stability of the ADI-FDTD method in lossy medium will be proven.

In this chapter, the “dispersion analysis” actually has a more general meaning, which includes the investigations of both the numerical loss and numerical dispersion, since the propagation constant is complex for the lossy media case. This is different from the previous dispersion analyses based on the assumption of lossless media, where the propagation constant is pure imaginary. So it is necessary to perform the dispersion analysis for the ADI-FDTD method in lossy media to reveal its individual characteristics, which will be meaningful for the application and further development of the ADI-FDTD method in lossy media. The real part of the numerical propagation constant derived from the dispersion analysis is the numerical loss constant, and its imaginary part is the numerical phase constant. The investigation on the numerical loss constant and phase constant will disclose the characteristics of numerical loss and dispersion respectively.

In the next section, the update equations for the ADI-FDTD method in lossy media are presented in a matrix form and the proof of the unconditional stability for the ADI-FDTD in lossy media is shown. For the simplicity of presentation, Section 3.3 starts with the dispersion analysis for the one-dimensional (1-D) ADI-FDTD method in lossy media. Subsequently it will be extended to the 3-D case. Some numerical results are provided, which reveal the numerical loss and dispersion

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characteristics of the ADI-FDTD method in lossy media.

3.2 Stability Analysis

The ADI-FDTD method in lossy media [72] can be written as

$$\begin{aligned} \epsilon_{I+1/2,J,K} \frac{E_x|_{I+1/2,J,K}^{n+1/2} - E_x|_{I+1/2,J,K}^n}{\Delta t/2} + \sigma_{I+1/2,J,K} \frac{E_x|_{I+1/2,J,K}^{n+1/2} + E_x|_{I+1/2,J,K}^n}{2} \\ = \frac{H_z|_{I+1/2,J+1/2,K}^{n+1/2} - H_z|_{I+1/2,J-1/2,K}^{n+1/2}}{\Delta y} - \frac{H_y|_{I+1/2,J,K+1/2}^n - H_y|_{I+1/2,J,K-1/2}^n}{\Delta z} \end{aligned} \quad (3-1a)$$

$$\begin{aligned} \epsilon_{I,J+1/2,K} \frac{E_y|_{I,J+1/2,K}^{n+1/2} - E_y|_{I,J+1/2,K}^n}{\Delta t/2} + \sigma_{I,J+1/2,K} \frac{E_y|_{I,J+1/2,K}^{n+1/2} + E_y|_{I,J+1/2,K}^n}{2} \\ = \frac{H_x|_{I,J+1/2,K+1/2}^{n+1/2} - H_x|_{I,J+1/2,K-1/2}^{n+1/2}}{\Delta z} - \frac{H_z|_{I+1/2,J+1/2,K}^n - H_z|_{I-1/2,J+1/2,K}^n}{\Delta x} \end{aligned} \quad (3-1b)$$

$$\begin{aligned} \epsilon_{I,J,K+1/2} \frac{E_z|_{I,J,K+1/2}^{n+1/2} - E_z|_{I,J,K+1/2}^n}{\Delta t/2} + \sigma_{I,J,K+1/2} \frac{E_z|_{I,J,K+1/2}^{n+1/2} + E_z|_{I,J,K+1/2}^n}{2} \\ = \frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} - H_y|_{I-1/2,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{H_x|_{I,J+1/2,K+1/2}^n - H_x|_{I,J-1/2,K+1/2}^n}{\Delta y} \end{aligned} \quad (3-1c)$$

$$\begin{aligned} \mu_{I,J+1/2,K+1/2} \frac{H_x|_{I,J+1/2,K+1/2}^{n+1/2} - H_x|_{I,J+1/2,K+1/2}^n}{\Delta t/2} + \sigma_{I,J+1/2,K+1/2}^* \frac{H_x|_{I,J+1/2,K+1/2}^{n+1/2} + H_x|_{I,J+1/2,K+1/2}^n}{2} \\ = \frac{E_y|_{I,J+1/2,K+1}^{n+1/2} - E_y|_{I,J+1/2,K}^{n+1/2}}{\Delta z} - \frac{E_z|_{I,J+1,K+1/2}^n - E_z|_{I,J,K+1/2}^n}{\Delta y} \end{aligned} \quad (3-1d)$$

$$\begin{aligned} \mu_{I+1/2,J,K+1/2} \frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} - H_y|_{I+1/2,J,K+1/2}^n}{\Delta t/2} + \sigma_{I+1/2,J,K+1/2}^* \frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} + H_y|_{I+1/2,J,K+1/2}^n}{2} \\ = \frac{E_z|_{I+1,J,K+1/2}^{n+1/2} - E_z|_{I,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{E_x|_{I+1/2,J,K+1}^n - E_x|_{I+1/2,J,K}^n}{\Delta z} \end{aligned} \quad (3-1e)$$

$$\begin{aligned} \mu_{I+1/2,J+1/2,K} \frac{H_z|_{I+1/2,J+1/2,K}^{n+1/2} - H_z|_{I+1/2,J+1/2,K}^n}{\Delta t/2} + \sigma_{I+1/2,J+1/2,K}^* \frac{H_z|_{I+1/2,J+1/2,K}^{n+1/2} + H_z|_{I+1/2,J+1/2,K}^n}{2} \\ = \frac{E_x|_{I+1/2,J+1,K}^{n+1/2} - E_x|_{I+1/2,J,K}^{n+1/2}}{\Delta y} - \frac{E_y|_{I+1,J+1/2,K}^n - E_y|_{I,J+1/2,K}^n}{\Delta x} \end{aligned} \quad (3-1f)$$

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$$\begin{aligned} \epsilon_{I+1/2,J,K} &= \frac{E_x|_{I+1/2,J,K}^{n+1} - E_x|_{I+1/2,J,K}^{n+1/2}}{\Delta t/2} + \sigma_{I+1/2,J,K} \frac{E_x|_{I+1/2,J,K}^{n+1} + E_x|_{I+1/2,J,K}^{n+1/2}}{2} \\ &= \frac{H_z|_{I+1/2,J+1/2,K}^{n+1/2} - H_z|_{I+1/2,J-1/2,K}^{n+1/2}}{\Delta y} - \frac{H_y|_{I+1/2,J,K+1/2}^{n+1} - H_y|_{I+1/2,J,K-1/2}^{n+1}}{\Delta z} \end{aligned} \quad (3-2a)$$

$$\begin{aligned} \epsilon_{I,J+1/2,K} &= \frac{E_y|_{I,J+1/2,K}^{n+1} - E_y|_{I,J+1/2,K}^{n+1/2}}{\Delta t/2} + \sigma_{I,J+1/2,K} \frac{E_y|_{I,J+1/2,K}^{n+1} + E_y|_{I,J+1/2,K}^{n+1/2}}{2} \\ &= \frac{H_x|_{I,J+1/2,K+1/2}^{n+1/2} - H_x|_{I,J+1/2,K-1/2}^{n+1/2}}{\Delta z} - \frac{H_z|_{I+1/2,J+1/2,K}^{n+1} - H_z|_{I-1/2,J+1/2,K}^{n+1}}{\Delta x} \end{aligned} \quad (3-2b)$$

$$\begin{aligned} \epsilon_{I,J,K+1/2} &= \frac{E_z|_{I,J,K+1/2}^{n+1} - E_z|_{I,J,K+1/2}^{n+1/2}}{\Delta t/2} + \sigma_{I,J,K+1/2} \frac{E_z|_{I,J,K+1/2}^{n+1} + E_z|_{I,J,K+1/2}^{n+1/2}}{2} \\ &= \frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} - H_y|_{I-1/2,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{H_x|_{I,J+1/2,K+1/2}^{n+1} - H_x|_{I,J-1/2,K+1/2}^{n+1}}{\Delta y} \end{aligned} \quad (3-2c)$$

$$\begin{aligned} \mu_{I,J+1/2,K+1/2} &= \frac{H_x|_{I,J+1/2,K+1/2}^{n+1} - H_x|_{I,J+1/2,K+1/2}^{n+1/2}}{\Delta t/2} + \sigma^*_{I,J+1/2,K+1/2} \frac{H_x|_{I,J+1/2,K+1/2}^{n+1} + H_x|_{I,J+1/2,K+1/2}^{n+1/2}}{2} \\ &= \frac{E_y|_{I,J+1/2,K+1/2}^{n+1/2} - E_y|_{I,J+1/2,K}^{n+1/2}}{\Delta z} - \frac{E_z|_{I,J+1/2,K+1/2}^{n+1} - E_z|_{I,J,K+1/2}^{n+1}}{\Delta y} \end{aligned} \quad (3-2d)$$

$$\begin{aligned} \mu_{I+1/2,J,K+1/2} &= \frac{H_y|_{I+1/2,J,K+1/2}^{n+1} - H_y|_{I+1/2,J,K+1/2}^{n+1/2}}{\Delta t/2} + \sigma^*_{I+1/2,J,K+1/2} \frac{H_y|_{I+1/2,J,K+1/2}^{n+1} + H_y|_{I+1/2,J,K+1/2}^{n+1/2}}{2} \\ &= \frac{E_z|_{I+1/2,J,K+1/2}^{n+1/2} - E_z|_{I,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{E_x|_{I+1/2,J,K+1/2}^{n+1} - E_x|_{I+1/2,J,K}^{n+1}}{\Delta z} \end{aligned} \quad (3-2e)$$

$$\begin{aligned} \mu_{I+1/2,J+1/2,K} &= \frac{H_z|_{I+1/2,J+1/2,K}^{n+1} - H_z|_{I+1/2,J+1/2,K}^{n+1/2}}{\Delta t/2} + \sigma^*_{I+1/2,J+1/2,K} \frac{H_z|_{I+1/2,J+1/2,K}^{n+1} + H_z|_{I+1/2,J+1/2,K}^{n+1/2}}{2} \\ &= \frac{E_x|_{I+1/2,J+1/2,K}^{n+1/2} - E_x|_{I+1/2,J,K}^{n+1/2}}{\Delta y} - \frac{E_y|_{I+1/2,J+1/2,K}^{n+1} - E_y|_{I,J+1/2,K}^{n+1}}{\Delta x} \end{aligned} \quad (3-2f)$$

where n is the time index and Δt is the time step. ϵ and μ are the permittivity and permeability of the medium, and the subscripts I , J , and K are the spatial indexes for Yee cell. σ and σ^* are the electric conductivity and equivalent magnetic loss.

Equations (3-1) and (3-2) represent the updating sub-procedures from n^{th} to $(n+1/2)^{\text{th}}$ time step and from $(n+1/2)^{\text{th}}$ to $(n+1)^{\text{th}}$ time step respectively. They can be

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expressed in matrix form as:

$$\left(\mathbf{U}_6 - \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{A} + \frac{\Delta t}{4} \mathbf{D}_d^{-1} \mathbf{D}_l \right) \vec{u}^{n+1/2} = \left(\mathbf{U}_6 + \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{B} - \frac{\Delta t}{4} \mathbf{D}_d^{-1} \mathbf{D}_l \right) \vec{u}^n \quad (3-3a)$$

$$\left(\mathbf{U}_6 - \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{B} + \frac{\Delta t}{4} \mathbf{D}_d^{-1} \mathbf{D}_l \right) \vec{u}^{n+1} = \left(\mathbf{U}_6 + \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{A} - \frac{\Delta t}{4} \mathbf{D}_d^{-1} \mathbf{D}_l \right) \vec{u}^{n+1/2} \quad (3-3b)$$

where $\mathbf{U}_6$ is a 6-by-6 identity matrix and $\mathbf{D}_l$ is diagonal matrix with positive real elements, which consist of the electric conductivity σ and equivalent magnetic loss σ^* . The notations $\vec{u}$, $\mathbf{D}_d$, $\mathbf{A}$, and $\mathbf{B}$ conform to the definition in (2-5). The matrix $\mathbf{D}_d$ is composed of the permittivity ϵ and permeability μ of the medium at corresponding positions. The matrices $\mathbf{A}$ and $\mathbf{B}$ are derived from the discretization of spatial derivative. $\vec{u}$ is the vector denoting electric and magnetic field components.

Here the stability analysis depends on a method based on the energy concept. This section is going to prove that a properly defined numerical energy will be bounded with bounded initial condition, which can also denote the bounded characteristics of the field components.

To facilitate the stability analysis of the ADI-FDTD method in lossy media, the finite integration technique (FIT) notations are introduced to the present analysis procedure. These notations are also adopted in [12] and [37], which are dedicated to the stability analysis of explicit FDTD in inhomogeneous lossy media and ADI-FDTD in lossless media respectively. Following the form of the update equation presented in FIT notations of [12] and [37], the ADI scheme for lossy media in (3-3) can be written as

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$$\begin{bmatrix} \mathbf{U}_3 + \frac{\Delta t}{4} \mathbf{M}_{\varepsilon^{-1}} \mathbf{M}_{\sigma} & -\frac{\Delta t}{2} \mathbf{M}_{\varepsilon^{-1}} \tilde{\mathbf{C}}_1 \\ \frac{\Delta t}{2} \mathbf{M}_{\mu^{-1}} \tilde{\mathbf{C}}_1^T & \mathbf{U}_3 + \frac{\Delta t}{4} \mathbf{M}_{\mu^{-1}} \mathbf{M}_{\sigma^*} \end{bmatrix} \begin{bmatrix} \hat{e}^{n+1/2} \\ \hat{h}^{n+1/2} \end{bmatrix} = \begin{bmatrix} \mathbf{U}_3 - \frac{\Delta t}{4} \mathbf{M}_{\varepsilon^{-1}} \mathbf{M}_{\sigma} & \frac{\Delta t}{2} \mathbf{M}_{\varepsilon^{-1}} \tilde{\mathbf{C}}_2 \\ -\frac{\Delta t}{2} \mathbf{M}_{\mu^{-1}} \tilde{\mathbf{C}}_2^T & \mathbf{U}_3 - \frac{\Delta t}{4} \mathbf{M}_{\mu^{-1}} \mathbf{M}_{\sigma^*} \end{bmatrix} \begin{bmatrix} \hat{e}^n \\ \hat{h}^n \end{bmatrix} \quad (3-4a)$$

$$\begin{bmatrix} \mathbf{U}_3 + \frac{\Delta t}{4} \mathbf{M}_{\varepsilon^{-1}} \mathbf{M}_{\sigma} & -\frac{\Delta t}{2} \mathbf{M}_{\varepsilon^{-1}} \tilde{\mathbf{C}}_2 \\ \frac{\Delta t}{2} \mathbf{M}_{\mu^{-1}} \tilde{\mathbf{C}}_2^T & \mathbf{U}_3 + \frac{\Delta t}{4} \mathbf{M}_{\mu^{-1}} \mathbf{M}_{\sigma^*} \end{bmatrix} \begin{bmatrix} \hat{e}^{n+1} \\ \hat{h}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{U}_3 - \frac{\Delta t}{4} \mathbf{M}_{\varepsilon^{-1}} \mathbf{M}_{\sigma} & \frac{\Delta t}{2} \mathbf{M}_{\varepsilon^{-1}} \tilde{\mathbf{C}}_1 \\ -\frac{\Delta t}{2} \mathbf{M}_{\mu^{-1}} \tilde{\mathbf{C}}_1^T & \mathbf{U}_3 - \frac{\Delta t}{4} \mathbf{M}_{\mu^{-1}} \mathbf{M}_{\sigma^*} \end{bmatrix} \begin{bmatrix} \hat{e}^{n+1/2} \\ \hat{h}^{n+1/2} \end{bmatrix} \quad (3-4b)$$

Here $\mathbf{U}_3$ is a 3-by-3 identity matrix. The state vectors $\hat{e}$ and $\hat{h}$ are accordingly composed of electric and magnetic grid voltages. $\mathbf{M}_{\varepsilon^{-1}}$, $\mathbf{M}_{\mu^{-1}}$, $\mathbf{M}_{\sigma}$ and $\mathbf{M}_{\sigma^*}$ are all diagonal and positive definite matrices. The matrices $\tilde{\mathbf{C}}_1$ and $\tilde{\mathbf{C}}_2$ are the discrete curl operators and T denotes the transpose of matrices. If the state vectors are transformed by

$$\begin{bmatrix} \hat{e}^n \\ \hat{h}^n \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\varepsilon^{-1}}^{1/2} & 0 \\ 0 & \mathbf{M}_{\mu^{-1}}^{1/2} \end{bmatrix} \begin{bmatrix} \hat{e}^n \\ \hat{h}^n \end{bmatrix} \quad (3-5)$$

The update equations can be rewritten as

$$\mathbf{M}_{L1} \vec{V}^{n+1/2} = \mathbf{M}_{R1} \vec{V}^n \quad (3-6a)$$

$$\mathbf{M}_{L2} \vec{V}^{n+1} = \mathbf{M}_{R2} \vec{V}^{n+1/2} \quad (3-6b)$$

where

$$\vec{V}^n = \begin{bmatrix} \hat{e}^n & \hat{h}^n \end{bmatrix}^T \quad (3-7a)$$

$$\mathbf{M}_{L1} = \begin{bmatrix} \mathbf{U}_3 + \frac{\Delta t}{4} \mathbf{M}_{\varepsilon^{-1}} \mathbf{M}_{\sigma} & -\frac{\Delta t}{2} \mathbf{M}_{\varepsilon^{-1}}^{1/2} \tilde{\mathbf{C}}_1 \mathbf{M}_{\mu^{-1}}^{1/2} \\ \frac{\Delta t}{2} \mathbf{M}_{\mu^{-1}}^{1/2} \tilde{\mathbf{C}}_1^T \mathbf{M}_{\varepsilon^{-1}}^{1/2} & \mathbf{U}_3 + \frac{\Delta t}{4} \mathbf{M}_{\mu^{-1}} \mathbf{M}_{\sigma^*} \end{bmatrix} \quad (3-7b)$$

$$\mathbf{M}_{R1} = \begin{bmatrix} \mathbf{U}_3 - \frac{\Delta t}{4} \mathbf{M}_{\varepsilon^{-1}} \mathbf{M}_{\sigma} & \frac{\Delta t}{2} \mathbf{M}_{\varepsilon^{-1}}^{1/2} \tilde{\mathbf{C}}_2 \mathbf{M}_{\mu^{-1}}^{1/2} \\ -\frac{\Delta t}{2} \mathbf{M}_{\mu^{-1}}^{1/2} \tilde{\mathbf{C}}_2^T \mathbf{M}_{\varepsilon^{-1}}^{1/2} & \mathbf{U}_3 - \frac{\Delta t}{4} \mathbf{M}_{\mu^{-1}} \mathbf{M}_{\sigma^*} \end{bmatrix} \quad (3-7c)$$

$$\mathbf{M}_{L2} = \begin{bmatrix} \mathbf{U}_3 + \frac{\Delta t}{4} \mathbf{M}_{\varepsilon^{-1}} \mathbf{M}_{\sigma} & -\frac{\Delta t}{2} \mathbf{M}_{\varepsilon^{-1}}^{1/2} \tilde{\mathbf{C}}_2 \mathbf{M}_{\mu^{-1}}^{1/2} \\ \frac{\Delta t}{2} \mathbf{M}_{\mu^{-1}}^{1/2} \tilde{\mathbf{C}}_2^T \mathbf{M}_{\varepsilon^{-1}}^{1/2} & \mathbf{U}_3 + \frac{\Delta t}{4} \mathbf{M}_{\mu^{-1}} \mathbf{M}_{\sigma^*} \end{bmatrix} \quad (3-7d)$$

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$$\mathbf{M}_{R2} = \begin{bmatrix} \mathbf{U}_3 - \frac{\Delta t}{4} \mathbf{M}_{\epsilon^{-1}} \mathbf{M}_{\sigma} & \frac{\Delta t}{2} \mathbf{M}_{\epsilon^{-1}}^{1/2} \tilde{\mathbf{C}}_1 \mathbf{M}_{\mu^{-1}}^{1/2} \\ -\frac{\Delta t}{2} \mathbf{M}_{\mu^{-1}}^{1/2} \tilde{\mathbf{C}}_1^T \mathbf{M}_{\epsilon^{-1}}^{1/2} & \mathbf{U}_3 - \frac{\Delta t}{4} \mathbf{M}_{\mu^{-1}} \mathbf{M}_{\sigma^*} \end{bmatrix} \quad (3-7e)$$

The numerical energy at n^{th} time step is defined as

$$W^n = \left(\mathbf{M}_{L2} \vec{V}^n \right)^T \mathbf{M}_{L2} \vec{V}^n \quad (3-8)$$

At the same time, if one denotes the left hand side and right hand side of (3-6a) as

$\overrightarrow{LHS}_1$ and $\overrightarrow{RHS}_1$, and the left hand side and right hand side of (3-6b) as $\overrightarrow{LHS}_2$ and

$\overrightarrow{RHS}_2$, it can be obtained that

$$\overrightarrow{LHS}_1^T \overrightarrow{LHS}_1 = \overrightarrow{RHS}_1^T \overrightarrow{RHS}_1 \quad (3-9a)$$

$$\overrightarrow{LHS}_2^T \overrightarrow{LHS}_2 = \overrightarrow{RHS}_2^T \overrightarrow{RHS}_2 \quad (3-9b)$$

Then it can be derived that

$$\begin{aligned} W^{n+1} - \overrightarrow{RHS}_1^T \overrightarrow{RHS}_1 &= \overrightarrow{LHS}_2^T \overrightarrow{LHS}_2 - \overrightarrow{RHS}_1^T \overrightarrow{RHS}_1 \\ &= \overrightarrow{RHS}_2^T \overrightarrow{RHS}_2 - \overrightarrow{LHS}_1^T \overrightarrow{LHS}_1 \\ &= \left(\vec{V}^{n+1/2} \right)^T \begin{bmatrix} -\Delta t \mathbf{M}_{\epsilon^{-1}} \mathbf{M}_{\sigma} & 0 \\ 0 & -\Delta t \mathbf{M}_{\mu^{-1}} \mathbf{M}_{\sigma^*} \end{bmatrix} \vec{V}^{n+1/2} \leq 0 \end{aligned} \quad (3-10)$$

and

$$W^n - \overrightarrow{RHS}_1^T \overrightarrow{RHS}_1 = \left(\vec{V}^n \right)^T \begin{bmatrix} \Delta t \mathbf{M}_{\epsilon^{-1}} \mathbf{M}_{\sigma} & 0 \\ 0 & \Delta t \mathbf{M}_{\mu^{-1}} \mathbf{M}_{\sigma^*} \end{bmatrix} \vec{V}^n \geq 0 \quad (3-11)$$

One can find that $W^{n+1} \leq W^n$ for any Δt , which indicates that the numerical energy will be bounded with a bounded initial field vector and so are the field components. Hence it can be concluded that the ADI method in lossy media is still unconditionally stable.

3.3 Dispersion Analysis

In this section the dispersion analysis will be performed for the ADI-FDTD method in lossy media. Such analysis has been completed for the conventional explicit

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FDTD method, such as the work in [73] and [74], where one can find that the dispersion characteristics in lossy media are different from those of lossless case. As a counterpart of the preceding numerical dispersion analysis of the explicit method, this section will reveal the individual characteristics of the ADI-FDTD method in lossy media, where it is also assumed that $\sigma^*=0$. To simplify the presentation, the analysis for 1-D ADI-FDTD in lossy media will be carried out first, and then it will be extended to the 3-D case.

