Array pattern synthesis using eigen-analysis

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Array Pattern Synthesis Using Eigen-Analysis

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Summary

One of the basic problems in array design is to synthesize desired array patterns that are independent of measured array data statistics. In this thesis, we consider the desired array patterns including the maximum-gain, minimum-gain, single look-direction and shaped patterns. For a given power of array weight, the maximum-gain pattern can be a spatial response in a target region or directions of interest that has maximum power, whereas the minimum-gain pattern has minimum power in a target region or directions of interest. For an acoustic source array, the maximum-gain and minimum-gain patterns represent the sound pressure level (SPL) distribution of a spatial acoustic hotspot and a quiet zone, respectively. A single look-direction pattern has a minimum-beamwidth mainlobe for a given sidelobe level. A shaped pattern has predefined responses in different directions of interest. A typical shaped pattern is the flat-top pattern, which has uniform response in the mainlobe region.

Despite the different definitions of desired array patterns, it is possible to study the pattern synthesis problems in one framework based on eigen-analysis. A general array system can be modeled as a multi-input multi-output (MIMO) system with the array weight being the multi-input (MI) and discrete samples in a target region or directions of interest being the multi-output (MO). By applying eigen-analysis, a series of basic modes of the MI can be found, which are useful in studying the potential of the MIMO system for producing desired MO. Since these modes are subject to different definitions of the target region or directions of interest, we name these modes the target-oriented array-modes (TOAMs). The TOAM series has some

interesting properties, and can be applied to solve the desired array-pattern-synthesis problems. Algorithms for three major applications are developed.

Firstly, an algorithm is formulated utilizing the maximum-gain TOAM to determine the optimal array weight for acoustical-hotspot generation using a source array. This solution is shown to be more effective in focusing acoustical energy in a target region than a widely used time-delay method.

Secondly, an analytical method employing lower-gain TOAMs to derive the optimal array weight is proposed for quiet-zone generation. The advantage of this method is that it is robust even when the transfer matrix is ill-conditioned, whereas an existing solution may show a weaker performance in this case.

Thirdly, the TOAM series expansion is applied to solve a nonlinear optimization problem in synthesizing single look-direction and shaped array patterns. This new method requires no multi-dimensional starting point, and therefore is more robust compared with a nonlinear least-square method which depends on a multi-dimensional starting point.

In addition, a new iterative method for the synthesis of single look-direction pattern and shaped pattern is developed by combining the above-mentioned new synthesis method with a peak-iteration strategy. This method is able to provide equal-level sidelobes and is effective in sidelobe suppression at the same time.

In general, this thesis presents original work on the following areas in synthesizing desired array patterns:

- (i) A target-oriented array-mode (TOAM) technique for analyzing general array-pattern-synthesis systems.

- (ii) An acoustical-hotspot generation method using the maximum-gain TOAM as the optimal array weight.
- (iii) A new analysis method using lower-gain TOAMs for deriving the optimal weights of secondary-sources in quiet-zone generation.
- (iv) A new method for the synthesis of single look-direction pattern and shaped array pattern based on a nonlinear optimization algorithm using TOAM series expansion.
- (v) A new iterative method for synthesizing single look-direction pattern and shaped array pattern with equal-level sidelobes.

The research work in this thesis is basically focused on the methodology of the desired array pattern synthesis. Theoretical analyses and simulation results are presented to show the effectiveness of the proposed algorithms. This work has also been published in technical journals and conferences.

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Chapter 1

Introduction

1.1 Research area and motivation

Array processing has played a very important role in many diverse applications. Human-computer-interface systems apply microphone arrays for capturing user speech [VHY95] [BW01] [BBK03]. Entertainment systems use loudspeaker arrays for delivering music or sound in certain spatial coverage [AMY96] [Kee02] [1Ltd05] [ARL00]. Communication systems utilize antenna arrays for signal receiving or transmitting [ST98] [Mai94]. Radio astronomy employs very large arrays (VLA) to achieve high-resolution plots [JD92]. A homeland security system may guard a region to be protected with a multi-sensor array [Duc98] [Sen04]. A noise cancellation system uses multiple secondary sources for controlling the noise in a target region [NCE92] [KM96] [Ell01].

Behind the various array applications are basically two classes of array processing: parameter estimation (also commonly referred as direction of arrival (DOA) estimation), and signal estimation (also referred as beamforming). In this thesis, we consider the problems related to the later class. The objective of beamforming is to determine an optimal array weight that enables the output of the array signal to have a higher signal-to-noise ratio (SNR) and signal-to-interference ratio (SIR) compared to a single element. The beamforming, in general can also be divided into two categories, consisting of data-dependent and -independent beamforming [VB88] [Sim98] [Her03]. Data-dependent beamforming requires the array weight to be properly chosen so as to obtain an optimal response of the

beamformer based on the measured array data statistics. The optimal response refers to the array output which contains minimal contribution due to noise or interference from undesired directions. Since the spatial and temporal characteristics of the source may be non-stationary in nature, adaptive processing techniques can be employed according to different source scenarios. Data-independent beamforming, on the other hand, aims to design an optimal array weight, such that the array pattern approximates to a desired response that is independent of the array data statistics. In this thesis, we focus on the desired array pattern synthesis that is independent of data statistics.

Array pattern synthesis (APS) usually refers to the synthesis of single or multiple look-direction patterns or shaped patterns. A single look-direction pattern has a minimum-beamwidth mainlobe in one look-direction with a given sidelobe level. Such a pattern can be typically synthesized using the Dolph-Chebyshev method [Dol46]. A multiple look-direction pattern has minimum-beamwidth mainlobes in multiple look-directions. Since single look-direction patterns are more often discussed and widely used than multiple look-direction patterns, we mainly study the synthesis of single look-direction patterns in this thesis. A shaped pattern refers to an amplitude response with pre-specified levels at different angles. One of the typical shaped patterns is flat-top pattern, which has uniform response in the mainlobe region. A flat-top pattern can be synthesized using, for example, the Woodward-Lawson [WL48], Parks-McClellan [PM72a] or adaptive array methods [ZI99]. Flat-top patterns can be regarded as single look-direction pattern. However, it has multiple look-directions in the mainlobe with equal response, and the synthesis is emphasized on the shape of the pattern. Therefore, it can be categorized as a shaped pattern.

According to the difference of cost functions used, there are two other widely used array patterns: the maximum-gain and minimum-gain patterns. For a given

power of array weight, the maximum-gain pattern can be a spatial response in a target region or directions of interest that has maximum power [CK02] [WYG03] [WYG05], whereas the minimum-gain pattern has minimum power [NCE92] [KM96] [Ell01].

In contrast to the APS problems that focus on the array directivity, the maximum/minimum-gain pattern syntheses concern with the efficiency of the energy transmitting between an array and a spatial target region. In spite of this difference, the APS and the maximum/minimum-gain pattern generation can be commonly regarded as spatial filtering. Therefore, it is possible to study these pattern synthesis problems together, and the desired array patterns can be extended to include the maximum/minimum-gain patterns, besides single look-direction and shaped patterns. In general, our research area is the desired array pattern synthesis categorized under data-independent beamforming, as highlighted in Fig. 1.1.

Fig. 1.1 Research area illustration.

This thesis focuses more on theoretical analysis and solutions to the synthesis problems than on the engineering design of the array. The potential of a general array for various array pattern syntheses is mainly studied. This study aims to provide valuable analysis results and better theoretical solutions than existing methods. The

general array is assumed to be formed by point-like omnidirectional elements without considering coupling effect among elements. According to the difference of the cost functions used and the shapes of patterns to be optimized, the desired array patterns discussed in this thesis include the following four categories:

- Maximum-gain pattern

The optimal array weight is obtained by

$$\mathbf{w}_{opt} = \arg \max_{\mathbf{w}} g(\mathbf{w}), \quad (1.1)$$

where $\mathbf{w}$ is the complex weight vector of the array. $g(\mathbf{w})$ is the gain function defined as $g(\mathbf{w}) = E_{out} / E_{in}$ with E_{out} being the output power in a target region and E_{in} being the power of the array weight.

- Minimum-gain pattern

The optimal array weight is obtained by

$$\mathbf{w}_{opt} = \arg \min_{\mathbf{w}} g(\mathbf{w}). \quad (1.2)$$

- Single look-direction pattern

The optimal array weight is obtained by

$$\mathbf{w}_{opt} = \arg \min_{\mathbf{w}} J(\mathbf{w}), \quad (1.3)$$

where error function $J(\mathbf{w})$ can be either the minimax error or square error between a desired pattern and synthesized pattern. In this thesis, the proposed methods in Chapters 4 and 5 are based on the square-error cost function.

- Shaped pattern (e.g. flat-top pattern)

Similar cost function as Eq. (1.3) is used. However, the cost function for shaped pattern synthesis also takes account of the factors such as mainlobe shape, mainlobe width/ripple, transition region, and so on.

Note: Section 2.1.4 in Chapter 2 will go into details of the construction of the functions mentioned above.

The synthesis of the maximum-gain, minimum-gain, single look-direction, and shaped patterns is applicable to both receiving and transmitting arrays. For example, consider a 15-element half-wavelength-spacing linear source array working at 1 kHz, as shown in Fig. 1.2. Using this array, we study the pattern synthesis in the following four different scenarios:

- (a) Synthesis of a maximum-gain pattern with the target region defined as a line segment from $(-0.3 \text{ m}, 5 \text{ m})$ to $(0.3 \text{ m}, 5 \text{ m})$. Fig. 1.3(a) shows the resulting pattern (using the method in [WYG05] or to be described in Section 3.2);
- (b) Synthesis of a minimum-gain pattern with the target region defined as a line segment from $(-0.3 \text{ m}, 5 \text{ m})$ to $(0.3 \text{ m}, 5 \text{ m})$. Fig. 1.3(b) shows the resulting pattern (using the method in [WYG06] or to be described in Section 3.3);
- (c) Synthesis of a single look-direction pattern with the look direction defined at $\theta = 0^\circ$ in the far field. Fig. 1.3(c) shows the resulting pattern (using Dolph-Chebyshev method [Dol46]);
- (d) Synthesis of a shaped pattern (flat-top pattern) with the mainlobe region defined at $[-30^\circ, 30^\circ]$ in the far field. Fig. 1.3(d) shows the resulting pattern (using the Woodward-Lawson method [WL48]).

Fig. 1.2 A linear array example.

(a) Maximum-gain pattern ($y = 5$ m)

(b) Minimum-gain pattern ($y = 5$ m)

(c) Single look-direction pattern

(d) Flat-top pattern

Fig. 1.3 Four examples of desired array patterns synthesized using a 15-element linear array (refer to Fig. 1.2 for coordinate system definition).

In the maximum-gain pattern (Fig. 1.3(a)), the array aims to transmit the maximum power to the line-segment region for a unit input power of the array. In the

minimum-gain pattern (Fig. 1.3(b)), on the contrary, the array is to transmit the minimum power to the line-segment region. In the single look-direction pattern (Fig. 1.3(c)), the array is used to transmit signal at $\theta = 0^\circ$ direction with a minimum beamwidth and a -40 dB sidelobe level. In the shaped pattern (Fig. 1.3(d)), the array is applied to transmit signal to the mainlobe region with unit response. It is interesting to note that if the source array is replaced by a receiving array and the coupling effect is ignored, the four synthesis problems remain the same. The only difference is that “transmitting signal from the array to a target region or look direction(s)” becomes “receiving signal from a target region or look direction(s) to the array”. For example, Fig. 1.3 (a) can be interpreted as a pattern for the array to receive the maximum energy from the target region, with the constraint of a unit-power array weight.

In regard to the above synthesis cases, we are motivated to present a general tool for the modeling and analyzing of different pattern synthesis systems. Two techniques are used in the tool, including multi-input multi-output (MIMO) modeling and eigen-analysis. Firstly, different pattern synthesis system can be uniformly modeled as a multi-input multi-output (MIMO) system. Consider a general array (either receiving or transmitting, uniform or non-uniform arrays), the multi-input (MI) can be the weights at the array elements, and the multi-output (MO) can be the responses at discrete samples in the target region or directions of interest. Specifically, for a source array used to control the acoustic power in a target region, the MI and the MO can be defined as the source strengths of array elements and the sound pressures at the discrete samples of the target region, respectively. For a sensor or antenna array, the MI and the MO can be the array weight and responses in the discrete look directions, respectively. The modeling of a target region or directions of interest by using discrete samples can be seen in many works [WZ93] [Wang03] [KN03]

[WGY05] [WGY05b]. Secondly, eigen-analysis is applied in the general tool. Eigen-analysis is a widely used technique in array processing. For example, in [DM03], the eigenvector technique is used to design eigen-filters for sensor arrays. In [YG00] [YTG02], the condition number, defined as the ratio of the maximum eigenvalue to minimum eigenvalue, is used for robustness measurement.

Combining the MIMO system modeling and the eigen techniques, we can develop the general tool, and we name this tool as *target-oriented array-mode* (TOAM) *analysis*. The *target* here represents a spatial target region or directions of interest. The *array-mode* refers to a series of orthogonal array weight vectors, named TOAM series, which can be obtained using eigen-analysis on a modeled MIMO system. The TOAM series determines some basic patterns of spatial response for an array to transmit or receive signals. Different TOAMs have different gains, where the gain is defined as the ratio of the power of the MO signal to the power of the MI signal. Hence, there are maximum-gain TOAM and minimum-gain TOAM for a given pattern synthesis system.

By applying the general tool, some specific pattern synthesis problems are studied in this thesis, including the acoustical-hotspot generation (as an example of the maximum-gain pattern synthesis), the quiet-zone generation (as an example of the minimum-gain pattern synthesis), the narrowband single look-direction pattern synthesis, and the narrowband flat-top pattern synthesis (as an example of the shaped pattern synthesis). In each case, we aim to present methods with improved or comparable performance compared to conventional methods. A brief introduction on these topics is given as follows.

Recently, with the advancement of digital signal processor and multi-channel signal processing techniques, sound beamforming or acoustical-hotspot generation

using a large source array becomes an increasingly active research topic [1Ltd05] [Yam05] [CK02] [WYG05]. For example, in the 1Ltd multi-media system [1Ltd05], a 256-speaker array is used to generate as many as eight sound beams at the same time. Usually, the classical time-delay method [VB88] [1Ltd05] can be used, which is realized by adjusting the phases of the source signals with different delays corresponding to the differences of the sound traveling time from individual sources to a target region. The advantage of the time-delay method is its simplicity and ease of implementation. However, a better performance in terms of higher gain in the target region can be achieved by using the maximum-gain TOAM as the optimal source strength vector [WYG05]. If time-delay method is regarded as a phase alignment, the new method is a combination of phase alignment and amplitude weighting. The amplitude weighting provides additional degree of flexibility to enhance the performance of acoustical-hotspot generation.

The topic of quiet-zone generation using multiple secondary sources for canceling the signal due to a primary source has been researched for decades. In many works [NCE92] [KM96] [Ell01], an analytical solution based on pseudo-inverse is used. However, this solution is not guaranteed to be effective when the secondary source transfer matrix is ill-conditioned [WYG06]. This problem can be overcome by considering the primary and secondary sources as one array, and applying lower-gain TOAMs to derive the optimal weights of the secondary sources. The advantage of this new method is that it is effective even when the transfer matrix is ill-conditioned.

The single look-direction pattern is probably the most frequently used array pattern. The synthesis of this pattern has been studied for many years. A state-of-the-art solution is the Dolph-Chebyshev method [Dol46], which is used in uniform linear arrays. For non-uniform arrays, there are many other methods, such as the minimax

methods [PM72a, b] [HTL72] [OES85] [DZ01], constrained methods [CT65] [CT66] [LLL66] [HJ67] [Cha71] [OES85] [Jun87], quadratic programming methods [Er92] [NEK93] [SE96] [NZQ01], adaptive array methods [Duf89] [OC90] [ZI98] [ZI99], and least-square methods [SF05]. In recent years, there is a trend of implementing more sophisticated optimization methods, such as the semidefinite programming [Wang03] and second-order cone programming [LB97]. In addition, evolutionary optimization methods, such as genetic programming are also investigated [YL97] [KHR03] [YGQ03].

The flat-top pattern, traditionally, can be synthesized by using the Woodward-Lawson method [WL48]. This method is simple and able to provide analytical solution. However, it is only applicable to uniform linear arrays and is not flexible in the control of sidelobe level (SLL) and mainlobe ripple (MLR). There is an analog between array pattern synthesis using a linear array and FIR filter design if we map the direction in pattern synthesis to the frequency in FIR design [VB88]. FIR filtering methods are possible to be used in the flat-top pattern synthesis. However, it is only applicable to equal-spaced linear array. There are some other synthesis methods that can be used to synthesize flat-top patterns for arbitrary linear arrays. For example, the adaptive array method [ZI99], the minimax method [Wang03] and the least-square method [SF05]. These methods show improved performance relative to the Woodward-Lawson method in terms of lower SLL and MLR.

The synthesis of single look-direction patterns has been commonly solved by using the existing methods [Dol46] [HTL72] [Er92] [TG92] [WZ93] [ZI99]. For example, the Dolph-Chebyshev method is able to guarantee a minimum beamwidth for a given SLL. However, to date there is no optimum solution to the flat-top pattern synthesis using a non-uniform linear array, in which case no method can guarantee a

smallest MLR for a given SLL, or a lowest SLL for a given MLR. There is still room to improve on the performance of flat-top pattern synthesis. Therefore, we are motivated to develop a better solution for this synthesis. Recently, in [Wang03] and [SF05], a new idea of constructing cost functions was presented in solving array pattern synthesis problems. The idea considers only the amplitude response without using the phase response in the cost function. For example, in [Wang03], only amplitude constraints are used in the minimax cost function. In [SF05], the algorithm utilizes the randomness of the phases of a desired pattern, and iteratively modifies the phases so that the error of the amplitude response is minimized. Similarly, we can construct a nonlinear least-square cost function that considers only amplitude response. This problem can be solved using the nonlinear least-square (NLS) method [WGY05]. However, the performance of this solution is dependent on different initializations of a multi-dimensional starting point in the optimization, especially when the array size is large. In this case, the robust nonlinear least-square (RNLS) method is preferred, which is independent of a multi-dimensional starting point. The RNLS method uses the TOAM series expansion from low to high order in a multi-stage optimization, and thus avoids the initialization of a starting point. The NLS and RNLS solutions have been found effective in sidelobe suppression for flat-top pattern synthesis [WGY05] [WGY05a]. In this thesis, more results and analyses are presented for the comparison of the two solutions and the comparisons of these solutions with other methods.

Many numerical methods for arbitrary array pattern synthesis are based on a two-step scheme [TG92] [WZ93] [ZI99] [SF05], where the first step is the calculation of an initial array pattern, and the second step usually uses iterative steps to optimize the synthesized pattern until a certain design requirement is met. Tseng and Griffiths

[TG92] used constraint quadratic programming to initial and optimize the array pattern. Wu and Zielinski [WZ93] presented a method by iteratively constructing and solving linear equations. Zhou and Ingram [ZI99] described the application of adaptive array theory for the synthesis. Wang et al [Wang03] proposed a minimax solution using semidefinite programming. Shi and Feng [SF04] introduced a two-step least-square method. In spite of the different cost functions and different optimization methods used in these works, the two-step scheme is commonly applied. In addition, in the second step, iterations are often performed on selected directions, such as directions associated with the peaks on the synthesized pattern, and the directions of interest. The advantages of using the iterative step are that the MLR and SLL can be further reduced after the initial step, and equal-level sidelobes can be obtained. Due to these advantages, we are motivated to develop a new iterative pattern synthesis method that combines the NLS/RNLS method with the two-step scheme. The new method aims to inherit the effectiveness of the NLS/RNLS method for sidelobe suppression, and at the same time, it to produce equal-level sidelobes.

1.2 Objective

The objective of this thesis lies in the following two aspects in the field of desired array pattern synthesis.

Firstly, we aim to establish a general tool that is applicable to various problems of desired array pattern synthesis, where the array can be either a receiving or transmitting array, and the desired patterns include the maximum-gain, minimum-gain, single look-direction, and shaped patterns. Basically, we will develop this tool by combining the MIMO system modeling and the eigen-analysis. The synthesis

methods for different patterns and arrays will be discussed under this general tool in this thesis.

Secondly, we aim to apply the general tool to some specific array pattern synthesis problems and seek improvement in these areas with reference to existing solutions. Three main applications are considered, including the acoustical-hotspot generation, the quiet-zone generation, narrowband single look-direction pattern synthesis, and narrowband flat-top pattern synthesis. On one hand, the effectiveness of the general tool can be examined in these applications. On the other hand, new algorithms for specific problems can be developed based on the general tool. In the acoustical-hotspot generation, we aim to develop a new method that has higher gain compared to the widely used time-delay method. In the quiet-zone generation, we aim to find a new analytical solution that is more robust than a frequently used analytical solution. In the narrowband array pattern synthesis, we aim to develop new algorithms that are applicable to either uniform or non-uniform linear arrays, and to either single look-direction or flat-top pattern synthesis. The new algorithms are expected to obtain comparable or better performance compared to existing methods.

1.3 Major contributions of the thesis

This thesis focuses on the desired array pattern synthesis. Its major contributions are highlighted as follows.

- **Target-oriented array-mode analysis as a general tool:**

A general tool, named the target-oriented array-mode (TOAM) analysis for modeling and analyzing general array pattern synthesis systems is developed. The TOAM

analysis models an array system as a multi-input multi-output (MIMO) system by regarding the array weight as the multi-input, and responses at discrete samples of a target region or interested directions as the multi-output. Based on this modeling, a gain function is defined, which is in the form of Rayleigh quotient [Str03]. Several eigenvector techniques are used to obtain a series of orthogonal patterns of the array weight, or the target-oriented array-modes (TOAMs). The TOAM series represents the capacity for an array to receive signal from or transmit signal to a target region or directions of interest. The TOAM analysis can be used in different cases of desired array pattern synthesis.

- **Maximum control-gain (MCG) method for acoustical-hotspot generation:**

The acoustical-hotspot generation aims to focus acoustical energy in a spatial target region using a source array. The MCG method is formulated by choosing the maximum-gain TOAM as the optimal array weight. This method is shown to be more effective than a widely used time-delay method [Mai94] [1Ltd05] in terms of higher gain of the array. However, the time-delay method has its merit of lower computational load and simpler implementation. Strategies for selecting the methods have been developed according to the positions of the target region, i.e. in far field or near field, and in the free field or an enclosure. Simulations show that in case of far field in a free field, the time-delay method can be used due to its good performance but lower computational load; whereas in the near field of a free field or in an enclosure, the MCG method should be used because of its better performance. This research work has been published in [WYG03] [WYG05]. This method is proposed for source arrays. However, it is applicable to receiving arrays.

- **Lower-gain TOAM analysis for quiet-zone generation:**

The quiet-zone generation aims to produce a spatial region with low acoustical energy. Two cases of quiet-zone generation are considered. The first case uses a single source array, whereas the second case utilizes secondary sources to cancel the acoustical power in a target region due to a primary source. For the first case, the minimum-gain TOAM is used as the solution. For the second case, the optimal secondary source weights are derived from the lower-gain TOAMs with iterative steps. Compared with the pseudo-inverse solution [Ell01] [NCE92], the solution using lower-gain TOAMs has higher robustness and leads to improved performance even when the transfer-function matrix from the secondary sources to the target region is ill-conditioned. This work is going to be published in [WYG06]. This method is most applicable to source arrays.

- **Nonlinear least-square (NLS) solution to narrowband APS:**

This method uses a cost function of the square amplitude difference between the synthesized pattern and the desired pattern. A complex-to-real transform is used to update the cost function so that complex array weight is applicable. The updated cost function is solved using some nonlinear optimization algorithms, such as the Levenberg-Marquardt algorithm [CBG99]. The NLS method is applicable to arbitrary linear array pattern synthesis with desired response to be either single look-direction pattern or flat-top pattern. The NLS method is especially effective in synthesizing flat-top patterns in terms of better sidelobe suppression, compared with some existing methods, such as the Woodward-Lawson [WL48], minimax [Wang03] and adaptive array [ZI99] methods. This work has been published in [WGY05]. This method is applicable to both receiving and transmitting arrays.

- **Robust NLS (RNLS) algorithm for narrowband APS:**

The RNLS method solves the nonlinear problem (same as the NLS method) with multi-stage optimizations, taking the advantage of the TOAM series expansion. The overall optimization is performed from low to high stages without dependence on a multi-dimensional starting point. By contrast, the optimal array pattern obtained by the NLS method is dependent on different starting points for optimization. Therefore, the RNLS method is more robust than the NLS method. Simulations have shown that the robustness of the RNLS method is more observable for a large array. In the case of a small array, these two methods tend to have similar performance. Since the RNLS is computationally more complex than the NLS method, it is recommended that the RNLS method be used for a large array (e.g. $N > 20$), whereas the NLS method be used for a small array. This work has been published in [WGY05a]. This method is applicable to both receiving and transmitting arrays.

- **Iterative NLS algorithm for narrowband APS:**

This method is developed from the NLS method combined with a peak-iteration strategy [WZ93] [ZI99] [SF05]. In this method, multiple NLS optimizations are iteratively performed on the peaks of synthesized patterns. Since the peaks determine the overall pattern, the iterations of optimization will eventually lead to an optimal pattern approximating to a desired pattern. This iterative method is effective in both single look-direction and flat-top pattern synthesis for arbitrary linear arrays. For single look-direction pattern synthesis, this method has the same performance with the Dolph-Chebyshev [Dol46], adaptive array [ZI99], and two-step least-square [SF05] methods. For flat-top pattern synthesis, this method shows a better performance than the minimax [Wang03], adaptive array, and two-step least-square methods in terms of

lower sidelobe level with similar mainlobe shaping. In addition, the proposed method is able to synthesize equal-level sidelobes. This method is applicable to both receiving and transmitting arrays.

1.4 Organization of the thesis

This thesis consists of six chapters. The organization of these chapters is as follows.

- The first chapter provides the motivation, objectives and major contributions of this thesis.
- Chapter 2 presents the TOAM analysis technique. Firstly, it introduces fundamental knowledge on the desired array pattern synthesis. Secondly, this chapter shows how a general array pattern synthesis system can be modeled as a MIMO system. Thirdly, the derivation of TOAM series for a MIMO system is presented. The properties of TOAM series are theoretically analyzed and demonstrated by simulation results. Finally, possible applications of the TOAM technique are briefly explored.
- Chapter 3 shows the applications of the maximum-gain TOAM in acoustical-hotspot generation, and lower-gain TOAMs in quiet-zone generation. The MCG method is compared with the time-delay method. Theoretical relations of the two methods are analyzed. The performances are compared by simulations in two typical acoustical fields, namely, the free field and a rectangular enclosure. Different strategies of choosing the methods are proposed according to the scenarios of far field or near field, enclosure or free field. In the quiet-zone generation, a new approach for deriving the secondary source array weight is presented. Rather than using the pseudo-inverse of the transfer matrix in a

classical solution, the new approach employs the lower-gain TOAMs to determine the optimal array weight. These two methods are discussed and compared by simulations.

- Chapter 4 focuses on the synthesis of narrowband single look-direction and flat-top patterns. The NLS and RNLS methods are introduced and compared. The NLS method is developed based on a nonlinear cost function and solved by a nonlinear optimization. The RNLS method uses the same cost function as the NLS method, but solves the problem by a multi-stage optimization using the TOAM series expansion. The advantages of these two methods over some conventional methods are shown by simulations in different synthesis scenarios: uniform/non-uniform linear arrays, single look-direction pattern, flat-top pattern, and multi-beam pattern.
- Chapter 5 proposes a new iterative array pattern synthesis method that combines the NLS method with a peak-iteration strategy. This new method is compared with some conventional iterative methods by simulations in different synthesis scenarios.
- Finally, Chapter 6 concludes the thesis and recommends future research.

Figure 1.4 shows how these chapters are related and linked. As illustrated in this figure, the main body of this thesis has two parts: general tool and applications. Chapter 2 introduces the general tool, namely the TOAM analysis. The application part consists of Chapters 3 and 4, presenting three specific applications of the TOAM analysis. In addition, the NLS solution presented in Chapter 4, can be extended into

an iterative NLS solution and applied to array pattern synthesis as discussed in Chapter 5.

Fig. 1.4 Links of thesis chapters.

Chapter 2

Target-oriented array-mode (TOAM) analysis for desired array pattern synthesis

This chapter presents the target-oriented array-mode (TOAM) analysis, which is used as a general tool for modeling and analyzing different array systems. The TOAM analysis consists of two parts: modeling an array system as a multi-input multi-output (MIMO) system, and analyzing the MIMO system using eigenvector technique.

The organization of this chapter is as follows. Section 2.1 provides the fundamental knowledge on the desired array pattern synthesis, which forms the background to the TOAM analysis to be presented in later sections. Section 2.2 introduces the MIMO system modeling, which shows how a general array system can be modeled as a MIMO system. Generally, for a given array and a spatial target region or directions of interest, the multi-input (MI) and multi-output (MO) can be defined as the weights on array elements and the responses at the discrete samples of the target region or directions of interest, respectively. Based on the MIMO system, a series of target-oriented array-modes (TOAMs) can be derived using eigen-analysis, as presented in Section 2.3. Some interesting properties of TOAM series are theoretically analyzed in Section 2.4, and numerically demonstrated in several simulation examples in Section 2.5. These properties enable the TOAM analysis to be applied in the synthesis of desired array patterns. Several possible applications are briefly explored in Section 2.6.

2.1 Fundamentals on desired array pattern synthesis

This section presents the fundamental knowledge on desired array pattern synthesis. Some basic descriptions on wave field, array categories and the definition of desired array pattern synthesis are briefly presented.

2.1.1 Wave field and transfer function

Wave field is the medium for signal transmission. A transfer function numerically describes the signal transmission between two points in a wave field. In the following sub-sections, the wave equations, far/near-field definitions and transfer functions are briefly reviewed.

2.1.1.1 Wave field

A space-time signal is written as $s(\mathbf{r}, t)$, where $\mathbf{r}$ is the 3-dimensional position of observation at time t . The position, $\mathbf{r}$ can be represented in a coordinate system, as shown in Fig. 2.1. Cartesian and spherical coordinate systems can be denoted as (x, y, z) and (r, θ, ϕ) , respectively. θ and ϕ are defined as the polar angle and horizontal angle, respectively. Figure 2.1 shows the relationship between the two coordinate systems.

Fig. 2.1 Relationship between Cartesian and spherical coordinate systems.

Consider a sound wave field, and assume the medium of the wave propagation is homogeneous, dispersion-free and lossless. Homogeneity assures a constant propagation speed throughout space and time. Dispersion happens in a nonlinear medium, where the wave interacts with the medium, and thus changes its amplitude related to its frequency contents. A lossless medium implies that the medium does not influence the amplitude attenuation of the propagating wave. Under these assumptions, a linear wave equation governing the wave propagation can be given as [Cro98] [Kut00] [Rai00]

$$\nabla^2 s(\mathbf{r}, t) = \frac{1}{c^2} \frac{\partial^2 s(\mathbf{r}, t)}{\partial t^2}, \quad (2.1)$$

where ∇^2 is the Laplacian operator, and c is the sound velocity (344 m/sec).

One solution to the wave equation of Eq. (2.1) is the monochromatic plane wave which can be generally described by the complex exponential

$$s(\mathbf{r}, t) = A \exp\{j(\omega_0 t - \mathbf{k}_0^T \mathbf{r})\}, \quad (2.2)$$

where $(.)^T$ denotes matrix or vector transpose, A is the amplitude, ω_0 is the frequency of the plane wave, and $\mathbf{k}_0$ is the wavenumber vector defined by

$$\mathbf{k}_0 = \frac{\omega_0}{c} \mathbf{a}(\theta, \phi) = \frac{2\pi}{\lambda_0} \mathbf{a}(\theta, \phi), \quad (2.3)$$

where λ_0 is the wavelength and $\mathbf{a}(\theta, \phi)$ is a unit vector. Assuming the propagation direction is defined from the spatial origin, $\mathbf{a}(\theta, \phi)$ is given as

$$\mathbf{a}(\theta, \phi) = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}. \quad (2.4)$$

The direction of $\mathbf{k}_0$ represents the propagation direction of the monochromatic plane wave, and the magnitude of $\mathbf{k}_0$ denotes the number of cycles in radians per meter in the propagation direction. Thus, $\mathbf{k}_0$ can be interpreted as the spatial frequency variable according to the temporal frequency ω_0 . The scalar product $\mathbf{k}_0^T \mathbf{r}$ in Eq. (2.2) is the propagation delay with the origin of the coordinate system as reference. Equation (2.2) also refers to a plane wave. For any time instant t , those points with equal amplitude of $s(\mathbf{r}, t)$ are lying on planes defined as $\mathbf{k}_0^T \mathbf{r} = b$, where b is a constant. The term monochromatic means that $s(\mathbf{r}, t)$ consists of one single harmonic with temporal frequency ω_0 .

A second solution of the linear wave equation is given by the monochromatic spherical wave, which describes the wave field of a point source located at the origin of the coordinate system. The solution is given as

$$s(r, t) = \frac{A}{r} \exp \{ j(\omega_0 t - |\mathbf{k}_0| r) \}, \quad (2.5)$$

where r is the distance of observation from the sources. In contrast to a plane wave (in Eq.(2.2)), the amplitude of the monochromatic spherical wave decreases hyperbolically with the distance r . The points with the same amplitude are lying on a sphere which is concentric to the spatial origin.

Due to the linearity of the wave equation, the principle of superposition holds for both the plane wave and spherical wave. More complicated wave fields [Cro98] [Rai00], such as the wave propagation of multiple sources, wideband sources or spatially continuous sources, can be expressed by Fourier integrals with respect to the temporal frequency ω_0 . Spatially continuous sources require spatial integrals as well.

2.1.1.2 Definitions of near field and far field

In the previous section, two solutions to the linear wave equation were given. In general, the radiation of point sources is modeled by spherical wave if the positions of observation are close to the source. In this case, the wavefront of the propagation wave is perceptibly curved with respect to the distance between the observation positions. The direction of propagation is dependent on the observation position. This kind of wave field is classified as a near field.

For a large distance, the wave field of a point source can be modeled by plane wave. Considering that the wavefront resembles a plane wave for decreasing curvature of the wavefront, the direction of propagation is approximately equal at all observation positions. This kind of wave field is classified as a far field.

The transition from the near field to the far field of a point source is dependent on the distance d between two adjacent observation positions. Generally, far field and near field are defined in accordance with

$$r > \frac{2d^2}{\lambda_0} \quad \text{for far field,} \quad (2.6)$$

and

$$r \leq \frac{2d^2}{\lambda_0} \quad \text{for near field,}$$

where r is the distance between the source and observation positions [Goo68] [KAW98].

2.1.1.3 Near-field transfer function

In the near field, the wavefront is spherical, and the Green's function can be used as the transfer function between a point source and an observation point. Supposing that the point source is located at point, $\mathbf{r}$ with a source strength or volume velocity of q , the complex sound pressure p at an observation point, $\mathbf{r}_o$ can be given as [HS96]

$$p = G(\mathbf{r}, \mathbf{r}_o)q, \quad (2.7)$$

where $G(\mathbf{r}, \mathbf{r}_o)$ is the Green's function, which is stated as

$$G(\mathbf{r}, \mathbf{r}_o) = \frac{-j\omega\rho_0}{4\pi |\mathbf{r} - \mathbf{r}_o|} e^{-j\frac{\omega}{c}|\mathbf{r} - \mathbf{r}_o|}, \quad (2.8)$$

where ρ_0 is the air density.

As discussed in Section 2.1.1.1, the principle of superposition holds because of the linearity of the wave field. Therefore, for an array of N sources located at

$\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$ with source strength vector of $\mathbf{q} = [q_1, q_2, \dots, q_N]^T$, the complex sound pressure at an observation point at $\mathbf{r}_o$ can be calculated as

$$p = \mathbf{G}\mathbf{q}, \quad (2.9)$$

where $\mathbf{G} = [G(\mathbf{r}_1, \mathbf{r}_o), G(\mathbf{r}_2, \mathbf{r}_o), \dots, G(\mathbf{r}_N, \mathbf{r}_o)]$ is the transfer-function vector.

In general, as shown in Fig. 2.2, the sound pressure at an observation point, $\mathbf{r}_o$ in the near field is determined by the spherical wavefront in the case of a single point source (Fig. 2.2 (a)), or by the superposition of the spherical wavefronts in the case of an array of point sources (Fig. 2.2 (b)).

Fig. 2.2 Near-field sound pressure observation for (a) a single point source, (b) a point-source array.

2.1.1.4 Far-field transfer function

In a far field, the wavefront can be modeled by a plane wave. The observed pressure is dependent on the arriving angle and difference of the sound traveling distance relative to a reference point, as illustrated in Fig. 2.3. Assume M observation points are arranged linearly with locations of $r_1, r_2, \dots, r_M$. The arrival angle of the plane wave is θ with reference to the vertical axis, and the first observation point (usually $r_1 = 0$) is chosen as the reference point. For this arrangement, the difference of sound traveling distance at the m -th observation point with reference to r_1 can be calculated as

$$d_m = (r_m - r_1) \sin \theta. \quad (2.10)$$

If the pressure at the reference point is p_1 , the pressure at the m -th observation point can be computed as

$$p_m = p_1 e^{-j\omega d_m / c}, \quad (2.11)$$

with c the wave propagation speed, and ω the frequency.

Fig. 2.3 Far field modeled by plane wave propagation.

In many applications, the signals received at observation points or a receiving array are weighted and summed to obtain a single signal, as shown in Fig. 2.4.

