Sliding mode observers and unknown input estimations for nonlinear systems

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Sliding Mode Observers and Unknown Input Estimations for Nonlinear Systems

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School of Electrical & Electronic Engineering

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*“jñeyah sa nitya-sanyāsi
yo na dveṣṭi na kāṅkṣati
nirdvandvo hi mahā-bāho
sukhaṁ bandhāt pramucyate ”*

-Bhagavad Gita

Translation

ONE WHO NEITHER HATES NOR DESIRES THE FRUITS OF HIS ACTIVITIES IS KNOWN TO BE ALWAYS RENOUNCED. SUCH A PERSON, FREE FROM ALL DUALITIES, EASILY OVERCOMES MATERIAL BONDAGE AND IS COMPLETELY LIBERATED.

**DEDICATED TO
MY PARENTS AND GRAND PARENTS**

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Summary

This thesis presents a new perspective for the analysis of sliding mode observers (SMO) and unknown input estimations of nonlinear uncertain systems. The main idea lies with exploiting the observability of the unknown inputs from the measured outputs. This allows us to design appropriate robust terms to estimate the unknown inputs from the sliding mode and thereby estimating the states in the presence of unknown inputs/disturbances.

The first part of the thesis addresses the continuous-time problems of linear and nonlinear uncertain systems. A sliding mode observer for linear MIMO systems with unknown inputs is proposed and the complete disturbance rejection properties of the observer are established. By the proposed method, the unknown inputs can be directly estimated from the corresponding sliding surfaces. The observer design of linear MIMO is then extended to a class of Lipschitz nonlinear systems. The stability conditions are established and the observability of unknown inputs from the sliding mode is further analyzed. To handle the state estimation problem for a more general class of SISO nonlinear systems perturbed by an unknown input, SMO is incorporated into the partially linearized system via nonlinear transformation. The measurable output estimation error is the sliding surface. In the sliding mode, the disturbance can be estimated from the equivalent control, and the convergence of the estimation error dynamics is proven. To further extend the development to a generic class of MIMO nonlinear systems perturbed by multiple unknown inputs, we propose a new nonlinear transformation to structurally classify the original nonlinear system into two subsystems so as to facilitate the design of multiple sliding mode observer. By construction, the unknown inputs are structurally reconstructible from the chosen sliding surfaces.

The second part of the thesis focuses on the development of discrete-time observers for the same continuous-time problems discussed earlier. The inherent problem associated with discrete-time sliding mode observers is solved by re-defining the existence of discrete-time sliding mode with a new strategy that completely eliminates chattering. Based on this strategy, the relation between sliding mode gain and boundary layer thickness that guarantees the existence of DSM and the elimination of chattering are explored. The continuous-time systems are discretized via *Taylor* series expansion. With the developed strategy for the existence of DSM, and the observability concepts of unknown inputs, the state and unknown input estimation problems of SISO nonlinear systems are solved. The design technique of discrete-time observer design of SISO nonlinear systems is also extended to the discrete-time multiple SMO of MIMO nonlinear systems.

The nonlinear state estimation methods developed in this thesis are useful tools that allow for efficient monitoring and control of nonlinear systems in the presence of unknown dynamics or modelling errors and/or unknown inputs. The methods have potential applications in fault detection and identification (FDI). Various applications (e.g. robotic arm, bioreactors) have been provided to demonstrate the performance characteristics and the validity of the theoretical results.

Table of Contents

Acknowledgments	iii
Summary	iv
Symbols and Acronyms	xv
1 Introduction	1
1.1 Motivations	1
1.2 Earlier Work	2
1.2.1 Observers for Nonlinear Systems	3
1.2.2 Sliding Mode Observers for Uncertain Systems	5
1.3 Objectives and Contributions	7
1.4 Organization of the Thesis	11
2 Sliding Mode Observers: A Review	13
2.1 Sliding Mode Theory	13
2.2 Sliding Mode Observers	17
2.2.1 Equivalent Control based SMO's	18

TABLE OF CONTENTS

vii

2.2.2	Lyapunov based SMO's	24
2.3	Nonlinear Lipschitz Observers	29
2.4	Uniform Observability and High-Gain Observers	33
2.4.1	Nonlinear State Transformation	34
2.4.2	Uniformly Observable Systems	35
2.4.3	High-Gain Observers	36
2.5	Summary	41
I	Continuous-Time Results	42
3	State and Unknown Inputs Estimations: A New Perspective	43
3.1	Observability of Unknown Inputs	43
3.2	Sliding Mode Observer for Uncertain Linear Systems	45
3.2.1	System	46
3.2.2	Observer Design	47
3.2.3	Existence of Sliding Mode	48
3.2.4	Error Dynamics in the Sliding Mode	49
3.2.5	Stability of the Reduced-order Error System in the Sliding Mode	50
3.2.6	Unknown Input Estimation	54
3.3	Sliding Mode Observer Design for Lipschitz Nonlinear Systems . . .	55
3.3.1	Observer Design	56
3.3.2	Existence of Sliding Mode and Gain Design	57

TABLE OF CONTENTS

viii

3.3.3	Convergence of Error Dynamics in the Sliding Mode	58
3.3.4	A Special Case of SMO Design	60
3.3.5	Design of Gain $\mathbf{L}$	61
3.3.6	Unknown Input Estimation	63
3.4	Application to Flexible Joint Robot Arm	63
3.4.1	Design of Nonlinear SMO	64
3.4.2	Design Method of [1]	66
3.4.3	Design of Linear SMO	67
3.4.4	Simulation Results	68
3.5	Summary	69
4	SMO Design for SISO Nonlinear Systems	72
4.1	Introduction	72
4.2	Robust Nonlinear State Estimator	73
4.2.1	Boundedness of Error Dynamics	75
4.2.2	Sliding Mode Gain Design	77
4.2.3	Convergence of Error Dynamics in the Sliding Mode	79
4.2.4	Observer in the Original Space	81
4.2.5	Unknown Input Estimation from Sliding Mode	81
4.3	Robust Nonlinear Lipschitz Observer	82
4.4	Application to Bioreactor	84
4.4.1	System Model in State-space	85

TABLE OF CONTENTS

ix

4.4.2	Robust Estimator	86
4.4.3	Simulation Results	86
4.5	Summary	88
5	Multiple SMO Design for MIMO Nonlinear Systems	90
5.1	Introduction	90
5.2	Preliminaries	92
5.2.1	Nonlinear Transformation	92
5.2.2	Transformed System	95
5.3	Multiple SMO Design for Class A Subsystems	97
5.3.1	Main Assumptions	98
5.3.2	Observer Design	100
5.3.3	Error Dynamics and Boundedness	100
5.3.4	Convergence of Estimation Error in the Sliding Mode	103
5.4	Sliding Mode Observer Design for Class B Subsystems	110
5.5	Unknown Input Estimation	112
5.6	Observer in the Original Space	113
5.7	Robust NLO with Multiple Sliding Modes	114
5.8	Illustrative Example	116
5.8.1	Design of Proposed Observer	116
5.8.2	Unknown Input Estimation	119
5.8.3	Simulation Results and Discussions	119

TABLE OF CONTENTS

x

5.9	Summary	121
II	Discrete-Time Results	126
6	Discrete-time Sliding Mode	127
6.1	Introduction	127
6.2	Existence of Discrete-time Sliding Mode	128
6.3	Boundary Layer	131
6.4	Chattering-free DSM with Boundary Layer	133
6.5	Summary	134
7	DSM Observer Design for SISO Nonlinear Systems	135
7.1	Introduction	135
7.2	Preliminaries	137
7.2.1	Discrete-time Plant Dynamics	137
7.2.2	Discrete-time Nonlinear State Transformation	138
7.3	DSM Nonlinear State Estimator	140
7.3.1	Boundedness of Error Dynamics	142
7.3.2	Main Result: Existence of DSM	143
7.3.3	Convergence of Error Dynamics in the Sliding Mode	149
7.3.4	Unknown Input Estimation from Sliding Mode	151
7.3.5	Selection of Sampling Period T_s and Boundary Layer Thick- ness ϵ	151
7.3.6	Discrete-time State Estimator in the Original Space	152

TABLE OF CONTENTS

xi

7.3.7	Design of Gain $\mathbf{L}$	153
7.4	Application to Bioreactor	155
7.4.1	Simulation Results and Discussions	156
7.5	Summary	161
8	Multiple DSM Observer for MIMO Nonlinear Systems	162
8.1	Introduction	162
8.2	Preliminaries	164
8.2.1	Transformed System	164
8.3	Multiple DSMO for Class A Subsystems	168
8.3.1	Observability of Unknown Inputs	169
8.3.2	Observer Design	170
8.3.3	Boundedness of Error Dynamics	171
8.3.4	Existence of DSM	172
8.3.5	Unknown Input Estimation from Multiple Sliding Mode	178
8.3.6	Design of Gain $\mathbf{L}_w$	179
8.4	Observer in the Original Space	180
8.5	Illustrative Example	181
8.5.1	DSMO Design	182
8.5.2	Simulation Results	183
8.6	Summary	186
9	Conclusions and Recommendations	188

TABLE OF CONTENTS

xii

9.1	Conclusions	188
9.2	Recommendations for Future Research	191
A	Design Procedures	193
A.1	Nonlinear Transformation	193
A.2	DSMO for Class B subsystems	196
B	Proofs of Relevant Lemmas and Theorems	198
B.1	Proof of Lemma 4.3	198
B.2	Proof of Theorem 4.2	199
B.3	Proof of Lemma 5.4	200
B.4	Proof of Theorem 5.2	201
B.5	Proof of Lemma 7.1	202
B.6	Proof of Theorem 7.1	204
B.7	Proof of Lemma 8.1	205
B.8	Proof of Theorem 8.1	207
B.9	Proof of Lemma 8.2	209
	Author's Publications	212
	Bibliography	215

List of Figures

1.1	Design of a Sliding Mode Observer	8
3.1	The estimation performance of the proposed sliding mode observer .	70
3.2	The estimation performance of the proposed sliding mode observer .	71
3.3	Disturbance estimation of d	71
3.4	Disturbance estimation of d	71
4.1	The estimation performance of the nonlinear estimator	87
4.2	The estimation performance of the robust nonlinear estimator . . .	87
4.3	The actual and estimated unknown input.	87
5.1	The estimation performance of the proposed sliding mode observer with HGO	122
5.2	The estimation of unknown inputs from multiple sliding mode . . .	123
5.3	The estimation performance of the proposed sliding mode observer with NLO	124
5.4	Comparison of estimation performance, HGO vs. NLO	125
6.1	Discrete-time sliding mode phenomenon.	133

LIST OF FIGURES

xiv

7.1	Disturbance for the bioreactor example.	157
7.2	The actual and estimated cell mass/substrate concentration by Discrete-time Nonlinear Estimator.	158
7.3	The actual and estimated cell mass/substrate concentration by DSM Nonlinear Estimator.	159
7.4	Phase trajectory of errors	160
8.1	The estimation performance of the proposed multiple DSM observer	184
8.2	The estimation of unknown inputs from multiple sliding mode . . .	184
8.3	Phase trajectory of errors	185

Symbols and Acronyms

Algebraic Operators

$\mathbf{A}^T$ Transpose of matrix $\mathbf{A}$

$\mathbf{A}^{-1}$ Inverse of matrix $\mathbf{A}$

$\det(\mathbf{A})$ Determinant of matrix $\mathbf{A}$

$\text{rank}(\mathbf{A})$ Rank of matrix $\mathbf{A}$

$\mathbf{A} > 0$ $\mathbf{A}$ is positive definite

$\mathbf{A} < 0$ $\mathbf{A}$ is negative definite

$\|\cdot\|$ Euclidian norm of vector or its induced matrix norm

$\lambda_{\min}(\cdot)$ the minimum eigenvalue of a matrix

$\lambda_{\max}(\cdot)$ the maximum eigenvalue of a matrix

$\sigma(\cdot)$ the condition number of matrix, given by $\sigma(\mathbf{A}) = \frac{\sqrt{\lambda_{\max}(\mathbf{A})}}{\sqrt{\lambda_{\min}(\mathbf{A})}}$

$\sigma_{\min}(\cdot)$ the minimum singular value of a matrix

$\sigma_{\max}(\cdot)$ the maximum singular value of a matrix

$\text{sign}(\cdot)$ signum function

$|\cdot|$ absolute value of a number

$L_{\mathbf{f}}h(\mathbf{x})$ the lie derivative, $\left[\frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} \right] \mathbf{f}(\mathbf{x})$

Sets and Set Operators

$\mathcal{R}(\mathcal{R}_+)$	Set of (non-negative) real numbers
$\mathcal{Z}(\mathcal{Z}_+)$	Set of (non-negative) integers
$\mathcal{N}$	Set of non-negative integers
$\mathcal{R}^n$	Set of real vectors with n elements
$\mathcal{R}^{n \times m}$	Set of real matrices with n rows and m columns
$\mathcal{B}(\mathcal{D})$	the space of bounded functions on $\mathcal{D}$
$\mathcal{C}^n(\mathcal{D})$	the space of $n \in \mathcal{Z}_+$ times continuously differentiable functions on $\mathcal{D}$
$\mathcal{M}$	$\mathcal{M} \subset \mathcal{R}^n$, a $\mathcal{C}^\infty$ connected manifold of dimension n

Others

$\mathbf{I}$	Identity matrix
$\mathbf{1}$	Vector of ones, $\mathbf{1} = [1 \ 1 \ \dots \ 1]^T$
$\mathbf{0}$	Vector of zeros, $\mathbf{0} = [0 \ 0 \ \dots \ 0]^T$

Acronyms

BIBS	Bounded Input Bounded States
DSM	Discrete-time Sliding Mode
DSMO	Discrete-time Sliding Mode Observer
HGO	High-Gain Observer
MIMO	Multi-Input Multi-Output
NLO	Nonlinear Lipschitz Observer
NUIO	Nonlinear Unknown Input Observer
SISO	Single Input Single Output
SPD	Symmetric Positive Definite
SMO	Sliding Mode Observer
PD	Positive Definite
UIO	Unknown Input Observer
USPD	Unique Symmetric Positive Definite

Chapter 1

Introduction

1.1 Motivations

State variables of a system are the variables that specify uniquely the state of the process or the system at any time, and thus effective control and monitoring of a process requires reliable real-time and accurate information on the state variables of the process. In many practical situations, only partial state information is available through measurements of the system/process. Thus, reliable on-line information of the unmeasurable state variables must be obtained through state estimator or observer.

In general, most physical systems are nonlinear in nature and exact modeling of the system behaviours is difficult. Thus, we need to include uncertainty terms in all the mathematical models of the physical systems. Further, the process's operating environment may also introduce uncertainties/disturbances into the system. The actual states of the system and the system dynamics that govern the state dynamics are not exactly known for most physical systems and have to be reconstructed based on the output measurements. The modeling of these state dynamics relies on several approximations of the actual system dynamics and are most often prone to

modeling errors or unmodeled dynamics. Thus, the robustness of state estimation in the presence of modeling uncertainties is a very important issue in the observer design.

Unknown inputs¹ are often present in the systems due to unmodeled dynamics and/or component failures. In many practical applications (e.g. avionics, robotics, biochemical processes), the unknown dynamics/uncertainties of the system may even dominate the known dynamics of the nominal model. Therefore, the knowledge of the unknown inputs/disturbances² is important for the diagnosis of the health of the system. Estimation of unknown inputs could provide valuable information about possible catastrophic failures and faults arising in the system.

The sliding mode observer, through sliding surface design and equivalent control concept, has proven to be an effective approach for handling uncertain systems with disturbances and/or modelling uncertainties.

In this context, the motivation for this research arises from the desire to design robust state and disturbance estimation methods that are capable of handling uncertainties and disturbances while providing accurate and reliable estimates of unmeasured states as well as unknown inputs of the uncertain nonlinear system. We shall consider a general class of nonlinear systems involving unknown inputs, and both the continuous-time and discrete-time observers will be examined.

1.2 Earlier Work

An observer is a dynamical system whose purpose is to estimate the states based on available measurements (or outputs) of a dynamical system/process. Apparently,

¹The term unknown input in general refers to all types of uncertainties/disturbances/unmodeled dynamics present in the system.

²Through out the thesis, the terms ‘unknown inputs’ or ‘disturbances’ are used interchangeably.

the term observer was introduced by Luenberger in [2].

The early foundations to statistical and deterministic estimation methods for linear systems were laid by Wiener [3], Kalman [4], Kalman and Bucy [5] and Luenberger [2, 6, 7]. By the development of Kalman filter [5] and its deterministic version known as Luenberger observer [2], the linear estimation problem in the presence of white noise is almost solved. The Kalman filter [5] for linear systems yields an optimal estimate with minimum variance for zero mean Gaussian noise/disturbance with known covariances. The theoretical properties of Kalman filter and Luenberger observer are well understood, and these have led to the development of various extensions for applications to nonlinear processes [8–12]. However, there has been several reports about the inadequacy of linear estimators for nonlinear processes [13], [14], which was proven analytically in [13]. While extended Kalman filter (EKF) and extended Luenberger observer have been successful in several implementations of nonlinear processes [15–18], however, when severe nonlinearities are present, observers designed based on linear approximation were inadequate [13]. In these cases, the successful usage of a direct nonlinear process model was reported in [19].

1.2.1 Observers for Nonlinear Systems

The first attempt to obtain linearizable error dynamics by nonlinear co-ordinate transformation was made in [20–22]. Later on, there have been several works on nonlinear observers based on linearization approach and geometric methods by nonlinear transformations [22–25]. The results in [26, 27] provide a good introduction to the geometric methods in the continuous-time domain by exact feedback linearization of nonlinear systems, which involved a state transformation through Lie derivatives. For the linearized system in the transformed domain, a linear

observer is then designed, which guarantees a globally asymptotically stable error dynamics for the nonlinear process. Methods based on bilinearization up to output injection for a class of nonlinear systems were provided in [28, 29]. Existing results for exact linearization by input-output injection were unified in [30]. All the designs of the observers discussed in this class rely on the existence of the nonlinear transformation that is determined from the invertibility of the Jacobian of the observability mapping which is obtained from higher order Lie derivatives of output measurements [22, 23, 31–35].

High gain Luenberger-type observers [36–40] were also developed for a class of nonlinear Lipschitz systems. In particular, uniform observability and global convergence conditions were discussed for Luenberger-type nonlinear observers in [37, 39]. These class of systems can achieve exponential convergence inspite of large Lipschitz constant. The development of these observers is further analyzed in Chapter 2. Recently, for systems in triangular forms obtained by injective mapping, the observers were designed in [40], [41]. Systematic treatment of output feedback and observer design for nonlinear systems, together with a set of tools customized for specific problems, were developed in [42–44].

The Lyapunov-based design for Lipschitz nonlinear observers was first initiated by Thau [45] and sufficient conditions for convergence, and existence were proved. Later, extensions to Thau’s work for special classes of nonlinear systems were also developed [1, 46–54]. The works in [52], [53] in particular detailed the Thau’s observer [45] by considering the distance to unobservability and the condition number that guarantees the stability of the error system. Sufficient conditions for gain design were also derived in [52], [53]. The work in [1] further extended the study of [52] and generalized the conditions for gain design. All the Lyapunov-based designs mainly rely strictly on Lipschitz or linear growth constraint assumptions, and convergence in most cases is only local and highly depends on nominal nonlinear

dynamics. Uncertainties in modeling can result in poor performance characteristics [25]. For the Lyapunov based design techniques, the gain design is not straight forward and only sufficient conditions were available. The detailed review of this class of observers is given in Chapter 2.

1.2.2 Sliding Mode Observers for Uncertain Systems

With the survey paper [55], much interest has been laid on the development of sliding mode controllers for uncertain systems [55–59]. Since then, sliding mode control has been an established method for handling disturbances and modeling uncertainties through the concepts of sliding surface design and equivalent control. Early extensions to SMO for linear systems were done in [60–62]. Later on, there has been several works on sliding mode observers in the literature. These can be classified into two forms based on the design methodology adopted. The designs are either by the Lyapunov method or the equivalent control concept. In comparison with linear observers, sliding mode observers can solve a wide variety of parameter estimation problems. Based on the Lyapunov method, we can design the SMO specifically for parameter estimation or simultaneous state and parameter estimation of linear systems [70]. Also, sliding mode observers can guarantee finite-time convergence of the estimation error. In many cases, the concepts of nonlinear observers were extended to sliding mode observers in order to incorporate robustness into the observer design.

State estimation for dynamic systems with bounded uncertainties/disturbances was dealt with by Walcott and Zak [23, 63, 64]. Extensions to a more general class of nonlinear systems were done in [25, 65, 66]. The design is based on Lyapunov min-max methods and also required matching conditions. The work in [67] designed an observer for nonlinear deterministic systems based on the gain design given

by [23]. The work in [68] developed a robust discontinuous observer based on Utkin observer [56]. In [65] and [69] the sliding mode observer were designed by appending switching functions to Luenberger-type observers. The work in [1] extended the SMO design to Lipschitz nonlinear systems based on matching condition [56] and the approach is similar to the development in [23, 63]. All the works to date relied on either structural conditions or matching assumptions such that the reduced-order system in the sliding mode is free from unknown inputs.

Sliding mode observers for linear systems based on the equivalent control method was given by [56] and extensions to SISO nonlinear systems were done in [70] and [71]. Higher order sliding modes which ensure better accuracy and finite time convergence were also developed [72], [73].

The idea of sliding mode observer design for nonlinear systems having triangular input form was given by [74]. Based on the works of [70, 71, 74], a SMO for a general class of MIMO nonlinear systems with unknown inputs, also termed as nonlinear unknown input observer (NUIO), was designed in [75]. This observer design relied on nonlinear transformation [34, 76] and Frobenius theorem [34, pg. 22] to decouple the unknown inputs in the transformed domain. Recent works [77, 78] for fault detection and isolation (FDI) also relied on decoupling the unknown inputs for the design of observers. All the above listed methods based on equivalent control relied on the availability of output estimation error and its higher-order derivatives. The methods available for sliding mode observers design either were only applicable to a limited class of nonlinear systems or the gain designs were not straightforward and were conservative. Extensive review of sliding mode observers is dealt with in Chapter 2.

In the recent years, the problem of unknown input observers³(UIO) [79–81] has

³In the literature unknown input observers refer to the state estimation in presence of unknown inputs

attracted significant attention in the area of fault detection and isolation (FDI). Sliding mode observers have been applied in fault diagnosis [82–84] for reconstructing the fault signal rather than detecting the fault through a residual signal. Several works [85], [75], [78] focused on the decoupling of the unknown inputs by nonlinear transformation, and by designing the observer for the states unaffected by unknown inputs in the transformed domain. Although there exist results of unknown input observers (UIO) for linear systems [86], [87], the unknown input estimation problem for nonlinear systems remained unsolved.

While there were significant developments of the continuous-time sliding mode observers, the discrete-time implementation of the sliding mode observer design has received limited attention. Discrete-time observer sliding mode designs for linear systems were given in [88], [89]. Note however that conditions for the existence of discrete-time sliding mode controllers were discussed in many works [90–97]. Unfortunately, extensions of these concepts to observer designs were not straightforward due to the inherent presence of chattering, and observers for nonlinear discrete-time systems have remained unsolved. In view of the many potential applications of nonlinear discrete-time observers in the fault detection and isolation (FDI) area, we attempt to address some of these issues in this thesis.

1.3 Objectives and Contributions

The focus of this thesis is on the development and stability analysis of sliding mode observers for robust state and unknown input estimations of uncertain nonlinear systems. Both the continuous-time and discrete-time domains observers will be examined. The systems considered in general represent the majority of problems considered in the literature. Throughout the thesis, much emphasis is on the problems of state estimation in the presence of unknown inputs and simultaneous

Figure 1.1: Design of a Sliding Mode Observer

unknown inputs estimations from the sliding modes.

We will present a new perspective to the design and analysis of sliding mode observers for state and unknown input estimations for nonlinear systems. The whole development relies on the idea that *“an unknown input/disturbance can be estimated when it is uniquely identifiable through the output measurements”*.

The approach to sliding mode observer design in the thesis can be as summarized in Fig. 1.1. Under some structural conditions for the uncertain part of the nonlinear systems, the sliding mode observer is designed to estimate the unknown inputs from the sliding modes. The observer design for the nominal part is performed with existing methods. The observer for the uncertain system is obtained by appending the sliding mode(s) observer to the existing observer for the nominal

system.

A general assumption about the considered systems is the Bounded-Input Bounded State (BIBS) property. Several other assumptions will be stated and discussed in the appropriate chapters of the thesis and these are mostly related to the bounds of the various functions and norms. Also, all the analyses of the nonlinear systems rely on Lipschitz assumptions of the nonlinear functions. For the analysis of discrete-time systems, discretization is obtained via *Taylor's* series expansion.

The major contributions of the thesis are as follows:

- (i) The design of a sliding mode observer that only requires the output information is proposed for an uncertain linear system. It is proved that the existence of a sliding mode observer is completely reliant on the observability of unknown inputs w.r.t. output measurements. Further, the robust terms or the switching terms are designed such that the unknown inputs are tracked by their robust terms and so can be reconstructed from the sliding mode. The minimum number of robust terms required is proven to be equal to the number of independent unknown inputs. The stability condition for the reduced order system is analyzed and the feedback gain is designed such that the reduced order system is stable.
- (ii) The concepts of unknown input observations of linear systems are extended to a class of Lipschitz nonlinear systems. Under some structural conditions for the unknown input distribution matrix, the unknown inputs are observable and accurate state estimation is achieved in the presence of unknown inputs.
- (iii) We proposed a continuous-time sliding mode observer for state and unknown input estimation for a class of SISO nonlinear uncertain systems. By linking the observability of the unknown input w.r.t. measured output, the so-called

matching condition is avoided. Sufficient conditions for the convergence of estimation error in the presence of unknown inputs are derived.

- (iv) A new nonlinear transformation is proposed for MIMO nonlinear uncertain systems, and multiple sliding mode observers are designed in the transformed space to tackle the unknown inputs. The conditions for complete observability or detectability of unknown inputs are analyzed and the unknown inputs are estimated from their corresponding sliding modes. Under the structural assumptions in the transformed space, the analysis is carried out and sufficient conditions for convergence are derived.
- (v) A new strategy is developed for the existence of discrete-time sliding mode for nonlinear systems. The relation between boundary layer thickness and the sliding mode gain is explored and chattering is completely eliminated.
- (vi) Discrete-time sliding mode observer for SISO nonlinear uncertain systems is developed for a transformed system obtained via nonlinear transformation and Taylor's series expansion. The estimation of unknown input is obtained through an approximation of the switching term. The existence conditions for the discrete-time sliding mode gain are derived based on the proposed strategy. Conditions for the convergence of estimation error and conditions for complete elimination of chattering are derived.
- (vii) Based on the proposed nonlinear transformation, a multiple sliding mode observer is designed for the discretized MIMO nonlinear system via Taylor's series expansion in the transformed space. The existence conditions for the discrete-time sliding mode are derived based on the new strategy developed. State and unknown input estimation problems are therefore solved concurrently.

1.4 Organization of the Thesis

The rest of the thesis is organized as follows:

Chapter 2 presents the concepts of sliding modes plus a general overview of the existing literature on sliding mode observers. The existing conditions that guarantee asymptotical convergence of the estimation error of Lipschitz nonlinear systems are reviewed. The uniform observability concepts of nonlinear systems and the design of high gain observers that guarantee exponential convergence of error are discussed.

Chapter 3 considers the design of continuous-time sliding mode observers for MIMO uncertain linear systems. The analysis of sliding mode observers for guaranteed complete observability of unknown inputs are discussed. The estimation of unknown inputs from the sliding mode and its effect on the stability of reduced-order error dynamics are analyzed. The design is then extended to a class of Lipschitz nonlinear systems. Application to a robotic-arm problem is provided to compare the proposed method with the existing design techniques.

Chapter 4 presents the design of continuous-time sliding mode observers for SISO nonlinear uncertain systems via nonlinear transformation. The disturbances are characterized by an unknown input distributed by a known state dependent vector, and the unknown input is re-constructed from the sliding mode. Sufficient conditions for convergence of the estimation error are analyzed. Application to a bioreactor is discussed and simulation results are presented.

Chapter 5 extends the study of chapter 4 to multiple sliding mode observers for continuous-time MIMO nonlinear uncertain systems with multiple unknown inputs. A new nonlinear transformation is formulated to partition the system

into two subsystems. Multiple sliding modes are introduced to deal with multiple unknown inputs. It is shown that, in the sliding mode, the unknown inputs can be reconstructed and an accurate state estimation is achievable. Sufficient conditions for the convergence of estimation error are analyzed in the sliding mode. Numerical studies support the theoretical development.

Chapter 6 examines the existing conditions for the discrete-time sliding mode (DSM) and proposes a new strategy to design the boundary layer that will guarantee the existence of DSM.

Chapter 7 extends the design of SMO design for SISO nonlinear uncertain systems in Chapter 4 to the discrete-time case. The discretization of the system is achieved through nonlinear transformation and *Taylor's* series expansion. Existence of DSM is achieved based on the new strategy developed for DSM in Chapter 6. State and unknown input estimations are discussed. Application to a bioreactor is given, and accurate estimation of the states and unknown inputs are obtained.

Chapter 8 proposes a discrete-time multiple sliding mode observer for MIMO nonlinear uncertain systems based on the development in Chapter 5 and Chapter 7. The analysis in presence of unknown inputs is carried out and the boundedness of error dynamics inside the boundary layer is proved. An application to DC motor for state and unknown input estimations is provided.

Chapter 9 provides the conclusions of the thesis and recommends several problems as future research directions.

Chapter 2

Sliding Mode Observers: A Review

The purpose of this chapter is to provide a comprehensive review and background knowledge on sliding mode observers for uncertain systems. Section 2.1 provides an introduction to the variable structure theory, followed by the conditions for the existence of sliding mode. This is followed in Section 2.2 by a review of approaches to sliding mode observers for continuous-time systems, with emphases on robustness properties to deal with unknown inputs or disturbances. Section 2.3 reviews the existing gain design methods for Lipschitz nonlinear systems. In Section 2.4, uniform observability concepts for nonlinear systems and the design of high-gain observers for uniform observable systems are discussed.

2.1 Sliding Mode Theory

A dynamical system in *sliding mode* is one in which the states of the system are restricted to lie on some surface or manifold, usually a subspace in the state space. The behavior of the dynamic system when confined to a specific surface can be

described as *sliding motion*. A sliding mode can appear in systems with variable structures or discontinuous structures.

Variable structure systems are characterized by systems with differential equations in which the forcing term is discontinuous across a submanifold of the state space. A variable structure system is said to be in *sliding mode* when, during a finite time interval, the system state encounters the surface of discontinuity infinitely often. Starting from an arbitrary initial state off the surface, the state reaches the surface in finite time and thereafter is constrained to lie on or slide along the surface. This sliding action imposes certain constraints on the dynamics of the system and is dependent on the defined sliding surface. Filippov's [98] solutions for differential equations of variable structure provided the basic understanding of various properties (existence, continuity, uniqueness, etc.). Filippov's analysis together with the equivalent control concepts of Utkin's [55, 56, 99] provided the tools for understanding the dynamics of systems in the sliding mode. The sliding dynamics are theoretically obtained in the limit as the switching frequency about the sliding surface or discontinuity surface becomes infinite.

In variable structure control systems, the control process varies, or switches, from one structure to another. A sliding mode arises in such a system if the switching process dictates that the system trajectory, after some finite time duration, is restricted to lie on a surface, or sub-manifold, of state space. The trajectory of such a system generally consist of two parts, representing the two modes of the system: the *reaching mode* in which the controlled trajectory lies off the sliding surface, and the *sliding mode* during which the trajectory is constrained to lie on the sliding surface. The design of sliding mode control systems is naturally carried out in two stages: the sliding surface is selected to achieve the desired sliding dynamics, and the switching control law is selected to make the trajectory reach the sliding surface.

The earlier works on sliding mode control [98, 100–105] consider the control problem for the following nominal part of a continuous-time SISO nonlinear system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u(\mathbf{x}) \quad (2.1)$$

where $\mathbf{x} \in \mathcal{M} \subseteq \mathcal{R}^n$, $\mathbf{f}, \mathbf{b} : \mathcal{R}^n \rightarrow \mathcal{R}^n$ are, for convenience, $\mathcal{C}^\infty$ vector-valued functions, and the scalar feedback control $u : \mathcal{R}^n \rightarrow \mathcal{R}$.

Let $s : \mathcal{R}^n \rightarrow \mathcal{R}$ denote a smooth function with nonzero gradient on $\mathcal{M}$. Define

$$\mathbf{S}_+ = \{\mathbf{x} : s(\mathbf{x}) > 0\}, \quad \mathbf{S} = \{\mathbf{x} : s(\mathbf{x}) = 0\}, \quad \mathbf{S}_- = \{\mathbf{x} : s(\mathbf{x}) < 0\} \quad (2.2)$$

Here, $\mathbf{S}$ is an $(n - 1)$ -dimensional submanifold of $\mathcal{M}$ and is oriented such that for any $\mathbf{x}^* \in \mathbf{S}$, the gradient $\frac{\partial s}{\partial \mathbf{x}}$ evaluated at $\mathbf{x}^*$ is directed from $\mathbf{x}$ into $\mathbf{S}_+$.

The sliding mode control law for $u(\mathbf{x})$ is evaluated based on the feedback values of $\text{sign}(s(\mathbf{x}))$. In short, it will be of the form

$$u(\mathbf{x}) = \begin{cases} u_+(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbf{S}_+ \\ u_-(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbf{S}_- \end{cases} \quad (2.3)$$

where $u_+ \neq u_-$ and $\mathbf{S}$ is called the *sliding manifold*. In order to establish a *sliding regime* on $\mathbf{S}$, the sliding mode control law (2.3) must be chosen such that the trajectories are directed towards the sliding surface [55], [33].

In general, the *sliding condition* [100], [33] can be represented as

$$\frac{1}{2} \frac{d}{dt} s^2(\mathbf{x}) \leq -\rho |s(\mathbf{x})| \quad \text{for } \mathbf{x} \in \mathcal{R}^n, \rho > 0 \quad (2.4)$$

Alternatively, one may also have the following general forms:

$$s(\mathbf{x})\dot{s}(\mathbf{x}) \leq 0 \quad (2.5)$$

$$\dot{s}(\mathbf{x}) \leq -\rho \text{sign}(s(\mathbf{x})) \quad (2.6)$$

where $\text{sign}(\cdot)$ is the signum function defined by

$$\text{sign}(s) = \begin{cases} +1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases} \quad (2.7)$$

The above conditions of (2.5)-(2.6) are the general conditions that guarantee the existence of continuous-time sliding mode. The conditions for the existence of the discrete-time case are considered in the second part of the thesis, in Chapter 6.

Over the last two decades, significant progress has been made in the development of sliding mode controllers [56–59, 88, 106–108] and more recently sliding mode observers [88], [97], [71] for handling disturbances and modeling uncertainties through the concepts of sliding surface design and equivalent control. The major advantage of employing sliding modes in the control and observation of systems is their inherent robustness to unmodeled dynamics or disturbances [33], [109].

Practically, ideal sliding cannot be achieved due to finite bandwidth and saturation limits of the switching mechanism. Due to these practical limitations, the control law (2.3) results in chattering which involves high control activity and may even excite high frequency unmodeled dynamics in the system. Several approximations have been proposed to overcome chattering through the usage of saturation functions also called the piecewise linear or smooth approximation of the switching element in a boundary layer [97, 99, 100, 110–112]. The implementation of boundary layer is generally through the following saturation function:

$$\text{sat}(\cdot, \epsilon) \triangleq \begin{cases} \cdot/\epsilon & \text{if } |\cdot| \leq \epsilon \\ \text{sign}(\cdot) & \text{if } |\cdot| > \epsilon \end{cases} \quad (2.8)$$

Inside the boundary layer, the switching function is approximated by a linear feedback gain. However, implementation of the boundary layer results in a trade-off between chattering and robustness of the controller. With the inclusion of a

boundary layer, the ideal sliding mode is lost, and the feedback system reduces to a system with no sliding mode [97]. In order for the system behavior to be close to that of ideal sliding mode in the presence of unknown disturbance, sufficient high gain is to be employed. In the absence of disturbance, a larger boundary layer would completely eliminate the chattering (or oscillatory behavior) around the sliding manifold.

With regards to observer design and implementation, however, the chattering issues are less severe as they relate only to numerical precision and the update rates of the digital implementation. The observer is not directly responsible for delivering the control signal and in general is not restrained by the physical properties of the plant.

The main problem in the observer design, as opposed to the control problem, is that not all states are available for measurements and hence a sliding surface that depends on all the states cannot be employed.

2.2 Sliding Mode Observers

This section reviews existing design methodologies that are important for the development in the later parts of the thesis. In the last two decades, there has been considerable research on the development of sliding mode observers [1, 23, 25, 51, 54, 60, 63, 65–75, 78, 88, 113–115] based on variable structure theory. Majority of these works can be classified into two categories:

- i) Equivalent control based SMO's [70, 71, 73–75, 78, 115]
- ii) Lyapunov based SMO's [1, 23, 25, 51, 54, 63, 65–68, 73, 113, 114]

In order to put into context the review of the design methodologies, we consider the following uncertain system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \Phi(\mathbf{x}, \mathbf{u}, t) + \mathbf{P}\mathbf{d}(\mathbf{x}, \mathbf{u}, t) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \quad (2.9)$$

where $\mathbf{x} \in \mathcal{R}^n$ is the state vector, $\mathbf{u} \in \mathcal{R}^q$ is the control input, $\mathbf{y} \in \mathcal{R}^p$ is the system output, and $\mathbf{d}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} d_1(\mathbf{x}, \mathbf{u}, t) & d_2(\mathbf{x}, \mathbf{u}, t) & \cdots & d_m(\mathbf{x}, \mathbf{u}, t) \end{bmatrix}^T$ is the unknown input vector and is assumed to be bounded as $\|\mathbf{d}(\mathbf{x}, \mathbf{u}, t)\| \leq \bar{d}$, $\forall \mathbf{x}, u$ and $t > 0$. $\Phi(\mathbf{x}, \mathbf{u}, t)$ is the vector field. $\mathbf{A} \in \mathcal{R}^{n \times n}$, $\mathbf{B} \in \mathcal{R}^{n \times q}$, $\mathbf{P} \in \mathcal{R}^{n \times m}$ and $\mathbf{C} \in \mathcal{R}^{p \times n}$ are constant matrices. Also $m \leq p \leq n$.

2.2.1 Equivalent Control based SMO's

Sliding mode observers for linear systems based on the equivalent control method was first given by Utkin [56] and its extensions to nonlinear systems were done in [70, 71, 74, 75]. Here, we first discuss the method of [56], followed by the developments for nonlinear systems [74, 75].

(a) Utkin SMO [56]

Referring to system (2.9), assuming that there are no unknown inputs and the nonlinear vector field, i.e. $\mathbf{P}\mathbf{d} = 0$ and $\Phi(\mathbf{x}, \mathbf{u}, t) = 0$. We also assume that the pair $(\mathbf{A}, \mathbf{C})$ is observable. Define the transformation matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_{n-p} & 0 \\ \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \quad (2.10)$$

where $\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}$, with $\mathbf{C}_1 \in \mathcal{R}^{p \times (n-p)}$, $\mathbf{C}_2 \in \mathcal{R}^{p \times p}$, and $\det(\mathbf{C}_2) \neq 0$. Clearly, the transformation is nonsingular. With the transformation $\mathbf{T}$, the system in the

transformed space can be obtained as

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{y} + \mathbf{B}_1u \\ \dot{\mathbf{y}} &= \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{y} + \mathbf{B}_2u\end{aligned}\tag{2.11}$$

where $\mathbf{T}\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{y} \end{bmatrix}$ and the transformed matrices are

$$\mathbf{TAT}^{-1} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{TB} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

Then according to [56], the following observer can be built to estimate the states of $\mathbf{y}$ subsystem:

$$\dot{\hat{\mathbf{y}}} = \mathbf{A}_{21}\hat{\mathbf{x}}_1 + \mathbf{A}_{22}\hat{\mathbf{y}} + \mathbf{B}_2u + \mathbf{L}_y\text{sign}(\mathbf{y} - \hat{\mathbf{y}})\tag{2.12}$$

where $\mathbf{L}_y$ is a constant feedback gain. By defining error $\mathbf{e}_1 = \mathbf{x}_1 - \hat{\mathbf{x}}_1$, $\mathbf{e}_y = \mathbf{y} - \hat{\mathbf{y}}$, one can obtain the error dynamics as

$$\dot{\mathbf{e}}_y = \mathbf{A}_{21}\mathbf{e}_1 + \mathbf{A}_{22}\mathbf{e}_y - \mathbf{L}_y\text{sign}(\mathbf{e}_y)\tag{2.13}$$

With a proper selection of the observer gain matrix $\mathbf{L}_y$, the sliding mode motion will occur along the manifold $\mathbf{e}_y = 0$ [56].

For the $\mathbf{x}_1$ subsystem, the observer of the following form can be built according to [56, 71] as

$$\dot{\hat{\mathbf{x}}}_1 = \mathbf{A}_{11}\hat{\mathbf{x}}_1 + \mathbf{A}_{12}\mathbf{y} + \mathbf{B}_1u + \mathbf{L}_1\mathbf{L}_y\text{sign}(\mathbf{e}_y)\tag{2.14}$$

The corresponding error dynamics will be

$$\dot{\mathbf{e}}_1 = \mathbf{A}_{11}\mathbf{e}_1 - \mathbf{L}_1\mathbf{L}_y\text{sign}(\mathbf{e}_y)\tag{2.15}$$

Based on equivalent control method, the system behaves as if $\mathbf{L}_y\text{sign}(\mathbf{e}_y)$ is replaced

by its equivalent value $(\mathbf{L}_y \text{sign}(\mathbf{e}_y))_{eq}$, which can be calculated from the subsystem (2.13), with $\mathbf{e}_y = 0$ and $\dot{\mathbf{e}}_y = 0$. Hence

$$(\mathbf{L}_y \text{sign}(\mathbf{e}_y))_{eq} = \mathbf{A}_{21} \mathbf{e}_1 \quad (2.16)$$

Substituting (2.16) into (2.15) we can obtain

$$\dot{\mathbf{e}}_1 = [\mathbf{A}_{22} - \mathbf{L}_1 \mathbf{A}_{12}] \mathbf{e}_1 \quad (2.17)$$

It can always be shown that if the pair $(\mathbf{A}, \mathbf{C})$ is observable then the pair $(\mathbf{A}_{11}, \mathbf{A}_{21})$ is also observable. By appropriate selection of $\mathbf{L}_1$, we can obtain $\mathbf{e}_1 \rightarrow 0$.

The Utkin SMO [56] uses the information from the measured states to build the observer. The equivalent control provides the additional information from the system for the design of the observer.

In usual practice, the evaluation of equivalent control is not straightforward. The term $(\text{sign}(\mathbf{e}_y))_{eq}$ denotes the average value of the chattering with high frequency variable $\text{sign}(\mathbf{e}_y)$. Thus, $(\text{sign}(\mathbf{e}_y))_{eq}$ has to be obtained via a low pass filter whose time constant is “sufficiently small” to pass the slow component of the motion (which is equivalent control) but is large enough to eliminate the high frequency component.

(b) Sliding Mode Observers of [74], [85]

In [74], the idea of equivalent control in state estimation was extended to nonlinear systems having triangular input form, i.e.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f_n(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} g_1(x_1, u) \\ g_2(x_1, x_2, u) \\ \vdots \\ g_{n-1}(x_1, x_2, \dots, x_{n-1}, u) \\ g_n(\mathbf{x}, u) \end{bmatrix}, \quad y = x_1 \quad (2.18)$$

where $f_n(\mathbf{x}), g_i(\mathbf{x}, u)$ are nonlinear functions for all $i = 1, \dots, n$. The system (2.18) is assumed to be BIBS. Based on the equivalent concept in [56], the SMO of the following form was developed:

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \vdots \\ \dot{\hat{x}}_{n-1} \\ \dot{\hat{x}}_n \end{bmatrix} = \begin{bmatrix} \hat{x}_2 + g_1(\hat{x}_1, u) + \lambda_1 \text{sign}(x_1 - \hat{x}_1) \\ \hat{x}_3 + g_2(\hat{x}_1, \tilde{x}_2, u) + \lambda_2 \text{sign}(\tilde{x}_2 - \hat{x}_2) \\ \vdots \\ \hat{x}_n + g_{n-1}(\hat{x}_1, \dots, \tilde{x}_{n-1}, u) + \lambda_{n-1} \text{sign}_{n-2}(\tilde{x}_{n-1} - \hat{x}_{n-1}) \\ f_n(\mathbf{x}) + g_n(\hat{x}_1, \tilde{x}_2, \dots, \tilde{x}_n, u) + \lambda_n \text{sign}(\tilde{x}_n - \hat{x}_n) \end{bmatrix} \quad (2.19)$$

where $\tilde{x}_i = \hat{x}_i + \lambda_{i-1} \text{sign}_{i-1}(x_{i-1} - \hat{x}_{i-1})$ for $i = 2, \dots, n-1$.

By defining error dynamics as $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, we can obtain

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n-1} \\ e_n \end{bmatrix} = \begin{bmatrix} e_2 + g_1(\mathbf{x}, u) - g_1(\hat{\mathbf{x}}, u) + \lambda_1 \text{sign}(\hat{x}_1 - x_1) \\ e_3 + g_2(\mathbf{x}, u) - g_2(\hat{\mathbf{x}}, u) + \lambda_2 \text{sign}((\lambda_1 \text{sign}(\hat{x}_1 - x_1))_{eq}) \\ \vdots \\ e_{n-1} + g_{n-1}(\mathbf{x}, u) - g_{n-1}(\hat{\mathbf{x}}, u) + \lambda_{n-1} \text{sign}_{n-2}(\tilde{x}_{n-1} - \hat{x}_{n-1}) \\ f_n(\mathbf{x}) - f_n(\hat{\mathbf{x}}) + g_n(\hat{x}_1, \tilde{x}_2, \dots, \tilde{x}_n, u) + \lambda_n \text{sign}_{n-1}(\tilde{x}_n - \hat{x}_n) \end{bmatrix}$$

In the standard form, we can have equivalent control as

$$\begin{aligned} & \lambda_i \text{sign}_{i-1}(\tilde{x}_{i-1} - \hat{x}_{i-1}) \\ = & \lambda_i \text{sign} \left(\lambda_{i-1} \text{sign} \left(\lambda_{i-2} \text{sign} (\dots (\lambda_1 \text{sign}(x_1 - \hat{x}_1))_{eq} \dots)_{eq} \right)_{eq} \right)_{eq} \end{aligned} \quad (2.20)$$

for all $i = 2, \dots, n$. By successive evaluation of equivalent controls, the sliding modes can be attained for all the states and the convergence of estimation error can be achieved. As the nonlinear part of the system satisfies the triangular structure and the corresponding states reach the sliding modes sequentially, we have equivalent control for the i^{th} error dynamics as

$$\lambda_i \text{sign}_{i-1}(\tilde{x}_{i-1} - \hat{x}_{i-1}) = e_{i+1} \quad (2.21)$$

The equivalent control signal is obtained through low pass filter [70] and anti-peaking structure [116]. The equivalent control signals $\lambda_i \text{sign}_{i-1}(\cdot)$ are the higher-

order derivatives of the output estimation error. The anti-peaking structure stems from the idea that we do not use the observation error information before reaching the sliding manifold linked with this information. Moreover, the sliding manifold is only reached one by one sequentially. Following this procedure, we obtain a subdynamic of order one and consequently the peaking phenomenon is avoided [116]. In the observer structure, the $\text{sign}(\cdot)$ function allows $\tilde{x}_i - \hat{x}_i$ to converge to zero if all the $\tilde{x}_j - \hat{x}_j$ for all $j < i$ have converged to zero. The SMO dependent on this procedure of evaluation of the system states was also termed as step-by-step SMO in the literature [74, 78].

Remark 2.1 *The equivalent control signal should only be used if its corresponding estimation error is in sliding mode. Since only the output estimation error is measurable and the sliding modes of estimation errors $e_2, \dots, e_{n-1}$ are not known, it is very difficult to implement the equivalent control. If the equivalent signal is used too early, peaking phenomenon is unavoidable. And if the equivalent control signal is used too late, correct estimation cannot be obtained. The implementation issues of low-pass filters, the relation between the estimation accuracy, the filter time constants and the sampling time were discussed in [117]. The accuracy of estimation decreases with the increase in the system order.*

Remark 2.2 *The major setback of the equivalent control based observer lies in the handling of the disturbances. If a disturbance or unknown input is present in the system, the equivalent control signal will carry the disturbance effect to all the states and the convergence of the error dynamics cannot be attained.*

The work in [115] extended the equivalent control method of [70], [74] for linear and nonlinear systems with disturbances and/or unknown inputs. For the case of linear systems, the results in [115] used a special co-ordinate basis [118] to transform the system (2.9)(without the nonlinear vector field) such that the unknown inputs only appear in the last equation of every subsystem. Then, the framework similar

to [74] is employed to design the observer.

In [75], the analysis is extended to a general class of nonlinear systems with unknown inputs. This observer design relies on a nonlinear transformation [34, 76] and Frobenius theorem [34, pg. 22] to decouple the unknown inputs in the transformed domain. By assuming the disturbance distribution vectors to be involutive, [75] transformed the nonlinear system to subsystems in which the unknown inputs only appear in the last equation of the subsystem. For the system in the transformed domain, the majority of the states are not directly affected by the unknown inputs and the equivalent control method similar to [74] is employed for the state estimation. The assumptions employed by [75] for the existence of relative degree and Frobenius theorem are conservative.

Recent work [77, 78] on fault estimation and detection also rely on decoupling the unknown inputs for the design of observers. All the above methods that are based on equivalent control relied on output estimation error and its higher-order derivatives.

(c) Higher-order Sliding Mode Observers

Over the last decade, higher-order sliding mode (HOSM) control [72, 119–121] has received significant attention due to the inherent absence of chattering [122, 123] and several other advantages over the traditional sliding mode. Also, higher-order sliding modes ensure better accuracy and finite time convergence [72]. The extension of HOSM concepts to observer design is not straight forward as the observer requires the knowledge of higher-order derivatives of the sliding manifold. Recently, the second-order sliding mode observers have been developed [73, 124, 125]. The work in [119] developed a robust exact differentiator with finite time convergence as an application of the super-twisting algorithm [72]. These observers

require the separation principle theorem [126] and the super-twisting algorithm [72]. One of the main advantages is that the HOSM observers can incorporate measurement delays. The class of systems the HOSM observers can currently handle is limited. Their robustness properties in the presence of unknown inputs are yet to be explored.

Due to the focus on the analysis of the sliding mode observers with unknown inputs in this thesis, the discussion relating to HOSM observers is kept brief.

2.2.2 Lyapunov based SMO's

SMO based on Lyapunov method was initiated by Walcott and Zak [23, 63, 64] for state estimation of uncertain systems with bounded uncertainties/disturbances. The design is based on Lyapunov min-max methods and also requires matching condition. This method was termed as Walcott-Zak Observer in the literature [68], [115]. The method proposed by Walcott and Zak was further developed by [66] and extended to a more general class of nonlinear systems in [25, 65, 66]. The work in [67] designed an observer for nonlinear deterministic and stochastic systems based on the gain design adopted by [23].

(a) Walcott-Zak SMO

The method of Walcott-Zak [23, 63, 64] guarantees the convergence of error to zero exponentially inspite of the presence of disturbances/unknown inputs. Also, the method only requires the bounds of uncertainties. We again consider the uncertain system (2.9) but without the nonlinear vector field, i.e. $\Phi(\mathbf{x}, \mathbf{u}, t) = 0$. The proposed observer is of the following form [23]:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) + \mathbf{P} \times \rho \text{sign}(\mathbf{W}\mathbf{C}\mathbf{e}) \quad (2.22)$$

where $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$, $\mathbf{S} = \{\mathbf{e} : \mathbf{W}\mathbf{C}\mathbf{e}\} = 0$, $\mathbf{W} \in \mathcal{R}^{m \times p}$ is a real matrix, $\mathbf{L}$ is the feedback gain matrix designed such that the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ are stable, and $\mathbf{Q}$ is a real symmetric positive definite (SPD) matrix satisfying:

$$\mathbf{Q}(\mathbf{A} - \mathbf{L}\mathbf{C}) + (\mathbf{A} - \mathbf{L}\mathbf{C})^T \mathbf{Q} = -\mathbf{R} \quad (2.23)$$

$$\mathbf{W}\mathbf{C} = \mathbf{P}^T \mathbf{Q} \quad (2.24)$$

for a SPD matrix $\mathbf{R} = \mathbf{R}^T > 0$.

Based on the conditions (2.23), (2.24), the convergence of the error can be proved based on standard Lyapunov analysis [23]. From the above, it is clear that the design copies the Luenberger observer [2] with an appended robust term or switching term to deal with uncertainties or unknown inputs in the system.

The major drawback of this method lies in finding matrices $\mathbf{Q}$ and $\mathbf{W}$ in order to satisfy the conditions (2.23), (2.24). An algorithm for finding the matrices by transforming the system to a canonical form was developed by [68].

The matching conditions were further discussed in [127] for sliding mode observer design for linear systems, and the canonical form was analyzed for the generalized observer. The work in [68] modified the observer of [56] and extended the discontinuous observer design for the uncertain systems. The Walcott-Zak SMO required complete observability of unknown inputs to compensate them in the observer design. In other words, each unknown input must be compensated by an output. The remaining outputs are used to construct the observer for the unknown input free subsystem.

Remark 2.3 *The conditions under which there exist matrices $\mathbf{Q}$ and $\mathbf{W}$ such that the SMO observer is totally unaffected by the disturbances is not known. The necessary and sufficient conditions of Walcott-Zak SMO have been a key problem in the design of SMO.*

Remark 2.4 *The major difference between the Walcott-Zak SMO and Equivalent control SMO lies in the manner in which they handle the disturbances. Walcott-Zak SMO requires the unknown inputs to be dealt with directly by the outputs, whereas the methods of [75], [78] require the outputs to have relative degree in order to decouple the unknown inputs for the design of SMO.*

(b) Discontinuous SMO [82, 128]

The class of observers developed in [68, 82, 109] used the outputs to observe the unknown inputs so as to design the sliding mode observer. The observer involves the concepts of Utkin SMO [56] and Walcott-Zak SMO [23].

Consider again the system (2.9) without the vector field, i.e. $\Phi = 0$ and assuming that the matrices $\mathbf{P} \in \mathcal{R}^{n \times m}$ and $\mathbf{C} \in \mathcal{R}^{p \times n}$ are full rank. It is also assumed that system (2.9) satisfies the following:

- $\text{rank}(\mathbf{CP}) = m$
- the invariant zeros of $(\mathbf{A}, \mathbf{P}, \mathbf{C})$ lie in $\mathcal{C}_-$.

Under the above conditions, there exists a transformation (See Lemma 6.1 [109]) similar to (2.10), such that the system in the transformed coordinates is of the form

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{12}\mathbf{y} + \mathbf{B}_1u \\ \dot{\mathbf{y}} &= \mathbf{A}_{21}\mathbf{x}_1 + \mathbf{A}_{22}\mathbf{y} + \mathbf{B}_2u + \mathbf{P}_2\mathbf{d}\end{aligned}\tag{2.25}$$

where $\mathbf{x}_1 \in \mathcal{R}^{(n-p)}$, $\mathbf{y} \in \mathcal{R}^p$ and the matrix $\mathbf{A}_{11}$ has stable eigenvalues. The following SMO was built to estimate the states in the presence of disturbances

$$\dot{\hat{\mathbf{x}}}_1 = \mathbf{A}_{11}\hat{\mathbf{x}}_1 + \mathbf{A}_{12}\hat{\mathbf{y}} + \mathbf{B}_1u - \mathbf{A}_{12}\mathbf{e}_y\tag{2.26}$$

$$\dot{\hat{\mathbf{y}}} = \mathbf{A}_{21}\hat{\mathbf{x}}_1 + \mathbf{A}_{22}\hat{\mathbf{y}} + \mathbf{B}_2u - (\mathbf{A}_{22} - \mathbf{L})\mathbf{e}_y + \nu\tag{2.27}$$

where $\mathbf{L}$ is a stable matrix of appropriate dimension and $\mathbf{e}_y = \hat{\mathbf{y}} - \mathbf{y}$. The discontinuous component or the sliding mode gain vector is defined by

$$\nu = \begin{cases} -\rho \|\mathbf{P}_2\| \frac{\mathbf{Q}_2 \mathbf{e}_y}{\|\mathbf{Q}_2 \mathbf{e}_y\|} & \text{if } \mathbf{e}_y \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.28)$$

where $\mathbf{Q}_2$ is a SPD matrix, and the scalar ρ is chosen such that $\|d(\mathbf{x}, \mathbf{u}, t)\| \leq \rho$.

The estimation error dynamics can be obtained from (2.27) and (2.25) as

$$\dot{\mathbf{e}}_1 = \mathbf{A}_{11} \mathbf{e}_1 \quad (2.29)$$

$$\dot{\mathbf{e}}_y = \mathbf{A}_{21} \mathbf{e}_1 + \mathbf{L} \mathbf{e}_y + \nu - \mathbf{P}_2 \mathbf{d} \quad (2.30)$$

The error system is proved to be quadratically stable and a sliding motion takes place, forcing $\mathbf{e}_y = 0$ in finite time [68].

Remark 2.5 *The method developed in [68, 109] relied on canonical form for decoupling the unknown input distribution matrix $\mathbf{P}$ such that the disturbances or unknown inputs only appear in the subsystem measured by the outputs and hence could be dealt with easily. This method also uses all the outputs to deal with the unknown inputs and so requires the reduced-order system itself to be stable.*

Remark 2.6 *One can clearly see the similarity of the above canonical form with the Utkin SMO. The observer of [68, 109] appends the Luenberger type feedback correction term together with the robust term to the Utkin SMO. The major difference with Utkin SMO is that the sliding mode gain design is done component wise, whereas the vector type design was adopted in [23, 68, 109].*

(c) Nonlinear Lipschitz SMO [1]

Recently, SMO design based on Walcott-Zak SMO [23] was extended to systems of the form (2.9) with nonlinear function $\Phi(\mathbf{x}, \mathbf{u}, t)$ satisfying the Lipschitz assumption. The sliding mode gain design needs to satisfy the so-called *matching*

condition for complete rejection of the disturbances or unknown inputs.

Consider the system (2.9), with the nonlinear function $\Phi(\mathbf{x}, \mathbf{u}, t)$ satisfying the Lipschitz condition

$$\|\Phi(\hat{\mathbf{x}}, \mathbf{u}, t) - \Phi(\mathbf{x}, \mathbf{u}, t)\| \leq l_\Phi \|\hat{\mathbf{x}} - \mathbf{x}\| \quad (2.31)$$

for some Lipschitz constant l_Φ . Under the assumption that $(\mathbf{A}, \mathbf{C})$ is observable, the following observer can be designed to estimate the states of system (2.9):

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) + \Phi(\hat{\mathbf{x}}, \mathbf{u}, t) + \Lambda \frac{\mathbf{W}\mathbf{C}\mathbf{e}}{\|\mathbf{C}\mathbf{e}\|} \quad (2.32)$$

where $\Lambda \in \mathcal{R}^{n \times m}$ is the feedforward injection map such that $\mathbf{C}\Lambda$ is non-singular. The feedback gain is to be designed such that the following conditions are satisfied.

$$\mathbf{Q}(\mathbf{A} - \mathbf{L}\mathbf{C}) + (\mathbf{A} - \mathbf{L}\mathbf{C})^T \mathbf{Q} + \frac{1}{\varepsilon} l_\Phi^2 \mathbf{I} + \varepsilon \mathbf{Q}^2 = -\mathbf{R} \quad (2.33)$$

$$\Lambda = \mathbf{Q}^{-1} \mathbf{C}^T \mathbf{W}^{-1}, \quad \mathbf{P} = \Lambda \mathbf{D} \quad (\text{matching condition [56]}) \quad (2.34)$$

where $\mathbf{W}$ is an $m \times m$ diagonal PD matrix, and $\mathbf{D}$ is an $m \times m$ matrix that satisfies the *matching condition* [56]. The details including the gain $\mathbf{L}$, will be discussed in the next section in relation to other works on nonlinear Lipschitz observers. Under this matching condition, the unknown inputs do not affect the reduced-order error system in the sliding mode.