Before the dispersion analysis, some notations are defined to facilitate the discussion.

$$b = \frac{\Delta t}{2\epsilon} \quad (3-12a)$$

$$d = \frac{\Delta t}{2\mu} \quad (3-12b)$$

$$R_\xi = -\frac{2b \sinh(\gamma_\xi \Delta \xi / 2)}{\Delta \xi} \quad (3-12c)$$

$$Q_\xi = -\frac{2d \sinh(\gamma_\xi \Delta \xi / 2)}{\Delta \xi} \quad (3-12d)$$

The complex number $\gamma = \alpha + j\beta$ is the numerical propagation constant in the lossy media, where α is the numerical loss constant, β is the numerical phase constant, and the subscript $\xi = x, y, z$ denotes the propagation direction. This is the main difference between the numerical dispersion characteristics of the lossy and lossless case, where the numerical propagation constant is pure imaginary. So when the numerical dispersion analysis is performed for the lossy case, both the numerical loss and phase constants need to be studied, which denote the characteristics of numerical loss and numerical dispersion respectively.

3.3.1 Analysis for 1-D Case

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To investigate the numerical loss and dispersion of the 1-D ADI-FDTD in lossy media, let us assume that the TEM wave is propagating in a lossy homogenous medium along z axis and the uniform mesh size is Δz . The numerical plane wave is represented as

$$E_y|_K^n = E_y^n(\omega) e^{-\gamma_z K \Delta z} \quad (3-13a)$$

$$H_x|_{K+1/2}^n = H_x^n(\omega) e^{-\gamma_z (K+1/2) \Delta z} \quad (3-13b)$$

where ω is the frequency and K is the spatial index in z direction. The update equations can be derived from (3-3) as

$$\mathbf{M}_{L1}^{1D} \begin{bmatrix} E_y^{n+1/2}(\omega) \\ H_x^{n+1/2}(\omega) \end{bmatrix} = \mathbf{M}_{R1}^{1D} \begin{bmatrix} E_y^n(\omega) \\ H_x^n(\omega) \end{bmatrix} \quad (3-14a)$$

$$\mathbf{M}_{L2}^{1D} \begin{bmatrix} E_y^{n+1}(\omega) \\ H_x^{n+1}(\omega) \end{bmatrix} = \mathbf{M}_{R2}^{1D} \begin{bmatrix} E_y^{n+1/2}(\omega) \\ H_x^{n+1/2}(\omega) \end{bmatrix} \quad (3-14b)$$

where

$$\mathbf{M}_{L1}^{1D} = \begin{bmatrix} 1+b\sigma/2 & -R_z \\ -Q_z & 1 \end{bmatrix} \quad (3-15a)$$

$$\mathbf{M}_{R1}^{1D} = \begin{bmatrix} 1-b\sigma/2 & 0 \\ 0 & 1 \end{bmatrix} \quad (3-15b)$$

$$\mathbf{M}_{L2}^{1D} = \begin{bmatrix} 1+b\sigma/2 & 0 \\ 0 & 1 \end{bmatrix} \quad (3-15c)$$

$$\mathbf{M}_{R2}^{1D} = \begin{bmatrix} 1-b\sigma/2 & R_z \\ Q_z & 1 \end{bmatrix} \quad (3-15d)$$

$$\begin{bmatrix} E_y^n(\omega) \\ H_x^n(\omega) \end{bmatrix} = \begin{bmatrix} E_{y0} e^{j\omega n \Delta t} \\ H_{x0} e^{j\omega n \Delta t} \end{bmatrix} \quad (3-15e)$$

Here E_{y0} , H_{x0} are the initial values of the field components. Based on (3-14) and (3-

15) one can obtain

$$\left(e^{j\omega \Delta t} \mathbf{U}_2 - \mathbf{M}_{1D} \right) \begin{bmatrix} E_{y0} \\ H_{x0} \end{bmatrix} = 0 \quad (3-16)$$

where $\mathbf{U}_2$ is a 2-by-2 identity matrix and

$$\mathbf{M}_{1D} = \left(\mathbf{M}_{L2}^{1D} \right)^{-1} \mathbf{M}_{R2}^{1D} \left(\mathbf{M}_{L1}^{1D} \right)^{-1} \mathbf{M}_{R1}^{1D} \quad (3-17)$$

By setting the determinant of the coefficient matrix in (3-16) to be zero, the dispersion relation for the 1-D ADI-FDTD method in lossy media can be

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represented as

$$\left| e^{j\omega\Delta t} \mathbf{U}_2 - \mathbf{M}_{1D} \right| = 0 \quad (3-18)$$

Based on the dispersion relation provided by (3-18), one can perform investigations on the dispersion characteristics of the 1-D ADI-FDTD method in lossy media numerically. Here the numerical loss error (*NLE*) and numerical phase error (*NPE*) will be used to represent the characteristics of numerical loss and numerical dispersion respectively. They are defined as

$$NLE = \frac{\alpha - \alpha_0}{\alpha_0} \quad (3-19a)$$

$$NPE = \frac{\beta - \beta_0}{\beta_0} \quad (3-19b)$$

where α_0 and β_0 are physical values of the loss constant and phase constant.

Let us assume that the frequency of the wave we are interested in is $f=5\text{GHz}$ and λ is the corresponding wavelength. Suppose that the mesh size is set to be λ/N_s , where N_s is the spatial sampling rate, and the Courant limit time step for such mesh size is Δt_C . Henceforward the mesh size and time step for various investigations on the numerical loss and dispersion will be described in terms of N_s and Δt_C .

The investigation begins with the *NLE* and *NPE* versus σ varying from 0.01 S/m to 1 S/m for different time steps. The permittivity and permeability are assumed to be the same as those of the free space, which will be the same for the following investigations. Figs. 3-1 and 3-2 present the *NLE* and *NPE* for the cases of $\Delta t=2\Delta t_C$, $4\Delta t_C$ with $N_s=60$. Some results from the numerical simulation of the numerical dispersion are also plotted on the figures (marked with circles) for demonstration, from which one can see that the analysis results of (3-18) basically coincide with the ones from the simulation. The simulation procedure is similar to the one in [13],

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where the loss constant and phase constant in frequency domain can be obtained from

$$e^{-(\alpha(\omega)+j\beta(\omega))L} = \frac{E(\omega, r_2)}{E(\omega, r_1)} \quad (3-20)$$

$E(\omega, r_1)$ and $E(\omega, r_2)$ are accordingly the Fourier transforms of the recorded electric field at two observation points along the propagation path, and L is the distance between the observation points r_1 and r_2 .

Here it can be found that the NLE always becomes more serious with the increase of time step, while the NPE sometimes may be lower with larger time step for some values of σ . This is different from the dispersion characteristics of the ADI-FDTD method for lossless case, where the numerical dispersion error always becomes serious with the increase of time step.

Fig. 3-1. Comparison of NLE for 1-D ADI-FDTD with different time steps

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Fig. 3-2. Comparison of NPE for 1-D ADI-FDTD with different time steps

Next, the investigation is on the NLE and NPE versus σ varying from 0.01 S/m to 1 S/m for different mesh sizes. Figs. 3-3 and 3-4 present the NLE and NPE for the cases of $N_s=20, 40$ with $\Delta t=2\Delta t_C$, where Δt_C is the Courant limit time step corresponding to $N_s=40$. It can be observed that the NLE can be reduced with the increase of spatial sampling rate. However, for some values of σ , the NPE can be larger even with smaller mesh size, which is different from the dispersion characteristics of the ADI-FDTD method for lossless media.

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Fig. 3-3. Comparison of NLE for 1-D ADI-FDTD with different mesh sizes

Fig. 3-4. Comparison of NPE for 1-D ADI-FDTD with different mesh sizes

From Figs. 3-1 to 3-4, it can be seen that the NLE is usually much larger than the NPE for the σ discussed here, especially when the time step or mesh size is

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increased. So the reduction of the *NLE* of the ADI method in lossy media is anticipated to be a meaningful future work and these analysis results can be used to evaluate the improvement.

In this subsection, the characteristics of the numerical loss and dispersion for 1-D ADI-FDTD in lossy media are studied. This subsection mainly concerns about the variation of these characteristics with different time steps, mesh sizes and electric conductivity σ . In the next subsection, the numerical loss and dispersion will be investigated for the 3-D case, where the variation of these characteristics with different propagation angles will be discussed.

3.3.2 Analysis for 3-D Case

For the 3-D case, the numerical plane wave can be represented as

$$E_x \Big|_{I+1/2, J, K}^n = E_{x0} e^{j\omega n \Delta t - (\gamma_x (I+1/2) \Delta x + \gamma_y J \Delta y + \gamma_z K \Delta z)} \quad (3-21a)$$

$$E_y \Big|_{I, J+1/2, K}^n = E_{y0} e^{j\omega n \Delta t - (\gamma_x I \Delta x + \gamma_y (J+1/2) \Delta y + \gamma_z K \Delta z)} \quad (3-21b)$$

$$E_z \Big|_{I, J, K+1/2}^n = E_{z0} e^{j\omega n \Delta t - (\gamma_x I \Delta x + \gamma_y J \Delta y + \gamma_z (K+1/2) \Delta z)} \quad (3-21c)$$

$$H_x \Big|_{I, J+1/2, K+1/2}^n = H_{x0} e^{j\omega n \Delta t - (\gamma_x I \Delta x + \gamma_y (J+1/2) \Delta y + \gamma_z (K+1/2) \Delta z)} \quad (3-21d)$$

$$H_y \Big|_{I+1/2, J, K+1/2}^n = H_{y0} e^{j\omega n \Delta t - (\gamma_x (I+1/2) \Delta x + \gamma_y J \Delta y + \gamma_z (K+1/2) \Delta z)} \quad (3-21e)$$

$$H_z \Big|_{I+1/2, J+1/2, K}^n = H_{z0} e^{j\omega n \Delta t - (\gamma_x (I+1/2) \Delta x + \gamma_y (J+1/2) \Delta y + \gamma_z K \Delta z)} \quad (3-21f)$$

where

$$\gamma_x = \gamma \sin \theta \cos \phi \quad (3-22a)$$

$$\gamma_y = \gamma \sin \theta \sin \phi \quad (3-22b)$$

$$\gamma_z = \gamma \cos \theta \quad (3-22c)$$

Here ϕ and θ are the azimuthal angle and elevation angle in a spherical coordinate system. Δx , Δy , and Δz are the uniform mesh sizes in x , y , and z directions. I , J , and K are the spatial indexes for Yee cell.

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Similar to the preceding analysis for the 1-D case, the dispersion relation for the 3-D ADI-FDTD method in lossy media can be derived, which is

$$\left| e^{j\omega\Delta t} \mathbf{U}_6 - \mathbf{M}_{3D} \right| = 0 \quad (3-23)$$

where

$$\mathbf{M}_{3D} = \left(\mathbf{M}_{L2}^{3D} \right)^{-1} \mathbf{M}_{R2}^{3D} \left(\mathbf{M}_{L1}^{3D} \right)^{-1} \mathbf{M}_{R1}^{3D} \quad (3-24a)$$

$$\mathbf{M}_{L1}^{3D} = \begin{bmatrix} 1 + \frac{b\sigma}{2} & 0 & 0 & 0 & 0 & -R_y \\ 0 & 1 + \frac{b\sigma}{2} & 0 & -R_z & 0 & 0 \\ 0 & 0 & 1 + \frac{b\sigma}{2} & 0 & -R_x & 0 \\ 0 & -Q_z & 0 & 1 & 0 & 0 \\ 0 & 0 & -Q_x & 0 & 1 & 0 \\ -Q_y & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3-24b)$$

$$\mathbf{M}_{R1}^{3D} = \begin{bmatrix} 1 - \frac{b\sigma}{2} & 0 & 0 & 0 & -R_z & 0 \\ 0 & 1 - \frac{b\sigma}{2} & 0 & 0 & 0 & -R_x \\ 0 & 0 & 1 - \frac{b\sigma}{2} & -R_y & 0 & 0 \\ 0 & 0 & -Q_y & 1 & 0 & 0 \\ -Q_z & 0 & 0 & 0 & 1 & 0 \\ 0 & -Q_x & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3-24c)$$

$$\mathbf{M}_{L2}^{3D} = \begin{bmatrix} 1 + \frac{b\sigma}{2} & 0 & 0 & 0 & R_z & 0 \\ 0 & 1 + \frac{b\sigma}{2} & 0 & 0 & 0 & R_x \\ 0 & 0 & 1 + \frac{b\sigma}{2} & R_y & 0 & 0 \\ 0 & 0 & Q_y & 1 & 0 & 0 \\ Q_z & 0 & 0 & 0 & 1 & 0 \\ 0 & Q_x & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3-24d)$$

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$$\mathbf{M}_{R2}^{3D} = \begin{bmatrix} 1 - \frac{b\sigma}{2} & 0 & 0 & 0 & 0 & R_y \\ 0 & 1 - \frac{b\sigma}{2} & 0 & R_z & 0 & 0 \\ 0 & 0 & 1 - \frac{b\sigma}{2} & 0 & R_x & 0 \\ 0 & Q_z & 0 & 1 & 0 & 0 \\ 0 & 0 & Q_x & 0 & 1 & 0 \\ Q_y & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3-24e)$$

For the 3-D case, a particular characteristic of the numerical loss and dispersion in contrast to the 1-D case is the anisotropy, which means that the numerical loss and dispersion depend on the propagation direction. Therefore this subsection will mainly discuss the variation of the *NLE* and *NPE* with propagation angle ϕ and θ based on the numerical investigation of the dispersion relation for the 3-D case in (3-23).

Figs. 3-5 and 3-6 present the *NLE* and *NPE* versus ϕ for different θ , where $0^\circ \leq \phi \leq 90^\circ$, $\theta = 22.5^\circ, 45^\circ, 67.5^\circ$. Here $N_s = 40$, $\Delta t = 2\Delta t_C$ and $\sigma = 0.05$ S/m. It can be found that both the *NPE* and *NLE* is symmetric about $\phi = 45^\circ$ and their minima appear at $\phi = 45^\circ$ for each θ . All these are similar to the lossless case, where the diagonal direction possesses the best numerical dispersion performance.

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Fig. 3-5. Comparison of NLE for 3-D ADI-FDTD with different propagation angles

Fig. 3-6. Comparison of NPE for 3-D ADI-FDTD with different propagation angles

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To make a more comprehensive investigation, another $\sigma=5$ S/m is considered in Figs. 3-7 and 3-8. One can observe that with the increase of the value of electric conductivity, $\phi=45^\circ$ is not the direction always featuring the least NLE and NPE anymore, which is different from the lossless case. On the other hand, the NPE and the NLE still keep the symmetry about $\phi=45^\circ$ for each θ .

Fig. 3-7. Comparison of NLE for 3-D ADI-FDTD with different propagation angles

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Fig. 3-8. Comparison of NPE for 3-D ADI-FDTD with different propagation angles

In order to further describe the anisotropy of the numerical loss and dispersion, two notations A_l and A_d are defined in (3-25a)-(3-25b) to denote their anisotropy in 3-D system respectively:

$$A_l = \frac{\alpha_{\max} - \alpha_{\min}}{\alpha_{\min}} \quad (3-25a)$$

$$A_d = \frac{\beta_{\max} - \beta_{\min}}{\beta_{\min}} \quad (3-25b)$$

$\alpha_{\max}$ and $\beta_{\max}$ are respectively the maxima of the numerical loss constants and numerical phase constants in various ϕ and θ , where $1^\circ \leq \phi \leq 90^\circ$ and $1^\circ \leq \theta \leq 90^\circ$. $\alpha_{\min}$ and $\beta_{\min}$ are the corresponding minima.

To illustrate the variation of the anisotropy with mesh size for different electric conductivity, Figs. 3-9 and 3-10 plot the A_l and A_d with different mesh sizes, where $N_s=40$ and 80, $\Delta t=2\Delta t_C$, $0.1 \leq \sigma \leq 1$. Here Δt_C is the Courant limit time step corresponding to $N_s=40$. One can observe that the anisotropy of both the numerical

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loss and dispersion becomes worse with the increase of mesh size for most of the electric conductivity values. However, this is not true for some electric conductivity values with larger mesh sizes. On the other hand, the anisotropy of the numerical loss A_l increases with the increase of the electric conductivity. However, the variation of the dispersion anisotropy A_d is not monotonic, which goes down first and then goes up with the increase of electric conductivity. The exact value of the electric conductivity achieving minimum A_d can be obtained by some searching techniques. For example, with the help of the optimization tools provided by Matlab, the electric conductivity corresponding to the minimum A_d for the case of $N_s=40$ and $\Delta t=2\Delta t_C$ can be obtained as 0.48 S/m.

Fig. 3-9. Variation of A_l with different mesh sizes

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Fig. 3-10. Variation of A_d with different mesh sizes

Figs. 3-11 and 3-12 present the A_l and A_d with different time steps, where $\Delta t=2\Delta t_C$ and $4\Delta t_C$, to illustrate the variation of the anisotropy with time step for different electric conductivity, where $0.1 \leq \sigma \leq 1$. Here $N_s=40$ and Δt_C is also the corresponding Courant limit time step. It can be observed that the anisotropy turns more serious with the increase of time step for all the electric conductivity values illustrated here. Similar to the preceding case, the anisotropy of the numerical loss A_l increases with the increase of the electric conductivity and the variation of the dispersion anisotropy A_d is not monotonic.

Compared with the dispersion analysis for lossless case, both the numerical loss and dispersion need to be studied for the lossy case since the propagation constant is complex. In addition, the electric conductivity also affects the numerical loss and dispersion characteristics besides the mesh size and time step, which leads to some

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individual characteristics of the lossy case. So the dispersion characteristics for the lossy case are more complicated than those of the lossless one. Fortunately, based on the numerical investigation of (3-23), one can conveniently achieve the performance with a set of specified parameters, which can be determined when the ADI-FDTD method is used for various applications. Such investigations provide a way to assess the accuracy of ADI-FDTD method for lossy media. Another application for the numerical loss and dispersion analyses is that the analysis results can be used to compare the accuracy of various numerical methods simulating lossy media, in case some other methods are developed for the improvement of accuracy in the future.

Fig. 3-11. Variation of A_l with different time steps

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Fig. 3-12. Variation of A_d with different time steps

3.4 Conclusion

This chapter proves the unconditional stability of the 3-D ADI-FDTD method in lossy medium, while the previous stability analyses for ADI method are usually with the assumption of lossless homogeneous medium. The stability analysis method based on the energy concept and FIT notations are adopted so as to remove these assumptions and facilitate the proof.

The dispersion analysis is also performed for the ADI-FDTD method in lossy media. Here the analysis of dispersion relation includes the investigations of both the numerical loss and numerical dispersion and therefore is more complicated, since the propagation constant is complex for the lossy media case. This is different from the previous dispersion analyses based on the assumption of lossless media, where

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the propagation constant is pure imaginary.

To simplify the presentation, the analysis for 1-D ADI-FDTD in lossy media is carried out first, and then it is extended to the 3-D case. The effects of time step, mesh size and conductivity value on the numerical loss and dispersion are studied for the 1-D method. The discussion about 3-D case is mainly about the variation of the numerical loss and dispersion with propagation angle, i.e. the anisotropy characteristics. One can also perform other investigations conveniently based on the numerical dispersion relation provided. All these will be meaningful for the evaluation and further development of the ADI-FDTD method in lossy media.

Chapter 4

Higher Order 3-D ADI-FDTD Method

4.1 Introduction

The ADI-FDTD method, which has been proven to be unconditionally stable, was proposed to eliminate the restriction on time step size due to the CFL condition. Although the time step size is no longer constrained by the CFL stability condition, the ADI method may still be limited by numerical errors or accuracy requirement. The numerical dispersion characteristics are often used to assess the accuracy of the FDTD method and many works of the numerical dispersion analysis for the ADI method have been carried out, such as [3], [12]-[19].

Using higher order schemes is a usual method to reduce the numerical dispersion error of the finite difference method. Some methods based on this idea have been developed for the conventional explicit method [75]-[86]. Similar works have also been carried out for the ADI-FDTD method [87]-[91]. However, most of the works are meant for two-dimensional (2-D) situations. The stability and dispersion analyses presented in this chapter can be generalized for a series of 3-D higher order

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ADI-FDTD methods, which not only justifies the improved dispersion performance achieved by the higher order ADI-FDTD methods but also can be useful for the selection and evaluation of these methods. This chapter will start with the stability consideration of the 4th order 3-D ADI-FDTD method in the next section, and prove that this method is unconditionally stable. Based on the deduction from the analysis of the 4th order method, Section 4.3 generalizes the stability analysis to other higher order 3-D methods adopting the 6th and 10th order cell-centered finite difference schemes. The generalized form of the dispersion relations for these unconditionally stable ADI-FDTD methods is presented in Section 4.4. Using the relations attained, the effects of the order of schemes, mesh size and time step on the dispersion are illustrated through the numerical results in Section 4.5. It will be seen that the higher order ADI schemes may obtain better accuracy even with coarser mesh. Such improved accuracy actually leads to other advantages of them, which are higher computation efficiency and lower memory requirement. Section 4.6 discusses the computational cost of the 2nd order and higher order schemes.

4.2 Stability Analysis of 4th Order ADI-FDTD Method

For simplicity, let us consider the 3-D wave propagation in a lossless isotropic medium with permittivity ϵ and permeability μ . Throughout the chapter, the finite difference approximation schemes for Maxwell's equations possess 2nd order accuracy in time domain. For the spatial differential operators, this section will start with the 4th order cell-centered finite difference approximation scheme.

