

Optimal policies for inventory models with multiple demand classes

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

**OPTIMAL POLICIES FOR INVENTORY
MODELS WITH MULTIPLE DEMAND CLASSES**

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NANYANG BUSINESS SCHOOL

2010

Optimal Policies for Inventory Models with Multiple Demand Classes

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Abstract

This dissertation consists of three essays that address issues in inventory management. We focus on the structural results, in particular on the structures of optimal policy for inventory systems with multiple demand classes.

In the first essay, we consider a finite horizon periodic review, single product inventory system, with a fixed setup cost and two stochastic demand classes that differ in their backordering costs. In each period, one must decide whether and how much to order, and how much demand from the lower class should be satisfied. We show that the optimal ordering policy can be characterized as a state dependent (s, S) policy, and we partially obtain the rationing structure based on the sub-convexity of the cost function. We then propose a simple heuristic rationing policy, which is easy to implement and close to optimal for a majority of our numerical examples. We study in more depth the case when the first demand class is deterministic and must be satisfied immediately. We show the optimality of the state dependent (s, S) ordering policy, and obtain additional rationing structural properties. Based on these properties, the optimal ordering and rationing policy for any state can be generated by finding the optimal policy of a finite set of states. For each state in this set, the optimal policy is obtained by simply choosing a policy from at most two alternatives. An efficient algorithm is then proposed.

In the second essay, we first consider a periodic review inventory system with a fixed setup cost and two demand classes: deterministic and stochastic, where the deterministic

demand must be satisfied immediately and the stochastic demand can be backlogged. Assuming that the stochastic demand is never backlogged if there is stock in the system, a modified (s, S) policy was proved optimal under certain conditions in a previous paper. The objective is to weaken one of the conditions in the literature while still obtaining the optimality of the (s, S) policy. We first present two properties that are each equivalent to the optimality of the (s, S) policy to the problem. These properties are instrumental in identifying the (s, S) - policy optimality conditions. We then propose one such sufficient condition that is weaker than that contained in the literature. As an application of this relaxation, we also study an inventory system where the stochastic demand is price sensitive and, thus, pricing and inventory decisions are made simultaneously. The relaxed condition above enables us to demonstrate that a modified (s, S, p) policy is optimal for additive demand functions and that a modified (s, S, A, p) policy is optimal for general demand functions.

In the third essay, we consider stock rationing of a single-item make-to-stock production/inventory system. Demand is classified into multiple classes. Each class of demand arrives as a Poisson process with a randomly distributed batch size. It is assumed that the batch demand can be partially satisfied. Production time follows an exponential distribution. The problem is formulated as a continuous-time Markov decision process (MDP). Both the lost-sales case and the backordering case are considered. In the backordering model, we further assume that the facility can produce a batch up to a certain capacity at the same time. For both models, we show that the optimal policy is characterized by a sequence of monotone stock rationing levels.

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Table of Contents

Abstract	I
Acknowledgements	III
Table of Contents	V
List of Tables	VIII
List of Figures	IX
1. Introduction	1
1.1 Introduction	1
1.2 Literature review	4
1.2.1 Inventory control with fixed ordering costs	4
1.2.2 Inventory rationing policy	6
1.2.3 Joint pricing and inventory control	8
1.3 Contributions	10
1.4 Organization	12
2. Optimal ordering/rationing policy for a periodic review system with two demand classes and backordering	14
2.1 Model formulation	15
2.2 Structure of the optimal policy for the basic model N/S/S	17
2.3 Structure of the optimal policy for the N/D/S model	33
2.4 Algorithm and heuristic	44
2.5 Extensions	49

2.6 Conclusions	53
2.7 Appendix	55
3. Inventory control and pricing strategies with two demand classes	58
3.1. Introduction	58
3.2. Inventory policies for systems with stochastic and deterministic demand.....	60
3.2.2. Model formulation	60
3.2.3 Properties equivalent to the optimality of the (s, S) policy	62
3.2.4. The optimal policy	64
3.2.5. The model with the tighter constraint.....	68
3.3. Joint pricing and inventory control with two demand classes.....	72
3.3.1. Introduction.....	72
3.3.2. Problem formulation	73
3.3.3. Additive demand functions	76
3.3.4. General demand functions.....	81
3.4. Conclusions.....	87
4. Optimal production and rationing policy of a make-to-stock production system with batch demand.....	89
4.1 Introduction	89
4.2 Lost-sales model	92
4.2.1 Model formulation	92
4.2.2 The optimal policy	94
4.2.3 Numerical studies.....	99

4.3 Backordering model.....	102
4.3.1 Model formulation	102
4.3.2 The optimal policy	106
4.3.3 Discussion and extensions.....	126
4.4 Conclusions.....	130
5. Conclusions and future research	131
Bibliography	133

List of Tables

Table 1.1 Common features among the three essays and closely related literature.....	12
Table 2.1 Example of the S/S/S model.....	31
Table 2.2 Example of the S/D/S model.....	43
Table 2.3 Algorithm (a) Distributions: I and II.....	48
Table 2.3 Algorithm (b) Distributions: III and IV.....	49
Table 2.4 Example of the S/S/S model with $LT=1$	50
Table 2.5 Example of the S/D/S model with $LT=1$	52
Table 3.1 A counterexample without additional condition.....	63
Table 4.1 MDP models for stock rationing problems.....	91
Table 4.2 Cost reduction for the lost-sales case with two demand classes.....	101
Table 4.3(a) Expected cost for the partially accepted case.....	127
Table 4.3(b) Expected cost for the all-or-none accepted case.....	128

List of Figures

Figure 2.1 The optimal structure for the N/S/S model.....	26
Figure 2.2 Proof of Lemma 2.9. Case I: $y \geq x$	36
Figure 2.3 Proof of Lemma 2.9. Case II: $0 < y < x$	37
Figure 2.4 The optimal structure for the N/D/S model.....	45
Figure 3.1 Sym- K -convex functions.....	82
Figure 4.1 Numerical Example 1, the lost-sales case.....	100
Figure 4.2 Graphical representation for Lemma 4.7.....	113
Figure 4.3 Graphical representation for Lemma 4.8.....	116
Figure 4.4 Graphical representation for Lemma 4.9.....	118

1. Introduction

1.1 Introduction

It is evident from current literature related to inventory control models, that the majority of existing approaches assign a uniform importance to the demand for a given item. In practice however, demand for an item can typically be subdivided into several different categories; each ranked in terms of their importance to a given entity/firm. For instance, demand from a customer may involve the customer's economic value to the firm, the volume of goods it purchases, at what price, and perhaps the contracts to which the customer commits. Thus, the demand for a single item can be seen to be generally heterogeneous, with possibly each subcategory being given a different importance.

The existence of heterogeneous demand is a common phenomenon in a wide range of industries. For example, in a service parts network, a customer can choose from among different contracts, each with a different cost and level of service (Arslan, Graves, & Roemer [2]). A "gold contract" might provide a 99% fill rate within twenty-four hours, while a "bronze contract" promises an 85% fill rate within two days. In other settings, a supplier segments its customers based on the delivery channel or the price they pay: the supplier recognizes some customers as deserving higher priority over other customers.

Another example is from an assemble-to-order system that has been previously discussed by Benjaafar & Mohsen [3], and Cheng, Ettl, Lu, & Yao [13], where IBM has recently restructured the manufacturing of its server computers into two stages. At the first stage, components with long manufacturing lead-times are built in a make-to-stock fashion.

While at the second stage, these components are assembled to order in response to customer demand. The demand for the servers arises from diverse corporate customers who can have different fulfillment time requirements and different penalties for delays.

Varying scenarios of customer heterogeneity can be found in rental businesses, or the airline and hotel industry. In the airline industry, the seats on an airplane are usually classified into different classes, such as First Class, Business Class, and Economy Class. They are sold to different customers according to their revenue contribution. Many rental companies separate their customer base into two groups (Noah & Sergei [40]). The first contract group consists of customers whose rentals are regulated by pre-negotiated contracts, which usually specify a fixed rental fee, as well as certain service obligations. The second customer group consists of walk-in customers, to whom a rental company has no long-term contractual obligations. Typically, these customers “shop for price” and do not expect a high degree of service.

For further examples of inventory systems with multiple demand classes, we refer readers to Kleijn & Dekker [33].

There are two basic approaches to dealing with heterogeneous demand: The capacity based control approach and the price based control approach (Zhou [62]). The capacity based control approach uses the capacity allocation to determine how the heterogeneous demand is served; whereas the price based control approach uses pricing to control the demand.

The practice of rationing inventory (or capacity) among different customer classes is an increasingly popular way for balancing supply with heterogeneous demand. When the inventory and production capacities are limited, it is reasonable to ration the limited

inventories among different demand classes. When a demand from a given class arrives, management has to decide between two alternatives: One, to satisfy this demand and avoid a small penalty cost, or two, to leave it unsatisfied in the hope of later using that extra unit of stock to satisfy a higher class of demand, and thus avoiding a high penalty cost.

One commonly used rationing policy that has been well described in the literature is the so called critical level policy, also known as threshold policy. In a system with multiple demand classes, the policy operates as follows: Demand from a particular class is satisfied from stock on hand if the inventory level exceeds the critical level associated with this class (Kleijn & Dekker [33]). Critical level policy is simple to implement, and in addition it has been shown to be optimal in a wide variety of rationing systems, though critical level policy may not be optimal in every case.

Pricing strategy has also been explored by many retail and manufacturing companies in an effort to improve their operations and ultimately their financial gain. In most retail and industrial business activities, firms use various forms of dynamic pricing-including personalized pricing, markdowns, display and trade promotions, coupons, discounts, clearance sales, and auctions and price negotiations, all to respond to market fluctuations and uncertainty in demand. The benefits can be significant, including potential increases in profit, improvements such as reduction in demand or production variability; resulting in more efficient supply chains (Chan, Shen, Simchi-Levi, & Swann [6]).

This thesis aims to study the structures of optimal policies for production/inventory systems with multiple demand classes. Both the capacity based control, and the price based control approaches are considered. In the first essay, we investigate the optimal ordering and

rationing policy for a finite horizon periodic review, single product inventory system with a fixed setup cost and two demand classes. In the second essay, we dissect a single product, finite-horizon periodic-review inventory system, also with two demand classes and fixed setup costs. Here the inventory control and pricing strategies will be under scrutiny. In the third essay, we consider stock rationing of a single-item, make-to-stock production/inventory system.

1.2 Literature review

Essentially we have chosen three related ranks of research for this dissertation and will elaborate on these below.

1.2.1 Inventory control with fixed ordering costs

An important development in the inventory theory has shown that (s, S) policies are optimal for a class of dynamic inventory models with random periodic demands and fixed ordering costs. Under an (s, S) policy, an order is placed to increase the item's inventory position (= inventory on-hand + orders outstanding - backlogs) to the levels S as soon as this inventory position drops below the level s . Scarf [47] introduces the notion of K -convex functions and use it to demonstrate the optimality of (s, S) policies for a class of finite-horizon periodic-review models in the presence of fixed ordering costs.

Iglehart [31] demonstrates the optimality of (s, S) policies for infinite horizon problems. Later, Veinott [54] discovers a forward formulation of the dynamic inventory problem and suggests a new proof of the optimality of (s, S) policies under new conditions. Sethi & Cheng [48] generalize the classical inventory models that exhibit (s, S) policies by assuming the

distribution of demands in successive periods is dependent on a Markov chain. They show that (s, S) policies are also optimal for the generalized model.

When replenishing inventory, there is often more than one supplier from which to choose. Porteus [43] analyzes inventory models with concave ordering costs, such as might arise from multiple suppliers. He demonstrates that a generalized (s, S) policy (a policy with multiple-threshold s s and multiple-target level S s) is optimal for the periodic-review n -period problem assuming mild cost conditions and demand that is a one-sided Pólya density. Porteus [44] later shows that the generalized (s, S) policy is optimal for uniform distributions of demand. Fox, Metters, & Semple [22] analyze an inventory model where the decision maker can buy from either of two suppliers, assuming that ordering costs are piecewise linear and concave. They show that a reduced form of generalized (s, S) policy is optimal for both finite and (discounted) infinite-horizon problems.

For the single-item, periodic-review inventory problem with a finite capacity per period and fixed setup costs, Chen & Lambrecht [8] show that the modified (s, S) policy is not optimal to the finite problems. The optimal policy, however, has an X - Y band structure: whenever the inventory level drops below X , order up to capacity; when the inventory level is above Y , do nothing. When the inventory level is between X and Y , the ordering pattern seems to be changing from problem to problem. Chen [9] studies the long-run limiting behavior of such capacitated problem. He shows that the optimal policy continues to exhibit the X - Y band structure.

Sobel & Zhang [49] consider a periodic-review inventory system with demand arriving simultaneously from a deterministic sources and a random source. The deterministic demand

has to be satisfied immediately and the stochastic demand can be backordered. They prove that a modified (s, S) policy is optimal under general conditions.

1.2.2 Inventory rationing policy

Past research work on the rationing problem can be broadly classified into two streams, depending on whether a fixed setup cost is present in the system studied.

When there is no setup cost, the critical level policy has been shown optimal in a wide variety of rationing systems. Veinott [55] is one of the first to investigate the inventory problem of several demand classes and introduced the concept of the critical level policy. Topkis [52] demonstrates the optimality of this policy both for the case of backordering and for the case of lost sales. Lee, Zhou, & Wu [36] study a capacitated make-to-stock production/inventory system with two demand classes. They show that the optimal production policy is a base stock policy, and the optimal rationing policy can be characterized by a base stock level for both finite and infinite horizon problems.

Rationing strategies have also been studied in make-to-stock queue models. Ha [26] uses an M/M/1 queue to study a single-item make-to-stock production system with multiple demand classes and lost sales. He shows that the optimal policy can be characterized by a sequence of monotone stock rationing levels. Ha [27] also considers the backordering case with two priority customer classes, and shows that the optimal policy can be characterized by a monotone curve. Ha [28] further extends previous results to the case of Erlang distributed processing time. de Vericourt, Karaesmen, & Dallery [15] consider an extension of Ha [27] with multiple-demand class, and show that the state dependent rationing policy established by

Ha [27] for two demand classes has a simpler structure, i.e., the rationing policy is constant and state independent. Recently, Huang & Iravani [29] extend the models in Ha [26] and Ha [27] to compound-Poisson demand cases. Benjaafar & Mohsen [3] consider an optimal production and inventory control of an assemble-to-order system with multiple customer classes. Zhao, Deshpande, & Ryan [61] study stock rationing in the decentralized dealer network systems.

When each production or replenishment incurs a fixed setup cost, the optimal rationing policy is more complicated. Nahmias & Demmy [39] consider a continuous review system with Poisson demand, backordering and two demand classes. They assume a (Q, R) replenishment policy and a critical level rationing policy. Deshpande, Morris, & Karen [17] analyze the same (Q, R) inventory rationing model with two demand classes without the restriction on the number of outstanding orders. Arslan et al. [2] consider a single-product inventory system without restriction on the number of demand classes. However, these papers focus on static, rather than dynamic, policies. Very little research exists on how to characterize the optimal rationing policy dynamically in the case of fixed setup costs.

One of the pioneering studies investigating the dynamic optimal structure of the ordering and rationing policies with fixed setup costs is Frank, Zhang, & Duenyas [23]. Frank et al. [23] consider the rationing problem with one deterministic and one stochastic demand class. The deterministic demand is a constant from period to period and must be satisfied immediately. The unmet stochastic demand is treated as lost sales. They show that the optimal structure of the ordering policy can be characterized as a state dependent (s, S) , while the rationing policy does not have a simple structure in general. They then propose a simple $(s, k,$

S) policy, where s and S (ordering policy) determine when and how much to order, while k (rationing policy) specifies how much of the lower-priority (i.e., stochastic) demand to satisfy.

There are also papers that consider inventory systems with multiple priority demand classes without using rationing policy. For example, Cohen, Kleindorfer, & Lee [14] present a model of an (s, S) inventory system with two demand classes. Stock is issued to meet high-priority demand first and the remaining stock is then made available to fill low-priority demand. Ding, Kouvelis, & Milner [18] consider a multiple-customer-class inventory system and use dynamic price discounts to encourage backlogging of demand for customer classes denied immediate service.

1.2.3 Joint pricing and inventory control

In recent years we have seen scores of retail and manufacturing companies exploring innovative pricing strategies in an effort to improve their operations and ultimately the capital gain. The joint pricing and inventory control problem has been studied by many researchers in operations management literature.

Whitin [56] is the first to add pricing decisions to inventory problem, where the selling price and order quantity are set simultaneously. Both Whitin [56] and Mills [38] address the single period version of the model. Subsequent work by Karlin & Carr [32], Zabel [60], Young [59], Polatoglu [42] and Lau & Lau [34] revise the same single period model under alternative specifications of the stochastic demand function.

Federgruen & Heching [19] study a problem where price and inventory decisions are coordinated under a linear production cost, stochastic demand and backlogging of excess demand. They show the optimality of the base-stock list-price policy for the nonstationary

finite-horizon model as well as the stationary infinite-horizon model. A base-stock list-price policy works as follows: If the starting inventory level is less than some level, then order up to this level, and charge a fixed list price; otherwise, do not order and offer a price discount.

Chen & Simchi-Levi [11] consider a finite-horizon model with complete backlogging, positive fixed costs. For the additive demand model, they show that an (s, S, p) policy is optimal, where the (s, S, p) policy is defined as follows: order nothing if inventory exceeds s ; order up to S otherwise. The price chosen depends on y , the inventory level after ordering, through a specified function $p(y)$. With the linear demand model, Chen & Simchi-Levi [10] show the optimality of the (s, S, p) policy in infinite-horizon models, both with the discounted-profit and the average-profit criteria. For the general demand model, Chen & Simchi-Levi [11] develop the notion of symmetric K -concavity, the generalization of K -concavity, and show that the profit function possesses this property. They show that the optimal policy has an (s, S, A, p) structure for the finite-horizon scenario. An (s, S, A, p) policy is characterized by two parameters s and S and a set $A \subset [s, (s + S) / 2]$. When the inventory level at the beginning of the period is less than s or in the set A , order up to S ; otherwise, no order is placed.

Yin & Rajaram [58] extend the results of Chen & Simchi-Levi [11] and Chen & Simchi-Levi [10] to Markovian environments. Chao & Zhou [7] provide algorithmic results for models with a Poisson demand process.

Feng & Chen [21] use fractional programming to establish (s, S, p) -optimality for the average-profit criterion, and provide an algorithm for computing the optimal parameters. The continuous-review extensions have been studied by Feng & Chen [20] and Chen &

Simchi-Levi [12].

Huh & Janakiraman [30] extend the optimality of the (s, S) -type structure for stationary systems by allowing a multidimensional sales level, and a less restrictive single-period expected profit function. For the model with multiplicative demand, lost sales and a fixed cost associated with any replenishment for finite selling horizon problems, Song, Ray, & Boyaci [50] show the optimality of the (s, S, A, p) policy.

1.3 Contributions

The key contributions of this thesis can be summarized as follows.

The research done for the first essay is largely motivated by, and closely related to, the work of Frank et al. [23]. This essay extends the model of Frank et al. [23] in the following significant ways. First, unsatisfied units of stochastic demand in Frank et al. [23] are considered as lost sales; while in our model, unsatisfied demand is backlogged. The issue of state-space in the backorder model becomes more complicated. Second, our model is more general, in that we assume both demand classes are stochastic; while Frank et al. [23] assume that the first demand class is deterministic. We show that the problem where one of the two demand classes is deterministic is a special case of the one with two stochastic classes. In addition, one restrictive assumption in Frank et al. [23] is that the deterministic demand value is a constant from period to period. In this essay, this assumption is relaxed and the optimal structure of the ordering/rationing policy can be entirely extended to include the non-stationary case. Third, for the one-deterministic-one-stochastic demand model, we reveal new insights and properties in the optimal structure of the rationing policy. Based on these

properties, a rationing region (i.e., only in this region do we need consider rationing) is determined, and consequently, the optimal ordering and rationing policy for any state can be generated by finding the optimal policy of only a finite set of states. For each state in this set, the optimal policy is obtained by choosing a policy from at most two alternatives. An efficient algorithm is then proposed. Fourth, for the model with two stochastic demand classes, we propose a heuristic algorithm based on critical level rationing policy. This heuristic is simple to implement and close to optimal.

The major contributions of the second essay include: (i) We relax one of the conditions in Sobel & Zhang [49] while still obtaining the optimality of the (s, S) policy. For a variation of the model with the only difference that one constraint is tighter, we further show that this condition can be removed entirely. (ii) We present two properties, each of which is equivalent to the optimality of the (s, S) policy to the problem with one deterministic and one stochastic demand class. (iii) As an application of the relaxation, we extend the Chen & Simchi-Levi [11] model to include two demand classes, and demonstrate the optimality of the (s, S) -type structure.

The third essay's contributions include: (i) We extend Ha [27] (two-class, unit demand), de Vericourt et al. [15] (multiple-class, unit demand) and Huang & Iravani [29] (two-class, batch demand) to address the multiple-class, batch demand model. (ii) We assume the facility can produce a batch up to a certain capacity at the same time; while all of the above papers consider unit-by-unit production. Their models however, do not apply to the batch production case, which is common in practice. (iii) We further extend our model to include the case with combined demand; i.e., one demand arrival may include demands from different classes. To

our knowledge, batch production and demand combination issues have not been well studied in the make-to-stock queue literature.

We present and highlight the common features of the three essays, in addition to the relationship with the closely related literature in Table 1.1.

Table 1.1 Common features among the three essays and closely related literature

Essay	Related literature	
Essay 1	Frank et al. [23]	
Periodic review	Periodic review	
Backordering	Lost-sales	
Finite-horizon	Finite-horizon	
Two stochastic demand classes	One-deterministic-one-stochastic	
Ordering/rationing policy	Ordering/rationing policy	
Fixed setup cost	Fixed setup cost	
Essay 2	Sobel & Zhang [49]	Chen & Simchi-Levi [11]
Periodic review	Periodic review	Periodic review
Backordering	Backordering	Backordering
Finite-horizon	Finite-horizon	Finite-horizon
One-deterministic-one-stochastic	One-deterministic-one-stochastic	One-deterministic-one-stochastic
Ordering/pricing policy	Ordering policy	Ordering/pricing policy
Fixed setup cost	Fixed setup cost	Fixed setup cost
Essay 3	de Vericourt et al. [15]	Huang & Iravani [29]
Continuous review	Continuous review	Continuous review
Backordering	Backordering	Backordering
Infinite-horizon	Infinite-horizon	Infinite-horizon
Multiple demand classes (Compound-Poisson)	Multiple demand classes (Poisson)	Two demand classes (Poisson with fixed batch)
Batch production	Unit production	Unit production
Production/rationing policy	Production/rationing policy	Production/rationing policy
No fixed setup cost	No fixed setup cost	No fixed setup cost

1.4 Organization

The remainder of the dissertation is organized as follows. In Chapter 2, we consider a finite horizon periodic review, single product inventory system, with a fixed setup cost and two demand classes. We first study the basic model where both demand classes are random

variables. We further study the model where the first demand class is deterministic and must be satisfied immediately. We show the optimality of the state dependent (s, S) ordering policy, and obtain structural properties for rationing. Based on these properties, we propose an algorithm for the basic model and a heuristic for the second model.

In Chapter 3, we first discuss sufficient conditions of the (s, S) -policy's optimality for the inventory system with fixed setup costs and two demand classes. We next study the joint inventory and pricing control for this system.

In Chapter 4, we take into account the stock rationing of a single-item make-to-stock production/inventory system. Demand arrives as a Poisson process with a randomly distributed batch size. It is assumed that the batch demand can be partially satisfied. We study the optimal ordering/rationing policy for both the lost-sales case and the backordering case.

In summary of our findings we present the results of where this research has led us and discuss directives for future endeavors in Chapter 5.

2. Optimal ordering/rationing policy for a periodic review system with two demand classes and backordering

In today's competitive market, stock-rationing is widely practiced where a firm maintains a common stock in order to satisfy different customers. Customers are differentiated in terms of their value to the firm, or supply contracts to which they commit. In a make-to-stock production system, for example, the firm produces a single product to satisfy multiple demand classes differing in their backordering cost. When the inventory level is low, it is reasonable to backlog demands with lower backordering costs and reserve stock for future demands with higher priorities. Numerous applications of rationing policy can also be found in service part supply chains, and in health care and airline/hotel industries.

This chapter studies the optimal ordering/rationing policy for a periodic review system with two demand classes. We first introduce the basic model for the two-stochastic-class backordering problem with fixed setup costs and present the dynamic programming formulation. We then study the structural properties of the optimal ordering/rationing policy. Next, we study additional properties of the optimal policy for a special case where the first class is deterministic and must be satisfied immediately. We then present an optimal algorithm for this special case, and propose a heuristic policy for the basic model. Finally, we consider some extensions of the basic model.

To simplify notations, we use abbreviations A/B/C to present models with different assumptions, where

A = S or N, specifies whether demands are stationary (S) or non-stationary (N);

B = S or D, specifies whether the first class demand is stochastic (S) or deterministic (D); (In this paper, we assume that if class 1 demand is deterministic, then it must be satisfied immediately.)

C = S or D, specifies whether the second class demand is stochastic (S) or deterministic (D).

2.1 Model formulation

In the basic model (N/S/S), we consider a periodic review inventory system with two stochastic demand classes that differ in their backordering costs. Demands for each class are independent from period to period, though the distributions can be different. Unsatisfied demands are backlogged and to be satisfied in the future periods. Every time an order is placed, the system incurs a fixed setup cost and a purchasing cost proportional to the order quantity. Denote:

x : inventory on hand (if $x \geq 0$) or units of class 1 backlogged (if $x < 0$) at the beginning of a period after demand arrivals,

y : units of class 2 backlogged (nonnegative integer) at the beginning of a period after demand arrivals,

n : planning horizon, numbered backward from the end of the horizon, i.e., the initial period is denoted as period n ,

k : setup cost per order,

h : holding cost per unit per period,

c : purchasing cost per unit,

π_i : backordering cost per unit per period of class i backorders (assume $\pi_1 > \pi_2 > c$),

D_i^j : class i demand in period j , random variable, nonnegative integer,

d_i^j : realized class i demand in period j ,

α : discount factor ($0 < \alpha \leq 1$),

Q : number of units to order, nonnegative integer, a decision variable,

ω : number of class 2 demand to satisfy, nonnegative integer, a decision variable,

$f_n(x, y, Q, \omega)$: optimal expected discounted cost for a n -period problem with (x, y) as the initial state given a policy (Q, ω) adopted in period n ,

$v_n(x, y)$: optimal n -period expected discounted cost, with (x, y) as the initial state.

The sequence of events and activities in every period is as follows:

1. Demand is observed at the beginning of the period.
2. Inventory on hand and/or backorders are then updated, followed by a decision on the order size Q .
3. The order (if any) arrives by the end of the period and at that point in time, class 1 demand with a higher backordering cost is satisfied as much as possible through on-hand inventory, and a decision is made on how much of class 2 demand to satisfy.

Note we assume that demand is observed before an order is placed. Therefore there is no demand uncertainty in the current period. This setting often happens in practice when advance demand information for a product is obtained when customers place orders in advance for a future delivery (See e.g., Gallego & Ozer [24]).

Our objective is to determine a policy (Q, ω) that minimizes the expected discounted cost.

$$v_n(x, y) = \min_{(Q, \omega)} f_n(x, y, Q, \omega)$$

The problem can be formulated by the following dynamic program:

$$\begin{aligned} v_n(x, y) = \min_{(Q, \omega)} \{ & k\delta(Q) + cQ + h(x + Q - \omega)^+ + \pi_1(x + Q - \omega)^- + \pi_2(y - \omega) \\ & + \alpha E v_{n-1}(x + Q - \omega - D_1^{n-1}, y - \omega + D_2^{n-1}) : \\ & Q \geq 0, 0 \leq \omega \leq \min\{y, (x + Q)^+\} \}; \end{aligned} \quad (1.1)$$

where $x^+ = x$ if $x > 0$, $x^+ = 0$ if $x \leq 0$, $x^- = 0$ if $x \geq 0$, $x^- = -x$ if $x < 0$, and $\delta(Q) = 1$ if $Q > 0$, and 0 otherwise. We also let $v_0(x, y) = 0$, that is, at the end of the planning horizon, the remaining inventory and/or backorders are discarded with zero cost.

The terms on the first line in (1.1) contain the total cost incurred in the current period including the setup cost, purchasing cost, holding cost, class 1 and 2 backordering costs. The term on the second line denotes the expected optimal cost from the next period forward. Constraints on the third line ensure that the order quantity is nonnegative, and the quantity satisfying class 2 demand is no more than class 2 backorders or on-hand inventory, whichever is smaller. Note when the holding cost is positive, class 1 backordering cost must be zero, and vice versa.

2.2 Structure of the optimal policy for the basic model N/S/S

In this sub-section, we first introduce the following lemmas, which help to characterize the optimal structure. We then summarize the main results, namely, the structural properties of the optimal ordering policy in Theorem 2.1.

Lemma 2.1 *If it is optimal to order at state (x, y) , then*

(1) $Q + x \geq 0$, i.e., all class 1 backorders must be satisfied.

(2) $\omega = y$, i.e., we always order enough to satisfy all class 2 backorders.

Proof First, for part (1), it is trivial to prove if $x \geq 0$. We only consider the case $x < 0$.

For $n = 1$, it is obvious that the optimal Q must satisfy $Q + x \geq 0$. By induction, suppose the result holds for period i , $1 \leq i \leq n - 1$. Now consider period n . we show that if $Q_n < -x$,

then policy $(Q_n, 0)$ cannot be optimal. For an arbitrary realized sample path of n period demand $(d_1^n, d_2^n; d_1^{n-1}, d_2^{n-1}; \dots; d_1^1, d_2^1)$, suppose the optimal policy is

$\Pi = (Q_n, 0; Q_{n-1}, \omega_{n-1}; \dots; Q_1, \omega_1)$, where $Q_n < -x$ (and thus ω_n must be 0), and suppose j is the next period after n such that an order is placed. If $j = 0$, i.e. there is no order after period n , since $Q_n < -x$ and $\omega_i = 0$ for $1 \leq i \leq n$, Π can be written as $\Pi = (Q_n, 0; 0, 0; \dots; 0, 0)$.

Now consider an alternative policy $\Pi_1 = (-x, 0; 0, 0; \dots; 0, 0)$. It is easy to show that

$$f_n^\Pi(x, y) - f_n^{\Pi_1}(x, y) \geq (\pi_2 - c)(-x - Q_n) > 0. \text{ Hence, } \Pi \text{ is not optimal.}$$

If $0 < j < n$, then Π can be written as $\Pi = (Q_n, 0; 0, 0; \dots; Q_j, \omega_j; \dots; Q_1, \omega_1)$ where $Q_j \geq -x_j$ and x_j ($x_j \leq x + Q_n < 0$) is the class 1 backorder in period j . Consider policy $\Pi_1 = (-x, 0; 0, 0; \dots; Q_j + x + Q_n, \omega_j; \dots; Q_1, \omega_1)$. Since $Q_j \geq -x_j \geq -x - Q_n > 0$, Π_1 is feasible. Compare the costs of Π and Π_1 ,

$$\begin{aligned} f_n^\Pi(x, y) - f_n^{\Pi_1}(x, y) &= -c(-x - Q_n) + \pi_1(-x - Q_n) + \alpha\pi_1(-x - Q_n) + \dots \\ &\quad + \alpha^{n-j-1}\pi_1(-x - Q_n) + \alpha^{n-j}c(-x - Q_n) > 0 \\ &= [\pi_1 - c + (\alpha + \alpha^2 + \dots + \alpha^{n-j-1})\pi_1 + \alpha^{n-j}c](-x - Q_n) > 0 \end{aligned}$$

Hence, Π is not optimal. Next, we consider part (2).

If $n = 1$. For any $Q > 0$ and $\omega < y$,

$$f_1(x, y, Q, \omega) = k + cQ + h(x + Q - \omega) + \pi_2(y - \omega)$$

We construct a policy $(Q + y - \omega, y)$,

$$\begin{aligned} f_1(x, y, Q + y - \omega, y) &= k + c(Q + y - \omega) + h(x + Q + y - \omega - y) + \pi_2(y - y) \\ &= f_1(x, y, Q, \omega) - (\pi_2 - c)(y - \omega) < f_1(x, y, Q, \omega) \end{aligned}$$

Hence, $f_1(x, y, Q, \omega)$ is not optimal.

Now consider $n > 1$. Suppose for any period i , $i \leq n - 1$, if $Q > 0$, then $\omega = y$. We show the result also true for period n . Otherwise, suppose (Q_n, ω_n) is optimal for state (x, y) , where $Q_n > 0$ and $\omega_n < y$. Consider an arbitrary realized sample path of the $n-1$ period demand $(d_1^{n-1}, d_2^{n-1}; \dots; d_1^1, d_2^1)$, and the optimal policy $\Pi = (Q_n, \omega_n; Q_{n-1}, \omega_{n-1}; \dots; Q_1, \omega_1)$ accordingly. Denote period j as the first period after period n such that all class 2 demand is satisfied, i.e. $\omega_j = y_j$ and $\omega_i < y_i$ for $i > j$. There are three cases:

Case I: $j = 0$.

It is obvious that there is no order from period $n - 1$ to 1, or $Q_i = 0$ ($1 \leq i \leq n - 1$). Because if $Q_i > 0$ in some period i ($1 \leq i \leq n - 1$), then $\omega_i = y_i$ and consequently $j = i > 0$. Now we construct a policy: order $Q_n + 1$ and satisfy $\omega_n + 1$ units of class 2 demand in period n , and same as Π in any other periods. Denote as $\Pi_1 = (Q_n + 1, \omega_n + 1; 0, \omega_{n-1}; \dots; 0, \omega_1)$. By the definition of j , this policy is also feasible for state (x, y) . Compare the costs of the two policies:

$$f_n^{\Pi_1}(x, y) - f_n^{\Pi}(x, y) = \pi_2 - c + \alpha\pi_2 + \dots + \alpha^{n-1}\pi_2 > 0$$

Case II: $j > 0$ and $Q_j > 0$.

If $j < n - 1$, it is obvious that there is no order from period $n - 1$ to $j + 1$. Now we construct a policy: order $Q_n + 1$ and satisfy $\omega_n + 1$ units of class 2 demand in period n ; from period $n - 1$ to $j + 1$, same as Π ; in period j , order $Q_j - 1$ units and satisfy $y_j - 1$ units of class 2 demand, where y_j is the class 2 backorders in period j according to the considered sample path and policy Π . Denote this policy as

$\Pi_1 = (Q_n + 1, \omega_n + 1; 0, \omega_{n-1}; \dots; Q_j - 1, y_j - 1; \dots; Q_1, \omega_1)$. By the definition of j , this policy is also feasible for state (x, y) . Compare the costs of the two policies:

$$\begin{aligned} f_n^\Pi(x, y) - f_n^{\Pi_1}(x, y) &= \pi_2 - c + \alpha\pi_2 + \dots + \alpha^{n-j-1}\pi_2 + \alpha^{n-j}c \\ &\quad + \alpha^{n-j}k - \alpha^{n-j}k\delta(Q_j - 1) > 0 \end{aligned}$$

If $j = n - 1$, the proof is similar.

Case III: $j > 0$ and $Q_j = 0$.

Because there is no order from period $n - 1$ to j , the inventory on hand before period j must be positive. Therefore, we can construct a feasible policy for state (x, y) : order Q_n and

satisfy $\omega_n + 1$ units of class 2 demand at period n ; from period $n - 1$ to $j + 1$

(Assume $j < n - 1$. For $j = n - 1$, it can be proved similarly), same as Π ; in period j , do not

order and satisfy $y_j - 1$, where y_j is the class 2 backorders in period j according to the

considered sample path and policy Π . Denote this policy as

$\Pi_1 = (Q_n, \omega_n + 1; 0, \omega_{n-1}; \dots; 0, y_j - 1; \dots; Q_1, \omega_1)$. Then

$$f_n^\Pi(x, y) - f_n^{\Pi_1}(x, y) = \pi_2 + \alpha\pi_2 + \dots + \alpha^{n-j-1}\pi_2 > 0$$

To summarize, we complete the proof for period n . ■

Lemma 2.2 *If $x > y$, then it is optimal not to order.*

Proof First consider $n = 1$. If $Q > 0$, from Lemma 1, $v_1(x, y) = f_1(x, y, Q, y)$. Thus,

$$f_1(x, y, Q, y) = k + cQ + h(x + Q - y) + \pi_2(y - y) > h(x - y) = f_1(x, y, 0, y).$$

Now suppose $n > 1$. Consider an arbitrary policy (Q_n, y) for state (x, y) , where $Q_n > 0$.

$$\begin{aligned} f_n(x, y, Q_n, y) &= k + cQ_n + h(x + Q_n - y) + \alpha E v_{n-1}(x + Q_n - y - D_1^{n-1}, D_2^{n-1}) \\ &= k + cQ_n + h(x + Q_n - y) \\ &\quad + \alpha E_{d_1^{n-1}, d_2^{n-1}} [k\delta(Q_{n-1}^*) + cQ_{n-1}^* + h(x + Q_n - y - d_1^{n-1} + Q_{n-1}^* - \omega_{n-1}^*)^+ \\ &\quad + \pi_1(x + Q_n - y - d_1^{n-1} + Q_{n-1}^* - \omega_{n-1}^*)^- + \pi_2(d_2^{n-1} - \omega_{n-1}^*) \\ &\quad + \alpha E_{D_1^{n-1}, D_2^{n-1}} [v_{n-2}(x + Q_n - y - d_1^{n-1} + Q_{n-1}^* - \omega_{n-1}^* - D_1^{n-1}, d_2^{n-1} - \omega_{n-1}^* + D_2^{n-1})]] \end{aligned}$$

where $(Q_{n-1}^*, \omega_{n-1}^*)$ is the optimal policy for period $n - 1$, given demands for the two classes in period $n - 1$ are d_1^{n-1} and d_2^{n-1} , respectively.