Suppose the weight vector of the receiving array is $\mathbf{w} = [w_1, w_2, \dots, w_M]^T$. The output signal can be expressed by

$$p = \sum_{m=1}^M w_m p_m = \sum_{m=1}^M p_1 w_m e^{-j\omega d_m / c} . \quad (2.12)$$

If we define $p_1 = 1$, Eq. (2.12) can be rewritten to

$$p = \mathbf{s}^H \mathbf{w} , \quad (2.13)$$

where the steering vector,

$$\mathbf{s} = [1, e^{j\omega d_2 / c}, \dots, e^{j\omega d_M / c}]^T \quad (2.14)$$

can be regarded as a transfer function vector, if we regard $\mathbf{w}$ and p as the input and output, respectively.

Fig. 2.4 Signal receiving using weighted sum.

2.1.2 Array categories

Some typical categories of array systems are shown in the following sections.

2.1.2.1 Receiving / transmitting array

According to the application of an array, it can be categorized as a receiving array or a transmitting array (source array). A typical receiving array can be a microphone array for speech capturing, or an antenna array for receiving radio frequency communication signal. A transmitting array can be a loudspeaker array for sound delivering.

Although a receiving array and a transmitting array are physically different, the basic array processing theory may be compatible due to the reciprocity of receiving and transmitting [MA00]. In Section 2.2, it will be shown that both arrays can be modeled by a multi-input multi-output (MIMO) system, and similar processing can be performed for the array pattern synthesis.

2.1.2.2 Array arrangement

According to the placement of array elements, an array can be roughly categorized as a one-dimension array (a linear array), a 2-dimension array (e.g. a circular array or a planar array) or a 3-dimension array. Some array configurations are presented in the following as examples.

For an N -element uniform linear array used in narrowband applications, the positions (x and y coordinates) of the elements can be given as

$$x_n = \beta \left\lfloor n - \frac{N+1}{2} \right\rfloor \frac{\lambda}{2}, \quad y_n = 0, \quad n = 1, 2, \dots, N, \quad (2.15)$$

where the $\lfloor \cdot \rfloor$ operator rounds the number within, β is a coefficient, and λ is the wave length. When $\beta = 1$, the element spacing is half-wavelength. Equation (2.15) is applicable to either even or odd element arrays.

A circular array consists of a number of N elements displayed in a concentric ring on the azimuth plane, as shown in Fig. 2.5. Since the elements are spaced evenly by an angle of $2\pi/N$ in the azimuth plane, the element positions can be expressed as

$$x_n = r \sin \left[(n-1) \cdot \frac{2\pi}{N} + \phi_0 \right], \quad y_n = r \cos \left[(n-1) \cdot \frac{2\pi}{N} + \phi_0 \right], \quad n = 1, 2, \dots, N, \quad (2.16)$$

where r is the radius of the array and ϕ_0 is the angle of the first element with respect to the x -axis. The radius is sometimes chosen according to the wavelength. For example, when

$$r = \frac{\lambda}{4 \sin \frac{\pi}{N}}, \quad (2.17)$$

the distance between neighboring elements is half wavelength.

Fig. 2.5 Example of $N = 8$ circular array.

A typical planar array, for instance, can be a rectangular array with equal number of elements in each row and column, respectively. Assume the array has $N = N_x \times N_y$ elements and the element spacing is $\beta\lambda/2$, the element positions can be given as

$$x_{n_x} = \beta \left[n_x - \frac{N_x + 1}{2} \right] \frac{\lambda}{2}, \quad y_{n_y} = \beta \left[n_y - \frac{N_y + 1}{2} \right] \frac{\lambda}{2} \quad (2.18)$$

with $n_x = 1, 2, \dots, N_x$ and $n_y = 1, 2, \dots, N_y$.

The above equations from (2.15) to (2.18) are used to describe uniform arrays. Non-uniform arrays are also widely used [TG92] [BW01]. In this thesis, both uniform and non-uniform arrays are considered.

2.1.3 Desired array pattern category

In this thesis, we study the synthesis of the following desired array patterns for a general array:

- Single look-direction pattern;
- Shaped pattern (e.g. flat-top pattern);
- Maximum-gain pattern;
- Minimum-gain pattern.

The general array is assumed to be an arbitrary array (either equal or unequal spacing) with point-like omnidirectional elements without considering coupling effect among elements.

2.1.3.1 Single look-direction pattern

Array pattern synthesis (APS) [Dol46] [Ell85] [TG92] [Tre02] aims to determine an optimal array weight for a given array configuration such that the synthesized pattern approximates to a predefined pattern. The predefined pattern can be a single or multiple look-direction pattern or a shaped pattern. A single look-direction pattern [Dol46] has a minimum-beamwidth mainlobe with a given sidelobe level or a minimum-sidelobe-level beam with a certain beamwidth. An example of the single look-direction pattern is shown in Fig. 2.6. This pattern is synthesized using a 15-element half-wavelength spacing uniform linear array. The desired sidelobe level is -40 dB. The beamwidth (half-power beamwidth for example), according to the theory of Dolph-Chebyshev method [Dol46], is the smallest for given settings of the array and sidelobe level. Correspondingly, if the same array is used to synthesize the same mainlobe as in the pattern shown in Fig. 2.6, the minimum sidelobe level obtainable is also -40 dB.

Fig. 2.6 Single look-direction for a 15-element uniform linear array with half wavelength spacing. The desired sidelobe level (SLL) is -40 dB.

2.1.3.2 Shaped pattern

A shaped pattern refers to an amplitude distribution with pre-specified levels at different angles. One of the typical shaped patterns is the flat-top pattern with uniform response in the mainlobe region. Figure 2.7 demonstrates an example of flat-top pattern synthesized using the Woodward-Lawson method [WL48] with Hamming-window filtering [OS89]. The array is a 30-element half-wavelength-spacing linear array. The designed beamwidth is 60 degree.

Fig. 2.7 A far-field flat-top pattern synthesized by Woodward-Lawson method with Hamming-window filtering (BW represents beamwidth).

The above two examples (Fig. 2.6 and Fig. 2.7) are given for uniform linear arrays. In many applications, non-uniform arrays are used to synthesize single look-direction patterns or flat-top patterns. Accordingly, the synthesis methods are different from those of the uniform arrays. This topic will be discussed in depth in Chapters 4 and 5.

2.1.3.3 Maximum-gain and minimum-gain patterns

Besides above synthesis problems, we also consider two other array pattern synthesis problems, namely, the syntheses of maximum-gain and minimum-gain patterns [WYG03][WYG05] [WYG06]. The gain here is defined as the ratio of the power measured in a target region or directions of interest to the input power of the array weight. For a given input power, the maximum-gain pattern can be a spatial response with maximum energy in a target region or directions of interest, whereas, the minimum-gain pattern can be a response having minimum energy. For example, consider a source array and a spatial target region R_T , as shown in Fig. 2.8. the maximum-gain and minimum-gain patterns generated by the system are shown in Fig. 2.9 (a) and (b), respectively. In Fig. 2.9 (a), the target region R_T has higher sound pressure level (SPL) than the neighboring region, whereas in Fig. 2.9 (b), the target region shows lower SPL than the neighboring region.

It is noted that unlike the single look-direction pattern and shaped pattern design methods, the maximum-gain and minimum-gain pattern syntheses put no constraint on the neighboring region around the target region. The synthesis of the maximum-gain and minimum-gain patterns will be discussed in more detail in Chapter 3.

Fig. 2.8 An example of multi-source array and target region.

(a)

(b)

Fig. 2.9 Examples of maximum-gain pattern (a) and minimum-gain pattern (b). The target region R_T is highlighted by a black circle in (a) and a white circle in (b).

2.1.4 Criteria for desired array pattern synthesis

In general, the criteria for desired array pattern synthesis can be simply shown as

$$\mathbf{w}_{opt} = \arg \min_{\mathbf{w}} c(\mathbf{w}), \quad (2.19)$$

where $\mathbf{w}$ is the complex array weight vector, and $c(\mathbf{w})$ represents the cost function.

$c(\mathbf{w})$ can be different according to specific synthesis problems.

- (a) For a maximum-gain pattern synthesis, the cost function, $c(\mathbf{w})$ can be given as $-g(\mathbf{w})$. The criteria can be presented as

$$\mathbf{w}_{opt} = \arg \min_{\mathbf{w}} (-g(\mathbf{w})), \quad (2.20)$$

where $g(\mathbf{w})$ is the gain function defined as the ratio of the power, E_{out} measured in a target region R_T or directions of interest Θ , to the power of the array, $E_{in} = \mathbf{w}^H \mathbf{w}$.

That is,

$$g(\mathbf{w}) = \frac{E_{out}}{E_{in}}. \quad (2.21)$$

E_{out} can be calculated as

$$E_{out} = \int_{\mathbf{r} \in R_T} p^*(\mathbf{r}) p(\mathbf{r}) d\mathbf{r}, \quad (2.22)$$

or

$$E_{out} = \int_{\theta \in \Theta} p^*(\theta) p(\theta) d\theta, \quad (2.23)$$

depending on the integration is over a spatial region or range of directions.

(b) For a minimum-gain pattern synthesis, the cost function, $c(\mathbf{w}) = g(\mathbf{w})$. The criteria can be shown as

$$\mathbf{w}_{opt} = \arg \min_{\mathbf{w}} g(\mathbf{w}). \quad (2.24)$$

(c) For the synthesis of a single look-direction pattern or a shaped pattern, the cost function $c(\mathbf{w})$ can be given as the square-error function $J_{se}(\mathbf{w})$. Thus, a possible criteria for the synthesis can be presented as

$$\mathbf{w}_{opt} = \arg \min_{\mathbf{w}} J_{se}(\mathbf{w}), \quad (2.25)$$

and $J_{se}(\mathbf{w})$ can be calculated as

$$J_{se}(\mathbf{w}) = \int_{\theta \in \Theta} |p(\theta) - p_d(\theta)|^2 d\theta, \quad (2.26)$$

where $|\cdot|$ denotes absolute value. $p(\theta)$ and $p_d(\theta)$ are the synthesized pattern and desired pattern, respectively. If only amplitude response is considered, the cost function can be given as

$$J_{se}(\mathbf{w}) = \int_{\theta \in \Theta} \|p(\theta) - p_d(\theta)\|^2 d\theta. \quad (2.27)$$

It is also possible to use minimax cost function, and the corresponding criteria can be stated as

$$\mathbf{w}_{opt} = \arg \min_{\mathbf{w}} \max_{\theta \in \Theta} |p(\theta) - p_d(\theta)|. \quad (2.28)$$

It is worth noting that the above criteria are applicable for a general array, regardless of equal or unequal spaced array. In later Chapters 3, 4 and 5, synthesis methods for a general array will be proposed and discussed in depth.

2.2 Multi-input multi-output (MIMO) system modeling

Various pattern synthesis systems have been briefly reviewed in the preceding section. This section will show that different array systems can all be modeled as a MIMO system.

2.2.1 Modeling of a transmitting array

A transmitting array is used to send signals to a spatial target region or range of directions, as shown in Fig. 2.10. Assume the array has N sources, and the target region or direction range is densely sampled by M samples. The source strengths on the N array elements can be defined as multi-input (MI), represented by $\mathbf{q} = [q_1, q_2, \dots, q_N]^T$, and the responses on the M discrete samples can be multi-output (MO), represented by $\mathbf{p} = [p_1, p_2, \dots, p_M]^T$. The MO can be calculated as the multiplication of the MI with a transfer-function matrix $\mathbf{G}$, that is, $\mathbf{p} = \mathbf{G}\mathbf{q}$. Thus, a MIMO system is obtained, as shown in Fig. 2.10.

For sampling a target region, the number of samples, M must satisfy the Nyquist sampling interval, so that the interval between adjacent samples is smaller than the Nyquist sampling interval. In practice, a dense sampling is preferred, which provides higher resolution for representing a target region. Importantly, the dense sampling will reduce the sensitivity of the MIMO system to the number of samples. In this thesis, dense sampling is always applied so that the pattern synthesis is not influenced much by the sampling.

Fig. 2.10 MIMO system modeling for a transmitting array with N elements.

The target region is densely sampled by M discrete samples. $\mathbf{G}$ is the transfer-function matrix between the multi-input (MI) and multi-output (MO).

2.2.2 Modeling of a receiving array

Similar to a transmitting array, a receiving array system can also be modeled as a MIMO system. Consider an N -element receiving array in the case of far field with a range of directions of interest, Θ . Suppose Θ is densely sampled by a number of M discrete directions, $\theta_1, \theta_2, \dots, \theta_M$, and the corresponding M steering vectors are represented by $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_M$. The MI can be defined as the array weight $\mathbf{w} = [w_1, w_2, \dots, w_N]^T$, and the MO can be defined as the responses in the M directions, represented as $\mathbf{p} = [p_1, p_2, \dots, p_M]^T$. The MO can be calculated as the multiplication of the MI with the steering matrix $\mathbf{S}$, namely $\mathbf{p} = \mathbf{S}^H \mathbf{w}$, where $\mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_M]$. Thus, the MIMO system for a receiving array is obtained, as shown in Fig. 2.11.

Fig. 2.11 MIMO system modeling for a receiving array with N elements.

The directions of interest are densely sampled by M discrete directions.

2.2.3 Modeling of a general array

The preceding Sections 2.2.1 and 2.2.2 have shown that both transmitting array and receiving array can be modeled as a MIMO system. Therefore, for a general array, as shown in Fig. 2.12, we can model the array weight as the MI and the signal at discrete samples of a spatial target region or directions of interest as the MO. The number of samples must be large so that the target region or directions of interest are well represented by the samples. The transfer function between the MI and MO is decided by the wave propagation property in the media.

Fig. 2.12 MIMO system modeling for a general array.

The modeling shown in Fig. 2.12 is applicable to both transmitting and receiving arrays. The only difference is that the transfer function matrix, $\mathbf{G}$ is “from the array elements to spatial samples” for a transmitting array, but “from the spatial samples to the array elements” for a receiving array. The dimension of matrix $\mathbf{G}$ is $M \times N$. A continuous target region can be densely sampled by a number of M points or directions so that $M > N$, and $\text{rank}(\mathbf{G}) = N$ can be satisfied. Without specially notation, we assume $M > N$ for a continuous target region in this thesis. For a discrete target region formed by M points or directions, $\text{rank}(\mathbf{G})$ may be smaller than N . Specially, when $M = 1$, $\text{rank}(\mathbf{G}) = 1$.

Matrix $\mathbf{G}$ can be constructed using the Green’s function (Eq.(2.8)) in the case of near field, which is given as

$$\mathbf{G} = \begin{bmatrix} G(\mathbf{a}_1, \mathbf{r}_1) & G(\mathbf{a}_2, \mathbf{r}_1) & \cdots & G(\mathbf{a}_N, \mathbf{r}_1) \\ G(\mathbf{a}_1, \mathbf{r}_2) & G(\mathbf{a}_2, \mathbf{r}_2) & \cdots & G(\mathbf{a}_N, \mathbf{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ G(\mathbf{a}_1, \mathbf{r}_M) & G(\mathbf{a}_2, \mathbf{r}_M) & \cdots & G(\mathbf{a}_N, \mathbf{r}_M) \end{bmatrix}, \quad (2.29)$$

where $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N$ represent the spatial locations of the N array elements. $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_M$ represent the discrete samples of a spatial region of interest. In the case of far field,

matrix $\mathbf{G}$ can be constructed by using the steering vectors (Eq. (2.14)) on M discrete directions $\theta_1, \theta_2, \dots, \theta_M$, and $\mathbf{G}$ is given as

$$\mathbf{G} = [\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_M]^H, \quad (2.30)$$

where $\mathbf{s}_m$ represents the steering vector for direction θ_m .

Using the general MIMO modeling, the output power of a system can be calculated as

$$E_{out} = \mathbf{p}^H \mathbf{p}, \quad (2.31)$$

which can be regarded as the discretized version of Eq. (2.22) or (2.23). Compared with Eqs. (2.22) and (2.23), Equation (2.31) uses no integration, and hence, the computational load is reduced by using Eq. (2.31). Furthermore, the cost functions in Eq. (2.26) and Eq. (2.27) can be modified as

$$J_{se}(\mathbf{w}) = \sum_{\theta=\theta_1}^{\theta_M} |p(\theta) - p_d(\theta)|^2, \quad (2.32)$$

and

$$J_{se}(\mathbf{w}) = \sum_{\theta=\theta_1}^{\theta_M} \|p(\theta) - p_d(\theta)\|^2, \quad (2.33)$$

respectively. By applying $\mathbf{p} = \mathbf{G}\mathbf{q}$, Equations (2.32) and (2.33) can be rewritten as

$$J_{se}(\mathbf{q}) = \|\mathbf{G}\mathbf{q} - \mathbf{p}_d\|^2, \quad (2.34)$$

and

$$J_{se}(\mathbf{q}) = \|\mathbf{G}\mathbf{q} - \mathbf{p}_d\|^2, \quad (2.35)$$

respectively, where $\|\cdot\|$ denotes norm, and $\mathbf{p}_d = [p_d(\theta_1), p_d(\theta_2), \dots, p_d(\theta_M)]^T$.

2.3 TOAM series definition and derivation

In the preceding section, the modeling of a general array into a MIMO system was introduced. In this section, a target-oriented array-mode (TOAM) analysis is presented based on the MIMO system modeling. The TOAM analysis studies the desired array pattern synthesis using a series of basic array modes of an array, namely the TOAM series.

The method for deriving TOAMs can be regarded as an extension of radiation mode analysis, which has been used in active structural acoustic control [EJ93]. Radiation modes are a series of independent radiation velocity distributions on the surface of an object, such as a beam, a plate, a polyvinylidene fluoride (PVDF) film or a wall, which are calculated as the eigenvectors of an elemental radiation resistance matrix [Ell01]. In both TOAM analysis and the radiation mode analysis, eigenvalue technique is used. However, the radiation mode analysis focuses on local control of the structural radiation, whereas the TOAM technique studies the array pattern (spatial response) in a spatial target region or directions of interest.

Since the derivation of TOAMs requires eigenvector technique, the fundamental knowledge on the eigenvector technique is briefly reviewed in Section 2.3.1 before the derivation of the TOAM series in Section 2.3.2.

2.3.1 Fundamental on eigenvector technique

The techniques of eigenvalue decomposition (EVD) [ND77] and the Rayleigh quotient [Str03] are important tools for analyzing array systems. These two techniques are briefly reviewed as follows.

2.3.1.1 Eigenvalue decomposition (EVD)

Let $\mathbf{A}$ be a square matrix. The non-zero vector $\mathbf{v}$ is an eigenvector of $\mathbf{A}$, if it satisfies

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \quad (2.36)$$

where scalar λ is the eigenvalue associated with vector $\mathbf{v}$. An eigenvector with unit inner product is usually used, that is $\mathbf{v}^H \mathbf{v} = 1$.

2.3.1.2 Rayleigh quotient and properties

The Rayleigh quotient [Str03] is defined as the ratio of two quadratic forms, which can be written as

$$r = \frac{\mathbf{v}^H \mathbf{A} \mathbf{v}}{\mathbf{v}^H \mathbf{B} \mathbf{v}}, \quad (2.37)$$

where $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices, and $\mathbf{B}$ must be positive definite. The simplest and most prevalent case occurs when $\mathbf{B}$ is an identity matrix. In this case, Eq. (2.37) is reduced to

$$r = \frac{\mathbf{v}^H \mathbf{A} \mathbf{v}}{\mathbf{v}^H \mathbf{v}}. \quad (2.38)$$

The Rayleigh quotient can be expressed in terms of the eigenvalues and eigenvectors of $\mathbf{A}$ as

$$r = \frac{\sum_{i=1}^N \lambda_i |\mathbf{v}^H \mathbf{q}_i|^2}{\mathbf{v}^H \mathbf{v}}, \quad (2.39)$$

where $\mathbf{q}_i$ is the i -th eigenvector of matrix $\mathbf{A}$.

Three frequently used properties of Rayleigh quotient are presented as follows:

- a) The maximum quotient corresponds to the maximum eigenvalue, $\lambda_{\max}$ of matrix $\mathbf{A}$, and thus one of the solutions leading to the maximum quotient is the eigenvector $\mathbf{q}_{\max}$, that is

$$\begin{cases} r_{\max} = \lambda_{\max} \\ \mathbf{v}_{\max} = \mathbf{q}_{\max} \end{cases} . \quad (2.40)$$

- b) The minimum quotient corresponds to the minimum eigenvalue, $\lambda_{\min}$ of matrix $\mathbf{A}$, and thus, one of the solutions leading to the minimum quotient is the eigenvector, $\mathbf{q}_{\min}$, that is

$$\begin{cases} r_{\min} = \lambda_{\min} \\ \mathbf{v}_{\min} = \mathbf{q}_{\min} \end{cases} . \quad (2.41)$$

- c) For any vector $\mathbf{v}$, the resulting quotient is within the range of $[\lambda_{\min}, \lambda_{\max}]$, that is

$$\lambda_{\min} \leq r \leq \lambda_{\max} . \quad (2.42)$$

2.3.2 TOAM derivation

Consider a general array system with N elements and a target region or range of directions of interest sampled by M discrete samples. The system can be modeled as a MIMO system according to Fig. 2.12 in Section 2.2.3. Assume the array weight and the spatial response at the discrete samples are represented as $\mathbf{q} = [q_1, q_2, \dots, q_N]^T$ and $\mathbf{p} = [p_1, p_2, \dots, p_M]^T$, respectively. Since $\mathbf{p} = \mathbf{G}\mathbf{q}$ with matrix $\mathbf{G}$ being the transfer-function matrix, Eq. (2.31) can be written as

$$E_{out} = \mathbf{p}^H \mathbf{p} = (\mathbf{G}\mathbf{q})^H \mathbf{G}\mathbf{q} = \mathbf{q}^H \mathbf{R}\mathbf{q} , \quad (2.43)$$

where $\mathbf{R} = \mathbf{G}^H \mathbf{G}$. Since the dimension of $\mathbf{G}$ is $M \times N$, the dimension of $\mathbf{R}$ is $N \times N$.

Applying Eq.(2.43) to Eq. (2.21), the gain function can be rewritten as

$$g(\mathbf{q}) = \frac{E_{out}}{E_{in}} = \frac{\mathbf{q}^H \mathbf{R} \mathbf{q}}{\mathbf{q}^H \mathbf{q}}. \quad (2.44)$$

Equation (2.44) is the gain function for a MIMO system.

The potential of a MIMO system for synthesizing an array pattern in a target region or directions of interest can be possibly revealed by the extremes of the gain function. To find the extremes, Eq. (2.44) can be rewritten to

$$g(\mathbf{q}) \mathbf{q}^H \mathbf{q} = \mathbf{q}^H \mathbf{R} \mathbf{q}. \quad (2.45)$$

Taking the derivative with respect to $\mathbf{q}$ leads to

$$\frac{\partial g}{\partial \mathbf{q}} \mathbf{q}^H \mathbf{q} + 2g\mathbf{q} = 2\mathbf{R}\mathbf{q}. \quad (2.46)$$

In the case of extremes, $\partial g / \partial \mathbf{q} = 0$. It gives

$$g\mathbf{q} = \mathbf{R}\mathbf{q}, \quad (2.47)$$

which implies that the eigenvalues of the matrix $\mathbf{R}$ are the extremes of function $g(\mathbf{q})$, and the eigenvectors are the array weight vectors correspondent to the extremes. Since $\mathbf{R}$ is an $N \times N$ Hermitian matrix, a total of N eigenvalues, $g_1 \geq g_2 \geq \dots \geq g_N$ and N orthonormal eigenvectors, $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$ can be obtained. These eigenvectors can be regarded as a series of basic modes of the array. Since these modes are subjected to different arrangements of a target region or directions of interest, they are named target-oriented array-modes (TOAMs).

2.4 TOAM series properties

The definition of TOAM has been introduced in the preceding section. In this section, the properties of TOAM series are discussed, which include:

- (I) Different TOAMs have different gains.
- (II) Condition number is determined by the maximum-gain and minimum-gain TOAMs.
- (III) An array weight vector can be expressed by the TOAM series.
- (IV) The output power E_{out} is independently determined by different TOAMs.
- (V) In solving inverse problems (defined later) using the TOAM series expansion with variable lengths, the longer the length is used, the lower the gain will be obtained. (We name it the decreasing property of gain function).
- (VI) In solving inverse problems, the longer the TOAM series length is used, the smaller the resulting square error will be. (We name it a decreasing property of square-error function).

These properties are discussed individually in Sections 2.4.1 to 2.4.6.

2.4.1 Property I: Different TOAMs have different gains

It follows from Eq. (2.47) that,

$$g(\mathbf{q}_i) = g_i, \quad (2.48)$$

where $g_1 \geq g_2 \geq \dots \geq g_N$ are the eigenvalues of matrix $\mathbf{R}$, and $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$ are the associated TOAMs. It is noted that the gain function in Eq. (2.44) is in the form of a Rayleigh quotient defined in Eq. (2.38). Therefore, according to the property of Rayleigh quotient given in Eq.(2.42), no higher gain than g_1 , and no lower gain than g_N can be obtained. Hence, $\mathbf{q}_1$ can be named the maximum-gain TOAM, and $\mathbf{q}_N$ as the minimum-gain TOAM. The gains of the rest of TOAMs are within the range of $[g_N, g_1]$. It is possible that some adjacent TOAMs have the same gain. In this case, these TOAMs are associated with the same eigenvalue.

2.4.2 Property II: Condition number is determined by the maximum-gain and minimum-gain TOAMs

The condition number is frequently used in the robustness measurement of a MIMO system when the minimum eigenvalue of the correlation matrix $\mathbf{R}$ is not equal to 0. The condition number is calculated as [YTG02] [YG00]

$$K = \lambda_{\max} / \lambda_{\min} , \quad (2.49)$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of the correlation matrix $\mathbf{R}$ of the system, respectively. According to the derivation of the TOAM, it holds that $\lambda_{\max} = g_1$, and $\lambda_{\min} = g_N$. It implies that the condition number of a system can be alternatively regarded as being determined by the maximum-gain and minimum-gain TOAMs.

2.4.3 Property III: An array weight vector can be expressed by the TOAM series

According to the derivation of TOAM series, $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$ are N orthonormal eigenvectors, which implies that this series can be taken as a complete orthonormal basis. Therefore, an arbitrary array weight vector, $\mathbf{q}$ can be expressed by a linear superposition of the TOAM series as

$$\mathbf{q} = v_1 \mathbf{q}_1 + v_2 \mathbf{q}_2 + \dots + v_N \mathbf{q}_N = \mathbf{Q} \mathbf{v}, \quad (2.50)$$

where $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N]$, and $\mathbf{v} = [v_1, v_2, \dots, v_N]^T$ is the weight vector of the TOAM series. The weight vector, $\mathbf{v}$ for expressing an array weight, $\mathbf{q}$ can be given as

$$\mathbf{v} = \mathbf{Q}^{-1} \mathbf{q}. \quad (2.51)$$

2.4.4 Property IV: E_{out} is determined by different TOAMs independently

It is noted that the Hermitian matrix, $\mathbf{R}$ can be diagonalized by matrix, $\mathbf{Q}$ as

$$\mathbf{R} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^H, \quad (2.52)$$

where the diagonal matrix

$$\mathbf{\Lambda} = \text{diag}[g_1, g_2, \dots, g_N]. \quad (2.53)$$

Applying Eqs. (2.50) and (2.52) to $E_{out} = \mathbf{q}^H \mathbf{R} \mathbf{q}$ (Eq.(2.43)) leads to

$$E_{out} = (\mathbf{Q} \mathbf{v})^H (\mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^H) \mathbf{Q} \mathbf{v} = \mathbf{v}^H \mathbf{\Lambda} \mathbf{v} = \sum_{i=1}^N |v_i|^2 g_i. \quad (2.54)$$

Equation (2.54) implies that the output power, E_{out} of the multi-output is independently determined by different TOAMs.

The above analysis is based on eigenvalue decomposition (EVD). Another widely used eigenvalue technique is singular value decomposition (SVD) [GV89] [LH74] [Par80] [Ste73] [Wil65]. SVD applied to an arbitrary matrix that is not necessarily to be symmetry. In the discussion of the properties of the TOAMs, $\mathbf{R} = \mathbf{G}^H \mathbf{G}$ is a Hermitian matrix, and therefore EVD is used.

2.4.5 Property V: Decreasing-gain property

2.4.5.1 TOAM series expansion with variable length

If Eq.(2.50) is regarded as a series expansion, we can consider constructing an array weight vector with incomplete TOAM series with length of L as

$$\mathbf{q}_{syn}(L) = \sum_{i=1}^L v_i \mathbf{q}_i = \mathbf{Q}_L \mathbf{v}_L \quad (L \leq N), \quad (2.55)$$

where $\mathbf{Q}_L = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_L]$ and $\mathbf{v}_L = [v_1, v_2, \dots, v_L]^T$. In Eq.(2.55), only a number of L higher-gain TOAMs, $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_L$ are retained in the synthesis of $\mathbf{q}_{syn}(L)$ without the rest $N - L$ lower-gain TOAMs.

The TOAM series expansion can be applied in solving inverse problems as described in the following section.

2.4.5.2 Inverse problem

For a given MIMO system, the inverse problem in general can be defined as the derivation of an optimal MI, $\mathbf{q}_{opt}$ for generating a predefined MO, $\mathbf{p}_d$. In a least-square-error sense, the problem is to minimize

$$J(\mathbf{q}) = \mathbf{e}^H(\mathbf{q})\mathbf{e}(\mathbf{q}), \quad (2.56)$$

where

$$\mathbf{e}(\mathbf{q}) = \mathbf{G}\mathbf{q} - \mathbf{p}_d \quad (2.57)$$

is error function. Equation (2.56) is equivalent to Eq.(2.34). Here Eq. (2.56) is used for the convenience of description. The solution to this problem can be given by inverting matrix $\mathbf{G}$ ($M \times N$) as

$$\mathbf{q}_{opt} = \mathbf{G}^{-1}\mathbf{p}_d, \quad (2.58)$$

when $\mathbf{G}$ is invertible; otherwise

$$\mathbf{q}_{opt} = \mathbf{G}^+\mathbf{p}_d, \quad (2.59)$$

where $\mathbf{G}^+$ denotes the pseudoinverse taken as $\mathbf{G}^+ = \mathbf{G}^H [\mathbf{G}\mathbf{G}^H]^{-1}$, if $\text{rank}(\mathbf{G}) = M$ or $\mathbf{G}^+ = [\mathbf{G}^H \mathbf{G}]^{-1} \mathbf{G}^H$, if $\text{rank}(\mathbf{G}) = N$. If $\mathbf{G}$ is square ($M = N$) and nonsingular, a unique exact solution is available, and the pseudoinverse reduces to $\mathbf{G}^{-1}$.

Theoretically, $\mathbf{q}_{opt}$ is the optimal solution to the inverse problem, but practically this solution may cause a low gain problem, meaning that a very large input power may be required for the approximation of the desired response $\mathbf{p}_d$. This problem will later be shown in simulation. The problem can be possibly solved by

using high-gain TOAMs, while prohibiting low-gain TOAMs to synthesize an optimal weight. For this purpose, the incomplete TOAM series expansion introduced in Eq. (2.55) can be applied to solve Eq.(2.56).

Applying Eq. (2.55) to Eq. (2.44) gives the gain function with respect to L as

$$\begin{aligned} g_{syn}(L) &= g[\mathbf{q}_{syn}(L)] \\ &= \frac{[\mathbf{q}_{syn}(L)]^H \mathbf{R} \mathbf{q}_{syn}(L)}{[\mathbf{q}_{syn}(L)]^H \mathbf{q}_{syn}(L)} \\ &= \frac{\mathbf{v}_L^H \mathbf{\Lambda}_L \mathbf{v}_L}{\mathbf{v}_L^H \mathbf{v}_L}, \end{aligned} \quad (2.60)$$

where $\mathbf{\Lambda}_L = diag[g_1, g_2, \dots, g_L]$, and

$$\mathbf{v}_L = \mathbf{Q}_L^{-1} \mathbf{G}^{-1} \mathbf{p}_d, \quad (2.61)$$

when $\mathbf{G}$ is invertible, or

$$\mathbf{v}_L = \mathbf{Q}_L^{-1} \mathbf{G}^+ \mathbf{p}_d, \quad (2.62)$$

when $\mathbf{G}$ is non-invertible.

In addition, applying Eq. (2.55) to Eq. (2.56) leads to

$$J_{syn}(L) = \mathbf{e}_{syn}^H(L) \mathbf{e}_{syn}(L), \quad (2.63)$$

where $\mathbf{e}_{syn}(L)$ is the error function given as

$$\mathbf{e}_{syn}(L) = \mathbf{G} \mathbf{Q}_L \mathbf{v}_L - \mathbf{p}_d. \quad (2.64)$$

The later Sections 2.4.5.3 and 2.4.6 will theoretically prove that both the gain function $g_{syn}(L)$ and the square-error function $J_{syn}(L)$ decrease with the increase of length L .

Since both the gain function $g_{syn}(L)$ and the square-error function $J_{syn}(L)$ are decreasing functions of L , a large square error may result if a small L is chosen, in order to maintain a relatively high gain when minimizing Eq. (2.63). On the other hand, if a large L is selected in order to suppress the square error, it may lead to a small gain. Consequently, a trade-off is inevitable for choosing a proper length, L . In practice, this trade-off can be balanced by introducing an error tolerance, J^* . Thus, an optimal length, L^* matching the error tolerance can be determined by

$$L^* = \min L \text{ subject to } \{J_{syn}(L) \leq J^*\}, \quad (2.65)$$

which is the smallest TOAM series length that keeps the square error smaller than J^* .

By applying L^* to Eq.(2.55), the optimal array weight can be obtained as

$$\mathbf{q}^* = \mathbf{q}_{syn}(L^*). \quad (2.66)$$

Equation (2.66) implies that the solution meeting the tolerance, J^* is a superposition of the first L^* TOAMs. The overall procedures for determining an optimal solution $\mathbf{q}^*$ can be summarized and illustrated by the flowchart presented in Fig. 2.13.

Fig. 2.13 Procedures for determining an optimal array weight.

It is noted that when $L^* = N$, $\mathbf{H}_L$ is an identity matrix. Thus, according to Eq.(2.55), it gives

$$\mathbf{q}_{syn}(N) = \mathbf{Q}\mathbf{v}. \quad (2.67)$$

Applying Eqs. (2.61) or (2.62) to Eq. (2.67) leads to $\mathbf{q}_{syn}(N) = \mathbf{G}^{-1}\mathbf{p}_d$ or $\mathbf{q}_{syn}(N) = \mathbf{G}^+\mathbf{p}_d$, respectively. Therefore,

$$\mathbf{q}_{syn}(N) = \mathbf{q}_{opt}, \quad (2.68)$$

which implies that the TOAM solution, $\mathbf{q}_{syn}(N)$ using the entire TOAM series ($L = N$) is identical to the solution, $\mathbf{q}_{opt}$ using the inversion of the transfer function.

2.4.5.3 Function $g_{syn}(L)$ decreases with the increasing of length L

The gain function, $g_{syn}(L)$ in Eq.(2.60) can be further written as:

$$g_{syn}(L) = \frac{\mathbf{v}_L^H \mathbf{\Lambda}_L \mathbf{v}_L}{\mathbf{v}_L^H \mathbf{v}_L} = \frac{\sum_{i=1}^L |v_i|^2 g_i}{\sum_{i=1}^L |v_i|^2}. \quad (2.69)$$

When $L = 1$, Eq. (2.69) becomes

$$g_{syn}(1) = g_1. \quad (2.70)$$

When $L = 2$,

$$\begin{aligned} & g_{syn}(2) - g_{syn}(1) \\ &= \frac{|v_1|^2 g_1 + |v_2|^2 g_2}{|v_1|^2 + |v_2|^2} - g_1 \\ &= \frac{(g_2 - g_1)|v_2|^2}{|v_1|^2 + |v_2|^2} \\ &\leq 0, \end{aligned} \quad (2.71)$$

which implies $g_{syn}(2) \leq g_{syn}(1)$, because $g_2 \leq g_1$.

In general, when $L = k$,

$$\begin{aligned} & g_{syn}(k) - g_{syn}(k-1) \\ &= \frac{|v_1|^2 g_1 + \dots + |v_{k-1}|^2 g_{k-1} + |v_k|^2 g_k}{|v_1|^2 + \dots + |v_{k-1}|^2 + |v_k|^2} - \frac{|v_1|^2 g_1 + \dots + |v_{k-1}|^2 g_{k-1}}{|v_1|^2 + \dots + |v_{k-1}|^2} \\ &= \frac{|v_1|^2 |v_k|^2 (g_k - g_1) + |v_2|^2 |v_k|^2 (g_k - g_2) + \dots + |v_{k-1}|^2 |v_k|^2 (g_k - g_{k-1})}{(|v_1|^2 + \dots + |v_{k-1}|^2 + |v_k|^2)(|v_1|^2 + \dots + |v_{k-1}|^2)} \\ &\leq 0, \end{aligned} \quad (2.72)$$

which means $g_{syn}(k) \leq g_{syn}(k-1)$, because $g_k \leq g_{k-1} \dots \leq g_2 \leq g_1$.

Therefore, it is proven that $g_{syn}(N) \leq g_{syn}(N-1) \leq \dots \leq g_{syn}(2) \leq g_{syn}(1)$. In other words, $g_{opt}(L)$ decreases with respect to the length L .

2.4.6 Property VI: Decreasing-error property

This section shows that the square-error function $J_{syn}(L)$ in Eq. (2.63) decreases with the increasing of the length L . Two cases are considered: (a) $\mathbf{G}$ is invertible, and (b) $\mathbf{G}$ is non-invertible.