Following the approach presented in [71], the computation for one discrete time

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step of the 4th order 3-D ADI-FDTD method is performed in two procedures. The first procedure is presented in (4-1a)-(4-1f) for time marching from n to $n+1/2$. The second procedure is presented in (4-2a)- (4-2f) for time marching from $n+1/2$ to $n+1$. They are

<First procedure>

$$E_x|_{I+1/2,J,K}^{n+1/2} = E_x|_{I+1/2,J,K}^n + b \left[P_y^{(4)} \left(H_z|_{I+1/2,J,K}^n \right) - P_z^{(4)} \left(H_y|_{I+1/2,J,K}^{n+1/2} \right) \right] \quad (4-1a)$$

$$E_y|_{I,J+1/2,K}^{n+1/2} = E_y|_{I,J+1/2,K}^n + b \left[P_z^{(4)} \left(H_x|_{I,J+1/2,K}^n \right) - P_x^{(4)} \left(H_z|_{I,J+1/2,K}^{n+1/2} \right) \right] \quad (4-1b)$$

$$E_z|_{I,J,K+1/2}^{n+1/2} = E_z|_{I,J,K+1/2}^n + b \left[P_x^{(4)} \left(H_y|_{I,J,K+1/2}^n \right) - P_y^{(4)} \left(H_x|_{I,J,K+1/2}^{n+1/2} \right) \right] \quad (4-1c)$$

$$H_x|_{I,J+1/2,K+1/2}^{n+1/2} = H_x|_{I,J+1/2,K+1/2}^n + d \left[P_z^{(4)} \left(E_y|_{I,J+1/2,K+1/2}^n \right) - P_y^{(4)} \left(E_z|_{I,J+1/2,K+1/2}^{n+1/2} \right) \right] \quad (4-1d)$$

$$H_y|_{I+1/2,J,K+1/2}^{n+1/2} = H_y|_{I+1/2,J,K+1/2}^n + d \left[P_x^{(4)} \left(E_z|_{I+1/2,J,K+1/2}^n \right) - P_z^{(4)} \left(E_x|_{I+1/2,J,K+1/2}^{n+1/2} \right) \right] \quad (4-1e)$$

$$H_z|_{I+1/2,J+1/2,K}^{n+1/2} = H_z|_{I+1/2,J+1/2,K}^n + d \left[P_y^{(4)} \left(E_x|_{I+1/2,J+1/2,K}^n \right) - P_x^{(4)} \left(E_y|_{I+1/2,J+1/2,K}^{n+1/2} \right) \right] \quad (4-1f)$$

<Second procedure>

$$E_x|_{I+1/2,J,K}^{n+1} = E_x|_{I+1/2,J,K}^{n+1/2} + b \left[P_y^{(4)} \left(H_z|_{I+1/2,J,K}^{n+1/2} \right) - P_z^{(4)} \left(H_y|_{I+1/2,J,K}^{n+1/2} \right) \right] \quad (4-2a)$$

$$E_y|_{I,J+1/2,K}^{n+1} = E_y|_{I,J+1/2,K}^{n+1/2} + b \left[P_z^{(4)} \left(H_x|_{I,J+1/2,K}^{n+1/2} \right) - P_x^{(4)} \left(H_z|_{I,J+1/2,K}^{n+1/2} \right) \right] \quad (4-2b)$$

$$E_z|_{I,J,K+1/2}^{n+1} = E_z|_{I,J,K+1/2}^{n+1/2} + b \left[P_x^{(4)} \left(H_y|_{I,J,K+1/2}^{n+1/2} \right) - P_y^{(4)} \left(H_x|_{I,J,K+1/2}^{n+1/2} \right) \right] \quad (4-2c)$$

$$H_x|_{I,J+1/2,K+1/2}^{n+1} = H_x|_{I,J+1/2,K+1/2}^{n+1/2} + d \left[P_z^{(4)} \left(E_y|_{I,J+1/2,K+1/2}^{n+1/2} \right) - P_y^{(4)} \left(E_z|_{I,J+1/2,K+1/2}^{n+1/2} \right) \right] \quad (4-2d)$$

$$H_y|_{I+1/2,J,K+1/2}^{n+1} = H_y|_{I+1/2,J,K+1/2}^{n+1/2} + d \left[P_x^{(4)} \left(E_z|_{I+1/2,J,K+1/2}^{n+1/2} \right) - P_z^{(4)} \left(E_x|_{I+1/2,J,K+1/2}^{n+1/2} \right) \right] \quad (4-2e)$$

$$H_z|_{I+1/2,J+1/2,K}^{n+1} = H_z|_{I+1/2,J+1/2,K}^{n+1/2} + d \left[P_y^{(4)} \left(E_x|_{I+1/2,J+1/2,K}^{n+1/2} \right) - P_x^{(4)} \left(E_y|_{I+1/2,J+1/2,K}^{n+1/2} \right) \right] \quad (4-2f)$$

Here, $b = \Delta t / (2\epsilon)$, $d = \Delta t / (2\mu)$, Δt is the time step size, n is the time index and $I, J,$

K are the space indexes for Yee cell. The difference operator $P_\xi^{(4)}$ is defined as

$$P_x^{(4)}(\psi|_{I,J,K}^n) = \gamma_{1\Delta x} \left(\psi|_{I+1/2,J,K}^n - \psi|_{I-1/2,J,K}^n \right) + \gamma_{2\Delta x} \left(\psi|_{I+3/2,J,K}^n - \psi|_{I-3/2,J,K}^n \right) \quad (4-3a)$$

$$P_y^{(4)}(\psi|_{I,J,K}^n) = \gamma_{1\Delta y} \left(\psi|_{I,J+1/2,K}^n - \psi|_{I,J-1/2,K}^n \right) + \gamma_{2\Delta y} \left(\psi|_{I,J+3/2,K}^n - \psi|_{I,J-3/2,K}^n \right) \quad (4-3b)$$

$$P_z^{(4)}(\psi|_{I,J,K}^n) = \gamma_{1\Delta z} \left(\psi|_{I,J,K+1/2}^n - \psi|_{I,J,K-1/2}^n \right) + \gamma_{2\Delta z} \left(\psi|_{I,J,K+3/2}^n - \psi|_{I,J,K-3/2}^n \right) \quad (4-3c)$$

where ψ can be any field component in (4-1)-(4-2), i.e. $E_x, E_y, E_z, H_x, H_y,$ or H_z ;

$\gamma_{1h} = 9/(8h), \gamma_{2h} = -1/(24h), h = \Delta x, \Delta y, \Delta z$. Equations (4-3a)-(4-3c) represent the

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4th order finite difference approximations to the first spatial derivative of electromagnetic fields.

To prove the unconditional stability of the 4th order ADI-FDTD method, let us resort to the von Neumann method, which has been mentioned in Chapter 2. Assuming k_x , k_y and k_z to be the spatial frequencies along x , y , and z directions, the field components in spectral domain at the n^{th} time step can be denoted as:

$$\psi|_{I,J,K}^n = \psi^n e^{-j(k_x I \Delta x + k_y J \Delta y + k_z K \Delta z)} \quad (4-4)$$

Here ψ conforms to the definition used in (4-3), and let the spectral field components ψ^n form the vector

$$\vec{F}^n = \begin{bmatrix} E_x^n & E_y^n & E_z^n & H_x^n & H_y^n & H_z^n \end{bmatrix}^T \quad (4-5)$$

Substituting (4-4) and (4-5) into (4-1a)-(4-1f), the time-marching relation in the first procedure can be represented in a matrix form as

$$\vec{F}^{n+1/2} = \mathbf{M}_1 \vec{F}^n \quad (4-6)$$

where

$$\mathbf{M}_1 = \begin{bmatrix} \frac{1}{A_z} & 0 & \frac{bdP_x P_z}{A_z} & 0 & -j\frac{bP_z}{A_z} & j\frac{bP_y}{A_z} \\ \frac{bdP_x P_y}{A_x} & \frac{1}{A_x} & 0 & j\frac{bP_z}{A_x} & 0 & -j\frac{bP_x}{A_x} \\ 0 & \frac{bdP_y P_z}{A_y} & \frac{1}{A_y} & -j\frac{bP_y}{A_y} & j\frac{bP_x}{A_y} & 0 \\ 0 & j\frac{dP_z}{A_y} & -j\frac{dP_y}{A_y} & \frac{1}{A_y} & \frac{bdP_x P_y}{A_y} & 0 \\ -j\frac{dP_z}{A_z} & 0 & j\frac{dP_x}{A_z} & 0 & \frac{1}{A_z} & \frac{bdP_y P_z}{A_z} \\ j\frac{dP_y}{A_x} & -j\frac{dP_x}{A_x} & 0 & \frac{bdP_x P_z}{A_x} & 0 & \frac{1}{A_x} \end{bmatrix} \quad (4-7)$$

Similarly, substituting (4-4) and (4-5) into (4-2a)-(4-2f), one can obtain the matrix

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form for the second procedure

$$\vec{F}^{n+1} = \mathbf{M}_2 \vec{F}^{n+1/2} \quad (4-8)$$

where

$$\mathbf{M}_2 = \begin{bmatrix} \frac{1}{A_y} & \frac{bdP_x P_y}{A_y} & 0 & 0 & -j \frac{bP_z}{A_y} & j \frac{bP_y}{A_y} \\ 0 & \frac{1}{A_z} & \frac{bdP_y P_z}{A_z} & j \frac{bP_z}{A_z} & 0 & -j \frac{bP_x}{A_z} \\ \frac{bdP_x P_z}{A_x} & 0 & \frac{1}{A_x} & -j \frac{bP_y}{A_x} & j \frac{bP_x}{A_x} & 0 \\ 0 & j \frac{dP_z}{A_z} & -j \frac{dP_y}{A_z} & \frac{1}{A_z} & 0 & \frac{bdP_x P_z}{A_z} \\ -j \frac{dP_z}{A_x} & 0 & j \frac{dP_x}{A_x} & \frac{bdP_x P_y}{A_x} & \frac{1}{A_x} & 0 \\ j \frac{dP_y}{A_y} & -j \frac{dP_x}{A_y} & 0 & 0 & \frac{bdP_y P_z}{A_y} & \frac{1}{A_y} \end{bmatrix} \quad (4-9)$$

In (4-7) and (4-9), the notations used are $A_\xi = 1 + bdP_\xi^2$, $\xi = x, y, z$; and

$$P_\xi = -2\gamma_{1\Delta\xi} \sin(k_\xi \Delta\xi / 2) / \Delta\xi - 2\gamma_{2\Delta\xi} \sin(3k_\xi \Delta\xi / 2) / \Delta\xi \quad (4-10)$$

Note that P_ξ is derived from the operation of the 4th order finite difference scheme defined in (4-3a)-(4-3c) onto the field components in (4-4). It is related to the difference operator $P_\xi^{(4)}$ by

$$P_\xi^{(4)} \left(V \Big|_{I,J,K}^n \right) = jP_\xi V \Big|_{I,J,K}^n \quad (4-11)$$

It will be seen later that the definition of P_ξ is important for leading towards the generalization discussed in Section 4.3.

Substituting (4-6) into (4-8) gives rise to

$$\vec{F}^{n+1} = \mathbf{M}_{21} \vec{F}^n = \mathbf{M}_2 \mathbf{M}_1 \vec{F}^n \quad (4-12)$$

where $\mathbf{M}_{21} = \mathbf{M}_2 \mathbf{M}_1$. The eigenvalues of $\mathbf{M}_{21}$ can be solved as

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$$\lambda_1 = \lambda_2 = 1 \quad (4-13a)$$

$$\lambda_3 = \lambda_4 = \frac{Q + \sqrt{-S^2}}{R} \quad (4-13b)$$

$$\lambda_5 = \lambda_6 = \frac{Q - \sqrt{-S^2}}{R} \quad (4-13c)$$

where

$$R = A_x A_y A_z = (1 + bdP_x^2)(1 + bdP_y^2)(1 + bdP_z^2)$$

$$Q = b^3 d^3 P_x^2 P_y^2 P_z^2 - b^2 d^2 (P_x^2 P_y^2 + P_x^2 P_z^2 + P_y^2 P_z^2) - bd (P_x P_y + P_x P_z + P_y P_z) + 1$$

$$S^2 = R^2 - Q^2 = 4bd (b^3 d^3 P_x^2 P_y^2 P_z^2 + 1) \cdot [P_x^2 + P_y^2 + P_z^2 + bd (P_x^2 P_y^2 + P_x^2 P_z^2 + P_y^2 P_z^2)]$$

Since P_x , P_y and P_z for the 4th order scheme are all real, b and d are both positive, and $R^2 \geq Q^2$, it is evident that the magnitude of each of the six eigenvalues is unity.

So one can conclude that the 4th order 3-D ADI-FDTD method here is unconditionally stable just like the 2nd order ADI method presented in [10].

4.3 Generalization of Stability Analysis

From the expressions in (4-13), one finds that the eigenvalues are only related to b , d and P_ξ ($\xi=x,y,z$). Since P_ξ can be derived from the higher order difference operator $P_\xi^{(N)}$ ($N=4$ in (4-11)), the unconditional stability of various higher order ADI methods will be related only to the higher order finite difference approximation scheme employed for the spatial differential operator. Hence the representation of the eigenvalues can be generalized to other higher order ADI-FDTD methods simply by replacing the P_ξ ($\xi=x,y,z$) corresponding to the 4th order scheme in (4-13) with those of other higher order finite difference approximation schemes. In this way one can also determine whether these higher order ADI FDTD

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methods are unconditionally stable through the generalized form of eigenvalues.

To investigate the stability of the higher order ADI methods, the P_ξ for the 4th, 6th and 10th order cell-centered finite difference approximation schemes is presented in a universal form. For subsequent dispersion comparison, the discussion includes as well the P_ξ for the 2nd order ADI method, which also represents one special case of the generalized discussion. According to the definition of the 2nd, 4th, 6th and 10th order cell-centered difference schemes and following the analysis procedure used previously for the 4th order ADI method, the N^{th} order cell-centered difference operator can be presented as

$$P_\xi^{(N)} \left(V \Big|_{I,J,K}^n \right) = j P_\xi V \Big|_{I,J,K}^n \quad (4-14)$$

$$P_\xi = -2 \sum_{m=1}^{N/2} \gamma_m \sin \left[(2m-1) k_\xi \Delta \xi / 2 \right] / \Delta \xi \quad (4-15)$$

The coefficients γ_m for $N=2, 4, 6, 10$ are given in Table 4-1.

Table 4-1. Coefficients of 2nd, 4th, 6th and 10th order cell-centered difference schemes

$N \backslash \gamma_m$	γ_1	γ_2	γ_3	γ_4	γ_5
2	1				
4	9/8	-1/24			
6	225/192	-25/384	3/640		
10	1.2112427	-0.0897217	0.0138428	-0.0017657	0.0001187

It is obvious that the values of P_ξ corresponding to the 2nd, 4th, 6th and 10th order cell-centered difference schemes are all real numbers, thus the eigenvalues associated with these four ADI-FDTD methods can be represented as (4-13), whose magnitudes are apparently no larger than one. Therefore it is proved that all of these methods are unconditionally stable.

4.4 Dispersion Analysis

To analyze the dispersion characteristic, the field is assumed to be a monochromatic wave with angular frequency ω . Then the field components of F^n in (4-5) become

$$E_\eta^n = E_\eta^0 e^{j\omega\Delta t n}, \quad H_\eta^n = H_\eta^0 e^{j\omega\Delta t n}, \quad \eta = x, y, z \quad (4-16)$$

and equation (4-12) reads

$$\left(e^{j\omega\Delta t} \mathbf{U}_6 - \mathbf{M}_{2l} \right) \vec{F}^n = 0 \quad (4-17)$$

Here $\mathbf{U}_6$ is a 6×6 identity matrix, $\mathbf{M}_{2l}$ is assumed to be the coefficient matrix of the

4th order 3-D ADI-FDTD method and $\vec{F}^n$ is related to the initial field vector $\vec{F}^0$:

$$\vec{F}^n = \vec{F}^0 e^{j\omega\Delta t n} \quad (4-18)$$

For a nontrivial solution of (4-17), the determinant of the coefficient matrix in (4-17)

should be zero. With reference to the eigenvalues of $\mathbf{M}_{2l}$ above, the dispersion

relationship of the 4th order 3-D ADI-FDTD method can be deduced as

$$\sin^2(\omega\Delta t) = \frac{4bd \left(b^3 d^3 P_x^2 P_y^2 P_z^2 + 1 \right) \left[P_x^2 + P_y^2 + P_z^2 + bd \left(P_x^2 P_y^2 + P_x^2 P_z^2 + P_y^2 P_z^2 \right) \right]}{\left(1 + bd P_x^2 \right)^2 \left(1 + bd P_y^2 \right)^2 \left(1 + bd P_z^2 \right)^2} \quad (4-19)$$

In line with the previous arguments on generalizing the eigenvalues from the 4th order ADI-FDTD method to other higher order ones, the dispersion relation in (4-19) can also be generalized to other higher order ADI methods. This is achieved simply by replacing P_x , P_y and P_z corresponding to the 4th order approximation scheme with those corresponding to other approximation schemes, specifically the 2nd, 6th and 10th order cell-centered schemes exemplified here. One finds that the dispersion relation of the 2nd order scheme in [14] is also included in the generalized form (4-19).

Letting the mesh size Δx , Δy , Δz approach zero, the dispersion relation of the ADI-

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FDTD method with infinitesimal mesh size can be written as

$$\sin^2(\omega\Delta t) = \frac{4bd(b^3d^3k_x^2k_y^2k_z^2+1)\left[bd(k_x^2k_y^2+k_x^2k_z^2+k_y^2k_z^2)+1\right]}{(1+bdk_x^2)^2(1+bdk_y^2)^2(1+bdk_z^2)^2} \quad (4-20)$$

This is the best dispersion performance that can be achieved by using higher order schemes for the spatial difference or refining mesh for a given time step size.

When Δx , Δy , Δz and Δt all approach zero, P_x , P_y and P_z will approach $-k_x$, $-k_y$ and $-k_z$ respectively. Equation (4-19) then simplifies to the theoretical dispersion relation

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 \quad (4-21)$$

4.5 Numerical Results

This section will investigate the dispersion characteristics of the 2nd, 4th, 6th and 10th order ADI-FDTD methods based on the generalized results. Assume that a wave is propagating along the azimuthal angle ϕ and the elevation angle θ in a spherical coordinate system. The characteristics of the numerical dispersion is illustrated for different ϕ and θ with various mesh size, time step or scheme order. The phase velocity error η will be used to denote the characteristics of the numerical dispersion error of various schemes. It is defined as

$$\eta = \frac{u_0 - u}{u_0} \times 100\% \quad (4-22)$$

where u is the numerical phase velocity in certain direction and u_0 is the corresponding physical phase velocity. In the following the case of propagation in free space is discussed and thus u_0 equals to the speed of light in free space. This study will be useful for the selection and evaluation of various higher order ADI methods.

In addition to the dispersion error, the anisotropy error is also one important aspect of the numerical dispersion characteristics, which has been mentioned in Chapter 3. Anisotropy denotes the dependency of the numerical dispersion on the propagation direction. For a given θ , the anisotropy error in ϕ direction can be defined as

$$Aniso = \frac{u_{\max} - u_{\min}}{u_{\min}} \times 100\% \quad (4-23)$$

where $u_{\max}$ and $u_{\min}$ are accordingly the maximum and minimum numerical phase velocity in various ϕ direction.

4.5.1 Effect of Higher Order Schemes on Numerical Dispersion

Figs. 4-1 to 4-3 present the phase velocity error of the 2nd, 4th, 6th and 10th order schemes versus propagation angle ϕ (in degrees) when the uniform mesh size $\Delta x = \Delta y = \Delta z = \delta = \lambda/20$ and time step $\Delta t = 1.5\Delta t_C$. Here $\Delta t_C = (\lambda/20)/(\sqrt{3}c)$ is equal to the CFL limit corresponding to the mesh size of $\lambda/20$ and θ is selected as 22.5°, 45°, 67.5° respectively for these three figures.

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Fig. 4-1. Phase velocity error versus wave propagation angle ϕ for the 2nd, 4th, 6th and 10th order schemes with $\Delta t=1.5\Delta t_c$, $\delta=\lambda/20$, $\theta=22.5^\circ$

Fig. 4-2. Phase velocity error versus wave propagation angle ϕ for the 2nd, 4th, 6th and 10th order schemes with $\Delta t=1.5\Delta t_c$, $\delta=\lambda/20$, $\theta=45^\circ$

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Fig. 4-3. Phase velocity error versus wave propagation angle ϕ for the 2nd, 4th, 6th and 10th order schemes with $\Delta t = 1.5\Delta t_c$, $\delta = \lambda/20$, $\theta = 67.5^\circ$

It is apparent that the dispersion error can be reduced when the order is increased. For example, the dispersion error of the 4th order scheme is reduced by more than 30% compared with the 2nd order one in Fig. 4-2. In addition, for the 6th and 10th order schemes, the dispersion error is almost the same as that of the 4th order scheme. One also can find that the phase velocity error is positive for all the cases discussed here, which means that the phase velocity of the numerical waves discussed here is less than the phase velocity of the corresponding wave in the physical world. Meanwhile, the numerical phase velocity depends on the wave propagation direction since the grid points are rectilinear and not spherically symmetric [92].

4.5.2 Effect of Mesh Size on Numerical Dispersion

This subsection uses the 2nd and 4th order schemes with different mesh sizes to

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illustrate their effect on numerical dispersion. Here the time step Δt is selected as $2\Delta t_C$, where Δt_C is the CFL limit corresponding to the mesh size of $\lambda/20$. As one will see, the higher order scheme may involve less CPU cost and lower memory requirement than the lower order case for certain required accuracy due to its higher spatial accuracy.

Figs. 4-4 to 4-6 compare the maximum dispersion error of the 4th order scheme when spatial sampling rate $N_s=r$ ($20 \leq r \leq 50$) with that of the 2nd order scheme when $N_s=r, 2r, 8r$, where θ is selected as $22.5^\circ, 45^\circ, 67.5^\circ$ respectively. The maximum dispersion error of the scheme with infinitesimal mesh size derived from (4-20) is also plotted for comparison. The spatial sampling rate N_s is defined as the number of sampling points per wavelength, which is equal to the ratio between the wavelength λ and the mesh size δ .

Fig. 4-4. Maximum phase velocity error versus spatial sampling rate for the 2nd, 4th order schemes and the one with infinitesimal mesh size when $\theta=22.5^\circ$

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Fig. 4-5. Maximum phase velocity error versus spatial sampling rate for the 2nd, 4th order schemes and the one with infinitesimal mesh size when $\theta=45^\circ$

Fig. 4-6. Maximum phase velocity error versus spatial sampling rate for the 2nd, 4th order schemes and the one with infinitesimal mesh size when $\theta=67.5^\circ$

It can be seen that the dispersion error is decreased by refining the mesh. This shows that refining mesh possesses the effect of reducing dispersion error like using

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higher order schemes. However, this method is only effective for the lower order scheme, especially that of 2nd order. Moreover, the dispersion error of the 4th order scheme is still no greater than that of the 2nd order one when the ratio of their spatial sampling rate equals to 1/2 or even 1/8.

Fig. 4-7. Anisotropy error versus spatial sampling rate for the 2nd, 4th order schemes and the one with infinitesimal mesh size when $\theta=22.5^\circ$

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Fig. 4-8. Anisotropy error versus spatial sampling rate for the 2nd, 4th order schemes and the one with infinitesimal mesh size when $\theta=45^\circ$

Fig. 4-9. Anisotropy error versus spatial sampling rate for the 2nd, 4th order schemes and the one with infinitesimal mesh size when $\theta=67.5^\circ$

Figs. 4-7 to 4-9 illustrates the anisotropy error defined in (4-23) for the cases

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studied in Figs. 4-4 to 4-6. One can find that the anisotropy error is also decreased by refining the mesh. It can be observed that the 4th order scheme achieves similar improvement of the anisotropy performance as that of the dispersion error performance. Its anisotropy error is still no greater than that of the 2nd order one when the ratio of their spatial sampling rate equals to 1/2 or even 1/8.

Such improvement of the dispersion realized by higher order schemes with coarser mesh leads to other advantages of the higher order schemes over the low order ones. In particular one knows that the CPU cost of solving banded systems for ADI schemes is proportional to w^2N [93], where w is the “half bandwidth” and N is the number of unknowns. If one expands and arranges the updating procedures described in (4-1) and (4-2) like [89], one can obtain $w=2$ and $w=4$ for the 2nd and 4th order scheme respectively. Furthermore, the number of field components for the 3D case discussed here is given by $N=6n_xn_y n_z+2n_xn_y+2n_xn_z+ 2n_y n_z+2(n_x+n_y+n_z)$, where n_x, n_y, n_z are the numbers of cells in x, y, z direction respectively. If the spatial sampling rate in each direction is halved, N will be reduced to about 1/8 of its original value when n_x, n_y, n_z are large enough. Then it can be found that the higher order scheme may result in higher computation efficiency than the lower order case for the same required accuracy. At the same time, the memory required by the updating procedures may also be reduced.

From Figs. 4-4 to 4-9, it is also found that the difference between both the dispersion and anisotropy errors of the 4th order scheme and the infinitesimal mesh size case can be neglected as compared to the improvement from the 2nd to 4th order scheme. This implies that the 4th order scheme should be good enough for most of

the applications.

4.5.3 Effect of Time Step on Numerical Dispersion

Figs. 4-10 to 4-12 show the dispersion error of the 2nd, 4th, 6th and 10th order schemes with time step $\Delta t = 2\Delta t_C$, $4\Delta t_C$, and uniform mesh size $\delta = \lambda/20$, where $\theta = 22.5^\circ$, 45° , 67.5° respectively. Here Δt_C is the CFL limit corresponding to the mesh size of $\lambda/20$. It can be seen that the higher order schemes achieve lower dispersion error compared with the 2nd order scheme for various time steps; meanwhile the dispersion error becomes more serious when the time step increases.

