Correlation based frequency estimation in the presence of multiplicative noise and additive colored noise

Wang, Zhi

2007

Wang, Z. (2007). Correlation based frequency estimation in the presence of multiplicative noise and additive colored noise. Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/39069>

<https://doi.org/10.32657/10356/39069>

Correlation Based Frequency Estimation in The Presence of Multiplicative Noise and Additive Colored Noise

Wang Zhi

School of Electrical and Electronics Engineering

A thesis submitted to the Nanyang Technological University
in fulfilment of the requirement for the degree of
Doctor of Philosophy

2007

Acknowledgments

I owe my gratitude to all the people who have made this thesis possible and because of whom my graduate experience has been one that I will cherish forever. First and foremost, I would like to thank my supervisor, Dr. Saman S. Abeysekera, for giving me an invaluable opportunity to work on challenging research over the past three years. He has always made himself available for help and advice. His guidance, curiosity and intuition in the sometimes confusing world of statistical signal processing has always encouraged me. It has been a pleasure to work with and learn from him. His encouragement and expert application of essential ideas are greatly acknowledged.

Thanks also to all the people in the information system research lab for the nice atmosphere we had.

Last but not least, thanks go to my parents who have always stood by me and have always supported me.

Abstract

This dissertation is devoted to the frequency estimation in the presence of multiplicative noise and colored noise. One is intrigued by the observation that the received signal is corrupted by a multiplicative noise in most practical applications, such as in multipath fading channel estimation, back scattered acoustic signal processing and array processing of spatially distributed signal, etc. While frequency estimation in additive white Gaussian noise has received intensive attention, neither frequency estimation in colored noise nor frequency estimation in multiplicative noise has received nearly as much attention. In this thesis, we investigate the correlation based single tone frequency estimation due to its computational simplicity and efficiency of the performance. The performance of this estimator is evaluated comprehensively in the presence of colored noise and multiplicative noise. Extension of the work to multiple frequency estimation in the presence of multiplicative noise and colored noise is also discussed in detail.

The dissertation consists of mainly four major parts. At first, we discuss the intrinsic limits of any unbiased frequency estimation in the presence of multiplicative noise (Chapter 2). The asymptotic Cramer-Rao Bound (CRB) for frequency estimation in the presence of multiplicative noise is discussed. To improve numerical stability, covariance matrix tapering is employed in time domain and frequency domain when the covariance matrix of the signal is singular at high SNR for certain fading models. Using the proposed techniques, the large sample CRB is evaluated in frequency domain. Unlike the CRB

derived in literature, the bound derived in this dissertation reveals the close relation between the parameters of the channel and its performance limits.

Following the evaluation of the bounds, the correlation based frequency estimation in broadband colored noise is proposed (Chapter 3). Different phase unwrapping techniques are compared and the symmetry property of the autocorrelation is applied to the frequency estimation in the presence of colored noise. Furthermore, a comprehensive performance evaluation of the correlation based frequency estimator under different fading channels is developed (Chapter 4). The efficiency of the estimator in terms of CRB is also investigated under different fading conditions. The optimal correlation lag value suitable for frequency estimation under different fading conditions is determined. Considering the practical use of the estimator, the Doppler spread associated with fading is also estimated. Frequency estimation in the presence of both multiplicative noise and colored noise is then discussed. The correlation based estimator is also used in multiple tone frequency estimation. Finally conclusions and future work are drawn.

Table of Contents

Abstract	i
List of Figures	xiv
1 Introduction	1
1.1 Background	3
1.2 Adopted Parameter Estimation Approach	10
1.3 The Signal Model	12
1.4 A Few Applications of Frequency Estimation	15
1.4.1 Pilot Symbol Assisted Modulation in Frequency Flat Fading	15
1.4.2 OFDM Carrier Frequency Estimation	16
1.5 Contributions	18
1.6 Organization of The Thesis	21

2 Asymptotic Bounds for Frequency Estimation in the Presence of Multiplicative Noise	22
2.1 Introduction	22
2.2 Bounds for Frequency Estimation Evaluated in the Time Domain	24
2.2.1 Exact CRB in the Presence of Multiplicative Noise	26
2.2.2 The CMT as a Regularization Technique for CRB Evaluation	27
2.3 Bounds for Frequency Estimation Evaluated in the Frequency Domain	30
2.3.1 Frequency Domain CRB with the CMT	31
2.3.2 Asymptotic Expressions for CRB for the Jakes' Model	34
2.4 Closed Form Expression for CRB in the Presence of Multiplicative Noise and Additive Colored Noise	40
2.5 Conclusions	44
3 Frequency Estimation via Multiple Lags of Autocorrelation in the Presence of Broadband Colored Noise	46
3.1 Introduction	46
3.2 Threshold Effects of the Correlation Based Frequency Estimation	49
3.2.1 Kay's Frequency Estimator	50
3.2.2 Correlation Based Estimator with Optimal Lag	53

3.2.3	Correlation Based Estimator with Down Sampling	54
3.2.4	Low SNR Threshold Effect of MLE in AWGN	56
3.2.5	Performance of the Correlation Based Estimator with Phase Un- wrapping	57
3.3	Frequency Estimation from Multiple Lags of Correlation	59
3.3.1	Approximate Relation to the MLE	59
3.3.2	Optimal Window Determination	61
3.3.3	Simulation Results for the Correlation Based Estimator in AWGN .	65
3.4	Frequency Estimation in Broadband Colored Noise	68
3.4.1	Performance Evaluation of Frequency Estimation in MA Colored Noise	68
3.4.2	Bias of the DFT Peak Search Frequency Estimation in Colored Noise	74
3.4.3	Phase Unwrapping via Initial Frequency Estimate	75
3.5	Conclusion	77
4	Performance of Correlation Based Frequency Estimation Methods in Fading Channels	80
4.1	Introduction	80
4.2	Correlation Based Frequency Estimator in Multiplicative Noise	83

4.2.1	Variance Expressions for the Correlation Based Frequency Estimator	83
4.2.2	Performance Evaluation Under Different Fading Conditions	87
4.3	Doppler Spread Estimation	91
4.4	Frequency Estimation from Multiple Lags of Correlations in Fading Channels	94
4.5	Simulations	96
4.6	Conclusions	99
5	Efficient Frequency Estimation in the Presence of Multiplicative Noise and Additive Colored Noise	104
5.1	Introduction	104
5.2	Doppler Spread Estimation	105
5.3	Order Estimation of the Colored Noise Process	110
5.4	Frequency Estimation in the Presence of Multiplicative Noise and Additive Colored Noise	112
5.4.1	Performance of the Estimator in Multiplicative Noise and AWGN .	113
5.4.2	Performance of the Estimator in Colored Noise-No Fading	116
5.4.3	Performance of the Estimator in Multiplicative Noise and Colored Noise	117
5.5	Suitable Phase Unwrapping Technique	119

5.6	Multiple Frequency Estimation Using Correlation Based Algorithm	121
5.7	Comparison of the Correlation Based Frequency Estimation with Other Estimation Techniques	123
5.7.1	Periodogram Based Frequency Estimation	123
5.7.2	MUSIC Algorithm	124
5.7.3	Iterative Filtering Algorithm	126
5.7.4	Comparison Examples	127
5.7.5	Computational Complexity of Various Frequency Estimators	129
5.8	Conclusions	133
6	Conclusions and Future Work	134
6.1	Conclusions	134
6.2	Future Work	137
	Author's Publications	140

List of Figures

1.1	Non-fading signal in additive white noise.	11
1.2	Non-fading signal in additive white noise.	11
1.3	Fading (multiplicative noise) signal in additive white noise.	12
1.4	Fading (multiplicative noise) signal in additive colored noise.	12
1.5	Signal model	14
2.1	The exact CRB compared with the CRB when CMT regularization is used for different SNR, $\text{SNR} = 25, 60, 100$ and the normalized Doppler frequency $f_d = 0.1, \mu = 0$	30
2.2	The periodogram based asymptotic CRB and exact CRB with respect to f_d . The results shown are for different values of N . $N = 32, 64, 128, 256$ from the top plot to the bottom plot, and $\kappa = 0, \beta = 10\text{dB}$	38
2.3	The periodogram based asymptotic CRB and exact CRB with respect to f_d , The results shown are for different values of N . $N = 32, 64, 128, 256$ from the top plot to the bottom plot and $\kappa = 4, \beta = 10\text{dB}$	39

2.4 The floor effect of the periodogram based asymptotic CRB and the exact CRB for different values of κ . $\kappa = 0, 0.1, 0.2, 0.3$ from top to bottom, $N = 128, f_d = 0.3$ 39

2.5 The asymptotic CRB with respect to N , The results shown are for different values of $f_d = 0.001, 0.003, 0.005, 0.011$ from top to bottom and $\kappa = 100$. $\beta = 20\text{dB}$ 40

2.6 The asymptotic CRB and exact CRB in the presence of first order AR colored noise and multiplicative noise for different values of N , $N = 32, 64, 128, 256$. SNR = 10dB. Doppler frequency $f_d = 0.1$. $\kappa = 1$ 44

3.1 The SNR threshold of Kay’s weighted linear predictor estimator. The normalized frequency $f_0 = 0.124$, $N = 6, 12, 24$ (from top to bottom), $m = 1$. 53

3.2 The threshold of single lag correlation based estimator with optimal lag, $m = 2N/3$. The normalized frequency $f_0 = 0.001$, $N = 32, 64, 128$ (from top to bottom) 54

3.3 The SNR threshold of single lag correlation based estimator with down sampling. The normalized frequency $f_0 = 0.124$, $N = 32, 128$ (from top to bottom) 56

3.4 The SNR threshold of single lag correlation based method with different phase unwrapping. The normalized frequency $f_0 = 0.124$, $N = 24, 64$ (from top to bottom) 59

3.5 The efficiency of the estimator with optimal window, triangular window, rectangular window and Fitz's window. $N = 25, 50$, normalized frequency $f_0 = 0.124$, SNR = 10dB. The lags are selected as $m = 1, 2, \dots, K$ 65

3.6 The Performance of the proposed estimator with different phase unwrapping techniques, $m = 1, 2, \dots, (N - 1)/2$, $f_0 = 0.124$, $N = 24, 64$ (from top to bottom plot) 67

3.7 The Performance of the proposed estimator with optimal window and DFT phase unwrapping as SNR varies, coefficients lags $m = (N + 1)/2, \dots, N - 1$. From top plot to bottom plot N varies as $N = 32, 64, 128, 256$, normalized frequency $f_0 = 0.124$ 67

3.8 The MSE of the frequency estimation in MA colored noise as SNR varies, compared with the exact CRB and approximate CRB of (3.4.55). $f_0 = 0.124$, $N = 64, 128, 256$ and 512 (from top to bottom plot).. 73

3.9 The bias of the frequency estimation in MA colored noise for DFT peak search and the covariance based estimator. $f_0 = 0.001$, $N = 32, 64, 128$ (from top to bottom plot). without phase unwrapping 78

3.10 The MSE of the frequency estimation in MA colored noise for DFT peak search and the correlation based estimator. $f_0 = 0.001$, $N = 32, 64, 128$ (from top to bottom plot). without phase unwrapping 78

3.11 The MSE of the frequency estimation in AWGN and MA colored noise as SNR varies for DFT peak search, $f_0 = 0.124, 0.324$, $N = 32$ 79

3.12 The MSE of the frequency estimation in MA colored noise as SNR varies using different phase unwrapping techniques. $m = 13, 17$, $f_0 = 0.124$, $N = 32, 64$ (from top to bottom plot) 79

4.1 The MSE, theoretical variance and the CRB of the correlation based estimator versus SNR for different values of κ . $N = 128$, $f_d = 0.1$ (very fast fading). $f_0 = 0.1234$, $\kappa = 0, 1, 10, 100$ (from top to bottom) 101

4.2 The MSE, theoretical variance and the CRB of the correlation based estimator versus SNR for different values of κ . $N = 128$, $f_d = 0.001$ (slow fading). $f_0 = 0.1234$, $\kappa = 0, 1, 10, 100$ 101

4.3 The MSE, theoretical variance and the CRB of the correlation based estimator versus data length N for different values of SNR. SNR = 0, 10, 20, 30dB (from top to bottom), $f_d = 0.05$. $f_0 = 0.1234$, $\kappa = 0$ 102

4.4 The MSE, theoretical variance and the CRB of the correlation based estimator versus data length N for different values of SNR. SNR = 0, 10, 20dB (from top to bottom). $f_d = 0.001$. $f_0 = 0.1234$, $\kappa = 0$ 102

4.5 The distribution of the Doppler spread estimation for various methods. $N = 128$, $f_d = 0.01$. $f_0 = 0$, $\kappa = 0$ 103

4.6 The performance comparison of the single lag estimate and multiple lag estimate for different values of f_d . $N = 128$, $f_d = 0.01, 0.006, 0.0001$ (from top to bottom). $f_0 = 0.1234$, $\kappa = 0$ 103

5.1 Expected value of the Doppler spread estimator in the presence of white noise versus SNR using different lag values. $N = 64, f_d = 0.0282$ 108

5.2 The MSE of Doppler estimator in the presence of white noise versus SNR using different lag values. $N = 64, f_d = 0.0282$ 108

5.3 Expected value at the Doppler spread estimator in the presence of colored noise versus SNR using different lag values. $N = 64, f_d = 0.0282$ 109

5.4 The MSE of Doppler estimator in the presence of colored noise versus SNR using different lag values. $N = 64, f_d = 0.0282$ 109

5.5 The correlation function in the presence of fading channels, the additive noise is white. 114

5.6 The MSE of estimator in the presence of multiplicative noise versus SNR using different lag values. $N = 32, m = 1, \dots, 16, m = 1, \dots, 6. f_d = 0.01. f_0 = 0.124$ 115

5.7 The MSE of estimator in the presence of multiplicative noise versus SNR using different lag values. $N = 32, m = 1, \dots, 16, m = 1, \dots, 24. f_d = 0.005. f_0 = 0.124$ 115

5.8 The correlation function in the presence of additive colored noise, no fading 116

5.9 The MSE of the estimator in the presence of colored noise using different lags of correlations 117

5.10 The correlation function in additive colored noise and multiplicative noise 118

5.11 The MSE of estimator in the presence of additive colored noise and multiplicative noise versus SNR. $N = 32$. $f_0 = 0.124$ 118

5.12 The IFA in AWGN, colored noise and multiplicative noise versus N . $N = 1 \dots 100$, $f_0 = 0.124$ 120

5.13 The MSE of the correlation based estimator in multiplicative noise for different phase unwrapping technique. $N = 32$, $f_d = 0.006$. $f_0 = 0.124$. . 120

5.14 The MSE of the correlation based multiple frequency estimation in AWGN 122

5.15 The MSE of the correlation based multiple frequency estimation in multiplicative noise 123

5.16 Comparison of several single frequency estimators in AWGN 130

5.17 Comparison of several single frequency estimators in additive colored noise 131

5.18 Comparison of several single frequency estimators in multiplicative noise and AWGN 131

5.19 Comparison of several single frequency estimators in the presence of multiplicative noise and additive colored noise 132

5.20 Comparison of correlation based estimator using DFT phase unwrapping and MUSIC algorithm in the presence of multiple frequency 132

Chapter 1

Introduction

The need to estimate the parameters of signals in the presence of noise arises in many applications, such as in radar, sonar, wireless communications etc. In these systems, we are all faced with the problem of extracting the values of parameters from the received waveforms. Commonly, a set of data which depends on the unknown parameters is observed. Our aim lies in developing methods to accurately estimate the parameters hidden in the data which is disturbed by the noise. Conceptually, the parameter estimation is concerned with the following. Consider that the random variable has a known cumulative distribution function, we would carry out an experiment during which a realization of the variable would be observed. From this realization, the value of parameters governing the distribution of the variable can be deduced [1]. Such a deduction is called estimation of the parameter. For example, in radar and sonar applications, where we need to estimate the range of the target, a signal pulse is transmitted, and the round trip delay from the transmitter to the target and back is related to the range, assuming that the propagating

speed is known. In this case, the estimation of range is equivalent to the estimation of a time delay. In most scenarios, the noise is modelled as additive white Gaussian noise (AWGN). Another application can be noted in a communication environment, where it is often necessary to accurately estimate the frequency of a monotone (single sinusoid) signal in the presence of noise. The additive noise model has been intensively studied and well understood in the literature. However, in most practical applications, the channel can not be simply modelled as additive, and it is very necessary to include a multiplicative noise and a colored noise model to represent all possible practical environments.

In the estimation, it is very useful to evaluate the performance of the estimator under the given conditions. The MSE (mean square error) is often used as a measurement of estimation inaccuracy, which theoretically should be higher than the well known CRB (Cramér-Rao Bound). CRB is a theoretical bound to compare the performance of any unbiased estimator, and it provides a limit to the achievable estimation variance. Although many other bounds exist, CRB by far is the easiest to determine, and is useful in signal processing feasibility tests. However, due to the associated matrix expressions in the CRB, the physical meaning of the bound is often unclear except in some simple cases such as single sinusoid frequency estimation in AWGN. As such, the calculation and interpretation of the CRB is always active in the parameter estimation fields, especially in the presence of colored noise and multiplicative noise.

Parameter estimation has mainly developed in two directions: the parametric and the non-parametric approaches [2]. The parametric approach assumes that the data obey a probabilistic model of known structure, but the model contains some unknown parame-

ters, and the data are used to estimate the unknown parameters. The main goal of the parametric approach is optimality, that is the ability to achieve the best accuracy in the estimated parameters. By comparison, the nonparametric methods attempt to build a model based on as few prior assumptions as possible. The well known non-parametric frequency estimate techniques are the periodogram based estimation and correlation based estimation which represent the spectral and covariance approaches, respectively. Of these the correlation based method requires a very low number of multiplications and it is proven that this method avoids the intensive computation as in the maximum likelihood estimation (MLE) while maintaining high effective performance. In this thesis, we apply the correlation based method to frequency estimation in the presence of additive colored noise and multiplicative noise. It provides a general investigation of the performance of the correlation based method for frequency estimation, in the presence of additive colored noise and multiplicative noise.

1.1 Background

The problem of estimating the parameters of sinusoids in noise is a classical one. Due to the long time span, we might assume that the problem has been solved long ago, but surprisingly, this is not so. Estimation of sinusoidal parameters in noise remains an active research area even today, with new estimation techniques continuing to emerge. We focus mainly on the recent contributions here. In 1974, Rife and Boorstyn [3] discussed the problem of estimation of parameters of frequency tones from a finite number of noise corrupted discrete time observations. They derived the exact CRB for the MLE

in AWGN, and described the properties of the MLE providing an investigation on the threshold effects of the MLE. They used the discrete Fourier transform (DFT) to coarsely search for the mean frequency. If the coarse search is located properly, then a fine search will accurately locate the global maximum of the periodogram. This method achieves the CRB asymptotically for a large data set. Their work gave direct insight into the problem of parameter estimation in the presence of AWGN, describing the relationship between the performance and SNR, and the data length. This is the touchstone case of parameter estimation from single cisoid in complex AWGN. Later, a straightforward algebraic result that approximately holds for the important case of two closely spaced cisoid in AWGN was shown in [4]. At heart, it is a modification of the single cisoid estimation in AWGN. Note that the MLE frequency estimator is obtained by minimizing a highly nonlinear function, which is unattractive in most practical applications. From a model based approach, since a sum of sinusoids can be modelled as the impulse response of a causal pole zero filter with poles on the unit circle, one can equivalently determine the frequencies from the pole locations. Furthermore, the pole locations can be considered as functions of autoregressive (AR) parameters [5]. Based on the AR frequency estimation, iterative filtering algorithm (IFA) is described in [6] which can be used to estimate the frequencies of sinusoids in AWGN if the number of sinusoids is known. The convergent properties of the IFA was investigated in [7–9]. The IFA repeatedly enhances the signal with a bandpass filter and estimates the frequency from the filtered data. As such, it performs better at low SNR and is computationally simple. However, the IFA has low accuracy compared with correlation based method and MLE.

Other approaches that are attractive from hardware implementation point of views are

the frequency estimation based on interpolation of the periodogram peak [10, 11] and correlation based frequency estimation [2, 12–16]. Using the former technique, frequency estimators possessing good performance accuracy can be obtained, but when the actual frequency does not fall on the exact DFT bin, the accuracy is rather poor. The correlation based method is much simpler in computation than the periodogram based frequency estimation, yet attains CRB with multiple lags of correlations [15] [17]. Frequency estimation can also be obtained using the phase vocoder [18] which is one of the methods based on the principle of the short time Fourier transform (STFT). Use of the phase vocoder can be found in the parametrization of speech, audio pitch estimation and classification of a signal into sinusoidal components [19]. It has also been traditionally used for both time-scaling and pitch-scaling of audio signals. Since almost every spectral estimation method employs an estimated autocorrelation function, it is of interest to note the behavior of the autocorrelation matrix for sinusoidal parameter estimation in AWGN. The key property of the autocorrelations matrix that can be exploited is that the noise subspace eigenvectors are orthogonal to the signal eigenvector. Based on this, the multiple signal classification (MUSIC) algorithm has been proposed. The MUSIC algorithm has shown to be an approximation to the MLE for large data samples [20].

It has been shown that the correlation based frequency estimation provides reliable frequency estimates, furthermore it is simple and the MSE is close to the CRB. However, phase unwrapping is required to resolve the phase ambiguity involved, furthermore, the higher threshold SNR level due to the phase unwrapping makes the algorithms less attractive. To decrease the threshold SNR level and improve the efficiency of the correlation based estimator, Fitz proposed the multiple lag correlation based estimator in [21] via the

expansion of the expression for periodogram. This algorithm uses multiple lags to estimate the frequency, and the MSE is quite close to the CRB when data length is large. The performance evaluation of Fitz's estimator was reported in [22]. An improved correlation based estimator with proper set of correlations was proposed in [15], where the estimator uses arbitrary correlation lags with an optimal window to achieve the CRB. The Chinese Remainder Theorem (CRT) has been utilized to resolve the phase unwrapping problem.