Remark 2.7 *The work in [1] requires similar matching conditions that are required to design the Walcott-Zak Observer. In fact, one also needs to design $\mathbf{L}$ to satisfy the Riccati equation (2.33). This methodology however becomes more conservative in light of the new condition (2.33), and the matching condition (2.34). The conditions under which the SMO can be designed are not exactly known.*

Remark 2.8 *From the nonlinear observer design point of view, the method [1] requires linear output equations. Also, the distribution matrix $\mathbf{P}$ of the unknown inputs is assumed to be a constant matrix.*

The shortcomings of the existing sliding mode observers that deal with unknown inputs can be summarized as follows:

- All the procedures rely on vector design for sliding mode switching terms to guarantee Lyapunov stability. The conditions under which the unknown inputs/disturbances are completely dealt with to achieve asymptotical convergence for the estimation error are not clear.
- All the developments have been for linear systems with unknown inputs, extensions of the concepts to nonlinear systems require conservative structural assumptions and higher-order derivatives of the output estimation errors in the case of equivalent control based SMO.
- So far, the unknown input distribution matrix and output distribution matrix was only considered to be a constant matrices. The procedures for SMO design for a general class of nonlinear systems for state and unknown input estimations are not available.

2.3 Nonlinear Lipschitz Observers

Lyapunov-based design for nonlinear observers was first initiated by Thau [45]. Thau [45] gave a sufficient condition for the convergence of estimation error for a class of nonlinear systems. The work of Thau [45] was further extended to the deterministic case by [46–48], and to the stochastic case by [49, 50], focusing on conditions for the existence of observer and convergence. These designs rely on the observer gain to satisfy the sufficient condition. However, the condition proposed by [45] serves only as a verification once the observer is designed.

Consider the system (2.9) without the unknown inputs, i.e. $\mathbf{P}\mathbf{d}(\mathbf{x}, \mathbf{u}, t) = 0$,

and the nonlinear function $\Phi(\mathbf{x}, \mathbf{u}, t)$ is assumed to satisfy the Lipschitz condition (2.31). The following observer can be designed according to [45]:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) + \Phi(\hat{\mathbf{x}}, \mathbf{u}, t) \quad (2.35)$$

where $\mathbf{A} - \mathbf{LC}$ is a stable matrix, and the feedback gain $\mathbf{L}$ is designed such that

$$\mathbf{Q}(\mathbf{A} - \mathbf{LC}) + (\mathbf{A} - \mathbf{LC})^T \mathbf{Q} = -\mathbf{R} \quad (2.36)$$

with $\mathbf{Q}$ and $\mathbf{R}$ being positive-definite (PD) matrices. The well-known result of Thau [45] for the stability of the system is given by

$$l_\Phi < \frac{\lambda_{\min}(\mathbf{R})}{2\lambda_{\max}(\mathbf{Q})} \quad (2.37)$$

The left-hand side of the condition (2.37) is maximized for $\mathbf{R} = \mathbf{I}$ [129], hence the condition (2.37) reduces to

$$l_\Phi < \frac{1}{2\lambda_{\max}(\mathbf{Q})} \quad (2.38)$$

The method for designing gain $\mathbf{L}$ such that the conditions (2.36), (2.37) are satisfied is not straightforward. By placing the eigenvalues of $(\mathbf{A} - \mathbf{LC})$ far in the left half plane will not guarantee that condition (2.37) is satisfied. Also, the clear relation between eigenvalues of the matrix $(\mathbf{A} - \mathbf{LC})$ and $\lambda_{\max}(\mathbf{Q})$ cannot be identified from the conditions (2.36), (2.38).

In the rest of the thesis, the observer gain design that depends on the conditions (2.36), (2.38) is termed as the nonlinear Lipschitz observer (NLO).

The problem of gain design was handled differently by several researchers. The work in [130], [51] altered the conditions (2.36), (2.38) for choosing the gain into the form of Riccati equation

$$\mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^T + \mathbf{Q}\left(l_\Phi^2 - \frac{1}{\varepsilon}\mathbf{C}^T\mathbf{C}\right)\mathbf{Q} + \mathbf{I} + \varepsilon\mathbf{I} = 0 \quad (2.39)$$

If there exists a PD solution $\mathbf{Q}$ for a small ε , then the following choice of gain

$$\mathbf{L} = \frac{1}{2\varepsilon} \mathbf{Q} \mathbf{C}^T \quad (2.40)$$

guarantees the stability of the system. The above condition also serves as a check for designing $\mathbf{L}$ but does not provide the conditions for the matrix $(\mathbf{A} - \mathbf{L}\mathbf{C})$ to meet the stability condition for the system.

In [64], it is proposed that if the gain is designed such that $\sigma_{\min}(\mathbf{A} - \mathbf{L}\mathbf{C}) > l_\Phi$, then the condition (2.38) is satisfied. However, in [52], [53], it was demonstrated that the above condition does not hold true for all cases.

An intuitively appealing procedure for the gain design is to consider the distance to unobservability¹ and the condition number that guarantees the stability of the error system, and these are given by Rajamani [52]. The following condition according to [52, 53] ensures the asymptotic stability of the observer (2.35):

$$\min_{\omega \in \mathbb{R}^+} \rho_{\min}(\mathbf{A} - \mathbf{L}\mathbf{C} - j\omega\mathbf{I}) > l_\Phi \quad (2.41)$$

If there exists a stable $(\mathbf{A} - \mathbf{L}\mathbf{C})$ matrix, then there exists a SPD solution $\mathbf{Q} = \mathbf{Q}^T$ for some $\varepsilon > 0$ to the Riccati equation

$$\mathbf{Q}(\mathbf{A} - \mathbf{L}\mathbf{C}) + (\mathbf{A} - \mathbf{L}\mathbf{C})^T \mathbf{Q} + l_\Phi^2 \mathbf{Q} \mathbf{Q} + \mathbf{I} + \varepsilon \mathbf{I} = 0 \quad (2.42)$$

In fact, the conditions (2.36), (2.37) of Thau [45] were generalized to the conditions (2.41), (2.42). Also, the design procedure for the computation of the gain $\mathbf{L}$ was also discussed in [52], [53].

Koshkouei and Zinober [1] further extended the study of [52] and generalized the condition (2.42) given by [52] to the algebraic Riccati equation (2.33). The condition (2.42) can be deduced from condition (2.33) with $\varepsilon = l_\Phi^2$. The above condition

¹The distance to unobservability of the pair $(\mathbf{A}, \mathbf{C})$ is the magnitude of the smallest perturbation that makes the pair unobservable [131], [53, Definition 2.1].

for the stability of the observer can be split into the following conditions:

$$\mathbf{Q}(\mathbf{A} - \mathbf{LC}) + (\mathbf{A} - \mathbf{LC})^T \mathbf{Q} + \frac{1}{\varepsilon} l_\Phi^2 \mathbf{I} + \mathbf{R} = 0 \quad (2.43)$$

$$\varepsilon < \frac{\lambda_{\min}(\mathbf{R})}{\lambda_{\max}^2(\mathbf{Q})} \quad (2.44)$$

The gain design, without the necessity to solve the Riccati equation, was also suggested in the same work.

Remark 2.9 *The major bottleneck in the Lyapunov-based design lies in solving the conditions (2.36), (2.38). In fact, the conditions are related to the ability of system $(\mathbf{A}, \mathbf{C})$ with gain $\mathbf{L}$ to provide extra degrees of freedom to account for the nonlinear function $\Phi(\mathbf{x}, \mathbf{u}, t)$ without losing stability.*

Remark 2.10 *The design procedures adopted by [51], [52], [1] combined the conditions (2.36), (2.38) in order to evaluate the parameter ε which in turn is related to the Lipschitz constant l_Φ such that the error system is stabilizable.*

Remark 2.11 *These design methods impose restrictions on the bound of the Lipschitz constant l_Φ such that the conditions for gain design (2.36), (2.38) are met. This means that the Lipschitz constant l_Φ cannot be arbitrary large.*

All the Lyapunov-based designs rely strictly on Lipschitz or linear growth constraint assumptions, and convergence in most cases is only local and highly dependent on nominal nonlinear dynamics. Uncertainties in modelling can result in poor performance characteristics [25].

In the following section, we discuss the high-gain observers that can accommodate the Lipschitz constant that is arbitrary large.

2.4 Uniform Observability and High-Gain Observers

In this section, we first review some basic concepts of observability. Then we discuss the class of uniformly observable systems [132] [37], [39] that are usually obtained via nonlinear transformation, the relevant assumptions, and the design of high-gain observers [37], [38], [133].

For linear systems, if the system is observable, the initial states can be reconstructed for any arbitrary control input. This property cannot be met for nonlinear systems in general. The observability of nonlinear systems in general is associated with inputs. The inputs that can make the system unobservable are classified as “bad inputs”. Hence, categorization of systems that are observable independent of inputs is crucial for the development of observers for systems with unknown inputs.

Consider the class of SISO nonlinear system described by

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u(t) \\ y(t) = h(\mathbf{x}) \end{cases} \quad (2.45)$$

where $\mathbf{x} \triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \in \mathcal{M}$, a $\mathcal{C}^\infty$ connected manifold of dimension n , the nonlinear functions $\mathbf{f}(\mathbf{x})$, $\mathbf{b}(\mathbf{x})$ are smooth vector fields on $\mathcal{M}$, $h(\mathbf{x})$ is a smooth function from $\mathcal{M}$ to $\mathcal{R}$. $u \in \mathcal{U} \subset \mathcal{R}$ is the control input and $y \in \mathcal{R}$ is the measurable output.

Definition 2.1 (*Observability*) [39] *The system (2.45) is said to be uniformly observable on $\mathcal{M}$ if, for every pair of initial conditions $(x_0, \bar{x}_0) \in \mathcal{M}$, $x_0 \neq \bar{x}_0$, and for all admissible inputs $u : [0, T] \rightarrow \mathcal{U}$, $y(x(x_0, u, t), u) \neq y(x(\bar{x}_0, u, t), u)$ on $[0, T]$.*

Definition 2.2 (*Observability for any input*) [39] *The system (2.45) is said to be*

uniformly observable if for any input $u(\cdot) \in \mathcal{U}$ and for any $(x_0, \bar{x}_0)$, $x_0 \neq \bar{x}_0$ there exists a time $t \geq 0$ such that $y(x_0, u(\cdot), t) \neq y(\bar{x}_0, u(\cdot), t)$.

Uniform observability of a system exists when the initial state can be constructed from the data formed by the output and its derivatives [37]. To characterize the systems that are uniformly observable, we first transform the nonlinear system (2.45) by the following nonlinear transformation.

2.4.1 Nonlinear State Transformation

By performing a nonlinear transformation

$$\mathbf{x} \rightarrow \tilde{\mathbf{x}} \triangleq \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = \Phi(\mathbf{x}) = \begin{bmatrix} h(\mathbf{x}) \\ L_{\mathbf{f}}h(\mathbf{x}) \\ \vdots \\ L_{\mathbf{f}}^{(n-1)}h(\mathbf{x}) \end{bmatrix} \quad (2.46)$$

we can obtain

$$\begin{aligned} \dot{\tilde{x}}_1 &= L_{\mathbf{f}}h(\mathbf{x}) + L_{\mathbf{b}}h(\mathbf{x})u = \tilde{x}_2 + L_{\mathbf{b}}h(\mathbf{x})u \\ \dot{\tilde{x}}_2 &= L_{\mathbf{f}}^{(2)}h(\mathbf{x}) + L_{\mathbf{b}}L_{\mathbf{f}}h(\mathbf{x})u = \tilde{x}_3 + L_{\mathbf{b}}L_{\mathbf{f}}h(\mathbf{x})u \\ \vdots &= \vdots \\ \dot{\tilde{x}}_{n-1} &= L_{\mathbf{f}}^{(n-1)}h(\mathbf{x}) + L_{\mathbf{b}}L_{\mathbf{f}}^{(n-2)}h(\mathbf{x})u = \tilde{x}_n + L_{\mathbf{b}}L_{\mathbf{f}}^{(n-2)}h(\mathbf{x})u \\ \dot{\tilde{x}}_n &= L_{\mathbf{f}}^{(n)}h(\mathbf{x}) + L_{\mathbf{b}}L_{\mathbf{f}}^{(n-1)}h(\mathbf{x})u \end{aligned}$$

The transformed system can be re-written as

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ L_{\mathbf{f}}^{(n)}h(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} L_{\mathbf{b}}h(\mathbf{x}) \\ L_{\mathbf{b}}L_{\mathbf{f}}h(\mathbf{x}) \\ \vdots \\ L_{\mathbf{b}}L_{\mathbf{f}}^{(n-2)}h(\mathbf{x}) \\ L_{\mathbf{b}}L_{\mathbf{f}}^{(n-1)}h(\mathbf{x}) \end{bmatrix} u \\ y = \mathbf{C}\tilde{\mathbf{x}} \end{cases} \quad (2.47)$$

where

$$\mathbf{A} \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \mathbf{C} \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \quad (2.48)$$

are the anti-shift constant matrix and the output matrix, respectively.

Assumption 2.1 *The mapping $\Phi(\mathbf{x})$ is a local diffeomorphism $\forall \mathbf{x}_0 \in \mathcal{M}$.*

Under **Assumption 2.1**, system (2.47) in the general form can be represented as

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \boldsymbol{\psi}(\tilde{\mathbf{x}}, u) \\ y = \mathbf{C}\tilde{\mathbf{x}} \end{cases} \quad (2.49)$$

where $\boldsymbol{\psi}(\tilde{\mathbf{x}}, u) = \begin{bmatrix} \psi_1(\tilde{\mathbf{x}}, u) & \psi_2(\tilde{\mathbf{x}}, u) & \cdots & \psi_n(\tilde{\mathbf{x}}, u) \end{bmatrix}^T \triangleq \boldsymbol{\alpha}(\tilde{\mathbf{x}}) + \boldsymbol{\gamma}(\tilde{\mathbf{x}})u$,
 $\boldsymbol{\alpha}(\tilde{\mathbf{x}}) \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & \phi(\tilde{\mathbf{x}}) \end{bmatrix}^T$, $\phi(\tilde{\mathbf{x}}) = L_{\mathbf{f}}^n h(\mathbf{x})$ and $\boldsymbol{\gamma}(\tilde{\mathbf{x}}) \triangleq \begin{bmatrix} \gamma_1(\tilde{\mathbf{x}}) & \gamma_2(\tilde{\mathbf{x}}) & \cdots & \gamma_n(\tilde{\mathbf{x}}) \end{bmatrix}^T$,
 $\gamma_i(\tilde{\mathbf{x}}) = L_{\mathbf{b}} L_{\mathbf{f}}^{(i-1)} h(\mathbf{x})$, for $i = 1, \dots, n$.

Remark 2.12 *For a system of the form (2.49), one may also design the nonlinear Lipschitz observer (NLO) as discussed in the last section under the Lipschitz assumption for the nonlinear function $\boldsymbol{\psi}(\tilde{\mathbf{x}}, u)$.*

2.4.2 Uniformly Observable Systems

Based on the work in [39], we have the following lemma that classifies the conditions under which the system observability can be independent of the inputs.

Lemma 2.1 [37, 39] *For the system (2.49), suppose that the input terms are of the form*

$$\begin{aligned} \psi_1(\tilde{\mathbf{x}}, u) &= \psi_1(\tilde{x}_1, u) \\ \psi_2(\tilde{\mathbf{x}}, u) &= \psi_2(\tilde{x}_1, \tilde{x}_2, u) \\ &\vdots = \vdots \end{aligned}$$

$$\begin{aligned}
\psi_{n-2}(\tilde{\mathbf{x}}, u) &= \psi_{n-2}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n-2}, u) \\
\psi_{n-1}(\tilde{\mathbf{x}}, u) &= \psi_{n-1}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n-1}, u) \\
\psi_n(\tilde{\mathbf{x}}, u) &= \psi_n(\tilde{\mathbf{x}}, u)
\end{aligned} \tag{2.50}$$

and the functions

$$1 + \frac{\partial \psi_i}{\partial \tilde{x}_{i+1}} \neq 0, i = 1, \dots, n-1 \tag{2.51}$$

Then, the system (2.49) is uniformly observable for the bounded input u .

It has been proven in [37] that the structure in **Lemma 2.1** is a sufficient condition but not a necessary condition to ensure uniform observability for any input. This lemma plays an important role in the development of unknown input observers.

2.4.3 High-Gain Observers

A high-gain observer was introduced in [37] for a class of uniformly observable SISO nonlinear systems that can guarantee the exponential convergence of the estimation error.

In the following, we discuss the construction and the convergence of analysis of high-gain observer for the SISO nonlinear system (2.45). To proceed with the design of the observer, we require the following assumption:

Assumption 2.2 *The function $\psi(\tilde{\mathbf{x}}, u)$ is globally Lipschitzian w.r.t. $\tilde{\mathbf{x}}$ and, locally w.r.t. bounded input $u \in \mathcal{U}$ such that*

$$\left\| \psi_i(\tilde{\mathbf{x}}', u) - \psi_i(\tilde{\mathbf{x}}, u) \right\| \leq l_{\psi_i} \|\tilde{\mathbf{x}}' - \tilde{\mathbf{x}}\| \tag{2.52}$$

$\forall \tilde{\mathbf{x}}', \tilde{\mathbf{x}} \in \mathcal{M}$ and for some Lipschitz constant l_{ψ_i} .

For the system (2.49) with the structure in **Lemma 2.1** satisfying the **Assump-**

tion 2.2, the observer can be designed as

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \boldsymbol{\psi}(\hat{\mathbf{x}}, u) - \mathbf{S}_\theta^{-1}\mathbf{C}^T (\mathbf{C}\hat{\mathbf{x}} - y) \quad (2.53)$$

where $\mathbf{S}_\theta$ the unique solution of the Lyapunov algebraic equation

$$\theta\mathbf{S}_\theta + \mathbf{A}^T\mathbf{S}_\theta + \mathbf{S}_\theta\mathbf{A} - \mathbf{C}^T\mathbf{C} = 0 \quad (2.54)$$

and θ is a positive parameter. The explicit solution of (2.54) can be obtained as

$$\mathbf{S}_\theta(i, j) = \frac{(-1)^{i+j}C_{i+j-2}^{j-1}}{\theta^{i+j-1}}, \quad 1 \leq i, j \leq n \text{ where } C_n^r = \frac{n!}{(n-r)!r!} \quad (2.55)$$

Furthermore, $\mathbf{S}_\theta$ is the symmetric positive definite (SPD) for every $\theta > 0$ (see [37]).

For the convergence analysis, we let Δ_θ to be the diagonal matrix:

$$\Delta_\theta = \text{diag} \left(1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}} \right) \quad (2.56)$$

From (2.54), we can directly obtain

$$\mathbf{S}_\theta = \frac{1}{\theta}\Delta_\theta\mathbf{S}_1\Delta_\theta \quad (2.57)$$

where $\mathbf{S}_1$ is the solution of (2.54) for $\theta = 1$. Also, we can deduce the following equalities:

$$\Delta_\theta\mathbf{A}\Delta_\theta^{-1} = \theta\mathbf{A} \quad (2.58)$$

$$\mathbf{C}\Delta_\theta = \mathbf{C}\Delta_\theta^{-1} = \mathbf{C} \quad (2.59)$$

By defining the error dynamics as $\mathbf{e} = \begin{bmatrix} e_1 & e_2 & \dots & e_n \end{bmatrix}^T \triangleq \hat{\mathbf{x}} - \tilde{\mathbf{x}}$, we have from (2.49) and (2.53):

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{S}_\theta^{-1}\mathbf{C}^T\mathbf{C}) \mathbf{e} + \boldsymbol{\psi}(\hat{\mathbf{x}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u) \quad (2.60)$$

To begin with the analysis of the high gain observer, we first define the exponential observer.

Definition 2.3 *The observer (2.53) is an exponential observer for the system (2.45) if*

$$\exists K_1 \geq 0 \text{ such that, } \forall t \geq 0, \quad \|\hat{\tilde{\mathbf{x}}} - \tilde{\mathbf{x}}\| \leq K_2 \exp(-K_1 t)$$

for some positive constants K_1 and K_2 .

The main result concerning high gain observer is as follow:

Theorem 2.1 *Assume the system (2.49) satisfies **Assumption 2.2**. Then there exists $\theta_0 > 0$ such that $\forall \theta > \theta_0$, the observer (2.53) is an exponential observer for the system (2.45).*

Proof: Set $\boldsymbol{\xi} = \Delta_\theta \mathbf{e}$, then

$$\dot{\boldsymbol{\xi}} = \Delta_\theta (\mathbf{A} - \mathbf{S}_\theta^{-1} \mathbf{C}^T \mathbf{C}) \Delta_\theta^{-1} \boldsymbol{\xi} + \Delta_\theta [\boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u)] \quad (2.61)$$

Using the equalities (2.56) - (2.59), we will obtain

$$\dot{\boldsymbol{\xi}} = \theta (\mathbf{A} - \mathbf{S}_1^{-1} \mathbf{C}^T \mathbf{C}) \boldsymbol{\xi} + \Delta_\theta [\boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u)] \quad (2.62)$$

Consider the Lyapunov function $V = \boldsymbol{\xi}^T \mathbf{S}_1 \boldsymbol{\xi}$, differentiating w.r.t. time, we have

$$\begin{aligned} \dot{V} &= 2\boldsymbol{\xi}^T \mathbf{S}_1 \dot{\boldsymbol{\xi}} \\ &= 2\boldsymbol{\xi}^T \mathbf{S}_1 \theta (\mathbf{A} - \mathbf{S}_1^{-1} \mathbf{C}^T \mathbf{C}) \boldsymbol{\xi} + 2\boldsymbol{\xi}^T \mathbf{S}_1 \Delta_\theta [\boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u)] \end{aligned}$$

From (2.54), we can deduce that $2\boldsymbol{\xi}^T \mathbf{S}_1 \mathbf{A} \boldsymbol{\xi} = -\boldsymbol{\xi}^T \mathbf{S}_1 \boldsymbol{\xi} + \boldsymbol{\xi}^T \mathbf{C}^T \mathbf{C} \boldsymbol{\xi}$. Therefore, we have

$$\begin{aligned} \dot{V} &= -\theta \boldsymbol{\xi}^T \mathbf{S}_1 \boldsymbol{\xi} - \theta \|\mathbf{C} \boldsymbol{\xi}\|^2 + 2\boldsymbol{\xi}^T \mathbf{S}_1 \Delta_\theta [\boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u)] \\ &\leq -\theta V + 2\boldsymbol{\xi}^T \mathbf{S}_1 \Delta_\theta [\boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u)] \end{aligned} \quad (2.63)$$

By assuming $\theta \geq 1$ and due to the triangular structure in **Lemma 2.1** and the

Lipschitz assumption (2.52), we have:

$$\begin{aligned}
\left\| \Delta_\theta \left(\boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u) \right) \right\| &\leq \sum_{i=1}^n \left| \frac{1}{\theta^{i-1}} \left(\psi_i(\hat{\tilde{\mathbf{x}}}, u) - \psi_i(\tilde{\mathbf{x}}, u) \right) \right| \\
&\leq l_\psi \sum_{i=1}^n \left| \frac{\bar{\mathbf{e}}_i}{\theta^{i-1}} \right| \leq nl_\psi \|\Delta_\theta \mathbf{e}\| \\
&\leq nl_\psi \|\boldsymbol{\xi}\|
\end{aligned} \tag{2.64}$$

where $l_\psi = \sup_i |l_{\psi_i}|$, the largest Lipschitz constant of $\boldsymbol{\psi}$ and $\bar{\mathbf{e}}_i = (e_1, \dots, e_i)$.

Therefore, we have from (2.63)

$$\begin{aligned}
\dot{V} &\leq -\theta V + 2nl_\psi \|\mathbf{S}_1 \boldsymbol{\xi}\| \|\boldsymbol{\xi}\| \\
&\leq -\theta V + 2nl_\psi \sigma(\mathbf{S}_1) V \\
&= -[\theta - 2nl_\psi \sigma(\mathbf{S}_1)] V
\end{aligned} \tag{2.65}$$

where $\sigma(\mathbf{S}_1)$ is the condition number of $\mathbf{S}_1$. By choosing $\theta > 2nl_\psi \sigma(\mathbf{S}_1)$, we have $\dot{V} < 0$. This completes the proof. $\blacksquare$

Although we can prove the convergence of the estimator (2.53), the main purpose is to produce the estimation of the original system (2.45). Hence we need an inverse state transformation to transfer the implementation of (2.53) into the original domain. Differentiating the state transformation of $\tilde{\mathbf{x}} = \Phi(\mathbf{x})$ with respect to time we obtain $\dot{\tilde{\mathbf{x}}} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]^{-1} \dot{\mathbf{x}}$. From the dynamics (2.49), we have

$$\dot{\tilde{\mathbf{x}}} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]^{-1} \dot{\mathbf{x}} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]^{-1} [\mathbf{A}\tilde{\mathbf{x}} + \boldsymbol{\psi}(\tilde{\mathbf{x}}, u)] = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u \tag{2.66}$$

Similar to (2.66), performing the inverse transformation of $\hat{\tilde{\mathbf{x}}} = \Phi(\hat{\mathbf{x}})$ for the estimator (2.53) produces the estimate $\hat{\mathbf{x}}$ of the original states $\mathbf{x}$:

$$\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}}^{-1} \dot{\hat{\tilde{\mathbf{x}}}} \\
&= \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{b}(\hat{\mathbf{x}})u + \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}}^{-1} \mathbf{S}_\theta^{-1} \mathbf{C}^T (y - h(\hat{\mathbf{x}}))
\end{aligned} \tag{2.67}$$

The approach was generalized to a more general class of nonlinear systems in [38–40, 134]. The design of other exponential observers such as the Gauthier-Kupka Observer [39] and the Kalman-Like Observer [38] also follows similar framework. Khalil and co-workers [126, 135–138] further explored the use of high gain observer for feedback control, numerical differentiation and sampled-data control. High-gain observers with sliding mode control have been used in the design of output feedback controllers due to their ability to robustly estimate the unmeasured states and to asymptotically attenuate the disturbances [126, 135, 138]. The works in [126, 135] also proved a nonlinear separation principle for the stabilization of nonlinear systems employing the high-gain observer. In [136], discrete-time implementation of HGO's was also explored. The work in [137] designed a discrete-time controller using high-gain observer with the separation principle.

The comparisons of nonlinear Lipschitz observer (NLO) with high-gain observer (HGO) are as follows:

- (i) For the Lipschitz nonlinear systems, the design of HGO requires the system to be uniformly observerable and the constant matrices $(\mathbf{A}, \mathbf{C})$ to be of the form (2.48), while the NLO only requires the pair $(\mathbf{A}, \mathbf{C})$ to be observable.
- (ii) The major difficulty of NLO is the gain design and it requires the Lipschitz constant to be small, whereas the HGO observer can be designed for any large Lipschitz constant by selecting large θ .
- (iii) HGO is very sensitive to output noise as the feedback gain values are high compared to NLO.

The selection of HGO or NLO will depend on the nonlinear process. If the nonlinearities and bounds are large, we may adopt the HGO method for the design of the robust observer. In the presence of output noise, a NLO may be preferred

over HGO due to less noise amplification. Also, HGO requires the system to be in the observable form which may not be satisfied for all nonlinear systems. In this thesis, we adopt both the HGO and NLO design techniques for the development of robust sliding mode observers for state and unknown input estimations. The comparative performance of the HGO and NLO are also discussed.

2.5 Summary

This chapter provides an introduction to the sliding mode theory and reviews the existing methods of sliding mode observer designs for linear and nonlinear systems in the presence of unknown inputs. Nonlinear Lipschitz observer (NLO) and the problems associated with its gain design for Lipschitz nonlinear systems are analyzed. Further, the design of high-gain observers (HGO) for a special class of uniformly observable systems that can handle any arbitrary Lipschitz constant is discussed. The nonlinear observer design procedures that are discussed in the last two sections are utilized in the later parts of thesis for the development of robust sliding mode(s) observer.

Part I

Continuous-Time Results

Chapter 3

State and Unknown Inputs

Estimations: A New Perspective

In this chapter, the analysis of sliding mode observers (SMO) for state and unknown inputs estimation through the perspective of observability concepts of unknown inputs w.r.t. output measurements will be discussed. The results are then used to propose a new SMO for a class of linear and nonlinear uncertain systems. Further, a fundamental insight into the design of sliding mode observers that only requires the output information is discussed.

3.1 Observability of Unknown Inputs

Essentially, we shall analyze the SMO from the perspective of estimating the unknown inputs/disturbances and ensuring that accurate state estimation is still achievable in the presence of unknown inputs/disturbances. The analysis is based on the detectability (observability) of disturbances or unknown inputs from the output measurements. The following example illustrates the main theme.

Example 3.1 Consider the following system with one state and one measurement.

$$\begin{aligned} \text{System: } \quad \dot{z} &= -az + f(z, d) \\ \text{Observer: } \quad \dot{\hat{z}} &= -a\hat{z} - l(\hat{z} - z) - \rho \text{sign}(\hat{z} - z) \end{aligned} \quad (3.1)$$

where $a > 0$, z is the state, $\hat{z}$ is the estimate, $f(z, d)$ is the unknown function and l is the feedback gain. The robust term $\rho \text{sign}(\hat{z} - z)$ in the observer could be viewed as a ‘tracking element’ for the uncertain function $f(z, d)$, since

$$(\rho \text{sign}(\hat{z} - z))_{eq} = f(z, d)$$

in the sliding mode of $e_z = \hat{z} - z = 0$. Once the system reaches the sliding surface, the robust term tracks the unknown function and hence it can be reconstructed from the sliding mode.

The problem of observability of the unknown inputs¹ is often discussed together with the design of SMO. When an unknown input is directly involved in the dynamics of the output derivative, i.e. if the unknown input is reconstructible from the equivalent control (see e.g. (3.1)), then the unknown input is considered to be observable² w.r.t. the output measurement. The following definition describes the observability of the unknown inputs.

Definition 3.1 The unknown inputs are said to be observable w.r.t. output measurements if they satisfy the conditions:

- (i) The unknown inputs are directly involved in the dynamics of the output derivative or the unknown inputs have relative degree one w.r.t. the output measurement.
- (ii) All the unknown inputs are uniquely reconstructible from the equivalent control (see e.g. (3.1)) of their respective sliding modes.

¹The observability of unknown inputs should not be confused with the observability concepts discussed in the earlier chapter (section 2.4).

²This implies that the unknown input can be reconstructed uniquely from the sliding mode.

The framework discussed in this chapter allows us to design the switching terms such that the unknown inputs are compensated by their corresponding robust terms. That is, there exists a robust term for every unknown input, such that the unknown inputs are reconstructible from their corresponding sliding modes. The observability of unknown inputs plays a crucial role in the development of SMO for accurate state estimation in the presence of unknown inputs. Our development clearly explains the notion of designing the observers to deal with the unknown inputs.

3.2 Sliding Mode Observer for Uncertain Linear Systems

So far, all the analyses of sliding mode observers with disturbances/unknown inputs [23], [63], [68], [1] have been in the framework of rendering the state-estimation insensitive to unknown inputs/disturbances, and are mainly limited to linear systems with a constant disturbance distribution matrix as discussed in the earlier chapter.

In this chapter, we shall design the SMOs from the perspective of observability of unknown inputs. The conditions under which a sliding mode observer exists are discussed and the observer design is applicable to a class of Lipschitz nonlinear systems. We shall show that the existence of a SMO is completely reliant on the observability of unknown inputs w.r.t. output measurements. Further, the robust terms or the switching terms are designed such that the unknown inputs are tracked by their robust terms and so can be reconstructed from the sliding mode.

In terms of the stability of the error dynamics, most analyses of the design of the

sliding mode gains [1,23,63] are based on the Lyapunov approach. The sliding mode gains and feedback gains are designed such that the error system is stable. Also the switching terms influence the states that are not effected by the unknown inputs. In the proposed method of observer design, the unknown inputs are structurally replaced by the robust terms, which also can be considered as the estimates of unknown inputs.

Here, we also emphasize the gain design using the outputs unaffected by disturbances to stabilize the reduced-order system. The main difference of the proposed method lies in the choice of the outputs to deal with the unknown inputs and to stabilize the system. The stability condition for the reduced order system is analyzed and the feedback gain is designed such that the system is stable. An application example to a robotic manipulator is presented to demonstrate the effectiveness of the proposed method.

3.2.1 System

We consider the uncertain system (2.9) but without the nonlinear vector field, i.e. $\Phi(\mathbf{x}, \mathbf{u}, t) = 0$ and is repeated here for convenience, i.e.

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{P}\mathbf{d}(\mathbf{x}, \mathbf{u}, t) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \quad (3.2)$$

where $\mathbf{x} \in \mathcal{R}^n$ is the state vector, $\mathbf{u} \in \mathcal{R}^q$ is the control input, and $\mathbf{y} \in \mathcal{R}^p$ is the system output, $\mathbf{d}(\mathbf{x}, \mathbf{u}, t) \in \mathcal{R}^m$ is the unknown input vector and assumed to be bounded as $\|\mathbf{d}(\mathbf{x}, \mathbf{u}, t)\| \leq \bar{d}$, $\forall \mathbf{x}, \mathbf{u}$ and $t > 0$. $\mathbf{A} \in \mathcal{R}^{n \times n}$, $\mathbf{B} \in \mathcal{R}^{n \times q}$, $\mathbf{P} \in \mathcal{R}^{n \times m}$ and $\mathbf{C} \in \mathcal{R}^{p \times n}$, where $m \leq p \leq n$.

To facilitate the observer design, we require the following assumptions:

Assumption 3.1 *The pair $(\mathbf{A}, \mathbf{C})$ is observable.*

Assumption 3.2 *The matrices $\mathbf{P} \in \mathcal{R}^{n \times m}$ and $\mathbf{C} \in \mathcal{R}^{p \times n}$ are full rank, i.e., $\text{rank}(\mathbf{CP})=m$.*

Assumption 3.2 is important and plays a crucial role in the complete observability of the unknown inputs. As will be shown in the later part of the chapter, the unknown input observer does not exist if $\text{rank}(\mathbf{CP}) < m$. The above condition is also a necessary condition for the existence of classical observers for linear systems with unknown inputs [86], [87].

Since $\mathbf{C}$ is a full rank matrix, without loss of generality, we assume that the first m rows (outputs) of matrix $\mathbf{C}$ form a full rank $m \times n$ matrix $\mathbf{C}_1$ such that

$$\mathbf{C}_1 = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \end{bmatrix}_{m \times n}, \text{ and } \text{rank}[\mathbf{C}_{11}]_{m \times m} = m \quad (3.3)$$

So $\text{rank}(\mathbf{C}_1)=m$ and $\text{rank}(\mathbf{C}_1\mathbf{P})=m$. With the partitioning of the output distribution matrix, we have

$$\mathbf{y} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} \mathbf{x}, \mathbf{y}_1 = \mathbf{C}_1 \mathbf{x} \quad (3.4)$$

3.2.2 Observer Design

For the system (3.2) satisfying **Assumptions 3.1-3.2**, the following form of sliding mode observer can effectively handle the disturbances/unknown inputs in the sliding mode:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) + \mathbf{P}\nu \quad (3.5)$$

where $\mathbf{L}$ is a constant feedback gain and its design will be discussed later, and ν is the sliding mode gain vector which is defined as

$$\nu = -(\mathbf{C}_1\mathbf{P})^{-1}\rho\text{sign}(\mathbf{e}_y) = -(\mathbf{C}_1\mathbf{P})^{-1}\rho\text{sign}(\mathbf{C}_1\hat{\mathbf{x}} - \mathbf{C}_1\mathbf{x}) \quad (3.6)$$

with $\mathbf{e}_y = \mathbf{C}_1 \hat{\mathbf{x}} - \mathbf{C}_1 \mathbf{x} = \begin{bmatrix} e_{y_1} & e_{y_2} & \cdots & e_{y_m} \end{bmatrix}$, ρ the sliding mode gain, and $\text{sign}(\cdot)$ defined component-wise. The selection of sliding mode gain ρ will be discussed in the next subsection, and it will be based on existence of the sliding mode.

Remark 3.1 *The observer defined by (3.5) and (3.6) copies the Luenberger observer, with disturbances replaced by their corresponding robust terms (3.6). The main aim is to show that state estimation will be totally insensitive to disturbances, if and only if there exists a robust term to track every unknown input. It is also clear that a linear observer (without robust terms) will be influenced by the effect of unknown inputs and the state estimation may become unstable.*

For the estimation error $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$, the error dynamics can be obtained from (3.2) and (3.5) as

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC}) \mathbf{e} + \mathbf{P}\nu - \mathbf{P}\mathbf{d} \quad (3.7)$$

3.2.3 Existence of Sliding Mode

In this section, we shall discuss the selection of sliding mode gain ρ to guarantee the existence of sliding mode. The main purpose of the robust term ν (3.6) in (3.5) is to deal with uncertain inputs or disturbances and to improve the estimation accuracy in the presence of unknown inputs. The procedure is enumerated as follows:

- (i) Define the following sliding surfaces,

$$\mathbf{e}_y = \mathbf{C}_1 \mathbf{e} = 0; \text{ i.e., } e_{y_1} = 0, e_{y_2} = 0, \dots, e_{y_m} = 0; \quad (3.8)$$

- (ii) Design the sliding mode estimation as (3.5) with ν given by (3.6) so that the system can reach and maintain the sliding mode.

- (iii) Ensure that the reduced order estimation error dynamics goes to zero on the sliding surfaces of $e_{y_1} = 0, e_{y_2} = 0, \dots, e_{y_m} = 0$.

In the following, we shall first prove the existence of sliding mode. Since $\mathbf{e}_y = \mathbf{C}_1 \mathbf{e}$, we have from (3.7) and (3.6), that

$$\dot{\mathbf{e}}_y = \mathbf{C}_1 \dot{\mathbf{e}} = \mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e} + \mathbf{C}_1 \mathbf{P} \nu - \mathbf{C}_1 \mathbf{P} \mathbf{d} \quad (3.9)$$

$$\begin{aligned} &= \mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e} - \mathbf{C}_1 \mathbf{P} (\mathbf{C}_1 \mathbf{P})^{-1} \rho \text{sign}(\mathbf{e}_y) - \mathbf{C}_1 \mathbf{P} \mathbf{d} \\ &= \mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e} - \rho \text{sign}(\mathbf{e}_y) - \mathbf{C}_1 \mathbf{P} \mathbf{d} \end{aligned} \quad (3.10)$$

With the Lyapunov function $V = \frac{1}{2} \mathbf{e}_y^T \mathbf{e}_y$, its derivative is given by

$$\dot{V} = \mathbf{e}_y^T \dot{\mathbf{e}}_y = \mathbf{e}_y^T [\mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e} - \mathbf{C}_1 \mathbf{P} \mathbf{d}] - \rho \|\mathbf{e}_y\| \quad (3.11)$$

If the sliding mode gain is chosen such that

$$\rho > \|\mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e} - \mathbf{C}_1 \mathbf{P} \mathbf{d}\| \quad (3.12)$$

then we will have $\dot{V} < 0$. Hence the subdynamics $\mathbf{C}_1 \mathbf{e}$ reaches the sliding mode in finite time and stays thereafter. In fact, condition (3.12) can be relaxed by separating the $\mathbf{e}_y$ dynamics from $\mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e}$ as will be shown in the analysis in the transformed domain.

In comparison with the works of [23], [1], the proposed method utilizes only the measurable estimation errors as the sliding surfaces.

3.2.4 Error Dynamics in the Sliding Mode

Since the states of subsystem $\mathbf{C}_1 \mathbf{x}$ are measured as outputs, the error dynamics (3.9) reach their corresponding sliding surfaces (3.8) in finite time, so in the sliding mode, we have $\mathbf{e}_y = 0$, i.e., $\mathbf{C}_1 \hat{\mathbf{x}} = \mathbf{C}_1 \mathbf{x}$. Therefore in the sliding mode of $\mathbf{e}_y = 0$,

by the virtue of equivalent control [56], ν_{eq} can be obtained from (3.9) as follows:

$$0 = \mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e}_d + \mathbf{C}_1 \mathbf{P} \nu_{eq} - \mathbf{C}_1 \mathbf{P} \mathbf{d} \quad (3.13)$$

Therefore,

$$\nu_{eq} = \mathbf{d} - (\mathbf{C}_1 \mathbf{P})^{-1} \mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e}_d \quad (3.14)$$

where the subscript $_d$ denotes the reduced order error dynamics in the sliding mode of $\mathbf{e}_y = 0$. From the above, it is clear that the equivalent control is directly related to the unknown input dynamics. The equivalent control information can be used to reconstruct the unknown inputs from the sliding mode, as will be discussed in later sections. In the following, we shall show that the error dynamics will be free from unknown inputs in the sliding mode.

Substituting (3.14) into the system error dynamics (3.7), we have

$$\begin{aligned} \dot{\mathbf{e}}_d &= (\mathbf{A} - \mathbf{LC}) \mathbf{e}_d + \mathbf{P} [\mathbf{d} - (\mathbf{C}_1 \mathbf{P})^{-1} \mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e}_d] - \mathbf{P} \mathbf{d} \\ &= [\mathbf{A} - \mathbf{LC} - \mathbf{P}(\mathbf{C}_1 \mathbf{P})^{-1} \mathbf{C}_1 (\mathbf{A} - \mathbf{LC})] \mathbf{e}_d \end{aligned} \quad (3.15)$$

$$= [\mathbf{I} - \mathbf{P}(\mathbf{C}_1 \mathbf{P})^{-1} \mathbf{C}_1] (\mathbf{A} - \mathbf{LC}) \mathbf{e}_d \quad (3.16)$$

Provided the matrix $[\mathbf{A} - \mathbf{LC} - \mathbf{P}(\mathbf{C}_1 \mathbf{P})^{-1} \mathbf{C}_1 (\mathbf{A} - \mathbf{LC})]$ is stable for a suitable choice of feedback gain $\mathbf{L}$, the observer system converges to the actual states asymptotically.

3.2.5 Stability of the Reduced-order Error System in the Sliding Mode

The stability of the reduced order dynamics can be analyzed in the transformed domain. From **Assumption 3.2** and (3.3), we have $\det(\mathbf{C}_{11}) \neq 0$, then there

exists a non-singular transformation

$$\mathbf{T} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ 0 & \mathbf{I}_{n-m} \end{bmatrix} \quad (3.17)$$

such that the system in the new co-ordinates $\tilde{\mathbf{x}} = \mathbf{T}\mathbf{x}$, with $\tilde{\mathbf{x}} = [\tilde{\mathbf{x}}_1^T, \tilde{\mathbf{x}}_2^T]^T$, can be obtained as

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}_1 &= \tilde{\mathbf{A}}_{11}\tilde{\mathbf{x}}_1 + \tilde{\mathbf{A}}_{12}\tilde{\mathbf{x}}_2 + \tilde{\mathbf{B}}_1u + \tilde{\mathbf{P}}_1\mathbf{d} \\ \dot{\tilde{\mathbf{x}}}_2 &= \tilde{\mathbf{A}}_{21}\tilde{\mathbf{x}}_1 + \tilde{\mathbf{A}}_{22}\tilde{\mathbf{x}}_2 + \tilde{\mathbf{B}}_2u + \tilde{\mathbf{P}}_2\mathbf{d} \end{aligned} \quad (3.18)$$

where the transformed matrices are:

$$\begin{aligned} \mathbf{TAT}^{-1} = \tilde{\mathbf{A}} &= \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{22} \end{bmatrix}, \quad \mathbf{TB} = \tilde{\mathbf{B}} = \begin{bmatrix} \tilde{\mathbf{B}}_1 \\ \tilde{\mathbf{B}}_2 \end{bmatrix} \\ \mathbf{TP} = \tilde{\mathbf{P}} &= \begin{bmatrix} \tilde{\mathbf{P}}_1 \\ \tilde{\mathbf{P}}_2 \end{bmatrix}, \quad \mathbf{C}_1\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{I}_m & 0 \end{bmatrix} \end{aligned} \quad (3.19)$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} \mathbf{T}^{-1}\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{y}_1 = \tilde{\mathbf{x}}_1 \quad (3.20)$$

Now, we have $(\mathbf{C}_1\mathbf{P}) = \tilde{\mathbf{P}}_1$. The observer according to (3.5) in the transformed domain will be of the form:

$$\dot{\hat{\tilde{\mathbf{x}}}} = \tilde{\mathbf{A}}\hat{\tilde{\mathbf{x}}} + \tilde{\mathbf{B}}u - \mathbf{H}\tilde{\mathbf{e}} + \tilde{\mathbf{P}}\nu, \quad \nu = -\tilde{\mathbf{P}}_1^{-1}\rho\text{sign}(\mathbf{e}_y) \quad (3.21)$$

where $\tilde{\mathbf{e}} = \begin{bmatrix} \tilde{\mathbf{e}}_1 & \tilde{\mathbf{e}}_2 \end{bmatrix}^T = \begin{bmatrix} \hat{\tilde{\mathbf{x}}}_1 - \tilde{\mathbf{x}}_1 & \hat{\tilde{\mathbf{x}}}_2 - \tilde{\mathbf{x}}_2 \end{bmatrix}^T$ and $\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix}$ is the feedback gain matrix in the transformed domain. Note that $\tilde{\mathbf{e}} = \mathbf{T}\mathbf{e} = \begin{bmatrix} \mathbf{e}_y & \tilde{\mathbf{e}}_2 \end{bmatrix}^T$, i.e. $\tilde{\mathbf{e}}_1 = \mathbf{e}_y$.

The error dynamics can be obtained from (3.18) and (3.21) as

$$\begin{aligned}\dot{\mathbf{e}}_y &= \left(\tilde{\mathbf{A}}_{11} - \mathbf{H}_{11}\right) \mathbf{e}_y + \left(\tilde{\mathbf{A}}_{12} - \mathbf{H}_{12}\right) \tilde{\mathbf{e}}_2 - \rho \text{sign}(\mathbf{e}_y) - \tilde{\mathbf{P}}_1 \mathbf{d} \\ \dot{\tilde{\mathbf{e}}}_2 &= \left(\tilde{\mathbf{A}}_{21} - \mathbf{H}_{21}\right) \mathbf{e}_y + \left(\tilde{\mathbf{A}}_{22} - \mathbf{H}_{22}\right) \tilde{\mathbf{e}}_2 + \tilde{\mathbf{P}}_2 \nu - \tilde{\mathbf{P}}_2 \mathbf{d}\end{aligned}\quad (3.22)$$

Since $\mathbf{H}_{11}$ is a full rank square matrix, one can arbitrarily choose high diagonal entries for $\mathbf{H}_{11}$ such that $\left(\tilde{\mathbf{A}}_{11} - \mathbf{H}_{11}\right)$ is negative definite, therefore by selecting

$$\rho > \left\| \left(\tilde{\mathbf{A}}_{12} - \mathbf{H}_{12}\right) \tilde{\mathbf{e}}_2 - \tilde{\mathbf{P}}_1 \mathbf{d} \right\| \quad (3.23)$$

and similar to the analysis in (3.11), the sliding mode can be guaranteed in finite time. In the sliding mode, we have $\mathbf{e}_y = 0$, $\dot{\mathbf{e}}_y = 0$. The virtual equivalent control signal can therefore be obtained as

$$\nu_{eq} = \mathbf{d} - \tilde{\mathbf{P}}_1^{-1} \left(\tilde{\mathbf{A}}_{21} - \mathbf{H}_{21}\right) \tilde{\mathbf{e}}_2 \quad (3.24)$$

The reduced order error system is obtained by substituting (3.24) into (3.22):

$$\begin{aligned}\dot{\tilde{\mathbf{e}}}_2 &= \left(\tilde{\mathbf{A}}_{22} - \mathbf{H}_{22}\right) \tilde{\mathbf{e}}_2 - \tilde{\mathbf{P}}_2 \tilde{\mathbf{P}}_1^{-1} \left(\tilde{\mathbf{A}}_{12} - \mathbf{H}_{12}\right) \tilde{\mathbf{e}}_2 \\ &= \left[\left(\tilde{\mathbf{A}}_{22} - \tilde{\mathbf{P}}_2 \tilde{\mathbf{P}}_1^{-1} \tilde{\mathbf{A}}_{12}\right) - \left(\mathbf{H}_{22} - \tilde{\mathbf{P}}_2 \tilde{\mathbf{P}}_1^{-1} \mathbf{H}_{12}\right) \right] \tilde{\mathbf{e}}_2\end{aligned}\quad (3.25)$$

If one can design the feedback gain matrix $\mathbf{H}$ such that the above system is stable, the reduced order error system will converge asymptotically to zero in the sliding mode.

- (i) For the case $p > m$, and $\text{rank}(\mathbf{C})=p$, the stability of the reduced order system relies on the design of the feedback gain $\left(\mathbf{H}_{22} - \tilde{\mathbf{P}}_2 \tilde{\mathbf{P}}_1^{-1} \mathbf{H}_{12}\right)$ such that the reduced order system is stable with the remaining $p - m$ measurements.
- (ii) If $p = m$, then we have $\mathbf{H}_{22} = 0$, and $\mathbf{H}_{12} = 0$, therefore the reduced order system itself should be stable. The reduced order matrix $\tilde{\mathbf{A}}_{22} - \tilde{\mathbf{P}}_2 \tilde{\mathbf{P}}_1^{-1} \tilde{\mathbf{A}}_{12}$ is stable if and only if the invariant zeros of the matrices $\left(\tilde{\mathbf{A}}, \tilde{\mathbf{P}}, \mathbf{C}\right)$ are in $\mathbb{C}_-$ [68].

The major difference of the proposed method from the method of [68], [82] lies in the choice of the robust terms to deal with unknown inputs. The method in [68] uses all the output information to deal with unknown inputs and so it requires the reduced order system itself to be stable in the sliding mode. In our proposed method, we require the robust terms to “reconstruct” all the unknown inputs from the sliding mode. The remaining outputs (i.e. $\mathbf{y}_2$) could be used to stabilize the reduced order system in the sliding mode.

Remark 3.2

- (i) *An unknown input or disturbance can only be rejected only if it is uniquely identifiable through the output measurements. In other words, the unknown inputs are observable through the output measurements, i.e., $\text{rank}(\mathbf{C}_1\mathbf{P}) = m$. From (3.9) and (3.10) it is clear that, unless the $\text{rank}(\mathbf{C}_1\mathbf{P}) = m$, the robust term ν cannot be designed and sliding mode cannot be guaranteed.*
- (ii) *If there exists a robust term for every unknown input such that the unknown inputs can be reconstructed from the sliding mode, then a disturbance free observer error system exists in the sliding mode. In our analysis, the number of robust terms is equal to the number of unknown inputs. Also, the number of independent unknown inputs (i.e. m) that the robust observer can identify uniquely and reconstruct is less than or equal to the number of outputs (i.e. p) or the dimension of the system (i.e. n) if all the states are measurable, i.e., $m \leq p \leq n$. The case where all the outputs are measurable was studied in [83] for fault detection applications.*
- (iii) *This development does not require the so-called matching condition [56] for the design of SMO. So the switching terms can be designed such that the unknown inputs are compensated by their corresponding robust terms, thereby avoiding unnecessary effect on the states that are free from unknown inputs.*

(iv) If $\text{rank}(\mathbf{C}_1\mathbf{P}) = p < m$, then the proposed observer can handle only p unknown inputs, and the remaining unknown inputs (i.e. $m - p$) will influence the reduced-order system in the sliding mode. The error system can be shown to be global asymptotically bounded, and it will depend on the bounds of unknown inputs that are not observable through output measurements.

3.2.6 Unknown Input Estimation

Once the trajectories reaches the sliding mode, and provided the reduced order system (3.15) is designed to be stable, all the system states converge to the actual states. After all the states have converged, we have

$$\mathbf{e} \rightarrow 0, \quad \Rightarrow \quad [(\mathbf{C}_1\mathbf{P})^{-1}\mathbf{C}_1(\mathbf{A} - \mathbf{LC})] \mathbf{e} \rightarrow 0$$

Hence, the equivalent control information can be used to reconstruct the unknown inputs from the sliding mode. The use of a low-pass filter for recovering the equivalent control signal was given by [56]. Continuous approximation of equivalent injection signal by using a small positive scalar δ was also implemented in [82]. Similar to the analysis of [82], in our approach, the unknown inputs are estimated from the equivalent control as follows:

$$\hat{\mathbf{d}} \approx ((\mathbf{C}_1\mathbf{P})^{-1}\rho \text{sign}(\mathbf{e}_y))_{eq} = -(\mathbf{C}_1\mathbf{P})^{-1}\rho \begin{bmatrix} \frac{e_{y1}}{|e_{y1}|+\delta} \\ \frac{e_{y2}}{|e_{y2}|+\delta} \\ \vdots \\ \frac{e_{ym}}{|e_{ym}|+\delta} \end{bmatrix} \quad (3.26)$$

The accuracy of the disturbance estimation will depend on δ . The estimation depends only on measurement error $\mathbf{e}_y$ and hence can be performed online with state estimation.

3.3 Sliding Mode Observer Design for Lipschitz Nonlinear Systems

In this section, we extend the earlier design of sliding mode observer to a class of Lipschitz nonlinear systems. Consider nonlinear systems of the following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \Phi(\mathbf{x}, \mathbf{u}) + \mathbf{P}(\mathbf{x}, \mathbf{u})\mathbf{d}(\mathbf{x}, \mathbf{u}, t) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \quad (3.27)$$

where $\mathbf{A} \in \mathcal{R}^{n \times n}$, $\mathbf{C} \in \mathcal{R}^{p \times n}$, with $m < p \leq n$, $\mathbf{x} \in \mathcal{M}$ is the state vector, $u \in \mathcal{R}^q$ is the control input, $\mathbf{y} \in \mathcal{R}^p$ is the system output, $\mathbf{d}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} d_1 & d_2 & \cdots & d_m \end{bmatrix} \in \mathcal{R}^m$ is the unknown input vector and is assumed to be bounded as $\|\mathbf{d}(\mathbf{x}, \mathbf{u}, t)\| \leq \bar{d}$, $\forall \mathbf{x}, u$ and $t > 0$, $\Phi(\mathbf{x}, \mathbf{u})$ is a real valued vector field on $\mathcal{R}^{n+q}$, and $\mathbf{P}(\mathbf{x}, \mathbf{u})$ is an $n \times m$ -matrix with real valued functions on $\mathcal{R}^{n+q}$. Further, without loss of generality, $\mathbf{C}$ is partitioned as in (3.3).

We assume the system (3.27) satisfies **Assumption 3.1**. We also need the following assumptions for nonlinear systems.

Assumption 3.3 *The known functions $\Phi(\mathbf{x}, \mathbf{u})$ and $\mathbf{P}(\mathbf{x}, \mathbf{u})$ are bounded with respect to their arguments. The inputs of the nonlinear system (3.27) are bounded with some upper bounds. Further, the system (3.27) is Bounded-Input-Bounded-States (BIBS) stable.*

Due to the nonlinear terms in the matrix $\mathbf{P}$, the results for linear systems cannot be directly extended to the nonlinear case. The following assumptions on the unknown input distribution matrix are the key requirement for the development of a SMO that can handle the unknown inputs in the sliding mode.

Assumption 3.4 *For the $\mathbf{C}_1$ output distribution matrix formed by choosing m outputs of the available p outputs, the inverse of the nonlinear matrix $\mathbf{C}_1\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u})$, exists $\forall \hat{\mathbf{x}} \in \mathcal{R}^n$ and $u \in \mathcal{R}^q$.*

Assumption 3.5 *The nonlinear functions $\Phi(\mathbf{x}, \mathbf{u})$ and $\mathbf{P}(\mathbf{x}, \mathbf{u})$ satisfy the Lipschitz conditions such that*

$$\|\Phi(\hat{\mathbf{x}}, \mathbf{u}) - \Phi(\mathbf{x}, \mathbf{u})\| \leq l_\Phi \|\hat{\mathbf{x}} - \mathbf{x}\| \quad (3.28)$$

$$\|\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}) (\mathbf{C}_1 \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1} - \mathbf{P}(\mathbf{x}, \mathbf{u}) (\mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u}))^{-1}\| \leq l_\gamma \|\hat{\mathbf{x}} - \mathbf{x}\| \quad (3.29)$$

for some Lipschitz constants l_Φ and l_γ .

3.3.1 Observer Design

For the system (3.27) satisfying **Assumptions 3.1, 3.4** and **3.5**, an observer of the following form can be designed to estimate the states:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \Phi(\hat{\mathbf{x}}, \mathbf{u}) + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) + \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u})\mathbf{u}_r(t) \quad (3.30)$$

where $\mathbf{L}$ is the feedback gain to be discussed later, and $\mathbf{u}_r(t)$ is the robust term given by the sliding mode estimation:

$$\mathbf{u}_r(t) = -(\mathbf{C}_1 \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1} \rho \text{sign}(\mathbf{C}_1 \hat{\mathbf{x}} - \mathbf{C}_1 \mathbf{x}) \quad (3.31)$$

The $\mathbf{u}_r(t)$ can also be termed as the unknown input or disturbance tracker.