(a) When $\mathbf{G}$ is invertible:

By applying Eq. (2.51) to Eq.(2.64), the error becomes

$$\begin{aligned}\mathbf{e}_{syn}(L) &= \mathbf{G}\mathbf{Q}_L\mathbf{v}_L - \mathbf{p}_d \\ &= \mathbf{G}\mathbf{Q}_L\mathbf{Q}_L^H\mathbf{G}^{-1}\mathbf{p}_d - \mathbf{p}_d \\ &= \mathbf{G}(\mathbf{Q}_L\mathbf{Q}_L^H - \mathbf{I})\mathbf{G}^{-1}\mathbf{p}_d.\end{aligned}\tag{2.73}$$

Applying Eq. (2.73) to Eq.(2.63), the square-error function can be written as

$$\begin{aligned}J_{syn}(L) &= \mathbf{e}_{syn}^H(L)\mathbf{e}_{syn}(L) \\ &= \mathbf{p}_d^H\mathbf{G}^{-H}(\mathbf{Q}_L\mathbf{Q}_L^H - \mathbf{I})\mathbf{G}^H\mathbf{G}(\mathbf{Q}_L\mathbf{Q}_L^H - \mathbf{I})\mathbf{G}^{-1}\mathbf{p}_d \\ &= \mathbf{p}_d^H\mathbf{G}^{-H}(\mathbf{Q}_L\mathbf{Q}_L^H - \mathbf{I})\mathbf{Q}\Lambda\mathbf{Q}^H(\mathbf{Q}_L\mathbf{Q}_L^H - \mathbf{I})\mathbf{G}^{-1}\mathbf{p}_d \\ &= \mathbf{p}_d^H\mathbf{G}^{-H}\mathbf{Q}[\mathbf{Q}^H(\mathbf{Q}_L\mathbf{Q}_L^H - \mathbf{I})\mathbf{Q}]\Lambda[\mathbf{Q}^H(\mathbf{Q}_L\mathbf{Q}_L^H - \mathbf{I})\mathbf{Q}]\mathbf{Q}^H\mathbf{G}^{-1}\mathbf{p}_d \quad (2.74) \\ &= \mathbf{b}^H\bar{\Lambda}_L\mathbf{b} \\ &= \sum_{i=L+1}^N |b_i|^2 g_i,\end{aligned}$$

where $\bar{\Lambda}_L = \text{diag}[0, \dots, 0, g_{L+1}, g_N]$, $\mathbf{b} = \mathbf{Q}^H\mathbf{G}^{-1}\mathbf{p}_d$ and b_i is the i -th element of $\mathbf{b}$.

Since $g_i > 0$, $J_{syn}(L)$ is a reducing function with respect to L .

(b) When $\mathbf{G}$ is non-invertible:

In this case, by replacing $\mathbf{G}^{-1}$ with $\mathbf{G}^+$, a similar expression of $J_{syn}(L)$ as

Eq.(2.74) can be obtained.

In general, it can be concluded that $J_{syn}(L)$ is a reducing function with respect to the length L .

2.5 Simulations for showing the TOAM series properties

The properties of TOAM series have been theoretically studied in preceding sections. In this section, these properties are demonstrated in numerical simulations using point-source arrays as examples, with the assumption of no coupling effect between array elements.

- Simulation I shows the array patterns in a target region when each TOAM is individually used as array weight. The distributions of the gain of each TOAM are illustrated. The maximum-gain and minimum-gain patterns are derived.
- Simulation II shows the properties of TOAM series in solving the inverse problem. The decreasing properties of the gain and the square-error functions are tested.

2.5.1 Simulation I

In order to show the effects of different TOAMs on the sound pressure distribution in a target region, let us consider an example system shown in Fig. 2.14. Without loss of the generality, the source array and target region are chosen to be in different shapes, and the two planes are not coaxial. Numerical settings are as follows. A number of $N = 16$ sources are arranged in a square array of $1.2 \times 1.2 \text{ m}^2$ with equal interval of 0.4 m . The circular target region is located at $O(0.5 \text{ m}, 0.25 \text{ m}, 5.0 \text{ m})$ and parallel to the

array in a distance of 5.0 m. The radius is 0.8 m. The target region is evenly sampled by 32 regularly spaced measuring points in the Cartesian coordinate system defined in Fig. 2.14. The simulation is conducted at 1 kHz in free field. According to Fig. 2.12, this array system can be modeled as a MIMO system, and a series of 16 TOAMs can be obtained using the derivation in Section 2.3.2. Figure 2.15 shows the simulated array patterns in the target region when each TOAM is applied as array weight. The sound pressures are calculated by

$$\mathbf{p}_i = \mathbf{G}\mathbf{q}_i, \quad i = 1, 2, \dots, 16, \quad (2.75)$$

where i is the index of the patterns in Fig. 2.15. $\mathbf{p}_1$ and $\mathbf{p}_{16}$ give the maximum-gain and minimum-gain patterns, respectively. The distributions of the eigenvalues, and the average sound pressure levels (SPLs) associated with different TOAMs are illustrated in Fig. 2.16 (a) and (b), respectively.

Fig. 2.14 An example array system with a square source array and a circular target region.

Fig. 2.15 Simulated array patterns (sound pressure level (SPL) in dB) in the target region.

Fig. 2.16 Simulation I results: (a) Gain distribution $(g_1, g_2, \dots, g_{16})$ in dB $(10\log_{10}(g_i))$; and (b) Average SPL in the target region by $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{16}$ in dB $(\bar{p} = 20\sum \log_{10} |\mathbf{G}_i \mathbf{q}| / M)$.

The following observations can be drawn from the simulation results:

- (a) As shown in Fig. 2.15, the array patterns associated with different TOAMs are different. It is interesting to note that different array patterns show different numbers of SPL peaks and valleys. The lower the index of the pattern is, the fewer SPL peaks and valleys are shown in the pattern. Specifically, the first pattern has a single peak in the centre without any valley, whereas the last pattern demonstrates a complex pattern filled with SPL peaks and valleys. A similar observation can be found in the radiation mode analysis by Elliott and Johnson [EJ93]. Their work showed that a thin beam or a panel vibrated according to a series of basic radiation patterns with different variations.
- (b) As shown in Fig. 2.16 (a), in general, different TOAMs have different gains. The gain $g(\mathbf{q}_i) = g_i$ (Eq.(2.48)) decreases with the increase of the indices of the TOAMs, that is, $g_1 \geq g_2 \geq \dots \geq g_N$.
- (c) As demonstrated in Fig. 2.16 (b), the resulting average SPL in the target region decreases when $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$ are individually used as array weights. This distribution is similar with the gain distribution as shown in Fig. 2.16 (a).
- (d) Among the 16 array patterns shown in Fig. 2.15, the maximum-gain pattern, $\mathbf{p}_1$ results in the highest average SPL of about 60 dB in the target region, whereas the minimum-gain pattern, $\mathbf{p}_N$ shows the lowest average SPL at about -60 dB. These results imply that $\mathbf{q}_1$ can be used for generating an acoustical-hotspot in a target region, whereas $\mathbf{q}_N$ can be applied to produce a quiet zone. Later in Chapter 3, the generations of acoustical hotspot and quiet zone will be discussed in depth.
- (e) Some adjacent TOAMs may share a same gain. As shown in Fig. 2.16, the gains

for the pairs of TOAMs as $(\mathbf{q}_2 \ \mathbf{q}_3)$, $(\mathbf{q}_7 \ \mathbf{q}_8)$, $(\mathbf{q}_9 \ \mathbf{q}_{10})$ and $(\mathbf{q}_{14} \ \mathbf{q}_{15})$ have the same value; and the associated average SPLs are also approximately equal. It is interesting to note that the array patterns for such pairs of modes are quite similar but orthogonal to each other, as shown in Fig. 2.15.

2.5.2 Simulation II

This simulation tests the decreasing properties of the gain function and the square-error function (refer to Properties V and VI in Section 2.4). Two simulation cases are considered with the configurations shown in Fig. 2.17. The square target regions 1 and 2 are used for Case 1 and 2, respectively, with following parameters:

- Case 1: target region 1, $L_a = 1.5$ m and $f = 1500$ Hz.
- Case 2: target region 2, $L_a = 2.5$ m and $f = 750$ Hz.

Fig. 2.17 Simulation arrangement. N -source square array with side length L_a parallel to the square target regions 1 and 2 with the same side length L_t , located at $O_1(0, 0, D_1)$ and $O_2(0, h_2, D_2)$, respectively. $N = 36$. $M = 100$. $L_t = 1.5$ m. $D_1 = 5$ m. $D_2 = 3$ m. Simulations conducted in free field with sound speed $c = 344$ m/s.

In each simulation case, the array is used to generate equal response in the target region, that is $\mathbf{p}_d = [1, 1, \dots, 1]^T_{M \times 1}$. By using the TOAM analysis, a series of $N = 16$ TOAMs can be obtained. The synthesized array weight $\mathbf{q}_{syn}(L)$ for $L = 1, 2, \dots, 36$ can be calculated by using Eq.(2.55). In order to demonstrate the effect of $\mathbf{q}_{syn}(L)$ at different values of L , the target region is sampled at every 0.01m, or $M' = 151 \times 151$ sample points. Four numerical results are calculated including:

- The gain $g_{syn}(L)$ (defined in Eq.(2.60));
- The average square error $\bar{J}_{syn}(L) = J_{syn}(L) / M'$;
- The average SPL in the target region $\bar{P}(L)$;
- The SPL variation in the target region $v(L)$.

The resulting quantities, $g_{syn}(L)$, $\bar{J}_{syn}(L)$, $\bar{P}(L)$ and $v(L)$ are shown in Fig. 2.18 (a), (b), (c) and (d), respectively.

From the distributions of the four quantities illustrated in Fig. 2.18 (a), (b), (c) and (d), following conclusions can be drawn. Firstly, the gain $g_{syn}(L)$ and average square error $\bar{J}_{syn}(L)$ are both decreasing functions with respect to the TOAM series length L . This observation is consistent with the theoretical analysis presented in Sections 2.4.5.3 and 2.4.6. As shown in Fig. 2.18, with the increasing of L from 1 to 36, the gain drops by around 17 dB from 97.4 to 80.9 dB, and the average square error decreases from 0.43 to around 0. Correspondingly, the average SPL falls from 28.7 to 14.1 dB, and the SPL variance changes from 8 to 0.03 dB.

(a)

(b)

(c)

(d)

Fig. 2.18 Case 1 simulation results: Distributions of

- (a) Gain function $g_{syn}(L)$ in dB;
- (b) Average square-error function $\bar{J}_{syn}(L)$;
- (c) Average SPL (dB) $\bar{P}(L)$;
- (d) SPL variance (dB) $v(L)$ in the target region, with respect to L (the variance is much higher than 8 dB for $L \leq 5$ and are not plotted).

Secondly, the trade-off between reducing the square error and maintaining the gain is exhibited, that is, the suppression of the square error results in a gain lost. However, it is interesting to note that the square error seems to drop faster than the gain, which implies that, at some values of L , an error tolerance J^* with a relative high gain can be met. For example, at $L = 10$, the gain only drops about 3 dB from 97.4 ($L = 1$) to 94.5 dB, whereas the average square error falls from 0.433 to 0.042. This interesting observation can also be demonstrated by the average SPL and SPL variance in the target region. As shown in Fig. 2.18 (c) and (d), by comparing the case of $L = 1$ and $L = 10$, it can be noticed that the SPL variance decreases from above 8 dB to only 1.1 dB, while the average SPL is nearly maintained.

Thirdly, regarding the inverse problem, the TOAM series solution shows an advantage of higher gain over the inverse solution (Eq.(2.58) or (2.59)) for a given error tolerance J^* . If we set an error tolerance of $J^* = 0.04$. The optimal length $L^* = 9$ can be obtained for Case 1 by performing the procedures shown in Fig. 2.13. Since the inverse solution equals to $\mathbf{q}_{syn}(N)$ as stated in Eq.(2.68), the comparison can be performed between $\mathbf{q}_{syn}(9)$ and $\mathbf{q}_{syn}(36)$. In the case of $L = 9$, the average SPL is 28.2 dB, 14 dB higher than that at $L = 36$; and the variance is only 1.1 dB, which implies that a uniform SPL distribution and a relative high SPL can be obtained simultaneously in the target region. The effectiveness of $\mathbf{q}_{syn}(9)$ over $\mathbf{q}_{syn}(36)$ is further illustrated by the sound field patterns developed over a $4.0 \times 4.0 \text{ m}^2$ region containing the target region in its centre, as shown in Fig. 2.19. In Fig. 2.19 (b), since the average SPL in the target region is about 15 dB lower than the neighboring region, the target region can be regarded as a quiet zone. In contrast, it is an acoustically hot region in Fig. 2.19 (a), as it has a higher SPL than the neighboring region. A hotter

target region is desirable for sound delivery, because it means that the sound energy is more focused in the target region than its neighboring region.

Fig. 2.19 Case 1 simulation results: SPL distributions using (a) $\mathbf{q}_{syn}(9)$ and (b) $\mathbf{q}_{syn}(36)$. (The square target region is highlighted by dashed lines.)

Similar conclusions can be drawn from Case 2 at a different frequency and target region located at O_2 , as shown in Fig. 2.17. The distributions of the gain, the average square error, the average SPL and the SPL variance with respect to the length, L are shown in Fig. 2.20 (a), (b), (c) and (d), respectively. From Fig. 2.20 (a) and (b), it can be observed that both the gain and the square error decrease with the increase of L . By setting a tolerance of $J^*=0.1$, an optimal length $L^*=12$ is selected by performing the procedures shown in Fig. 2.13. The resulting array patterns, correspondent to $L=12$ and 36 are presented in Fig. 2.21 (a) and (b), respectively. As shown in Fig. 2.20 (c), about a 18 dB SPL difference (24.8 against 6.4 dB) in the target region can be achieved by choosing $\mathbf{q}_{syn}(12)$ over $\mathbf{q}_{syn}(36)$. The average SPL

does not constantly decrease when $L = 1, \dots, 5$, whereas the gain does. This is because a region with a highest gain may not correspond to a highest average SPL in the region. The TOAM property, as proven in Section 2.4.5 guarantees a decreasing gain with the increase of L . Furthermore, the target region is an acoustical hotspot (SPL higher than neighboring region) for $\mathbf{q}_{syn}(12)$ as shown in Fig. 2.21 (a), whereas it is a quiet zone for $\mathbf{q}_{syn}(36)$ as shown in Fig. 2.21 (b).

In general, the observations from the simulation results are consistent with the theoretical analysis on the properties of the TOAM series expansion. The TOAM solution can be used to improve the gain for solving inverse problems. In Chapter 4, the application of the TOAM series expansion is extended to the solution of a nonlinear optimization problem in the synthesis of single look-direction and flat-top array patterns.

Fig. 2.20 Case 2 simulation results: Distributions of

- (a) Gain function $g_{syn}(L)$ in dB;
- (b) Average square-error function $\bar{J}_{syn}(L)$ (the square error is much higher than 0.25 for $L \leq 5$ and are not plotted);
- (c) Average SPL (dB) $\bar{P}(L)$;
- (d) SPL variance (dB) $v(L)$ in the target region, with respect to L (the variance is much higher than 15 dB for $L \leq 5$ and are not plotted).

Fig. 2.21 Case 2 simulation results: SPL distribution using (a) $\mathbf{q}_{syn}(12)$ and (b) $\mathbf{q}_{syn}(36)$. (The target region is denoted by dashed lines.)

2.6 Application exploration

Based on the TOAM analysis, three applications can be possibly developed, as briefly introduced in this section. These applications will be discussed in depth in Chapters 3 to 5.

2.6.1 Application of maximum-gain TOAM

According to the analysis in Section 2.4 and simulations in Section 2.5, the maximum-gain array pattern for a given array is determined by the maximum-gain TOAM, $\mathbf{q}_1$. Therefore, this mode can be used as the optimal weight of a source array for the generation of an acoustical hotspot in a target region. This application will be presented in detail in Chapter 3.

2.6.2 Application of lower-gain TOAM

The TOAM analysis and simulation also shows that lower-gain TOAMs are associated with lower gain or response in a target region or directions of interest. Therefore, these modes can be potentially used to produce a quiet zone in a target region. An interesting application is to use an array as secondary-source to cancel the signal in a target region due to a primary source. A new method based on this idea is developed, which will be introduced in Chapter 3.

2.6.3 Application of TOAM series expansion

It has been shown that the TOAM series expansion has some interesting properties in solving inverse problem. For example, the square-error function, $J_{syn}(L)$ is decreasing function of L . In Chapter 4, the TOAM series expansion is applied in the solving of a nonlinear optimization problem for synthesizing single look-direction and flat-top array patterns.

2.7 Conclusions

This chapter has introduced a target-oriented array-mode (TOAM) technique for analyzing desired array-pattern-synthesis systems. By using this technique, a general array can be modeled as a MIMO system. The multi-input (MI) and multi-output (MO) are defined as the weights on array elements and the responses at the discrete samples of a target region or directions of interest, respectively. For a given MIMO system, a series of TOAMs can be derived using eigen-analysis. The TOAM series show some interesting properties, which are theoretically discussed and numerically demonstrated in several simulations. Three possible applications of the TOAM technique have been briefly explored, which will be individually studied in the succeeding chapters.

Chapter 3

Application of TOAM analysis in maximum-gain and minimum-gain pattern synthesis

In Chapter 2, the target-orient array-mode (TOAM) analysis was introduced, and three main applications were previewed. This chapter presents two applications using the TOAM analysis. The first application is the acoustical-hotspot generation, which is an example of the maximum-gain pattern synthesis. The other application is the quiet-zone generation, which is an example of the minimum-gain pattern synthesis.

3.1 Overview

As presented in Section 2.2 of Chapter 2, a source-array system used for controlling the sound pressure distribution in a target region can be modeled as a multi-input multi-output (MIMO) system, and the maximum-gain and minimum-gain TOAMs of this MIMO system can be obtained using eigen-analysis. These two TOAMs determine the potential of the array for controlling the target region to be acoustically hot or quiet. Accordingly, two applications can be developed as presented in this chapter.

Section 3.2 focuses on the acoustical-hotspot generation using the maximum-gain TOAM. Since this mode corresponds to the maximum gain for controlling a target region, we name it the maximum-control-gain (MCG) method [WYG05]. The

MCG method is compared with a widely used time-delay (TD) method [Mai94] [1Ltd05] in two typical acoustical fields, namely, the free field and a rectangular enclosure. For each of the fields, the far-field and near-field cases are individually considered. Based on theoretical analysis and simulation results, strategies for the acoustical-hotspot generation are developed according to different sound fields where the target region is defined. In addition, the wideband implementations of the TD and MCG solutions are introduced and compared.

Section 3.3 focuses on the quiet-zone generation using lower-gain TOAMs. Two cases of the quiet-zone generation are discussed. In the first case, a source array is used to generate a quiet zone in a target region [CK02b]. In the second case, a source array is utilized as secondary sources to cancel the signal in a target region produced by a primary source, so that a quiet zone can be generated in the target region [Ell01] [NCE92] [WYG06]. The solutions for each case are developed based on the feature of the TOAM analysis. Simulations are conducted to show the performance of the proposed method.

Rather than discuss the engineering design of the array for maximum/minimum-gain pattern synthesis, this chapter focuses more on theoretical analysis and solutions. The potential of a general array for controlling a spatial target region is mainly studied. The aim is to provide valuable analysis results and better theoretical solutions than some existing methods. The general array is assumed to be formed by point-like omnidirectional elements without considering coupling effect among elements.

3.2 Acoustical-hotspot generation

3.2.1 Introduction

Contrary to the noise control for generating a quiet zone [NE92][TRE00], and different from sound equalization for producing a zone of equalization [San01], the acoustical-hotspot generation aims to yield a spatial region with a higher acoustical energy than its neighboring region by using multiple sources. It is useful in many applications. For example, in a big recreation center, we may want to generate a desired listening area for music delivery; or we may wish to transmit personal message to a certain area in a public hall.

Two methods are discussed in this chapter. The first method is the classical time-delay (TD) method [Mai94] [1Ltd05] [VB88], which is realized by emitting the same signal with different time delays. The delay for each source is determined according to the time difference of the sound from the source to a target point compared with the sound from a reference source. This method is also known as the phased-array method [Mai94]. The main advantage of the time-delay method is its simplicity for computation and application. The time delays for all array elements can be calculated according to the spatial arrangement of the array and target region. By attaching different delays for the corresponding sources, the delayed signals enhance each other in the target region, and thus, generate an acoustical hotspot. This method has been applied in some commercial products, such as 1-Limit sound beam generation system [1Ltd05], in which more than 200 sources are attached with different controllable time delays in order to deliver the sound to a certain target region.

The other method is the maximum-control-gain (MCG) method [WYG05], which uses the maximum-gain TOAM as the optimal source strength vector. A similar method was presented in [CK02] [WYG03], but these works only discussed the free-field case, and did not perform the comparison with the time-delay method. In [WYG05], more analyses were presented and the discussion was extended to an enclosed environment. This section is mainly based on the work in [WYG05]. In addition, the wideband implementation in free field is introduced.

The MCG method is more computational complex compared to the TD method. However, the MCG method has a better performance in terms of higher gain. It can be theoretically proven that when the two methods are individually used to generate an acoustical hotspot with the target region defined as a point in free field, the solution derived from the MCG method is actually the multiplication of the TD solution with an amplitude weighting. The amplitude weighting plays an additional role in improving the gain. This explains why the MCG method is always able to obtain a higher gain than the TD method. A detailed discussion will be given in Section 3.2.4.

The selection of the two methods is largely dependent on the acoustical field where the sources and the target region are located. In free field, the sound field is determined by the direct sound from the source array. The further away the target region is from the source array, the closer is the performance of the two methods. Therefore, for far-field case, the TD method can be chosen due to its simplicity but close performance to the MCG method. However, for near-field case, the MCG method is preferred because of its better performance.

In an enclosed environment, such as a rectangular room, the sound field becomes much more complicated than the free field due to the existence of

reverberant sound and direct sound at the same time. The sound pressure at a measuring point is not solely dependent on the distances to the sources, but also on the spatial position in the enclosure [MI86] [Mou85]. For a given array and target region, the sound field can be considered as near field or far field, determined according to the critical distance [Kut00]. In the near field, the direct sound dominates, whereas in the far field, the reverberant sound prevails. As a result, the TD method is still applicable in near field, but no longer effective in the far field. In contrast, the MCG method can be used in both far and near field. Therefore, the MCG method should be applied for far-field application in an enclosed environment.

The rest of this section is arranged as follows. The formulations of the TD and MCG methods are presented in Section 3.2.2 and Section 3.2.3, respectively. The theoretical link between the two methods is analyzed in Section 3.2.4. The performances of the two methods are compared for the free field and enclosed environment in Section 3.2.5 and Section 3.2.6, respectively. The wideband solution to the acoustical-hotspot generation is discussed in Section 3.2.7.

3.2.2 The time-delay (TD) method

In the case of a point target region, the TD method requires the input strength of each source to be delayed in such a way that the sound pressure attributed to each of the sources at the target point has the same phase, and consequently, the overall pressure is enhanced. Suppose there are a total of N sources, and the first source is selected as the reference. The time delays applied to the corresponding sources are $\tau_1, \tau_2, \tau_3, \dots, \tau_N$, where $\tau_1 = 0$ (reference) and τ_i can be calculated as

$$\tau_i = \frac{r_i - r_1}{c}, \quad (3.1)$$

with c the sound speed of 344 m/sec, r_i the distance from the i -th source to the target point, as illustrated in Fig. 3.1.

Fig. 3.1 Target region is a point.

The complex strength of the i -th source can be shown as

$$q_i = q_0 e^{j\omega\tau_i}, \quad (3.2)$$

where q_0 is the amplitude of the source strength. Thus, the unit-power ($\mathbf{q}_{TD}^H \mathbf{q}_{TD} = 1$) source strength vector can be given as

$$\mathbf{q}_{TD} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & e^{j\omega\tau_2} & \dots & e^{j\omega\tau_N} \end{bmatrix}^T, \quad (3.3)$$

which is the TD solution for the acoustical-hotspot generation with a point target.

When $\mathbf{q}_{TD}$ is applied to the array, the complex sound pressure at the target point can be calculated as

$$p = \sum_{i=1}^N G(r_i) q_i = \mathbf{G} \mathbf{q}_{TD}, \quad (3.4)$$

where $\mathbf{G}$ is the transfer-function vector ($1 \times N$) from the i th source to the target point.

$G(r_i)$ can be given by the Green's function in free field as [HS96]

$$G(r_i) = \frac{-j\omega\rho_0}{4\pi r_i} e^{-j\frac{\omega}{c}r_i}, \quad (3.5)$$

where ρ_0 is the air density. The output energy at the target point can be shown by the squared pressure as

$$E_{TD} = \|\mathbf{G} \mathbf{q}_{TD}\|^2. \quad (3.6)$$

In the case that the target region is not a point, the problem can be approximately solved by finding out a virtual target point that is behind the target region and is on the sound projection path from the source array to the target region. The array works as if it aims to deliver the sound to the virtual target point, as illustrated in Fig. 3.2. The points $\mathbf{O}_1$ and $\mathbf{O}_2$ are the orthocenters of the array and the target region, respectively. Thus, the TD solution formulated in Eq. (3.3) can be applied for this case. The position of such a virtual target point can be determined as

$$\mathbf{O}_v = \frac{-\sqrt{S_2}}{\sqrt{S_1} - \sqrt{S_2}} \mathbf{O}_1 + \frac{\sqrt{S_1}}{\sqrt{S_1} - \sqrt{S_2}} \mathbf{O}_2, \quad (3.7)$$

where S_1 and S_2 are the square measures of the array and the target region, respectively.

Fig. 3.2 Illustration of the TD method for hotspot generation. $\mathbf{O}_1$ and $\mathbf{O}_2$ are the orthocenters of the array and the target region, respectively.

3.2.3 The MCG method

To maximize the acoustic energy in a target region using a source array is equivalent to maximizing the gain function defined in Eq. (2.44). Suppose that a number of N sources are used, the target region is sampled by M measuring points, and $\mathbf{G}$ is the $M \times N$ transfer-function matrix between the array and the measuring points. According to Section 2.4, the MCG solution can be derived as the eigenvector $\mathbf{q}_{\max}$ corresponding to the maximum eigenvalue $g_{\max}$ of $\mathbf{G}^H \mathbf{G}$. The unit-power solution ($\|\mathbf{q}_{MCG}\| = 1$) can be computed as

$$\mathbf{q}_{MCG} = \frac{\mathbf{q}_{\max}}{\|\mathbf{q}_{\max}\|}, \quad (3.8)$$

where $\|\cdot\|$ denotes norm operation.

3.2.4 Theoretical link between the TD and MCG methods

This section aims to find out the theoretical connection between the MCG solution and the TD solution in free field when the target region is reduced to a point.

For a point target, the transfer-function matrix $\mathbf{G}$ is reduced to a $1 \times N$ vector, where N is the number of elements. Thus, the rank of the $N \times N$ Hermitian matrix $\mathbf{G}^H \mathbf{G}$ is one. Consequently, vector $\mathbf{G}^H$ is an eigenvector of matrix $\mathbf{G}^H \mathbf{G}$, which implies that the MCG solution can be expressed analytically by normalizing $\mathbf{G}^H$ as

$$\begin{aligned}
 \mathbf{q}_{MCG} &= \frac{\mathbf{G}^H}{\|\mathbf{G}\|} \\
 &= \frac{1}{\sqrt{\sum_{n=1}^N r_n^{-2}}} \cdot \begin{bmatrix} \frac{1}{r_1} & \frac{e^{-j\frac{\omega}{c}(r_2-r_1)}}{r_2} & \dots & \frac{e^{-j\frac{\omega}{c}(r_N-r_1)}}{r_N} \end{bmatrix}^H \\
 &= \underbrace{\frac{1}{\sqrt{\frac{1}{N} \cdot \sum_{n=1}^N r_n^{-2}}} \cdot \begin{bmatrix} r_1^{-1} & 0 & \dots & 0 \\ 0 & r_2^{-1} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & r_N^{-1} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ e^{j\frac{\omega}{c}(r_2-r_1)} \\ \vdots \\ e^{j\frac{\omega}{c}(r_N-r_1)} \end{bmatrix}}_{\mathbf{q}_{TD}} \\
 &= \mathbf{A} \mathbf{q}_{TD} , \tag{3.9}
 \end{aligned}$$

where $\mathbf{q}_{TD}$ is the TD solution as given in Eq. (3.3). Since matrix $\mathbf{A}$ is only related to the values of $r_1, r_2, \dots, r_N$, and independent of frequency ω , it implies that matrix $\mathbf{A}$ determines the amplitudes of the source strengths, which can be regarded as an amplitude weighting. Therefore, the MCG solution is actually the multiplication of the TD solution, $\mathbf{q}_{TD}$ with an amplitude weighting matrix, $\mathbf{A}$.

Furthermore, we can use the ratio of the gains of the two solutions to indicate their performance, that is

$$\eta = \frac{g(\mathbf{q}_{MCG})}{g(\mathbf{q}_{TD})} , \tag{3.10}$$

where $g(\cdot)$ is the gain function defined in Eq. (2.44). The gain ratio, η can be further expressed as

$$\eta = \frac{\mathbf{q}_{MCG}^H \mathbf{G}^H \mathbf{G} \mathbf{q}_{MCG} / (\mathbf{q}_{MCG}^H \mathbf{q}_{MCG})}{\mathbf{q}_{TD}^H \mathbf{G}^H \mathbf{G} \mathbf{q}_{TD} / (\mathbf{q}_{TD}^H \mathbf{q}_{TD})}. \quad (3.11)$$

Since $\mathbf{q}_{MCG}^H \mathbf{q}_{MCG} = 1$ and $\mathbf{q}_{TD}^H \mathbf{q}_{TD} = 1$, Eq. (3.11) can be rewritten as

$$\eta = \frac{\mathbf{q}_{MCG}^H \mathbf{G}^H \mathbf{G} \mathbf{q}_{MCG}}{\mathbf{q}_{TD}^H \mathbf{G}^H \mathbf{G} \mathbf{q}_{TD}}. \quad (3.12)$$

Applying Eq. (3.3) and Eq.(3.9) to Eq. (3.12) leads to

$$\eta = \frac{\sum_{n=1}^N r_n^{-2}}{\frac{1}{N} \left(\sum_{n=1}^N r_n^{-1} \right)^2}. \quad (3.13)$$

According to the Cauchy inequality [JJ88], two important conclusions can be drawn from Eq. (3.13):

1) $\eta \geq 1$ for any positive values of $r_1, r_2, \dots, r_N$, implying $g(\mathbf{q}_{MCG}) \geq g(\mathbf{q}_{TD})$.

2) $\eta = 1$ holds only when $r_1 = r_2 = \dots = r_N$, implying $g(\mathbf{q}_{MCG}) = g(\mathbf{q}_{TD})$.

Therefore, the MCG method always has a better performance than the TD method in terms of higher gain. When $r_1 = r_2 = \dots = r_N$, the two methods have the same performance.

In the far-field case, considering that $r_1 \approx r_2 \approx \dots \approx r_N$, the gain ratio $\eta \approx 1$, implying similar performance for both TD and MCG method. In contrast, in the near-field case, $r_1, r_2, \dots, r_N$ are different (except that all sources are symmetric with respect to the target point). Hence, the gain ratio $\eta > 1$, which indicates that the MCG method has a better performance.

The performance of the two methods has been analyzed as presented above. It is noted that the two methods have different computational complexity. As shown in Section 3.2.2, the TD method only needs N subtractions and N divisions (or equivalently one division and $N-1$ multiplications) to compute the time delay. In contrast, the MCG method needs to compute one $N \times N$ correlation matrix $\mathbf{G}^H \mathbf{G}$, and one eigenvector decomposition as presented in Section 3.2.3. Therefore, the MCG method is computationally more complex than the TD method.

The above analysis leads to the necessity for differentiating the far field and near field. In the far field, the TD method should be used due to its low computational complexity and similar gain with the MCG method. However, in the near field, the MCG method should be used due to its higher gain compared with the TD method. Generally, the following far-field criterion can be used [Goo68] [Her03]. The far field is assumed if

$$D > D^* = \frac{2l^2}{\lambda}, \quad (3.14)$$

where D is the average distance from the array to the target point, λ is the wavelength, and l is the distance between two most distant points of the array. D^* can be regarded as a critical distance for far-field and near-field separation. At such a distance ($D > D^*$), the wavefront is approximately a plane wave. However, this criterion may not be applicable for determining far field or near field in acoustical-hotspot generation, because a critical distance calculated by using Eq.(3.14) is usually big. For example, when $l = 1$ m and $\lambda = 0.2$ m, $D^* = 10$ m is obtained. At such a distance, the gain ratio η may be very close to 1, making it insignificant for deciding which method to use. For acoustical-hotspot generation, we propose a critical distance determined by

$$D^* = D|_{\eta=1+\gamma}, \quad (3.15)$$

where $\gamma \geq 0$ is an indicator showing how much higher gain obtained by the MCG against the TD method. For example, if a 5% higher gain is desirable to differentiate whether the MCG or TD method should be used, the parameter $\gamma = 5\%$ can be set, and a critical distance D^* can be found accordingly. When $D > D^*$, the far field is identified, and the TD method should be used due to the similar performance ($\eta < 1 + \gamma$) of the two method; when $D < D^*$, the near field is identified and the MCG method should be chosen in order obtain a higher gain with $\eta > 1 + \gamma$.

3.2.5 Free field comparison

In the preceding section, it has been theoretically proven that the MCG method has a better performance than the TD method. In this section, their performances are compared in free field simulations, including a point target simulation and a circular target region simulation. The array elements are assumed to be point sources without coupling effect.

3.2.5.1 Point target simulation

This simulation is designed for the point target case. A linear array is used to generate an acoustical hotspot at a target point. The simulation configuration is shown in Fig. 3.3. The 20-element linear array is placed along the x -axis and symmetry to the axis origin with a half-wavelength element-spacing. The target point is located at $P(-1.5 \text{ m}, D)$, where D is the distance between P and the x -axis. r_i represents the distance from

the i -th source to P . The measuring line is parallel to the x -axis and includes the point P . The excitation frequency is $f = 1000$ Hz, and the wavelength is $\lambda = c / f$ with the speed of sound, $c = 344$ m/s. The critical distance, D^* can be determined using Eq. (3.15). As an example, when $\gamma = 0.05$ is applied, η equals to 1.05 at D^* , corresponding to a 5% higher gain by using the MCG method. The resulting critical distance is $D^* = 1.7$ m.

Three test cases are considered:

- a) $D = 3$ m (far field);
- b) $D = D^* = 1.7$ m (critical distance);
- c) $D = 0.3$ m (near field).

The distributions of the sound pressure levels (SPLs) on the measuring line are shown in Fig. 3.4 (a), (b) and (c) for the three test cases, respectively. Numerical results, such as the gain ratio η and the SPL difference ΔP of the two methods at the target point are listed in Table 3.1.

Fig. 3.3 Configuration for point target simulation

(a)

(b)

(c)

Fig. 3.4 SPL pattern in (a) far field (b) critical distance, and (c) near field.

Table 3.1 Numerical results for point target simulation.

	D (m)	η	ΔP (dB)
Far field	3.0	1.01	0.06
D^*	1.7	1.05	0.24
Near field	0.3	1.72	2.37

The following observations can be drawn from the simulation results:

- (1) In the far-field case, the two methods obtain similar array patterns along the measuring line, as shown in Fig. 3.4 (a). The peak SPL obtained by the MCG method at the target point is only 0.06 dB higher than the TD method. In the sidelobe region, although lower SPLs are obtained by the MCG method compared to the TD method, the differences are small (< 2 dB). Similar performance is also indicated by the gain ratio $\eta = 1.01$, representing only 1% higher gain of the MCG method compared with the TD method.
- (2) In the case of $D = 1.7$ m, $\Delta P = 0.24$ dB, higher than in the far field case. Furthermore, as shown in Fig. 3.4 (b), the difference of the two resulting curves is easier to observe, compared with that in Fig. 3.4 (a). It means that the better performance of the MCG method is more apparent than in the far-field case.
- (3) In the near-field case, the MCG method again demonstrates a better performance. As shown in the SPL distributions in Fig. 3.4 (c), the peak SPL using the MCG method is about 2.37 dB higher than that using the TD method. In addition, the SPLs in the sidelobes are much better attenuated (around 8 dB) compared to the TD method.

- (4) All three test cases show that the MCG method has a better performance in terms of higher SPL at the target point, lower SPL in the neighboring region and higher gain. This observation matches the theoretical conclusion in Section 3.2.4 that the MCG always has a better performance than the TD method.

The above analysis has shown that the advantage of the MCG over the TD method is related to the location of the target point. The relation can be further illustrated by testing the gain ratio η with respect to the distance D . Based on the simulation configuration in Fig. 3.3, the ratio η is calculated for D from 0.1 m to 6 m for every 0.05 m interval, and the resulting distribution is demonstrated in Fig. 3.5, in which the critical distance is $D^* = 1.7$ m (refer to Eq.(3.15)). It can be observed that η reduces with the increase of the distance D . In the near field, η is relatively high ($\eta > 1.05$), whereas in the far field, the η is small and slowly approximates to 1 when D is big. It is noted that $\eta > 1$ constantly holds in the simulation, which is consistent with the theoretical analysis in Section 3.2.4. In general, the MCG method has a better performance than the TD method for acoustical hotspot generation with a point target, especially in the near field.

It is worth noting that the conventional critical distance defined in Eq.(3.14) can not be to used to determine which method to use. Consider the configuration in Fig. 3.3. A critical distance of 62 m is calculated by using Eq.(3.14). However, in a distance of $D = 3$ m, the two methods have already shown similar SPL distributions, as demonstrated in Fig. 3.4 (a). When $D = 62$ m, it is hard to tell the difference of their performance, and the critical distance in Eq.(3.14) becomes no long “critical” for choosing the methods.

Fig. 3.5 Gain ratio η distribution with respect to the distance D .

At $D^* = 1.7$ m, $\eta = 1.05$.