Fig. 4-10. Phase velocity error versus wave propagation angle ϕ for the 2nd, 4th, 6th and 10th order schemes with $\Delta t = 2\Delta t_C$ and $\Delta t = 4\Delta t_C$, $\theta = 22.5^\circ$

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Fig. 4-11. Phase velocity error versus wave propagation angle ϕ for the 2nd, 4th, 6th and 10th order schemes with $\Delta t = 2\Delta t_c$ and $\Delta t = 4\Delta t_c$, $\theta = 45^\circ$

Fig. 4-12. Phase velocity error versus wave propagation angle ϕ for the 2nd, 4th, 6th and 10th order schemes with $\Delta t = 2\Delta t_c$ and $\Delta t = 4\Delta t_c$, $\theta = 67.5^\circ$

Next the anisotropy error of the 2nd, 4th, 6th and 10th order schemes with various

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time step sizes is considered in Figs. 4-13 to 4-15. The time step size equals to Δt_C times the time step size factor, which is the horizontal axis variable in these figures and ranging from 2 to 4. The mesh size is the same as that of the preceding dispersion error discussion and the anisotropy error defined in (4-23) is plotted for $\theta=22.5^\circ, 45^\circ, 67.5^\circ$. It can be observed that the higher order schemes possess lower anisotropy error compared with the 2nd order scheme for various time steps. In addition, the anisotropy error becomes serious with the increase of time step size.

Fig. 4-13. Anisotropy error versus various time step size factors for the 2nd, 4th, 6th and 10th order schemes with $\theta=22.5^\circ$

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Fig. 4-14. Anisotropy error versus various time step size factors for the 2nd, 4th, 6th and 10th order schemes with $\theta=45^\circ$

Fig. 4-15. Anisotropy error versus various time step size factors for the 2nd, 4th, 6th and 10th order schemes with $\theta=67.5^\circ$

From the above investigations, one can conclude that higher order methods can

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achieve lower dispersion and anisotropy errors, however, the improvement is not apparent above 6th order. Actually the spatial accuracy of the 4th order ADI method is good enough for most applications as demonstrated in the numerical results. One also finds that refining mesh is not as effective for the higher (e.g. 4th) order ADI method as for the 2nd order one. When the time step is increased, the dispersion and anisotropy errors of each higher order ADI method becomes more serious and thus the scheme to decrease the effect of temporal discretization on dispersion error will also be needed.

4.6 Computational Cost

As mentioned in Section 4.5.2, the higher order ADI-FDTD schemes involve coefficient matrices with greater bandwidth. Therefore, the computational cost of the higher order schemes should be more than the 2nd order ADI-FDTD method for the simulation with the same grid and iteration numbers. A numerical experiment is performed to compare the CPU time of the 2nd and higher order schemes. Since the 4th order scheme is good enough for most applications, only the CPU time of the 4th order scheme will be compared with the 2nd order scheme.

The numerical experiment performed here is similar to the numerical dispersion demonstration experiment [13]. The computational domain is $0.5 \times 0.5 \times 50 \text{ cm}^3$, where the lengths in x , y and z directions are accordingly 0.5cm, 0.5cm and 50cm. The source is at the center of this domain and TEM wave is propagating along z direction. The uniform mesh size is set to be 0.5mm and the time step equals to four times the Courant limit time step size. The Matlab programs for both the 2nd and 4th order schemes run on a SUN FIRE 15K work station. The CPU time are recorded

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for 200 iterations and the numerical phase velocity error defined in (4-22) is derived from these two schemes. Table 4-2 lists the CPU time and phase velocity error. One can find that the computational cost of the 2nd order scheme is about 30% lower than the higher order scheme. If we increase the mesh size to 1mm and run the 4th order ADI-FDTD method again, the CPU time can be reduced since the grid number will be reduced. This result is also listed in Table 4-2. One can find that the 4th order ADI-FDTD method with the coarser mesh size can achieve less computational cost than the 2nd order scheme for a given computational domain, while the accuracy of the higher order scheme is still better.

Table 4-2. Comparison of the CPU time and numerical velocity error between 2nd order and 4th order schemes

Scheme	Step number	Total time used (s)	Phase velocity error
2 nd order with mesh size 0.5mm	200	76.4	5.6%
4 th order with mesh size 0.5mm	200	113.5	4.8%
4 th order with coarser mesh size 1mm	200	49.8	5%

4.7 Conclusion

Through the generalized analysis of stability, the unconditional stability for a series of higher order 3D ADI-FDTD methods based on higher order cell-centered finite difference schemes have been proven. The generalized form of the dispersion relations has also been provided for these higher order unconditionally stable ADI methods.

From the numerical dispersion results of the 2nd, 4th, 6th and 10th order ADI methods,

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one finds that both the dispersion and anisotropy errors can be reduced by using higher order approximation schemes. The improved numerical dispersion performance realized by the higher order ADI schemes actually leads to other advantages of them, which are higher computation efficiency and lower memory requirement, since they may obtain better dispersion performance even with coarser mesh. When the time step increases, the effect of spatial discretization on dispersion and anisotropy errors, which is related to the mesh size and the order of spatial finite difference scheme, may become diminished. Then the time step size turns out to be the dominant factor for the dispersion error, and becomes a real bottleneck for improving dispersion characteristic. Thus it is anticipated that ADI-FDTD methods with high time resolution will be meaningful future work to be investigated.

Chapter 5

A Parameter Optimized ADI-FDTD Method Based on the (2,4) Stencil

5.1 Introduction

To improve the numerical dispersion performance of the ADI-FDTD method, the higher order schemes have been introduced to this method, which has been discussed in Chapter 4. Besides employing the higher order schemes, optimizing the coefficients in the finite difference scheme is also a usual method to improve the dispersion performance. Such works have been carried out for the explicit FDTD method [94]-[99], and a parameter optimized method has also been proposed for the ADI-FDTD method based on the (2,2) stencil (denoted as (2,2) PO-ADI-FDTD) [100]. This method is used to minimize the dispersion error for different incident angles and different time step sizes.

To further improve the dispersion performance, this chapter will present a parameter optimized ADI-FDTD method based on the (2,4) stencil. This is an extension of the work in [98], where the parameter optimized method has been proposed for the explicit FDTD method based on the (2,4) stencil. Through the investigation of the

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numerical dispersion, it can be demonstrated that the (2,4) PO-ADI-FDTD further reduces the dispersion error compared to the previous (2,2) PO-ADI-FDTD method. In addition, it can achieve better dispersion performance than the 4th order ADI-FDTD method [89], which is also based on the (2,4) stencil.

To be concise, this chapter will present in detail the parameter optimized method based on a 2-D ADI FDTD method, and discuss in brief the extension to 3-D ADI FDTD case. The updating procedure as well as the stability and dispersion analyses is provided in Section 5.2. One can find that the (2,4) PO-ADI-FDTD method is unconditionally stable for any real parameters. This is different from the (2,2) PO-ADI-FDTD method [100], whose parameters are required to satisfy some conditions to maintain the unconditional stability. Based on the dispersion relation obtained, Section 5.3 presents the schemes to optimize the parameters to satisfy different requirements, such as zero numerical dispersion error (ZNDE) in the axes direction, ZNDE in the diagonal direction, zero anisotropy dispersion, ZNDE for two arbitrary angles, and minimum average dispersion error. The performance of some of these schemes is compared with that of the (2,2) PO-ADI-FDTD method. The extension to 3-D (2,4) PO-ADI-FDTD method is also discussed in this section. It can be found that the dispersion relation studied here does not rely on the assumption of the same mesh size in different directions as in [98]. Therefore all the schemes presented here are also useful for the case with different mesh sizes in different directions, which will be illustrated in Section 5.3 as well.

When the mesh size and time step have been determined for the simulation procedure, the signal at different frequencies will possess different performance due

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to numerical dispersion. Thus the variation of the performance of the (2,4) PO-ADI-FDTD method for different frequencies is studied in Section 5.4. The comparison with the (2,2) PO-ADI-FDTD method is also made in this section. In view of the advantages of the (2,4) PO-ADI-FDTD over the (2,2) PO-ADI-FDTD method summarized in the conclusion, one will find that it is very meaningful to propose the parameter optimized method for the ADI-FDTD method based on the (2,4) stencil.

5.2 (2,4) PO-ADI-FDTD Method: Procedure, Stability and Dispersion Analyses

Let us consider the 2-D TE_z wave propagating in a lossless homogenous medium with permittivity ϵ and permeability μ . The finite difference approximation scheme for Maxwell's equations possesses 2nd order accuracy in time domain. The spatial differential operators are approximated by the cell-centered finite difference scheme with four stencils, which features 2nd order accuracy in general.

The updating procedure of the 2-D (2,4) PO-ADI-FDTD method is performed in two sub-procedures. The first sub-procedure is presented in (5-1a)-(5-1c) for time marching from n to $n+1/2$. The second sub-procedure is presented in (5-2a)-(5-2c) for time marching from $n+1/2$ to $n+1$. They are

<First sub-procedure>

$$E_x|_{I+1/2,J}^{n+1/2} = E_x|_{I+1/2,J}^n + bP_y^{(4)} \left(H_z|_{I+1/2,J}^{n+1/2} \right) \quad (5-1a)$$

$$E_y|_{I,J+1/2}^{n+1/2} = E_y|_{I,J+1/2}^n - bP_x^{(4)} \left(H_z|_{I,J+1/2}^n \right) \quad (5-1b)$$

$$H_z|_{I+1/2,J+1/2}^{n+1/2} = H_z|_{I+1/2,J+1/2}^n + d \left[P_y^{(4)} \left(E_x|_{I+1/2,J+1/2}^{n+1/2} \right) - P_x^{(4)} \left(E_y|_{I+1/2,J+1/2}^n \right) \right] \quad (5-1c)$$

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<Second sub-procedure>

$$E_x|_{I+1/2,J}^{n+1} = E_x|_{I+1/2,J}^{n+1/2} + bP_y^{(4)} \left(H_z|_{I+1/2,J}^{n+1/2} \right) \quad (5-2a)$$

$$E_y|_{I,J+1/2}^{n+1} = E_y|_{I,J+1/2}^{n+1/2} - bP_x^{(4)} \left(H_z|_{I,J+1/2}^{n+1/2} \right) \quad (5-2b)$$

$$H_z|_{I+1/2,J+1/2}^{n+1} = H_z|_{I+1/2,J+1/2}^{n+1/2} + d \left[P_y^{(4)} \left(E_x|_{I+1/2,J+1/2}^{n+1/2} \right) - P_x^{(4)} \left(E_y|_{I+1/2,J+1/2}^{n+1/2} \right) \right] \quad (5-2c)$$

Here, $b = \Delta t / (2\varepsilon)$, $d = \Delta t / (2\mu)$, Δt is the time step, n is the time index and I, J are

the space indexes for Yee cell. The difference operator $P_\xi^{(4)}$ is defined as

$$P_x^{(4)}(\psi|_{I,J}^n) = c_x \frac{\psi|_{I+1/2,J}^n - \psi|_{I-1/2,J}^n}{\Delta x} + (1-c_x) \frac{\psi|_{I+3/2,J}^n - \psi|_{I-3/2,J}^n}{3\Delta x} \quad (5-3a)$$

$$P_y^{(4)}(\psi|_{I,J}^n) = c_y \frac{\psi|_{I,J+1/2}^n - \psi|_{I,J-1/2}^n}{\Delta y} + (1-c_y) \frac{\psi|_{I,J+3/2}^n - \psi|_{I,J-3/2}^n}{3\Delta y} \quad (5-3b)$$

where ψ can be any field component in (5-1)-(5-2), i.e. E_x , E_y , or H_z . c_x and c_y are

the parameters to be optimized. Equations (5-3a)-(5-3b) represent the finite difference approximations to the first spatial derivative of electric and magnetic fields. One can find that their accuracy is generally of 2nd order, and becomes 4th order specially when $c_x=c_y=9/8$ that leads to the 4th order ADI-FDTD method.

Following the stability analysis procedure in the last chapter, one can represent the updating procedure for one time step in a matrix form and solve the eigenvalues as

$$\lambda_1 = 1 \quad (5-4a)$$

$$\lambda_2 = \frac{Q + \sqrt{-S^2}}{R} \quad (5-4b)$$

$$\lambda_3 = \frac{Q - \sqrt{-S^2}}{R} \quad (5-4c)$$

where

$$R = (1 + bdP_x^2)(1 + bdP_y^2)$$

$$Q = 1 - bdP_xP_y - b^2d^2P_x^2P_y^2 \quad S^2 = R^2 - Q^2 = 4bd(P_x^2 + P_y^2 + bdP_x^2P_y^2)$$

$$P_\xi = -2c_\xi \sin(k_\xi \Delta \xi / 2) / \Delta \xi - 2(1 - c_\xi) \sin(3k_\xi \Delta \xi / 2) / (3\Delta \xi) \quad (5-5)$$

Since P_ξ ($\xi=x,y$) are real, b and d are both positive, and $R^2 \geq Q^2$, it is evident that the

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magnitude of each of the three eigenvalues is unity for any real c_x and c_y . So one can conclude that this (2,4) PO-ADI-FDTD method is unconditionally stable just like the 2nd order ADI FDTD method in [3], which is based on the (2,2) stencil. In contrast, the parameters of the (2,2) PO-ADI-FDTD method are required to satisfy certain conditions [100] in order to maintain the unconditional stability. This is one advantage of the (2,4) PO-ADI-FDTD over the (2,2) PO-ADI-FDTD method.

For the dispersion relation, it can also be deduced by a similar procedure in the last chapter, which reads

$$\sin^2(\omega\Delta t) = \frac{4bd(P_x^2 + P_y^2 + bdP_x^2P_y^2)}{(1+bdP_x^2)^2(1+bdP_y^2)^2} \quad (5-6)$$

Based on the optimization of this dispersion relation in different senses, one can get several schemes and obtain the parameters to satisfy different requirements. These schemes will be presented in the next section.

5.3 Schemes for Optimizing Parameter

From the dispersion relation in (5-6), one can see that it is generally difficult and tedious to solve the optimized parameters by analytical derivation like the axes-optimized method (AOM), diagonally optimized method (DOM) and isotropic optimization method (IOM) in [98]. Fortunately the numerical search algorithm can be applied easily to the (2,4) PO-ADI-FDTD method for optimizing the parameters without demanding much caution on the time step size due to its unconditional stability. This is an advantage over the method in [98], which requires the time step size to be bounded by the Courant-Friedrich-Levy (CFL) condition. Thus the schemes of the (2,4) PO-ADI-FDTD method can be achieved simply by specifying the desired optimization objectives.

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The schemes with ZNDE in the axes direction, ZNDE in the diagonal direction and zero anisotropy dispersion proposed here correspond to the AOM, DOM and IOM in [98]. In addition this section will also present the schemes with ZNDE for two arbitrary angles, and minimum average dispersion error. Actually the schemes for these two requirements have already been proposed in [100] for the ADI-FDTD with the (2,2) stencil. It will be demonstrated that the (2,4) PO-ADI-FDTD achieves better performance. Apart from the five schemes that represent the often desired performance in practice, one can also construct some other schemes to optimize the parameters by specifying other optimization objectives.

In practical simulation, the mesh size and time step of FDTD method are determined based on the highest frequency and the structure geometry to be analyzed. Consider that the wave with highest frequency ω_0 is propagating in free space and the uniform mesh size is related to the wavelength by $\Delta x = \Delta y = \delta = \lambda/20$, where λ is the wavelength corresponding to ω_0 . The CFL limit corresponding to the mesh size is $\Delta t_C = (\lambda/20)/(\sqrt{2}c)$, where c is the speed of light in free space, and the time step is set to be $\Delta t = 2\Delta t_C$. The mesh size δ and time step Δt will be the default values (unless otherwise stated) for the optimization schemes discussed below. With the mesh size and time step adopted for the simulation procedure, the signal at different frequencies will possess different performance due to numerical dispersion. Hence the performance of the (2,4) PO-ADI-FDTD for different frequencies of interest is also necessary to be studied and this will be discussed in the next section.

To describe the optimization objectives, some notations are to be introduced. Let the

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normalized phase velocity along propagation angle ϕ (in degree) be denoted as $u(\phi)$, $0^\circ \leq \phi \leq 90^\circ$. The normalized phase velocity refers to the numerical phase velocity v_p normalized by c . v_p can be derived from the ratio of frequency and wave number [15], where

$$v_p = \frac{\omega}{k_{final}} \quad (5-7)$$

k_{final} is the final result solved from the dispersion relation.

To specify the numerical dispersion error, an operator notation Err is defined in terms of $u(\phi)$ as:

$$Err(u(\phi), 1) = |u(\phi) - 1| \quad (5-8a)$$

$$Err(u(\phi_1), u(\phi_2)) = |u(\phi_1) - u(\phi_2)| \quad (5-8b)$$

$$Err(u, 1) = \frac{1}{91} \sum_{\phi=0^\circ}^{90^\circ} |u(\phi) - 1| \quad (5-8c)$$

The $Err(u, 1)$ is used to denote the average dispersion error. In order to quantify the anisotropy error of the numerical dispersion, it is defined that

$$Aniso(u) = \frac{u_{\max} - u_{\min}}{u_{\min}} \times 100\% \quad (5-9)$$

where $u_{\max}$ and $u_{\min}$ are accordingly the maxima and minima of the normalized numerical phase velocity obtained by the dispersion analysis. This definition of anisotropy error is similar to the one in (4-23).

5.3.1 Scheme with ZNDE in the Axes Direction

To achieve ZNDE along x and y axes direction simultaneously, one can set the optimization objective to be the minimization of $Err(u(0^\circ), 1) + Err(u(90^\circ), 1)$. Upon carrying out the optimization, one obtains $c_x = c_y = 1.6428$.

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Fig. 5-1. Numerical dispersion of the 2nd order ADI-FDTD, (2,2) PO-ADI-FDTD, 4th order ADI-FDTD and (2,4) PO-ADI-FDTD with ZNDE in the axes direction

Fig. 5-1 presents the phase velocity error versus propagation angle ϕ for the 2nd order ADI-FDTD, (2,2) PO-ADI-FDTD, 4th order ADI-FDTD and (2,4) PO-ADI-FDTD methods. The dispersion of the (2,2) PO-ADI-FDTD method has been included here especially to highlight the advantage of the (2,4) PO-ADI-FDTD method. All the dispersion of the 2nd order ADI-FDTD, (2,2) PO-ADI-FDTD, and 4th order ADI-FDTD is investigated under the same situation.

From the figure it can be observed that the (2,2) PO-ADI-FDTD achieves little improvement in anisotropy compared with the 2nd order ADI-FDTD method. Based on (5-9) one can find that the anisotropy error changes from 0.85% to 0.84% when the parameter optimized method is applied. However, the (2,4) PO-ADI-FDTD obtains the improvement over the 4th order ADI-FDTD method not only in average dispersion error, but also in anisotropy with considerable decrease from 0.63% to

0.19%.

5.3.2 Scheme with ZNDE in the Diagonal Direction

The ZNDE along the diagonal direction can be achieved by minimizing $Err(u(45^\circ), 1)$. To illustrate the case with different mesh size in different directions as mentioned in the introduction, here let Δx and Δy be $\lambda/8$ and $\lambda/40$ respectively, while the time step size retains the default value. For this case, the dispersion of the 4th order ADI-FDTD method turns out to be asymmetric about the diagonal. If the symmetry of the dispersion is required as another condition for this scheme, one can set the minimization of $Err(u(45^\circ), 1) + Err(u(0^\circ), u(90^\circ))$ as the optimization objective and obtain $c_x = 1.2391$, $c_y = 3.5664$. The dispersion of this scheme is compared with that of the 4th order ADI-FDTD method in Fig. 5-2. It can be found that the scheme leads to ZNDE in the diagonal direction and much improvement in average dispersion error and anisotropy of dispersion.

Fig. 5-2. Numerical dispersion of the 4th order ADI-FDTD and (2,4) PO-ADI-FDTD with ZNDE in the diagonal direction

5.3.3 Scheme with Zero Anisotropy Dispersion

The zero anisotropy dispersion can be achieved by minimizing $Err(u(0^\circ), u(45^\circ)) + Err(u(90^\circ), u(45^\circ))$, where $c_x=1.5154$ and $c_y=1.5154$ are obtained. Fig. 5-3 compares the dispersion of the zero anisotropy dispersion scheme with that of the 4th order ADI-FDTD method. It shows that the anisotropy of dispersion disappears with this scheme. Although the dispersion cannot achieve the ideal case, i.e. the normalized numerical phase velocity equals to one, it is still better than the dispersion of the 4th order ADI-FDTD method.

Fig. 5-3. Numerical dispersion of the 4th order ADI-FDTD and (2,4) PO-ADI-FDTD with zero anisotropy

5.3.4 Scheme with ZNDE for Two Arbitrary Angles

The ZNDE for two angles ϕ_1 and ϕ_2 can be achieved by minimizing $Err(u(\phi_1), 1) + Err(u(\phi_2), 1)$. Actually the schemes with ZNDE in the axes direction and diagonal direction can be regarded as the special cases of this scheme.

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Fig. 5-4 shows the dispersion of the schemes for the (2,2) PO-ADI-FDTD and (2,4) PO-ADI-FDTD methods, comparing with that of the 2nd order ADI-FDTD and 4th order ADI-FDTD methods. The optimization is performed for two angles, for example 0° and 45°, and it is obtained that $c_x=1.6428$ and $c_y=1.8852$. With these parameters, one can see that there is indeed no numerical dispersion error in 0° and 45° direction. However, the anisotropy of dispersion is affected and that of the (2,2) PO-ADI-FDTD is worse than that of the (2,4) PO-ADI-FDTD method. More precisely, for the 2nd order ADI-FDTD and (2,2) PO-ADI-FDTD method, the anisotropy error increases from 0.85% to 1.91%. For the 4th order ADI-FDTD and (2,4) PO-ADI-FDTD method, it only changes from 0.63% to 0.84%.

Fig. 5-4. Numerical dispersion of the 2nd order ADI-FDTD, (2,2) PO-ADI-FDTD, 4th order ADI-FDTD and (2,4) PO-ADI-FDTD with ZNDE in two angles

5.3.5 Scheme with Minimum Average Dispersion Error

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When $Err(u,1)$ is minimized, one can obtain the parameters to produce the minimum average dispersion error, which read $c_x=1.6686$, $c_y=1.6687$. The dispersion of both the (2,2) PO-ADI-FDTD and (2,4) PO-ADI-FDTD methods with minimum average dispersion error are compared in Fig. 5-5. Also shown is the dispersion of the 2nd order ADI-FDTD and 4th order ADI-FDTD methods so as to highlight the improvement of this parameter optimized method.

Fig. 5-5. Numerical dispersion of the 2nd order ADI-FDTD, (2,2) PO-ADI-FDTD, 4th order ADI-FDTD and (2,4) PO-ADI-FDTD with minimum average dispersion error

It can be observed clearly again that the (2,2) PO-ADI-FDTD achieves little improvement in anisotropy compared with the 2nd order ADI-FDTD. However, the (2,4) PO-ADI-FDTD leads to much improvement not only in average dispersion error, but also in anisotropy over the 4th order ADI-FDTD.

One can find that the above five schemes for the (2,4) PO-ADI-FDTD method can satisfy different requirements and they can be adopted for different applications. It

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is also seen that the (2,4) PO-ADI-FDTD method achieves much better performance than the (2,2) PO-ADI-FDTD method for the same requirements. Although these schemes are proposed for 2-D case, they are readily extended to 3-D case.