Remark 3.3 *In the above observer design, the disturbance inputs of the system are exactly replaced by robust terms. The number of robust terms in $\mathbf{u}_r(t)$ is equal to the number of unknown inputs d_i .*

By defining the estimation error or residual as $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, the estimation error dynamics can be computed from (3.27) and (3.30) as

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e} + \Phi(\hat{\mathbf{x}}, \mathbf{u}) - \Phi(\mathbf{x}, \mathbf{u}) + \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u})\mathbf{u}_r(t) - \mathbf{P}(\mathbf{x}, \mathbf{u})\mathbf{d} \quad (3.32)$$

3.3.2 Existence of Sliding Mode and Gain Design

By defining $\mathbf{e}_y = \mathbf{C}_1 \mathbf{e}$, we have

$$\begin{aligned} \dot{\mathbf{e}}_y &= \mathbf{C}_1 \dot{\mathbf{e}} = \mathbf{C}_1 (\mathbf{A} - \mathbf{L}\mathbf{C}) + \mathbf{C}_1 \Phi(\hat{\mathbf{x}}, \mathbf{u}) - \mathbf{C}_1 \Phi(\mathbf{x}, \mathbf{u}) + \mathbf{C}_1 \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}) \mathbf{u}_r(t) \\ &\quad - \mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u}) \mathbf{d} \end{aligned} \quad (3.33)$$

Substituting (3.31) into (3.33) we obtain

$$\begin{aligned} \dot{\mathbf{e}}_y &= \mathbf{C}_1 (\mathbf{A} - \mathbf{L}\mathbf{C}) \mathbf{e} + \mathbf{C}_1 \Phi(\hat{\mathbf{x}}, \mathbf{u}) - \mathbf{C}_1 \Phi(\mathbf{x}, \mathbf{u}) - \rho \text{sign}(\mathbf{e}_y) \\ &\quad - \mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u}) \mathbf{d} \end{aligned} \quad (3.34)$$

To analyze whether that the sliding mode gain can guarantee the sliding mode, we transform the error dynamics (3.34) with the transformation (3.17). Similar to the development in (3.22), we have

$$\begin{aligned} \dot{\mathbf{e}}_y &= \left(\tilde{\mathbf{A}}_{11} - \mathbf{H}_{11} \right) \mathbf{e}_y + \left(\tilde{\mathbf{A}}_{12} - \mathbf{H}_{12} \right) \tilde{\mathbf{e}}_2 + \Gamma_1(\hat{\mathbf{x}}, \mathbf{u}) - \Gamma_1(\mathbf{x}, \mathbf{u}) \\ &\quad - \rho \text{sign}(\mathbf{e}_y) - \tilde{\mathbf{P}}_1(\hat{\mathbf{x}}, \mathbf{u}) \mathbf{d} \end{aligned} \quad (3.35)$$

$$\begin{aligned} \dot{\tilde{\mathbf{e}}}_2 &= \left(\tilde{\mathbf{A}}_{21} - \mathbf{H}_{21} \right) \mathbf{e}_y + \left(\tilde{\mathbf{A}}_{22} - \mathbf{H}_{22} \right) \tilde{\mathbf{e}}_2 + \Gamma_2(\hat{\mathbf{x}}, \mathbf{u}) - \Gamma_2(\mathbf{x}, \mathbf{u}) \\ &\quad + \tilde{\mathbf{P}}_2(\hat{\mathbf{x}}, \mathbf{u}) \nu - \tilde{\mathbf{P}}_2(\mathbf{x}, \mathbf{u}) \mathbf{d} \end{aligned} \quad (3.36)$$

where the transformed matrices are

$$\mathbf{T}\Phi(\mathbf{x}, \mathbf{u}) = \Gamma(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \Gamma_1(\mathbf{x}, \mathbf{u}) \\ \Gamma_2(\mathbf{x}, \mathbf{u}) \end{bmatrix}, \quad \mathbf{T}\mathbf{P}(\mathbf{x}, \mathbf{u}) = \tilde{\mathbf{P}}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \tilde{\mathbf{P}}_1(\mathbf{x}, \mathbf{u}) \\ \tilde{\mathbf{P}}_2(\mathbf{x}, \mathbf{u}) \end{bmatrix}$$

Consider the Lyapunov function $\mathbf{V}_2 = \frac{1}{2} \mathbf{e}_y^T \mathbf{e}_y$, differentiating w.r.t. time and substituting (3.36), we have

$$\begin{aligned} \mathbf{e}_y^T \dot{\mathbf{e}}_y &= \mathbf{e}_y^T \left[\left(\tilde{\mathbf{A}}_{12} - \mathbf{H}_{12} \right) \tilde{\mathbf{e}}_2 - \tilde{\mathbf{P}}_1(\mathbf{x}, \mathbf{u}) \mathbf{d} + \Gamma_1(\hat{\mathbf{x}}, \mathbf{u}) - \Gamma_1(\mathbf{x}, \mathbf{u}) \right] \\ &\quad + \mathbf{e}_y^T \left(\tilde{\mathbf{A}}_{11} - \mathbf{H}_{11} \right) \mathbf{e}_y - \rho \|\mathbf{e}_y\| \end{aligned} \quad (3.37)$$

By selecting

$$\rho > \left\| \left(\tilde{\mathbf{A}}_{12} - \mathbf{H}_{12} \right) \tilde{\mathbf{e}}_2 - \tilde{\mathbf{P}}_1(\mathbf{x}, \mathbf{u})\mathbf{d} + \Gamma_1(\hat{\mathbf{x}}, \mathbf{u}) - \Gamma_1(\mathbf{x}, \mathbf{u}) \right\| \quad (3.38)$$

we get

$$\mathbf{e}_y^T \dot{\mathbf{e}}_y < \mathbf{e}_y^T \left(\tilde{\mathbf{A}}_{11} - \mathbf{H}_{11} \right) \mathbf{e}_y$$

Since $\mathbf{H}_{11}$ is full rank, we can select $\mathbf{H}_{11}$ such that $\left(\tilde{\mathbf{A}}_{11} - \mathbf{H}_{11} \right) = -\mathbf{Q}_1$, where $\mathbf{Q}_1 > 0$ is positive definite. Therefore

$$\dot{\mathbf{V}}_2 < -\mathbf{e}_y^T \mathbf{Q}_1 \mathbf{e}_y < 0$$

Hence the sliding mode can be attained in finite time.

3.3.3 Convergence of Error Dynamics in the Sliding Mode

The error dynamics in the sliding mode can be obtained by substituting the virtual equivalent control signal in the error dynamics (3.32). In the sliding mode of $\mathbf{e}_y = 0$, we have $\dot{\mathbf{e}}_y = 0$. Therefore, the virtual equivalent control of $\mathbf{u}_r(t)$, $\mathbf{u}_{eq}$ can be obtained from (3.33) according to [56] as follows:

$$\begin{aligned} 0 &= \mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e} + \mathbf{C}_1 \Phi(\hat{\mathbf{x}}, \mathbf{u}) - \mathbf{C}_1 \Phi(\mathbf{x}, \mathbf{u}) + \mathbf{C}_1 \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}) \mathbf{u}_{eq} \\ &\quad - \mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u}) \mathbf{d} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{u}_{eq} &= (\mathbf{C}_1 \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1} \mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u}) \mathbf{d} \\ &\quad - (\mathbf{C}_1 \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1} [\mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e} + \mathbf{C}_1 \Phi(\hat{\mathbf{x}}, \mathbf{u}) - \mathbf{C}_1 \Phi(\mathbf{x}, \mathbf{u})] \quad (3.39) \end{aligned}$$

By substituting the equivalent control signal into the error dynamics (3.32), we have the following estimation error dynamics in the sliding mode of $\mathbf{e}_y = 0$:

$$\begin{aligned}\dot{\mathbf{e}} &= (\mathbf{A} - \mathbf{LC})\mathbf{e} + \Phi(\hat{\mathbf{x}}, \mathbf{u}) - \Phi(\mathbf{x}, \mathbf{u}) \\ &\quad + \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u})(\mathbf{C}_1\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1}\mathbf{C}_1\mathbf{P}(\mathbf{x}, \mathbf{u})\mathbf{d} - \mathbf{P}(\mathbf{x}, \mathbf{u})\mathbf{d} \\ &\quad - \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u})(\mathbf{C}_1\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1}[\mathbf{C}_1(\mathbf{A} - \mathbf{LC})\mathbf{e} + \mathbf{C}_1\Phi(\hat{\mathbf{x}}, \mathbf{u}) - \mathbf{C}_1\Phi(\mathbf{x}, \mathbf{u})]\end{aligned}\quad (3.40)$$

From the above error dynamics, it is clear that the disturbance becomes an increment of the Lipschitz function and so can be dealt with by proper design of the feedback gain.

Lemma 3.1 *For the system (3.27) satisfying Assumptions 3.1 and 3.4 and using the estimator (3.30), the sliding mode gain (3.38) ensures that the estimation error is asymptotically stable in the multiple sliding mode of $e_{y_1} = 0, \dots, e_{y_m} = 0$ provided the gain $\mathbf{L}$ satisfies*

$$(\mathbf{A} - \mathbf{LC})^T \mathbf{Q}_2 + \mathbf{Q}_2 (\mathbf{A} - \mathbf{LC}) = -\mathbf{I} \quad (3.41)$$

$$\lambda_{\max}(\mathbf{Q}_2) \leq \frac{1}{2} [b_{\gamma_3} + l_\phi + b_{\gamma_1}l_\phi + b_{\gamma_2}l_\gamma\bar{d}]^{-1} \quad (3.42)$$

for Lipschitz constants l_γ , l_ϕ and some upper bounds b_{γ_1} , b_{γ_2} , b_{γ_3} and $\bar{d}$.

Proof:

Since the known functions are bounded according to Assumption 3.3, we have

$$\begin{aligned}\|\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u})(\mathbf{C}_1\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1}\mathbf{C}_1\| &\leq b_{\gamma_1}, \quad \|\mathbf{C}_1\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u})\| \leq b_{\gamma_2} \\ \|\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u})(\mathbf{C}_1\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1}\mathbf{C}_1(\mathbf{A} - \mathbf{LC})\| &\leq b_{\gamma_3}\end{aligned}\quad (3.43)$$

for some upper bounds b_{γ_1} , b_{γ_2} and b_{γ_3} .

Now, we choose the Lyapunov function candidate as $V_1 = \mathbf{e}^T \mathbf{Q}_2 \mathbf{e}$. Differentiating with respect to time, and using (3.40), we have

$$\begin{aligned}\dot{V}_1 &= \mathbf{e}^T \mathbf{Q}_2 \dot{\mathbf{e}} + \dot{\mathbf{e}}^T \mathbf{Q}_2 \mathbf{e}\end{aligned}$$

$$\begin{aligned}
&= \mathbf{e}^T \left[(\mathbf{A} - \mathbf{LC})^T \mathbf{Q}_2 + \mathbf{Q}_2 (\mathbf{A} - \mathbf{LC}) \right] \mathbf{e} + 2\mathbf{e}^T \mathbf{Q}_2 [\Phi(\hat{\mathbf{x}}, \mathbf{u}) - \Phi(\mathbf{x}, \mathbf{u})] \\
&\quad - 2\mathbf{e}^T \mathbf{Q}_2 [\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u})(\mathbf{C}_1 \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1} [\mathbf{C}_1 (\mathbf{A} - \mathbf{LC}) \mathbf{e} + \mathbf{C}_1 \Phi(\hat{\mathbf{x}}, \mathbf{u}) - \mathbf{C}_1 \Phi(\mathbf{x}, \mathbf{u})]] \\
&\quad + 2\mathbf{e}^T \mathbf{Q}_2 \left\{ \left[\mathbf{P}(\hat{\mathbf{x}}, \mathbf{u})(\mathbf{C}_1 \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1} - \mathbf{P}(\mathbf{x}, \mathbf{u})(\mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u}))^{-1} \right] \mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u}) \mathbf{d} \right\}
\end{aligned}$$

Using the results (3.28) and (3.29) of **Assumption 3.5**, and (3.43), we have

$$\dot{V}_1 \leq -\|\mathbf{e}\|^2 + 2\lambda_{\max}(\mathbf{Q}_2) [b_{\gamma_3} + l_\phi + b_{\gamma_1} l_\phi + b_{\gamma_2} l_\gamma \bar{d}] \|\mathbf{e}\|^2$$

If the estimation gain $\mathbf{L}$ is designed such that conditions (3.41) and (3.42) are satisfied, then the error dynamics in the sliding mode will be asymptotically stable, and rejecting all the disturbances in the sliding mode. ■

Remark 3.4 *If the unknown input distribution matrix $\mathbf{P}$ which is a constant matrix or just involves functions of outputs, i.e., $\mathbf{P}(\mathbf{x}, \mathbf{u}) = \mathbf{P}(\mathbf{y}, \mathbf{u})$, then the reduced order error dynamics will be completely free from unknown inputs. The conditions that will guarantee the stability are reduced to the form:*

$$(\mathbf{A} - \mathbf{LC})^T \mathbf{Q}_2 + \mathbf{Q}_2 (\mathbf{A} - \mathbf{LC}) = -\mathbf{I} \quad (3.44)$$

$$\lambda_{\max}(\mathbf{Q}_2) \leq \frac{1}{2} [b_{\gamma_3} + l_\phi + b_{\gamma_1} l_\phi]^{-1} \quad (3.45)$$

The reduced order error dynamics of the proposed observer in the sliding mode is free from disturbances and the stability is governed by the feedback gain to deal with the nonlinear Lipschitz function. The feedback gain design is a standard problem of gain design of the Lipschitz nonlinear system discussed in Section 2.3.

3.3.4 A Special Case of SMO Design

Assumption 3.4 is crucial for the existence of the proposed SMO, but it requires the computation of the inverse of $\mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u})$. If there exists the following lower-

triangular structure:

$$\mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} p_{11}^c(\mathbf{x}, \mathbf{u}) & 0 & \cdots & 0 \\ p_{21}^c(\mathbf{x}, \mathbf{u}) & p_{22}^c(\mathbf{x}, \mathbf{u}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}^c(\mathbf{x}, \mathbf{u}) & p_{m2}^c(\mathbf{x}, \mathbf{u}) & \cdots & p_{mm}^c(\mathbf{x}, \mathbf{u}) \end{bmatrix} \quad (3.46)$$

where $p_{ii}^c(\mathbf{x}, \mathbf{u}) \neq 0, \forall, i = 1, \dots, m$, then we can design the sliding mode observer (3.30) directly without the evaluation of $(\mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u}))^{-1}$. It is clear that the condition (3.46) is a special case of **Assumption 3.4**. Also, the reconstruction of unknown inputs will be easier with the lower triangular structure in (3.46).

In this case, the sliding mode gain $\mathbf{u}_r(t)$ of (3.31) of the proposed observer (3.30) can be designed as

$$\mathbf{u}_r(t) = - \begin{bmatrix} \frac{\rho_1}{p_{11}^c(\hat{\mathbf{x}}, \mathbf{u})} \text{sign}(e_{y_1}) \\ \frac{\rho_2}{p_{22}^c(\hat{\mathbf{x}}, \mathbf{u})} \text{sign}(e_{y_2}) \\ \vdots \\ \frac{\rho_m}{p_{mm}^c(\hat{\mathbf{x}}, \mathbf{u})} \text{sign}(e_{y_m}) \end{bmatrix} \quad (3.47)$$

where $\mathbf{e}_y = \mathbf{C}_1 \hat{\mathbf{x}} - \mathbf{C}_1 \mathbf{x} = \begin{bmatrix} e_{y_1} & e_{y_2} & \cdots & e_{y_m} \end{bmatrix}^T$ and ρ_i is designed individually. The lower triangular structure guarantees the existence of sliding mode for the error dynamics $\mathbf{e}_y$ and can be proven readily. A multiple sliding mode observer for a general class of nonlinear systems based on the lower-triangular structure (3.46) will be discussed in Chapter 5.

3.3.5 Design of Gain L

The gain should be designed such that it meets the two conditions (3.41) and (3.42). If there are no disturbances in the system, the gain design of the system (3.40) will be similar to the known result of Thau's observer (2.37).

The gain is to be designed such that the distance to unobservability (see Section

2.3 and [131]) of the linear matrix $(\mathbf{A} - \mathbf{LC})$ is greater than the Lipschitz constants and bounded functions (right hand side of inequality in (3.42)) in order to achieve stability for the system.

From the gain design point of view, the problem is similar to the problem discussed in (2.43) and (2.44), except that the development is for general nonlinear systems with multiple sliding modes and involves the Lipschitz constants as well as bounds resulting from nonlinear sliding mode observer.

The result presented in (3.42) can be conservative as the bounds can be large. If the unknown input distribution matrix $\mathbf{P}$ is a constant matrix or just functions of the outputs, then the conditions (3.41) and (3.42) reduce to the form of (3.44) and (3.45).

In the absence of unknown inputs, i.e., if all the inputs are known, then condition (3.42) becomes

$$\lambda_{\max}(\mathbf{Q}_2) \leq \frac{1}{2l_\phi} \quad (3.48)$$

The above condition together with (3.41) is exactly the same as that proposed in Thau's observer (2.35). The presence of disturbances inherently includes the constraints into the system. One may use the procedures adopted by [1], [53] for the gain design under similar conditions as discussed in Chapter 2.

Remark 3.5 *The drawback of the proposed design methodology is that, the feedback gain design depends on the Lipschitz constants and bounds which cannot be large. The gain design is only feasible for the smaller bounds and Lipschitz constants.*

3.3.6 Unknown Input Estimation

Once the reduced order error system (3.40) converges to zero, the equivalent control $\mathbf{u}_{eq}$ will approach the unknown inputs. After all the states are converged, we have

$$\mathbf{e} \rightarrow 0, \quad \mathbf{C}_1 \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}) \equiv \mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u}) \text{ and } \Phi(\hat{\mathbf{x}}, \mathbf{u}) \equiv \Phi(\mathbf{x}, \mathbf{u}) \quad (3.49)$$

Therefore, from (3.39), we have

$$\mathbf{u}_{eq} \rightarrow \mathbf{d} \quad (3.50)$$

Thus, similar to (3.26), the unknown inputs can be estimated as

$$\hat{\mathbf{d}} \approx (\mathbf{C}_1 \mathbf{P}(\hat{\mathbf{x}}, \mathbf{u}))^{-1} (\mathbf{C}_1 \mathbf{P}(\mathbf{x}, \mathbf{u})) (\rho \text{sign}(\mathbf{e}_y))_{eq} \approx \rho \begin{bmatrix} \frac{e_{y1}}{|e_{y1}|+\delta} \\ \frac{e_{y2}}{|e_{y2}|+\delta} \\ \vdots \\ \frac{e_{ym}}{|e_{ym}|+\delta} \end{bmatrix} \quad (3.51)$$

In practical situations, high frequency system dynamics will play an important role in the signal reconstruction. The above approximation can achieve fairly good reconstruction if high frequency dynamics are not present in the system.

3.4 Application to Flexible Joint Robot Arm

We consider the laboratory model of a single-link flexible joint robot [51], [53], [1] defined by the following nonlinear system equations:

$$\begin{aligned} \dot{\theta}_m &= \omega_m \\ \dot{\omega}_m &= \frac{k}{J_m}(\theta_l - \theta_m) - \frac{S}{J_m}\omega_m + \frac{K_\tau}{J_m}u \\ \dot{\theta}_l &= \omega_l \\ \dot{\omega}_l &= -\frac{k}{J_l}(\theta_l - \theta_m) - \frac{mgb}{J_l}\sin(\theta_l) \end{aligned} \quad (3.52)$$

where θ_m and θ_l are the angular rotations of the motor and the link respectively, with ω_m and ω_l being their angular velocities, J_m represents the inertia of the dc

motor, J_l the inertia of the controlled link, k the elastic constant, m the link mass, g the acceleration due to gravity, b the centre of mass, K_τ the amplifier gain, and S the viscous friction coefficient.

In order to evaluate the effectiveness of the proposed method, we consider the following perturbations with the nominal model rewritten in the following form with $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} \theta_m & \omega_m & \theta_l & \omega_l \end{bmatrix}^T$:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/J_m & -B/J_m & k/J_m & 0 \\ 0 & 0 & 0 & 1 \\ k/J_l & 0 & -k/J_l & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} u \\ 0 \\ -\frac{mgb}{J_l} \sin(x_3) \end{bmatrix} + \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} d \quad (3.53)$$

where d is the unknown input. We assume x_1 and x_4 to be measurable as outputs. In the following, we discuss the design of linear and nonlinear sliding mode observers for the above system.

3.4.1 Design of Nonlinear SMO

If the disturbance distribution matrix is a constant matrix, the design of nonlinear SMO will be similar to the design of linear SMO except that the feedback gain should stabilize the error system in the presence of the nonlinear function.

The dynamical system (3.53) can be represented in the form of (3.27) as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \Phi(\mathbf{x}, \mathbf{u}) + \mathbf{P}d \quad (3.54)$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} \quad (3.55)$$

$$\text{where } \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/J_m & -B/J_m & k/J_m & 0 \\ 0 & 0 & 0 & 1 \\ k/J_l & 0 & -k/J_l & 0 \end{bmatrix}, \Phi(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} u \\ 0 \\ -\frac{mgb}{J_l} \sin(x_3) \end{bmatrix}, \text{ and } \mathbf{P} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}.$$

As the unknown input is involved in the dynamics of the output derivative, the unknown input is observable through output y_1 . Therefore $\mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$. The other output $y_2 = x_4$ can provide added stability to the reduced order system in the sliding mode. The following form of observer with one robust term can be designed according to (3.30):

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \Phi(\hat{\mathbf{x}}, \mathbf{u}) + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) + \mathbf{P}\mathbf{u}_r(t) \quad (3.56)$$

$$\mathbf{u}_r(t) = -(\mathbf{C}_1\mathbf{P})^{-1}\rho\text{sign}(\mathbf{C}_1\hat{\mathbf{x}} - \mathbf{C}_1\mathbf{x}) = -\frac{1}{\eta_1}\rho\text{sign}(\hat{x}_1 - x_1) \quad (3.57)$$

The error dynamics can be computed from (3.54) and (3.56) as

$$\begin{aligned} \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/J_m & -B/J_m & k/J_m & 0 \\ 0 & 0 & 0 & 1 \\ k/J_l & 0 & -k/J_l & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{mgb}{J_l} \end{bmatrix} (\sin(\hat{x}_3) - \sin(x_3)) \\ &+ \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} (\mathbf{u}_r(t) - d) - \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \\ l_{31} & l_{32} \\ l_{41} & l_{42} \end{bmatrix} \begin{bmatrix} \hat{x}_1 - x_1 \\ \hat{x}_4 - x_4 \end{bmatrix} \end{aligned} \quad (3.58)$$

where $\mathbf{L} = \begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ l_{12} & l_{22} & l_{23} & l_{24} \end{bmatrix}^T$ is the feedback gain. In the sliding mode of $e_1 = \hat{x}_1 - x_1 = 0$, we have

$$\mathbf{u}_{eq} = \frac{1}{\eta_1} [-e_2 + l_{12}e_4] + d \quad (3.59)$$

From (3.32), we have the reduced order system in the sliding mode that is free from unknown inputs/disturbances by substituting (3.59) in (3.58) as

$$\begin{aligned} \begin{bmatrix} \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} &= \begin{bmatrix} -B/J_m - \frac{\eta_2}{\eta_1} & k/J_m & \frac{\eta_2}{\eta_1}l_{12} - l_{22} \\ -\frac{\eta_3}{\eta_1} & 0 & 1 + \frac{\eta_3}{\eta_1}l_{12} - l_{32} \\ -\frac{\eta_4}{\eta_1} & -k/J_l & \frac{\eta_4}{\eta_1}l_{12} - l_{42} \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \\ e_4 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ -\frac{mgb}{J_l} \end{bmatrix} (\sin(\hat{x}_3) - \sin(x_3)) \end{aligned} \quad (3.60)$$

The feedback gain $\mathbf{L}$ is designed such that the system is stable and satisfies the conditions (3.44) and (3.45). Since $\Phi(\mathbf{x}, \mathbf{u}) = \frac{mgb}{J_l} \sin(x_3)$, $l_\Phi = \frac{mgb}{J_l}$. If $\mathbf{Q}_2$ is the Lyapunov matrix for the system, then the condition $\lambda_{\max}(\mathbf{Q}_2) \leq \frac{J_l}{2mgb}$ should be met in order for the system to be stable. The unknown input can be reconstructed from the sliding mode according to (3.51) as

$$\hat{d} = -\frac{1}{\eta_1} \rho \frac{e_1}{|e_1| + \delta} \quad (3.61)$$

We choose two outputs to demonstrate the observer design when $\mathbf{C}_1 \neq \mathbf{C}$. The proposed observer can also be designed similarly with one output measurement $y_1 = x_1$.

3.4.2 Design Method of [1]

To design the observer, we consider one output measurement $y_1 = x_1$. The observer according to [1] will be of the form:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \Phi(\hat{\mathbf{x}}, \mathbf{u}) + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) + \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}^T W \text{sign}(\hat{x}_1 - x_1) \quad (3.62)$$

where $\mathbf{L} = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 \end{bmatrix}^T$ is the feedback gain, W is a positive real number, and $\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}^T$ which is the feedforward injection map, are selected such that the matching condition [56] is fulfilled. The robust term $\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}^T \times W \text{sign}(\hat{x}_1 - x_1)$ replaces the term $\mathbf{P}\mathbf{u}_r(t)$ in our proposed observer (3.57). The error dynamics will be similar to the form of (3.58). The equivalent control for the switching term $W \text{sign}(\hat{x}_1 - x_1)$ in the sliding mode of $e_1 = \hat{x}_1 - x_1 = 0$ can be obtained as

$$\mathbf{u}_{eq} = -\frac{e_2}{\lambda_1} + \frac{\eta_1}{\lambda_1} d \quad (3.63)$$

Similar to the analysis in (3.60), the reduced order error system is of the form:

$$\begin{aligned} \begin{bmatrix} \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} &= \begin{bmatrix} -B/J_m - \frac{\lambda_2}{\lambda_1} & k/J_m & 0 \\ -\frac{\lambda_3}{\lambda_1} & 0 & 1 \\ -\frac{\lambda_4}{\lambda_1} & -k/J_l & 0 \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \\ e_4 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ -\frac{mgb}{J_l} \end{bmatrix} (\sin(\hat{x}_3) - \sin(x_3)) + \begin{bmatrix} \frac{\lambda_2}{\lambda_1} \eta_1 d - \eta_2 d \\ \frac{\lambda_3}{\lambda_1} \eta_1 d - \eta_3 d \\ \frac{\lambda_4}{\lambda_1} \eta_1 d - \eta_4 d \end{bmatrix} \end{aligned} \quad (3.64)$$

In this method, the switching terms or robust terms are appended to all the states. In fact, the sliding mode gain design is based on Lyapunov approach and is similar to the method of [63]. By the *matching condition* the unknown inputs do not affect the reduced-order system.

Remark 3.6 *The main purpose of this illustration is to show that, the states that are not influenced by unknown inputs/disturbances are also affected by the switching terms. In our proposed method, the unknown inputs are structurally replaced by switching terms, and so the states that are free from disturbances are not affected by the switching terms.*

3.4.3 Design of Linear SMO

To design the linear observer for the robot-arm, we treat the nonlinear function $-\frac{mgb}{J_l} \sin(x_3)$ as unknown input/disturbance. The system (3.54) can be re-written as a linear system as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{P}_L \mathbf{d} \quad (3.65)$$

$$\text{where } \mathbf{B} = \begin{bmatrix} 0 & \frac{K_\tau}{J_m} u & 0 & 0 \end{bmatrix}^T, \mathbf{P}_L = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ 0 & 0 & 0 & \zeta_1 \end{bmatrix}^T, \text{ and } \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

Since there are two unknown inputs in the system, we require both the outputs for building the robust terms to estimate the unknown inputs. It can be easily verified that $\text{rank}(\mathbf{C}_1 \mathbf{P}_L) = 2$, therefore $\mathbf{C}_1 = \mathbf{C}$. The sliding mode observer will be

of the form:

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) + \mathbf{P}_L\nu \\ \nu &= -(\mathbf{C}_1\mathbf{P}_L)^{-1}\rho\text{sign}(\mathbf{C}_1\hat{\mathbf{x}} - \mathbf{C}_1\mathbf{x}) = -\begin{bmatrix} \frac{1}{\eta_1} & 0 \\ -\frac{\eta_4}{\eta_1\zeta_1} & \frac{1}{\zeta_1} \end{bmatrix} \begin{bmatrix} \rho\text{sign}(\hat{x}_1 - x_1) \\ \rho\text{sign}(\hat{x}_2 - x_2) \end{bmatrix}\end{aligned}$$

$$\text{Also, we have from (3.16), } [\mathbf{I} - \mathbf{P}(\mathbf{C}_1\mathbf{P})^{-1}\mathbf{C}_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{\eta_2}{\eta_1} & 1 & 0 & 0 \\ -\frac{\eta_3}{\eta_1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \text{Provided}$$

$\left|\frac{\eta_2}{\eta_1}\right| \leq 1$ and $\left|\frac{\eta_3}{\eta_1}\right| \leq 1$, the matrix $[\mathbf{I} - \mathbf{P}(\mathbf{C}_1\mathbf{P})^{-1}\mathbf{C}_1] \geq 0$. Therefore, by designing $\mathbf{L}$ such that $\mathbf{A} - \mathbf{L}\mathbf{C}$ is stable, the reduced order error system converges asymptotically. Based on this method, the unknown inputs can be approximated from the sliding mode as:

$$\begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{bmatrix} \approx -\begin{bmatrix} \frac{1}{\eta_1} & 0 \\ -\frac{\eta_4}{\eta_1\zeta_1} & \frac{1}{\zeta_1} \end{bmatrix} \begin{bmatrix} \rho\frac{e_1}{|e_1|+\delta} \\ \rho\frac{e_4}{|e_4|+\delta} \end{bmatrix} \quad (3.66)$$

3.4.4 Simulation Results

For the simulation results, we considered the following values: $J_m = 0.0037 \text{ kg} - \text{m}^2$, $J_l = 0.0093 \text{ kg} - \text{m}^2$, $m = 0.21 \text{ kg}$, $2b = 0.3 \text{ m}$, $k = 0.18 \text{ Nm/rad}$, $B = 0.0083 \text{ Nm/volt}$, $K_\tau = 0.08 \text{ Nm/volt}$, $u = 0.001$ and $g = 9.81 \text{ m/sec}^2$.

To study the robustness of the proposed method, we introduce $4\sin(0.6\pi t) - 2\cos(0.35\pi t)$ as unknown input. The unknown input distribution vector elements are chosen as $\eta_1 = 1.2$, $\eta_2 = 0.4$, $\eta_3 = 0.8$, $\eta_4 = 0.2$. The sliding mode gain is chosen to be $\rho = 10$. The initial conditions for plant and estimator are set as $\mathbf{x}(0) = \begin{bmatrix} 6 & 3 & 6 & 2 \end{bmatrix}^T$ and $\hat{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$. For the reconstruction of the unknown input from the sliding mode, the parameter δ in (3.61) is chosen to be 0.015.

The gain is designed such that the reduced-order system is stable. We select the feedback gain $\mathbf{L} = \begin{bmatrix} 2 & 0.38 & 1.2 & 0.2 \\ 0.1 & -5 & -5 & 6 \end{bmatrix}^T$. For the error system (3.58), the positive definite matrix $\mathbf{Q}_2$ that satisfies the ARE (3.44) is obtained to be $\mathbf{Q}_2 = \begin{bmatrix} 0.1494 & 0.3346 & 0.2873 \\ 0.3346 & 2.1031 & 0.8597 \\ 0.2873 & 0.8597 & 1.1891 \end{bmatrix}$ with $\lambda_{\max}(\mathbf{Q}_2) = 2.6941$. The Lipschitz constant for the chosen values was calculated to be $l_\phi = \frac{mgb}{J_t} = 0.138$. Now, we can verify that condition (3.45) is satisfied. Alternatively, we may also follow the gain procedure suggested in [1], [52].

Simulation results for the state estimation with the nonlinear SMO are shown in Fig. 3.1. Despite the presence of a large unknown input, which created large oscillations in the robot arm motion, the observer was able to track the states. The robust term was also able to track the unknown input. The reconstructed unknown input from the sliding mode is shown in Fig. 3.3. Also, to check the estimation accuracy in the presence of high frequency components, we introduce a square wave disturbance of amplitude 6 and period 13.33 sec. The estimation performance is shown in Fig. 3.2. The high frequency system dynamics can be seen in the reconstructed unknown input from the sliding mode in Fig. 3.4.

3.5 Summary

In this chapter, a fundamental insight into the observability of the unknown inputs and its relation to state estimation in the presence of unknown inputs are discussed. State and unknown input estimation are handled from a different perspective. This method clearly specifies when an unknown input can be uniquely identified from the sliding mode thereby producing accurate state-estimates in the presence of unknown inputs in the sliding mode. The stability of the error system in the sliding mode is also discussed for a class of Lipschitz nonlinear systems

Figure 3.1: The estimation performance of the proposed sliding mode observer and relevant conditions are established. Application to a robotic arm problem is given to validate the theoretical development. In the next chapters, the analysis is extended to a more general class of nonlinear systems that can handle arbitrary large Lipschitz constants and bounds.

(a): x_1 and $\hat{x}_1$ (b): x_2 and $\hat{x}_2$ (c): x_3 and $\hat{x}_3$ (d): x_4 and $\hat{x}_4$

Figure 3.2: The estimation performance of the proposed sliding mode observer

Figure 3.3: Disturbance estimation of d Figure 3.4: Disturbance estimation of d

Chapter 4

SMO Design for SISO Nonlinear Systems

In the last chapter, the design of SMO is for a class of Lipschitz nonlinear systems with a constant output distribution matrix. The feedback gain design is based on the nonlinear Lipschitz observer (NLO) and it depends on the Lipschitz constants. In this chapter, we shall extend the observability concepts of unknown inputs for SMO design to a more general class of nonlinear systems. The problem associated with feedback gain design of NLO is solved by employing a high-gain observer (HGO). The sliding mode observer is combined with HGO to build the robust estimator for state and unknown input estimations.

4.1 Introduction

To handle the state estimation of a nonlinear system perturbed by a scalar disturbance distributed by a known nonlinear vector, in this chapter, we incorporate a SMO into the nonlinear observer. The nonlinear system is partially linearized via nonlinear transformation that produces a constant output distribution matrix.

By imposing a structural assumption on the unknown input distribution matrix, the observability of the disturbance w.r.t. the output is safeguarded. Once the unknown input is observable, the disturbance can be estimated from the sliding surface. In the sliding mode, the disturbance under an equivalent control becomes an increment of a Lipschitzian function, and the convergence of the estimation error dynamics can be proven similar to the analysis of the high-gain observer. Under a Lipschitz condition for the nonlinear part, the nonlinear observers are designed under the structural assumption that the system is observable with respect to any input. The proposed robust nonlinear estimator is applied to state and unknown input estimation in a bioreactor. The simulation results demonstrate the effectiveness of the proposed method.

The rest of the chapter is organized as follows: Section 4.2 presents the design and analysis of a robust nonlinear observer that incorporates a sliding mode observer. Section 4.3 discusses the estimation of unknown input from sliding mode and the convergence analysis. In Section 4.4, the gain design based on NLO is discussed. In Section 4.5, the robust nonlinear observer is applied to bioreactor and simulation results demonstrate the effectiveness of the proposed observer design. Section 4.6 concludes the chapter.

4.2 Robust Nonlinear State Estimator

We consider the following nonlinear uncertain system where modeling uncertainties and disturbance are represented as a scalar-valued disturbance or unknown input $d(\mathbf{x}, t)$ distributed by a known vector $\mathbf{p}(\mathbf{x})$:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u(t) + \mathbf{p}(\mathbf{x})d(\mathbf{x}, t) \\ y(t) = h(\mathbf{x}) \end{cases} \quad (4.1)$$

where $\mathbf{x} \triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \in \mathcal{M}$. The nonlinear functions $\mathbf{f}(\mathbf{x})$, $\mathbf{b}(\mathbf{x})$ are smooth vector fields on $\mathcal{M}$, $h(\mathbf{x})$ is a smooth function from $\mathcal{M}$ to $\mathcal{R}$. u is the bounded control input and y is the measurable output. The unknown input is assumed to be bounded for some upper bound $\bar{d}$, i.e. $|d(\mathbf{x}, t)| \leq \bar{d}$.

We assume that the nominal plant is of the form (2.45) and satisfies **Assumptions 2.1** and **2.2**, and the structure in **Lemma 2.1**. Under **Assumption 2.1**, we transform (4.1) using the mapping of $\tilde{\mathbf{x}} = \Phi(\mathbf{x})$ in (2.46) to the following:

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \boldsymbol{\psi}(\tilde{\mathbf{x}}, u) + \tilde{\mathbf{p}}(\tilde{\mathbf{x}})d(\mathbf{x}, t) \quad (4.2)$$

where $y = \mathbf{C}\tilde{\mathbf{x}}$ and $\tilde{\mathbf{p}}(\tilde{\mathbf{x}}) = \frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \mathbf{p}(\mathbf{x})$. To design a robust nonlinear estimator, we need the following additional assumptions:

Assumption 4.1 *The known functions $\mathbf{f}(\mathbf{x})$, $\mathbf{b}(\mathbf{x})$ and $\mathbf{p}(\mathbf{x}, t)$ are bounded with respect to their arguments on $\mathcal{M}$. The input of the nonlinear system (4.1) is bounded such that $u \leq u_{max}$ for some upper bound u_{max} . Moreover, the system (4.1) is Bounded-Input-Bounded-States (BIBS) stable.*

Assumption 4.2 *The distribution vector $\tilde{\mathbf{p}}(\tilde{\mathbf{x}})$ is a Lipschitz function w.r.t. $\tilde{\mathbf{x}}$ and has the following form*

$$\tilde{\mathbf{p}}(\tilde{\mathbf{x}}) = \begin{bmatrix} 1 & \tilde{p}_2(\tilde{x}_1, \tilde{x}_2) & \tilde{p}_2(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) & \cdots & \tilde{p}_n(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \end{bmatrix}^T \quad (4.3)$$

For system (4.2) satisfying **Assumptions 4.1-4.2**, a nonlinear estimator with the robust term can be designed as

$$\dot{\hat{\tilde{\mathbf{x}}}} = \mathbf{A}\hat{\tilde{\mathbf{x}}} + \boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}, u) + \mathbf{L}(y - \mathbf{C}\hat{\tilde{\mathbf{x}}}) + \tilde{\mathbf{p}}(\hat{\tilde{\mathbf{x}}})u_r(t) \quad (4.4)$$

In the above equation,

$$\mathbf{L} = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}^T \quad (4.5)$$

is a properly chosen constant feedback estimation gain, and the design of the

scalar-valued robust term u_r is based on sliding mode theory, and is given as

$$u_r(t) = -\rho \text{sign}(e_1) = \rho \text{sign}(y - h(\hat{\mathbf{x}})) \quad (4.6)$$

where ρ is a finite constant sliding mode estimation gain to be given in the ensuing **Lemma 4.2**.

Remark 4.1 *The robust term (4.6) is similar to the general switching term employed in the literature. The structure in (4.3) can be obtained by normalizing $\tilde{\mathbf{p}}(\tilde{\mathbf{x}})$ with $\tilde{p}_1(\tilde{\mathbf{x}})$, its first component. If **Assumption 4.2** does not hold, i.e., if $\tilde{p}_1(\tilde{\mathbf{x}}) = 0$, then the unknown input is not observable, and the observer cannot overcome the unknown input/disturbance in the estimation. Also by this proposed method, we no longer require the so-called matching condition [56] to be satisfied in the design.*

Remark 4.2 *The design of $\mathbf{L}$ by HGO presented in Section 2.4.3 will be discussed in the following subsections, together with the conditions for the convergence of the estimation error and the design of $\mathbf{L}$. In the later part of the chapter in Section 4.3, the design of $\mathbf{L}$ by NLO is discussed with the conditions that guarantee the convergence of estimation error.*

4.2.1 Boundedness of Error Dynamics

For the error dynamics $\mathbf{e} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}^T \triangleq \hat{\mathbf{x}} - \tilde{\mathbf{x}}$, we can obtain from (4.2) and (4.4) that

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \boldsymbol{\psi}(\hat{\mathbf{x}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u) + \mathbf{L}(\mathbf{C}\tilde{\mathbf{x}} - \mathbf{C}\hat{\mathbf{x}}) + \tilde{\mathbf{p}}(\hat{\mathbf{x}})u_r - \tilde{\mathbf{p}}(\tilde{\mathbf{x}})d(\mathbf{x}, t) \quad (4.7)$$

$$= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e} + \boldsymbol{\psi}(\hat{\mathbf{x}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u) + \tilde{\mathbf{p}}(\hat{\mathbf{x}})u_r - \tilde{\mathbf{p}}(\tilde{\mathbf{x}})d(\mathbf{x}, t) \quad (4.8)$$

The boundedness of $\mathbf{e}$ is proved in the following lemma based on Lyapunov function analysis. To proceed with the analysis, we select the gain as

$$\mathbf{L} = \mathbf{S}_\theta^{-1}\mathbf{C}^T \quad (4.9)$$

similar to the design of high-gain observer, where $\mathbf{S}_\theta$ is the unique solution of Lyapunov equation (2.54). ‘ θ ’ is an independent parameter that deals with the bounds of nonlinearities and guarantees the exponential convergence of the estimation error.

The error dynamics will be of the form

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{S}_\theta^{-1} \mathbf{C}^T \mathbf{C}) \mathbf{e} + \boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u) + \tilde{\mathbf{p}}(\hat{\tilde{\mathbf{x}}})u_r - \tilde{\mathbf{p}}(\tilde{\mathbf{x}})d(\mathbf{x}, t) \quad (4.10)$$

Lemma 4.1 *Assume the system (4.1) satisfies **Assumptions 4.1-4.2**. For the estimator (4.4), there exists $\theta_0 > 0$ such that $\forall \theta > \theta_0$, $\hat{\tilde{\mathbf{x}}}$ and $\mathbf{e}$ remain bounded.*

Proof: The analysis is similar to the proof of **Theorem 2.1**. By letting $\boldsymbol{\xi} = \Delta_\theta \mathbf{e}$ and selecting Lyapunov function $V = \boldsymbol{\xi}^T \mathbf{S}_1 \boldsymbol{\xi}$, then on similar grounds as the proof of **Theorem 2.1**, we can obtain according to (2.63)

$$\dot{V} \leq -\theta V + 2\boldsymbol{\xi}^T \mathbf{S}_1 \Delta_\theta [\boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u) + \tilde{\mathbf{p}}(\hat{\tilde{\mathbf{x}}})u_r - \tilde{\mathbf{p}}(\tilde{\mathbf{x}})d(\mathbf{x}, t)] \quad (4.11)$$

From (2.64), we have

$$\left\| \Delta_\theta [\boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u)] \right\| \leq nl_\psi \|\boldsymbol{\xi}\|$$

As $|d(\mathbf{x}, t)| \leq \bar{d}$ and $|u_r| = \rho$. Under **Assumptions 2.1** and **4.1**, due to the existence of $\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}}$, $\tilde{\mathbf{p}}(\tilde{\mathbf{x}})$ and $\tilde{\mathbf{p}}(\hat{\tilde{\mathbf{x}}})$ are bounded, hence we have

$$\left\| \tilde{\mathbf{p}}(\hat{\tilde{\mathbf{x}}})u_r - \tilde{\mathbf{p}}(\tilde{\mathbf{x}})d(\mathbf{x}, t) \right\| \leq b_{\tilde{\mathbf{p}}} (\bar{d} + \rho) \quad (4.12)$$

for some upper bound $b_{\tilde{\mathbf{p}}}$. Using the above inequalities, we have from (4.11)

$$\begin{aligned} \dot{V} &\leq -\theta V + 2nl_\psi \|\mathbf{S}_1 \boldsymbol{\xi}\| \|\boldsymbol{\xi}\| + 2\|\mathbf{S}_1 \boldsymbol{\xi}\| b_{\tilde{\mathbf{p}}} (\bar{d} + \rho) \\ &\leq -(\theta - 2nl_\psi \sigma(\mathbf{S}_1))V + 2\lambda_{\max}(\mathbf{S}_1) b_{\tilde{\mathbf{p}}} (\bar{d} + \rho) \|\boldsymbol{\xi}\| \end{aligned} \quad (4.13)$$

$$= -c_1 \|\boldsymbol{\xi}\|^2 + c_2 \|\boldsymbol{\xi}\| \quad (4.14)$$

where $c_1 \triangleq \theta - 2nl_\psi \sigma(\mathbf{S}_1)$, $c_2 \triangleq 2\lambda_{\max}(\mathbf{P}) b_{\tilde{\mathbf{p}}} (\bar{d} + \rho)$. By selecting $\theta > 2nl_\psi \sigma(\mathbf{S}_1) =$

θ_0 , we have $c_1 < 0$. Hence $\mathbf{e}$ is bounded such that $\|\mathbf{e}\| \leq c_2/c_1$. Since $\mathbf{x}$ is bounded, then $\hat{\mathbf{x}}$ is also bounded. Also, the bound of $\|\mathbf{e}\|$ can be made small by selecting a large θ . ■

4.2.2 Sliding Mode Gain Design

From the estimation error dynamics (4.8), we can notice that if $d(\mathbf{x}, t) = 0$, then a normal estimator (4.4) where $u_r = 0$ will make the right hand side of (4.8) the increments of Lipschitzian functions under **Assumption 4.2**, hence we can prove $\mathbf{e} \rightarrow \mathbf{0}$ by a proper $\mathbf{L}$. However, for modeling uncertainties from a scalar disturbance $d(\mathbf{x}, t)$ not necessarily Lipschitzian, the normal estimator can only achieve a bounded $\mathbf{e}$ instead of $\mathbf{e} \rightarrow \mathbf{0}$.

In our design, to improve estimation accuracy, we use SMO with the term $\tilde{\mathbf{p}}(\tilde{\mathbf{x}})u_r$ in (4.4). The rationale of this solution is two-fold:

- (a) With the sliding surface

$$e_1 = 0 \tag{4.15}$$

the aim is to design the sliding mode estimation as (4.6) to reach and maintain in the sliding mode.

- (b) Ensure that the term $[u_r - d(\mathbf{x}, t)]$ of the estimation error dynamics (4.8) in the sliding mode $e_1 = 0$, i.e., a zero dynamics, can be substituted by an increment of Lipschitzian function through an equivalent control signal, hence we can prove the asymptotic convergence of $\mathbf{e}$.

The following **Lemma 4.2** and **Theorem 4.1** are devoted to the above mentioned points (a) and (b), respectively.

Lemma 4.2 *For the system (4.1) satisfying **Assumptions 4.1-4.2** and the estimator (4.4), the sliding mode estimation (4.6) ensures that the sliding surface $e_1 = 0$ can be reached and maintained provided that the sliding mode gain satisfies*

$$\rho > \sup |e_2(t) - d(\mathbf{x}, t)| \quad (4.16)$$

and

$$l_1 > l_{\gamma_1} u_{max} \quad (4.17)$$

where l_{γ_1} is the Lipschitz constant of $\gamma_1(\cdot)$.

Proof: First, we analyze the dynamics of e_1 of the error dynamics $\mathbf{e}$ from (4.8).

We can obtain the first dynamics e_1 from (4.7) as follows:

$$\dot{e}_1 = e_2 - l_1 e_1 + \left[\gamma_1(\hat{x}_1) - \gamma_1(\tilde{x}_1) \right] u + u_r - d(\mathbf{x}, t) \quad (4.18)$$

For the Lyapunov function $V_1 = \frac{1}{2}e_1^2$, using the above and the sliding mode estimation (4.6) we have

$$\dot{V}_1 = e_1 \dot{e}_1 = -l_1 e_1^2 + e_1 \left[\gamma_1(\hat{x}_1) - \gamma_1(\tilde{x}_1) \right] u + [e_2(t) - d(\mathbf{x}, t)] e_1 - \rho |e_1|$$

Since $\mathbf{x}$ is bounded according to the BIBS property in **Assumption 4.1**, under the condition in **Lemma 4.1** that ensures the boundedness of e_2 , there exists a finite constant gain satisfying (4.16) such that, if $e_1 \neq 0$, we have

$$\dot{V}_1 < -l_1 e_1^2 + e_1 \left[\gamma_1(\hat{x}_1) - \gamma_1(\tilde{x}_1) \right] u$$

Under the Lipschitzian condition in **Assumption 2.2** and the bounded input condition in **Assumption 4.1**, we have

$$\dot{V}_1 < -l_1 e_1^2 + l_{\gamma_1} u_{max} e_1^2$$

for some Lipschitz constant l_{γ_1} and some maximum input u_{max} . Hence a proper design of feedback estimation gain l_1 such that $l_1 > l_{\gamma_1} u_{max}$ ensures that $\dot{V}_1 < 0$ if

$e_1 \neq 0$. The robust term (4.6) using the gain (4.16) ensures that the sliding surface $e_1 = 0$ can be reached in a finite time and maintained thereafter. ■

4.2.3 Convergence of Error Dynamics in the Sliding Mode

Since the estimator design (4.4) using the robust term (4.6) ensures the sliding mode, we only need to examine the convergence of the dynamics of $\mathbf{e}$ during the sliding mode.

In the sliding mode when $e_1 = 0$ and $\dot{e}_1 = 0$, $\hat{x}_1 = \tilde{x}_1$, the equivalent control of u_r can be obtained from (4.18) as in [56]:

$$u_{eq} = d(\mathbf{x}, t) - e_{2,d} \quad (4.19)$$

where the subscript $_d$ denotes the estimated $\hat{\mathbf{x}}$ -related variables in the sliding mode, i.e., $\mathbf{e}_d = \begin{bmatrix} e_{1,d} & e_{2,d} & \cdots & e_{n,d} \end{bmatrix}^T \triangleq \hat{\mathbf{x}}_d - \tilde{\mathbf{x}}$.

Substituting the above equivalent control (4.19) into (4.8), we have the estimation error dynamics in the sliding mode of $e_1 = 0$,

$$\begin{aligned} \dot{\mathbf{e}}_d &= (\mathbf{A} - \mathbf{S}_\theta^{-1} \mathbf{C}^T \mathbf{C}) \mathbf{e}_d + \boldsymbol{\psi}(\hat{\mathbf{x}}_d, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u) + \left(\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d) - \tilde{\mathbf{p}}(\tilde{\mathbf{x}}) \right) u_{eq} \\ &\quad + \tilde{\mathbf{p}}(\tilde{\mathbf{x}}) [u_{eq} - d(\mathbf{x}, t)] \\ &= (\mathbf{A} - \mathbf{S}_\theta^{-1} \mathbf{C}^T \mathbf{C}) \mathbf{e}_d + \boldsymbol{\psi}(\hat{\mathbf{x}}_d, u) - \boldsymbol{\psi}(\tilde{\mathbf{x}}, u) \\ &\quad + \left(\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d) - \tilde{\mathbf{p}}(\tilde{\mathbf{x}}) \right) u_{eq} - \tilde{\mathbf{p}}(\tilde{\mathbf{x}}) e_{2,d} \end{aligned} \quad (4.20)$$

The equivalent control in the sliding mode clearly cancels the disturbance effect in the state estimation. Obviously, the equilibrium point of the error dynamics (4.10) is $\mathbf{e}_d = \mathbf{0}$, i.e., $\hat{\mathbf{x}}_d = \tilde{\mathbf{x}}$.

When the error e_1 is in the sliding mode, the robust term can be viewed as a ‘tracking element’ for the unknown input or disturbance input. So the robust term in the sliding mode can be viewed as an estimate of the disturbance $d(\mathbf{x}, t)$ to

improve the accuracy of estimation and so the convergence of estimation of error to zero can be achieved. The reconstruction of the disturbance from the sliding mode is discussed in the next section.

The following theorem proves the asymptotic stability of the estimation error.

Theorem 4.1 *Assume that system (4.1) satisfies **Assumptions 4.1-4.2**. For the estimator (4.4) with the robust term (4.6) and the sliding mode gain (4.16), there exists $\theta_1 > 0$ such that $\forall \theta > \theta_1$, the estimation error is asymptotically stable in the sliding mode of $e_1 = 0$.*

Proof: The proof follows the similar pattern in **Lemma 4.1**. By selecting $\xi_d = \Delta_\theta \mathbf{e}_d$ and Lyapunov function $V_2 = \xi_d^T \mathbf{S}_1 \xi_d$, we can obtain with (4.20) similar to (2.63):

$$\begin{aligned} \dot{V}_2 \leq & -\theta V_2 + 2\xi_d^T \mathbf{S}_1 \Delta_\theta \left[\psi(\hat{\mathbf{x}}_d, u) - \psi(\tilde{\mathbf{x}}_d, u) \right] \\ & + 2\xi_d^T \mathbf{S}_1 \Delta_\theta \left[\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d) - \tilde{\mathbf{p}}(\tilde{\mathbf{x}}_d) \right] u_{eq} - 2\xi_d^T \mathbf{S}_1 \Delta_\theta \tilde{\mathbf{p}}(\tilde{\mathbf{x}}) e_{2,d} \end{aligned} \quad (4.21)$$

Since e_2 , $\mathbf{x}$ and $d(\mathbf{x}, t)$ are bounded, then according to (4.19), $u_{eq} \leq \bar{u}_{eq}$ for some upper bound $\bar{u}_{eq}$. From (4.12), we have $\|\tilde{\mathbf{p}}(\tilde{\mathbf{x}})\| \leq b_{\tilde{\mathbf{p}}}$. Under **Assumption 4.2**, $\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d)$ is a Lipschitz function. Similar to (2.64), due to the triangular structure of $\tilde{\mathbf{p}}(\tilde{\mathbf{x}})$ in **Assumption 4.2**, we can obtain

$$\left\| \Delta_\theta \left(\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d) - \tilde{\mathbf{p}}(\tilde{\mathbf{x}}) \right) \right\| \leq n l_{\tilde{\mathbf{p}}} \|\xi_d\| \quad (4.22)$$

where $l_{\tilde{\mathbf{p}}}$ is the largest Lipschitz constant of $\tilde{\mathbf{p}}$.

Using the above results, we can obtain from (4.21)

$$\begin{aligned} \dot{V}_2 \leq & -\theta V_2 + 2n l_\psi \|\mathbf{S}_1 \xi_d\| \|\xi_d\| + 2n l_{\tilde{\mathbf{p}}} \|\mathbf{S}_1 \xi_d\| \|\xi_d\| \bar{u}_{eq} + 2b_{\tilde{\mathbf{p}}} \|\mathbf{S}_1 \xi_d\| \|\Delta_\theta \mathbf{e}_d\| \\ \leq & -\theta V_2 + [2n l_\psi \sigma(\mathbf{S}_1) + 2n l_{\tilde{\mathbf{p}}} \sigma(\mathbf{S}_1) \bar{u}_{eq} + 2b_{\tilde{\mathbf{p}}} \sigma(\mathbf{S}_1)] V_2 \end{aligned} \quad (4.23)$$

By selecting $\theta > 2n l_\psi \sigma(\mathbf{S}_1) + 2n l_{\tilde{\mathbf{p}}} \sigma(\mathbf{S}_1) \bar{u}_{eq} + 2b_{\tilde{\mathbf{p}}} \sigma(\mathbf{S}_1) = \theta_1$, we have $\dot{V}_2 < 0$. Hence,

the robust estimator guarantees asymptotic stability of the estimation error. ■

4.2.4 Observer in the Original Space

To estimate the states in the original system (4.1), we transform (4.4) back to the original space:

$$\dot{\hat{\mathbf{x}}} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}}^{-1} \dot{\hat{\mathbf{x}}}$$

The final observer in the original space is obtained as follows:

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{b}(\hat{\mathbf{x}})u + \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}}^{-1} \mathbf{S}_\theta^{-1} \mathbf{C}^T (y - h(\hat{\mathbf{x}})) + \mathbf{p}(\hat{\mathbf{x}}) \rho \text{sign}(y - h(\hat{\mathbf{x}})) \quad (4.24)$$

To avoid the switching caused by the $\text{sign}(\cdot)$ function, we may approximate $\text{sign}(\cdot)$ by a saturation function $\text{sat}(\cdot, \varepsilon) \triangleq \begin{cases} \cdot/\varepsilon & \text{if } |\cdot| \leq \varepsilon \\ \text{sign}(\cdot) & \text{if } |\cdot| > \varepsilon \end{cases}$, where $\varepsilon > 0$ is the threshold.

Remark 4.3 By replacing the $\text{sign}(\cdot)$ function with $\text{sat}(\cdot)$, the error dynamics finally settle to a bound inside the boundary layer rather than converging to zero in the sliding plane. As shown in the last subsection, the convergence of $\mathbf{e}$ occurs after e_1 reaches the sliding mode. Decreasing the boundary layer thickness ε in $\text{sat}(\cdot)$ increases the accuracy in estimation.

4.2.5 Unknown Input Estimation from Sliding Mode

Once the trajectories reach the sliding mode and all the states converge to the true states, the equivalent control u_{eq} information can be used to reconstruct the unknown input. Once the system attains the sliding mode, and states converge to the true states, we have $\hat{\mathbf{x}}_d \rightarrow \tilde{\mathbf{x}}$. Therefore,

$$e_{2,d} \approx 0$$

So, from (4.19) we can approximate

$$u_{eq} \approx d(\mathbf{x}, t) \quad (4.25)$$

Similar to the approximation of equivalent control given in (3.26), for some small positive scalar δ , the unknown input can be estimated as

$$\begin{aligned} \hat{d}(\mathbf{x}, t) &\approx (\rho \text{sign}(e_1))_{eq} \\ &\approx \rho \frac{e_1}{(|e_1| + \delta)} \end{aligned} \quad (4.26)$$

4.3 Robust Nonlinear Lipschitz Observer

In this section, we shall discuss the design of robust NLO for the system (4.1). The observer design is similar to the design in the last section, except for the design of feedback gain $\mathbf{L}$. The boundedness of estimation error and asymptotic stability of the error dynamics for the robust nonlinear Lipschitz observer are examined. The analysis of robust NLO is established by replacing **Lemma 4.1** and **Theorem 4.1** with the following lemma and theorem, respectively.

The boundedness of $\mathbf{e}$ in the error dynamics (4.8) is proved in the following lemma based on Lyapunov function analysis. First, we need to design $\mathbf{L}$ such that the eigenvalues of $\mathbf{A} - \mathbf{LC}$ are stable, then we can define Lyapunov function $V_0 \triangleq \mathbf{e}^T \mathbf{Q} \mathbf{e}$, where the positive symmetric matrix $\mathbf{Q} = \mathbf{Q}^T > 0$ is such that

$$(\mathbf{A} - \mathbf{LC})^T \mathbf{Q} + \mathbf{Q} (\mathbf{A} - \mathbf{LC}) = -\mathbf{I} \quad (4.27)$$

Lemma 4.3 *For the system (4.1) satisfying Assumptions 4.1-4.2 and the estimator (4.4), $\hat{\mathbf{x}}$ and $\mathbf{e}$ are all bounded provided that*

$$\lambda_{\max}(\mathbf{Q}) \leq \frac{1}{2 l_{\psi}} \quad (4.28)$$

for Lipschitz constant l_{ψ} .

Proof: See Appendix B.1. ■

The asymptotical stability of the error dynamics in the sliding mode is proven in the following theorem.

Theorem 4.2 *For the system (4.1) satisfying Assumptions 4.1-4.2, the estimator (4.4) with the robust term (4.6) and the gain (4.16) ensures that the estimation error is globally asymptotically stable in the sliding mode $e_1 = 0$ provided that the gain $\mathbf{L}$ satisfies $l_1 > l_{\gamma_1} u_{max}$, condition (4.28), and*

$$\lambda_{max}(\mathbf{Q}) \leq \frac{1}{2} \left[l_{\boldsymbol{\psi}} + l_{\tilde{\mathbf{p}}} \bar{u}_{eq} + b_{\tilde{\mathbf{p}}}^* \right]^{-1} \quad (4.29)$$

for some Lipschitz constants $l_{\boldsymbol{\psi}}$, $l_{\tilde{\mathbf{p}}}$ and upper bounds $\bar{u}_{eq}$, $b_{\tilde{\mathbf{p}}}^*$.

Proof: See Appendix B.2. ■

The gain $\mathbf{L}$ should be designed such that it satisfies the two conditions of (4.28) and (4.29), and the Lyapunov condition (4.27).

If the unknown input distribution matrix $\tilde{\mathbf{p}}(\tilde{\mathbf{x}})$ is a constant matrix or functions of the outputs, i.e. $\tilde{\mathbf{p}}(\tilde{\mathbf{x}}) = \tilde{\mathbf{p}}(y)$, we have $l_{\tilde{\mathbf{p}}} = l_{\tilde{p}_1} = 0$. So, the condition (4.29), reduces to the form

$$\lambda_{max}(\mathbf{P}) \leq \frac{1}{2 \left[l_{\boldsymbol{\psi}} + b_{\tilde{\mathbf{p}}}^* \right]} \quad (4.30)$$

In the absence of unknown inputs, i.e., if all the inputs are known, the condition (4.29) will be reduced to the form:

$$\lambda_{max}(\mathbf{P}) \leq \frac{1}{2 \left[l_{\boldsymbol{\psi}} + l_{\tilde{\mathbf{p}}} \bar{d} \right]} \quad (4.31)$$

The gain design can be carried out as discussed earlier in Section 3.3.5 under similar conditions.