When recent work has turned attention to the more challenging fields of parameter estimation in the presence of colored noise, the complexity of the problem seems to multiply. MLE method which estimates both the signal and noise parameters in additive colored noise, can often attain the optimal performance corresponding to the asymptotic CRB, as the sample length increases [23,24]. In Reference [24], a computationally much simpler nonlinear least squares method, which only estimates the signal parameters, is shown to achieve similar performance to MLE with larger data records, while [23] considered the problem of estimating the complex exponentials in the presence of complex additive noise with unknown covariance function. The conditional approximate CRB and the exact CRB which is related to the Gaussian circular noise and Autoregressive(AR) noise have been deduced. Direction finding in the presence of colored noise where covariance matrix is unknown has been investigated in [25] and the noise model estimate for a series of arrays in the presence of signal sources is explored concisely in [26].

In certain applications, it is necessary to include a multiplicative noise model. Examples can be cited in modelling the fading process in mobile communications, coherent radar processing, array processing of spatially distributed signals, and backscattered acous-

tic signal processing [27–30]. Frequency estimation in slow fading multipath channels was considered in [31], where the MLE algorithm in the presence of AWGN and Rayleigh fading channel has been derived. Multiplicative noise was also encountered in the spatial distributed array signal processing, such as in the direction of arrival estimation applications [32–34]. The problem of estimating the frequency of harmonics in multiplicative and additive noise has been addressed using the cyclic statistics [28] and subspace approaches [35]. In [29], it was shown that the correlation based frequency estimator showed very good performance even at very low SNR, however, the threshold SNR and the phase unwrapping effects were not reported in [29]. [36] also investigated the correlation based frequency estimation in flat fading channel, however, the frequency range decreased with the data length, and the Doppler spread is assumed to be known a priori. In [37], the optimal lag value for the single lag correlation based frequency estimator was proposed. In [38], parameter estimation in the application of time synchronization of continuous phase modulation signals was discussed under different fading environments including slow fading, fast fading conditions. The corresponding approximate CRB is derived to get the dependence of bounds on the key parameters [38]. Other parameter estimation algorithms and performance evaluation can be found in the papers [28, 39–41]. Parameter estimation algorithms in frequency domain was discussed in [32–34, 42–45].

As noted earlier, any unbiased frequency estimator's performance can be compared with the CRB. In the evaluation of the CRB, the key term in the Fisher information matrix (FIM) is the inverse of the data covariance matrix. A closed form expression for the CRB has been derived in the presence of AWGN [3]. The CRB in the presence of colored noise was fully investigated in [46], where it was demonstrated that for narrowband signals,

the FIM does not depend linearly on the number of data points, thus the exact CRB cannot be achieved. Computation of the exact information matrix of Gaussian time series with stationary random components was discussed in [47], in which an algorithm for efficient recursive computation of the Fisher information matrix of Gaussian time series was developed. The CRB computation methods mentioned above are considered in the time domain. Using frequency domain expressions, Ariela derived the asymptotic expression for frequency domain CRB on the parameters of Gaussian processes [48]. The result is an extension of Whittle's formula [49]. For the deterministic mean part of the FIM, the covariance matrix can be diagonalized through the Fourier transform matrix [50]. The Fourier transform matrix is utilized to diagonalize the covariance matrix, which is more accurate for large data length. Based on the Whittle's theorem, a simple derivation of the CRB was presented for delay estimation problems related to a single source, multiple sources and multipath propagations in [51]. The FIM in time domain and frequency domain are mathematically equivalent, and one can be used to derive the other. However, the frequency domain expression would be more convenient and insightful.

From frequency domain viewpoint of the estimation problem, we can make a conclusion that in the presence of AWGN, the CRB of frequency estimate is independent of the mean frequency. This was proven in [52] in which the exact CRB of the parameters of a complex Gaussian random process was related to Doppler radar parameter estimation in AWGN. The approximate CRB was also derived by taking advantage of the fact that the DFT of the data series has the same information as the original data series contained. When the DFT coefficients were mutually uncorrelated then so were the periodogram coefficients. In this case, the periodogram contain the same information as the original time series.

This method was also discussed in [27] and in [50].

The drawback of CRB evaluation mentioned above is that even for simple scenarios, the exact results are usually presented in matrix form which provides only little insight into the CRB's relationship with the number of data points, SNR etc. Therefore, approximate expressions for the CRB have also been derived in literature. Estimation of narrowband signal frequency from short data records is the most common. As such, D. N. Swingler investigated the frequency estimation in colored noise for short data points [53]. In this paper, an AR or MA process is considered to represent the additive colored noise effect. The author computed the CRB on a fixed frequency and substitute the AR spectral density at the fixed sinusoid frequency into the CRB in AWGN. Similarly, CRB for frequency estimation from closely spaced sinusoids problem was considered in [54].

It is noted here that by considering the performance evaluation and hardware implementation, the correlation based estimators are advantageous due to its simplicity and excellent accuracy even in the presence of multiplicative noise. The correlation based method can also be found in many applications such as in carrier and timing synchronization for MSK modulation, digital phase modulation [55], global satellite synchronization [27], acoustic backscattered signal processing [30], array signal processings [35] and Doppler spread estimation in fading channels [29], etc. This thesis mainly develops the correlation based frequency estimation algorithms in the presence of AWGN, additive colored noise, and multiplicative noise environments. The performance of the developed methods is investigated comprehensively.

1.2 Adopted Parameter Estimation Approach

The problem of estimating frequencies of sinusoidal components embedded in noise arises in many fields. In general, frequency f_0 , phase ϕ and amplitude A need to be estimated. Estimation of frequency f_0 is independent of estimation ϕ and A . Once f_0 is estimated, ϕ and A can be easily obtained [3] [56]. Therefore, the thesis primarily considers frequency estimation starting with single sinusoids but extending to multiple sinusoids at the later Sections.

The periodogram based frequency estimation method is widely used in practical applications. As a compliment to the periodogram, low complexity frequency estimators are of interest. We will investigate the correlation based frequency estimator, initially for the case of sinusoids in the white additive Gaussian noise. The correlation based approach is also extended to the frequency estimation in the presence of multiplicative noise and additive colored noise environments.

Figure 1.1-Figure 1.4 show the correlation function for different noise conditions. It is seen from Figure 1.1 that the noise correlation is a delta function, and any lags value of correlations can be used to retrieve the information of the signal from white noise. Figure 1.2 is the signal correlation function in broad-band colored noise, and any correlation lag m from L_1 to N can be used to estimate frequency as will be discussed in Chapter 3. Figure 1.3 shows the correlation function of the signal in the presence of multiplicative noise (e. g. in fading channels, therefore, 1 to L_2 correlation lags can be used. If L_2 is very small compared with the length of data sample N , the fading is fast, otherwise, the fading is slow. Fast fading only occurs when the coherence time of the channel is smaller

than the symbol period of the transmitted signal. In frequency domain, when the Doppler spread is larger than the bandwidth of the transmitted signal, the channel is fast fading channel.

The correlation based frequency estimation in different fading channels is discussed in Chapter 4. In Figure 1.4, the correlation function in the presence of multiplicative noise, and colored noise is shown. For this case, the order of colored noise and Doppler spread are estimated as discussed in Chapter 5. The performance of the correlation based estimators are evaluated in Chapters 3 to 5.

Figure 1.1: Non-fading signal in additive white noise.

Figure 1.2: Non-fading signal in additive white noise.

Figure 1.3: Fading (multiplicative noise) signal in additive white noise.

Figure 1.4: Fading (multiplicative noise) signal in additive colored noise.

1.3 The Signal Model

Figure 1.5 shows the signal model in the presence of multiplicative noise and additive colored noise, which represents the various scenarios from Figure 1.1 to Figure 1.4. This model has a broad range of applications for frequency estimation such as in flat fading wireless communications, backscattered acoustic signal processing, and array signal processing. For example, consider a general, discrete time complex time varying channel in wireless communications, having a frequency offset between the transmitter and the

receiver. The data samples at the receiver can be expressed as

$$x(n) = (\mu e^{j\phi} + a(n))e^{j\omega_0 n} + v(n), \quad n = 0, 1 \dots N - 1. \quad (1.3.1)$$

where μ and ϕ are the amplitude and phase of the signal propagating along the direct path, $a(n)$ is the fading causing multiplicative noise, ω_0 is the frequency offset between the transmitter and the receiver, and $v(n)$ is the additive white noise or colored noise. N is the number of data samples. The following assumptions are placed on the signal model:

1. $a(n)$ is a stationary complex Gaussian process which is circular symmetric with zero mean and variance σ_a^2 . A process is circular if the autocorrelation of the real part is identical to that of the imaginary part, and the cross correlation of the real part and imaginary part is antisymmetrical. (As noted in [30, 56, 57] circularity is a necessary condition for a realistic bandpass channel to be wide sense stationary (WSS) [56]). The normalized autocorrelation function of $a(n)$ is defined as $r_a(m) = E[a(n+m)a^*(n)]/\sigma_a^2$, thus $r_a(0) = 1$. The Gaussian assumption of $a(n)$ gives rise to the well known Rayleigh distributed amplitude fading when $\mu = 0$, while if $\mu \neq 0$, it is the Rician fading.
2. $v(n)$ is a sequence of independent, identically distributed complex zero mean Gaussian variable with variance σ_v^2 , and is independent of $a(n)$. The SNR is defined as $\text{SNR} = \frac{\mu^2 + \sigma_a^2}{\sigma_v^2} = \frac{\mu^2/\sigma_a^2 + 1}{\sigma_v^2/\sigma_a^2} = \beta(1 + \kappa)$, where $\kappa = \mu^2/\sigma_a^2$ is the Rician factor, and $\beta = \sigma_a^2/\sigma_v^2$. Thus it can be seen for Rayleigh fading ($\mu = 0, \sigma_a^2 \neq 0$), $\text{SNR} = \beta$, while in the classical additive white Gaussian noise environment, $\text{SNR} = \frac{\mu^2}{\sigma_v^2} = \kappa\beta$.

3. In addition to circularity, it is also assumed that the real part and imaginary part of $a(n)$ are independent of each other. Therefore, $r_a(m)$ is real valued which implies that the power spectrum density of $a(n)$ is symmetric around zero frequency. This will suffice to ensure a consistent frequency estimation via the algorithms proposed in this thesis. Otherwise, the phase of $r_a(m)$ has to be estimated prior to the frequency estimation. The above assumption has been made implicitly by many authors, e.g., in ionospheric channels for mobile cellular communications [58]. The assumption could be relaxed if one can estimate the parameters of $r_a(m)$ without the knowledge of frequency. Note that the correlation of the fading process $a(n)$ is commonly given by $J_0(2\pi f_d m)$, where $J_0(\cdot)$ is the zeroth-order Bessel function of the first kind and f_d is the Doppler spread [58].

Some applications of the use of the model of Figure 1.5 are discussed in the next section.

Figure 1.5: Signal model

1.4 A Few Applications of Frequency Estimation

1.4.1 Pilot Symbol Assisted Modulation in Frequency Flat Fading

Employing a large modulation alphabet in a mobile digital communication system is a way to achieve high bandwidth efficiency. However, a coherent demodulation is required to obtain the frequency offset correction. One significant problem for coherent demodulation is the time varying multiplicative noise which will cause the fading of the channel. This fading effect will impose severe accuracy specifications on the radio frequency oscillators. The uncompensated frequency offset will result in a significant performance loss.

Assume a pilot symbol assisted modulation based communication system with linear modulations. The transmitted signal has the form

$$s(t) = \sum_i d_i u(t - iT) \quad (1.4.2)$$

where $u(t)$ represents unit energy square root Nyquist pulse shape and T is the symbol duration, and d_i is the transmitted modulation symbol having M levels (MPSK or M-QSM). Known pilot symbols are inserted in the transmitted symbols to help the demodulator to obtain a channel state estimate.

Assume that the channel model is a flat Rayleigh fading channel having a frequency offset

between the transmitter and receiver. The input to the receiver is then given by [59]

$$z(t) = e^{j2\pi\tilde{f}_0 t} a(t)s(t) + n(t) \quad (1.4.3)$$

where $a(t)$ is a complex multiplicative noise with correlation function $r_a(\tau) = E[a(t)a^*(t - \tau)] = E_b J_0(2\pi f_d \tau)$ [60], f_d is the Doppler spread and E_b is the average received energy per bit, $\tilde{f}_0$ is the unknown frequency offset, and $n(t)$ is a complex AWGN. Assume perfect symbol timing recovery and approximately stationary multiplicative noise for a pulse duration, if the Inter-Symbol Interference (ISI) is negligible, then the sampled output of the matched filter takes the form

$$z(kT_s) = e^{j2\pi\tilde{f}_0 kT_s} a(kT_s)s(kT_s) + n(kT_s) \implies z(k) = e^{j2\pi f_0 k} a(k)s(k) + n(k) \quad (1.4.4)$$

where T_s is the sampling interval and f_0 is the normalized frequency corresponding to $\tilde{f}_0$. For simplicity of notation, $z(k)$, $a(k)$, $s(k)$ have been used to denote $z(kT_s)$, $a(kT_s)$, $s(kT_s)$. Thus (1.4.4) results in a problem of frequency estimation in the presence of multiplicative noise.

1.4.2 OFDM Carrier Frequency Estimation

Orthogonal frequency division multiplexing (OFDM) is an effective transmission technology to combat multipath fading. By inserting a guard interval (which is called cyclic prefix) between symbols, the ISI can be mitigated. OFDM has been adopted as the modulation scheme for DAB (Digital Audio Broadcasting) systems and ADSL (Asymmetry

Digital Subscriber Loop). It is also proposed for the terrestrial HDTV transport in Europe. OFDM systems are very sensitive to carrier frequency offsets, and even a small frequency offset can greatly degrade the performance of the OFDM receiver [61].

OFDM input signals are complex numbers of the transmitter signal constellation (e.g., PSK, QAM). Let X_k denote the complex signal modulated onto the k th subcarrier. The OFDM samples at the outputs of the IFFT are given by

$$x(n) = \frac{1}{N} \sum_{k=-K}^K X_k e^{j\frac{2\pi nk}{N}} \quad n = 0, 1, \dots, N-1, N \geq 2K+1 \quad (1.4.5)$$

where N is the OFDM block length. Sequence $x(n)$ consists of $2K+1$ complex sinusoids which have been modulated with $2K+1$ complex values X_k .

The OFDM samples are pulse-shape filtered and the filter output is up converted for signal transmission. Assume that the matched filter frequency response is flat within the transmitter bandwidth and the symbol timing is obtained within the guard interval region. If an OFDM transmission symbol has been repeated, one can obtain, in the absence of noise, a $2N$ point sequence given by

$$y(n) = \frac{1}{N} \sum_{k=-K}^K X_k H_k e^{j\frac{2\pi n(k+\epsilon)}{N}} \quad n = 0, 1, \dots, 2N-1 \quad (1.4.6)$$

where H_k is a complex term resulting from the transfer function of the k th subcarrier channel, and ϵ is the carrier frequency offset. Let us assume that the channel is time invariant during one OFDM symbol duration, then the k th element of a N point DFT of the first N points of (1.4.6) is given by

$$Y_1(k) = \frac{1}{N} \sum_{n=0}^{N-1} y(n) e^{-j \frac{2\pi n k}{N}} \quad k = 0, 1, \dots, N-1 \quad (1.4.7)$$

and the k th element of a N point DFT of the second half of the sequence is

$$Y_2(k) = \frac{1}{\sqrt{N}} \sum_{n=N}^{2N-1} y(n) e^{-j \frac{2\pi n k}{N}} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y(n+N) e^{-j \frac{2\pi n k}{N}} \quad k = 0, 1, \dots, N-1 \quad (1.4.8)$$

From (1.4.6), we can get $y(n+N) = y(n)e^{2\pi j\epsilon}$, thus $Y_2(k) = Y_1(k)e^{2\pi j\epsilon}$. Observe that the first DFT $Y_1(k)$ and second DFT $Y_2(k)$ are very similar, except for a phase shift proportional to frequency offset. Therefore, from the two DFT sequences $Y_1(k)$ and $Y_2(k)$, the frequency offset ϵ can be estimated. Note that OFDM is sensitive to Doppler and carrier frequency errors, which destroy the subcarrier orthogonality and result in inter-channel interference (ICI). When the OFDM block duration is much smaller than the channel coherence time, the channel can be assumed to be approximately constant over an OFDM block. However, this assumption on block time-invariance may not be valid with high Doppler spread, hence, the problem of retrieving the frequency offset ϵ from the discrete OFDM samples is one application of frequency estimation in the presence of multiplicative noise. Frequency estimation using the signal phase has been discussed in detail in this thesis.

1.5 Contributions

The research described in this thesis mainly focuses on frequency estimation in the presence of multiplicative noise and additive colored noise. The contributions of the reported

work consist of efficient algorithm development via the correlation based frequency estimation as well as theoretical analysis of the existing and proposed new methods.

In Chapter 2, a general framework for performance evaluation in the presence of additive colored noise as well as multiplicative noise is provided via time domain analysis and frequency domain analysis. The asymptotic CRB for frequency estimation in the frequency domain provides a general framework for the CRB evaluation in different fading channels. The closed form expression for different fading channels is originally proposed, and the explanation and performance studies contribute new insight to frequency estimation in different channel environments.

The original contribution of the work in Chapter 3 can be summarized as follows. The performance of frequency estimation from correlations with multiple lags, both in additive white Gaussian noise and broad-band colored noise is evaluated. The SNR threshold of estimator is fully investigated and different phase unwrapping techniques for correlation based estimation are compared. To decrease the threshold SNR of the estimator in [15], it is proposed to use the DFT coarse search as an initial estimate to demodulate the frequency. An optimal window is also derived to achieve the CRB in frequency estimation. The mirror coefficients of the optimal window is applied in frequency estimation in the presence of colored noise. It is shown that the proposed estimator can achieve the CRB in the presence of the colored noise, and has a low complexity compared with MLE. However, the DFT peak search is shown to be a biased estimate in the presence of colored noise. As such, we propose a new phase unwrapping technique by using the higher lags of correlations, and it is shown that by using the higher lags correlations and a proper phase

unwrapping technique, the correlation based estimator achieves CRB asymptotically while maintaining a low SNR threshold level.

In Chapter 4, we derive the closed form expressions for the frequency estimation in different fading conditions. These formulations are more robust and provide direct insight into the frequency estimation problems. Based on this, the optimal lag value which is critical in the correlation based frequency estimation is determined. In addition, this chapter also considers the frequency estimation using multiple lags of correlations in fading channels. When multiple lags are used, phase unwrapping is necessary to resolve the phase ambiguity. It is noted that the phase unwrapping can be avoided at the price of reducing the frequency estimation range as in [36] and [29] which is not usually practical. Here we use DFT based phase unwrapping technique which is optimal considering the threshold effect of the frequency estimation. The proposed algorithm possesses a wide frequency acquisition range and can operate in low SNR environments.

Chapter 5 investigates the performance of the correlation based frequency estimator in the presence of multiplicative noise and additive colored noise. Before implementing the correlation based algorithm, the Doppler bandwidth is estimated. The spectral bandwidth of the broadband colored noise is also estimated under fading conditions. These are necessary to determine the range of useful correlation lags. We also compare different multiple tone frequency estimation methods in different fading channels. It is shown that the correlation based estimator with DFT phase unwrapping is efficient even at low SNR. It is demonstrated that the implementation of the proposed frequency estimators is simple compared with other techniques.

1.6 Organization of The Thesis

The remaining Chapters of this thesis are organized as follows:

In Chapter 2, the CRB in the presence of multiplicative noise and colored noise is evaluated. A general framework for performance evaluation in the presence of additive colored noise as well as multiplicative noise is provided via time domain analysis and frequency domain analysis. Asymptotic CRB for frequency estimation in the presence of multiplicative noise is investigated.

In Chapter 3, correlation based frequency estimator in the presence of colored noise is proposed, and threshold effect of the estimator is analyzed comprehensively. The different phase unwrapping techniques are compared. It is shown that the proposed technique outperforms the DFT peak search in the presence of colored noise with the use of a proper phase unwrapping technique.

Chapter 4 discusses the correlation based frequency estimator in the presence of multiplicative noise. The closed form expressions to determine performance evaluation are given and optimal lag values are determined for different fading channels.

In Chapter 5, the correlation based frequency estimator in the presence of both multiplicative noise and additive colored noise is discussed. The performance of the estimator with Doppler spread and colored noise bandwidth estimation is illustrated via simulations. The correlation based estimator is also used in multiple tone frequency estimation. Different other multiple frequency estimation methods are compared with the correlation based estimator. Finally, conclusions and future works are drawn in Chapter 6.

Chapter 2

Asymptotic Bounds for Frequency Estimation in the Presence of Multiplicative Noise

2.1 Introduction

The problem of frequency estimation from noisy signals is of fundamental importance in a variety of applications. Although, the performance of frequency estimation in the presence of additive noise is rather well understood, the same can not be stated for frequency estimation in the presence of multiplicative noise. Recently, frequency estimation in the presence of multiplicative noise has received much attention, especially in fading multipath channels, backscatter radar signal processing, and array processing of spatial distributed signals [27,33,34]. A preliminary step in the development of estimation algorithms in these

environments is to identify the fundamental limits of their performance. The Cramer Rao lower bound (CRB) is a such fundamental lower bound on the variance of any unbiased estimate [56], and is also known to be asymptotically achievable when the number of observations is large.

Computation of the exact CRB in the presence of multiplicative noise has been discussed in [27] [29]. Although one can see the bounds on each parameter by examining the corresponding diagonal element, the exact result usually offers little insight on how the CRB depends on data length and different channel conditions, such as SNR, etc. Furthermore, it is noticed that under high signal noise ratio (SNR) conditions, the covariance matrix involved in CRB evaluation tends to be singular which makes the computation of the exact CRB numerically unstable. This effect is more prominent when the number of observations is large, and in certain multiplicative noise models (e.g. Jakes' model) the effect is quite apparent even at a low number of data samples. Similar singularity problems were also encountered in fading channel simulation, minimum mean square error (MMSE) multiuser detection [62,63]. Parameter estimation with singular information matrices was also discussed in [64], and commonly the singularity is caused by high dimensional parameter estimation problems via the use of matrix pseudo-inverse. In this chapter, to improve numerical stability in the CRB computation, we propose to use covariance matrix tapering (CMT) technique when the covariance matrix of the signal is singular. The CMT technique was previously proposed to modify the array pattern in the application of adaptive beamforming [65]. Here in this chapter, we shall use CMT to resolve the problem of covariance matrix singularity.