5.3.6 Extension to 3-D (2,4) PO-ADI-FDTD method

The two sub-procedures of the 3-D (2,4) PO-ADI-FDTD method can be represented in (5-10)-(5-11):

$$E_x|_{I+1/2,J,K}^{n+1/2} = E_x|_{I+1/2,J,K}^n + b \left[P_y^{(4)} \left(H_z|_{I+1/2,J,K}^n \right) - P_z^{(4)} \left(H_y|_{I+1/2,J,K}^{n+1/2} \right) \right] \quad (5-10a)$$

$$E_y|_{I,J+1/2,K}^{n+1/2} = E_y|_{I,J+1/2,K}^n + b \left[P_z^{(4)} \left(H_x|_{I,J+1/2,K}^n \right) - P_x^{(4)} \left(H_z|_{I,J+1/2,K}^{n+1/2} \right) \right] \quad (5-10b)$$

$$E_z|_{I,J,K+1/2}^{n+1/2} = E_z|_{I,J,K+1/2}^n + b \left[P_x^{(4)} \left(H_y|_{I,J,K+1/2}^n \right) - P_y^{(4)} \left(H_x|_{I,J,K+1/2}^{n+1/2} \right) \right] \quad (5-10c)$$

$$H_x|_{I,J+1/2,K+1/2}^{n+1/2} = H_x|_{I,J+1/2,K+1/2}^n + d \left[P_z^{(4)} \left(E_y|_{I,J+1/2,K+1/2}^n \right) - P_y^{(4)} \left(E_z|_{I,J+1/2,K+1/2}^{n+1/2} \right) \right] \quad (5-10d)$$

$$H_y|_{I+1/2,J,K+1/2}^{n+1/2} = H_y|_{I+1/2,J,K+1/2}^n + d \left[P_x^{(4)} \left(E_z|_{I+1/2,J,K+1/2}^n \right) - P_z^{(4)} \left(E_x|_{I+1/2,J,K+1/2}^{n+1/2} \right) \right] \quad (5-10e)$$

$$H_z|_{I+1/2,J+1/2,K}^{n+1/2} = H_z|_{I+1/2,J+1/2,K}^n + d \left[P_y^{(4)} \left(E_x|_{I+1/2,J+1/2,K}^n \right) - P_x^{(4)} \left(E_y|_{I+1/2,J+1/2,K}^{n+1/2} \right) \right] \quad (5-10f)$$

$$E_x|_{I+1/2,J,K}^{n+1} = E_x|_{I+1/2,J,K}^{n+1/2} + b \left[P_y^{(4)} \left(H_z|_{I+1/2,J,K}^{n+1} \right) - P_z^{(4)} \left(H_y|_{I+1/2,J,K}^{n+1/2} \right) \right] \quad (5-11a)$$

$$E_y|_{I,J+1/2,K}^{n+1} = E_y|_{I,J+1/2,K}^{n+1/2} + b \left[P_z^{(4)} \left(H_x|_{I,J+1/2,K}^{n+1} \right) - P_x^{(4)} \left(H_z|_{I,J+1/2,K}^{n+1/2} \right) \right] \quad (5-11b)$$

$$E_z|_{I,J,K+1/2}^{n+1} = E_z|_{I,J,K+1/2}^{n+1/2} + b \left[P_x^{(4)} \left(H_y|_{I,J,K+1/2}^{n+1} \right) - P_y^{(4)} \left(H_x|_{I,J,K+1/2}^{n+1/2} \right) \right] \quad (5-11c)$$

$$H_x|_{I,J+1/2,K+1/2}^{n+1} = H_x|_{I,J+1/2,K+1/2}^{n+1/2} + d \left[P_z^{(4)} \left(E_y|_{I,J+1/2,K+1/2}^{n+1} \right) - P_y^{(4)} \left(E_z|_{I,J+1/2,K+1/2}^{n+1/2} \right) \right] \quad (5-11d)$$

$$H_y|_{I+1/2,J,K+1/2}^{n+1} = H_y|_{I+1/2,J,K+1/2}^{n+1/2} + d \left[P_x^{(4)} \left(E_z|_{I+1/2,J,K+1/2}^{n+1} \right) - P_z^{(4)} \left(E_x|_{I+1/2,J,K+1/2}^{n+1/2} \right) \right] \quad (5-11e)$$

$$H_z|_{I+1/2,J+1/2,K}^{n+1} = H_z|_{I+1/2,J+1/2,K}^{n+1/2} + d \left[P_y^{(4)} \left(E_x|_{I+1/2,J+1/2,K}^{n+1} \right) - P_x^{(4)} \left(E_y|_{I+1/2,J+1/2,K}^{n+1/2} \right) \right] \quad (5-11f)$$

where

$$P_x^{(4)}(\psi|_{I,J,K}^n) = c_x \frac{\psi|_{I+1/2,J,K}^n - \psi|_{I-1/2,J,K}^n}{\Delta x} + (1-c_x) \frac{\psi|_{I+3/2,J,K}^n - \psi|_{I-3/2,J,K}^n}{3\Delta x} \quad (5-12a)$$

$$P_y^{(4)}(\psi|_{I,J,K}^n) = c_y \frac{\psi|_{I,J+1/2,K}^n - \psi|_{I,J-1/2,K}^n}{\Delta y} + (1-c_y) \frac{\psi|_{I,J+3/2,K}^n - \psi|_{I,J-3/2,K}^n}{3\Delta y} \quad (5-12b)$$

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$$P_z^{(4)}(\psi|_{I,J,K}^n) = c_z \frac{\psi|_{I,J,K+1/2}^n - \psi|_{I,J,K-1/2}^n}{\Delta z} + (1 - c_z) \frac{\psi|_{I,J,K+3/2}^n - \psi|_{I,J,K-3/2}^n}{3\Delta z} \quad (5-12c)$$

Here ψ can be any field component in (5-10)-(5-11), i.e. E_x , E_y , E_z , H_x , H_y , or H_z . c_x , c_y , and c_z are the parameters to be optimized.

Following the stability and dispersion analysis procedure in the last chapter, the 3-D scheme can also be proven to be unconditionally stable for any parameters and the dispersion relation for the 3-D (2,4) PO-ADI-FDTD method can be deduced as

$$\sin^2(\omega\Delta t) = \frac{4bd(b^3d^3P_x^2P_y^2P_z^2+1)\left[P_x^2+P_y^2+P_z^2+bd(P_x^2P_y^2+P_x^2P_z^2+P_y^2P_z^2)\right]}{(1+bdP_x^2)^2(1+bdP_y^2)^2(1+bdP_z^2)^2} \quad (5-13)$$

where P_ξ ($\xi=x,y,z$) conforms to the definition in (5-5). The parameter optimized method proposed here can also be applied based on this dispersion relation so that one can get the 3-D (2,4) PO-ADI-FDTD method. The detailed optimization procedure for the 3-D case is a bit different, but the philosophy is quite similar.

5.4 Variation of Performance for Different Frequencies

In this section, the variation of the performance of the parameter optimized method for different frequencies is discussed. To make the presentation concise, this section will only discuss some representative schemes, which include the ones with ZNDE in the axes direction, zero anisotropy dispersion and minimum average dispersion error.

Let us assume that the highest frequency we are interested in is $\omega_0=12\pi\times10^9$ rad/s, the uniform mesh size is $\Delta x=\Delta y=\delta=\lambda/20$ and the time step is $\Delta t=2\Delta t_C$. Here λ is the wavelength corresponding to 6GHz and $\Delta t_C=(\lambda/20)/(\sqrt{2}c)$ is the CFL limit

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corresponding to the mesh size of $\lambda/20$. The variation of the numerical dispersion performance of the various parameter optimized schemes for different frequencies will be shown below.

5.4.1 Performance of the Scheme with ZNDE in the Axes Direction for Different Frequencies

The dispersion of the scheme with ZNDE in the axes direction for $0.1\omega_0$, $0.5\omega_0$, ω_0 , $1.5\omega_0$, and $2\omega_0$ is shown in Fig. 5-6. It can be found that the scheme basically preserves ZNDE in the axes direction for frequencies lower than ω_0 , and achieves even better dispersion performance than the case with frequency of ω_0 . This means that it features lower average dispersion error and anisotropy for frequencies below ω_0 .

On the other hand, for frequencies higher than ω_0 the scheme becomes far from ZNDE in the axes direction, and its dispersion performance also deteriorates at the same time. If taking the average dispersion error and anisotropy error for the scheme with frequency ω_0 and $2\omega_0$ as an example, it can be found that the average dispersion error changes from 0.00095 to 0.0079 with the increase of frequency, and simultaneously the anisotropy error grows from 0.19% to 0.36%.

For comparison and further investigation, the dispersion of the (2,2) PO-ADI-FDTD method for the same set of frequencies is presented in Fig. 7. It can be observed that the ZNDE in the axes direction only works with frequency equal to ω_0 . For frequencies higher or lower than ω_0 , the dispersion error always becomes worse. This reveals another advantage of the (2,4) PO-ADI-FDTD. It is easy to

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demonstrate that the (2,2) PO-ADI-FDTD method for other requirements possesses the similar performance, which will not be repeated here.

Fig. 5-6. Numerical dispersion of the (2,4) PO-ADI-FDTD with ZNDE in the axes direction for different frequencies

Fig. 5-7. Numerical dispersion of the (2,2) PO-ADI-FDTD with ZNDE in the axes direction for different frequencies

5.4.2 Performance of the Scheme with Zero Anisotropy Dispersion for Different Frequencies

Fig. 5-8 presents the dispersion of the (2,4) PO-ADI-FDTD with zero anisotropy dispersion in a similar way as Fig. 5-6. The variation of the performance for different frequencies is also found to be similar. The scheme for frequencies lower than ω_0 basically preserves zero anisotropy dispersion, and achieves even better dispersion performance than the case with ω_0 . However, for frequencies higher than ω_0 , the scheme does not possess zero anisotropy dispersion any more and its dispersion performance also deteriorates at the same time.

Fig. 5-8. Numerical dispersion of the (2,4) PO-ADI-FDTD with zero anisotropy for different frequencies

5.4.3 Performance of the Scheme with Minimum Average Dispersion Error for Different Frequencies

The frequency effect on the performance is also studied in a similar way for the (2,4) PO-ADI-FDTD with minimum average dispersion error. As shown in Fig. 5-9, the

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dispersion performance of this scheme becomes better with frequencies lower than ω_0 and turns worse with frequencies higher than ω_0 .

Fig. 5-9. Numerical dispersion of the (2,4) PO-ADI-FDTD with minimum average dispersion error for different frequencies

From the above, one can conclude that the (2,4) PO-ADI-FDTD method basically preserves the desired performance within a band below the frequency where one performs the optimization of the parameters. This is an advantage over the (2,2) PO-ADI-FDTD, which is relatively a narrow band method. So in practice, we should perform the optimization at the highest frequency we are interested in when the (2,4) PO-ADI-FDTD method is applied.

5.5 Conclusion

This chapter has presented a parameter optimized ADI-FDTD method based on the (2,4) stencil to achieve better dispersion performance. The (2,4) PO-ADI-FDTD method has been proven to be unconditionally stable for any real parameters. By

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setting different optimization objectives, the method proposed here can be applied to satisfy different requirements, such as ZNDE in the axes direction, ZNDE in the diagonal direction, zero anisotropy dispersion, ZNDE for two arbitrary angles, and minimum average dispersion error. The variation of the performance of the (2,4) PO-ADI-FDTD method for different frequencies has also been studied. One can find that the (2,4) PO-ADI-FDTD method can basically preserve the desired performance within a band below the frequency where the optimization of parameters is performed. Although this method is proposed for 2-D case, it is readily extended to 3-D case, which has been discussed as well.

All these PO-ADI-FDTD schemes proposed in this chapter are based on the (2,4) stencil, whose complexity and computational cost are the same as the 4th order ADI-FDTD method discussed in Chapter 4. Meanwhile, the previous (2,2) PO-ADI-FDTD method features the same complexity and computational cost as the 2nd ADI-FDTD. Therefore, the (2,4) PO-ADI-FDTD method may be less efficient than the (2,2) PO-ADI-FDTD based on the discussion in Chapter 4. However, compared with the previous (2,2) PO-ADI-FDTD method, the (2,4) PO-ADI-FDTD method possesses the following advantages:

- 1) The (2,4) PO-ADI-FDTD method is unconditionally stable for any real parameters. The (2,2) PO-ADI-FDTD requires the parameters to satisfy some conditions to maintain the unconditional stability.
- 2) The (2,4) PO-ADI-FDTD method achieves much better performance than the (2,2) PO-ADI-FDTD method for the same requirements.

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3) The (2,4) PO-ADI-FDTD method basically preserves the desired performance within a band below certain frequency. The (2,2) PO-ADI-FDTD method only maintains the performance within a narrow band.

These advantages imply that it is meaningful to propose the parameter optimized method for the ADI-FDTD method based on the (2,4) stencil.

Chapter 6

ADI-FDTD Method Including Passive Lumped Elements

6.1 Introduction

Due to the flexibility and versatility of FDTD method, there have been many efforts made to extend it for the global simulation of RF circuits and EM structures as mentioned in Chapter 1. The stability of the extended FDTD including passive lumped elements like resistors, capacitors and inductors, which was developed from the conventional explicit FDTD method, is related to either the mesh size, or both the mesh size and the values of the elements. Hence the time step for each updating procedure of this explicit method must be constrained in a stable range determined by the mesh size or both the mesh size and the values of the elements.

The ADI-FDTD method was proposed to eliminate constraint on the time step due to the CFL condition, and the incorporation of lumped elements to this method has been carried out [101], where the stability has not been studied. This chapter will present a 3-D ADI-FDTD method including lumped elements, whose stability is proven to be neither related to the mesh size, nor related to the values of the elements. Since it is difficult to analyze the magnitudes of the eigenvalues of the

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update matrix, the von Neumann method, which has been used to study the stability of the ADI-FDTD method, is not suitable here and therefore another stability analysis method based on the energy concept [69] is adopted to prove the unconditional stability. This unconditional stability may lead to considerable reduction of the simulation iteration number and higher computation efficiency, and thus should be a distinguished advantage over the conventional explicit FDTD method including lumped elements. At the end, the ADI-FDTD including linear lumped network is proposed as the extension of the preceding schemes, which can also be demonstrated to possess higher computation efficiency.

The next section starts with the schemes for 3-D ADI-FDTD method including resistors, capacitors and inductors. The stability analysis for these schemes is provided in Section 6.3. Some numerical experiments are performed in Section 6.4 for the demonstration of the higher computation efficiency and validation of these schemes. Section 6.5 presents the ADI scheme including general linear lumped network and the corresponding numerical demonstration.

6.2 3-D ADI-FDTD Schemes Including Lumped Elements

In this section the 3-D ADI-FDTD schemes including passive lumped elements will be presented. Assuming that Δx , Δy , Δz are the uniform mesh sizes in x , y , z axis direction respectively, the update equations for these schemes can be expressed as:

$$\begin{aligned} \mathcal{E}_{I+1/2,J,K} &= \frac{E_x|_{I+1/2,J,K}^{n+1/2} - E_x|_{I+1/2,J,K}^n}{\Delta t / 2} \\ &= \frac{H_z|_{I+1/2,J+1/2,K}^{n+1/2} - H_z|_{I+1/2,J-1/2,K}^{n+1/2}}{\Delta y} - \frac{H_y|_{I+1/2,J,K+1/2}^n - H_y|_{I+1/2,J,K-1/2}^n}{\Delta z} - J_{dx1}|_{I+1/2,J,K}^{n+1/2} \end{aligned} \quad (6-1a)$$

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$$\begin{aligned} \mathcal{E}_{I,J+1/2,K} &= \frac{E_y|_{I,J+1/2,K}^{n+1/2} - E_y|_{I,J+1/2,K}^n}{\Delta t/2} \\ &= \frac{H_x|_{I,J+1/2,K+1/2}^{n+1/2} - H_x|_{I,J+1/2,K-1/2}^{n+1/2}}{\Delta z} - \frac{H_z|_{I+1/2,J+1/2,K}^n - H_z|_{I-1/2,J+1/2,K}^n}{\Delta x} - J_{dy1}|_{I,J+1/2,K}^{n+1/2} \end{aligned} \quad (6-1b)$$

$$\begin{aligned} \mathcal{E}_{I,J,K+1/2} &= \frac{E_z|_{I,J,K+1/2}^{n+1/2} - E_z|_{I,J,K+1/2}^n}{\Delta t/2} \\ &= \frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} - H_y|_{I-1/2,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{H_x|_{I,J+1/2,K+1/2}^n - H_x|_{I,J-1/2,K+1/2}^n}{\Delta y} - J_{dz1}|_{I,J,K+1/2}^{n+1/2} \end{aligned} \quad (6-1c)$$

$$\begin{aligned} \mu_{I,J+1/2,K+1/2} &= \frac{H_x|_{I,J+1/2,K+1/2}^{n+1/2} - H_x|_{I,J+1/2,K+1/2}^n}{\Delta t/2} \\ &= \frac{E_y|_{I,J+1/2,K+1}^{n+1/2} - E_y|_{I,J+1/2,K}^{n+1/2}}{\Delta z} - \frac{E_z|_{I,J+1,K+1/2}^n - E_z|_{I,J,K+1/2}^n}{\Delta y} \end{aligned} \quad (6-1d)$$

$$\begin{aligned} \mu_{I+1/2,J,K+1/2} &= \frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} - H_y|_{I+1/2,J,K+1/2}^n}{\Delta t/2} \\ &= \frac{E_z|_{I+1,J,K+1/2}^{n+1/2} - E_z|_{I,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{E_x|_{I+1/2,J,K+1}^n - E_x|_{I+1/2,J,K}^n}{\Delta z} \end{aligned} \quad (6-1e)$$

$$\begin{aligned} \mu_{I+1/2,J+1/2,K} &= \frac{H_z|_{I+1/2,J+1/2,K}^{n+1/2} - H_z|_{I+1/2,J+1/2,K}^n}{\Delta t/2} \\ &= \frac{E_x|_{I+1/2,J+1,K}^{n+1/2} - E_x|_{I+1/2,J,K}^{n+1/2}}{\Delta y} - \frac{E_y|_{I+1,J+1/2,K}^n - E_y|_{I,J+1/2,K}^n}{\Delta x} \end{aligned} \quad (6-1f)$$

$$\begin{aligned} \mathcal{E}_{I+1/2,J,K} &= \frac{E_x|_{I+1/2,J,K}^{n+1} - E_x|_{I+1/2,J,K}^{n+1/2}}{\Delta t/2} \\ &= \frac{H_z|_{I+1/2,J+1/2,K}^{n+1/2} - H_z|_{I+1/2,J-1/2,K}^{n+1/2}}{\Delta y} - \frac{H_y|_{I+1/2,J,K+1/2}^{n+1} - H_y|_{I+1/2,J,K-1/2}^{n+1}}{\Delta z} - J_{dx2}|_{I+1/2,J,K}^{n+1/2} \end{aligned} \quad (6-2a)$$

$$\begin{aligned} \mathcal{E}_{I,J+1/2,K} &= \frac{E_y|_{I,J+1/2,K}^{n+1} - E_y|_{I,J+1/2,K}^{n+1/2}}{\Delta t/2} \\ &= \frac{H_x|_{I,J+1/2,K+1/2}^{n+1/2} - H_x|_{I,J+1/2,K-1/2}^{n+1/2}}{\Delta z} - \frac{H_z|_{I+1/2,J+1/2,K}^{n+1} - H_z|_{I-1/2,J+1/2,K}^{n+1}}{\Delta x} - J_{dy2}|_{I,J+1/2,K}^{n+1/2} \end{aligned} \quad (6-2b)$$

$$\begin{aligned} \mathcal{E}_{I,J,K+1/2} &= \frac{E_z|_{I,J,K+1/2}^{n+1} - E_z|_{I,J,K+1/2}^{n+1/2}}{\Delta t/2} \\ &= \frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} - H_y|_{I-1/2,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{H_x|_{I,J+1/2,K+1/2}^{n+1} - H_x|_{I,J-1/2,K+1/2}^{n+1}}{\Delta y} - J_{dz2}|_{I,J,K+1/2}^{n+1/2} \end{aligned} \quad (6-2c)$$

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$$\begin{aligned} \mu_{I,J+1/2,K+1/2} &= \frac{H_x|_{I,J+1/2,K+1/2}^{n+1} - H_x|_{I,J+1/2,K+1/2}^{n+1/2}}{\Delta t / 2} \\ &= \frac{E_y|_{I,J+1/2,K+1}^{n+1/2} - E_y|_{I,J+1/2,K}^{n+1/2}}{\Delta z} - \frac{E_z|_{I,J+1,K+1/2}^{n+1} - E_z|_{I,J,K+1/2}^{n+1}}{\Delta y} \end{aligned} \quad (6-2d)$$

$$\begin{aligned} \mu_{I+1/2,J,K+1/2} &= \frac{H_y|_{I+1/2,J,K+1/2}^{n+1} - H_y|_{I+1/2,J,K+1/2}^{n+1/2}}{\Delta t / 2} \\ &= \frac{E_z|_{I+1,J,K+1/2}^{n+1/2} - E_z|_{I,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{E_x|_{I+1/2,J,K+1}^{n+1} - E_x|_{I+1/2,J,K}^{n+1}}{\Delta z} \end{aligned} \quad (6-2e)$$

$$\begin{aligned} \mu_{I+1/2,J+1/2,K} &= \frac{H_z|_{I+1/2,J+1/2,K}^{n+1} - H_z|_{I+1/2,J+1/2,K}^{n+1/2}}{\Delta t / 2} \\ &= \frac{E_x|_{I+1/2,J+1,K}^{n+1/2} - E_x|_{I+1/2,J,K}^{n+1/2}}{\Delta y} - \frac{E_y|_{I+1,J+1/2,K}^{n+1} - E_y|_{I,J+1/2,K}^{n+1}}{\Delta x} \end{aligned} \quad (6-2f)$$

where n is the time index and Δt is the time step. J_{dx1} , J_{dx2} , J_{dy1} , J_{dy2} , J_{dz1} and J_{dz2} are the current density terms and the subscripts I , J , and K are the spatial indexes for Yee cell. One can find that the update equations of the magnetic field components in this method are the same as those of the conventional ADI-FDTD method [10]. The relation between the current density terms and the electric field components can be accordingly derived from (6-5), (6-8), and (6-11) for different applications, which will be discussed below.