4.4 Application to Bioreactor

As an application example, we investigate the on-line monitoring of bioreactors [139], [37]. The bioreactor system with uncertain parameters belongs to the generic form of the nonlinear uncertain system considered for the proposed robust estimator design. Consider the following continuous biological system - a bioreactor, which has been discussed in many works [37], [16], [140]:

$$\begin{cases} \dot{X} = \frac{\mu_m X S}{k_c X + S} - X D \\ \dot{S} = -\frac{1}{Y} \frac{\mu_m X S}{k_c X + S} + (S_f - S) D \end{cases} \quad (4.32)$$

In the above equations, X represents the cell mass concentration that can be measured “on-line”, S represents the substrate concentration that is not measured, S_f is the substrate concentration in the feed stream, Y is the yield of cell mass, $D > 0$ is the dilution rate, and also the plant’s control input. S_f and Y are known constants.

The system satisfies: (1) the dilution rate is bounded as $D(t) \geq D_{min} > 0, \forall t$, where D_{min} is a constant; (2) the feed rate S_f is bounded; and (3) each reaction involves at least one reactant which is neither a catalyst nor an autocatalyst, hence according to *Theorem 1.1* in [16], we can conclude that the state variables $X(t) > 0$, $S(t) > 0$ and $D(t) > 0$ and bounded for all t . Hence **Assumption 4.1** is satisfied.

The cell mass concentration $X(t)$ is measured by a biosensor. The target is to estimate the unavailable $S(t)$ from the measurable $X(t)$. The growth rate function is described by the *Contois model*: $\mu(X, S) \triangleq \frac{\mu_m S}{k_c X + S}$, where $\mu_m > 0$ denotes the uncertain maximum growth, and $k_c > 0$ denotes the uncertain saturation constant. In a practical situation, μ_m and k_c may be uncertain and time-varying. Hence we model them into $\mu_m = \mu_m^0 + d_1(t)$, $k_c = k_c^0 + d_2(t)$, where μ_m^0 and k_c^0 are the *known nominal* parameters, and $d_1(t)$ and $d_2(t)$ model the bounded additive time-varying

parametric uncertainties.

4.4.1 System Model in State-space

Define a state vector as $\mathbf{x} \triangleq \begin{bmatrix} X & S \end{bmatrix}^T$. The system can be re-written into the form of (4.1), where

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{\mu_m^0 X S}{k_c^0 X + S} \\ -\frac{1}{Y} \frac{\mu_m^0 X S}{k_c^0 X + S} \end{bmatrix}, \quad \mathbf{b}(\mathbf{x}) = \begin{bmatrix} -X \\ S_f - S \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 1 \\ -\frac{1}{Y} \end{bmatrix}, \quad L_{\mathbf{p}} h(\mathbf{x}) = 1$$

$$d(\mathbf{x}, t) = \frac{\mu_m X S}{k_c X + S} - \frac{\mu_m^0 X S}{k_c^0 X + S}, \quad y = h(\mathbf{x}) = X, \quad \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The state transformation $\tilde{\mathbf{x}} = \Phi(\mathbf{x}) = \begin{bmatrix} h(\mathbf{x}) & L_{\mathbf{f}} h(\mathbf{x}) \end{bmatrix}^T$ produces

$$\Phi(\mathbf{x}) = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} X \\ \frac{\mu_m^0 X S}{k_c^0 X + S} \end{bmatrix}$$

Hence, we have

$$\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} 1 & 0 \\ \frac{\mu_m^0 S^2}{(\mu_m^0 X + S)^2} & \frac{\mu_m^0 k_c^0 X^2}{(\mu_m^0 X + S)^2} \end{bmatrix}, \quad \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{S^2}{k_c^0 X^2} & \frac{(\mu_m^0 X + S)^2}{\mu_m^0 k_c^0 X^2} \end{bmatrix}$$

Therefore **Assumption 2.1** is satisfied. Also, $\tilde{\mathbf{p}}(\tilde{\mathbf{x}}) = \frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \mathbf{p}(\mathbf{x}) = \begin{bmatrix} 1 & \frac{\mu_m^0 (S^2 - \frac{k_c^0 X^2}{Y})}{(\mu_m^0 X + S)^2} \end{bmatrix}^T$.

It can be readily computed that the transformed system becomes

$$\begin{bmatrix} \dot{\tilde{X}} \\ \dot{\tilde{S}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{S} \end{bmatrix} + \begin{bmatrix} 0 \\ \phi(\tilde{X}, \tilde{S}) \end{bmatrix} + \begin{bmatrix} -\tilde{X}(t) \\ \gamma(\tilde{X}, \tilde{S}) \end{bmatrix} D(t) \\ + \begin{bmatrix} 1 \\ \zeta(\tilde{X}, \tilde{S}) \end{bmatrix} d(\mathbf{x}, t) \quad (4.33)$$

It can be verified that $\phi(\tilde{X}, \tilde{S}), \gamma(\tilde{X}, \tilde{S})$ are differentiable with respect to their

arguments, this together with the boundedness of the states X , S , $\tilde{X}$, $\tilde{S}$ ensures Lipschitzian ϕ and γ . Moreover, the above equation belongs to the structure given in (2.49). Hence both **Assumption 2.1** and the structure in **Lemma 2.1** are satisfied. Due to the boundedness of the states X , S , $\tilde{X}$, $\tilde{S}$, it can also be verified that $\tilde{\mathbf{p}}(\tilde{\mathbf{x}})$ is a Lipschitz function with $\tilde{p}_1 = L_{\mathbf{p}}h(\mathbf{x}) = 1$. Hence **Assumption 4.2** is also satisfied.

4.4.2 Robust Estimator

In the following, we design the gain using the high-gain approach. From (2.55), we can compute $\mathbf{S}_\theta$, $\mathbf{S}_\theta^{-1}$ and feedback gain $\mathbf{L}$ as

$$\mathbf{S}_\theta = \begin{bmatrix} \frac{1}{\theta} & -\frac{1}{\theta^2} \\ -\frac{1}{\theta^2} & \frac{2}{\theta^3} \end{bmatrix}, \quad \mathbf{S}_\theta^{-1} = \begin{bmatrix} 2\theta & \theta^2 \\ \theta^2 & \theta^3 \end{bmatrix}, \quad \mathbf{L} = \mathbf{S}_\theta^{-1} \mathbf{C}^T = \begin{bmatrix} 2\theta \\ \theta^2 \end{bmatrix}$$

A proper sliding mode estimation gain ρ is also required. Since $L_{\mathbf{p}}h(\mathbf{x}) = 1$, the proposed robust estimator is designed as

$$\dot{\hat{X}} = \frac{\mu_m^0 \hat{X} \hat{S}}{k_c^0 \hat{X} + \hat{S}} - \hat{X} D + 2\theta(y - \hat{X}) + \rho \times \text{sat}(y - \hat{X}, \varepsilon) \quad (4.34)$$

$$\begin{aligned} \dot{\hat{S}} &= -\frac{1}{Y} \frac{\mu_m^0 \hat{X} \hat{S}}{k_c^0 \hat{X} + \hat{S}} + (S_f - \hat{S}) D + \left[\frac{(\mu_m^0 X + S)^2 \theta^2}{\mu_m^0 k_c^0 X^2} - \frac{2\theta S^2}{k_c^0 X^2} \right] (y - \hat{X}) \\ &\quad - \frac{\rho}{Y} \times \text{sat}(y - \hat{X}, \varepsilon) \end{aligned} \quad (4.35)$$

The unknown input or disturbance can be estimated according to (4.26) as

$$\hat{d}(\mathbf{x}, t) \approx \frac{\rho(\hat{X} - X)}{|\hat{X} - X| + \delta} \quad (4.36)$$

4.4.3 Simulation Results

The following parameters of the bioreactor described by (4.32) were chosen for the simulation: $\mu_m^0 = 1$ 1/hour (1/hr), $k_c^0 = 1$, $s_f = 1$ gram/litre (g/l), $Y = 1$. The

Figure 4.1: The estimation performance of the nonlinear estimator

Figure 4.2: The estimation performance of the robust nonlinear estimator

Figure 4.3: The actual and estimated unknown input.

additive parametric uncertainties of μ_m and k_c are $\sin(1.5\pi t)$ and $0.6 \cos(\pi t)$, respectively. The initial cell mass concentration is $X(t_0) = 0.02$ (g/l), and the initial substrate concentration is $S(t_0) = 0.03$ (g/l). The initial value of the estimated cell mass concentration is $\hat{X}(t_0) = 0.01$ (g/l), and the initial value of the estimated substrate concentration is $\hat{S}(t_0) = 0.01$ (g/l). The control input, i.e. the dilution rate is chosen as $D = 0.5$ (l/hr).

For the simulation, we select $\theta = 4$, therefore estimation gains will be $\mathbf{L} = \begin{bmatrix} 8 & 16 \end{bmatrix}^T$. We use gain $\rho = 1$ and $\varepsilon = 0.01$. First, we applied the nonlinear estimator without the robust terms in accordance with (2.53):

$$\begin{aligned}\dot{\hat{X}} &= \frac{\mu_m^0 \hat{X} \hat{S}}{k_c^0 \hat{X} + \hat{S}} - \hat{X} D + 2\theta(y - \hat{X}) \\ \dot{\hat{S}} &= -\frac{1}{Y} \frac{\mu_m^0 \hat{X} \hat{S}}{k_c^0 \hat{X} + \hat{S}} + (S_f - \hat{S}) D + \left[\frac{(\mu_m^0 X + S)^2 \theta^2}{\mu_m^0 k_c^0 X^2} - \frac{2\theta S^2}{k_c^0 X^2} \right] (y - \hat{X})\end{aligned}$$

Fig. 4.1 shows the estimation errors due to the imposed parametric uncertainties. Clearly, the estimator tracks the actual state but the accuracy is not very good. To improve the estimation accuracy, we applied the proposed robust nonlinear estimator described by (4.4) and (4.6), and implemented by (4.34) and (4.35) for this bioreactor application. Fig. 4.2 shows the improved estimation performance.

To estimate the unknown input according to (4.36), δ is chosen to be 0.001. The unknown input reconstructed from the sliding mode is shown in Fig. 4.3. The estimated unknown input converges to the actual disturbance after all the states have converged.

4.5 Summary

In this chapter, we have designed a SMO for a class of SISO nonlinear systems by incorporating a sliding mode term into the nonlinear observer to improve the esti-

mation accuracy. The sliding surface uses only the measurable output estimation error. The unknown input is estimated from the measurable sliding surface. Hence the proposed method can be easily implemented. It has been proven that the estimation error is asymptotically convergent in the sliding mode. The gain design is based on HGO that guarantees the exponential convergence of the estimation error for any Lipschitz constant. The proposed robust nonlinear estimator is applied to the state estimation of a bioreactor and the simulation results demonstrate the effectiveness of the proposed approach.

Chapter 5

Multiple SMO Design for MIMO Nonlinear Systems

5.1 Introduction

In this chapter, we shall extend the SMO design technique of the last chapter to MIMO uncertain nonlinear systems perturbed by multiple disturbances. To deal with unknown inputs directly, we formulate new nonlinear transformations. Through the nonlinear transformation, the original system is partitioned into two classes of interconnected subsystems. The existence of outputs of relative degree ‘one’ [33] for the outputs w.r.t. unknown inputs guarantees the observability of unknown inputs, and so accurate state and unknown input estimation become feasible in the transformed domain.

Under structural assumptions for the unknown input distribution matrix, the observer is designed to tackle completely the unknown inputs. Multiple sliding modes are then introduced to compensate for multiple unknown input terms by appending them to a constant gain observer in the transformed domain. The observer is designed separately for each of the transformed subsystems. The conditions for

asymptotic stability of estimation error dynamics are derived based on Lipschitz assumptions for nonlinear functions similar to the high-gain observer (HGO) presented in Chapter 3. The observer in the original space is obtained by means of inverse transformation. We will also discuss the design of the robust nonlinear Lipschitz observer.

The available methods for state estimation in the presence of unknown inputs rely on decoupling the unknown inputs through nonlinear transformations and relative degree [75], [78]. This imposes strict conditions on the disturbance distribution matrix and unknown input structures. There is no existing method for unknown input estimation of general MIMO uncertain nonlinear systems. The following example provides the motivation behind our work.

Example 5.1 *Consider a simple nonlinear system with three states:*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 u_1 \\ \sin(x_1) - 2x_2 x_3 + x_2 u_2 \\ 2x_2 - 2x_3^2 + u_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ 0.6d_1 + 0.4d_2 \\ d_1 - 0.25d_2 \end{bmatrix} \quad (5.1)$$

$$y_1 = h_1(\mathbf{x}) = x_1 \quad y_2 = h_2(\mathbf{x}) = x_2$$

where u_1 and u_2 are known inputs, d_1 and d_2 are unknown inputs, and y_1 and y_2 are measurements. For this class of systems with outputs having relative degree one, the observer design is not feasible with existing methods for state and unknown input estimations.

In this chapter, we consider a more general class of nonlinear systems than those considered in [75] and we shall remove the involutive conditions on the disturbance distribution vectors in the development of multiple SMO.

5.2 Preliminaries

We consider the class of continuous-time nonlinear systems that can be represented as

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x}, \mathbf{u}) + \sum_{i=1}^m \mathbf{p}_i(\mathbf{x}) d_i(\mathbf{x}, \mathbf{u}, t) \\ y_j = h_j(\mathbf{x}), \quad \text{for } j = 1, \dots, s \end{cases} \quad (5.2)$$

where $\mathbf{x} \in \mathcal{M}$, $\mathbf{f}(\mathbf{x})$ and $\mathbf{p}_i(\mathbf{x})$, $i = 1, \dots, m$, are smooth vector fields on $\mathcal{M}$, $h_j(\mathbf{x})$, $j = 1, \dots, s$, are smooth functions from $\mathcal{M}$ to $\mathcal{R}$, and $\mathbf{G}(\mathbf{x}, \mathbf{u})$ is a vector field with smooth functions. The disturbance or unknown input vector is represented as $\mathbf{d}(\mathbf{x}, \mathbf{u}, t) = [d_1, \dots, d_m]^T$ with $d_i(\mathbf{x}, \mathbf{u}, t)$ denoting uncertainties and disturbances in the system, and we assume $d_i(\mathbf{x}, \mathbf{u}, t)$ to be bounded, i.e. $|d_i(\mathbf{x}, \mathbf{u}, t)| \leq \bar{d}_i$ for some upper bound $\bar{d}_i$.

5.2.1 Nonlinear Transformation

The traditional nonlinear transformation [34], [75] uses the structural properties of the system to decouple the known or unknown inputs by transforming the system into another domain. In order to design the nonlinear unknown input observer (NUIO), the system distribution vectors $\mathbf{p}_1(\mathbf{x}), \dots, \mathbf{p}_m(\mathbf{x})$ must satisfy the involutive property [75], [34]. The outputs should also have vector relative degree corresponding to $\mathbf{G}(\mathbf{x}, \mathbf{u})$ at each point $\mathbf{x}_o \in \mathcal{M}$. These assumptions in general are conservative. Also, to complete the transformation, one needs to find $n - a$ terms to decouple the unknown inputs for the remaining states in the transformed domain which is not straight-forward, where a is the sum of relative degrees of the outputs chosen for transformation [75], [34].

Instead of decoupling the unknown inputs, we shall define a new transformation to deal with the disturbances directly.

Assumption 5.1 *From the s outputs, there are at least q outputs with relative degrees $r_j = 1$ w.r.t. unknown inputs, $j = 1, \dots, q$.*

Our proposed transformation is developed in two steps.

- (I) We choose q outputs with relative degrees $r_j = 1$ to form the first part of the transformation. Suppose that such outputs are given by the first q outputs $y_1, \dots, y_q$. For each of these outputs y_j , we define a k_j and then form the following transformation

$$\phi_{w_j} = \begin{bmatrix} h_j(\mathbf{x}) \\ L_{\mathbf{f}} h_j(\mathbf{x}) \\ \vdots \\ L_{\mathbf{f}}^{k_j-1} h_j(\mathbf{x}) \end{bmatrix}, \quad \phi_w = \begin{bmatrix} \phi_{w_1} \\ \phi_{w_2} \\ \vdots \\ \phi_{w_q} \end{bmatrix} \quad (5.3)$$

for $j = 1, \dots, q$.

- (II) The remaining outputs with relative degrees $r_j \geq 1$ are used to form the second part of the transformation via

$$\phi_{w_0} = \begin{bmatrix} h_{q+1}(\mathbf{x}) \\ \vdots \\ h_s(\mathbf{x}) \end{bmatrix} \quad (5.4)$$

The selection of outputs q and the values of k_j in step I are to be done such that the desired transformation holds. That is, the outputs with relative degree 1 are divided between steps I and II so as to form n functions to complete the transformation.

Remark 5.1 *The first step of selecting q outputs with relative degree one allows us to deal with the disturbance inputs directly in our design of the observer. The*

proposed methodology exploits the structural properties when designing the multiple sliding mode observer to tackle the disturbance inputs.

Remark 5.2 *For outputs with relative degree $r_i > 2$ (if exist), the disturbance inputs are decoupled and are absent in the transformed domain. So, the observer can be designed easily as the states are measurable directly as outputs.*

Our complete transformation is given by

$$\tilde{\mathbf{x}} = \Phi(\mathbf{x}) = \begin{bmatrix} \phi_{w_1}^T & \cdots & \phi_{w_q}^T & \phi_{w_0}^T \end{bmatrix}^T \quad (5.5)$$

The transformed coordinates are represented as

$$\tilde{\mathbf{x}}_{w_j} = \begin{bmatrix} x_1^{w_j} \\ x_2^{w_j} \\ \vdots \\ x_{k_j}^{w_j} \end{bmatrix} = \phi_{w_j}, \quad \tilde{\mathbf{x}}_{w_0} = \begin{bmatrix} x_1^{w_0} \\ x_2^{w_0} \\ \vdots \\ x_c^{w_0} \end{bmatrix} = \phi_{w_0}$$

for $j = 1, \dots, q$, and $c = s - q$.

If the transformation cannot be found for the complete observability of the unknown inputs, the proposed method can only guarantee the boundedness of the estimation error. The transformation formulated in this section is more suitable when the number of unknown inputs are close to the number of outputs with relative degree one. If there are more outputs with relative degree greater than two, the transformation defined above may not exist. In that scenario, we can use the outputs with relative degree greater than two to form a new class of inter-linked subsystems. The unknown inputs will be absent in these subsystems and the observer can be designed with the existing techniques. The new transformation and the observer design are discussed in Appendix A.1.

5.2.2 Transformed System

System (5.2) can be transformed into the new coordinates with the transformation (5.5) so that

$$\dot{\tilde{\mathbf{x}}} = \begin{bmatrix} \dot{\tilde{\mathbf{x}}}_{w_1}^T & \cdots & \dot{\tilde{\mathbf{x}}}_{w_q}^T & \dot{\tilde{\mathbf{x}}}_{w_0}^T \end{bmatrix}^T = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right] \dot{\mathbf{x}} \quad (5.6)$$

In the transformed domain, we have two classes of subsystems, namely Class A and Class B as follows:

$$\tilde{\mathbf{x}} = \Phi(\mathbf{x}) = \underbrace{\begin{bmatrix} \phi_{w_1}^T & \cdots & \phi_{w_q}^T \end{bmatrix}}_{\text{Class A}} \underbrace{\begin{bmatrix} \phi_{w_0}^T \end{bmatrix}}_{\text{Class B}}^T$$

Class A Subsystems:

For the subsystems under the transformations $\phi_{w_1}, \dots, \phi_{w_q}$, the following structure can be obtained

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}_{w_j} &= \begin{bmatrix} \dot{x}_1^{w_j} \\ \vdots \\ \dot{x}_{k_j-1}^{w_j} \\ \dot{x}_{k_j}^{w_j} \end{bmatrix} = \begin{bmatrix} \dot{x}_2^{w_j} \\ \vdots \\ \dot{x}_{k_j}^{w_j} \\ 0 \end{bmatrix} + \begin{bmatrix} \mu_1^{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \\ \vdots \\ \mu_{k_j-1}^{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \\ \alpha_{w_j}(\tilde{\mathbf{x}}) + \mu_{k_j}^{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \end{bmatrix} + \begin{bmatrix} \xi_1^{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) \\ \vdots \\ \xi_{k_j-1}^{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) \\ \xi_{k_j}^{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) \end{bmatrix} \\ y_{w_j} &= x_1^{w_j} \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} \alpha_{w_j}(\tilde{\mathbf{x}}) &= L_{\mathbf{f}}^{k_j} h_j(\Phi^{-1}(\tilde{\mathbf{x}})), \quad \mu_i^{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) = \frac{\partial L_{\mathbf{f}}^i h_j}{\partial \mathbf{x}} \mathbf{G}(\Phi^{-1}(\tilde{\mathbf{x}}), \mathbf{u}) \\ \xi_i^{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) &= \sum_{v=1}^m L_{\mathbf{P}_v} L_{\mathbf{f}}^i h_j(\Phi^{-1}(\tilde{\mathbf{x}})) d_v \end{aligned}$$

for $j = 1, \dots, q$, and $i = 1, \dots, k_j$. Its more general form can be represented as

$$\begin{cases} \dot{\tilde{\mathbf{x}}}^{w_j} = \mathbf{A}_{w_j} \tilde{\mathbf{x}}_{w_j} + \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) + \boldsymbol{\xi}_{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) \\ y_{w_j} = x_1^{w_j} = \mathbf{C}_{w_j} \tilde{\mathbf{x}}_{w_j} \end{cases} \quad (5.8)$$

where $\mathbf{A}_{w_j} = \begin{bmatrix} 0 & \mathbf{I}_{(k_j-1) \times (k_j-1)} \\ \mathbf{0}_{1 \times (k_j)} \end{bmatrix}$ is an anti-shift constant matrix, and

$$\begin{aligned} \mathbf{C}_{w_j} &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \\ \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) &= \begin{bmatrix} \mu_1^{w_j} & \cdots & \mu_{k_j-1}^{w_j} & \alpha_{w_j} + \mu_{k_j}^{w_j} \end{bmatrix}^T \end{aligned} \quad (5.9)$$

$$\boldsymbol{\xi}_{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) = \begin{bmatrix} \xi_1^{w_j} & \cdots & \xi_{k_j}^{w_j} \end{bmatrix}^T \quad (5.10)$$

for $j = 1, \dots, q$.

Class B Subsystems:

For the subsystem under the transformation ϕ_{w_0} , the transformed system has the following structure

$$\dot{x}_i^{w_0} = \varphi_i(\tilde{\mathbf{x}}, \mathbf{u}) + \gamma_i(\tilde{\mathbf{x}}, \mathbf{d}), \quad y_{w_0,i} = x_i^{w_0} \quad (5.11)$$

where

$$\begin{aligned} \varphi_i(\tilde{\mathbf{x}}, \mathbf{u}) &= L_{\mathbf{f}} h_{q+i}(\Phi^{-1}(\tilde{\mathbf{x}})) + \frac{\partial h_{q+i}}{\partial \mathbf{x}} \mathbf{G}(\Phi^{-1}(\tilde{\mathbf{x}}), \mathbf{u}) \\ \gamma_i(\tilde{\mathbf{x}}, \mathbf{d}) &= \sum_{v=1}^m L_{\mathbf{p}_v} h_{q+i}(\Phi^{-1}(\tilde{\mathbf{x}})) d_v \end{aligned}$$

for $i = 1, \dots, (s - q)$. The state of the above subsystem is measurable from the output and so the observer design problem can be handled with ease. For the system outputs with relative degree $r_i = 2$, the disturbance terms do not exist and so $\gamma_i(\tilde{\mathbf{x}}, \mathbf{d}) = 0$.

To establish our results, we require the following additional assumptions.

Assumption 5.2 *The mapping $\Phi(\mathbf{x})$ is a diffeomorphism.*

Assumption 5.3 *The known functions $\mathbf{f}(\mathbf{x})$, $\mathbf{G}(\mathbf{x}, \mathbf{u})$ and $\mathbf{p}_i(\mathbf{x})$ are bounded with respect to their arguments. The inputs of the nonlinear system (5.2) are bounded*

with some upper bounds. Further, the system (5.2) is Bounded-Input-Bounded-States (BIBS) stable.

Assumption 5.4 The transferred system (5.7) has the following structure:

$$\begin{bmatrix} \dot{x}_1^{w_j} \\ \vdots \\ \dot{x}_{k_j-1}^{w_j} \\ \dot{x}_{k_j}^{w_j} \end{bmatrix} = \begin{bmatrix} \dot{x}_1^{w_j} \\ \vdots \\ \dot{x}_{k_j}^{w_j} \\ 0 \end{bmatrix} + \begin{bmatrix} \mu_1^{w_j} (x_1^{w_j}, \bar{\mathbf{x}}_{w_j}^*, \tilde{\mathbf{x}}_{w_0}, \mathbf{u}) \\ \mu_2^{w_j} (x_1^{w_j}, x_2^{w_j}, \bar{\mathbf{x}}_{w_j}, \tilde{\mathbf{x}}_{w_0}, \mathbf{u}) \\ \dots \\ \mu_{k_j}^{w_j} (x_1^{w_j}, x_2^{w_j}, \dots, x_{k_j}^{w_j}, \bar{\mathbf{x}}_{w_j}, \tilde{\mathbf{x}}_{w_0}, \mathbf{u}) \end{bmatrix} \\ + \begin{bmatrix} \xi_1^{w_j} (x_1^{w_j}, \bar{\mathbf{x}}_{w_j}^*, \tilde{\mathbf{x}}_{w_0}, \mathbf{d}) \\ \xi_2^{w_j} (x_1^{w_j}, x_2^{w_j}, \bar{\mathbf{x}}_{w_j}, \tilde{\mathbf{x}}_{w_0}, \mathbf{d}) \\ \dots \\ \xi_{k_j}^{w_j} (x_1^{w_j}, x_2^{w_j}, \dots, x_{k_j}^{w_j}, \bar{\mathbf{x}}_{w_j}, \tilde{\mathbf{x}}_{w_0}, \mathbf{d}) \end{bmatrix}$$

for all $j = 1, \dots, q$, where $\bar{\mathbf{x}}_{w_j}^* = \{x_1^{w_1}, x_2^{w_2}, \dots, x_1^{w_{j-1}}\}$, $\bar{\mathbf{x}}_{w_j} = \{\tilde{\mathbf{x}}_{w_1}, \tilde{\mathbf{x}}_{w_2}, \dots, \tilde{\mathbf{x}}_{w_{j-1}}\}$, and $\bar{\mathbf{x}}_{w_j}$ and $\tilde{\mathbf{x}}_{w_0}$ are regarded as inputs to the subsystem under consideration. Then the $\tilde{\mathbf{x}}_{w_j}$ subsystem is uniformly observable for all inputs $\bar{\mathbf{x}}_{w_j}, \tilde{\mathbf{x}}_{w_0}, \mathbf{u}$ and $\mathbf{d}$.

Assumption 5.4 according to **Lemma 2.1** ensures uniform observability for the subsystems irrespective of known and unknown inputs.

For **Class B** systems, the observer design is trivial, as all the states are measured as outputs. In the following section, the design of the observer for **Class A** systems, which is the major concern of this chapter, is addressed.

5.3 Multiple SMO Design for Class A Subsystems

The **Class A** subsystems described by (5.8) can be re-written as

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = \mathbf{A}_{w_j} \tilde{\mathbf{x}}_{w_j} + \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) + \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) d_i(\tilde{\mathbf{x}}, \mathbf{u}, t) \\ y_{w_j} = x_1^{w_j} = \mathbf{C}_{w_j} \tilde{\mathbf{x}}_{w_j} \end{cases} \quad (5.12)$$

where $\xi_{w_j}(\tilde{\mathbf{x}}, \mathbf{d}) = \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) d_i(\tilde{\mathbf{x}}, \mathbf{u}, t)$.

For ease of analysis, all the subsystems $\tilde{\mathbf{x}}_{w_1}, \dots, \tilde{\mathbf{x}}_{w_q}$ of **Class A** can be grouped together to form a single system as

$$\begin{cases} \dot{\tilde{\mathbf{x}}}_w = \mathbf{A}_w \tilde{\mathbf{x}}_w + \boldsymbol{\mu}_w(\tilde{\mathbf{x}}, \mathbf{u}) + \sum_{i=1}^m \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) d_i(\tilde{\mathbf{x}}, \mathbf{u}, t) \\ \mathbf{y}_w = \begin{bmatrix} x_1^{w_1} & \dots & x_1^{w_q} \end{bmatrix}^T = \mathbf{C}_w \tilde{\mathbf{x}}_w \end{cases} \quad (5.13)$$

where

$$\tilde{\mathbf{x}}_w = \begin{bmatrix} \tilde{\mathbf{x}}_{w_1} \\ \vdots \\ \tilde{\mathbf{x}}_{w_{q-1}} \\ \tilde{\mathbf{x}}_{w_q} \end{bmatrix}, \quad \boldsymbol{\mu}_w(\tilde{\mathbf{x}}, \mathbf{u}) = \begin{bmatrix} \boldsymbol{\mu}_{w_1} \\ \vdots \\ \boldsymbol{\mu}_{w_{q-1}} \\ \boldsymbol{\mu}_{w_q} \end{bmatrix}, \quad \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) = \begin{bmatrix} \tilde{\mathbf{Z}}_i^{w_1}(\tilde{\mathbf{x}}) \\ \vdots \\ \tilde{\mathbf{Z}}_i^{w_{q-1}}(\tilde{\mathbf{x}}) \\ \tilde{\mathbf{Z}}_i^{w_q}(\tilde{\mathbf{x}}) \end{bmatrix}$$

and $\mathbf{A}_w = \text{diag} [\mathbf{A}_{w_1}, \mathbf{A}_{w_2}, \dots, \mathbf{A}_{w_q}]$, $\mathbf{C}_w = \text{diag} [\mathbf{C}_{w_1}, \mathbf{C}_{w_2}, \dots, \mathbf{C}_{w_q}]$.

In the following subsection, the structural assumptions that guarantee complete detectability of unknown inputs are discussed, while the observer design, convergence of the subsystems in the sliding mode, and unknown input estimation are dealt with in the later subsections.

5.3.1 Main Assumptions

Arising from the nonlinear transformation, the first state of every **Class A** subsystems (5.7) is measurable as an output. These system dynamics play a crucial role in the complete observability of the unknown inputs. From the system dynamics (5.12), we have the first dynamics as follows:

$$\dot{x}_1^{w_j} = x_2^{w_j} + \mu_1^{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) + z_{11}^{w_j}(\tilde{\mathbf{x}}) d_1 + \dots + z_{1m}^{w_j}(\tilde{\mathbf{x}}) d_m \quad (5.14)$$

for $j = 1, \dots, q$, where $z_{1i}^{w_j}(\tilde{\mathbf{x}})$, $i = 1, \dots, m$, are smooth functions of $\tilde{\mathbf{Z}}_i^{w_j}$ corresponding to dynamics of $x_1^{w_i}$. All these first dynamics of subsystems can be grouped

together for ease of analysis.

The following **Assumption** is the key requirement in the development of sliding mode observer for **Class A** subsystem.

Assumption 5.5 *The number of disturbance inputs m in the transformed domain are less than or equal to the number of outputs q , i.e., $(m \leq q)$. The dynamics of states that are measured as outputs (5.14) of ‘ m ’ subsystems have the following structure:*

$$\begin{bmatrix} \dot{x}_1^{w_1} \\ \dot{x}_1^{w_2} \\ \vdots \\ \dot{x}_1^{w_m} \end{bmatrix} = \begin{bmatrix} x_2^{w_1} \\ x_2^{w_2} \\ \vdots \\ x_2^{w_m} \end{bmatrix} + \begin{bmatrix} \mu_1^{w_1}(\tilde{\mathbf{x}}, \mathbf{u}) \\ \mu_1^{w_2}(\tilde{\mathbf{x}}, \mathbf{u}) \\ \vdots \\ \mu_1^{w_m}(\tilde{\mathbf{x}}, \mathbf{u}) \end{bmatrix} + \begin{bmatrix} z_{11}^{w_1}(\tilde{\mathbf{x}}) & 0 & \cdots & 0 \\ z_{11}^{w_2}(\tilde{\mathbf{x}}) & z_{12}^{w_2}(\tilde{\mathbf{x}}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_{11}^{w_m}(\tilde{\mathbf{x}}) & z_{12}^{w_m}(\tilde{\mathbf{x}}) & \cdots & z_{1m}^{w_m}(\tilde{\mathbf{x}}) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix} \quad (5.15)$$

where $z_{1i}^{w_i}(\tilde{\mathbf{x}}) \neq 0, i = 1, \dots, m$.

Remark 5.3 *The selection of q outputs is done such that the system (5.7) has the structure according to **Assumption 5.5** in the transformed domain. The number of unknown inputs in the transformed domain is less than or equal to the q outputs.*

Remark 5.4 *The above structure guarantees the complete observability of the unknown inputs with respect to the outputs. The unknown inputs can be tackled in the state estimation and can be reconstructed from multiple sliding modes. The problem of state estimation and unknown input estimation are therefore solved simultaneously through multiple sliding modes.*

The structure of the form (5.15) is instrumental in the development of SMO for the case of multiple disturbance terms. The main idea is to deal with each disturbance term or unknown input separately. Therefore, we need multiple robust terms to compensate for multiple disturbance inputs, i.e. we have multiple sliding modes in a single subsystem.

Assumption 5.6 *The distribution vector $\tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}})$ and the function $\mu_w(\tilde{\mathbf{x}}, u)$ are global Lipschitz functions on the argument $\tilde{\mathbf{x}}$ for all $i = 1, \dots, m$.*

5.3.2 Observer Design

For the subsystem (5.12) satisfying **Assumptions 5.4 - 5.6**, an observer of the following form can be designed to estimate the states:

$$\dot{\hat{\mathbf{x}}}_{w_j} = \mathbf{A}_{w_j} \hat{\mathbf{x}}_{w_j} + \boldsymbol{\mu}_{w_j}(\hat{\mathbf{x}}, \mathbf{u}) + \mathbf{L}_{w_j} (y_{w_j} - \mathbf{C}_{w_j} \hat{\mathbf{x}}_{w_j}) + \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_j}(\hat{\mathbf{x}}) u_r^{w_i}(t) \quad (5.16)$$

where

$$\mathbf{L}_{w_j} = \begin{bmatrix} l_1^{w_j} & l_2^{w_j} & \cdots & l_{k_j-1}^{w_j} & l_{k_j}^{w_j} \end{bmatrix}^T \quad (5.17)$$

for all $j = 1, \dots, q$.

In (5.16), $u_r^{w_i}(t)$ is a scalar-valued robust term similar to (4.6), and is given by the sliding mode estimation:

$$u_r^{w_i}(t) = -\frac{\rho_{w_i}}{z_{1i}^{w_i}(\hat{\mathbf{x}})} \text{sign}(y_{w_i} - \hat{x}_1^{w_i}) \quad (5.18)$$

for all $i = 1, \dots, m$.

Remark 5.5 *In the observer design, the disturbance inputs of the system are exactly replaced by the robust terms. The number of robust terms $u_r^{w_i}(t)$ is equal to the number of unknown inputs $\tilde{d}_i$. Clearly, we have multiple sliding modes in the observer system.*

5.3.3 Error Dynamics and Boundedness

Now, we can obtain the error dynamics from (5.16) and (5.12) as

$$\begin{aligned} \dot{\mathbf{e}}_{w_j} &= (\mathbf{A}_{w_j} - \mathbf{L}_{w_j} \mathbf{C}_{w_j}) \mathbf{e}_{w_j} + \boldsymbol{\mu}_{w_j}(\hat{\mathbf{x}}, \mathbf{u}) - \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \\ &\quad + \sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i^{w_j}(\hat{\mathbf{x}}) u_r^{w_i}(t) - \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) d_i(\tilde{\mathbf{x}}, \mathbf{u}, t) \right) \end{aligned} \quad (5.19)$$

Here, $u_r^{w_i}$ is introduced to increase the robustness of the system and to handle the uncertainties. Clearly, multiple robust terms are required to handle the multiple

disturbances or unknown inputs present in the system.

For the combined **Class A** subsystem, the error will be of the form

$$\begin{aligned}\dot{\mathbf{e}}_w &= (\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w) \mathbf{e}_w + \boldsymbol{\mu}_w(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \boldsymbol{\mu}_w(\tilde{\mathbf{x}}, \mathbf{u}) \\ &\quad + \sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i(\hat{\tilde{\mathbf{x}}}) u_r^{w_i}(t) - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) d_i(\tilde{\mathbf{x}}, \mathbf{u}, t) \right)\end{aligned}\quad (5.20)$$

where

$$\begin{aligned}\mathbf{L}_w &= \text{diag} \begin{bmatrix} \mathbf{L}_{w_1} & \mathbf{L}_{w_2} & \cdots & \mathbf{L}_{w_q} \end{bmatrix} \\ &= \text{diag} \begin{bmatrix} \mathbf{S}_{\theta_1}^{-1} \mathbf{C}_{w_1}^T & \mathbf{S}_{\theta_2}^{-1} \mathbf{C}_{w_2}^T & \cdots & \mathbf{S}_{\theta_q}^{-1} \mathbf{C}_{w_q}^T \end{bmatrix}\end{aligned}\quad (5.21)$$

is designed according to high-gain observer, and $\mathbf{S}_{\theta_j}$ is the unique solution of Lyapunov equation

$$\theta_j \mathbf{S}_{\theta_j} + \mathbf{A}_{w_j}^T \mathbf{S}_{\theta_j} + \mathbf{S}_{\theta_j} \mathbf{A}_{w_j} - \mathbf{C}_{w_j}^T \mathbf{C}_{w_j} = 0 \quad (5.22)$$

Then, (5.19) is replaced by

$$\begin{aligned}\dot{\mathbf{e}}_{w_j} &= \left(\mathbf{A}_{w_j} - \mathbf{S}_{\theta_j}^{-1} \mathbf{C}_{w_j}^T \mathbf{C}_{w_j} \right) \mathbf{e}_{w_j} + \boldsymbol{\mu}_{w_j}(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \\ &\quad + \sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i^{w_j}(\hat{\tilde{\mathbf{x}}}) u_r^{w_i}(t) - \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) d_i(\tilde{\mathbf{x}}, \mathbf{u}, t) \right)\end{aligned}\quad (5.23)$$

The following Lemma analyzes the conditions for the boundedness of estimation error dynamics of the **Class A** subsystem. To proceed with the lemma, we let Δ_{θ_j} and Δ_{θ} be the diagonal matrices:

$$\Delta_{\theta_j} = \text{diag} \left(1, \frac{1}{\theta_j}, \cdots, \frac{1}{\theta_j^{k_j-1}} \right) \quad (5.24)$$

$$\Delta_{\theta} = \text{diag} (\Delta_{\theta_1}, \Delta_{\theta_2}, \cdots, \Delta_{\theta_q}) \quad (5.25)$$

Similar to (2.58) and (2.59), we can directly obtain

$$\mathbf{S}_{\theta_j} = \frac{1}{\theta_j} \Delta_{\theta_j} \mathbf{S}_{1_j} \Delta_{\theta_j}$$

$$\Delta_{\theta_j} \mathbf{A}_{w_j} \Delta_{\theta_j}^{-1} = \theta_j \mathbf{A}_{w_j} \quad (5.26)$$

$$\mathbf{C}_{w_j} \Delta_{\theta_j} = \mathbf{C} \Delta_{\theta_j}^{-1} = \mathbf{C}_{w_j} \quad (5.27)$$

where $\mathbf{S}_{1_j}$ is the solution of (5.22) for $\theta_j = 1$.

Lemma 5.1 *Assume that the subsystem (5.12) satisfies **Assumptions 5.3 - 5.6**. Then there exists $\theta_0 > 0$ such that $\forall \theta_j > \theta_0$, the estimator (5.16) will ensure that $\tilde{\mathbf{x}}_{w_j}$ and $\mathbf{e}_{w_j}$ are all bounded.*

Proof: Set $\boldsymbol{\zeta}_{w_j} = \Delta_{\theta_j} \mathbf{e}_{w_j}$, $\boldsymbol{\zeta}_w = \Delta_{\theta} \mathbf{e}_w$ and then

$$\begin{aligned} \dot{\boldsymbol{\zeta}}_{w_j} &= \Delta_{\theta} \left(\mathbf{A}_{w_j} - \mathbf{S}_{\theta_j}^{-1} \mathbf{C}_{w_j}^T \mathbf{C}_{w_j} \right) \Delta_{\theta_j}^{-1} \boldsymbol{\zeta}_{w_j} + \Delta_{\theta_j} \left[\boldsymbol{\mu}_{w_j}(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \right] \\ &\quad + \Delta_{\theta_j} \sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i^{w_j}(\hat{\tilde{\mathbf{x}}}) u_r^{w_i}(t) - \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) d_i(\tilde{\mathbf{x}}, \mathbf{u}, t) \right) \end{aligned} \quad (5.28)$$

Using the equalities (5.24) - (5.27), similar to the analysis in **Theorem 2.1**, for the Lyapunov function $V_{w_j} = \boldsymbol{\zeta}_{w_j}^T \mathbf{S}_{1_j} \boldsymbol{\zeta}_{w_j}$, we can obtain

$$\begin{aligned} \dot{V}_{w_j} &= -\theta_j \boldsymbol{\zeta}_{w_j}^T \mathbf{S}_{1_j} \boldsymbol{\zeta}_{w_j} - \theta \|\mathbf{C}_{w_j} \boldsymbol{\zeta}_{w_j}\|^2 + 2 \boldsymbol{\zeta}_{w_j}^T \mathbf{S}_{1_j} \Delta_{\theta_j} \left[\boldsymbol{\mu}_{w_j}(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \right] \\ &\quad + 2 \boldsymbol{\zeta}_{w_j}^T \mathbf{S}_{1_j} \Delta_{\theta_j} \sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i^{w_j}(\hat{\tilde{\mathbf{x}}}) u_r^{w_i}(t) - \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) d_i(\tilde{\mathbf{x}}, \mathbf{u}, t) \right) \end{aligned} \quad (5.29)$$

By assuming $\theta_j \geq 1$ and due to the triangular structure in **Assumption 5.4** and the Lipschitz **Assumption 5.6**, we have:

$$\begin{aligned} \left\| \Delta_{\theta_j} \left(\boldsymbol{\mu}_{w_j}(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \right) \right\| &\leq \left\| \Delta_{\theta} \left(\boldsymbol{\mu}_w(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \boldsymbol{\mu}_w(\tilde{\mathbf{x}}, \mathbf{u}) \right) \right\| \\ &\leq l_{\boldsymbol{\mu}_w} \sum_{i=1}^{n_k} |\Delta_{\theta} \bar{\mathbf{e}}_w| \leq n_k l_{\boldsymbol{\mu}_w} \|\boldsymbol{\zeta}_w\| \end{aligned} \quad (5.30)$$

where $n_k = k_1 + \dots + k_q$, and $l_{\boldsymbol{\mu}_w}$ is the Lipschitz constant of $\boldsymbol{\mu}_w$.

Since, the states of Class B subsystems are measured as outputs, and due to the triangular structure in **Assumption 5.4**, we have $\bar{\mathbf{e}}_w = (\mathbf{e}^{w_1}, \dots, \mathbf{e}^{w_q}, 0)$.

Since disturbances and states are bounded, we have $|d_i(\tilde{\mathbf{x}}, \mathbf{u}, d)| \leq \bar{d}_i$ for some upper bound $\bar{d}_i$. Since $z_{1i}^{w_i}(\hat{\tilde{\mathbf{x}}})$ and sliding mode gain are bounded, we have $u_r^{w_i}$

bounded as $|u_r^{w_i}| \leq b_{u_r}^i$. Therefore, we have

$$\left\| \tilde{\mathbf{Z}}_i^{w_j}(\hat{\mathbf{x}})u_r^{w_i} - \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}})d_i(\tilde{\mathbf{x}}, \mathbf{u}, t) \right\| \leq b_{\tilde{\mathbf{Z}}_i}^j(\bar{d}_i + b_{u_r}^i) \quad (5.31)$$

for some upper bound $b_{\tilde{\mathbf{Z}}_i}^j$. Therefore, we have from (5.29)

$$\dot{V}_{w_j} \leq -\theta_j V_{w_j} + 2n_k l \boldsymbol{\mu}_w \left\| \mathbf{S}_{1_j} \boldsymbol{\zeta}_{w_j} \right\| \left\| \boldsymbol{\zeta}_w \right\| + 2 \left\| \mathbf{S}_{1_j} \boldsymbol{\zeta}_{w_j} \right\| \sum_{i=1}^m b_{\tilde{\mathbf{Z}}_i}^j(\bar{d}_i + b_{u_r}^i) \quad (5.32)$$

Then, we can obtain

$$\begin{aligned} \dot{V}_w &= \sum_{j=1}^q \dot{V}_{w_j} \\ &\leq -\theta_{j0} V_w + 2qn_k l \boldsymbol{\mu}_w \left\| \mathbf{S}_{1_w} \right\| \left\| \boldsymbol{\zeta}_w \right\|^2 + 2q \left\| \mathbf{S}_{1_w} \right\| \left\| \boldsymbol{\zeta}_w \right\| b_{\mathbf{Z}d} \end{aligned} \quad (5.33)$$

where $\theta_{j0} = \inf \{\theta_1, \dots, \theta_q\}$, $\mathbf{S}_{1_w} = \sup \{\mathbf{S}_{1_1}, \dots, \mathbf{S}_{1_q}\}$ and $b_{\mathbf{Z}d} = \sup_j \left\{ \sum_{i=1}^m b_{\tilde{\mathbf{Z}}_i}^j(\bar{d}_i + b_{u_r}^i) \right\}$.

Finally, we have

$$\begin{aligned} \dot{V}_w &\leq -\theta_{j0} V_w + 2qn_k l \boldsymbol{\mu}_w \sigma(\mathbf{S}_{1_w}) V_w + 2q \boldsymbol{\lambda}_{max}(\mathbf{S}_{1_w}) b_{\mathbf{Z}d} \left\| \boldsymbol{\zeta}_w \right\| \\ &= -c_1 \left\| \boldsymbol{\zeta}_w \right\|^2 + c_2 \left\| \boldsymbol{\zeta}_w \right\| \end{aligned}$$

where $c_1 \triangleq \theta_{j0} - 2qn_k l \boldsymbol{\mu}_w \sigma(\mathbf{S}_{1_w})$, $c_2 \triangleq 2q \boldsymbol{\lambda}_{max}(\mathbf{S}_{1_w}) b_{\mathbf{Z}d}$. By selecting $\theta_{j0} > 2qn_k l \boldsymbol{\mu}_w \sigma(\mathbf{S}_{1_w}) = \theta_{j0}$, we have $c_1 < 0$. Hence $\mathbf{e}_w$ is bounded such that $\|\mathbf{e}_w\| \leq c_2/c_1$. Also, the bound of $\|\mathbf{e}_w\|$ can be made small by selecting a large θ_j . By choosing, $\theta_j > 2qn_k l \boldsymbol{\mu}_w \sigma(\mathbf{S}_{1_w}) = \theta_0$, we have $\dot{V}_w < 0$. Hence this completes the proof. ■

5.3.4 Convergence of Estimation Error in the Sliding Mode

This subsection details the crucial steps in the estimation algorithm for sliding mode gain design and convergence analysis. Although the estimation error is bounded, the main purpose of $u_r^{w_i}$ in (5.16) is to deal with uncertain inputs or disturbances and to improve the estimation accuracy in the presence of unknown

inputs. The procedure is itemized as follows:

- (i) Define the following sliding surfaces,

$$e_1^{w_i} \triangleq \hat{x}_1^{w_i} - x_1^{w_i} = 0 \quad (5.34)$$

for all $i = 1, \dots, m$ for the subsystems $\tilde{\mathbf{x}}_{w_1}, \dots, \tilde{\mathbf{x}}_{w_q}$.

- (ii) Design the sliding mode estimation as (5.18) to reach and maintain the sliding mode for all the subsystems $\tilde{\mathbf{x}}_{w_1}, \dots, \tilde{\mathbf{x}}_{w_q}$.

- (iii) Ensure the estimation error $\mathbf{e}_w$ goes to zero in the sliding surface of $e_1^{w_1} = 0, e_1^{w_2} = 0, \dots, e_1^{w_m} = 0$.

The above procedure guarantees the conditions for reachability and existence of sliding mode for the error trajectories. The following procedure for gain design guarantees the reaching phase, as well as the existence for sliding surfaces (5.34).

(a) Sliding Mode Gain Design

The sliding mode gain is designed sequentially for all the subsystems in **Class A**. The gain ρ_{w_j} of $\tilde{\mathbf{x}}_{w_j}$ subsystem relies on the sliding mode dynamics of the subsystems $\tilde{\mathbf{x}}_{w_1}, \dots, \tilde{\mathbf{x}}_{w_{j-1}}$. The dynamics of the measurement errors will be obtained from (5.15) according to (5.19) and **Assumption 5.4** as follows:

$$\begin{aligned} \begin{bmatrix} \dot{e}_1^{w_1} \\ \dot{e}_1^{w_2} \\ \vdots \\ \dot{e}_1^{w_m} \end{bmatrix} &= \begin{bmatrix} e_2^{w_1} \\ e_2^{w_2} \\ \vdots \\ e_2^{w_m} \end{bmatrix} + \begin{bmatrix} \mu_1^{w_1}(\hat{x}_1^{w_1}, \hat{\mathbf{x}}_{w_0}, \mathbf{u}) - \mu_1^{w_1}(x_1^{w_1}, \tilde{\mathbf{x}}_{w_0}, \mathbf{u}) \\ \mu_1^{w_2}(\hat{x}_1^{w_1}, \hat{x}_1^{w_2}, \hat{\mathbf{x}}_{w_0}, \mathbf{u}) - \mu_1^{w_2}(x_1^{w_1}, x_1^{w_2}, \tilde{\mathbf{x}}_{w_0}, \mathbf{u}) \\ \vdots \\ \mu_1^{w_m}(\hat{x}_1^{w_1}, \dots, \hat{x}_1^{w_m}, \hat{\mathbf{x}}_{w_0}, \mathbf{u}) - \mu_1^{w_m}(x_1^{w_1}, \dots, x_1^{w_m}, \tilde{\mathbf{x}}_{w_0}, \mathbf{u}) \end{bmatrix} \\ &+ \begin{bmatrix} z_{11}^{w_1}(\hat{\mathbf{x}}) & 0 & \cdots & 0 \\ z_{11}^{w_2}(\hat{\mathbf{x}}) & z_{12}^{w_2}(\hat{\mathbf{x}}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_{11}^{w_m}(\hat{\mathbf{x}}) & z_{12}^{w_m}(\hat{\mathbf{x}}) & \cdots & z_{1m}^{w_m}(\hat{\mathbf{x}}) \end{bmatrix} \begin{bmatrix} u_r^{w_1} \\ u_r^{w_2} \\ \vdots \\ u_r^{w_m} \end{bmatrix} \end{aligned} \quad (5.35)$$

$$- \begin{bmatrix} z_{11}^{w_1}(\tilde{\mathbf{x}}) & 0 & \cdots & 0 \\ z_{11}^{w_2}(\tilde{\mathbf{x}}) & z_{12}^{w_2}(\tilde{\mathbf{x}}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_{11}^{w_m}(\tilde{\mathbf{x}}) & z_{12}^{w_m}(\tilde{\mathbf{x}}) & \cdots & z_{1m}^{w_m}(\tilde{\mathbf{x}}) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$

The error dynamics reach their corresponding sliding modes one-by-one sequentially. The reaching phase of the sliding modes is dependent on the structural **Assumption 5.4**. The states of $\tilde{\mathbf{x}}_{w_0}$ of Class B subsystems are all measured as outputs, and hence in a finite time we have $\hat{\tilde{\mathbf{x}}}_{w_0} \equiv \tilde{\mathbf{x}}_{w_0}$ (as will be shown in the next section).

The following lemma analyzes the sufficient condition for the sliding mode gain design for existence of its corresponding sliding mode.

Lemma 5.2 *For the subsystem (5.12) satisfying **Assumptions 5.3 - 5.6** and using the estimator (5.16), the sliding mode estimation (5.18) ensures that the sliding surfaces (5.34) can be reached and maintained provided the gains satisfy*

$$\rho_{w_j} > \sup \left| e_2^{w_j}(t) + \sum_{i=1}^{j-1} \left[z_{1i}^{w_j}(\hat{\tilde{\mathbf{x}}}) u_r^{w_i} - z_{1i}^{w_j}(\tilde{\mathbf{x}}) d_i \right] + z_{1j}^{w_j}(\tilde{\mathbf{x}}) d_j \right| \quad (5.36)$$

and $l_1^{w_j} > l_{\mu_1}^{w_j}$ for some Lipschitz constant $l_{\mu_1}^{w_j}$ of $\mu_1^{w_j}(\cdot)$.

Proof: Here, we analyze the dynamics of $e_1^{w_j}$ of the error dynamics $\mathbf{e}_{w_j}$ from (5.35), i.e.

$$\begin{aligned} \dot{e}_1^{w_j} &= -l_1^{w_j} e_1^{w_j} + e_2^{w_j}(t) + \mu_1^{w_j}(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \mu_1^{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \\ &\quad + \sum_{i=1}^{j-1} \left[z_{1i}^{w_j}(\hat{\tilde{\mathbf{x}}}) u_r^{w_i} - z_{1i}^{w_j}(\tilde{\mathbf{x}}) d_i \right] + z_{1j}^{w_j}(\hat{\tilde{\mathbf{x}}}) u_r^{w_j} - z_{1j}^{w_j}(\tilde{\mathbf{x}}) d_j \end{aligned} \quad (5.37)$$

By using the Lyapunov function $V_{j1} = \frac{1}{2}(e_1^{w_j})^2$, and sliding mode estimation (5.18), we have

$$\begin{aligned} \dot{V}_{j1} &= \dot{e}_1^{w_j} e_1^{w_j} \\ &= -l_1^{w_j} (e_1^{w_j})^2 + \left[\mu_1^{w_j}(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \mu_1^{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \right] e_1^{w_j} \end{aligned}$$

$$+ \left[e_2^{w_j}(t) + \sum_{i=1}^{j-1} \left[z_{1i}^{w_j}(\hat{\mathbf{x}}) u_r^{w_i} - z_{1i}^{w_j}(\tilde{\mathbf{x}}) d_i \right] - z_{1j}^{w_j}(\tilde{\mathbf{x}}) d_j \right] e_1^{w_j} - \rho_{w_j} |e_1^{w_j}|$$

There exists a finite gain satisfying (5.36) such that, if $e_1^{w_j} \neq 0$, we have

$$\dot{V}_{j1} < -l_1^{w_j} (e_1^{w_j})^2 + [\mu_1^{w_j}(\hat{\mathbf{x}}, \mathbf{u}) - \mu_1^{w_j}(\tilde{\mathbf{x}}, \mathbf{u})] e_1^{w_j} \quad (5.38)$$

Based on Lipschitz **Assumption 5.6**, we have

$$\dot{V}_{j1} < -l_1^{w_j} (e_1^{w_j})^2 + l_{\mu_1}^{w_j} |\mathbf{e}_{w_j}| |e_1^{w_j}| \quad (5.39)$$

Since $\mu_1^{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) = \mu_1^{w_j}(x_1^{w_1}, \dots, x_1^{w_j}, \tilde{\mathbf{x}}_{w_0}, \mathbf{u})$, and as the states of the subsystems converge sequentially in the sliding modes of $e_1^{w_1}, \dots, e_1^{w_{j-1}}$, plus the states of **Class B** subsystems $\tilde{\mathbf{x}}_{w_0}$ also converge, we eventually have $\mathbf{e}_{w_j} \equiv [e_1^{w_j} \ 0 \ \dots \ 0]^T$. Therefore, $\dot{V}_{j1} < -l_1^{w_j} (e_1^{w_j})^2 + l_{\mu_1}^{w_j} (e_1^{w_j})^2$. Hence $l_1^{w_j} > l_{\mu_1}^{w_j}$ ensures that the sliding surface $e_1^{w_j} = 0$ can be reached asymptotically and maintain the state on it thereafter. ■

(b) Dynamics in the Sliding Mode

Since conditions (5.36) guarantee the sliding mode, the corresponding subsystems error dynamics $\mathbf{e}_{w_j}$ also converge sequentially in the sliding modes of $e_1^{w_1} = 0, e_1^{w_2} = 0, \dots, e_1^{w_q} = 0$. The equivalent control for the sliding dynamics is evaluated sequentially. In the following analysis, the superscript d denotes the estimated $\hat{\mathbf{x}}_{w_j}$ -related variables in the sliding mode, i.e., $\mathbf{e}_{w_j}^d = \begin{bmatrix} e_{1,d}^{w_j} & e_{2,d}^{w_j} & \dots & e_{k_j,d}^{w_j} \end{bmatrix} \triangleq \hat{\mathbf{x}}_{w_j}^d - \tilde{\mathbf{x}}_{w_j}$.