Now we construct a policy: in period n , do not order and satisfy y units of class 2 demand; in period $n - 1$, order $Q_n + Q_{n-1}^*$ and satisfy ω_{n-1}^* units of demand. Then,

$$\begin{aligned} f_n(x, y, 0, y) &= h(x - y) + \alpha E_{d_1^{n-1}, d_2^{n-1}} [k + c(Q_{n-1}^* + Q_n) + h(x + Q_n - y - d_1^{n-1} + Q_{n-1}^* - \omega_{n-1}^*)^+ \\ &\quad + \pi_1(x + Q_n - y - d_1^{n-1} + Q_{n-1}^* - \omega_{n-1}^*)^- + \pi_2(d_2^{n-1} - \omega_{n-1}^*) \\ &\quad + \alpha E_{D_1^{n-1}, D_2^{n-1}} [v_{n-2}(x + Q_n - y - d_1^{n-1} + Q_{n-1}^* - \omega_{n-1}^* - D_1^{n-1}, d_2^{n-1} - \omega_{n-1}^* + D_2^{n-1})]] \\ &= f_n(x, y, Q_n, y) - k - \alpha k \delta(Q_{n-1}^*) + \alpha k - hQ_n - cQ_n + \alpha cQ_n < f_n(x, y, Q_n, y). \end{aligned}$$

Therefore, it is not optimal to place an order if the inventory on hand is greater than the class 2 backorders. ■

Lemma 2.1 and 2.2 state that if it is optimal to place an order, then we always order sufficient units to clear all class 1 and 2 backorders. On the other hand, if the on-hand inventory after clearing class 1 demand is enough to satisfy class 2 backorders, then it is optimal not to order in the current period. This result is quite intuitive, since placing an order in later periods rather than now will reduce the setup and holding costs without incurring additional backordering cost. Lemma 2.1 and 2.2 are also used to demonstrate the next three lemmas.

Lemma 2.3 *If it is optimal to order $Q > 0$ at state (x, y) , then it is optimal to order $Q + 1$ at state $(x, y + 1)$, i.e., if $v_n(x, y) = f_n(x, y, Q, y)$, then $v_n(x, y + 1) = f_n(x, y + 1, Q + 1, y + 1)$.*

Proof First, we demonstrate by contradiction that it is optimal to place an order in period n . To this end, suppose $v_n(x, y + 1) = f_n(x, y + 1, 0, \omega_n)$. Since it is optimal to order at state (x, y) , from Lemma 2, $x \leq y$. So $\omega_n \leq y$ and $(0, \omega_n)$ is feasible at state (x, y) .

Now consider an arbitrary realized sample path of the $n - 1$ periods

demand $(d_1^{n-1}, d_2^{n-1}; \dots; d_1^1, d_2^1)$. Suppose under the optimal policy, a sequence of decision rules that is associated with this sample path starting from state $(x, y+1)$ in period n is $\Pi = (0, \omega_n; Q_{n-1}, \omega_{n-1}; \dots; Q_1, \omega_1)$. Further, assume that period j ($n > j \geq 0$) is the first period after period n during which an order is placed. By Lemma 2.1, all class 2 backorders must be cleared, so that $\omega_j = y_j \geq y+1 > 0$. Now we construct a policy for state (x, y) : order $Q_j - 1$ and satisfy $\omega_j - 1$ units of class 2 demand in period j , and same as Π in any other periods. Denote this policy as $\Pi_1 = (0, \omega_n; Q_{n-1}, \omega_{n-1}; \dots; Q_j - 1, \omega_j - 1; \dots; Q_1, \omega_1)$. Because $\omega_j > 0$ and $Q_j > 0$, this policy is feasible for state (x, y) . Note if $j = 0$, then $\Pi = \Pi_1$. Comparing costs of $f_n^\Pi(x, y+1)$ and $f_n^{\Pi_1}(x, y)$, it is easy to identify that $f_n^\Pi(x, y+1) - f_n^{\Pi_1}(x, y) = \pi_2 + \alpha\pi_2 + \dots + \alpha^{n-j-1}\pi_2 + \alpha^{n-j}c + \alpha^{n-j}k - \alpha^{n-j}k\delta(Q_j - 1) > c$.

Therefore, $v_n(x, y+1) > f_n^{\Pi_1}(x, y) + c \geq v_n(x, y) + c$. On the other hand,

$$\begin{aligned} f_n(x, y+1, Q+1, y+1) &= k + c(Q+1) + h(x+Q+1-y-1)^+ + \pi_1(x+Q+1-y-1)^- \\ &\quad + \pi_2(y+1-y-1) + \alpha E v_{n-1}(x+Q+1-y-1-D_1^{n-1}, y+1-y-1+D_2^{n-1}) \\ &= f_n(x, y, Q, y) + c = v_n(x, y) + c \end{aligned}$$

Thus $v_n(x, y+1) > f_n(x, y+1, Q+1, y+1) = v_n(x, y) + c$, which is contradicted.

Next we show that given $v_n(x, y) = f_n(x, y, Q, y)$, the optimal policy for state $(x, y+1)$ is exactly $(Q+1, y+1)$. It is sufficient to show that for an arbitrary policy $(Q', y+1)$ for state $(x, y+1)$, where $Q' > 0$, $f_n(x, y+1, Q', y+1) \geq f_n(x, y+1, Q+1, y+1)$. In view of

$$\begin{aligned} f_n(x, y+1, Q', y+1) &= k + cQ' + h(x+Q'-y-1) + \alpha E v_{n-1}(x+Q'-y-1-D_1^{n-1}, D_2^{n-1}) \\ &= k + cQ' + h(x+Q'-1-y-1+1) + \alpha E v_{n-1}(x+Q'-1-y-1+1-D_1^{n-1}, D_2^{n-1}) \\ &= f_n(x, y, Q'-1, y) + c + k - k\delta(Q'-1) \\ &\geq f_n(x, y, Q, y) + c = v_n(x, y) + c, \end{aligned}$$

the proof is completed. ■

Lemma 2.4 *If it is optimal to order $Q > 0$ at state $(x+1, y)$, then it is optimal to order*

$Q+1$ at state (x, y) , i.e., if $v_n(x+1, y) = f_n(x+1, y, Q, y)$, then $v_n(x, y) = f_n(x, y, Q+1, y)$.

Proof We demonstrate that in period n it is optimal to place an order at state (x, y) . Suppose

by contrary $v_n(x, y) = f_n(x, y, 0, \omega')$.

If $x < 0$, then $\omega' = 0$,

$$\begin{aligned} v_n(x, y) &= f_n(x, y, 0, 0) \geq f_n(x+1, y, 0, 0) + \pi_1 > f_n(x+1, y, 0, 0) + c \\ &\geq f_n(x+1, y, Q, y) + c = v_n(x+1, y) + c \end{aligned}$$

The first inequality can be explained as follows: Consider an arbitrary realized sample path of the demands appearing in following n periods $(d_1^n, d_2^n; d_1^{n-1}, d_2^{n-1}; \dots; d_1^1, d_2^1)$. Under the

optimal policy a sequence of decision rules that is associated with this sample path starting from state (x, y) of period n is $\Pi = (0, 0; 0, 0; \dots; Q_j, y_j; \dots; Q_1, \omega_1)$, assuming that j is the

first period after n during which Π places an order. Now construct a policy Π_1 for state $(x+1, y)$: (a) if $j = 0$, same as Π . In this case, it is obvious that

$$f_n^\Pi(x, y) - f_n^{\Pi_1}(x+1, y) \geq \pi_1.$$

(b) If $j > 0$, except order $Q_j + 1$ and satisfy $y_j + 1$ in period j , the rest is the same as Π . It is easy to identify that $f_n^\Pi(x, y) - f_n^{\Pi_1}(x+1, y) \geq \pi_1$.

On the other hand, $f_n(x, y, Q+1, y) = f_n(x+1, y, Q, y) + c = v_n(x+1, y) + c$, which is a contradiction and implies that $f_n(x, y, 0, \omega')$ is not optimal.

If $x \geq 0$, then $\omega' \leq x$.

$$\begin{aligned} v_n(x, y) &= f_n(x, y, 0, \omega') = h(x - \omega') + \pi_2(y - \omega') + \alpha E v_{n-1}(x - \omega' - D_1^{n-1}, y - \omega' + D_2^{n-1}) \\ &= h(x+1 - \omega' - 1) + \pi_2(y+1 - \omega' - 1) + \alpha E v_{n-1}(x+1 - \omega' - 1 - D_1^{n-1}, y+1 - \omega' - 1 + D_2^{n-1}) \\ &= f_n(x+1, y+1, 0, \omega'+1) \end{aligned}$$

From Lemma 2.3, if $v_n(x+1, y) = f_n(x+1, y, Q, y)$ where $Q > 0$, then it is optimal to order at $(x+1, y+1)$, and $v_n(x+1, y+1) = f_n(x+1, y+1, Q+1, y+1) = v_n(x+1, y) + c$.

Therefore,

$$\begin{aligned}
 f_n(x, y, 0, \omega') &= f_n(x+1, y+1, 0, \omega'+1) \geq v_n(x+1, y+1) \\
 &= f_n(x+1, y+1, Q+1, y+1) \\
 &= f_n(x+1, y, Q, y) + c = v_n(x+1, y) + c
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 f_n(x, y, Q+1, y) &= k + c(Q+1) + h(x+Q+1-y)^+ + \pi_1(x+Q+1-y)^- \\
 &\quad + \alpha E v_{n-1}(x+Q+1-y - D_1^{n-1}, D_2^{n-1}) \\
 &= f_n(x+1, y, Q, y) + c = v_n(x+1, y) + c
 \end{aligned}$$

Since $v_n(x, y) = f_n(x, y, 0, \omega') \geq v_n(x+1, y) + c = f_n(x, y, Q+1, y)$,

$v_n(x, y) = f_n(x, y, Q+1, y)$, and the policy $(Q+1, y)$ is optimal. Thus, it is optimal to order at state (x, y) .

Next, we show that $v_n(x, y) = f_n(x, y, Q+1, y)$. Consider an arbitrary policy (Q', y) with $Q' > 0$,

$$\begin{aligned}
 f_n(x, y, Q', y) &= k + cQ' + h(x+Q'-y) + \alpha E v_{n-1}(x+Q'-y - D_1^{n-1}, D_2^{n-1}) \\
 &= f_n(x+1, y, Q'-1, y) + k - k\delta(Q'-1) + c \\
 &\geq f_n(x+1, y, Q'-1, y) + c \\
 &\geq f_n(x+1, y, Q, y) + c = v_n(x+1, y) + c
 \end{aligned}$$

Together with the fact that $f_n(x, y, Q+1, y) = v_n(x+1, y) + c$, we have

$$v_n(x, y) = f_n(x, y, Q+1, y) \quad \blacksquare$$

Lemma 2.5 *If it is optimal to order $Q > 0$ at state $(x+1, y+1)$, then it is optimal to order Q at state (x, y) , i.e., if $v_n(x+1, y+1) = f_n(x+1, y+1, Q, y+1)$,*

then $v_n(x, y) = f_n(x, y, Q, y)$.

Proof From Lemma 2.1,

$$\begin{aligned}
 v_n(x+1, y+1) &= f_n(x+1, y+1, Q, y+1) \\
 &= k + cQ + h(x+Q-y) + \alpha E v_{n-1}(x+Q-y - D_1^{n-1}, D_2^{n-1}). \\
 &= f_n(x, y, Q, y) \geq v_n(x, y)
 \end{aligned}$$

On the other hand, it can be easily proved that $v_n(x+1, y+1) \leq v_n(x, y)$. Therefore,

$$v_n(x, y) = f_n(x, y, Q, y) \quad \blacksquare$$

Lemmas 2.3 and 2.4 imply that a threshold ordering policy is optimal. That is, given class 2 backorders, there exists a threshold such that it is optimal to order if the inventory on hand is not greater than this threshold, and not to order otherwise. Given the inventory on hand, it is optimal to order if class 2 backorder is not less than the threshold, and not to order otherwise. Furthermore, at states where it is optimal to order, we always order sufficient units to bring inventory up to the same order-up-to level and clear all backorders.

The following theorem summarizes the form and property of the optimal ordering policy:

Theorem 2.1 *For the N/S/S model, a state-dependent (s, S) ordering policy is optimal. That is, there exists a switching curve $x = s(y)$, such that it is optimal to order to an order-up-to level S if $x \leq s(y)$, and do not order otherwise. In addition, (1) $0 \leq s(y+1) - s(y) \leq 1$; (2) $s(y) \leq y$.*

Proof From Lemma 2.3 and 2.4, obviously there exist a switching curve $x = s(y)$ and an order-up-to level S , such that it is optimal to order if $x \leq s(y)$, and not to order otherwise. $0 \leq s(y+1) - s(y)$ is from Lemma 2.4, $s(y+1) - s(y) \leq 1$ is from Lemma 2.5, and $s(y) \leq y$ is from Lemma 2.2. ■

The form of the optimal ordering policy is illustrated in Figure 2.1. The switching curve $s(y)$ divides the state space into two regions. In any period n , for state (x, y) in region I, it is optimal to order up to the state $(S, 0)$. On the other hand, if state (x, y) is in region II, then it is optimal not to order. In addition, the switching curve $s(y)$ is non-decreasing in y and must be above the line $y = x$.

Note we use the term *state-dependent (s, S) ordering policy*, because the order point, s , is

dependent on the state (x, y) , and thus is characterized by a curve $x = s(y)$. However, the order-up-to level, S , is state independent.

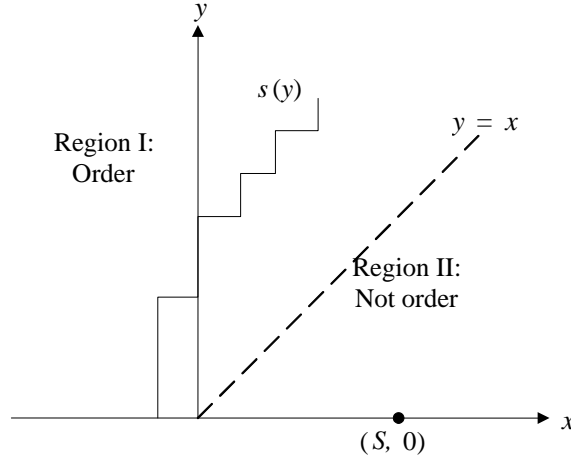


Figure 2.1 The optimal structure for the N/S/S model

Lemma 2.6 *If $y - x > k / (\pi_2 - c)$, then it is optimal to order.*

Proof We prove by induction. For state (x, y) in period $n = 1$, where $y > x$, if $x > 0$, then

$$f_1(x, y, y - x, y) = k + c(y - x) < \pi_2(y - x) = \min_{0 \leq \omega \leq \min\{x, y\}} \{\pi_2(y - \omega) + h(x - \omega)\} = f_1(x, y, 0, \omega^*)$$

where ω^* is the optimal quantity to satisfy class 2 backorder if it does not place an order.

Clearly it is optimal to order. If $x \leq 0$, then

$$f_1(x, y, y - x, y) = k + c(y - x) < \pi_2(y - x) \leq -\pi_1 x + \pi_2 y = f_1(x, y, 0, 0).$$

Thus it is optimal to order. Now suppose the result holds from period $n - 1$ to 1, and consider

period n . Let $0 \leq \omega \leq \min\{x, y\}$. If $x > 0$, we compare $f_n(x, y, 0, \omega)$ and

$f_n(x, y, y - \omega, y)$.

$$\begin{aligned} f_n(x, y, 0, \omega) &= h(x - \omega) + \pi_2(y - \omega) + \alpha E v_{n-1}(x - \omega - D_1^{n-1}, y - \omega + D_2^{n-1}) \\ &= h(x - \omega) + \pi_2(y - \omega) + \alpha E [k + cQ_{n-1}^* + h(x - D_1^{n-1} + Q_{n-1}^* - y - D_2^{n-1}) \\ &\quad + \alpha E v_{n-2}(x - D_1^{n-1} + Q_{n-1}^* - y - D_2^{n-1} - D_1^{n-2}, D_2^{n-2})] \end{aligned}$$

Where the second equality is because $y - \omega + D_2^{n-1} - (x - \omega - D_1^{n-1}) \geq y - x > k / (\pi_2 - c)$,

and Q_{n-1}^* is the optimal order quantity in period $n-1$.

$$\begin{aligned} f_n(x, y, y - \omega, y) &= h(x - \omega) + k + c(y - \omega) + \alpha E v_{n-1}(x - \omega - D_1^{n-1}, D_2^{n-1}) \\ &\leq h(x - \omega) + k + c(y - \omega) + \alpha E [k + c(Q_{n-1}^* - y + \omega) + h(x - D_1^{n-1} + Q_{n-1}^* - y - D_2^{n-1}) \\ &\quad + \alpha E v_{n-2}(x - D_1^{n-1} + Q_{n-1}^* - y - D_2^{n-1} - D_1^{n-2}, D_2^{n-2})] \end{aligned}$$

Where the inequality is because that ordering $Q_{n-1}^* - y + \omega$ units may not be optimal. Hence,

$$\begin{aligned} f_n(x, y, y - \omega, y) - f_n(x, y, 0, \omega) &= k + (1 - \alpha)c(y - \omega) - \pi_2(y - \omega) \\ &\leq k + c(y - \omega) - \pi_2(y - \omega) \leq k + c(y - x) - \pi_2(y - x) < 0 \end{aligned}$$

If $x \leq 0$, similarly we can obtain

$$\begin{aligned} f_n(x, y, y - x, y) - f_n(x, y, 0, 0) &\leq k + c(y - x) + \pi_1 x - \pi_2 y \\ &\leq k + c(y - x) + \pi_2 x - \pi_2 y < 0 \end{aligned}$$

Hence, when $y - x > k / (\pi_2 - c)$, it is optimal to order. ■

Lemma 2.6 provides a sufficient condition when it is optimal to order. Here $y - x$ can be considered as the total number of backorders. Intuitively, a decision to place an order accrues setup cost and purchasing cost, but may reduce backordering cost. A decision not to order may incur backordering cost, but save setup cost and purchasing cost. When the total number of backorders is so large such that the backordering cost when no order is placed exceeds the sum of the setup cost and the purchasing cost when an order is placed, the optimal decision must be to place an order.

Theorem 2.1 only tells us that there exists a switching curve and an order-up-to level such that as long as the state is on the left side of the switching curve, it is optimal to order up to this level and clear all backorders. However, how much class 2 demand will be satisfied in case it is optimal not to order remains unclear. Our numerical results show that the optimal rationing structure is quite complicated and it is hard to characterize the optimal rationing policy in general. The rationing structure depends on not only the state (x, y) , demand

distribution of D_1^{n-1} and D_2^{n-1} , but also the cost parameters k, π_1, π_2, h and c .

Lemma 2.7 *If $x \leq y$, then $v_n(x, y+1) \geq v_n(x, y) + c$.*

Proof For $n = 1$, $v_1(x, y+1) = v_1(x, y) + c$ if it is optimal to order at (x, y) , and $v_1(x, y+1) = v_1(x, y) + \pi_2 > v_1(x, y) + c$ if it is optimal not to order at (x, y) . Suppose the result holds for period 1 to $n - 1$. Now consider $v_n(x, y+1) - v_n(x, y)$. If it is optimal to order Q units at $(x, y+1)$, then, as $x + Q \geq y+1, x + Q - 1 \geq y$ and policy $(Q-1, y)$ is feasible at state (x, y) . Thus,

$$\begin{aligned} v_n(x, y+1) - v_n(x, y) &\geq f_n(x, y+1, Q, y+1) - f_n(x, y, Q-1, y) \\ &= k + cQ - k\delta(Q-1) - c(Q-1) \geq c \end{aligned}$$

If it is optimal not to order at $(x, y+1)$, and suppose $v_n(x, y+1) = f_n(x, y+1, 0, \omega)$ where $\omega \leq y$, then

$$\begin{aligned} v_n(x, y+1) - v_n(x, y) &\geq f_n(x, y+1, 0, \omega) - f_n(x, y, 0, \omega) \\ &= \pi_2 + \alpha E[v_{n-1}(x - \omega - D_1^{n-1}, y+1 - \omega + D_2^{n-1}) - v_{n-1}(x - \omega - D_1^{n-1}, y - \omega + D_2^{n-1})] \\ &\geq \pi_2 > c \end{aligned}$$

This completes the induction and the lemma follows. ■

Based on Lemma 2.7, we can partially characterize the property of rationing based on the sub-convexity of the cost function.

Lemma 2.8 *Sub-convexity. $v_n(x+1, y+1) - v_n(x, y) \geq v_n(x+1, y) - v_n(x, y-1)$ for all $y > 0$.*

Proof We prove by induction. It is easy to check that $v_1(x, y-1) + v_1(x+1, y+1) \geq v_1(x, y) + v_1(x+1, y)$. Assume that the result holds for period $n-1$. Now consider period n . Obviously $v_n(x+1, y+1) - v_n(x, y) \leq 0$.

If $v_n(x+1, y+1) - v_n(x, y) = 0$, then the result is immediate since $v_n(x, y-1) \geq v_n(x+1, y)$.

If $v_n(x+1, y+1) < v_n(x, y)$, then it is optimal not to order at $(x+1, y+1)$. Otherwise, it is also optimal to order at (x, y) according to Lemma 2.5, and thus $v_n(x+1, y+1) = v_n(x, y)$. In addition, the fact $v_n(x+1, y+1) < v_n(x, y)$ implies that if $x \geq 0$, then the optimal quantity to satisfy class 2 backorder at $(x+1, y+1)$ must be 0.

Hence,

$$v_n(x+1, y+1) = h(x+1)^+ + \pi_1(x+1)^- + \pi_2(y+1) + \alpha E v_{n-1}(x+1 - D_1^{n-1}, y+1 + D_2^{n-1}).$$

By Lemma 2.3, it is also optimal not to order at $(x+1, y)$.

We first show that $v_n(x+1, y) < v_n(x, y-1)$. If $x < 0$ then this relation holds straightforwardly. Now we show it in the case $x \geq 0$ by contradiction. To this end, suppose by contrary, that $v_n(x+1, y) = v_n(x, y-1)$. Thus, it is optimal not to order at $(x, y-1)$ and meanwhile, allocate positive ω units to state $(x, y-1)$ and $\omega+1$ units to state $(x+1, y)$.

Further, because for $x \geq 0$,

$$\begin{aligned} f_n(x, y, 0, \omega) - v_n(x+1, y+1) &= h(x-\omega) + \pi_2(y-\omega) + \alpha E v_{n-1}(x-\omega - D_1^{n-1}, y-\omega + D_2^{n-1}) \\ &\quad - [h(x+1) + \pi_2(y+1) + \alpha E v_{n-1}(x+1 - D_1^{n-1}, y+1 + D_2^{n-1})] \\ &\leq h(x-\omega) + \pi_2(y-\omega) + \alpha E v_{n-1}(x-\omega - D_1^{n-1}, y-\omega-1 + D_2^{n-1}) \\ &\quad - [h(x+1) + \pi_2(y+1) + \alpha E v_{n-1}(x+1 - D_1^{n-1}, y + D_2^{n-1})] \\ &\leq v_n(x, y-1) - v_n(x+1, y) = 0 \end{aligned}$$

and $f_n(x, y, 0, \omega) - v_n(x+1, y+1) \geq v_n(x, y) - v_n(x+1, y+1) > 0$, a contradiction takes place. Hence, $v_n(x+1, y) < v_n(x, y-1)$.

Next, denote $L(x, y) = h(x)^+ + \pi_1(x)^- + \pi_2 y$. If it is optimal not to order at $(x, y-1)$, suppose $v_n(x, y-1) = f_n(x, y-1, 0, \omega)$, then

$$\begin{aligned}
 v_n(x, y) + v_n(x+1, y) &\leq L(x - \omega, y - \omega) + \alpha E v_{n-1}(x - \omega - D_1^{n-1}, y - \omega + D_2^{n-1}) \\
 &\quad + L(x+1, y) + \alpha E v_{n-1}(x+1 - D_1^{n-1}, y + D_2^{n-1}) \\
 &\leq L(x - \omega, y - 1 - \omega) + \alpha E v_{n-1}(x - \omega - D_1^{n-1}, y - 1 - \omega + D_2^{n-1}) \\
 &\quad + L(x+1, y+1) + \alpha E v_{n-1}(x+1 - D_1^{n-1}, y+1 + D_2^{n-1}) \\
 &= v_n(x+1, y+1) + v_n(x, y-1)
 \end{aligned}$$

Where the second inequality is due to the sub-convexity in period $n - 1$ and the fact that

$$L(x - \omega, y - \omega) + L(x+1, y) = L(x - \omega, y - 1 - \omega) + L(x+1, y+1).$$

If it is optimal to order at $(x, y - 1)$, then by Lemma 2.3, it is also optimal to order at (x, y) .

In addition, $v_n(x, y) = v_n(x, y - 1) + c$. By Lemma 2.7, $v_n(x+1, y+1) \geq v_n(x+1, y) + c$,

hence, $v_n(x, y - 1) + v_n(x+1, y+1) \geq v_n(x, y) + v_n(x+1, y)$. ■

The sub-convexity of the optimal cost function may be explained as: It is less expensive to allocate inventory to future class 1 demand for a small amount of backorders of class 2 than for a large amount. The rationing properties described in corollary 2.1 are straightforward from Lemma 2.8.

Corollary 2.1 *Let $\omega_{(x,y)}$ be the optimal ω at (x, y) , then $\omega_{(x,y+1)} \geq \omega_{(x,y)}$ and $\omega_{(x+1,y)} \leq \omega_{(x,y)} + 1$.*

We provide the following numerical example to illustrate the structure of the optimal policy. In Table 2.1, $\alpha = 0.95$, $k = 100$, $\pi_1 = 10$, $\pi_2 = 3$, $c = 2$, $h = 0.5$ and $n = 3$. For each period, both class 1 and 2 demands are discrete uniformly distributed between 0 and 9. The bold line represents $s(y)$. In this example, the order-up-to level is 16. At state (6, 8), it is optimal not to order and satisfy 5 units of class 2 backorders. Thus the state after decision is (1, 3). For state (7, 8) to (10, 8), it is optimal not to order and satisfy only 4 units of class 2 backorders. Similarly, at state (8, 10), it is optimal not to order and satisfy 7 units of class 2 backorders. Whereas at state (10, 10), it is optimal not to order and satisfy only 6 units of

class 2 backorders.

Table 2.1 Example of the S/S/S model.

Parameters: $\alpha = 0.95, k = 100, h = 0.5, c = 2, \pi_1 = 10, \pi_2 = 3, n = 3$
 D_1, D_2 : discrete uniform distributed between 0 and 9

xy	0	1	2	3	4	5	6	7	8	9	10
-3	(19 ,0)	(20 ,1)	(21 ,2)	(22 ,3)	(23 ,4)	(24 ,5)	(25 ,6)	(26 ,7)	(27 ,8)	(28 ,9)	(29 ,10)
-2	(0 ,0)	(19 ,1)	(20 ,2)	(21 ,3)	(22 ,4)	(23 ,5)	(24 ,6)	(25 ,7)	(26 ,8)	(27 ,9)	(28 ,10)
-1	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(22 ,5)	(23 ,6)	(24 ,7)	(25 ,8)	(26 ,9)	(27 ,10)
0	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(25 ,9)	(26 ,10)
1	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(25 ,10)
2	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)
3	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,2)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,3)
4	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,1)	(0 ,3)	(0 ,4)	(0 ,4)	(0 ,4)	(0 ,4)
5	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,4)	(0 ,5)	(0 ,5)	(0 ,5)
6	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,5)	(0 ,6)	(0 ,6)
7	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,6)	(0 ,7)
8	(0 ,0)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,7)
9	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)
10	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)

A special case of the N/S/S model is $k = 0$. Since the decision is made after demand arrival in every period, it is obvious that a lot-for-lot policy is optimal: the system entails inventory requirements equaling the demand requirements that were not satisfied by existing inventory levels in every period. Therefore, only purchasing cost incurs.

Another special case of the model considers the ordering policy without rationing: satisfy class 1 and 2 demands as much as possible and do not allow any backorder whenever there is inventory on hand. To formulate this problem, we need only modify the second constraint in equation (1), i.e., replacing $0 \leq \omega \leq \min\{y, (x + Q)^+\}$ by $\omega = \min\{y, (x + Q)^+\}$. The optimality of the state dependent (s, S) policy can be easily extended to the case when no rationing is considered.

The next theorem states the concavity of the cost function in any of the cost parameters.

Theorem 2.2 For the N/S/S model, if fixing state (x, y) , period n and other parameters, the optimal expected cost function $v_n(x, y)$ is a nondecreasing and concave function of h (or c , k , π_1 , π_2) when else are fixed.

Proof We only show that $v_n(x, y)$ is nondecreasing and concave in h , as proofs for other parameters are similar. Denote the optimal expected cost function $v_n(x, y)$ as $v_n(h, x, y)$, or simply as $v(h)$, and the cost function $f_n^\Pi(x, y)$ associated with policy Π as $f^\Pi(h)$. Consider any h_1 and h_2 . Without loss of generality, $h_1 \leq h_2$.

(1) Suppose $v(h_1) = f^{\Pi_1}(h_1)$ and $v(h_2) = f^{\Pi_2}(h_2)$. Because Π_1 is also feasible for h_2 , and Π_2 is feasible for h_1 , $v(h_1) = f^{\Pi_1}(h_1) \leq f^{\Pi_2}(h_1) \leq f^{\Pi_2}(h_2) = v(h_2)$. Hence, v_n is nondecreasing in h .

(2) Suppose $v(h) = f^{\Pi_1}(h)$. Increase h by Δh every time. Suppose

$v(h + \Delta h) = f^{\Pi_2}(h + \Delta h)$ and $v(h + 2\Delta h) = f^{\Pi_3}(h + 2\Delta h)$. Because Π_1 is feasible for h , $h + \Delta h$ and $h + 2\Delta h$ (Π_2 and Π_3 , too), and obviously,

$$f^{\Pi_i}(h) \leq f^{\Pi_i}(h + \Delta h) \leq f^{\Pi_i}(h + 2\Delta h), \text{ for } i = 1, 2, 3,$$

$$f^{\Pi_2}(h + \Delta h) - f^{\Pi_2}(h) \leq f^{\Pi_2}(h + \Delta h) - f^{\Pi_1}(h) = v(h + \Delta h) - v(h),$$

$$f^{\Pi_2}(h + 2\Delta h) - f^{\Pi_2}(h + \Delta h) \geq f^{\Pi_3}(h + 2\Delta h) - f^{\Pi_2}(h + \Delta h) = v(h + 2\Delta h) - v(h + \Delta h).$$

Given a policy, the change of h does not affect the inventory on hand and backorders in each period. Therefore, $f^{\Pi_2}(h + \Delta h) - f^{\Pi_2}(h) = f^{\Pi_2}(h + 2\Delta h) - f^{\Pi_2}(h + \Delta h)$, and thus

$$v(h + \Delta h) - v(h) \geq v(h + 2\Delta h) - v(h + \Delta h).$$

$$v(h + 2\Delta h) - v(h + \Delta h) \geq v(h + 3\Delta h) - v(h + 2\Delta h),$$

$$v(h + 3\Delta h) - v(h + 2\Delta h) \geq v(h + 4\Delta h) - v(h + 3\Delta h), \dots$$

We conclude that $v_n(h, x, y)$ is concave in h . ■

The findings of Theorem 2.2 are quite intuitive: a large marginal holding cost induces to hold less stock on average, and vice versa. Hence, an increase in a large holding cost will incur smaller additional cost than that in a small holding cost. A large setup cost leads to order less frequently, and vice versa. This explains why the incremental cost by an increase in a large setup cost is smaller than that of a small setup cost. The results for c, π_1 and π_2 can be explained similarly.

2.3 Structure of the optimal policy for the N/D/S model

In this section, we study the optimal structure of the N/D/S model. We will show that N/D/S is a special case of the N/S/S model. Thus all results in previous section still hold. For the N/D/S model, we further characterize the structure of the optimal rationing policy, which may not hold for the general N/S/S model.

Define λ_j as the deterministic demand in period j . The N/D/S model can be formulated as:

$$\begin{aligned} v_n(x, y) = \min_{(Q, \omega)} \{ & k\delta(Q) + cQ + h(x + Q - \omega) + \pi_2(y - \omega) \\ & + \alpha E v_{n-1}(x + Q - \omega - \lambda_{n-1}, y - \omega + D_2^{n-1}) : \\ & Q \geq \max\{0, (x)^-\}, 0 \leq \omega \leq \min\{y, x + Q\} \} \end{aligned} \quad (2.2)$$

The first constraint guarantees that deterministic demand must be satisfied immediately. For the S/D/S model, λ_{n-1} and D_2^{n-1} in (2) are replaced by λ and D_2 respectively.

Recall that for the N/D/S model: (a) class 1 demand is deterministic; (b) class 1 demand must be satisfied immediately. To prove that the optimal structure of the ordering/rationing policy in section 2.2 holds for the N/D/S model, we first remove assumption (b), allowing deterministic demand to be backlogged. Thus the optimal structure of N/S/S will apply, since deterministic demand can be considered as a random variable with a special distribution

where demand is a constant. Next, it is sufficient to show that under a particular situation, the optimal policy does not permit any class 1 backorder. In fact, the following theorem states that N/D/S is a special case of N/S/S, and thus a state dependent (s, S) policy is also optimal for N/D/S.

Theorem 2.3 *For the N/D/S model, a state-dependent (s, S) ordering policy is optimal. That is, there exists a switching curve $x = s(y)$, such that it is optimal to order to an order-up-to level $(S, 0)$ if $x \leq s(y)$, and do not order otherwise.*

Proof We only need to show that N/D/S is a special case of N/S/S. First, we remove the constraint that deterministic demand must be satisfied immediately. Assume the backordering cost for deterministic demand per period is π_1 ($\pi_1 > \pi_2 > c$). Because deterministic demand (a constant) can be considered as a random variable with a special distribution that demand in every period is a constant with probability 1, this problem is a special case of the N/S/S model. Next, it is sufficient to show that under particular condition, i.e. if π_1 is sufficient large, even if class 1 backorder is allowed, backorder will never occur under an optimal policy.

Consider any policy Π that allows class 1 backorder, if one unit backorder of deterministic demand occurs in some period, say n , we construct an alternative policy Π_1 : order one more unit and clear this unit of backorder. Comparing costs of the two policies, it is easy to identify that at the worst case, Π_1 incurs $k + c - \pi_1$ more cost in period n . After period n , one more unit of class 1 backorder will never provide any benefit to reduce the total expected cost, which implies that the total cost of Π_1 from period $n - 1$ to 1 is less than that of Π . Therefore, if $\pi_1 > k + c$, Π_1 is always better than Π , and class 1 backorder will never happen under an optimal policy. This is just the case of N/D/S. ■

As we have pointed out in section 2.3, it is hard to characterize an optimal rationing policy under a general condition. For the S/D/S lost-sales model, however, Frank et al. [23] showed that when it is optimal not to order, the rationing policy either satisfies all the stochastic demand or reserves inventory as an integer multiple of deterministic demand. Next, we extend this result to the N/D/S backordering case.