3.2.5.2 Circular target region simulation and discussion

This simulation compares the performance of the MCG and TD methods when the target region is a circular region.

The simulation arrangement is shown in Fig. 3.6. The circular array has 25 elements arranged in eight branches. The centers of the array and the target region are at $O_1(0, 0, 0)$ and $O_2(0, 0, D)$, respectively. The array radius is $r_1 = 0.52$ m, and the target region radius is $r_2 = 0.2$ m. The measuring plane and measuring line are parallel to x - and y - axis and cover the target region.

Two test cases are considered:

- d) Far field with $D = 2$ m

The simulated array patterns on the measuring plane and measuring line are given in Fig. 3.7 and Fig. 3.8, respectively.

e) Near field with $D = 0.3$ m.

The simulated array patterns on the measuring plane and measuring line are given in Fig. 3.9 and Fig. 3.10, respectively.

Numerical results as the gain ratio and average SPL difference ΔP in the target region are listed in Table 3.2.

Fig. 3.6 Configuration for circular target region simulation.

Fig. 3.7 Simulated SPL patterns for the far-field test by (a) MCG method and (b) TD method. The target region in each pattern is highlighted by the black circle.

Fig. 3.8 Far-field simulation: SPL pattern on the measuring line.

Fig. 3.9 Simulated SPL patterns for the near-field test by (a) MCG method and (b) TD method. The target region in each pattern is highlighted by the black circle.

Fig. 3.10 Near-field simulation: SPL pattern on the measuring line.

Table 3.2 Numerical results in circular target simulation.

	D (m)	η	ΔP (dB)
Far field	2.0	1.03	0.13
Near field	0.3	1.43	2.50

In the far-field case, both methods successfully generate an acoustical hotspot in the target region, as shown in Fig. 3.7 (a) and (b). It can be observed that resulting array patterns are quite similar, implying that the two methods have similar performance. The similar performance is further demonstrated by the SPL distributions on the measuring line in Fig. 3.8. It can be seen that the two methods result in similar SPL distributions near $x = 0$, where the target region is defined. Despite the similar performance, the MCG method obtains about 0.13 dB higher average SPL in the target region than the TD method. In addition, the ratio $\eta = 1.03$ indicates that the resulting gains using the MCG methods is slightly (3%) higher than using the TD method.

In the near-field case, a better performance is again obtained using the MCG method, as shown by the SPL patterns in Fig. 3.9 and Fig. 3.10. The MCG method results in a 2.5 dB higher average SPL in the target region, and about 10 dB lower sidelobes compared to the TD method. In addition, as listed in Table 3.2, a gain ratio of $\eta = 1.43$ is obtained, representing a 43% higher gain using the MCG method.

The relation between the gain ratio η and the distance D is shown in Fig. 3.11. The critical distance is found to be $D^* = 1.54$ m. That is, $\eta > 1.05$ when $D^* < 1.54$ m. Similarly with the result illustrated in Fig. 3.5 for the linear array simulation, Fig. 3.11 shows that $\eta > 1$ constantly and η decreases with the increase of the distance. In addition, it is noticed that the patterns of the curve in Fig. 3.5 and Fig. 3.11 are similar. When $D < D^*$, η decreases very fast with the increase of D , but slow down when $D > D^*$.

Fig. 3.11 Gain ratio η distribution with respect to the distance D for circular array simulation. At $D^* = 1.54$ m, $\eta = 1.05$.

3.2.5.3 Summary of the free field simulation

The following conclusions can be drawn for the free-field simulations:

- (1) The MCG method always has a better performance than the TD method in terms of higher gain.
- (2) In the far field, the two methods tend to have similar performance, indicated by a gain ratio close to one; whereas in the near field, the MCG has a prominent performance compared with the TD method, indicated by a gain ratio much bigger than one.
- (3) The simulation results agree with the theoretical analysis presented in Section 3.2.4.
- (4) Since the two methods have close performance in the far field, the TD method can be chosen in this case due to its computational simplicity. However, in the near field, the MCG method is preferred because of its better performance.

3.2.6 Comparison in a rectangular enclosure

In the previous section, the performances of the MCG method and TD method are compared in free field. It is also interesting to compare the two methods in a reverberation field. In this section, we consider the problem in an empty rectangular enclosure, which is modeled using the Green's function in an enclosure [MI86]. Based on this new transfer function, simulations are conducted for the comparison of the two methods.

3.2.6.1 Room acoustics for simple sources

The transfer function from a simple source to a measuring point in an empty rectangular enclosure can be given by the Green's function [MI86] as

$$G(\vec{r}_l, \vec{r}_0) = j\omega\rho_0 \sum_n \frac{\varphi_n(\vec{r}_l)\varphi_n(\vec{r}_0)}{K_n(k^2 - k_n^2 - 2j\frac{\delta_n}{c}k_n)}, \quad (3.16)$$

where

$\vec{r}_l = (x_l, y_l, z_l)$ is the location of the observing point.

$\vec{r}_0 = (x_0, y_0, z_0)$ is the location of the point source.

L_x , L_y and L_z are the length, width and height of the room, respectively.

$$K_n = \int_0^{L_x} \cos^2\left(\frac{n_x \pi x_l}{L_x}\right) dx \int_0^{L_y} \cos^2\left(\frac{n_y \pi y_l}{L_y}\right) dy \int_0^{L_z} \cos^2\left(\frac{n_z \pi z_l}{L_z}\right) dz = \frac{L_x L_y L_z}{8}.$$

$k = \omega / c$ is wave number.

$n = (n_x, n_y, n_z)$ represents three nonnegative integers.

δ_n is the damping constant.

$$k_n = \pi \sqrt{\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2} \text{ are the eigenvalues.}$$

$$\varphi_n(\vec{r}) = \cos\left(\frac{n_x \pi x}{L_x}\right) \cos\left(\frac{n_y \pi y}{L_y}\right) \cos\left(\frac{n_z \pi z}{L_z}\right) \text{ are the eigen- functions at } \vec{r}.$$

By applying the transfer function, the sound pressure at point $\vec{r}_l$ produced by a simple source at $\vec{r}_0$ can be calculated as

$$p(\vec{r}_l, \vec{r}_0) = G(\vec{r}_l, \vec{r}_0) \cdot q, \quad (3.17)$$

where q is the source strength. For an N -source array with source strength vector $\mathbf{q}$, the sound pressures on the M measuring points in a spatial region can be calculated by $\mathbf{p} = \mathbf{G}\mathbf{q}$, where $\mathbf{G}$ is the $M \times N$ transfer-function matrix. Since dense sampling is used in this thesis, $M > N$ usually holds for a continuous target region.

3.2.6.2 Hotspot generation method and critical distance

The MCG method for the acoustical-hotspot generation in an enclosure is basically the same as that in free field. The difference is to construct the gain function, $g(\mathbf{q})$ using the transfer-function matrix $\mathbf{G}$ of the enclosure. The optimal source strength vector can be obtained as

$$\mathbf{q}_{MCG} = \frac{\mathbf{q}_{\max}}{\|\mathbf{q}_{\max}\|}, \quad (3.18)$$

where $\mathbf{q}_{\max}$ can be calculated by eigenvalue decomposition of $\mathbf{G}^H \mathbf{G}$. Equation (3.18) is in the same form as in Eq. (3.8), but based on different transfer functions.

It is noted that the TD method is usually under the assumption of a free field or a field dominated by direct sound. Therefore, if the target region is defined within the near field where the sound field is more determined by the direct sound than by the reverberant sound, the TD method is still effective. However, in the far field where the reverberant sound is stronger than the direct sound, the assumption of the TD

method cannot be satisfied, and thus, it is no longer practical in this case. In contrast, the MCG method is independent on this assumption, because the derivation of $\mathbf{q}_{\max}$ is on the basis of the transfer function of a sound field instead of the time differences from different sources. Therefore, the MCG method is applicable to both near field and far field, if the transfer function is known.

The far field and near field in an enclosure regarding the hotspot generation problem can be divided by the critical distance (also called reverberation distance), defined as a distance from the source where the energy densities of the direct sound and reverberant sound are the same. The critical distance can be roughly estimated by [Kut00]

$$r_h = 0.1 \sqrt{\frac{V}{\pi T}}, \quad (3.19)$$

where V is the volume of the enclosure, and T is the reverberation time. In the region where $r < r_h$, the direct sound prevails; whereas in the place where $r > r_h$, the reverberant sound dominates.

3.2.6.3 Far-field simulation

In this section, the performances of the MCG and TD methods are compared in far-field simulation.

The system configuration is illustrated in Fig. 3.12. The 13-point-source array is used to generate an acoustical hotspot in the circular target region. The interval of two adjacent array elements is 0.17 m. The source array is located at $O_1(1.5 \text{ m}, 2.25 \text{ m}, 1.5 \text{ m})$. A critical distance of $r_h = 0.61 \text{ m}$ is calculated by using

Eq.(3.19). The circular target region is located at $O_2(1.5 \text{ m}, 2.25 \text{ m}, D+1.5 \text{ m})$ with a radius of 0.2 m , where D is the distance between the array and the target region. Distance D is set to 2.5 m ($D > r_h$), representing a far-field case. The excitation frequency is 500 Hz . Assume that the damping constant $\delta_n = 10$, and the reverberation time $T = 0.7 \text{ s}$.

Fig. 3.12 Configuration of the hotspot generation system in a rectangular enclosure with $L_x \times L_y \times L_z = 3 \times 4.5 \times 6 \text{ m}^3$.

Based on the above configuration, the array pattern (SPL in dB) on the measuring plane ($2 \times 2 \text{ m}^2$) is simulated. The resulting patterns are shown in Fig. 3.13 (a) and (b) for the MCG and TD methods, respectively. The target region is highlighted by a circle in each pattern. Furthermore, the SPL distributions along the

measuring line (with a 2 m length) parallel to x -axis are shown in Fig. 3.14. In addition, numerical results, such as the average SPL in the target region and the gain ratio are listed in Table 3.3.

Fig. 3.13 Far-field simulation: SPL patterns on the measuring plane by the use of (a) MCG method and (b) TD method. The black circle in the center highlights the target region.

Fig. 3.14 Far-field simulation: SPL distribution on the measuring line.

Table 3.3 Far-field simulation. Average SPLs in the target region: $\bar{P}_{MCG}$ and $\bar{P}_{TD}$.

$\bar{P}_{MCG}$ (dB)	$\bar{P}_{TD}$ (dB)	η (dB)
-0.48	-7.17	7.17

The following observations can be made from the simulation results:

- (1) Comparing the two patterns in Fig. 3.13 (a) and (b), it is observed that an acoustical-hotspot can be successfully synthesized in the circular target region using the MCG method, as shown in Fig. 3.13 (a). However, the TD method fails to generate such a hotspot, as shown in Fig. 3.13 (b). Moreover, the SPL derived by the TD method in the target region is lower than in the neighboring region.
- (2) Furthermore, the SPL distribution in Fig. 3.14 shows a similar pattern. The MCG method results in a SPL peak at the target region ($1.3 < x < 1.7$ m), whereas the TD method gives a valley in the target region. The center SPL difference is about 7.70 dB.
- (3) The better performance of the MCG method is also indicated by the gain ratio η , which is 7.17 dB in this simulation, representing a 7.17 dB higher gain in the MCG method compared with the TD method. This result means that the MCG method is more effective in focusing acoustical energy in the target region.

In general, in the far-field case, the MCG method is effective in acoustical hotspot generation, whereas the TD method is not applicable.

3.2.6.4 Near-field simulation

In this section, the near-field simulation is presented. The same configuration as in the far-field simulation is used, except that the target region is set to $D = 0.5$ m away from the array. Since $D < r_h = 0.61$ m, this simulation represents a near-field case. The resulting patterns are shown in Fig. 3.15 (a) and (b) correspondent to the MCG and TD methods, respectively. The SPL distributions along the measuring line by the two methods are demonstrated in Fig. 3.16. Numerical results are listed in Table 3.4.

Fig. 3.15 Near-field simulation: SPL patterns on the measuring plane by the use of (a) MCG method and (b) TD method. The target region is highlighted by the black circle in the center.

Fig. 3.16 Near-field simulation: SPL distribution on the measuring line. Peak-peak difference of the two curves is 1.37dB.

Table 3.4 Near-field simulation. Average SPLs in the target region: $\bar{P}_{MCG}$ and $\bar{P}_{TD}$.

$\bar{P}_{MCG}$ (dB)	$\bar{P}_{TD}$ (dB)	η (dB)
-0.30	-1.37	1.31

As shown in the two pattern plots of Fig. 3.15 (a) and (b), an acoustical hotspot is successfully synthesized in the circular region using both methods, and the overall patterns are nearly the same. According to Table 3.4, the average SPL in the target region is -0.30 dB using the MCG method, slightly higher than the SPL (-1.37 dB) using TD method. This similar performance is also demonstrated in the SPL distribution on the measuring line, as shown in Fig. 3.16. These results imply that both methods are applicable in the near-field case in the enclosure. However, it is noted that the MCG method has a better performance than the TD method.

3.2.6.5 Discussions on the hotspot generation strategy

Based on the above simulations, the strategies for hotspot generation in an enclosure can be concluded as follows.

- (1) In the case of far field, the MCG method is recommended over the TD method.
- (2) In the case of near field, if computational load is an important factor for choosing the algorithm, the TD method can be used due to its computational simplicity and close performance to the MCG method.
- (3) The critical distance can be used for separating the near field and far field. It is important to understand the relationship between the critical distance r_h and the enclosure size and absorption coefficient α . Since the critical distance is used for separating the near and far fields, it is important to understand the relationship between the critical distance r_h and the condition of the enclosure in terms of enclosure size and absorption coefficient α . Using the Sabine's equation [Kut00],

$$T \approx 0.163V / (S\alpha), \quad (3.20)$$

Equation (3.19) can be rewritten to

$$r_h = 0.248\sqrt{S\alpha / \pi}, \quad (3.21)$$

where S is the surface of the enclosure. Equation (3.21) implies that for a given absorption coefficient, the bigger the surface, the longer is the critical distance.

Specifically, for a cubic enclosure with length of L , $S = 6L^2$. Thus,

$$r_h = 0.248L\sqrt{6\alpha / \pi}, \quad (3.22)$$

which means that the critical distance is in proportion to the length of the cubic enclosure.

3.2.7 Wideband implementation

The TD and MCG methods for narrowband applications have been introduced in preceding sections. In this section, the extension of the methods for wideband application is discussed.

3.2.7.1 Wideband TD solution

Basically, the TD method can be implemented by using a number of delays on the signal lines connected to the sources, as shown in Fig. 3.17. For the point target case, the delays can be calculated directly using Eq. (3.1). Whereas for a region target, a virtual target point can be determined using Eq. (3.7) first, and then the delays are obtained using Eq. (3.1).

From Fig. 3.17, it can be seen that the implementation of the TD method for wideband solution is straight forward. Only N delays, $\tau_1, \tau_2, \dots, \tau_N$ are needed without other processing. This simple implementation is the advantage of the TD method.

Fig. 3.17 Wideband solution using TD method

3.2.7.2 Wideband MCG solution

In Section 3.2.3, the MCG solution for a single frequency has been stated in Eq. (3.8). This narrowband solution can be developed into a wideband solution by applying either a delay-and-weight processing or a FFT processing. The basic schemes for the two kinds of processing are illustrated in Fig. 3.18 (a) and (b), respectively.

Fig. 3.18 Wideband solution using the MCG method. (a) Delay-and-weight processing (b) FFT processing.

For the delay-and-weight processing shown in Fig. 3.18 (a), the delays for each signal line can be calculated using Eq. (3.1). If the target is a region, a virtual target point determined using Eq. (3.7) can be used instead. The amplitude weights $A_{11}, A_{22}, \dots, A_{NN}$ are the diagonal elements of matrix $\mathbf{A}$ defined in Eq. (3.9), that is

$$A_{nn} = \frac{r_n^{-1}}{\sqrt{\frac{1}{N} \cdot \sum_{n=1}^N r_n^{-2}}}, \quad (3.23)$$

where r_n represents the distance between the n -th source to the target point.

The FFT processing is realized by applying a complex weight, $\mathbf{w}_n$ after FFT and before IFFT in n -th signal line, as shown in Fig. 3.18 (b). Suppose a K -point FFT is used. There are K frequency bins after the FFT. The calculation of the weight vector $\mathbf{w}_n$ requires solving narrowband MCG solutions for each frequency bin, and combining these solutions to generate a wideband solution. It is noted that the idea of formulating wideband solution from narrowband solutions has been used in many works [Tre02] [BW01].

For the k -th bin with frequency range of $[\omega_{L,k}, \omega_{H,k}]$, the MCG solution, $\mathbf{q}_{MCG,k}$ can be found by solving the optimization problem

$$\mathbf{q}_{MCG,k} = \max_{\mathbf{q}} g_k(\mathbf{q}), \quad (3.24)$$

where $g_k(\mathbf{q})$ is the gain function for k -th frequency bin. Function $g_k(\mathbf{q})$ is defined as

$$g_k(\mathbf{q}) = \frac{E_{out,k}(\mathbf{q})}{E_{in,k}(\mathbf{q})}, \quad (3.25)$$

where $E_{out,k}(\mathbf{q})$ and $E_{in,k}(\mathbf{q})$ are the output power in the target region and the input power of the array, respectively. They can be calculated using

$$E_{out,k}(\mathbf{q}) = \int_{\omega_{L,k}}^{\omega_{H,k}} \|\mathbf{G}(\omega)\mathbf{q}\|^2 d\omega = \mathbf{q}^H \left[\int_{\omega_{L,k}}^{\omega_{H,k}} \mathbf{G}^H(\omega)\mathbf{G}(\omega) d\omega \right] \mathbf{q}, \quad (3.26)$$

and

$$E_{in,k}(\mathbf{q}) = \mathbf{q}^H \mathbf{q}, \quad (3.27)$$

respectively, where $\mathbf{G}(\omega)$ is the transfer-function matrix from the multiple sources to the discrete samples of the target region. If we define

$$\mathbf{R}_k = \int_{\omega_{L,k}}^{\omega_{H,k}} \mathbf{G}^H(\omega)\mathbf{G}(\omega) d\omega, \quad (3.28)$$

$g_k(\mathbf{q})$ becomes

$$g_k(\mathbf{q}) = \frac{\mathbf{q}^H \mathbf{R}_k \mathbf{q}}{\mathbf{q}^H \mathbf{q}}. \quad (3.29)$$

It is interesting to note that Eq. (3.29) is a Rayleigh-quotient [Str03], which is also similar to the single frequency gain function defined in Eq. (2.44). Thus, the MCG solution for the k -th frequency bin, $\mathbf{q}_{MCG,k}$ can be given as the eigenvector associated with the maximum eigenvalue of matrix $\mathbf{R}_k$. Therefore, for the K frequency bins, K solutions can be obtained as $\mathbf{q}_{MCG,1}, \mathbf{q}_{MCG,2}, \dots, \mathbf{q}_{MCG,K}$.

After the narrowband solutions are obtained, the complex weight, $\mathbf{w}_n$ can be determined by

$$\mathbf{w}_n = [\mathbf{q}_{MCG,1}(n) \quad \mathbf{q}_{MCG,2}(n) \quad \dots \quad \mathbf{q}_{MCG,K}(n)]^T, \quad (3.30)$$

where $\mathbf{q}_{MCG,k}(n)$ represents the n -th element of $\mathbf{q}_{MCG,k}$. By using Eq. (3.30) for $n = 1, 2, \dots, N$, the N weigh vectors, $\mathbf{w}_1, \mathbf{w}_2, \dots$, and $\mathbf{w}_N$ can be calculated. Thus, the wideband MCG solution using FFT processing is obtained.

The following comments can be drawn by comparing the two solutions using delay-and-weight processing and FFT processing.

Firstly, the solution using the delay-and-weight processing is much simpler than using the FFT processing. The former only needs some simple calculation as presented in Eq. (3.23). In contrast, the later requires K times of integration with respect to frequency ω , as shown in Eq. (3.28). These integrations require computational load, especially when the array size or the number of discrete samples of the target region are large, or when high-resolution integrations are used.

Secondly, the solution using the delay-and-weight processing is much easier to implement. Only N delays and amplitude weights are needed, as shown in Fig. 3.18 (a). In contrast, the FFT processing requires one FFT, N IFFT and N weights, which result in a higher computational load compared to the delay-and-weight processing.

Thirdly, the FFT processing is actually equivalent to the optimum MCG solution with delay-and-weight processing for each frequency bin, as shown in Eq. (3.30).

Based on above comments, it is recommended to use the MCG solution with the delay-and-weight processing for wideband acoustical-hotspot generation.

In the following sections 3.2.7.3 and 3.2.7.4, the MCG method using the delay-and-weight processing is compared with the TD method for wideband simulations in near field and far field, respectively.

3.2.7.3 Near-field wideband simulation

Assume the same simulation configuration as illustrated in Fig. 3.3. A 20-element uniform linear array with element spacing of $d = 0.172$ m is used to focus the sound at the near-field target point located at (1.5 m, 0.3 m). Consider a frequency band of 250~3400 Hz. The TD and the MCG solutions are compared based on these arrangements.

The time delays, $[\tau_1, \tau_2, \dots, \tau_{20}] = [0, -0.0001, 0.0001, 0.0005, 0.0009, 0.0013, 0.0018, 0.0023, 0.0028, 0.0032, 0.0037, 0.0042, 0.0047, 0.0052, 0.0057, 0.0062, 0.0067, 0.0072, 0.0077, 0.0082]$ sec can be calculated using Eq. (3.1). Thus, the TD solution is obtained.

The MCG solution requires the amplitude weights of $A_{11}, A_{22}, \dots, A_{NN}$, as shown in Fig. 3.18 (a). These weights can be calculated using Eq. (3.23), obtained as $[A_{11}, A_{22}, \dots, A_{NN}] = [2.1238, 2.3076, 1.9056, 1.4367, 1.1076, 0.8883, 0.7370, 0.6279, 0.5461, 0.4828, 0.4323, 0.3913, 0.3573, 0.3286, 0.3042, 0.2832, 0.2648, 0.2487, 0.2344, 0.2216]$. Thus, the MCG solution is accomplished.

By using the above solutions, the array pattern on the measuring line (Fig. 3.3) for the frequency band, 250~ 3400 Hz can be simulated, as shown in Fig. 3.19 and Fig. 3.20 for 3D and 2D patterns, respectively.

(a)

(b)

Fig. 3.19 Simulated 3D array patterns using (a) MCG solution, and (b) TD solution.

(a)

(b)

Fig. 3.20 Simulated 2D array patterns using (a) MCG solution, and (b) TD solution.

Fig. 3.21 SPL difference by the MCG and TD solutions with respect to the frequency.

The following conclusions can be drawn from the simulated array patterns:

- (I) Both the TD and MCG solutions successfully generate an acoustical hotspot at the target point. As shown in Fig. 3.19 (a) and (b), both patterns show a SPL peak at $x = 1.5$ m for almost the entire frequency range of $f = 250$ to 3400 Hz. When f is low ($f < 1000$ Hz), the peak is a bit flat along the x -axis.
- (II) A better performance is obtained by the MCG solution. Comparing the two 2D patterns in Fig. 3.20 (a) and (b), it can be observed that the SPL contrast between the target region ($x = 1.5$ m) and its neighboring region ($x \neq 1.5$ m) is more observable in pattern (a) than in pattern (b). The neighboring region in (a) is noticeably darker than that in (b). This result implies that the MCG solution is more able to focus the acoustical energy in a target region. This conclusion can be further shown by the SPL difference of the two solutions at the target point with respect to the frequency, as illustrated in Fig. 3.21. The SPL obtained by the MCG solution is about 2.4 dB higher than that obtained by the TD solution throughout the frequency range.

In general, the simulation results show that both the MCG and TD solutions can be used in wideband acoustical-hotspot generation in near field. However, a better

performance in terms of higher SPL in the target region is obtained by the MCG solution.

It is worth noting that the simulation is under the assumption of omnidirectional point sources. In a real application, the calibration of the sources and varying radiation patterns of individual sources have to be considered. For an array of paper-cone loudspeakers widely used today, the radiation patterns of each speaker may be very different, and thus it is not easy to calibrate the array for wideband signals. This difficulty makes the MCG method too expensive to apply. However, the method will be useful in the future when very uniform and small-sized sources are available.

3.2.7.4 Far-field wideband simulation

Consider a far-field hotspot generation scenario, as illustrated in Fig. 3.22. A 20-element linear array with element-spacing of 0.172 m is arranged along the x -axis and symmetry with respect to the origin of the coordinate. The target region is a line segment with $x = [-0.1 \text{ m}, 0.1 \text{ m}]$ at $y = 5 \text{ m}$. Consider the frequency band of $[250, 2500] \text{ Hz}$. Since the distance from the target region to the array is 5 m, it can be regarded as a far-field case according to the criterion in Eq. (3.15). By using Eq. (3.7), a virtual target point is determined at $O(0, 5.70 \text{ m})$.

Based on above settings, the time delays can be calculated using Eq. (3.1), and given as $[\tau_1, \tau_2, \dots, \tau_{20}] = [0, -0.1311, -0.2485, -0.3518, -0.4409, -0.5156, -0.5755, -0.6206, -0.6507, -0.6658, -0.6658, -0.6507, -0.6206, -0.5755, -0.5156, -0.4409, -0.3518, -0.2485, -0.1311, 0] \text{ msec}$. Thus, the TD is obtained. The amplitude weights

for the MCG solution are calculated using Eq. (3.23), and obtained as $[A_{11}, A_{22}, \dots, A_{NN}] = [0.9754, 0.9829, 0.9896, 0.9957, 1.0010, 1.0055, 1.0091, 1.0118, 1.0137, 1.0146, 1.0146, 1.0137, 1.0118, 1.0091, 1.0055, 1.0010, 0.9957, 0.9896, 0.9829, 0.9754]$.

By using the TD and MCG solutions, the wideband array pattern on the measuring line (Fig. 3.22) can be simulated, as shown in Fig. 3.23 and Fig. 3.24 for 3D and 2D patterns, respectively. The difference of the average SPL in the target region by the two solutions with respect to the frequency is illustrated in Fig. 3.25.

The following observations can be drawn from the simulated array patterns:

- (I) Both the TD and MCG solutions successfully generate acoustical hotspot at the target region. As shown by the 3D patterns in Fig. 3.23 and 2D patterns in Fig. 3.24, both solutions produce higher SPL around the target region of $x = [-0.1 \text{ m}, 0.1 \text{ m}]$, compared to the SPL in the neighboring region.
- (II) As shown in Fig. 3.23 and Fig. 3.24, the two solutions arrive at the same array patterns. The resulting average SPLs in the target region obtained by the two solutions are very close. Throughout the frequency band, the SPL difference is smaller than 0.04 dB, as demonstrated in Fig. 3.25.

In general, the simulation results have shown that both the MCG and TD solutions can be used in far-field wideband acoustical-hotspot generation, and they have similar performance. Therefore, for the far-field case in the free field, the TD method is recommended due to its simpler implementation.

Fig. 3.22 Acoustical-hotspot generation system configuration.

(a)

(b)

Fig. 3.23 Simulated 3D array patterns using (a) MCG solution, and (b) TD solution.

(a)

Fig. 3.24 Simulated 2D array patterns using (a) MCG solution, and (b) TD solution.

Fig. 3.25 SPL difference by the MCG and TD solutions with respect to the frequency.

3.3 Quiet-zone generation

The method for generating an acoustical hotspot has been introduced in preceding sections. In this section, the topic of quiet-zone generation is discussed. Rather than using the maximum-gain TOAM, the optimal solution to quiet-zone generation is determined by lower-gain TOAMs. Two cases of quiet-zone generation are considered, and the solutions for each case are provided. Simulations are performed for performance analysis. This work is going to be published in [WYG06].

The organization of this section is as follows. A brief introduction is given in Section 3.3.1. Sections 3.3.2 and 3.3.3 discuss the two cases of quiet-zone generation, respectively.

3.3.1 Introduction

A quiet zone refers to a predefined spatial region that receives minimum acoustical energy from a source array with a given input power. There are two cases of quiet-zone generation. In the first case, a source array is used to generate a quiet zone directly with a given input power of the array [CK02b]. In the second case, a source array is utilized as secondary sources or a noise barrier to cancel the sound energy in a target region due to a primary source [Ell01] [NCE92]. These two cases can be illustrated in Fig. 3.26 (a) and (b), respectively. The difference is that the source array plays different roles in the two systems.

Fig. 3.26 Illustration of quiet-zone generation systems: (a) Single array system; (b) Secondary source array used as a noise barrier.

It has been shown in Sections 2.3 and 2.4 that the potential for a source array to produce minimum gain in a quiet zone is actually determined by the minimum-gain TOAM of the system. Therefore, for the first case, the minimum-gain TOAM can be readily chosen as the optimal source-strength vector.

For the second case of quiet-zone generation problem, an existing analytical solution is based on minimizing the cost function of the output power in the sensors or measuring points in the target region [Ell01] [NCE92]. Since this solution uses matrix inversion, a potential problem is that the performance may be affected when the correlation matrix of the transfer impedance from the secondary sources to the target region is ill-conditioned. In this section, a new approach is proposed by applying TOAM analysis. The optimal solution is derived from some lower-gain TOAMs. Simulation will show that when the correlation matrix is ill-conditioned, the new approach has a better performance; otherwise, similar performance can be obtained by the two methods.

It is noted that this research does not aim to implement a new noise barrier system, but to propose a new theoretical analysis method that has better performance than the existing theoretical solution using matrix-inversion (shown later in Section 3.3.3.1). In addition, similar to the matrix-inversion solution, this new approach assumes point sources and is used for narrowband analysis.

3.3.2 First case of quiet-zone generation

This section focuses on the first case of quiet-zone generation. The solution is given in Section 3.3.2.1, followed by simulations and discussion in Section 3.3.2.2.

3.3.2.1 Solution

Consider a quiet-zone generation system using an N -element source array. According to the TOAM analysis presented in Chapter 2, this system can be modeled as a MIMO

(multi-input multi-output) system, with the source strengths being the multi-input, and SPLs at the M discrete sample points in the target region being the multi-output. For this MIMO system, a number of N TOAMs, $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$ exist, among which the minimum gain is determined by the minimum-gain TOAM, $\mathbf{q}_N$. Since $\mathbf{q}_N$ corresponds to the minimum reception of the acoustic energy in the target region for a unit-power input, it can be used as the optimal solution to the quiet-zone generation. That is,

$$\mathbf{q}_{opt} = \mathbf{q}_N. \quad (3.31)$$

The effectiveness of this solution will be shown in the following simulations.

3.3.2.2 Simulation and discussion

The simulation system is shown in Fig. 3.27. The N -element square source array is used to generate a quiet-zone in the circular target region with a radius of r , where $N = 9$, $r = 1.0$ m, and the length of the array is $l = 0.2$ m. The sources are assumed to be point sources without mutual coupling. The target region is evenly sampled by M points with $M = 1250$. The sound field is assumed to be free field with sound speed $c = 344$ m/sec.

Two simulations are designed:

- Simulation I: Target region placed at $O_1 (0, 0, 5 \text{ m})$.

Testing frequency at $f = 500, 1000, 2500$ and 5000 Hz.

- Simulation II: Target region placed at $O_2 (-1 \text{ m}, -1 \text{ m}, 5 \text{ m})$.

Testing frequency at $f = 500, 1000, 2500$ and 5000 Hz.

Fig. 3.27 Simulation system for the first case quiet-zone generation.

$N = 9$. $l = 0.2$ m. $r = 1.0$ m. $O_1 = (0, 0, 5 \text{ m})$ for Simulation I , and

$O_2 = (-1 \text{ m}, -1 \text{ m}, 5 \text{ m})$. $M = 1250$.

Based on the above configurations, optimal solutions of each test can be calculated using Eq.(3.31). The simulated array patterns are shown in Fig. 3.28 and Fig. 3.29 for Simulations I and II, respectively. The numerical results of the average SPLs in the target and neighboring regions are listed in Table 3.5 and Table 3.6 for the two simulations, respectively.

(a) $f = 500$ Hz

(b) $f = 1000$ Hz

(c) $f = 2500$ Hz

(d) $f = 5000$ Hz

Fig. 3.28 Resulting array patterns for Simulation I at $f = 500$, 1000, 2500 and 5000 Hz, respectively. The target region is highlighted by a white circle.

(a) $f = 500$ Hz

(b) $f = 1000$ Hz

(c) $f = 2500$ Hz

(d) $f = 5000$ Hz

Fig. 3.29 Resulting array patterns for Simulation II at $f = 500$, 1000, 2500 and 5000 Hz, respectively. The target region is highlighted by a white circle.

Table 3.5 Numerical results for Simulation I.

	$f = 500$ (Hz)	$f = 1000$ (Hz)	$f = 2500$ (Hz)	$f = 5000$ (Hz)
$g_{\min}$	2.6×10^{-4}	0.26	2.5×10^3	2.3×10^6
$\bar{p}_T$ (dB)	-61.96	-61.42	-56.46	-40.25
$\bar{p}_N$ (dB)	-22.39	-22.34	-19.64	-12.43
$\Delta\bar{p}$ (dB)	39.57	38.08	36.82	27.82

Note: $\bar{p}_T$ and $\bar{p}_N$ represent the average SPL in the target region and neighboring region, respectively. $\Delta\bar{p} = \bar{p}_N - \bar{p}_T$. $g_{\min}$ is the eigenvalue correspondent to the optimum solution $\mathbf{q}_{opt}$ in Eq. (3.31).

Table 3.6 Numerical results for Simulation II.

	$f = 500$ (Hz)	$f = 1000$ (Hz)	$f = 2500$ (Hz)	$f = 5000$ (Hz)
$g_{\min}$	1.4×10^{-4}	0.14	1.3×10^3	1.3×10^6
$\bar{p}_T$ (dB)	-75.51	-74.64	-64.11	-43.11
$\bar{p}_N$ (dB)	-32.57	-31.83	-24.23	-21.76
$\Delta\bar{p}$ (dB)	42.94	42.81	39.88	21.35

The following observations can be drawn from the simulation results:

- (1) Quiet zones are successfully generated in the target region at different testing frequencies. As shown by the patterns in Fig. 3.28 (a)~(d) of Simulation I, the circular target region has a lower SPL than the neighboring region. This observation is also demonstrated by the numerical results in Table 3.5. The average SPL of the target region, $\bar{p}_T$ is at least 27.82 dB lower than $\bar{p}_N$ of the

neighboring region at the four testing frequencies. For example, at $f = 1000$ Hz, $\bar{p}_T$ is about 39 dB lower than $\bar{p}_N$. These results imply that the solution using the minimum-gain TOAM is effective in producing quiet zones.

- (2) The solution is also successful in quiet-zone generation for different target-region definitions. Simulations I and II have different locations of target region, but quiet zones are generated in both simulations. In Simulation II, $\bar{p}_T$ is at least 21.35 dB lower than $\bar{p}_N$ at the four test frequencies, as shown in the patterns in Fig. 3.29 and numerical results in Table 3.6.
- (3) In both simulations, the performance of the solution varies with the frequency, where the performance is denoted by the SPL difference, $\Delta\bar{p}$. For example, in Simulation I, $\Delta\bar{p}$ is 39.57 dB at $f = 500$ Hz, as shown in Table 3.5. This observation can be further illustrated by the distributions of $\bar{p}_T$, $\bar{p}_N$ and $\Delta\bar{p}$ in the frequency band of 200~8000Hz, using the same simulation arrangements. The resulting distributions are given in Fig. 3.30 and Fig. 3.31, for Simulations I and II, respectively. In both figures, $\Delta\bar{p}$ decreases with the increase of f . In lower frequency band ($f < 2000$ Hz), $\Delta\bar{p}$ is around 40 dB, representing that $\bar{p}_T$ is about 40 dB lower than $\bar{p}_N$. However, in higher frequency band ($f > 7000$ Hz), $\Delta\bar{p}$ is only about 10 dB, implying a weaker performance compared with the lower band. These results imply that the array arrangement favors the lower frequency band than the higher frequency band. If a tolerance of $\Delta\bar{p} < 20$ dB is set for the quiet-zone generation, the array arrangement can be used for the band of $f < 5000$ Hz. For higher frequency band, such a tolerance requires a smaller array. This requirement can be explained by the relationship between

wavelength and the array pattern. It is known that different sizes of arrays are needed in order to generate the same array pattern for different frequency bands [BW01]. The purpose of this section is to show the effectiveness of the minimum-gain TOAM solution for quiet-zone generation. The discussion on the array size and array pattern can be found in [BW01] [JD92].

In general, the simulation has shown that minimum-gain TOAM solution is effective in the first case of quiet-zone generation. The solution to the second case is introduced in the next section.

Fig. 3.30 SPL distributions with respect to f , using Simulation I arrangement.

(a) Average SPLs in the target/neighboring region, $\bar{p}_T$, $\bar{p}_N$;

(b) SPL difference $\Delta\bar{p} = \bar{p}_N - \bar{p}_T$.

Fig. 3.31 SPL distributions with respect to f , using Simulation II arrangement.

(a) Average SPLs in the target/neighboring region, $\bar{p}_T$, $\bar{p}_N$;

(b) SPL difference $\Delta\bar{p} = \bar{p}_N - \bar{p}_T$.

3.3.3 Second case of quiet-zone generation

This section focuses on the second case of quiet-zone generation. A conventional solution is briefly introduced in Section 3.3.3.1. A new solution is proposed in Section 3.3.3.2, followed by simulations and discussions in Section 3.3.3.3.