Consider the error dynamics $e_1^{w_1}$ in the sliding mode. Once the trajectory reaches the sliding mode and stays in its vicinity, we have $e_1^{w_1} = 0$ and $\dot{e}_1^{w_1} = 0$, $\hat{x}_1^{w_1} = x_1^{w_1}$, the equivalent control of $u_r^{w_i}$ can be obtained from (5.37) as follows [56]:

$$0 = 0 + e_{2,d}^{w_1} + \mu_1^{w_1}(\hat{x}_1^{w_1}, \hat{\mathbf{x}}_{w_0}, \mathbf{u}) - \mu_1^{w_1}(x_1^{w_1}, \tilde{\mathbf{x}}_{w_0}, \mathbf{u}) + z_{11}^{w_1}(\hat{\mathbf{x}}) u_{eq}^{w_1} - z_{11}^{w_1}(\tilde{\mathbf{x}}) d_1$$

where $u_{eq}^{w_1}$ represents the equivalent control dynamics of $u_r^{w_1}$. According to struc-

tural **Assumption 5.4**, in the sliding mode of $e_1^{w_1} = 0$, we have $z_{11}^{w_1}(\tilde{\mathbf{x}}) \equiv z_{11}^{w_1}(\hat{\tilde{\mathbf{x}}})$ and $\mu_1^{w_1}(\hat{x}_1^{w_1}, \hat{\tilde{\mathbf{x}}}_{w_0}, \mathbf{u}) \equiv \mu_1^{w_1}(x_1^{w_1}, \tilde{\mathbf{x}}_{w_0}, \mathbf{u})$. Therefore

$$u_{eq}^{w_1} = -\frac{e_{2,d}^{w_1}}{z_{11}^{w_1}(\hat{\tilde{\mathbf{x}}})} + d_1 \quad (5.40)$$

Similarly for the trajectory $e_1^{w_2}$, in the sliding mode $e_1^{w_1} = 0$ and $e_1^{w_2} = 0$, we may evaluate

$$u_{eq}^{w_2} = -\frac{e_{2,d}^{w_2}}{z_{12}^{w_2}(\hat{\tilde{\mathbf{x}}})} - \frac{1}{z_{12}^{w_2}(\hat{\tilde{\mathbf{x}}})} \left[z_{11}^{w_2}(\hat{\tilde{\mathbf{x}}}) u_{eq}^{w_1} - z_{11}^{w_2}(\tilde{\mathbf{x}}) d_1 \right] + \frac{z_{12}^{w_2}(\tilde{\mathbf{x}})}{z_{12}^{w_2}(\hat{\tilde{\mathbf{x}}})} d_2$$

Substituting $u_{eq}^{w_1}$ (5.40) into the above equation, we have

$$u_{eq}^{w_2} = -\frac{e_{2,d}^{w_2}}{z_{12}^{w_2}(\hat{\tilde{\mathbf{x}}})} + \frac{z_{11}^{w_2}(\hat{\tilde{\mathbf{x}}})}{z_{12}^{w_2}(\hat{\tilde{\mathbf{x}}}) z_{11}^{w_1}(\hat{\tilde{\mathbf{x}}})} e_{2,d}^{w_1} - \frac{1}{z_{12}^{w_2}(\hat{\tilde{\mathbf{x}}})} \left[z_{11}^{w_2}(\hat{\tilde{\mathbf{x}}}) d_1 - z_{11}^{w_2}(\tilde{\mathbf{x}}) d_1 \right] + \frac{z_{12}^{w_2}(\tilde{\mathbf{x}})}{z_{12}^{w_2}(\hat{\tilde{\mathbf{x}}})} d_2$$

In the sliding mode of $e_1^{w_1} = 0$ and $e_1^{w_2} = 0$, based on **Assumption 5.4**, we have $z_{11}^{w_2}(\tilde{\mathbf{x}}) \equiv z_{11}^{w_2}(\hat{\tilde{\mathbf{x}}})$ and $z_{12}^{w_2}(\tilde{\mathbf{x}}) \equiv z_{12}^{w_2}(\hat{\tilde{\mathbf{x}}})$. Hence, we have

$$u_{eq}^{w_2} = d_2 - \frac{e_{2,d}^{w_2}}{z_{12}^{w_2}(\hat{\tilde{\mathbf{x}}})} + \frac{z_{11}^{w_2}(\hat{\tilde{\mathbf{x}}})}{z_{12}^{w_2}(\hat{\tilde{\mathbf{x}}}) z_{11}^{w_1}(\hat{\tilde{\mathbf{x}}})} e_{2,d}^{w_1} \quad (5.41)$$

By substitution of successive equivalent control elements, we can obtain the equivalent control for all the measurement errors. Therefore in the sliding modes of $e_1^{w_1} = 0, \dots, e_1^{w_m} = 0$, the equivalent control of $u_{eq}^{w_m}$ can be evaluated, which is similar to the evaluation of (5.41) and also contains the second error dynamics $e_{2,d}^{w_1}, e_{2,d}^{w_2}, \dots, e_{2,d}^{w_m}$ of all the subsystems. In short, the equivalent control of $e_1^{w_i} = 0$ can be represented as

$$u_{eq}^{w_i} = d_i + \chi_i \left(e_{2,d}^{w_1}, e_{2,d}^{w_2}, \dots, e_{2,d}^{w_i}, \hat{\tilde{\mathbf{x}}}_w \right) \quad (5.42)$$

for all $i = 1, \dots, m$, where $\chi_i(\cdot)$ is a function of second error dynamics of the subsystems.

Remark 5.6 The sliding mode gain design ' ρ_{w_j} ' (5.36) depends on the robust terms of other subsystems. In fact, the condition (5.36) can be relaxed by replacing

the robust terms with their corresponding equivalent control elements as the system is in the sliding mode of $e_1^{w_1}, \dots, e_1^{w_{j-1}}$. Therefore, we have

$$\rho_{w_j} > \sup \left| e_2^{w_j}(t) + \sum_{i=1}^{j-1} z_{1i}^{w_j} \chi_i(\cdot) + z_{1j}^{w_j}(\tilde{\mathbf{x}}) d_j \right| \quad (5.43)$$

So, finally the robust terms only depend on their respective unknown input and second error dynamics of the equivalent controls (5.42).

Remark 5.7 The equivalent controls evaluated here are only for the purpose of analysis. In the final observer design, we need not compute the dynamics in the sliding mode as the disturbances will be dealt with by the corresponding robust terms automatically.

(c) Convergence of Error

Substituting the above equivalent controls (5.40)-(5.42) into (5.19) we have the estimation error dynamics in the sliding mode as

$$\begin{aligned} \dot{\mathbf{e}}_{w_j}^d &= \left(\mathbf{A}_{w_j} - \mathbf{S}_{\theta_j}^{-1} \mathbf{C}_{w_j}^T \mathbf{C}_{w_j} \right) \mathbf{e}_{w_j}^d + \boldsymbol{\mu}_{w_j}(\hat{\mathbf{x}}, \mathbf{u}) - \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \\ &\quad + \sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i^{w_j}(\hat{\mathbf{x}}) - \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) \right) u_{eq}^{w_i}(t) \\ &\quad - \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) \chi_i(e_{2,d}^{w_1}, e_{2,d}^{w_2}, \dots, e_{2,d}^{w_i}) \end{aligned} \quad (5.44)$$

From the above, it can be clearly seen that equivalent control completely cancels the unknown inputs. Finally, the error dynamics in the sliding mode is free from the unknown inputs. Based on the structural **Assumptions 5.4 - 5.6**, we can see that the equilibrium point of error dynamics (5.44) is $\mathbf{e}_w^d = 0$, i.e., $\hat{\mathbf{x}}_w^d = \tilde{\mathbf{x}}_w$.

The following theorem examines further the condition for the asymptotical stability of the estimator error of the subsystems sequentially.

Theorem 5.1 For the subsystem (5.12) satisfying **Assumptions 5.3 - 5.6**, and the estimator (5.16) with the sliding mode gain (5.36), there exists $\theta_1 > 0$ such

that $\forall \theta_j > \theta_1$, the estimator will ensure that the estimation error is asymptotically stable in the multiple sliding modes of $e_1^{w_1} = 0, \dots, e_1^{w_q} = 0$.

Proof: The proof is an extension of that in **Lemma 5.1**. By selecting $\bar{\zeta}_{w_j} = \Delta_{\theta_j} \mathbf{e}_{w_j}^d$, $\bar{\zeta}_w = \Delta_{\theta} \mathbf{e}_w^d$ and Lyapunov function $V_{w_j}^d = \bar{\zeta}_{w_j}^T \mathbf{S}_{1_j} \bar{\zeta}_{w_j}$, and with (5.44), we can obtain similar to (5.29) the following derivative:

$$\begin{aligned} \dot{V}_{w_j}^d &= -\theta_j \bar{\zeta}_{w_j}^T \mathbf{S}_{1_j} \bar{\zeta}_{w_j} - \theta_j \|C_{w_j} \bar{\zeta}_{w_j}\|^2 + 2\bar{\zeta}_{w_j}^T \mathbf{S}_{1_j} \Delta_{\theta_j} \left[\mu_{w_j}(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \mu_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \right] \\ &\quad + 2\bar{\zeta}_{w_j}^T \mathbf{S}_{1_j} \Delta_{\theta_j} \sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i^{w_j}(\hat{\tilde{\mathbf{x}}}) - \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) \right) u_{eq}^{w_i}(t) \\ &\quad - 2\bar{\zeta}_{w_j}^T \mathbf{S}_{1_j} \Delta_{\theta_j} \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) \chi_i(e_{2,d}^{w_1}, e_{2,d}^{w_2}, \dots, e_{2,d}^{w_i}) \end{aligned} \quad (5.45)$$

By assuming $\theta_j \geq 1$ and due to the triangular structure in **Assumption 5.4** and the Lipschitz condition in **Assumption 5.6**, we have:

$$\begin{aligned} \left\| \Delta_{\theta_j} \left(\mu_{w_j}(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \mu_{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \right) \right\| &\leq \sum_{i=1}^{k_j} \left| \frac{1}{\theta_j^{i-1}} \left(\mu_i^{w_j}(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \mu_i^{w_j}(\tilde{\mathbf{x}}, \mathbf{u}) \right) \right| \\ &\leq l_{\mu_j} \sum_{i=1}^{k_j} \frac{1}{\theta_j^{i-1}} |\bar{\mathbf{e}}_{w_j}| \leq k_j l_{\mu_j} \|\Delta_{\theta_j} \mathbf{e}_{w_j}^d\| \\ &\leq k_j l_{\mu_j} \|\bar{\zeta}_{w_j}\| \end{aligned} \quad (5.46)$$

where l_{μ_j} is the largest Lipschitz constant of μ_{w_j} . Since the states of Class B subsystems are measured as outputs, and due to the triangular structure in **Assumption 5.4**, we have $\bar{\mathbf{e}}_{w_j} = (\mathbf{e}^{w_1}, \dots, \mathbf{e}^{w_j}, \mathbf{0})$. Also, as the subsystems converge sequentially, the j^{th} subsystem converges after all $j-1$ subsystems converge to the true states. Therefore we have $\bar{\mathbf{e}}_{w_j} \equiv [0 \ \dots \ 0, \mathbf{e}_{w_j}, \mathbf{0}]$.

From (5.40), as the second dynamics $e_{2,d}^{w_j}$ and $\hat{\tilde{\mathbf{x}}}_w$ are bounded, we have $|u_{eq}^{w_i}| \leq b_{u_i}$ for some upper bound b_{u_i} . Similar to (5.46), we can obtain

$$\left\| \Delta_{\theta_j} \left(\tilde{\mathbf{Z}}_i^{w_j}(\hat{\tilde{\mathbf{x}}}) - \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) \right) u_{eq}^{w_i} \right\| \leq k_j b_{u_i} l_{\mathbf{Z}_j}^i \|\bar{\zeta}_{w_j}\| \quad (5.47)$$

where $l_{\mathbf{Z}_j}^i$ is the largest Lipschitz constant of $\tilde{\mathbf{Z}}_i^{w_j}$. Based on the boundedness of

the functions $z_{ij}(\hat{\mathbf{x}})$, $\forall i, j$, we can have

$$\|\tilde{\mathbf{Z}}_i^{w_j} \chi_i(e_{2,d}^{w_1}, e_{2,d}^{w_2}, \dots, e_{2,d}^{w_i}, \hat{\mathbf{x}}_w)\| \leq b_{\mathbf{Z}_i}^j b_{\chi_i} \|\mathbf{e}_w^d\| \quad (5.48)$$

for some upper bound $b_{\mathbf{Z}_i}^j$, and b_{χ_i} is the bounded constant for all the successive bounds in $\chi_i(\cdot)$.

Using the above inequalities, from (5.45), we can obtain

$$\begin{aligned} \dot{V}_{w_j}^d &\leq -\theta_j V_{w_j}^d + 2k_j l_{\boldsymbol{\mu}_j} \|\bar{\boldsymbol{\zeta}}_{w_j} \mathbf{S}_{1_j}\| \|\bar{\boldsymbol{\zeta}}_{w_j}\| + 2k_j \|\bar{\boldsymbol{\zeta}}_{w_j} \mathbf{S}_{1_j}\| \sum_{i=1}^m b_{u_i} l_{\mathbf{Z}_j} \|\bar{\boldsymbol{\zeta}}_{w_j}\| \\ &\quad + 2 \|\bar{\boldsymbol{\zeta}}_{w_j} \mathbf{S}_{1_j}\| \Delta_\theta \sum_{i=1}^m b_{\mathbf{Z}_i}^j b_{\chi_i} \|\mathbf{e}_w^d\| \end{aligned} \quad (5.49)$$

For the Lyapunov function $\dot{V}_w^d = \sum_{j=1}^q \dot{V}_{w_j}^d$, we have

$$\begin{aligned} \dot{V}_w^d &\leq -\theta_{j0} V_w^d + 2qk_d l_{\boldsymbol{\mu}_d} \|\mathbf{S}_{1_w}\| \|\bar{\boldsymbol{\zeta}}_w\|^2 + 2qk_d l_{Z_u} \|\mathbf{S}_{1_w}\| \|\bar{\boldsymbol{\zeta}}_w\|^2 \\ &\quad + 2qb_{Z_\chi} \|\mathbf{S}_{1_w}\| \|\bar{\boldsymbol{\zeta}}_w\|^2 \\ &\leq -(\theta_{j0} - [2qk_d l_{\boldsymbol{\mu}_d} \sigma(\mathbf{S}_{1_w}) + 2qk_d l_{Z_u} \sigma(\mathbf{S}_{1_w}) + 2qb_{Z_\chi} \sigma(\mathbf{S}_{1_w})]) V_w^d \end{aligned} \quad (5.50)$$

where $\theta_{j0} = \inf \{\theta_1, \dots, \theta_q\}$, $k_d = \sup \{k_1, \dots, k_q\}$, $l_{Z_u} = \sup_j \left\{ \sum_{i=1}^m b_{u_i} l_{\mathbf{Z}_j} \right\}$, $b_{Z_\chi} = \sup_j \left\{ \sum_{i=1}^m b_{\mathbf{Z}_i}^j b_{\chi_i} \right\}$, $l_{\boldsymbol{\mu}_d} = \sup_j \{l_{\boldsymbol{\mu}_j}\}$.

Setting $\theta_{j0} > [2qk_d l_{\boldsymbol{\mu}_d} \sigma(\mathbf{S}_{1_w}) + 2qk_d l_{Z_u} \sigma(\mathbf{S}_{1_w}) + 2qb_{Z_\chi} \sigma(\mathbf{S}_{1_w})] = \theta_1$, we have $\dot{V}_w^d < 0$. Hence, in the multiple sliding modes, an appropriate selection of θ_j can guarantee the exponential convergence of the estimator. ■

5.4 Sliding Mode Observer Design for Class B Subsystems

For the transformed **Class B** subsystem, the observer can be designed as follows:

$$\dot{\hat{x}}_i^{w_0} = \varphi_i(\hat{\mathbf{x}}, \mathbf{u}) + l_{i,w_0} (y_{i,w_0} - \hat{x}_i^{w_0}) - \rho_{i,w_0} \text{sign}(y_{i,w_0} - \hat{x}_i^{w_0}) \quad (5.51)$$

for $i = 1, \dots, (s-q)$. With proper selection of sliding mode gain ρ_{i,w_0} , we can ensure the asymptotic convergence of the above estimator. The following lemma shows that the estimation error will converge asymptotically if we select the following sliding surface:

$$e_i^{w_0} = \hat{x}_i^{w_0} - x_i^{w_0} = 0 \quad (5.52)$$

Lemma 5.3 *The subsystem described by (5.11) with sliding mode estimator of the form (5.51) ensures that the sliding surface $e_i^{w_0} = 0$ can be reached and maintained provided that the gain satisfies*

$$\rho_{i,w_0} > \sup \left| \varphi_i(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \varphi_i(\tilde{\mathbf{x}}, \mathbf{u}) - \gamma_i(\tilde{\mathbf{x}}, \mathbf{d}) \right| \quad (5.53)$$

and $l_{i,w_0} > 0$ for all $i = 1, \dots, (s-q)$.

Proof: The error dynamics from (5.11) and (5.51) can be formulated as

$$\dot{e}_i^{w_0} = \varphi_i(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \varphi_i(\tilde{\mathbf{x}}, \mathbf{u}) - \gamma_i(\tilde{\mathbf{x}}, \mathbf{d}) - l_{i,w_0} e_i^{w_0} - \rho_{i,w_0} \text{sign}(e_i^{w_0}) \quad (5.54)$$

By choosing Lyapunov function candidate as $V_i = \frac{1}{2}(e_i^{w_0})^2$, and using the above error dynamics (5.54), we have

$$\dot{V}_i = \dot{e}_i^{w_0} e_i^{w_0} = -l_{i,w_0} (e_i^{w_0})^2 - \rho_{i,w_0} |e_i^{w_0}| + \left[\varphi_i(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \varphi_i(\tilde{\mathbf{x}}, \mathbf{u}) - \gamma_i(\tilde{\mathbf{x}}, \mathbf{d}) \right] e_i^{w_0}$$

There exists a finite constant gain ρ_{i,w_0} satisfying (5.53) such that, if $e_i^{w_0} \neq 0$, we have

$$\dot{V}_i < -l_{i,w_0} (e_i^{w_0})^2$$

Hence, with $l_{i,w_0} > 0$, we have $\dot{V}_i < 0$ if $e_i^{w_0} \neq 0$. The sliding mode estimator (5.51) using the gain (5.53) ensures that the sliding surface $e_i^{w_0} = 0$ can be reached in a finite time and maintained thereafter. ■

Remark 5.8 *For the case of outputs with relative degree $r_i = 2$, the design of gain*

and observer will be the same with $\gamma_i(\tilde{\mathbf{x}}, \mathbf{d}) = 0$ in the analysis of **Lemma 5.3**.

Remark 5.9 In practice, as all the states of **Class B** are measured as outputs, the observer design for state estimation is not necessary. The above design of sliding mode observer also incorporates the unknown input estimation for outputs with relative degree one $r_i = 1$.

5.5 Unknown Input Estimation

Once the trajectories reach the sliding mode and all the states converge to the true states, the equivalent control $u_{eq}^{w_i}$ becomes equal to the unknown input or disturbances in the system. After all the states are converged, we have,

$$e_{2,d}^{w_1} \rightarrow 0, e_{2,d}^{w_2} \rightarrow 0 \cdots, e_{2,d}^{w_m} \rightarrow 0$$

Therefore

$$\chi_i \left(e_{2,d}^{w_1}, e_{2,d}^{w_2}, \cdots, e_{2,d}^{w_i}, \hat{\tilde{\mathbf{x}}}_w \right) \rightarrow 0$$

for all $i = 1, \cdots, m$.

From the above, it is clear that the equivalent control (5.42) converges to the respective unknown input. Therefore,

$$\begin{aligned} u_{eq}^{w_1} &\rightarrow d_1 \\ u_{eq}^{w_2} &\rightarrow d_2 \\ \cdots &\quad \cdots \\ u_{eq}^{w_m} &\rightarrow d_m \end{aligned} \tag{5.55}$$

Similar to (3.26), the unknown inputs can be estimated from their respective equiv-

alent controls signals as:

$$\hat{d}_i \approx \left(\frac{\rho_{w_i}}{z_{ii}^{w_i}(\hat{\mathbf{x}})} \text{sign}(e_1^{w_i}) \right)_{eq} = \frac{\rho_{w_i}}{z_{ii}^{w_i}(\hat{\mathbf{x}})} \frac{e_1^{w_i}}{|e_1^{w_i}| + \delta} \quad (5.56)$$

for all $i = 1, \dots, m$.

Remark 5.10 *The estimations of the unknown inputs rely only on the output estimation errors and hence the estimations can be performed online together with state estimation.*

5.6 Observer in the Original Space

Although the convergence of the estimator has been proven in the transformed space, the main purpose is to produce the estimation of the original system. Hence, we need to transfer the observer for subsystems of **Classes A, B** back into the original space. Under **Assumption 5.2**, using (5.6), we have

$$\dot{\mathbf{x}} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]^{-1} \dot{\hat{\mathbf{x}}} \quad (5.57)$$

Similarly, for the combined observer in the transformed space described by (5.16) and (5.51) by means of inverse transformation, we can obtain the observer in the original space as

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{G}(\hat{\mathbf{x}}, \mathbf{u}) + \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}}^{-1} \mathbf{L}_{tr} \quad (5.58)$$

where the $\mathbf{L}_{tr}$ matrix stands for the observer gains together with feedback terms designed in the transformed space. The $\mathbf{L}_{tr}$ matrix constitutes the correction terms

from all the subsystems and is given as follows:

$$\mathbf{L}_{tr} = \begin{bmatrix} \mathbf{L}_{w_1}(y_{w_1} - \hat{x}_1^{w_1}) - \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_1} \frac{\rho_{w_i}}{z_{1i}^{w_i}(\hat{\mathbf{x}})} \text{sign}(y_{w_i} - \hat{x}_1^{w_i}) \\ \vdots \\ \mathbf{L}_{w_q}(y_{w_q} - \hat{x}_1^{w_q}) - \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_q} \frac{\rho_{w_i}}{z_{1i}^{w_i}(\hat{\mathbf{x}})} \text{sign}(y_{w_i} - \hat{x}_1^{w_i}) \\ l_{1,w_0}(y_{1,w_0} - \hat{x}_1^{w_0}) - \rho_{1,w_0} \text{sign}(y_{1,w_0} - \hat{x}_1^{w_0}) \\ \vdots \\ l_{s,w_0}(y_{s-q,w_0} - \hat{x}_{(s-q)}^{w_0}) - \rho_{(s-q),w_0} \text{sign}(y_{s-q,w_0} - \hat{x}_{(s-q)}^{w_0}) \end{bmatrix} \quad (5.59)$$

5.7 Robust NLO with Multiple Sliding Modes

Earlier, the feedback gain design has followed the procedure of HGO. In this section, we discuss the gain design based on the design of NLO. The observer design is similar to the development presented earlier, except for the design of the feedback gain $\mathbf{L}_w$.

For the combined **Class A** subsystem, the error will be of the form:

$$\begin{aligned} \dot{\mathbf{e}}_w &= (\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w) \mathbf{e}_w + \boldsymbol{\mu}_w(\hat{\mathbf{x}}, \mathbf{u}) - \boldsymbol{\mu}_w(\tilde{\mathbf{x}}, \mathbf{u}) \\ &+ \sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i(\hat{\mathbf{x}}) u_r^{w_i}(t) - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) d_i(\tilde{\mathbf{x}}, \mathbf{u}, t) \right) \end{aligned} \quad (5.60)$$

where

$$\mathbf{L}_w = \text{diag} \begin{bmatrix} \mathbf{L}_{w_1} & \mathbf{L}_{w_2} & \cdots & \mathbf{L}_{w_q} \end{bmatrix} \quad (5.61)$$

is a properly chosen estimation gain such that the eigenvalues of $\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w$ are stable, and there exists a positive symmetric matrix $\mathbf{Q}_w = \mathbf{Q}_w^T > 0$ satisfying

$$(\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w)^T \mathbf{Q}_w + \mathbf{Q}_w (\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w) = -\mathbf{I} \quad (5.62)$$

The result of robust NLO is established by replacing **Lemma 5.1** and **Theorem 5.1** with the following lemma and theorem that analyze the conditions for the

boundedness of the estimation errors dynamics of the combined Class A system, and establish the convergence of estimation error.

Lemma 5.4 *For the subsystem (5.12) satisfying Assumptions 5.3 - 5.6, the estimator (5.16) ensures that $\tilde{\mathbf{x}}$ and $\mathbf{e}_{w_j}$ are all bounded provided that*

$$\lambda_{\max}(\mathbf{Q}_w) \leq \frac{1}{2l_{\boldsymbol{\mu}}} \quad (5.63)$$

for some Lipschitz constant $l_{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}_w(\cdot)$

Proof: See Appendix B.3. ■

The asymptotical stability of error dynamics in the sliding mode is proven by the following theorem.

Theorem 5.2 *For the subsystem (5.12) satisfying Assumptions 5.3 - 5.6 and using the estimator (5.16), the sliding mode gain (5.36) ensures that the estimation error is asymptotically stable in the multiple sliding modes of $e_1^{w_1} = 0, \dots, e_1^{w_q} = 0$ provided the gain $\mathbf{L}_w$ satisfies (5.62), (5.63), and*

$$\lambda_{\max}(\mathbf{Q}_w) \leq \frac{1}{2} \left[l_{\boldsymbol{\mu}} + \sum_{i=1}^m \left[l_{\mathbf{z}_i} b_{u_i} + b_{\tilde{\mathbf{z}}_i}^* b_{\chi_i} \right] \right]^{-1} \quad (5.64)$$

for some Lipschitz constants $l_{\boldsymbol{\mu}}$, $l_{\mathbf{z}_i}$ and some upper bounds b_{u_i} , $b_{\tilde{\mathbf{z}}_i}^*$, b_{χ_i} .

Proof: See Appendix B.4. ■

The condition (5.64) is stronger than condition (5.63). If the unknown input distribution matrices $\tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}})$ are constant matrices or functions of the outputs, i.e. $\tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) = \tilde{\mathbf{Z}}_i(x_1^{w_1}, \dots, x_2^{w_q}, \tilde{\mathbf{x}}_{w_0})$, we have $l_{\mathbf{z}_i} = 0$. So, the condition (5.64) reduces to the form

$$\lambda_{\max}(\mathbf{Q}_w) \leq \frac{1}{2 \left[l_{\boldsymbol{\mu}} + \sum_{i=1}^m b_{\tilde{\mathbf{z}}_i}^* b_{\chi_i} \right]} \quad (5.65)$$

In the absence of unknown inputs, i.e., if all the inputs are known, the condition

(5.64) will be reduced to the form:

$$\begin{aligned}\lambda_{max}(\mathbf{Q}_w) &\leq \frac{1}{2[l_{\boldsymbol{\mu}} + \sum_{i=1}^m l_{\mathbf{z}_i}]} \\ &= \frac{1}{2(l_{\boldsymbol{\mu}}^*)}\end{aligned}\quad (5.66)$$

where $l_{\boldsymbol{\mu}}^*$ is the equivalent Lipchitz constant for the nonlinear part in the absence of disturbances. The gain design will be similar to the analysis carried in Section 3.3.5.

5.8 Illustrative Example

In this section, we choose the motivating example (5.1) to illustrate the effectiveness of the proposed method for state and unknown input estimations, i.e.

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ \sin x_1 - 2x_2x_3 \\ 2x_2 - 2x_3^2 \end{bmatrix} + \begin{bmatrix} -x_1 & 0 \\ 0 & x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0.6 & 0.4 \\ 1 & -0.25 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad (5.67)$$

The output functions are $y_1 = h_1(\mathbf{x}) = x_1$ and $y_2 = h_2(\mathbf{x}) = x_3$. u_1 and u_2 are known inputs whereas d_1 and d_2 are unknown inputs to the system, and they can also be regarded as the disturbances in the system.

5.8.1 Design of Proposed Observer

Both outputs have relative degree equal to one. The system contains two disturbances. Both the subsystems under transformation belong to Class A. For the subsystem under $h_1(\mathbf{x})$ we select $k_1 = 1$, and therefore $k_2 = 2$. We may also choose $k_1 = 2$ and $k_2 = 1$. The transformation exists for both scenarios.

The transformation for change of co-ordinates is selected as follows:

$$x_1^{w_1} = x_1$$

$$\begin{aligned}
x_1^{w2} &= x_3 \\
x_2^{w2} &= L_f h_2(\mathbf{x}) = 2x_2 - 2x_3^2
\end{aligned} \tag{5.68}$$

The transformation matrix together with Jacobian and inverse Jacobian of transformation matrix can be formulated as

$$\tilde{\mathbf{x}} = \Phi(\mathbf{x}) = \begin{bmatrix} x_1 \\ x_3 \\ 2x_2 - 2x_3^2 \end{bmatrix}, \quad \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -4x_3 \end{bmatrix}, \quad \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2x_3 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

System (5.67) under the new coordinates can be expressed as follows:

$$\begin{aligned}
\dot{x}_1^{w1} &= \frac{x_2^{w2} + 2(x_1^{w2})^2}{2} - x_1^{w1}u_1 + d_2 \\
\dot{x}_1^{w2} &= x_2^{w2} + u_2 + d_1 - 0.25d_2 \\
\dot{x}_2^{w2} &= 2[\sin(x_1^{w1}) - x_2^{w2}x_1^{w2} + 2(x_1^{w2})^3] - 4x_1^{w2}x_2^{w2} + [x_2^{w2} + 2(x_1^{w2})^2 - 4x_1^{w1}]u_2 \\
&\quad + [1.2 - 4x_1^{w2}]d_1 + [0.8 + x_1^{w2}]d_2
\end{aligned}$$

In the general form according to (5.12), we have

$$\begin{aligned}
\dot{x}_1^{w1} &= \mu_1^{w1}(\tilde{\mathbf{x}}, u) + d_2 \\
\begin{bmatrix} \dot{x}_1^{w2} \\ \dot{x}_2^{w2} \end{bmatrix} &= \begin{bmatrix} x_2^{w2} \\ 0 \end{bmatrix} + \begin{bmatrix} \mu_1^{w2}(\tilde{\mathbf{x}}, u) \\ \mu_1^{w2}(\tilde{\mathbf{x}}, u) \end{bmatrix} + \begin{bmatrix} 1 \\ 1.2 - 4x_1^{w2} \end{bmatrix} d_1 \\
&\quad + \begin{bmatrix} -0.25 \\ (0.8 + x_1^{w2}) \end{bmatrix} d_2
\end{aligned} \tag{5.69}$$

where $\mu_1^{w1}(\tilde{\mathbf{x}}, u) = \frac{x_2^{w2} + 2(x_1^{w2})^2}{2} - x_1^{w1}u_1$, $\mu_1^{w2}(\tilde{\mathbf{x}}, u) = u_2$, and $\mu_2^{w2}(\tilde{\mathbf{x}}, u) = 2[\sin(x_1^{w1}) - 2[x_2^{w2}x_1^{w2} + 2(x_1^{w2})^3] - 4x_1^{w2}x_2^{w2} + [x_2^{w2} + 2(x_1^{w2})^2 - 4x_1^{w1}]u_2]$. In the transformed coordinates, the observer is designed separately for the two subsystems under Class A. With proper estimation gain l_1^{w1} and sliding mode gain ρ_{w1} , the sliding mode estimate for the first equation of Class A subsystem can be designed according to (5.16) as follows:

$$\dot{\hat{x}}_1^{w1} = \mu_1^{w1}(\hat{\tilde{\mathbf{x}}}, u) + l_1^{w1}(y_{w1} - \hat{x}_1^{w1}) + \rho_{w1} \text{sign}(y_{w1} - \hat{x}_1^{w1}) \tag{5.70}$$

The estimation gains for the two subsystems are designed by the high-gain observer

method, and they are of the form:

$$\begin{aligned} l_1^{w_1} &= \theta_1 \\ \mathbf{L}_{w_2} &= \begin{bmatrix} l_1^{w_2} \\ l_2^{w_2} \end{bmatrix} = \mathbf{S}_{\theta_2}^{-1} \mathbf{C}_{w_2}^T = \begin{bmatrix} 2\theta_2 \\ \theta_2^2 \end{bmatrix} \end{aligned} \quad (5.71)$$

Now, for the w_2 subsystem in (5.69), the system satisfies **Assumption 5.5** and the observer can be designed according to (5.16). By selecting an appropriate sliding mode gain ρ_{w_2} , the sliding mode observer for the above system can be designed according to (5.16) as

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1^{w_2} \\ \dot{\hat{x}}_2^{w_2} \end{bmatrix} &= \begin{bmatrix} \hat{x}_2^{w_1} \\ 0 \end{bmatrix} + \begin{bmatrix} u_2 \\ \mu_2^{w_2}(\hat{\mathbf{x}}, u) \end{bmatrix} + \begin{bmatrix} l_1^{w_2} \\ l_2^{w_2} \end{bmatrix} (y_{w_2} - \hat{x}_1^{w_2}) \\ &+ \begin{bmatrix} 1 \\ 1.2 - 4\hat{x}_1^{w_2} \end{bmatrix} \rho_{w_2} \text{sign}(y_{w_2} - \hat{x}_1^{w_2}) + \begin{bmatrix} -0.25 \\ 0.8 + \hat{x}_1^{w_2} \end{bmatrix} \rho_{w_1} \text{sign}(y_{w_1} - \hat{x}_1^{w_1}) \end{aligned}$$

We notice that the observer copies the system structure while replacing the unknown inputs with robust terms. To avoid excessive chattering, we can replace $\text{sign}(\cdot)$ by $\text{sat}(\cdot)$ defined in (2.8) with a small boundary layer ϵ . By means of the inverse transformation, the observer can be transformed back to the original space as follows:

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} \hat{x}_2 \\ \sin(\hat{x}_1) - a_1 \hat{x}_3 \hat{x}_2 \\ 2\hat{x}_2 - 2\hat{x}_3^2 \end{bmatrix} + \begin{bmatrix} -\hat{x}_1 u_1 \\ \hat{x}_2 u_2 \\ u_2 \end{bmatrix} + \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}}^{-1} \mathbf{L}_{tr} \quad (5.72)$$

where

$$\mathbf{L}_{tr} = \begin{bmatrix} \theta_1(y_{w_1} - \hat{x}_1) + \rho_{w_1} \text{sat}(y_{w_1} - \hat{x}_1, \epsilon) \\ 2\theta_2(y_{w_2} - \hat{x}_3) + \rho_{w_2} \text{sat}(y_{w_2} - \hat{x}_3, \epsilon) - 0.25\rho_{w_1} \text{sat}(y_{w_1} - \hat{x}_1, \epsilon) \\ \theta_2^2(y_{w_2} - \hat{x}_3) + (1.2 - 4\hat{x}_1^{w_2}) \rho_{w_2} \text{sat}(y_{w_2} - \hat{x}_3, \epsilon) + (0.8 + \hat{x}_1^{w_2}) \rho_{w_1} \text{sat}(y_{w_1} - \hat{x}_1, \epsilon) \end{bmatrix}$$

5.8.2 Unknown Input Estimation

Based on the development in section 5.5, the unknown inputs can be estimated from the robust terms. Once the system state trajectories converge to the true states, the unknown inputs can be estimated from the robust terms similar to (5.56) as follows:

$$\begin{aligned}\hat{d}_1 &\approx \rho_{w_1} \frac{e_1^{w_1}}{|e_1^{w_1}| + \epsilon} \\ \hat{d}_2 &\approx \rho_{w_1} \frac{e_1^{w_2}}{|e_1^{w_2}| + \epsilon}\end{aligned}\tag{5.73}$$

where $\hat{d}_1$ and $\hat{d}_2$ are the estimates of unknown inputs obtained through multiple sliding modes.

5.8.3 Simulation Results and Discussions

The following values are chosen for simulation: $u_1 = 0.8$, $u_2 = 1.2$. The initial conditions for plant and estimator are $\mathbf{x}_o = \begin{bmatrix} 3 & 5 & 2 \end{bmatrix}^T$ and $\hat{\mathbf{x}}_o = \begin{bmatrix} 0.5 & 0 & 1 \end{bmatrix}^T$ respectively. The unknown input d_2 is selected to be a square wave of amplitude 1.8 and period 1.21, whereas $d_1 = 2 \times \cos(2.5\pi t)$. The sliding mode gains are chosen to be $\rho_{w_1} = \rho_{w_2} = 4$.

(a) Gain Design by HGO

We choose $\theta_1 = 8$, and $\theta_2 = 6$, therefore the estimation gains are computed to be $l_1^{w_1} = 8$, $l_1^{w_2} = 12$, and $l_2^{w_2} = 36$.

Fig. 5.1 shows the tracking performance when both disturbances are present. States which are measured quickly tracks the actual trajectories, state $\hat{x}_1$ converges first in the sliding mode of $e_1^{w_1} = 0$, whereas $\hat{x}_3$ converges to the sliding surfaces of $e_1^{w_1} = 0$ and $e_1^{w_2} = 0$ after the state $\hat{x}_1$ has converged. The robust estimate

$\hat{x}_2$ converges to the actual trajectory after both $\hat{x}_1$ and $\hat{x}_3$ have converged in the sliding mode of $e_1^{w_1} = 0$ and $e_1^{w_2} = 0$. For the state $\hat{x}_2$ in subsystem w_2 , we have multiple sliding modes where x_2 overcomes the uncertainties in the system and track the actual trajectory as shown in Fig. 5.1(b).

As the trajectories track the true states, the robust terms track the unknown inputs accurately. The boundary layer thickness for implementation of saturation function (2.8) is selected to be $\epsilon = 0.015$. The unknown inputs are reconstructed from the sliding modes and are shown in Fig. 5.2(a) and Fig. 5.2(b). The unknown input d_1 is reconstructed from the output estimation error $e_1^{w_1} = 0$ which in fact is influenced by the disturbance d_1 alone and the accuracy is high. For the case of d_2 , the output estimation error $e_1^{w_2} = 0$ involves both d_1 and d_2 , so the reconstruction is prone to influence by d_1 as seen in Fig. 5.2(b). The disturbances with high frequency components are selected to show the effectiveness of the proposed method in reconstruction. When the saturation function (2.8) is absent in the state estimation, the reconstruction of unknown inputs is as shown in Fig. 5.2(c) and Fig. 5.2(d). The state estimation remains the same and does not reflect any chattering.

(b) Gain Design by NLO

Under the same conditions, for the estimation gains $l_1^{w_1} = 1.2$, $l_1^{w_2} = 1.6$, and $l_2^{w_2} = 1.8$, the robust nonlinear Lipschitz observer is implemented and similar results are obtained, as shown in Fig. 5.3. From the implementation point of view, the performance can be guaranteed only for the local case, as the Lipschitz constants cannot be arbitrarily large. For this example, based on the calculated bounds and Lipschitz constants, the condition (5.64) is satisfied for the selected feedback gain $\mathbf{L}_w$.

(c) Relative Performance

Comparison of the estimation errors for the gain design of NLO and HGO are shown in Fig. 5.4. It is clear that with observer gain based on HGO one achieves convergence earlier than the NLO.

5.9 Summary

This chapter extends the results in Chapter 4 to a general case of uncertain MIMO nonlinear systems. A new methodology for the design of sliding mode observer with multiple sliding modes for the nonlinear uncertain systems is systematically developed through the proposed nonlinear transformation. Multiple sliding modes were introduced to handle the multiple unknown inputs. The unknown inputs or disturbances inputs are reconstructed from multiple sliding modes and the observer is made robust w.r.t. disturbances. The conditions for asymptotical stability of error dynamics are established.

(a): x_1 and $\hat{x}_1$ (b): x_2 and $\hat{x}_2$ (c): x_3 and $\hat{x}_3$

Figure 5.1: The estimation performance of the proposed sliding mode observer with HGO

(a): d_1 and $\hat{d}_1$

(b): d_2 and $\hat{d}_2$

(c): d_1 and $\hat{d}_1$ without $\text{sat}(\cdot)$ function

(d): d_2 and $\hat{d}_2$ without $\text{sat}(\cdot)$ function

Figure 5.2: The estimation of unknown inputs from multiple sliding mode

(a): x_1 and $\hat{x}_1$ (b): x_2 and $\hat{x}_2$ (c): x_3 and $\hat{x}_3$

Figure 5.3: The estimation performance of the proposed sliding mode observer with NLO

Figure 5.4: Comparison of estimation performance, HGO vs. NLO

Part II

Discrete-Time Results

Chapter 6

Discrete-time Sliding Mode

The second part of the thesis focuses on the discrete-time counterparts of the sliding mode observers. While our developments are for the design of discrete-time SMOs for continuous-time systems, the developed results can be readily applied to discrete-time systems.

In this chapter, we shall review the existing results on the existence of discrete-time sliding mode (DSM) and later, propose a new strategy to overcome the problems in the traditional DSM observer designs.

6.1 Introduction

For the continuous time system, the conditions (2.5), (2.6) ensures the existence of sliding mode. The condition (2.6) ensures that the direction of motion of the state trajectory is towards the switching surface and so ensures that the sliding mode is attained and maintained. For the existence of condition (2.6), we require infinitely fast switching frequency. It is obvious that the definition for continuous-time sliding modes cannot be applied to the discrete-time sliding modes since the concept of velocity vectors of the system trajectories is not available. The switching frequency

is actually equal to or lower than the sampling frequency. The comparatively low switching frequency causes the discrete-time system to move about the switching surface in a zigzag manner, eventually resulting in chattering about the sliding surface. The extension of continuous-time sliding mode to the discrete-time counterpart is not straight forward and there have been several attempts to study the nature and existence conditions of DSM [56, 88, 90–96, 141, 142].

6.2 Existence of Discrete-time Sliding Mode

Discrete-time sliding modes were introduced as “quasi-sliding modes” by Milosavljevic [141]. The necessary condition for the existence of quasi-sliding mode was proposed by [141], by replacing the derivative term in the well-known condition

$$\lim_{s \rightarrow 0} \dot{s} < 0 \quad (6.1)$$

with a forward difference $\lim_{s(k) \rightarrow 0^+} \nabla s(k) < 0$ and $\lim_{s(k) \rightarrow 0^-} \nabla s(k) > 0$. Here, s is the sliding trajectory and $s = 0$ is the sliding surface. However, as the sampling period in a discrete-time system is increased, the similarity with continuous-time system is lost, and the system trajectory moves in a zigzag manner within a bounded domain. In the subsequent development, the quasi-sliding modes were later called the pseudo-sliding modes in the literature. The following condition is a slightly modified version of definition given by [141]:

Definition 6.1 [88, pg. 136] *The pseudo-sliding mode is said to exist if in an open neighbourhood of the manifold $s(k) = 0$, denoted by Ω_s , the condition*

$$\nabla s(k) s(k) < 0 \quad (6.2)$$

holds where $\nabla s(k) = s(k+1) - s(k)$.

The condition (6.2) is only a necessary condition but not sufficient for the existence of the pseudo-sliding mode. The condition guarantees only switching across the sliding plane, an increase in amplitude of the trajectory may lead eventually to instability.

The work in [90] modified the condition (6.2) and proposed the following necessary and sufficient condition for the existence of convergent pseudo-sliding mode:

$$|s(k+1)| < |s(k)| \quad (6.3)$$

The condition (6.3) can be decomposed into the following two inequalities

$$[s(k+1) - s(k)] \operatorname{sign}(s(k)) < 0 \quad (6.4)$$

$$[s(k+1) + s(k)] \operatorname{sign}(s(k)) > 0 \quad (6.5)$$

This condition imposes upper and lower bounds on the discrete variable structure control systems and these bounds also depend on the distance of the system state from the sliding surface. Condition (6.4) ensures switched motion about the hyperplane or convergence to the hyperplane if switching does not occur on the sliding manifold. Condition (6.5) is required to ensure convergence onto the hyperplane if switching occurs. Condition (6.5) is not necessary if switching does not occur on the sliding manifold. Spurgeon [92] later showed that the condition (6.3) is not a necessary condition as claimed by [90], but is a sufficient condition for the existence of pseudo-sliding mode.

The following conditions for existence of sliding mode are equivalent to the condition (6.3) [91], [93]:

$$\begin{aligned} s(k)\nabla s(k+1) &< -\frac{1}{2}\nabla s^2(k+1) \\ |s(k)s(k+1)| &< s^2(k) \\ s^2(k+1) &< s^2(k) \end{aligned} \quad (6.6)$$

Finally, one may define the DSM based on the existing conditions in the following form:

Definition 6.2 [88, pg. 138] *A discrete-time system (e.g. $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), u(k))$, $\mathbf{x} \in \mathbb{R}^n$) is said to exhibit pseudo-sliding mode, if there exists an integer $K > 0$, such that for all $k > K$, $\mathbf{x}(k) \in \Omega_s$. It follows that $\mathbf{f}(\mathbf{x}(k), u) \in \Omega_s$.*

The above definition states that, the states need not reside in the DSM. It also states that in an open neighbourhood Ω_s , the states need not always approach the sliding surface as long as they does not leave Ω_s . The neighbourhood can be defined as $\Omega_s = \{\mathbf{x} : |s(k)| < \varepsilon; \varepsilon > 0\}$. The above definition also states that the trajectory only stays in a small layer around the sliding manifold and does not relate to the conditions for the existence of DSM. In fact, one may also consider the small layer around the sliding surface as the “boundary layer”. The existence of DSM and its dependency on the boundary layer will be discussed in the next section.

The concept of equivalent control has played an important role in the analysis of variable structure systems. When a system is in sliding condition, its dynamics can be considered to be driven by an equivalent control [56]. The existence of equivalent control for discrete-time systems was proved in [93]. In an ideal case, we can evaluate the equivalent control $u_{eq}(k)$, the discrete-time equivalent control, such that

$$s(k) = 0; s(k+1) = 0 \quad (6.7)$$

Following this approach, several researchers have proposed methods for the design and analysis of sliding mode controllers for linear and nonlinear controller problems [71, 91, 93, 143–147]. Recently, discretization behaviors of equivalent control based sliding mode control are analyzed and the bounds are established in [142].

Chattering in sliding mode controllers was further analyzed in [148].

However, the discrete-time sliding mode observer design has not received much attention. The DSM observer design for linear systems was given in [88]. In [89], the concept of sliding lattice for discrete-time systems was introduced and discrete-time sliding mode observer (DSMO) was designed for SISO linear systems using the Lyapunov min-max method. Other than the above mentioned works, the extension to nonlinear systems remained a challenging problem due to the problems associated with the existence of DSM.

6.3 Boundary Layer

In standard sliding mode, a boundary layer is generally used to avoid excessive chattering over the sliding manifold in the continuous-time domain. A standard implementation of boundary layer in the sliding mode can be given by a saturation function

$$\text{sat}(\cdot, \varepsilon) \triangleq \begin{cases} \cdot/\varepsilon & \text{if } |\cdot| \leq \varepsilon \\ \text{sign}(\cdot) & \text{if } |\cdot| > \varepsilon \end{cases} \quad (6.8)$$

where ε defines the boundary layer. Inside the boundary layer, the switching function can be approximated by a linear feedback gain [97], [110]. The boundary layer has been implemented in several works with the sole purpose of elimination of chattering in discrete-time domain.

To date, most of the analyses of discrete-time sliding mode have been done in the absence of disturbances. The presence of disturbance adds extra complexity to the existence of DSM. When uncertainties and disturbances are present in the system, DSM cannot assure invariance or asymptotic convergence of the estimation error. Due to these reasons, the developments of discrete-time robust observer design for

nonlinear uncertain systems remained unsolved.

To overcome the existing problems of chattering, we propose a new strategy to guarantee the existence of DSM in the presence of disturbances. To begin with, we state the following definition that is a modified form of **Definition 6.2**.

Definition 6.3 *A trajectory (e.g. $e(k)$) is said to exhibit DSM if the trajectory $e(k)$ satisfies conditions (6.4) and (6.5) outside the boundary layer (i.e. $|e(k)| > \epsilon$) and that the trajectory $e(k)$ is finally confined within the boundary layer once it enters the boundary layer, i.e. if there exists an integer $K > 0$ such that $|e(K)| < \epsilon$, then $\forall k > K, |e(k)| < \epsilon$.*

The convergence of trajectory to a final ultimate bound around the sliding surface, where the system overcomes the disturbance, will be called the DSM. For convergent switching to occur and the DSM to exist, the conditions (6.4), (6.5) need to be satisfied.

The DSM as defined above states that the sliding trajectory enters a small boundary layer on the sliding surface and stays inside the boundary layer forever. The trajectory, once enters the boundary layer, is locked inside the boundary layer. Also, the trajectory is not allowed to cross the other side of the boundary layer and therefore the chattering is completely avoided. Compared to **Definition 6.2**, **Definition 6.3** imposes stronger conditions on the trajectory in order to attain and maintain DSM.

Existing analyses of the discrete-time sliding mode are performed outside the boundary layer and there is no available analytical method for the design of boundary layer. In the next section, we propose an implementation strategy of the proposed definition to guarantee DSM, and it will be used for the observer design in the subsequent chapters.

Figure 6.1: Discrete-time sliding mode phenomenon.

6.4 Chattering-free DSM with Boundary Layer

Our approach to establish the DSM is to avoid switching on the sliding manifold. The trajectory is made to reach the sliding manifold, and once it enters the boundary layer, it is retained inside the boundary layer. The proposed methodology can be described in two steps:

- (i) The sliding mode gain is designed such that the trajectory is directed towards the sliding surface, and to reach the boundary layer in finite time and maintains the sliding mode thereafter. The trajectory satisfies both the conditions (6.4) and (6.5) outside the boundary layer.
- (ii) An appropriate selection of the boundary layer and the sliding mode gain constrains the trajectory to the pre-defined boundary layer. The trajectory never crosses outside the boundary layer as illustrated in Fig. 6.1.

The confinement of the trajectory to the boundary layer, after the trajectory enters the boundary layer, prevents chattering. This is crucial for the existence of DSM. The implementation of the boundary layer is through the saturation function (6.8). In our proposed design methodology, the boundary layer forms an integral part

of the observer design in our proposed design methodology. The convergence of trajectory to an ultimate bound inside the boundary is considered as DSM and its existence is based on the boundary layer. Further, an analytical solution for the design of sliding mode gain and the boundary layer thickness that completely eliminates the chattering and guarantees the existence of DSM will be developed. The boundedness of the trajectory within the boundary layer of the sliding manifold after the trajectory enters the boundary layer is also established.

6.5 Summary

In this chapter, we have analyzed the conditions for the existence of discrete-time sliding mode. The importance of boundary layer in the sliding mode is discussed, and then the discrete-time sliding mode that is dependent on the boundary layer is proposed. When the boundary layer forms an integral part of the DSM, chattering free DSM can be guaranteed. The implementation issues of the proposed chattering free DSM for the discrete-time nonlinear observer are discussed.

Chapter 7

DSM Observer Design for SISO Nonlinear Systems

In this chapter, we shall develop a discrete-time nonlinear sliding mode observer for state and unknown input estimations of SISO nonlinear uncertain continuous-time systems. The analysis in this chapter will follow that of its continuous-time counterpart in Chapter 4 except for the existence of DSM, sliding mode gain design and feedback gain design.

7.1 Introduction

Although there has been significant development of sliding mode controllers and observers for nonlinear continuous-time systems, its discrete-time counterpart has received limited attention apart from the works of SMO for linear systems [88], [89]. The main hindrance to the development of discrete-time SMO was the existence of DSM for nonlinear systems. In the following development, the proposed chattering free DSM in the previous chapter will be applied for the existence of DSM in the SMO.

Through a nonlinear change of coordinates, partial linearization is achieved by output and output derivative injections. The discrete-time plant model is obtained via a *Taylor* series expansion on the transformed system. The obtained discrete-time plant dynamic model “copies” the first order dynamics of the nonlinear model. To overcome the problem of disturbances/uncertainties in estimation, we combine the nonlinear observer with the sliding mode term, similar to the development in Chapter 4 to effectively handle the uncertainties. Instead of the prediction estimation, the current state estimation technique [149] is employed in the design of estimator. We combined the discretized observer with SMO, which is a robust term introduced to handle the parametric uncertainties and disturbances. The disturbances in the system are also estimated through the sliding mode term as discussed in the earlier chapters.

The estimator design analysis is performed on the transformed system obtained through coordinate mapping by a diffeomorphism. Through an inverse state transformation, we can readily obtain the estimator for the original system. It is shown that a proper design of the feedback estimation gain will ensure the ultimate boundedness of the estimation error. The ultimate bound becomes smaller with a higher sampling rate, and the quality of convergence also increases as DSM approximates the continuous sliding mode. The estimation performance is significantly improved with the incorporation of the sliding mode term.

As an application, we will study the state estimation of a bioreactor discussed in Chapter 4. It will be shown that a good estimation accuracy is obtained by using our proposed approach. The simulation results show good performance in the presence of uncertainties.

The rest of the chapter is organized as follows: Section 7.2 presents the main idea of the discrete-time sliding mode phenomenon. Section 7.3 discusses the discrete-time

estimator which includes preliminaries on discrete-time nonlinear state transformation. Section 7.4 contains the main result of nonlinear discrete-time observer design, the relevant analysis, existence of DSM, unknown input estimation, and the design of DSMO. Section 7.5 presents the application to a bioreactor with simulation results to demonstrate the effectiveness of the proposed approach. Section 7.6 concludes the chapter.

7.2 Preliminaries

We consider SISO nonlinear continuous-time systems described by (4.1). The system is re-written here for convenience, i.e.

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})u(t) + \mathbf{p}(\mathbf{x})d(\mathbf{x}, u, t) \\ y(t) = h(\mathbf{x}) \end{cases} \quad (7.1)$$

7.2.1 Discrete-time Plant Dynamics

As the functions $\mathbf{f}(\cdot)$ and $\mathbf{b}(\cdot)$ are nonlinear in nature, the ordinary differential equation (ODE) (7.1) cannot be solved exactly, and hence the exact form of the discrete-time difference equation is difficult to obtain. To overcome this problem we make use of Taylor's series expansion to discretize the system.

Denoting the discrete-time index (k) as the variable at time $t = kT_s$, where T_s is the sampling period, we use the following *Taylor* series expansion of $\mathbf{x}$ about the variable " k " :

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \dot{\mathbf{x}}(t) \big|_{t=kT_s} T_s + \mathcal{O}_1(T_s^2) \quad (7.2)$$

where $\mathcal{O}_1(T_s^2)$ is the higher order terms of the above expansion, i.e.,

$$\mathcal{O}_1(T_s^2) = \frac{1}{2!} \mathbf{x}^{(2)}(kT_s) T_s^2 + \cdots + \frac{1}{v!} \mathbf{x}^{(v)}(kT_s) T_s^v + \frac{1}{(v+1)!} \mathbf{x}^{(v+1)}(\zeta) T_s^{v+1}$$

$$\zeta \in (kT_s, kT_s + T_s)$$

where $v \geq 2$. The higher the sampling rate, the smaller is the higher order term of $\mathcal{O}_1(T_s^2)$. Substituting (7.1) into (7.2), we have the discrete-time model

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{x}(k) + T_s \mathbf{f}(\mathbf{x}(k)) + T_s \mathbf{b}(\mathbf{x}(k))u(k) \\ \quad + T_s \mathbf{p}(\mathbf{x}(k))d(\mathbf{x}(k), u(k), k) + \mathcal{O}_1(T_s^2) \\ y(k) = h(\mathbf{x}(k)) \end{cases} \quad (7.3)$$

where $d(\mathbf{x}(k), u(k), k)$ is represented as $d(k)$ for ease of analysis. A similar discretization for robust control of nonlinear plants is given in [150]. In the above discrete-time model, $\mathbf{x}(k)$ is the value of $\mathbf{x}(t)$ at time kT_s . Our discrete-time estimator design will be based on the above discrete-time model obtained via *Taylor* series expansion. In here, we only consider the first order expansion, and treat the higher order terms as disturbance.

7.2.2 Discrete-time Nonlinear State Transformation

To facilitate the design of nonlinear observer, we use discrete-time nonlinear state transformation, similar to the continuous-time nonlinear state transformation (2.46), i.e.

$$\mathbf{x}(k) \rightarrow \tilde{\mathbf{x}}(k) \triangleq \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \\ \vdots \\ \tilde{x}_n(k) \end{bmatrix} = \Phi(\mathbf{x}(k)) = \begin{bmatrix} h(\mathbf{x}(k)) \\ L_{\mathbf{f}}h(\mathbf{x}(k)) \\ \vdots \\ L_{\mathbf{f}}^{(n-1)}h(\mathbf{x}(k)) \end{bmatrix} \quad (7.4)$$

where $L_{\mathbf{f}}h(\mathbf{x}(k)) \triangleq \left(\left[\frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} \right] \mathbf{f}(\mathbf{x}) \right)_{t=kT_s} = \left[\frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} \right]_{t=kT_s} \mathbf{f}(\mathbf{x}(k))$.

Similar to (7.2), using the *Taylor* series expansion we have

$$\tilde{\mathbf{x}}(k+1) - \tilde{\mathbf{x}}(k) = \dot{\tilde{\mathbf{x}}}(t) \big|_{t=kT_s} T_s + \mathcal{O}_2(T_s^2) \quad (7.5)$$

where $\mathcal{O}_1(T_s^2)$ stands for higher order terms. It can be evaluated from (7.1) that

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}(t) \big|_{t=kT_s} &= \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{t=kT_s} \dot{\mathbf{x}}(t) \big|_{t=kT_s} \\ &= \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{t=kT_s} [\mathbf{f}(\mathbf{x}(t)) + \mathbf{b}(\mathbf{x}(t))u(t) + \mathbf{p}(\mathbf{x}(t))d(\mathbf{x}(t), u(t), t)]_{t=kT_s} \\ &= \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{t=kT_s} [\mathbf{f}(\mathbf{x}(k)) + \mathbf{b}(\mathbf{x}(k))u(k) + \mathbf{p}(\mathbf{x}(k))d(k)]\end{aligned}\quad (7.6)$$

We can further obtain the following general form

$$\dot{\tilde{\mathbf{x}}}(t) \big|_{t=kT_s} = \mathbf{A}\tilde{\mathbf{x}}(k) + \boldsymbol{\alpha}(\tilde{\mathbf{x}}(k)) + \boldsymbol{\gamma}(\tilde{\mathbf{x}}(k))u(k) + \tilde{\mathbf{p}}(\tilde{\mathbf{x}}(k))d(k) \quad (7.7)$$

so that from the *Taylor* series expansion (7.5), we have the following transformed system:

$$\begin{cases} \tilde{\mathbf{x}}(k+1) = (\mathbf{I} + \mathbf{A}T_s)\tilde{\mathbf{x}}(k) + T_s\boldsymbol{\psi}(\tilde{\mathbf{x}}(k), u(k)) + T_s\tilde{\mathbf{p}}(\tilde{\mathbf{x}}(k))d(k) + \mathcal{O}_1(T_s^2) \\ y(k) = \mathbf{C}\tilde{\mathbf{x}}(k) \end{cases} \quad (7.8)$$

where $\mathbf{A} \triangleq \begin{bmatrix} 0 & I_{(n-1) \times (n-1)} \\ \mathbf{0}_{1 \times n} \end{bmatrix}$ is an anti-shift constant matrix, $\mathbf{C} \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ is the output matrix,

$$\begin{aligned}\boldsymbol{\psi}(\tilde{\mathbf{x}}(k), u(k)) &\triangleq \boldsymbol{\alpha}(\tilde{\mathbf{x}}(k)) + \boldsymbol{\gamma}(\tilde{\mathbf{x}}(k))u(k) \\ \boldsymbol{\alpha}(\tilde{\mathbf{x}}(k)) &\triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & \xi(\tilde{\mathbf{x}}(k)) \end{bmatrix}^T \\ \boldsymbol{\gamma}(\tilde{\mathbf{x}}(k)) &\triangleq \begin{bmatrix} \gamma_1(\tilde{\mathbf{x}}(k)) & \gamma_2(\tilde{\mathbf{x}}(k)) & \cdots & \gamma_{n-1}(\tilde{\mathbf{x}}(k)) & \gamma_n(\tilde{\mathbf{x}}(k)) \end{bmatrix}^T \\ \tilde{\mathbf{p}}(\tilde{\mathbf{x}}(k)) &\triangleq \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right] \mathbf{p}(\mathbf{x}(k))\end{aligned}$$

with $\xi(\tilde{\mathbf{x}}(k)) = L_{\mathbf{f}}^n h(\mathbf{x}(k))$ and $\gamma_i(\tilde{\mathbf{x}}) = L_{\mathbf{b}} L_{\mathbf{f}}^{(i-1)} h(\mathbf{x}(k))$, for all $i = 1, \dots, n$.

In order to design the DSM observer, we need the following assumptions that are similar to the development in Chapter 4.

Assumption 7.1 *The mapping $\Phi(\mathbf{x})$ is a diffeomorphism.*

Assumption 7.2 *The transferred system (7.8) has the structure*

$$\boldsymbol{\gamma}(\tilde{\mathbf{x}}(k)) \triangleq \begin{bmatrix} \gamma_1(\tilde{x}_1(k)) & \gamma_2(\tilde{x}_1(k), \tilde{x}_2(k)) & \cdots & \gamma_n(\tilde{x}_1(k), \tilde{x}_2(k), \dots, \tilde{x}_n(k)) \end{bmatrix}^T.$$

Assumption 7.3 The functions $\tilde{\mathbf{p}}(\tilde{\mathbf{x}})$, $\xi(\tilde{\mathbf{x}}(k))$ and $\gamma_i(\tilde{x}_1(k), \tilde{x}_2(k), \dots, \tilde{x}_i(k))$, $i = 1, \dots, n$, are global Lipschitz functions w.r.t. $\tilde{\mathbf{x}}(k)$.