From Lemma 2.1, we only need to consider the region where it is optimal not to order. Assume it is optimal not to order at state (x, y) in period n . For the rationing decision, (I) we either satisfy as much as possible class 2 demand, i.e., $\omega = y$ if $x \geq y > 0$, or $\omega = x$ if $y > x \geq 0$, or (II) we reserve some inventory on hand and leave part or all class 2 demand backlogged. Define integer set M_n for period n ($n \geq 1$):

$$M_n = \{0, \lambda_{n-1}, \lambda_{n-1} + \lambda_{n-2}, \lambda_{n-1} + \lambda_{n-2} + \lambda_{n-3}, \dots, \sum_{j=1}^{n-1} \lambda_j\}.$$

Let $m_0 = 0, m_1 = \lambda_{n-1}, m_2 = \lambda_{n-1} + \lambda_{n-2}, \dots, m_{n-1} = \sum_{j=1}^{n-1} \lambda_j$, then M_n can be written as

$M_n = \{m_0, m_1, m_2, \dots, m_{n-1}\}$, where m_i is nondecreasing in i . M_n presents the set of cumulated future deterministic demand. For case (II), since reserved inventory is only for satisfying the future deterministic demand, the quantity we need reserve should belong to M_n .

Lemma 2.9 *For the N/D/S model, if it is optimal not to order at state (x, y) in period n , then the optimal policy is $(0, \omega)$, where $\omega = \min\{y, x - m\}$, $m \in M_n$ and $x - m \geq 0$.*

Proof If $y = 0$ or $x \leq 0$ then $\omega = 0$ and the result holds trivially. Now consider $xy > 0$.

There are two cases:

Case I: $0 < x \leq y$

To simplify, assume deterministic demand in any period is positive. Define $m_n = x$ if $x > m_{n-1}$, and $m_n = m_{n-1} + 1$ otherwise. Because m_i is increasing in i , there exists a unique $0 \leq i \leq n-1$, such that $m_i < x \leq m_{i+1}$, as illustrated in Figure 2.2.

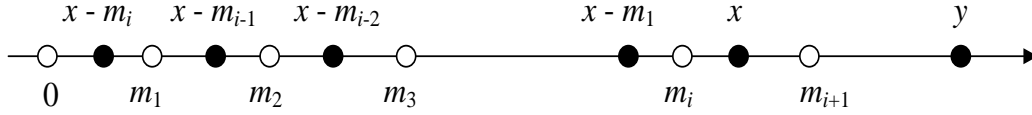


Figure 2.2 Proof of Lemma 2.9. Case I: $y \geq x$.

In this case, we need show that $\omega = x - m$ is optimal for some $m \in \{0, m_1, m_2, \dots, m_i\}$. We prove by contradiction. Suppose the optimal decision in period n is $(0, x - m_r - l)$ for some $m_r \in \{0, m_1, m_2, \dots, m_i\}$ and $0 < l < m_{r+1} - m_r$ if $r < i$, or $0 < l < x - m_i$ if $r = i$. Now consider an arbitrary sample path $(d^n, d^{n-1}, \dots, d^1)$ of the stochastic demand from period n to 1. Let Π denote the optimal policy and assume period j is the first period after n ($0 \leq j < n$) during which an order is placed. Since $\omega_n = x - m_r - l$ and $y \geq x$, at least l unit inventories can be reserved in period n to fulfill stochastic demands in following periods. Suppose that the state of period j resulting from implementing policy Π is (x_j, y_j) where $x_j = m_r + l - m_j - l'$ is nonnegative and $l' \geq 0$. If $l' > 0$ then Π allocates positive inventories to stochastic demand in periods $n-1$ through $j+1$. Otherwise, $l' = 0$ and $\omega_{n-1} = \omega_{n-2} = \dots = \omega_{j-1} = 0$. We claim in both cases that Π can be improved by satisfying one more unit of class 2's backorder in period n to reduce the cost of holding/backlogging in periods n through j . As a matter of fact, if $l' > 0$, without loss of generality, let $\Pi = (0, \omega_n, 0, \omega_{n-1}, 0, \dots, Q_j, y_j, \dots, Q_1, y_1)$ and $\omega_{n-1} > 0$. We adapt Π into policy $\Pi_1 = (0, \omega_n + 1, 0, \omega_{n-1} - 1, 0, \dots, Q_j, y_j, \dots, Q_1, y_1)$, which obviously is feasible. In addition,

$f_n^{\Pi}(x,y) - f_n^{\Pi_1}(x,y) > h + \pi_2 > 0$, which implies that Π is not optimal, a contradiction. If $l=0$, then $\Pi = (0, \omega_n, 0, 0, \dots, 0, 0, Q_j, y_j, \dots, Q_1, y_1)$ can be proven to be outperformed by policy $\Pi_1 = (0, \omega_n + 1, 0, 0, \dots, 0, 0, Q_j, y_j - 1, \dots, Q_1, y_1)$ in a similar spirit.

Case II: $0 < y < x$

Similarly, there exists unique a and b , $0 \leq b \leq a \leq n-1$ such that $m_a < x \leq m_{a+1}$ and $m_b < y \leq m_{b+1}$, as illustrated in Figure 2.3.

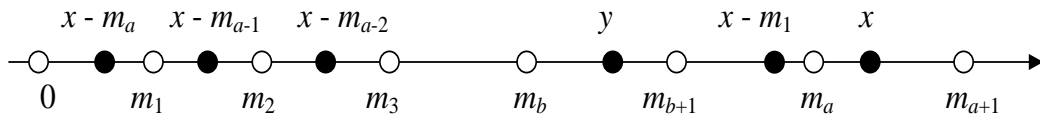


Figure 2.3 Proof of Lemma 2.9. Case II: $0 < y < x$.

Consider two sub cases: (1) $y = x - m_c$ for some $m_c \in \{m_1, m_2, \dots, m_a\}$ and (2) there exists a unique $m_c \in \{m_1, m_2, \dots, m_a\}$ such that $x - m_c < y < x - m_{c-1}$ or $0 < y < x - m_a$.

For (1), by contradiction, suppose $\omega = x - m_r - l$ is optimal, where $r \geq c$ and $0 < l < m_{r+1} - m_r$. We can construct a new policy $\omega = x - m_r$ with cost lower than the optimal one, which implies $\omega = x - m_r - l$ is not optimal. We omit the details since the proof is similar as in case I.

For (2), consider three possibilities: $x - m_c < \omega < y$, or $x - m_{r+1} < \omega < x - m_r$ for some r , where $c \leq r < a-1$, or $0 \leq \omega < x - m_a$. If $x - m_c < \omega < y$, similarly as the proof in case I, we can improve the rationing policy by constructing a policy with $\omega = y$. If $x - m_{r+1} < \omega < x - m_r$, we can improve the policy by constructing a policy with $\omega = x - m_r$. If $0 \leq \omega < x - m_a$, we can improve the policy by constructing a policy with $\omega = x - m_a$. Here we omit the details.

To summarize, the optimal policy must satisfy $\omega = \min\{y, x - m\}$ for some $m \in M_n$ and

$$x - m \geq 0. \quad \blacksquare$$

Corollary 2.2 For the S/D/S model, if it is optimal not to order at state (x, y) in period n , then the optimal policy is $(0, \omega)$, where $\omega = \min\{y, x - k_{x,y}\lambda\}$ and $k_{x,y}$ is a nonnegative integer.

Corollary 2.2 states that for the S/D/S model, if it is optimal not to order, then the optimal rationing decision is either to satisfy all stochastic demand, or to reserve an integer multiple of deterministic demand after making the rationing decision at the beginning of the period.

Lemma 2.10 For the N/D/S model, (1) if the optimal policy for state (m_j, y) , $j = 0, 1, \dots, n-1$, in period n is (Q, ω) , then the optimal policy for $(x, y + x - m_j)$, where $m_j \leq x < m_{j+1}$ (assume $m_n = +\infty$), is $(Q, \omega + x - m_j)$. In addition, $v_n(m_j, y) = v_n(x, y + x - m_j)$.

(2) If the optimal policy for state $(x, 0)$, where $m_j \leq x < m_{j+1}$, in period n is (Q, ω) , then the optimal policy for $(x + i, i)$, $i = 1, \dots, m_{j+1} - x - 1$, is $(Q, \omega + i)$. In addition,

$$v_n(x, 0) = v_n(x + i, i).$$

Proof We only show (1) since the proof of (2) is similar. By Lemma 2.5, if it is optimal not to order at (m_j, y) , then it is optimal not to order at $(x, y + x - m_j)$, where $m_j < x < m_{j+1}$.

Suppose $v_n(m_j, y) = f_n(m_j, y, 0, \omega)$, then $(0, \omega + x - m_j)$ is also a feasible policy for $(x, y + x - m_j)$, and thus $v_n(x, y + x - m_j) \leq v_n(m_j, y)$. The case $v_n(x, y + x - m_j) < v_n(m_j, y)$ will happen only if $v_n(x, y + x - m_j) = f_n(x, y + x - m_j, 0, \omega')$ where $\omega' < \omega + x - m_j$. However, this is impossible due to Lemma 2.9. Hence, $v_n(x, y + x - m_j) = v_n(m_j, y)$ and the optimal policy for $(x, y + x - m_j)$ is $(0, \omega + x - m_j)$. If $v_n(m_j, y) = f_n(m_j, y, Q, y)$ where $Q > 0$, we need show that $v_n(x, y + x - m_j) = f_n(x, y + x - m_j, Q, y + x - m_j)$. Otherwise, suppose

$v_n(x, y + x - m_j) = f_n(x, y + x - m_j, 0, \omega')$. By Lemma 2.9, $\omega' \geq x - m_j$. Thus

$(0, \omega' - x + m_j)$ is also a feasible policy for (m_j, y) . Since it is optimal to order at (m_j, y) ,

$$v_n(x, y + x - m_j) = f_n(x, y + x - m_j, 0, \omega') = f_n(m_j, y, 0, \omega' - x + m_j) > f_n(m_j, y, Q, y) = v_n(m_j, y).$$

However, $v_n(x, y + x - m_j) \leq f_n(x, y + x - m_j, Q, y + x - m_j) = f_n(m_j, y, Q, y) = v_n(m_j, y)$.

Thus, it is optimal to order Q units at $(x, y + x - m_j)$ and $v_n(x, y + x - m_j) = v_n(m_j, y)$.

■

Remarks (1) Lemma 2.10 implies that to study the optimal rationing policy and the expected cost of the non-order region, we only need to consider the states in the sets: $\{(x, y) : x = m_j, j = 0, 1, \dots, n-1; s(y) < x\}$ and $\{(x, y) : x \geq 0; y = 0\}$. For example, once the optimal policy and the expected cost for the non-ordering states (m_1, y) and $(x, 0)$, where $m_1 \leq x < m_2$, are known, then the optimal policy and costs for all the non-ordering states with $m_1 \leq x < m_2$ are determined. (2) If the optimal policy and expect costs for all the states on line $\{x = m_{n-1}, y \geq 0\}$ and $\{x \geq m_{n-1}, y = 0\}$ are determined, then so they are for all states with $x \geq m_{n-1}$. (3) If it is optimal to order at (m_j, y) , $j = 0, 1, \dots, n-2$, then it is optimal to order the same quantity at $(m_j + i, y + i)$, $i = 1, 2, \dots, \lambda_{n-j-1} - 1$, and vice verse. If it is optimal to order at (m_{n-1}, y) , then it is optimal to order the same quantity at $(m_{n-1} + i, y + i)$, $i = 1, 2, 3, \dots$, and vice verse.

The next lemma provides a sufficient condition such that if it is optimal to order at (m_j, y) , $j = 0, 1, \dots, n-2$, then it is also optimal to order at $(m_{j+1}, y + \lambda_{n-j-1})$.

Lemma 2.11 *For the N/D/S model, if it is optimal to order at (x, y) in period n , where $x \geq m_{i-1}$, $i = 1, \dots, n-1$, then it is also optimal to order at $(x+1, y+1)$ if*

$$k < \left(\sum_{l=0}^{i-1} \alpha^l / \alpha^i \right) (h + \pi_2) \lambda_{n-i} + [\pi_2 - (1 - \alpha)c] \lambda_{n-i}.$$

Proof From Lemma 2.10, we only need to consider states with $x = m_{i-1}$, $i = 1, \dots, n$. That is, it suffices to prove that if it is optimal to order at (m_{i-1}, y) , then it is also optimal to order at $(m_i, y + \lambda_{n-i})$ if $k < (\sum_{l=0}^{i-1} \alpha^l / \alpha^i)(h + \pi_2)\lambda_{n-i} + [\pi_2 - (1 - \alpha)c]\lambda_{n-i}$. We construct the sufficient condition as follows. Note that the value of m_{i-1} is dependent on period n , throughout this proof, we use m_{i-1}^j to denote the value of m_{i-1} with respect to period j , $j = 1, \dots, n$. Since it is optimal to order at (m_{i-1}^n, y) , from Lemma 2.2, $y \geq m_{i-1}^n$. Suppose it is optimal not to order at $(m_i^n, y + \lambda_{n-i})$, then $v_n(m_i^n, y + \lambda_{n-i}) < v_n(m_{i-1}^n, y)$, which implies that $v_n(m_i^n, y + \lambda_{n-i}) = f_n(m_i^n, y + \lambda_{n-i}, 0, 0)$. Thus, a sufficient condition to ensure to place an order at $(m_i^n, y + \lambda_{n-i})$ is $f_n(m_i^n, y + \lambda_{n-i}, 0, 0) > f_n(m_{i-1}^n, y, 0, 0)$, i.e.,

$$hm_i^n + \pi_2(y + \lambda_{n-i}) + \alpha E v_{n-1}(m_{i-1}^{n-1}, y + \lambda_{n-i} + D_2^{n-1}) > hm_{i-1}^n + \pi_2 y + \alpha E v_{n-1}(m_{i-2}^{n-1}, y + D_2^{n-1}),$$

$$\text{or } (h + \pi_2)\lambda_{n-i} > \alpha E[v_{n-1}(m_{i-2}^{n-1}, y + D_2^{n-1}) - v_{n-1}(m_{i-1}^{n-1}, y + \lambda_{n-i} + D_2^{n-1})].$$

Here, $m_{i-1}^{n-1} = m_i^n - \lambda_{n-1}$ and $m_{i-2}^{n-1} = m_{i-1}^n - \lambda_{n-1}$. Consider the right hand side, for any realization of D_2^{n-1} , there are two possibilities:

$$(1) \quad v_{n-1}(m_{i-2}^{n-1}, y + D_2^{n-1}) = v_{n-1}(m_{i-1}^{n-1}, y + \lambda_{n-i} + D_2^{n-1}).$$

$$(2) \quad v_{n-1}(m_{i-2}^{n-1}, y + D_2^{n-1}) > v_{n-1}(m_{i-1}^{n-1}, y + \lambda_{n-i} + D_2^{n-1}).$$

We only need to consider case (2), since for case (1), $(h + \pi_2)\lambda_{n-i} > 0$ is obvious. Case (2) happens if and only if $v_{n-1}(m_{i-1}^{n-1}, y + \lambda_{n-i} + D_2^{n-1}) = f_{n-1}(m_{i-1}^{n-1}, y + \lambda_{n-i} + D_2^{n-1}, 0, 0)$. Hence, considering $v_{n-1}(m_{i-2}^{n-1}, y + D_2^{n-1}) \leq f_{n-1}(m_{i-2}^{n-1}, y + D_2^{n-1}, 0, 0)$, a sufficient condition to ensure to place an order at $(m_i^n, y + \lambda_{n-i})$ is:

$$\begin{aligned} (h + \pi_2)\lambda_{n-i} &> \alpha E[f_{n-1}(m_{i-2}^{n-1}, y + D_2^{n-1}, 0, 0) - f_{n-1}(m_{i-1}^{n-1}, y + \lambda_{n-i} + D_2^{n-1}, 0, 0)] \\ &= \alpha E[h \frac{m_{i-2}^{n-1}}{i-2} + \pi_2(y + D_2^{n-1}) + \alpha E v_{n-2}(m_{i-3}^{n-2}, y + D_2^{n-1} + D_2^{n-2}) \\ &\quad - hm_{i-1}^{n-1} - \pi_2(y + \lambda_{n-i} + D_2^{n-1}) - \alpha E v_{n-2}(m_{i-2}^{n-2}, y + \lambda_{n-i} + D_2^{n-1} + D_2^{n-2})] \end{aligned}$$

Where the equation is due to the facts that $m_{i-3}^{n-2} = m_{i-2}^{n-1} - \lambda_{n-2}$ and $m_{i-2}^{n-2} = m_{i-1}^{n-1} - \lambda_{n-2}$.

Because $m_{i-1}^{n-1} = m_{i-2}^{n-1} + \lambda_{n-i}$, we have

$$(1 + \alpha)(h + \pi_2)\lambda_{n-i} > \alpha^2 E[v_{n-2}(m_{i-3}^{n-2}, y + D_2^{n-1} + D_2^{n-2}) - v_{n-2}(m_{i-2}^{n-2}, y + \lambda_{n-i} + D_2^{n-1} + D_2^{n-2})].$$

We continue this process and finally we can obtain a sufficient condition as:

$$\sum_{l=0}^{i-2} \alpha^l (h + \pi_2)\lambda_{n-i} > \alpha^{i-1} E[v_{n-i+1}(0, y + \sum_{l=1}^{i-1} D_2^{n-l}) - v_{n-i+1}(\lambda_{n-i}, y + \lambda_{n-i} + \sum_{l=1}^{i-1} D_2^{n-l})].$$

Similarly, a sufficient condition for this inequality is

$$\sum_{l=0}^{i-1} \alpha^l (h + \pi_2)\lambda_{n-i} > \alpha^i E[f_{n-i}(-\lambda_{n-i}, y + \sum_{l=1}^i D_2^{n-l}, \lambda_{n-i}, 0) - f_{n-i}(0, y + \lambda_{n-i} + \sum_{l=1}^i D_2^{n-l}, 0, 0)]$$

If $i = n$ then the right side of the inequality is 0, implying that if it is optimal to order at (m_{n-1}, y) , then it is also optimal to order at $(x, y + x - m_i)$ for any $x \geq m_{n-1}$. If $i < n$, then

$$\begin{aligned} \sum_{l=0}^{i-1} \alpha^l (h + \pi_2)\lambda_{n-i} &> \alpha^i [k + c\lambda_{n-i} + \pi_2(y + \sum_{l=1}^i D_2^{n-l}) + \alpha E v_{n-i-1}(-\lambda_{n-i}, y + \sum_{l=1}^{i+1} D_2^{n-l}) \\ &\quad - \pi_2(y + \lambda_{n-i} + \sum_{l=1}^i D_2^{n-l}) - \alpha E v_{n-i-1}(-\lambda_{n-i}, y + \lambda_{n-i} + \sum_{l=1}^{i+1} D_2^{n-l})] \\ &= \alpha^i [k + c\lambda_{n-i} - \pi_2\lambda_{n-i} - \alpha c\lambda_{n-i}] \end{aligned}$$

Hence, a sufficient condition that it is optimal to order at $(m_i, y + \lambda_{n-i})$ is

$$k < (\sum_{l=0}^{i-1} \alpha^l / \alpha^i)(h + \pi_2)\lambda_{n-i} + [\pi_2 - (1 - \alpha)c]\lambda_{n-i}. \quad \blacksquare$$

For the S/D/S model, $\lambda_{n-i} = \lambda$. Since $\sum_{l=0}^{i-1} \alpha^l / \alpha^i$ is increasing when i increases, then

by letting $i = 1$, we obtain directly a sufficient condition under which only the difference between x and y decides whether to order.

Corollary 2.3 For the S/D/S model, if it is optimal to order at (x, y) in period n , where $x \geq 0$, then it is also optimal to order at $(x + 1, y + 1)$ if $k < (h + \pi_2)\lambda / \alpha + [\pi_2 - (1 - \alpha)c]\lambda$.

Remark A slightly restrictive condition than the one in Corollary 3 is $(\pi_2 + h - \alpha c)\lambda > \alpha k$, which is similar to the condition for the lost-sales stationary case in Frank et al. [23].

From Lemma 2.9, we see that if it is optimal not to order at (x, y) in period n and $x < \lambda_{n-1}$, then there is no benefit to reserve any on-hand inventory to the next period, since

there must be an order in period $n-1$. Therefore, $x = \lambda_{n-1}$ can be considered as a lower bound of the rationing region. When $x - y \geq m_{n-1}$, the on-hand inventory is sufficient to satisfy all the future deterministic demand. Then $x - y = m_{n-1}$ can be considered as a bound of the rationing region. To identify a finite rationing region, we also need the following result:

Lemma 2.12 *For the N/D/S model, if it is optimal not to order in period n and $y - x > k / (\pi_2 - c) - \lambda_{n-1}$, then the optimal policy will satisfy as much as possible class 2 backorder if there is any on-hand inventory.*

Proof Suppose $v_n(x, y) = f_n(x, y, 0, \omega)$. Then at the beginning of period $n-1$ before decision, the state is $(x - \omega - \lambda_{n-1}, y - \omega + D_2^{n-1})$. Thus,

$$(y - \omega + D_2^{n-1}) - (x - \omega - \lambda_{n-1}) \geq y - x + \lambda_{n-1} > k / (\pi_2 - c).$$

From Lemma 2.6, it is optimal to order in period $n-1$. Consequently, it is optimal to satisfy class 2 demand as much as possible in period n . ■

Lemma 2.12 provides an upper bound of the rationing region in y . Now we summarize the optimal structure of the rationing policy for the N/D/S model in the following theorem.

Theorem 2.4 *Define $\Omega = \{(x, y) : \lambda_{n-1} \leq x \text{ and } x - m_{n-1} \leq y \leq x + k / (\pi_2 - c) - \lambda_{n-1}\}$, then the rationing region of the N/D/S model must belong to Ω . Specifically,*

(1) *If $(x, y) \notin \Omega$, then it is optimal either to order and satisfy all backorder, or not to order and satisfy as much backorder as possible.*

(2) *If $(x, y) \in \Omega$ and it is optimal not to order at (x, y) , then the optimal rationing structure has the following properties: (a) Let $\omega_{(x,y)}$ be the optimal ω at (x, y) , then*

$$\omega_{(x,y+1)} \geq \omega_{(x,y)} \text{ and } \omega_{(x+1,y)} \leq \omega_{(x,y)} + 1; \text{ (b) } \omega_{(x,y)} = \min\{y, x - m\}, \text{ where } m \in M_n \text{ and}$$

$$x - m \geq 0.$$

Proof (1) is from Lemma 2.9 and 2.12. Part (a) of (2) is from Corollary 2.1 and part (b) is from Lemma 2.9. ■

Table 2.2 Example of the S/D/S model.

Parameters: $\alpha = 1, k = 30, h = 1, c = 1, \pi_2 = 2, \lambda = 3, n = 5$ D_2 discrete uniform distributed between 1 and 10											
x\y	0	1	2	3	4	5	6	7	8	9	10
-2	(12 ,0)	(13 ,1)	(14 ,2)	(15 ,3)	(16 ,4)	(17 ,5)	(18 ,6)	(19 ,7)	(20 ,8)	(21 ,9)	(22 ,10)
-1	(11 ,0)	(12 ,1)	(13 ,2)	(14 ,3)	(15 ,4)	(16 ,5)	(17 ,6)	(18 ,7)	(19 ,8)	(20 ,9)	(21 ,10)
0	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(20 ,10)
1	(0 ,0)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)
2	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)
3	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,3)
4	(0 ,0)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,4)	(0 ,4)	(0 ,4)	(0 ,4)	(0 ,4)	(0 ,4)	(0 ,4)
5	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,5)	(0 ,5)	(0 ,5)	(0 ,5)	(0 ,5)	(0 ,5)
6	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,3)	(0 ,3)	(0 ,6)	(0 ,6)	(0 ,6)	(0 ,6)	(0 ,6)
7	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,4)	(0 ,4)	(0 ,7)	(0 ,7)	(0 ,7)	(0 ,7)
8	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,5)	(0 ,5)	(0 ,8)	(0 ,8)	(0 ,8)
9	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)	(0 ,6)	(0 ,6)	(0 ,9)	(0 ,9)
10	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)	(0 ,7)	(0 ,7)	(0 ,7)	(0 ,10)
11	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)	(0 ,7)	(0 ,8)	(0 ,8)	(0 ,8)

We provide the following numerical example to illustrate the structure of the optimal policy. In Table 2.2, $\alpha = 1, k = 30, \pi_2 = 2, c = 1, h = 1$ and $n = 5$. For each period, the deterministic demand $\lambda = 3$, and the stochastic demand is discrete uniformly distributed between 1 and 10. The order region is $x < 0$ or $y \geq x + 10$. The order-up-to level is 10. At state (6, 6), it is optimal not to order and satisfy all class 2 backorders. Thus the state after decision is (0, 0). At state (7, 6), it is optimal not to order and satisfy 4 units of class 2 backorders. 3 units of inventory are reserved for the deterministic demand of the next period, such that the system does not have to place an order in the next period. The optimal policy at (7, 6) can be determined once we know that the optimal policy at (6, 5) is (0, 3). Similarly, at state (10, 10), it is optimal not to order and satisfy 10 units of class 2 backorders. Whereas at state (11, 10), it is optimal not to order and satisfy only 8 units of class 2 backorders. Thus the

inventory on hand after the decision is 3, the exact value of the deterministic demand in the next period.

2.4 Algorithm and heuristic

Based on the properties of the N/D/S model, we propose a backward induction algorithm to compute the optimal reorder points, $s_n(y)$, order up to level, S_n , and the quantity of class 2 demand to satisfy, ω_n . Starting from the last period and moving backward, the algorithm computes the optimal policy and the expected costs for the entire time horizon within the state space $\{(x, y) : \underline{x} \leq x \leq \bar{x}, 0 \leq y \leq \bar{y}\}$, where \underline{x}, \bar{x} and \bar{y} are the lower and upper bound of x , upper bound of y respectively. At the beginning of period n , the optimal policy (Q, ω) and the expected costs $v_j(x, y)$ are known for period $j = 1, 2, \dots, n-1$. The algorithm in period n follows several steps:

(1) Compute $v_n(x, 0) = f_n(x, 0, 0, 0)$ for $0 \leq x \leq \bar{x}$. The order up to level

$$S_n = \arg \min_{x \geq 0} \{x : v_n(x, 0) + cx\}.$$

(2) Starting from $(0, 0)$, compute $v_n(0, y)$. This can be done by comparing $f_n(0, y, 0, 0)$

and $f_n(0, y, S_n + y, y)$, and choosing the smaller one. Find the smallest y , denoted

as $y(0)$, such that it is optimal to order. By Lemma 2.10, the optimal policy and costs for

all (x, y) , $0 < x < m_1$, are then determined.

(3) For $j = 1, 2, \dots, n-1$, compute $v_n(m_j, y)$ and find the optimal policy for (m_j, y) .

This can be done by comparing $v_n(m_j - 1, y - 1)$ and $f_n(m_j, y, 0, 0)$. Find the smallest

y , denoted as $y(m_j)$, such that it is optimal to order. Similarly, the optimal policy and

costs for all (x, y) , $m_j < x < m_{j+1}$, are then determined.

(4) The optimal policy and costs for all (x, y) , $x > m_{n-1}$ and $x < 0$ can be determined directly by Lemma 2.10 and Theorem 2.3, respectively.

The detailed algorithm is provided in the Appendix. From the properties obtained in section 2.3, we only need to find the optimal policy for states in the sets: $\{(x, y) : x = m_j\}$, $j = 1, 2, \dots, n-1$, and $\{(x, y) : x > 0, y = 0\}$. For each state in the sets, the optimal policy can be obtained by choosing one from at most two alternatives. States in these sets are represented in Figure 2.4 by black dots. Grey dots represent the states in the order region, and white dots represent the states in the non-order region besides the black dots. Line (1) and (3) in Figure 2.4 are two bounds of the order region and rationing region respectively, which help to reduce the computation.

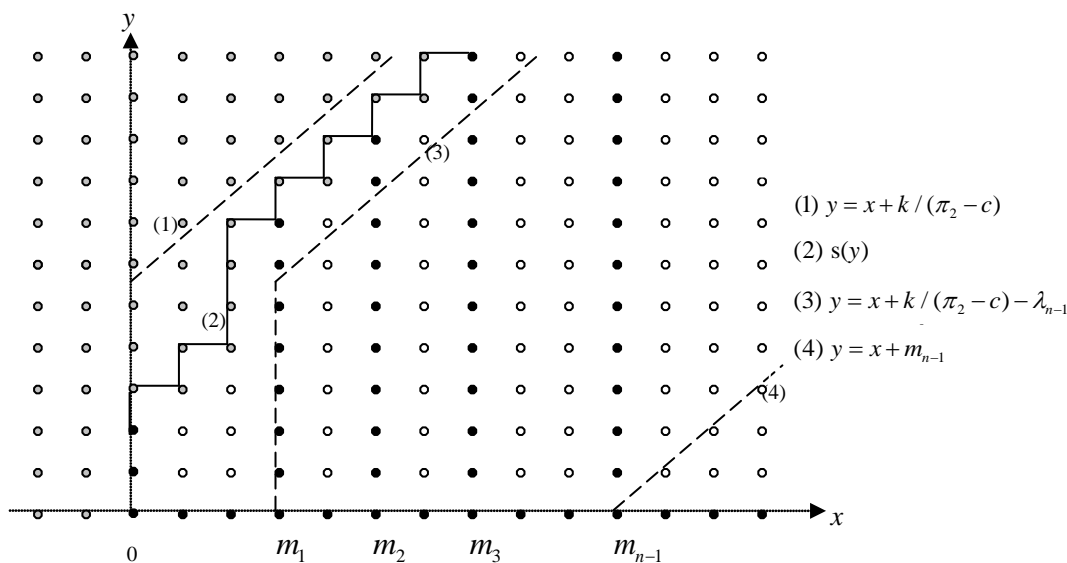


Figure 2.4 The optimal structure for the N/D/S model

How much effort does the algorithm require? When the bounds of the state space are \underline{x}, \bar{x} and \bar{y} , the time horizon is N , and the upper bound of the stochastic demand is \bar{D} , the total number of calculations to determine the optimal policy is less than $(N-1)(\bar{x} + 2(N-1)\bar{y})\bar{D}$. In our numerical examples with $N = 20, \bar{x} = \bar{y} = 1000$ and

$\bar{D} = 50$, the algorithm takes less than one minute on a Pentium III computer.

The algorithm, however, does not apply for the N/S/S model. As seen from section 2.3, when both demand classes are stochastic, the optimal ordering policy is state-dependent and the rationing policy is complicated. We therefore consider a simple heuristic policy for practice. Our heuristic is based on a modified (s, S) ordering policy and a critical level rationing policy. We simplify the ordering policy by assuming that the reorder point only depends on the difference between x and y for any $x \geq 0$. Let $y(x_0)$ denote the smallest y to place an order when $x = x_0$. Then for $x \geq 0$, we only need to compute $y(0)$, and $y(x)$ can be given by $y(x) = y(0) + x$. When $x < 0$, it is inappropriate to assume $y(x) = y(0) + x$, since $y(x)$ depends on both π_1 and π_2 . In this case, we decide whether to order by comparing $f_n(x, y, 0, 0)$ and $f_n(x, y, S_n - x + y, y)$ for (x, y) . This does not increase the computation complexity significantly (unless $k / (\pi_2 - c)$ is extremely large) due to two reasons: (1) The number of states we need to compute the cost is less than $(y(0))^2$. (2) For each state we need to compute the cost, the optimal policy is chosen from only two alternatives. Roughly speaking, the number of states we need compare is about $(y(0))^2 \cdot \pi_2 / (2\pi_1)$. When $k / (\pi_2 - c)$ is large, we can simplify the ordering policy by using a line segment, which is determined by $(0, y(0))$ and $(y(0)\pi_2 / \pi_1, 0)$, to separate the order region and the non-order region for $x < 0$. We simplify the rationing policy by assuming that it is determined by a critical level u . That is, when $x \leq u$, reserve all inventory on hand for future class 1 demand. Otherwise, satisfy class 2 demand as much as possible until the inventory level decreases to u . The critical level is determined by states (x, x) , $x > 0$: starting from $(1, 1)$, find the smallest x such that $f_n(x, x, 0, 0) > f_n(x, x, 0, 1)$. We

summarize the heuristic policy as follows:

- (0) At the beginning of period n , $v_{n-1}(x, y)$ is obtained.
- (1) Find the order-up-to level: $S_n = \arg \min_{x \geq 0} \{x : v_n(x, 0) + cx\}$.
- (2) Determine the order region. (i) find $y(0)$; (ii) $y(x) = y(0) + x$ for $x \geq 0$; (iii) for $x < 0$, starting from -1 , find $y(x)$ until the largest x such that $y(x) = 0$.
- (3) Find the critical level u for rationing: $u = x^* - 1$, where x^* is the smallest x such that $f_n(x, x, 0, 0) > f_n(x, x, 0, 1)$.

We use the following experiments to test the heuristic. For all our examples, we assume that parameters are stationary from period to period. Let $n = 5$, $c = 1$ and $\alpha = 0.95$. We consider different values of $k = 10, 20, 50, 100, 200$ and 500 ; $h = 0.5, 1, 2, 5$ and 10 ; $\pi_1 = 3, 5, 10, 20$ and 50 ; $\pi_2 = 2, 3, 5, 10$ and 20 . This gives a total of 450 combinations (note $\pi_1 > \pi_2$). For each combination, we test four groups of distributions: (I) $p\{D_1 = 1\} = p\{D_1 = 9\} = 0.5$, denoted by $B[1, 9]$, and D_2 is discrete uniformly distributed between 1 and 10, denoted by $U[1, 10]$; (II) $D_1 \sim U[1, 10]$ and $D_2 \sim U[1, 10]$; (III) $D_1 \sim T[0, 6]$ and $D_2 \sim T[0, 6]$, where $T[0, 6]$ denotes a discrete triangular distribution with $(p_0, p_1, p_2, p_3, p_4, p_5, p_6) = (0.0625, 0.125, 0.1875, 0.25, 0.1875, 0.125, 0.0625)$; (IV) $D_1 \sim U[1, 10]$ and $D_2 \sim T[0, 6]$. The state space is truncated to $\{-100 \leq x \leq 150, 0 \leq y \leq 250\}$. The heuristic algorithm is implemented on a Pentium III computer using Excel. It takes only seconds to find the heuristic solution. We compute the percentage difference in costs between the optimal solution and the heuristic solution for each state and take the maximum over all possible states.

Some of the numerical results are reported in Table 2.3. Here, $s_{(x,0)}$ is the largest

reorder point on $y = 0$ and $s_{(0,y)}$ is the smallest reorder point on $x = 0$. We found that the heuristic performs well for a large majority of the cases. When the setup cost is small ($k \leq 50$), the heuristic is actually optimal for all the cases. The percentage difference increases when the setup cost increases. The largest percentage difference of our examples is 9.27% in a case when $k = 500$. Even for those cases when the percentage difference is not near to zero (e.g., those between 6 and 10), we found that for most of the possible states, the percentage difference between the optimal and heuristic solutions is zero or near to zero. Therefore, considering the fact that the percentage difference we tested is only the worst case of all possible states, the percentage difference in costs on average is much less than the number reported in Table 2.3.