For large sample CRB evaluation, for computational simplicity, we also propose a DFT based frequency domain CRB evaluation in conjunction with the CMT technique. In this approach, the asymptotic CRB has been derived using the periodogram of the data in frequency domain [52]. It is noted that this technique is accurate under the condition that the data record length is much larger than the correlation time of the multiplicative noise [66]. Finally, a closed form expression of the large samples CRB for the Jakes' model is given and a general expression for an asymptotic CRB in the presence of multiplicative noise and colored additive noise is provided. The unified approach of the use of CMT in time domain and frequency domain CRB evaluation and the proposal of an improved periodogram based CRB and its subsequent use to obtain a general closed form expression for the CRB for the Jakes' model are the novel contributions reported in the chapter.

2.2 Bounds for Frequency Estimation Evaluated in the Time Domain

In this section, we consider the CRB for frequency estimation evaluated in the time domain. The CMT is used to regularize the possible ill conditioning of the covariance matrix in time domain. To begin with, consider the signal model presented in Section 1.3. Note here that in the initial discussion, we assume white noise, which is relaxed in Section 2.4. Recall that the variance of an unbiased estimate $\hat{\theta}_i$ of the parameter θ_i has a lower bound that is given by [56]

$$E[(\hat{\theta}_i - \theta_i)^2] \geq \mathbf{J}_i^{-1} \quad (2.2.1)$$

where $E[\cdot]$ denotes ensemble average and $\mathbf{J}_{ii}^{-1}$ is the i th element of the diagonal of the inverse of the Fisher information matrix (FIM) $\mathbf{J}$ for a Gaussian process with its (i, j) th element [56]

$$\mathbf{J}_{ij} = 2\text{Re}\left(\frac{\partial \mathbf{m}^H}{\partial \theta_i} \mathbf{R}^{-1} \frac{\partial \mathbf{m}}{\partial \theta_j}\right) + \text{tr}\left(\mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial \theta_i} \mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial \theta_j}\right) \quad (2.2.2)$$

In the above, $[\cdot]^H$ denotes the Hermitian transpose, θ is the vector consisting of parameters θ_i . $\text{Re}(u)$ is the real part of u , tr denotes trace of the matrix, $\mathbf{R}$ and $\mathbf{m}$ are the covariance matrix and mean vector of the received signal. Representing the signal model in (1.3.1) in terms of vectors, we obtain

$$\mathbf{x} = \Lambda(\mu e^{j\phi} \mathbf{1} + \mathbf{a}) + \mathbf{v} \quad (2.2.3)$$

where $\mathbf{x} = [x[0], x[1], \dots, x[N-1]]^T$, $\mathbf{v} = [v[0], v[1], \dots, v[N-1]]^T$, $\mathbf{1} = [1, 1, \dots, 1]^T$, $\mathbf{a} = [a[0], a[1], \dots, a[N-1]]^T$ and $[\cdot]^T$ denoting vector transpose. $\Lambda = \text{diag}[1, e^{j\omega_0}, \dots, e^{j\omega_0(N-1)}]$.

Suppose that the covariance matrix of multiplicative noise $a(n)$ is expressed as $\sigma_a^2 \mathbf{R}_a$. then mean vector and covariance matrix of the data vector $\mathbf{x}$ can be expressed as

$$\mathbf{m} = \mu e^{j\phi} \Lambda \mathbf{1} \quad (2.2.4)$$

$$\mathbf{R} = \sigma_a^2 \Lambda \mathbf{R}_a \Lambda^{-1} + \sigma_v^2 \mathbf{I} \quad (2.2.5)$$

2.2.1 Exact CRB in the Presence of Multiplicative Noise

Substitute (2.2.4) and (2.2.5) into (2.2.2), the entries of the FIM can be written as [29]

$$\mathbf{J}_{\omega_0, \omega_0} = 2\mu^2 \mathbf{1}^T \mathbf{D} \mathbf{R}_z^{-1} \mathbf{D} \mathbf{1} + 2\text{tr}(\mathbf{R}_z^{-1} \mathbf{D} \mathbf{R}_z \mathbf{D} - \mathbf{D}^2) \quad (2.2.6)$$

$$\mathbf{J}_{\omega_0, \phi} = 2\mu^2 \text{Re}\{\mathbf{1}^T \mathbf{D} \mathbf{R}_z^{-1} \mathbf{1}\} \quad (2.2.7)$$

$$\mathbf{J}_{\phi, \phi} = 2\mu^2 \mathbf{1}^T \mathbf{R}_z^{-1} \mathbf{1} \quad (2.2.8)$$

where $\mathbf{R}_z = \sigma_a^2 \mathbf{R}_a + \sigma_v^2 \mathbf{I}$. $\mathbf{D}$ is a diagonal matrix having the form $\mathbf{D} = \text{diag}[0, 1, 2, \dots, N-1]$.

The FIM entry for μ is decoupled from the entries for frequency offset and phase of the multiplicative noise [27], thus we have the CRB for frequency estimation

$$\text{CRB}(\omega_0) = \frac{\mathbf{J}_{\phi, \phi}}{\mathbf{J}_{\omega_0, \omega_0} \mathbf{J}_{\phi, \phi} - \mathbf{J}_{\omega_0, \phi} \mathbf{J}_{\omega_0, \phi}} \quad (2.2.9)$$

Specifically, if $\mu \neq 0$, $\sigma_a^2 = 0$, the covariance matrix becomes proportional to identity matrix, the CRB for frequency estimation is simplified as below

$$\text{CRB}(\omega_0) = \frac{2\sigma_v^2}{\mu^2 \mathbf{1}^H \mathbf{D} \mathbf{D} \mathbf{1}} = \frac{12\sigma_v^2}{\mu^2 N(N-1)(2N-1)} \approx \frac{6\sigma_v^2}{\mu^2 N^3}. \quad (2.2.10)$$

The approximation in (2.2.10) is valid when N is very large. This is the classical expression for CRB in additive white Gaussian noise [56]. If $\mu = 0$, $\sigma_a^2 \neq 0$, which represents the Rayleigh fading in wireless communications, the CRB for frequency estimation is

simplified to [27]

$$\text{CRB}(\omega_0) = \frac{1}{2\text{tr}(\mathbf{R}_z^{-1}\mathbf{D}\mathbf{R}_z\mathbf{D} - \mathbf{D}^2)}. \quad (2.2.11)$$

Note that the denominator of (2.2.11) vanishes for temporally white multiplicative noise which will lead to an infinite CRB, but it is also known that under this condition, that the parameters are unidentifiable.

2.2.2 The CMT as a Regularization Technique for CRB Evaluation

The CRB obtained in (2.2.9) and (2.2.11) is exact for even a finite length of data N . However, for certain fading models, the covariance matrix $\mathbf{R}_z$ can be singular especially when the SNR is high. This singularity is caused by the rapid time variation of the fading resulting in an information singular process. Information-singular processes are simply those having zero Kolmogorov entropy (or equivalently, those processes which are completely determined by their infinite past) [67]. Hence such processes are deterministic which will cause some eigenvalues of the covariance matrix to be zero according to the prediction theory [62]. Note that, in wireless communication applications, the Jakes' model is widely used to represent the Doppler power spectrum. The Jakes' model is based on summing a large number of sinusoids resulting from scatters assumed to be uniformly distributed around a circle. The Jakes' model is also deterministic and stationary, and its covariance function is given by $r(\tau) = J_0(\omega_d\tau)$, where $J_0(\cdot)$ is the zeroth order Bessel function of the first kind and ω_d is the maximum Doppler frequency. (The Bessel functions of the first kind are solutions of Bessel's differential equation. The graphs of Bessel

functions look roughly like oscillating sine or cosine functions that decay proportionally to $1/\sqrt{\tau}$). For the Jakes' model, the covariance matrix tends to be ill conditioned even when the data length is small. The effect is more prominent with the increase in the data length. In the following, we shall regularize this information singularity using the CMT. The basic idea of CMT is to multiply the elements of the covariance matrix with different weights, in order to attenuate those elements situated away from the main diagonal. With the use of CMT, the CRB for frequency estimation in (2.2.11) can be re-expressed as

$$\text{CRB}_T(\omega_0) = \frac{1}{2\text{tr}((\mathbf{R}_z \circ \mathbf{T})^{-1} \mathbf{D}(\mathbf{R}_z \circ \mathbf{T})\mathbf{D} - \mathbf{D}^2)}. \quad (2.2.12)$$

$\mathbf{T}$ denoting a tapering matrix. $\mathbf{T}$ is real, symmetric and Toeplitz. The symbol $\circ$ representing the Hadamard element wise product between matrices. Hadamard element wise product is defined as $\mathbf{C} = \mathbf{A} \circ \mathbf{B}$, where $c_{ij} = a_{ij}b_{ij}$, and a_{ij} , b_{ij} , c_{ij} are the element of matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. The following theorems provide useful insight into the underlying stochastic properties of the CMT technique. For proofs and more details, see [65].

Theorem 1: If $\mathbf{A}, \mathbf{B} \in \mathcal{C}^{N \times N}$ are both positive semidefinite matrices, so is $\mathbf{A} \circ \mathbf{B}$. Moreover, if $\mathbf{A}$ is positive definite and $\mathbf{B}$ is positive semidefinite with no zero diagonal entries, then $\mathbf{A} \circ \mathbf{B}$ is positive definite.

Theorem 2: Let $\mathcal{P}^N$ denotes the space of complex valued $N \times N$ positive semidefinite covariance matrices. If $\mathbf{A}, \mathbf{B} \in \mathcal{P}^N$ and with additional property $d_i = 1, \forall i : i = 1, \dots, N$, where d_i denotes the i th diagonal entry of $\mathbf{A}$ and $\mathbf{B}$. Then $\chi(\mathbf{A}) \geq \chi(\mathbf{A} \circ \mathbf{B})$, where $\chi(\mathbf{A})$ is the eigenvalue spread of $\mathbf{A}$. Eigenvalue spread is the ratio of the largest eigenvalue over the smallest eigenvalue.

Suppose we choose $\mathbf{B}$ as $\mathbf{T}$, with $\mathbf{T}$ being positive definite. Then Theorem 2 suggests $(\mathbf{A} \circ \mathbf{T})$ is positive definite and that the eigenvalue spread has been reduced due to the tapering. That is a singular matrix can be regularized via the use of $\mathbf{T}$. Though Theorem 1 and Theorem 2 provide the fundamental property of the CMT as a regularization technique, it does not provide a general guideline to choose an optimal tapering matrix. In [65], the tapering matrix is chosen as

$$\mathbf{T}_{mn} = \frac{\sin(\alpha\pi|m-n|)}{\alpha\pi|m-n|} = \text{sinc}(\alpha(m-n)) \quad (2.2.13)$$

where $\mathbf{T}_{mn}$ is the (m, n) th element of matrix $\mathbf{T}$ and $0 \leq \alpha \leq 1$. Note that for the special case $\alpha = 0$, the tapering matrix defined in (2.2.13) becomes a matrix with all ones and thus no regularization is obtained. While for the case $\alpha = 1$, the tapering matrix becomes a identity matrix and the covariance matrix $\mathbf{R}_z$ becomes a diagonal matrix. As an example, Figure 2.1 illustrates the regularization property of the CMT when the covariance matrix is singular. Here we consider a Rayleigh fading channel with normalized Doppler frequency $f_d = 0.1$, and the multiplicative noise process is described via the Jakes' model. (The Normalized Doppler spread is defined as $f_d = B_d T_s / 2$, where T_s is the symbol duration, and B_d is the frequency range where the received power spectrum is nonzero when a pure sinusoidal tone is transmitted.) The SNR varies from 25 dB, 60 dB to 100 dB. It can be seen that when α decreases from 0.003, 0.002 to 0.001, the CRB gradually meets the exact $\text{CRB}(\alpha = 0)$. Therefore, it is evident (from the plot of SNR=25 dB) that if $\alpha_1 < \alpha_2$, then $\text{CRB}(\alpha_1) < \text{CRB}(\alpha_2)$, and $\alpha = 0$ provide no regularization to the covariance matrix. At high SNR = 100 dB and for large data samples, the singularity of the covariance matrix is

prominent which can be seen from the discontinuities of the lowest curve. We note from the figure that, the bound obtained using the tapering matrix with $\alpha = 0.001$ is very close to the exact CRB while maintaining regular conditions.

Figure 2.1: The exact CRB compared with the CRB when CMT regularization is used for different SNR, $\text{SNR} = 25, 60, 100$ and the normalized Doppler frequency $f_d = 0.1$, $\mu = 0$

2.3 Bounds for Frequency Estimation Evaluated in the Frequency Domain

The time domain CRB evaluation discussed in the previous section requires matrix inversion. When the data lengths are extremely large even with the use of CMT, the technique is prone to numerical instability. In this section, we elaborate the use of CMT in the evaluation of the CRB in the frequency domain. The major advantage of the frequency

domain evaluation is that it avoids matrix inversion and thus it is especially useful when the data lengths are quite large. In particular, we shall show that the periodogram based CRB discussed in [52] is a special case of frequency domain CRB evaluation via the CMT with $\alpha = 1$. It is further noted that, the periodogram based CRB can be related to the power spectrum of the signal, thereby providing more insight into characteristics of the estimation problem. We also discuss a method of improving the periodogram based CRB and use this to derive closed form expressions for large samples CRB for the Jakes' model.

2.3.1 Frequency Domain CRB with the CMT

By performing the DFT operation on the data vector $\mathbf{x}$, we obtain,

$$\mathbf{y} = \mathbf{F}\mathbf{x} \quad (2.3.14)$$

where $\mathbf{F}$ is the normalized Fourier transform unitary matrix, ($\mathbf{F}^H \mathbf{F} = \mathbf{I}$), which has the form $\mathbf{F} = [\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{N-1}]$, where

$$\mathbf{e}_k = \frac{1}{\sqrt{N}} [1, e^{-j\frac{2\pi k}{N}}, e^{-j\frac{4\pi k}{N}} \dots e^{-j\frac{2\pi k(N-1)}{N}}], k = 0, 1, \dots, N-1. \quad (2.3.15)$$

$\mathbf{y}$ is the transformed data vector having length N . Substituting (2.2.3) into (2.3.14), we obtain

$$\mathbf{y} = \sqrt{N}\mathbf{F}\mathbf{A}(\mu e^{j\phi}\mathbf{1} + \mathbf{a}) + \sqrt{N}\mathbf{F}\mathbf{v}. \quad (2.3.16)$$

Consequently, the mean vector $\mathbf{m}_y$ and the covariance matrix $\mathbf{R}_y$ of $\mathbf{y}$ can be expressed as

$$\mathbf{m}_y = \sqrt{N}\mu e^{j\phi} \mathbf{F} \mathbf{\Lambda} \mathbf{1} \quad (2.3.17)$$

$$\mathbf{R}_y = N\sigma_a^2 \mathbf{F} \mathbf{\Lambda} \mathbf{R}_a \mathbf{\Lambda}^H \mathbf{F}^H + \sigma_v^2 N \mathbf{I} \quad (2.3.18)$$

Note that the DFT is a linear reversible operator, and thus it can not be directly used to avoid the singularity associated with $\mathbf{R}_y$. Again, CMT can be employed in the frequency domain to avoid the singularity problem. Equation (2.3.18) can be then rewritten as

$$\mathbf{R}_t = N \mathbf{F} \mathbf{\Lambda} \mathbf{R}_z \mathbf{\Lambda}^H \mathbf{F}^H \circ \mathbf{T} \quad (2.3.19)$$

where $\mathbf{T}$ is as given in (2.2.13). The (m, n) th element of $\mathbf{R}_t$ can be expressed via matrix expansion

$$\begin{aligned} \mathbf{R}_{t_{mn}} &= \frac{e^{j\frac{2\pi}{N}(m-n)}}{N} \sum_{t=0}^{N-1} \sum_{k=0}^{N-1} R_z(t-k, \omega_0) \\ &\quad \times e^{-j\frac{2\pi}{N}(t-k)} e^{-j\frac{2\pi}{N}(mk-nt)} \text{sinc}(\alpha(m-n)). \end{aligned} \quad (2.3.20)$$

where $R_z(t, \omega_0) = \sigma_a^2 e^{j\omega_0 t} R_a(t) + \sigma_v^2$. After further manipulation, (2.3.20) is expressed as

$$\begin{aligned} \mathbf{R}_{t_{mn}} &= \frac{1}{N} e^{j\pi(m-n)} \text{sinc} \alpha(m-n) \sum_{l=0}^{N-1} R_z(l, \omega_0) \\ &\quad \times e^{j\frac{\pi(m+n)l}{N}} \frac{\sin\left(\frac{\pi(m-n)(N-l)}{N}\right)}{\sin \frac{\pi(m-n)}{N}}. \end{aligned} \quad (2.3.21)$$

It can be seen from (2.3.20) that using the tapering matrix $\mathbf{T}$, with $\alpha = 1$, $\mathbf{R}_t$ becomes a diagonal matrix

$$\mathbf{R}_t = \text{diag}\{P(\omega_1, \omega_0), P(\omega_2, \omega_0), \dots, P(\omega_{N-1}, \omega_0)\} \quad (2.3.22)$$

with the diagonal element $P(\omega_k, \omega_0)$ given by

$$P(\omega_k, \omega_0) = \sum_{l=-(N-1)}^{N-1} w_B(l) R_z(\omega_0, l) e^{-j\omega_k l}. \quad (2.3.23)$$

where $\omega_k = \frac{2\pi k}{N}$ and $R_z(\omega_0, l) = \sigma_a^2 R_a(l) e^{j\omega_0 l} + \sigma_v^2 \delta(l)$, and $\delta(l)$ is the Kronecker Delta function. $w_B(l)$ is the Bartlett (triangular) window which is given by

$$w_B(l) = \begin{cases} 1 - \frac{|l|}{N} & -N+1 \leq l \leq N-1 \\ 0 & \text{elsewhere} \end{cases} \quad (2.3.24)$$

Substituting (2.3.17) and (2.3.22) into FIM in (2.2.2), we obtain the entries of the FIM as

$$\mathbf{J}_{\omega_0, \omega_0} \approx \frac{2\mu^2 \mathbf{1}^H \mathbf{D} \mathbf{D} \mathbf{1}}{P(\omega_\gamma, \omega_0)} + \sum_{k=0}^{N-1} \left(\frac{P'(\omega_k, \omega_0)}{P(\omega_k, \omega_0)} \right)^2 \quad (2.3.25)$$

$$\mathbf{J}_{\omega_0, \phi} \approx \frac{2\mu^2 \mathbf{1}^H \mathbf{D} \mathbf{1}}{P(\omega_\gamma, \omega_0)} \quad (2.3.26)$$

$$\mathbf{J}_{\phi, \phi} \approx \frac{2\mu^2 \mathbf{1}^H \mathbf{1}}{P(\omega_\gamma, \omega_0)} \quad (2.3.27)$$

where $\gamma = \frac{N\omega_0}{2\pi}$ and $P'(\omega_k, \omega_0)$ is the derivative of $P(\omega_k, \omega_0)$ with respect to ω_0 . The frequency domain evaluation of the CRB obtained via CMT with $\alpha = 1$ in fact is the periodogram based CRB discussed in [52]. The accuracy of the periodogram based CRB

increases with the data samples N . From (2.3.21), when $N \rightarrow \infty$, $\mathbf{R}_{\mathbf{t}_{mn}} \rightarrow 0$ for $m \neq n$ and $\alpha < 1$. Hence, $\mathbf{R}_{\mathbf{t}}$ asymptotically is a diagonal matrix and the periodogram based CRB asymptotically approaches the exact CRB. Note that by decreasing α the accuracy can be increased but by then it requires matrix inversion, losing the advantage of evaluating the CRB in the frequency domain. Also note that when $\alpha = 0$ the CRB evaluation in the frequency domain yields the same results as the evaluation in the time domain. As seen in the simulation of Figure 2.2 and Figure 2.3, even when the data length is very small (e.g. $N = 32$), the asymptotic CRB is still close to the true CRB, and when data length is increased, the asymptotic CRB is very closer to the true CRB.

2.3.2 Asymptotic Expressions for CRB for the Jakes' Model

Using the periodogram based CRB evaluation discussed in the previous section, how asymptotic expressions for CRB can be evaluated will be shown in here. These expressions are useful because they provide direct insight into how bounds vary with parameters such as number of data points, the Doppler frequency or the SNR. We consider the Jakes' model in wireless communication and assume that the time spanned by the length of data samples is much larger than the correlation time of the multiplicative noise. In this case, the power spectrum associated with the Jake's model covariance function can be written at discrete frequency points as [68]

$$P(\omega_k) = \begin{cases} \frac{2\sigma_a^2}{\omega_d \sqrt{1 - (\omega_k - \omega_0)/\omega_d)^2}} + \sigma_v^2 & |\omega_k - \omega_0| < \omega_d \\ 0 & \text{elsewhere} \end{cases} \quad (2.3.28)$$

where $\omega_0 = 2\pi f_0$ and $\omega_d = 2\pi f_d$. Substituting (2.3.28) into (2.3.25) (2.3.26) and (2.3.27), the FIM entries can be expressed as

$$\mathbf{J}_{\omega_0, \omega_0} \approx \frac{2/3\kappa\beta N^3}{\frac{2\beta}{\omega_d} + 1} + \sum_{k=0}^{N-1} \left(\frac{2\beta\omega_k(1 - (\omega_k/\omega_d)^2)^{-1}}{2\beta\omega_d^2 + \omega_d^2(1 - (\omega_k/\omega_d)^2)^{1/2}} \right)^2 \quad (2.3.29)$$

$$\mathbf{J}_{\omega_0, \phi} \approx \frac{\kappa\beta N^2}{\frac{2\beta}{\omega_d} + 1} \quad (2.3.30)$$

$$\mathbf{J}_{\phi, \phi} \approx \frac{2\kappa\beta N}{\frac{2\beta}{\omega_d} + 1} \quad (2.3.31)$$

Recall that $\kappa = \mu^2/\sigma_a^2$ and $\beta = \sigma_a^2/\sigma_v^2$. Therefore, the large sample CRB for frequency estimation is obtained as

$$\text{CRB}(\omega_0) = \frac{6}{N^3\xi + 6\zeta} \quad (2.3.32)$$

where $\xi = \frac{\kappa\beta}{\frac{2\beta}{\omega_d} + 1}$ $\zeta = \sum_{k=0}^{N-1} \left(\frac{2\beta\omega_k(1 - (\omega_k/\omega_d)^2)^{-1}}{2\beta\omega_d^2 + \omega_d^2(1 - (\omega_k/\omega_d)^2)^{1/2}} \right)^2$. Eq. (2.3.32) is a closed form expression for the CRB at large samples related to Doppler frequency, data length, and the SNR. It is worth noting that the large sample (asymptotic) CRB can be obtained by approximating the average periodogram by the power spectrum in the continuous form and subsequently using the Whittle's formula [29] [48]. However, the approach used here is direct and more appropriate as the discussed frequency estimate is obtained using discrete data.