Assumption 7.4 The known functions $\mathbf{f}(\cdot)$, $\mathbf{b}(\cdot)$ and $\mathbf{p}(\cdot)$ are bounded with respect to their arguments. The input of the nonlinear system (7.1) is bounded such that $u \leq u_{\max}$ for some upper bound $u_{\max}$. Further, the system (7.1) is Bounded-Input-Bounded-States (BIBS) stable.

Assumption 7.5 The disturbance input $d(\cdot)$ is observable from the output measurement, i.e., $L_{\mathbf{p}}h(\mathbf{x}) \neq 0, \forall \mathbf{x}$.

7.3 DSM Nonlinear State Estimator

For system (7.1) satisfying **Assumptions 7.1-7.5**, a discrete-time nonlinear state estimator with the robust term incorporating current estimation can be designed as follows:

- Prediction at $t = kT_s$.

$$\tilde{\tilde{\mathbf{x}}}(k+1) = (\mathbf{I} + \mathbf{A}T_s)\hat{\tilde{\mathbf{x}}}(k) + T_s\boldsymbol{\psi}(\hat{\tilde{\mathbf{x}}}(k), u(k)) + \tilde{\mathbf{p}}(\hat{\tilde{\mathbf{x}}}(k))u_r(k) \quad (7.9)$$

- Full estimation at $t = (k+1)T_s$

$$\hat{\tilde{\mathbf{x}}}(k+1) = \tilde{\tilde{\mathbf{x}}}(k+1) + \mathbf{L}[y(k+1) - \mathbf{C}\tilde{\tilde{\mathbf{x}}}(k+1)] \quad (7.10)$$

The implementation of the current estimator is similar to [149], where the output correction is based on the current sample instead of the previous sample. That is,

- (i) At time $t=kT_s$, (7.9) predicts the partially estimated states ($\bar{\tilde{\mathbf{x}}}(k+1)$) based on the previous sample's fully estimated states ($\hat{\tilde{\mathbf{x}}}(k)$).
- (ii) At time $t = kT_s + T_s$, when the output $y(k+1)$ is measured, (7.10) gives a fully estimated $\hat{\tilde{\mathbf{x}}}(k+1)$ based on $\bar{\tilde{\mathbf{x}}}(k+1)$ and $y(k+1)$.

Here,

$$\mathbf{L} = \begin{bmatrix} l_1 & l_2 & \cdots & l_{n-1} & l_n \end{bmatrix}^T \quad (7.11)$$

is a properly chosen constant feedback estimation gain.

The design of the robust term $u_r(k)$ is given as

$$u_r(k) = -\frac{\rho}{L_{\mathbf{p}}h(\hat{\tilde{\mathbf{x}}}(k))} \text{sat}(e_1, \epsilon) = \frac{\rho}{L_{\mathbf{p}}h(\hat{\tilde{\mathbf{x}}}(k))} \text{sat}(y(k) - h(\hat{\tilde{\mathbf{x}}}(k)), \epsilon) \quad (7.12)$$

where $\tilde{p}_1(\hat{\tilde{\mathbf{x}}}(k)) = L_{\mathbf{p}}h(\hat{\tilde{\mathbf{x}}}(k)) \triangleq \left[\frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\tilde{\mathbf{x}}}(k)} \mathbf{p}(\hat{\tilde{\mathbf{x}}}(k))$ is the first element of the distribution vector $\tilde{\mathbf{p}}(\tilde{\mathbf{x}}(k)) = \begin{bmatrix} \tilde{p}_1 & \tilde{p}_2 & \cdots & \tilde{p}_n \end{bmatrix}^T$ in the transformed domain. According to **Assumption 7.5**, $\tilde{p}_1(\tilde{\mathbf{x}}(k)) = L_{\mathbf{p}}h(\hat{\tilde{\mathbf{x}}}(k)) \neq 0$.

The $\text{sat}(\cdot)$ function in (6.8) plays a crucial role in the existence of DSM as will be shown in the later part of the chapter. In (7.12), ρ is a finite sliding mode estimation gain and ϵ defines the boundary layer of the sliding manifold, both are to be given later in Section 7.3.2.

Remark 7.1 *The robust term (7.12) is similar to the switching term (4.6), but the $\text{sign}(\cdot)$ is replaced by $\text{sat}(\cdot)$ and is normalized w.r.t. $\tilde{p}_1(\tilde{\mathbf{x}})$, the first element of the transformed distribution vector $\tilde{\mathbf{p}}(\tilde{\mathbf{x}})$. If **Assumption 7.5** does not hold, i.e., if $\tilde{p}_1(\tilde{\mathbf{x}}(k)) = 0$, then the unknown input is not observable, and the observer cannot overcome the unknown input/disturbance in the estimation.*

In what follows, the boundedness of the estimation error system is proved and later the conditions for the existence of discrete-time sliding mode are established. A

relation between sliding mode gain and boundary layer thickness is also established. The estimation system rejects the disturbances/uncertainties in the sliding mode and robustness of the system is increased.

7.3.1 Boundedness of Error Dynamics

The estimation residual or error is defined as

$$\mathbf{e}(k) \triangleq \hat{\mathbf{x}}(k) - \tilde{\mathbf{x}}(k) \triangleq \begin{bmatrix} e_1(k) & e_2(k) & \cdots & e_n(k) \end{bmatrix}^T \quad (7.13)$$

We begin with the error analysis by computing the difference of (7.8) and (7.9) as

$$\begin{aligned} \bar{\tilde{\mathbf{x}}}(k+1) - \tilde{\mathbf{x}}(k+1) &= (\mathbf{I} + \mathbf{A}T_s) \left[\hat{\tilde{\mathbf{x}}}(k) - \tilde{\mathbf{x}}(k) \right] + T_s \phi \left(\tilde{\mathbf{x}}(k), \hat{\tilde{\mathbf{x}}}(k), u(k) \right) \\ &\quad - \mathcal{O}_1(T_s^2) + \tilde{\mathbf{p}}(\hat{\tilde{\mathbf{x}}}(k))u_r(k) - T_s \tilde{\mathbf{p}}(\tilde{\mathbf{x}}(k))d(k) \end{aligned} \quad (7.14)$$

where $\phi \left(\tilde{\mathbf{x}}(k), \hat{\tilde{\mathbf{x}}}(k), u(k) \right) \triangleq \psi(\hat{\tilde{\mathbf{x}}}(k), u(k)) - \psi(\tilde{\mathbf{x}}(k), u(k))$. From (7.10), we have

$$\begin{aligned} \hat{\tilde{\mathbf{x}}}(k+1) - \tilde{\mathbf{x}}(k+1) &= \bar{\tilde{\mathbf{x}}}(k+1) - \tilde{\mathbf{x}}(k+1) + \mathbf{L}[y(k+1) - C\bar{\tilde{\mathbf{x}}}(k+1)] \\ &= (\mathbf{I} - \mathbf{L}\mathbf{C}) \left[\bar{\tilde{\mathbf{x}}}(k+1) - \tilde{\mathbf{x}}(k+1) \right] \end{aligned} \quad (7.15)$$

Substituting (7.14) into (7.15), we obtain

$$\begin{aligned} \mathbf{e}(k+1) &= \mathbf{M}\mathbf{e}(k) - \mathbf{N} \left(T_s \phi \left(\tilde{\mathbf{x}}(k), \hat{\tilde{\mathbf{x}}}(k), u(k) \right) - \mathcal{O}_1(T_s^2) \right) \\ &\quad + \mathbf{N} \left[\tilde{\mathbf{p}}(\hat{\tilde{\mathbf{x}}}(k))u_r(k) - T_s \tilde{\mathbf{p}}(\tilde{\mathbf{x}}(k))d(k) \right] \end{aligned} \quad (7.16)$$

where $\mathbf{M} = [\mathbf{I} + \mathbf{A}T_s - \mathbf{L}(\mathbf{C} + \mathbf{C}\mathbf{A}T_s)]$ and $\mathbf{N} = [\mathbf{I} - \mathbf{L}\mathbf{C}]$.

In the following **Lemma 7.1**, we analyze the boundedness of the estimation error dynamics (7.16) using the Lyapunov function analysis. To proceed with the analysis, we need to design $\mathbf{L}$ such that the eigenvalues of $\mathbf{M}$ are within the unit circle, then we can define a discrete-time Lyapunov function $V(k) \triangleq \mathbf{e}^T(k)\mathbf{Q}\mathbf{e}(k)$, where

the constant matrix $\mathbf{Q} = \mathbf{Q}^T > 0$ is such that

$$\mathbf{M}^T \mathbf{Q} \mathbf{M} - \mathbf{Q} = -\mathbf{I} \quad (7.17)$$

Lemma 7.1 *For the system (7.1) satisfying Assumptions 7.1-7.5, the estimator (7.9) - (7.10) will yield bounded $\hat{\mathbf{x}}(k)$ and $\mathbf{e}(k)$ provided that*

$$\lambda_{\max}(\mathbf{Q}) < \frac{1}{C_1} \quad (7.18)$$

where $C_1 \triangleq \|\mathbf{N}\|^2 \left[T_s l_\phi + l_{\hat{\mathbf{p}}} b_\rho \right]^2 + 2\|\mathbf{M}\| \|\mathbf{N}\| \left[T_s l_\phi + l_\phi b_\rho \right]$, for some Lipschitz constants l_ϕ , $l_{\hat{\mathbf{p}}}$ and some upper bound b_ρ .

Proof: Refer to Appendix B.5. ■

7.3.2 Main Result: Existence of DSM

In here, we take two steps to improve estimation accuracy by DSM design:

- (a) Define the sliding surface,

$$e_1(k) = 0 \quad (7.19)$$

and design the sliding mode estimation as in (7.12) to reach and remain in the sliding mode.

- (b) Ensure the estimation error $\mathbf{e}$ goes to a bound in the sliding mode.

The following **Lemma 7.2** and **Theorem 7.1** are devoted to these two steps.

Lemma 7.2 proves the convergence of $e_1(k)$ to a bound ϵ inside the boundary layer.

Lemma 7.2 is the main result. It proves the existence of DSM and guarantees the confinement of trajectory within the boundary layer once it enters it.

To obtain **Lemma 7.2**, we shall first analyze the dynamics of $e_1(k)$ of the error dynamics $\mathbf{e}(k)$ derived in (7.16). From (7.12), we already know the corresponding function of $\tilde{\mathbf{p}}(\mathbf{x}(k))$ for $e_1(k+1)$ as $\tilde{p}_1(\tilde{\mathbf{x}}) = L_{\mathbf{p}}h(\mathbf{x})$. With $\mathbf{A}$, $\mathbf{C}$ defined in (7.8), and $\mathbf{L}$ in (7.11) we can compute

$$(\mathbf{I} - \mathbf{LC}) = \begin{bmatrix} 1 - l_1 & 0 & \cdots & 0 \\ -l_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_n & 0 & \cdots & 1 \end{bmatrix}, \quad (\mathbf{I} + \mathbf{AT}_s) = \begin{bmatrix} 1 & T_s & 0 & \cdots & 0 \\ 0 & 1 & T_s & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (7.20)$$

From the above, we can compute the first row of $\mathbf{M}$ matrix to evaluate the first dynamics $e_1(k+1)$ from (7.16) as follows:

$$\begin{aligned} e_1(k+1) = & \begin{bmatrix} (1 - l_1) & (1 - l_1)T_s & 0 & \cdots & 0 \end{bmatrix} \mathbf{e}(k) \\ & + (1 - l_1) \left[0 + T_s \left(\gamma_1 \left(\hat{\tilde{\mathbf{x}}}(k) \right) - \gamma_1 \left(\tilde{\mathbf{x}}(k) \right) \right) u(k) - \mathcal{O}_2^*(T_s^2) \right] \\ & + (1 - l_1) [L_{\mathbf{p}}h(\hat{\tilde{\mathbf{x}}}(k))u_r - T_s L_{\mathbf{p}}h(\mathbf{x}(k))d(k)] \end{aligned}$$

Here, $\mathcal{O}_2^*(T_s^2)$ represents the higher order terms corresponding to $e_1(k+1)$. Rearranging, and substituting (7.12), we have

$$\begin{aligned} e_1(k+1) = & (1 - l_1)e_1(k) + (1 - l_1) \left[T_s \left(\gamma_1 \left(\hat{\tilde{\mathbf{x}}}(k) \right) - \gamma_1 \left(\tilde{\mathbf{x}}(k) \right) \right) u(k) \right] \\ & + (1 - l_1) [T_s e_2(k) - T_s L_{\mathbf{p}}h(\mathbf{x}(k))d(k) - \mathcal{O}_2^*(T_s^2)] \\ & - (1 - l_1) \rho_{\text{sat}}(e_1(k), \epsilon) \end{aligned} \quad (7.21)$$

Lemma 7.2 *Consider the system (7.1) satisfying **Assumptions 7.1-7.5** and the estimator (7.9) - (7.10) satisfying (7.18). The sliding mode estimation (7.12) ensures that the sliding trajectory $e_1(k)$ satisfies conditions (6.4) and (6.5) if outside the boundary layer (i.e. $|e_1(k)| > \epsilon$) and that the trajectory $e_1(k)$ is finally confined within the boundary layer once it enters the boundary layer (i.e. if $|e_1(k)| < \epsilon$, then $|e_1(k+i)| < \epsilon, \forall i \in N$), provided the gain in (7.12) satisfies*

$$\rho^- < \rho < 2\epsilon - \rho^- \quad (7.22)$$

and

$$\frac{T_s l_{\gamma_1} u_{max}}{1 + T_s l_{\gamma_1} u_{max}} < l_1 < 1 \quad (7.23)$$

where l_{γ_1} is the global Lipschitz constant of $\gamma_1(\cdot)$ in (7.9), and

$$\rho^- = \sup |\varsigma(k)|, \quad \varsigma(k) = T_s e_2(k) - T_s L_{\mathbf{p}} h(\mathbf{x}(k)) d(k) - \mathcal{O}_2^*(T_s^2) \quad (7.24)$$

Proof: We apply the strategy proposed in section 6.4 to guarantee the DSM and to design the sliding mode gain. The proof is divided into two parts. In the first part (I), the analysis and the design of sliding mode gain for the existence of discrete-time sliding mode are performed outside the boundary layer. Outside the boundary layer, the trajectory is required to satisfy the conditions (6.4) and (6.5) for the existence of DSM. The conditions (6.4) and (6.5) requires the sliding mode gain ρ to have lower and upper bounds. The second part (II) proves the boundedness of error dynamics inside the boundary layer.

(I) Outside the Boundary Layer

In our case, $\mathbf{s}(k) = e_1(k)$. For the trajectory $e_1(k)$ outside the boundary, i.e. $|e_1(k)| > \epsilon$, we have from (6.8)

$$\text{sat}(e_1(k), \epsilon) = \text{sign}(e_1(k)) \quad (7.25)$$

- **Lower bound for Sliding Mode Gain ρ :** From (7.21) with (7.12) and (7.25), we get

$$\begin{aligned} & [e_1(k+1) - e_1(k)] \text{sign}(e_1(k)) \\ = & -l_1 |e_1(k)| + (1 - l_1) T_s \left[\gamma_1(\hat{\mathbf{x}}(k)) - \gamma_1(\tilde{\mathbf{x}}(k)) \right] u(k) \text{sign}(e_1(k)) \\ & + (1 - l_1) [T_s e_2(k) - T_s L_{\mathbf{p}} h(\mathbf{x}(k)) d(k) - \mathcal{O}_2^*(T_s^2)] \text{sign}(e_1(k)) \\ & - (1 - l_1) \rho \end{aligned} \quad (7.26)$$

Since $\mathbf{x}$ is bounded according to the BIBS property in **Assumption 7.4**, under the condition of **Lemma 7.1** that ensures the boundedness of e_2 , there exists a finite gain

$$\rho > \rho^-; \quad \rho^- = \sup |T_s e_2(k) - T_s L_p h(\mathbf{x}(k)) d(k) - \mathcal{O}_2^*(T_s^2)| \quad (7.27)$$

such that when $e_1(k) \neq 0$, we have

$$\begin{aligned} & [e_1(k+1) - e_1(k)] \text{sign}(e_1(k)) \\ & < -l_1 |e_1(k)| + (1-l_1) T_s \left[\gamma_1 \left(\hat{\mathbf{x}}(k) \right) - \gamma_1 \left(\tilde{\mathbf{x}}(k) \right) \right] u(k) \text{sign}(e_1(k)) \end{aligned}$$

Under the globally Lipschitzian condition in **Assumption 7.3** and the bounded input condition in **Assumption 7.4**, we have

$$[e_1(k+1) - e_1(k)] \text{sign}(e_1(k)) < -l_1 |e_1(k)| + (1-l_1) T_s l_{\gamma_1} u_{max} |e_1(k)|$$

for some Lipschitz constant l_{γ_1} and upper bound u_{max} . With a proper design of the feedback estimation gain l_1 such that $l_1 > \frac{T_s l_{\gamma_1} u_{max}}{1 + T_s l_{\gamma_1} u_{max}}$, the inequality (6.4) for the existence of DSM can be met.

- **Upper bound for Sliding Mode Gain ρ :** Similar to (7.26), we can show that

$$\begin{aligned} & [e_1(k+1) + e_1(k)] \text{sign}(e_1(k)) \\ & = (2-l_1) |e_1(k)| + (1-l_1) \varsigma(k) \text{sign}(e_1(k)) - (1-l_1) \rho \\ & \quad + (1-l_1) T_s \left[\gamma_1 \left(\hat{\mathbf{x}}(k) \right) - \gamma_1 \left(\tilde{\mathbf{x}}(k) \right) \right] u(k) \text{sign}(e_1(k)) \end{aligned}$$

In order for the above equation to meet condition (6.5), we need to have

$$\rho < \min \left\{ \frac{2-l_1}{1-l_1} |e_1(k)| + T_s \left[\gamma_1 \left(\hat{\mathbf{x}}(k) \right) - \gamma_1 \left(\tilde{\mathbf{x}}(k) \right) \right] u(k) \text{sign}(e_1(k)) + \varsigma(k) \text{sign}(e_1(k)) \right\} \quad (7.28)$$

From the above analysis, we already have

$$\begin{aligned} \left| \left[\gamma_1 \left(\hat{\mathbf{x}}(k) \right) - \gamma_1 \left(\tilde{\mathbf{x}}(k) \right) \right] u(k) \text{sign}(e_1(k)) \right| &\leq T_s l_{\gamma_1} u_{max} |e_1(k)| \\ &\leq \frac{l_1}{1-l_1} |e_1(k)| \\ |\varsigma(k) \text{sign}(e_1(k))| &\leq \rho^- \end{aligned}$$

where $\rho^- = \sup |\varsigma(k)|$. The right-hand side of (7.28) satisfies:

$$\begin{aligned} &\left\{ \frac{(2-l_1)}{(1-l_1)} |e_1(k)| + T_s \left[\gamma_1 \left(\hat{\mathbf{x}}(k) \right) - \gamma_1 \left(\tilde{\mathbf{x}}(k) \right) \right] u(k) \text{sign}(e_1(k)) + \varsigma(k) \text{sign}(e_1(k)) \right\} \\ &\geq \frac{2-l_1}{1-l_1} |e_1(k)| - \frac{l_1}{1-l_1} |e_1(k)| - \rho^- \\ &= 2|e_1(k)| - \rho^- \end{aligned}$$

For the trajectory outside the boundary layer, i.e. $|e_1(k)| > \epsilon$, the upper bound for the sliding mode gain ρ^+ can be derived as

$$\rho^+ = 2\epsilon - \rho^- \quad (7.29)$$

So finally the gain should be designed as $\rho^- < \rho < \rho^+$ to meet both conditions (6.4) and (6.5). The above conditions ensure the convergence of $e_1(k)$ until $e_1(k)$ reaches the boundary layer.

The selection of ρ forms the basis for preventing switching across the sliding manifold. Once the trajectory enters the boundary layer, the switching is changed to feedback correction and the trajectory stays inside the boundary layer forever as discussed in the section 6.4.

(II) Inside the Boundary Layer

Inside the boundary layer, we have $|e_1(k)| < \epsilon$ and $\text{sat}(\cdot, \epsilon) = (\cdot)/\epsilon$. The error dynamics (7.21) can be reformulated as

$$\begin{aligned} e_1(k+1) = & (1-l_1)e_1(k) + (1-l_1)\varsigma(k) - (1-l_1)\rho \frac{e_1(k)}{\epsilon} \\ & + (1-l_1)T_s \left[\left(\gamma_1(\hat{\mathbf{x}}(k)) - \gamma_1(\tilde{\mathbf{x}}(k)) \right) u(k) \right] \end{aligned}$$

Since $\rho^- > \sup|\varsigma(k)|$ and $l_1 > (1-l_1)T_sl_{\gamma_1}u_{max}$, the following can be shown easily:

$$\begin{aligned} |e_1(k+1)| & < \left| (1-l_1) \left[\frac{\rho}{\epsilon} - 1 \right] e_1(k) \right| + l_1|e_1(k)| + (1-l_1)\rho^- \\ & < (1-l_1)|\rho - \epsilon| + l_1\epsilon + (1-l_1)\rho^- \end{aligned}$$

From condition (7.22), we need to have $\epsilon > \rho^-$, and from (7.29), $\rho^+ - \epsilon = \epsilon - \rho^-$.

Thus, $|\rho - \epsilon| < \epsilon - \rho^-$, and it can be readily shown that

$$\begin{aligned} |e_1(k+1)| & < (1-l_1)[\epsilon - \rho^-] + l_1\epsilon + (1-l_1)\rho^- \\ & = \epsilon \end{aligned}$$

Hence, condition (7.22) confines the error e_1 to an ultimate bound inside the boundary layer. ■

In the ideal case of $T_s = 0$, $b_0 = 0$ and the system will remain on the sliding surface. But due to limited sampling rate (T_s^{-1}) the error will finally remain in a small bound that is determined by the disturbance $d(k)$, the higher order dynamics $\mathcal{O}_1(T_s^2)$ and the sampling period T_s .

Remark 7.2 From the definition of ρ^- in (7.24) we note that ρ^- is a (polynomial)

function of T_s , hence ϵ can be made sufficiently small when sampling rate $1/T_s$ is sufficiently high.

7.3.3 Convergence of Error Dynamics in the Sliding Mode

In the ideal sliding mode, $e_1(k+1) = e_1(k) = 0$ and $\hat{\mathbf{x}}_1(k) = \tilde{\mathbf{x}}_1(k)$, the discrete equivalent control $u_r^*(k)$ can be obtained from (7.21) and (7.12) as

$$\begin{aligned} 0 &= (1 - l_1)T_s \times e_{2d}(k) - (1 - l_1)\mathcal{O}_2^*(T_s^2) \\ &\quad + (1 - l_1) [L_{\mathbf{p}}h(\hat{\mathbf{x}}_d(k))u_r^*(k) - T_s L_{\mathbf{p}}h(\mathbf{x}(k))d(\mathbf{x}(k))] \end{aligned}$$

Hence

$$u_r^*(k) = \frac{\mathcal{O}_2^*(T_s^2)}{L_{\mathbf{p}}h(\hat{\mathbf{x}}_d(k))} - \frac{T_s \times e_{2d}(k)}{L_{\mathbf{p}}h(\hat{\mathbf{x}}_d(k))} + \frac{T_s L_{\mathbf{p}}h(\mathbf{x}(k))d(k)}{L_{\mathbf{p}}h(\hat{\mathbf{x}}_d(k))} \quad (7.30)$$

where the subscript $_d$ denotes the estimated $\hat{\mathbf{x}}$ -related variables in the sliding mode, i.e., $\mathbf{e}_d(k) = \begin{bmatrix} e_{1,d}(k) & e_{2,d}(k) & \cdots & e_{n,d}(k) \end{bmatrix}^T \triangleq \hat{\mathbf{x}}_d(k) - \tilde{\mathbf{x}}(k)$ and $\hat{\mathbf{x}}_d = \Phi^{-1}(\hat{\mathbf{x}}_d(k))$.

Remark 7.3 From (7.30), it can be clearly seen that the equivalent control of the robust term is directly related to the unknown input/disturbance. Once the system reaches the sliding mode, the unknown input can be tackled by the state estimation. The unknown input can also be estimated from the above equivalent control (7.30), and its estimation will be discussed in the next subsection.

Substituting (7.30) into (7.16), we obtain the estimation error dynamics in the sliding mode of $e_1 = 0$, i.e.

$$\begin{aligned} \mathbf{e}_d(k+1) &= \mathbf{M}\mathbf{e}_d(k) - \mathbf{N} \left[T_s \phi(\tilde{\mathbf{x}}(k), \hat{\mathbf{x}}_d(k), u(k)) - \mathcal{O}_1(T_s^2) \right] \\ &\quad + \mathbf{N} [\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d(k)) - \tilde{\mathbf{p}}(\mathbf{x}(k))] u_r^*(k) + \mathbf{N}\tilde{\mathbf{p}}(\mathbf{x}(k)) [u_r^*(k) - T_s d(k)] \end{aligned} \quad (7.31)$$

Obviously, the equilibrium point of the error dynamics (7.31) is $\mathbf{e}_d(k) = \mathbf{0}$, i.e., $\hat{\mathbf{x}}_d(k) = \mathbf{x}(k)$. The following theorem defines the condition for asymptotic stability of the estimation error.

Theorem 7.1 *Consider the system (7.1) satisfying Assumptions 7.1-7.5 and the estimator (7.9) - (7.10) with sliding mode estimation (7.12) and gain (7.22). The estimation error is globally asymptotically stable in the sliding mode $e_1(k+1) = e_1(k) = 0$ provided the gain $\mathbf{L}$ satisfies*

$$\frac{T_s l_{\gamma_1} u_{max}}{1 + T_s l_{\gamma_1} u_{max}} < l_1 < 1 \quad (7.32)$$

and

$$\lambda_{max}(\mathbf{Q}) < \min \left\{ C_1^{-1}, \left\{ 2\|\mathbf{N}\| \|\mathbf{M}\| C_b + \|\mathbf{N}\|^2 C_b^2 \right\}^{-1} \right\} \quad (7.33)$$

where C_1 is defined in **Lemma 7.1** and $C_b = T_s l_{\phi} + l_{\tilde{\mathbf{p}}} \bar{u}_r^* + T_s b_{\tilde{\mathbf{p}}} b_{\frac{1}{\tilde{p}_1}} (1 + l_{\tilde{p}_1} \bar{d})$ for Lipschitz constants l_{ϕ} , $l_{\tilde{\mathbf{p}}}$, $l_{\tilde{p}_1}$, and upper bounds $\bar{u}_r^*$, $\bar{d}$, $b_{\tilde{\mathbf{p}}}$, $b_{\frac{1}{\tilde{p}_1}}$.

Proof: See Appendix B.6. ■

From inequality (B.24), if $\mathcal{O}_1(T_s^2) = 0$, then $b_{\mathcal{O}} = 0$, and condition (7.33) ensures global asymptotic stability of $\mathbf{e}(k)$. Due to limited sampling rate, $\mathcal{O}_1(T_s^2) \neq 0$, condition (7.33) can only ensure the global ultimate boundedness of $\mathbf{e}(k)$. A small $\mathcal{O}_1(T_s^2)$ will yield a smaller ultimate bound. It is also clear from the proof of **Theorem 7.1** that the effect of unknown input is cancelled in the state estimation by the presence of robust term.

Remark 7.4 *The positive constants C_1 in Lemma 7.1 and C_b in Theorem 7.1 are dependent on the sampling period T_s . In fact, they can be rewritten as $C_1 = T_s \times f_{c_1}(\cdot)$ and $C_b = T_s \times f_{c_b}(\cdot)$ where $f_{c_1}(\cdot)$ and $f_{c_b}(\cdot)$ are polynomial functions of T_s . Hence, the selection of a lower sampling period T_s decreases the bounds in inequalities (7.18) and (7.33).*

7.3.4 Unknown Input Estimation from Sliding Mode

Once the trajectories reach the sliding mode and the states converge to the true states, the equivalent control u_{eq} information can be used to reconstruct the unknown input to a certain level of accuracy. Once the system attains DSM, and states converge to the true states, we have $\hat{\mathbf{x}}_d(k) \rightarrow \tilde{\mathbf{x}}(k)$. Therefore

$$e_{2,d} \approx 0, \quad L_{\mathbf{p}}h(\hat{\mathbf{x}}_d(k)) \approx L_{\mathbf{p}}h(\tilde{\mathbf{x}}(k))$$

Therefore, from (7.30) we can approximate

$$u_r^*(k) \approx \frac{\mathcal{O}_2^*(T_s^2)}{L_{\mathbf{p}}h(\hat{\mathbf{x}}_d(k))} + T_s d(k) \quad (7.34)$$

Although ideal sliding mode cannot be attained in DSM, we are still able to approximate the unknown input. By neglecting the higher order terms, we have

$$\hat{d}(k) \approx \frac{(u_r^*(k))_{eq}}{T_s} \quad (7.35)$$

Similar to the earlier analysis, for some small positive scalar δ , the unknown input can be estimated as

$$\begin{aligned} \hat{d}(k) &\approx \frac{1}{T_s} \left(\frac{\rho}{L_{\mathbf{p}}h(\hat{\mathbf{x}}(k))} \text{sat}(e_1, \epsilon) \right)_{eq} \\ &\approx \frac{\rho}{T_s L_{\mathbf{p}}h(\hat{\mathbf{x}}(k))} \frac{e_1}{(|e_1| + \delta)} \end{aligned} \quad (7.36)$$

7.3.5 Selection of Sampling Period T_s and Boundary Layer Thickness ϵ

The selection of sampling period in the proposed observer design plays a crucial role in the accuracy of state and unknown input estimations. The proposed DSM determines the sliding mode gain ρ based on the boundary layer thickness ϵ , which in turn is related to the magnitude of the disturbance and the sampling period T_s .

For a given sampling period, a boundary layer can be selected according to (7.22), such that within a sampling period, the trajectory cannot cross the boundary layer despite the presence of unknown input. It also reveals that the estimation accuracy is dependent on the frequency of the unknown input. In order to have good accuracy in the proposed DSM, the sampling frequency ($\frac{1}{T_s}$) should be higher than the highest frequency component of the disturbance/unknown input. This ensures the confinement of the error trajectory within a small boundary layer compared to the unknown input dynamics.

If the sampling frequency of the system is small compared to the highest frequency components of unknown input, then a large boundary layer will be required, and the fast dynamics of the unknown inputs may not be accurately estimated. The accuracy of the state estimation will be poor. The sampling period should be chosen appropriately such that the robust term is able to track the unknown input accurately. The accuracy of the unknown input estimation (7.34) also relies on the sampling period T_s as the higher-order dynamics $\mathcal{O}_2^*(T_s^2)$ is dependent on T_s .

7.3.6 Discrete-time State Estimator in the Original Space

In the previous sections, the design of the nonlinear observer is performed in the transformed domain so that the analysis can be readily carried out. However, our purpose is to produce the estimation of the original system (7.1). Hence we need to transfer (7.9) from the transformed domain back to the original space. As the transformation matrix $\Phi(\mathbf{x}(k))$ is a diffeomorphism according to **Assumption 7.1**, the Jacobian matrix $\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}}$ and the inverse Jacobian matrix $\left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}}\right]^{-1}$ exist. From (7.6) and (7.7) we can have

$$\begin{aligned} & \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}}\right]_{t=kT_s}^{-1} (\mathbf{A}T_s\tilde{\mathbf{x}}(k) + \boldsymbol{\psi}(\tilde{\mathbf{x}}(k)) + \boldsymbol{\gamma}(\tilde{\mathbf{x}}(k))u(k)) \\ &= T_s\mathbf{f}(\mathbf{x}(k)) + T_s\mathbf{b}(\mathbf{x}(k))u(k) \end{aligned} \quad (7.37)$$

For the current estimator described by (7.9) and (7.10) in the combined form, we have (7.37)

$$\begin{aligned}\hat{\mathbf{x}}(k+1) - \hat{\mathbf{x}}(k) &= \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}(k)} [T_s \mathbf{f}(\hat{\mathbf{x}}(k)) + T_s \mathbf{b}(\hat{\mathbf{x}}(k))u(k) + \mathbf{p}(\hat{\mathbf{x}}(k))u_r(k)] \\ &\quad + \mathbf{L} [y(k+1) - h(\bar{\mathbf{x}}(k+1))] \end{aligned} \quad (7.38)$$

Since $\left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}(k)}^{-1} [\hat{\mathbf{x}}(k+1) - \hat{\mathbf{x}}(k)] = \hat{\mathbf{x}}(k+1) - \hat{\mathbf{x}}(k)$, from (7.38) we have the following estimator in the original space:

$$\begin{aligned}\hat{\mathbf{x}}(k+1) - \hat{\mathbf{x}}(k) &= T_s \mathbf{f}(\hat{\mathbf{x}}(k)) + T_s \mathbf{b}(\hat{\mathbf{x}}(k))u(k) + \mathbf{p}(\hat{\mathbf{x}}(k))u_r(k) \\ &\quad + \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}(k)}^{-1} \mathbf{L} [y(k+1) - h(\bar{\mathbf{x}}(k+1))] \end{aligned} \quad (7.39)$$

The estimator in the original space can therefore be achieved in 2 steps:

1) *Prediction of partial states at instant “k”:*

$$\begin{aligned}\bar{\mathbf{x}}(k+1) &= \hat{\mathbf{x}}(k) + T_s \mathbf{f}(\hat{\mathbf{x}}(k)) + T_s \mathbf{b}(\hat{\mathbf{x}}(k))u(k) \\ &\quad + \mathbf{p}(\hat{\mathbf{x}}(k)) \frac{\rho}{L_p h(\hat{\mathbf{x}}(k))} \text{sat} [y(k) - h(\hat{\mathbf{x}}(k)), \epsilon] \end{aligned} \quad (7.40)$$

2) *Full estimation incorporating current estimator at instant “k+1”:*

$$\hat{\mathbf{x}}(k+1) = \bar{\mathbf{x}}(k+1) + \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}(k)}^{-1} \mathbf{L} [y(k+1) - h(\bar{\mathbf{x}}(k+1))] \quad (7.41)$$

Remark 7.5 *In practice, the observer design can be directly performed in the original space. The system is transformed into the observation space for ease of analysis.*

7.3.7 Design of Gain $\mathbf{L}$

The gain $\mathbf{L}$ should be designed to meet the Lyapunov condition (7.17) and the two conditions in (7.33). In our design, we need to calculate the Lipschitz constants

and bounds for the functions in the transformed space. Then, we need to design the observer such that the distance to unobservability of the linear matrix $\mathbf{I} + \mathbf{A}T_s - \mathbf{LC} - \mathbf{LCAT}_s$ is greater than the Lipschitz constants and the bounded functions in (7.33) in order to achieve stability for the system.

As an alternative approach, we may use the LMI approach [151] to solve the gain design problem. The iterative procedure is as follows:

- (i) We first design gain $\mathbf{L}$ such that $1 > l_1 > \frac{T_s l_{\gamma_1} u_{max}}{1 + T_s l_{\gamma_1} u_{max}}$ and the eigenvalues of $\mathbf{M}$ lie within the unit circle (to ensure stability).
- (ii) The Lyapunov condition (7.17) can be rewritten in the form of matrix inequality in variable $\mathbf{Q}$ as follows:

$$\mathbf{M}^T \mathbf{Q} \mathbf{M} - \mathbf{Q} < 0 \quad (7.42)$$

$$\mathbf{Q} > 0 \quad (7.43)$$

The matrix $\mathbf{Q}$ should be designed such that it satisfies the inequalities in (7.33).

- (iii) Once the gain $\mathbf{L}$ and Lipschitz constants are known, the inequalities in (7.33) can be evaluated. The inequalities in (7.33) can be formed in variable $\mathbf{Q}$ as

$$\mathbf{Q} < \mathbf{I} \times \min \left\{ \frac{1}{C_1}, \frac{1}{K_2} \right\} \quad (7.44)$$

where $K_2 \triangleq \{2\|\mathbf{N}\|\|\mathbf{M}\|C_b + \|\mathbf{N}\|^2 C_b^2\}$.

- (iv) From the above, we can express the linear matrix inequality in the variable

$\mathbf{Q}$ as

$$\mathbf{Q} > 0; \begin{bmatrix} \mathbf{Q} & \mathbf{M}^T \mathbf{Q} \\ \mathbf{Q} \mathbf{M} & \mathbf{Q} \end{bmatrix} > 0; \mathbf{Q} < \mathbf{I} \times \min \left\{ \frac{1}{C_1}, \frac{1}{K_2} \right\} \quad (7.45)$$

- (v) If there does not exist a $\mathbf{Q}$ satisfying the above LMI problem, the sampling rate will be increased and the above steps are repeated.

7.4 Application to Bioreactor

To verify the proposed discrete-time sliding mode observer design, we consider the bioreactor system (4.32) discussed in section 4.4 as an example. From the development in section 4.4, the discrete-time system in the observation space is obtained via *Taylor* series expansion of (4.33):

$$\begin{bmatrix} \tilde{X}(k+1) \\ \tilde{S}(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{X}(k) \\ \tilde{S}(k) \end{bmatrix} + \begin{bmatrix} 0 & T_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X}(k) \\ \tilde{S}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \phi(\tilde{X}(k), \tilde{S}(k)) \end{bmatrix} \\ + \begin{bmatrix} -T_s \tilde{X}(k) \\ \gamma(\tilde{X}(k), \tilde{S}(k)) \end{bmatrix} D(k) + T_s \begin{bmatrix} 1 \\ \zeta(\tilde{X}(k), \tilde{S}(k)) \end{bmatrix} d(k) + \mathcal{O}_1(T_s^2)$$

It can be easily verified that **Assumptions 7.2, 7.3** and **7.5** are satisfied as discussed earlier in subsection 4.4.1.

Based on the design of feedback gain in section 7.3.7, we can select a proper estimation gain as $\mathbf{L} = \begin{bmatrix} l_1 & l_2 \end{bmatrix}^T$. The sliding mode estimation gain ρ and boundary layer thickness ϵ are selected according to condition (7.22). The proposed discrete time robust estimator is designed in accordance with (7.40) and (7.41) as

- (i) *Prediction phase:*

$$\begin{aligned} \bar{X}(k+1) &= \hat{X}(k) + \frac{T_s \mu_m^0 \hat{X}(k) \hat{S}(k)}{k_c^0 \hat{X}(k) + \hat{S}(k)} - T_s \hat{X}(k) D(k) \\ &\quad + \rho \text{sat}[y(k) - h(\hat{\mathbf{x}}(k), \epsilon)] \end{aligned} \quad (7.46)$$

$$\begin{aligned}\bar{S}(k+1) &= \hat{S}(k) - \frac{T_s}{Y} \frac{\mu_m^0 \hat{X}(k) \hat{S}(k)}{k_c^0 \hat{X}(k) + \hat{S}(k)} + T_s (S_f - \hat{S}(k)) D(k) \\ &\quad - \frac{1}{Y} \rho \text{sat}[y(k) - h(\hat{\mathbf{x}}(k)), \epsilon]\end{aligned}\quad (7.47)$$

(ii) *Full Estimation phase:*

$$\hat{X}(k+1) = \bar{X}(k+1) + l_1(y(k+1) - \bar{X}(k+1)) \quad (7.48)$$

$$\begin{aligned}\hat{S}(k+1) &= \bar{S}(k+1) \\ &\quad + \left[\frac{(\mu_m^0 \hat{X}(k) + \hat{S}(k))^2}{\mu_m^0 k_c^0 \hat{X}^2(k)} l_2 - \frac{\hat{S}^2(k)}{k_c^0 \hat{X}^2(k)} l_1 \right] (y(k+1) - \bar{X}(k+1))\end{aligned}\quad (7.49)$$

The unknown input or disturbances can be estimated according to (7.36) as

$$\hat{d}(\mathbf{x}(k)) \approx \frac{1}{T_s} \frac{(\hat{X}(k) - X(k))}{(\hat{X}(k) - X(k)) + \delta} \quad (7.50)$$

7.4.1 Simulation Results and Discussions

The bioreactor parameter values for numerical simulation in section 4.4 are unaltered. The sampling period T_s for simulation is chosen to be 0.05 hr. The additive parametric uncertainties $d_1(t)$ and $d_2(t)$ are chosen to be $0.4\sin(\pi t)$ and $0.3\sin(\frac{3}{4}\pi t)$ respectively. The uncertainties chosen are relatively high compared to their nominal values, and disturbance $d(k)$ is shown in Fig. 7.1. The control input, i.e. the dilution rate is chosen as

$$D(t) = \begin{cases} 0.2 & 0 < t \leq 10 \\ 0.6 & 10 < t \leq 20 \\ 0.2 & 20 < t \leq 30 \end{cases} \quad (7.51)$$

Based on the bounds of disturbances and states available, the Lipschitz constants and bounds of functions are evaluated. We obtain $l_{\gamma_1} = 0.05$, $u_{max} = 0.6$, $|e_2(k)| \leq 0.04$ and $|d(\mathbf{x})| \leq 0.3$. So, we have $\frac{T_s l_{\gamma_1} u_{max}}{1 + T_s l_{\gamma_1} u_{max}} = 0.003$, $\rho^- = 0.016$. Hence according to **Lemma 7.2**, we choose $l_1 = 0.12$, $\rho = 0.032$ and $\epsilon = 0.025$ to satisfy (7.22) in order to ensure the existence of discrete-time sliding mode. We select $l_2 =$

Figure 7.1: Disturbance for the bioreactor example.

0.1. Both eigenvalues of $\mathbf{I} + \mathbf{A}T_s - \mathbf{L}\mathbf{C} - \mathbf{L}\mathbf{C}\mathbf{A}T_s$ are within the unit circle. The maximum eigenvalue of $\mathbf{Q}$ from the solution of (7.17) is 25. Hence, C_1 and $\{2\|\mathbf{N}\|\|\mathbf{M}\|C_b + \|\mathbf{N}\|^2C_b^2\}$ must be less than 0.04 to satisfy (7.33) and (7.18) for convergence. We evaluated the bounds and Lipschitz constants used in the analysis of **Lemma 7.1** and **Theorem 7.1** and verified that they satisfy the conditions (7.18) and (7.33).

We first applied the discrete-time nonlinear estimator without the robust terms, i.e., with $\rho = 0$ in (7.46) - (7.49). The performance is shown in Fig. 7.2. Using the DSM approach, where the robust estimator is described by (7.46) - (7.49), there is a significant improvement in the tracking performance, as shown in Fig. 7.3.

The proposed robust nonlinear estimator uses $e_1 = X(k) - \hat{X}(k) = 0$ as the sliding surface. It demonstrates that the sliding surface is reached in finite time and the trajectory is maintained inside the boundary layer thereafter. From Fig. 7.4(a), it is evident that the errors e_1 and e_2 converge to the boundary layer in a finite time and stays in the vicinity of the sliding plane inside the boundary layer.

Figure 7.2: The actual and estimated cell mass/substrate concentration by Discrete-time Nonlinear Estimator.

To estimate the unknown input according to (7.50), δ is chosen to be 0.001. The unknown input reconstructed from the sliding mode is shown in Fig. 7.1. The estimated unknown input converges to the actual disturbance after all the states have converged. The accuracy of the unknown input estimation relies on the bound of higher-order dynamics and the sampling period T_s .

Since the ultimate boundedness of the error depends on the sampling period, the selection of sampling period should be appropriate to maintain accuracy in estimation, such that the robust term can track the unknown input/disturbance accurately. When sampling period is changed to $T_s = 0.1$, the error bound increases slightly as shown in Fig. 7.4(b). For $T_s = 0.4$, the ultimate bound increases and estimation accuracy is decreased as the robust term cannot track the unknown input as depicted in Fig. 7.4(c).

We next examine the effect of changing the sliding mode gain and boundary layer

Figure 7.3: The actual and estimated cell mass/substrate concentration by DSM Nonlinear Estimator.

thickness. With $\rho^- = 0.016$, $\rho = 0.032$ and $\epsilon = 0.025$, the condition (7.22), $\rho < 2\epsilon - \rho^-$ is satisfied. Chattering is completely eliminated and trajectory is confined inside the boundary layer as shown in Fig. 7.4(a). If $\rho > 2\epsilon - \rho^-$, the condition (7.22) is violated, chattering occurs across the sliding manifold. With $\rho = 0.042$ and $\epsilon = 0.025$, chattering starts to occur but the trajectory is finally confined to the boundary layer as shown in Fig. 7.4(d). When $\rho = 0.051$ and $\epsilon = 0.025$, more severe chattering occurs as depicted in Fig. 7.4(e). When $\rho = 0.053$ and $\epsilon = 0.025$, the chattering does not converge as shown in Fig. 7.4(f). If condition (7.22) does not hold, the system cannot eliminate chattering and discrete-time sliding mode does not exist. A slight variation in the sliding mode gain can alter the system stability. Hence, the condition (7.22) plays a crucial role in the design of DSM.

7.4 Application to Bioreactor

160

(a): $\rho < 2\epsilon - \rho^-$; $\rho = 0.032$ and $\epsilon = 0.025$

(b): $T_s = 0.1$

(c): $T_s = 0.4$

(d): $\rho > 2\epsilon - \rho^-$ with $\rho = 0.045$

(e): $\rho > 2\epsilon$ with $\rho = 0.051$

(f): $\rho > 2\epsilon$ with $\rho = 0.053$

Figure 7.4: Phase trajectory of errors

7.5 Summary

In this chapter, we have considered the design of a discrete-time sliding mode observer for a class of nonlinear systems to robustly estimate the system states when uncertainties or unknown input is present in the system. With the proposed DSM design, the uncertainties and disturbances can be readily handled and a good estimation accuracy can be obtained. The unknown input or disturbances can be estimated through sliding mode. The effect of sampling period on accuracy of state and unknown input estimations are also discussed. A convergence analysis has been performed in the sliding mode and necessary conditions for the existence of sliding mode are obtained. A relation between sliding mode gain and boundary layer thickness is also established for the existence of DSM. Our analysis proves that the estimation error converges under estimation feedback gain and sliding mode gain. An application to bioreactor is given and the simulation results demonstrate the desired convergence properties. In the next chapter, we extend the design technique to MIMO nonlinear systems with multiple unknown inputs.

Chapter 8

Multiple DSM Observer for MIMO Nonlinear Systems

8.1 Introduction

This chapter addresses the discrete-time state and unknown input estimation for continuous-time nonlinear systems with multiple unknown inputs. It extends the development of SISO nonlinear systems in the last chapter to the case of MIMO nonlinear uncertain systems.

The *Taylor* series expansion technique as discussed in the last chapter is used to obtain the discrete-time plant dynamic model. Then the nonlinear transformation developed in chapter 5 is used to divide the original system into two interconnected subsystems. Under a structural assumption for the unknown inputs, we combined the discretized observer with multiple sliding modes, which are robust terms introduced to handle the unknown inputs. The novelty lies in the formulation of multiple sliding modes estimator for the states that are directly influenced by unknown inputs, which cannot be decoupled. The framework allows for the estimation of unknown inputs from the multiple sliding mode to a certain level of

accuracy. Discrete-time sliding is guaranteed by the strategy discussed in Chapter 6, and the relation between the boundary layer thickness and the sliding mode gain design that eliminates chattering is formulated similar to the development in Chapter 7.

The involutive condition for the disturbance distribution vectors in [75] is also removed in the design. The proposed methodology relies only on the measurable output estimation error, and the observer can be directly designed in the original space. Comparing with the existing methods in continuous-time domain [75], the design is performed under less conservative conditions. It is shown that a proper design of the feedback estimation gain ensures the ultimate boundedness of the estimation error. The ultimate bound becomes smaller for a higher sampling rate. The estimation performance is significantly improved with the incorporation of sliding mode terms. Through an inverse state transformation, the estimator for the original system is readily obtained.

Most of the development in this chapter will be similar to the development of its continuous-time counterpart in Chapter 5. In section 8.2, the continuous-time system is discretized by Taylor series expansion, and then nonlinear transformation transforms the system into the new domain. In section 8.3, we design the discrete-time SMO similar to the continuous-time multiple SMO discussed in chapter 5. The existence of DSM for the multiple sliding mode observer together with the conditions for observability of unknown inputs are formulated. The estimation of unknown inputs from the multiple DSM is also discussed within this section. Section 8.5 applies the proposed method to state and unknown input estimation of three phase current motor model to validate the development. Section 8.6 concludes the chapter.

8.2 Preliminaries

Here, we consider the continuous time uncertain nonlinear systems (5.2), and is re-written here for convenience, i.e.

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x}, \mathbf{u}) + \sum_{i=1}^m \mathbf{p}_i(\mathbf{x}) d_i(\mathbf{x}, u, t) \\ y_j = h_j(\mathbf{x}), \quad \text{for } j = 1, \dots, s \end{cases} \quad (8.1)$$

8.2.1 Transformed System

In this section, we transform the nonlinear system by the nonlinear transformation developed in section 5.2.1 of Chapter 5. *Taylor* series expansion is then adopted to discretize the continuous-time system similar to the development (section 7.2.1) in the last chapter.

The complete transformation according to (5.5) is given by

$$\tilde{\mathbf{x}} = \Phi(\mathbf{x}) = \begin{bmatrix} \phi_{w_1}^T & \cdots & \phi_{w_q}^T & \phi_{w_0}^T \end{bmatrix}^T \quad (8.2)$$

System (8.1) can be transformed into the new coordinates with the transformation (8.2) so that

$$\dot{\tilde{\mathbf{x}}} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right] \dot{\mathbf{x}} \quad (8.3)$$

Similar to (7.2), using the *Taylor* series expansion, we have

$$\tilde{\mathbf{x}}(k+1) - \tilde{\mathbf{x}}(k) = \dot{\tilde{\mathbf{x}}}(t) \big|_{t=kT_s} T_s + \mathcal{O}_1(T_s^2) \quad (8.4)$$

where $\tilde{\mathbf{x}}(k)$ represents the discrete-time value of $\tilde{\mathbf{x}}(t)$ at $t = kT_s$ and $\mathcal{O}_1(T_s^2)$ stands for the higher order terms. It can be evaluated from (8.1) that

$$\dot{\tilde{\mathbf{x}}}(t) \big|_{t=kT_s} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{t=kT_s} \dot{\mathbf{x}}(t) \big|_{t=kT_s} \quad (8.5)$$

$$= \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{t=kT_s} \left[\mathbf{f}(\mathbf{x}(k)) + \mathbf{G}(\mathbf{x}(k), \mathbf{u}(k)) + \sum_{i=1}^m \mathbf{p}_i(\mathbf{x}) d_i(\mathbf{x}(k), \mathbf{u}(k)) \right]$$

With the formulation (8.5) and *Taylor* series expansion (8.4), the transformed system that can be divided into the two classes with the transformed coordinates represented by

$$\begin{aligned} \tilde{\mathbf{x}}_{w_j} &= \begin{bmatrix} x_1^{w_j}(k) & x_2^{w_j}(k) & \cdots & x_{k_j}^{w_j}(k) \end{bmatrix}^T = \phi_{w_j} \\ \tilde{\mathbf{x}}_{w_0} &= \begin{bmatrix} x_1^{w_0}(k) & x_2^{w_0}(k) & \cdots & x_c^{w_0}(k) \end{bmatrix}^T = \phi_{w_0} \end{aligned}$$

for $j = 1, \dots, q$, and $c = s - q$. Denote $\tilde{\mathbf{x}}_w(k) = [\tilde{\mathbf{x}}_{w_1}^T(k), \dots, \tilde{\mathbf{x}}_{w_q}^T(k)]^T$, then $\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{\mathbf{x}}_w^T(k) & \tilde{\mathbf{x}}_{w_0}^T(k) \end{bmatrix}^T$.

(a) Class A Subsystems

For the subsystems under the transformation $\phi_{w_1}, \dots, \phi_{w_q}$, the following structure can be obtained

$$\begin{aligned} x_1^{w_j}(k+1) &= x_1^{w_j}(k) + T_s x_2^{w_j}(k) + T_s \mu_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \\ &\quad + T_s \xi_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) + \mathbf{o}_1^{w_j}(T_s^2) \\ &\vdots \\ x_{k_j-1}^{w_j}(k+1) &= x_{k_j-1}^{w_j}(k) + T_s x_{k_j}^{w_j}(k) + T_s \mu_{k_j-1}^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \\ &\quad + T_s \xi_{k_j-1}^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) + \mathbf{o}_{k_j-1}^{w_j}(T_s^2) \\ x_{k_j}^{w_j}(k+1) &= x_{k_j}^{w_j}(k) + T_s \alpha_{w_j}(\tilde{\mathbf{x}}(k)) + T_s \mu_{k_j}^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \\ &\quad + T_s \xi_{k_j}^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) + \mathbf{o}_{k_j}^{w_j}(T_s^2) \\ y_{w_j}(k) &= x_1^{w_j}(k) \end{aligned}$$

where

$$\begin{aligned} \alpha_{w_j}(\tilde{\mathbf{x}}(k)) &= L_{\mathbf{f}}^{k_j} h_{w_j}(\Phi^{-1}(\tilde{\mathbf{x}}(k))), \quad \mu_i^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) = \frac{\partial L_{\mathbf{f}}^{i-1} h_{w_j}}{\partial \mathbf{x}} \mathbf{G}(\Phi^{-1}(\tilde{\mathbf{x}}(k)), \mathbf{u}(k)) \\ \xi_i^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) &= \sum_{v=1}^m L_{\mathbf{p}_v} L_{\mathbf{f}}^{i-1} h_{w_j}(\Phi^{-1}(\tilde{\mathbf{x}}(k))) d_v(k) \end{aligned}$$

for $j = 1, \dots, q$, and $i = 1, \dots, k_j$, and $\mathbf{o}_i^{w_j}(T_s^2)$, $i = 1, \dots, k_j$ are the higher-order terms from *Taylor* series expansion.

Its more general form can be represented as

$$\begin{cases} \tilde{\mathbf{x}}_{w_j}(k+1) = [\mathbf{I} + \mathbf{A}_{w_j}T_s] \tilde{\mathbf{x}}_{w_j}(k) + T_s\boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}) \\ \quad + T_s\boldsymbol{\xi}_{w_j}(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) + \mathcal{O}_{w_j}(T_s^2) \\ y_{w_j}(k) = x_1^{w_j}(k) = \mathbf{C}_{w_j}\tilde{\mathbf{x}}_{w_j}(k) \end{cases} \quad (8.6)$$

where for all $j = 1, \dots, q$, $\mathbf{A}_{w_j} = \begin{bmatrix} 0 & \mathbf{I}_{(k_j-1) \times (k_j-1)} \\ \mathbf{0}_{1 \times (k_j)} \end{bmatrix}$, $\mathbf{C}_{w_j} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$, and

$$\boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) = \begin{bmatrix} \mu_1^{w_j} & \dots & \mu_{k_j-1}^{w_j} & \alpha_{w_j} + \mu_{k_j}^{w_j} \end{bmatrix}^T \quad (8.7)$$

$$\boldsymbol{\xi}_{w_j}(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) = \sum_{i=1}^m \mathbf{Z}_i^{w_j}(\tilde{\mathbf{x}}(k))d_i(k) = \begin{bmatrix} \xi_1^{w_j} & \dots & \xi_{k_j}^{w_j} \end{bmatrix}^T \quad (8.8)$$

and $\mathcal{O}_{w_j}(T_s^2)$ corresponds to higher-order terms in the *Taylor* series expansion. For the analysis of subsystem $\tilde{\mathbf{x}}_{w_j}(k)$, structural assumptions will be imposed in order to guarantee the convergence of each subsystem.

(b) Class B Subsystems

For the subsystem under the transformation $\boldsymbol{\phi}_{w_0}$, the disturbance terms exist, and the transformed system has the following structure

$$\begin{aligned} x_i^{w_0}(k+1) &= x_i^{w_0}(k) + T_s\varphi_i(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) + T_s\gamma_i(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) + \mathbf{o}_i^{w_0}(T_s^2) \\ y_{w_0,i}(k) &= x_i^{w_0}(k) \end{aligned} \quad (8.9)$$

where

$$\begin{aligned} \varphi_i(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) &= L_{\mathbf{f}}h_{q+i}(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}})) + \frac{\partial h_{q+i}}{\partial \mathbf{x}}\mathbf{G}(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}), \mathbf{u}(k)) \\ \gamma_i(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) &= \sum_{v=1}^m L_{\mathbf{p}_v}h_{q+i}(\boldsymbol{\Phi}^{-1}(\tilde{\mathbf{x}}))d_v(k) \end{aligned}$$

for $i = 1, \dots, c$. Disturbance inputs may appear in the **Class B** subsystems but are directly observable from each output. So, the observer design problem can be handled with ease. For the system outputs with relative degree $r_i \geq 2$, the disturbance terms do not exist and so $\gamma_i(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) = 0$. The higher order terms that are the result of *Taylor* series expansion are represented by $\mathbf{o}(T_s^2)$ with corresponding superscripts and subscripts.

In usual practice, the design of Class B subsystems is not a problem as the output measurements (i.e. Class B states) can be used as inputs to Class A subsystems. The design of observer for Class B is discussed in Appendix A.2.

To establish our results, we require the following assumptions similar to those discussed for SMO design in Chapter 5.

Assumption 8.1 *The mapping $\Phi(\mathbf{x})$ is a diffeomorphism.*

Assumption 8.2 *The known functions $\mathbf{f}(\mathbf{x})$, $\mathbf{G}(\mathbf{x}, \mathbf{u})$ and $\mathbf{p}_i(\mathbf{x})$ are bounded with respect to their arguments. The inputs of the nonlinear system (8.1) are bounded with some upper bounds. Further, the system (8.1) is Bounded-Input-Bounded-States (BIBS) stable.*

Assumption 8.3 *The distribution vector $\tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}(k))$ and $\boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k))$ are Lipschitzian on the argument $\tilde{\mathbf{x}}(k)$ for all $i = 1, \dots, m$ and $j = 1, \dots, q$.*

Assumption 8.4 *The nonlinear functions (8.7) and (8.8) in the transferred system (8.10) are of the form:*

$$\begin{aligned} \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) &\triangleq \begin{bmatrix} \mu_1^{w_j}(x_1^{w_j}(k), \bar{\tilde{\mathbf{x}}}_{w_j}^*(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) \\ \mu_2^{w_j}(x_1^{w_j}(k), x_2^{w_j}(k), \bar{\tilde{\mathbf{x}}}_{w_j}(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) \\ \dots \\ \mu_{k_j}^{w_j}(x_1^{w_j}(k), x_2^{w_j}(k), \dots, x_{k_j}^{w_j}(k), \bar{\tilde{\mathbf{x}}}_{w_j}(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) \end{bmatrix} \\ \boldsymbol{\xi}_{w_j}(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) &\triangleq \begin{bmatrix} \xi_1^{w_j}(x_1^{w_j}(k), \bar{\tilde{\mathbf{x}}}_{w_j}(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{d}(k)) \\ \xi_2^{w_j}(x_1^{w_j}(k), x_2^{w_j}(k), \bar{\tilde{\mathbf{x}}}_{w_j}(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{d}(k)) \\ \dots \\ \xi_{k_j}^{w_j}(x_1^{w_j}(k), x_2^{w_j}(k), \dots, x_{k_j}^{w_j}(k), \bar{\tilde{\mathbf{x}}}_{w_j}(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{d}(k)) \end{bmatrix} \end{aligned}$$

for all $j = 1 \cdots q$, where $\tilde{\mathbf{x}}_{w_j}^*(k) = \{x_1^{w_1}(k), x_1^{w_2}(k), \dots, x_1^{w_{j-1}}(k)\}$, $\tilde{\mathbf{x}}_{w_j}(k) = \{\tilde{\mathbf{x}}_{w_1}(k), \tilde{\mathbf{x}}_{w_2}(k), \dots, \tilde{\mathbf{x}}_{w_{j-1}}(k)\}$, and $\tilde{\mathbf{x}}_{w_j}(k)$ and $\tilde{\mathbf{x}}_{w_0}(k)$ are regarded as inputs to the subsystem under consideration. Then according to [39], $\tilde{\mathbf{x}}_{w_j}(k)$ subsystem is uniformly observable for all inputs $\tilde{\mathbf{x}}_{w_j}(k)$, $\tilde{\mathbf{x}}_{w_0}(k)$, $\mathbf{u}(k)$ and $\mathbf{d}(k)$.