3.3.3.1 Conventional solution

Consider a quiet-zone generation system for theoretical analysis as shown in Fig. 3.32. The secondary sources work as a noise barrier for controlling the noise from the primary source, and thus generating a quiet zone in the target region. It is noted that a practical noise cancellation system may be very different with the theoretical modal. In practice, the primary source is not a loudspeaker but a noise source generating diffuse wavefield measured by reference microphones. The secondary sources disturb the sound field of the primary source and feed back into the reference microphones for measuring the primary source. In this thesis, we study the theoretical modal and aim to present a better solution than a conventional solution [Ell01] [NCE92]. The conventional solution is given as

$$\mathbf{q}_{s0} = -(\mathbf{G}_{se}^H \mathbf{G}_{se})^{-1} \mathbf{G}_{se}^H \mathbf{G}_{pe} q_p \text{ when } M > N, \quad (3.32)$$

and

$$\mathbf{q}_{s0} = -\mathbf{G}_{se}^{-1} \mathbf{G}_{pe} q_p \text{ when } M = N, \quad (3.33)$$

where q_p is the primary source strength, $\mathbf{q}_{s0}$ is an $N \times 1$ vector of secondary source strength, $\mathbf{G}_{pe}$ is an $M \times 1$ vector of the transfer impedance from the primary sources to the error sensors, and $\mathbf{G}_{se}$ is an $M \times N$ matrix of transfer impedance from the secondary sources to the sample points in the target region. The basic idea of this method is to minimize the output power or the cost function J defined as

$$J = \mathbf{p}^H \mathbf{p} = (\mathbf{G}_{pe} q_p + \mathbf{G}_{se} \mathbf{q}_s)^H (\mathbf{G}_{pe} q_p + \mathbf{G}_{se} \mathbf{q}_s), \quad (3.34)$$

where

$$\mathbf{p} = \mathbf{G}_{pe} q_p + \mathbf{G}_{se} \mathbf{q}_s, \quad (3.35)$$

is the complex pressures at the M sample points. It is noted that the solution in Eq. (3.32) requires the assumption of invertibility of $\mathbf{G}_{se}^H \mathbf{G}_{se}$, which is a potential problem when the matrix is poorly conditioned.

Fig. 3.32 Example system for the second-case quiet-zone generation.

3.3.3.2 New solution

In this section, a new solution is developed using the TOAM analysis. Suppose there are one primary source and N secondary sources in the system. If these $N+1$ sources are considered as one array, the problem is reduced to finding an $(N+1)$ -dimension source strength vector that minimizes the gain in the target region. Mathematically, the optimal vector can be obtained by

$$\mathbf{q}^o = \arg \min_{\mathbf{q}} g(\mathbf{q}) \quad \text{subject to } q_1^o = q_p, \quad (3.36)$$

where q_1^o is the first element of vector $\mathbf{q}^o = [q_1^o, q_2^o, q_3^o, \dots, q_{N+1}^o]^T$, corresponding to the primary source. Thus, the secondary source strengths can be determined by $\mathbf{q}_s = [q_2^o, q_3^o, \dots, q_{N+1}^o]^T$. The Hermitian matrix for constructing the gain function $g(\mathbf{q})$ (defined in Eq. (2.44)) is updated by

$$\mathbf{R} = \begin{bmatrix} \mathbf{G}_{pe} & \mathbf{G}_{se} \end{bmatrix}^H \begin{bmatrix} \mathbf{G}_{pe} & \mathbf{G}_{se} \end{bmatrix}. \quad (3.37)$$

Since the system is an $(N+1)$ -source system, a total of $N + 1$ TOAMs can be derived, and the minimum-gain TOAM is $\mathbf{q}_{N+1}$. Therefore, the solution to Eq.(3.36) can be developed from $\mathbf{q}_{N+1}$ using a linear transform to meet the requirement of $q_1^o = q_p$, which is given as

$$\mathbf{q}^o = \frac{q_p}{q_{N+1,1}} \cdot \mathbf{q}_{N+1}, \quad (3.38)$$

where $q_{N+1,1}$ represents the first element of vector $\mathbf{q}_{N+1}$.

There is a possibility that $q_{N+1,1}$ in Eq. (3.38) may be a very small number or even zero. In this case, another lower-gain TOAM, $\mathbf{q}_k$ can be selected by searching from high to low indices as $k = N, N-1, \dots, 1$. The search can be stopped once $q_{k,1}$ is larger than some pre-defined small number, say 0.0001. Then, the optimal source strength $\mathbf{q}^o$ can be calculated as

$$\mathbf{q}^o = \frac{q_p}{q_{k,1}} \cdot \mathbf{q}_k. \quad (3.39)$$

Since matrix $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{N+1}]$ is full ranked, it is guaranteed to obtain a candidate mode $\mathbf{q}_k$ satisfying $q_{k,1} \neq 0$. Otherwise, if $q_{i,1} = 0$, for at least one $i = 1, 2, \dots, N+1$, the rank of $\mathbf{Q}$ would be smaller than $N+1$, which is not possible. Finally, the optimal secondary source strength can be obtained as

$$\mathbf{q}_s = [q_2^o, q_3^o, \dots, q_{N+1}^o]^T = \frac{q_p}{q_{k,1}} \cdot [q_{k,2}, q_{k,3}, \dots, q_{k,N+1}]^T. \quad (3.40)$$

The above procedures can be summarized by the flow chart shown in Fig. 3.33.

Fig. 3.33 The flow chart for determining the optimal solution to the

quiet-zone generation of the second case. $\mathbf{q}^o_2 = [q^o_2, q^o_3, \dots, q^o_{N+1}]^T$

and σ is a small number (for example, 0.0001).

3.3.3.3 Simulation and discussion

In this section, free-field simulations are performed to test the effectiveness of the new method by comparing it with the conventional method (refer to Eq. (3.32)). For the convenience of description, the new method is named TOAM method, whereas the conventional method is called the Min-J method. The simulation configuration is shown in Fig. 3.34. Four simulations are performed as follows:

- Simulations I : $f = 400$ Hz, $N = 25$;
 - Simulations II : $f = 400$ Hz, $N = 9$;
 - Simulations III: $f = 1000$ Hz, $N = 25$;
 - Simulations IV: $f = 1500$ Hz, $N = 25$.
- } Same frequency, different source numbers
 } Different frequencies, same source number

Based on the configuration, the secondary source strength vectors using the two methods can be calculated using Eqs. (3.32) and (3.40), respectively. The SPL distributions are simulated over a $3 \times 3 \text{ m}^2$ measuring region containing the target region at its center. The resulting SPL distributions are shown in Fig. 3.35 and Fig. 3.36. Numerical results are listed in Table 3.7.

Fig. 3.34 Arrangement of a quiet-zone generation system with multiple secondary sources. N secondary sources, evenly distributed in the $L_s \times L_s$ region are used to control the $L_e \times L_e$ target region evenly sampled by M sensors in a distance of D_{se} . The primary source is placed at a distance of D_{ps} from the secondary source plane. $M = 36$, $L_e = 1.2 \text{ m}$, $O(0, 0, D_{se})$, $L_s = 0.6 \text{ m}$, $D_{ps} = 3 \text{ m}$, $D_{se} = 5 \text{ m}$, $q_p = 1 \text{ m}^3/\text{s}$, and the sound speed $c = 344 \text{ m/s}$. $N = 25$ is used in Simulations I, III and IV. $N = 9$ is used in Simulation II.

Fig. 3.35 Resulting array patterns for Simulation I and II: SPL distribution over the $3 \times 3 \text{ m}^2$ measuring region sampled evenly by 1024 points. (Target region is highlighted by dashed lines. Ave. SPL refers to the average SPL in the target region.)

- (a) Simulation I: Min-J method ($N=25, f=400 \text{ Hz}$);
- (b) Simulation I: TOAM method ($N=25, f=400 \text{ Hz}$);
- (c) Simulation II: Min-J method ($N=9, f=400 \text{ Hz}$);
- (d) Simulation II: TOAM method ($N=9, f=400 \text{ Hz}$).

Fig. 3.36 Resulting array patterns for Simulations III and IV: SPL distribution over the $3 \times 3 \text{ m}^2$ measuring region sampled evenly by 1024 points. (Target region is highlighted by dashed lines. Ave. SPL refers to the average SPL in the target region.)

- (a) Simulation III: Min-J method ($N = 25$, $f = 1000 \text{ Hz}$);
- (b) Simulation III: TOAM method ($N = 25$, $f = 1000 \text{ Hz}$);
- (c) Simulation IV: Min-J method ($N = 25$, $f = 1500 \text{ Hz}$);
- (d) Simulation IV: TOAM method ($N = 25$, $f = 1500 \text{ Hz}$).

Table 3.7 Table of simulation results on input / output power and average SPL in the target region by Min-J and TOAM methods, respectively. k is condition number of matrix $\mathbf{G}_{se}^H \mathbf{G}_{se}$. Input power: $E_{in} = \mathbf{q}^H \mathbf{q}$. Output power: $J = \mathbf{p}^H \mathbf{p}$. Average SPL in the quiet zone: $\bar{p}$ (dB).

	Simulation I ($k = 6.26 \times 10^{16}$)		Simulation II ($k = 5.86 \times 10^8$)	
	Min-J	TOAM	Min-J	TOAM
E_{in}	56.184	56.889	8.3181	9.615
J	0.027	2.559×10^{-5}	0.0159	0.018
$\bar{p}$ (dB)	-27.6	-60.9	-33.5	-33.5

	Simulation III ($k = 8.15 \times 10^{13}$)		Simulation IV ($k = 1.07 \times 10^{17}$)	
	Min-J	TOAM	Min-J	TOAM
E_{in}	63.538	63.716	27.994	28.108
J	0.061	8.403×10^{-11}	0.155	1.702×10^{-5}
$\bar{p}$ (dB)	-22.4	-29.2	-20.4	-42.3

Fig. 3.37 Resulting average SPL (dB) in the simulations using the Min-J and TOAM methods, respectively.

The following observations can be drawn from the simulation results.

Firstly, in all simulations, the TOAM method successfully generates a quiet zone in the target region. For example, a quiet zone with an average SPL of -60.9 dB is produced in Simulation I, and a -33.5 dB SPL is obtained in Simulation II, as shown in Fig. 3.35 (b) and (d), respectively. In contrast, the SPLs in the neighboring region are roughly all above 0 dB, much higher than the SPL in the target region.

Secondly, a better performance in terms of a lower average SPL in the target region is obtained by the TOAM method, as illustrated in Fig. 3.37. For example, in Simulation I the TOAM method shows an average SPL of -60.9 dB, compared to the average SPL of -27.6 dB using the Min-J method. Thus, a 33.3 dB SPL difference is obtained. In Simulation III and IV, the SPL differences are 6.8 and 21.9 dB, respectively. However, in Simulation II the same average SPL is obtained by both TOAM and Min-J methods. This difference of performance can be possibly explained by analyzing the condition number of matrix $\mathbf{G}_{se}^H \mathbf{G}_{se}$ (refer to Eq. (3.32)) in each simulation. The condition number is as large as 6.26×10^{16} in Simulation I, but only 5.86×10^8 in Simulation II. It means that $\mathbf{G}_{se}^H \mathbf{G}_{se}$ is more ill-conditioned in Simulation I. This ill-condition causes the difficulty of the matrix inversion in Eq. (3.32), although the matrix-inversion can also be realized by eigenvalue decomposition. Assume $\mathbf{G}_{se}^H \mathbf{G}_{se}$ can be expressed by eigenvalue decomposition as $\mathbf{G}_{se}^H \mathbf{G}_{se} = \mathbf{Q}_{se} \mathbf{\Lambda}_{se} \mathbf{Q}_{se}^H$, where matrix $\mathbf{Q}_{se}$ is the transform matrix formed with the eigenvectors of $\mathbf{G}_{se}^H \mathbf{G}_{se}$, and $\mathbf{\Lambda}_{se} = \text{diag}[\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_N]$ with λ_1 , λ_2 and λ_N being the eigenvalues of $\mathbf{G}_{se}^H \mathbf{G}_{se}$ in descending order. The inversion can be obtained as

$$\left[\mathbf{G}_{se}^H \mathbf{G}_{se} \right]^{-1} = \mathbf{Q}_{se}^{-H} \mathbf{\Lambda}_{se}^{-1} \mathbf{Q}_{se}^{-1}. \quad (3.41)$$

When $\mathbf{G}_{se}^H \mathbf{G}_{se}$ is ill-conditioned, the lower order eigenvalues, such as λ_N may be very small, causing $\mathbf{\Lambda}_{se}^{-1}$ to have a very large element. As a result, the inversion in (3.41) becomes inaccurate. Compared with the Min-J method, the TOAM method uses no matrix inversion, and hence it has better performance even when $\mathbf{G}_{se}^H \mathbf{G}_{se}$ is ill-conditioned.

Thirdly, for the same array working at different frequencies, the performance difference of the two methods may also be different, depending on the condition number k of $\mathbf{G}_{se}^H \mathbf{G}_{se}$. A small performance difference of 6.7 dB (−29.2 against −22.4 dB) is obtained in Simulation III, but a 21.9 dB (−42.3 against −20.4 dB) difference results in Simulation IV. The correspondent condition numbers are 8.15×10^{13} and 1.07×10^{17} , respectively. These results imply that the larger the condition number is, the more effective the TOAM method is, compared with the Min-J method. This conclusion can also be drawn from the comparison of Simulation I with II, which represent the case of different arrays working at the same frequency. In general, the advantage of the TOAM method over the Min-J method is dependent on the condition number of matrix $\mathbf{G}_{se}^H \mathbf{G}_{se}$.

3.4 Conclusions

The applications of the maximum-gain TOAM for acoustical-hotspot generation and the minimum/lower-gain TOAMs for quiet-zone generation have been studied in this chapter. Conclusions are summarized as below.

3.4.1 Conclusions on the acoustical-hotspot generation

In the acoustical-hotspot generation, the following conclusions can be drawn based on the theoretical analysis and simulation results on the TD and MCG methods.

- f) Firstly, the merit of the TD method over MCG method is its simplicity. The main advantage of the MCG method over TD is its better performance in terms of higher gain in either free field or a rectangular enclosure.
- g) Secondly, in the free field, the TD method is preferred in the case of far field, due to its simplicity and comparable performance to the MCG method. In the case of near field, the MCG method can be used because of its better performance.
- h) Thirdly, in a rectangular enclosure, the sound field is complicated due to the existence of reverberant sound. In the near field, the direct sound dominates, and therefore, the TD method is still applicable. However, in the far field, the reverberant sound prevails. Thus, the TD method is no longer effective. In contrast, the MCG method can be used in both far and near fields.
- i) Finally, the wideband implementation of the TD and MCG methods in the free field is studied in this chapter. Similar to the conclusion drawn in the narrowband case, the MCG solution is recommended in the near field due to its better performance; whereas the TD solution is preferred in the far field for its simpler implementation but comparable performance to the MCG solution.

3.4.2 Conclusions on the quiet-zone generation

Two cases of quiet-zone generation are studied in this chapter.

- In the first case, a source array is used to generate a quiet-zone in a target region directly. The minimum-gain TOAM can be used as the optimal array weight.
- In the second case, a source array works as secondary sources to cancel the acoustical energy in a target region due to the primary source. The optimal strengths of the secondary sources can be derived by analyzing lower-gain TOAMs. This new method has been shown to be more robust than the conventional solution in that the new method is effective even when the matrix, $\mathbf{G}_{se}^H \mathbf{G}_{se}$ is ill-conditioned; whereas the conventional solution has a weaker performance in this case.

In general, this chapter has shown that the TOAM analysis is useful in sound-field control of either acoustical-hotspot generation or quiet-zone generation.

Chapter 4

Nonlinear least-square solution to the synthesis of single look-direction and shaped array patterns in the far field

Array pattern synthesis (APS) usually refers to the generation of single/multiple look-direction and shaped array patterns. In this chapter, the APS problems in the far field and free field are investigated. Although these problem has been studied for many years and there have been many classical solutions [Dol46] [WL48] [App76] [TG92], APS continues to attract researchers' attention. In recent years, new methods are developed [ZI99] [Wang03] [SF05], aiming to provide better solutions in terms of better mainlobe shaping and sidelobe suppression. In this chapter, two nonlinear least-square solutions are presented. The first solution which was published in [WGY05] was known as the nonlinear least-square (NLS) method. The second solution applies the TOAM series expansion, introduced in Chapter 2. Due to robustness of the second solution in solving nonlinear optimization problem, this solution is called the robust nonlinear least-square (RNLS) method [WGY05a]. In this chapter, these two methods are compared, and they are also compared with some other existing methods [Dol46] [WL48] [ZI99] [Wang03].

4.1 Overview

Array pattern synthesis (APS) aims to determine an optimal array weight that can generate single look-direction (SLD) and shaped array patterns. The shaped array pattern, typically, can be a flat-top (FT) pattern. Both types of patterns can be

synthesized with either uniform or non-uniform linear arrays. It is noted there are some other design goals, such as multi-beam pattern, which aims to provide multiple beams with different look directions. This thesis more focuses on SLD and FT pattern syntheses, because they are more widely applied, studied and frequently seen in many publications [Dol46] [Ell76a] [Ell76b] [Ell77] [ES84] [OC90] [TG92] [WZ93] [NEK93] [ZI99].

The SLD pattern synthesis concerns how the beamwidth can be reduced for a specified sidelobe level (SLL), or how the SLL for a given beamwidth can be attenuated. In the past decades, many methods [Dol46] [Rib47] [Ell85] have been proposed. For uniform linear arrays, the Dolph-Chebyshev method [Dol46] is the most commonly used. The advantage is that it produces equal-level sidelobes and a minimum beamwidth for a pre-defined SLL. Since the original method introduced in [Dol46] is only optimum for the case of $d \geq \lambda/2$ element spacing, Riblet [Rib47] proposed an improved procedure for the spacing of $d \leq \lambda/2$. The similarity of these two methods is that they are both based on Chebyshev polynomial. In addition, the two methods provide the same results when $d = \lambda/2$. The array pattern of a linear equally spaced array can be expressed as the multiplication of single element pattern and a factor, called array factor. Array polynomial is an alternative representation for array factor by using the transform of $z = e^{j\theta}$, where θ is the look direction of the array. There are some other methods that take the advantage of array polynomial for pattern synthesis [Ell76a] [Ell76b] [Ell77] [ES84].

For non-uniform or arbitrary arrays, several algorithms [OC90] [TG92] [WZ93] [NEK93] [ZI99] are formulated based on minimum mean-square-error criterion with linear constraints. These algorithms usually use quadratic programming to derive the

solution. For example, Tseng et al. [TG92] proposed an iterative method under this scheme, which updates the linear constraint sequentially to ensure the SLL meets the specification of a desired pattern. A similar method [NEK93] was developed by Ng et al. in 1993, whose work offered more results for different beam definitions (single/multiple beam, null region, unequal sidelobes) and different arrays including linear, rectangular and circular arrays. In Wu's work [WZ93], the set of discrete angle directions, called the "constraint directions" is used. This algorithm increases the number of the constraint directions, or increases the weights in the regions where small synthesis error is required. Sim and Er [Er92] [SE96] proposed a new weighting function and solve the problem with quadratic programming, which is flexible in sidelobe control according to sidelobe locations. Adaptive array [App76] [TFN76] is another technique that can be used in APS. The basic idea is to design an array weight that maximizes the signal-interference-plus-noise power ratio (SINR) through maximizing the pattern in the desired directions and minimizing the pattern in interference directions. Typical methods can be seen in [OC90] proposed by Olen and Compton and a recent work by Zhou and Ingram [ZI99]. These two methods have similar performance in synthesizing SLD pattern, but Zhou's method can be extended to FT pattern synthesis. Many other methods are based on minimax technique, developed by Parks and McClellan [PM72a] [PM72b] and subsequent contributions in [MPR73], [MP73], [MR79] and [RMP75]. In 1972, Hersey et al. [HTL72] published a paper on minimax technique and gave a more general design, under which arbitrary upper and lower constraint functions can be used. The results of this work were implemented in linear arrays by Mucci et al. [MTL75]. Recently, some new optimization methods are applied to solve minimax problems, such as convex

optimization [LB97] and semi-definite programming [Wang03]. One of the advantages of minimax methods is that equal-level sidelobes can be obtained.

The FT pattern synthesis considers the factors of mainlobe ripple (MLR), SLL and width of transition region at the same time. The trade-off among these factors has to be taken into account in the design of an optimal array weight. Conventionally, the Woodward-Lawson (WL) method [WL48] and the Fourier transform method [OS89] [Orf96] [Tre02] are used for the synthesis. These methods are simple and able to provide analytical solutions. However, a window function, such as Hamming window is required to reduce the ripples in the mainlobe, which leads to an undesirable large transition region. In addition, these methods are only applicable to uniform arrays. For non-uniform linear array, the adaptive array [ZI99] and minimax methods [PM72a,b] [Wang03] can be used. These methods have been shown to be effective in shaping patterns according to design specifics.

Some of the methods mentioned above will be briefly introduced in Section 4.2. In summary, the SLD pattern can be effectively synthesized by using the Dolph-Chebyshev method for uniform linear array, and minimax method or adaptive array method for non-uniform linear array. However, for FT pattern synthesis, there is no method that has been finalized as a benchmark. Therefore, this chapter aims to propose a new solution that is effective in sidelobe suppression for FT pattern synthesis. In the case only amplitude response is concerned in the synthesis, it is possible to further suppress the sidelobes by releasing phase constraint in the cost function. This mathematically leads to a nonlinear optimization problem, and quadratic programming used in LSE problems [ESK93] can not be applied for the solution. Therefore, a nonlinear least-square (NLS) method is proposed [WGY05]. On the other hand, the NLS method requires a multi-dimensional starting point, making

the optimization dependent on initializations. To overcome this problem, an initial-value independent method is proposed using the TOAM series expansion introduced in Chapter 2. This method takes multi-stage optimizations from low order to high order successively, and in each stage the starting point is constructed from the optimal solution obtained in the previous stage. The overall algorithm requires no initial value, and thus is more robust. This method is named the robust nonlinear least-square (RNLS) method. This method has been published in [WGY05a].

This chapter is organized as follows. Section 4.2 presents some classical and recent methods, such as the Dolph-Chebyshev, Woodward-Lawson, and minimax methods. Section 4.3 presents the NLS method. Simulations are performed to test the performance of this method. Section 4.4 discusses the RNLS method. The robustness of the RNLS method is shown in several simulations by comparing the RNLS and NLS methods in solving same synthesis problems. Finally, conclusions are presented in Section 4.5. Since most array pattern synthesis methods are introduced based on the analysis of linear arrays, simulations presented in this chapter are mainly carried out for uniform or non-uniform linear arrays.

4.2 Existing methods

In this section, some classical methods for array pattern synthesis are briefly reviewed, including the Dolph-Chebyshev, Fourier series, Woodward-Lawson, adaptive array, minimax, and quadratic programming methods. It is noted all these methods assume free field and far field.

4.2.1 Dolph-Chebyshev method

The Dolph-Chebyshev method is based on the Chebyshev polynomials [Riv90] and was introduced by Dolph [Dol46], [Rib47]. This method has been well known for its explicit control of the SLL for uniform linear arrays, which produces the narrowest beamwidth for a given SLL.

The synthesized pattern for an array of N elements with an element spacing of $\lambda/2 \leq d$ at broadside is

$$F(z) = T_M(z), \quad (4.1)$$

where $M = N - 1$ and $T_M(z)$ is the Chebyshev polynomial of order M given as

$$T_M(z) = \begin{cases} \cos(M \cos^{-1} z) & \text{for } |z| \leq 1 \\ \cosh(M \cosh^{-1} z) & \text{for } |z| \geq 1 \end{cases}, \quad (4.2)$$

and

$$z = z_0 \cos[(\pi d / \lambda) \sin \theta], \quad (4.3)$$

with

$$z_0 = \cosh(1/M \cosh^{-1} r), \quad (4.4)$$

where r is the mainlobe to sidelobe amplitude ratio. Usually, $r > 1$, such that the sidelobe level $\text{SLL}_{dB} = -20 \log_{10} r$ is a negative number. If $F(z)$ is forced to match the Chebyshev polynomial in such a way that the array sidelobe region occupies the range $|z| \leq 1$ and the mainlobe peak at $\theta = 0$ is in the region $z_0 > 1$, then

$$T_M(z_0) = r. \quad (4.5)$$

Figure 4.1 shows the synthesized patterns using an 8-element linear array with Chebyshev weight and desired SLL of -20 , -30 and -40 dB as examples.

The optimal weights for element spacing $d \geq \lambda / 2$ can be given as

$$w_m = \frac{1}{N} \left[r + 2 \sum_{k=1}^{\frac{N-1}{2}} T_m(z_0 \cos[k\pi / N]) \cos(2mk\pi / N) \right] \quad m = 0, 1, 2, \dots, (N-1)/2, (4.6)$$

for odd N , and

$$w_m = \frac{1}{N} \left[r + 2 \sum_{k=1}^{\frac{N-1}{2}} T_m(z_0 \cos[k\pi / N]) \cos[(2m-1)k\pi / N] \right] \quad m = 0, 1, 2, \dots, (N/2-1), (4.7)$$

for even N , where N is the number of elements.

The above derivation of the optimal array weight requires an element spacing of $d \geq \lambda / 2$. For $d < \lambda / 2$, the optimal solution can be found in [Rib47] [Orf04], which was first introduced by Riblet [Rib47]. This solution is also based on Chebyshev polynomial, but it uses a modified variable in the polynomial.

Fig. 4.1 Chebyshev array patterns for an 8-element linear array with SLL = -20, -30 and -40 dB, respectively.

4.2.2 Fourier series method

Consider an array of N elements at locations, x_m along the x -axis with an element spacing of d . If N is odd, namely, $N = 2M + 1$, and the array is symmetric, the element location x_m can be written as

$$x_m = md, \text{ where } m = 0, \pm 1, \pm 2, \dots, \pm M. \quad (4.8)$$

If N is even, namely $N = 2M$, the element positions can be written as

$$x_{\pm m} = (m \mp \frac{1}{2})d, \text{ where } m = 1, 2, \dots, M. \quad (4.9)$$

An array pattern can be expressed as the multiplication of a single element pattern and a factor, called array factor. The array factor in general can be presented as

$$p(\theta) = \sum_m w_m e^{j\omega m d \sin \theta / c}. \quad (4.10)$$

In the case $N = 2M + 1$, the array factor can be written as a discrete-space Fourier transform as

$$p(\psi) = \sum_{m=-M}^M w_m e^{jm\psi} = w_0 + \sum_{m=-M}^M \left[w_m e^{jm\psi} + w_{-m} e^{-jm\psi} \right], \quad (4.11)$$

or as a spatial z -transform:

$$p(z) = \sum_{m=-M}^M w_m z^m = w_0 + \sum_{m=-M}^M \left[w_m z^m + w_{-m} z^{-m} \right], \quad (4.12)$$

where $\psi = md \sin \theta / c$, and $z = e^{j\psi}$. Similarly, in the case $N = 2M$, the array factor becomes

$$p(\psi) = \sum_{m=-M}^M w_m e^{jm\psi} = \sum_{m=-M}^M \left[w_m e^{jm\psi} + w_{-m} e^{-jm\psi} \right], \quad (4.13)$$

as a discrete-space Fourier transform, or

$$p(z) = \sum_{m=-M}^M w_m z^m = \sum_{m=-M}^M [w_m z^m + w_{-m} z^{-m}], \quad (4.14)$$

as a spatial z-transform.

The Fourier series design method is based on the inverse discrete-space Fourier transforms of the array factor. Equations (4.11) and (4.13) can be regarded as the truncated or windowed version of the corresponding infinite Fourier series. Assuming an infinite and convergent series, for the case of odd N , the array factor can be presented as

$$p(\psi) = w_0 + \sum_{m=1}^{\infty} [w_m e^{jm\psi} + w_{-m} e^{-jm\psi}]. \quad (4.15)$$

Then, the corresponding inverse transform becomes

$$w_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\psi) e^{-jm\psi} d\psi, \quad \text{where } m = 0, \pm 1, \pm 2, \dots \pm M. \quad (4.16)$$

Similarly, in the case where N is even, the array factor can be reduced to

$$p(\psi) = \sum_{m=1}^{\infty} [w_m e^{j(m-1/2)\psi} + w_{-m} e^{-j(m-1/2)\psi}], \quad (4.17)$$

with its inverse transform of

$$w_{\pm m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\psi) e^{\mp jm\psi} d\psi, \quad \text{where } m = 1, 2, \dots M. \quad (4.18)$$

In general, to synthesize a desired array pattern requires an infinite number of coefficients, w_m to be represented exactly. Keeping only a finite number (M in Eq.(4.18)) of coefficients in the Fourier series introduces unwanted ripples in the

desired response, known as the Gibbs phenomenon [OS89] [Orf96]. Such ripples can be minimized using an appropriate window, but at the expense of wider transition region.

The Fourier series method is summarized as follows. Given a desired response, $p_d(\psi)$, pick an odd or even window length, for example $N = 2M + 1$, and calculate the N ideal weights by evaluating the inverse transform,

$$w_d(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_d(\psi) e^{-jm\psi} d\psi, \text{ where } m = 0, \pm 1, \pm 2, \dots, \pm M. \quad (4.19)$$

Afterwards, the final weights are obtained by windowing with a length- N window $h(m)$ as

$$w(m) = h(m)w_d(m), \text{ where } m = 0, \pm 1, \pm 2, \dots, \pm M, \quad (4.20)$$

where $h(m)$, for example can be a Hamming window [Tre02].

4.2.3 Woodward-Lawson method

The Woodward-Lawson method [Woo46] [WL48] uses array pattern sampling design, which overcomes the problem of the Fourier transform method introduced in Section 4.2.2. For an N -element array, the method performs an inverse N -point DFT. Assume N samples of the desired array factor $p(\psi)$ are available, which are $p(\psi_i)$ with $i = 0, 1, \dots, N-1$. ψ_i , $i = 0, 1, \dots, N-1$ are the N -point DFT frequencies at

$$\psi_i = \frac{2\pi i}{N}, \quad i = 0, 1, \dots, N-1. \quad (4.21)$$

The frequency samples, $p(\psi_i)$ are related to the array weights via the N -point DFT obtained by evaluating Eqs. (4.11) and (4.13) at the N -point DFT frequencies as

$$p(\psi_i) = w_0 + \sum_{m=1}^M \left[w_m e^{jm\psi_i} + w_{-m} e^{-jm\psi_i} \right], \text{ for } N = 2M + 1, \quad (4.22)$$

or

$$p(\psi_i) = \sum_{m=1}^M \left[w_m e^{j(m-1/2)\psi_i} + w_{-m} e^{-j(m-1/2)\psi_i} \right], \text{ for } N = 2M. \quad (4.23)$$

The corresponding inverse N -point DFTs are as follows. For $N = 2M + 1$,

$$w_m = \frac{1}{N} \sum_{i=0}^{N-1} p(\psi_i) e^{-jm\psi_i}, \quad m = 0, \pm 1, \pm 2, \dots, \pm M, \quad (4.24)$$

and for $N = 2M$,

$$w_{\pm m} = \frac{1}{N} \sum_{i=0}^{N-1} p(\psi_i) e^{\mp j(m-1/2)\psi_i}, \quad m = 1, 2, \dots, M. \quad (4.25)$$

The Woodward-Lawson method can be summarized as follows. Given a set of N frequency response values $p(\psi_i)$, $i = 0, 1, \dots, N-1$, calculate the N array weights, $w(m)$ using the inverse DFT formulas given in (4.24) or (4.25). Subsequently, these weights are replaced by their windowed versions using any symmetric length- N window. The final expressions for the windowed weights are given as

$$w(m) = h(m) \frac{1}{N} \sum_{i=0}^{N-1} p(\psi_i) e^{-jm\psi_i}, \quad m = 0, \pm 1, \pm 2, \dots, \pm M, \quad (4.26)$$

for odd $N = 2M + 1$, and

$$w(\pm m) = h(\pm m) \frac{1}{N} \sum_{i=0}^{N-1} p(\psi_i) e^{\mp j(m-1/2)\psi_i}, \quad m = 1, 2, \dots, M, \quad (4.27)$$

for $N = 2M$.

4.2.4 Minimax method

This section discusses the array pattern synthesis using minimax design. Consider the problem shown in Fig. 4.2. In the mainlobe region, the pattern is unity. Thus, a design constraint can be employed that $p(\psi)$ must lie between $1 - \delta_p$ and $1 + \delta_p$ in the range $[0, \psi_p]$ and between $+\delta_s$ and $-\delta_s$ in the range $[\psi_s, \pi]$. This problem is the spatial filter analogous to the optimum minimax error design problems for FIR filters.

For a given desired array pattern $p_d(\psi)$, a weighted error can be defined as

$$e_{pm}(\psi) = W_{pm}(\psi)[p_d(\psi) - p(\psi)], \quad (4.28)$$

where it assumes that $p_d(\psi)$ is a real symmetric function. The functions $e_{pm}(\psi)$, $W_{pm}(\psi)$, and $p_d(\psi)$ are only defined over sub-intervals of $0 \leq \psi \leq \pi$. For the case of odd N , the functions are defined over $[0, \psi_p]$ and $[\psi_s, \pi]$. Assuming N , ψ_p and ψ_s are fixed design parameters, the model specification in Fig. 4.2 is stated as

$$p_d(\psi) = \begin{cases} 1, & 0 \leq \psi \leq \psi_p, \\ 0, & \psi_s \leq \psi \leq \pi, \end{cases} \quad (4.29)$$

and the weighting function is

$$W_{pm}(\psi) = \begin{cases} \frac{1}{K}, & 0 \leq \psi \leq \psi_p, \\ 1, & \psi_s \leq \psi \leq \pi, \end{cases} \quad (4.30)$$

where $K = \delta_p / \delta_s$.

Fig. 4.2 Tolerance scheme for minimax method. [OS89]

The aim of minimax criterion is to seek an optimal array weight such that the resulting array pattern minimizes the maximum weighted approximation error. Such a weight can be found by optimizing

$$\mathbf{w}_{opt} = \min_{\mathbf{w}=[w_1 \ w_2 \ \dots \ w_N]^T} \left(\max_{\psi \in F} |e_{pm}(\psi)| \right), \quad (4.31)$$

where F is a closed subset of $0 \leq \psi \leq \pi$, which can be presented as

$$[0 \leq \psi \leq \psi_p] \cup [\psi_s \leq \psi \leq \pi]. \quad (4.32)$$

Park and McClellan [PM72a] formulated the problem as a polynomial approximation problem. Applying $z = \cos \psi$ to Eq. (4.2) leads to

$$\cos(m\psi) = T_m(\cos \psi), \quad (4.33)$$

where $T_m(x)$ is the m -th order Chebyshev polynomial. Therefore,

$$p(e^{j\psi}) = \sum_{k=0}^L c_k (\cos \psi)^k, \quad (4.34)$$

where c_k are constants related to the original weights w_n and $L = (N-1)/2$. The optimum pattern $p_o(e^{j\psi})$ will satisfy the following set of equations,

$$W_{pm}(\psi_i) [p_d(e^{j\psi_i}) - p_o(e^{j\psi_i})] = (-1)^{i+1} \delta, \text{ for } i = 1, 2, \dots, L+2, \quad (4.35)$$

where δ is the optimal error.

Parks and McClellan utilized the following iterative algorithm for finding the $p_o(e^{j\psi})$ [OS89]:

Step 1: Select an initial set of ψ_i , $i = 1, 2, \dots, L+2$. ψ_p and ψ_s are included in the set.

Step 2: The set of equations (4.35) could be solved for c_k and δ . However, Parks and McClellan found a more efficient approach using a polynomial approximation.

(a) For a given set of ψ_i , δ is given as

$$\delta = \frac{\sum_{k=1}^{L+2} b_k p_d(e^{j\psi_k})}{\sum_{k=1}^{L+2} \frac{b_k (-1)^{k+1}}{W_{pm}(\psi_k)}}, \quad (4.36)$$

where

$$b_k = \prod_{i=1, i \neq k}^{L+2} \frac{1}{x_k - x_i}, \quad (4.37)$$

and

$$x_i = \cos \psi_i. \quad (4.38)$$

(b) Since $p_o(e^{j\psi})$ is an L -th order trigonometric polynomial, we can interpolate it through $L+1$ of the known $L+2$ values. Parks and McClellan used a Lagrange interpolation formula to obtain

$$p_o(e^{j\psi}) = p(\cos \psi) = \frac{\sum_{k=1}^{L+1} \left[\frac{d_k}{x - x_k} \right] f_k}{\sum_{k=1}^{L+1} \left[\frac{d_k}{x - x_k} \right]}, \quad (4.39)$$

where $x = \cos \psi$ and $x_k = \cos \psi_k$,

$$f_k = p_d(e^{j\psi_k}) - \frac{(-1)^{k+1} \delta}{W_{pm}(\psi_k)}, \quad (4.40)$$

and

$$d_k = \prod_{i=1, i \neq k}^{L+1} \frac{1}{x_k - x_i} = \frac{b_k}{x_k - x_L + 2}, \quad (4.41)$$

where we use $\psi_1, \psi_2, \dots, \psi_{L+1}$ to find the polynomial. The value at ψ_{L+2} will

be correct because $p_o(e^{j\psi})$ in Eq. (4.39) satisfies Eq. (4.35).

Step 3: The original set $\psi_1, \psi_2, \dots, \psi_{L+1}$ is exchanged for a completely new set

$\psi'_1, \psi'_2, \dots, \psi'_{L+1}$ (ψ_p and ψ_s are still included). The new ψ_i are defined by the

$(L+2)$ largest peaks in the pattern. There are at most $L-1$ local minima

and maxima in the open intervals $(0 < \psi < \psi_p)$ and $(\psi_s < \psi < \pi)$. If there

are $L-1$, the remaining point can be either 0 or π , and the largest error point is chosen for the next iteration.

Step 4: The iteration is continued in this manner until the change in δ between

iterations falls below some small pre-defined value. The result is $p_o(e^{j\psi})$.