It can be seen that, when $\beta = 0, \kappa \neq 0$, Eq. (2.3.32) provides the CRB in additive white Gaussian noise. When $\kappa = 0, \beta \neq 0$, the CRB is entirely determined by (Rayleigh fading) ζ , and the decay rate of the CRB with Doppler spread is in the order of ω_d^{-4} . Furthermore, when $\beta \gg 1$, ξ is simplified to $\xi = \frac{\kappa\omega_d}{2}$, while ζ can be written as $\zeta = \sum_{k=0}^{N-1} \frac{\omega_k^2}{\omega_d^4(1 - (\omega_k/\omega_d)^2)}$.

In this case, the CRB is only determined by ω_d and κ , and not by SNR, which is known as the floor effect of the fading channels.

Note that under the extremely fast fading condition for Rayleigh fading channel, the multiplicative noise process $a(n)$ can be seen as a case of completely uncorrelated process, i.e., $r_a(m) = \sigma_a^2 \delta_m$, where δ_m is the Kronecker Delta function, which is expressed as

$$\delta_m = \begin{cases} 1, & \text{if } m = 0 \\ 0, & \text{if } m \neq 0 \end{cases}$$

so that $S'_a(\omega_k, \omega_0) = 0$, $\zeta = 0$. Thus CRB becomes infinite, that means in the presence of temporarily white multiplicative noise, no parameters are identifiable. The CRB in the time invariant (slow) fading channel was derived in [27] as

$$\text{CRB}(\omega_0) \approx \frac{6}{\beta(1 + \kappa)N^3} \quad (2.3.33)$$

The CRB in the Rician fast fading channel when κ is not too small was approximately derived in [27] as following

$$\begin{aligned} \text{CRB}(\omega_0) &\approx \frac{2(\kappa + 1 + \rho S_a(\omega_0))}{\beta(\kappa + 1)\kappa N^3} \\ &\approx \frac{2}{\beta(\kappa + 1)N^3} + \frac{2S_a(\omega_0)}{\kappa N^3} \end{aligned} \quad (2.3.34)$$

Eq.(2.3.34) is obtained approximately, and we can see that when $\beta \rightarrow +\infty$, (2.3.34) has the same floor effect with (2.3.32). However, (2.3.32) also can be used in Rayleigh fading and is a general expression for fast fading channels.

In Figure 2.2, we show that for the multiplicative noise with zero mean ($\kappa = 0$) (Rayleigh fading), the CRB increases with the Doppler frequency monotonically. While in Figure 2.3, $\kappa \neq 0$, (Rician fading), the CRB first increase with Doppler frequency at small values, while beyond that, the CRB decreases with Doppler frequency. The dashed lines are the exact CRB and the solid lines represent the large sample asymptotic expressions obtained from (2.3.32). It can be seen that the asymptotic CRB meet the exact CRB only at high Doppler frequency and large data samples. This verifies that the periodogram based CRB expressed in (2.3.32) is an asymptotic result mostly suitable for fast fading channels. Also notice that in Figure 2.3, the CRB is lower than in Figure 2.2, this is because the SNR is increased in Rician channels due to the presence of μ . Figure 2.4 demonstrates the floor effect of the CRB when $\beta \gg 1$ in the presence of fading channels. The floor effect of Rician channels has been discussed in the literature [27] [29]. But here we emphasize that the floor effect is caused by the multiplicative noise, and would be present even in Rayleigh channels.

Figure 2.5 demonstrates the asymptotic CRB in the presence of multiplicative noise with respect to the data length N and different values of f_d . It can be seen that when $f_d = 0.001$, the fading effect is almost negligible, and the channel is similar to an additive white Gaussian noise channel. For large f_d , for example, $f_d = 0.011$, the fading effect is determined by the data length N , ξ and ζ as shown in (2.3.32). In this case, it is seen that when N is smaller, the decay rate of CRB is around N^{-3} , and the channel behaves like a slowly fading channel. With increasing of N , the fading relatively becomes fast, and the decay rate of CRB is around N^{-1} which is mainly determined by ζ in (2.3.32). In this situation, the channel can be seen as a Rayleigh fast fading channel.

When N is very large, the decay rate of the CRB is around N^{-3} as seen in Figure 2.5. This is also confirmed by (2.3.32) where ξ dominates and the channel behaves as a Rician Fast fading channel. Thus knowing N , ξ and ζ , and using (2.3.32) we can determine whether the channel behaves as a slow fading, Rayleigh fast fading or Rician fast fading channel. That is the channel can be easily characterized with the use of (2.3.32). (See also the results in Table 4.1 in Chapter 4.)

Figure 2.2: The periodogram based asymptotic CRB and exact CRB with respect to f_d . The results shown are for different values of N . $N = 32, 64, 128, 256$ from the top plot to the bottom plot, and $\kappa = 0$. $\beta = 10\text{dB}$

Figure 2.3: The periodogram based asymptotic CRB and exact CRB with respect to f_d . The results shown are for different values of N . $N = 32, 64, 128, 256$ from the top plot to the bottom plot and $\kappa = 4$. $\beta = 10\text{dB}$

Figure 2.4: The floor effect of the periodogram based asymptotic CRB and the exact CRB for different values of κ . $\kappa = 0, 0.1, 0.2, 0.3$ from top to bottom, $N = 128$, $f_d = 0.3$

Figure 2.5: The asymptotic CRB with respect to N , The results shown are for different values of $f_d = 0.001, 0.003, 0.005, 0.011$ from top to bottom and $\kappa = 100$. $\beta = 20\text{dB}$.

2.4 Closed Form Expression for CRB in the Presence of Multiplicative Noise and Additive Colored Noise

So far in the discussion we have considered the additive noise as a white process. Through this discussion, the asymptotic CRB expressions for frequency estimation in multiplicative noise and additive white noise has been obtained. In this section, we seek asymptotic CRB expressions in the presence of multiplicative noise and additive colored noise. Without loss of generality, suppose that the colored noise $v(n)$ can be modeled as an order p autoregressive (AR) process, which is expressed via the AR coefficients a_k in the following

manner.

$$v(n) = - \sum_{k=1}^p a_k v(n-k) + e(n) \quad (2.4.35)$$

Here $e(n)$ is a white Gaussian noise process with variance σ^2 . We define the SNR = $(\mu^2 + \sigma_a^2)/\sigma^2$, where σ_a^2 is the variance of the multiplicative noise defined previously. Let us first assume that only the colored noise is present (i.e. $\sigma_a^2 = 0$). Then the inversion of the data covariance matrix $\mathbf{R}_x$ is given by [53]

$$\mathbf{R}_x^{-1} = \frac{1}{\sigma^2} (\mathbf{A}_1 \mathbf{A}_1^H - \mathbf{A}_2 \mathbf{A}_2^H) \quad (2.4.36)$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are N by N lower triangular Toeplitz matrices given by

$$[\mathbf{A}_1]_{ij} = \begin{cases} 1, & i = j \\ a_{i-j}, & i > j \\ 0, & i < j \end{cases}$$

$$[\mathbf{A}_2]_{ij} = \begin{cases} 0, & i < j \\ a_{N-i+j}^*, & i \geq j \end{cases}$$

where $a_k = 0$ for $k > p$. The form in (2.4.36) is useful to calculate the exact CRB in the presence of AR colored noise. Substituting (2.4.36) into (2.2.2), and performing the matrix expansion, the FIM entries in the presence of the AR colored noise are obtained as

$$\mathbf{J}_{\mu,\mu} \approx \frac{2}{\sigma^2} |A(e^{j\omega_0})|^2 (N-3)$$

$$\mathbf{J}_{\omega_0, \omega_0} \approx \frac{2\mu^2}{\sigma^2} (|A(e^{j\omega_0})|^2 \sum_{n=0}^{N-1} (n-2)^2 + (N-1)^2)$$

$$\mathbf{J}_{\phi, \phi} \approx \frac{2\mu^2}{\sigma^2} |A(e^{j\omega_0})|^2 (N-1)$$

$$\mathbf{J}_{\omega_0, \phi} = \mathbf{J}_{\phi, \omega_0} \approx \frac{2\mu^2}{\sigma^2} |A(e^{j\omega_0})|^2 \sum_{n=0}^{N-1} n$$

$$\mathbf{J}_{\mu, \omega_0} = \mathbf{J}_{\omega_0, \mu} = 0$$

$$\mathbf{J}_{\mu, \phi} = \mathbf{J}_{\phi, \mu} = 0$$

where $|A(e^{j\omega_0})|^{-2}$ is the normalized spectrum of the AR colored noise and $A(e^{j\omega_0}) = 1 + \sum_{k=1}^p a_k e^{-jk\omega_0}$, ($k = 1, 2 \dots p$).

It can be easily shown that if the following condition is satisfied

$$|A(e^{j\omega_0})|^{-2} \ll N/3 \quad (2.4.37)$$

the second term in the right hand side of equation for $\mathbf{J}_{\omega_0, \omega_0}$ is small and can be negligible.

This condition would be satisfied for large N . Then, after the matrix inversion we obtain the asymptotic CRB for frequency estimation in AR colored noise as

$$\text{CRB}(\omega_0) = \frac{6\sigma^2 |A(e^{j\omega_0})|^{-2}}{\mu^2 N(N^2 - 1)} \quad (2.4.38)$$

This is in accordance with the asymptotic CRB for short data length as discussed in [53]. Further investigation has revealed that for other colored noise, such as MA colored noise, the results are entirely similar as that of the AR colored noise, provided that the normalized spectrum of the colored noise are much smaller than data length, i.e. the

condition in (2.4.37) is satisfied. Hence we are led to the conjecture that the CRB in colored noise can be obtained by modifying the variance of white noise to accommodate the true AR spectral noise density at the sinusoid frequency. Following the above, the obvious modification to (2.3.32) for the case of colored AR noise is given by replacing β by $\beta = \frac{\sigma_a^2 |A(e^{j\omega_0})|^2}{\sigma^2}$, and making $\kappa\beta = \frac{\mu^2 |A(e^{j\omega_0})|^2}{\sigma^2}$ while keeping κ unchanged. The CRB in the time invariant (slow) fading channel in additive colored noise then is given by

$$\text{CRB}(\omega_0) \approx \frac{6|A(e^{j\omega_0})|^{-2}}{\beta(1+\kappa)N^3} \quad (2.4.39)$$

Note that the exact bound can also be obtained from the FIM, where the mean vector is as same as what appeared in (2.2.4). Assuming the multiplicative noise and additive colored noise are independent, the data covariance matrix can be expressed as

$$\mathbf{R}_x = \sigma_a^2 \mathbf{\Lambda} \mathbf{R}_a \mathbf{\Lambda}^{-1} + \sigma_v^2 \mathbf{R}_v, \quad (2.4.40)$$

where $\sigma_v^2 \mathbf{R}_v$ is the covariance matrix of the colored noise. Eq. (2.4.40) can be used to evaluate the exact CRB. Figure 2.6 compares the exact CRB and asymptotic CRB expression in multiplicative noise and additive first order AR noise. In Figure 2.6, CRB versus frequency offset is shown with the data samples N varying from 32, 64, 128 and 256 from the top plot to the bottom plot. The AR coefficient is set as $a_1 = 0.9e^{j0.6\pi}$. It can be seen that the asymptotic CRB is close to the exact CRB when N is large. It is noted that the AR coefficient is chosen as complex valued as this will allow a more general non-symmetric spectrum for the AR process.

Notice that the CRB in the presence of additive colored noise varies with the frequency, although in the presence of additive white noise, it is independent of the frequency. We

can also see from Figure 2.6 that the the CRB peaks at a frequency offset corresponding to the phase of the AR coefficient a_1 , a fact due to the spectral characteristics of the colored noise.

Figure 2.6: The asymptotic CRB and exact CRB in the presence of first order AR colored noise and multiplicative noise for different values of N , $N = 32, 64, 128, 256$. SNR = 10dB. Doppler frequency $f_d = 0.1$. $\kappa = 1$

2.5 Conclusions

Computation of the CRB in the presence of multiplicative noise has been addressed in detail in this chapter. It is noted that under high SNR conditions, and in certain multiplicative noise models (e.g. Jakes' model), the covariance matrix involved in CRB evaluation tends to be singular which makes the evaluation of the CRB numerically unstable. In this chapter, we propose to use CMT technique when the covariance matrix of the signal is singular so as to improve the numerical stability. This technique evaluates the CRB via

time domain and is robust and accurate when the number of data samples is small.

For the CRB evaluation when the number of data samples are large, we propose a computationally simple, DFT based, frequency domain CRB evaluation method in conjunction with the CMT technique. In this approach, the CRB has been derived using the periodogram of the data. It is noted that this technique is accurate under the condition that the time spanned by the data record length is larger than the correlation time of the multiplicative noise. Finally, large sample approximations to the CRB for the Jakes' model is given and a general expression for an asymptotic CRB in the presence of multiplicative noise and colored additive noise is provided.

Chapter 3

Frequency Estimation via Multiple Lags of Autocorrelation in the Presence of Broadband Colored Noise

3.1 Introduction

Estimating the frequency of a single sinusoid in additive white Gaussian noise (AWGN) arises in many signal processing applications [56]. The maximum likelihood estimate (MLE) working on the principal of locating the peak of the periodogram, is known as an excellent frequency estimator especially at low signal noise ratio (SNR). However, the computation burden involved in the peak search is quite large and often sub-optimal

methods with a low computational load are preferred. A simple algorithm for frequency estimation of a sinusoid is the correlation based algorithm, originally proposed by Benham et. al. in the application of Doppler spectrum estimation [69]. Since then, it is widely used in radar and sonar applications. One of the important parameters required in this estimator is the lag value for computation of correlations. Using the first lag correlation value, an optimal window required to obtain the CRB at relatively high SNR, can be developed through least square processing of the first order phase differences between signal samples [14]. Though this Kay's estimator achieves CRB, its threshold SNR level (i. e. the SNR requires to achieve CRB) is quite high. Besides, the threshold SNR of this estimator increases with the number of signal samples N . This anomalous behavior was also reported in [13]. Techniques using lag values other than one are also reported in the literature. The optimal lag value which attains the best performance is given by $2N/3$ [30, 70], however, the performance improvement is obtained at the expense of the introduction of a phase ambiguity. A simple technique to resolve the phase ambiguity was introduced in [12] [71], but it too has high SNR threshold characteristics. Frequency estimation from proper sets of correlations was investigated in [15]. By invoking the Chinese remainder theorem (CRT) and using multiple correlation lags, the accuracy of the frequency estimator is improved, however, it still suffers from a high threshold level. As such, at low SNR, the correlation based methods do not perform as well as the MLE.

In this chapter, phase unwrapping error in frequency estimation using correlation based techniques is investigated via the analysis of probability of outliers. It is shown that in the presence of AWGN the DFT coarse estimation is most suitable for phase unwrapping. Based on this, a frequency estimator using multiple lags of correlation is proposed. It

is shown that the estimate is efficient even for short length of data samples and has a low SNR threshold. Simulations shows that the estimator outperform all the correlation based estimators in the presence of AWGN.

The chapter also investigates correlation based frequency estimation in the presence of colored noise. Estimation of parameters in autoregressive (AR) colored noise was discussed in [72]. It was shown that the MLE of parameters using this model is approximately efficient. But the method leads to an involved computational problem as discussed in [24], and thus the nonlinear least squares (NLS) technique was proposed as a simple frequency estimate which achieves the CRB, asymptotically. In this chapter, we model the broadband colored noise as an moving average (MA) process in order to obtain an optimal estimator. It is shown that the variance of the developed estimator asymptotically approaches CRB. The estimator is also shown to be unbiased for low and medium SNR, whereas the estimation using the DFT based peak search is biased and has a high SNR threshold level.

The chapter is organized as follows. In Section 3.2 we present the threshold analysis of the correlation based frequency estimation methods. Based on the error analysis, in Section 3.3, frequency estimation from multiple lags of correlations with DFT phase unwrapping is proposed. The optimal linear weights are derived which can be used to achieve CRB at any length of data samples. Several alternative windows are also compared with the optimal window for reduced computational complexity. In Section 3.4, the performance of the proposed method in the presence of MA colored noise is considered. It is shown that the estimator achieves the CRB asymptotically while maintaining a low SNR threshold

level. Some final remarks conclude the chapter.

The original contribution of the work reported in this chapter can be summarized as follows. (i) Giving the detailed SNR threshold analysis of the correlation based methods and showing that the DFT coarse estimation is most suitable for phase unwrapping in the presence of AWGN (Section 3.2); (ii) Relating the MLE to the use of multiple lags of correlation for frequency estimation. Subsequent proposal of an efficient frequency estimation algorithm which uses second half of the correlation lags with optimal weighting (Section 3.3); (iii) Derivation of the asymptotic CRB in the presence of colored noise and using it to obtain SNR threshold for DFT peak search. Demonstrating that better estimation to DFT fine peak search is possible with the use of a suitable phase un-wrapping technique (Section 3.4).

3.2 Threshold Effects of the Correlation Based Frequency Estimation

Consider the following signal received in an AWGN channel

$$x(n) = Ae^{j\phi}e^{j\omega_0 n} + v(n), \quad n = 1, 2, \dots, N-1. \quad (3.2.1)$$

where $v(n)$ is a white Gaussian process with variance σ^2 , ϕ is uniformly distributed in $[0, 2\pi)$, ω_0 is the angular frequency, $\omega_0 \in (0, \pi]$ [see also the discussion in Section 1.3 for the model]. A is the signal amplitude and N is the number of data samples observed.

SNR is defined as A^2/σ^2 . Using the signal model in (3.2.1), the frequency estimation with a single lag of autocorrelation is expressed as

$$\hat{\omega}_m = \frac{1}{m} \arg\{r(m)\} \quad (3.2.2)$$

where $\hat{\omega}_m$ is the frequency estimate from the m th lag of correlation, and $\arg\{\cdot\}$ represents the argument of a complex number. $r(m)$ is the correlation of the received signal which can be expressed as

$$r(m) = \frac{1}{N-m} \sum_{n=1}^{N-m} d(n)x(n+m)x^*(n) \quad (3.2.3)$$

and $d(n)$ is a data windowing function. Note that when $m > 1$, the estimator in (3.2.2) needs phase unwrapping, and when SNR decreases to a certain value, the performance of this estimator deteriorates rapidly, this is the threshold effect at low SNR. In the following, we investigate this low SNR threshold effect of a number of commonly used frequency estimators.

3.2.1 Kay's Frequency Estimator

For the estimate with lag $m = 1$, no phase unwrapping is needed, and an optimal weight is required to achieve the CRB [14]. Here we will investigate the low SNR threshold effect of this estimator. First, rewrite equation (3.2.1) as

$$x(n) = (A + s(n))e^{j\phi}e^{j\omega_0 n} \quad (3.2.4)$$

where now, $s(n)$ is a complex zero mean random variable with $v(n) = s(n)e^{j(\omega_0 n + \phi)}$, and $s(n)$ have the same distribution with $v(n)$. Then the frequency estimate can be written as:

$$\begin{aligned}\hat{\omega}_1 &= \arg \sum_{n=1}^{N-1} d(n)(A + s(n+1))(A + s^*(n))e^{j\omega_0} \\ &= \omega_0 + \arg \sum_{n=1}^{N-1} d(n)(A + s(n+1))(A + s^*(n)).\end{aligned}\quad (3.2.5)$$

Now, let $s(n) = A(z_1(n) + jz_2(n))$. Then $z_1(n)$ and $z_2(n)$ are zero mean, independent, identically distributed, each with variance σ^2/A^2 . Thus, the error of the estimator can be expressed as:

$$\omega - \omega_0 = \varepsilon = \arg(1 + X + jY) = \arctan \frac{Y}{1 + X} \approx Y \quad (3.2.6)$$

with $X = \sum_{n=1}^{N-1} d(n)(z_1(n+1) + z_1(n) + z_1(n)z_1(n+1) + z_2(n)z_2(n+1))$ and

$Y = \sum_{n=1}^{N-1} d(n)(z_2(n+1) - z_2(n) - z_1(n)z_2(n+1) + z_2(n)z_1(n+1))$. In this case $d(n)$, the optimal weighting to achieve CRB is given by

$$d(n) = \frac{6[(N-1) + (N-2)n - n^2]}{N(N^2 - 1)} \quad n = 0, 1, \dots, N-2, \quad (3.2.7)$$

It can be proven that Y is Gaussian distributed with zero mean and its variance is given by [73]

$$\sigma_Y^2 = E[Y^2] = \frac{6}{\text{SNR}N(N^2 - 1)} \left\{ 1 + \frac{N^2 + 1}{10\text{SNR}} \right\} \quad (3.2.8)$$

The expression in (3.2.8) assumes that no outliers have occurred and thus is true only at high SNR. In order to obtain a good approximation to the variance at entire range

of SNR, we note that at extremely low SNR, when the outlier occurs, the frequency estimate is uniformly distributed in the range of $[-\pi, \pi]$, and thus the variance will be $\pi^2/3$. Hence, we may identify two mutually exclusive events, one being the event that given the measurement sequence an outlier occurs and the other being the event that no outlier occurs. The frequency error variance may then be expressed as

$$E(\hat{\omega} - \omega)^2 = (1 - P_r)E[Y^2] + P_r \frac{\pi^2}{3} \quad (3.2.9)$$

where P_r is the probability of the occurrence of an outlier which as defined by

$$P_r = P\{|Y| > \alpha\} = 2Q\left(\frac{\alpha}{\sqrt{2}\sigma_Y}\right) \quad (3.2.10)$$

here α is selected as $1/2$, and Q is the complementary error function as defined in [60]. The value of α is chosen because when $\alpha > 1/2$, the non-linearity in the $\arg()$ function becomes prominent. The choice of $\alpha = 0.5$ would be justified from the following simulations. In Figure 3.1, the analytical results obtained from (3.2.9) and the simulations obtained via 1000 iterations for the estimator with parabolic window, $f_0 = 0.124$ and $\alpha = 0.5$ are compared. It can be seen that the theoretical analysis meets the simulation results very well.