For Class B subsystems, the observer design is trivial, as all the states are measured as outputs. In the following section, the design of the observer for Class A subsystems, which is of major concern, is addressed. Most of the development will follow closely to its continuous-time counter-part design except for the existence of discrete-time sliding mode.

8.3 Multiple DSMO for Class A Subsystems

The **Class A** subsystems described by (8.6) - (8.8) can be re-written as

$$\begin{cases} \tilde{\mathbf{x}}_{w_j}(k+1) = [\mathbf{I} + \mathbf{A}_{w_j}T_s] \tilde{\mathbf{x}}_{w_j}(k) + T_s \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}) \\ \quad + T_s \sum_{i=1}^m \mathbf{Z}_i^{w_j}(\tilde{\mathbf{x}}(k)) d_i(k) + \mathcal{O}_{w_j}(T_s^2) \\ y_{w_j}(k) = x_1^{w_j}(k) = \mathbf{C}_{w_j} \tilde{\mathbf{x}}_{w_j}(k) \end{cases} \quad (8.10)$$

for all $j = 1, \dots, q$.

For ease of analysis, all the subsystems $\tilde{\mathbf{x}}_{w_1}(k) \cdots \tilde{\mathbf{x}}_{w_q}(k)$ of **Class A** can be grouped together to form a single system as

$$\begin{cases} \tilde{\mathbf{x}}_w(k+1) = \mathbf{A}_w \tilde{\mathbf{x}}_w(k) + T_s \boldsymbol{\mu}_w(\tilde{\mathbf{x}}(k), \mathbf{u}) \\ \quad + T_s \sum_{i=1}^m \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}(k)) d_i(k) + \mathcal{O}_w(T_s^2) \\ \mathbf{y}_w(k) = \mathbf{C}_w \tilde{\mathbf{x}}_w(k) \end{cases} \quad (8.11)$$

where $\tilde{\mathbf{x}}_w(k) = \begin{bmatrix} \tilde{\mathbf{x}}_{w_1}^T(k) & \tilde{\mathbf{x}}_{w_2}^T(k) & \cdots & \tilde{\mathbf{x}}_{w_q}^T(k) \end{bmatrix}^T$, $\mathbf{C}_w = \text{diag} [\mathbf{C}_{w_1} \mathbf{C}_{w_2}, \dots, \mathbf{C}_{w_q}]$, $\mathbf{A}_w = \text{diag} [(\mathbf{I} + \mathbf{A}_{w_1}T_s), (\mathbf{I} + \mathbf{A}_{w_2}T_s), \dots, (\mathbf{I} + \mathbf{A}_{w_q}T_s)]$, $\boldsymbol{\mu}_w(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) = \begin{bmatrix} \boldsymbol{\mu}_{w_1}^T & \cdots & \boldsymbol{\mu}_{w_{q-1}}^T & \boldsymbol{\mu}_{w_q}^T \end{bmatrix}^T$,

$$\text{and } \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}(k)) = \begin{bmatrix} \tilde{\mathbf{Z}}_i^{w_1}(\tilde{\mathbf{x}}(k))^T & \cdots & \tilde{\mathbf{Z}}_i^{w_{q-1}}(\tilde{\mathbf{x}}(k))^T & \tilde{\mathbf{Z}}_i^{w_q}(\tilde{\mathbf{x}}(k))^T \end{bmatrix}^T.$$

In the following subsection, the structural assumptions that guarantee complete observability of unknown inputs are discussed, while the observer design, convergence of the subsystems in the sliding mode, and unknown input estimation are dealt with in the later subsections.

8.3.1 Observability of Unknown Inputs

The first state of every **Class A** subsystem is measurable as output. These system dynamics play a crucial role in the complete observability of the unknown inputs. From the system dynamics (8.10), we have the first dynamics given by:

$$\begin{aligned} x_1^{w_j}(k+1) &= x_1^{w_j}(k) + T_s x_2^{w_j}(k) + T_s \mu_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \\ &\quad + T_s z_{11}^{w_j}(\tilde{\mathbf{x}}(k)) d_1(k) + \cdots + T_s z_{1m}^{w_j}(\tilde{\mathbf{x}}(k)) d_m(k) + \mathbf{o}_1^{w_j}(T_s^2) \end{aligned} \quad (8.12)$$

for $j = 1, \dots, q$, where $z_{1i}^{w_i}(\tilde{\mathbf{x}})$, $i = 1, \dots, m$ are functions of $\tilde{\mathbf{Z}}_i^{w_j}(k)$ corresponding to dynamics of $x_1^{w_i}(k)$. In the following Assumption, all these first dynamics of subsystems are grouped together for ease of analysis.

The following **Assumption** is the key requirement in the development of sliding mode observer for **Class A** subsystem.

Assumption 8.5 *The number of disturbance inputs m in the transformed domain are less than or equal to the number of outputs q , i.e., ($m \leq q$). The dynamics of states that are measured as outputs (8.12) of ‘ m ’ subsystems have the following structure:*

$$\begin{bmatrix} x_1^{w_1}(k+1) \\ x_1^{w_2}(k+1) \\ \vdots \\ x_1^{w_m}(k+1) \end{bmatrix} = \begin{bmatrix} x_1^{w_1}(k) \\ x_1^{w_2}(k) \\ \vdots \\ x_1^{w_m}(k) \end{bmatrix} + T_s \begin{bmatrix} x_2^{w_1}(k) \\ x_2^{w_2}(k) \\ \vdots \\ x_2^{w_m}(k) \end{bmatrix} + T_s \begin{bmatrix} \mu_1^{w_1}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \\ \mu_1^{w_2}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \\ \vdots \\ \mu_1^{w_m}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \end{bmatrix}$$

$$+T_s \begin{bmatrix} 1 & 0 & \cdots & 0 \\ z_{11}^{w_2}(\tilde{\mathbf{x}}(k)) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_{11}^{w_m}(\tilde{\mathbf{x}}(k)) & z_{12}^{w_m}(\tilde{\mathbf{x}}(k)) & \cdots & 1 \end{bmatrix} \begin{bmatrix} d_1(k) \\ d_2(k) \\ \vdots \\ d_m(k) \end{bmatrix} + \begin{bmatrix} \mathbf{o}_1^{w_1}(T_s^2) \\ \mathbf{o}_1^{w_2}(T_s^2) \\ \vdots \\ \mathbf{o}_1^{w_m}(T_s^2) \end{bmatrix}$$

Note that the diagonal entries $z_{1i}^{w_i}(\tilde{\mathbf{x}}(k))$ can be made unity by normalizing the original elements in each column of the original disturbance distribution matrix.

Remark 8.1 Comparing **Assumption 8.5** with **Assumption 5.5**, the former can be obtained by normalizing the original elements of unknown input distribution matrix with $z_{1i}^{w_i}(\tilde{\mathbf{x}}(k))$. By setting the diagonal entries unity, the analysis will be easier as will be shown in the following development.

This structure is instrumental in the development of SMO for the case of multiple disturbance terms. The main idea is to deal with each disturbance term or unknown input separately.

8.3.2 Observer Design

For the subsystem (8.10) satisfying **Assumptions 8.3 - 8.5**, a discrete-time non-linear state estimator with the robust terms can be designed as follows:

$$\begin{aligned} \hat{\mathbf{x}}_{w_j}(k+1) &= [\mathbf{I} + \mathbf{A}_{w_j}T_s] \hat{\mathbf{x}}_{w_j}(k) + T_s \boldsymbol{\mu}_{w_j}(\hat{\mathbf{x}}(k), \mathbf{u}(k)) + \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_j}(\hat{\mathbf{x}}(k)) u_r^{w_i}(k) \\ &\quad + \mathbf{L}_{w_j} \left(y_{w_j}(k) - \mathbf{C}_{w_j} \hat{\mathbf{x}}_{w_j}(k) \right) \end{aligned} \quad (8.13)$$

where

$$\mathbf{L}_{w_j} = \begin{bmatrix} l_1^{w_j} & l_2^{w_j} & \cdots & l_{k_j-1}^{w_j} & l_{k_j}^{w_j} \end{bmatrix}^T \quad (8.14)$$

$l_1^{w_j}, l_2^{w_j}, \dots, l_{k_j-1}^{w_j}, l_{k_j}^{w_j} > 0$, are properly chosen estimation gains such that the eigenvalues of $\mathbf{I} + \mathbf{A}_{w_j}T_s - \mathbf{L}_{w_j}\mathbf{C}_{w_j}$ are within the unit circle. Here, $u_r^{w_i}(k)$ is a scalar-

valued robust term similar to the robust term in (7.12) and is given by

$$u_r^{w_i}(k) = -\rho_{w_i} \text{sat}(e_1^{w_i}(k), \epsilon_{w_i}) = \rho_{w_i} \text{sat}(y_{w_i}(k) - x_1^{w_i}(k), \epsilon_{w_i}) \quad (8.15)$$

for all $i = 1, \dots, m$, where ρ_{w_i} and ϵ_{w_i} are the sliding mode gain and the boundary layer to be addressed in the section 8.3.4.

In the following subsections, we shall first prove the boundedness of the estimation error dynamics, then establish the DSM. The asymptotical convergence of the estimation error to a small bound in the sliding mode is also proved.

8.3.3 Boundedness of Error Dynamics

To obtain the error dynamics, we first compute the difference of (8.13) and (8.10) as

$$\begin{aligned} \mathbf{e}_{w_j}(k+1) &= \hat{\mathbf{x}}_{w_j}(k+1) - \tilde{\mathbf{x}}_{w_j}(k+1) \\ &= [\mathbf{I} + \mathbf{A}_{w_j}T_s - \mathbf{L}_{w_j}\mathbf{C}_{w_j}] \mathbf{e}_{w_j}(k) \\ &\quad + \left[T_s \boldsymbol{\mu}_{w_j}(\hat{\mathbf{x}}(k), \mathbf{u}(k)) - T_s \boldsymbol{\mu}_{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) - \mathcal{O}_{w_j}(T_s) \right] \\ &\quad + \left[\sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_j}(\hat{\mathbf{x}}(k)) u_r^{w_i}(k) - T_s \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_j}(\tilde{\mathbf{x}}) d_i(k) \right] \end{aligned} \quad (8.16)$$

$$\text{where } \mathbf{e}_{w_j}(k+1) = \begin{bmatrix} e_1^{w_j}(k) & e_2^{w_j}(k) & \dots & e_{k_j}^{w_j}(k) \end{bmatrix}^T.$$

For the combined **Class A** subsystem, the error will be of the form

$$\begin{aligned} \mathbf{e}_w(k+1) &= [\mathbf{A}_w - \mathbf{L}_w\mathbf{C}_w] \mathbf{e}_w(k) + T_s \boldsymbol{\mu}_w(\hat{\mathbf{x}}(k), \mathbf{u}(k)) - T_s \boldsymbol{\mu}_w(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \\ &\quad - \mathcal{O}_w(T_s) + \left[\sum_{i=1}^m \tilde{\mathbf{Z}}_i(\hat{\mathbf{x}}(k)) u_r^{w_i}(k) - T_s \sum_{i=1}^m \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) d_i(k) \right] \end{aligned} \quad (8.17)$$

where

$$\mathbf{L}_w = \text{diag} \begin{bmatrix} \mathbf{L}_{w_1} & \mathbf{L}_{w_2} & \dots & \mathbf{L}_{w_q} \end{bmatrix} \quad (8.18)$$

are properly chosen estimation gains such that the eigenvalues of $\mathbf{M}_w \triangleq [\mathbf{A}_w - \mathbf{L}_w\mathbf{C}_w]$

are within the unit circle, and there exists a positive symmetric constant matrix $\mathbf{Q}_w = \mathbf{Q}_w^T > 0$ is such that

$$\mathbf{M}_w^T \mathbf{Q}_w \mathbf{M}_w - \mathbf{Q}_w = -\mathbf{I} \quad (8.19)$$

The following **Lemma 8.1** ensures the boundedness of the error dynamics by the Lyapunov function analysis.

Lemma 8.1 *For the subsystem (8.10) satisfying **Assumptions 8.3 - 8.5** and the estimator (8.13), $\hat{\mathbf{x}}(k)$ and $\mathbf{e}_{w_j}(k)$ are all bounded provided that*

$$\lambda_{\max}(\mathbf{Q}_w) \leq \frac{1}{2} \left[2\|\mathbf{M}_w\| \times l_{\boldsymbol{\mu}Z} + (l_{\boldsymbol{\mu}Z})^2 \right]^{-1} \quad (8.20)$$

for some bounded function $l_{\boldsymbol{\mu}Z} = T_s l_{\boldsymbol{\mu}} + \sum_{i=1}^q l_{\mathbf{Z}}^i \rho_{w_i}$ with Lipschitz constants $l_{\boldsymbol{\mu}}$, $l_{\mathbf{Z}}^i$ and some upper bound ρ_{w_i} .

Proof: See Appendix B.7. ■

8.3.4 Existence of DSM

To establish the DSM, we adopt the following approach which is similar to the earlier development to improve accuracy:

- (i) Defining the following sliding surfaces:

$$e_1^{w_i}(k) = 0 \quad (8.21)$$

for all $i = 1, \dots, m$ for all the subsystems $\tilde{\mathbf{x}}_{w_1}(k) \dots \tilde{\mathbf{x}}_{w_q}(k)$.

- (ii) Design the sliding mode estimation as (8.15) to reach and maintain the sliding mode.

- (iii) Ensure the estimation error $\mathbf{e}_w(k)$ goes to zero in the sliding mode of $e_1^{w_1}(k) = 0, e_1^{w_2}(k) = 0, \dots, e_1^{w_m}(k) = 0$.

The following **Lemma 8.2** and **Theorem 8.1** are devoted to these steps.

(a) Sliding Mode Gain Design

The sliding mode gain is designed sequentially for all the subsystems in **Class A**. The dynamics of the measurement errors will be obtained from (8.12) according to (8.16) and **Assumption 8.5** as follows:

$$\begin{aligned}
 \begin{bmatrix} e_1^{w_1}(k+1) \\ e_1^{w_2}(k+1) \\ \vdots \\ e_1^{w_m}(k+1) \end{bmatrix} &= \begin{bmatrix} e_1^{w_1}(k) \\ e_1^{w_2}(k) \\ \vdots \\ e_1^{w_m}(k) \end{bmatrix} + T_s \begin{bmatrix} e_2^{w_1}(k) \\ e_2^{w_2}(k) \\ \vdots \\ e_2^{w_m}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{o}_1^{w_1}(T_s^2) \\ \mathbf{o}_1^{w_2}(T_s^2) \\ \vdots \\ \mathbf{o}_1^{w_m}(T_s^2) \end{bmatrix} \\
 &+ T_s \begin{bmatrix} \mu_1^{w_1}(\hat{x}_1^{w_1}(k), \hat{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) - \mu_1^{w_1}(x_1^{w_1}(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) \\ \mu_1^{w_2}(\hat{x}_1^{w_1}(k), \hat{x}_1^{w_2}(k), \hat{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) - \mu_1^{w_2}(x_1^{w_1}(k), x_1^{w_2}(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) \\ \vdots \\ \mu_1^{w_m}(\hat{x}_1^{w_1}(k), \dots, \hat{x}_1^{w_m}(k), \hat{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) - \mu_1^{w_m}(x_1^{w_1}(k), \dots, x_1^{w_m}(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) \end{bmatrix} \\
 &+ \begin{bmatrix} 1 & 0 & \dots & 0 \\ z_{11}^{w_2}(\hat{\mathbf{x}}(k)) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_{11}^{w_m}(\hat{\mathbf{x}}(k)) & z_{12}^{w_m}(\hat{\mathbf{x}}(k)) & \dots & 1 \end{bmatrix} \begin{bmatrix} u_r^{w_1}(k) \\ u_r^{w_2}(k) \\ \vdots \\ u_r^{w_m}(k) \end{bmatrix} \\
 &- T_s \begin{bmatrix} 1 & 0 & \dots & 0 \\ z_{11}^{w_2}(\tilde{\mathbf{x}}(k)) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_{11}^{w_m}(\tilde{\mathbf{x}}(k)) & z_{12}^{w_m}(\tilde{\mathbf{x}}(k)) & \dots & 1 \end{bmatrix} \begin{bmatrix} d_1(k) \\ d_2(k) \\ \vdots \\ d_m(k) \end{bmatrix} \quad (8.22)
 \end{aligned}$$

The error dynamics reach their corresponding sliding modes one-by-one sequentially. The reaching phase of the sliding modes is dependent on the structural **Assumptions 8.4** and **8.5**. Since, the states of $\tilde{\mathbf{x}}_{w_0}(k)$ of Class B subsystems are all measured as outputs, in a finite time, we have $\hat{\tilde{\mathbf{x}}}_{w_0}(k) \equiv \tilde{\mathbf{x}}_{w_0}(k)$ (See Appendix A.2).

The following **Lemma 8.2** establishes the DSM and proves the convergence of

$e_1^{w_i}(k)$ to a bound inside the boundary layer ϵ_{w_j} . The proof is similar to the proof of **Lemma 7.2**.

Lemma 8.2 *Consider the subsystem described by (8.10), the sliding mode estimator of the form (8.13) ensures that the sliding surface $e_1^{w_j}(k)$ satisfies (6.4) and (6.5) outside the boundary layer, i.e., $|e_1^{w_j}(k)| > \epsilon_{w_j}$ and the trajectory $e_1^{w_j}(k)$ is finally confined within the boundary layer once it enters the boundary layer, i.e., if $|e_1^{w_j}(k)| < \epsilon_{w_j}$, then $|e_1^{w_j}(k+i)| < \epsilon_{w_j}$, $\forall i \in N$ provided that the gain in (8.15) satisfies*

$$\rho_{w_j}^- < \rho_{w_j} < 2(1 - l_1^{w_j})\epsilon_{w_j} - \rho_{w_j}^- \quad (8.23)$$

and

$$T_s l_{\mu_1}^{w_j} < l_1^{w_j} < 1 \quad (8.24)$$

for Lipschitz constants $l_{\mu_1}^{w_j}$ of μ_{w_j} , for all $j = 1, \dots, m$ subsystems with

$$\rho_{w_j}^- = \sup |\delta_{w_j}(k)| \quad (8.25)$$

where

$$\delta_{w_j}(k) = T_s e_2^{w_j}(k) + \sum_{i=1}^{j-1} \left[z_{1i}^{w_j}(\hat{\mathbf{x}}(k)) u_r^{w_i} - T_s z_{1i}^{w_j}(\tilde{\mathbf{x}}(k)) d_i(k) \right] - T_s d_j(k) - \mathbf{o}_2^{w_j}(T_s)$$

Proof: Refer to Appendix B.9. ■

Remark 8.2 *Due to limitation in sampling rate T_s^{-1} the error is finally confined to a bound inside the boundary layer ϵ_{w_j} , which is a function of sampling period T_s , disturbance $d_j(k)$ and higher order dynamics $\mathcal{O}_{w_j}(T_s^2)$. Also, from (8.23), we notice that $\rho_{w_j}^-$ is a function of T_s , hence ρ_{w_j} can be sufficiently small when sampling rate $\frac{1}{T_s}$ is sufficiently high.*

(b) Dynamics in the Sliding Mode

Since conditions in **Lemma 8.2** guarantee the sliding mode, the corresponding subsystems error dynamics $\mathbf{e}_{w_j}(k)$ also converge sequentially in the sliding modes of $e_1^{w_1}(k) = 0, e_1^{w_2}(k) = 0, \dots, e_1^{w_q}(k) = 0$. The equivalent control for the sliding dynamics is evaluated sequentially. In the following analysis, the superscript d denotes the estimated $\hat{\mathbf{x}}_{w_j}(k)$ -related variables in the sliding mode, i.e., $\mathbf{e}_{w_j}^d(k) = \begin{bmatrix} e_{1,d}^{w_j}(k) & e_{2,d}^{w_j}(k) & \dots & e_{k_j,d}^{w_j}(k) \end{bmatrix} \triangleq \hat{\mathbf{x}}_{w_j}^d(k) - \tilde{\mathbf{x}}(k)$.

Consider the error dynamics $e_1^{w_j}(k)$ in the sliding mode. Once the trajectory reaches the sliding mode and stays in its vicinity, we have $e_1^{w_1}(k+1) = 0$ and $e_1^{w_1}(k) = 0$, $\hat{x}_1^{w_1} = x_1^{w_1}$, the equivalent control of $u_r^{w_1}(k)$ can be obtained from (8.22) as follows [56]:

$$\begin{aligned} 0 &= 0 + T_s e_{2,d}^{w_1}(k) + \mu_1^{w_1}(\hat{x}_1^{w_1}(k), \hat{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) - \mu_1^{w_1}(x_1^{w_1}(k), \tilde{\mathbf{x}}(k), \mathbf{u}(k)) \\ &\quad + u_{eq}^{w_1}(k) - T_s d_1(k) - \mathbf{o}_2^{w_1}(T_s^2) \end{aligned}$$

where $u_{eq}^{w_1}(k)$ represents the equivalent control dynamics of $u_r^{w_1}(k)$. According to the structural **Assumption 8.4**, in the sliding mode of $e_1^{w_1}(k) = 0$, we have $\mu_1^{w_1}(\hat{x}_1^{w_1}(k), \hat{\mathbf{x}}_{w_0}(k), \mathbf{u}(k)) \equiv \mu_1^{w_1}(x_1^{w_1}(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{u}(k))$. Therefore

$$u_{eq}^{w_1}(k) = T_s d_1(k) - T_s e_{2,d}^{w_1}(k) + \mathbf{o}_2^{w_1}(T_s^2) \quad (8.26)$$

Similarly for the trajectory $e_1^{w_2}(k)$, in the sliding mode of $e_1^{w_1}(k) = 0$ and $e_1^{w_2}(k) = 0$, we may evaluate

$$u_{eq}^{w_2}(k) = -T_s e_{2,d}^{w_2}(k) - z_{11}^{w_2}(\hat{\mathbf{x}}(k)) u_{eq}^{w_1} - T_s z_{11}^{w_2}(\tilde{\mathbf{x}}(k)) d_1(k) + T_s d_2(k) + \mathbf{o}_2^{w_2}(T_s^2)$$

Substituting $u_{eq}^{w_1}(k)$ (8.26) into the above equation, we have

$$\begin{aligned} u_{eq}^{w_2}(k) &= T_s d_2(k) - T_s e_{2,d}^{w_2}(k) - T_s z_{11}^{w_2}(\hat{\mathbf{x}}(k)) e_{2,d}^{w_1}(k) + z_{11}^{w_2}(\hat{\mathbf{x}}(k)) d_1(k) \\ &\quad - z_{11}^{w_j}(\tilde{\mathbf{x}}(k)) d_1(k) + T_s z_{11}^{w_2}(\hat{\mathbf{x}}(k)) \mathbf{o}_2^{w_1}(T_s^2) + \mathbf{o}_2^{w_2}(T_s^2) \end{aligned}$$

In the sliding mode of $e_1^{w_1}(k) = 0$ and $e_1^{w_2}(k) = 0$, based on **Assumption 8.4**, we have $z_{11}^{w_2}(\tilde{\mathbf{x}}(k)) \equiv z_{11}^{w_2}(\hat{\tilde{\mathbf{x}}}(k))$. Hence, we have

$$\begin{aligned} u_{eq}^{w_2}(k) &= T_s d_2(k) - T_s e_{2,d}^{w_2}(k) - T_s z_{11}^{w_2}(\hat{\tilde{\mathbf{x}}}(k)) e_{2,d}^{w_1}(k) \\ &\quad + T_s z_{11}^{w_2}(\hat{\tilde{\mathbf{x}}}(k)) \mathbf{o}_2^{w_1}(T_s^2) + \mathbf{o}_2^{w_2}(T_s^2) \end{aligned} \quad (8.27)$$

By substitution of successive equivalent control elements, we can obtain the equivalent control for all the measurement errors. Therefore in the sliding modes of $e_1^{w_1}(k) = 0, e_1^{w_2}(k) = 0, \dots, e_1^{w_m}(k) = 0$, the equivalent control of $u_{eq}^{w_m}(k)$ can be evaluated, which is similar to the evaluation of (8.27) and also contains the second error dynamics $e_{2,d}^{w_1}(k), e_{2,d}^{w_2}(k), \dots, e_{2,d}^{w_m}(k)$ of all the subsystems. In short, the equivalent control of $e_1^{w_i}(k) = 0$ can be represented as

$$u_{eq}^{w_i}(k) = T_s d_i(k) - T_s \chi_i \left(e_{2,d}^{w_1}(k), e_{2,d}^{w_2}(k), \dots, e_{2,d}^{w_i}(k), \hat{\tilde{\mathbf{x}}}_w(k) \right) + \mathbf{o}_i^{eq}(T_s^2) \quad (8.28)$$

for all $i = 1, \dots, m$, where $\chi_i(\cdot, k)$ is function of second error dynamics of the subsystems and $\mathbf{o}_i^{eq}(T_s^2)$ accounts for the higher-order terms that results from the successive evaluation.

Remark 8.3 *The sliding mode gain design ρ_{w_j} in (8.25) depends on the robust terms of other subsystems. In fact, the condition (8.25) can be relaxed by replacing the robust terms with their corresponding equivalent control elements as the system is in the sliding mode of $e_1^{w_1}(k) = 0, \dots, e_1^{w_{j-1}}(k) = 0$. Therefore, we have*

$$\rho_{w_j}^- = \sup \left| T_s e_2^{w_j}(k) + \sum_{i=1}^{j-1} [\mathbf{o}_i^{eq}(T_s^2) - T_s \chi_i(\cdot, k)] - T_s d_j(k) - \mathbf{o}_2^{w_j}(T_s^2) \right| \quad (8.29)$$

So, finally the robust terms depend only on the bounds of respective unknown input, second error dynamics of the equivalent controls (8.28) and the higher-order terms of the Taylor series expansion.

(c) Error Convergence in the Sliding Mode

The error dynamics (8.17) in the multiple sliding mode of $e_1^{w_1}(k) = 0, e_1^{w_2}(k) = 0, \dots, e_1^{w_m}(k) = 0$ will be of the form:

$$\begin{aligned} & \mathbf{e}_w^d(k+1) \\ = & \mathbf{M}_w \mathbf{e}_w^d(k) + \left[T_s \boldsymbol{\mu}_w(\hat{\mathbf{x}}(k), \mathbf{u}(k)) - T_s \boldsymbol{\mu}_w(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) - \mathcal{O}_w(T_s^2) \right] \\ & + \left[\sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i(\hat{\mathbf{x}}(k)) - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}(k)) \right) u_{eq}^{w_i}(k) \right] + \sum_{i=1}^m \tilde{\mathbf{Z}}_i(\hat{\mathbf{x}}(k)) [u_{eq}^{w_i}(k) - T_s d_i(k)] \end{aligned}$$

Substituting the evaluated equivalent control elements (8.28) into the above error dynamics, we have the following estimation error dynamics in the multiple sliding mode

$$\begin{aligned} & \mathbf{e}_w^d(k+1) \\ = & \mathbf{M}_w \mathbf{e}_w^d(k) + \left[T_s \boldsymbol{\mu}_w(\hat{\mathbf{x}}(k), \mathbf{u}(k)) - T_s \boldsymbol{\mu}_w(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) - \mathcal{O}_w(T_s^2) \right] \\ & + \left[\sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i(\hat{\mathbf{x}}(k)) - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}(k)) \right) u_{eq}^{w_i}(k) \right] + \sum_{i=1}^m \tilde{\mathbf{Z}}_i(\hat{\mathbf{x}}(k)) [\mathbf{o}_i^{eq}(T_s^2) - \chi_i(\cdot, k)] \end{aligned} \quad (8.30)$$

Remark 8.4 From the error dynamics (8.30), we can see that the equivalent control completely cancels the disturbance terms and the error dynamics in the sliding mode is totally free from unknown inputs.

Since the robust terms track the unknown dynamics in the sliding mode, we require the system to be stable in the sliding mode. The following theorem further examines the condition for the asymptotical boundedness of the estimator error.

Theorem 8.1 For the subsystem (8.10) satisfying **Assumptions 8.3 - 8.5**, the estimator (8.13) with the sliding mode gain (8.15) ensures that the estimation error is asymptotically bounded in the sliding modes of $e_1^{w_i}(k) = 0, \forall i = 1, \dots, m$, provided the gain $\mathbf{L}_w$ satisfies $T_s l_{\mu_1}^{w_j} < l_1^{w_j} < 1$, (8.19), (8.20) and

$$\lambda_{\max}(\mathbf{Q}_w) \leq \Delta_1^{-1} \quad (8.31)$$

for some positive constant Δ_1 . See (B.32) of Appendix B.8 for details.

Proof: See Appendix B.8. ■

The above ensures the asymptotical stability of $\mathbf{e}_w(k)$ if $\mathcal{O}_w(T_s^2) = 0$. Owing to the limited sampling rate $\mathcal{O}_w(T_s^2) \neq 0$, the above can only ensure the ultimate boundedness of $\mathbf{e}_w(k)$. The smaller the $\mathcal{O}_w(T_s^2)$, the smaller is the ultimate bound.

Remark 8.5 *The positive constant Δ_1 is dependent on the sampling period T_s and it can be rewritten as $\Delta_1 = T_s \times f_{\Delta_1}(\cdot)$ where $f_{\Delta_1}(\cdot)$ is polynomial function of T_s . Hence, the selection of a lower sampling period T_s decreases the bound in inequality (8.31) as well as the ultimate bound for error convergence.*

8.3.5 Unknown Input Estimation from Multiple Sliding Mode

Once the trajectories reach the sliding mode and all the states converge to the true states, the equivalent control $u_{eq}^{w_i}(k)$ information can be used to reconstruct the unknown inputs to a certain level of accuracy. Once the measurable states attain DSM and remain inside the boundary layer, we have $\hat{\mathbf{x}}_w^d(k) \rightarrow \tilde{\mathbf{x}}_w(k)$. Therefore

$$e_{2,d}^{w_1}(k) \approx 0, \quad e_{2,d}^{w_2}(k) \approx 0 \quad \cdots \quad e_{2,d}^{w_m}(k) \approx 0$$

Thus,

$$\chi_i \left(e_{2,d}^{w_1}(k), e_{2,d}^{w_2}(k), \dots, e_{2,d}^{w_i}(k), \hat{\mathbf{x}}_w(k) \right) \rightarrow 0$$

for all $i = 1, \dots, m$.

Therefore from (8.26), (8.27) and (8.28) we can approximate

$$\begin{aligned} u_{eq}^{w_1}(k) &\approx T_s d_1(k) + \mathbf{o}_2^{w_1}(T_s^2) \\ u_{eq}^{w_2}(k) &\approx T_s d_2(k) + T_s z_{11}^{w_2}(\hat{\mathbf{x}}(k)) \mathbf{o}_2^{w_1}(T_s^2) + \mathbf{o}_2^{w_2}(T_s^2) \\ &\dots \quad \dots \end{aligned}$$

$$u_{eq}^{w_m}(k) \approx T_s d_m(k) + \mathbf{o}_m^{eq}(T_s^2)$$

As the equivalent control (8.28) is directly involved with its respective unknown input, the unknown input estimation can be reconstructed from the multiple sliding mode. Although the ideal sliding mode cannot be attained in DSM, we are still able to approximate the unknown input. By neglecting the higher order terms, we have

$$\hat{d}_i(k) \approx \frac{u_{eq}^{w_i}(k)}{T_s}, \quad \forall i = 1, \dots, m. \quad (8.32)$$

Similar to the approximation (7.36) given in Chapter 7, for some small positive scalar δ , the unknown inputs can be estimated as

$$\hat{d}_i(k) \approx \frac{1}{T_s} (\rho_{w_i} \text{sat}(e_1^{w_i}(k), \epsilon_{w_i}))_{eq} = \frac{1}{T_s} \frac{\rho_{w_i} e_1^{w_i}(k)}{(|e_1^{w_i}(k)| + \delta)} \quad (8.33)$$

for all $i = 1, \dots, m$.

8.3.6 Design of Gain $\mathbf{L}_w$

The gain $\mathbf{L}_w$ should be designed such that the two conditions of (8.20) and (8.31), and the Lyapunov condition (8.19) are satisfied. We need to design the gain such that the distance to unobservability of linear matrix $\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w$ is greater than the Lipschitz constants and bounded functions (right hand side of inequalities in (8.20) and (8.31)) in order to achieve stability for the subsystem. We may use LMI approach as discussed in the last chapter.

Remark 8.6 *The conditions (8.20), (8.31) are dependent on Lipschitz constants and bounds of nonlinear functions that explicitly involve the sampling period T_s . So, even if the Lipschitz constants and bounds are larger, a smaller sampling period will make the bounds smaller and hence the design of gain $\mathbf{L}_w$ is feasible.*

8.4 Observer in the Original Space

To produce the estimation of the original system, we transfer the observer for subsystems of **Classes A, B** back into the original space. Under **Assumption 8.1**, using (8.3), we have

$$\dot{\mathbf{x}} = \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]^{-1} \dot{\hat{\mathbf{x}}} \quad (8.34)$$

Similarly, for the combined observer in the transformation space described by (8.13) and (A.7), by means of inverse transformation, we can obtain the observer in the original space as

$$\hat{\mathbf{x}}(k+1) = \hat{\mathbf{x}}(k) + T_s [\mathbf{f}(\hat{\mathbf{x}}(k)) + \mathbf{G}(\hat{\mathbf{x}}(k), \mathbf{u}(k))] + \left[\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right]_{\mathbf{x}=\hat{\mathbf{x}}(k)}^{-1} \mathbf{L}_{tr} \quad (8.35)$$

where $\mathbf{L}_{tr}$ matrix stands for the observer gains together with feedback terms designed in the transformed space. The $\mathbf{L}_{tr}$ matrix constitutes the correction terms from all the subsystems as follows:

$$\mathbf{L}_{tr} = \begin{bmatrix} \mathbf{L}_{w_1}(y_{w_1}(k) - \hat{x}_1^{w_1}(k)) + \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_1} \rho_{w_i} \text{sat}(y_{w_i}(k) - \hat{x}_1^{w_i}(k), \epsilon_{w_i}) \\ \vdots \\ \mathbf{L}_{w_q}(y_{w_q}(k) - \hat{x}_1^{w_q}(k)) + \sum_{i=1}^m \tilde{\mathbf{Z}}_i^{w_q} \rho_{w_i} \text{sat}(y_{w_i}(k) - \hat{x}_1^{w_i}(k), \epsilon_{w_i}) \\ l_{w_0,1}(y_{w_0,1}(k) - \hat{x}_1^{w_0}(k)) + \rho_{w_0,1} \text{sat}(y_{w_0,1}(k) - \hat{x}_1^{w_0}(k), \epsilon_{w_0,1}) \\ \vdots \\ l_{w_0,c}(y_{w_0,c}(k) - \hat{x}_c^{w_0}(k)) + \rho_{w_0,c} \text{sat}(y_{w_0,c}(k) - \hat{x}_c^{w_0}(k), \epsilon_{w_0,c}) \end{bmatrix} \quad (8.36)$$

8.5 Illustrative Example

We consider the three-phase current motor model [75, 152] described by following nonlinear equations:

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -A_1x_2 - A_2x_3\sin x_1 - A_3x_3\sin(2x_1) \\ -D_1x_3 + D_2\cos(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0.4 & 0.3 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad (8.37)$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$, with x_1 , x_2 and x_3 representing the rotor angle, speed deviation and field flux linkage, respectively, u_1 and u_2 are known inputs and correspond to mechanical power input and field voltage respectively. The unknown inputs d_i represent uncertainties or faults in the system.

The output functions are $y_1 = h_1(\mathbf{x}) = x_1$ and $y_2 = h_2(\mathbf{x}) = x_3$. Both the outputs have relative degree one. Both the outputs, according to the proposed transformation, belong to **Class A**. For the subsystem under $h_1(\mathbf{x})$ we select $k_1 = 2$, and therefore $k_2 = 1$.

The transformation for change of co-ordinates is selected as

$$\begin{aligned} x_1^{w_1} &= x_1 \\ x_2^{w_1} &= L_{\mathbf{f}}h_1(\mathbf{x}) = x_2 \\ x_1^{w_2} &= x_3 \end{aligned} \quad (8.38)$$

The outputs in the transformed domain are $y_{w_1} = x_1^{w_1}$ and $y_{w_2} = x_1^{w_2}$. As the output equations are linear, the transformation is linear, the Jacobian and Inverse Jacobian of transformation are identity matrices.

The discrete-time system in the observation space for system (8.37) under the new coordinates can be expressed as follows:

$$\begin{bmatrix} x_1^{w_1}(k+1) \\ x_2^{w_1}(k+1) \end{bmatrix} = \begin{bmatrix} x_1^{w_1}(k) \\ x_2^{w_1}(k) \end{bmatrix} + \begin{bmatrix} 0 & T_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{w_1}(k) \\ x_2^{w_1}(k) \end{bmatrix} + \begin{bmatrix} T_s\mu_1^{w_1}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \\ T_s\mu_2^{w_1}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \end{bmatrix}$$

$$\begin{aligned}
& +T_s \begin{bmatrix} 1 \\ 0.3 \end{bmatrix} d_1(k) + T_s \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} d_2(k) + \mathcal{O}_{w_1}(T_s) \\
x_1^{w_2}(k+1) &= x_1^{w_2}(k) + T_s \mu^{w_2}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) + 0.2T_s d_1(k) + T_s d_2(k) + \mathcal{O}_{w_2}(T_s)
\end{aligned}$$

where

$$\begin{aligned}
\mu_1^{w_1}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) &= x_2^{w_1}(k) \\
\mu_2^{w_2}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) &= -A_1 x_2^{w_1}(k) - A_2 x_1^{w_2}(k) \sin(x_1^{w_1}(k)) - A_3 \sin(2x_1^{w_1}(k)) + u_1(k) \\
\mu^{w_2}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) &= -D_1 x_1^{w_2}(k) + D_2 \cos(x_1^{w_1}(k)) + u_2(k)
\end{aligned}$$

From the above structure, it is clear that the system satisfies the structural condition in **Assumption 8.4** and the output measurement equations satisfy the observability conditions in **Assumption 8.5**. $x_1^{w_1}$ is influenced by unknown input d_1 whereas $x_1^{w_2}$ is affected by both the unknown inputs. In the following, we shall design the observer such that we are able to estimate both the states as well as unknown inputs with the proposed multiple SMO.

8.5.1 DSMO Design

The observer is designed separately for the two subsystems under **Class A**. The sliding mode observer for the system can be designed according to (8.13) as

$$\begin{aligned}
\begin{bmatrix} \hat{x}_1^{w_1}(k+1) \\ \hat{x}_2^{w_1}(k+1) \end{bmatrix} &= \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1^{w_1}(k) \\ \hat{x}_2^{w_1}(k) \end{bmatrix} + \begin{bmatrix} T_s \mu_1^{w_1}(\hat{\tilde{\mathbf{x}}}(k), \mathbf{u}(k)) \\ T_s \mu_2^{w_1}(\hat{\tilde{\mathbf{x}}}(k), \mathbf{u}(k)) \end{bmatrix} \\
&+ \begin{bmatrix} l_1^{w_1}(y_{w_1}(k) - \hat{x}_1^{w_1}(k)) \\ l_2^{w_1}(y_{w_1}(k) - \hat{x}_1^{w_1}(k)) \end{bmatrix} + \begin{bmatrix} 1 \\ 0.3 \end{bmatrix} u_r^{w_1}(k) + \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} u_r^{w_2}(k) \\
\hat{x}_1^{w_2}(k+1) &= \hat{x}_1^{w_2}(k) + T_s \mu^{w_2}(\hat{\tilde{\mathbf{x}}}(k), \mathbf{u}(k)) + l_1^{w_2}(y_{w_2}(k) - \hat{x}_1^{w_2}(k)) \\
&+ 0.2 * u_r^{w_1}(k) + u_r^{w_2}(k)
\end{aligned}$$

Comparing the observer with the system, the unknown inputs are replaced by their corresponding robust terms and appended with a feedback term. The robust

terms are designed according to (8.15):

$$\begin{aligned} u_r^{w_1}(k) &= \rho_{w_1} \text{sat}(y_{w_1}(k) - \hat{x}_1^{w_1}(k), \epsilon_{w_1}) \\ u_r^{w_2}(k) &= \rho_{w_2} \text{sat}(y_{w_2}(k) - \hat{x}_1^{w_2}(k), \epsilon_{w_2}) \end{aligned}$$

Since the inverse transformation is identity, the observer system can be written directly in the original system equations as follows:

$$\begin{aligned} & \hat{\mathbf{x}}(k+1) \\ = & \hat{\mathbf{x}}(k) + T_s \begin{bmatrix} \hat{x}_2(k) \\ -A_1 \hat{x}_2(k) - A_2 \hat{x}_3(k) \sin \hat{x}_1(k) - A_3 \hat{x}_3(k) \sin(2\hat{x}_1(k)) + u_1(k) \\ -D_1 \hat{x}_3(k) + D_2 \cos(\hat{x}_1(k)) + u_2(k) \end{bmatrix} + \mathbf{L}_{tr} \end{aligned}$$

where $\mathbf{L}_{tr} =$

$$\begin{bmatrix} l_1^{w_1}(y_1 - \hat{x}_1(k)) + \rho_{w_1} \text{sat}(y_1(k) - \hat{x}_1(k), \epsilon_{w_1}) \\ l_2^{w_1}(y_1(k) - \hat{x}_1(k)) + 0.3 * \rho_{w_1} \text{sat}(y_1(k) - \hat{x}_1(k), \epsilon_{w_1}) + 0.4 * \rho_{w_2} \text{sat}(y_2(k) - \hat{x}_3(k), \epsilon_{w_2}) \\ l_1^{w_2}(y_2(k) - \hat{x}_3(k)) + 0.2 * \rho_{w_1} \text{sat}(y_1(k) - \hat{x}_1(k), \epsilon_{w_1}) + \rho_{w_2} \text{sat}(y_2(k) - \hat{x}_3(k), \epsilon_{w_2}) \end{bmatrix}$$

The unknown inputs can be estimated from the sliding mode according to (8.33) as follows:

$$\begin{aligned} \hat{d}_1(k) &\approx \frac{1}{T_s} \frac{\rho_{w_1}(y_1(k) - \hat{x}_1(k))}{(|y_1(k) - \hat{x}_1(k)| + \delta)} \\ \hat{d}_2(k) &\approx \frac{1}{T_s} \frac{\rho_{w_2}(y_2(k) - \hat{x}_3(k))}{(|y_2(k) - \hat{x}_3(k)| + \delta)} \end{aligned}$$

8.5.2 Simulation Results

The following values are chosen for the simulation: $A_1 = 0.2703$, $A_2 = 1.01$, $A_3 = -1.5$, $D_1 = 1.5$, $D_2 = 1.5$, $u_1 = 5$, $u_2 = 1$, $\mathbf{x}(0) = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix}^T$, $\hat{\mathbf{x}}(0) = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix}^T$, $d_1(t) = 0.8 \sin(0.4\pi t)$, $d_2(t)$ is a square signal of amplitude 1.0 with a period of 6 seconds. The sampling period (T_s) is chosen to be 0.04. The estimation gains for the subsystems are $l_1^{w_1} = 0.4$, $l_2^{w_1} = 0.3$, and $l_1^{w_2} = 0.4$.

Figure 8.1: The estimation performance of the proposed multiple DSM observer

Figure 8.2: The estimation of unknown inputs from multiple sliding mode

8.5 Illustrative Example

185

Figure 8.3: Phase trajectory of errors

Based on the bounds of states and inputs, the Lipschitz constants and bounds are calculated. In the sliding mode of $e_1^{w_1} = 0$ and $e_1^{w_2} = 0$, we have the reduced-order $\tilde{\mathbf{Z}}_1 \equiv [0 \ 0.3 \ 0]^T$ and $\tilde{\mathbf{Z}}_2 \equiv [0 \ 0.4 \ 0]^T$. So, the bounds are calculated to be $b_{\tilde{\mathbf{Z}}_1} = 0.3$, $b_{\tilde{\mathbf{Z}}_2} = 0.4$. Similarly, $\boldsymbol{\mu}_w \equiv [x_1^{w_2}, -A_1 x_1^{w_2}, 0]^T$, and $l_{\boldsymbol{\mu}} \geq \left\| \frac{\partial \boldsymbol{\mu}_w(\tilde{\mathbf{x}}, u)}{\partial \tilde{\mathbf{x}}} \right\| = 1.0359$. Also $l_{\tilde{\mathbf{Z}}_1} = l_{\tilde{\mathbf{Z}}_2} = 0$, $b_{\chi_1}(\cdot) \leq \rho_{w_1}^-$, $b_{\chi_2}(\cdot) \leq \rho_{w_2}^-$, $\rho_{w_1}^- = 0.06$, $\rho_{w_2}^- = 0.05$. The eigenvalues of $\mathbf{I} + \mathbf{A}_w T_s - \mathbf{L}_w \mathbf{C}_w$ are well within the unit circle. The maximum eigenvalue of $\mathbf{Q}_w$, the solution of (8.19), is 32.12. Based on the bounds obtained, both the inequalities in (8.20) and (8.31) are satisfied.

Fig. 8.1 shows the tracking performance when disturbances are present. Clearly, states which are measured quickly track the actual trajectories, while the robust estimate x_2 converges to the actual trajectory after both x_1 and x_3 converge in the sliding modes of $e_1^{w_1} = 0$ and $e_1^{w_2} = 0$. For the x_2 subsystem, we have multiple sliding modes where x_2 overcomes the uncertainties in the system and tracks the actual trajectory as shown in Fig. 8.1(b). The unknown inputs are reconstructed from the multiple DSM and are shown in Fig. 8.2.

We shall also examine the effects of different choices sliding mode gain and boundary layer thickness. With $\rho_{w_1} = \rho_{w_2} = 0.26$, $\epsilon_{w_1} = \epsilon_{w_2} = 0.3$, and $\rho_{w_1}^- = 0.06$, $\rho_{w_2}^- = 0.05$, the condition (8.23) is satisfied. Chattering is completely eliminated and the trajectory is confined inside the boundary layer as shown in Fig. 8.3(a) - Fig. 8.3(c). If condition (8.23) is not satisfied, e.g. by selection of $\rho_{w_1} = \rho_{w_2} = 0.45$, chattering occurs across the sliding manifold as depicted in Fig. 8.3(d) - Fig. 8.3(f). Hence, the condition (8.23) plays a crucial role in the design of DSM.

8.6 Summary

In this chapter, we have extended the design of DSMO design of SISO nonlinear systems to MIMO uncertain nonlinear systems. The nonlinear transformation to-

gether with the Taylor's series expansion is used to discretize the system. With the discrete-time multiple sliding modes design, the uncertainties and disturbances can be handled and a good estimation can be obtained. The proposed framework allows us to directly estimate the unknown inputs from the multiple sliding mode. Existence conditions for the sliding mode are established. Detailed analysis shows that the estimation converges under estimation feedback gain, sliding mode gain and the boundary layer condition. Finally, simulation results are given to demonstrate the effectiveness of the proposed approach. This chapter also concludes the theoretical development in the thesis.

Chapter 9

Conclusions and Recommendations

9.1 Conclusions

Observer designs for uncertain nonlinear systems have been an active research area over the last two decades. Recently, the estimation of uncertainties/unknown inputs that result from faults is also receiving increased attention for fault detection and isolation (FDI) of uncertain linear and nonlinear systems. In these problems, sliding mode observers are effective technologies to deal with accuracy and robustness.

In this thesis, we have considered the problems of state and unknown input estimations of linear and nonlinear systems in both the continuous-time and discrete-time domains. The uncertainties can be structural or external disturbances to the system. In particular, we analyzed the observability properties of unknown inputs and designed sliding mode observers to add robustness to the nonlinear observers. The observability concepts are analyzed and a new perspective for designing sliding mode observers to deal with unknown inputs is presented. The proposed sliding

mode observers can be appended to many available nonlinear observers in the literature to provide robustness and to estimate the disturbances/unknown inputs.

The contributions of the thesis can be summarized as follows:

- (i) The elucidation of observability conditions that guarantee complete observability of unknown inputs and their connections to accurate state estimation in the presence of unknown inputs/disturbances. The unknown inputs can be estimated from the sliding mode, and the observer design does not require the derivatives of the output. The design methodology is applicable to a class of Lipschitz nonlinear systems and the observer can be designed under less conservative conditions than the existing sliding mode observers.
- (ii) The proposal of a sliding mode observer for a class of uniformly observable SISO nonlinear systems. Based on the observability of the unknown input, a robust term is designed to track/estimate the unknown input. The unknown input/disturbance in the sliding mode becomes an increment of the Lipschitz term. The gain design follows the gain design of high-gain observer (HGO) to handle any arbitrary large Lipschitz constant. We establish the exponential convergence of the estimation error. The design of robust NLO with SMO is also shown to follow similar design pattern. In this case, the gain design is only applicable for small Lipschitz constants in order to guarantee asymptotical stability.
- (iii) The proposal of a multiple mode sliding observer for state and unknown input estimation of a general class of nonlinear systems influenced by multiple unknown inputs. The structural conditions that guarantee complete observability of unknown inputs are discussed and a multiple SMO is proposed to deal with the unknown inputs. The robust terms structurally replace the unknown inputs in the observer and it is proved that the number of ro-

bust terms is equal to the number of unknown inputs. Through the proposed framework, the robust terms estimate the unknown inputs and the high-gain observer is designed to guarantee the asymptotical stability of estimation error in the sliding mode. Compared to SMO based on equivalent control methods, the observer is established with fewer structural conditions, and the involutive condition for the unknown input distribution vectors is also removed.

- (iv) The inherent chattering problem associated with discrete-time sliding mode (DSM) observer is solved. The relation between sliding mode gain and boundary layer thickness that guarantees the existence of DSM is revealed. Through the proposed method, chattering is completely eliminated and the trajectory is locked inside the pre-defined boundary layer.
- (v) The discrete-time sliding mode observer for SISO nonlinear systems proposed is based on *Taylor* series expansion and nonlinear transformation. For the new strategy, the existence of DSM is proved. The unknown input estimation is obtained through an approximation of the robust term. The feedback gain design is based on nonlinear Lipschitz observer and convergence conditions are obtained. With the aid of nonlinear transformation developed for continuous-time observer design, the design of SISO nonlinear systems is extended to discrete-time observer for MIMO nonlinear uncertain systems with multiple unknown inputs. The observability conditions for unknown inputs are established. The conditions for the convergence of estimation error and existence of DSM are obtained. The unknown inputs are also estimated from multiple sliding modes.

The robust observer design problems considered in the thesis are generic and have been studied over the last two decades. Several design methods have been for-

ulated over the last decade and only sufficient conditions for the rejection of unknown inputs/ disturbances are formulated. The significance of our proposed design methodologies is that the necessary conditions for the observability of unknown inputs to completely track the disturbances through robust terms, and their relations to state estimation and unknown input estimation are provided. Also, our proposed new sliding mode observers can provide estimates of states as well as unknown inputs and they outperform many of the existing works in the literature that require the uncertainties to be structural. Further, the discrete-time state and unknown inputs estimation problems for nonlinear uncertain systems are solved, and they have great potential in the FDI area.

9.2 Recommendations for Future Research

Although we have solved several state and unknown input estimation problems of uncertain nonlinear continuous-time and discrete-time systems, there remain several problems that should be addressed in the future. In this section, we recommend the following topics for future research.

- (i) Our proposed sliding mode observers can be appended to a wide class of nonlinear observers [39], [38] that satisfy the Lipschitz condition for the nonlinear parts. It would be interesting and useful to extend the design technique to a more general class of nonlinear systems without the Lipschitz conditions.
- (ii) In most physical systems, the states are constrained in a particular space (e.g. the biological systems have positive states). The design of nonlinear observers for constrained state estimation is a challenging problem. The sliding mode observers have the potential to restrict the trajectories to certain subspace. This could provide some interesting results for constrained state estimation

of nonlinear systems.

- (iii) In practical situations (e.g. bioreactors), delays in measurements are inevitable. It will be interesting to analyze the robust estimation procedures for the nonlinear systems with delayed measurements.
- (iv) The DSM observer in Chapter 5 may be extended to design discrete-time robust multi-rate estimators [153], [154] that could incorporate measurements with different sampling intervals. By establishing the sliding mode for the trajectories that are measurable at the higher-sampling rate, the observer can be made robust to unknown inputs/disturbances.
- (v) Although the focus in the thesis was limited to state and unknown input observation, many of the results can indeed be extended to robust sliding mode controller design for nonlinear uncertain systems. Further research is required to design a robust controller that relies only on output measurements to tackle the disturbances in the system.
- (vi) The state and unknown input estimation results in the thesis can be applied to observer-based fault detection schemes. The estimation of unknown inputs can be used to generate residual signals for detection. The estimation of unknown inputs/disturbances can provide a good measure of the health of the system.

Appendix A

Design Procedures

A.1 Nonlinear Transformation

By selecting $r_i - 1$ terms, (where r_i stands for the relative degree of output y_i) instead of r_i terms in the traditional nonlinear transformation, we can avoid the disturbance inputs in the transformed space that normally appear in the r_i^{th} differentiation. A class of transformed subsystems is then not affected by the unknown inputs, and that is desirable.

The step II of the transformation in Section 5.2.1, is divided into Steps (II) and (III) as follows:

- (II) The remaining outputs with relative degree $r_j = 1$ and some outputs with relative degree $r_j \geq 2$ are used to form the transformation. Suppose that we have $b - q$ such outputs and are given by $y_{q+1}, \dots, y_b$, we define the following transformation

$$\phi_{w_0} = \begin{bmatrix} h_{q+1}(\mathbf{x}) & \cdots & h_b(\mathbf{x}) \end{bmatrix} \quad (\text{A.1})$$

(III) For the remaining $s - b$ outputs with relative degree $r_j > 2$, we denote for each of the outputs

$$\phi_j = \begin{bmatrix} h_i(\mathbf{x}) & L_{\mathbf{f}} h_i(\mathbf{x}) & \cdots & L_{\mathbf{f}}^{r_j-2} h_i(\mathbf{x}) \end{bmatrix}^T \quad (\text{A.2})$$

for $j = 1, \dots, s - b$ and $i = b, \dots, s$.

The selection of outputs q and the value of k_j in step I and the selection of outputs with relative degree $r_j > 2$ in Step II and Step III yield n functions such that the desired transformation holds.

Combined transformation resulting from the three steps is then given by

$$\tilde{\mathbf{x}} = \Phi(\mathbf{x}) = \begin{bmatrix} \underbrace{\phi_{w_1}^T \cdots \phi_{w_q}^T}_{\text{Class A}} & \underbrace{\phi_{w_0}^T}_{\text{Class B}} & \underbrace{\phi_1^T \phi_2^T \cdots \phi_{s-b}^T}_{\text{Class C}} \end{bmatrix}^T \quad (\text{A.3})$$

The transformed coordinates for Step III are represented as

$$\tilde{\mathbf{x}}_j = \begin{bmatrix} x_1^j & x_2^j & \cdots & x_{r_j-1}^j \end{bmatrix}^T = \phi_j \quad (\text{A.4})$$

for $j = 1, \dots, s - b$

Similar to (5.6), system (5.2) can be transformed into the new coordinates with the transformation in (A.3) to form three classes of subsystems, namely Class A, Class B and Class C. Class A and Class B subsystems are obtained similar to (5.8) and (5.11) respectively.

Class C Subsystems:

The coordinates under the transformation $\phi_1, \dots, \phi_{s-b}$ have the following structure, and the disturbance inputs are completely absent in the subsystems described below:

$$\dot{x}_1^i = x_2^i + \beta_1^i(\tilde{\mathbf{x}}, u)$$

$$\begin{aligned}
& \dots \quad \dots \\
\dot{x}_{r_i-2}^i &= x_{r_i-1}^i + \beta_{r_i-2}^i(\tilde{\mathbf{x}}, u) \\
\dot{x}_{r_i-1}^i &= \alpha_i(\tilde{\mathbf{x}}) + \beta_{r_i-1}^i(\tilde{\mathbf{x}}, u) \\
y_i &= x_1^i
\end{aligned} \tag{A.5}$$

where $\beta_j^i(\tilde{\mathbf{x}}, u) = \frac{\partial(L_{\mathbf{f}}^{j-1}h_i)}{\partial \mathbf{x}} \mathbf{G}(\Phi^{-1}(\tilde{\mathbf{x}}), u)$ and $\alpha_i(\tilde{\mathbf{x}}) = L_{\mathbf{f}}^{r_i-1}h_i(\Phi^{-1}(\tilde{\mathbf{x}}))$, for all $j = 1, \dots, r_i - 1$ and for all $i = 1, \dots, s - b$.

From the above, it is clear that the unknown disturbance inputs do not appear and so the observer can be designed for these subsystems by existing techniques based on the structural assumptions for the nonlinear part.

The subsystem can be represented in the following general form

$$\begin{aligned}
\dot{\tilde{\mathbf{x}}}_i &= \mathbf{A}_i \tilde{\mathbf{x}}_i + \boldsymbol{\beta}^i(\tilde{\mathbf{x}}, u) \\
y_i &= \mathbf{C}_i \tilde{\mathbf{x}}_i
\end{aligned} \tag{A.6}$$

for all $i = 1, \dots, s - b$. $\mathbf{A}_i = \begin{bmatrix} 0 & \mathbf{I}_{(r_i-1) \times (r_i-1)} \\ 0 & \mathbf{0}_{1 \times (r_i-1)} \end{bmatrix}$, $\boldsymbol{\beta}_i = [\beta_1^i, \dots, \beta_{r_i-2}^i, \alpha_i + \beta_{r_i-1}^i]^T$ and $\mathbf{C}_i = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$.

We assume the functions $\beta_1^i, \dots, \beta_{r_i-2}^i, \alpha_i + \beta_{r_i-1}^i$ are Lipschitzian on argument $\mathbf{x}$, and each $\boldsymbol{\beta}^i$, $i = 1, \dots, s$ has the block triangular input structure which ensures uniform observability.

Remark A.1 *The Step II in the transformation provides some flexibility to customize the transformation for estimation of required unknown inputs and states in the system.*

A.2 DSMO for Class B subsystems

Disturbance inputs may appear in the **Class B** subsystems but are directly observable from each output which involves only one state variable. So, the states involved in **Class B** subsystems are all measured as outputs. The observer can therefore be designed as follows:

$$\begin{aligned}\hat{x}_i^{w_0}(k+1) &= \hat{x}_i^{w_0}(k) + T_s \varphi_i(\hat{\mathbf{x}}(k), \mathbf{u}(k)) + l_{w_0,i} (y_{w_0,i}(k) - \hat{x}_i^{w_0}(k)) \\ &\quad + \rho_{w_0,i} \text{sat} [(y_{w_0,i}(k) - \hat{x}_i^{w_0}(k)), \epsilon_{w_0,i}]\end{aligned}\quad (\text{A.7})$$

for $i = 1, \dots, c$. By proper selection of sliding mode gain $\rho_{w_0,i}$, we can ensure the asymptotic convergence of the above estimator.