Table 2.3 Algorithm (a) Distributions: I and II

Parameters				Distributions: I		Distributions: II	
k	h	π_1	π_2	$(s_{(x,0)}, s_{(0,y)}, u, S)$	Error (%)	$(s_{(x,0)}, s_{(0,y)}, u, S)$	Error (%)
10	1	5	3	(-3, 4, 0, 0)	0	(-3, 4, 0, 0)	0
10	5	3	2	(-4, 5, 0, 0)	0	(-4, 6, 0, 0)	0
20	1	3	2	(-6, 10, 0, 0)	0	(-7, 10, 0, 0)	0
20	5	5	3	(-4, 7, 0, 0)	0	(-4, 7, 0, 0)	0
50	1	5	3	(-8, 13, 0, 10)	0	(-8, 13, 0, 10)	0
50	2	3	2	(-12, 18, 0, 9)	0	(-12, 18, 0, 9)	0
100	1	5	3	(-12, 20, 0, 19)	0	(-9, 16, 0, 33)	0
100	1	20	3	(-3, 16, 1, 33)	3.23	(-3, 17, 0, 38)	0
100	2	10	3	(-6, 20, 0, 19)	0.05	(-7, 22, 0, 18)	0
100	5	10	3	(-7, 24, 0, 9)	0	(-8, 24, 0, 8)	0
200	1	5	3	(-10, 17, 0, 32)	0	(-11, 17, 0, 34)	0
200	1	10	3	(-6, 19, 1, 36)	2.36	(-6, 19, 0, 37)	0.17
200	1	20	3	(-3, 20, 1, 39)	3.02	(-3, 20, 0, 39)	0.05
200	1	20	10	(-4, 7, 0, 43)	0	(-7, 13, 0, 14)	0
200	5	20	3	(-6, 37, 1, 10)	2.59	(-7, 42, 5, 10)	6.48
500	1	20	5	(-4, 16, 1, 41)	4.81	(-4, 16, 0, 43)	0.08
500	1	20	10	(-4, 9, 0, 44)	0.08	(-4, 8, 0, 45)	0.29
500	5	20	5	(-10, 38, 1, 27)	2.54	(-10, 38, 2, 28)	2.13

Table 2.3 Algorithm (b) Distributions: III and IV

Parameters				Distributions: III		Distributions: IV	
k	h	π_1	π_2	$(s_{(x,0)}, s_{(0,y)}, u, S)$	Error (%)	$(s_{(x,0)}, s_{(0,y)}, u, S)$	Error (%)
10	1	5	3	(-2, 4, 0, 6)	0	(-3, 4, 0, 0)	0
10	5	3	2	(-4, 5, 0, 0)	0	(-4, 6, 0, 0)	0
20	1	3	2	(-6, 9, 0, 0)	0	(-7, 10, 0, 0)	0
20	5	5	3	(-4, 7, 0, 0)	0	(-4, 7, 0, 0)	0
50	1	5	3	(-5, 9, 0, 22)	0	(-7, 11, 0, 23)	0
50	2	3	2	(-10, 15, 0, 10)	0	(-12, 18, 0, 9)	0
100	1	5	3	(-8, 14, 0, 14)	0	(-8, 13, 0, 27)	0
100	1	20	3	(-3, 17, 0, 18)	0.31	(-3, 15, 0, 32)	0.06
100	2	10	3	(-5, 15, 0, 17)	0	(-6, 21, 0, 22)	0
100	5	10	3	(-8, 27, 3, 11)	6.93	(-7, 23, 0, 8)	0
200	1	5	3	(-7, 11, 3, 18)	4.16	(-9, 15, 0, 27)	0.02
200	1	10	3	(-4, 12, 5, 20)	7.04	(-5, 16, 0, 30)	1.03
200	1	20	3	(-2, 13, 0, 22)	3.00	(-3, 17, 0, 33)	1.13
200	1	20	10	(-6, 11, 0, 17)	0	(-3, 6, 0, 36)	0
200	5	20	3	(-5, 31, 3, 13)	6.40	(-7, 42, 3, 18)	4.23
500	1	20	5	(-3, 11, 9, 23)	8.19	(-9, 33, 6, 24)	7.94
500	1	20	10	(-3, 6, 0, 25)	0.12	(-4, 7, 0, 36)	0
500	5	20	5	(-6, 23, 7, 15)	9.27	(-9, 33, 6, 22)	7.94

2.5 Extensions

In this section, we discuss some extensions of the N/S/S and N/D/S model.

In the basic N/S/S model, we have assumed that an order placed in a given period arrives at the same period, i.e., replenishments are immediate. A positive lead time is often incurred between the placement of the order and its receipt. Therefore, one interesting question we are interested in is whether assuming positive order lead time will change our insights.

Consider the S/S/S model with an order lead time of one period. We assume that in every period we receive our delivery of the order from the previous period, observe demand, and then make the ordering and rationing decisions. Therefore, x now denotes the inventory on hand, which includes the amount left over from the previous period plus the delivered order

from the previous period. We can then write the optimality equation as:

$$v_n(x, y) = \min_{(Q, \omega)} \{ k\delta(Q) + cQ + h(x - \omega)^+ + \pi_1(x - \omega)^- + \pi_2(y - \omega) \\ + \alpha E v_{n-1}(x + Q - \omega - D_1^{n-1}, y - \omega + D_2^{n-1}) : \\ Q \geq 0, 0 \leq \omega \leq \min\{y, x^+\} \};$$

Table 2.4 illustrates the results of a numerical example with the same parameters as that in Table 2.1, except that the order lead time is one. As we can see, the order point can still be characterized by a monotone switch curve (the bold line in Table 2.4). However, since one order can only arrive at the beginning of the next period, when $x \leq 0$, class 2 backorder cannot be satisfied in the current period. Therefore, different with the example in Table 2.1, $\omega = 0$ now for all $x \leq 0$. The order-up-to level is 14, which now represents the inventory position (= inventory on hand + inventory on order – class 1 and 2 backorders).

Table 2.4 Example of the S/S/S model with LT=1.

Parameters: $\alpha = 0.95, k = 100, h = 0.5, c = 1, \pi_1 = 10, \pi_2 = 3, n = 3, LT = 1$
 D_2 discrete uniform distributed between 0 and 9

x\y	0	1	2	3	4	5	6	7	8	9	10
-6	(20 ,0)	(21 ,0)	(22 ,0)	(23 ,0)	(24 ,0)	(25 ,0)	(26 ,0)	(27 ,0)	(28 ,0)	(29 ,0)	(30 ,0)
-5	(19 ,0)	(20 ,0)	(21 ,0)	(22 ,0)	(23 ,0)	(24 ,0)	(25 ,0)	(26 ,0)	(27 ,0)	(28 ,0)	(29 ,0)
-4	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(22 ,0)	(23 ,0)	(24 ,0)	(25 ,0)	(26 ,0)	(27 ,0)	(28 ,0)
-3	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(24 ,0)	(25 ,0)	(26 ,0)	(27 ,0)
-2	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(26 ,0)
-1	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)
0	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)
1	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)
2	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)
3	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)
4	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)	(0 ,0)
5	(0 ,0)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)	(0 ,1)
6	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)	(0 ,2)

In the N/D/S model, we have assumed that class 1 demand λ_j ($j = 1, 2, \dots, n$) is known at the beginning of period n . The assumption can be relaxed. For example, in period n , rather than all demands in future periods are known, we may only know demand of class 1 for the

next L periods ($L \leq n$). This often happens when advance demand information for a product is obtained when customers place orders in advance for a future delivery (See e.g., Özer & Gallego [41]).

We next extend the N/D/S model by considering both the positive order lead time and the advance demand information. We make the following assumptions: (1) An order from one period arrives at the beginning of the next period; (2) class 1 demand information arrives one period in advance of its requested due date; (3) class 1 demand must be satisfied on the due date.

At the beginning of period n , the order from the previous period arrives. Class 1 demand for the next period, d_1^{n-1} , and class 2 demand for the current period are observed. The current period class 1 demand will be satisfied first. Then decisions are made on how much of class 2 demand to satisfy, and if an order should be placed. By assumption, d_1^{n-1} must be satisfied in period $n-1$. An order therefore must be placed if the current inventory level is less than d_1^{n-1} . The order, if any, will arrive at the beginning of the next period. Let x and y denote the inventory level and class 2 backorder before decisions are made. Then the problem can be formulated as:

$$\begin{aligned} v_n(x, y, d_1^{n-1}) = \min_{(Q, \omega)} \{ & k\delta(Q) + cQ + h(x - \omega) + \pi_2(y - \omega) \\ & + \alpha E v_{n-1}(x + Q - \omega - d_1^{n-1}, y - \omega + D_2^{n-1}, D_1^{n-2}) : \\ & x - \omega + Q \geq d_1^{n-1}, 0 \leq \omega \leq \min\{x, y\} \} \end{aligned}$$

for $n \geq 2$. Here, we assume $D_1^0 = D_2^0 = 0$ and $v_0(\cdot, \cdot, \cdot) = 0$. Obviously, it is optimal not to order in period 1 and satisfy as much as possible class 2 backorder, i.e., $v_1(x, y, 0) = h(x - \omega) + \pi_2(y - \omega)$, where $\omega = \min\{x, y\}$.

Table 2.5 illustrates the results of a numerical example with the same parameters as that

in Table 2.2, except that the supply lead time is one. We can see that the optimal structure of the N/D/S model still holds. $s(y)$ is represented by the bold line. Because the supply lead time is one, when $x < 3$, an order must be placed such that the deterministic demand of the next period can be satisfied. Comparing with Table 2.2, the order-up-to level is increased (10 and 18 for Table 2.2 and 2.5 respectively).

Table 2.5 Example of the S/D/S model with $LT=1$.

Parameters: $\alpha = 1, k = 30, h = 1, c = 1, \pi_2 = 2, \lambda = 3, n = 5, LT = 1$ D_2 discrete uniform distributed between 1 and 10											
xy	0	1	2	3	4	5	6	7	8	9	10
0	(22 ,0)	(23 ,0)	(24 ,0)	(25 ,0)	(26 ,0)	(27 ,0)	(28 ,0)	(29 ,0)	(30 ,0)	(31 ,0)	(32 ,0)
1	(21 ,0)	(22 ,1)	(23 ,1)	(24 ,1)	(25 ,1)	(26 ,1)	(27 ,1)	(28 ,1)	(29 ,1)	(30 ,1)	(31 ,1)
2	(20 ,0)	(21 ,1)	(22 ,2)	(23 ,2)	(24 ,2)	(25 ,2)	(26 ,2)	(27 ,2)	(28 ,2)	(29 ,2)	(30 ,2)
3	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,3)	(0 ,3)	(25 ,3)	(26 ,3)	(27 ,3)	(28 ,3)	(29 ,3)
4	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,4)	(0 ,4)	(25 ,4)	(26 ,4)	(27 ,4)	(28 ,4)
5	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,5)	(0 ,5)	(25 ,5)	(26 ,5)	(27 ,5)
6	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)	(0 ,6)	(0 ,6)	(25 ,6)	(26 ,6)
7	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)	(0 ,7)	(0 ,7)	(0 ,7)	(25 ,7)
8	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)	(0 ,7)	(0 ,8)	(0 ,8)	(0 ,8)
9	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)	(0 ,7)	(0 ,8)	(0 ,9)	(0 ,9)
10	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)	(0 ,7)	(0 ,8)	(0 ,9)	(0 ,10)
11	(0 ,0)	(0 ,1)	(0 ,2)	(0 ,3)	(0 ,4)	(0 ,5)	(0 ,6)	(0 ,7)	(0 ,8)	(0 ,9)	(0 ,10)

We note that the analytical results obtained in section 2.2 and 2.3 cannot be simply extended to the model with one period supply lead time. The difficulty is that Lemma 2.2 does not hold any more, even we revise the lemma as: If $x > y + d_1^{n-1}$, then it is optimal not to order. The rationale is, even if $x > y + d_1^{n-1}$, it is still possible to place an order in case that the expected amount of class 2 demand in period $n - 1$ is large, since a large number of class 2 backorders will incur a high backordering cost. The problem becomes more complicated when the supply lead time is greater than one period. As mentioned in Frank et al. [23], we have to keep track of orders at all periods. The knowledge of the inventory position or the inventory on hand is not sufficient to determine the optimal policy. However, our

numerical examples show that a state dependent (s, S) policy carries over to the case with one period lead time.

In the basic N/S/S model, we have assumed that there are two prioritized demand classes. One interesting generalization is the case with multiple demand classes. Considering the problem with three demand classes as an example, let x_1 be the inventory on hand if $x_1 \geq 0$, or class 1 backorder if $x_1 < 0$, x_2 and x_3 be the class 2 and 3 backorders respectively. It is easy to show that demand from a lower priority class should be satisfied only if all demands from higher priority classes are cleared up.

In this case, a state dependent (s, S) ordering policy remains optimal. This can be proved by using similar approach as in section 2.3 while Lemmas, e.g., Lemma 2.1-2.4, are adapted as follows:

(1) If it is optimal to order at (x_1, x_2, x_3) , then all backorders must be satisfied. (2) If $x_1 > x_2 + x_3$, then it is optimal not to order. (3) If it is optimal to order $Q > 0$ at state (x_1, x_2, x_3) , then it is optimal to order $Q+1$ at state (x_1, x_2+1, x_3) or (x_1, x_2, x_3+1) . (4) If it is optimal to order $Q > 0$ at state (x_1+1, x_2, x_3) , then it is optimal to order $Q+1$ at state (x_1, x_2, x_3) .

2.6 Conclusions

In this chapter, we consider a periodic review, single inventory system with a fixed setup cost. There are two stochastic demand classes, which differ in their backordering costs. We show that the optimal ordering policy can be characterized by a state dependent (s, S) policy. The optimal rationing policy is partially obtained based on the property of sub-convexity. The

optimal structure still holds for the model with one deterministic and one stochastic demand class, with the assumption that the deterministic demand must be satisfied immediately. We further obtain additional rationing structural properties for the one-deterministic-one-stochastic model. Based on these properties, the optimal ordering and rationing policy for any state can be generated by finding the optimal policy of only a finite set of states. For each state in this set, the optimal policy is obtained by simply choosing a policy from at most two alternatives. An efficient algorithm is then proposed. For the general case with two stochastic demand classes, we propose a simple heuristic rationing policy, which is easy to implement. The heuristic is close to optimal for a large majority of our numerical examples.

2.7 Appendix

Algorithm for N/D/S

Step 0: (Initialization)

For $x=0$ *to* \bar{x}_0 , $y=0$ *to* \bar{y}_0 ,

$$v_0(x, y) = 0$$

Next

Step 1: (Iterative Step)

For $n=1$ *to* N

For $x=0$ *to* \bar{x}

$$v_n(x, 0) = f_n(x, 0, 0, 0), Q_{(x,0)} = 0, \omega_{(x,0)} = 0$$

Next

$$S_n = \arg \min_{x \geq 0} \{x : v_n(x, 0) + cx\}$$

$$y = 0$$

Do while $y < k / (\pi_2 - c)$ *and* $y(m_0) = 0$

If $f_n(0, y, 0, 0) < f_n(0, y, S_n + y, y)$ *then*

$$v_n(0, y) = f_n(0, y, 0, 0), Q_{(0,y)} = 0, \omega_{(0,y)} = 0$$

Else

$$v_n(0, y) = f_n(0, y, S_n + y, y), Q_{(0,y)} = S_n + y, \omega_{(0,y)} = y; y(m_0) = y$$

End if

$$y = y + 1$$

Loop

For $y = y(m_0) + 1$ *to* \bar{y}

$$v_n(0, y) = v_n(0, y(m_0)) + c(y - y(m_0)), Q_{(0,y)} = S_n + y, \omega_{(0,y)} = y$$

Next

For $x = m_0 + 1$ *to* $m_1 - 1$ (*to* \bar{x}_n *if* $n = 1$)

For $y = 1$ *to* \bar{y}

If $y < x$ *then*

$$v_n(x, y) = f_n(x - y, 0, Q_{(x-y,0)}, y), Q_{(x,y)} = Q_{(x-y,0)}, \omega_{(x,y)} = y$$

Else

$$v_n(x, y) = f_n(0, y - x, Q_{(0,y-x)}, x), Q_{(x,y)} = Q_{(0,y-x)}, \omega_{(x,y)} = x$$

End if

Next

Next

If $n > 1$ then

For $i = 1$ to $n - 1$

$$y = y(m_{i-1}) + \lambda_{n-i}$$

Do while $y < k / (\pi_2 - c)$ and $y(m_i) = 0$

If $v_n(m_i - 1, y - 1) < f_n(m_i, y, 0, 0)$ then

$$v_n(m_i, y) = f_n(m_i, y, S_n + y - m_i, y),$$

$$Q_{(m_i,y)} = S_n + y - m_i, \omega_{(m_i,y)} = y, y(m_i) = y$$

Else

$$v_n(m_i, y) = f_n(m_i, y, 0, 0), Q_{(m_i,y)} = 0, \omega_{(m_i,y)} = 0$$

End if

$$y = y + 1$$

Loop

For $y = y(m_i) + 1$ to \bar{y}

$$v_n(m_i, y) = v_n(m_i, y(m_i)) + c(y - y(m_i)),$$

$$Q_{(m_i,y)} = S_n + y - m_i, \omega_{(m_i,y)} = y$$

Next

For $y = 1$ to $y(m_{i-1}) + \lambda_{n-i} - 1$

If $\omega_{(m_i,y-1)} > 0$ then

$$v_n(m_i, y) = f_n(m_i, y, 0, \omega_{(m_i-1,y-1)} + 1),$$

$$Q_{(m_i,y)} = 0, \omega_{(m_i,y)} = \omega_{(m_i-1,y-1)} + 1$$

Else

If $f_n(m_i, y, 0, 0) < v_n(m_i - 1, y - 1)$ then

$$v_n(m_i, y) = f_n(m_i, y, 0, 0), Q_{(m_i, y)} = 0, \omega_{(m_i, y)} = 0$$

Else

$$v_n(m_i, y) = f_n(m_i, y, 0, \omega_{(m_i-1, y-1)} + 1),$$

$$Q_{(m_i, y)} = 0, \omega_{(m_i, y)} = \omega_{(m_i-1, y-1)} + 1$$

End if

End if

Next

For $x = m_i + 1$ *to* $m_{i+1} - 1$

For $y = 1$ *to* \bar{y}

If $y < x - m_i$ *then*

$$v_n(x, y) = f_n(x - y, 0, Q_{(x-y, 0)}, y), Q_{(x, y)} = 0, \omega_{(x, y)} = y$$

Else

$$v_n(x, y) = f_n(m_i, y - x + m_i, Q_{(m_i, y-x+m_i)}, \omega_{(m_i, y-x+m_i)} + x - m_i),$$

$$Q_{(x, y)} = Q_{(m_i, y-x+m_i)}, \omega_{(x, y)} = \omega_{(m_i, y-x+m_i)} + x - m_i$$

End if

Next

Next

Next

End if

For $x = \underline{x}$ *to* -1

For $y = 0$ *to* \bar{y}

$$v_n(x, y) = f_n(x, y, S_n + y - x, y), Q_{(x, y)} = S_n + y - x, \omega_{(x, y)} = y$$

Next

Next

Next

3. Inventory control and pricing strategies with two demand classes

3.1. Introduction

In production/inventory systems, it is very common that demand arises in various forms including booked orders and unscheduled orders. Booked orders may be from customers who have quantity commitment contracts or from internal demand of the manufacturer (for detailed analysis of quantity commitment contracts, see Anupindi & Bassok [1]). These contracts require a buyer to purchase a fixed amount in each period. These orders are known in advance and must be satisfied without delay due to significant penalties of late delivery. This type of demand can be considered as deterministic. On the other hand, unscheduled order is stochastic in nature. These orders may be from walk-in customers who request occasionally, or even from contract customers who ask for additional orders besides the booked orders. The stochastic demand has a low priority and can be backlogged.

The practice described above has been observed in various settings. For example, a large glass manufacturer in the Detroit area that signs contracts with the Big Three Auto Companies to provide glass for their current-year models (Frank et al. [23]). These long-term contracts require just-in-time deliveries and the manufacturer faces significant penalties if it misses a delivery. The manufacturer also faces demand from the after-market segment, i.e., automotive glass replacement for installation into old vehicles. The manufacturer has the option to accept or reject the after-market orders.

In most of the literature on inventory and production models, demand is treated either as a deterministic process or as a stochastic process. The (s, S) policy has been well studied for various model constructs of the problem with only the stochastic demand and a fixed setup cost. The question is: when the demand contains both a deterministic and a stochastic component, will an (s, S) policy continue to be optimal?

One of the major research contributions to studying this problem is presented by Sobel & Zhang [49]. Sobel & Zhang [49] considered a finite horizon periodic review inventory system with demand arriving simultaneously from a deterministic source and a random source. The deterministic demand has to be satisfied immediately and the stochastic demand can be backlogged. Under certain conditions, they proved that the cost function is K -convex and an (s, S) policy is optimal (Sobel & Zhang [49] call it as a modified (s, S) policy, since s is dependent on the deterministic demand in the current period).

However, one of the conditions (Equation (17)) in [41] implies that demands in consecutive periods are not independent. This condition may not be satisfied for a class of applications with fluctuating demand environment, e.g., (1) seasonal demand; (2) obsolescent demand: demand is now healthy, but there is a sizeable change that demand will drop precipitously in the future; (3) Markovian demand; and (4) price sensitive demand. Because fluctuating demand is very common in practice, it is important to study whether this condition in S&Z can be relaxed, and whether the (s, S) policy is optimal for the inventory problems with two demand classes and fluctuating demand.

In this section, we show that this condition (Equation (17)) in [41] can be relaxed. Based on the results derived in Sobel & Zhang [49], we first present two properties in section 3.2.3,

each of which is equivalent to the optimality of the (s, S) policy to the problem. These properties help to identify the (s, S) -policy optimality conditions. We then propose one such sufficient condition that is weaker than that of Sobel & Zhang [49] in section 3.2.4.

Sobel & Zhang [49] also considered a model similar to their basic one, with the only difference that one constraint is tighter. This tighter constraint forces immediate satisfaction of the deterministic demand as well as the backlogs from the previous period if there is positive deterministic demand. They concluded without proof that one can obtain results similar to their basic model. For the model with the tighter constraint, we show in section 3.2.5 that even if the restrictive condition is removed, the optimality of the ordering policy can still be characterized as an (s, S) type, but with a modified S .

As an application of this relaxation, in section 3.3, we will show how the results in section 3.2 enable us to analyze the joint pricing and inventory control problems with two demand classes.

3.2. Inventory policies for systems with stochastic and deterministic demand

3.2.2. Model formulation

Consider a periodic review inventory system with two demand classes: one deterministic and one stochastic. The deterministic demand must be satisfied immediately, while the stochastic demand can be backlogged. At the beginning of each period, the inventory is reviewed and a decision on how much to order is made. We assume that the order is delivered immediately, i.e., the lead-time is 0. The stochastic demand is then observed and will be satisfied as much

as possible from the on hand inventory after all the deterministic demand in the current period is satisfied. Unmet stochastic demand is backlogged and will be satisfied in future periods. Every time an order is placed, the system incurs a fixed setup cost as well as a variable cost proportional to the order quantity. Let

T = the planning horizon,

x_t = net inventory level at the start of period t ,

y_t = total supply of goods available to satisfy demand in period t ,

r_t = the deterministic demand in period t ,

D_t = total of the deterministic and the stochastic demands in period t ,

K_t = fixed setup cost in period t ,

c_t = variable unit cost in period t ,

$\gamma_t(\cdot)$ = inventory-related cost in period t , a convex function,

β = single-period discount factor, $\beta \geq 0$,

$f_t(\cdot)$ = the value function of the costs.

The objective is to choose an ordering policy so as to minimize the total expected cost. The original formulation of this problem given by Sobel & Zhang [49] is:

$$f_t(x) = \inf\{K_t\delta(y-x) + J_t(y) : y \geq x \vee [-(-x)^+ + r_t]\}, \quad x \in \mathfrak{R} \quad (3.1)$$

$$J_t(y) = G_t(y) + \beta E[f_{t+1}(y - D_t)], \quad y \in \mathfrak{R} \quad (3.2)$$

$$G_t(y) = (c_t - \beta c_{t+1})y + E[\gamma_t(y - D_t)], \quad y \in \mathfrak{R} \quad (3.3)$$

for $t = 1, \dots, T$, where $a \vee b$ denotes $\max\{a, b\}$, $(x)^+ = \max(x, 0)$ and $f_{T+1}(\cdot) = 0$.

Sobel & Zhang [49] also consider the following problem that is similar to (3.1).

$$f_t(x) = \inf\{K_t\delta(y-x) + J_t(y) : y \geq x \text{ if } r_t = 0; y \geq x \vee r_t \text{ if } r_t > 0\} \quad (3.4)$$

The only difference between (3.1) and (3.4) is: the constraint $y \geq x \vee [-(-x)^+ + r_t]$ in (3.1)

is replaced by a similar but tighter one

$$y \geq \begin{cases} x & \text{if } r_t = 0, \\ x \vee r_t & \text{if } r_t > 0. \end{cases} \quad (3.5)$$

That is, for (3.1), deterministic demand must be satisfied without backlogging, while stochastic demand does not have to be met immediately. For (3.4), however, constraint (3.5) forces immediate satisfaction of the deterministic demand as well as the backlogs from the previous period if there is positive deterministic demand. Note when $r_t > 0$, we correct this constraint in Sobel & Zhang [49] ($y \geq r_t$) to $y \geq x \vee r_t$. This is reasonable to prevent the case $y < x$.

To facilitate discussion, we will call (3.1) the basic model in this section. In section 3.1.3 and 3.1.4, we will consider the basic model, where stochastic demand does not have to be met immediately.

3.2.3 Properties equivalent to the optimality of the (s, S) policy

Let S_t be the largest minimizer of $J_t(y)$ on \mathfrak{R} and s_t be the smallest value of y such that $J_t(y) \leq K_t + J_t(S_t)$, i.e.,

$$S_t = \sup\{y : J_t(y) \leq K_t + J_t(S_t), \text{ for all } a \in \mathfrak{R}\}$$

$$s_t = \inf\{y : J_t(y) \leq K_t + J_t(S_t)\}$$

An (s, S) policy to this problem is defined by:

$$y_t = \begin{cases} S_t & \text{if } x_t < s_t \vee r_t \\ x_t & \text{if } x_t \geq s_t \vee r_t \end{cases} \quad (3.6)$$

To ensure the optimality of an (s, S) policy, the following conditions are often assumed in the literature:

- (i) $G_t(y) \rightarrow \infty$ as $|y| \rightarrow \infty$, $t = 1, \dots, T$.
- (ii) $K_t \geq \beta K_{t+1}$, $t = 1, \dots, T - 1$.

However, without some additional condition, an (s, S) policy may not be optimal, as demonstrated in the following counterexample.

Example 1 $T = 3$, $\beta = 1$, $c_2 = c_3 = c_4 = 1$, $c_1 = 50$. $K_t = 100$, $\gamma_t(x) = 1$ if $x \geq 0$, $\gamma_t(x) = x^2$ if $x < 0$, $r_t = 5$ for $t = 1, 2, 3$. The stochastic demand is stationary, discrete uniformly distributed within $[1, 10]$. The optimal ordering policy in period 1 is shown in Table 3.1. Clearly it is not an (s, S) policy.

Table 3.1 A counterexample without additional condition

x	-3	-2	-1	0	1	2	3	4	5	6
Q	5	5	5	5	4	3	2	1	0	0

x : period beginning inventory level. Q : order quantity.

To study what additional condition is required, we present two properties, each of which is shown to be equivalent to the optimality of the (s, S) policy.

Property A For any period t with $r_t > 0$, if it is optimal to place an order, then it is optimal to clear up all the backlogs from previous periods.

Property B $S_t \geq r_t$ for any period t with $r_t > 0$.

Theorem 3.1. Property A and Property B are each equivalent to the optimality of the (s, S) policy to the problem.

Proof We show first the equivalence of Property A and Property B. Assume $S_t \geq r_t$ for period t with $r_t > 0$. Then starting at a negative inventory level, it is optimal to order up to S_t . The order quantity will be large enough to satisfy the required deterministic demand r_t and clear up all the backlogs. Conversely, if Property A holds, then we must have $S_t \geq r_t$ if $r_t > 0$. Otherwise if $S_t < r_t$, then for any inventory level $x \leq S_t - r_t$, it has to order because of the deterministic demand and the optimal order-up-to level is S_t . However, the corresponding order quantity will not be enough to clear up all the backlogs after satisfying the r_t units of deterministic demand.

Next we show the equivalence of Property B and the optimality of the (s, S) policy. Sobel & Zhang [49] have proven that when $S_t \geq r_t$, then an (s, S) policy is optimal. If we relax the condition by assuming $S_t \geq r_t$ only when $r_t > 0$, then the optimality of the (s, S) policy can be proved similarly. Assume on the other hand an (s, S) policy defined in (3.6) is optimal. Since when $x < r_t$, it has to order because of the requirement of the deterministic demand, as such it has to be $S_t = S \geq s \geq r_t$. ■

3.2.4. The optimal policy

Property A and Property B are instrumental in identifying the conditions for the optimality of the (s, S) policy. In other words, to ensure the optimality of the (s, S) policy, it suffices to find any condition, which satisfies Property A or Property B. For example, the following condition in Sobel & Zhang [49] is a sufficient condition of Property B:

(iii) $0 \leq \underline{S}_t - r_t \leq \underline{S}_{t+1}$, $t = 1, \dots, T-1$, where \underline{S}_t minimizes $G_t(\cdot)$ on \mathfrak{R} .

Note that \underline{S}_t is the minimizer of the one-period inventory cost $G_t(\cdot)$, while S_t is the minimizer of the value function $J_t(\cdot)$.

We note that the left inequality, $0 \leq \underline{S}_t - r_t$, should be satisfied for $t = 1, \dots, T$ instead of $t = 1, \dots, T - 1$ in Sobel & Zhang [49]. Otherwise, counterexamples can be easily found.

Although (iii) is a relaxed one on Veinott [55], it does not hold for many practical applications. Even if the holding cost and penalty cost are linear and stationary, as discussed in Sobel & Zhang [49], (iii) precludes a model in which the stochastic demands are stochastically decreasing over time too rapidly or unit production costs are sharply concave and increasing with respect to t . When the inventory-related cost is not stationary, (iii) can be easily violated, as can be seen in the following example.

Example 2 $h_t = h_{t+1} = 1$, $p_t = 3$, $p_{t+1} = 1.5$, $c_t = \beta c_{t+1}$, $r_t = r_{t+1} = 1$. D_t and D_{t+1} have the same distribution with $P(2) = 0.4$, $P(4) = 0.2$, $P(6) = 0.15$, $P(8) = 0.25$.

It can be shown (by a direct application of Lemma 3.1) that the optimal policy for this problem is of (s, S) type. However, by a simple calculation, one can obtain that $\underline{S}_t = 6$ and $\underline{S}_{t+1} = 4$. Thus, (iii) does not hold.

It is not hard to see that for problems with any of the four fluctuating demand types mentioned in section 3.1, (iii) may not hold. For example, price sensitive demand has been studied in joint inventory and pricing control models (see e.g., Chen & Simchi-Levi [11], Huh & Janakiraman [30]), where a basic assumption is: demands in consecutive periods are independent, but their distributions depends on price, a decision variable. Clearly, (iii) does not hold, since demands in consecutive periods are not independent. (iii) may also not hold

when demand is seasonal or obsolescent, since demand may stochastically decrease over time rapidly.

Denote $\gamma'_t(x) = \frac{d}{dy} E[\gamma_t(y - D_t)] \Big|_{y=x^-}$. Since $D_t \geq r_t$, when $x \leq r_t$, $-\gamma'_t(x)$ is the

marginal expected penalty cost at x . By the convexity of $\gamma_t(\cdot)$, $-\gamma'_t(r_t)$ is the least unit penalty cost.

Now we propose a relaxed one of condition (iii) as follows.

$$(iv) \quad c_t - \beta c_{t+1} + \gamma'_t(r_t) \leq 0 \quad \text{for } t = 1, \dots, T.$$

One special case of (iv) is that the shortage penalty cost function is linear. Denote the unit penalty cost as p_t , then condition (iv) becomes:

$$(v) \quad p_t \geq c_t - \beta c_{t+1} \quad \text{for } t = 1, \dots, T.$$

Lemma 3.1 (iv) is a sufficient condition for Property A.

Proof Suppose $c_T - \beta c_{T+1} + \gamma'_T(r_T) \leq 0$, and it is optimal to place an order in period T . For each unit of backlogged demand at the beginning of period T , it can be either satisfied with a variable unit cost c_T , or backlogged to period $T+1$ with a salvage cost βc_{T+1} and a penalty cost that is not less than $-\gamma'_T(r_T)$ (due to the convexity of $\gamma_T(\cdot)$). Since $c_T \leq \beta c_{T+1} - \gamma'_T(r_T)$, it is optimal to clear up all the backlogs from previous periods in period T . Suppose under (iv), Property A holds for period i , $t+1 \leq i \leq T$. Now consider the case with $x < 0$ at the beginning of period t , and suppose, contrary to the proposition, it is optimal to place an order with at least one unit of demand unsatisfied (denote this policy as Π), then there are two possibilities. (1) If t_1 ($t < t_1 \leq T$) is the next ordering period after t , then by induction, all the backlogged units must be satisfied in period t_1 ; (2) no order is placed after period t and all the backlogged units are carried forward with penalty costs from period t to T

and a salvage cost in period $T + 1$. For case (1), we consider an alternative policy Π_1 : in period t , order one more unit and reduce one more unit of backlogs; in period t_1 , order one less unit and clear up all backlogs, and same as Π in other periods. It is easy to check that the difference of the expected costs by adopting Π and Π_1 is:

$$f_t^\Pi(x) - f_t^{\Pi_1}(x) \geq -\gamma'_t(r_t) - \beta\gamma'_{t+1}(r_{t+1}) - \dots - \beta^{t_1-t-1}\gamma'_{t_1-1}(r_{t_1-1}) + \beta^{t_1-t}c_{t_1} - c_t \geq 0,$$

where the first inequality is due to the convexity of $\gamma_t(\cdot)$, and the second inequality is from (iv). For case (2), Π_1 is designed as: order one more unit and reduce one more unit of backlogs in period t , and no order afterwards. Similarly, the difference of the expected costs by adopting Π and Π_1 is:

$$f_t^\Pi(x) - f_t^{\Pi_1}(x) \geq -\gamma'_t(r_t) - \beta\gamma'_{t+1}(r_{t+1}) - \dots - \beta^{T-t}\gamma'_T(r_T) + \beta^{T-t+1}c_{T+1} - c_t \geq 0.$$

When $f_t^\Pi(x) - f_t^{\Pi_1}(x) > 0$, Π is not optimal and a contradiction incurs. Therefore, under condition (iv), it is optimal to clear up all the backlogs from previous periods whenever an order is placed. ■

Lemma 3.2 *Condition (iv) is weaker than condition (iii).*

Proof Let $G'_t(r_t-)$ be the left side derivative of $G_t(\cdot)$ at r_t . Then

$G'_t(r_t-) = c_t - \beta c_{t+1} + \gamma'_t(r_t)$. Due to the convexity of $G_t(\cdot)$ and the definition of \underline{S}_t , (iv) is a necessary condition of $0 \leq \underline{S}_t - r_t$, which itself is a necessary condition of (iii). On the other hand, there are examples (such as Example 2) that satisfy (iv) but not (iii). This completes the proof. ■

We summarize the main result for the basic Model in the following theorem.

Theorem 3.2 *For problem (3.1), under conditions (i), (ii) and (iv), the (s, S) policy defined in (3.6) is optimal.*

Proof The result is immediate from Theorem 3.1 and Lemma 3.1. ■

3.2.5. The model with the tighter constraint

In section 3.2.4, we show that for the basic model, when condition (iii) is replaced by a weaker condition (iv), the (s, S) policy defined in (3.6) is still optimal. In this section, we consider the model with the tighter constraint defined in (3.4). We show that condition (iii) or (iv) is not required for the optimality of the (s, S) policy. Specifically, under conditions (i) and (ii), an (s, S) policy is optimal. However, s and S here may be different with that defined in (3.6).

Definition 3.1 (Scarf [47]) A real-valued function f is called K -convex functions for $K \geq 0$, if for any $z \geq 0$, $b > 0$, and any y we have

$$K + f(z + y) \geq f(y) + \frac{z}{b}(f(y) - f(y - b))$$

A function f is called K -concave if $-f$ is K -convex.

An equivalent definition of K -convexity is provided by Porteus [43]:

Definition 3.2 A real-valued function f is called K -convex functions for $K \geq 0$, if for any $x_0 \leq x_1$, and $\lambda \in [0, 1]$,

$$f((1 - \lambda)x_0 + \lambda x_1) \leq (1 - \lambda)f(x_0) + \lambda f(x_1) + \lambda K$$

We summarize properties of K -convex functions as follows. Please see Bertsekas [4] for details.

Lemma 3.3 (a) A real-valued convex function is also 0-convex and, hence, K -convex for all $K \geq 0$. A K_1 -convex function is also a K_2 -convex function for $K_1 \leq K_2$.

(b) If $g_1(y)$ and $g_2(y)$ are K_1 -convex and K_2 -convex, respectively, then for $\alpha, \beta \geq 0$,

$\alpha g_1(y) + \beta g_2(y)$ is $(\alpha K_1 + \beta K_2)$ -convex.

(c) If $g(y)$ is K -convex, and ω is a random variable, then $E[g(y - \omega)]$ is also K -convex, provided $E[|g(y - \omega)|] < \infty$ for all y .

(d) If g is a continuous K -convex function and $g(y) \rightarrow \infty$ as $|y| \rightarrow \infty$, then there exists scalars s and S with $s \leq S$ such that

(i) $g(s) \leq g(y)$ for all y ,

(ii) $g(s) + K = g(s) < g(y)$ for all $y < s$,

(iii) $g(y)$ is a decreasing function on $(-\infty, s)$, and

(iv) $g(y) \leq g(z) + K$ for all y, z with $s \leq y \leq z$.

Let S_t and s_t be the same as defined in section 3.2.3, and

$$S'_t = \sup\{y : J_t(y) \leq J_t(a), \text{ for all } a \geq r_t\}. \quad (3.7)$$

By definition, when $r_t > S_t$, S'_t is the largest minimizer of $J_t(y)$ on $\{y : y \geq r_t\}$. Define

the (s, S) policy for (3.4) as:

If $r_t \leq S_t$,

$$y_t = \begin{cases} S_t, & \text{if } x_t < s_t \vee r_t \\ x_t, & \text{if } x_t \geq s_t \vee r_t \end{cases} \quad (3.8)$$

If $r_t > S_t$,

$$y_t = \begin{cases} S'_t, & \text{if } x_t < r_t \\ x_t, & \text{if } x_t \geq r_t \end{cases} \quad (3.9)$$

We will show that the ordering policy specified in (3.8) or (3.9) is optimal. To do this, we need the following lemma.