Figure 3.1: The SNR threshold of Kay's weighted linear predictor estimator. The normalized frequency $f_0 = 0.124$, $N = 6, 12, 24$ (from top to bottom), $m = 1$

3.2.2 Correlation Based Estimator with Optimal Lag

In general, if $\omega_0 < (2\pi m)^{-1}$, the estimate in (3.2.2) does not need phase unwrapping. In this case, using a rectangular window, the variance of the estimator in AWGN is shown to be [30, 70]

$$E(\varepsilon_m^2) = \frac{1}{m^2} \frac{\min(m, N-m)}{(N-m)^2 \text{SNR}} + \frac{1}{2(N-m)\text{SNR}^2} \quad (3.2.11)$$

where $\min(x, y)$ denotes the minimum of x and y . It can be shown that for a given N , when SNR is very large, the second term of (3.2.11) vanishes, and the variance of the estimator in AWGN achieves a minimum value for $m = N/3$ or $m = 2N/3$. In the thesis, $m = 2N/3$ will be used subsequently. Figure 3.2 is similar with Figure 3.1, but a rectangular window has been used with the optimal lag value $m = 2N/3$, to avoid phase

unwrapping, the frequency is selected as $f_0 = 0.001$ and α is chosen as 0.5. It is evident from Figure 3.2 that when SNR is very low, the probability of an outlier will be almost one, as such, the variance expression from (3.2.9) will be a constant. At high SNR, the estimator achieve the CRB and no outlier occurs.

Figure 3.2: The threshold of single lag correlation based estimator with optimal lag, $m = 2N/3$. The normalized frequency $f_0 = 0.001$, $N = 32, 64, 128$ (from top to bottom)

3.2.3 Correlation Based Estimator with Down Sampling

To avoid the phase unwrapping problem in (3.2.2), computational efficient, down sampling based algorithm is proposed in [74]. The down sampling method is briefly described in the following:

1. Estimate the frequency using the correlation based method with lag $m = 1$
2. Use the estimated frequency to demodulate the signal and down sampled by a factor 2

3. Obtain a new estimate using down sampled signal via the correlation based method at lag $m = 1$.
4. Repeat steps 2)-3) until signal length is 2.

The mean square error resulting from down sampling method can be evaluated using (3.2.11). To do this, we note that down sampling by $N/2$ increases the SNR by a factor of $N/2$. Using this argument, we get

$$E(\varepsilon^2) = \frac{4}{N^2} \left(\frac{2}{N\text{SNR}} + \frac{2}{N^2\text{SNR}^2} \right) \quad (3.2.12)$$

Thus when $\text{SNR} \gg 1/N$, we get

$$E(\varepsilon^2) = \frac{8}{N^3\text{SNR}} = \frac{6}{N^3\text{SNR}} \times \frac{4}{3} = \text{CRB} + 1.25\text{dB} \quad (3.2.13)$$

Figure 3.3 shows the performance of the down sampling method compared with simulations, with $f = 0.124$, $\alpha = 0.5$, $N = 32, 128$. Compared with Figure 3.2 and Figure 3.1, it can be seen that the down sampling method using the lag value of $m = 1$ improves the SNR threshold performance over the Kay's method, furthermore, the full frequency range from $[-\pi, \pi]$ can be utilized without the phase ambiguity, and the variance is above the CRB by 1.25dB.

Figure 3.3: The SNR threshold of single lag correlation based estimator with down sampling. The normalized frequency $f_0 = 0.124$, $N = 32, 128$ (from top to bottom)

3.2.4 Low SNR Threshold Effect of MLE in AWGN

Rife and Boorstyn described an algorithm for single tone MLE frequency estimation. The algorithm is comprised of two stages, a coarse search followed by a fine search. The coarse search is generally performed by passing the received signal through a DFT routine. The coarse frequency estimate is then taken to be the bin frequency corresponding to the maximum of the magnitude of the output. The fine search then uses the coarse frequency estimate as its initial condition. The analysis of Rife and Boorstyn presented a means of computing the frequency error variance below threshold. This essentially reduces to the problem of determining the probability of an outlier. It is shown in [3] that the outliers occurs only at the stage of the coarse search, since the fine search serves merely to provide

a more accurate determination of the MLE in the immediate neighborhood of the coarse frequency estimate. The frequency error variance may then be expressed as

$$E(\hat{\omega} - \omega_0)^2 = (1 - P_r) \frac{6}{\text{SNR}N(N^2 - 1)} + P_r \frac{\pi^2}{3} \quad (3.2.14)$$

The analytical expressions for P_r was derived and the theoretical performance curves agree well with simulation results. The SNR threshold is approximately given by $\text{SNR} = 24/N$. Comparing the results of [3] with the results of Figure 3.1-Figure 3.3, we note that the DFT peak search procedure has superior performance over the correlation based method in terms of the low SNR threshold. Therefore, in the next section, we will use the DFT coarse search as the initial condition to solve the phase unwrapping problem of correlation based estimators.

3.2.5 Performance of the Correlation Based Estimator with Phase Unwrapping

It is known that the correlation based estimator will achieve the minimum variance when the lag value $m = 2/3N$ [30, 70]. But when the lag values higher than 1 are used, phase unwrapping is required to estimate in the frequency range $[-\pi, \pi]$. Intuitively, one may use the first lag estimate to correct the phase ambiguity [12] [71], but as can be seen from Figure 3.1, the SNR threshold level deteriorates with increasing value of the data length N . In [15], Chinese remainder theorem (CRT) is used to resolve phase ambiguity. however, the error caused by phase unwrapping is large which finally leads to a high

SNR threshold. In [75], two prime lags are used to resolve the phase ambiguity, it has been shown that the phase unwrapping solution is unique and the probability of the error drops very fast from medium SNR to high SNR [75], but it still has a higher threshold level compared with the MLE. As such, we shall use the DFT coarse search as an initial condition to perform phase unwrapping for the correlation based estimators. Figure 3.4 shows a final comparison between different phase unwrapping techniques. The lag value $m = 2N/3$ is used to achieve the optimal performance. We can see that DFT phase unwrapping has the lowest SNR threshold level compared with other phase unwrapping techniques. However, also note that even at high SNR, frequency estimators in Figure 3.4 do not achieve CRB.

Considering the threshold effects and efficiency of the estimates, and to achieve CRB, in the next section we propose an estimator based on correlations which uses multiple lags in combination with DFT phase unwrapping. It will be shown that the SNR threshold and efficiency of this estimate with multiple lags combining with the DFT phase unwrapping will have same performance compared with MLE in AWGN, but has a low computational load.

Figure 3.4: The SNR threshold of single lag correlation based method with different phase unwrapping. The normalized frequency $f_0 = 0.124$, $N = 24, 64$ (from top to bottom)

3.3 Frequency Estimation from Multiple Lags of Correlation

3.3.1 Approximate Relation to the MLE

The optimal solution of the MLE frequency estimate amounts to locating the peak of the periodogram. That is

$$\hat{\omega}_0 = \operatorname{argmax}\{I(\omega)\} \quad (3.3.15)$$

where $I(\omega)$ is expressed as

$$I(\omega) = \left| \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right|^2 = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(n) x^*(m) e^{-j\omega(n-m)} \quad (3.3.16)$$

To find the maximum of the periodogram, we take the derivative of (3.3.16) with respect to ω and set it to zero to obtain

$$\sum_{n=0}^{N-1} \sum_{m=0}^{N-1} (n-m) x(n) x^*(m) e^{-j\hat{\omega}_0(n-m)} = 0 \quad (3.3.17)$$

rearranging the terms, we can get

$$\sum_{m=0}^{N-1} m(N-m) r(m) e^{-j\hat{\omega}_0 m} = \sum_{m=0}^{N-1} m(N-m) \text{Im}\{r(m) e^{-j\hat{\omega}_0 m}\} = 0 \quad (3.3.18)$$

where $\text{Im}\{\cdot\}$ is the imaginary part of a complex number, $r(m)$ is the autocorrelation function as defined in (3.2.3). Let $r(m)$ be expressed as

$$r(m) = A e^{j(m\omega_0 + \varepsilon_m)}, \quad (3.3.19)$$

then

$$\begin{aligned} \text{Im}\{r(m) e^{-j\hat{\omega}_0 m}\} &= A \sin(\varepsilon_m + m\omega_0 - m\hat{\omega}_0) \\ &\approx A \{\arg(r(m)) - m\hat{\omega}_0\} \end{aligned} \quad (3.3.20)$$

In the above approximation, it is assumed that the SNR is high and thus $\hat{\omega}_0 \approx \omega_0$ and ε_m is very small. Inserting (3.3.20) into (3.3.18), a weighted autocorrelation based frequency

estimate can be obtained as

$$\hat{\omega}_0 = \frac{\sum_{m=0}^{N-1} (N-m)m \arg\{r(m)\}}{\sum_{m=0}^{N-1} (N-m)m^2} \quad (3.3.21)$$

Thus a weighting window suitable for the autocorrelation based estimator to achieve performance similar to the MLE is given by

$$d_m = \frac{(N-m)m^2}{\sum_{m=0}^{N-1} (N-m)m^2} = \frac{12(N-m)m^2}{N^2(N^2-1)} \quad (3.3.22)$$

A similar estimator by examining the periodogram was also discussed in [21]. But the weighting window obtained in [21] is not optimal and the variance of the estimator is above the CRB. We note that (3.3.22) approximately is a parabolic window, especially for large N . In the next section, we will show that the parabolic window is optimal and the variance of the estimator is equal to the CRB. As such, it can be stated that the estimator with (3.3.22) obtained by examining the periodogram is approximately efficient for large N , while the estimator proposed with the optimal window in the next section is efficient for any length of data samples.

3.3.2 Optimal Window Determination

With proper phase unwrapping, the frequency can be estimated unbiasedly from every correlation lag. Assume that the frequency estimated after correct phase unwrapping is $\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_K, K \leq N-1$. Restricts the estimate to be a linear combination of the

estimated frequencies, and a weighted estimate is then defined as

$$\widehat{\omega}_0 = \sum_{m=1}^K d_m \hat{\omega}_m, \quad m = 1, 2, \dots, K, \quad K \leq N - 1. \quad (3.3.23)$$

where d_m 's are coefficients yet to be determined. The optimal weighting window can be determined as [56]

$$\mathbf{d} = \frac{\mathbf{R}^{-1} \mathbf{1}}{\mathbf{1}^T \mathbf{R}^{-1} \mathbf{1}} \quad (3.3.24)$$

where $\mathbf{d} = [d_1, d_2, \dots, d_K]^T$, $\mathbf{1} = [1, 1, \dots, 1]^T$ and $\mathbf{R}$ is the covariance matrix of the frequency estimates from different lags which was derived in [22] (for large SNR). The (m, l) th element of $\mathbf{R}$ is given by

$$R_{m,l} = \frac{\min(K - m, K - l, m, l)}{(K - m)m(K - l)l\text{SNR}} \quad (3.3.25)$$

Without loss of generality, we assume N to be odd, and note that $\mathbf{R}$ is rank deficient for $K \geq (N + 1)/2$. But due to the symmetrical property of $\mathbf{R}$, we may first obtain the half of the coefficients $d_m, m = 1, 2, \dots, (N - 1)/2$ by inverting $\mathbf{R}(m, l \leq (N - 1)/2)$, then the other half of the coefficients can be obtained symmetrically around $(N - 1)/2$. Using a polynomial curve fitting technique as given in [56], a_m can be shown to be exactly a parabolic window when $K = N - 1$, i.e.,

$$d_m = \frac{6m(N - m)}{N(N^2 - 1)} \quad m = 1, \dots, N - 1. \quad (3.3.26)$$

The variance for $K = N - 1$ becomes

$$E[(\omega_0 - \hat{\omega})^2] = \sum_{m=1}^{N-1} \frac{d_m}{m(N-m)} \sum_{l=1}^{N-1} \frac{d_l}{l(N-l)} R_{m,l} \quad (3.3.27)$$

$$E(\varepsilon^2) = \frac{36}{\text{SNR}N^2(N^2-1)^2} \sum_{m=1}^{N-1} \sum_{l=1}^{N-1} \min(N-m, N-l, m, l) \quad (3.3.28)$$

as it can be easily proven that

$$\sum_{m=1}^{N-1} \sum_{l=1}^{N-1} \min(N-m, N-l, m, l) = \frac{N(N^2-1)}{6} \quad (3.3.29)$$

we are finally led to

$$E(\varepsilon^2) = \frac{6}{\text{SNR}N(N^2-1)} \quad (3.3.30)$$

This is the CRB for frequency estimation in the presence of AWGN. In general, the window for an arbitrary set of lag values K can be obtained as

$$d_m = \frac{3m(2K+1-m)}{K(K+1)(2K+1)}, \quad m = 1, 2, \dots, K, K \leq N/2 \quad (3.3.31)$$

this is the same weighting window recursively derived in [15]. Considering the symmetry property of the weighting and computational complexity, we may only use the $K = (N-1)/2$ lag values to achieve the CRB, that is

$$d_m = \frac{12m(N-m)}{N(N^2-1)} \quad m = 1, \dots, (N-1)/2. \quad (3.3.32)$$

Substituting (3.3.32) into (3.3.27) and considering the symmetry property, we get

$$E(\varepsilon^2) = \frac{144}{\text{SNR}N^2(N^2 - 1)^2} \sum_{m=1}^{(N-1)/2} \sum_{l=1}^{(N-1)/2} \min(m, l) = \frac{6}{\text{SNR}N(N^2 - 1)} \quad (3.3.33)$$

Furthermore, it can be easily proven that the estimator with the following mirror coefficients of (3.3.32) will also achieve CRB

$$d_m = \frac{12m(N - m)}{N(N^2 - 1)} \quad m = \frac{N + 1}{2}, \frac{N + 3}{2}, \dots, N - 1. \quad (3.3.34)$$

This property will be quite useful for the frequency estimation in the presence of colored noise as will be shown in the next section. From (3.3.33), we can see that with $(N - 1)/2$ number of linear weights, the estimator achieves the CRB. For reasons of computational efficiency, we may use less number of weights and different weighting windows to obtain a suboptimal estimator, for example, a triangular window with a window length $K \leq N/2$ can be used as an approximation to the optimal window which is expressed as

$$d_m = \frac{2m}{K(K + 1)}, m = 1, 2 \dots K, K \leq N/2 \quad (3.3.35)$$

whereas a rectangular window is expressed as

$$d_m = \frac{1}{K}, m = 1, 2 \dots K, K \leq N/2 \quad (3.3.36)$$

As another example, in [21], a weighting window is derived approximately as

$$d_m = \frac{6m^2}{N(N + 1)(2N + 1)} \quad (3.3.37)$$

Figure 3.5 shows the efficiency (defined as variance/CRB in dB) of the use of various weighting window functions versus the window length. It is seen that $K = (N - 1)/2$ is required to achieve the CRB for the optimal window, and the triangular window is a very good approximation to the optimal window, while the rectangular window and Fitz's window [21] cause a large deviation from the CRB.

Figure 3.5: The efficiency of the estimator with optimal window, triangular window, rectangular window and Fitz's window. $N = 25, 50$, normalized frequency $f_0 = 0.124$, SNR = 10dB. The lags are selected as $m = 1, 2 \dots K$

3.3.3 Simulation Results for the Correlation Based Estimator in AWGN

Based on the discussion in the previous section, we propose a novel frequency estimator using an optimal window based on the mirror coefficients in (3.3.34). Simulation also confirms that the estimator with weighting window in (3.3.34) achieves CRB. At this

point, we note that estimator in (3.3.23), and thus both estimators using (3.3.32) and (3.3.34) need proper phase unwrapping. Based on the discussion in Section 3.2, we shall use the DFT as the initial coarse search to resolve phase unwrapping due to superior threshold characteristics of DFT coarse search. Note that the use of (3.3.32) and (3.3.34) will lead to the same SNR threshold. More importantly, we shall show in the next section that coefficients in (3.3.34) can also be used efficiently in the frequency estimation in colored noise. Figure 3.6 shows the simulation results for the mean square error (MSE) of the estimator using multiple correlations ($m = 1, 2, \dots, (N - 1)/2$) using different phase unwrapping techniques. 1000 realizations are used in the simulation of Figure 3.6 and subsequent Figures. It can be seen that the estimator with DFT phase unwrapping is efficient and has a lower SNR threshold compared to CRT in [15] and Crozies phase unwrapping methods in [12]. Figure 3.6 is similar to Figure 3.4, but use multiple lags for the frequency estimation. Figure 3.7 shows the simulation results for the MSE of the estimator obtained using the mirror coefficients in (3.3.34) together with the use of DFT phase unwrapping, and the CRB values are also plotted in dashed lines as a comparison. It is seen that the performance of this estimate considering the MSE and the SNR threshold is comparable to the MLE in AWGN. From Figure 3.7, we conclude that the MSE results using the mirror coefficients (3.3.34) with DFT phase unwrapping is as same as MLE results, and the use of the coefficients in (3.3.32) results in identical performance.

Figure 3.6: The Performance of the proposed estimator with different phase unwrapping techniques, $m = 1, 2, \dots, (N - 1)/2$, $f_0 = 0.124$, $N = 24, 64$ (from top to bottom plot)

Figure 3.7: The Performance of the proposed estimator with optimal window and DFT phase unwrapping as SNR varies, coefficients lags $m = (N + 1)/2, \dots, N - 1$. From top plot to bottom plot N varies as $N = 32, 64, 128, 256$, normalized frequency $f_0 = 0.124$

3.4 Frequency Estimation in Broadband Colored Noise

3.4.1 Performance Evaluation of Frequency Estimation in MA Colored Noise

Assume that the noise term in signal model (3.2.1) is broadband colored, i.e. the correlation of the noise process is time limited. Let the noise process be assumed as an MA process expressed as

$$v(n) = \sum_{k=1}^q b_k u(n-k) + u(n) \quad (3.4.38)$$

where $u(n)$ is a white Gaussian process with zero mean and variance σ^2 . The process $v(n)$ can be obtained by passing AWGN through a filter with a system function $B(z) = 1 + b_1 z^{-1} + \dots b_q z^{-q}$. The spectral density of $v(n)$ is

$$\psi(\omega) = \left| 1 + \sum_{k=1}^q b_k e^{j\omega k} \right|^2 \quad (3.4.39)$$

For the convenience of presentation, before we derive the variance of the frequency estimate in the presence the MA colored noise, define:

$$\mathbf{b} = [1 \ b_1 e^{-j\omega_0} \dots b_q e^{-jq\omega_0}]^T,$$

$$\mathbf{1} = [1 \ 1 \dots 1]^T,$$

$$\mathbf{u} = [u(n) \ u(n-1) \dots u(n-q)]^T, \text{ where } [\cdot]^T \text{ denoting transposition.}$$

Under the same modelling assumptions, we can show that the signal model in additive

MA colored noise is statistically equivalent to that in

$$x(n) = Ae^{j(\omega_0 n + \phi)} + v_1(n) = (Ae^{j\phi} + e(n))e^{j\omega_0 n} \quad (3.4.40)$$

which facilitates the analysis in the thesis. Here $e(n) = \mathbf{b}^T \mathbf{u}$, $e(n)e^{j\omega_0 n} = v_1(n)$, and $v_1(n)$ is expressed as

$$v_1(n) = u(n)e^{j\omega_0 n} + \sum_{k=1}^q b_k u(n-k)e^{j\omega_0(n-k)} \quad (3.4.41)$$

It is seen that the process $v_1(n)$ can be obtained by passing the AWGN $u(n)e^{j\omega_0 n}$ through a filter with a system function $B(z) = 1 + b_1 z^{-1} + \dots b_q z^{-q}$. Therefore $v_1(n)$ has the same statistics with $v(n)$, and $e(n)$ has the same statistics with $v(n)$. Then the estimated correlation function is given by

$$\begin{aligned} r(m) &= \frac{1}{N-m} \sum_{n=1}^{N-m} x(n+m)x^*(n) \\ &= \frac{1}{N-m} \sum_{n=1}^{N-m} (A^2 + Ae(n+m) + Ae^*(n) + e^*(n)e(n+m))e^{j\omega_0 m} \end{aligned} \quad (3.4.42)$$

The error of the frequency estimate from the m th lag is given by

$$\begin{aligned} \varepsilon_m &= \frac{1}{m} \arg \left\{ \sum_{n=1}^{N-m} (1 + t(n+m) + t^*(n) + t(n)t^*(n+m)) \right\} \\ &= \frac{1}{m} \arg \left\{ 1 + \frac{X_m}{N-m} + j \frac{Y_m}{N-m} \right\} \end{aligned} \quad (3.4.43)$$

where $t(n) = e(n)/A^2$, which is also a colored process, and $t(n) = t_1(n) + jt_2(n)$.

$$X_m = \sum_{n=1}^{N-m} t_1(n+m) + t_1(n) + t_1(n)t_1(n+m) + t_2(n)t_2(n+m)$$

$$Y_m = \sum_{n=1}^{N-m} t_2(n+m) - t_2(n) - t_1(n)t_2(n+m) + t_2(n)t_1(n+m).$$

Since $t_1(n)$ and $t_2(n)$ are both MA colored process, $E(\varepsilon_m) \neq 0$ when $m \leq q$. It is easily proven that when $m > q$, $E(X_m) = E(Y_m) = 0$. Note that $t_1(n) = \mathbf{b}^T \mathbf{w} / 2A^2$ and denote $t_1(n+m) = \mathbf{b}^T \mathbf{w} z^m / 2A^2$. We then arrive at

$$\varepsilon_m = \frac{Y_m}{m(N-m)} \quad \text{for } m > q \quad (3.4.44)$$

The autocorrelation of Y_m is required to proceed with the estimator performance evaluation.