Now, we can obtain the error dynamics from system (8.9) and (A.7) as

$$\begin{aligned}e_i^{w_0}(k+1) &= (1 - l_d^i) e_i^{w_0}(k) + \left[T_s \varphi_i(\hat{\mathbf{x}}(k), \mathbf{u}(k)) - T_s \varphi_i(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \right. \\ &\quad \left. - T_s \gamma_i(\tilde{\mathbf{x}}(k), \mathbf{d}) - \mathbf{o}_i^{w_0}(T_s) \right] - \rho_{w_0,i} \text{sat} (e_i^{w_0}(k), \epsilon_{w_0,i})\end{aligned}\quad (\text{A.8})$$

The following **Lemma A.1** shows that by the selection of the following sliding surface, the estimation error can converge asymptotically, i.e.

$$e_i^{w_0}(k) = \hat{x}_i^{w_0}(k) - x_i^{w_0}(k) = 0 \quad (\text{A.9})$$

Lemma A.1 *For the subsystem described by (8.9), the sliding mode estimator of the form (A.7) ensures that the sliding surface $e_i^{w_0}(k)$ satisfies (6.4) and (6.5) outside the boundary layer, i.e., $|e_i^{w_0}(k)| > \epsilon_{w_0,i}$ and the trajectory $e_i^{w_0}(k)$ is finally confined within the boundary layer once it enters the boundary layer, i.e., if $|e_i^{w_0}(k)| < \epsilon_{w_0,i}$, then $|e_i^{w_0}(k+j)| < \epsilon_{w_0,i}$, $\forall j \in N$ provided that the gain in (A.7) satisfies*

$$\rho_{w_0,i}^- < \rho_{w_0,i} < 2(1 - l_{w_0,i})\epsilon_{w_0,i} - \rho_{w_0,i}^- \quad (\text{A.10})$$

where $\rho_{w_0,i}^- \triangleq \sup |\varsigma_{i,d}(k)|$; $\varsigma_{i,d}(k) = T_s \varphi_i(\hat{\mathbf{x}}(k), \mathbf{u}(k)) - T_s \varphi_i(\tilde{\mathbf{x}}(k), \mathbf{u}(k))$

$-T_s \gamma_i(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) - \mathbf{o}_i^{w_0}(T_s)$ and $1 > l_{w_0,i} > 0$ for all $i = 1, \dots, c$.

Proof: Similar to the Proof of **Lemma 8.2** in Appendix B.9 . ■

For the case of outputs with relative degree $r_i \geq 2$, the design of gain and observer for the respective subsystems will be the same except that $\gamma_i(\tilde{\mathbf{x}}(k), \mathbf{d}(k)) = 0$ in the analysis of **Lemma A.1**.

Appendix B

Proofs of Relevant Lemmas and Theorems

B.1 Proof of Lemma 4.3

Under **Assumptions 2.1**, $\psi(\tilde{\mathbf{x}}, u)$ is a Lipschitz function. So, we have

$$\left\| \psi(\hat{\tilde{\mathbf{x}}}, u) - \psi(\tilde{\mathbf{x}}, u) \right\| \leq l_{\psi} \|\mathbf{e}\| \quad (\text{B.1})$$

for some Lipschitz constant l_{ψ} . Also, $|d(\mathbf{x}, t)| \leq \bar{d}$ and $|u_r| = \rho$. Under **Assumptions 2.1** and **4.1**, due to the existence of $\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}}$, $\tilde{\mathbf{p}}(\tilde{\mathbf{x}})$ and $\tilde{\mathbf{p}}(\hat{\tilde{\mathbf{x}}})$ are bounded, hence we have

$$\left\| \tilde{\mathbf{p}}(\hat{\tilde{\mathbf{x}}})u_r - \tilde{\mathbf{p}}(\tilde{\mathbf{x}})d(\mathbf{x}) \right\| \leq b_{\tilde{\mathbf{p}}} (\bar{d} + \rho) \quad (\text{B.2})$$

for some upper bound $b_{\tilde{\mathbf{p}}}$.

Differentiating the Lyapunov function $V_0 = \mathbf{e}^T \mathbf{P} \mathbf{e}$ with respect to time and using (4.8), we have

$$\dot{V}_0 = \mathbf{e}^T \mathbf{P} \dot{\mathbf{e}} + \dot{\mathbf{e}}^T \mathbf{P} \mathbf{e} = \mathbf{e}^T \left[(\mathbf{A} - \mathbf{L}\mathbf{C})^T \mathbf{P} + \mathbf{P} (\mathbf{A} - \mathbf{L}\mathbf{C}) \right] \mathbf{e}$$

$$+2\mathbf{e}^T \mathbf{P} \left[\psi(\hat{\mathbf{x}}, u) - \psi(\tilde{\mathbf{x}}, u) + \tilde{\mathbf{p}}(\hat{\mathbf{x}})u_r - \tilde{\mathbf{p}}(\tilde{\mathbf{x}})d(\mathbf{x}) \right]$$

Under Lyapunov equation (4.27) and the derived results in (B.1) and (B.2), we have

$$\begin{aligned} \dot{V}_0 &\leq -\|\mathbf{e}\|^2 + 2\lambda_{\max}(\mathbf{P})l_{\psi}\|\mathbf{e}\|^2 + 2\lambda_{\max}(\mathbf{P})b_{\tilde{\mathbf{p}}}(\bar{d} + \rho)\|\mathbf{e}\| \\ &= -c_1\|\mathbf{e}\|^2 + c_2\|\mathbf{e}\| \end{aligned}$$

where $c_1 \triangleq 1 - 2\lambda_{\max}(\mathbf{P})l_{\psi}$, $c_2 \triangleq 2\lambda_{\max}(\mathbf{P})b_{\tilde{\mathbf{p}}}(\bar{d} + \rho)$. The estimation gain $\mathbf{L}$ shall be designed such that the condition (4.28) is satisfied. Hence $\mathbf{e}$ is bounded. Since $\mathbf{x}$ is bounded, then $\hat{\mathbf{x}}$ is also bounded.

B.2 Proof of Theorem 4.2

Since e_2 , $\mathbf{x}$ and $d(\mathbf{x})$ are bounded, then according to (4.19), $u_{eq} \leq \bar{u}_{eq}$ for some upper bound $\bar{u}_{eq}$. Similar to (4.12), $\|\tilde{\mathbf{p}}(\tilde{\mathbf{x}})\| \leq b_{\tilde{\mathbf{p}}}^*$ where $b_{\tilde{\mathbf{p}}}^*$ is the bound of reduced-order $\tilde{\mathbf{p}}$ in the sliding mode. Under **Assumptions 2.2** and **4.2**, $\psi(\tilde{\mathbf{x}}, u)$ and $\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d)$ are Lipschitz functions, so we have

$$\left\| \psi(\hat{\mathbf{x}}_d, u) - \psi(\tilde{\mathbf{x}}, u) \right\| \leq l_{\psi}\|\mathbf{e}_d\| \quad (\text{B.3})$$

$$\left\| \left(\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d) - \tilde{\mathbf{p}}(\tilde{\mathbf{x}}) \right) u_{eq} \right\| \leq l_{\tilde{\mathbf{p}}}\bar{u}_{eq}\|\mathbf{e}_d\| \quad (\text{B.4})$$

for Lipschitz constants l_{ψ} and $l_{\tilde{\mathbf{p}}}$.

By choosing a Lyapunov function $V_2 = \mathbf{e}_d^T \mathbf{P} \mathbf{e}_d$ and using (4.10), we have

$$\begin{aligned} \dot{V}_2 &= \mathbf{e}_d^T \mathbf{P} \dot{\mathbf{e}}_d + \dot{\mathbf{e}}_d^T \mathbf{P} \mathbf{e}_d \\ &= \mathbf{e}_d^T \left[(\mathbf{A} - \mathbf{L}\mathbf{C})^T \mathbf{P} + \mathbf{P} (\mathbf{A} - \mathbf{L}\mathbf{C}) \right] \mathbf{e}_d \\ &\quad + 2\mathbf{e}_d^T \mathbf{P} \left[\psi(\hat{\mathbf{x}}_d, u) - \psi(\tilde{\mathbf{x}}, u) + \left(\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d) - \tilde{\mathbf{p}}(\tilde{\mathbf{x}}) \right) u_{eq} \right] + 2\mathbf{e}_d^T \mathbf{P} [-\tilde{\mathbf{p}}(\tilde{\mathbf{x}})e_{2,d}] \end{aligned}$$

Under Lyapunov equation (4.27) and the derived results in (B.3) and (B.4), we

have

$$\dot{V}_2 \leq -\|\mathbf{e}_d\|^2 + 2\lambda_{\max}(\mathbf{P}) \left[l_{\mathbf{p}} + l_{\tilde{\mathbf{p}}} \bar{u}_{eq} + b_{\tilde{\mathbf{p}}}^* \right] \|\mathbf{e}_d\|^2$$

Hence the condition (4.29) guarantees the convergence of the robust estimator.

B.3 Proof of Lemma 5.4

Based on **Assumption 5.6**, $\mu_w(\tilde{\mathbf{x}}, \mathbf{u})$, $\tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}})$ are Lipschitzian such that

$$\begin{aligned} \left\| \mu_w(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \mu_w(\tilde{\mathbf{x}}, \mathbf{u}) \right\| &\leq l_{\mu} \|\mathbf{e}\| \\ \left\| \tilde{\mathbf{Z}}_i(\hat{\tilde{\mathbf{x}}}) - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) \right\| &\leq l_{\mathbf{Z}_i} \|\mathbf{e}\| \end{aligned} \quad (\text{B.5})$$

As all the states of **Class B** subsystems are measured as outputs, and the estimation states converge, we have $\|\mathbf{e}\| \equiv \|\mathbf{e}_w\|$. Since disturbances and states are bounded, we have $|d_i(\tilde{\mathbf{x}}, \mathbf{u}, d)| \leq \bar{d}_i$ for some upper bound $\bar{d}_i$. Since $z_{1i}^{w_i}(\hat{\tilde{\mathbf{x}}})$ and sliding mode gain are bounded, we have $u_r^{w_i}$ bounded as $|u_r^{w_i}| \leq b_{u_r}^i$. Therefore, we have

$$\left\| \tilde{\mathbf{Z}}_i(\hat{\tilde{\mathbf{x}}}) u_r^{w_i} - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) d_i(\tilde{\mathbf{x}}, \mathbf{u}, t) \right\| \leq b_{\tilde{\mathbf{Z}}_i} (\bar{d}_i + b_{u_r}^i) \quad (\text{B.6})$$

for some upper bound $b_{\tilde{\mathbf{Z}}_i}$.

Now, we choose a Lyapunov function candidate as $V_1^w = \mathbf{e}_w^T \mathbf{Q}_w \mathbf{e}_w$. Differentiating with respect to time, and using (5.20) we have

$$\begin{aligned} \dot{V}_1^w &= \mathbf{e}_w^T \mathbf{Q}_w \dot{\mathbf{e}}_w + \dot{\mathbf{e}}_w^T \mathbf{Q}_w \mathbf{e}_w \\ &= \mathbf{e}_w^T \left[(\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w)^T \mathbf{Q}_w + \mathbf{Q}_w (\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w) \right] \mathbf{e}_w \\ &\quad + 2\mathbf{e}_w^T \mathbf{Q}_w \left[\mu_w(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \mu_w(\tilde{\mathbf{x}}, \mathbf{u}) + \sum_{i=1}^q \left(\tilde{\mathbf{Z}}_i(\hat{\tilde{\mathbf{x}}}) u_r^{w_i} - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) d_i \right) \right] \end{aligned} \quad (\text{B.7})$$

From the above results, we have

$$\dot{V}_1^w \leq -\|\mathbf{e}_w\|^2 + 2\lambda_{\max}(\mathbf{Q}_w)l_{\boldsymbol{\mu}}\|\mathbf{e}_w\|^2 + 2\lambda_{\max}(\mathbf{Q}_w)\sum_{i=1}^q b_{\tilde{\mathbf{z}}_i}(\bar{d}_i + b_{u_r}^i)\|\mathbf{e}_w\|$$

The estimation gain $\mathbf{L}_w$ should be designed such that condition (5.63) is satisfied.

Hence $\mathbf{e}_w$ is bounded. Since $\tilde{\mathbf{x}}_w$ is bounded, $\hat{\tilde{\mathbf{x}}}_w$ is also bounded.

B.4 Proof of Theorem 5.2

Substituting the equivalent control dynamics (5.42) into the error dynamics (5.44), we obtain

$$\begin{aligned} \dot{\mathbf{e}}_w^d &= (\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w) \mathbf{e}_w^d + \boldsymbol{\mu}_w(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \boldsymbol{\mu}_w(\tilde{\mathbf{x}}, \mathbf{u}) \\ &\quad + \sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i(\hat{\tilde{\mathbf{x}}}) - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) \right) u_{eq}^{w_i}(t) + \sum_{i=1}^m \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) \chi_i(e_{2,d}^{w_1}, e_{2,d}^{w_2}, \dots, e_{2,d}^{w_i}) \end{aligned} \quad (\text{B.8})$$

Based on **Assumption 5.6**, $\boldsymbol{\mu}_w(\tilde{\mathbf{x}}, \mathbf{u})$ and $\tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}})$ are Lipschitz functions. And as the **Class B** subsystems converge, similar to (B.5) we have

$$\left\| \boldsymbol{\mu}_w(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \boldsymbol{\mu}_w(\tilde{\mathbf{x}}, \mathbf{u}) \right\| \leq l_{\boldsymbol{\mu}} \|\mathbf{e}\| = l_{\boldsymbol{\mu}} \|\mathbf{e}_w^d\| \quad (\text{B.9})$$

$$\left\| \tilde{\mathbf{Z}}_i(\hat{\tilde{\mathbf{x}}}, \mathbf{u}) - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}, \mathbf{u}) \right\| \leq l_{\mathbf{Z}_i} \|\mathbf{e}\| = l_{\mathbf{Z}_i} \|\mathbf{e}_w^d\| \quad (\text{B.10})$$

From (5.40), as the second dynamics $e_{2,d}^{w_j}$ and $\hat{\tilde{\mathbf{x}}}_w$ are bounded, we have $|u_{eq}^{w_i}| \leq b_{u_i}$ for some upper bound b_{u_i} . From (5.42), we can obtain

$$u_{eq}^{w_i} - d_i = \chi_i(e_{2,d}^{w_1}, e_{2,d}^{w_2}, \dots, e_{2,d}^{w_i}, \hat{\tilde{\mathbf{x}}}_w)$$

So, based on boundedness of the functions $z_{ij}(\hat{\tilde{\mathbf{x}}}), \forall i, j$, we have

$$\|u_{eq}^{w_i} - d_i\| \leq b_{\chi_i} \mathbf{e}_w^d \quad (\text{B.11})$$

where b_{χ_i} is the equivalent bounded constant of all the successive bounds.

Now, we choose a Lyapunov function as $V_2^w = (\mathbf{e}_w^d)^T \mathbf{Q}_w \mathbf{e}_w^d$. Using (5.44) we can obtain

$$\begin{aligned}
\dot{V}_2^w &= (\mathbf{e}_w^d)^T \mathbf{Q}_w \dot{\mathbf{e}}_w^d + (\dot{\mathbf{e}}_w^d)^T \mathbf{Q}_w \mathbf{e}_w^d \\
&= (\mathbf{e}_w^d)^T \left[(\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w)^T \mathbf{Q}_w + \mathbf{Q}_w (\mathbf{A}_w - \mathbf{L}_w \mathbf{C}_w) \right] \mathbf{e}_w^d \\
&\quad + 2 (\mathbf{e}_w^d)^T \mathbf{Q}_w \left[\boldsymbol{\mu}_w(\hat{\mathbf{x}}, \mathbf{u}) - \boldsymbol{\mu}_w(\tilde{\mathbf{x}}, \mathbf{u}) + \sum_{i=1}^q \left(\tilde{\mathbf{Z}}_i(\hat{\mathbf{x}}) - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) \right) u_{eq}^{w_i}(t) \right] \\
&\quad + 2 (\mathbf{e}_w^d)^T \mathbf{Q}_w \left[\sum_{i=1}^q \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}) [u_{eq}^{w_i} - d_i] \right]
\end{aligned} \tag{B.12}$$

From the Lyapunov equation (5.62) and using the derived results (B.9), (B.10) and (B.11), we have

$$\dot{V}_2^{w_j} \leq -\|\mathbf{e}_w^d\|^2 + 2\lambda_{\max}(\mathbf{Q}_w) \|\mathbf{e}_w^d\|^2 \left[l_{\boldsymbol{\mu}} + \sum_{i=1}^q \left[l_{\mathbf{Z}_i} b_{u_i} + b_{\tilde{\mathbf{Z}}_i}^* b_{\chi_i} \right] \right] \tag{B.13}$$

where $b_{\tilde{\mathbf{Z}}_i}^*$ is the bound of the reduced order $\tilde{\mathbf{Z}}_i$ in the multiple sliding mode. Hence the condition (5.64) guarantees the asymptotical convergence of the robust estimator.

B.5 Proof of Lemma 7.1

For the system satisfying **Assumption 7.4**, $|u(k)| \leq u_{\max}$, the system states $\tilde{\mathbf{x}}(k)$ are all bounded, and $\|\mathcal{O}_1(T_s^2)\| \leq b_{\mathcal{O}}$ for some upper bounds $u_{\max}$ and $b_{\mathcal{O}}$. Since $L_{\mathbf{p}h}(\mathbf{x}(k)) \neq 0$, $\forall(\mathbf{x})$, then $|u_r(k)| \leq \frac{\rho}{L_{\mathbf{p}h}(\mathbf{x}(k))} \leq b_{\rho}$ for some upper bound b_{ρ} . Under **Assumption 7.3**, $\psi(\tilde{\mathbf{x}}(k), u(k))$ and $\tilde{\mathbf{p}}(\tilde{\mathbf{x}}(k))$ are globally Lipschitzian, so we have

$$\left\| \psi(\hat{\mathbf{x}}(k), u(k)) - \psi(\tilde{\mathbf{x}}(k), u(k)) \right\| \leq l_{\boldsymbol{\phi}} \|\mathbf{e}(k)\| \tag{B.14}$$

$$\left\| \left(\tilde{\mathbf{p}}(\hat{\mathbf{x}}(k)) - \tilde{\mathbf{p}}(\tilde{\mathbf{x}}(k)) \right) u_r(k) \right\| \leq l_{\tilde{\mathbf{p}}} b_{\rho} \|\mathbf{e}(k)\| \tag{B.15}$$

for Lipschitz constants l_ϕ and $l_{\tilde{\mathbf{p}}}$. Under **Assumptions 7.1** and **7.4**, $\tilde{\mathbf{p}}(\tilde{\mathbf{x}}(k))$ is bounded due to the existence of $\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}}$ and the boundedness of $\mathbf{x}(k)$. So, we have

$$\|\tilde{\mathbf{p}}(\mathbf{x}(k)) [T_s d(k) - u_r(k)]\| \leq b_{\tilde{\mathbf{p}}} (T_s \bar{d} + b_\rho) \quad (\text{B.16})$$

for some upper bound $b_{\tilde{\mathbf{p}}}$. Using the above results (B.14) - (B.16), we have

$$\begin{aligned} & \|[\tilde{\mathbf{p}}(\hat{\mathbf{x}}(k))u_r(k) - T_s \tilde{\mathbf{p}}(\mathbf{x}(k))d(k)]\| \\ & \leq \|\tilde{\mathbf{p}}(\hat{\mathbf{x}}(k))u_r(k) - \tilde{\mathbf{p}}(\mathbf{x}(k))u_r(k)\| + \|\tilde{\mathbf{p}}(\mathbf{x}(k))u_r(k) - T_s \tilde{\mathbf{p}}(\mathbf{x}(k))d(k)\| \\ & \leq l_\phi b_\rho \|\mathbf{e}(k)\| + b_{\tilde{\mathbf{p}}} (T_s \bar{d} + b_\rho) \end{aligned} \quad (\text{B.17})$$

Using the Lyapunov function $V(k) = \mathbf{e}^T(k) \mathbf{Q} \mathbf{e}(k)$, and the obtained results (B.14) - (B.17), together with (7.17), we may evaluate the difference of Lyapunov function as

$$\begin{aligned} & V(k+1) - V(k) \\ & = \mathbf{e}^T(k+1) \mathbf{Q} \mathbf{e}(k+1) - \mathbf{e}^T(k) \mathbf{Q} \mathbf{e}(k) \\ & \leq [\mathbf{e}^T(k) [\mathbf{M}^T \mathbf{P} \mathbf{M}] \mathbf{e}(k) - \mathbf{e}^T(k) \mathbf{Q} \mathbf{e}(k)] \\ & \quad + \lambda_{\max}(\mathbf{Q}) C_1 \times \|\mathbf{e}(k)\|^2 + \lambda_{\max}(\mathbf{Q}) C_2 \times \|\mathbf{e}(k)\| + \lambda_{\max}(\mathbf{Q}) C_3 \\ & \leq -[1 - \lambda_{\max}(\mathbf{Q}) C_1] \|\mathbf{e}(k)\|^2 + \lambda_{\max}(\mathbf{Q}) C_2 \times \|\mathbf{e}(k)\| + \lambda_{\max}(\mathbf{Q}) C_3 \end{aligned} \quad (\text{B.18})$$

for some positive constants C_2 and C_3 defined by

$$\begin{aligned} C_2 & \triangleq \|\mathbf{M}\| \|\mathbf{N}\| b_\Theta + T_s \|\mathbf{N}\|^2 b_\Theta l_\phi + \|\mathbf{N}\|^2 l_{\tilde{\mathbf{p}}} b_\rho + \|\mathbf{M}\| \|\mathbf{N}\| b_{\frac{\partial T}{\partial \mathbf{x}} \mathbf{P}} (T_s \bar{d} + b_\rho) \\ & \quad + T_s \|\mathbf{N}\|^2 l_\phi b_{\frac{\partial T}{\partial \mathbf{x}} \mathbf{P}} (T_s \bar{d} + b_\rho) + \|\mathbf{N}\|^2 l_{\tilde{\mathbf{p}}} b_\rho b_{\frac{\partial T}{\partial \mathbf{x}} \mathbf{P}} (T_s \bar{d} + b_\rho) \\ C_3 & \triangleq \|\mathbf{N}\|^2 \left[b_\Theta^2 + (b_{\frac{\partial T}{\partial \mathbf{x}} \mathbf{P}} (T_s \bar{d} + b_\rho))^2 + 2b_\Theta b_{\frac{\partial T}{\partial \mathbf{x}} \mathbf{P}} (T_s \bar{d} + b_\rho) \right] \end{aligned}$$

Feedback gain is designed such that $\lambda_{\max}(\mathbf{Q}) < \frac{1}{C_1}$, i.e. condition (7.18) is satisfied. Hence the quadratic term is negative and so $\mathbf{e}(k)$ is bounded. As $\mathbf{x}(k)$ is bounded, $\hat{\mathbf{x}}(k)$ is also bounded.

From (B.18), it is clear that the constants C_1 , C_2 and C_3 are functions of the

sampling period. Due to discretization and limited sampling period (T_s), the error will finally settle to a bound dictated by $b_{\mathcal{O}}$, T_s and disturbance bound $\bar{d}$. Provided the sampling period (T_s) is low, the error bound can be largely reduced.

B.6 Proof of Theorem 7.1

Since $e_2(k)$, $\mathbf{x}(k)$ and $d(k)$ are bounded, then according to (7.30), $|u_r^*| \leq \bar{u}_r^*$ for some upper bound $\bar{u}_r^*$. Under **Assumption 7.3**, similar to (B.14) and (B.15), we have

$$\left\| \phi \left(\tilde{\mathbf{x}}(k), \hat{\mathbf{x}}_d(k), u(k) \right) \right\| \leq l_{\phi} \|\mathbf{e}_d(k)\| \quad (\text{B.19})$$

$$\|(\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d(k)) - \tilde{\mathbf{p}}(\mathbf{x}(k))) u_r^*(k)\| \leq l_{\tilde{\mathbf{p}}} \bar{u}_r^* \|\mathbf{e}_d(k)\| \quad (\text{B.20})$$

Similar to (B.16), $\|\tilde{\mathbf{p}}(\mathbf{x}(k))\| \leq b_{\tilde{\mathbf{p}}}$ and $\frac{1}{L_{\mathbf{p}}h(\mathbf{x}(k))} \leq b_{\frac{1}{\tilde{\mathbf{p}}_1}}$. According to **Assumption 7.3**, $\tilde{p}_1(\tilde{\mathbf{x}}) = L_{\mathbf{p}}h(\mathbf{x}(k))$ is globally Lipschitzian. Therefore

$$\|L_{\mathbf{p}}h(\hat{\mathbf{x}}_d(k)) - L_{\mathbf{p}}h(\mathbf{x}(k))\| = \|\tilde{p}_1(\hat{\mathbf{x}}_d(k)) - \tilde{p}_1(\tilde{\mathbf{x}}(k))\| \leq l_{\tilde{p}_1} \|\mathbf{e}_d(k)\| \quad (\text{B.21})$$

From (7.30) with (B.21), we may evaluate

$$\begin{aligned} & \| [u_r^*(k) - T_s d(k)] \| \\ & \leq \left\| \left[\frac{\mathcal{O}_1(T_s^2)}{L_{\mathbf{p}}h(\hat{\mathbf{x}}_d(k))} - \frac{T_s e_{2,d}(k)}{L_{\mathbf{p}}h(\hat{\mathbf{x}}_d(k))} \right] + \left[\frac{L_{\mathbf{p}}h(\mathbf{x}(k))}{L_{\mathbf{p}}h(\hat{\mathbf{x}}_d(k))} - 1 \right] T_s d(k) \right\| \\ & \leq b_{\mathcal{O}} b_{\frac{1}{L_{\mathbf{p}}h}} + T_s b_{\frac{1}{L_{\mathbf{p}}h}} \|\mathbf{e}_d(k)\| + T_s b_{\frac{1}{L_{\mathbf{p}}h}} l_{L_{\mathbf{p}}h} \bar{d} \|\mathbf{e}_d(k)\| \\ & = b_{\frac{1}{L_{\mathbf{p}}h}} b_{\mathcal{O}} + T_s b_{\frac{1}{L_{\mathbf{p}}h}} (1 + l_{L_{\mathbf{p}}h} \bar{d}) \|\mathbf{e}_d(k)\| \end{aligned} \quad (\text{B.22})$$

The error dynamics (7.31) could be further simplified using the above results of (B.20) - (B.22) to obtain

$$\begin{aligned} & \|\mathbf{e}_d(k+1)\| \\ & \leq \|\mathbf{M} \mathbf{e}_d(k)\| + \|\mathbf{N}\| \left\| T_s \phi \left(\tilde{\mathbf{x}}(k), \hat{\mathbf{x}}_d(k), u(k) \right) + \mathcal{O}_1(T_s^2) \right\| \\ & \quad + \|\mathbf{N}\| \|\tilde{\mathbf{p}}(\hat{\mathbf{x}}_d(k)) - \tilde{\mathbf{p}}(\mathbf{x}(k))\| \|u_r^*(k)\| + \|\mathbf{N}\tilde{\mathbf{p}}(\mathbf{x}(k))\| \|u_r^*(k) - T_s d(k)\| \\ & \leq \|\mathbf{M}\| \|\mathbf{e}_d(k)\| + T_s l_{\phi} \|\mathbf{N}\| \|\mathbf{e}_d(k)\| + \|\mathbf{N}\| b_{\mathcal{O}} + l_{\tilde{\mathbf{p}}} \bar{u}_r^* \|\mathbf{N}\| \|\mathbf{e}_d(k)\| \end{aligned}$$

$$\begin{aligned}
& + \|\mathbf{N}\| b_{\tilde{\mathbf{p}}} b_{\frac{1}{L_{\mathbf{p}}h}} b_{\mathcal{O}} + T_s b_{\tilde{\mathbf{p}}} b_{\frac{1}{L_{\mathbf{p}}h}} (1 + l_{L_{\mathbf{p}}h} \bar{d}) \|\mathbf{N}\| \|\mathbf{e}_d(k)\| \\
= & \|\mathbf{M}\| \|\mathbf{e}_d(k)\| + \|\mathbf{N}\| \|\mathbf{e}_d(k)\| C_b + \|\mathbf{N}\| b_{\mathcal{O}} C_l
\end{aligned} \tag{B.23}$$

where $C_b = T_s \left(l_{\phi} + l_{\mathbf{g}} u_{max} \right) + l_{\frac{\partial T}{\partial \mathbf{x}} \mathbf{p}} \bar{u}_r^* + T_s b_{\tilde{\mathbf{p}}} b_{\frac{1}{L_{\mathbf{p}}h}} (1 + l_{L_{\mathbf{p}}h} \bar{d})$, $C_l = b_{\frac{\partial T}{\partial \mathbf{x}} \mathbf{p}} b_{\frac{1}{L_{\mathbf{p}}h}} + 1$.

Now, we choose Lyapunov function candidate as $V_3(k) = \mathbf{e}_d^T(k) \mathbf{Q} \mathbf{e}_d(k)$, and from (7.31) together with (B.23), we have the difference of Lyapunov function as

$$\begin{aligned}
& V_3(k+1) - V_3(k) \\
= & \mathbf{e}_d^T(k+1) \mathbf{Q} \mathbf{e}_d(k+1) - \mathbf{e}_d^T(k) \mathbf{Q} \mathbf{e}_d(k) \\
\leq & [\mathbf{e}_d^T(k) [\mathbf{M}^T \mathbf{Q} \mathbf{M}] \mathbf{e}_d(k) - \mathbf{e}_d^T(k) \mathbf{Q} \mathbf{e}_d(k)] + 2 \lambda_{max}(\mathbf{Q}) \|\mathbf{N}\| \|\mathbf{M}\| C_b \|\mathbf{e}_d(k)\|^2 \\
& + 2 \lambda_{max}(\mathbf{Q}) [\|\mathbf{M}\| \|\mathbf{N}\| b_{\mathcal{O}} C_l + \|\mathbf{N}\|^2 C_b b_{\mathcal{O}} C_l] \|\mathbf{e}_d(k)\| \\
& + \lambda_{max}(\mathbf{Q}) \|\mathbf{N}\|^2 b_{\mathcal{O}}^2 C_l^2 + \lambda_{max}(\mathbf{Q}) \|\mathbf{N}\|^2 C_b^2 \|\mathbf{e}_d(k)\|^2 \\
\leq & - [1 - \lambda_{max}(\mathbf{Q}) \{2 \|\mathbf{N}\| \|\mathbf{M}\| C_b + \|\mathbf{N}\|^2 C_b^2\}] \|\mathbf{e}_d(k)\|^2 \\
& + \lambda_{max}(\mathbf{Q}) b_{\mathcal{O}} \{2 \|\mathbf{M}\| \|\mathbf{N}\| C_l + \|\mathbf{N}\|^2 C_b C_l\} \|\mathbf{e}_d(k)\| \\
& + \lambda_{max}(\mathbf{Q}) b_{\mathcal{O}}^2 \{\|\mathbf{N}\|^2 C_l^2\}
\end{aligned} \tag{B.24}$$

Hence the condition (7.33) guarantees the convergence of the robust estimator to a bound dictated by the sampling period (T_s).

B.7 Proof of Lemma 8.1

Based on **Assumption 8.3**, $\mu_w(\tilde{\mathbf{x}}(k), \mathbf{u}(k))$, $\tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}(k))$ are Lipschitz functions such that

$$\begin{aligned}
\left\| \mu_w(\hat{\tilde{\mathbf{x}}}(k), \mathbf{u}(k)) - \mu_w(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \right\| & \leq l_{\mu} \|\mathbf{e}_w(k)\| \\
\left\| \tilde{\mathbf{Z}}_i(\hat{\tilde{\mathbf{x}}}(k)) - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}(k)) \right\| & \leq l_{\tilde{\mathbf{Z}}_i} \|\mathbf{e}_w(k)\|
\end{aligned} \tag{B.25}$$

Since disturbances and states are bounded, we have $|d_i(k)| \leq \bar{d}_i$, $\|\mathcal{O}_w(T_s^2)\| \leq b_{\mathcal{O}}^w$ for upper bounds $\bar{d}_i$ and $b_{\mathcal{O}}^w$ respectively. Therefore, with **Assumption 8.2**, we

have

$$\left\| \tilde{\mathbf{Z}}_i(\hat{\mathbf{x}}(k)) [u_r^{w_i}(k) - T_s d_i(k)] \right\| \leq b_{\tilde{\mathbf{Z}}_i}(\bar{d}_i T_s + \rho_{w_i}) \quad (\text{B.26})$$

for some upper bound $b_{\tilde{\mathbf{Z}}_i}$. Using the Lyapunov function $V_1^w = \mathbf{e}_w^T(k) \mathbf{Q}_w \mathbf{e}_w(k)$, we may evaluate the difference of Lyapunov function as

$$\begin{aligned} V_1^w(k+1) - V_1^w(k) &= \mathbf{e}_w^T(k+1) \mathbf{Q}_w \mathbf{e}_w(k+1) - \mathbf{e}_w^T(k) \mathbf{Q}_w \mathbf{e}_w(k) \\ &= \mathbf{e}_w^T(k) \mathbf{M}_w^T \mathbf{Q}_w \mathbf{M}_w \mathbf{e}_w(k) - \mathbf{e}_w^T(k) \mathbf{Q}_w \mathbf{e}_w(k) \\ &\quad + 2\mathbf{e}_w^T(k) \mathbf{M}_w^T \mathbf{Q}_w \Omega(k) + \Omega^T(k) \mathbf{Q}_w \Omega(k) \end{aligned} \quad (\text{B.27})$$

where $\Omega(k) = T_s \boldsymbol{\mu}_w(\hat{\mathbf{x}}(k), \mathbf{u}) - T_s \boldsymbol{\mu}_w(\tilde{\mathbf{x}}(k), \mathbf{u}) - \mathcal{O}_w(T_s) + \sum_{i=1}^q \tilde{\mathbf{Z}}_i(\hat{\mathbf{x}}(k)) u_r^{w_i}(k) - \sum_{i=1}^q \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}(k)) d_i(k) T_s$ according to (8.17).

Using the evaluated bounds (B.25) and (B.26) we have:

$$\begin{aligned} \Omega(k) &\leq \left(T_s l_{\boldsymbol{\mu}} + \sum_{i=1}^q l_{\tilde{\mathbf{Z}}_i} \rho_{w_i} \right) \|\mathbf{e}(k)\| + \sum_{i=1}^q b_{\tilde{\mathbf{Z}}_i}(\bar{d}_i T_s + \rho_{w_i}) + b_{\mathcal{O}}^w \\ &= l_{\boldsymbol{\mu}Z} \|\mathbf{e}(k)\| + b_{\tilde{Z}d} + b_{\mathcal{O}}^w \end{aligned} \quad (\text{B.28})$$

where $l_{\boldsymbol{\mu}Z} = T_s l_{\boldsymbol{\mu}} + \sum_{i=1}^q l_{\tilde{\mathbf{Z}}_i} \rho_{w_i}$ and $b_{\tilde{Z}d} = \sum_{i=1}^q b_{\tilde{\mathbf{Z}}_i}(\bar{d}_i T_s + \rho_{w_i})$. Based on the above results (B.28) together with (8.19), we can simplify the difference of Lyapunov function (B.27) as

$$\begin{aligned} V_1(k+1) - V_1(k) &\leq -\|\mathbf{e}_w(k)\|^2 + 2\lambda_{\max}(\mathbf{Q}_w) \|\mathbf{M}_w\| \|\mathbf{e}_w(k)\| [l_{\boldsymbol{\mu}Z} \|\mathbf{e}(k)\| + b_{\tilde{Z}d} + b_{\mathcal{O}}^w] \\ &\quad + \lambda_{\max}(\mathbf{Q}_w) \times \left[(l_{\boldsymbol{\mu}Z} \|\mathbf{e}(k)\|)^2 + (b_{\tilde{Z}d} + b_{\mathcal{O}}^w)^2 + 2(l_{\boldsymbol{\mu}Z} \|\mathbf{e}(k)\|) \times (b_{\tilde{Z}d} + b_{\mathcal{O}}^w) \right] \end{aligned}$$

As the states in **Class B** subsystems converge to the actual states asymptotically, we have $\hat{\mathbf{x}}_{w_0}(k) = \tilde{\mathbf{x}}_{w_0}(k)$ and eventually we have $\|\mathbf{e}(k)\| \triangleq \|\mathbf{e}_w(k)\|$. Hence

$$\begin{aligned} V_1(k+1) - V_1(k) &\leq -[1 - \lambda_{\max}(\mathbf{Q}_w) K_1] \|\mathbf{e}_w(k)\|^2 + \lambda_{\max}(\mathbf{Q}_w) (b_{\tilde{Z}d} + b_{\mathcal{O}}^w)^2 \\ &\quad + 2\lambda_{\max}(\mathbf{Q}_w) \|\mathbf{e}_w(k)\| \{ \|\mathbf{M}_w\| (b_{\tilde{Z}d} + b_{\mathcal{O}}^w) + l_{\boldsymbol{\mu}Z} (b_{\tilde{Z}d} + b_{\mathcal{O}}^w) \} \\ &= -C_1 \|\mathbf{e}_w(k)\|^2 + C_2 \|\mathbf{e}_w(k)\| + C_3 \end{aligned} \quad (\text{B.29})$$

where $K_1 \triangleq \{2\|\mathbf{M}_w\| \times l_{\mu Z} + (l_{\mu Z})^2\}$, $C_1 \triangleq [1 - \lambda_{\max}(\mathbf{Q}_w)K_1]$, $C_2 \triangleq 2\lambda_{\max}(\mathbf{Q}_w) \left\{ \|\mathbf{M}_w\| (b_{\tilde{Z}d} + b_{\mathcal{O}}^w) + l_{\mu Z} (b_{\tilde{Z}d} + b_{\mathcal{O}}^w) \right\}$ and $C_3 \triangleq \lambda_{\max}(\mathbf{Q}_w) (b_{\tilde{Z}d} + b_{\mathcal{O}}^w)^2$ are positive constants. The estimation gain $\mathbf{L}_w$ should be designed such that condition (8.20) is satisfied and so $C_1 > 0$. Hence the quadratic equation is asymptotically bounded. Hence $\mathbf{e}_w(k)$ is bounded. Since $\tilde{\mathbf{x}}_w(k)$ is bounded, $\hat{\tilde{\mathbf{x}}}_w(k)$ is also bounded.

Due to discretization and limited sampling period (T_s), the error will finally settle to a bound dictated by $b_{\mathcal{O}}^w$, T_s and disturbance bound $\bar{d}_i$. If the sampling period T_s is small, the error bound can be largely reduced.

B.8 Proof of Theorem 8.1

For the error dynamics (8.30), we make the following denotations for ease of analysis:

$$\begin{aligned} \Theta(k) &= T_s \boldsymbol{\mu}_w(\hat{\tilde{\mathbf{x}}}(k), \mathbf{u}(k)) - T_s \boldsymbol{\mu}_w(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) \\ &\quad + \sum_{i=1}^m \left(\tilde{\mathbf{Z}}_i(\hat{\tilde{\mathbf{x}}}(k)) - \tilde{\mathbf{Z}}_i(\tilde{\mathbf{x}}(k)) \right) u_{eq}^{w_i}(k) - \mathcal{O}_{w_j}(T_s) \\ \Gamma(k) &= \sum_{i=1}^m \tilde{\mathbf{Z}}_w^i(\hat{\tilde{\mathbf{x}}}(k)) [\mathbf{o}_i^{eq}(T_s^2) - T_s \chi_i(\cdot, k)] \end{aligned} \quad (\text{B.30})$$

Since $e_{2,d}^{w_i}, \hat{\tilde{\mathbf{x}}}(k)$ and d_i are bounded, we have $|u_{eq}^{w_i}| \leq b_{u_i}$ for some upper bound b_{u_i} . Also $\chi_i(\cdot, k)$ corresponds to second-error dynamics of all subsystems, therefore from (8.28)

$$\left\| T_s \sum_{i=1}^m \tilde{\mathbf{Z}}_i(\hat{\tilde{\mathbf{x}}}(k)) \chi_i(\cdot, k) \right\| \leq T_s \sum_{i=1}^m b_{\tilde{\mathbf{Z}}_i}^* b_{\chi_i} \mathbf{e}_w^d(k) \quad (\text{B.31})$$

for some upper bound $b_{\tilde{\mathbf{Z}}_i}^*$ of the reduced-order $\tilde{\mathbf{Z}}_i(\hat{\tilde{\mathbf{x}}}(k))$ in the multiple sliding mode, and b_{χ_i} is the equivalent bounded constant of $\chi_i(\cdot, k)$.

Similar to (B.26) and using the above evaluations, we have

$$\begin{aligned}\|\Theta(k)\| &\leq \left[\left(T_s l_{\mu} + \sum_{i=1}^m l_{\mathbf{z}_i} b_{u_i} \right) \|\mathbf{e}(k)\| + b_{\odot}^w \right] = [l_{\mu Z} \|\mathbf{e}(k)\| + b_{\odot}^w] \\ \|\Gamma(k)\| &\leq T_s \sum_{i=1}^m b_{\mathbf{z}_i}^* b_{\chi_i} \mathbf{e}_w^d(k) + \sum_{i=1}^j b_{\mathbf{z}_i}^* b_{\mathbf{o}}^{w_i} = b_{\mu Z} \|\mathbf{e}_w^d(k)\| + b_{\odot Z}\end{aligned}$$

where $l_{\mu Z} = T_s l_{\mu} + \sum_{i=1}^m l_{\mathbf{z}_i} b_{u_i}$, $b_{\mu Z} = T_s \sum_{i=1}^m b_{\mathbf{z}_i}^* b_{\chi_i}$ and $b_{\odot Z} = \sum_{i=1}^j b_{\mathbf{z}_i}^* b_{\mathbf{o}}^{w_i}$.

Now, we choose a Lyapunov function as $V_2(k) = (\mathbf{e}_w^d(k))^T \mathbf{Q}_w \mathbf{e}_w^d(k)$. Similar to the analysis of (B.29) we have $\|\mathbf{e}(k)\| \triangleq \|\mathbf{e}_w(k)\|$. In the multiple DSM of $e_1^{w_1} = 0, \dots, e_1^{w_m} = 0$, we have the reduced order error dynamics, i.e., $\|\mathbf{e}(k)\| \triangleq \|\mathbf{e}_w^d(k)\|$. Using (8.30) and the above results, we can obtain the difference of Lyapunov function similar to analysis of **Lemma 8.1** as follows:

$$\begin{aligned}V_2(k+1) - V_2(k) &\leq -\|\mathbf{e}_w^d(k)\|^2 + \|\mathbf{e}_w^d(k)\|^2 \lambda_{\max}(\mathbf{Q}_w) \left[(l_{\mu Z})^2 + 2\|\mathbf{M}_w\| (l_{\mu Z} + b_{\mu Z}) \right. \\ &\quad \left. + [(b_{\mu Z})^2 + 2l_{\mu Z} b_{\mu Z}] \right] + \|\mathbf{e}_w^d(k)\| \lambda_{\max}(\mathbf{Q}_w) \left[2\|\mathbf{M}_w\| [b_{\odot}^w + b_{\odot Z}] + 2(b_{\odot}^w) l_{\mu Z} \right. \\ &\quad \left. + 2b_{\odot Z} + 2[l_{\mu Z} b_{\odot Z} + b_{\mu Z} b_{\odot}^w] \right] + \lambda_{\max}(\mathbf{Q}_w) [(b_{\odot}^w)^2 + b_{\odot Z}^2 + 2b_{\odot}^w b_{\odot Z}] \quad (\text{B.32}) \\ &\leq -[1 - \lambda_{\max}(\mathbf{Q}_w) \Delta_1] \|\mathbf{e}_w^d(k)\|^2 + \lambda_{\max}(\mathbf{Q}_w) \Delta_2 \|\mathbf{e}_w^d(k)\| + \Delta_3 \quad (\text{B.33})\end{aligned}$$

where $\Delta_1 \triangleq \left[(l_{\mu Z})^2 + 2\|\mathbf{M}_w\| (l_{\mu Z} + b_{\mu Z}) + [(b_{\mu Z})^2 + 2l_{\mu Z} b_{\mu Z}] \right]$, $\Delta_2 \triangleq \left[2\|\mathbf{M}_w\| [b_{\odot}^w + b_{\odot Z}] + 2(b_{\odot}^w) l_{\mu Z} + 2b_{\odot Z} + 2[l_{\mu Z} b_{\odot Z} + b_{\mu Z} b_{\odot}^w] \right]$ and $\Delta_3 \triangleq \lambda_{\max}(\mathbf{Q}_w) [(b_{\odot}^w)^2 + b_{\odot Z}^2 + 2b_{\odot}^w b_{\odot Z}]$ are positive constants that explicitly depend on sampling period T_s . Hence the condition (8.31) guarantees the convergence of the robust estimator to a bound dictated by the sampling period (T_s).

Remark B.1 The positive constants C_1, C_2 in Appendix B.7 and Δ_1, Δ_2 and Δ_3 in Appendix B.8 are dependent on the sampling period T_s . In fact, they can be rewritten as $C_1 = T_s \times f_{c_1}(\cdot)$, $C_2 = T_s \times f_{c_2}(\cdot)$ and $\Delta_i = T_s \times f_{\Delta_i}(\cdot)$, for $i = 1, 2, 3$ where $f_{c_1}(\cdot)$, $f_{c_2}(\cdot)$ and $f_{\Delta_i}(\cdot)$ are polynomial functions of T_s . Hence, the selection of a lower sampling period T_s decreases the bounds in inequalities (8.20) and (8.31) as well as the ultimate bounds for error convergence.

B.9 Proof of Lemma 8.2

Here, we analyze the dynamics of $e_1^{w_j}(k)$ of the error dynamics $\mathbf{e}_{w_j}(k)$ from (8.22).

$$\begin{aligned} e_1^{w_j}(k+1) &= (1 - l_1^{w_j})e_1^{w_j}(k) + T_s e_2^{w_j}(k) + T_s \mu_1^{w_j}(\hat{\mathbf{x}}(k), \mathbf{u}(k)) \\ &\quad - T_s \mu_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) + \sum_{i=1}^{j-1} \left[z_{1i}^{w_j}(\hat{\mathbf{x}}(k)) u_r^{w_i} - T_s z_{1i}^{w_j}(\tilde{\mathbf{x}}(k)) d_i(k) \right] \\ &\quad + u_r^{w_j}(k) - T_s d_j(k) - \mathbf{o}_2^{w_j}(T_s) \end{aligned} \quad (\text{B.34})$$

The proof is divided into two parts and the analysis inside and outside the boundary layer are separately addressed.

Outside the Boundary Layer: In our case, $\mathbf{s}(k) = e_1^{w_j}(k)$. For the trajectory $e_1^{w_j}(k)$ outside the boundary, i.e. $|e_1^{w_j}(k)| > \epsilon_{w_j}$, we have from (6.8)

$$\text{sat}(e_1^{w_j}(k), \epsilon_{w_j}) = \text{sign}(e_1^{w_j}(k)) \quad (\text{B.35})$$

- **Lower bound for Sliding Mode Gain ρ_{w_j} :** From (B.34), we can obtain according to (6.4):

$$\begin{aligned} &[e_1^{w_j}(k+1) - e_1^{w_j}(k)] \text{sign}(e_1^{w_j}(k)) \\ &= -l_1^{w_j} |e_1^{w_j}(k)| + \text{sign}(e_1^{w_j}(k)) \delta_{w_j}(k) - \rho_{w_j} \\ &\quad + T_s [\mu_1^{w_j}(\hat{\mathbf{x}}(k), \mathbf{u}(k)) - \mu_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k))] \text{sign}(e_1^{w_j}(k)) \end{aligned} \quad (\text{B.36})$$

There exists a finite constant gain such that $\rho_{w_j} > \rho_{w_j}^-$; $\rho_{w_j}^- = \sup |\delta_{w_j}(k)|$ such that when $e_1^{w_j}(k) \neq 0$, we have

$$\begin{aligned} &[e_1^{w_j}(k+1) - e_1^{w_j}(k)] \text{sign}(e_1^{w_j}(k)) \\ &< -l_1^{w_j} |e_1^{w_j}(k)| + T_s [\mu_1^{w_j}(\hat{\mathbf{x}}(k), \mathbf{u}(k)) - \mu_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k))] \text{sign}(e_1^{w_j}(k)) \end{aligned}$$

Based on Lipschitz **Assumption 8.3**, we have

$$[e_1^{w_j}(k+1) - e_1^{w_j}(k)] \text{sign}(e_1^{w_j}(k)) < -l_1^{w_j} |e_1^{w_j}(k)| + T_s l_{\mu_1}^{w_j} |\mathbf{e}_{w_j}|$$

For Lipschitz constant $l_{\mu_1}^{w_j}$ of $\mu_1^{w_j}(\cdot)$. Since $\mu_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k)) = \mu_1^{w_j}(x_1^{w_1}(k), \dots, x_1^{w_j}(k), \tilde{\mathbf{x}}_{w_0}(k), \mathbf{u}(k))$, and as the states of the subsystems converge sequentially in the sliding modes of $e_1^{w_1}(k), \dots, e_1^{w_{j-1}}(k)$, plus the states of **Class B** subsystems $\tilde{\mathbf{x}}_{w_0}(k)$ also converge, we eventually have $\mathbf{e}_{w_j}(k) \equiv e_1^{w_j}(k)$. Therefore,

$$[e_1^{w_j}(k+1) - e_1^{w_j}(k)] \text{sign}(e_1^{w_j}(k)) < -l_1^{w_j}|e_1^{w_j}(k)| + T_s l_{\mu_1}^{w_j}|e_1^{w_j}(k)|$$

Therefore by proper design of feedback estimation gain $T_s l_{\mu_1}^{w_j} < l_1^{w_j} < 1$, the inequality (6.4) can be met.

- **Upper bound for Sliding Mode Gain ρ_{w_j} :** Similar to (B.36) we can obtain

$$\begin{aligned} & [e_1^{w_j}(k+1) + e_1^{w_j}(k)] \text{sign}(e_1^{w_j}(k)) \\ = & (2 - l_1^{w_j})|e_1^{w_j}(k)| + \delta_{w_j}(k) \text{sign}(e_1^{w_j}(k)) - \rho_{w_j} \\ & + T_s [\mu_1^{w_j}(\hat{\tilde{\mathbf{x}}}(k), \mathbf{u}(k)) - \mu_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k))] \text{sign}(e_1^{w_j}(k)) \end{aligned}$$

In order to meet Condition (6.5) for the above equation, we need to have

$$\begin{aligned} \rho_{w_j} < \min \left\{ (2 - l_1^{w_j})|e_1^{w_j}(k)| + \text{sign}(e_1^{w_j}(k))\delta_{w_j}(k) \right. \\ \left. + T_s [\mu_1^{w_j}(\hat{\tilde{\mathbf{x}}}(k), \mathbf{u}(k)) - \mu_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k))] \text{sign}(e_1^{w_j}(k)) \right\} \end{aligned} \quad (\text{B.37})$$

From the above analysis we already have $|T_s [\mu_1^{w_j}(\hat{\tilde{\mathbf{x}}}(k), \mathbf{u}(k)) - \mu_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k))]| \leq T_s l_{\mu_1}^{w_j}|e_1^{w_j}(k)| < l_1^{w_j}|e_1^{w_j}(k)|$ and $|\text{sign}(e_1^{w_j}(k))\delta_{w_j}(k)| \leq \rho_{w_j}^-$. Therefore, we can show that the right-hand side of (B.37) satisfies:

$$\begin{aligned} (2 - l_1^{w_j})|e_1^{w_j}(k)| + \text{sign}(e_1^{w_j}(k))\delta_{w_j}(k) & \geq (2 - l_1^{w_j})|e_1^{w_j}(k)| - l_1^{w_j}|e_1^{w_j}(k)| - \rho_{w_j}^- \\ & = 2(1 - l_1^{w_j})|e_1^{w_j}(k)| - \rho_{w_j}^- \end{aligned}$$

For the trajectory outside the boundary layer, i.e., $|e_1^{w_j}(k)| > \epsilon_{w_j}$, the upper bound for the sliding mode gain $\rho_{w_j}^+$ can be derived as

$$\rho_{w_j}^+ = 2(1 - l_1^{w_j})\epsilon_{w_j} - \rho_{w_j}^- \quad (\text{B.38})$$

So finally the gain should be designed as $\rho_{w_j}^- < \rho_{w_j} < \rho_{w_j}^+$ to meet both conditions (6.4) and (6.5). The above conditions ensures the convergence of $e_1^{w_j}(k)$ until it reaches the boundary layer.

Inside the Boundary Layer: Inside the boundary layer, we have $|e_1^{w_j}(k)| < \epsilon_{w_j}$ and $\text{sat}(\cdot, \epsilon_{w_j}) = (\cdot)/\epsilon_{w_j}$. The error dynamics (B.34) can be reformulated as

$$\begin{aligned} e_1^{w_j}(k+1) &= (1 - l_1^{w_j})e_1^{w_j}(k) + \delta_{w_j}(k) - \rho_{w_j} \frac{e_1^{w_j}(k)}{\epsilon_{w_j}} \\ &\quad + T_s[\mu_1^{w_j}(\hat{\mathbf{x}}(k), \mathbf{u}(k)) - \mu_1^{w_j}(\tilde{\mathbf{x}}(k), \mathbf{u}(k))] \end{aligned}$$

Since $\rho_{w_j}^- > |\delta_{w_j}(k)|$ and $l_1^{w_j} > T_s l_{\mu_1}^{w_j}$, we can obtain

$$\begin{aligned} |e_1^{w_j}(k+1)| &< \left| \left[(1 - l_1^{w_j}) - \frac{\rho_{w_j}}{\epsilon_{w_j}} \right] e_1^{w_j}(k) \right| + l_1^{w_j} |e_1^{w_j}(k)| + \rho_{w_j}^- \\ &< \left| (1 - l_1^{w_j})\epsilon_{w_j} - \rho_{w_j} \right| + l_1^{w_j} \epsilon_{w_j} + \rho_{w_j}^- \end{aligned}$$

From condition (B.38), we have $\rho_{w_j}^+ - (1 - l_1^{w_j})\epsilon_{w_j} = (1 - l_1^{w_j})\epsilon_{w_j} - \rho_{w_j}^-$, and eventually we have $|\rho_{w_j} - (1 - l_1^{w_j})\epsilon_{w_j}| < (1 - l_1^{w_j})\epsilon_{w_j} - \rho_{w_j}^-$. Now, it can be easily shown that

$$\begin{aligned} |e_1^{w_j}(k+1)| &< (1 - l_1^{w_j})\epsilon_{w_j} - \rho_{w_j}^- + l_1^{w_j} \epsilon_{w_j} + \rho_{w_j}^- \\ &= \epsilon_{w_j} \end{aligned} \quad (\text{B.39})$$

Hence, the condition (8.23) confines the estimation error to an ultimate bound inside the boundary layer.

Author's Publications

International Conferences or Proceedings:

- (1) W. Cao, Y. C. Soh and **K. C. Veluvolu**, *An integrated nonlinear observer with sliding mode estimation for a class of nonlinear uncertain systems*, in Proceedings of the 42nd IEEE Conference on Decision and Control, Hawaii, USA, 2003, pp. 5741-5746.
- (2) **K. C. Veluvolu**, Y. C. Soh and W. Cao, *Discrete-time Sliding Mode Observer Design for a Class of Uncertain Nonlinear Systems with Application to Bioprocess*, in Proceedings of ICARCV, Kunming - China, December 2004, pp. 855-860.
- (3) **K. C. Veluvolu**, Y. C. Soh, W. Cao, and Z. Y. Liu, *Observer with Multiple Sliding Modes for a Class of Nonlinear Uncertain Systems*, in Proceeding of American Control Conference, USA, 2005, 2445 -2450.
- (4) **K. C. Veluvolu**, Y. C. Soh, W. Cao, and Z. Y. Liu, *Discrete-time Observers with Multiple Sliding Modes for a Class of Nonlinear Uncertain Systems*, in Proceedings of the 1st IEEE Conference on Industrial Electronics and Applications, Singapore, May 2006, 197 -202.
- (5) **K. C. Veluvolu**, S. Pavuluri, Y. C. Soh, W. Cao, and Z. Y. Liu,, *Observers with Multiple Sliding Modes for Uncertain Linear MIMO Systems*, in Proceedings of the 1st IEEE Conference on Industrial Electronics and Applications, Singapore, May 2006, 185 -190.
- (6) **K. C. Veluvolu**, Y. C. Soh, W. Cao, and Z. Y. Liu, *Discrete-time Sliding Mode Observers for a Class of Nonlinear Uncertain Systems*, in Proceedings of American

Control Conference, USA, June 2006, 2605 -2610.

- (7) **K. C. Veluvolu**, and Y. C. Soh, *Nonlinear Sliding Mode State and Unknown Input Estimations: A New Perspective*, *International Conference on Advances in Control and Optimization of Dynamical Systems (ACODS'2007)*, Bangalore, India, February 1 - 2, 2007, 332 - 339.
- (8) **K. C. Veluvolu**, Y. C. Soh, W. Cao, and Z. Y. Liu, *Nonlinear Discrete-time Multiple Sliding Mode Observations with Application to Three-phase Current Motor*, *IEEE Conference on Industrial Electronics and Applications*, Harbin, China, May 2007, 531 - 536.
- (9) **K. C. Veluvolu** and Y. C. Soh, *Nonlinear Sliding Mode Observers for State and Unknown Input Estimations*, submitted to, IEEE conference on Decision and Control, USA, December 2007.
- (10) **K. C. Veluvolu** and Y. C. Soh, *High-Gain Observers with Sliding Mode for State and Unknown Input Estimations*, submitted to, IEEE conference on Decision and Control, USA, December 2007.
- (11) **K. C. Veluvolu** and Y. C. Soh, *High-gain Observer with Multiple Sliding Modes for State and Unknown Input Estimations*, for submission to, IFAC world congress, Seoul, Korea, 2008.

International Journals:

- (1) **K. C. Veluvolu**, Y. C. Soh and W. Cao, *Robust Discrete-time Nonlinear Sliding Mode State Estimation for a Class of Uncertain Nonlinear Systems*, *International Journal of Robust and Nonlinear Control*, vol. 17, no. 9, pp. 803 - 828, 2007.
- (2) **K. C. Veluvolu**, Y. C. Soh and W. Cao, *Robust Observer with Sliding Mode Estimation for Nonlinear Uncertain Systems*, *IET Proceedings on Control Theory and Applications*, vol. 1, no. 5, pp. 1533 - 1540, 2007.
- (3) **K. C. Veluvolu**, Y. C. Soh, *High Gain Observer with Sliding Mode for State and Unknown Input Estimations*, submitted to, *IEEE transactions on Automatic*

Control, 2007.

- (4) **K. C. Veluvolu**, Y. C. Soh, *High Gain Multiple Sliding Mode State and Unknown Input Observations for Nonlinear Uncertain Systems*, submitted to *SIAM Journal on Control and Optimization*, 2007.
- (5) **K. C. Veluvolu**, and Y. C. Soh, *Multiple Sliding Mode State and Unknown Input Estimations For Nonlinear Uncertain Systems*, under review, *International Journal of Control*, 2007.
- (6) **K. C. Veluvolu**, and Y. C. Soh, *Sliding Mode Observers and Unknown Input Observations: A new Perspective*, under review, *International Journal of Robust and Nonlinear Control*, 2007.
- (7) **K. C. Veluvolu**, Y. C. Soh, W. Cao and Z. Y. Liu, *Discrete-time Multiple Sliding Mode State and Unknown Input Estimations for Nonlinear Systems*, under review, *International Journal of Control*, 2007.

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