4.2.5 Quadratic programming method

This section briefly introduces the constrained optimization approach based on the quadratic programming for designing a general array pattern. Example methods using quadratic programming can be seen in [Er91], [Er92] and [NEK93].

Consider a general array of N isotropic elements. Assume a narrowband signal arrives at an angle of (ϕ, θ) . The far-field pattern can be given as

$$p(f, \phi, \theta) = \mathbf{s}^T(f, \phi, \theta) \mathbf{w}, \quad (4.42)$$

where f is the frequency of interest. The vector

$$\mathbf{s}(f, \phi, \theta) = [1, e^{j\omega\tau_2}, \dots, e^{j\omega\tau_N}]^T \quad (4.43)$$

is the steering vector, where τ_i , $i = 1, 2, \dots, N$, are the propagation delays between the plane wavefront and the array elements. The vector

$$\mathbf{w} = [w_1, w_2, \dots, w_N]^T \in C^N \quad (4.44)$$

is the complex weight vector.

The normalized mean-square error between the desired response and the response of the array system over a certain mainlobe width (defined by $\Delta\phi$ and $\Delta\theta$) is given as

$$\begin{aligned} e^2 &= \frac{1}{\beta} \int_{\theta_0 - \frac{\Delta\theta}{2}}^{\theta_0 + \frac{\Delta\theta}{2}} \int_{\phi_0 - \frac{\Delta\phi}{2}}^{\phi_0 + \frac{\Delta\phi}{2}} |p_d(f, \phi, \theta) - \mathbf{s}^T(f, \phi, \theta) \mathbf{w}|^2 d\phi d\theta \\ &= \mathbf{w}^H \mathbf{Q}_1 \mathbf{w} - \mathbf{w}^H \mathbf{P} - \mathbf{P}^H \mathbf{w} + 1, \end{aligned} \quad (4.45)$$

where $p_d(f, \phi, \theta)$ is the desired response and (θ_0, ϕ_0) is the look direction. $\Delta\phi$ and $\Delta\theta$ are the mainlobe width of interest. $\mathbf{Q}_1$ is the $N \times N$ -dimensional positive definite Hermitian matrix given as

$$\mathbf{Q}_1 = \frac{1}{\beta} \int_{\theta_0 - \frac{\Delta\theta}{2}}^{\theta_0 + \frac{\Delta\theta}{2}} \int_{\phi_0 - \frac{\Delta\phi}{2}}^{\phi_0 + \frac{\Delta\phi}{2}} \mathbf{s}(f, \phi, \theta) \mathbf{s}^H(f, \phi, \theta) d\phi d\theta, \quad (4.46)$$

and $\mathbf{P}$ is the N -dimensional vector given as

$$\mathbf{P} = \frac{1}{\beta} \int_{\theta_0 - \frac{\Delta\theta}{2}}^{\theta_0 + \frac{\Delta\theta}{2}} \int_{\phi_0 - \frac{\Delta\phi}{2}}^{\phi_0 + \frac{\Delta\phi}{2}} p_d^*(f, \phi, \theta) \mathbf{s}(f, \phi, \theta) d\phi d\theta, \quad (4.47)$$

where β is a scalar given as

$$\beta = \int_{\theta_0 - \frac{\Delta\theta}{2}}^{\theta_0 + \frac{\Delta\theta}{2}} \int_{\phi_0 - \frac{\Delta\phi}{2}}^{\phi_0 + \frac{\Delta\phi}{2}} p_d^*(f, \phi, \theta) p_d(f, \phi, \theta) d\phi d\theta. \quad (4.48)$$

Note that Eq. (4.45) can be factorized as

$$e^2 = (\mathbf{w}_0 - \mathbf{w})^H \mathbf{Q}_1 (\mathbf{w}_0 - \mathbf{w}) + \alpha, \quad (4.49)$$

where α is a scalar given as

$$\alpha = 1 - \mathbf{w}_0^H \mathbf{Q}_1 \mathbf{w}_0, \quad (4.50)$$

and $\mathbf{w}_0$ is the N -dimensional vector satisfying

$$\mathbf{Q}_1 \mathbf{w}_0 = \mathbf{P}. \quad (4.51)$$

The mean-square value of the array response over the sidelobe regions is given as

$$\rho = \frac{1}{\Delta\phi_1\Delta\theta_1} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} |p(f, \phi, \theta)|^2 d\phi d\theta + \frac{1}{\Delta\phi_2\Delta\theta_2} \int_{\theta_3}^{\theta_4} \int_{\phi_3}^{\phi_4} |p(f, \phi, \theta)|^2 d\phi d\theta, \quad (4.52)$$

$$= \mathbf{w}^H \mathbf{Q}_2 \mathbf{w}$$

where $[\phi_1, \phi_2]$, $[\phi_3, \phi_4]$, $[\theta_1, \theta_2]$, and $[\theta_3, \theta_4]$ define the sidelobe region. $\Delta\phi_1 = |\phi_2 - \phi_1|$, $\Delta\phi_2 = |\phi_4 - \phi_3|$, $\Delta\theta_1 = |\theta_2 - \theta_1|$, and $\Delta\theta_2 = |\theta_4 - \theta_3|$. $\mathbf{Q}_2$ is the $N \times N$ -dimensional Hermitian matrix given as

$$\mathbf{Q}_2 = \frac{1}{\Delta\phi_1\Delta\theta_1} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \mathbf{s}(f, \phi, \theta) \mathbf{s}^H(f, \phi, \theta) d\phi d\theta$$

$$+ \frac{1}{\Delta\phi_2\Delta\theta_2} \int_{\theta_3}^{\theta_4} \int_{\phi_3}^{\phi_4} \mathbf{s}(f, \phi, \theta) \mathbf{s}^H(f, \phi, \theta) d\phi d\theta. \quad (4.53)$$

To match the array response to a desired response over the mainlobe width, we minimize the weighted mean square value of the array response over the sidelobe regions. The optimal weight vector, $\mathbf{w}_{opt}$ is the solution to the following constrained optimization problem:

$$\min_{\mathbf{w}} \mathbf{w}^H \mathbf{Q}_2 \mathbf{w}$$

$$\text{subject to } \mathbf{w}^H \mathbf{Q}_1 \mathbf{w} - \mathbf{w}^H \mathbf{P} - \mathbf{P}^H \mathbf{w} + 1 \leq \xi, \quad (4.54)$$

where $0 < \xi < 1$ defines a normalized error over the mainlobe. The closed form solution to the constrained optimization problem can be obtained by applying the duality theorem [ESK93], and this solution is given as

$$\mathbf{w}_{opt} = \hat{\lambda} (\mathbf{Q}_2 + \hat{\lambda} \mathbf{Q}_1)^{-1} \mathbf{P}. \quad (4.55)$$

For a specified value of ξ , $\hat{\lambda}$ can be determined by solving the following transcendental equation,

$$\hat{\lambda}^2 \mathbf{P}^H (\mathbf{Q}_2 + \hat{\lambda} \mathbf{Q}_1)^{-1} \mathbf{Q}_1 (\mathbf{Q}_2 + \hat{\lambda} \mathbf{Q}_1)^{-1} \mathbf{P} - 2\hat{\lambda} \mathbf{P}^H (\mathbf{Q}_2 + \hat{\lambda} \mathbf{Q}_1)^{-1} \mathbf{P} + 1 = \xi. \quad (4.56)$$

The solution can be seen [Lue84], in which $\hat{\lambda}$ is called the optimal Lagrange multiplier.

4.2.6 Adaptive array method

There are several methods that can be categorized as “adaptive array method”, such as the methods introduced in [OC90] and [ZI99]. Since Chapter 5 will go into more details, the review of the adaptive array method is waived in this chapter. However, this method will be compared with the proposed NLS method in the simulation part of this chapter.

4.3 Nonlinear least-square (NLS) method

This section introduces the nonlinear least-square (NLS) method for array pattern synthesis. We have previously proposed this method for solving flat-top array patterns [WGY05]. In this section, more results using the NLS method are presented, including the synthesis of single look-direction patterns. The organization of this section is as follows. The nonlinear optimization problem for array pattern synthesis is defined in Section 4.3.1. The solution is provided in Section 4.3.2. Finally, the method is compared with other methods by simulations in Section 4.3.3.

4.3.1 Problem formulation

Consider an arbitrary linear array of N isotropic elements located at d_n ($n = 1, 2, \dots, N$) with $d_1 = 0$ as reference. Assume a narrowband signal arrives at an angle of θ . The far-field pattern can be given as

$$p(\theta) = \mathbf{s}^T(\theta) \mathbf{w}, \quad (4.57)$$

where $\mathbf{s}(\theta) = [1, e^{j\alpha d_n \sin \theta / c}, \dots, e^{j\alpha d_N \sin \theta / c}]^T$ is the steering vector, and $\mathbf{w} = [w_1, w_2, \dots, w_N]^T \in \mathbb{C}^N$ is the complex weight vector. A widely used cost function for describing the square error between a synthesized pattern, $p(\theta)$ and a defined pattern, $p_d(\theta)$ can be formulated as [WZ93]

$$J = \sum_{\theta \in \Theta} [f(\theta)(\mathbf{s}^T(\theta) \mathbf{w} - p_d(\theta))]^2, \quad (4.58)$$

or [DM03]

$$J = \int_{\theta \in \Theta} [f(\theta)(\mathbf{s}^T(\theta) \mathbf{w} - p_d(\theta))]^2 d\theta, \quad (4.59)$$

depending on Θ is a set of discrete angles or a region of continuous angles. $f(\theta)$ is used to weight the errors of $|p(\theta) - p_d(\theta)|$ with respect to θ .

The optimization of Eqs. (4.58) or (4.59) leads to an optimal pattern, $p(\theta)$ which approximates towards $p_d(\theta)$ in both amplitude and phase. However, in many applications of array pattern synthesis, only amplitude response is concerned, and a phase constraint may not be necessary [SE86] [Wang03]. In this case, the aim of the synthesis becomes to determine an optimal weight vector $\mathbf{w}_{opt}$, so that the amplitude

response $|p(\theta)|$ best approximates to the desired amplitude pattern, $|p_d(\theta)|$ [SE86] [Wang03]. Thus, the cost function can be formulated as

$$J = \int_{\theta \in \Theta} [f(\theta)(|\mathbf{s}^T(\theta)\mathbf{w}| - |p_d(\theta)|)]^2 d\theta, \quad (4.60)$$

where continuous direction, θ is used.

According to the TOAM analysis in Chapter 2, the pattern synthesis can be modeled as a multi-input multi-output (MIMO) system. The multi-input is the array weight $\mathbf{w}$, and the multi-output is the response at the discrete samples in the directions of interest. That is, Θ is sampled by M directions, $\theta_1, \theta_2, \dots, \theta_M$, and the response in θ_m is $p_m = \mathbf{s}^T(\theta_m)\mathbf{w}$. Thus, the cost function can be written as

$$J = \sum_{m=1}^M \left[f(\theta_m)(|\mathbf{s}^T(\theta_m)\mathbf{w}| - |p_d(\theta_m)|) \right]^2. \quad (4.61)$$

The advantage of using discrete angles is that the integration in Eq. (4.60) can be avoided and the computational load can be reduced. It is noted that minimization of cost function, J is a nonlinear optimization problem due to the absolute operation.

The definitions of $p_d(\theta)$ for synthesizing SLD and FT patterns are illustrated in Fig. 4.3 (a) and (b), respectively. In Fig. 4.3 (a), θ_0 represents the look direction, which is also the peak direction of the mainlobe. The sidelobe region includes $[-90^\circ, \theta_{s1}]$ and $[\theta_{s2}, 90^\circ]$. The parameter, β represents the desired sidelobe amplitude, which is usually a small value. For example, $\beta = 10^{-3}$, corresponding to a -60 dB SLL, if the peak mainlobe level is defined as 0 dB. For the FT pattern, the angular axis of $[-90^\circ, 90^\circ]$ is divided into three regions, including the mainlobe

region of $[\theta_{p1}, \theta_{p2}]$, the sidelobe regions of $[-90^\circ, \theta_{s1}]$ and $[\theta_{s2}, 90^\circ]$, and the transition regions of $[\theta_{s1}, \theta_{p1}]$ and $[\theta_{p2}, \theta_{s2}]$, as shown in Fig. 4.3 (b).

(a)

(b)

Fig. 4.3 Definition of desired pattern, $p_d(\theta)$ for (a) SLD pattern, and (b)

FT pattern. Parameter β represents the desired sidelobe amplitude.

The weighting function, $f(\theta)$ is defined according to the angles in specific regions. For a SLD pattern, it is defined as

$$f(\theta) = \begin{cases} 1 & \theta = \theta_0 \\ \alpha & \theta \in [-90^\circ, \theta_{s1}] \cup [\theta_{s2}, 90^\circ] \\ \text{Not defined} & \text{otherwise} \end{cases} \quad (4.62)$$

Parameter α is a penalty factor for weighting sidelobe errors. For a FT pattern, it is defined as

$$f(\theta) = \begin{cases} 1 & \text{mainlobe region} \\ \alpha & \text{sidelobe region} \\ \text{Not defined} & \text{transition region} \end{cases} \quad (4.63)$$

After these definitions, the next step is to solve the cost function in Eq. (4.61), as discussed in the following section.

4.3.2 Solution

As shown in Eq. (4.61), there are two absolute-value operations. Therefore, solving Eq. (4.61) requires a nonlinear optimization technique, for example, the Levenberg-Marquardt algorithm [Lev44] [Mar63] [CBG99]. For the convenience of computation using the Matlab optimization toolbox, Eq. (4.61) is rewritten to a matrix form as

$$J = \|\mathbf{f} \bullet (|\mathbf{S}\mathbf{w}| - |\mathbf{p}_d|)\|^2, \quad (4.64)$$

where the operation “ $\mathbf{a} \bullet \mathbf{b}$ ” represents element-by-element multiplications of the two vectors, $\mathbf{a}$ and $\mathbf{b}$. The vector parameter

$$\mathbf{f} = [f(\theta_1), f(\theta_2), \dots, f(\theta_M)]^T, \quad (4.65)$$

is the error weights in discrete angles, $\theta_1, \theta_2, \dots, \theta_M$. $\mathbf{S}$ is the steering matrix, presented as

$$\mathbf{S} = [\mathbf{s}(\theta_1), \mathbf{s}(\theta_2), \dots, \mathbf{s}(\theta_M)]^T, \quad (4.66)$$

with $\mathbf{s}(\theta_m)$ being the steering vector at θ_m .

Usually, optimization algorithms, such as the Levenberg-Marquardt algorithm, use real-valued variables. Therefore, to solve Eq. (4.64), a complex-to-real transform of the complex weight is necessary. This transform can be simply realized by introducing a matrix,

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & \cdots & 0 & j & 0 & \cdots & 0 \\ 0 & 1 & & 0 & 0 & j & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & j \end{bmatrix}_{N \times 2N}, \quad (4.67)$$

and constructing a real weight vector as

$$\mathbf{y} = [\text{Re}(\mathbf{w}^T), \text{Im}(\mathbf{w}^T)]^T. \quad (4.68)$$

Therefore, it holds that

$$\mathbf{w} = \mathbf{T}\mathbf{y}. \quad (4.69)$$

Applying Eq. (4.69) to Eq. (4.64) leads to

$$J = \|\mathbf{f} \cdot (\|\mathbf{S}\mathbf{T}\mathbf{y} - \mathbf{p}_d\|)\|^2 = \|\mathbf{B}\mathbf{y} - \mathbf{z}\|^2, \quad (4.70)$$

where

$$\mathbf{B} = [\text{diag}(\mathbf{f})]\mathbf{S}\mathbf{T}, \quad (4.71)$$

and

$$\mathbf{z} = \mathbf{f} \bullet |\mathbf{p}_d|. \quad (4.72)$$

The gradient vector $\mathbf{g}$ and Hessian matrix $\mathbf{H}$ of J can be used to improve the performance of the optimization. A full derivation is given in Appendix A, and stated as

$$\mathbf{g} = \frac{\partial J}{\partial \mathbf{y}} = 2 \operatorname{Re} \left[\operatorname{diag}(\mathbf{x}^H) \mathbf{\Lambda}_{ev} \mathbf{B} \right], \quad (4.73)$$

and

$$\mathbf{H} = \frac{\partial^2 J}{\partial \mathbf{y}^2} = 2 \operatorname{Re} \left[\mathbf{B}^H \mathbf{\Lambda}_{ev} \mathbf{B} \right] + 2 \mathbf{K}^H \mathbf{\Lambda}_{zv} \mathbf{K}, \quad (4.74)$$

where

$$\mathbf{x} = \mathbf{B} \mathbf{y},$$

$$\mathbf{\Lambda}_{ev} = \operatorname{diag}([e_m / v_m]_{1 \times M}),$$

$$\mathbf{\Lambda}_{zv} = \operatorname{diag}([z_m / v_m^3]_{1 \times M}),$$

$$\mathbf{K} = \operatorname{Re}[\operatorname{diag}(\mathbf{x}^H) \mathbf{B}],$$

with z_m , v_m and e_m being the elements of $\mathbf{z}$, $\mathbf{v} = |\mathbf{x}|$ and $\mathbf{e} = \mathbf{v} - \mathbf{z}$, respectively.

The optimization requires a $2N$ -dimension starting point, which can be initialized as

$$\mathbf{y}_0 = [\operatorname{Re}(\mathbf{w}_0); \operatorname{Im}(\mathbf{w}_0)], \quad (4.75)$$

where $\mathbf{w}_0$ can be obtained through pseudo-inverse initialization as

$$\mathbf{w}_0 = (\mathbf{S}^H \mathbf{S})^{-1} \mathbf{S}^H \mathbf{p}_d . \quad (4.76)$$

After $\mathbf{B}$, $\mathbf{z}$, $\mathbf{g}$, $\mathbf{H}$ and $\mathbf{y}_0$ are computed, the Levenberg-Marquardt algorithm can be performed to solve Eq.(4.64), which gives an optimal real weight, $\mathbf{y}_{opt}$. The Levenberg-Marquardt algorithm has been proved an effective method to solve nonlinear least squares problems. It is more robust than the gradient descent algorithm [CBG99], which means that in many cases it finds a solution even if it starts very far off the final minimum. On the other hand, it tends to be a bit slower than the gradient descent algorithm.

Finally, the corresponding optimal complex weight can be given as

$$\mathbf{w}_{opt} = \mathbf{T} \mathbf{y}_{opt} . \quad (4.77)$$

The above algorithm is named the nonlinear least-square (NLS) method.

4.3.3 Simulations on flat-top (FT) pattern synthesis

In this section, simulations are performed to test the performance of NLS method for synthesizing FT patterns. The array elements are assumed to be isotropic and without mutual coupling.

4.3.3.1 Sidelobe control by penalty factor adjustment

This simulation illustrates the control of sidelobes by adjusting the penalty factor, α , when the pre-specified sidelobe amplitude, β is set to zero.

Consider a 20-element half-wavelength spacing linear array. A desired FT pattern is defined with the following configurations: $\theta_{p2} = -\theta_{p1} = 30^\circ$, $\theta_{s2} = -\theta_{s1} = 35^\circ$, and $\Delta\theta = 0.25^\circ$ for sampling the interested region Θ (with reference to Fig. 4.3 (b)). The NLS method is applied for different penalty factors of $\alpha = 1, 10, 100$ and 1000 . The resulting patterns are shown in Fig. 4.4 for different penalty factors.

As demonstrated in Fig. 4.4, with the increase of α , the SLL decreases, whereas the mainlobe ripple (MLR) increases. The SLL varies from -28 to -70 dB, showing that the SLL can be effectively controlled by adjusting the penalty factor, α . This point is further illustrated in Fig. 4.5 (a) and (b), which illustrates the variations of the MLR and SLL with respect to α . It is interesting to note that the decrease of the SLL is in some degree proportional to the increase of $\log(\alpha)$, whereas the MLR roughly increases from 0.4 to 1.5 dB. Specially, at $\alpha = 200$, $\text{MLR} = 1$ dB and $\text{SLL} = -55$ dB. In Fig. 4.5 (b), the MLR does not increase monotonically. It is because solving Eq.(4.64) is a nonlinear optimization and the MLR is not monotonically associated to the SLL. It is possible that sometimes both the MLR and the SLL are reducing (for example, at $\alpha = 10$). However, the MLR generally increases with the decrease of the SLL.

Based on the simulation results and analysis, it can be concluded that the SLL of a FT pattern can be controlled through adjusting the penalty factor, α .

Fig. 4.4 Simulated array patterns for a 20-element linear array, when $\alpha = 1, 10, 100$ and 1000 , respectively.

Fig. 4.5 SLL (a) and MLR (b) with respect to factor α .

4.3.3.2 NLS compared with Woodward-Lawson (WL) method

This simulation compares the NLS method with the Woodward-Lawson (WL) method [WL48], where the WL method is considered in two cases: with and without Hamming-window filtering.

Two simulation configurations (with reference to Fig. 4.3 (b)) are considered:

- Test 1: Element number $N = 20$. Desired FT pattern: $\theta_{p2} = -\theta_{p1} = 30^\circ$,

$$\theta_{s2} = -\theta_{s1} = 35^\circ, \Delta\theta = 0.25^\circ.$$

- Test 2: Element number $N = 25$. Desired FT pattern: $\theta_{p2} = -\theta_{p1} = 20^\circ$,

$$\theta_{s2} = -\theta_{s1} = 25^\circ, \Delta\theta = 0.25^\circ.$$

The factors of $\alpha = 100$ and $\beta = 0$ are used for the NLS method. The simulated array patterns using the WL and the NLS methods are shown in Fig. 4.6 (a) and (b) for Test 1 and 2, respectively. Numerical results of the MLR and SLL are listed in Table 4.1.

In Test 1, compared with the WL method (dotted line), the NLS method (solid line) achieves a 33 dB lower SLL (-18 vs. -51 dB), and a 0.4 dB lower MLR (1.2 vs. 0.9 dB). These results show that the NLS method has a better performance in sidelobe suppression than the WL method. Furthermore, compared to the WL method with Hamming window (dashed line), the NLS method obtains a 5 dB lower SLL (-46 vs. -51 dB), but a 0.8 dB lower MLR (0.9 vs. 0.1 dB). However, it is noted that the WL method with Hamming window results in an undesirable expansion of the transition region. The first null beside the mainlobe is located at $\pm 42^\circ$ as shown in Fig. 4.6 (a).

This result implies that the WL method with Hamming window is not applicable for the pattern synthesis with narrow transition requirement, say 5° transition.

Similar observations can be drawn from Test 2. The WL method without Hamming window is poor in sidelobe suppression. On the other hand, the WL method with Hamming window causes noticeable expansion of the transition region beside the mainlobe. Moreover, the WL method can only be used for uniform linear arrays. (Later simulation will show the application of the NLS method for non-uniform linear arrays.) In general, the NLS method is more suitable to FT pattern synthesis, compared with the WL method with or without Hamming window.

(a) $N = 20$, 60° -wide mainlobe, 5° -wide transition region;

(b) $N = 30$, 40° -wide mainlobe, 5° -wide transition region

Fig. 4.6 Array patterns obtained using the Woodward-Lawson with/without Hamming window and NLS methods, respectively.

Table 4.1 Numerical results obtained using the WL, WL with Hamming window, and NLS methods in Test 1 and 2. (HM represents Hamming window.)

	Test 1			Test 2		
	WL	WL+HM	NLS	WL	WL+HM	NLS
MLR(dB)	1.2	0.1	0.9	0.7	0.1	0.8
SLL(dB)	-18	-46	-51	-15	-48	-52

4.3.3.3 NLS compared with Parks-McClellan method

This simulation compares the NLS method with the Parks-McClellan method [PM72a] [PM72b] [Iee79]. Three simulation configurations (with reference to Fig. 4.3 (b)) are considered:

- Test 1: Element number $N = 20$. Desired FT pattern: $\theta_{p2} = -\theta_{p1} = 30^\circ$,

$$\theta_{s2} = -\theta_{s1} = 35^\circ , \Delta\theta = 0.25^\circ .$$

- Test 2: Element number $N = 30$. Desired FT pattern: $\theta_{p2} = -\theta_{p1} = 20^\circ$,

$$\theta_{s2} = -\theta_{s1} = 25^\circ , \Delta\theta = 0.25^\circ .$$

- Test 3: Element number $N = 40$. Desired FT pattern: $\theta_{p2} = -\theta_{p1} = 30^\circ$,

$$\theta_{s2} = -\theta_{s1} = 35^\circ , \Delta\theta = 0.25^\circ .$$

The simulated array patterns are shown in Fig. 4.7 (a) (b) and (c) for Tests 1, 2 and 3, respectively. The dotted line and solid line are for the Parks-McClellan and NLS methods, respectively. Numerical results of the MLR and SLL are listed in Table 4.2.

In Test 1, compared with the Parks-McClellan method, the NLS method obtains a 9 dB lower SLL (−23 vs. −32 dB) with almost the same MLR (0.5 dB). The sidelobes in the NLS pattern becomes lower with the angle further away from 0° . By contrast, the Parks-McClellan pattern shows equal-level sidelobes. Similar conclusions can be drawn from Tests 2 and 3. These results show that the NLS method has a better performance in sidelobe suppression than the Parks-McClellan method.

It is worth noting that the Parks-McClellan method can only be used for uniform linear arrays, whereas the NLS method can be applied for non-uniform linear arrays

(later simulations will show the application of the NLS method on non-uniform linear array). The advantage of the Parks-McClellan method is that it gives equal ripples.

(a) Test 1 results. $N = 20$, 60° -wide FT pattern synthesis;

(b) Test 2 results. $N = 30$, 40° -wide FT pattern synthesis;

(c) Test 3 results. $N = 40$, 60° -wide FT pattern synthesis.

Fig. 4.7 Array pattern synthesis using the Parks-McClellan and NLS methods.

Table 4.2 Numerical results obtained using the Parks-McClellan (PM) and NLS methods in Test 1, 2 and 3.

	Test 1		Test 2		Test 3	
	PM	NLS	PM	NLS	PM	NLS
MLR(dB)	0.6	0.5	0.2	0.2	0.1	0.1
SLL(dB)	-23	-32	-32	-33	-35	-40

4.3.3.4 NLS compared with a minimax method using semidefinite programming

In this simulation, the NLS method is compared with a recent minimax method based on semidefinite programming [Wang03].

Consider a FT pattern synthesis problem that has been discussed in [Wang03]. A 41-element non-uniform linear array has element positions of $\mathbf{d} = [0, \pm 0.3749, \pm 0.6299, \pm 1.5302, \pm 1.8494, \pm 2.3497, \pm 2.8973, \pm 3.2995, \pm 3.8098, \pm 4.6065, \pm 5, \pm 5.3749, \pm 5.6299, \pm 6.5302, \pm 6.8494, \pm 7.3497, \pm 7.8973, \pm 8.2995, \pm 8.8098, \pm 9.6065, \pm 10]\lambda$, where λ is the wavelength. The mainlobe and sidelobe regions are set to $|\theta| \leq \theta_p = 20^\circ$, and $|\theta| \geq \theta_s = 25^\circ$, respectively. Factors $\alpha = 10$ and $\beta = 0$ are used for this simulation.

By using the NLS method, an optimal array weight can be obtained, and the resulting patterns are shown in Fig. 4.8. The reference array pattern is obtained using the minimax method using semidefinite programming [Wang03].

As demonstrated by the resulting patterns, the NLS method results in a much lower SLL of -47 dB, against the -30 dB SLL obtained using the minimax method. In addition, a slightly smaller MLR of 0.36 dB is observed as compared to the MLR of 0.44 dB obtained by the minimax method. These simulation results show a better performance obtained using the proposed NLS method, compared with the minimax method. Notice that solving Eq.(4.64) is a nonlinear optimization, and the minimax method also applies a nonlinear minimax cost function that only takes account of the amplitude response [Wang03]. In this case, multiple local minima exist and the global minimum is actually unknown. Thus, the better performance of the NLS method can be explained as its better ability in approximation the global minimum, compared with

the minimax method. In addition, it is noted that in above simulation the two methods use the same initial value of the array weight.

Fig. 4.8 Simulated array patterns using the NLS method and the minimax method [Wang03], respectively.

4.3.3.5 NLS compared with adaptive array method

This simulation compares the NLS method with the adaptive array method [ZI99] for synthesizing double flat-top beams.

Consider a 41-element non-uniform linear array as discussed in [ZI99]. The array is used to synthesize a double-beam FT pattern. The desired pattern is illustrated in Fig. 4.9. Based on the array and pattern configuration with a setting of $\beta = 0$, the NLS method is applied, and a pattern is synthesized, as shown in Fig. 4.10. The reference pattern with dashed line is synthesized using the adaptive array method [ZI99]. Compared with the reference pattern, the resulting pattern obtained using the

NLS method demonstrates a 15 dB lower SLL (-48 vs. -33 dB) with nearly the same double mainlobes. Although equal-level sidelobes are not resulted, the effectiveness of sidelobe suppression using the NLS method is demonstrated.

Fig. 4.9 Desired array pattern definition.

Fig. 4.10 Simulated array patterns using the NLS and adaptive array methods.

4.3.3.6 Effect of the parameter β for sidelobe control

In previous simulations, the parameter β is set to 0 in the NLS method, representing a desired sidelobe amplitude of 0. β is useful in controlling the SLL because it is defined as the desired sidelobe amplitude. This simulation illustrates the effect of non-zero β values for synthesizing FT patterns.

Consider the same non-uniform linear array and mainlobe configuration as shown in Section 4.3.3.4. Five cases are tested with $\beta = 10^{-1}$, $10^{-1.5}$, 10^{-2} , $10^{-2.5}$ and 10^{-3} (correspondent to SLLs of -20 , -30 , -40 , -50 and -60 dB, respectively). The synthesized patterns are illustrated in Fig. 4.11. As shown in Fig. 4.11, the resulting sidelobes are very close to the pre-specified level for each of the simulation cases, especially when β is not relatively small, such as $\beta = 10^{-1}$, $10^{-1.5}$ and 10^{-2} . When β is small, such as $\beta = 10^{-2.5}$ and 10^{-3} , low SLLs of -46 and -48 dB are obtained, although these SLLs are not exactly equal to the pre-specified levels.

Numerical results of the MLR and SLL with respect to β are listed in Table 4.3. With the decrease of β , the resulting MLR increases from 0.1 to 0.4 dB, whereas the SLL goes down from -20 to -48 dB. Specifically, when $\beta = 10^{-2}$, a -39 dB SLL is obtained with a small MLR of 0.2 dB. These simulation results demonstrate that the NLS method is effective in synthesis pre-specified FT patterns. In addition, the sidelobes are shown to be more and more uneven with the decrease of β . It is because that to synthesize sidelobes with their amplitude approximating to β becomes more and more difficult when β decreases. The square error increases in this case.

Fig. 4.11 Simulation for synthesizing FT patterns with pre-specified SLLs.

Table 4.3 Numerical results of the MLR and SLL with respect to β .

β	10^{-1}	$10^{-1.5}$	10^{-2}	$10^{-2.5}$	10^{-3}
MLR(dB)	0.12	0.15	0.2	0.3	0.4
SLL(dB)	-20	-30	-39	-46	-48

In Section 4.3.3.1, it has been shown that the sidelobe can be controlled by adjusting the parameter α . A higher α emphasizes the sidelobe error weight. Thus, α can be used to control the SLL. In this section, the sidelobe control is realized by adjusting the parameter β . Since β defines the sidelobe amplitude directly, the synthesized sidelobe automatically approximates to the defined SLL in the optimization. In practice, both parameters can be used to control the SLL, in order to achieve desired array patterns.

4.3.4 Simulation on single look-direction pattern synthesis

The effectiveness of the NLS method for FT pattern synthesis has been shown in the previous section. In this section, the NLS method is applied to SLD pattern synthesis with reference to the Dolph-Chebyshev method, which is known as the optimal solution to the SLD pattern synthesis for uniform linear arrays.

Consider a 20-element uniform linear array with a half-wavelength spacing. Suppose the desired SLL is -40 dB. By using Dolph-Chebyshev method, a SLD array pattern can be obtained as shown in Fig. 4.12 with dotted line. To synthesize a similar pattern using the NLS method, the following parameter settings can be used: $\beta = 10^{-2}$, $\theta_{s2} = -\theta_{s1} = 10^\circ$, $\alpha = 1$. The resulting array pattern is demonstrated in Fig. 4.12 with sold line.

Fig. 4.12 Synthesized SLD patterns with Dolph-Chebyshev and NLS methods. $N = 20$. Desired SLL = -40 dB.

Compared with the Dolph-Chebyshev method, the NLS method has a weaker performance in terms of higher SLL for synthesizing the same mainlobe, as shown in Fig. 4.12. The resulting SLL using the NLS method is -36.6 dB, about 3.4 dB higher than that obtained using the Dolph-Chebyshev method. Similar results are obtained in two other simulation cases presented below:

- Case I: Use the same desired array pattern (desired SLL = -40 dB), but the array element is changed to $N = 31$;
- Case II: Use the same number ($N = 20$) of elements, but the desired SLL is set to -60 dB.

The resulting array patterns for the two test cases are shown in Fig. 4.13 and Fig. 4.14, respectively. Both simulations show that the SLLs obtained by the NLS method is slightly higher compared to the Dolph-Chebyshev method (-37.5 against -40 dB for Case I, and -50.7 against -60 dB for Case II).

The Dolph-Chebyshev method is shown to be more suitable than the NLS method for SLD pattern synthesis using a uniform linear array. The Dolph-Chebyshev solution is analytical, and it is actually the optimum solution to the synthesis problem for a uniform linear array. By contrast, the NLS method is based on nonlinear optimization, and the solution is not guaranteed to be the optimum. This is why a weaker performance is shown by the NLS method in Fig. 4.12, compared to the Dolph-Chebyshev method.

Fig. 4.13 Case I: Synthesized SLD patterns with Dolph-Chebyshev and NLS methods. $N = 31$. Desired SLL = -40 dB.

Fig. 4.14 Case II: Synthesized SLD patterns with Dolph-Chebyshev and NLS methods, respectively. $N = 20$. Desired SLL = -60 dB.

4.3.5 Summary of the NLS method for array pattern synthesis

From the simulation results in Section 4.3.3 on FT pattern synthesis and Section 4.3.4 on SLD pattern synthesis, the following observations can be drawn.

- The NLS method is flexible in sidelobe control. In the synthesis of FT patterns, the NLS method shows a better performance in terms of lower SLL for synthesizing the same mainlobe, compared to the Woodward-Lawson method

[WL48], Parks-McClellan method [Iee79], minimax method [Wang03] and adaptive array method [ZI99].

- For SLD pattern synthesis, however, the NLS method shows a weaker performance compared to the classical Dolph-Chebyshev method. This is shown by a higher SLL when the NLS method is used. In general, the NLS method is recommended to be used in FT pattern synthesis.

4.4 Robust nonlinear least-square (RNLS) method

The NLS method has been introduced in the previous section. It is noted that the optimization of Eq. (4.64) is a nonlinear problem requiring a multi-dimensional starting point, which makes the optimization dependent on different initializations (it will be shown by the simulation in the later Section 4.4.3.2). Therefore, it is interesting to develop a robust method that is independent on initializations. Such a method can be developed by applying the TOAM series expansion in the optimization as presented in the following.

4.4.1 RNLS solution

According to the TOAM series derivation in Section 2.3, given an N -element array and directions of interest for synthesizing a desired array pattern, a number of N TOAMs, $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$ can be obtained. The weight vector, $\mathbf{w}$ can be expressed using the TOAM series expansion as

$$\mathbf{w} = v_1 \mathbf{q}_1 + v_2 \mathbf{q}_2 + \dots + v_N \mathbf{q}_N = \mathbf{Q} \mathbf{v}, \quad (4.78)$$

where $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N]$, and $\mathbf{v} = [v_1, v_2, \dots, v_N]^T \in C^N$. The vector $\mathbf{v}$ represents the weights on $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$. We denote $\mathbf{v}$ as eigen weight in this chapter to avoid confusion with the array weight vector, $\mathbf{w}$.

Applying $\mathbf{w} = \mathbf{Q} \mathbf{v}$ to Eq. (4.64) and defining $\mathbf{A} = \mathbf{S} \mathbf{Q}$ lead to

$$J = \left\| \mathbf{f} \cdot (|\mathbf{A} \mathbf{v}| - |\mathbf{p}_d|) \right\|^2. \quad (4.79)$$

Equation (4.79) can be solved by optimizations conducted from low to high order (1 to N) using L -order TOAM series expansion as

$$\mathbf{w}^{(L)} = v_1 \mathbf{q}_1 + v_2 \mathbf{q}_2 + \dots + v_L \mathbf{q}_L = \mathbf{Q}^{(L)} \mathbf{v}^{(L)} \quad (L = 1, 2, \dots, N), \quad (4.80)$$

where

$$\mathbf{Q}^{(L)} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_L], \quad (4.81)$$

and

$$\mathbf{v}^{(L)} = [v_1, v_2, \dots, v_L]^T \in C^L. \quad (4.82)$$

Applying Eq. (4.80) to Eq. (4.61) and defining $\mathbf{A}^{(L)} = \mathbf{S} \mathbf{Q}^{(L)}$ produce

$$J^{(L)} = \left\| \mathbf{f} \cdot (|\mathbf{A}^{(L)} \mathbf{v}^{(L)}| - |\mathbf{p}_d|) \right\|^2. \quad (4.83)$$

For $L = 1, 2, \dots, N$, a series of N cost functions, $J^{(1)}, J^{(2)}, \dots, J^{(N)}$ are obtained according to Eq.(4.83). It is noted that $J^{(N)}$ is equivalent to the cost function J in Eq. (4.64) when $L = N$. Hence, solving $J^{(1)}, J^{(2)}, \dots, J^{(N)}$ sequentially will eventually produce a solution to Eq. (4.64).