$$\begin{aligned} E[Y_m Y_l] &= \frac{1}{2A^2} \sum_{n_1=1}^{N-m} \sum_{n_2=1}^{N-l} E(\mathbf{b}^T \mathbf{w}_1 z^m - \mathbf{b}^T \mathbf{w}_1)(\mathbf{b}^T \mathbf{w}_2 z^l - \mathbf{b}^T \mathbf{w}_2) \quad (3.4.45) \\ &= \frac{1}{A^2} \sum_{n_1=1}^{N-m} \sum_{n_2=1}^{N-l} E(\mathbf{b}^T (\mathbf{w}_1 z^m)(\mathbf{w}_2 z^l)^T \mathbf{b} - \mathbf{b}^T (\mathbf{w}_1 z^m) \mathbf{w}_2^T \mathbf{b}) \\ &= \frac{1}{A^2} \mathbf{b}^T \left(\sum_{n_1=1}^{N-m} \sum_{n_2=1}^{N-l} E((\mathbf{w}_1 z^m)(\mathbf{w}_2 z^l)^T - (\mathbf{w}_1 z^m) \mathbf{w}_2^T) \right) \mathbf{b} \end{aligned}$$

where $E[(\mathbf{w}_1 z^m)(\mathbf{w}_2 z^l)^T - (\mathbf{w}_1 z^m) \mathbf{w}_2^T]$ is a $q \times q$ matrix, which can be expressed as

$$E((\mathbf{w}_1 z^m)(\mathbf{w}_2 z^l)^T - (\mathbf{w}_1 z^m) \mathbf{w}_2^T) = \mathbf{D}_1 - \mathbf{D}_2 \quad (3.4.46)$$

with

$$\mathbf{D}_1 = \begin{pmatrix} \delta_{n_1+m, n_2+l} & \delta_{n_1+m, n_2+l-1} & \cdots & \delta_{n_1+m, n_2+l-q} \\ \delta_{n_1+m-1, n_2+l} & \delta_{n_1+m-1, n_2+l-1} & \cdots & \delta_{n_1+m-1, n_2+l-(q-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n_1+m-q, n_2+l} & \delta_{n_1+m-(q-1), n_2+l} & \cdots & \delta_{n_1+m-q, n_2+l-q} \end{pmatrix} \quad (3.4.47)$$

$$\mathbf{D}_2 = \begin{pmatrix} \delta_{n_1+m, n_2} & \delta_{n_1+m, n_2-1} & \cdots & \delta_{n_1+m, n_2-q} \\ \delta_{n_1+m-1, n_2} & \delta_{n_1+m-1, n_2-1} & \cdots & \delta_{n_1+m-1, n_2-(q-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n_1+m-q, n_2} & \delta_{n_1+m-(q-1), n_2} & \cdots & \delta_{n_1+m-q, n_2-q} \end{pmatrix} \quad (3.4.48)$$

Define $\mathbf{M} = \sum_{n_1=1}^{N-m} \sum_{n_2=1}^{N-l} (\mathbf{D}_1 - \mathbf{D}_2)$ and after some manipulation, $\mathbf{M}$ can be expressed as

$$[\mathbf{M}]_{ij} = \begin{cases} \min(N-m, m, N-l, l), & i = j \\ \min(N-m+t, m-t, N-l, l), & |i-j| = t, \end{cases} \quad (3.4.49)$$

where $t = 1, 2, \dots, q-1$. Then the autocorrelation of Y_m in the presence of MA colored noise is given by

$$E[Y_m Y_l] = \frac{1}{A^2} \mathbf{b}^T \mathbf{M} \mathbf{b} \quad (3.4.50)$$

In Section 3.3.2, we have derived the optimal window with K lags $K \leq (N-1)/2$, and the window is expressed in (3.3.31). However, in the presence of MA colored noise, to achieve the best linear unbiased estimator, (3.3.34) needs to be used as the optimal window. That is, we use the second half of the correlation lags. Note that if $(N+1)/2 > q$, which can be made satisfied in most cases by choosing N , the use of the second half of lags would

results in an unbiased frequency estimator. The corresponding variance is given by

$$E(\varepsilon^2) = \frac{144}{N^2(N^2 - 1)^2} \sum_{m=(N+1)/2}^{N-1} \sum_{l=(N+1)/2}^{N-1} E[Y_m Y_l] \quad (3.4.51)$$

$$= \frac{144}{A^2 N^2 (N^2 - 1)^2} \sum_{m=(N+1)/2}^{N-1} \sum_{l=(N+1)/2}^{N-1} \mathbf{b}^T \mathbf{M} \mathbf{b} \quad (3.4.52)$$

where $\mathbf{M}$ is the $q \times q$ square matrix defined in (3.4.49). Assume $N \gg q$, $\mathbf{M}$ approximately can be expressed as

$$\mathbf{M} = \min(N - m, m, N - l, l) \mathbf{E}. \quad (3.4.53)$$

where $\mathbf{E}$ is a square matrix with all its elements equal to 1, thus

$$\begin{aligned} E(\varepsilon^2) &\approx \frac{144}{\text{SNR} N^2 (N^2 - 1)^2} \sum_{m=(N+1)/2}^{N-1} \sum_{l=(N+1)/2}^{N-1} \min(N - m, N - l) \mathbf{1}^T \mathbf{b} \mathbf{b}^T \mathbf{1} \quad (3.4.54) \\ &= \frac{6}{\text{SNR} N (N^2 - 1)} |(1 + b_1 e^{j\omega_0} + b_2 e^{j2\omega_0} \dots + b_q e^{jq\omega_0})|^2 = \frac{6\psi(\omega_0)}{\text{SNR} N^2 (N^2 - 1)} \end{aligned}$$

The general technique of evaluating the CRB of a complex Gaussian process is given in [56]. Because of the complexity involved when calculating the exact CRB and the lack of direct insight into the parameter problems, the asymptotic CRB is often more interested and insightful. In [53], a simple formulae is obtained for the CRB for frequency estimation in colored noise. This is obtained by, using the CRB for frequency estimation in AWGN, and replacing the variance of AWGN by the power spectral density of the colored noise to get

$$\text{CRB}(\hat{\omega}_0) = \frac{6\psi(\omega_0)}{\text{SNR} N^2 (N^2 - 1)} \quad (3.4.55)$$

where $\psi(\omega)$ is the power spectral density of colored noise as expressed in (3.4.39). Thus

we prove that (3.4.54) asymptotically equals to (3.4.55). That is, the correlation based estimate with optimal lags asymptotically achieves the CRB in MA colored noise. In Figure 3.8, we show the estimation of normalized frequency at $f_0 = 0.124$, corrupted by colored noise. The noise is generated by passing AWGN through a MA filter with system function $B(z) = 1 - 1.8z^{-1} + 1.6z^{-2} - 2.4z^{-3}$. It can be seen that, the MSE of the estimate approaches the asymptotic CRB. It is noted here that the estimator for different frequency will also achieve CRB as proven in (3.4.54), and the MSE of the frequency estimator in MA colored noise is insensitive to the order of the MA process if $q \ll N$.

Figure 3.8: The MSE of the frequency estimation in MA colored noise as SNR varies, compared with the exact CRB and approximate CRB of (3.4.55). $f_0 = 0.124$, $N = 64, 128, 256$ and 512 (from top to bottom plot).

3.4.2 Bias of the DFT Peak Search Frequency Estimation in Colored Noise

It was shown in Section 3.2 that in AWGN, the DFT has a lower SNR threshold value compared with other phase unwrapping techniques. However, this is no longer true in the presence of colored noise. Assume that we have N data points, to estimate the frequency, and the periodogram is computed as

$$P_{xx}(f) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j2\pi f n} \right|^2 \quad (3.4.56)$$

where $x(n)$ is the signal corrupted by the colored noise which is expressed in (3.4.38). Assuming that N is sufficiently larger, the expectation value of (3.4.56) can be expressed as

$$\lim_{N \rightarrow \infty} E[P_{xx}(f)] = \frac{1}{N} \left| \frac{\sin \pi N(f - f_0)}{\sin \pi(f - f_0)} \right|^2 + \psi(f) \quad (3.4.57)$$

where $\psi(f)$ is the spectrum of the colored noise. Taking derivative of the above equation and expanding the sine and cosine functions into first order and second order Taylor series and assuming that the estimate $\hat{f}$ is in the vicinity of f_0 , we obtain

$$\frac{2 \sin \pi N(\hat{f} - f_0)}{N \sin \pi(\hat{f} - f_0)} \frac{\pi N \cos \pi N(\hat{f} - f_0) \sin \pi(\hat{f} - f_0) - \pi \sin \pi N(\hat{f} - f_0) \cos \pi(\hat{f} - f_0)}{\sin^2 \pi(\hat{f} - f_0)} + \psi'(f) = 0 \quad (3.4.58)$$

$$N\pi^2(1 - N^2)(\hat{f} - f_0) + \psi'(f) = 0 \quad (3.4.59)$$

$$\hat{f} = f_0 + \frac{\psi'(f)}{\pi^2 N(N^2 - 1)} \quad (3.4.60)$$

where $\psi'(f)$ is the derivative of $\psi(f)$. Thus we can see that in the presence of AWGN, $\hat{f} = f_0$ and the estimator via periodogram peak search is unbiased, however, in the presence of colored noise, the estimation is affected by the spectrum of the colored noise, Eq. (3.4.60) also shows that the estimation bias is decreasing with data length N , as $O(N^{-3})$.

3.4.3 Phase Unwrapping via Initial Frequency Estimate

Due to the bias shown in Section 3.4.2, the DFT peak search in the presence of colored noise is not attractive as an initial condition for phase unwrapping. In searching for another phase unwrapping technique, we consider the iterative filtering technique which is proposed as a favorable approach for developing computational efficient frequency estimates. This method employs an autoregressive bandpass filter endowed with a bandwidth parameters. It was shown that if the bandwidth is properly adjusted with iterations, the estimate can accommodate a poor initial guess and would converge rapidly to the correct frequency [9]. However, the final estimation accuracy does not achieve the CRB. Here, we use iterative filtering as an initial estimate in colored noise followed with the multiple lags of correlation based frequency estimator. The use of the bandwidth filter will enhance the signal, as such, the threshold SNR level of the final estimate is improved around 5dB better than DFT peak search in the presence of colored noise. Furthermore using the multiple correlations, the frequency estimator achieves CRB. Another well established technique used in the pulse Doppler radar to resolve the phase ambiguity problem is de-ghosting. This technique involves transmitting bursts of pulses for resolving range am-

biguities, normally, several pulses are used, but the concept is easily explained with two pulses. The true target range is determined from the coincidence of received pulses after de-ghosting. Similarly, in the frequency estimation applications, through the selection of two correlations with relatively high prime lags, frequency can be obtained correctly [75].

Figure 3.9 shows the simulation results for the bias of the DFT peak search and the correlation based estimate with multiple lags using the window in (3.3.34). The signal has normalized frequency at $f_0 = 0.001$ (thus no need of phase unwrapping), which is corrupted by a MA colored noise. The noise is generated by passing AWGN through a MA filter with system function $B(z) = 1 - 2.8z^{-1} + 1.6z^{-2} - 1.4z^{-3}$. It is shown that the DFT peak search has a large bias in the presence of MA colored noise, and the correlation based frequency estimation is unbiased using the higher lag correlations with weighting window in (3.3.34). Increasing the data length will decrease the bias for DFT peak search, but the correlation based estimate is unbiased even for very short data length ($N = 32$). Figure 3.10 is the MSE of the estimate for DFT peak search and correlation based estimate for the same signal and noise conditions. It is observed that the DFT peak search has very high level of SNR threshold, which is caused by the bias of the estimate. Figure 3.11 is the MSE of the DFT peak search in the presence of AWGN and colored noise with the same system function of $B(z) = 1 - 2.8z^{-1} + 1.6z^{-2} - 1.4z^{-3}$ for two different frequencies f_0 . The solid line in Figure 3.11 shows the performance results in the presence of AWGN. It is seen that the colored noise deteriorates the performance of the estimate due to the bias, and the performance of the estimate depends on the frequency f_0 . Figure 3.9 and Figure 3.10 are obtained without phase unwrapping, thus only valid for a very small frequency range. To increase the frequency range, the iterative filtering initial

estimate and frequency de-ghosting technique are used as shown in Figure 3.12 to resolve the phase unwrapping. Figure 3.12 shows that using the iterative filtering technique result in a lower SNR threshold level compared with DFT in the presence of MA colored noise, but de-ghosting technique's threshold SNR level is high. The use of CRT of [15] for phase unwrapping is also similar to de-ghosting and results in worse performance than the use of DFT coarse search.

3.5 Conclusion

We propose a frequency estimation method in the presence of AWGN and MA colored noise based on the multiple lags of correlations. The SNR threshold effect of the estimator is investigated comprehensively. It is shown that this estimator is efficient for short length of data samples while maintaining a low SNR threshold. The performance of the estimator has similar threshold level with MLE in the presence of AWGN and outperforms other correlation based method. In the presence of colored noise, with the use of a proper phase unwrapping technique, the estimator can outperform frequency estimation obtained from a DFT fine peak search. For the phase un-wrapping in the presence of colored noise, the method of iterative filtering has shown superior performances. However, in the presence of AWGN the most suitable phase unwrapping technique is the use of DFT coarse search. In the next chapter, we will investigate the effect of multiplicative noise in the frequency estimation.

Figure 3.9: The bias of the frequency estimation in MA colored noise for DFT peak search and the covariance based estimator. $f_0 = 0.001$, $N = 32, 64, 128$ (from top to bottom plot). without phase unwrapping

Figure 3.10: The MSE of the frequency estimation in MA colored noise for DFT peak search and the correlation based estimator. $f_0 = 0.001$, $N = 32, 64, 128$ (from top to bottom plot). without phase unwrapping

Figure 3.11: The MSE of the frequency estimation in AWGN and MA colored noise as SNR varies for DFT peak search, $f_0 = 0.124, 0.324$, $N = 32$

Figure 3.12: The MSE of the frequency estimation in MA colored noise as SNR varies using different phase unwrapping techniques. $m = 13, 17$, $f_0 = 0.124$, $N = 32, 64$ (from top to bottom plot)

Chapter 4

Performance of Correlation Based Frequency Estimation Methods in Fading Channels

4.1 Introduction

Frequency estimation from a finite data samples has been a classical problem in signal processing, especially in the areas of wireless communication, acoustics signal processing and radar and sonar engineering [3, 30, 56, 70]. The traditional methods used in the analysis and development of frequency estimators employ an additive white Gaussian noise model with the signal. However, the additive noise model is not sufficiently representative in most practical applications. As an example, consider a digital communication system with burst transmission using coherent signal detection over a radio link where

the transmitted signal is corrupted by a time varying multiplicative distortion. In such conditions, frequency estimators based on additive Gaussian models do not perform well, and a low accuracy carrier frequency estimation for the fading channel will cause serious performance degradation. As such, efficient and accurate estimation of the carrier frequency offset between transmitter and receiver is necessary to improve the performance of the receiver [36]. Another application is in global satellite mobile communication systems employing low earth orbit satellites or medium earth orbit satellites. In these systems, the Doppler shift of the carrier frequency varies randomly and should be estimated and compensated for by the receiver [76]. It is well known that the peak picking of the periodogram of the received signal leads to an excellent frequency estimation, however, due to the computational complexity employing the peak search, simple techniques are often preferred. One simple technique for frequency estimation is the correlation based method which is computationally efficient with a complexity of $O[N]$, N being the number of data samples. On the other hand, the discrete Fourier transform (DFT) based periodogram technique has a complexity of $O[N \log N]$ [69] [30]. It is reported that even in Rayleigh fading channels, the correlation based frequency estimator is almost efficient with regard to the Cramér-Rao lower bound (CRB) [29]. In this chapter, we show that, by proper selection of lags, the correlation based frequency estimator is efficient in all kind of channel fading environments.

The primary aim of this chapter is to provide a comprehensive performance evaluation for the correlation based frequency estimator under various fading channel conditions. The correlation based method using a single lag in multiplicative noise has been discussed in [30] [29], but the relative efficiency in terms of CRB was not adequately addressed in

detail under different fading conditions. Major contributions of the work appearing in this chapter are stated as below.

In the chapter, closed form expressions for the variance of frequency estimation errors under different fading conditions are derived. These formulations are accurate and provide direct insight into the frequency estimation problem. Based on this, the optimal lag value, which is essential for obtaining a correlation based frequency estimation, is determined. When optimum lags are used, phase unwrapping is necessary to resolve the phase ambiguity of the estimate. The phase unwrapping can be avoided at the price of reducing the operating frequency range as done in [36] [29], but this is a highly unattractive solution. Here, as done in the previous chapter, we use a simple DFT based phase unwrapping technique which is superior to other phase unwrapping techniques in considering the threshold effect at low signal to noise ratios [77]. Furthermore, the chapter also addresses the frequency estimation using multiple lags of correlations in fading channels. The use of multiple lags improves the performance at low signal to noise ratios, but requires an optimum weighting function for linearly combining the multiple estimates. The optimum weighting function selection is also discussed in the chapter. The optimal lag determination and optimum window selection requires the knowledge of the Doppler spread and thus a novel Doppler spread estimator is also presented in the Chapter.

To the best of our knowledge we consider the following papers in the literature as the main contributions in the area of correlation based frequency estimation under fading conditions [36], [29] and [15]. Of these, [29] and [36] ignores the Doppler spread estimation and the phase unwrapping problem. Reference [36] uses multiple lags for estimation although an

optimal lag value suitable for a given fading condition has not been derived. In [15] phase unwrapping is obtained via the use of complicated Chinese remainder theorem. But the results are only derived for an additive noise model. In comparison with the work in the literature, it is highlighted that the proposed algorithm considers all the aspects necessary for proper frequency estimation under all kind of fading environments and, possesses a wide range of frequency acquisition even in low SNR environments. Implementation of the frequency estimator is simple compared with the MLE based techniques [78], nevertheless the estimation accuracy approaches the CRB.

The rest of the chapter is organized as follows. The performance evaluation of the single optimal lag correlation based estimator in different fading channels is developed in Section 4.2. Section 4.3 shows a simple technique to determine the bandwidth of the multiplicative noise. Frequency estimation using multiple lags of correlations is discussed in Section 4.4 followed by the illustrative examples given in Section 4.5. Section 4.6 of the chapter provides some concluding remarks.

4.2 Correlation Based Frequency Estimator in Multiplicative Noise

4.2.1 Variance Expressions for the Correlation Based Frequency Estimator

Consider the signal model (1.3.1), and it is noted that both the multiplicative noise and additive white noise are normal circular complex processes. That is $a(n) = a_1(n) + ja_2(n)$

and $v(n) = v_1(n) + jv_2(n)$, where $a_1(n)$, $a_2(n)$, $v_1(n)$, $v_2(n)$ are all zero mean independent identically distributed real processes. Furthermore, $E[a_1^2(n)] = E[a_2^2(n)] = \sigma_a^2/2$, $E[v_1^2(n)] = E[v_2^2(n)] = \sigma_v^2/2$. A model with real multiplicative noise was considered in [30], and it was shown that the real multiplicative noise and the complex circular multiplicative noise differ fundamentally in how the noise affect in the frequency estimation, and the real multiplicative noise would not cause the floor effect of estimation in fading channels.