For the convenience of further discussion, (6-1)-(6-2) can be written in a matrix form as (6-3). The updating procedures of these schemes are composed of two sub-procedures as in (6-3a) and (6-3b), which are for n to $(n+1/2)$ time step and $(n+1/2)$ to $(n+1)$ time step respectively:

$$\begin{bmatrix} \mathbf{D}_\epsilon & -\frac{\Delta t}{2} \mathbf{C}_1 \\ \frac{\Delta t}{2} \mathbf{C}_1^T & \mathbf{D}_\mu \end{bmatrix} \begin{bmatrix} \vec{E}^{n+1/2} \\ \vec{H}^{n+1/2} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_\epsilon & \frac{\Delta t}{2} \mathbf{C}_2 \\ -\frac{\Delta t}{2} \mathbf{C}_2^T & \mathbf{D}_\mu \end{bmatrix} \begin{bmatrix} \vec{E}^n \\ \vec{H}^n \end{bmatrix} - \frac{\Delta t}{2} \begin{bmatrix} \vec{J}_{d1}^{n+1/2} \\ 0 \end{bmatrix} \quad (6-3a)$$

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$$\begin{bmatrix} \mathbf{D}_\varepsilon & -\frac{\Delta t}{2}\mathbf{C}_2 \\ \frac{\Delta t}{2}\mathbf{C}_2^T & \mathbf{D}_\mu \end{bmatrix} \begin{bmatrix} \vec{E}^{n+1} \\ \vec{H}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_\varepsilon & \frac{\Delta t}{2}\mathbf{C}_1 \\ -\frac{\Delta t}{2}\mathbf{C}_1^T & \mathbf{D}_\mu \end{bmatrix} \begin{bmatrix} \vec{E}^{n+1/2} \\ \vec{H}^{n+1/2} \end{bmatrix} - \frac{\Delta t}{2} \begin{bmatrix} \vec{J}_{d2}^{n+1/2} \\ 0 \end{bmatrix} \quad (6-3b)$$

Here $\vec{J}_{d1}$ and $\vec{J}_{d2}$ are the vectors composed by the current density terms in the two sub-procedures. For the place where there is no lumped element, the current density term will not be present. $\vec{E}$ and $\vec{H}$ are the vectors of electric and magnetic field components. The matrices $\mathbf{D}_\varepsilon$ and $\mathbf{D}_\mu$ consist of the permittivity ε and permeability μ of the media at corresponding positions. They are diagonal matrices with positive real elements. The matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ are derived from the discretization of spatial derivative and T denotes the transpose of matrices. $\vec{E}$, $\vec{H}$, $\mathbf{D}_\varepsilon$, $\mathbf{D}_\mu$, $\mathbf{C}_1$ and $\mathbf{C}_2$, can be related to the notations in (2-5) by

$$\vec{E}^n = \begin{bmatrix} \vec{E}_x^n & \vec{E}_y^n & \vec{E}_z^n \end{bmatrix}^T \quad (6-4a)$$

$$\vec{H}^n = \begin{bmatrix} \vec{H}_x^n & \vec{H}_y^n & \vec{H}_z^n \end{bmatrix}^T \quad (6-4b)$$

$$\mathbf{D}_\varepsilon = \begin{bmatrix} \mathbf{D}_{\varepsilon x} & 0 & 0 \\ 0 & \mathbf{D}_{\varepsilon y} & 0 \\ 0 & 0 & \mathbf{D}_{\varepsilon z} \end{bmatrix} \quad (6-4c)$$

$$\mathbf{D}_\mu = \begin{bmatrix} \mathbf{D}_{\mu x} & 0 & 0 \\ 0 & \mathbf{D}_{\mu y} & 0 \\ 0 & 0 & \mathbf{D}_{\mu z} \end{bmatrix} \quad (6-4d)$$

$$\mathbf{C}_1 = \begin{bmatrix} 0 & 0 & \mathbf{D}_{Ay} \\ \mathbf{D}_{Az} & 0 & 0 \\ 0 & \mathbf{D}_{Ax} & 0 \end{bmatrix} \quad (6-4e)$$

$$\mathbf{C}_2 = \begin{bmatrix} 0 & \mathbf{D}_{Bz} & 0 \\ 0 & 0 & \mathbf{D}_{Bx} \\ \mathbf{D}_{By} & 0 & 0 \end{bmatrix} \quad (6-4f)$$

When $\vec{J}_{d1}$ and $\vec{J}_{d2}$ in (6-3) are expressed as in (6-5), (6-8), and (6-11), the schemes of ADI-FDTD including passive lumped elements can be deduced accordingly for resistors, capacitors and inductors.

6.2.1 Resistors

$$\vec{J}_{d2}^{n+1/2} = \vec{J}_{d1}^{n+1/2} = \mathbf{D}_R \vec{E}^{n+1/2} \quad (6-5)$$

Here

$$\vec{E} = [\vec{E}_x \quad \vec{E}_y \quad \vec{E}_z]^T \quad (6-6a)$$

$$\vec{J}_{d2} = \vec{J}_{d1} = [\vec{J}_{Rx} \quad \vec{J}_{Ry} \quad \vec{J}_{Rz}]^T \quad (6-6b)$$

$$\mathbf{D}_R = \begin{bmatrix} \mathbf{D}_{Rx} & 0 & 0 \\ 0 & \mathbf{D}_{Ry} & 0 \\ 0 & 0 & \mathbf{D}_{Rz} \end{bmatrix} \quad (6-6c)$$

$\vec{E}_x$, $\vec{E}_y$, $\vec{E}_z$ are the vectors of electric field in x, y, z direction, and $\vec{J}_{Rx}$, $\vec{J}_{Ry}$, $\vec{J}_{Rz}$

are the corresponding vectors of current density terms. $\mathbf{D}_{Rx}$, $\mathbf{D}_{Ry}$, $\mathbf{D}_{Rz}$ are diagonal matrices forming the coefficient matrix $\mathbf{D}_R$. The elements of these three matrices can be represented respectively as

$$\mathbf{D}_{Rx}(m_x, m_x) = \frac{\Delta x}{\Delta y \Delta z R_{m_x}} \quad (6-7a)$$

$$\mathbf{D}_{Ry}(m_y, m_y) = \frac{\Delta y}{\Delta x \Delta z R_{m_y}} \quad (6-7b)$$

$$\mathbf{D}_{Rz}(m_z, m_z) = \frac{\Delta z}{\Delta x \Delta y R_{m_z}} \quad (6-7c)$$

where R_m ($m=m_x, m_y, m_z$) is the resistance at corresponding position.

6.2.2 Capacitors

$$\vec{J}_{d1}^{n+1/2} = \mathbf{D}_C (\vec{E}^{n+1/2} - \vec{E}^n) \quad (6-8a)$$

$$\vec{J}_{d2}^{n+1/2} = \mathbf{D}_C (\vec{E}^{n+1} - \vec{E}^{n+1/2}) \quad (6-8b)$$

Here the definition of E can be found in (6-6a) and

$$\vec{J}_{d1} = [\vec{J}_{Cx1} \quad \vec{J}_{Cy1} \quad \vec{J}_{Cz1}]^T \quad (6-9a)$$

$$\vec{J}_{d2} = [\vec{J}_{Cx2} \quad \vec{J}_{Cy2} \quad \vec{J}_{Cz2}]^T \quad (6-9b)$$

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$$\mathbf{D}_C = \begin{bmatrix} \mathbf{D}_{Cx} & 0 & 0 \\ 0 & \mathbf{D}_{Cy} & 0 \\ 0 & 0 & \mathbf{D}_{Cz} \end{bmatrix} \quad (6-9c)$$

$\vec{J}_{Cx1}$, $\vec{J}_{Cy1}$, $\vec{J}_{Cz1}$ are the vectors of current density terms in x , y , z direction, and so

are $\vec{J}_{Cx2}$, $\vec{J}_{Cy2}$, $\vec{J}_{Cz2}$. $\mathbf{D}_{Cx}$, $\mathbf{D}_{Cy}$, $\mathbf{D}_{Cz}$ are diagonal matrices forming the coefficient matrix $\mathbf{D}_C$. The elements of these three matrices can be represented respectively as

$$\mathbf{D}_{Cx}(m_x, m_x) = \frac{2\Delta x C_{m_x}}{\Delta y \Delta z \Delta t} \quad (6-10a)$$

$$\mathbf{D}_{Cy}(m_y, m_y) = \frac{2\Delta y C_{m_y}}{\Delta x \Delta z \Delta t} \quad (6-10b)$$

$$\mathbf{D}_{Cz}(m_z, m_z) = \frac{2\Delta z C_{m_z}}{\Delta x \Delta y \Delta t} \quad (6-10c)$$

where C_m ($m=m_x, m_y, m_z$) is the capacitance at corresponding position.

6.2.3 Inductors

$$\vec{J}_{d1}^{n+1/2} - \vec{J}_{d2}^{n-1/2} = \mathbf{D}_L \left(\vec{E}^{n+1/2} + \vec{E}^{n-1/2} \right) \quad (6-11a)$$

$$\vec{J}_{d2}^{n+1/2} = \vec{J}_{d1}^{n+1/2} \quad (6-11b)$$

Here the definition of E can also be found in (6-6a) and

$$\vec{J}_{d2} = \vec{J}_{d1} = \begin{bmatrix} \vec{J}_{Lx} & \vec{J}_{Ly} & \vec{J}_{Lz} \end{bmatrix}^T \quad (6-12a)$$

$$\mathbf{D}_L = \begin{bmatrix} \mathbf{D}_{Lx} & 0 & 0 \\ 0 & \mathbf{D}_{Ly} & 0 \\ 0 & 0 & \mathbf{D}_{Lz} \end{bmatrix} \quad (6-12b)$$

$\vec{J}_{Lx}$, $\vec{J}_{Ly}$, $\vec{J}_{Lz}$ are the vectors of current density terms in x , y , z direction. $\mathbf{D}_{Lx}$, $\mathbf{D}_{Ly}$,

$\mathbf{D}_{Lz}$ are diagonal matrices forming the coefficient matrix $\mathbf{D}_L$. The elements of these

three matrices can be represented respectively as

$$\mathbf{D}_{Lx}(m_x, m_x) = \frac{\Delta x \Delta t}{2\Delta y \Delta z L_{m_x}} \quad (6-13a)$$

$$\mathbf{D}_{Ly}(m_y, m_y) = \frac{\Delta y \Delta t}{2\Delta x \Delta z L_{m_y}} \quad (6-13b)$$

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$$\mathbf{D}_{Lz}(m_z, m_z) = \frac{\Delta z \Delta t}{2 \Delta x \Delta y L_{m_z}} \quad (6-13c)$$

where L_m ($m=m_x, m_y, m_z$) is the inductance at corresponding position.

According to the proposed explicit FDTD method including lumped elements [35] and the way of the implementation of the current term in ADI-FDTD method [68], one can find that different schemes can be obtained for the incorporation of the same type of elements, e.g. [101], when the current density term is discretized in different ways. Here the main target is to present one group of unconditionally stable schemes, which can be justified to be valid and achieve higher computation efficiency. Other different schemes that may be achieved and the comparison of their stability and numerical distortion characteristics are beyond the present scope.

It can be noticed that the discretized current density term in the ADI schemes is usually related to the electric field components at the half integer time step, which are found to be accurate to only first order in time rather than second order as the electric field components at the integer time step in the conventional ADI scheme [102]. This will affect the accuracy of the ADI scheme including lumped elements. Fortunately, the ADI method is usually applied to some problems with fine scale structures, where the mesh size is much smaller than the wavelength. So the time step can be a few times larger than the Courant limit, which is still small enough to provide good accuracy. This will be demonstrated by the numerical experiments in Section 6.4.

6.3 Stability Analysis for 3-D ADI-FDTD Including Passive Lumped Elements

This section analyzes the stability of the 3-D ADI-FDTD method including lumped elements by the analysis method based on the energy concept, which has already been applied to the stability analysis of the conventional ADI-FDTD method [11] and mentioned in Chapter 2. It is to be proven that certain properly defined numerical energy, which is related to the magnitude of field components, will be bounded with initial bounded field components for each scheme.

6.3.1 Stability Analysis for the Scheme Including Resistors

Combining (6-3) and (6-5), one can get the update equations for the scheme including resistors. The equations can be written in matrix form as

$$\overrightarrow{LHS}_R^{n+1/2} = \mathbf{M}_{R2} \overrightarrow{u}^{n+1/2} = \mathbf{M}_{R1} \overrightarrow{u}^n = \overrightarrow{RHS}_R^n \quad (6-14a)$$

$$\overrightarrow{LHS}_R^{n+1} = \mathbf{M}_{R4} \overrightarrow{u}^{n+1} = \mathbf{M}_{R3} \overrightarrow{u}^{n+1/2} = \overrightarrow{RHS}_R^{n+1/2} \quad (6-14b)$$

where

$$\overrightarrow{u}^n = \begin{bmatrix} \overrightarrow{E}^n & \overrightarrow{H}^n \end{bmatrix}^T \quad (6-15)$$

is the field component vector and

$$\mathbf{M}_{R1} = \begin{bmatrix} \mathbf{D}_\varepsilon^{1/2} & \frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} \mathbf{C}_2 \\ -\frac{\Delta t}{2} \mathbf{D}_\mu^{-1/2} \mathbf{C}_2^T & \mathbf{D}_\mu^{1/2} \end{bmatrix} \quad (6-16a)$$

$$\mathbf{M}_{R2} = \begin{bmatrix} \mathbf{D}_\varepsilon^{1/2} + \frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} \mathbf{D}_R & -\frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} \mathbf{C}_1 \\ \frac{\Delta t}{2} \mathbf{D}_\mu^{-1/2} \mathbf{C}_1^T & \mathbf{D}_\mu^{1/2} \end{bmatrix} \quad (6-16b)$$

$$\mathbf{M}_{R3} = \begin{bmatrix} \mathbf{D}_\varepsilon^{1/2} - \frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} \mathbf{D}_R & \frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} \mathbf{C}_1 \\ -\frac{\Delta t}{2} \mathbf{D}_\mu^{-1/2} \mathbf{C}_1^T & \mathbf{D}_\mu^{1/2} \end{bmatrix} \quad (6-16c)$$

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$$\mathbf{M}_{R4} = \begin{bmatrix} \mathbf{D}_\varepsilon^{1/2} & -\frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} \mathbf{C}_2 \\ \frac{\Delta t}{2} \mathbf{D}_\mu^{-1/2} \mathbf{C}_2^T & \mathbf{D}_\mu^{1/2} \end{bmatrix} \quad (6-16d)$$

It can be found that

$$\left(\overrightarrow{LHS}_R^{n+1/2} \right)^T \overrightarrow{LHS}_R^{n+1/2} = \left(\overrightarrow{RHS}_R^n \right)^T \overrightarrow{RHS}_R^n \quad (6-17a)$$

$$\left(\overrightarrow{LHS}_R^{n+1} \right)^T \overrightarrow{LHS}_R^{n+1} = \left(\overrightarrow{RHS}_R^{n+1/2} \right)^T \overrightarrow{RHS}_R^{n+1/2} \quad (6-17b)$$

Here the numerical energy at n^{th} time step is defined as

$$W_R^n = \left(\vec{E}^n \right)^T \mathbf{w}_\varepsilon \vec{E}^n + \left(\vec{H}^n \right)^T \mathbf{w}_\mu \vec{H}^n \quad (6-18)$$

where

$$\mathbf{w}_\varepsilon = \mathbf{D}_\varepsilon + \frac{\Delta t^2}{4} \mathbf{C}_2 \mathbf{D}_\mu^{-1} \mathbf{C}_2^T \quad (6-19a)$$

$$\mathbf{w}_\mu = \mathbf{D}_\mu + \frac{\Delta t^2}{4} \mathbf{C}_2^T \mathbf{D}_\varepsilon^{-1} \mathbf{C}_2 \quad (6-19b)$$

Then one can obtain

$$\begin{aligned} W_R^{n+1} - W_R^n &= \left(\overrightarrow{LHS}_R^{n+1} \right)^T \overrightarrow{LHS}_R^{n+1} - \left(\overrightarrow{RHS}_R^n \right)^T \overrightarrow{RHS}_R^n \\ &= \left(\overrightarrow{RHS}_R^{n+1/2} \right)^T \overrightarrow{RHS}_R^{n+1/2} - \left(\overrightarrow{LHS}_R^{n+1/2} \right)^T \overrightarrow{LHS}_R^{n+1/2} \\ &= -2\Delta t \left(\vec{E}^{n+1/2} \right)^T \mathbf{D}_R \vec{E}^{n+1/2} \leq 0 \end{aligned} \quad (6-20)$$

One can find that the numerical energy will be bounded with a bounded initial field

vector for any Δt , which indicates that the field components will also be bounded.

Hence it can be concluded that the scheme including resistors is unconditionally stable.

6.3.2 Stability Analysis for the Scheme Including Capacitors

The update equation in matrix form for this scheme can be similarly deduced from

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(6-3) and (6-8):

$$\overrightarrow{LHS}_C^{n+1/2} = \mathbf{M}_{C2} \overrightarrow{u}^{n+1/2} = \mathbf{M}_{C1} \overrightarrow{u}^n = \overrightarrow{RHS}_C^n \quad (6-21a)$$

$$\overrightarrow{LHS}_C^{n+1} = \mathbf{M}_{C4} \overrightarrow{u}^{n+1} = \mathbf{M}_{C3} \overrightarrow{u}^{n+1/2} = \overrightarrow{RHS}_C^{n+1/2} \quad (6-21b)$$

where

$$\mathbf{M}_{C1} = \begin{bmatrix} \mathbf{D}_{\varepsilon C}^{1/2} & \frac{\Delta t}{2} \mathbf{D}_{\varepsilon C}^{-1/2} \mathbf{C}_2 \\ -\frac{\Delta t}{2} \mathbf{D}_{\mu}^{-1/2} \mathbf{C}_2^T & \mathbf{D}_{\mu}^{1/2} \end{bmatrix} \quad (6-22a)$$

$$\mathbf{M}_{C2} = \begin{bmatrix} \mathbf{D}_{\varepsilon C}^{1/2} & -\frac{\Delta t}{2} \mathbf{D}_{\varepsilon C}^{-1/2} \mathbf{C}_1 \\ \frac{\Delta t}{2} \mathbf{D}_{\mu}^{-1/2} \mathbf{C}_1^T & \mathbf{D}_{\mu}^{1/2} \end{bmatrix} \quad (6-22b)$$

$$\mathbf{M}_{C3} = \begin{bmatrix} \mathbf{D}_{\varepsilon C}^{1/2} & \frac{\Delta t}{2} \mathbf{D}_{\varepsilon C}^{-1/2} \mathbf{C}_1 \\ -\frac{\Delta t}{2} \mathbf{D}_{\mu}^{-1/2} \mathbf{C}_1^T & \mathbf{D}_{\mu}^{1/2} \end{bmatrix} \quad (6-22c)$$

$$\mathbf{M}_{C4} = \begin{bmatrix} \mathbf{D}_{\varepsilon C}^{1/2} & -\frac{\Delta t}{2} \mathbf{D}_{\varepsilon C}^{-1/2} \mathbf{C}_2 \\ \frac{\Delta t}{2} \mathbf{D}_{\mu}^{-1/2} \mathbf{C}_2^T & \mathbf{D}_{\mu}^{1/2} \end{bmatrix} \quad (6-22d)$$

$$\mathbf{D}_{\varepsilon C} = \mathbf{D}_{\varepsilon} + \frac{\Delta t}{2} \mathbf{D}_C \quad (6-22e)$$

Here the numerical energy at n^{th} time step is defined as

$$W_C^n = \left(\overrightarrow{E}^n \right)^T \mathbf{w}_{\varepsilon C} \overrightarrow{E}^n + \left(\overrightarrow{H}^n \right)^T \mathbf{w}_{\mu} \overrightarrow{H}^n \quad (6-23)$$

where

$$\mathbf{w}_{\varepsilon C} = \mathbf{D}_{\varepsilon C} + \frac{\Delta t^2}{4} \mathbf{C}_2 \mathbf{D}_{\mu}^{-1} \mathbf{C}_2^T \quad (6-24)$$

Then one can obtain

$$\begin{aligned} & W_C^{n+1} - W_C^n \\ &= \left(\overrightarrow{LHS}_C^{n+1} \right)^T \overrightarrow{LHS}_C^{n+1} - \left(\overrightarrow{RHS}_C^n \right)^T \overrightarrow{RHS}_C^n \\ &= \left(\overrightarrow{RHS}_C^{n+1/2} \right)^T \overrightarrow{RHS}_C^{n+1/2} - \left(\overrightarrow{LHS}_C^{n+1/2} \right)^T \overrightarrow{LHS}_C^{n+1/2} \\ &= 0 \end{aligned} \quad (6-25)$$

One can find that the numerical energy will be bounded with a bounded initial field

vector for any Δt , which indicates that the field components will also be bounded.

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Hence it can be concluded that the scheme including capacitors is unconditionally stable.

6.3.3 Stability Analysis for the Scheme Including Inductors

Based on (6-3) and (6-11), the update equation in matrix form can be achieved:

$$\overrightarrow{LHS}_L^{n+1/2} = \mathbf{M}_{L2} \overrightarrow{u}^{n+1/2} + \mathbf{D}_{L1} \overrightarrow{u}_L^{n+1/2} = \mathbf{M}_{L1} \overrightarrow{u}^n = \overrightarrow{RHS}_L^n \quad (6-26a)$$

$$\overrightarrow{LHS}_L^{n+1} = \mathbf{M}_{L4} \overrightarrow{u}^{n+1} = \mathbf{M}_{L3} \overrightarrow{u}^{n+1/2} - \mathbf{D}_{L1} \overrightarrow{u}_L^{n+1/2} = \overrightarrow{RHS}_L^{n+1/2} \quad (6-26b)$$

where

$$\mathbf{M}_{L1} = \begin{bmatrix} \mathbf{D}_\varepsilon^{1/2} & \frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} \mathbf{C}_2 \\ -\frac{\Delta t}{2} \mathbf{D}_\mu^{-1/2} \mathbf{C}_2^T & \mathbf{D}_\mu^{1/2} \end{bmatrix} \quad (6-27a)$$

$$\mathbf{M}_{L2} = \begin{bmatrix} \mathbf{D}_\varepsilon^{1/2} & -\frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} \mathbf{C}_1 \\ \frac{\Delta t}{2} \mathbf{D}_\mu^{-1/2} \mathbf{C}_1^T & \mathbf{D}_\mu^{1/2} \end{bmatrix} \quad (6-27b)$$

$$\mathbf{M}_{L3} = \begin{bmatrix} \mathbf{D}_\varepsilon^{1/2} & \frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} \mathbf{C}_1 \\ -\frac{\Delta t}{2} \mathbf{D}_\mu^{-1/2} \mathbf{C}_1^T & \mathbf{D}_\mu^{1/2} \end{bmatrix} \quad (6-27c)$$

$$\mathbf{M}_{L4} = \begin{bmatrix} \mathbf{D}_\varepsilon^{1/2} & -\frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} \mathbf{C}_2 \\ \frac{\Delta t}{2} \mathbf{D}_\mu^{-1/2} \mathbf{C}_2^T & \mathbf{D}_\mu^{1/2} \end{bmatrix} \quad (6-27d)$$

$$\mathbf{D}_{L1} = \begin{bmatrix} \frac{\Delta t}{2} \mathbf{D}_\varepsilon^{-1/2} & 0 \\ 0 & 0 \end{bmatrix} \quad (6-27e)$$

$$\overrightarrow{u}_L^{n+1/2} = \begin{bmatrix} \overrightarrow{J}_L^{n+1/2} \\ 0 \end{bmatrix}^T \quad (6-27f)$$

$$\overrightarrow{J}_L^{n+1/2} = \overrightarrow{J}_L^{n-1/2} + \mathbf{D}_L \left(\overrightarrow{E}^{n+1/2} + \overrightarrow{E}^{n-1/2} \right) \quad (6-27g)$$

It can be found that

$$\begin{aligned} & \left(\overrightarrow{LHS}_L^{n+1} \right)^T \overrightarrow{LHS}_L^{n+1} - \left(\overrightarrow{RHS}_L^n \right)^T \overrightarrow{RHS}_L^n \\ &= \left(\overrightarrow{RHS}_L^{n+1/2} \right)^T \overrightarrow{RHS}_L^{n+1/2} - \left(\overrightarrow{LHS}_L^{n+1/2} \right)^T \overrightarrow{LHS}_L^{n+1/2} \\ &= -2\Delta t \left(\overrightarrow{J}_L^{n+1/2} \right)^T \overrightarrow{E}^{n+1/2} \end{aligned} \quad (6-28)$$