In the first stage where $L = 1$, the cost function is

$$J^{(1)} = \left\| \mathbf{f} \cdot (|\mathbf{v}^{(1)} \mathbf{A}^{(1)}| - |\mathbf{p}_d|) \right\|^2, \quad (4.84)$$

where $\mathbf{A}^{(1)} = \mathbf{S}\mathbf{q}^{(1)}$. Since the dimension of $\mathbf{v}^{(1)}$ is one, it can be solved analytically, and one solution can be given as

$$\mathbf{v}_{opt}^{(1)} = \frac{|\mathbf{A}^{(1)}|^T |\mathbf{p}_d|}{|\mathbf{A}^{(1)}|^T |\mathbf{A}^{(1)}|}. \quad (4.85)$$

In the L -th stage ($L > 1$), $J^{(L)}$ can be solved using some nonlinear optimization methods, such as the Levenberg-Marquardt algorithm [CBG99], with the starting point constructed by

$$\mathbf{v}_0^{(L)} = [(\mathbf{v}_{opt}^{(L-1)})^T, 0 + 0j]^T, \quad (4.86)$$

where $\mathbf{v}_{opt}^{(L-1)}$ is the optimal solution to $J^{(L-1)}$ obtained in the $(L-1)$ -th stage, and $0 + 0j$ is the initial eigen-weight of $\mathbf{q}_L$. The optimal solution, $\mathbf{v}_{opt}^{(L)}$ can then be used to construct the starting point, $\mathbf{v}_0^{(L+1)}$ of the $(L+1)$ -th stage.

After the N -th optimization, an optimal eigen-weight vector, $\mathbf{v}_{opt} = \mathbf{v}_{opt}^{(N)}$ can be obtained, which is the solution to Eq. (4.79). Finally, the optimal array weight, $\mathbf{w}_{opt}$ to Eq. (4.64) can be constructed using Eq. (4.78), that is,

$$\mathbf{w}_{opt} = \mathbf{Q}\mathbf{v}_{opt}. \quad (4.87)$$

This solution is independent on the initial value, and thus it is more robust than the NLS method (which will be shown by simulations in later sections). Therefore, this solution is named robust nonlinear-least-square (RNLS) method for array pattern synthesis.

The overall algorithm can be summarized as below:

Step (i): Initialize $L = 1$, $\mathbf{Q}^{(1)} = \mathbf{q}_1$, and compute $\mathbf{v}_{opt}^{(1)}$ according to Eq. (4.85).

Step (ii): Update $L = L + 1$, $\mathbf{Q}^{(L)} = [\mathbf{Q}^{(L-1)}, \mathbf{q}_L]$, $\mathbf{A}^{(L)} = \mathbf{S}\mathbf{Q}^{(L)}$ and

$$\mathbf{v}_0^{(L)} = [(\mathbf{v}_{opt}^{(L-1)})^T, 0 + 0j]^T.$$

Step (iii): Solve $J^{(L)}$ in Eq. (4.83) with the starting point of $\mathbf{v}_0^{(L)}$ by implementing the

Levenberg-Marquardt algorithm. The solution is $\mathbf{v}_{opt}^{(L)}$.

Step (iv): If $L < N$, go to Step (ii), otherwise, construct the optimal weight

$$\mathbf{w}_{opt} = \mathbf{Q}\mathbf{v}_{opt}^{(N)}.$$

Note: In Step (iii), a complex-to-real transform is required. Since $J^{(L)}$ is in the same form with J in Eq. (4.79), except the dimension of the eigen weight, which is N and L in Eq. (4.79) and Eq. (4.83), respectively. A simple transform can be easily realized by introducing $\mathbf{T}^{(L)} = [\mathbf{T}]_{L \times 2L}$, and Eq. (4.83) can be written to

$$J^{(L)} = \left\| \mathbf{B}^{(L)} \mathbf{y}^{(L)} - \mathbf{z} \right\|^2, \quad (4.88)$$

where $\mathbf{B}^{(L)} = \text{diag}(\mathbf{f})\mathbf{A}^{(L)}\mathbf{T}^{(L)}$, $\mathbf{y}^{(L)} = [\text{Re}(\mathbf{v}^{(L)})^T, \text{Im}(\mathbf{v}^{(L)})^T]^T$, and the eigen weight is constructed by $\mathbf{v}^{(L)} = \mathbf{T}^{(L)}\mathbf{y}^{(L)}$. The new starting point for optimizing Eq. (4.88) is generated as $\mathbf{y}_0^{(L)} = [\text{Re}(\mathbf{v}_0^{(L)})^T, \text{Im}(\mathbf{v}_0^{(L)})^T]^T$, where $\mathbf{v}_0^{(L)}$ is obtained in Step (ii).

4.4.2 Comments on the RNLS method

The following comments on the RNLS method can be given by comparing the RNLS method with the NLS method.

- As shown in the preceding section, the RNLS method requires no multi-dimensional initial value. It only depends on the steering matrix, $\mathbf{S}$ and desired pattern, $\mathbf{p}_d$. In contrast, the NLS method is dependent on a multi-dimensional initial value, which can be, for instance, $\mathbf{w}_0$ in Eq. (4.76) based on pseudo-inverse. There may be some other available initial values, such as,

- a. $\mathbf{w}_0 = [a + bj]_{N \times 1}$, where a and b are small values;
- b. $\mathbf{w}_0 = [1 + 1j]_{N \times 1}$;
- c. $w_{0,i} = x_i + y_i j$ ($i = 1, 2, \dots, N$) with x_i and y_i being random values.

It is not easy to tell which initialization is better, because for the nonlinear least-square optimization, there are multiple local minimums, and different initializations may lead to different results.

- Compared with the NLS method which needs only one iteration of optimization, the RNLS method requires N iterations of optimization, where N is the number of array elements. It means that the RNLS method has higher computational load than the NLS method. In other words, the robustness of the RNLS method is at the cost of computational load.
- The RNLS method can be alternatively explained by “vector space” concept, where $\mathbf{q}_L$ is the L -th vector in the N -dimensional vector space. The optimal

solution is obtained after N -stage of optimization. In the L -th stage, the optimization is limited to the use of L vectors, and $\mathbf{v}_{opt}^{(L)}$ is independent from the other $N - L + 1$ vectors in the vector space. In the $(L+1)$ -th stage, since new vector, $\mathbf{q}_{L+1}$ is fed in the optimization, the square error is driven to reduce or at least remains the same. If the resulting square error reduces, it means that the new vector contributes to the optimization; otherwise, there is no contribution. In general, the more vectors are used, the smaller the square error tends to be. The reducing of the square error with respect to the number of TOAMs will be shown by simulations in Section 4.4.3.

- The RNLS method has a property of reducing array gain during the multi-stage optimization, where the gain in the L -th stage is defined as

$$g^{(L)} = \frac{E_{out}^{(L)}}{E_{in}^{(L)}}. \quad (4.89)$$

The output power $E_{out}^{(L)}$ and the input power $E_{in}^{(L)}$ can be calculated as

$$E_{out}^{(L)} = (\mathbf{p}^{(L)})^H \mathbf{p}^{(L)} = (\mathbf{S}\mathbf{Q}^{(L)}\mathbf{v}^{(L)})^H (\mathbf{S}\mathbf{Q}^{(L)}\mathbf{v}^{(L)}), \quad (4.90)$$

and

$$E_{in}^{(L)} = (\mathbf{w}^{(L)})^H \mathbf{w}^{(L)}, \quad (4.91)$$

respectively. This property is proven in Appendix B.

4.4.3 Simulations

In this section, several simulations are performed to show the performance of the RNLS method. The robustness (i.e. the independence of initial values) is

demonstrated by comparing the RNLS method with the NLS method. The array elements are assumed isotropic without mutual coupling.

4.4.3.1 Test on the square error and array gain

This simulation is to show that both the square error, $J^{(L)}$ and gain, $g^{(L)}$ reduce with respect to the index L in the RNLS method.

Consider a 30-element linear array with half-wavelength spacing. This array is used to synthesize a desired FT pattern as shown in Fig. 4.15. The mainlobe is defined in the region from -5° to 25° with a 5° transition region on both sides of the mainlobe. The distributions of the resulting gain $g^{(L)}$ (Eq. (4.89)) and square error $J^{(L)}$ (Eq. (4.83)) with respect to L are shown in Fig. 4.16 (a) and (b), respectively.

Figure 4.16 (a) denotes that $g^{(L)}$ decreases with respect to the increase of L . This observation is consistent with the theoretical analysis in the preceding section. Furthermore, it can be observed that the decreasing of the array gain becomes slower with the increase of L . It implies that the array gain tends to be stable when L is close to N .

Figure 4.16 (b) demonstrates that the square error reduces constantly during the optimization and reaches to the minimum square error when $L = N$. It implies that in each stage of the optimization, the feeding in of new TOAM, in general leads to the reduction of the square error. In addition, the optimization process can be shown by the array patterns during some middle-stage optimizations such as $L = 20, 23, 27$ and 30 , as shown in Fig. 4.17 (a), (b), (c) and (d), respectively. It can be observed that the synthesized array patterns approximate to the predefined array pattern gradually with

the increase of L . Specially, when $L = N = 30$, an optimal pattern is obtained with a 1 dB MLR and a -42 dB SLL, as shown in Fig. 4.17 (d).

Fig. 4.15 Predefined FT array pattern.

Fig. 4.16 Simulation results. (a) Gain distribution, and (b) square-error distribution, with respect to L .

Fig. 4.17 Array patterns at $L = 20, 23, 27$ and 30 , respectively.

4.4.3.2 Dependence of the NLS method on initializations

This simulation is to show the dependence of the performance of the NLS method on different initializations. Consider the following four initializations:

(a) $\mathbf{w}_0 = (\mathbf{S}^H \mathbf{S})^{-1} \mathbf{S}^H \mathbf{p}_d$,

(b) $\mathbf{w}_0 = [0.01 + 0.01j]_{N \times 1}$,

(c) $\mathbf{w}_0 = [1 + 1j]_{N \times 1}$ and

(d) $w_{0,i} = x_i + y_i j$ ($i=1,2,\dots,N$) with x_i and y_i being random values between ± 0.5 .

Assume a 41-element non-uniform linear array with element positions at $\mathbf{x} = [0, \pm 0.3749, \pm 0.6299, \pm 1.5302, \pm 1.8494, \pm 2.3497, \pm 2.8973, \pm 3.2995, \pm 3.8098, \pm 4.6065, \pm 5, \pm 5.3749, \pm 5.6299, \pm 6.5302, \pm 6.8494, \pm 7.3497, \pm 7.8973, \pm 8.2995, \pm 8.8098, \pm 9.6065, \pm 10] \lambda$, where λ is the wavelength. This array is to synthesize a FT pattern with mainlobe region at $|\theta| \leq \theta_p = 20^\circ$, sidelobe region at $|\theta| \geq \theta_s = 25^\circ$ and the rest region being the transition region. By using these settings and applying the NLS method introduced in Section 4.3, the corresponding patterns can be obtained, as shown in Fig. 4.18.

Fig. 4.18 FT pattern synthesis using the NLS method with different initializations. (NLS-a represents the NLS method with initialization (a).)

The patterns in Fig. 4.18 demonstrated that different initializations result in different patterns when the NLS method is applied. In other words, the performance of the NLS method is dependent on different initializations. Table 4.4 shows some numerical results of the simulation, including the square error (SE), SLL and MLR. It can be observed that different initializations correspond to different results. For example, the NLS-a results in a 0.57 SE, -45.0 dB SLL and 0.6 dB MLR, whereas the NLS-c leads to a 0.12 SE, -45.9 dB SLL and 0.2 dB MLR. Although NLS-c shows the best performance in terms of the smallest SE, SLL and MLR, it does not mean that this initialization is also effective for other synthesis problems.

Table 4.4 Numerical results of the square error (SE), SLL and MLR in dB.

(NLS-a represents the NLS method with initialization (a).)

	NLS-a	NLS-b	NLS-c	NLS-d
SE	0.57	0.56	0.12	0.69
SLL (dB)	-45.0	-43.7	-45.9	-42.6
MLR (dB)	0.6	0.5	0.2	0.6

4.4.3.3 Comparison Test 1

In this simulation, three methods, including the minimax [Wang03], the NLS (with initialization of $\mathbf{w}_0 = (\mathbf{S}^H \mathbf{S})^{-1} \mathbf{S}^H \mathbf{p}_d$), and the RNLS methods are compared on the same FT pattern synthesis problem (same array and desired array pattern) as presented

in the previous section. The resulting patterns obtained by different methods are shown in Fig. 4.19. Numerical results of the square error, SLL and MLR are given in Table 4.5.

From the simulation results, following observations can be drawn:

- Compared with the minimax method, the RNLS method results in a lower SLL (-47 dB) and smaller MLR (0.2 dB) at the same time. It implies that the RNLS method has a better performance than the minimax method for sidelobe suppression.
- Compared with the NLS method, the RNLS method also shows a better performance. As shown by the numerical results, the RNLS method results in a smaller SE (0.10 vs. 0.57), lower MLR (0.2 vs. 0.6 dB), and lower SLL (-47 vs. -45 dB).
- Although it is not guaranteed to have better performance than the NLS method with any possible initialization, the RNLS wins in most cases. This observation will be further shown in later simulations. The better performance of the RNLS method is due to its independence on a multi-dimensional starting point in the optimization. In contrast, the NLS method is subjected to the initializations as discussed in the simulations in the previous section.

Fig. 4.19 FT pattern synthesis using the minimax, NLS and RNLS methods

Table 4.5 Numerical results of the square error (SE), SLL and MLR in dB.

(Note: “---” means that SE is not applicable to the minimax method.)

	Minimax	NLS	RNLS
SE	---	0.57	0.10
SLL (dB)	-30.0	-45.0	-47.0
MLR (dB)	0.4	0.6	0.2

4.4.3.4 Comparison Test 2

In the preceding section, the NLS and RNLS methods are compared for a 41-element array. In this section, more non-uniform linear arrays are used for the comparison.

Consider 91 arrays with the element number, N varying from 10 to 100. Assume the element positions of each array are determined by

$$x_i = (i + \varepsilon) \cdot \lambda / 2, \quad (4.92)$$

where $i = -(N-1)/2 : 1 : (N-1)/2$, λ is the wavelength, and ε is a random number between $\pm 5\%$. The desired array pattern is a FT pattern with the mainlobe located at $|\theta| \leq \theta_p = 20^\circ$ and sidelobe region at $|\theta| \geq \theta_s = 25^\circ$. The transition region on each side of the mainlobe is 5° wide. Other parameters are: $\alpha = 10$, $\beta = 0$ and $\Delta\theta = 1^\circ$. Based on these settings, the optimal solution using the NLS and RNLS methods for each array can be obtained. The resulting square errors with respect to the element number, N are illustrated in Fig. 4.20.

The following observations can be drawn from the simulation results:

- As shown in Fig. 4.20, the square error obtained using the RNLS method decreases stably with the increase of N . In contrast, the decreasing of the square error resulted from the NLS method fluctuates wildly as N increases. It implies that the RNLS method is more robust than the NLS method.
- Although in some cases of N , for instance, $N = 30$, a smaller square error is obtained by the NLS method, the RNLS method results in smaller square errors in most cases, as shown in Fig. 4.20.

- When N is small ($N < 20$), the two methods show similar performance. This is due to the fact that all the local minima are close, and thus the influence of different initial values on the NLS optimization is small. For bigger N , especially when N is large ($N > 75$), the better performance of the RNLS method in terms of smaller square error is easy to be observed. These results imply that the two methods may have similar performance when the array is small, whereas for large arrays, the RNLS method tends to have better performance.

Fig. 4.20 Square errors with respect to the number of elements
(non-uniform linear array test)

4.4.3.5 Comparison test 3

The previous simulation is based on non-uniform arrays. It is also interesting to compare the NLS and RNLS methods in the case of uniform linear arrays. Similarly, consider 91 arrays with the element number, N varying from 10 to 100. The positions of each array can be presented by $x_i = i \cdot \lambda / 2$, with $i = -(N-1)/2 : 1 : (N-1)/2$. The desired array pattern is also the same as in the previous simulation. That is, $\theta_p = 20^\circ$, $\theta_s = 25^\circ$, $\alpha = 10$, $\beta = 0$ and $\Delta\theta = 1^\circ$. The variations of the resulting square errors correspondent to different methods are shown in Fig. 4.21.

From the simulation results, similar conclusions as in the previous section can be drawn. Firstly, the RNLS shows a more robust performance than the NLS method in terms of a more stable decreasing of the square error with respect to the increase of N , and smaller square error in most cases, as shown in Fig. 4.21.

Secondly, the smaller square errors by the RNLS method is easier to be observed when N is big, for example, $N > 40$. When $N < 40$, the two methods tend to have similar performance.

In spite of the similar conclusions, there is a noticeable difference in the two simulations. The robustness of RNLS over the NLS is observed when $N > 20$ in the previous simulation, but $N > 40$ in this simulation. It implies that in the cases of non-uniform linear arrays, the RNLS method tends to have a better performance, compared to the NLS method.

Fig. 4.21 Square errors with respect to the number of elements for uniform linear array test.

4.5 Conclusions

This chapter has presented two nonlinear least-square solutions, the NLS and RNLS methods for the problems of the array pattern synthesis. Simulations were performed to compared these two methods with some existing methods [Dol46] [WL48] [ZI99] and [Wang03]. From the simulation results, the following conclusions can be drawn:

Firstly, for SLD pattern synthesis, compared to the Dolph-Chebyshev method, the NLS and RNLS methods have a slightly poorer performance in terms of higher SLL for synthesizing the same mainlobe beam. (In the next chapter, a NLS based

iterative method will be introduced, which is able to achieve the same performance as the Dolph-Chebyshev and the adaptive array methods for SLD pattern synthesis.)

Secondly, for FT pattern synthesis, the NLS and RNLS methods are shown to be more effective in sidelobe suppression than some existing methods, such as the Woodward-Lawson method (with and without Hamming-window smoothing), the Parks-McClellan method, the adaptive array method, and the minimax method using semidefinite programming.

Thirdly, the NLS and RNLS methods are flexible in controlling the sidelobes. Both the parameters of the sidelobe penalty factor and the predefined sidelobe amplitude can be used to control the synthesized SLL.

Fourthly, compared to the NLS method, the RNLS method is shown to be more robust in terms of independence of multi-dimensional initializations. The resulting square error obtained by the RNLS method tends to be smaller than the NLS method for most synthesis cases. This advantage of the RNLS method over the NLS method is especially prominent when the array is large or non-uniform.

Finally, for FT pattern synthesis using a small array (for example, element number smaller than 20), the NLS method is preferred over the RNLS method, due to lower computational load of the NLS method but similar performance. Whereas, for a large array, the RNLS method is recommended in order to obtain a small square error.

In conclusion, the NLS and RNLS methods can be used to solve array pattern synthesis problems for either uniform or non-uniform linear arrays. The solutions are especially effective in FT pattern synthesis.

Chapter 5

A new iterative array pattern synthesis method

In this chapter, a new iterative array pattern synthesis method is proposed. It combines the NLS/RNLS method presented in Chapter 4 with a widely used peak-iteration strategy [WZ93] [ZI99] [SF05]. This new method inherits the effectiveness of sidelobe suppression of the NLS/RNLS method, and provides equal-level sidelobes. Simulations are performed to compare the performance of this method with the Dolph-Chebyshev [Dol46], minimax [Wang03], adaptive array [ZI99] and two-step least-square [SF05] methods. Both single look-direction (SLD) pattern and flat-top (FT) pattern syntheses are tested in both uniform and non-uniform linear arrays.

5.1 Overview

Many array pattern synthesis methods applicable to arbitrary linear arrays utilize peak-iteration strategy in their optimizations [TG92] [WZ93] [ZI99] [SF05]. The peaks in an array pattern refer to the maximum amplitude responses in each lobes, in other words, every lobe has one peak. The look directions associated with the peaks are chosen as constraint directions [WZ93]. In each iteration, by choosing constraint directions and putting weights on these directions, the peaks of the synthesized pattern can be controlled to approximate to desired levels. The overall pattern will be optimized after a number of iterations. It is noted that, besides peaks in an array pattern, some other directions of interest desirable to be emphasized in the

optimization can also be included in the constraint directions [SF05]. The advantage of using the peak-iteration is that the computational load is lower compared to the iterations using all directions. The effectiveness of this strategy has been testified by many works, such as [TG92], [WZ93], [ZI99] and [SF05]. These works all use the peak-iteration strategy, but differ in the formulation of cost functions and optimization methods. Section 5.2 will briefly review the formulation of these methods. As to the searching of the peaks in a pattern, some methods are available. For example, the Newton's method [GW84] is repeatedly used to locate the peaks in [WZ93]. An easier method is to compare the neighboring values in a pattern [ZI99].

In addition to the peak-iteration strategy, a two-step scheme is commonly used in various iterative methods. The first step is initialization, in which a cost function is defined, and an initial array pattern is obtained. The second step performs iterative optimizations as a post-trimming of the initial pattern. In each iteration, the cost function is iteratively updated and solved according to the constraint directions selected from the pattern synthesized in the previous iteration. The iteration can be stopped when either the array weight or synthesized pattern converges.

The above scheme implies that it is possible to develop a new iterative method by employing the NLS/RNLS method presented in Chapter 4 in the initialization step, and applying the peak-iteration strategy in the iteration step. This new method is expected to inherit the effectiveness of sidelobe suppression of the NLS/RNLS method, and at the same time, to be able to produce equal-level sidelobes.

The rest of this chapter is organized as follows. Section 5.2 briefly reviews some existing iterative methods. Section 5.3 presents the new iterative method. The performance is tested by simulations including both single look-direction (SLD)

pattern and flat-top (FT) pattern synthesis in Section 5.4. Conclusions are given in Section 5.5.

5.2 Existing iterative methods

This section briefly reviews some existing iterative array pattern synthesis methods. The purpose is to provide a background for introducing the proposed iterative method.

5.2.1 Linearly constrained iterative algorithm

Tseng and Griffiths use linearly constrained least-square criterion for array pattern synthesis, which minimizes the sidelobe power defined as [TG92]

$$\min_{\mathbf{w}} \mathbf{w}^H \mathbf{A} \mathbf{w}, \quad (5.1)$$

subject to a number of M linear constraints,

$$\mathbf{C}^H \mathbf{w} = \mathbf{f}, \quad (5.2)$$

where, $\mathbf{w}$ is complex weight vector of the array, $\mathbf{A}$ is an $N \times N$ positive-definite matrix with N being the element number ($N \geq M$), and $\mathbf{C}$ is an $N \times M$ full-rank matrix. Matrix $\mathbf{A}$ can be calculated as

$$\mathbf{A} = \mathbf{S}^H \mathbf{S}, \quad (5.3)$$

where $\mathbf{S}$ is the steering matrix. The solution to Eq. (5.1) is

$$\mathbf{w} = \mathbf{A}^{-1} \mathbf{C} (\mathbf{C}^H \mathbf{A}^{-1} \mathbf{C})^{-1} \mathbf{f}. \quad (5.4)$$

Equation (5.4) provides the solution to an initial weight, and thus an initial pattern is obtained. Afterwards, iterative steps are taken to further optimize the pattern. The updating of the array weight can be simply shown as

$$\mathbf{w} \leftarrow \mathbf{w} + \Delta \mathbf{w}, \quad (5.5)$$

where $\Delta \mathbf{w}$ is a residual weight vector. This vector can be obtained by optimizing

$$\min_{\Delta \mathbf{w}} \Delta \mathbf{w}^H \mathbf{A} \Delta \mathbf{w}, \quad (5.6)$$

subject to

$$\begin{aligned} \mathbf{v}_s^H \Delta \mathbf{w} &= 0 \\ \text{Re}(\mathbf{v}_d^H \Delta \mathbf{w}) &= 0 \\ \mathbf{v}_i^H \Delta \mathbf{w} &= f_i \text{ for } i = 1, 2, \dots, m, \end{aligned} \quad (5.7)$$

where m is the number of peaks located according to the initial pattern. Vectors, $\mathbf{v}_s$ and $\mathbf{v}_d$ are the steering vector and its derivative with respect to the direction, respectively. Vector $\mathbf{v}_i$ is the steering vector at the i -th peak direction. Scalar f_i is the associated constraint. The iteration steps are executed until a certain small error is reached, and finally result in an optimal array weight.

5.2.2 Minimum approximation error method

Wu and Zielinski proposed a very simple iterative method for an array with $2N$ elements [WZ93]. The method uses real weights and is based on a minimum approximation error criterion with the error defined as

$$J = \sum_m v_m |p(\theta_m) - p_d(\theta_m)|^2, \quad (5.8)$$

where v_m denotes weighting values on the errors associated with the selected directions. In the k -th iteration, a number of N linear equations are formulated according to the peak locations, $\theta_m^{(k-1)}$ in the synthesized pattern of the $(k-1)$ -th iteration. These equations can be given as

$$\sum_{i=1}^N w^{(k)} g_{i,n} = D_n, \quad n = 1, 2, \dots, N \quad (5.9)$$

with

$$g_{i,n} = \sum_m g_i(\theta_m^{(k-1)}) g_n(\theta_m^{(k-1)}), \quad (5.10)$$

$$D_n = \sum_m g_n(\theta_m^{(k-1)}) p_d(\theta_m^{(k-1)}) \quad (5.11)$$

and

$$g_i(\theta) = f_i(\theta) \cos(2\pi\theta d_i / \lambda), \quad (5.12)$$

where $f_i(\theta)$ is the directivity function of the i -th element, and d_i is the distance between the i -th element and the array center.

The solution to the linear equations, Eq. (5.9) results in the optimal weight of the k -th iteration. Subsequently, the $(k+1)$ -th iteration can be started. The iteration can be stopped when

$$\|\mathbf{w}^{(k+1)} - \mathbf{w}^{(k)}\| < \delta, \quad (5.13)$$

where $\mathbf{w}^{(k)}$ and $\mathbf{w}^{(k+1)}$ are the optimal array weight vectors at the k and $(k+1)$ -th iterations, respectively. The tolerance δ is a small value, which typically, can be 10^{-3} .

5.2.3 Synthesis method based on adaptive array theory

The peak-iteration was also implemented by Zhou and Ingram in their proposed array pattern synthesis method [ZI99] based on adaptive array theory [App76] [Com88].

The method utilizes a weighted error define as

$$J = \sum_{i=1}^N f(\theta_i) \left| \mathbf{s}^H(\theta_i) \mathbf{w} - p_d(\theta_i) \right|^2, \quad (5.14)$$

where N is the number of sampling angles. J represents the average output power of a “sidelobe canceller” with main channel response $p_d(\theta_i)$ to be a collection of jammers, where the i -th jammer is weighted by $f(\theta_i)$. The main idea is to adjust the jammer power to emphasize selected parts of the achieved pattern, particularly the mainlobe and sidelobe peaks. The solution to Eq. (5.14) for minimizing the error J is given as

$$\mathbf{w}_{opt} = \mathbf{R}_s^{-1} \mathbf{R}_d, \quad (5.15)$$

where $\mathbf{R}_s$ is the covariance matrix and $\mathbf{R}_d$ is the cross-correlation vector defined as

$$\mathbf{R}_s = \sum_{n=1}^N f(\theta_n) \mathbf{s}(\theta_n) \mathbf{s}^H(\theta_n), \quad (5.16)$$

and

$$\mathbf{R}_d = \sum_{n=1}^N f(\theta_n) p_d(\theta_n) \mathbf{s}(\theta_n), \quad (5.17)$$

respectively. In addition, if constraints are needed, the pattern synthesis problem is formulated as

$$\min_{\mathbf{w}} \sum_{i=1}^N f(\theta_i) \left| \mathbf{s}^H(\theta_i) \mathbf{w} - p_d(\theta_i) \right|^2, \quad (5.18)$$

subject to

$$\mathbf{C}\mathbf{w} = \mathbf{h}, \quad (5.19)$$

where $\mathbf{C}$ is the constraint matrix and $\mathbf{h}$ is the constraint vector. This constraint is similar to Eq. (5.2). For the same synthesis problem, the constraint can be the same.

The solution to Eq.(5.18) can be given as

$$\mathbf{w}_{opt} = \mathbf{R}_s^{-1}\mathbf{R}_d + \mathbf{R}_s^{-1}\mathbf{C}^H(\mathbf{C}\mathbf{R}_s^{-1}\mathbf{C}^{-1})^{-1}(\mathbf{h} - \mathbf{C}\mathbf{R}_s^{-1}\mathbf{R}_d). \quad (5.20)$$

Equation (5.20) provides an optimal solution to the minimization problem in Eq. (5.18). However, in order to obtain uniform sidelobes, iteration steps are required.

Zhou and Ingram's algorithm updates the weight vector in the $(k+1)$ -th iteration as

$$\mathbf{w}(k+1) = \mathbf{R}_s^{-1}(k+1)\mathbf{R}_d(k+1), \quad (5.21)$$

where

$$\mathbf{R}_s(k+1) = \mathbf{R}_s(k) + \tilde{\mathbf{R}}_s(k), \quad (5.22)$$

and

$$\mathbf{R}_d(k+1) = \mathbf{R}_d(k) + \tilde{\mathbf{R}}_d(k). \quad (5.23)$$

The residual covariance matrix $\tilde{\mathbf{R}}_s(k)$ and the residual cross-correlation vector $\tilde{\mathbf{R}}_d(k)$

can be calculated as

$$\tilde{\mathbf{R}}_s(k) = \sum_{n=1}^N r_k(n) \mathbf{s}(\theta_n) \mathbf{s}^H(\theta_n), \quad (5.24)$$

and

$$\tilde{\mathbf{R}}_d(k) = \sum_{n=1}^N r_k(n) p_r(\theta_n) \mathbf{s}(\theta_n), \quad (5.25)$$

respectively, where

$$r_k(n) = \begin{cases} s_k(n) & \theta_n \text{ in mainlobe,} \\ \max\{0, K_p [p_{yk}(\theta_n) - p_d(\theta_n)]\} & \theta_n \text{ in sidelobe peak,} \end{cases} \quad (5.26)$$

with

$$s_k(n) = \begin{cases} 0 & \text{if } |p_{yk}(\theta_n) - p_d(\theta_n)| < \varepsilon \\ K_m [p_{yk}(\theta_n) - p_d(\theta_n)] & \text{otherwise,} \end{cases} \quad (5.27)$$

where K_p and K_m are iteration gains set as constants [ZI99].

The above optimization algorithm is based on the adaptive array scheme [App76] [Com88], which is different from the methods presented in Sections 5.2.1 and 5.2.2. However, the same idea of using peak-iteration is utilized.

5.2.4 Two-step least-square method

Recently, Shi and Feng presented a two-step least-square method that optimizes the amplitude response without considering the phase response [SF05]. This is feasible, because in many applications of antenna arrays, there is no phase constraint on the directional response of the array.

The synthesis problem is formulated as

$$\min_{\mathbf{w}, \mathbf{p}_{dP}} \|\mathbf{p} - \mathbf{p}_d\|^2 = \min_{\mathbf{w}, \mathbf{p}_{dP}} \|\mathbf{S}^H \mathbf{w} - \mathbf{p}_{dP} \text{diag}(\mathbf{p}_{dM})\|^2, \quad (5.28)$$

where $\mathbf{S}$ is the steering matrix, $\mathbf{w}$ is the array weight, $\mathbf{p}_d$ is the desired response, $\mathbf{p}_{dM} = |\mathbf{p}_d|$ is the desired amplitude and $\mathbf{p}_{dP} = \exp(j\angle \mathbf{p}_d)$ is the phase. The two-step least-square method consists of an initial step and iteration step. In the first step, an initial weight vector $\mathbf{w}_0$ can be obtained using some classical pattern synthesis

method, such as the Capon's method [Hay91]. In the k -th iteration of the second step, an intermediate solution is obtained as

$$\mathbf{p}_{dP,k} = \mathbf{w}_{k-1}^H \mathbf{S} [\text{diag}(\mathbf{p}_{dM,k-1})]^{-1}. \quad (5.29)$$

Subsequently, project all elements of $\mathbf{p}_{dP,k}$ to the closest values on the unit circle to produce a new vector $\mathbf{p}_{dP0,k}$. Finally, the optimal weight vector of this iteration can be obtained as

$$\mathbf{w}_k = (\mathbf{S}\mathbf{S}^H)^{-1} \mathbf{S} [\text{diag}(\mathbf{p}_{dM,k})] \mathbf{p}_{dP0,k}. \quad (5.30)$$

The basic idea of this method is to update the phase response, $\mathbf{p}_{dP}$ continuously in the optimization, so that the pattern synthesis is only dependent on the amplitude approximation to a given desired pattern. The iteration stops when $\mathbf{w}_k$ and $\mathbf{w}_{k-1}$ are close enough (for example, $\|\mathbf{w}_k - \mathbf{w}_{k-1}\| < 10^{-3}$).

5.3 New method: Iterative NLS/RNLS method

In this section, we present a new iterative array pattern synthesis method. Consider an arbitrary linear array of N isotropic elements located at d_n ($n=1,2,\dots,N$) apart, where $d_1=0$ is the reference element. Assume a narrowband signal arrives at an angle of θ . The far-field pattern is given as $p(\theta) = \mathbf{s}^T(\theta) \mathbf{w}$, where $\mathbf{s}(\theta) = [1, e^{j\omega d_1 \sin \theta / c}, \dots, e^{j\omega d_N \sin \theta / c}]^T$ is the steering vector, and $\mathbf{w} = [w_1, w_2, \dots, w_N]^T$ is the complex-weight vector. According to Chapter 4, the synthesis problem can be formulated by a nonlinear cost function based on the least-square-error criterion as

$$J = \sum_{\theta \in \Theta} [f(\theta)(|\mathbf{s}^T(\theta)\mathbf{w}| - |p_d(\theta)|)]^2, \quad (5.31)$$

or

$$J = \|\mathbf{f} \cdot (|\mathbf{S}\mathbf{w}| - |\mathbf{p}_d|)\|^2, \quad (5.32)$$

in a matrix form, where the region of interest, $\Theta = \{\Theta_p, \Theta_s\}$ is a dense set of M angles evenly sampled in the range of mainlobe, $\{\Theta_p\}$ and sidelobe, $\{\Theta_s\}$. The weighting function $f(\theta) = 1$ in Θ_p , and $f(\theta) = \alpha$ in Θ_s , where α is a penalty factor for weighting sidelobe errors. The definitions of $p_d(\theta)$ for SLD and FT patterns were shown in Fig. 4.3.

In Chapter 4, the pattern synthesis problems are solved using the NLS/RNLS method. However, the sidelobes are not guaranteed to have equal level, especially when the desired sidelobe level (SLL) is very low. This problem can be solved by the iterative method presented below, which comprises two steps of the initialization and iteration of the NLS/RNLS method.

5.3.1 Initialization step

In this step, the initial array weight, $\mathbf{w}^{(0)}$ is calculated by using the NLS or RNLS method. For a given synthesis problem, a cost function as shown in Eq. (5.32) can be formulated. Suppose the solution to the cost function is $\mathbf{w}_{opt,NLS}$. The initial weight vector can be given as

$$\mathbf{w}^{(0)} = \mathbf{w}_{opt,NLS}. \quad (5.33)$$

Accordingly, the initial pattern can be obtained as

$$|\mathbf{p}^{(0)}| = |\mathbf{S}^{(0)} \mathbf{w}^{(0)}|, \quad (5.34)$$

where $\mathbf{S}^{(0)}$ is the steering matrix for the set of look directions of interest, Θ .

By performing peak searching of the initial pattern, a number of peaks, say L peaks are located at directions of $\Theta^{(0)} = \{\theta_1, \theta_2, \dots, \theta_L\}$. Correspondingly, the initial steering matrix, $\mathbf{S}^{(1)}$ can be constructed, and the desired pattern, $|\mathbf{p}_d|$ is updated to

$$|\mathbf{p}_d^{(1)}| = [|p_d(\theta_1)|, |p_d(\theta_2)|, \dots, |p_d(\theta_L)|]^T. \quad (5.35)$$

The initialization step is accomplished when $\mathbf{w}^{(0)}$, $\mathbf{S}^{(1)}$ and $|\mathbf{p}_d^{(1)}|$ are calculated, and the next step can be started.

5.3.2 Iteration step

Basically, the iteration means performing the NLS method iteratively. The optimal weight at each iteration is used as the initial weight to the next iteration.

In the first iteration, the cost function is formulated as

$$J^{(1)} = \|\mathbf{f} \cdot (|\mathbf{S}^{(1)} \mathbf{w}^{(0)}| - |\mathbf{p}_d^{(1)}|)\|^2. \quad (5.36)$$

Optimizing Eq. (5.36) leads to the optimal weight, $\mathbf{w}^{(1)}$ at the first iteration. Thus, the array pattern is obtained as $|\mathbf{p}^{(1)}| = |\mathbf{S} \mathbf{w}^{(1)}|$. Subsequently, the peak searching can be conducted to determine the constraint direction set, $\Theta^{(1)}$. Afterwards, $\mathbf{S}^{(2)}$ and $\mathbf{p}_d^{(2)}$ can be updated.

In the t -th iteration, the cost function is updated to

$$J^{(t)} = \left\| \mathbf{f} \cdot (|\mathbf{S}^{(t)} \mathbf{w}^{(t-1)}| - |\mathbf{p}_d^{(t)}|) \right\|^2, \quad (5.37)$$

which gives the optimal weight of the t -th iteration. Similarly, the peak searching and cost-function updating can be performed. The convergence of the iterations can be described by the squared difference of the patterns in two successive iterations, which is given as

$$E = \mathbf{e}^T \mathbf{e}, \quad (5.38)$$

with

$$\mathbf{e} = |\mathbf{p}^{(t)}| - |\mathbf{p}^{(t-1)}|. \quad (5.39)$$

A small error tolerance, ε can be set to control the iterations. That is, the iteration stops when

$$E < \varepsilon. \quad (5.40)$$

The overall algorithm can be presented as below.