The investigated estimator in this section exploits the sample correlations given by

$$r(m) = \frac{1}{N-m} \sum_{n=1}^{N-m} x(n+m)x^*(n) \quad (4.2.1)$$

where $x(n)$ is the received signal as expressed in (1.3.1), m is the lag value of the correlation function. N is the number of data samples used in the estimation. Substituting (1.3.1) into (4.2.1), $r(m)$ can be expressed as

$$r(m) = \frac{1}{N-m} \sum_{n=1}^{N-m} (\mu + a(n+m) + v(n+m))(\mu + a^*(n) + v^*(n))e^{j\omega_0 m} \quad (4.2.2)$$

The frequency estimator from the m th lag correlation can be expressed as

$$\hat{\omega}_0 = \frac{1}{m} \arg r(m) \quad (4.2.3)$$

Note here, phase unwrapping via DFT is needed for the estimation in (4.2.3). Further-

more, the error of the estimate from the m th lag is given by

$$\begin{aligned}\varepsilon_m &= \frac{1}{m} \arg \left(\sum_{n=1}^{N-m} (\mu + a(n+m) + v(n+m))(\mu + a^*(n) + v^*(n)) \right) \\ &= \frac{1}{m} \arg \left(\sum_{n=1}^{N-m} (\mu^2 + X_m + X_a) + jY_m \right)\end{aligned}\quad (4.2.4)$$

where $X_a = a_1((n+m)a_1(n) + a_2(n+m)a_2(n)$,

$$Y_m = Y_{m1} + Y_{m2} + Y_{m3},$$

$$Y_{m1} = \mu[a_2(n+m) + v_2(n+m) - a_2(n) - v_2(n)],$$

$$Y_{m2} = a_2(n+m)v_1(n) - a_1(n+m)v_2(n) + v_2(n+m)a_1(n) - v_1(n+m)a_2(n) + v_1(n)v_2(n+m) - v_1(n+m)v_2(n).$$

$$Y_{m3} = a_2((n+m)a_1(n) - a_1(n+m)a_2(n),$$

Eq. (4.2.4) can be further arranged as

$$\begin{aligned}\varepsilon_m &= \frac{1}{m} \arg \left((\mu^2 + \sigma_a^2 r_a(m))(N-m) + \sum_{n=1}^{N-m} (X_m + jY_m) \right) \\ &= \frac{1}{m} \arg \left(1 + \frac{\sum_{n=1}^{N-m} (X_m + jY_m)}{(\mu^2 + \sigma_a^2 r_a(m))(N-m)} \right) \\ &\approx \frac{\sum_{n=1}^{N-m} Y_m}{(\mu^2 + \sigma_a^2 r_a(m))m(N-m)}\end{aligned}\quad (4.2.5)$$

Let $\sum_{n=1}^{N-m} Y_m = \sum_{n=1}^{N-m} (Y_{m1} + Y_{m2} + Y_{m3}) = Z_{m1} + Z_{m2} + Z_{m3}$. Then the correlation of

the frequency estimation error with different lag values can be expressed as

$$E[\varepsilon_m \varepsilon_l] = \frac{E[Z_{m1}Z_{l1}] + E[Z_{m2}Z_{l2}] + E[Z_{m3}Z_{l3}]}{(\mu^2 + \sigma_a^2 r_a(m))(\mu^2 + \sigma_a^2 r_a(l))ml(N-m)(N-l)} \quad (4.2.6)$$

where

$$\begin{aligned} E[Z_{m1}Z_{l1}] &= \sum_{n_1=1}^{N-m} \sum_{n_2=1}^{N-l} \mu^2 E[(a_2(n_1+m) + v_2(n_1+m) - a_2(n_1) - v_2(n_1)) \\ &\quad \times (a_2(n_2+l) + v_2(n_2+l) - a_2(n_2) - v_2(n_2))] \\ &= 2\mu^2 \sum_{n_1=1}^{N-m} \sum_{n_2=1}^{N-l} \{E[a_2(n_1)a_2(n_2)] - E[a_2(n_1)a_2(n_2+l)]\} \\ &\quad + \{E[v_2(n_1)v_2(n_2)] - E[v_2(n_1)v_2(n_2+l)]\} \\ &= \mu^2 \sigma_v^2 \min(m, l, N-m, N-l) + \mu^2 \sigma_a^2 \sum_{n_1=1}^{N-m} \sum_{n_2=1}^{N-l} \{r_a(n_1 - n_2) - r_a(n_1 - n_2 - l)\} \\ \\ E[Z_{m2}Z_{l2}] &= \sum_{n_1=1}^{N-m} \sum_{n_2=1}^{N-l} 2E[a_1(n_1+m)a_1(n_2+l)v_2(n_1)v_2(n_2) - a_1(n_1+m)a_1(n_2)v_2(n_1)v_2(n_2+l)] \\ &\quad + 2E[a_1(n_1)a_1(n_2)v_2(n_1+m)v_2(n_2+l) - a_1(n_1)a_1(n_2+l)v_2(n_1+m)v_2(n_2)] \\ &\quad + 2E[v_1(n_1)v_1(n_2)v_2(n_1+m)v_2(n_2+l)] \\ &= \sigma_a^2 \sigma_v^2 [\min(N-m, N-l) - \sigma_a^2 \sigma_v^2 \sum_{k=1}^{N-l} r_a(k+l) \max(N-m-l, 0)] + \frac{\sigma_v^4}{2} (N-m) \delta_{m,l} \\ \\ E[Z_{m3}Z_{l3}] &= \sum_{n_1=1}^{N-m} \sum_{n_2=1}^{N-l} 2E[a_2(n_1+m)a_2(n_2+l)a_1(n_1)a_1(n_2) - a_1(n_1+m)a_1(n_2)a_2(n_1)a_2(n_2+l)] \\ &= \sigma_a^4/2 \sum_{n_1=1}^{N-m} \sum_{n_2=1}^{N-l} r_a(n_1 - n_2)r_a(n_1+m - (n_2+l)) - r_a(n_1+m - n_2)r_a(n_2+l - n_1) \end{aligned}$$

Using the above, the variance of the single lag correlation based frequency estimator can be expressed as

$$E[\varepsilon_m^2] = \frac{E[Z_{m1}^2] + E[Z_{m2}^2] + E[Z_{m3}^2]}{(\mu^2 + \sigma_a^2 r_a(m))^2 m^2 (N - m)^2} \quad (4.2.7)$$

4.2.2 Performance Evaluation Under Different Fading Conditions

Eq.(4.2.7) is quite cumbersome, and to gain an insight into the performance of the frequency estimator in fading channels, a few approximations are made in this subsection for some special cases.

No Fading (same as an AWGN channel), $\kappa \neq 0$, $\beta = 0$, where $\kappa = \frac{\mu^2}{\sigma_a^2}$, and $\beta = \frac{\sigma_v^2}{\sigma_r^2}$.
($\kappa\beta = \frac{\mu^2}{\sigma_a^2} \frac{\sigma_v^2}{\sigma_r^2} = \mu^2/\sigma_v^2 \neq 0$)

Eq.(4.2.7) reduces to

$$E[\varepsilon_m^2] = \frac{\min(m, N - m)}{\kappa\beta m^2 (N - m)^2} + \frac{1}{2(\kappa\beta)^2 m^2 (N - m)} \quad (4.2.8)$$

This is the variance expression in the presence of AWGN. When $m = 2N/3$, it achieves a minimum value of 0.51dB above the CRB at high SNR [17, 70].

Rayleigh Fading, $\kappa = 0$, $\beta \neq 0$

Eq. (4.2.7) reduces to

$$E[\varepsilon_m^2] = \frac{\beta(N-m) - \beta \sum_{k=1}^{N-m} r_a(k+m) \max(N-2m, 0) + 1/2(N-m)}{\beta^2 r_a^2(m) m^2 (N-m)^2} (4.2.9) \\ + \frac{\sum_{k=0}^{N-m-1} (N-m-k) r_a^2(k) - \sum_{k=m}^{N-1} (N-k) r_a^2(k)}{2 r_a^2(m) m^2 (N-m)^2}$$

Assuming that the channel is fast fading, that is, the coherence time is much smaller than the data length, we get $r_a(k+m) \approx 0$. Then at large SNR, i.e. $\beta > 0.5$. (4.2.9) can be expressed as

$$E[\varepsilon_m^2] \approx \frac{1}{\beta r_a^2(m) m^2 (N-m)} + \frac{\sum_{k=0}^{N-m-1} (N-m-k) r_a^2(k)}{2 r_a^2(m) m^2 (N-m)^2} (4.2.10)$$

Define channel SNR = $\beta r_a^2(m)$. It is seen that in this Rayleigh fading channel, the SNR decreases with the square of the normalized autocorrelation function of the multiplicative noise, which is determined by the Doppler spread of the fading process. As such we can see that a fast fading channel seriously deteriorates the SNR, thus small values of m is preferred for frequency estimation. From (4.2.10) it is seen that for small m , the variance is approximately of order $1/N$. Furthermore, it is seen when $\beta \gg 1$, the first term of (4.2.10) vanishes, while the second term is independent of β . This is the floor effect of the fading channel and it is caused by the time varying nature of the fading channel.

It is clear that fast fading channels lead to a rapid channel SNR deterioration, and in the extreme of having a temporally white Gaussian noise, $r_a(m) = \delta_{m0}$, the frequency can not be estimated via the correlation based technique.

In Rayleigh slow fading channels, the coherence time is larger than the data length. Extremely slow fading can be seen as a case of completely correlated process $a(n)$, thus $r_a(m) \approx 1$, and the second term of (4.2.9) will be equal to zero. Then (4.2.9) reduces to

$$E[\varepsilon_m^2] \approx \frac{\min(m, N-m)}{\beta m^2 (N-m)^2} \quad (4.2.11)$$

The variance has a decay rate of N^{-3} . It can be shown that (4.2.11) has the same decay rate as that in (2.3.33) when the optimal lag value is chosen as $m \approx 2N/3$.

Rician fading, $\kappa \neq 0$, $\beta \neq 0$

Assume that the Rician fading is very fast, then (4.2.7) can be simplified as

$$E[\varepsilon_m^2] \approx \frac{\mu^2 \sigma_v^2 \min(m, N-m) + \sigma_a^2 \sigma_v^2 (N-m) + \mu^2 \sigma_a^2 \sum_{k=0}^{N-m-1} r_a(k) + \sigma_a^4 / 2 (N-m)}{(\mu^2 + \sigma_a^2 r_a(m))^2 m^2 (N-m)^2} \quad (4.2.12)$$

Further, assuming $\kappa \gg 1$, (4.2.12) can be written as

$$\begin{aligned} E[\varepsilon_m^2] &\approx \frac{\sigma_v^2 \min(m, N-m) + \sigma_a^2 \sum_{k=0}^{N-m-1} r_a(k)}{\mu^2 m^2 (N-m)^2} \\ &= \frac{\min(m, N-m)}{\beta \kappa m^2 (N-m)^2} + \frac{\sum_{k=0}^{N-m-1} r_a(k)}{\kappa m^2 (N-m)^2} \end{aligned} \quad (4.2.13)$$

Thus it is clear that the variance of the estimator is related to both κ and β for fixed N and m . It is also apparent that when $\beta \rightarrow \infty$, the variance is mainly determined by κ . We can also see that the variance of the estimator for fast fading Rician channel in (4.2.12) has the same floor effect (i.e., when $\beta \rightarrow 0$) as that of the CRB in (2.3.32) when κ is large.

Though in this case, the fading is still prominent, but since κ is large, the SNR is not seriously deteriorated and the lag m can be selected as the optimal value (corresponding to AWGN) around $2N/3$.

When $\kappa \ll 1$, (4.2.12) can be simplified as

$$E[\varepsilon_m^2] \approx \frac{1}{\beta(\kappa + r_a(m))^2 m^2 (N - m)} + \frac{\kappa \sum_{k=0}^{N-m-1} r_a(k) + (N - m)}{(\kappa + r_a(m))^2 m^2 (N - m)^2} \quad (4.2.14)$$

From (4.2.14) it can be seen that the decay rate of the variance in the Rician fast fading channel is around N^{-1} when $\kappa \ll 1$ and a small lag value is used due to fast fading. Further we can see, when $\kappa = 0$, the channel behaves as Rayleigh fading, and (4.2.14) approximates (4.2.10).

In Rician time invariant slow fading, (4.2.7) is simplified as

$$E[\varepsilon_m^2] \approx \frac{\min(m, N - m)}{\beta(1 + \kappa)m^2(N - m)^2}$$

The CRB is expressed in (2.3.33) and the optimal lag value $m \approx 2N/3$ can be selected to achieve its minimum variance. Based on above discussions, as a generalization, the optimal lag value m for different fading channels and the decay rate of variance of the estimate with respect to N are shown in Table 4.1. Note that Figure 2.6 in Chapter 2 illustrates clearly the behavior noted in Table 4.1.

Table 4.1: The optimal lag value m for different fading channels

Fading Condition	AWGN	Rayleigh fast	Rayleigh slow
Optimal Lag m	$2N/3$	Determined by Eq.(4.2.10)	$\approx 2N/3$
Decay Rate	$\approx 1/N^3$	$\approx 1/N$	$\approx 1/N^3$
Fading Condition	Rician fast($\kappa \gg 1$)	Rician fast($\kappa \ll 1$)	Rician slow
Optimal Lag m	Determined by Eq.(4.2.13)	Determined by Eq.(4.2.14)	$\approx 2N/3$
Decay Rate	$\approx 1/N^3$	$\approx 1/N$	$\approx 1/N^3$

4.3 Doppler Spread Estimation

From Section 4.2.2, it is seen that the performance of the estimator depends on the selection of lag m , which in turn depends on Doppler spread. Thus it follows that the Doppler spread needs to be estimated before implementing the correlation based frequency estimator. We note here that the classical spectral width estimation can be based either on power spectral density (PSD) or the correlation function [2]. Due to the intrinsic simplicity of the correlation approach, it is more preferred in practical applications.

The autocorrelation function can be expressed as

$$r(m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{jm\omega} d\omega \quad (4.3.15)$$

where the power spectral density $S(\omega)$ is the Fourier transform of $r(m)$. Eq. (4.3.15) can be written as

$$r(m) = \frac{1}{2\pi} e^{j\omega_0 m} \int_{-\infty}^{\infty} S(\omega) e^{jm(\omega-\omega_0)} d\omega \quad (4.3.16)$$

where ω_0 is the first order moment of $S(\omega)$. Expanding $e^{jm(\omega-\omega_0)}$ via the Taylor series, and taking the first three terms, yields the approximation,

$$\begin{aligned} r(m)e^{-j\omega_0 m} &\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)(1 + jm(\omega - \omega_0) - m^2(\omega - \omega_0)^2)d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)d\omega + \frac{jm}{2\pi} \int_{-\infty}^{\infty} (\omega - \omega_0)S(\omega)d\omega - \frac{m^2}{4\pi} \int_{-\infty}^{\infty} (\omega - \omega_0)^2 S(\omega)d\omega \\ &= r(0) + jmr(0)\mathcal{M}_1(S(\omega)) - \frac{m^2}{2}r(0)\mathcal{M}_2(S(\omega)) \end{aligned} \quad (4.3.17)$$

where $\mathcal{M}_1(S(\omega))$ and $\mathcal{M}_2(S(\omega))$ are the first and second order central moments, as given by

$$\mathcal{M}_1(S(\omega)) = \frac{\frac{1}{2\pi} \int_{\omega_0-\omega_B}^{\omega_0+\omega_B} (\omega - \omega_0)S(\omega)d\omega}{\frac{1}{2\pi} \int_{\omega_0-\omega_B}^{\omega_0+\omega_B} S(\omega)d\omega} \quad (4.3.18)$$

$$\mathcal{M}_2(S(\omega)) = \frac{\frac{1}{2\pi} \int_{\omega_0-\omega_B}^{\omega_0+\omega_B} (\omega - \omega_0)^2 S(\omega)d\omega}{\frac{1}{2\pi} \int_{\omega_0-\omega_B}^{\omega_0+\omega_B} S(\omega)d\omega} \quad (4.3.19)$$

$$r(0) = \frac{1}{2\pi} \int_{\omega_0-\omega_B}^{\omega_0+\omega_B} S(\omega)d\omega. \quad (4.3.20)$$

The integrals limits in (4.3.18), (4.3.19) and (4.3.20) are changed from $\pm\infty$ to $\omega_0 \pm \omega_B$ assuming that $2\omega_B$ is the bandwidth of $S(\omega)$. The higher order terms of (4.3.17) are neglected on the assumption that $m(\omega - \omega_0) \ll 1$. This condition is satisfied if $m < 1/\omega_B$ since $(\omega - \omega_0) < \omega_B$ within the integral limit. This condition is in consistent with the discussions in Section 4.2, where in the presence of fading, only smaller lag values within the correlation length of the multiplicative noise have been used.

Equating the imaginary part of (4.3.17), we can get

$$\text{Im}(r(m)e^{-j\omega_0 m}) = mr(0)\mathcal{M}_1(S(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\omega - \omega_0)S(\omega)d\omega = 0 \quad (4.3.21)$$

The right-hand side of (4.3.21) is equal to zero as ω_0 is the first order moment of $S(\omega)$.

Therefore, we have

$$\text{Im}(r(m)e^{-j\omega_0 m}) = \text{Im}(|r(m)|e^{j\arg(r(m)) - j\omega_0 m}) = 0 \Rightarrow \omega_0 = \arg(r(m))/m \quad (4.3.22)$$

Eq.(4.3.22) is the same expression as (4.2.3). It is also known that the second spectral moment of the power spectrum is the spectral width estimator $\hat{f}_d^2$ [2]. Equating the real part of (4.3.17) and from (4.3.22), we can get

$$\text{Re}(r(m)e^{-j\omega_0 m}) \approx \text{Re}(r(m)) = r(0) - \frac{m^2}{2}r(0)4\pi^2\hat{f}_d^2 \quad (4.3.23)$$

therefore, we have

$$\hat{f}_d = \frac{1}{\sqrt{2\pi m}} \sqrt{1 - \text{Re}\left(\frac{r(m)}{r(0)}\right)} \quad (4.3.24)$$

Choosing $m = 1$, we can obtain the Doppler spread estimate as

$$\hat{f}_d = \frac{1}{\pi\sqrt{2}} \sqrt{1 - \text{Re}\left(\frac{r(1)}{r(0)}\right)} \quad (4.3.25)$$

This is the spectral width estimate obtained in [2], and it was shown that this estimate is slightly biased in the presence of AWGN [2]. After further manipulation, we can obtain an estimate using the first lag and the second lag of the correlation function as

$$\hat{f}_d = \frac{1}{\pi\sqrt{2}} \sqrt{\frac{\text{Re}(r(1) - r(2))}{\text{Re}(4r(1) - r(2))}} \quad (4.3.26)$$

This spectral width estimate avoids using $r(0)$, and thus it is more robust in the presence

of AWGN, a fact that will be demonstrated by simulation in Section 4.5.

Note that any lag value $m \leq K$ can be used to estimate $\hat{f}_d$, where K is the maximum lag value where $r_a(m)$ is nonzero. K can be found by making $r(m) = 0$ for $m = K$, to obtain

$$\hat{f}_d = \frac{1}{\sqrt{2\pi}K} \Rightarrow K = \frac{1}{\sqrt{1 - \text{Re}(r(1)/r(0))}} \quad (4.3.27)$$

Since the Doppler spread has been estimated, and using the other system parameters such as κ and β , the channel model can be determined and thus the optimal lag value to be used in the covariance based frequency estimation can also be obtained. Furthermore, to improve the accuracy of the frequency estimation, multiple lags can be used in frequency estimation, as will be show in Section 4.4. Note that estimated frequency ω_0 and Doppler spread f_d are channel dependent while κ and β commonly are system dependent which could be assumed to be known.

4.4 Frequency Estimation from Multiple Lags of Correlations in Fading Channels

Frequency estimation from multiple lag of correlations has been developed in the presence of AWGN [15]. In Chapter 3, the frequency estimation in additive colored noise has been discussed using multiple lags of correlations. Here, we will investigate the correlation based frequency estimator with multiple lags in the presence of multiplicative noise.

Assume that the estimate from multiple lags of correlations are a linear combination of

the estimates from correlations of single lags, that is

$$\hat{\omega}_0 = \sum_{m=1}^K d_m \hat{\omega}_m, \quad m = 1, 2, \dots, K, K \leq N - 1. \quad (4.4.28)$$

where d_m 's are weighting coefficients yet to be determined. Note that d_m are real coefficients satisfying $\sum_{m=1}^K d_m = 1$. The optimal weight in AWGN to achieve CRB is obtained in [15] as,

$$d_m = \frac{3m(2K + 1 - m)}{K(K + 1)(2K + 1)}, \quad m = 1, 2, \dots, K, K = N/2 \quad (4.4.29)$$

However, in fading channels, d_m is related to the Doppler spread. The length of the weighting window K can be determined. Using this value of K , the optimum choice of the weighting window is given by [56]

$$\mathbf{d} = \frac{\mathbf{C}^{-1} \mathbf{1}}{\mathbf{1}^T \mathbf{C}^{-1} \mathbf{1}} \quad (4.4.30)$$

where $\mathbf{d} = [d_1, d_2, \dots, d_K]^T$ and $\mathbf{1} = [1, 1, \dots, 1]^T$, $\mathbf{C}$ is the covariance matrix of the frequency estimate from single lags, which is given in (4.2.6). The corresponding minimum variance of the estimate is

$$E[(\omega_0 - \hat{\omega}_0)^2] = \frac{1}{\mathbf{1}^T \mathbf{C}^{-1} \mathbf{1}} \quad (4.4.31)$$

In practice, a rectangular weighting window can be used considering the computational complexity of the algorithm. Simulation shows that this will not cause much performance loss (within 0.5dB).

In time invariant slow fading channels, $r_a(m) \approx 1$, in this case, when $K \approx (N - 1)/2$ and

the optimal weighting in (4.4.29) is used, (4.4.31) can be simplified as

$$\begin{aligned} E(\varepsilon^2) &= \frac{144\sigma_v^2}{(\mu^2 + \sigma_a^2)N^2(N^2 - 1)^2} \sum_{m=1}^{(N-1)/2} \sum_{l=1}^{(N-1)/2} \min(m, l) \\ &= \frac{6\sigma_v^2}{(\mu^2 + \sigma_a^2)N(N^2 - 1)} \end{aligned} \quad (4.4.32)$$

This is identical to the CRB of a time invariant slow fading channel as obtained in (2.3.33).

The simulations in the following section show that at low SNR, the performance of the estimator using multiple lags is also improved.

4.5 Simulations

This section provides simulation results for the performance of correlation based frequency estimation under different fading conditions. The multiplicative noise process $a(n)$ is often modelled as Jake's model [79]. However, it is widely recognized that the actual shape of $S_a(\omega)$ has little impact on the overall system performance. The sole really influential parameter being the Doppler spread, so in this chapter the standard practice of using a low pass filter with a reasonable sharp 3 dB cutoff is used for fading simulations. The bandlimiting of the fading process can be approximated by a finite impulse response (FIR) low pass filter. Fading process can also be simulated, for instance, in [27], via the use of a low pass autoregressive filter. In the simulations, we have assumed that the multiplicative noise obeys a Gaussian shaped autocorrelation, that is $r_a(m) = \sigma_a^2 e^{-2\pi f_d^2 m^2/4}$.

First consider the performance of the single lag correlation based frequency estimator as a function of SNR ($\text{SNR} = (\mu^2 + \sigma_a^2)/\sigma_v^2 = \beta(1 + \kappa)$) for different values of the Rice factor

κ . The number of data samples is $N = 128$, Doppler spread is assumed to be $f_d = 0.1$ (fast fading). As noted in the Introduction, fast fading occurs when the coherence time of the channel is smaller than the symbol period of the transmitted signal. In frequency domain, when the Doppler spread is larger than the bandwidth of the transmitted signal, the channel is fast fading channel. In Figure 4.1 and $f_d = 0.001$ in Figure 4.2. Inspection of Figure 4.1 reveals that in the presence of fast fading (κ is small), both CRB and the frequency estimator has a floor effect in terms of SNR. When κ is very large (for example, $\kappa = 100$), the fading is negligible, the optimal lag value around $m = 2N/3$ can be chosen, and it behaves as the correlation based frequency estimator in AWGN. The optimal lag value m can be determined from (4.2.12), and in Figure 4.1, $m = 3.6, N/3, 2N/3$ from the top plots to bottom plots. It is seen that the fading deteriorates the performance seriously, especially in fast fading channels as shown in Figure 4.1, but the variance of the estimator is very close to the CRB. The theoretical result derived in (4.2.12) also satisfies the simulation results. Figure 4.2 shows that for slow fading channel with different value of $\kappa = 0, 1, 10$, the variance of the estimator is not too sensitive to the variation in κ . the decay rate of the variance and the CRB is approximately of order $1/N$ when the optimal lag value $2N/3$ is used.