Lemma 3.4 Suppose $f(x)$ is a K -convex function. If $\lim_{\varepsilon \rightarrow 0^+} f(x - \varepsilon)$, denoted as $f(x^-)$,

exists for all $x \in \mathfrak{X}$, then $f(x^-) \geq f(x)$ holds for all $x \in \mathfrak{X}$.

Proof If $f(x)$ is continuous, then the result is obvious. Suppose $f(x)$ is not continuous at x_0 and to the contrary of the lemma, $f(x_0) = f(x_0^-) + c$, where $c > 0$. Because $f(x)$ is K -convex, for any $\alpha > 0$ and $\theta > 0$, we have

$\frac{\alpha}{\theta}(f(x_0) - f(x_0 - \theta)) \leq K + f(x_0 + \alpha) - f(x_0)$, but this inequality cannot hold when we let $\theta \rightarrow 0$ with a fixed α value. ■

Note that $f(x)$ may not be a continuous function. Lemma 3.4 is needed to ensure the existence of S'_t , s_t and S_t .

Theorem 3.3 For problem (3.4), under conditions (i) and (ii),

(a) $J_t(\cdot)$ and $f_t(\cdot)$ are K_t -convex.

(b) The (s, S) policy defined in (3.8) or (3.9) is optimal, where specifically,

$$(s, S) = \begin{cases} (s_t, S_t), & \text{if } r_t \leq s_t \\ (r_t, S_t), & \text{if } s_t < r_t \leq S_t \\ (r_t, S'_t), & \text{if } r_t > S_t \end{cases} \quad (3.10)$$

Proof We prove by induction. For $t = T + 1$, $f_{T+1}(x) = 0$. Assume that $f_{t+1}(\cdot)$ is K_{t+1} -convex, right-continuous and $f_{t+1}(x^-)$ exists for all $x \in \mathfrak{X}$. Thus, $J_t(\cdot)$ is K_t -convex, right-continuous and $J_t(x^-)$ exists for all $x \in \mathfrak{X}$. By Lemma 3.4, we have $J_t(x^-) \geq J_t(x)$ for all $x \in \mathfrak{X}$. This, together with condition (i), ensures the existence of S'_t , s_t and S_t .

Now consider $f_t(\cdot)$. There are three cases: 1. $S_t < r_t$, 2. $0 < r_t < s_t$ and 3. $0 < s_t \leq r_t \leq S_t$ or $s_t \leq 0 < r_t \leq S_t$.

For case 1, if $x < r_t$, an order has to be placed. Clearly, for all $x < r_t$, the optimal order-up-to level is S'_t defined in (3.7). If $x \geq r_t$, since $J_t(x)$ is K_t -convex, it is optimal not to order at x . Thus,

$$f_t(x) = \begin{cases} K_t + J_t(S'_t), & \text{if } x < r_t \\ J_t(x), & \text{if } x \geq r_t \end{cases} \quad (3.11)$$

For any $\omega: \mathfrak{R} \rightarrow \mathfrak{R}$, $\alpha \geq 0$ and $\theta > 0$, define

$$\Delta_\omega(x, \alpha, \theta) = K_t + \omega(x + \alpha) - \omega(x) - (\alpha / \theta)[\omega(x) - \omega(x - \theta)].$$

To prove that $f_t(x)$ in (3.11) is K_t -convex, we consider four different possibilities:

I. If $x + \alpha < r_t$, $\Delta_{f_t}(x, \alpha, \theta) = K_t \geq 0$.

II. If $x - \theta \geq r_t$, $\Delta_{f_t}(x, \alpha, \theta) = \Delta_{J_t}(x, \alpha, \theta) \geq 0$ because $J_t(\cdot)$ is K_t -convex.

III. If $x < r_t < x + \alpha$, $\Delta_{f_t}(x, \alpha, \theta) = J_t(x + \alpha) - J_t(S'_t) \geq 0$.

IV. If $x - \theta < r_t \leq x$, $\Delta_{f_t}(x, \alpha, \theta) = K_t + J_t(x + \alpha) - J_t(x) - \frac{\alpha}{\theta}(J_t(x) - K_t - J_t(S'_t))$

If $J_t(x) - K_t - J_t(S'_t) \leq 0$, then $\Delta_{f_t}(x, \alpha, \theta) \geq K_t + J_t(x + \alpha) - J_t(x) \geq 0$.

If $J_t(x) - K_t - J_t(S'_t) > 0$, since $x - \theta < r_t < x$ or $0 < x - r_t < \theta$, and $S_t < r_t \leq S'_t$,

$$\begin{aligned} \Delta_{f_t}(x, \alpha, \theta) &\geq K_t + J_t(x + \alpha) - J_t(x) - \frac{\alpha}{x - r_t}(J_t(x) - K_t - J_t(S'_t)) \\ &\geq K_t + J_t(x + \alpha) - J_t(x) - \frac{\alpha}{x - r_t}(J_t(x) - J_t(r_t)) \geq 0 \end{aligned}$$

Therefore, $f_t(\cdot)$ is K_t -convex. From (3.11), it is obvious that $f_t(\cdot)$ is right-continuous and $f_t(x^-)$ exists for all $x \in \mathfrak{R}$.

For case 2 and 3, the proof is very much the same as that in Sobel & Zhang [49].

Accordingly, it can also be easily shown that $f_t(\cdot)$ is right-continuous and $f_t(x^-)$ exists for all $x \in \mathfrak{R}$. ■

Remark It can be seen that (3.6) and (3.8) are identical. This is not surprise, since condition (iv) ensures $S_t \geq r_t$. Consequently, when condition (iv) is satisfied, (3.1) and (3.4) are equivalent problems.

3.3. Joint pricing and inventory control with two demand classes

3.3.1. Introduction

We have shown that under a weaker condition, an (s, S) policy is still optimal for one-deterministic-one-stochastic problem studied in Sobel & Zhang [49]. Our results are useful to apply the Sobel & Zhang [49] model to problems with fluctuating demand. Obviously, under the weaker condition, the optimality of the modified (s, S) policy still holds for the two demand-class problems with obsolescent demand or seasonal demand.

As an application, in this sub-section, we will show how the results in section 3.2 enable us to analyze the joint pricing and inventory control problems with two demand classes.

In recent years, scores of retail and manufacturing companies have started exploring innovative pricing strategies in an effort to improve their operations. Firms are employing methods such as dynamically adjusting price over time based on inventory levels or production schedules, as well as segmenting customers based on their sensitivity to price and lead time. For example, Dell Computers uses pricing strategies to target particular customer segments. On Dell's website, the exact same product is sold at different prices, depending on whether the purchase is made by a private consumer, a small, medium or large business, or partners. The price of the same product for the same industry may change significantly over time. The ability of price to affect consumer decision and its flexibility makes pricing strategies important in meeting Dell's objectives in a competitive environment.

One basic assumption of periodic review, joint pricing and inventory control models in the literature is that demand in consecutive periods are independent, but their distributions depend on the item's price in accordance with general stochastic demand functions. (See, e.g.,

Federgruen & Heching [19], Chen & Simchi-Levi [11], Huh & Janakiraman [30]). Obviously, condition (iii) does not satisfy the basic assumption, since demand in consecutive periods are dependent.

Our model generalizes the results in Chen & Simchi-Levi [11] to the problems with two demand classes. Based on the results in section 3.2, we will show that the optimal policy still has an (s, S, p) type for the additive demand functions, and an (s, S, A, p) type for the general demand functions.

3.3.2. Problem formulation

We consider a periodic review inventory/production system over a finite horizon with T periods. There are two types of demand classes: deterministic and stochastic. The deterministic demand (class 1) is from customers who have long-term contracts and therefore must be satisfied immediately without any delay. Prices of the deterministic demand are determined by the contracts at the beginning of the time horizon. Stochastic demands (class 2) in consecutive periods are independent, but their distributions depend on the item's price in accordance with general stochastic demand functions. Unsatisfied demand is backlogged with a penalty cost. The model requires the following notations:

T = the planning horizon,

x = the inventory level (inventory on-hand or class 2 backorders) at the start of a period, just before placing an order, if any,

y = the inventory level (inventory on-hand or class 2 backorders) at the start of a period, after placing an order, if any,

r_t = the deterministic demand in period t ,

ω_t = the stochastic demand in period t ,

$\tilde{D}_t = r_t + \omega_t$, total demand in period t ,

K_t = fixed setup cost in period t ,

c_t = unit purchasing cost in period t ,

p_t = selling price of stochastic demand in period t ,

$\underline{p}_t, \bar{p}_t$ = lower and upper bounds on p_t , respectively,

$h_t(\cdot)$ = inventory holding or penalty cost in period t , a convex function,

γ = discount factor, $0 < \gamma \leq 1$.

The sequence of events in each period, $t = 1, 2, \dots, T$, is as follows. At the beginning of period t , the inventory level is observed. The decision on the ordering quantity and selling price for stochastic demand is determined. The order, if any, arrives and the deterministic demand is satisfied. The stochastic demand is realized and is satisfied as much as possible. Finally, an inventory-related cost (holding cost for inventory on-hand or penalty cost for class 2 backorders) is charged at the end of this period.

The objective is to maximize expected profit over finite horizon by selecting the optimal inventory ordering policy and pricing strategy.

A general stochastic demand function is defined as:

$$\omega_t = D_t(p_t, \varepsilon_t) = \alpha_t D_t(p_t) + \beta_t \quad (3.12)$$

where $\varepsilon_t = (\alpha_t, \beta_t)$, α_t and β_t are random variables with $E\{\alpha_t\} = 1$ and $E\{\beta_t\} = 0$ (see, e.g., Federgruen & Heching [19], Chen & Simchi-Levi [11]). $D_t(p_t)$ has a strictly decreasing inverse function, D_t^{-1} . Furthermore, we assume that the expected revenue,

$R_t(d_t) = d_t D_t^{-1}(d_t)$, is concave in d_t , where $d_t = D_t(p_t)$ is the expected demand in period t . Hence, the total demand can be written as $\tilde{D}_t = r_t + \alpha_t D_t(p_t) + \beta_t$.

From these assumptions, the price p_t is uniquely determined by the mean demand d_t . $d_t \in [\underline{d}_t, \bar{d}_t]$, where $\underline{d}_t = D_t(\underline{p}_t)$ and $\bar{d}_t = D_t(\bar{p}_t)$. Therefore, our decision problem is equivalent to determining the ordering quantity and the expected demand level in period t .

Let $G_t(y, p) = E\{h_t(y - r_t - D_t(p_t, \varepsilon_t))\}$ be the expected inventory and backordering cost for period t . Similarly as Federgruen & Heching [19] and Chen & Simchi-Levi [11], we make the following assumptions regarding $G_t(\cdot, \cdot)$, $h_t(\cdot)$ and the stochastic demand function.

Assumption 3.1 (1) $\lim_{y \rightarrow \infty} G_t(y, p) = \lim_{y \rightarrow \infty} [c_t y + G_t(y, p)] = \lim_{y \rightarrow \infty} [(c_t - c_{t+1})y + G_t(y, p)] = \infty$

for all $p \in [\underline{p}_t, \bar{p}_t]$.

(2) $0 \leq G_t(y, p) = O(|y|^\rho)$ for some integer ρ .

(3) $E\{D_t(p, \varepsilon_t)\}^\rho < \infty$ for all $p \in [\underline{p}_t, \bar{p}_t]$.

Assumption 3.2 $K_t \geq \gamma K_{t+1}$.

Let $v_t(x)$ denote maximum expected discounted profit for period $t, t+1, \dots, T$ when starting from period t at state x . The problem can be formulated as:

$$v_t(x) = \max\{-K_t \delta(y - x) + g_t(y, d_t(y)) + c_t x : y \geq x \text{ if } r_t = 0; y \geq x \vee r_t \text{ if } r_t > 0\} \quad (3.13)$$

where

$$g_t(y, d) = R_t(d) - c_t y + E\{-h_t(y - r_t - \alpha_t d - \beta_t)\} + \gamma v_{t+1}(y - r_t - \alpha_t d - \beta_t) \quad (3.14)$$

$d_t(y)$ is the expected demand associated with the best selling price for a given inventory level y , $\delta(y - x) = 1$ if $y > x$, $= 0$ if $y = x$, and $x \vee r_t$ denotes $\max\{x, r_t\}$. For all x , we assume $v_{T+1}(x) = 0$.

Throughout our discussion, we assume that all assumptions stated in this section are satisfied.

3.3.3. Additive demand functions

One special case of (3.12) is the additive function, with $\alpha_t = 1$. For the additive function, the total demand becomes $\tilde{D}_t = r_t + D_t(p_t) + \beta_t$. In this sub-section, we prove that a modified (s, S, p) policy is optimal. This is based on the results that $g_t(y, d)$ and $v_t(x)$ are K_t -convex functions.

Let \mathfrak{R} denote the set of real number. Define

$$S_t = \sup\{y : g_t(y, d_t(y)) \geq g_t(x, d_t(x)), \text{ for all } x \in \mathfrak{R}\} \quad (3.15)$$

$$s_t = \inf\{y : g_t(y, d_t(y)) \geq g_t(S_t, d_t(S_t)) - K_t, \text{ for all } x \in \mathfrak{R}\} \quad (3.16)$$

$$S'_t = \sup\{y : g_t(y, d_t(y)) \geq g_t(x, d_t(x)), \text{ for all } x \geq r_t\} \quad (3.17)$$

When $r_t \leq S_t$, S'_t is just S_t . When $r_t > S_t$, S'_t is the largest x on $\{x : x \geq r_t\}$ that maximizes $g_t(x, d_t(x))$.

The following result is needed to prove the K_t -concavity of $v_t(x)$.

Lemma 3.5 *Suppose that $v_{t+1}(x)$ is K_{t+1} -concave, then there exists a $d_t(y)$ that maximizes $g_t(y, d)$ for any given y , such that $y - d_t(y)$ is a nondecreasing function of y .*

Proof Define

$$\begin{aligned} \tilde{g}_t(y, d) &= g_t(y, y - d) \\ &= R_t(y - d) - c_t y + E[-h_t(d - r_t - \beta_t) + v_{t+1}(d - r_t - \beta_t)] \end{aligned}$$

Then, Assumption 3.1 implies that $\tilde{g}_t(y, d)$ has increasing differences in y and d . The lemma thus follows from Topkis [53] (Theorem 2.4.3 and Lemman 2.8.1). ■

The proof of Lemma 3.5 is similar as of Chen & Simchi-Levi [11], with $d - \beta_t$

replaced by $d - r_t - \beta_t$. We note here for Lemma 3.5, it is not required that $g_t(y, d)$ is jointly continuous in (y, d) . As can be seen later, $g_t(y, d)$ is not necessarily a continuous function for our problem.

We shall prove that a modified (s, S, p) policy is optimal, where the order-up-to level is defined as:

If $r_t \leq S_t$,

$$y = \begin{cases} S_t, & \text{if } x < s_t \vee r_t \\ x, & \text{if } x \geq s_t \vee r_t \end{cases} \quad (3.18)$$

If $r_t > S_t$,

$$y = \begin{cases} S'_t, & \text{if } x < r_t \\ x, & \text{if } x \geq r_t \end{cases} \quad (3.19)$$

Theorem 3.4 For $t = T, T-1, \dots, 1$,

(a) $g_t(y, d) = O(|y|^\rho)$ and $v_t(x) = O(|x|^\rho)$.

(b) $g_t(y, d)$ is continuous in d and right continuous in y . $\lim_{|y| \rightarrow \infty} g_t(y, d) = -\infty$ for any $d \in [\underline{d}_t, \bar{d}_t]$. Hence, for any fixed y , $g_t(y, d)$ has a finite maximizer, $d_t(y)$.

(c) $g_t(y, d_t(y))$ and $v_t(x)$ are K_t -concave.

(d) A modified (s, S, p) policy defined in (3.18) and (3.19) is optimal. That is, when $x \geq s_t \vee r_t$, it is optimal not to order and set $d_t = d_t(x)$; otherwise, order to an order-up-to level S_t when $r_t \leq S_t$ and set $d_t = d_t(S_t)$, or order up to S'_t and set $d_t = d_t(S'_t)$ when $r_t > S_t$.

Proof By induction. $g_1(y, d) = O(|y|^\rho)$ by Assumption 3.1. Assume now that for some $t+1 = 2, 3, \dots, T$, $g_{t+1}(\cdot, \cdot) = O(|y|^\rho)$. It is easy verified that $v_{t+1}(x) = O(|x|^\rho)$, i.e., a constant M exists such that $v_{t+1}(x) \leq M(|x|^\rho + 1)$ for all x . Thus,

$$\begin{aligned} v_{t+1}(y - r_t - \alpha_t d - \beta_t) &\leq M(|y - r_t - \alpha_t d - \beta_t|^\rho + 1) \\ &\leq M(|y - r_t| + (\alpha_t d + \beta_t))^\rho + M \end{aligned}$$

And hence employing the Binomial expansion of the right-hand side and Assumption 3.1,

$$\begin{aligned} E v_{t+1}(y - r_t - d - \beta_t) &\leq M E(|y - r_t| + (d + \beta_t))^\rho + M \\ &\leq M \sum_{l=0}^{\rho} \binom{\rho}{l} |y - r_t|^l \max_d E[(d + \beta_t)]^{\rho-l} + M \end{aligned}$$

By Assumption 3.1, $g_t(\cdot, \cdot) = O(|y|^\rho)$ as well.

Assume that parts (a), (b), (c) and (d) hold for period T to $t+1$. Given y , the continuity of $g_t(y, d)$ is easy to check. Because $v_{t+1}(x)$ is right continuous in x , $g_t(y, d)$ is also a right continuous function in y . From part (d),

$$v_{t+1}(x) = \begin{cases} -K_{t+1} + g_{t+1}(S_{t+1}, d_{t+1}(S_{t+1})) + c_{t+1}x & \text{if } x < s_{t+1} \vee r_{t+1} \text{ and } r_{t+1} \leq S_{t+1} \\ -K_{t+1} + g_{t+1}(S'_{t+1}, d_{t+1}(S'_{t+1})) + c_{t+1}x & \text{if } x < r_{t+1} \text{ and } r_{t+1} > S_{t+1} \\ g_{t+1}(x, d_{t+1}(x)) + c_{t+1}x & \text{if } (x \geq s_{t+1} \vee r_{t+1} \text{ and } r_{t+1} \leq S_{t+1}) \text{ or } x \geq r_{t+1} > S_{t+1} \end{cases}$$

This implies that $E[v_{t+1}(y - r_t - d - \beta_t) - c_{t+1}(y - r_t - d - \beta_t)] \leq v_{t+1}(S_{t+1}) - c_{t+1}S_{t+1}$.

Because $G_t(y, D_t^{-1}(d))$ is jointly concave in (y, d) , we have that $\lim_{|y| \rightarrow \infty} g_t(y, d) = -\infty$

for any $d \in [\underline{d}_t, \bar{d}_t]$ uniformly. Hence, for any fixed y , $g_t(y, d)$ has a finite maximizer $d_t(y)$.

For (c), we first show that $g_t(y, d_t(y))$ is K_t -concave.

For any $y \leq y'$ and $\lambda \in [0, 1]$, we have by Lemma 3.5 and the assumption that

$v_{t+1}(x)$ are K_{t+1} -concave that

$$\begin{aligned} &v_{t+1}((1 - \lambda)(y - d_t(y) - \beta_t) + \lambda(y' - d_t(y') - \beta_t)) \\ &\geq (1 - \lambda)v_{t+1}(y - d_t(y) - \beta_t) + \lambda v_{t+1}(y' - d_t(y') - \beta_t) - \lambda K_{t+1} \end{aligned}$$

In addition, the concavity of $R_t(d)$ implies that

$$R_t((1 - \lambda)d_t(y) + \lambda d_t(y')) \geq (1 - \lambda)R_t(d_t(y)) + \lambda R_t(d_t(y'))$$

Because $h_t(x)$ is convex, we also have

$$\begin{aligned} & -h_{t+1}((1-\lambda)(y-d_t(y)-\beta_t)+\lambda(y'-d_t(y')-\beta_t)) \\ & \geq -(1-\lambda)h_{t+1}(y-d_t(y)-\beta_t)-\lambda h_{t+1}(y'-d_t(y')-\beta_t) \end{aligned}$$

Add the last inequalities and taking expectation we get

$$\begin{aligned} & g_t((1-\lambda)y+\lambda y', d_t((1-\lambda)y+\lambda y')) \\ & \geq (1-\lambda)g_t(y, d_t(y))+\lambda g_t(y', d_t(y'))-\lambda\gamma K_{t+1} \\ & \geq (1-\lambda)g_t(y, d_t(y))+\lambda g_t(y', d_t(y'))-\lambda K_t \end{aligned}$$

That is, $g_t(y, d_t(y))$ is a K_t -concave function in y . Thus, S_t , s_t and S'_t defined in (3.15)-(3.17) exist.

We now show the K_t -concavity of $v_t(x)$. If $r_t = 0$, then the problem becomes a single-demand-class problem. So we only consider $r_t > 0$. There are three different cases:

1. If $0 < r_t < s_t$, we have,

$$v_t(x) = \begin{cases} -K_t + g_t(S_t, d_t(S_t)) + c_t x & \text{if } x < s_t \\ g_t(x, d_t(x)) + c_t x & \text{if } x \geq s_t \end{cases}$$

The K_t -concavity of $v_t(x)$ can be proved directly.

2. If $0 < s_t \leq r_t$ or $s_t \leq 0 < r_t \leq S_t$, then

$$v_t(x) = \begin{cases} -K_t + g_t(S_t, d_t(S_t)) + c_t x & \text{if } x < s_t \\ g_t(x, d_t(x)) + c_t x & \text{if } x \geq s_t \end{cases}$$

For any $x < x'$ and $\lambda \in [0, 1]$, let $x_\lambda = (1-\lambda)x + \lambda x'$ and

$$\Delta v = v_t(x_\lambda) - (1-\lambda)v_t(x) - \lambda v_t(x') + \lambda K_t.$$

Consider the following cases:

I. $x' < r_t$.

$$\Delta v = -K_t + g_t(S_t, d_t(S_t)) + c_t x_\lambda - [-K_t + g_t(S_t, d_t(S_t)) + c_t(1-\lambda)x + c_t \lambda x'] + \lambda K_t \geq 0$$

II. $r_t \leq x < x'$

$$\Delta v = g_t(x_\lambda, d_t(x_\lambda)) + c_t x_\lambda - (1-\lambda)[g_t(x, d_t(x)) + c_t x] - \lambda[g_t(x', d_t(x')) + c_t x'] + \lambda K_t \geq 0$$

III. $x \leq x_\lambda < r_t \leq x'$

$$\begin{aligned}\Delta v &= -K_t + g_t(S_t, d_t(S_t)) + c_t x_\lambda - (1-\lambda)[-K_t + g_t(S_t, d_t(S_t)) + c_t x] \\ &\quad - \lambda[g_t(x', d_t(x')) + c_t x'] + \lambda K_t \\ &= \lambda[g_t(S_t, d_t(S_t)) - g_t(x', d_t(x'))] \geq 0\end{aligned}$$

IV. $x < r_t \leq x_\lambda$

$$\begin{aligned}\Delta v &= g_t(x_\lambda, d_t(x_\lambda)) + c_t x_\lambda - (1-\lambda)[-K_t + g_t(S_t, d_t(S_t)) + c_t x] \\ &\quad - \lambda[g_t(x', d_t(x')) + c_t x'] + \lambda K_t \\ &= (1-\lambda)[g_t(x_\lambda, d_t(x_\lambda)) - g_t(S_t, d_t(S_t)) + K_t] \\ &\quad + \lambda[g_t(x_\lambda, d_t(x_\lambda)) - g_t(x', d_t(x')) + K_t]\end{aligned}$$

If $g_t(x_\lambda, d_t(x_\lambda)) - g_t(S_t, d_t(S_t)) + K_t \geq 0$ then

$$\Delta v \geq \lambda[g_t(x_\lambda, d_t(x_\lambda)) - g_t(x', d_t(x')) + K_t] \geq 0.$$

If $g_t(x_\lambda, d_t(x_\lambda)) - g_t(S_t, d_t(S_t)) + K_t < 0$, then $S_t < x_\lambda$. Hence, there exists $\mu < \lambda$, such

that $x_\lambda = (1-\mu)S_t + \mu x'$.

$$\begin{aligned}\Delta v &= g_t(x_\lambda, d_t(x_\lambda)) - (1-\lambda)[-K_t + g_t(S_t, d_t(S_t))] - \lambda g_t(x', d_t(x')) + \lambda K_t \\ &= g_t(x_\lambda, d_t(x_\lambda)) - (1-\mu)[-K_t + g_t(S_t, d_t(S_t))] - \mu g_t(x', d_t(x')) + \mu K_t \\ &\quad + (\lambda - \mu)[g_t(S_t, d_t(S_t)) - g_t(x', d_t(x'))] + (1-\mu)K_t \\ &\geq (\lambda - \mu)[g_t(S_t, d_t(S_t)) - g_t(x', d_t(x'))] + (1-\mu)K_t \\ &\geq 0\end{aligned}$$

3. If $r_t > S_t$, then

$$v_t(x) = \begin{cases} -K_t + g_t(S_t, d_t(S_t)) + c_t x & \text{if } x < r_t \\ g_t(x, d_t(x)) + c_t x & \text{if } x \geq r_t \end{cases}.$$

Similarly as 2, consider the following cases:

I. $x' < r_t$. $\Delta v = \lambda K_t \geq 0$

II. $r_t \leq x < x'$

$$\Delta v = g_t(x_\lambda, d_t(x_\lambda)) + c_t x_\lambda - (1-\lambda)[g_t(x, d_t(x)) + c_t x] - \lambda[g_t(x', d_t(x')) + c_t x'] + \lambda K_t \geq 0$$

III. $x \leq x_\lambda < r_t \leq x'$

$$\begin{aligned}
 \Delta v &= -K_t + g_t(S'_t, d_t(S'_t)) + c_t x_\lambda - (1-\lambda)[-K_t + g_t(S'_t, d_t(S'_t)) + c_t x] \\
 &\quad - \lambda[g_t(x', d_t(x')) + c_t x'] + \lambda K_t \\
 &= \lambda[g_t(S'_t, d_t(S'_t)) - g_t(x', d_t(x'))] \\
 &\geq 0
 \end{aligned}$$

IV. $x < r_t \leq x_\lambda$

$$\begin{aligned}
 \Delta v &= g_t(x_\lambda, d_t(x_\lambda)) + c_t x_\lambda - (1-\lambda)[-K_t + g_t(S'_t, d_t(S'_t)) + c_t x] \\
 &\quad - \lambda[g_t(x', d_t(x')) + c_t x'] + \lambda K_t \\
 &= (1-\lambda)[g_t(x_\lambda, d_t(x_\lambda)) - g_t(S'_t, d_t(S'_t)) + K_t] \\
 &\quad + \lambda[g_t(x_\lambda, d_t(x_\lambda)) - g_t(x', d_t(x')) + K_t]
 \end{aligned}$$

If $g_t(x_\lambda, d_t(x_\lambda)) - g_t(S'_t, d_t(S'_t)) + K_t \geq 0$ then

$$\Delta v \geq \lambda[g_t(x_\lambda, d_t(x_\lambda)) - g_t(x', d_t(x')) + K_t] \geq 0.$$

If $g_t(x_\lambda, d_t(x_\lambda)) - g_t(S'_t, d_t(S'_t)) + K_t < 0$, then $S'_t < x_\lambda \leq x'$. Hence, there exists $\mu < \lambda$,

such that $x_\lambda = (1-\mu)S'_t + \mu x'$.

$$\begin{aligned}
 \Delta v &= g_t(x_\lambda, d_t(x_\lambda)) + c_t x_\lambda - (1-\lambda)[-K_t + g_t(S'_t, d_t(S'_t)) + c_t x] \\
 &\quad - \lambda[g_t(x', d_t(x')) + c_t x'] + \lambda K_t \\
 &= g_t(x_\lambda, d_t(x_\lambda)) - (1-\mu)g_t(S'_t, d_t(S'_t)) - \mu g_t(x', d_t(x')) + \mu K_t \\
 &\quad + (\lambda - \mu)[g_t(S'_t, d_t(S'_t)) - g_t(x', d_t(x'))] + (1-\lambda)K_t + \lambda K_t - \mu K_t \\
 &\geq (\lambda - \mu)[g_t(S'_t, d_t(S'_t)) - g_t(x', d_t(x')) + K_t] \\
 &\geq 0
 \end{aligned}$$

This completes the proof of part (c).

Part (d) is immediate from part (c). ■

3.3.4. General demand functions

For the general function, the total demand is $\tilde{D}_t = r_t + \alpha_t D_t(p_t) + \beta_t$. For the single demand-class problem, Chen & Simchi-Levi [11] show that $v_t(x)$ may not be k -convex and an (s, S, p) policy may not be optimal. To address the problem, they introduce one new concept termed *sym-K-convex*, and show that the optimal policy satisfies a more general structure. Specifically, an (s, S, A, p) policy is optimal: For period t , there exist s_t and S_t

with $s_t \leq S_t$ and a set $A_t \subset [s_t, (s_t + S_t)/2]$ such that it is optimal to order $S_t - x$ and set the expected demand level $d_t = d_t(S_t)$ when $x_t \leq s_t$ or $x_t \in A_t$, and not to order anything and set $d_t = d_t(x_t)$ otherwise.

For completeness, we summarize the definition and properties of sym- K -convexity. For an overall review of sym- K -convexity, we refer readers to Chen & Simchi-Levi [11].

Definition 3.3 (Chen & Simchi-Levi [11]). A real-valued function f is called sym- K -convex for $K \geq 0$, if for any x_0, x_1 and $\lambda \in [0,1]$,

$$f((1-\lambda)x_0 + \lambda x_1) \leq (1-\lambda)f(x_0) + \lambda f(x_1) + \max\{\lambda, 1-\lambda\}K.$$

A function f is called sym- K -concave if $-f$ is sym- K -convex.

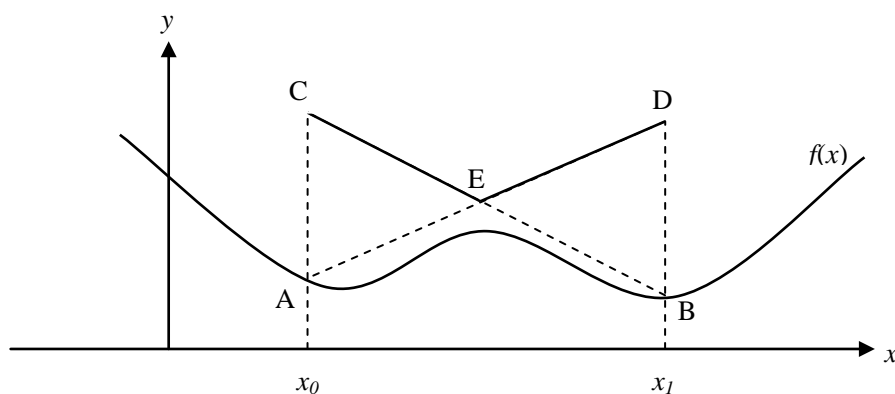


Figure 3.1 Sym- K -convex functions

As shown in Figure 3.1, let $A = (x_0, f(x_0))$, $B = (x_1, f(x_1))$, $C = (x_0, f(x_0) + K)$, and $D = (x_1, f(x_1) + K)$. Denote E as the intersection point of the lines connecting A to D and B to C . Graphically, the function $f(x)$ is sym- K -convex if for any $x_0 \leq x_1$, $f(x)$ between x_0 and x_1 is always above the solid line \overline{CED} .

Lemma 3.6 (a) A real valued convex function is also sym-0-convex and, hence, sym- K -convex for all $K \geq 0$. A sym- K_1 -convex function is also a sym- K_2 -convex function for $K_1 \leq K_2$.

(b) If $g_1(y)$ and $g_2(y)$ are sym- K_1 -convex and sym- K_2 -convex, respectively, then for $\alpha, \beta \geq 0$, $\alpha g_1(y) + \beta g_2(y)$ is sym- $(\alpha K_1 + \beta K_2)$ -convex.

(c) If $g(y)$ is sym- K -convex and ω is a random variable, then $E\{g(y - \omega)\}$ is also sym- K -convex, provided $E\{|g(y - \omega)|\} < \infty$ for all y .

(d) If g is a continuous sym- K -convex function and $g(y) \rightarrow \infty$ as $|y| \rightarrow \infty$, then there exist s and S with $s \leq S$ such that

(i) $g(S) \leq g(y)$ for all y ,

(ii) s is the smallest value x such that $g(x) = g(S) + K$; therefore, $g(y) > g(s)$ for all $y < s$,

(iii) $g(y) = g(z) + K$ for all y, z with $(s + S)/2 \leq y \leq z$.

We will prove that for the one-deterministic-one-stochastic problem, $g_t(y, d)$ and $v_t(x)$ are sym- K_t -convex functions, and consequently, a modified (s, S, A, p) policy is optimal.

Theorem 3.5 For $t = T, T-1, \dots, 1$,

(a) $g_t(y, d) = O(|y|^\rho)$ and $v_t(x) = O(|x|^\rho)$.

(b) $g_t(y, d)$ is continuous in d and right continuous in y . $\lim_{|y| \rightarrow \infty} g_t(y, d) = -\infty$ for any $d \in [\underline{d}_t, \bar{d}_t]$. Hence, for any fixed y , $g_t(y, d)$ has a finite maximizer, $d_t(y)$.

(c) $g_t(y, d)$ and $v_t(x)$ are sym- K_t -concave.

(d) A modified (s, S, A, p) policy is optimal, where s and S are defined by (3.18) and (3.19), and A is a set with $A \subset [s_t, (s_t + S_t)/2]$. That is, when $x \geq s_t \vee r_t$ and $x \notin A_t$, it is optimal not to order and set $d_t = d_t(x)$; otherwise, order to an order-up-to level S_t when $r_t \leq S_t$ and set $d_t = d_t(S_t)$, or order up to S_t' and set $d_t = d_t(S_t')$ when $r_t > S_t$.

Proof By induction. $g_1(y, d) = O(|y|^\rho)$ by Assumption 3.1. Assume now that for some $t+1 = 2, 3, \dots, T$, $g_{t+1}(\cdot, \cdot) = O(|y|^\rho)$. It is easy verified that $v_{t+1}(x) = O(|x|^\rho)$, i.e., a

constant M exists such that $v_{t+1}(x) \leq M(|x|^\rho + 1)$ for all x . Thus,

$$\begin{aligned} v_{t+1}(y - r_t - \alpha_t d - \beta_t) &\leq M(|y - r_t - \alpha_t d - \beta_t|^\rho + 1) \\ &\leq M(|y - r_t| + (\alpha_t d + \beta_t))^\rho + M \end{aligned}$$

And hence employing the Binomial expansion of the right-hand side and Assumption 3.1,

$$\begin{aligned} E v_{t+1}(y - r_t - \alpha_t d - \beta_t) &\leq M E(|y - r_t| + (\alpha_t d + \beta_t))^\rho + M \\ &\leq M \sum_{l=0}^{\rho} \binom{\rho}{l} |y - r_t|^l \max_d E[(\alpha_t d + \beta_t)]^{\rho-l} + M \end{aligned}$$

By Assumption 3.1, $g_t(\cdot, \cdot) = O(|y|^\rho)$ as well.

Assume that parts (a), (b), (c) and (d) hold for period T to $t+1$. Given y , the continuity of $g_t(y, d)$ is easy to check. Because $v_{t+1}(x)$ is right continuous in x , $g_t(y, d)$ is also a right continuous function in y . Similarly as the proof of Theorem 3.4, we can show that $E[v_{t+1}(y - r_t - \alpha_t d - \beta_t) - c_{t+1}(y - r_t - \alpha_t d - \beta_t)] \leq v_{t+1}(S_{t+1}) - c_{t+1} S_{t+1}$.

Because $G_t(y, D_t^{-1}(d))$ is jointly concave in (y, d) , we have that $\lim_{|y| \rightarrow \infty} g_t(y, d) = -\infty$ for any $d \in [\underline{d}_t, \bar{d}_t]$ uniformly. Hence, for any fixed y , $g_t(y, d)$ has a finite maximizer $d_t(y)$.

We now show part (c) by induction. $v_{T+1}(x) = 0$ is sym- K_{T+1} -concave. Assuming that $v_{t+1}(x)$ is sym- K_{t+1} -concave. We need to show that both $g_t(y, d_t(y))$ and $v_t(x)$ are sym- K_t -concave.