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Similarly it is achieved that

$$\left(\overrightarrow{LHS}_L^n\right)^T \overrightarrow{LHS}_L^n - \left(\overrightarrow{RHS}_L^{n-1}\right)^T \overrightarrow{RHS}_L^{n-1} = -2\Delta t \left(\overrightarrow{J}_L^{n-1/2}\right)^T \overrightarrow{E}^{n-1/2} \quad (6-29)$$

From (6-27g), it can be deduced that

$$\begin{aligned} & \left(\overrightarrow{J}_L^{n+1/2}\right)^T \mathbf{D}_L^{-1} \overrightarrow{J}_L^{n+1/2} + \left(\overrightarrow{E}^{n+1/2}\right)^T \mathbf{D}_L \overrightarrow{E}^{n+1/2} - \left(\overrightarrow{J}_L^{n-1/2}\right)^T \mathbf{D}_L^{-1} \overrightarrow{J}_L^{n-1/2} - \left(\overrightarrow{E}^{n-1/2}\right)^T \mathbf{D}_L \overrightarrow{E}^{n-1/2} \\ &= 2 \left(\overrightarrow{J}_L^{n+1/2}\right)^T \overrightarrow{E}^{n+1/2} + 2 \left(\overrightarrow{J}_L^{n-1/2}\right)^T \overrightarrow{E}^{n-1/2} \end{aligned} \quad (6-30)$$

Here the numerical energy at n^{th} time step is defined as

$$\begin{aligned} W_L^n &= \left(\overrightarrow{E}^n\right)^T \mathbf{w}_\epsilon \overrightarrow{E}^n + \left(\overrightarrow{H}^n\right)^T \mathbf{w}_\mu \overrightarrow{H}^n + \left(\overrightarrow{E}^{n-1}\right)^T \mathbf{w}_\epsilon \overrightarrow{E}^{n-1} + \left(\overrightarrow{H}^{n-1}\right)^T \mathbf{w}_\mu \overrightarrow{H}^{n-1} \\ &+ \Delta t \left(\overrightarrow{J}_L^{n-1/2}\right)^T \mathbf{D}_L^{-1} \overrightarrow{J}_L^{n-1/2} + \Delta t \left(\overrightarrow{E}^{n-1/2}\right)^T \mathbf{D}_L \overrightarrow{E}^{n-1/2} \end{aligned} \quad (6-31)$$

Considering (6-28)-(6-30), it can be obtained that

$$\begin{aligned} & W_L^{n+1} - W_L^n \\ &= \left(\overrightarrow{LHS}_L^{n+1}\right)^T \overrightarrow{LHS}_L^{n+1} - \left(\overrightarrow{RHS}_L^n\right)^T \overrightarrow{RHS}_L^n \\ &+ \left(\overrightarrow{LHS}_L^n\right)^T \overrightarrow{LHS}_L^n - \left(\overrightarrow{RHS}_L^{n-1}\right)^T \overrightarrow{RHS}_L^{n-1} \\ &+ \Delta t \left(\overrightarrow{J}_{L2}^{n+1/2}\right)^T \mathbf{D}_L^{-1} \overrightarrow{J}_{L2}^{n+1/2} + \Delta t \left(\overrightarrow{E}^{n+1/2}\right)^T \mathbf{D}_L \overrightarrow{E}^{n+1/2} \\ &- \Delta t \left(\overrightarrow{J}_{L2}^{n-1/2}\right)^T \mathbf{D}_L^{-1} \overrightarrow{J}_{L2}^{n-1/2} - \Delta t \left(\overrightarrow{E}^{n-1/2}\right)^T \mathbf{D}_L \overrightarrow{E}^{n-1/2} \\ &= 0 \end{aligned} \quad (6-32)$$

One can find that the numerical energy will be bounded with a bounded initial field

vector for any Δt , which indicates that the field components will also be bounded.

Hence it can be concluded that the scheme including inductors is unconditionally stable.

6.4 Numerical Experiments

In order to demonstrate the unconditional stability and also validate the schemes proposed here and their improved computation efficiency, a few numerical

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experiments will be performed using both the schemes provided here and the conventional explicit schemes [20]-[21] [35].

To compare with the stability of the preceding explicit schemes [35], let us start with 1-D TEM wave propagation in one area loaded with resistors. Assume that all the resistors, whose values equal to 100Ω , are loaded in y axis direction and wave is propagating in z axis. All the mesh sizes are $\Delta z=0.5\text{mm}$ and the length of the area is 1cm. One side of this area is the Gaussian pulse source and another side is terminated by PEC. The continuous function of the pulse is $g(t)$,

$$g(t) = 100e^{-\left(\frac{t-100\Delta t_C}{20\Delta t_C}\right)^2} \quad (6-33)$$

which is also adopted in the subsequent experiments. Δt_C is the Courant limit time step. Setting the time step $\Delta t=1.1\Delta t_C$, the values of E_y at the observation point located at the middle of this area is recorded for the simulation of both the explicit scheme [35] and the ADI scheme.

Fig. 6-1. Record of the values of electric field E_y for 1800 time steps for the explicit scheme including resistors

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Fig. 6-2. Record of the values of electric field E_y for 1800 time steps for the ADI scheme including resistors

The electric field values from the previous explicit scheme including resistors are shown in Fig. 6-1. One can find that the values explode with such a time step size larger than the Courant limit. However, the ADI scheme still works well with this time step size according to the waveform given by Fig. 6-2. Such stability verification can also be performed for the schemes including other elements, which will not be repeated here.

The stability can also be demonstrated from the pole-zero point of view. For simplicity, let us consider five field components in the above simulation, which comprise the vector $\vec{u}_{R1}^n(K_R)$:

$$\vec{u}_{R1}^n(K_R) = \begin{bmatrix} H_x^n(K_R - 1/2) & E_y^n(K_R) & H_x^n(K_R + 1/2) & E_y^n(K_R + 1) & H_x^n(K_R + 3/2) \end{bmatrix}^T \quad (6-34)$$

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Here K_R is the spatial index. Based on the discussion in [103], the zeros of coefficient matrix of the explicit FDTD scheme including resistors discussed above can be plotted in Fig. 6-3. The scheme will be stable if and only if the determinant of the coefficient matrix does not have zeros outside the unit circle [103], and thus one can find that this scheme is unstable with the time step greater than Courant limit since there is one zero outside the unit circle.

Fig. 6-3. Plot of zeros of the explicit FDTD method

The update equations of the ADI-FDTD scheme including resistors can be represented in matrix form as

$$\vec{u}_{R1}^{n+1/2}(K_R) = \mathbf{A}_{R1} \vec{u}_{R1}^n(K_R) \quad (6-35a)$$

$$\mathbf{A}_{R2} \vec{u}_{R1}^{n+1/2}(K_R) = \vec{u}_{R1}^n(K_R) \quad (6-35b)$$

and it can be obtained that

$$\vec{u}_{R1}^{n+1}(K_R) = \mathbf{A}_{R21} \vec{u}_{R1}^n(K_R) = \mathbf{A}_{R2}^{-1} \mathbf{A}_{R1} \vec{u}_{R1}^n(K_R) \quad (6-36)$$

where

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$$\mathbf{A}_{R1} = \begin{bmatrix} 1 + \frac{\Delta t}{2\epsilon R \Delta z} & 0 & -\frac{\Delta t}{2\epsilon \Delta z} & \frac{\Delta t}{2\epsilon \Delta z} & 0 \\ 0 & 1 + \frac{\Delta t}{2\epsilon R \Delta z} & 0 & -\frac{\Delta t}{2\epsilon \Delta z} & \frac{\Delta t}{2\epsilon \Delta z} \\ \frac{\Delta t}{2\mu \Delta z} & 0 & 1 & 0 & 0 \\ -\frac{\Delta t}{2\mu \Delta z} & \frac{\Delta t}{2\mu \Delta z} & 0 & 1 & 0 \\ 0 & -\frac{\Delta t}{2\mu \Delta z} & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_{R2} = \begin{bmatrix} 1 - \frac{\Delta t}{2\epsilon R \Delta z} & 0 & \frac{\Delta t}{2\epsilon \Delta z} & -\frac{\Delta t}{2\epsilon \Delta z} & 0 \\ 0 & 1 - \frac{\Delta t}{2\epsilon R \Delta z} & 0 & \frac{\Delta t}{2\epsilon \Delta z} & -\frac{\Delta t}{2\epsilon \Delta z} \\ -\frac{\Delta t}{2\mu \Delta z} & 0 & 1 & 0 & 0 \\ \frac{\Delta t}{2\mu \Delta z} & -\frac{\Delta t}{2\mu \Delta z} & 0 & 1 & 0 \\ 0 & \frac{\Delta t}{2\mu \Delta z} & 0 & 0 & 1 \end{bmatrix}$$

Equation (6-36) can be transformed into z domain and after some arrangement we have

$$(z\mathbf{I} - \mathbf{A}_{R21})\vec{u}_{R1}^n(z) = 0 \quad (6-37)$$

The ADI scheme will be stable if and only if the determinant of coefficient matrix in (6-37) does not have zeros outside the unit circle [103]. Fig 6-4 plots the zeros of the coefficient matrix in (6-37) and one can find it is stable since there are no zeros outside the unit circle.

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Fig. 6-4. Plot of zeros of the ADI-FDTD method

To validate the ADI scheme, the recorded electric and magnetic field values at the observation point are used to derive the impedance of the load and compare with the exact value. If the space index of the recorded electric field component is denoted as K_{Ll} , the voltage and current at the n^{th} time step can be calculated as

$$V_{y1}^n = E_y|_{K_{Ll}}^n \Delta y \quad (6-38a)$$

$$I_{y1}^n = (H_x|_{K_{Ll}+1/2}^n - H_x|_{K_{Ll}-1/2}^n) \Delta x \quad (6-38b)$$

Then the total numerical impedance can be obtained as

$$Z_t = \frac{DFT(V_{y1}^n)}{DFT(I_{y1}^n)} \quad (6-39)$$

If one removes the load at the observation point and also records the electric and magnetic field components, the intrinsic impedance associated with the cell can be achieved similarly, which is denoted as Z_C . Thus the final numerical impedance of the load, Z_L , is given by

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$$Z_L = \left(\frac{1}{Z_t} - \frac{1}{Z_C} \right)^{-1} \quad (6-40)$$

Figs. 6-5 (a) and (b) present the magnitude and phase of the numerical impedance obtained by the ADI scheme, compared with the exact value. One can find that the numerical results are consistent with the exact values quite well so that it is difficult to distinguish the curves from the figures.

(a)

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(b)

Fig. 6-5. Comparison between the numerical results and the exact value of the impedance in their (a) magnitudes and (b) phases

In order to quantify the difference between the numerical results and exact value, the error of the numerical results is defined in decibel for each sampling point:

$$ErrdB(f_m) = 20 \log \left(\frac{|F(f_m) - F_0(f_m)|}{|F_0(f_m)|} \right) \quad (6-41)$$

$F(f_m)$ is the numerical result at m-th frequency sampling point f_m while $F_0(f_m)$ is the corresponding exact value.

From the calculation results of Z_L , the $ErrdB$ of the numerical impedance can be obtained and plotted in Fig 6-6.

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Fig. 6-6 The *ErrdB* of the numerical impedance from the ADI-FDTD method

The next experiment simulates a finite-width parallel-plate waveguide loaded with one inductor, which is dedicated for the demonstration of the improved computation efficiency and validation of the scheme including inductors. Here the waveguide structure has a length of 16mm in z axis direction, a width of 8mm in x axis direction, and a height of 3mm in y axis direction. The wave is propagating in z direction. The inductor is located at the middle of the width and the cross section of this structure at the middle of x axis direction is shown in Fig. 6-7. The structure is truncated by Mur's ABC [104] during simulation. Assume that the inductor is 8mm away from both sides of z axis direction and its value is $L=1\text{nH}$. $\Delta x=0.5\text{mm}$, $\Delta y=1\text{mm}$, $\Delta z=0.5\text{mm}$ are the mesh sizes in x , y , z axes direction respectively. Both the explicit [21] and ADI schemes will be run with different time steps to demonstrate the higher efficiency and validate the ADI scheme.

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Fig. 6-7. Configuration for comparison calculations

Here the implementation of Mur's ABC in the ADI scheme will be illustrated by the update equations involving E_y on the boundary $z=0$ and $z=16$. From (6-1), the equation of E_y updating from n^{th} to $(n+1/2)^{\text{th}}$ time step on the plane $k=1$ within the computational domain reads:

$$\begin{aligned}
 & -\frac{\Delta t^2}{4\mu\epsilon\Delta z^2} E_y|_{I,J+1/2,0}^{n+1/2} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta z^2}\right) E_y|_{I,J+1/2,1}^{n+1/2} - \frac{\Delta t^2}{4\mu\epsilon\Delta z^2} E_y|_{I,J+1/2,2}^{n+1/2} \\
 & = E_y|_{I,J+1/2,1}^n + \frac{\Delta t}{2\epsilon} \frac{H_x|_{I+1/2,J,3/2}^n - H_x|_{I+1/2,J,1/2}^n}{\Delta z} - \frac{\Delta t}{2\epsilon} \frac{H_z|_{I+1/2,J+1/2,1}^n - H_z|_{I-1/2,J+1/2,1}^n}{\Delta x} \\
 & \quad - \frac{\Delta t^2}{4\mu\epsilon\Delta z\Delta y} (E_z|_{I,J+1,3/2}^n - E_z|_{I,J,3/2}^n - E_z|_{I,J+1,1/2}^n + E_z|_{I,J,1/2}^n) - \frac{\Delta t}{2\epsilon} J_{dy1}|_{I,J+1/2,1}^{n+1/2} \quad (6-42)
 \end{aligned}$$

Here ϵ and μ are set to be the permittivity and permeability of free space. $E_y|_{I,J+1/2,0}^{n+1/2}$

can be obtained from the Mur's ABC [104]:

$$E_y|_{I,J+1/2,0}^{n+1/2} = E_y|_{I,J+1/2,1}^n + \frac{v\Delta t/2 - \Delta z}{v\Delta t/2 + \Delta z} [E_y|_{I,J+1/2,1}^{n+1/2} - E_y|_{I,J+1/2,0}^n] \quad (6-43)$$

where v is the speed of the wave propagation. Substituting (6-43) into (6-42), one

can obtain

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$$\begin{aligned}
 & \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta z^2} - \frac{\Delta t^2}{4\mu\epsilon\Delta z^2} \frac{v\Delta t/2 - \Delta z}{v\Delta t/2 - \Delta z}\right) E_y \Big|_{I,J+1/2,1}^{n+1/2} - \frac{\Delta t^2}{4\mu\epsilon\Delta z^2} E_y \Big|_{I,J+1/2,2}^{n+1/2} \\
 &= E_y \Big|_{I,J+1/2,1}^n + \frac{\Delta t}{2\epsilon} \frac{H_x \Big|_{I+1/2,J,3/2}^n - H_x \Big|_{I+1/2,J,1/2}^n}{\Delta z} - \frac{\Delta t}{2\epsilon} \frac{H_z \Big|_{I+1/2,J+1/2,1}^n - H_z \Big|_{I-1/2,J+1/2,1}^n}{\Delta x} \\
 &\quad - \frac{\Delta t^2}{4\mu\epsilon\Delta z\Delta y} (E_z \Big|_{I,J+1,3/2}^n - E_z \Big|_{I,J,3/2}^n - E_z \Big|_{I,J+1,1/2}^n + E_z \Big|_{I,J,1/2}^n) - \frac{\Delta t}{2\epsilon} J_{dy1} \Big|_{I,J+1/2,1}^{n+1/2} \\
 &\quad + \frac{\Delta t^2}{4\mu\epsilon\Delta z^2} (E_y \Big|_{I,J+1/2,1}^n - \frac{v\Delta t/2 - \Delta z}{v\Delta t/2 - \Delta z} E_y \Big|_{I,J+1/2,0}^n) \quad (6-44)
 \end{aligned}$$

Also from (6-1), the equation of E_y updating from n^{th} to $(n+1/2)^{\text{th}}$ time step on the plane $k=31$ within the computational domain reads:

$$\begin{aligned}
 & -\frac{\Delta t^2}{4\mu\epsilon\Delta z^2} E_y \Big|_{I,J+1/2,30}^{n+1/2} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta z^2}\right) E_y \Big|_{I,J+1/2,31}^{n+1/2} - \frac{\Delta t^2}{4\mu\epsilon\Delta z^2} E_y \Big|_{I,J+1/2,32}^{n+1/2} \\
 &= E_y \Big|_{I,J+1/2,31}^n + \frac{\Delta t}{2\epsilon} \frac{H_x \Big|_{I+1/2,J,31\frac{1}{2}}^n - H_x \Big|_{I+1/2,J,30\frac{1}{2}}^n}{\Delta z} - \frac{\Delta t}{2\epsilon} \frac{H_z \Big|_{I+1/2,J+1/2,31}^n - H_z \Big|_{I-1/2,J+1/2,31}^n}{\Delta x} \\
 &\quad - \frac{\Delta t^2}{4\mu\epsilon\Delta z\Delta y} (E_z \Big|_{I,J+1,31\frac{1}{2}}^n - E_z \Big|_{I,J,31\frac{1}{2}}^n - E_z \Big|_{I,J+1,30\frac{1}{2}}^n + E_z \Big|_{I,J,30\frac{1}{2}}^n) - \frac{\Delta t}{2\epsilon} J_{dy1} \Big|_{I,J+1/2,31}^{n+1/2} \quad (6-45)
 \end{aligned}$$

$E_y \Big|_{I,J+1/2,32}^{n+1/2}$ can also be obtained from the Mur's ABC:

$$E_y \Big|_{I,J+1/2,32}^{n+1/2} = E_y \Big|_{I,J+1/2,31}^n + \frac{v\Delta t/2 - \Delta z}{v\Delta t/2 - \Delta z} [E_y \Big|_{I,J+1/2,31}^{n+1/2} - E_y \Big|_{I,J+1/2,32}^n] \quad (6-46)$$

Substituting (6-46) into (6-45), one can obtain

$$\begin{aligned}
 & -\frac{\Delta t^2}{4\mu\epsilon\Delta z^2} E_y \Big|_{I,J+1/2,30}^{n+1/2} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta z^2} - \frac{\Delta t^2}{4\mu\epsilon\Delta z^2} \frac{v\Delta t/2 - \Delta z}{v\Delta t/2 - \Delta z}\right) E_y \Big|_{I,J+1/2,31}^{n+1/2} \\
 &= E_y \Big|_{I,J+1/2,31}^n + \frac{\Delta t}{2\epsilon} \frac{H_x \Big|_{I+1/2,J,31\frac{1}{2}}^n - H_x \Big|_{I+1/2,J,30\frac{1}{2}}^n}{\Delta z} - \frac{\Delta t}{2\epsilon} \frac{H_z \Big|_{I+1/2,J+1/2,31}^n - H_z \Big|_{I-1/2,J+1/2,31}^n}{\Delta x} \\
 &\quad - \frac{\Delta t^2}{4\mu\epsilon\Delta z\Delta y} (E_z \Big|_{I,J+1,31\frac{1}{2}}^n - E_z \Big|_{I,J,31\frac{1}{2}}^n - E_z \Big|_{I,J+1,30\frac{1}{2}}^n + E_z \Big|_{I,J,30\frac{1}{2}}^n) - \frac{\Delta t}{2\epsilon} J_{dy1} \Big|_{I,J+1/2,31}^{n+1/2} \\
 &\quad + \frac{\Delta t^2}{4\mu\epsilon\Delta z^2} (E_y \Big|_{I,J+1/2,31}^n - \frac{v\Delta t/2 - \Delta z}{v\Delta t/2 - \Delta z} E_y \Big|_{I,J+1/2,32}^n) \quad (6-47)
 \end{aligned}$$

Combining the update equations (6-44) and (6-47), which involve the field components E_y at the boundary, with those involving only the components inside the boundary, the tri-diagonal coefficient matrix can be constructed for updating E_y

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from n^{th} to $(n+1/2)^{\text{th}}$ time step. Hence E_y at $(n+1/2)^{\text{th}}$ time step can be solved like the conventional ADI-FDTD method introduced in Chapter 2. The derivation for the update equations involving other electric field components at the boundary marching from n^{th} to $(n+1/2)^{\text{th}}$ time step is similar. Subsequently the update equations for the field components marching from $(n+1/2)^{\text{th}}$ to $(n+1)^{\text{th}}$ time step can be achieved in the same way.

The explicit scheme runs with the Courant time limit Δt_C , whereas $6\Delta t_C$ is applied to the ADI one. The numerical admittance derived from these two schemes is compared with the exact value in Fig. 6-8. The procedure to achieve the numerical admittance is similar to the preceding procedure to obtain numerical impedance. Based on the recorded values of the electric and magnetic field components associated with the observation point, whose space index is $(I_{L2}, J_{L2}+1/2, K_{L2})$, the voltage and current derived from the ADI scheme can be written as

$$V_{y2}^n = E_y \Big|_{I_{L2}, J_{L2}+1/2, K_{L2}}^n \Delta y \quad (6-48a)$$

$$I_{y2}^n = (H_x \Big|_{I_{L2}, J_{L2}+1/2, K_{L2}+1/2}^n - H_x \Big|_{I_{L2}, J_{L2}+1/2, K_{L2}-1/2}^n) \Delta x \\ - (H_z \Big|_{I_{L2}+1/2, J_{L2}+1/2, K_{L2}}^n - H_z \Big|_{I_{L2}-1/2, J_{L2}+1/2, K_{L2}}^n) \Delta z \quad (6-48b)$$

Then the total numerical admittance at the loaded position can be obtained by

$$Y_t = \frac{DFT(I_{y2}^n)}{DFT(V_{y2}^n)} \quad (6-49)$$

In order to achieve the final numerical admittance of the load, the intrinsic admittance associated with the cell capacitance need to be deducted from the total one.

For the explicit scheme, since there is a half time step shift between the electric field and magnetic field, there is also the same time step shift between the voltage

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and current, which can be presented as

$$\tilde{V}_{y2}^{n+1} = \tilde{E}_y \Big|_{I_{L2}, J_{L2}+1/2, K_{L2}}^{n+1} \Delta y \quad (6-50a)$$

$$\begin{aligned} \tilde{I}_{y2}^{n+1/2} = & (\tilde{H}_x \Big|_{I_{L2}, J_{L2}+1/2, K_{L2}+1/2}^{n+1/2} - \tilde{H}_x \Big|_{I_{L2}, J_{L2}+1/2, K_{L2}-1/2}^{n+1/2}) \Delta x \\ & - (\tilde{H}_z \Big|_{I_{L2}+1/2, J_{L2}+1/2, K_{L2}}^{n+1/2} - \tilde{H}_z \Big|_{I_{L2}-1/2, J_{L2}+1/2, K_{L2}}^{n+1/2}) \Delta z \end{aligned} \quad (6-50b)$$

Therefore a factor $e^{j\omega\Delta t/2}$ is necessary when deriving the numerical admittance for this scheme:

$$\tilde{Y}_t = \frac{DFT(\tilde{I}_{y2}^{n+1/2})}{DFT(\tilde{V}_{y2}^{n+1})} e^{j\omega\Delta t/2} \quad (6-51)$$

(a)

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(b)

Fig. 6-8. Comparison between the numerical results and the exact value of the admittance in their (a) magnitudes and (b) phases

Fig. 6-9. Comparison between the $ErrdB$ of numerical admittance from the explicit and ADI methods

Figs. 6-8 (a) and (b) compare the magnitude and phase of the numerical admittance

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obtained by these two schemes with the exact value. One can find that they agree well with each other so that it is difficult to distinguish them from the figures. Hence the *ErrdB* of the numerical admittance are plotted in Fig. 6-9 to show the error of numerical results from the explicit and implicit schemes.