Algorithm:

Step 1. Initialization:

Set $t = 0$.

Formulate the cost function in Eq. (5.32).

Perform the NLS or RNLS method to obtain $|\mathbf{p}^{(0)}|$, $\mathbf{w}^{(0)}$, $\mathbf{S}^{(1)}$ and $|\mathbf{p}_d^{(1)}|$.

Step 2. Iteration step:

(i) $t = t + 1$.

(ii) Use $\mathbf{w}^{(t-1)}$ as the initial weight and solve Eq. (5.37) to obtain $\mathbf{w}^{(t)}$.

(iii) Calculate array pattern $|\mathbf{p}^{(t)}| = |\mathbf{S} \mathbf{w}^{(t)}|$.

(iv) Compute the error $E = \mathbf{e}^T \mathbf{e}$ with $\mathbf{e} = |\mathbf{p}^{(t)}| - |\mathbf{p}^{(t-1)}|$, and check convergence:

If $E < \varepsilon$ or $t < T_{\max}$ ($T_{\max}$ is the maximum iteration), jump to (viii);

Else go on to (v).

- (v) Perform peak searching to find out new angle set of the peaks, $\Theta^{(t+1)}$.
- (vi) Update steering matrix $\mathbf{S}^{(t+1)}$ and desired amplitude response $|\mathbf{p}_d^{(t+1)}|$.
- (vii) Go back to (i).
- (viii) $\mathbf{w}_{opt} = \mathbf{w}^{(t-1)}$ is the optimal weight, and $|\mathbf{p}| = |\mathbf{p}^{(t)}|$ is the synthesized array pattern.

5.3.3 Comments on the algorithm

- (I) The initialization step is to find an initial weight and initial array pattern. Although various other methods can be employed for this purpose, the NLS or RNLS method is proposed in this step, due to their effectiveness in sidelobe suppression as shown in Chapter 4. A good initialization in terms of low SLL and mainlobe ripple (MLR) is important to the overall solution. This is because the iteration step works as a post-trimming of the initial array pattern. If the initial pattern has poor sidelobe suppression, the iteration step may result in an undesired SLL.
- (II) The parameter $\mathbf{f}$ in the iteration step is not necessarily to be the same with that in the initialization step. As introduced in Chapter 4, $\mathbf{f}$ contains the penalty factor α that is used to weight the approximation errors, and thus, control the patterns in the mainlobe and sidelobe regions. Factor α can be set to a different value in the iteration step in order to obtain an optimal initial pattern. For example, for flat-top pattern synthesis, factor α can be reduced if the

synthesized mainlobe is not flat enough. The setting of α can be determined by multiple trials. On the contrary, if the resulting SLL is not low enough, factor α can be increased so that more weight is put on the sidelobes. There is no need to set α for each iteration. It can be set once for the initialization step and once for the iteration step. An example will be shown in Section 5.4.5.

- (III) The proposed iterative method converges very fast. In many cases, only around 10 iterations are needed, which will be shown in simulations in the next section.
- (IV) The advantage of the iterative NLS method over the NLS method is that the iterative NLS method is able to synthesize equal-level sidelobes, and the sidelobe level is usually lower than that using the NLS method.

5.4 Simulations

In this section, several simulations are performed to test the performance of the proposed iterative NLS method, including:

- Comparison of the iterative NLS method with the Dolph-Chebyshev [Dol46] method for SLD pattern synthesis using uniform linear arrays.
- Comparison of the iterative NLS method with the adaptive array [ZI99] and two-step least-square [SF05] methods for predefined SLD pattern synthesis using a non-uniform linear array.
- Comparison of the iterative NLS method with the minimax [Wang03] and two-step least-square [SF05] methods for FT pattern synthesis using a non-uniform linear array.

- Comparison of the iterative NLS method with the Parks-McClellan method [PM72a] [PM72b] [Iee79] for FT pattern synthesis using a uniform linear array.
- Synthesis of very low sidelobe FT patterns using the iterative NLS method.

5.4.1 SLD pattern synthesis

Consider a problem of synthesizing a SLD pattern with a -42 dB SLL using an $N = 15$ element half-wavelength spacing linear array. By applying the iterative NLS method, array patterns in each iteration can be obtained. Figure 5.1 (a) to (f) show the synthesized patterns in the first 6 iterations. The standard SLD pattern synthesized using the Dolph-Chebyshev method is shown by dotted line as reference. For the simplicity of discussion, we note the standard SLD pattern as Chebyshev pattern. The convergence of the algorithm is illustrated by the error E distribution with respect to iterations, as demonstrated in Fig. 5.2. (The error E is defined in Eq. (5.38).)

(a)

(b)

(c)

(d)

(e)

(f)

Fig. 5.1 Synthesized patterns by the 15-element uniform linear array for iterations from 1 to 6, corresponding to (a)~(f). Desired SLL is -42 dB. The dotted line denotes the standard SLD pattern.

Fig. 5.2 Convergence: error E distribution with respect to the number of iterations.

Based on the simulation results, two observations can be drawn. Firstly, the synthesized pattern by the proposed method converges to the Chebyshev pattern. As shown in Fig. 5.1 (f), the pattern derived by the iterative NLS method overlaps with the standard Chebyshev pattern. This result implies that the proposed method has the same performance as the Dolph-Chebyshev method. Secondly, the convergence of the algorithm is very fast. If an error tolerance of $\varepsilon = 10^{-3}$ is set, the algorithm converges at the 6-th iteration, with a resulting error of $E = 0.0001$. This fast convergence can also be demonstrated by the error E distribution with respect to the iteration index k , as shown in Fig. 5.2. The decreasing of the square error from about 5 to 10^{-32} takes only 8 iterations.

The effectiveness of the iterative NLS method for synthesizing single look-direction patterns is also shown by two other simulations. The simulations use half-wavelength-spacing arrays with $N = 20$ and 25, respectively. The desired SLL is set to -50 dB. The resulting array patterns are demonstrated in Fig. 5.3 (a) and (b) for $N = 20$ and $N = 25$, respectively. In both simulations, it takes only 6 iterations for the patterns to converge to the -50 dB SLL. The synthesized patterns (solid line) overlap with the standard Chebyshev patterns (dotted line).

In general, the proposed method is effective for synthesizing single look-direction pattern. In other words, it has comparable performance with the Dolph-Chebyshev method [Dol46].

(a) Iteration = 6, SLL = -50 dB, $N = 20$

(b) Iteration = 6, SLL = -50 dB, $N = 25$

Fig. 5.3 Array patterns with desired SLL = -50 dB. Synthesis using the iterative NLS method (solid line) and Dolph-Chebyshev method (dotted line). Half-wavelength-spacing linear array with element number (a) $N=20$ and (b) $N=25$.

5.4.2 Single beam synthesis using a non-uniform linear array

In this section, the proposed method is applied to the synthesis of a single beam pattern with a desired mainlobe response of $|p(\theta)| = \cos^2(7\theta)$. A 21-element non-uniform linear array is used with the element locations given in Table 5.1. The synthesized patterns with desired SLL = -36 and -38.8 dB are shown in Fig. 5.4 (a) and (b), respectively. The same synthesis task has been formerly studied in [SF05] using the two-step least-square method, and in [ZI99] using adaptive array method. In Fig. 5.4, the published results in [ZI99] and [SF05] are used as references for comparison.

As shown in Fig. 5.4 (a), both of the mainlobes obtained by the proposed method and the two-step LSE method overlap with the desired mainlobe. Moreover, both methods give equal-level sidelobes at -36 dB, although the sidelobe patterns are different.

Figure 5.4 (b) shows the patterns synthesized by the proposed method and the adaptive array method. The desired SLL is -38.8 dB. Both methods obtain a similar mainlobe and equal-level sidelobes at -38.8 dB. In addition, it can be observed that the resulting two patterns do not overlap with the predefined mainlobe completely. This is due to the trade-off between mainlobe approximation and sidelobe suppression. The slightly mismatch in the mainlobe is due to the cost for producing the -38.8 dB SLL.

In general, the simulations show that the three methods have similar performance in synthesizing a single beam pattern with a predefined mainlobe shape and SLL requirement.

Table 5.1 21-element non-uniform array arrangement. λ represents the wavelength. The 21 elements are indexed from 1 to 21.

Element No.	1, 21	2, 20	3, 19	4, 18	5,17
Position	$\pm 5.0000\lambda$	$\pm 4.6065\lambda$	$\pm 3.8098\lambda$	$\pm 3.2995\lambda$	$\pm 2.8973\lambda$

Element No.	6, 16	7, 15	8, 14	9, 13	10, 12	11
Position	$\pm 2.3497\lambda$	$\pm 1.8494\lambda$	$\pm 1.5302\lambda$	$\pm 0.6299\lambda$	$\pm 0.3749\lambda$	0λ

(a) -36 dB SLL

(b) – 38.8 dB SLL

Fig. 5.4 Synthesis of a single beam pattern with desired mainlobe amplitude as $\cos^2(7\theta)$ using a 21-element non-uniform linear array. Resulting patterns: (a) –36 dB SLL obtained by the two-step method [SF05] and proposed method, respectively; (b) –38.8 dB SLL obtained by the adaptive array method [ZI99] and proposed method.

5.4.3 Flat-top pattern synthesis using a non-uniform linear array

In this section, three iterative methods: the minimax, two-step LSE and the proposed methods are compared in the flat-top pattern synthesis. Consider the 41-element non-uniform linear array that has been studied in [Wang03] and [SF05]. The positions of the elements are shown in Table 5.2. Two simulation examples are discussed. In the first example, the three methods are used to synthesize patterns with the same SLL,

and we compare the resulting MLRs. In the second example, we compare the resulting SLLs when a same MRL requirement is applied. The synthesized patterns are illustrated in Fig. 5.5 and Fig. 5.6 for the two examples, respectively.

As shown in Fig. 5.5 (a), a -30 dB SLL is obtained by the three methods. However, different MLRs are produced. The proposed method results in a very small MLR of only 0.09 dB, compared to the 0.4 dB MLRs by the minimax and two-step methods, as illustrated by the zoom-in pattern in Fig. 5.5 (b). These simulation results show that the iterative NLS method offers a lowest MLR for synthesizing a FT pattern with a given SLL among the three methods.

In the second example, the three methods produce the same MLR of about 0.4 dB, as shown in Fig. 5.6 (b). However, the proposed method provides a -54 dB SLL, which is 24 dB lower compared to the other two methods, as shown in Fig. 5.6 (a). These simulation results show that among the three methods, the iterative NLS method is able to provide a lowest SLL for synthesizing a FT pattern with a same MLR limitation.

Table 5.2 41-element non-uniform array arrangement.

Element No.	1, 41	2, 40	3, 39	4, 38	5,37	
Position	$\pm 10\lambda$	$\pm 9.6065\lambda$	$\pm 8.8098\lambda$	$\pm 8.2995\lambda$	$\pm 7.8973\lambda$	
Element No.	6, 36	7, 35	8, 34	9, 33	10, 32	
Position	$\pm 7.3497\lambda$	$\pm 6.8494\lambda$	$\pm 6.5302\lambda$	$\pm 5.6299\lambda$	$\pm 5.3749\lambda$	
Element No.	11, 31	12, 30	13, 29	14, 28	15,27	
Position	$\pm 5.0000\lambda$	$\pm 4.6065\lambda$	$\pm 3.8098\lambda$	$\pm 3.2995\lambda$	$\pm 2.8973\lambda$	
Element No.	16, 26	17, 25	18, 24	19, 23	20, 22	21
Position	$\pm 2.3497\lambda$	$\pm 1.8494\lambda$	$\pm 1.5302\lambda$	$\pm 0.6299\lambda$	$\pm 0.3749\lambda$	0λ

(a) Overall patterns

(b) Zoom-in mainlobe patterns

Fig. 5.5 Flat-top pattern synthesis using the minimax, two-step LSE and proposed methods. SLL = -30 dB. (a) Overall patterns; (b) Zoom-in mainlobe patterns. Resulting MLR = 0.4 , 0.4 and 0.09 dB obtained by the minimax, two-step LSE and proposed methods, respectively.

(a) Overall patterns

(b) Zoom-in mainlobe patterns

Fig. 5.6 Flat-top pattern synthesis using minimax, two-step LSE and proposed methods, respectively. $MLR = 0.4$ dB. (a) Overall patterns; (b) Zoom-in mainlobe patterns. Resulting SLL = -30 , -30 and -54 dB obtained by the minimax, two-step LSE and proposed methods, respectively.

5.4.4 Flat-top pattern synthesis for a uniform linear array

In this section, the proposed method is compared with the Parks-McClellan method for flat-top pattern synthesis. Consider a 30-element uniform linear array with half-wavelength element-spacing. The desired pattern has a flat-top mainlobe at $[-20^\circ, 20^\circ]$ with 5° transition region on both sides. We compare the resulting SLLs when the two methods obtain the same MLR. The synthesized patterns are illustrated in Fig. 5.7. Numerical results, SLL and MLR are listed in Table 5.3.

As shown in Fig. 5.7 and Table 5.3, the synthesized flat-top mainlobes using the two methods are very similar in terms of same beamwidth (40°), transition region (5°) and MLR (0.1 dB). Furthermore, both of the methods produce equal-level sidelobes. However, the iterative NLS method obtains 8 dB lower SLL (-45 vs. -37 dB). These results show that the proposed iterative NLS method has a better performance than the Parks-McClellan method in FT pattern synthesis. It is worth noting that in Section 4.3.3.3 the NLS method is not able to synthesize equal-level sidelobes, although it has a better performance in terms of sidelobe suppression compared with the Parks-McClellan method. Therefore, the iterative NLS method in this chapter can be regarded as an upgraded NLS method, which is effective in sidelobe suppression and in synthesizing equal-level sidelobes at the same time.

(a)

(b)

Fig. 5.7 Flat-top pattern synthesis using the Parks-McClellan and proposed methods. MLR = 0.1 dB. (a) Overall patterns; (b) Zoom-in mainlobe patterns.

Table 5.3 Resulting MLR and SLL (dB) for Flat-top pattern synthesis using the Parks-McClellan and proposed methods.

	Parks-McClellan	Iterative NLS
MLR(dB)	0.1	0.1
SLL(dB)	−37	−45

5.4.5 Flat-top pattern synthesis with very low SLL

In the preceding simulation, the proposed iterative NLS method has shown its effectiveness in sidelobe suppression. It is interesting to further test its performance for synthesizing very low SLL.

There is always a trade-off between the mainlobe approximation and sidelobe suppression. To produce a low SLL will inevitably cause the increase of the error of mainlobe approximation. For very low sidelobe synthesis, the balance of the trade-off is sometimes difficult. The difficulty lies in the finding of a “good” initial weight vector in the initialization step. Here the term “good” implies that the initial array pattern already has a small MLR and low SLL, even before the iteration step is performed. We utilize the NLS/RNLS method for initialization, because this method has been shown to be effective in sidelobe suppression, as demonstrated in Chapter 4. In the NLS/RNLS method, the parameters α and β can be individually used for controlling the sidelobes. In this iterative NLS method, these two parameters can be

used together. In the initialization and iteration steps, the parameters can be configured differently, in order to produce desired low SLL. In the initialization step, β can be set to 0, and α is used to control the weight on the sidelobe. In the iteration step, β can be set to $10^{\text{SLL}/20}$, where “SLL” is the desired SLL. α can be configured to 1 and then adjusted by a few try-and-errors. Roughly speaking, the lower the desired SLL is, the bigger α is required.

The simulation uses following settings. Mainlobe region: $|\theta| \leq \theta_p = 20^\circ$. Sidelobe region: $|\theta| \geq \theta_s = 25^\circ$. Desired SLL: -50 , -60 , -70 and -80 dB. The same 41-element non-uniform linear array as in the preceding section is used.

The numerical parameters, α and β , resulting SLLs and MLRs are listed in Table 5.4. The synthesized array patterns are shown in Fig. 5.8.

Table 5.4 Numerical parameters and results for flat-top pattern synthesis.

Desired SLL = -50 , -60 , -70 and -80 dB, respectively.

Desired SLL (dB)			-50	-60	-70	-80
Parameters	Initialization	α	0.5	100	100	200
		β	0	0	0	0
	Iteration	α	0.5	1	5	5
		β	$10^{-2.5}$	10^{-3}	$10^{-3.5}$	10^{-4}
Results	SLL (dB)		49.9	-59.7	-68.8	-78.6
	MLR (dB)		0.15	0.53	0.67	1.50

(a)

(b)

(c)

(d)

Fig. 5.8 Synthesized flat-top patterns using the iterative NLS method, with desired SLLs of (a) -50 dB; (b) -60 dB; (c) -70 dB; and (d) -80 dB.

The following observations can be drawn from the simulation results shown in Table 5.4 and Fig. 5.8:

- (I) Flat-top patterns with desired low SLLs are successfully synthesized. The patterns in Fig. 5.8 (a) to (d) show SLLs of -49.9 , -59.7 , -68.8 and -78.6 dB, respectively, which are very close to the predefined SLLs of -50 , -60 , -70 and -80 dB, respectively. However, if equal ripple of mainlobe is an important concern for choosing synthesis method, the iterative NLS method may not be so effective. For example, in Fig. 5.8 (a) to (d), the mainlobe does not show equal ripples.
- (II) The resulting flat-top patterns have small MLRs. For example, when $\text{SLL} = -49.9$ dB, the MLR is only 0.09 dB; and when $\text{SLL} = -68.8$ dB, the MLR is 0.67 dB.
- (III) The iterative NLS method provides equal-level sidelobes, as shown in Fig. 5.8 (a) to (d). It is noted that this property is not shown by the NLS or RNLS methods introduced in Chapter 4. This advantage is due to the iteration step involved in the optimization, which works as a post-trimming of the initial array pattern, and thus produces equal-level sidelobes.
- (IV) The trade-off between the MLR and SLL is revealed in this simulation. With the reduction of the SLL from -49.9 to -78.6 , the MLR increases from 0.09 dB to 1.50 dB. Therefore, the cost of the sidelobe suppression is the increase of the MLR.

In general, this simulation shows that the proposed iterative NLS method is effective in synthesizing flat-top patterns with very low sidelobes.

5.5 Conclusions

A new iterative APS method has been successfully developed, which combines the peak-iteration strategy with the NLS/RNLS method presented in Chapter 4. The new method uses a two-step scheme. In the initiation step, the NLS or RNLS method is used to obtain an initial array pattern. The initial pattern is further optimized in the second step by using iterative nonlinear optimizations. The iteration is stopped when a small synthesis error or a certain number of iteration is reached.

The similarity among the proposed method and existing iterative synthesis methods [WZ93] [ZI99] [Wang03] [SF05] is that they all use the two-step scheme and the peak-iteration strategy. Moreover, these methods all provide equal-level sidelobes. The difference is that the proposed method utilizes the nonlinear cost function, and applies the NLS method iteratively to derive the optimal solution.

Simulation results have shown the effectiveness of the iterative NLS method in both SLD and FT pattern synthesis for arbitrary linear arrays. For SLD pattern synthesis, the proposed method has the same performance with the Dolph-Chebyshev [Dol46], adaptive array [ZI99] and two-step least-square [SF05] methods. For FT pattern synthesis, the proposed method shows a better performance in terms of lower SLL with similar mainlobe shaping, compared with the minimax [Wang03], the adaptive array [ZI99], the two-step least-square [SF05] and the Parks-McClellan methods.

Chapter 6

Conclusions and future work

6.1 Conclusions

The research background, motivation and objective of the thesis have been introduced. The research topic is mainly focused on the desired array pattern synthesis. A general analysis technique, namely, the target-oriented array-mode (TOAM) analysis has been presented for modeling and analyzing a general array system with predefined spatial target region or directions of interest. Mainly, three applications using the TOAM analysis are developed, including the acoustical-hotspot generation using maximum-gain TOAM, quiet-zone generation using lower-gain TOAM and the synthesis of single look-direction and shaped patterns using TOAM series expansion. In conclusion, the main contributions of the thesis are presented as follows.

The TOAM technique has been developed as a general tool for studying the desired array pattern synthesis problems, as presented in Chapter 2. This tool models a general array system as a multi-input multi-output (MIMO) system by regarding the complex array weight as the multi-input, and responses at the discrete samples of a target region or directions of interest as the multi-output. A gain function defined as the ratio of the power at the discrete samples to the power of the array weight is employed to describe the energy-transmission efficiency of the array system. Since this function is in the form of a Rayleigh quotient [Str03], eigen-analysis can be used to derive a series of basic weight vectors of the array, which are defined as target-oriented array-mode (TOAM). The TOAM series has some interesting properties.

Firstly, the maximum eigenvalue denotes the maximum gain of the system, and the associated TOAM can be chosen as the optimal array weight for maximum-gain pattern synthesis. Secondly, lower-gain TOAMs correspond to lower energy received at the target region, and thus can be applied in minimum-gain pattern synthesis. Thirdly, the TOAM series can be used to express an array weight as a series expansion. With the increase of the length of the series, the square error between a synthesized pattern and a desired pattern decreases. This property enables the application of the TOAM series expansion in the synthesis of pre-defined array patterns.

The generation of acoustical-hotspot as an example of maximum-gain pattern synthesis has been investigated in Chapter 3. The maximum control-gain (MCG) method uses the maximum-gain TOAM as the optimal array weight [WYG03] [WYG05]. This method is compared with a widely used time-delay (TD) method [Mai94] [1Ltd05]. A theoretical analysis has shown that a better performance in terms of higher gain is always obtainable by the MCG method over the TD method. The MCG solution can be regarded as a multiplication of the TD solution with an amplitude weighting. This weighting enhances the acoustical energy in the target region, and thus contributes to the better performance of the MCG method. However, the TD method has the advantage of lower computational load and easy implementation. Our research has shown that the choice between the two methods is largely dependent on the acoustical field, where the sources and the target region are located. In the free field, the sound field is solely determined by the direct sound from the sources. As the target region is further apart from the source array, these two methods converge to a similar performance. Therefore, the TD method can be chosen in the far field for its simplicity and comparable performance; whereas the MCG

method is preferred in the near field because of its better performance. In an enclosed environment, such as a rectangular room, the sound field becomes complicated due to the existence of reverberant sound and direct sound at the same time. For a given array, target region and enclosure arrangement, the sound field can be considered in near-field and far-field cases according to the critical distance. In the near field, the direct sound dominates, whereas in the far field, the reverberant sound prevails. As a result, the TD method is still applicable in near field, but no longer effective in the far field. In contrast, the MCG method can be used in both cases.

In Chapter 3, the quiet-zone generation is used as an example of minimum-gain pattern synthesis. Two cases of quiet-zone generation are discussed. In the first case, a source array is used to generate a quiet zone directly with a given input power of the array. The minimum-gain TOAM can be readily chosen as the solution. In the second case, a source array is utilized as secondary sources to cancel the acoustical energy in a target region due to the primary source. An existing analytical method solves this problem by minimizing the cost function of the output power of the sensors or measuring points in the target region [Ell01] [NCE92]. However, since this solution requires matrix inversion, a potential problem is that the performance may be affected when the correlation matrix of the transfer impedance from the secondary sources to the target region is ill-conditioned. A new approach is therefore proposed. The optimal solution is derived from some lower-gain TOAMs by taking iterative steps starting from the lowest-gain TOAM. Simulation results have shown that the new solution has a better performance when the correlation matrix is ill-conditioned.

The synthesis of single look-direction pattern and flat-top pattern has been investigated in Chapter 4. Two solutions have been proposed. The first solution is known as the nonlinear least-square (NLS) method. This method uses a nonlinear cost

function of the square amplitude difference between the synthesized pattern and the desired pattern. The cost function is solved by using the Levenberg-Marquardt algorithm [CBG99]. Under this NLS method, the sidelobe level (SLL) can be flexibly controlled by changing parameters of either the sidelobe weight or the predefined amplitude value. This method is applicable for arbitrary linear array pattern synthesis. It is especially effective in synthesizing flat-top patterns in terms of better sidelobe suppression, compared with some existing methods, such as the Woodward-Lawson [WL48], minimax [Wang03] and adaptive array [ZI99] methods. The second solution is known as the robust nonlinear least-square (RNLS) method. This method takes advantage of the TOAM series expansion, and solves the nonlinear problem with multi-stage optimizations corresponding to different lengths of series used. Since the solution to the first stage optimization is analytical, and each higher stage optimization is initialized with the optimal solution in the previous stage, the overall optimization requires no multi-dimensional starting point. In contrast, the NLS method is dependent on initial values. Consequently, the RNLS method is more robust than the NLS method. Simulation results have shown that the robustness of the RNLS method over the NLS method is especially prominent when the array size is large (20 elements and above). For a small array, the two methods tend to have similar performance. It is noted that the robustness of the RNLS method is in the cost of higher computational cost compared to the NLS method. Therefore, for a small array (elements fewer than 20), the NLS method is preferred; otherwise, the RNLS method is recommended.

In Chapter 5, the NLS/RNLS method is further developed to become an iterative APS method by applying a peak-iteration strategy. Although this strategy requires multiple optimizations iteratively, the computational load is effectively

reduced by operating the optimizations on the peaks of the synthesized array pattern in each iteration. The proposed iterative method has shown to be effective in both single look-direction and flat-top pattern synthesis for arbitrary linear arrays. For single look-direction pattern synthesis, this proposed iterative method has the same performance with the Dolph-Chebyshev method [Dol46] for a uniform linear array. Furthermore, the proposed method has comparable performance with the adaptive array [ZI99] and two-step least-square-error [SF05] methods in synthesizing a single look-direction pattern for a non-uniform linear array. For flat-top pattern synthesis, the proposed method shows a better performance in terms of better sidelobe suppression, compared with the minimax [Wang03], adaptive array and two-step least-square-error methods.

6.2 Future work

This section presents several investigations that are worthy to be pursued for further research based on the developed theories and algorithms in the thesis.

In the acoustical-hotspot generation, the proposed MCG method assumes that all sources are point sources with omni-directional radiation pattern. This assumption is for the simplicity of theoretical analysis. However, in practice, this assumption may cause inaccuracy in the optimal solution. The radiation pattern of each source has to be taken into account. This problem can be solved by multiplying a directional radiation pattern with the transfer function from a source to a measuring point.

In the application of the MCG method for acoustical-hotspot generation in an enclosure, the Green's function is used as the transfer function. It is also possible to

use a transfer function based on the measurement. The implementation of the measured transfer function may lead to interesting audio/acoustical applications. In addition, the time reversal technique [Fin92][WTF92][CF92][Fin97] can be used in focusing sound in one point [DF97][YTF03][YTF03a]. An examination of the theoretical link between the time reversal solution and the MCG method can provide a useful research topic.

In the synthesis of single look-direction pattern and shaped pattern, the proposed nonlinear least-square solutions (including the NLS, RNLS and iterative NLS methods) have shown their effectiveness for synthesizing desired array patterns. However, the method is introduced without considering the coupling effect. In the future work, the proposed methods can be extended to consider the coupling effect among array elements.

The iterative NLS method has shown good performance in synthesizing shaped array patterns. In the future, this method can be extended to the applications of acoustical arrays for generating desired sound field distribution in a target region.

Author's publications

Journal papers:

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Appendix A

This appendix derives the gradient vector and Hessian matrix of the cost function,

$J = \|\mathbf{By} - \mathbf{z}\|^2$, which can be alternatively shown by $J = \mathbf{e}^T \mathbf{e} = \mathbf{v}^T \mathbf{v} - 2\mathbf{v}^T \mathbf{z} + \mathbf{z}^T \mathbf{z}$ with $\mathbf{e} = \mathbf{By} - \mathbf{z}$ and $\mathbf{v} = \mathbf{By}$.

Define $\mathbf{x} = \mathbf{By}$, $\mathbf{v} = \mathbf{x}$, and assume that x_m , y_j , z_m , v_m , e_m and B_{mn} are the elements of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{v}$, $\mathbf{e}$ and $\mathbf{B}$, respectively. Thus, $x_m = \sum_{j=1}^{2N} B_{mj} y_j$, $v_m = \sqrt{x_m^* x_m}$ and $e_m = v_m - z_m$. Since $\partial x_m / \partial y_n = B_{mn}$ and $\partial x_m^* / \partial y_n = B_{mn}^*$, supposing $v_m \neq 0$, we get

$$\frac{\partial v_m}{\partial y_n} = \frac{1}{2v_m} \cdot \left(x_m^* \frac{\partial x_m}{\partial y_n} + x_m \frac{\partial x_m^*}{\partial y_n} \right) = \frac{1}{v_m} \cdot \text{Re} \left[x_m^* B_{mn} \right]. \quad (\text{A1})$$

Therefore,

$$\begin{aligned} \frac{\partial J}{\partial y_n} &= \frac{\partial(\mathbf{v}^T \mathbf{v})}{\partial y_n} - 2 \frac{\partial(\mathbf{v}^T \mathbf{z})}{\partial y_n} = 2 \sum_{m=1}^M v_m \frac{\partial v_m}{\partial y_n} - 2 \sum_{m=1}^M z_m \frac{\partial v_m}{\partial y_n} \\ &= 2 \sum_{m=1}^M \frac{e_m}{v_m} \cdot \text{Re} \left[x_m^* B_{mn} \right], \end{aligned} \quad (\text{A2})$$

and

$$\begin{aligned} \frac{\partial^2 J}{\partial y_n \partial y_l} &= \frac{\partial^2(\mathbf{v}^T \mathbf{v})}{\partial y_n \partial y_l} - 2 \frac{\partial^2(\mathbf{v}^T \mathbf{z})}{\partial y_n \partial y_l} \\ &= \sum_{m=1}^M \left(\frac{\partial x_m^*}{\partial y_l} B_{mn} + \frac{\partial x_m}{\partial y_l} B_{mn}^* \right) - 2 \left(- \sum_{m=1}^M \frac{z_m}{2v_m^2} \cdot (x_m^* B_{mn} + x_m B_{mn}^*) \cdot \frac{\partial v_m}{\partial y_l} \right. \\ &\quad \left. + \sum_{m=1}^M \frac{z_m}{2v_m} \cdot \left(\left(\frac{\partial x_m}{\partial y_l} \right)^* B_{mn} + \frac{\partial x_m}{\partial y_l} B_{mn}^* \right) \right) \\ &= 2 \sum_{m=1}^M \frac{e_m}{v_m} \cdot \text{Re} \left[B_{ml}^* B_{mn} \right] + 2 \sum_{m=1}^M \frac{z_m}{v_m^3} \cdot \text{Re} \left[x_m^* B_{mn} \right] \cdot \text{Re} \left[x_m^* B_{ml} \right]. \end{aligned} \quad (\text{A3})$$

From Eq. (A2), the gradient vector $\mathbf{g} = \partial J / \partial \mathbf{y}$ can be presented as

$$\mathbf{g} = 2 \operatorname{Re} \left[\operatorname{diag}(\mathbf{x}^H) \mathbf{\Lambda}_{ev} \mathbf{B} \right], \quad (\text{A4})$$

where $\mathbf{\Lambda}_{ev} = \operatorname{diag}([e_m / v_m]_{1 \times M})$. From Eq. (A3), the Hessian matrix $\mathbf{H} = \partial^2 J / \partial \mathbf{y}^2$ can be obtained as

$$\mathbf{H} = 2 \operatorname{Re} \left[\mathbf{B}^H \mathbf{\Lambda}_{ev} \mathbf{B} \right] + 2 \mathbf{K}^H \mathbf{\Lambda}_{zv} \mathbf{K}, \quad (\text{A5})$$

where $\mathbf{\Lambda}_{zv} = \operatorname{diag}([z_m / v_m^3]_{1 \times M})$ and $\mathbf{K} = \operatorname{Re}[\operatorname{diag}(\mathbf{x}^H) \mathbf{B}]$. Note that Eqs. (A4) and (A5) are based on matrix operations, making the computation easier in Matlab.

It is noted that $\partial J / \partial \mathbf{y}$ is not continuous at $v_m = 0$, where $\partial^2 J / \partial \mathbf{y}^2$ does not exist. In this case, we can simply use the values of the derivatives in the previous iteration instead. This approximation, according to our simulations, does not influence the convergence of the optimization. This is due to three reasons. Firstly, $v_m = 0$ seldom happens. Secondly, the optimization uses multiple iterations, and thus some occasional approximation does not change the overall convergence. Thirdly, the derivatives are used to improve the convergence of the Levenberg-Marquardt algorithm [CBG99], but not absolutely necessary.

Appendix B

This appendix is to prove the statement that when the $(L+1)$ -th stage optimization starts, using $\mathbf{q}_{L+1}$ will cause the array gain to decrease.

The array gain at L -th stage is defined as $g^{(L)} = E_{out}^{(L)} / E_{in}^{(L)}$, where the input power of the array $E_{in}^{(L)} = (\mathbf{w}^{(L)})^H \mathbf{w}^{(L)}$, and the output power of the pattern $E_{out}^{(L)} = (\mathbf{p}^{(L)})^H \mathbf{p}^{(L)}$. Assume that the output power always converges to $E_{out} \approx \mathbf{p}_d^H \mathbf{p}_d$ during the multi-stage optimization, where $\mathbf{p}_d$ is the desired pattern. Under this assumption, an equivalent statement can be given that the gain function decreases when the eigen-weight v_{L+1} for $\mathbf{q}_{L+1}$ is introduced in the stage of $L+1$. In other words, the derivative $\partial g^{(L+1)} / \partial (v_{L+1}^* v_{L+1}) \leq 0$.

At the moment when the $(L+1)$ -th stage is started, suppose v_{L+1} is changed from the initial value of 0 to a small number v . The calculation of $\partial g / \partial (v_L^* v_L)$ can be given as

$$\frac{\partial g^{(L+1)}}{\partial (v_{L+1}^* v_{L+1})} \approx \frac{\Delta g^{(L+1)}}{\Delta (v_{L+1}^* v_{L+1})} = \frac{g^{(L+1)} - g^{(L)}}{v^* v}, \quad (\text{B1})$$

where $g^{(L+1)} = E_{out}^{(L+1)} / E_{in}^{(L+1)}$. Calculation of $E_{in}^{(L)}$, $E_{out}^{(L)}$, $E_{in}^{(L+1)}$ and $E_{out}^{(L+1)}$ can be given as

$$\begin{aligned} E_{in}^{(L)} &= (\mathbf{w}^{(L)})^H \mathbf{w}^{(L)} \\ &= (\mathbf{Q}^{(L)} \mathbf{v}^{(L)})^H (\mathbf{Q}^{(L)} \mathbf{v}^{(L)}) \\ &= (\mathbf{v}^{(L)})^H (\mathbf{Q}_L^H \mathbf{Q}_L) \mathbf{v}^{(L)} \\ &= (\mathbf{v}^{(L)})^H \mathbf{v}^{(L)} \\ &= \sum_{i=1}^L v_i^* v_i, \end{aligned} \quad (\text{B2})$$

$$\begin{aligned}
 E_{out}^{(L)} &= (\mathbf{p}^{(L)})^H \mathbf{p}^{(L)} \\
 &= (\mathbf{S}\mathbf{Q}^{(L)}\mathbf{v}^{(L)})^H (\mathbf{S}\mathbf{Q}^{(L)}\mathbf{v}^{(L)}) \\
 &= (\mathbf{v}^{(L)})^H (\mathbf{Q}^{(L)})^H (\mathbf{S}^H \mathbf{S}) \mathbf{Q}^{(L)} \mathbf{v}^{(L)} \\
 &= (\mathbf{v}^{(L)})^H (\mathbf{Q}^{(L)})^H (\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H) \mathbf{Q}^{(L)} \mathbf{v}^{(L)} \\
 &= (\mathbf{v}^{(L)})^H \mathbf{\Lambda}^{(L)} \mathbf{v}^{(L)} \\
 &= \sum_{i=1}^L \lambda_i v_i^* v_i,
 \end{aligned} \tag{B3}$$

$$E_{in}^{(L+1)} = (\mathbf{w}^{(L+1)})^H \mathbf{w}^{(L+1)} = (\mathbf{v}^{(L+1)})^H \mathbf{v}^{(L+1)} = v^* v + \sum_{i=1}^L v_i^* v_i, \tag{B4}$$

and

$$E_{out}^{(L+1)} = (\mathbf{p}^{(L+1)})^H \mathbf{p}^{(L+1)} = \lambda_{L+1} v^* v + \sum_{i=1}^L \lambda_i v_i^* v_i. \tag{B5}$$

Then $g^{(L+1)} - g^{(L)}$ can be calculated as

$$\begin{aligned}
 g^{(L+1)} - g^{(L)} &= \frac{E_{out}^{(L+1)}}{E_{in}^{(L+1)}} - \frac{E_{out}^{(L)}}{E_{in}^{(L)}} \\
 &= \frac{\lambda_{L+1} v^* v + \sum_{i=1}^L \lambda_i v_i^* v_i}{v^* v + \sum_{i=1}^L v_i^* v_i} - \frac{\sum_{i=1}^L \lambda_i v_i^* v_i}{\sum_{i=1}^L v_i^* v_i} \\
 &= \frac{v^* v \sum_{i=1}^L (\lambda_{L+1} - \lambda_i) v_i^* v_i}{\left(v^* v + \sum_{i=1}^L v_i^* v_i \right) \sum_{i=1}^L v_i^* v_i}.
 \end{aligned} \tag{B6}$$

Applying Eq. (B6) to (B1) leads to

$$\frac{\partial g^{(L+1)}}{\partial (v_{L+1}^* v_{L+1})} \approx \frac{\sum_{i=1}^L (\lambda_{L+1} - \lambda_i) v_i^* v_i}{\left(v^* v + \sum_{i=1}^L v_i^* v_i \right) \sum_{i=1}^L v_i^* v_i} \leq 0. \quad (\lambda_{L+1} \leq \lambda_L \leq \dots \leq \lambda_1). \tag{B7}$$

Thus, the statement is proven.