For any $y \leq y'$ and $\lambda \in [0, 1]$, we have by the assumption that $v_{t+1}(x)$ is sym- K_{t+1} -concave that

$$\begin{aligned} &v_{t+1}((1-\lambda)(y - d_t(y) - \beta_t) + \lambda(y' - d_t(y') - \beta_t)) \\ &\geq (1-\lambda)v_{t+1}(y - d_t(y) - \beta_t) + \lambda v_{t+1}(y' - d_t(y') - \beta_t) - \lambda \max\{\lambda, 1-\lambda\} K_{t+1} \end{aligned}$$

In addition, the concavity of $R_t(d)$ implies that

$$R_t((1-\lambda)d_t(y) + \lambda d_t(y')) \geq (1-\lambda)R_t(d_t(y)) + \lambda R_t(d_t(y'))$$

Because $h_t(x)$ is convex, we also have

$$\begin{aligned} & -h_{t+1}((1-\lambda)(y-d_t(y)-\beta_t) + \lambda(y'-d_t(y')-\beta_t)) \\ & \geq -(1-\lambda)h_{t+1}(y-d_t(y)-\beta_t) - \lambda h_{t+1}(y'-d_t(y')-\beta_t) \end{aligned}$$

Add the last inequalities and taking expectation we get

$$\begin{aligned} & g_t((1-\lambda)y + \lambda y', d_t((1-\lambda)y + \lambda y')) \\ & \geq (1-\lambda)g_t(y, d_t(y)) + \lambda g_t(y', d_t(y')) - \lambda \gamma \max(\lambda, 1-\lambda)K_{t+1} \\ & \geq (1-\lambda)g_t(y, d_t(y)) + \lambda g_t(y', d_t(y')) - \lambda \max(\lambda, 1-\lambda)K_t \end{aligned}$$

That is, $g_t(y, d_t(y))$ is a sym- K_t -concave function in y .

It remains to prove that $v_t(x)$ is sym- K_t -concave.

1. If $0 < r_t < s_t$, we have,

$$v_t(x) = \begin{cases} -K_t + g_t(S_t, d_t(S_t)) + c_t x & \text{if } x < s_t \\ g_t(x, d_t(x)) + c_t x & \text{if } x \geq s_t \end{cases}$$

The sym- K_t -concavity of $v_t(x)$ can be proved directly.

2. If $0 < s_t \leq r_t$ or $s_t \leq 0 < r_t \leq S_t$, then consider two cases: I. $s_t < r_t \leq (s_t + S_t)/2 < S_t$,

and II. $s_t < (s_t + S_t)/2 \leq r_t < S_t$. We only show case I since the proof for case II is similar.

Define $v_t^*(x) = v_t(x) - c_t x$,

I. $s_t < r_t \leq (s_t + S_t)/2 < S_t$

Let $I_t = \{y \leq s_t : g_t(y, d_t(y)) \leq g_t(S_t, d_t(S_t)) - K_t \text{ or } y \leq r_t\}$, then

$$v_t^*(x) = \begin{cases} -K_t + g_t(S_t, d_t(S_t)) & \text{if } x \in I_t \\ g_t(x, d_t(x)) & \text{if } x \notin I_t \end{cases}$$

Define $\Delta v = v_t^*(x_\lambda) - (1-\lambda)v_t^*(x) - \lambda v_t^*(x') + \max\{1-\lambda, \lambda\}K_t$. Consider four different

sub-cases.

I.I. $x < x' \leq r_t$.

$$\Delta v = [-K_t + g_t(S_t, d_t(S_t))] - [-K_t + g_t(S_t, d_t(S_t))] + \max\{1-\lambda, \lambda\}K_t > 0$$

I.II. $x \leq x_\lambda \leq r_t < x'$. If $x' \notin I_t$, then,

$$\begin{aligned}\Delta v &= [-K_t + g_t(S_t, d_t(S_t))] - (1 - \lambda)[-K_t + g_t(S_t, d_t(S_t))] \\ &\quad - \lambda g_t(x', d_t(x')) + \max\{1 - \lambda, \lambda\}K_t \\ &= \lambda[-K_t + g_t(S_t, d_t(S_t)) - g_t(x', d_t(x'))] + \max\{1 - \lambda, \lambda\}K_t > 0\end{aligned}$$

Otherwise, if $x' \in I_t$, then,

$$\Delta v = \lambda[-K_t + g_t(S_t, d_t(S_t))] - \lambda[-K_t + g_t(S_t, d_t(S_t))] + \max\{1 - \lambda, \lambda\}K_t > 0$$

I.III. $x \leq r_t \leq x_\lambda < x'$.

If $x_\lambda \in I_t$ and $x' \in I_t$, then $\Delta v = \max\{1 - \lambda, \lambda\}K_t > 0$.

If $x_\lambda \in I_t$ and $x' \notin I_t$, then

$$\begin{aligned}\Delta v &= [-K_t + g_t(S_t, d_t(S_t))] - (1 - \lambda)[-K_t + g_t(S_t, d_t(S_t))] \\ &\quad - \lambda g_t(x', d_t(x')) + \max\{1 - \lambda, \lambda\}K_t > 0\end{aligned}$$

If $x_\lambda \notin I_t$ and $x' \in I_t$, then

$$\begin{aligned}\Delta v &= g_t(x_\lambda, d_t(x_\lambda)) - [-K_t + g_t(S_t, d_t(S_t))] + \max\{1 - \lambda, \lambda\}K_t \\ &= [K_t + g_t(x_\lambda, d_t(x_\lambda)) - g_t(S_t, d_t(S_t))] + \max\{1 - \lambda, \lambda\}K_t > 0\end{aligned}$$

If $x_\lambda \notin I_t$ and $x' \notin I_t$, then

$$\begin{aligned}\Delta v &= g_t(x_\lambda, d_t(x_\lambda)) - (1 - \lambda)[-K_t + g_t(S_t, d_t(S_t))] \\ &\quad - \lambda g_t(x', d_t(x')) + \max\{1 - \lambda, \lambda\}K_t \\ &\geq [-K_t + g_t(S_t, d_t(S_t))] - (1 - \lambda)[-K_t + g_t(S_t, d_t(S_t))] \\ &\quad - \lambda g_t(x', d_t(x')) + \max\{1 - \lambda, \lambda\}K_t \\ &\geq \lambda[g_t(S_t, d_t(S_t)) - g_t(x', d_t(x'))] \geq 0\end{aligned}$$

I.IV. $r_t \leq x < x'$, the proof follows Chen & Simchi-Levi [11].

3. If $r_t > S_t$, then

$$v_t(x) = \begin{cases} -K_t + g_t(S'_t, d_t(S'_t)) + c_t x & \text{if } x < r_t \\ g_t(x, d_t(x)) + c_t x & \text{if } x \geq r_t \end{cases}$$

Similarly as 2, consider the following cases:

I. $x' < r_t$. $\Delta v = \max\{\lambda, 1 - \lambda\}K_t \geq 0$

II. $r_t \leq x < x'$

$$\Delta v = g_t(x_\lambda, d_t(x_\lambda)) - (1 - \lambda)[g_t(x, d_t(x))] - \lambda[g_t(x', d_t(x'))] + \max\{\lambda, 1 - \lambda\}K_t \geq 0$$

III. $x \leq x_\lambda < r_t \leq x'$

$$\begin{aligned} \Delta v &= -K_t + g_t(S'_t, d_t(S'_t)) - (1-\lambda)[-K_t + g_t(S'_t, d_t(S'_t))] \\ &\quad - \lambda[g_t(x', d_t(x'))] + \max\{\lambda, 1-\lambda\}K_t \\ &\geq \lambda[g_t(S'_t, d_t(S'_t)) - g_t(x', d_t(x'))] \geq 0 \end{aligned}$$

IV. $x < r_t \leq x_\lambda$

$$\begin{aligned} \Delta v &= g_t(x_\lambda, d_t(x_\lambda)) - (1-\lambda)[-K_t + g_t(S'_t, d_t(S'_t))] \\ &\quad - \lambda[g_t(x', d_t(x'))] + \max\{\lambda, 1-\lambda\}K_t \\ &= (1-\lambda)[g_t(x_\lambda, d_t(x_\lambda)) - g_t(S'_t, d_t(S'_t)) + K_t] \\ &\quad + \lambda[g_t(x_\lambda, d_t(x_\lambda)) - g_t(x', d_t(x'))] + \max\{\lambda, 1-\lambda\}K_t \end{aligned}$$

If $g_t(x_\lambda, d_t(x_\lambda)) - g_t(S'_t, d_t(S'_t)) + K_t \geq 0$ then

$$\Delta v \geq \lambda[g_t(x_\lambda, d_t(x_\lambda)) - g_t(x', d_t(x')) + K_t] \geq 0.$$

If $g_t(x_\lambda, d_t(x_\lambda)) - g_t(S'_t, d_t(S'_t)) + K_t < 0$, then $S'_t < x_\lambda \leq x'$. Hence, there exists $\mu < \lambda$,

such that $x_\lambda = (1-\mu)S'_t + \mu x'$.

$$\begin{aligned} \Delta v &= g_t(x_\lambda, d_t(x_\lambda)) - (1-\mu)g_t(S'_t, d_t(S'_t)) - \mu g_t(x', d_t(x')) + \max\{\mu, 1-\mu\}K_t \\ &\quad + (\lambda-\mu)[g_t(S'_t, d_t(S'_t)) - g_t(x', d_t(x'))] \\ &\quad + (1-\lambda)K_t + \max\{\lambda, 1-\lambda\}K_t - \max\{\mu, 1-\mu\}K_t \\ &\geq (1-\lambda)K_t + \max\{\lambda, 1-\lambda\}K_t - \max\{\mu, 1-\mu\}K_t \end{aligned}$$

If $\mu \leq 1/2$, then $\Delta v \geq (1-\lambda + \lambda - 1 + \mu)K_t \geq 0$.

If $\mu > 1/2$, then $\Delta v \geq (1-\lambda + \lambda - \mu)K_t \geq 0$.

Therefore, $v_t(x)$ is sym- K_t -concave.

Part (d) follows from part (c) by defining $A_t = I_t \cap [s_t, (s_t + S_t)/2]$. ■

3.4. Conclusions

In chapter 3, we first consider a periodic review inventory system with a fixed setup cost and two demand classes: deterministic and stochastic, where the deterministic demand must be satisfied immediately and the stochastic demand can be backlogged. Assuming that the

stochastic demand is never backlogged if there is stock in the system, a modified (s, S) policy is proved optimal by a previous paper under certain conditions. We first relax one of the conditions in literature that is restrictive and precludes many practical applications. We present two properties that each is equivalent to the optimality of the (s, S) policy to the problem. These properties are instrumental in identifying the (s, S) -policy optimality conditions. We then propose one such sufficient condition that is weaker than that in literature. As an application, in the second part, we study a joint pricing and inventory system where the stochastic demand is price sensitive and thus pricing and inventory decisions are made simultaneously. The relaxed condition above enables us to show that a modified (s, S, p) policy is optimal for additive demand functions and a modified (s, S, A, p) policy is optimal for general demand functions.

4. Optimal production and rationing policy of a make-to-stock production system with batch demand

4.1 Introduction

In this chapter, we consider a dynamic production system that maintains inventory of a single product to satisfy demands of different classes of customers. Customers are classified into classes according to their priorities which can be expressed in a variety of ways e.g., economic value to the manufacturer, terms of supply contracts etc. In periods of high demand a commonly used practice is to ration the limited stock among different demand classes. Stock rationing strategies have been widely used in many industries, including manufacturing and retail industries for allocating goods to the contracted versus walk-in customers, airlines for rationing seat inventories, as well as hotels for renting rooms. Since customers arrive sequentially, an important decision problem for managers is to decide whether or not they should fill a current demand with a lower priority in favor of reserving the stock for later demand with a higher priority. Determining the optimal policy for stock rationing, and furthermore, analyzing simple structures for the optimal policy is therefore a key decision problem with real life significance.

We consider the stock rationing of a production/inventory system in which a random batch-size demand arrives in accordance with Poisson process. Batch demand is very common in the wholesale market and manufacturing industries such as electronic and automobile. In this essay, we assume that batch arrivals can be partially accepted, i.e., given a

demand for $a \geq 1$ units, we can sell any quantity k in the range $0 \leq k \leq a$, and the customer is willing to buy any amount in this range. We believe that this is a reasonable assumption. For instance, an electronic equipment manufacturer produces spare parts that are used to replace defective components when repairing its specified equipment. The production lead-time is stochastic due to uncertain production environment. Since some equipment is vital while some others are auxiliary, spare parts installed in the equipment should be treated differently. Thus, demand can be classified by multiple classes with different priority. When a batch demand of the parts arises due to the failure of some auxiliary equipment, the manager will decide how much the batch of demand should be satisfied, or reject this non-critical demand and reserve inventory in the hope to satisfy later demand from vital equipment. However, if only part of the demand is accepted, partial fulfillment of the order is possible. The customer would like to accept any amount the manufacturer provides such that she can repair as much as possible the defective equipment. In addition, the rest of the demand may be sent by emergency supply from another warehouse that serves as a backup. For the stock at the manufacturer, the demand for the non-satisfied sets is considered as lost.

In another example, air-conditioner retailers often place batch orders to air-conditioner manufacturers. In a hot season, a manufacturer may not be able to satisfy all demand. It is reasonable to ration inventory among retailers with different economic value to the manufacturer. Therefore, even though the current inventory is adequate, the manufacturer may reject or only partially accept a batch of demand from a retailer with low priority. However, the retailers would like to accept the partial offer and try to purchase the unmet amount of air-conditioners from other manufacturers.

This behavior can also be observed among retailers of companies like Apple Computers that market trendy products. When Apple releases its new line of iPhones or iPods, typically a huge demand builds up in the initial launch period which for reasons described above cannot be satisfied immediately. Independent retailers are accorded a lower priority as compared to Apple operated company stores. Since Apple can only satisfy partial orders from independent retailers, the balance demand is backordered and is fulfilled in future periods (Wingfield [57]).

We study both the lost-sales model and the backordering model. For the lost-sales model, our work parallels to Huang & Iravani [29]. For the backordering model, our work is closely related with Ha [27], de Vericourt et al. [15] and Huang & Iravani [29]. Different with previous papers, we consider the stock rationing production/inventory systems with batch production, multiple demand classes and batch demand for each arrival. We seek to investigate the optimality of the critical level policy under more general scenarios. The relationship between this essay and the relevant literature is illustrated in Table 4.1.

Table 4.1 MDP models for stock rationing problems

	Lost-sales, Poisson demand	Lost-sales, Compound Poisson	Backorders, Poisson demand	Backorders, Compound Poisson
2 demand classes			Ha [27]	Huang & Iravani [29] *
N demand classes	Ha [26]	Huang & Iravani [29], this essay	de Vericourt et al. [15]	This essay

*: fixed and deterministic batch size

First, as seen, we extend Ha [27] (two-class, unit demand), de Vericourt et al. [15] (multiple-class, unit demand) and Huang & Iravani [29] (two-class, batch demand) to address the multiple-class, batch demand problem. Second, we assume the facility can produce a

batch up to a certain capacity at the same time, while all the above papers consider producing unit-by-unit. These models do not apply to the batch production case, which is common in practice. To our knowledge, batch production has not been well studied in the make-to-order queue literature.

Similar extension from unit-demand case to batch-demand case has also been found in airline yield management where Brumelle & Walczak [5] consider a batch-size demand with the semi-Markov arrival process. Our work differs significantly from theirs in the following ways. First, we consider both stock rationing and production in an infinite-horizon problem, while they consider a finite-period problem in context of seat capacity allocation. Their model is a one-dimension semi-Markov decision process and our model (the backordering case) is a multi-dimension Markov decision process. Second, structures of the optimal control policies are essentially different. In their paper, the switching curve, called booking curve therein, is non-stationary and monotone with respect to time. Here, the optimal rationing policy is characterized by multiple rationing levels.

4.2 Lost-sales model

4.2.1 Model formulation

Consider a production facility that produces a single product to stock. There are N different demand classes, which differ in their lost-sales costs c_i ($1 \leq i \leq N$) with $c_1 > c_2 > \dots > c_N$. For class i , arrivals occur according to a Poisson process with rate λ_i ($1 \leq i \leq N$) and each arrival requests a batch size a ($1 \leq a < +\infty$) with probability p_{ia} ,

where $\sum_{a=1}^{\infty} p_{ia} = 1$. When a batch-size demand arrives, it is either satisfied immediately (fully

or partially) from on-hand inventory or rejected. Accordingly, each rejected demand incurs a lost-sales cost c_i . We assume that the batch arrivals can be partially satisfied – that is, given demand with $a \geq 1$ units, we can satisfy any quantity k in the range $0 \leq k \leq a$, provided that inventory on hand is sufficient. The production time is exponentially distributed with mean $1/\mu$. The objective is to minimize the total expected costs through stock rationing and production control.

Let $X(t)$ be the inventory level at time t and $h(X(t))$ be the convex holding cost function. Let α be the continuous-time discount rate. When a batch demand of class i with size a arrives, it could be satisfied by units ranging from 0 to $\min(k, X(t))$, the minimum of k and $X(t)$. If j units are rejected, a lost sales cost $c_i j$ is incurred. Formally, let $u_i(t) = u_i(X(t))$ denote the action function and a policy $\{u_0(t), \dots, u_N(t) : t \geq 0\}$ by u . Let $N_i^u(t)$ be the number of class i customers who have been rejected up to time t , given a policy u . Let $X^u(t)$ be the inventory level at time t given a policy u is adopted. We seek to find the control policy u that minimizes the following expected discounted cost over an infinite horizon:

$$J^u(x) = E_x^u \left[\int_0^\infty e^{-\alpha t} h(X^u(t)) dt + \sum_{i=1}^N \int_0^\infty e^{-\alpha t} c_i dN_i^u(t) \right] \quad (4.1)$$

where x is the initial inventory level $X(0)$. E_x^u denotes the expected value over demand, given an initial inventory level x and a policy u . Let $v(x) = \min_u J^u(x)$ be the optimal cost

function with initial inventory x . By the uniformization technique in Lippman [37],

let $\gamma = \mu + \sum_{i=1}^N \lambda_i$. Because it is always possible to redefine the time scale, without loss of

generality, we can define the time unit so that $\alpha + \gamma = 1$. Thus $\lambda_i = \lambda_i / (\alpha + \mu + \sum_{i=1}^N \lambda_i)$

and $\mu = \mu / (\alpha + \mu + \sum_{i=1}^N \lambda_i)$ become, respectively, the probability that the next uniformized transition is a type- i arrival or production completion.

Note that the uniformation rate includes the discount factor, α . In fact, it is well known that discounting at rate α is equivalent to including a constant intensity at which the process terminates. Thus, one may think of α as the per-period probability that the next transition is a terminating one. (See, e.g., Puterman [45] §5.3 and Savin, Cohen, Gans, & Katalan [46]).

Then the dynamic programming equation is: $v(x) = Tv(x)$, where T is the operator on the set of real-valued functions v defined on the set of nonnegative integers Z^+ with

$$Tv(x) = h(x) + \mu \min \{v(x+1), v(x)\} + \sum_{i=1}^N \lambda_i \left[\sum_{a=1}^{\infty} p_{ia} H_{ia} v(x) \right]$$

where H_{ia} is the operator with $H_{ia} v(x) = \min_{0 \leq k \leq \min\{a, x\}} [v(x-k) + c_i(a-k)]$.

4.2.2 The optimal policy

Define V as the set of functions on Z^+ such that if $v \in V$, then:

$$(1) \quad v(x+1) - v(x+2) \leq v(x) - v(x+1)$$

$$(2) \quad v(x+1) - v(x) \geq -c_1$$

By definition, V is the set of convex functions with first difference bounded below by $-c_1$.

Lemma 4.1 Suppose $g(\cdot)$ is a convex function in $x \geq 0$. Define function f as

$$f(x) = \min_{0 \leq k \leq \min(a, x)} \{g(x-k) + c(a-k)\}$$

where c is a nonnegative constant and k is a nonnegative integer. Then $f(x)$ is convex in $x \geq 0$.

Proof The proof follows the approach in Lemma 1 of Lautenbacher & Stidham [35]. Consider two cases:

(i) $0 < k \leq x$, let $t = x - j$, then

$$\begin{aligned} f(x) &= \min_{0 \leq j \leq k \leq x} \{g(x-j) + c(k-j)\} = ck + \min_{0 \leq j \leq k \leq x} \{g(x-j) - cj\} \\ &= c(k-x) + \min_{0 \leq x-k \leq t \leq x} \{g(t) + ct\} = c(k-x) + \tilde{f}(x) \end{aligned}$$

where $\tilde{f}(x) = \min_{0 \leq x-k \leq t \leq x} \{g(t) + ct\}$.

Let $t^* = \arg \min_{t \geq 0} \{g(t) + ct\}$. By the convexity of g , we have

$$\tilde{f}(x) = \begin{cases} g(x) + cx & \text{if } t^* \geq x \\ g(t^*) + ct^* & \text{if } x-k < t^* < x \\ g(x-k) + c(x-k) & \text{if } t^* \leq x-k \end{cases}$$

For $x < t^*$,

$$\begin{aligned} \tilde{f}(x-1) - \tilde{f}(x) &= g(x-1) + c(x-1) + x \cdot c - g(x) - cx \\ &= g(x-1) - g(x) - c \\ &\geq g(x) - g(x+1) - c \\ &= \tilde{f}(x) - \tilde{f}(x+1). \end{aligned}$$

For $t^* \leq x \leq t^* + k$, it follows from the definition of t^* that

$$\tilde{f}(x-1) \geq g(t^*) + ct^* = \tilde{f}(x) \leq \tilde{f}(x+1)$$

So we have

$$\tilde{f}(x-1) - \tilde{f}(x) \geq 0 \geq \tilde{f}(x) - \tilde{f}(x+1).$$

For $x > t^* + k$,

$$\begin{aligned} \tilde{f}(x-1) - \tilde{f}(x) &= g(x-k-1) - g(x-k) - c \\ &\geq g(x-k) - g(x-k+1) - c \\ &= \tilde{f}(x) - \tilde{f}(x+1). \end{aligned}$$

Thus $\tilde{f}(x-1) - \tilde{f}(x)$ is non-increasing in x , and \tilde{f} is convex for $x \geq 0$. Therefore,

$f(x) = \tilde{f}(x) + c(k-x)$ is also convex for $x \geq 0$.

(ii) $0 \leq x \leq k$. $f(x)$ can be written as

$$\begin{aligned} f(x) &= \min_{0 \leq j \leq x \leq k} \{g(x-j) + c(k-j)\} = ck + \min_{0 \leq j \leq x \leq k} \{g(x-j) - cj\} \\ &= c(k-x) + \min_{0 \leq t \leq x} \{g(t) + ct\} = c(k-x) + \tilde{f}(x) \end{aligned}$$

where $\tilde{f}(x) = \min_{0 \leq t \leq x} \{g(t) + ct\}$.

Let $t^* = \arg \min_{t \geq 0} \{g(t) + ct\}$. By the convexity of g ,

$$\tilde{f}(x) = \begin{cases} g(x) + cx & \text{if } t^* \geq x \\ g(t^*) + ct^* & \text{if } 0 \leq t^* < x \end{cases}$$

For $x < t^*$,

$$\begin{aligned} \tilde{f}(x-1) - \tilde{f}(x) &= g(x-1) + c(x-1) - g(x) - cx \\ &= g(x-1) - g(x) - c \\ &\geq g(x) - g(x+1) - c \\ &= \tilde{f}(x) - \tilde{f}(x+1). \end{aligned}$$

For $t^* \leq x$, it follows from the definition of t^* that

$$\tilde{f}(x-1) \geq g(t^*) + ct^* = \tilde{f}(x) \leq \tilde{f}(x+1)$$

So we have

$$\tilde{f}(x-1) - \tilde{f}(x) \geq 0 \geq \tilde{f}(x) - \tilde{f}(x+1)$$

Thus $\tilde{f}(x-1) - \tilde{f}(x)$ is non-increasing in x , and \tilde{f} is convex for $x \geq 0$. Therefore,

$$f(x) = \tilde{f}(x) + c(k-x) \text{ is also convex for } x \geq 0. \quad \blacksquare$$

Lemma 4.2 *If $v \in V$, $h(x)$ is convex and nondecreasing in x , then $Tv \in V$.*

Proof We first prove the convexity. From Lemma 4.1, if $v \in V$, then $H_{ia}v(x)$ is convex

for $1 \leq i < N$, $1 \leq a < \infty$. Therefore, $\sum_{i=1}^N \lambda_i [\sum_{a=1}^{\infty} p_{ia} H_{ia}v(x)]$ is convex. Also from the

convexity of $h(x)$ and $\mu \min\{v(x+1), v(x)\}$, we have that Tv is convex.

To prove the lower bound for the difference of the cost functions, firstly, we show

$$\text{that } H_{ia}v(x+1) - H_{ia}v(x) \geq -c_1.$$

For $H_{ia}v(x) = \min_{0 \leq k \leq \min\{a, x\}} [v(x-k) + c_i(a-k)]$ and

$$H_{ia}v(x+1) = \min_{0 \leq k \leq \min\{a, x+1\}} [v(x+1-k) + c_i(a-k)], \text{ consider two cases}$$

(i) $a \leq x$

$$\begin{aligned} H_{ia}v(x+1) &= \min_{0 \leq k \leq \min\{a, x+1\}} [v(x+1-k) + c_i(a-k)] \\ &= \min\{v(x+1), v(x) - c_i, \dots, v(x+1-a) - c_i a\} + c_i a \end{aligned}$$

$$H_{ia}v(x) - c_1 = \min\{v(x) - c_1, v(x-1) - c_i - c_1, \dots, v(x-a) - c_i a - c_1\} + c_i a$$

Because $v(x+1-k) - c_i k \geq v(x-k) - c_i k - c_1$ for $0 \leq k \leq a$, we have

$$H_{ia}v(x+1) \geq H_{ia}v(x) - c_1.$$

(ii) $a > x$

$$\begin{aligned} H_{ia}v(x+1) &= \min_{0 \leq k \leq \min\{a, x+1\}} [v(x+1-k) - c_i k] + c_i a \\ &= \min\{v(x+1), v(x) - c_i, \dots, v(1) - c_i x, v(0) - c_i(x+1)\} + c_i a \end{aligned}$$

$$H_{ia}v(x) - c_1 = \min\{v(x) - c_1, v(x-1) - c_i - c_1, \dots, v(0) - c_i x - c_1\} + c_i a$$

Because $v(x+1) \geq v(x) - c_1$, $v(x) - c_i \geq v(x-1) - c_i - c_1, \dots, v(1) - c_i x \geq v(0) - c_i x - c_1$,

and $v(0) - c_i(x+1) \geq v(0) - c_i x - c_1$, we have $H_{ia}v(x+1) \geq H_{ia}v(x) - c_1$.

Since v and Tv are convex and $h(x)$ is increasing in x , we have

$$\begin{aligned} Tv(x+1) - Tv(x) &= h(x+1) - h(x) + \mu[\min\{v(x+2), v(x+1)\} - \min\{v(x+1), v(x)\}] \\ &\quad + \sum_{i=1}^N \lambda_i \left[\sum_{a=1}^{\infty} p_{ia} (H_{ia}v(x+1) - H_{ia}v(x)) \right] \\ &\geq \mu[\min\{v(x+2), v(x+1)\} - \min\{v(x+1), v(x)\}] \\ &\quad + \sum_{i=1}^N \lambda_i \left[\sum_{a=1}^{\infty} p_{ia} (H_{ia}v(x+1) - H_{ia}v(x)) \right] \\ &\geq -\mu c_1 - \sum_{i=1}^N \lambda_i \sum_{a=1}^{\infty} p_{ia} \cdot c_1 = -(\mu + \sum_{i=1}^N \lambda_i) \cdot c_1 \geq -c_1 \end{aligned}$$

This completes the proof. ■

Based on Lemma 4.2, we have the following results for the lost-sales problem.

Theorem 4.1 (a) *The optimal cost function $v(x)$ is convex and its first difference is bounded below by $-c_1$.*

(b) *A base-stock policy is optimal for controlling production, i.e., there exists a critical stock level S such that it is optimal to produce if the inventory level drops below S and not to*

produce otherwise.

A stock-rationing policy is optimal for rationing inventory, i.e., there exist rationing levels R_1, \dots, R_N such that it is optimal to reject the whole batch of class i demand if the inventory level is below R_i , or to satisfy it as much as possible until the inventory level reaches R_i . Moreover, $S \geq R_N \geq \dots \geq R_1 = 0$.

Proof Part (a) is an immediate result from Lemma 4.1 and 4.2. For Part (b), define

$$S = \min\{x \in Z^+ : v(x+1) - v(x) > 0\} \quad \text{and} \quad R_i = \min\{x \in Z^+ : v(x+1) - v(x) > -c_i\} .$$

Since $v \in V$, $v(x+1) - v(x)$ is increasing in x . As x increases, $v(x+1) - v(x)$ will first cross $-c_1$, then cross $-c_2$, and cross $-c_N$, and finally cross 0. Therefore, for demand of class i with a batch size a , we have four cases: (i) If inventory level x is below R_i , it is optimal to reject the demand since $v(x) - v(x-1) \leq -c_i$ for any x . (ii) If inventory level $x \geq R_i + a$, it is optimal to accept all the a units of the demand since $v(x - (a-1)) - v(x-a) > -c_i$. (iii) If inventory level x is $R_i < x < R_i + a$, it is optimal to partially accept the demands until the inventory decreases to R_i and reject the remainder of the demands. (iv) If inventory level x is below S , then $v(x+1) - v(x) \leq 0$, it is optimal to produce; otherwise stop production.

Hence, the structure of the optimal rationing and production policy can be characterized as the critical stock levels $S \geq R_N \geq \dots \geq R_1 = 0$. ■

Here, $R_1 = 0$, that is $v(x) - v(x-1) \geq \dots \geq v(1) - v(0) > -c_1$ for any $x > 0$, implying that it is always optimal to satisfy class 1 demand if there is any inventory on hand. This is intuitive since class 1 has the highest priority.

4.2.3 Numerical studies

As seen from section 4.2.2, the structures of the optimal production and rationing policy for the Poisson demand models continue to hold when demand arrival follows a compound Poisson process, provided that the batch demand can be partially accepted.

On the other hand, problems could be more complicated when the batch demand must be accepted on an all-or-none basis – that is, given a request for $a > 1$ units we can only sell all a units or none at all. This seemingly modest change has a profound impact on the structure of optimal allocation policies leading to the loss of convexity of the optimal cost functions. Brumelle & Walczak [5] confirm such kind of loss of concavity of optimal value function in a general demand arrival case. To illustrate this situation, we revise our operators on the basis of the all-or-none control. $H_{ia}v(x)$ becomes

$$H_{ia}v(x) = \begin{cases} \min[v(x) + a c_i v(x-a)] & \text{if } a \leq x \\ v(x) + ac_i & \text{if } a > x \end{cases}$$

The next example illustrates the optimal structure when the batch can be partially accepted, and shows that the optimal structure may lose when demand must be accepted on an all-or-none basis.

Example The state space is truncated to $[0, 50]$. There are five demand classes with lost-sales cost 3, 2.5, 2, 1.5 and 1 respectively. The holding cost is $h=1$. The production rate is $\mu=10$. The demand arrival rates for the five classes are $\lambda_i = 1, i = 1, 2, \dots, 5$. Let discount rate be $\alpha = 0.05$. Batches for different demand classes have the same distribution with $p_{i2} = p_{i10} = 0.5, i = 1, 2, \dots, 5$.

The cost functions for the two cases, all-or-none accepted and partially accepted, are shown in Figure 4.1 respectively. Clearly for the partially accepted case, the cost function is

convex. For the all-or-none accepted case, however, the convexity of cost function is destroyed.

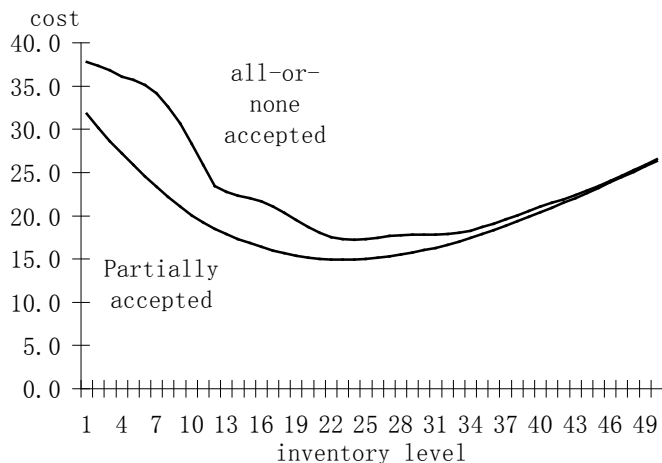


Figure 4.1 Numerical Example: the lost-sales case

Next, we present the results of a set of numerical examples in which we evaluate the benefits of implementing an optimal rationing policy under different parameters. The state space is truncated to $[0, 50]$. We compare the costs of optimal policies with and without rationing decision. The cost reduction rate (CR) is defined as

$$\max_{x \in [0, 50]} \frac{\text{cost without rationing} - \text{cost with rationing}}{\text{cost with rationing}}$$

That is, we compute the cost reduction rate for each initial state and take the maximum over all possible states.

Some of the numerical results are reported in Table 4.2. We consider two demand classes with different values of $h = 0.1, 0.2, 0.5, 1, 5$ and 10 ; $c_1 = 1.5, 2, 5, 10$ and 20 ; $\mu = 10, 20, 50$ and 100 ; $\lambda_1 / \lambda_2 = 0.1, 0.2, 1, 5, 10$ and 20 . In all these examples, $\alpha = 0.95$ and $c_2 = 1$. We test two distributions of batch: (I) Batches for both classes are discrete uniformly distributed within $[1, 10]$; (II) $p_{i4} = p_{i6} = 0.5$ for $i = 1, 2$.

Table 4.2 Cost reduction for the lost-sales case with two demand classes

h	c_1	c_2	μ	λ_1	λ_2	Batch (I)	Batch (II)
						CR (%)	CR (%)
0.1	1.5	1	50	5	5	0.468	0.598
0.1	2	1	50	5	5	0.468	0.598
0.1	5	1	50	5	5	17.745	20.329
0.1	10	1	50	5	5	35.161	38.107
0.1	20	1	50	5	5	50.380	53.884
0.2	5	1	50	5	5	16.839	19.305
0.5	5	1	50	5	5	14.562	16.819
1	5	1	50	5	5	11.727	13.723
5	5	1	50	5	5	2.555	3.716
10	5	1	50	5	5	0.086	0.449
0.1	5	1	50	1	10	0.246	0.322
0.1	5	1	50	1	5	0	0
0.1	5	1	50	5	1	5.132	5.869
0.1	5	1	50	10	1	7.070	8.044
0.1	5	1	50	20	1	6.278	7.140
0.1	5	1	10	5	5	31.844	34.370
0.1	5	1	20	5	5	30.185	33.265
0.1	5	1	100	5	5	3.641	2.986
0.1	5	1	200	5	5	0	0

In the first group (example 1-5) in Table 4.2, we investigate the cost reduction by increasing the lost-sales cost of the first class. As we can see, when the lost-sales cost of the first class is close to that of the second class, the cost reduction rate almost equals to zero. Hence, there is not much benefit from rationing. When c_1 / c_2 increases, the cost reduction rate increases significantly and thus much benefit is gained from rationing. For example, when $c_1 / c_2 = 20$, the cost reduction exceeds 50%. In the second group (example 6-10), the cost reduction is decreasing when the holding cost is increasing. This result is intuitive since rationing incurs less lost-sales cost but more holding cost when the holding cost is high. In the third group (example 11-15), we change λ_1 / λ_2 from 0.1 to 20. We find that when λ_1 is small compared with λ_2 , there is not much benefit from rationing. On the other hand, when

λ_1 is larger, it is more valuable to implement the rationing policy. It seems that the cost reduction is not monotone in the holding cost.

In the fourth group (example 16-19), the results show that cost saving of the optimal rationing policy is larger when the production rate becomes smaller. The results are intuitive: when the system has excessive capacity to meet demand, it is optimal to hold less inventory and hence the effect of inventory rationing is not significant. When capacity becomes small relative to demand, stockout becomes more frequent and the optimal rationing policy yields significant benefit. It is worth noting that when capacity continues to fall, eventually inventory is depleted to zero and demand is lost most of the time and the effect of the rationing policy diminishes. This has been shown in the literature (See e.g., Ha [26] pp1100-1101). Therefore, a rationing is most critical when the traffic intensity $((\lambda_1 + \lambda_2) / \mu)$ of the system takes some intermediate values.

4.3 Backordering model

4.3.1 Model formulation

Consider a production facility that produces a single product. The nature of the production technology is such that products are made in bulk: relatively large, discrete batches are produced, and finished product becomes available only when an entire batch is complete. Furthermore, the time required to make a batch depends only slightly on the size of the batch, either because the total time is dominated by setup time, or because the processing itself affects the entire batch at once. This is called bulk service problem in the literature (see Gross & Harris [25] pp122).