To justify the higher computation efficiency achieved by the ADI scheme, the Matlab programs for both schemes are run on a SUN FIRE 15K work station. The error and CPU time for various schemes are compared in Table 6-1. It can be found that the ADI scheme achieves a considerable reduction of simulation time (more than 50%) and the results from the ADI scheme should be accurate enough, although there is some accuracy degeneration compared with the ones from the explicit scheme. Thus the application of the ADI-FDTD method including lumped element is justified to improve the computation efficiency with good accuracy.

Table 6-1. Comparison of CPU time and error for Explicit and ADI schemes including inductors

Method	Time step	Step number	Total time used (s)
Explicit FDTD	Δt_C	900	2.64
ADI-FDTD	$6\Delta t_C$	150	1.18

Finally a structure involving capacitors is simulated for the demonstration of the computation efficiency and validation of the corresponding scheme. The structure studied here is also a finite-width parallel-plate waveguide, whose dimension is the same as the preceding one. A plane loaded with identical capacitors $C=1\text{pF}$ in each cell is located at the $z=8\text{mm}$ plane, whose configuration is shown in Fig. 6-10. Both the explicit and ADI schemes will be used to study the reflection coefficient on this plane. A similar structure has also been discussed in [33]. The interest to such a structure is due to the fact that it allows analytical solution.

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Fig. 6-10. Configuration of the plane loaded with capacitors at $z=8\text{mm}$

When the plane loaded with capacitor is not present at $z=8\text{mm}$, the recorded electric value on this plane is the incident wave E_i . If the plane loaded with capacitor is present, the recorded electric value is the sum of the incident wave and reflected wave, from which the reflected wave E_r can be obtained. Thus the reflection coefficient can be achieved from the ratio of the Fourier transform of E_i and E_r :

$$\Gamma = \frac{DFT(E_r)}{DFT(E_i)} \quad (6-52)$$

From Figs. 6-11 (a) and (b), one can see that the numerical results match the exact value very well. To specify the error better, the $ErrdB$ of the numerical reflection coefficients from explicit and ADI methods are presented in Fig. 6-12.

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(a)

(b)

Fig. 6-11. Comparison between the numerical results and the exact value of the reflection coefficients in their (a) magnitudes and (b) phases

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Fig. 6-12. Comparison between the *ErrdB* of the numerical reflection coefficients from the explicit and ADI methods

The Matlab programs for these two schemes are also run on a SUN FIRE 15K work station. The CPU time and error for these two schemes are listed in Table 6-2 for comparison. Similar improved computation efficiency with good accuracy as that of the preceding scheme is justified. Hence it can be concluded that much time will be saved when these schemes are used for some large scale EM problems requiring much more iterations.

Table 6-2. Comparison of CPU time and Error for Explicit and ADI schemes including capacitors

Method	Time step	Step number	Total time used (s)
Explicit FDTD	Δt_C	2700	9.12
ADI-FDTD	$6\Delta t_C$	450	4.23

6.5 ADI-FDTD Method Including Linear Lumped Network

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For the extension of the preceding works, the ADI-FDTD method including lumped network will be discussed. Based on the discussion in [68], the 3-D ADI-FDTD method including current density term can be presented as

$$(\mathbf{I} - \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{A}) \vec{u}^{n+1/2} = (\mathbf{I} + \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{B}) \vec{u}^n - \frac{\Delta t}{2} \mathbf{D}_d^{-1} \vec{J}^{n+1/2} \quad (6-53a)$$

$$(\mathbf{I} - \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{B}) \vec{u}^{n+1} = (\mathbf{I} + \frac{\Delta t}{2} \mathbf{D}_d^{-1} \mathbf{A}) \vec{u}^{n+1/2} - \frac{\Delta t}{2} \mathbf{D}_d^{-1} \vec{J}^{n+1/2} \quad (6-53b)$$

where $\vec{u}$, $\mathbf{D}_d$, $\mathbf{A}$ and $\mathbf{B}$ conform to the definitions in (2-5). $\vec{J}$ is the vector denoting current density term. Equation (6-53) can be represented as

$$\begin{aligned} \epsilon_{I+1/2,J,K} & \frac{E_x|_{I+1/2,J,K}^{n+1/2} - E_x|_{I+1/2,J,K}^n}{\Delta t / 2} \\ &= \frac{H_z|_{I+1/2,J+1/2,K}^{n+1/2} - H_z|_{I+1/2,J-1/2,K}^{n+1/2}}{\Delta y} - \frac{H_y|_{I+1/2,J,K+1/2}^n - H_y|_{I+1/2,J,K-1/2}^n}{\Delta z} - J_{dx}|_{I+1/2,J,K}^{n+1/2} \end{aligned} \quad (6-54a)$$

$$\begin{aligned} \epsilon_{I,J+1/2,K} & \frac{E_y|_{I,J+1/2,K}^{n+1/2} - E_y|_{I,J+1/2,K}^n}{\Delta t / 2} \\ &= \frac{H_x|_{I,J+1/2,K+1/2}^{n+1/2} - H_x|_{I,J+1/2,K-1/2}^{n+1/2}}{\Delta z} - \frac{H_z|_{I+1/2,J+1/2,K}^n - H_z|_{I-1/2,J+1/2,K}^n}{\Delta x} - J_{dy}|_{I,J+1/2,K}^{n+1/2} \end{aligned} \quad (6-54b)$$

$$\begin{aligned} \epsilon_{I,J,K+1/2} & \frac{E_z|_{I,J,K+1/2}^{n+1/2} - E_z|_{I,J,K+1/2}^n}{\Delta t / 2} \\ &= \frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} - H_y|_{I-1/2,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{H_x|_{I,J+1/2,K+1/2}^n - H_x|_{I,J-1/2,K+1/2}^n}{\Delta y} - J_{dz}|_{I,J,K+1/2}^{n+1/2} \end{aligned} \quad (6-54c)$$

$$\begin{aligned} \mu_{I,J+1/2,K+1/2} & \frac{H_x|_{I,J+1/2,K+1/2}^{n+1/2} - H_x|_{I,J+1/2,K+1/2}^n}{\Delta t / 2} \\ &= \frac{E_y|_{I,J+1/2,K+1}^{n+1/2} - E_y|_{I,J+1/2,K}^{n+1/2}}{\Delta z} - \frac{E_z|_{I,J+1,K+1/2}^n - E_z|_{I,J,K+1/2}^n}{\Delta y} \end{aligned} \quad (6-54d)$$

$$\begin{aligned} \mu_{I+1/2,J,K+1/2} & \frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} - H_y|_{I+1/2,J,K+1/2}^n}{\Delta t / 2} \\ &= \frac{E_z|_{I+1,J,K+1/2}^{n+1/2} - E_z|_{I,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{E_x|_{I+1/2,J,K+1}^n - E_x|_{I+1/2,J,K}^n}{\Delta z} \end{aligned} \quad (6-54e)$$

$$\begin{aligned} \mu_{I+1/2,J+1/2,K} & \frac{H_z|_{I+1/2,J+1/2,K}^{n+1/2} - H_z|_{I+1/2,J+1/2,K}^n}{\Delta t / 2} \\ &= \frac{E_x|_{I+1/2,J+1,K}^{n+1/2} - E_x|_{I+1/2,J,K}^{n+1/2}}{\Delta y} - \frac{E_y|_{I+1,J+1/2,K}^n - E_y|_{I,J+1/2,K}^n}{\Delta x} \end{aligned} \quad (6-54f)$$

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$$\begin{aligned} \epsilon_{I+1/2,J,K} &= \frac{E_x|_{I+1/2,J,K}^{n+1} - E_x|_{I+1/2,J,K}^{n+1/2}}{\Delta t/2} \\ &= \frac{H_z|_{I+1/2,J+1/2,K}^{n+1/2} - H_z|_{I+1/2,J-1/2,K}^{n+1/2}}{\Delta y} - \frac{H_y|_{I+1/2,J,K+1/2}^{n+1} - H_y|_{I+1/2,J,K-1/2}^{n+1}}{\Delta z} - J_{dx}|_{I+1/2,J,K}^{n+1/2} \end{aligned} \quad (6-55a)$$

$$\begin{aligned} \epsilon_{I,J+1/2,K} &= \frac{E_y|_{I,J+1/2,K}^{n+1} - E_y|_{I,J+1/2,K}^{n+1/2}}{\Delta t/2} \\ &= \frac{H_x|_{I,J+1/2,K+1/2}^{n+1/2} - H_x|_{I,J+1/2,K-1/2}^{n+1/2}}{\Delta z} - \frac{H_z|_{I+1/2,J+1/2,K}^{n+1} - H_z|_{I-1/2,J+1/2,K}^{n+1}}{\Delta x} - J_{dy}|_{I,J+1/2,K}^{n+1/2} \end{aligned} \quad (6-55b)$$

$$\begin{aligned} \epsilon_{I,J,K+1/2} &= \frac{E_z|_{I,J,K+1/2}^{n+1} - E_z|_{I,J,K+1/2}^{n+1/2}}{\Delta t/2} \\ &= \frac{H_y|_{I+1/2,J,K+1/2}^{n+1/2} - H_y|_{I-1/2,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{H_x|_{I,J+1/2,K+1/2}^{n+1} - H_x|_{I,J-1/2,K+1/2}^{n+1}}{\Delta y} - J_{dz}|_{I,J,K+1/2}^{n+1/2} \end{aligned} \quad (6-55c)$$

$$\begin{aligned} \mu_{I,J+1/2,K+1/2} &= \frac{H_x|_{I,J+1/2,K+1/2}^{n+1} - H_x|_{I,J+1/2,K+1/2}^{n+1/2}}{\Delta t/2} \\ &= \frac{E_y|_{I,J+1/2,K+1}^{n+1/2} - E_y|_{I,J+1/2,K}^{n+1/2}}{\Delta z} - \frac{E_z|_{I,J+1,K+1/2}^{n+1} - E_z|_{I,J,K+1/2}^{n+1}}{\Delta y} \end{aligned} \quad (6-55d)$$

$$\begin{aligned} \mu_{I+1/2,J,K+1/2} &= \frac{H_y|_{I+1/2,J,K+1/2}^{n+1} - H_y|_{I+1/2,J,K+1/2}^{n+1/2}}{\Delta t/2} \\ &= \frac{E_z|_{I+1,J,K+1/2}^{n+1/2} - E_z|_{I,J,K+1/2}^{n+1/2}}{\Delta x} - \frac{E_x|_{I+1/2,J,K+1}^{n+1} - E_x|_{I+1/2,J,K}^{n+1}}{\Delta z} \end{aligned} \quad (6-55e)$$

$$\begin{aligned} \mu_{I+1/2,J+1/2,K} &= \frac{H_z|_{I+1/2,J+1/2,K}^{n+1} - H_z|_{I+1/2,J+1/2,K}^{n+1/2}}{\Delta t/2} \\ &= \frac{E_x|_{I+1/2,J+1,K}^{n+1/2} - E_x|_{I+1/2,J,K}^{n+1/2}}{\Delta y} - \frac{E_y|_{I+1,J+1/2,K}^{n+1} - E_y|_{I,J+1/2,K}^{n+1}}{\Delta x} \end{aligned} \quad (6-55f)$$

where J_{dx} , J_{dy} , and J_{dz} are the current density terms. The definitions of other symbols conform to (6-1)-(6-2).

Assume that the admittance functions of the lumped networks loaded in the x , y and z axis directions are denoted accordingly as $Y_x(s)$, $Y_y(s)$, and $Y_z(s)$ in s domain. Then the relation between the current density and the electric field in z domain can be derived by bi-linear transform [108], which reads:

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$$\frac{J_{dx}(z)}{E_x(z)} = \frac{\Delta x}{\Delta y \Delta z} Y_x(s) \Big|_{s=\frac{2}{\Delta t} \frac{1-z^{-1}}{1+z^{-1}}} = \frac{\sum_{n_1=0}^{N_{x1}} p_{xn_1} z^{-n_1}}{\sum_{n_2=0}^{N_{x2}} q_{xn_2} z^{-n_2}} \quad (6-56a)$$

$$\frac{J_{dy}(z)}{E_y(z)} = \frac{\Delta y}{\Delta x \Delta z} Y_y(s) \Big|_{s=\frac{2}{\Delta t} \frac{1-z^{-1}}{1+z^{-1}}} = \frac{\sum_{n_1=0}^{N_{y1}} p_{yn_1} z^{-n_1}}{\sum_{n_2=0}^{N_{y2}} q_{yn_2} z^{-n_2}} \quad (6-56b)$$

$$\frac{J_{dz}(z)}{E_z(z)} = \frac{\Delta z}{\Delta x \Delta y} Y_z(s) \Big|_{s=\frac{2}{\Delta t} \frac{1-z^{-1}}{1+z^{-1}}} = \frac{\sum_{n_1=0}^{N_{z1}} p_{zn_1} z^{-n_1}}{\sum_{n_2=0}^{N_{z2}} q_{zn_2} z^{-n_2}} \quad (6-56c)$$

where N_{x1} , N_{x2} , N_{y1} , N_{y2} , N_{z1} and N_{z2} are nonnegative integers related to the order of the loaded lumped networks.

If the lumped network is loaded only in y direction, the current density at the $(n+1/2)^{\text{th}}$ time step, $J_{dy}^{n+1/2}$, can be conveniently calculated by inverse z transform of (6-56b):

$$J_{dy}^{n+1/2} = \frac{1}{q_{y0}} \left(\sum_{n_1=0}^{N_{y1}} p_{yn_1} E_y^{n+1/2-n_1} - \sum_{n_2=1}^{N_{y2}} q_{yn_2} J_{dy}^{n+1/2-n_2} \right) \quad (6-57)$$

Combining (6-54b) and (6-57), the equations for updating E_y from n^{th} to $(n+1/2)^{\text{th}}$ time step can be achieved. In particular, (6-57) needs to be substituted into (6-54b) to eliminate $J_{dy}^{n+1/2}$ so that one can obtain the equation about $E_y^{n+1/2}$, which reads:

$$\begin{aligned} & -\left(\frac{\Delta t^2}{4\mu\epsilon\Delta z^2}\right) E_y \Big|_{I,J+1/2,K-1}^{n+1/2} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta z^2} + \frac{p_{y0}}{q_{y0}}\right) E_y \Big|_{I,J+1/2,K}^{n+1/2} - \left(\frac{\Delta t^2}{4\mu\epsilon\Delta z^2}\right) E_y \Big|_{I,J+1/2,K+1}^{n+1/2} \\ & = E_y \Big|_{I,J+1/2,K}^n + \frac{\Delta t}{2\epsilon\Delta z} (H_x \Big|_{I,J+1/2,K+1/2}^n - H_x \Big|_{I,J+1/2,K-1/2}^n) \\ & \quad - \frac{\Delta t}{2\epsilon\Delta x} (H_z \Big|_{I+1/2,J+1/2,K}^n - H_z \Big|_{I-1/2,J+1/2,K}^n) \\ & \quad - \frac{\Delta t^2}{4\mu\epsilon\Delta z\Delta y} (E_z \Big|_{I,J+1,K+1/2}^n - E_z \Big|_{I,J,K+1/2}^n - E_z \Big|_{I,J+1,K-1/2}^n + E_z \Big|_{I,J,K-1/2}^n) \\ & \quad - \frac{1}{q_{y0}} \left(\sum_{n_1=1}^{N_{y1}} p_{yn_1} E_y^{n+1/2-n_1} - \sum_{n_2=1}^{N_{y2}} q_{yn_2} J_{dy}^{n+1/2-n_2} \right) \end{aligned} \quad (6-58)$$

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Here ε and μ are set to be the permittivity and permeability of free space. It is obvious that (6-58) can be solved implicitly like the conventional ADI-FDTD. After $E_y^{n+1/2}$ is obtained, $J_{dy}^{n+1/2}$ can be deduced from (6-57) and substituted into (6-55b) directly. Subsequently E_y^{n+1} can be solved. If there are loads in other directions, the update equation can be presented and then solved in a similar way. Thus the scheme for the 3-D ADI-FDTD method including lumped networks can be implemented. A numerical experiment will be performed to validate the scheme proposed here and also demonstrate the higher computation efficiency that can be achieved.

The structure simulated here is similar to the one adopted in the preceding section, which is a finite-width parallel-plate waveguide loaded with identical lumped networks in the transverse plane. This structure is selected also due to the fact that it allows an analytical exact solution. This waveguide structure to be simulated has the same dimension as the one used in the previous experiment, which is also truncated by Mur's ABC during simulation. Assume that the wave is propagating in the z axis direction and the plane loaded with identical lumped networks in each cell is located at the middle of the structure, where $z=8\text{mm}$, and transverse to the propagation direction. Here the lumped networks are selected as series RLC networks, where $R=10\Omega$, $L=1\text{nH}$, and $C=1\text{pF}$.

Both the ADI scheme and previous explicit one are applied to study the reflection coefficient on the plane loaded with lumped networks. The time step size adopted by the explicit scheme is the Courant limit time step Δt_C , while that of the ADI scheme is $6\Delta t_C$.

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(a)

(b)

Fig. 6-13. Comparison between the numerical results from explicit and ADI schemes and the exact value of the reflection coefficients in their (a) magnitudes and (b) phases

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Figs. 6-13 (a) and (b) show the magnitudes and phases of the reflection coefficients from these two schemes compared with the exact value. It can be seen that the numerical results from the ADI scheme basically agree well with the exact value. To highlight the difference between the numerical results and exact values, the $ErrdB$ of the numerical reflection coefficients from explicit and ADI methods are compared in Fig. 6-14.

Fig. 6-14. Comparison between the $ErrdB$ of the numerical reflection coefficients from the explicit and ADI methods

Based on the Matlab programs for the explicit and ADI schemes running on a SUN FIRE 15K work station, the CPU time and memory size for these two schemes are listed in Table 6-3 for comparison. It can be found that the ADI scheme achieves a considerable reduction of simulation time (more than 50%). Hence a lot of time will also be saved when this scheme is used for some large scale EM problems requiring more iterations. So the application of the ADI-FDTD method including lumped networks will improve the computation efficiency with acceptable accuracy.

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Table 6-3. Comparison of CPU time and memory size for Explicit and ADI schemes including lumped networks

Method	Time step	Step number	Total time used (s)	Memory size (Kb)
Explicit FDTD	Δt_C	2400	7.78	186.4
ADI-FDTD	$6\Delta t_C$	400	3.45	308.6

6.6 Conclusion

This chapter has presented the 3-D ADI-FDTD method including passive lumped elements, i.e. resistors, capacitors and inductors, whose stability is proven to be neither related to the mesh size, nor related to the values of the elements. This should be a distinguished advantage over the conventional explicit FDTD method including lumped elements. Some experiments are performed for the demonstration of higher computation efficiency and validation of the ADI schemes.

According to preceding discussion, when the current density term is discretized in different ways, there are different schemes for the incorporation of the same type of elements. In this chapter the main target is to present one group of unconditionally stable schemes, which has been justified to be valid and achieves higher computation efficiency. Other different schemes that may be achieved and the comparison of their stability and numerical distortion characteristics are anticipated to be meaningful future works.

Finally, the ADI-FDTD method is extended to incorporate general linear lumped networks, which is also demonstrated to feature higher computation efficiency with acceptable accuracy.

Chapter 7

Conclusion and Recommendations

7.1 Conclusion

This thesis has discussed about the investigation and development of the ADI-FDTD method, which is anticipated to be a promising and useful method for electromagnetic computation thanks to its unconditional stability. The investigation and development works presented here consist of four parts:

1) The stability and dispersion analyses are performed for the ADI-FDTD method in lossy media. The stability analysis theoretically proves the unconditional stability of the ADI-FDTD method in lossy media. The dispersion analysis reveals the numerical loss and dispersion characteristics of this method. This will be meaningful for the evaluation and further development of the ADI-FDTD method in lossy media.

2) A series of 3-D higher order ADI-FDTD method is discussed and their stability and dispersion analyses are presented simultaneously. The higher order scheme is one usual way to achieve improved dispersion performance, and the study on

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stability and dispersion will be useful for the selection and evaluation of various higher order ADI methods. From the stability analysis, this series of higher order ADI-FDTD methods is proven to be unconditionally stable. The dispersion analysis reveals the numerical dispersion characteristics and also demonstrates the improved dispersion performance achieved by the higher order scheme.

3) The (2,4) PO-ADI-FDTD method is developed also to achieve improved dispersion performance and its stability and dispersion analyses are presented as well. From the stability analysis, the (2,4) PO-ADI-FDTD method has been proven to be unconditionally stable for any real parameters. The dispersion investigation verifies the improved dispersion performance achieved by the parameter optimized scheme. By setting different optimization objectives, the method can be applied to satisfy different requirements. The variation of the performance of the (2,4) PO-ADI-FDTD method for different frequencies has also been studied. One can find that the (2,4) PO-ADI-FDTD method can basically preserve the desired performance within a band below the frequency where the optimization of parameters is performed. Comparing with the existing (2,2) PO-ADI-FDTD method, it can be seen that this (2,4) PO-ADI-FDTD method outperforms the preceding method in several aspects.

4) The ADI-FDTD method including passive lumped elements is proposed and the stability analysis is also performed. It has been proven that all the ADI schemes including resistors, capacitors and inductors are unconditionally stable. So the stability of the proposed schemes is neither related to the mesh size, nor related to the values of the elements, which is therefore a distinguished advantage over the

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previous extended FDTD, whose stability is related to either the mesh size, or both the mesh size and the values of the elements. Some numerical experiments have been performed for the demonstration of higher efficiency and validation of these schemes. This thesis also incorporates the general linear lumped network into the ADI-FDTD method to extend the preceding work, and the higher efficiency achieved has been demonstrated by the numerical experiment as well.

7.2 Recommendations for Future Research

Based on the thesis work presented above, there are some recommendations for future research, which include:

1) From the numerical dispersion analysis for the ADI-FDTD method in lossy media, one can find that the numerical loss error is usually much more serious than the numerical dispersion error, especially when the time step or mesh size becomes larger. So the scheme to improve the dispersion performance of the ADI-FDTD method in lossy media is anticipated to be a meaningful future work. The dispersion analysis that has been presented in this thesis can provide a foundation for the development and also be used to evaluate the improvement.

2) Based on the discussion about higher order ADI-FDTD method, it can be seen that to propose an ADI-FDTD method with higher temporal order is also very meaningful.

3) The (2,4) PO-ADI-FDTD method is presented with the assumption of homogenous media. It will be more valuable if this method can be extended to the

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case of inhomogeneous media.

4) From the discussion of the ADI-FDTD method including lumped elements, it can be found that different ADI schemes can be obtained for the incorporation of the same type of elements, when the current density term is discretized in different ways. The study on the potential schemes and the comparison of the stability and numerical distortion of all these schemes will be helpful when they are to be used in various practical applications.

In long term, all the investigation and development results presented in this thesis, as well as the future work recommended above, can be applied to analyze various problems in different areas, such as integrated antennas, Microwave Integrated-Circuits, RF IC, and biological electromagnetics. This will facilitate the research work in different research areas.

Author's Publications

Published or accepted:

- [1] W. Fu, and E. L. Tan, "Stability and dispersion analysis for ADI-FDTD method in lossy media," *IEEE Trans. Antennas and Propagation*, vol. 55, no. 4, pp. 1095-1102, April 2007.
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