Such bulk processing can be thought of as "cooking," and indeed many food and beverage products are made in this way. The same features characterize parts of several other industries, including the blending of gasoline and other petrochemicals; the mixing, baking, granulation and quality control of pharmaceuticals; and the coating, etching and finishing of semi-conductor (Zipkin [63]).

We consider a single-channel production facility whose production time is exponentially distributed with mean $1/\mu$. The facility can produce up to M units at a time. If less than M units are produced, new production up to limit M can be started immediately, and all units will finish together, regardless of the starting time. To avoid infinite backorders, we assume

$$M\mu > \sum_{l=1}^N \lambda_l \sum_{a=1}^{\infty} ap_{la}.$$

The finished items are placed in a common inventory with unit holding costs of h per unit time. There are N demand classes, which differ in their backordering costs b_l ($1 \leq l \leq N$), where $b_1 > b_2 > \dots > b_N$. For class l , arrivals occur according to a Poisson process with rate λ_l ($1 \leq l \leq N$) and each arrival requests a batch size a ($1 \leq a < +\infty$) with probability p_{la} , where $\sum_{a=1}^{\infty} p_{la} = 1$. When a batch demand arrives, it is either satisfied immediately from on-hand inventory or backlogged. Accordingly, each backlogged demand incurs a backordering cost b_l per unit time. We assume that the batch arrivals can be partially satisfied – that is, given demand with $a \geq 1$ units, we can satisfy it with any quantity k in the range $0 \leq k \leq a$, provided that inventory on hand is sufficient. Since $b_1 > b_2 > \dots > b_N$, we only consider satisfying class l after all the backorders from more valuable classes are delivered.

Define the state variable of the system as $x(t) = (x_1(t), x_2(t), \dots, x_N(t))$ with $x_1(t) \in Z$ and $x_l(t) \in Z^+$ for $2 \leq l \leq N$. Notation $x_1^+(t) = \max(0, x_1(t))$ is the on-hand inventory at time t and $x_1^-(t) = -\min(0, x_1(t))$ is the number of class 1 backorders at time t .

We let $x_l(t)$ ($2 \leq l \leq N$) be the number of backorders of class l at time t .

$IN(t) = x_1 - \sum_{l=2}^N x_l$ denotes the net inventory at time t . Let

$C^\pi(x) = (C_0^\pi(x), C_1^\pi(x), \dots, C_N^\pi(x))$ denote the control action associated with a policy π .

Specifically,

$C_0^\pi(x) = (k_1, k_2, \dots, k_N)$ to produce $\sum_{l=1}^N k_l (\leq M)$ and allocate k_1 to the on-hand

inventory (or to satisfy class 1 backorder), and allocate $k_l (0 \leq k_l \leq x_l)$ to the

backorder of class l , $2 \leq l \leq N$.

$C_1^\pi(x) = (a \wedge x_1^+)$ to allocate $(a \wedge x_1^+)$ units from the on-hand inventory to an order of class 1 with size a .

$C_l^\pi(x) = k_l$ to allocate k_l units from the on-hand inventory to an order of class l with size a , $0 \leq k_l \leq (a \wedge x_l^+)$, $2 \leq l \leq N$.

where $a \wedge b = \min(a, b)$.

Let α be the discount rate. We seek to find the optimal control policy π that minimizes the following expected discounted costs over an infinite horizon:

$$J^\pi(x) = E_{x(0)}^\pi \left[\int_0^\infty e^{-\alpha t} (hx_1^+(t) + b_1 x_1^-(t) + \sum_{l=2}^N b_l x_l(t)) dt \right] \quad (4.2)$$

Define the optimal cost function $v(x) = \min_{\pi} J^\pi(x)$. Following the uniformization

technique in Lippman [37], let $\gamma = \mu + \sum_{l=1}^N \lambda_l$. Without loss of generality, we have $\alpha + \gamma = 1$

by rescaling the time. Thus, the dynamic programming equation is $v(x) = Tv(x)$, where

$$Tv(x) = c(x) + \mu T_0 v(x) + \sum_{l=1}^N \lambda_l \sum_{a=1}^{\infty} p_{la} T_{la} v(x) \quad (4.3)$$

In (4.3), $c(x)$ is the inventory holding and backordering cost function defined by

$$c(x) = \begin{cases} hx_1^+ + b_1 x_1^- + \sum_{l=2}^N b_l x_l & \text{if } x_l \geq 0 \text{ for } 2 \leq l \leq N \\ +\infty & \text{otherwise} \end{cases}$$

and

$$T_0 v(x) = \min_{k_1, k_2, \dots, k_N} [v(x + k_1 e_1 - \sum_{l=2}^N k_l e_l) : 0 \leq \sum_{l=1}^N k_l \leq M; k_l \geq 0; 0 \leq k_l \leq x_l, 2 \leq l \leq N]$$

$$T_{1a} v(x) = v(x - a e_1)$$

$$T_{la} v(x) = \begin{cases} \min_{0 \leq k \leq (a \wedge x_1^+)} v(x + (a - k) e_l - k e_1) & \text{if } x_1 > 0 \\ v(x + a e_l) & \text{if } x_1 \leq 0 \end{cases} \quad 2 \leq l \leq N$$

where e_l is the unit vector of dimension l ($1 \leq l \leq N$).

Note in $T_0 v(x)$, $\sum_{l=1}^N k_l (\leq M)$ is the total units to produce, and k_l represents the units

of production allocated to class l . When $\sum_{l=1}^N k_l = 0$, it implies not to produce. Term $T_{1a} v(x)$

implies that it is always to satisfy class 1 demand, the one with the highest priority, if there is

on-hand inventory. $T_{la} v(x)$ is illustrated as follows: Given class l demand with batch size a

and the current inventory level x_1 , we seek to find the optimal amount to satisfy class l

demand so as to minimize the total cost. If k ($0 \leq k \leq (a \wedge x_1^+)$) units of class l demand are

satisfied, the inventory on hand will reduce to $x_1 - k$, while the class l backorders will

increase to $x_1 + a - k$. If the current inventory level is negative, all demand of class l has to be

backordered with x_l increasing to $x_l + a$.

4.3.2 The optimal policy

We first define the critical level policy for the N -demand-class, batch production problem. We will show that the optimal production policy is a modified base-stock policy and the optimal rationing policy can be characterized by multiple critical levels.

A critical level policy π , is a policy characterized by an $N+1$ dimensional rationing level vector $z = (z_1, z_2, \dots, z_{N+1})$ where $z_1 = 0 \leq z_2 \leq \dots \leq z_{N+1}$ such that

$$(1) \text{ For production } C_0^\pi(x) : \sum_{i=1}^N k_i = (z_{N+1} - IN)^+ \wedge M$$

$$k_l = \left(\sum_{i=1}^N k_i + x_1 - \sum_{i=2}^{l-1} x_i - z_l \right)^+ \wedge x_l, \quad 2 \leq l \leq N$$

$$k_1 = (z_{N+1} - IN)^+ \wedge M - \sum_{i=2}^N k_i$$

$$(2) \text{ For rationing } C_l^\pi(x) = a \wedge (x_l - z_l)^+, \quad 1 \leq l \leq N.$$

The critical level policy works as follows. The production follows a modified base stock policy: Produce to bring the net inventory up to z_{N+1} if possible, or produce to the capacity M if the production quantity exceeds M . For rationing, backorders are satisfied in order of priority. Class l backorder will be satisfied as much as possible whenever the inventory level is above z_l .

Note this critical level policy is a generalized form of the ML policy in de Vericourt et al. [15] and Huang & Iravani [29]. When $M = 1$ and $a = 1$, it becomes the ML policy in de Vericourt et al. [15]. When $M = 1$ and $N = 2$, it becomes the ML policy in Huang & Iravani [29].

Define operators Δ_l and Δ_{ll} of any function v on Z^N .

$$\Delta_l v(x) = v(x + e_l) - v(x) \quad 1 \leq l \leq N$$

$$\Delta_{ll} v(x) = v(x + e_l) - v(x - e_l) \quad 2 \leq l \leq N$$

Define $m(x)$ as the most expensive class that has backlogged demand, i.e.,

$$m(x) = \begin{cases} 1 & \text{if } x_1 < 0 \\ \min_l(l : x_l > 0) & \text{if } x_1 \geq 0 \text{ and } \exists l, x_l > 0, 2 \leq l \leq N \\ N+1 & \text{otherwise} \end{cases}$$

Let V be the set of functions defined on Z^N such that if $v \in V$ then

(a) For all $x_1 < 0$

$$\Delta_1 v(x) \leq 0 \quad (4.4)$$

$$\Delta_{ll} v(x) \leq 0, \quad 2 \leq l \leq N \quad (4.5)$$

(b) For any $2 \leq l \leq N$, $x_l > 0$,

$$\Delta_l v(x - e_l) \geq 0 \quad (4.6)$$

(c) Sub-modularity/sub-convexity. If $m(x) = l$, $2 \leq l \leq N$, then

$$\Delta_1 v(x + e_l) \leq \Delta_1 v(x) \quad (4.7)$$

$$\Delta_{ll} v(x + e_l) \geq \Delta_{ll} v(x) \quad (4.8)$$

$$\Delta_{ll} v(x + e_l) \geq \Delta_{ll} v(x); \text{ The equality holds when } \Delta_{ll} v(x) < 0 \quad (4.9)$$

It is easy to verify that the following properties can be obtained from (4.7) – (4.9).

(d) Convexity. If $m(x) = l$, $2 \leq l \leq N$,

$\Delta_1 v(x + e_l) \geq \Delta_1 v(x)$, including states where $x_l = 0$, and

$$\Delta_l v(x + e_l) \geq \Delta_l v(x) \quad (4.10)$$

$$\Delta_{ll} v(x + e_1 + e_l) \geq \Delta_{ll} v(x) \quad (4.11)$$

Property (d) states that if l is the most valuable class of demand which has backorder,

then functions in V are convex in any x_l , and in the diagonal direction $e_1 + e_l$ as well. The convexity property is important to characterize the optimal structure of the production and rationing policies. For example, if the cost function is convex in x_l , then a modified base stock production policy is optimal, i.e., there exists a base stock level S such that for the states with $x_l = 0$ for all $2 \leq l \leq N$, it is optimal to produce as close as possible to S when the inventory on-hand does not exceed S and do not produce otherwise. If the cost function is convex in the diagonal direction $e_1 + e_l$, then considering the rationing policy for any initial state x , there exists a threshold S , such that it is optimal to satisfy class l demand as much as possible when $x_l > S$, otherwise, reserve inventory for future more valuable demand.

A rationing policy is *sequential* if a backorder is satisfied only when there is no backorder from any more important class. Since $b_1 > b_2 > \dots > b_N$, if a policy π is optimal, then π is sequential.

The sequential property of the optimal policy implies that we always firstly consider satisfying the demand from a more important class. The property will be adapted to the rationing policy. Also from (4.11), the following result is obvious:

Suppose $v \in V$. For the initial state x ($x_1 > 0$), $m(x) = l$, if $\Delta_l v(x) \leq 0$, then it is optimal not to satisfy any backorders from class i for $i \geq l$.

To prove that a critical level policy is optimal, we need to show that (I) If $v \in V$, i.e., the cost function retains properties (4.4) - (4.11), then a critical level policy is optimal; (II) Tv preserves properties (4.4) - (4.11).

We present the main results of the backordering model in the following theorem.

Theorem 4.2 *If $v \in V$, then a critical level policy is optimal. Specifically, the critical level*

vector $z = (z_1, z_2, \dots, z_{N+1})$ is defined by

$$z_{N+1} = \min_{x_1} \{\Delta_1 v(x_1 e_1) > 0\};$$

$$z_l = 0; z_l = \min_{x_1} \{\Delta_l v(x_1 e_1 + e_l) > 0\}, \quad l = m(x), \quad 2 \leq l \leq N,$$

where z_l ($2 \leq l \leq N$) is the critical rationing level associated with class l , and z_{N+1} is the base stock level for production.

Proof From (4.10), there exists a z_{N+1} that minimizes $v(x_1 e_1)$ for all $x_1 \in Z$, i.e., $z_{N+1} = \min_{x_1} \{\Delta_1 v(x_1 e_1) > 0\}$. From (4.6), the global minimizer of v must be at some x where $x_i = 0$ for $2 \leq i \leq N$. Thus, $x^* = z_{N+1} e_1$ is the global minimizer of v . For $2 \leq l \leq N$,

$$\begin{aligned} \Delta_l v(z_{N+1} e_1 + e_l) &= v((z_{N+1} + 1)e_1 + e_l) - v(z_{N+1} e_1) \\ &\geq v((z_{N+1} + 1)e_1) - v(z_{N+1} e_1) = \Delta_1 v(z_{N+1} e_1) \end{aligned}$$

where the inequality is from (4.6). Thus $z_l \leq z_{N+1}$. In addition, for $1 < i < j \leq N$ and any x , it is obvious that $v(x + e_i) \geq v(x + e_j)$. Therefore,

$$\begin{aligned} \Delta_i v(z_j e_1 + e_i) &= v((z_j + 1)e_1 + e_i) - v(z_j e_1) \\ &\geq v((z_j + 1)e_1 + e_j) - v(z_j e_1) = \Delta_{1j} v(z_j e_1 + e_j) \end{aligned}$$

This implies $z_i \leq z_j$ for $1 < i < j \leq N$. $z_1 = 0$ is obvious, since class 1 is the most important class.

Define $z_l(x) = \min_{x_1} \{\Delta_l v(x + e_l) > 0\}$. We need show that $z_l(x) = z_l$, i.e., the threshold is independent of any state. From (4.11), we can easily verify that $z_l(x)$ is independent of x_l (when fix x_i , $i > l$). It suffices to show that $z_l(x)$ is independent of x_i , $i = l+1, \dots, N$.

Consider the initial state x at time $t_0 = 0$. Let $v^\pi(x)$ denote the expected cost when a policy $\pi = (\pi_1, \dots, \pi_N)$ is adopted (where π_i represents the policy associated with class i).

Because an optimal policy is sequential, if $x_l > 0$, we first consider to satisfy backorder from class l before satisfying any backorder from class i , $i > l$. Suppose \tilde{t}_0 ($\tilde{t}_0 > t_0$) is the first

time at which $x_j = 0$ ($2 \leq j \leq l$) and we allocate inventory to class i ($i > l$) backorder. This process continues until some new demand from class j ($j \leq l$) arrives at some time, t_1 ($t_1 > \tilde{t}_0$), and we then start to consider class j ($j \leq l$) demand. If we define the two status as *on* (consider to satisfy class j ($j \leq l$) backorder) and *off* (consider to satisfy class i ($i > l$) backorder), then the whole process is an alternating renewal process. $v^\pi(x)$ can be written as

$$v^\pi(x) = E_{t_0} [E_{t_1} [\int_{t_0}^{\tilde{t}_0} e^{-at} c(x(t)) dt + \int_{\tilde{t}_0}^{t_1} e^{-at} c(x(t)) dt + e^{-at_1} v^\pi(x(t_1)) | \tilde{t}_0]]]$$

We only consider the cost function in the first cycle: $[t_0, \tilde{t}_0)$ and $[\tilde{t}_0, t_1)$, since $[t_1, \infty)$ can be analyzed similarly. The result $z_l(x) = z_l$ can be obtained from the following facts: (1) \tilde{t}_0 is dependent of $z_{l+1}(x)$, independent of $\pi_l(z_l(x))$ and any π_j ($j < l$); (2) Given \tilde{t}_0 , t_1 only depends on demand arrival, independent of the rationing decision; (3) π_l only affects the cost function in $[t_0, \tilde{t}_0)$, i.e., the cost in $[\tilde{t}_0, t_1)$ is independent of π_l ; (4) In $[t_0, \tilde{t}_0)$, π_l is independent of x_i ($l < i \leq N$), since class i backorder is not considered in this period.

Therefore, the critical level z_l is independent of state x . Now consider the optimal policy. For rationing, when a batch demand of class i arrives with size a , from (4.11) and the definition of z_i , it is optimal to satisfy as much demand as possible whenever the on-hand inventory is greater than z_i . When $x_1 \leq z_i$, it is optimal not to satisfy any more. For production, there are the following cases: (1) If $IN \geq z_{N+1}$, it implies that all backlogged demands are satisfied. It is optimal not to produce. (2) If $z_{N+1} - M \leq IN < z_{N+1}$, obviously, it is optimal to produce $z_{N+1} - IN$ and satisfy all backorders. (3) If $IN < z_{N+1} - M$, we show that it is optimal to produce M units. For any policy that produces less than M , there must be some backorder that cannot be satisfied. Hence, this policy can be improved by producing one

more unit and satisfying one more backorder. From (4.10) and (4.11), the rationing policy works as follows: allocate production to the backorders in order, starting from the most important class. Stop to allocate any backorder when at some class l , the inventory on-hand drops to z_l , or finally all production units are allocated. ■

We next show that part (II) is true, i.e., Tv preserves properties (4.4) - (4.11). We first present some preliminary results in the following lemmas (from Lemma 4.3 to Lemma 4.6).

Obviously, Lemma 4.3 is immediate from (4.10) and (4.11).

Lemma 4.3 For any nonnegative integer k and state x , consider $v(x + ie_1 + ie_l)$, $i = 0, \dots, k$,

$2 \leq l \leq N$. Define

$$j^* = \min\{\arg \min_{0 \leq j \leq k} v(x + je_1 + je_l)\}$$

and

$$j^{**} = \min\{\arg \min_{1 \leq j \leq k+1} v(x + je_1 + je_l)\}$$

Then (1) $v(x + ie_1 + ie_l) \downarrow i$ for $0 \leq i \leq j^*$ and $v(x + ie_1 + ie_l) \uparrow i$ for $j^* \leq i \leq k$, where

\uparrow and \downarrow denote nondecreasing and nonincreasing, respectively.

(2) $j^{**} = j^*$ or $j^{**} = j^* + 1$.

To obtain the optimal structure of the production and rationing policy, we need only to prove that T preserves these structural properties (4.4) – (4.9). By (4.3), it suffices to show that if $v \in V$, then $c \in V$, $T_0v \in V$, $T_{1a}v \in V$ and $T_{la}v \in V$. Since c is similar as those defined in de Vericourt et al. [15], it has been proved therein that $c \in V$. The proof for T_{1a} can follow that in Huang & Iravani [29] with slight revision. In the following part, we focus on proving $T_{la}v \in V$ ($2 \leq l \leq N$) and $T_0v \in V$, i.e., $T_{la}v$ and T_0v preserve properties (4.4) – (4.9).

Lemma 4.4 Suppose $n = a \leq x_1$. For any demand class l , $2 \leq l \leq N$, define

$$i_1^* = \max \{ \arg \min_{0 \leq i \leq n} v(x - ie_1 + (a - i + 1)e_l) \},$$

$$i_2^* = \max \{ \arg \min_{0 \leq i \leq n} v(x - ie_1 + (a - i)e_l) \},$$

$$i_3^* = \max \{ \arg \min_{0 \leq i \leq n} v(x - (i - 1)e_1 + (a - i + 1)e_l) \},$$

$$i_4^* = \max \{ \arg \min_{0 \leq i \leq n} v(x - (i - 1)e_1 + (a - i)e_l) \}.$$

We have,

(1) If $i_2^* = i_3^* = n$, then $i_1^* = i_4^* = n$.

If $i_2^* = i_3^* = 0$, then $i_1^* = i_4^* = 0$.

(2) If $i_2^* \neq n$ and $i_3^* \neq 0$, then $i_3^* = i_4^* = i_1^* + 1 = i_2^* + 1$.

Lemma 4.4 is used to show that $T_{la}v$ satisfies (4.7) for the case $x_1 \geq a$. It indicates the relationships among the following four sets of state:

$$Z_1 = \{ (x - ie_1 + (a - i + 1)e_l) : 0 \leq i \leq n \},$$

$$Z_2 = \{ (x - ie_1 + (a - i)e_l) : 0 \leq i \leq n \},$$

$$Z_3 = \{ (x - (i - 1)e_1 + (a - i + 1)e_l) : 0 \leq i \leq n \},$$

$$Z_4 = \{ (x - (i - 1)e_1 + (a - i)e_l) : 0 \leq i \leq n \}.$$

To make our proof easier for readers to follow, we illustrate the relationships among the states in Lemma 4.4 in Figure 4.2. Here, Q_i , R_i and S_i , $0 \leq i \leq n + 1$ are used to denote the possible state x . Let S_{n-i} denote $x - ie_1 + (a - i)e_l$, R_{n-i} denote $x - ie_1 + (a - i + 1)e_l$ and Q_{n-i} denote $x - ie_1 + (a - i - 1)e_l$. As an example, S_n denotes $x + ae_l$, R_n denotes $x + (a + 1)e_l$, and Q_n denotes $x + (a - 1)e_l$. Thus, the four sets can be represented as $Z_1 = \{R_0, R_1, \dots, R_n\}$, $Z_2 = \{S_0, S_1, \dots, S_n\}$, $Z_3 = \{S_1, S_2, \dots, S_{n+1}\}$ and $Z_4 = \{Q_1, Q_2, \dots, Q_{n+1}\}$, respectively.

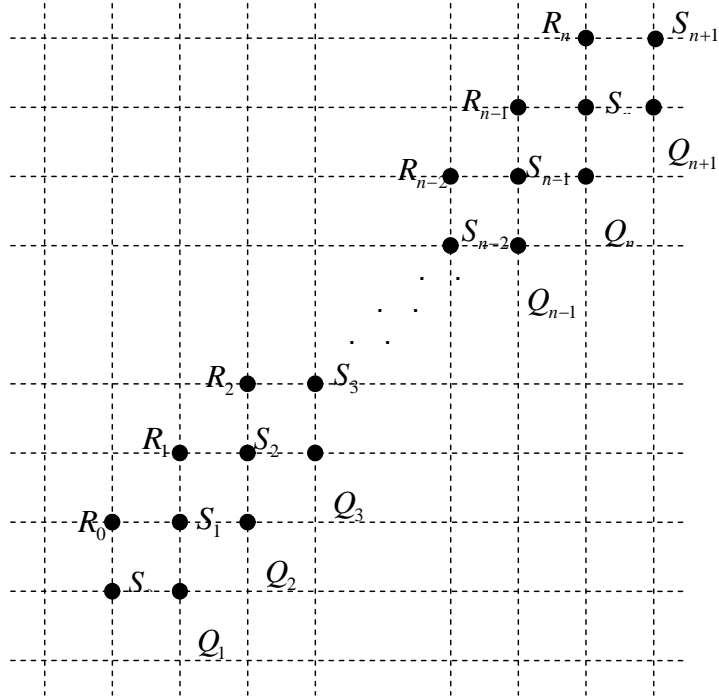


Figure 4.2 Graphical representation for Lemma 4.4

Define integers i_1, i_2, i_3, i_4 , states P_1, P_2, P_3, P_4 and state set M as:

$$i_1 = \min_{0 \leq i \leq n} \{\arg \min v(R_i)\}, \quad i_2 = \min_{0 \leq i \leq n} \{\arg \min v(S_i)\}, \quad i_3 = \min_{1 \leq i \leq n+1} \{\arg \min v(S_i)\},$$

$$i_4 = \min_{1 \leq i \leq n+1} \{\arg \min v(Q_i)\}, \quad P_1 = R_{i_1}, \quad P_2 = S_{i_2}, \quad P_3 = S_{i_3}, \quad P_4 = Q_{i_4},$$

and $M = \{P_1, P_2, P_3, P_4\}$.

It is easy to verify that $i_1 = n - i_1^*$, $i_2 = n - i_2^*$, $i_3 = n + 1 - i_3^*$ and $i_4 = n + 1 - i_4^*$ hold. Thus,

Lemma 4.4 can be presented graphically as follows:

$$(1) \text{ If } P_2 = S_0, \quad P_3 = S_1, \text{ then } P_1 = R_0 \text{ and } P_4 = Q_1 \quad (4.12)$$

$$\text{If } P_2 = S_n, \quad P_3 = S_{n+1}, \text{ then } P_1 = R_n \text{ and } P_4 = Q_{n+1} \quad (4.13)$$

(2) If $P_2 \neq S_0$ and $P_3 \neq S_{n+1}$, then $P_2 = P_3$. Furthermore,

$$\text{If } P_2 = P_3 = S_k \quad (1 \leq k \leq n), \text{ then } P_1 = R_k \text{ and } P_4 = Q_k \quad (4.14)$$

Remarks To show that $T_{la}v$ satisfies (4.7) for the case $0 \leq x_1 < a$, we need revise the four

sets as: $Z_1 = \{R_0, R_1, \dots, R_n\}$, $Z_2 = \{S_0, S_1, \dots, S_n\}$, $Z_3 = \{S_0, S_1, \dots, S_{n+1}\}$ and

$Z_4 = \{Q_0, Q_1, \dots, Q_{n+1}\}$. We have

(1) If $P_2 = S_n$, $P_3 = S_{n+1}$, then $P_1 = R_n$ and $P_4 = Q_{n+1}$

(2) If $P_2 = P_3 = S_k$ ($0 \leq k \leq n$), then $P_1 = R_k$ and $P_4 = Q_k$

The proof for the case $0 < x_1 < a$ is similar and thus is omitted.

Proof It is equivalent to verify (4.12), (4.13) and (4.14). For (4.12) and (4.13), we only prove

(4.12), since the proof for (4.13) is similar. As shown in Figure 4.2, if $P_2 = S_0$, $P_3 = S_1$ and $P_1 \neq R_0$, suppose $P_1 = R_k$, $1 \leq k \leq n$, by the definition of P_1 and Lemma 4.3,

$$v(R_0) \geq v(R_1) \geq \dots \geq v(R_{k-1}) > v(R_k),$$

and because $v(S_0) \leq v(S_1) \leq \dots \leq v(S_k)$, we have

$$v(S_0) + v(R_k) < v(R_0) + v(S_k). \quad (4.15)$$

On the other hand, by (4.9), we have $v(S_{i-1}) + v(R_i) \geq v(R_{i-1}) + v(S_i)$ for $1 \leq i \leq k$.

Therefore,

$$v(S_0) + v(R_k) \geq v(R_0) + v(S_k)$$

which contradicts with (4.15). Similarly, $P_4 = Q_1$ can be obtained by (4.8).

For (4.14), if $P_2 \neq S_0$ and $P_3 \neq S_{n+1}$, then $P_2 = P_3$ can be obtained directly by the definition of P_2 and P_3 . Now suppose $P_2 = P_3 = S_k$, $1 \leq k \leq n$. By (4.9) and the definition of P_2 , P_1 can only be some R_j , $j \geq k$. However, it is easy to verify that the case $j > k$ can be precluded since it contradicts with (4.9). Similarly, we have $P_4 = Q_k$. ■

The following properties in Lemma 4.5 and Lemma 4.6 are needed to show that $T_{1a}v$ satisfies (4.8) and (4.9), respectively.

Lemma 4.5 Suppose $n = a \leq x_1$. For any demand class l , $2 \leq l \leq N$, define

$$i_1^* = \max \{ \arg \min_{0 \leq i \leq n} v(x - (i-2)e_1 + (a-i)e_l) \},$$

$$i_2^* = \max\{\arg \min_{0 \leq i \leq n} v(x - (i-1)e_1 + (a-i-1)e_l)\},$$

$$i_3^* = \max\{\arg \min_{0 \leq i \leq n} v(x - (i-1)e_1 + (a-i)e_l)\},$$

$$i_4^* = \max\{\arg \min_{0 \leq i \leq n} v(x - ie_1 + (a-i-1)e_l)\}.$$

Then,

(1) If $i_3^* = i_4^* = n$, then $i_1^* = i_2^* = n$.

If $i_1^* = i_2^* = 0$, then $i_3^* = i_4^* = 0$.

(2) If $i_1^* = i_2^* = n$, then either $i_3^* = i_4^* = n$, or $i_3^* = n$ and $i_4^* = n-1$.

If $i_3^* = i_4^* = 0$, then either $i_1^* = i_2^* = 0$, or $i_1^* = 1$ and $i_2^* = 0$.

We put graphical interpretation as follows: in Figure 4.3, we use R_i and S_i , $0 \leq i \leq n+1$

to denote the possible x , where

$$T_2 v(x + 2e_1) = \min_{0 \leq k \leq n} v(x - (k-2)e_1 + (a-k)e_l) = \min_{1 \leq i \leq n+1} v(S_i)$$

$$T_2 v(x + e_1 - e_l) = \min_{0 \leq k \leq n} v(x - (k-1)e_1 + (a-k-1)e_l) = \min_{0 \leq i \leq n} v(S_i)$$

$$T_2 v(x + e_1) = \min_{0 \leq k \leq n} v(x - (k-1)e_1 + (a-k)e_l) = \min_{1 \leq i \leq n+1} v(R_i)$$

$$T_2 v(x - e_l) = \min_{0 \leq k \leq n} v(x - ke_1 + (a-k-1)e_l) = \min_{0 \leq i \leq n} v(R_i)$$

Define integers i_1, i_2, i_3, i_4 , states P_1, P_2, P_3, P_4 and state set M as

$$i_1 = \min_{1 \leq i \leq n+1} \{\arg \min v(S_i)\}, \quad i_2 = \min_{0 \leq i \leq n} \{\arg \min v(S_i)\}, \quad i_3 = \min_{1 \leq i \leq n+1} \{\arg \min v(R_i)\},$$

$$i_4 = \min_{0 \leq i \leq n} \{\arg \min v(R_i)\}, \quad P_1 = S_{i_1}, \quad P_2 = S_{i_2}, \quad P_3 = R_{i_3}, \quad P_4 = R_{i_4} \text{ and } M = \{P_1, P_2, P_3, P_4\}$$

Then, Lemma 4.5 is equivalent to the follows:

$$(1) \text{ If } P_3 = R_1 \text{ and } P_4 = R_0, \text{ then } P_1 = S_1 \text{ and } P_2 = S_0. \quad (4.16)$$

$$\text{If } P_1 = S_{n+1} \text{ and } P_2 = S_n, \text{ then } P_3 = R_{n+1} \text{ and } P_4 = R_n. \quad (4.17)$$

$$(2) \text{ If } P_1 = S_1 \text{ and } P_2 = S_0, \text{ then } P_3 = R_1 \text{ and } P_4 = R_0, \text{ or } P_3 = P_4 = R_1. \quad (4.18)$$

$$\text{If } P_3 = R_{n+1} \text{ and } P_4 = R_n, \text{ then } P_1 = S_{n+1} \text{ and } P_2 = S_n, \text{ or } P_1 = P_2 = S_n. \quad (4.19)$$

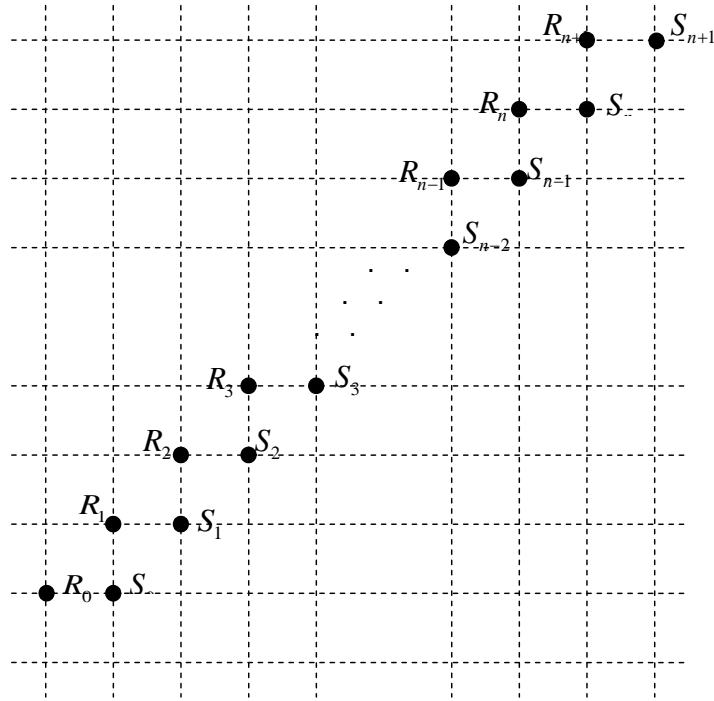


Figure 4.3 Graphical representation for Lemma 4.5

Remarks To show that $T_{la}v$ satisfies (4.8) for the case $0 \leq x_1 < a$, we make the following

revision: $T_{la}v(x + 2e_1) = \min_{-1 \leq i \leq n+1} v(S_i)$, $T_{la}v(x + e_1 - e_l) = \min_{-1 \leq i \leq n} v(S_i)$,

$T_{la}v(x + e_1) = \min_{0 \leq i \leq n+1} v(R_i)$ and $T_{la}v(x - e_l) = \min_{0 \leq i \leq n} v(R_i)$. Similarly, we can prove

(1) If $P_1 = S_{n+1}$, $P_2 = S_n$, then $P_3 = R_{n+1}$ and $P_4 = R_n$.

(2) If $P_3 = R_{n+1}$, $P_4 = R_n$, then $P_1 = S_{n+1}$ and $P_2 = S_n$, or $P_1 = P_2 = S_n$.

(3) If $P_1 = P_2 = S_k$ ($-1 \leq k \leq n-1$), then $P_3 = P_4 = R_{k+1}$.

Proof We only need to verify (4.16) - (4.19). As shown in Figure 4.3, for (4.16), if $P_2 \neq S_0$,

suppose $P_2 = S_k$, $1 \leq k \leq n$. By the definition of P_2 and Lemma 4.3,

$$v(S_0) \geq v(S_1) \geq \dots \geq v(S_{k-1}) > v(S_k)$$

and because $v(R_0) \leq v(R_1) \leq \dots \leq v(R_k)$, we have

$$v(R_0) + v(S_k) < v(S_0) + v(R_k)$$

On the other hand, by (4.8),

$$\begin{aligned}
 v(R_0) + v(S_1) &\geq v(S_0) + v(R_1) \\
 v(R_1) + v(S_2) &\geq v(S_1) + v(R_2) \\
 &\dots \\
 v(R_{k-1}) + v(S_k) &\geq v(S_{k-1}) + v(R_k)
 \end{aligned}$$

Therefore, $v(R_0) + v(S_k) \geq v(S_0) + v(R_k)$, which contradicts the above derived result

$v(R_0) + v(S_k) < v(S_0) + v(R_k)$. Similarly, we have (4.17).

For (4.18), suppose $P_3 = R_k$, $2 \leq k \leq n$. Then we have

$$v(R_1) \geq v(R_2) \geq \dots \geq v(R_{k-1}) > v(R_k)$$

Because

$$v(S_0) \leq v(S_1) \leq \dots \leq v(S_{k-1})$$

We have

$$v(S_0) + v(R_k) < v(R_1) + v(S_{k-1})$$

On the other hand, by (4.9)

$$\begin{aligned}
 v(S_0) + v(R_2) &\geq v(R_1) + v(S_1) \\
 v(S_1) + v(R_3) &\geq v(R_2) + v(S_2) \\
 &\dots \\
 v(S_{k-2}) + v(R_k) &\geq v(R_{k-1}) + v(S_{k-1})
 \end{aligned}$$

Therefore

$$v(S_0) + v(R_k) \geq v(R_1) + v(S_{k-1})$$

which contradicts the above result $v(S_0) + v(R_k) < v(R_1) + v(S_{k-1})$. Similarly, we have

(4.19). ■

Lemma 4.6 Suppose $n = a \leq x_1$. For any demand class l , $2 \leq l \leq N$, define

$$\begin{aligned}
 i_1^* &= \max \{ \arg \min_{0 \leq i \leq n} v(x - (i-1)e_1 + (a-i)e_l) \}, \\
 i_2^* &= \max \{ \arg \min_{0 \leq i \leq n} v(x - ie_1 + (a-i-1)e_l) \}, \\
 i_3^* &= \max \{ \arg \min_{0 \leq i \leq n} v(x - (i-1)e_1 + (a-i+1)e_l) \},
 \end{aligned}$$

$$i_4^* = \max_{0 \leq i \leq n} \{ \arg \min v(x - ie_1 + (a - i)e_l) \}.$$

Then,

(1) If $i_3^* = i_4^* = 0$ or $i_3^* = i_4^* = n$, then $i_1^* = i_2^* = i_3^* = i_4^*$.

If $i_1^* = i_2^* = 0$ or $i_1^* = i_2^* = n$, then $i_3^* = i_4^* = i_1^* = i_2^*$.

(2) If $i_1^* = i_2^* + 1$, then $i_3^* = i_1^*$ and $i_4^* = i_2^*$.

If $i_3^* = i_4^* + 1$, then $i_1^* = i_3^*$ and $i_2^* = i_4^*$.

Graphical interpretation is as follow: in Figure 4.4, we use R_i and S_i , $0 \leq i \leq n + 1$ to denote the possible x , where

$$\begin{aligned} T_{la} v(x + e_1) &= \min_{0 \leq k \leq (x \wedge a)} v(x - (k - 1)e_1 + (a - k)e_l) = \min_{1 \leq i \leq n+1} v(S_i) \\ T_{la} v(x - e_1) &= \min_{0 \leq k \leq (x \wedge a)} v(x - ke_1 + (a - k - 1)e_l) = \min_{0 \leq i \leq n} v(S_i) \\ T_{la} v(x + e_1 + e_l) &= \min_{0 \leq k \leq (x \wedge a)} v(x - (k - 1)e_1 + (a - k + 1)e_l) = \min_{1 \leq i \leq n+1} v(R_i) \\ T_{la} v(x) &= \min_{0 \leq k \leq (x \wedge a)} v(x - ke_1 + (a - k)e_l) = \min_{0 \leq i \leq n} v(R_i) \end{aligned}$$

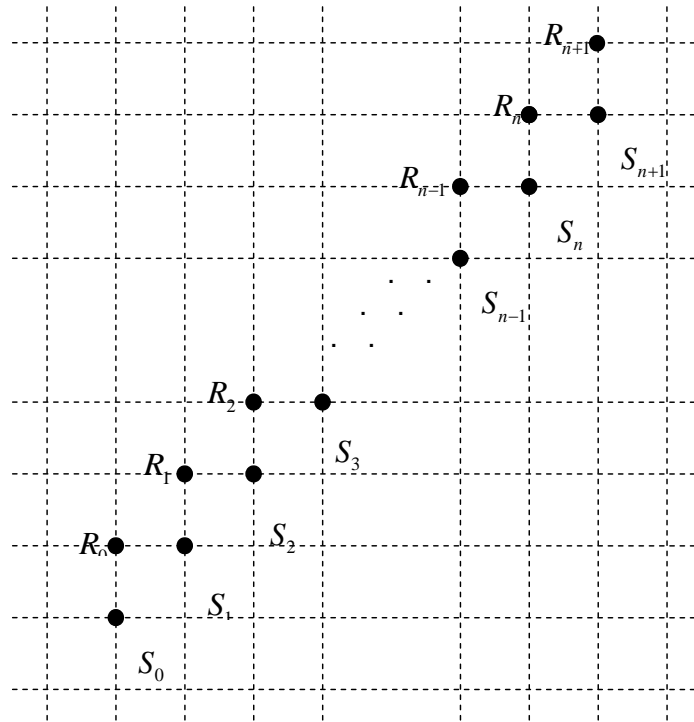


Figure 4.4 Graphical representation for Lemma 4.6

Define integers i_1, i_2, i_3, i_4 , states P_1, P_2, P_3, P_4 and state set M as

$$i_1 = \min_{1 \leq i \leq n+1} \{\arg \min v(S_i)\}, \quad i_2 = \min_{0 \leq i \leq n} \{\arg \min v(S_i)\}, \quad i_3 = \min_{1 \leq i \leq n+1} \{\arg \min v(R_i)\},$$

$$i_4 = \min_{0 \leq i \leq n} \{\arg \min v(R_i)\}, \quad P_1 = S_{i_1}, \quad P_2 = S_{i_2}, \quad P_3 = R_{i_3}, \quad P_4 = R_{i_4} \quad \text{and} \quad M = \{P_1, P_2, P_3, P_4\}.$$

Then Lemma 4.6 is equivalent to the follows:

$$(1) \quad P_3 = R_1 \quad \text{and} \quad P_4 = R_0 \Leftrightarrow P_1 = S_1 \quad \text{and} \quad P_2 = S_0 \quad (4.20)$$

$$P_3 = R_{n+1} \quad \text{and} \quad P_4 = R_n \Leftrightarrow P_1 = S_{n+1} \quad \text{and} \quad P_2 = S_n \quad (4.21)$$

$$(2) \quad P_1 = P_2 = S_k \Leftrightarrow P_3 = P_4 = R_k, \quad 0 < k < n+1 \quad (4.22)$$

Remarks To show that $T_{la}v$ satisfies (4.9) for the case $0 \leq x_1 < a$, we make the following

$$\text{revision: } T_{la}v(x+e_1) = \min_{0 \leq i \leq n+1} v(S_i), \quad T_{la}v(x-e_1) = \min_{0 \leq i \leq n} v(S_i),$$

$$T_{la}v(x+e_1+e_l) = \min_{0 \leq i \leq n+1} v(R_i) \quad \text{and} \quad T_{la}v(x) = \min_{0 \leq i \leq n} v(R_i). \quad \text{Similarly, we can prove}$$

$$(1) \quad P_3 = R_{n+1} \quad \text{and} \quad P_4 = R_n \Leftrightarrow P_1 = S_{n+1} \quad \text{and} \quad P_2 = S_n.$$

$$(2) \quad P_1 = P_2 = S_k \Leftrightarrow P_3 = P_4 = R_k, \quad 0 \leq k < n+1.$$

Proof As shown in Figure 4.4: For (4.20) and (4.21), we only verify (4.20).

\Rightarrow : First, if $P_1 = S_1$, then from (4.9) we have $P_2 = S_0$. We proceed to prove $P_1 = S_1$.

Otherwise, if $P_1 = S_k$, $2 \leq k \leq n$, then from Lemma 4.3, we have

$$v(S_0) \geq v(S_1) \geq \dots \geq v(S_{k-1}) > v(S_k). \quad \text{Since } v(R_0) \leq v(R_1) \leq \dots \leq v(R_k), \text{ we have}$$

$$v(R_{k-2}) + v(S_k) < v(R_{k-1}) + v(S_{k-1}), \text{ which contradicts (4.8). If } P_1 = S_0, \text{ it implies}$$

$$v(S_0) < v(S_1) \quad \text{and thus } v(R_0) < v(R_1), \text{ which is a contradiction to } P_3 = R_1.$$

\Leftarrow : First, if $P_3 = R_1$, then from (4.9), we have $P_4 = R_0$. We proceed to prove $P_3 = R_1$.

Otherwise, if $P_3 = R_k$, $2 \leq k \leq n$, similarly, we have $v(S_0) + v(R_k) < v(S_k) + v(R_0)$,

which contradicts (4.9). If $P_3 = R_0$, it implies $P_1 = S_0$, thus a contradiction follows.

Further, (4.22) can be easily obtained from (4.9). ■

Now we are ready to prove part (II).

Theorem 4.3 *If $v \in V$, then $Tv \in V$.*

Proof We need show that (A) $T_{la}v \in V$ ($2 \leq l \leq N$), and (B) $T_0v \in V$.

(A) Proof of $T_{la}v \in V$ ($2 \leq l \leq N$)

For (4.4) and (4.5), since $x_1 < 0$, $x_1 + 1 \leq 0$, we have $T_{la}v(x) = v(x + a \varrho)$,

$T_{la}v(x + e_1) = v(x + e_1 + a \varrho)$, and $T_{la}v(x - e_1) = v(x + (a - 1)e_1)$. Since $v \in V$, we have

$$\begin{aligned} T_{la}v(x + e_1) &= v(x + e_1 + a \varrho) \leq v(x + a \varrho) = T_{la}v(x) \\ T_{la}v(x + e_1) &= v(x + e_1 + a \varrho) \leq v(x + (a - 1)e_1) = T_{la}v(x - e_1) \end{aligned}$$

For (4.6), consider two cases as follow:

(i) $x_1 \leq 0$. We have $T_{la}v(x) = v(x + a \varrho)$ and $T_{la}v(x - e_1) = v(x + (a - 1)e_1)$. From (4.6),

we have $v(x + (a - 1)e_1) \leq v(x + a \varrho)$. Thus $T_{la}v(x - e_1) \leq T_{la}v(x)$.

(ii) $x_1 > 0$. From (4.7), we have $v(x - ke_1 + (a - k - 1)e_1) \leq v(x - ke_1 + (a - k)e_1)$ for

$k = 0, \dots, (x \wedge a)$. Thus,

$$T_{la}v(x - e_1) = \min_{0 \leq k \leq (x \wedge a)} v(x - ke_1 + (a - k - 1)e_1) \leq \min_{0 \leq k \leq (x \wedge a)} v(x - ke_1 + (a - k)e_1) = T_{la}v(x).$$

For (4.7), we consider three cases:

(i) $x_1 \geq a$. Following the notations in the proof of Lemma 4.4 (see Figure 4.2), we enumerate

all the cases of $M = \{P_1, P_2, P_3, P_4\}$:

- (1) $P_1 = R_0, P_2 = S_0, P_3 = S_1, P_4 = Q_1$
- (2) $P_1 = R_n, P_2 = S_n, P_3 = S_{n+1}, P_4 = Q_{n+1}$
- (3) $P_2 = P_3 = S_k, P_1 = R_k, P_4 = Q_k, 1 \leq k \leq n$

For each case, by the convexity of v and (4.7) - (4.9), it is straightforward to show that $T_{la}v$

satisfies (4.7).

(ii) $0 \leq x_1 < a$. we enumerate all the cases

- (1) $P_1 = R_n$, $P_2 = S_n$, $P_3 = S_{n+1}$ and $P_4 = Q_{n+1}$
 (2) $P_2 = P_3 = S_k$, $P_1 = R_k$ and $P_4 = Q_k$, $0 \leq k \leq n$

Similarly, we have that $T_{la}v$ satisfies (4.7).

(iii) $x_1 < 0$. We have

$$\begin{aligned} T_{la}v(x) &= v(x + a \varrho) \\ T_{la}v(x + e_l) &= v(x + (a+1)e_l) \\ T_{la}v(x + e_1) &= \begin{cases} \min[v(x + e_1 + a \varrho), v(x + (a-1)e_l)] & \text{if } x_1 = 0 \\ v(x + e_1 + ae_l) & \text{if } x_1 < 0 \end{cases} \\ T_{la}v(x + e_1 + e_l) &= \begin{cases} \min[v(x + e_1 + (a+1)e_l), v(x + ae_l)] & \text{if } x_1 = 0 \\ v(x + e_1 + (a+1)e_l) & \text{if } x_1 < 0 \end{cases} \end{aligned}$$

If $T_{la}v(x + e_1) = v(x + (a-1)e_l)$, then $x_1 = 0$ and

$$\begin{aligned} T_{la}v(x + e_1 + e_l) + T_{la}v(x) &\leq v(x + a \varrho) + v(x + a \varrho) \\ &\leq v(x + (a+1)e_l) + v(x + (a-1)e_l) = T_{la}v(x + e_l) + T_{la}v(x + e_1) \end{aligned}$$

where the second inequality follows from the convexity of v .

If $T_{la}v(x + e_1) = v(x + e_1 + a \varrho)$, then

$$\begin{aligned} T_{la}v(x + e_1 + e_l) + T_{la}v(x) &\leq v(x + e_1 + (a+1)e_l) + v(x + a \varrho) \\ &\leq v(x + e_1 + a \varrho) + v(x + (a+1)e_l) = T_{la}v(x + e_1) + T_{la}v(x + e_l) \end{aligned}$$

where the second inequality follows from the fact that v satisfies (4.7).

For (4.8) we consider three cases as follow:

- (i) $x_1 > a$. Following the notations in the proof of Lemma 4.5 (see Figure 4.3), we enumerate all the cases of $M = \{P_1, P_2, P_3, P_4\}$: (1) $P_1 = S_1$, $P_2 = S_0$, $P_3 = R_1$, $P_4 = R_0$;
 (2) $P_1 = S_{n+1}$, $P_2 = S_n$, $P_3 = R_{n+1}$, $P_4 = R_n$; (3) $P_1 = S_1$, $P_2 = S_0$, $P_3 = P_4 = R_1$;
 (4) $P_1 = P_2 = S_n$, $P_3 = R_{n+1}$, $P_4 = R_n$; (5) $P_1 = P_2 = S_k$, $P_3 = P_4 = R_{k+1}$, $1 \leq k \leq n$.

For each case, it is straightforward to show that $T_{la}v$ satisfies (4.8).

(ii) $0 \leq x_1 < a$. we enumerate all the cases

$$(1) P_1 = S_{n+1}, P_2 = S_n, P_3 = R_{n+1} \text{ and } P_4 = R_n.$$

$$(2) P_1 = P_2 = S_n, P_3 = R_{n+1} \text{ and } P_4 = R_n.$$

$$(3) P_1 = P_2 = S_k, P_3 = P_4 = R_{k+1}, -1 \leq k \leq n-1.$$

For each case, we have that $T_{la}v$ satisfies (4.8).

(iii) $x_1 < 0$. We have

$$T_{la}v(x - e_l) = v(x + (a-1)e_l)$$

$$T_{la}v(x + 2e_1) = \begin{cases} \min[v(x + 2e_1 + a \varrho), v(x + e_1 + (a-1)e_l), v(x + (a-2)e_l)] & \text{if } x_1 = 0 \\ \min[v(x + 2e_1 + a \varrho), v(x + e_1 + (a-1)e_l)] & \text{if } x_1 = -1 \\ v(x + 2e_1 + ae_l) & \text{if } x_1 \leq -2 \end{cases}$$

If $T_{la}v(x + 2e_1) = v(x + 2e_1 + a \varrho)$, then

$$\begin{aligned} T_{la}v(x + e_1 - e_l) + T_{la}v(x + e_1) &\leq v(x + e_1 + (a-1)e_l) + v(x + e_1 + a \varrho) \\ &\leq v(x + 2e_1 + a \varrho) + v(x + (a-1)e_l) = T_{la}v(x + 2e_1) + T_{la}v(x - e_l) \end{aligned}$$

where the first inequality follows from the definition of T_{la} , and the second one is from the fact that v satisfies (4.8).

If $T_{la}v(x + 2e_1) = v(x + e_1 + (a-1)e_l)$ and $x_1 = -1$, then

$$\begin{aligned} T_{la}v(x + e_1 - e_l) + T_{la}v(x + e_1) &= v(x + e_1 + (a-1)e_l) + v(x + e_1 + a \varrho) \\ &\leq v(x + e_1 + (a-1)e_l) + v(x + (a-1)e_l) = T_{la}v(x + 2e_1) + T_{la}v(x - e_l) \end{aligned}$$

where the first inequality follows from the fact that v satisfies (4.5).

If $x_1 = 0$, then

$$\begin{aligned} T_{la}v(x + e_1 - e_l) &= \min[v(x + e_1 + (a-1)e_l), v(x + (a-2)e_l)] \\ T_{la}v(x + e_1) &= \min[v(x + e_1 + a \varrho), v(x + (a-1)e_l)] \end{aligned}$$

Hence, if $T_{la}v(x + 2e_1) = v(x + e_1 + (a-1)e_l)$, then

$$\begin{aligned} T_{la}v(x + e_1 - e_l) + T_{la}v(x + e_1) &\leq v(x + e_1 + (a-1)e_l) + v(x + (a-1)e_l) \\ &= T_{la}v(x + 2e_1) + T_{la}v(x - e_l) \end{aligned}$$

If $T_{la}v(x + 2e_1) = v(x + (a-2)e_l)$, we have

$$\begin{aligned} T_{la}v(x + e_1 - e_l) + T_{la}v(x + e_l) &\leq v(x + (a - 2)e_l) + v(x + (a - 1)e_l) \\ &= T_{la}v(x + 2e_l) + T_{la}v(x - e_l) \end{aligned}$$

For (4.9), we consider three cases as follow:

(i) $x_1 \geq a$. From Lemma 4.6, it is straightforward to show that $T_{la}v$ satisfies (4.9).

(ii) $0 \leq x_1 < a$. It is straightforward that $T_{la}v$ satisfies (4.9).

(iii) $x_1 < 0$. We have

$$\begin{aligned} T_{la}v(x) &= v(x + a - \rho) \\ T_{la}v(x - e_l) &= v(x + (a - 1)e_l) \\ T_{la}v(x + e_1 + e_l) &= \begin{cases} \min[v(x + a - \rho), v(x + e_1 + (a + 1)e_l)] & \text{if } x_1 = 0 \\ v(x + e_1 + (a + 1)e_l) & \text{if } x_1 \leq -1 \end{cases} \end{aligned}$$

If $T_{la}v(x + e_1 + e_l) = v(x + e_1 + (a + 1)e_l)$, then

$$\begin{aligned} T_{la}v(x + e_1) + T_{la}v(x) &\leq v(x + e_1 + a - \rho) + v(x + a - \rho) \\ &\leq v(x + e_1 + (a + 1)e_l) + v(x + (a - 1)e_l) = T_{la}v(x + e_1 + e_l) + T_{la}v(x - e_l) \end{aligned}$$

where the first inequality follows from the definition of T_{la} , and the second one is from the fact that v satisfies (4.8).

If $T_{la}v(x + e_1 + e_l) = v(x + a - \rho)$ then $x_1 = 0$, and

$$T_{la}v(x + e_1) + T_{la}v(x) \leq v(x + (a - 1)e_l) + v(x + a - \rho) = T_{la}v(x - e_l) + T_{la}v(x + e_1 + e_l)$$

where the inequality follows from the fact that

$$T_{la}v(x + e_l) = \min[v(x + (a - 1)e_l), v(x + e_1 + a - \rho)]$$

(B) Proof of $T_0v \in V$

For (4.4), we need to show $T_0v(x + e_1) \leq T_0v(x)$. This is obvious, since for the optimal policy π at x (note the production quantity is positive), adopting π except for one less unit production at $x + e_1$ results in the same cost.

For (4.5), suppose (k_1, \dots, k_N) is the optimal policy for $x - e_l$. We can construct a suitable policy (k'_1, \dots, k'_N) for $x + e_l$ (not necessary optimal): Let $k'_1 = k_1 + 1$, and others same as

(k_1, \dots, k_N) . This leads to the same cost at the two states.

For (4.6), suppose (k_1, \dots, k_N) is the optimal policy for x . Consider the following cases:

(i) No production at x . It implies also no production at $x - e_l$. The result is immediate since no allocation.

(ii) Produce and $k_l < x_l$. We can construct a suitable policy (k'_1, \dots, k'_N) for $x - e_l$ (not necessarily optimal): $k'_l = k_l - 1$ and others same as (k_1, \dots, k_N) . From (4.6), this leads to a higher (or equal) cost at x .

(iii) Produce and $k_l = x_l$. There are two possibilities: (a) $T_0 v(x - e_l) = v(x'_1 e_1)$ and $T_0 v(x) = v((x'_1 - 1)e_1)$ for some $x'_1 \in Z$. Notice there must be $x'_1 \leq z_{N+1}$, from (4.10), we have $T_0 v(x - e_l) \leq T_0 v(x)$. (b) $T_0 v(x - e_l) = v(x' - e_j)$ and $T_0 v(x) = v(x')$ for some $x' = (x'_1, 0, \dots, 0, x'_j, \dots, x'_n)$, where $l < j \leq N$ and $x'_j > 0$. The result follows from (4.6).

For (4.7), denote $P = x = (x_1, 0, \dots, 0, x_l, x_{l+1}, \dots, x_N)$, $P_1 = x + e_1$, $P_l = x + e_l$ and

$P_{ll} = x + e_1 + e_l$. We need show $T_0 v(P_{ll}) + T_0 v(P) \leq T_0 v(P_1) + T_0 v(P_l)$. We only discuss the case $0 < x_1 + 1 \leq z_l$, because any state with $x_1 + 1 > z_l$ is transient. Suppose

$T_0 v(P) = v(P')$, i.e., after the optimal decision, P moves to $P' = x'$. Similarly,

$T_0 v(P_1) = v(P'_1)$, $T_0 v(P_l) = v(P'_l)$ and $T_0 v(P_{ll}) = v(P'_{ll})$. Consider the following cases:

(i) $x_1 + 1 + M \leq z_l$. Then we have $P' = x + M e_1$, $P'_1 = x + e_1 + M e_1$, $P'_l = x + e_l + M e_1$, $P'_{ll} = x + e_1 + e_l + M e_1$. The result follows directly from (4.7).

(ii) $z_l < x_1 + 1 + M \leq z_l + x_1$. It is easy to verify that

$$P' = P'_{ll} = x + (z_l - x_1)e_1 - (M - z_l + x_1)e_l,$$

$$P'_1 = (x + e_1) + (z_l - x - 1)e_1 - (M - z_l + x_1 + 1)e_l = x + (z_l - x)e_1 - (M - z_l + x_1 + 1)e_l,$$

$$P'_l = (x + e_l) + (z_l - x)e_1 - (M - z_l + x_1)e_l = x + (z_l - x)e_1 - (M - z_l + x_1 - 1)e_l.$$

The result follows directly from (4.10).

(iii) $z_l + x_l + 1 = x_1 + 1 + M$. It is easy to verify that $P' = P_{1l}' = x + (z_l - x_1)e_1 - x_l e_l$,

$P_1' = x + (z_l - x_1 + 1)e_1 - x_l e_l$, and $P_l' = x + (z_l - x_1)e_1 - (x_l - 1)e_l$.

Let $P'' = x + (z_l - x_1 + 1)e_1 - (x_l - 1)e_l$. Obviously, $v(P'') \geq v(P')$. We have

$v(P') + v(P_{1l}') \leq v(P'') + v(P_{1l}') \leq v(P_1') + v(P_l')$, where the second inequality is from (4.7).

(iv) $z_l + x_l + 1 < x_1 + 1 + M \leq z_{N+1} + \sum_{i=l}^N x_i$. It can be shown that for all the four states, it is

optimal to produce M units, satisfy all the class l backorder: First, P moves to

$P'' = x + (M - x_l)e_1 - x_l e_l$, P_1 moves to $P_1'' = x + (1 + M - x_l)e_1 - x_l e_l$, P_l moves to

$P_l'' = x + (M - x_l - 1)e_1 - x_l e_l$, P_{1l} moves to $P_{1l}'' = x + (M - x_l)e_1 - x_l e_l$, and then it need

to consider allocating production to the backorders from less valuable classes. From (4.10),

we can show that $v(P'') + v(P_{1l}'') \leq v(P_1'') + v(P_l'')$ by following similar analysis in Lemma 4.4

- 4.6.

(v) $x_1 + 1 + M = z_{N+1} + \sum_{i=l}^N x_i + 1$. It can be shown that it is optimal to produce M units and

satisfy all backorders at P and P_{1l} , to produce $M - 1$ units and satisfy all backorders at

P_1 , and to produce M units and leave one unit of backorder unsatisfied at P_l . Finally,

$P' = P_{1l}' = P_1' = z_{N+1}e_1$, $P_l' = z_{N+1}e_1 + e_j$ for some $j(j > l)$. The result is obvious.

(vi) $x_1 + M \geq z_{N+1} + \sum_{i=l}^N x_i + 1$. It can be shown that $P' = P_{1l}' = P_1' = P_l' = z_{N+1}e_1$.

For (4.8), denote $P = x = (x_1, 0, \dots, 0, x_l, x_{l+1}, \dots, x_N)$, $P_1 = x + e_1$, $P_{1l} = x + e_1 + e_l$ and

$P_{2l} = x + 2e_1 + e_l$. We need show $T_0 v(P_{1l}) + T_0 v(P_1) \leq T_0 v(P) + T_0 v(P_{2l})$. We will discuss

the case $0 < x_1 + 2 \leq z_l$, as the case $z_l < 2$ or $x_1 + 2 > z_l$ is trivial. Suppose

$T_0 v(P) = v(P')$, $T_0 v(P_1) = v(P_1')$, $T_0 v(P_{1l}) = v(P_{1l}')$ and $T_0 v(P_{2l}) = v(P_{2l}')$. Consider the

following cases:

(i) $x_1 + 2 + M \leq z_l$. Then we have $P' = x + Me_1$, $P'_1 = x + e_1 + Me_1$,

$P'_{1l} = x + e_l + (M + 1)e_1$, $P'_{2l} = x + e_l + (M + 2)e_1$. The result follows directly from (4.8).

(ii) $x_1 + 2 + M = z_l + 1$. It is easy to verify that $P'_1 = P'_{2l} = x + (M + 1)e_1$, $P' = x + Me_1$,

$P'_{1l} = x + e_l + (M + 1)e_1$. Since $x_1 + M < z_l$, we have $v(P') \geq v(P'_{1l})$, and thus the result holds.

(iii) $x_1 + 2 + M > z_l + 1$. It can be shown that $P'_1 = P'_{2l}$ and $P' = P'_{1l}$.

For (4.9), denote $P = x = (x_1, 0, \dots, 0, x_l, x_{l+1}, \dots, x_N)$, $P_l = x + e_l$, $P_{1l} = x + e_1 + e_l$ and

$P_{1,2l} = x + e_1 + 2e_l$. We need show $T_0v(P_{1l}) + T_0v(P_l) \leq T_0v(P) + T_0v(P_{1,2l})$. We will assume

$0 < x_1 + 1 \leq z_l$, as the case $z_l < 1$ or $x_1 + 1 > z_l$ is trivial. Suppose $T_0v(P) = v(P')$,

$T_0v(P_l) = v(P'_l)$, $T_0v(P_{1l}) = v(P'_{1l})$ and $T_0v(P_{1,2l}) = v(P'_{1,2l})$. Consider the following cases:

(i) $x_1 + 1 + M \leq z_l$. Then we have $P' = x + Me_1$, $P'_l = x + e_l + Me_1$,

$P'_{1l} = x + e_l + (M + 1)e_1$, $P'_{1,2l} = x + (M + 1)e_1 + 2e_l$. The result follows directly from (4.9).

(ii) $x_1 + 1 + M > z_l$. It is easy to verify that $P'_l = P'_{1,2l}$, $P' = P'_{1l}$. ■

4.3.3 Discussion and extensions

We have shown that a critical level policy is optimal for the N-class, batch production and compound Poisson demand model when the batch demand can be partially accepted. On the other hand, problems could be more complicated when the batch demand must be accepted on an all-or-none basis – that is, given a request for $a > 1$ units we can only sell all a units or none at all. This seemingly modest change has a profound impact on the structure of optimal allocation policies leading to the loss of convexity of the optimal cost functions. Brumelle &

Walczak [5] confirm such kind of loss of concavity of the optimal value function in a general demand arrival case.

To illustrate this case, we modify our operators on the basis of the all-or-none control. For simplicity, we consider a two-class problem and assume that class 1 demand can be partially accepted and class 2 follows the all-or-none basis. We also assume that previous backorders can be satisfied by production unit by unit, regardless of the class of the backorder. This assumption will simplify the model since we do not need to track all the previous backorders. Then we only need to change $T_{2a}v(x)$ to:

$$T_{2a}v(x) = \begin{cases} \min[v(x - ae_1), v(x + ae_2)] & \text{if } x_1 \geq a \\ v(x + ae_2) & \text{if } x_1 < a \end{cases}$$

The next example illustrates the structure of the optimal policy when the batch can be partially accepted, and shows that the structure may lose when demand must be accepted on an all-or-none basis.

Example The parameters are as follows: $h = 1$, $b_1 = 3$, $b_2 = 2$, $\mu = 20$, $\lambda_1 = 3$, $\lambda_2 = 2$, $\alpha = 0.01$. The batches for two demand classes have the same distribution with $p_{i2} = p_{i10} = 0.5$, $i = 1, 2$.

Table 4.3(a) Expected cost for the partially accepted case

$x_2 \backslash x_1$	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
7	8.89	8.04	7.67	7.34	7.17	7.01	6.90	6.80	6.71	6.63	6.57	6.53	6.66	6.80
6	8.12	7.29	6.94	6.61	6.45	6.29	6.18	6.08	5.99	5.91	5.86	5.83	5.97	6.12
5	7.37	6.58	6.24	5.94	5.77	5.62	5.51	5.41	5.32	5.25	5.21	5.20	5.34	5.51
4	6.66	5.92	5.61	5.32	5.17	5.02	4.91	4.81	4.73	4.67	4.64	4.66	4.81	4.98
3	6.00	5.33	5.04	4.79	4.64	4.50	4.39	4.29	4.22	4.17	4.16	4.21	4.37	4.55
2	5.41	4.81	4.55	4.35	4.20	4.07	3.97	3.88	3.81	3.78	3.79	3.86	4.03	4.22
1	4.89	4.39	4.18	4.00	3.86	3.75	3.66	3.58	3.53	3.51	3.53	3.62	3.80	4.01
0	4.47	4.09	3.90	3.76	3.65	3.55	3.47	3.41	3.38	3.37	3.40	3.50	3.68	3.89

Table 4.3(b) Expected cost for the all-or-none accepted case

$x_2 \backslash x_1$	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
7	8.89	8.04	7.84	7.67	7.46	7.44	7.42	7.45	7.48	7.54	7.61	7.70	6.94	7.16
6	8.12	7.29	7.10	6.93	6.74	6.72	6.70	6.73	6.76	6.82	6.89	6.98	6.25	6.47
5	7.37	6.58	6.40	6.24	6.06	6.05	6.03	6.06	6.10	6.16	6.24	6.29	5.62	5.84
4	6.66	5.93	5.75	5.62	5.45	5.44	5.43	5.46	5.50	5.57	5.66	5.67	5.08	5.29
3	6.01	5.35	5.19	5.07	4.92	4.91	4.90	4.94	4.99	5.06	5.16	5.13	4.63	4.84
2	5.43	4.87	4.72	4.62	4.49	4.49	4.48	4.52	4.57	4.65	4.76	4.69	4.30	4.48
1	4.95	4.49	4.37	4.27	4.18	4.18	4.18	4.22	4.28	4.37	4.44	4.36	4.07	4.25
0	4.57	4.25	4.14	4.06	4.01	4.01	4.01	4.06	4.12	4.22	4.23	4.13	3.95	4.13

The expected costs with initial states ranging from $-1 \leq x \leq 12$, $0 \leq y \leq 7$ are shown in Table 4.3. For the partially accepted case in Table 4.3(a), $z_1 = z_2 = 0$ and $z_3 = 8$. For the all-or-none case in Table 4.3(b), it is easy to verify that the optimal cost function is not convex and the mentioned structure does not hold. We highlight the cost values that destroy the structure of the optimal policy.

We now consider an interesting extension of the original model. In section 4.3.1, we have assumed that each arrival of the demand has a random batch size and demands in the batch are equally important. However, practically, customers may order a batch that includes heterogeneous demands. For example, consider a customer who replenishes spare parts from a central warehouse to repair defective equipment. There are two types of equipment that one is vital and the other is auxiliary. The customer may place an order consisting of demands for different equipment. In this case, the demand segmentation for the system is based on the equipment criticality (see e.g., Teunter & Haneveld [51], Dekker, Kleijn, & de Rooij [16]).

We formulate the demand as a combination of different classes with a batch size for each class. Suppose there are N demand classes, which differ in their backordering costs b_l ($1 \leq l \leq N$), where $b_1 > b_2 > \dots > b_N$. There are m types of customers, which differ in their

combination of demands and different batch sizes for each demand class. To simplify the problem, we assume that demand segmentation is only based on the equipment criticality (demand class), not based on customer type. Demand from the i th ($1 \leq i \leq m$) type customer arrives according to a Poisson process with the arrival rate λ_i and a fixed batch for each demand class. Denote $a_i = (a_{i1}, a_{i2}, \dots, a_{iN})$ as the batch vector for the i th customer. Other settings are the same as those in section 4.3.1, we seek to find the optimal control policy π that minimizes the discounted costs over an infinite horizon.

Given the batch vector a_i from the i th ($1 \leq i \leq m$) customer, the rationing decision is to determine how much demand from each class should be satisfied. Given the arrival of the i th type customer and the on-hand inventory x_1 , then either there is a unique l' such that

$$\sum_{j=1}^{l'} a_{ij} < x_1 \leq \sum_{j=1}^{l'+1} a_{ij} \quad (\text{If } l' = N, \text{ then we define } a_{i,N+1} = \infty), \text{ or } l' = 0 \text{ if } x_1 \leq a_{i1}. \text{ The}$$

dynamic programming equation is $v(x) = Tv(x)$, and the operator T can be formulated as

$$Tv(x) = c(x) + \mu T_0 v(x) + \sum_{i=1}^m \lambda_i [T_i v(x)]$$

where $c(x)$ and $T_0 v(x)$ are the same as those defined in section 4.3.1.

$$T_i v(x) = \begin{cases} \min\{T_{i1}' v(x), T_{i2}' v(x), \dots, T_{il'}' v(x)\} & \text{if } l' > 0 \\ v(x - a_{i1} e_1 + \sum_{j=2}^N a_{ij} e_j) & \text{if } l' = 0 \end{cases}$$

$$T_{i1}' v(x) = \min_{0 \leq k \leq a_{i2}} v(x - (a_{i1} + k) e_1 + (a_{i2} - k) e_2 + \sum_{j=3}^N a_{ij} e_j)$$

$$T_{i2}' v(x) = \min_{0 \leq k \leq a_{i3}} v(x - (a_{i1} + a_{i2} + k) e_1 + (a_{i3} - k) e_3 + \sum_{j=4}^N a_{ij} e_j)$$

.....

$$T_{i'}^{l'} v(x) = \begin{cases} \min_{0 \leq k \leq x_1 - \sum_{j=1}^{l'} a_{ij}} v(x - (\sum_{j=1}^{l'} a_{ij} + k)e_1 + (a_{i,l'+1} - k)e_{l'+1} + \sum_{j=l'+2}^N a_{ij}e_j) & \text{if } l' < N \\ v(x - \sum_{j=1}^N a_{ij}e_1) & \text{if } l' = N \end{cases}$$

Following the same technique in section 4.3.2, we can show that properties (4.4) - (4.11) still hold for such problem with combined demands. Thus, a critical level policy is still optimal. Specifically, the optimal rationing policy works as follows: Consider satisfying demands in order starting from the most valuable class: (1) Satisfy class 1 demand as much as possible; (2) Satisfy class 2 demand as much as possible whenever the inventory on-hand is greater than z_2 ; (3) The process continues until the inventory on-hand drops to z_l for some l when it is starting to satisfy class l demand, or all demands are satisfied finally if the inventory on-hand is greater than z_N .

4.4 Conclusions

In this chapter, we have studied stock rationing problems for a single-item, make-to-stock inventory/production system. In both the lost-sales and backordering models, demands are classified as several classes and each class is a Poisson arrival with a random batch size. The batch demands can be partially accepted. For both the lost-sales and backordering cases, we show that the structures of the optimal policies are characterized as critical levels.

We have considered the bulk service problem in the backordering model but not in the lost-sales model. In fact, following similar technique, the optimal structure for the lost-sales model can be generalized to the case when products are made in bulk.

5. Conclusions and future research

In this chapter, we summarize the main findings and conclusions in each essay as follows.

In the first essay, we consider a finite horizon periodic review, single product inventory system with a fixed setup cost and two stochastic demand classes that differ in their backordering costs. In each period, one must decide whether and how much to order, and how much demand from the lower class should be satisfied. We show that the optimal ordering policy can be characterized as a state dependent (s, S) policy, and we partially obtain the optimal rationing structure based on the sub-convexity of the cost function. We then propose a simple heuristic rationing policy, which is easy to implement and close to optimal for a large set of numerical examples. We study in depth the case when the first demand class is deterministic and must be satisfied immediately. We show the optimality of the state dependent (s, S) ordering policy, and obtain additional rationing structural properties. Based on these properties, the optimal ordering and rationing policy for any state can be generated by finding the optimal policy of a finite set of states. For each state in this set, the optimal policy is obtained by simply choosing a policy from at most two alternatives. An efficient algorithm is then proposed.

In the second essay, we consider a periodic review inventory system with a fixed setup cost and two demand classes: deterministic and stochastic. The deterministic demand must be satisfied immediately, while the stochastic demand can be backlogged. Under certain conditions, a modified (s, S) policy was proved optimal in a previous paper. This essay is to weaken one of the conditions in the literature while still obtaining the optimality of the (s, S) policy. We present two properties, each is equivalent to the optimality of the (s, S) policy to

the problem. These properties are instrumental in identifying conditions for the optimality of an (s, S) policy. We then propose one such sufficient condition that is weaker than that contained in the literature. As an application of the relaxation, we study a joint pricing and inventory model where the stochastic demand is price sensitive and pricing and inventory decisions are made simultaneously. The weaker condition above enables us to prove that a modified (s, S, p) policy is optimal for additive demand functions, and a modified (s, S, A, p) policy is optimal for general demand functions.

In the third essay, we consider the stock rationing of a single-item, make-to-stock production/inventory system with multiple demand classes. Each class of demand arrives as a Poisson process with a randomly distributed batch size. It is assumed that the batch demand can be partially satisfied. Production time follows an exponential distribution. The problem is formulated as a continuous-time Markov decision process (MDP). Both the lost-sales case and the backordering case are considered. For both models, we show that the optimal policy is characterized by a sequence of monotone stock rationing levels.

There are several points to be explored in the future research.

In the first essay, to simplify the problem, we have assumed that an order placed in a given period arrives at the same period, i.e., replenishments are immediate. A positive lead time is often incurred between the placement of the order and its receipt. The problem with a positive lead time is more challenging and will be left for future research.

In the second essay, we have assumed that the stochastic demand is price sensitive, while the deterministic demand has fixed prices. It is interesting and challenging if both the demand classes are stochastic and controlled by dynamic pricing policies.

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