Figure 4.3 shows the performance of the frequency estimator in a Rayleigh fast fading channel for various data length N . The optimal lag value for the given fading conditions is determined from (4.2.10) as $m = 6$. As expected, the decay rate of the variance is of order N^{-1} which is in agreement with (4.2.10) when m is small, as well as the CRB expression in (2.3.32). It is evident from Figure 4.3 that the frequency estimate does not improve as the SNR is increased beyond 20dB, this is the floor effect of the frequency

estimation in Rayleigh fading channels. Figure 4.4 shows the performance in a Rayleigh slow fading channel, where the decay rate is approximately N^{-3} as shown in (4.2.11). The optimal lag value $m = 2N/3$ is used in the simulation. As such, it can be seen that in the presence of slow fading and Rician fast fading with large κ , the estimator with optimal lag value $m = 2N/3$ is efficient, having a decay rate of N^{-3} . When the fading is fast and Rayleigh, the preferred lag value is small. Even in this case, the estimator is still efficient with regard to the CRB of fading channels but has a decay rate of N^{-1} .

So far in the simulations, it is assumed that the Doppler spread is prior known. In practical applications, this approach might prove inadequate because of the unknown channel conditions. Furthermore, if multiple lags correlations are used in the frequency estimation, the useful maximum lag value and the length of the optimal weighting window need to be estimated. These are dependent on the Doppler spread of the fading process, thus, the Doppler spread need to be estimated. Under the Gaussian Doppler shape correlation assumption, the Doppler spread can be estimated using (4.3.24) and (4.3.26). To determine the accuracy of the Doppler spread estimator, Figure 4.5 shows the distribution of the Doppler spread estimate error at a high SNR(20dB), obtained from simulating 100,000 estimates. It is assumed that $N = 128$, $\kappa = 0$ and $f_d = 0.01$. Similar distributions can be obtained for other fading models such as exponential model. It is seen that the estimates of (4.3.24) and (4.3.26) are slightly biased, and the use of $m = 2$ has a smaller variance than the use of $m = 1$ in (4.3.24), while (4.3.26) is a better estimate compared to (4.3.24). Figure 4.6 compares the performance of the frequency estimators based on a single optimal lag and multiple lags as function of SNR for different values of Doppler spread f_d , for $\kappa = 0$.

The data length $N = 32$, $f_d = 0.01, 0.006, 0.0001$ from the top plots to the bottom plots. The optimal lag values $m = 6, N/3, 2N/3$ have been used respectively. It is seen that when using the optimal lag, the variance of the estimate is very close to the CRB, and using multiple lags will increase the performance at medium and low SNR. It is also seen that the performance loss of the frequency estimator using optimal lag compared with multiple lags is within 0.5dB at high SNR. On the other hand, it can also be seen that large f_d (fast fading) deteriorates the performance of the estimator considerably, regardless of the use of a single lag or multiple lags, however, the correlation based frequency estimation is still asymptotically efficient.

4.6 Conclusions

This chapter has investigated the method of correlation based frequency estimation under fading conditions. Closed form formulae for the performance evaluation are derived under different fading conditions. Simple asymptotic CRB expressions obtained using a frequency domain approach is used in order to perform comparisons. It is seen that the fast fading channel deteriorate the performance of the frequency estimation dramatically, although, the relative efficiency of the estimator in comparison with the CRB is not degraded. The Doppler spread of the fading channel is also estimated because the optimal frequency estimator implementation requires the spectral bandwidth of the signal. It is shown that for all kind of fading channels (either Rician or Rayleigh fading), by proper selection of the optimal lag value of the correlation function, the variance of the frequency estimator can be made closer to the CRB. Simulation results are used to verify

the theoretical analysis developed in the chapter. Note that this Chapter has described the frequency estimation in a fading channel in the presence of additive white noise. In the next Chapter we will generalize the fading channel to incorporate additive colored noise. Note also that the effect of colored noise under no fading conditions have been discussed in detail in Chapter 3.

Figure 4.1: The MSE, theoretical variance and the CRB of the correlation based estimator versus SNR for different values of κ . $N = 128$, $f_d = 0.1$ (very fast fading). $f_0 = 0.1234$. $\kappa = 0, 1, 10, 100$ (from top to bottom)

Figure 4.2: The MSE, theoretical variance and the CRB of the correlation based estimator versus SNR for different values of κ . $N = 128$, $f_d = 0.001$ (slow fading). $f_0 = 0.1234$. $\kappa = 0, 1, 10, 100$

Figure 4.3: The MSE, theoretical variance and the CRB of the correlation based estimator versus data length N for different values of SNR. SNR = 0, 10, 20, 30dB (from top to bottom). $f_d = 0.05$. $f_0 = 0.1234$. $\kappa = 0$

Figure 4.4: The MSE, theoretical variance and the CRB of the correlation based estimator versus data length N for different values of SNR. SNR = 0, 10, 20dB (from top to bottom), $f_d = 0.001$. $f_0 = 0.1234$. $\kappa = 0$

Figure 4.5: The distribution of the Doppler spread estimation for various methods. $N = 128$, $f_d = 0.01$. $f_0 = 0$, $\kappa = 0$

Figure 4.6: The performance comparison of the single lag estimate and multiple lag estimate for different values of f_d . $N = 128$, $f_d = 0.01, 0.006, 0.0001$ (from top to bottom). $f_0 = 0.1234$, $\kappa = 0$

Chapter 5

Efficient Frequency Estimation in the Presence of Multiplicative Noise and Additive Colored Noise

5.1 Introduction

The frequency estimation in the presence of additive colored noise was described in Chapter 3, and the performance of the correlation based frequency estimator in the presence of multiplicative noise was investigated in Chapter 4. In this chapter, we will discuss the frequency estimation in the presence of both multiplicative noise and additive colored noise. To achieve efficient frequency estimation in the presence of multiplicative noise and colored noise, the Doppler spread is firstly estimated in the presence of additive colored noise assuming that the order of the colored noise is prior known. This is followed with the

estimation of the order of the colored noise assuming that the data length is much smaller than the correlation length of the multiplicative noise. Justification for this assumption is also provided. Finally, the performance of the correlation based estimator with the use of Doppler spread and colored noise information is illustrated via simulation by selecting suitable lags of correlations. However, the proposed frequency estimators require accurate phase unwrapping. DFT as an efficient and accurate phase unwrapping technique has been discussed previously in Chapter 3. Here, for completeness, the iterative filtering technique as a phase unwrapping technique is also discussed, and it is shown that the DFT phase unwrapping, in the presence of multiplicative noise, has the best performance considering the low SNR threshold.

Finally, we discuss the multiple frequency estimation using the correlation based technique, and a comparison is made with other commonly used multiple frequency estimation methods.

5.2 Doppler Spread Estimation

The performance of the frequency estimation in the presence of multiplicative noise and colored noise depends on the range of correlation lags used in the estimation. This range is determined by the Doppler spread [37] as well as the correlation length of the noise process (see also the discussion in Section 1.2). Thus it follows that the Doppler spread needs to be estimated before implementing an accurate correlation based frequency estimator. In Chapter 4, using the spectral moments, we have noted an algorithm to estimate the

Doppler spread in the presence of AWGN, which is given by

$$\hat{f}_d^2 = \frac{1}{2\pi^2 m^2} \text{Re} \left(1 - \frac{r(m)}{r(0)} \right) \quad (5.2.1)$$

where $\text{Re}(u)$ is the real part of u . The investigated Doppler spread estimator exploits the sample autocorrelations given by

$$r(m) = \frac{1}{N-m} \sum_{n=1}^{N-m} x(n+m)x^*(n) \quad (5.2.2)$$

where $x(n)$ is the received signal as expressed in (1.3.1), m is the lag value of the correlation function, and N is the number of data samples used in the estimation. From (5.2.1), we can see that any lag value $m \leq K$ can be used to estimate $\hat{f}_d$, where K is the maximum lag value that is determined by the fading channel. Choosing $m = 1$, we can obtain the square of the Doppler spread estimate as

$$\hat{f}_d^2 = \frac{1}{2\pi^2} \text{Re} \left(1 - \frac{r(1)}{r(0)} \right) \quad (5.2.3)$$

This is the spectral width estimate obtained in [2], and it was noted that this estimate is slightly biased in the presence of AWGN [2]. In this chapter, we assume that the additive noise is broad band colored and that it can be modeled as a MA process. Therefore, any lag value lower than the MA process order would incur a bias to the Doppler spread estimation. Thus, it is necessary that high lag values should be used to reduce the bias of the estimator. On the other hand, using two lags, m and n , the Doppler spread estimate

can also be obtained in a similar form to (5.2.1) as shown in the following

$$\hat{f}_d^2 = \frac{1}{2\pi^2} \text{Re} \left(\frac{r(m) - r(n)}{n^2 r(m) - m^2 r(n)} \right) \quad (5.2.4)$$

The above Doppler spread estimate avoids $r(0)$, and thus is more robust at low SNR. Figure 5.1 illustrates the bias of the Doppler estimate in the presence of additive white noise. From the top plot to the bottom plot, values of $m = 1, 2, 3$ have been used in the expression of (5.2.1). It is seen the estimator with $m = 3$ has better performance than $m = 1$. We also show the estimation from (5.2.4) by selecting $m = 1, n = 2$ and $m = 2, n = 3$ respectively. It is seen that the estimate in (5.2.4) is better than (5.2.1) at low SNR in the presence of AWGN. The corresponding MSE of the Doppler spread estimate is shown in Figure 5.2. In the presence of broad band colored noise, we have shown that the lag value should be larger than the order of the colored noise process to obtain an unbiased estimate. Figure 5.3 shows the bias of the estimate in colored noise. The MA coefficients are chosen as $B = [1, -0.48, 1.6, -1.4]$. From the top plot to the bottom plot, $m = 1, 3$, and 5 are separately used in (5.2.1) to estimate the Doppler spread. It is seen that the estimate with $m = 1$ has a larger bias, and with increasing lag value, the bias of the estimate is reduced. Figure 5.4 shows the corresponding MSE of the estimator in the presence of colored noise. The Doppler spread $f_d = 0.0282$ and data length $N = 64$ are used in the simulations. From the simulations, we conclude that (5.2.4) is of advantage in AWGN, and (5.2.1) can be used to estimate the Doppler spread unbiasedly in colored noise when the lag value is selected larger than the order of the colored noise correlation length.

Figure 5.1: Expected value of the Doppler spread estimator in the presence of white noise versus SNR using different lag values. $N = 64$, $f_d = 0.0282$

Figure 5.2: The MSE of Doppler estimator in the presence of white noise versus SNR using different lag values. $N = 64$, $f_d = 0.0282$

Figure 5.3: Expected value at the Doppler spread estimator in the presence of colored noise versus SNR using different lag values. $N = 64$, $f_d = 0.0282$

Figure 5.4: The MSE of Doppler estimator in the presence of colored noise versus SNR using different lag values. $N = 64$, $f_d = 0.0282$.

5.3 Order Estimation of the Colored Noise Process

In the previous section, prior knowledge of the colored noise process order is assumed for proper estimation of the Doppler spread. Here, we shall consider estimating the order of the MA colored noise in the presence of multiplicative noise. To do this, note that by choosing the data length much smaller than the correlation length of the multiplicative noise, the channel can be assumed to behave like a slow fading channel (This has been shown in Chapter 2 with Figure 2.6). Several information criteria have been proposed for model order estimation, such as Akaike information criteria, final prediction error criteria, etc. [80] [81]. However, these methods do not yield consistent estimates of the model order. An approach for selecting the order of the ARMA process based on the eigenvalues of the covariance matrix was proposed in [82]. This algorithm for estimating the order of ARMA process for complex sequences can be modified to deal with the MA order estimation.

Recall that the signal model in (3.2.1), and now the noise is modelled as an MA process, which can be represented as follows:

$$x(n) - Ae^{j\phi}e^{j\omega_0 n} = \sum_{k=1}^q b_k u(n-k) + u(n), \quad n = 1, 2, \dots, N-1. \quad (5.3.5)$$

where ϕ is uniformly distributed in $[0, 2\pi)$, ω_0 is the angular frequency, $\omega_0 \in (0, \pi]$. A is the signal amplitude and N is the number of data samples observed. The above relationship

can be written in matrix form as

$$\begin{bmatrix} x(0) - Ae^{j\phi}e^{j\omega_0 0} & u(0) & \cdots & 0 \\ x(1) - Ae^{j\phi}e^{j\omega_0 1} & u(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x(N-1) - Ae^{j\phi}e^{j\omega_0 N-1} & u(N-1) & \cdots & u(N-1-q) \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_q \end{bmatrix} = \begin{bmatrix} \epsilon(0) \\ \epsilon(1) \\ \vdots \\ \epsilon(N-1) \end{bmatrix} \quad (5.3.6)$$

where $\epsilon(n)$ is assumed to be zero mean white Gaussian noise that represents modelling error. Equation (5.3.6) can be expressed in a compact form as

$$\mathbf{D}_{N,q+1} \mathbf{b}_{q+1,1} = \boldsymbol{\epsilon}_{N,1} \quad (5.3.7)$$

where $\mathbf{D}_{N,q+1}$ is the data matrix, and $\mathbf{b}_{q+1,1}$ is the MA coefficients vector.

The minimum description length criteria was given by [82]

$$f(\boldsymbol{\epsilon}) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \boldsymbol{\epsilon}^T \boldsymbol{\epsilon}\right) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2} \mathbf{b}^T \mathbf{R} \mathbf{b}\right) \quad (5.3.8)$$

where $\mathbf{R} = \mathbf{D}_{N,q+1}^T \mathbf{D}_{N,q+1}$ is the data covariance matrix and T means matrix transpose. $\boldsymbol{\epsilon} = [\epsilon(0), \epsilon(1), \dots, \epsilon(N-1)]^T$. $\epsilon(n)$ is a zero mean white Gaussian noise, σ^2 is the variance of $\epsilon(n)$. It can be shown that (5.3.8) reduces to [82]

$$J(p, q, \mathbf{b}) = \frac{N}{2} \log \sigma^2 + \frac{N}{2} \log 2\pi + \frac{1}{2\sigma^2} \mathbf{b}^T \mathbf{R} \mathbf{b} + \frac{1}{2} (p + q + 1) \log N \quad (5.3.9)$$

for the problem discussed in this section, $p = 1$, and for fixed q , the choice of $\mathbf{b}$ that minimizes the criterion is found to be the eigenvector associated with the minimum eigenvalue

of $\mathbf{R}$, $\lambda_{\min}$. After some further manipulation and when N is large, the criterion is simplified as

$$J(p, q) = \lambda_{\min}(N^{1/N}) \approx \lambda_{\min} \quad (5.3.10)$$

This indicates the model selection is asymptotically simplified to examining the minimum eigenvalues of the covariance matrix for different values of p and q . Using this technique, the order of the MA colored noise can be estimated. As an example, consider a non-fading signal model $x(n) = e^{j2\pi 0.124n} + 0.432u(n-4) + 0.632u(n-3) + 0.232u(n-2) + 0.12u(n-1) + u(n)$, where $u(n)$ is AWGN with variance $\sigma^2 = 0.01$. (i.e. the SNR = 20dB) In the presence of fading channel, the above signal model becomes $x(n) = a(n)e^{j2\pi 0.124n} + 0.432u(n-4) + 0.632u(n-3) + 0.232u(n-2) + 0.12u(n-1) + u(n)$, with $a(n)$ being modelled as a process by passing white noise through an AR filter. Suppose we choose the matrix size N to be much smaller than the correlation length of $a(n)$, then the channel is slowly fading. Here, we choose the correlation length $L = 512$ and data matrix size $N = 32$. In this slow fading case, it was found that the MA order can be estimated accurately.

5.4 Frequency Estimation in the Presence of Multiplicative Noise and Additive Colored Noise

In this section, the performance of the correlation based estimator using multiple lags in both additive colored noise and multiplicative noise is illustrated via simulations. The colored noise is modelled as a broadband process represented by a MA model with order

q. Suppose we choose the data length N much smaller than the correlation length of the fading process, then the channel is slowly fading, and thus the order estimation of the colored noise is possible. We use multiple lags of correlations to obtain accurate frequency estimation. For completeness, in the following discussion, we present the non fading and additive colored noise scenarios as well.

5.4.1 Performance of the Estimator in Multiplicative Noise and AWGN

Figure 5.5 shows typical correlation functions of the signal and noise. For this case, assume that the correlation length of the multiplicative noise is smaller than half of the data length, that is $L_2 < N/2$. In this case, the fading caused by the multiplicative noise is fast, as such, only the lower lags of correlations can be used for frequency estimation. If lags greater than L_2 are used, then the estimation will cause large variance due to the fast fading. In Figure 5.6, the simulation results of frequency estimation in fast fading channel is shown. Here $f_d = 0.01$ corresponding $L_2 = 6$ and the data length $N = 32$. The circled line is obtained with correlation lags $m = 1, 2, \dots, N/2$ while the line with rectangular symbols is obtained using the lags $m = 1, 2, \dots, L_2$. It is shown that at medium to high SNR, using lower lags of correlations, the estimate achieves CRB, whereas the estimate using half of the correlation lags is around 5dB poorer in performance. This illustrates the importance of the Doppler spread estimation to achieve the CRB. Figure 5.7 shows the performance of the estimator in the presence of slowly fading channel, the correlation length $L_2 > N/2$ ($L_2 = 24, N = 32, f_d = 0.005$) and the larger lag values of correlations

can be used to estimate the frequency. As shown in Figure 5.7, we use the lag values $m = 1, \dots, 24$ and $m = 1, \dots, 16$ separately to estimate the frequency, the weighting window (3.3.31) is used in simulation. The plots using the two different set of correlation lags are quite close with each other. Note that in Chapter 4, we have used the correlation at optimal lag to estimate the frequency, in this section the estimate with multiple lags of correlations are used to demonstrate the efficiency of the algorithm in slow fading and fast fading channels.

Figure 5.5: The correlation function in the presence of fading channels, the additive noise is white.

Figure 5.6: The MSE of estimator in the presence of multiplicative noise versus SNR using different lag values. $N = 32$, $m = 1, \dots, 16$, $m = 1, \dots, 6$. $f_d = 0.01$. $f_0 = 0.124$

Figure 5.7: The MSE of estimator in the presence of multiplicative noise versus SNR using different lag values. $N = 32$, $m = 1, \dots, 16$, $m = 1, \dots, 24$. $f_d = 0.005$. $f_0 = 0.124$

5.4.2 Performance of the Estimator in Colored Noise-No Fading

In this section, we compare the performance of the estimator in the presence of colored noise using different sets of correlation lags. In Figure 5.8, it is shown that the colored noise to be broadband and is modelled as an MA process with correlation length L_1 . With information of the MA order of the colored noise, L_1 can be determined and the correlation lag values required for estimation are correspondingly determined. Figure 5.9 shows the simulation results of the frequency estimation in the presence of colored noise using multiple lags, with data length $N = 32$, $L_1 = 24$, and we use different set of correlation lags with the optimal window in (3.3.34). The line with rectangular sign is the MSE of the estimator in additive colored noise, without the knowledge of order of the MA colored noise. In this case, as noted in Chapter 3, half of the correlation lags $m = N/2 + 1, \dots, N$ is used to obtain the estimate. The circled line represent the MSE of the estimator after the order estimation of the colored noise, and thus the set $m = 24, \dots, 32$ is used to obtain a better estimate in the presence of broad band colored noise.

Figure 5.8: The correlation function in the presence of additive colored noise, no fading

Figure 5.9: The MSE of the estimator in the presence of colored noise using different lags of correlations

5.4.3 Performance of the Estimator in Multiplicative Noise and Colored Noise

In this section, the most general signal model is considered. The typical correlation function is shown in Figure 5.10, where L_1 is the correlation length of the colored noise and L_2 is the correlation length of the multiplicative noise. The condition $L_1 < L_2$ must be satisfied to guarantee unbiased frequency estimation. Optimally, the correlations with lag values $m = L_1, \dots, L_2$ could be used to achieve the best performance. This is achieved by proper estimation of the order of the MA colored noise and the Doppler spread. In the simulation shown in Figure 5.11, the performance of the estimator with optimal lag values is illustrated, the frequency was selected as $f = 0.124$, and $f_d = 0.002$ corresponding to $L_2 = 28$. The order of MA noise was estimated as $q = 8$. The line with circled sign

is obtained using the lag values $m = 8, 9 \dots 28$ and the circled line is obtained using the correlations with lag values $m = 1, 2, 16$. The CRB in the presence of multiplicative noise and colored noise is also shown.

Figure 5.10: The correlation function in additive colored noise and multiplicative noise

Figure 5.11: The MSE of estimator in the presence of additive colored noise and multiplicative noise versus SNR. $N = 32$. $f_0 = 0.124$

5.5 Suitable Phase Unwrapping Technique

In the previous discussions, the correlation based frequency estimation has assumed correct phase unwrapping. As discussed in Chapter 3, this is achieved via the DFT coarse search as the initial estimation. It is also shown in Chapter 3 that although the DFT phase unwrapping has the lowest threshold in AWGN, in the presence of colored noise, it is not so. The iterative filtering algorithm was a good initial estimate for phase unwrapping considering the threshold effect. In this section, we will revisit and discuss the iterative filtering algorithm as a phase unwrapping method in the presence of multiplicative noise.

Figure 5.12 shows the convergent property of the iterative filtering algorithm (IFA) in the presence of AWGN, colored noise, and multiplicative noise separately. The frequency is $f_0 = 0.114$ and $N = 100$. The colored noise is obtained by passing white Gaussian noise through a filter $B(z) = 1 - 1.8z^{-1} + 1.6z^{-2} - 1.4z^{-3}$. The Doppler spread for the multiplicative noise is $f_d = 0.002$. It was shown that in the presence of colored noise, though the algorithm converges rapidly with iterations, an inherent bias is present due to the presence of colored noise, and in the presence of multiplicative noise, the iterative filtering algorithm converge slower compared with AWGN. It has been shown in Chapter 3 that the iterative filtering as an initial estimate has good SNR threshold property compared with DFT. In Figure 5.13, the performance of the correlation based estimator using DFT and iterative filtering algorithm is compared. It is seen that in the presence of multiplicative noise, the DFT coarse search has a lower SNR threshold.

Figure 5.12: The IFA in AWGN, colored noise and multiplicative noise versus N . $N = 1, \dots, 100$, $f_0 = 0.124$

Figure 5.13: The MSE of the correlation based estimator in multiplicative noise for different phase unwrapping technique. $N = 32$, $f_d = 0.006$. $f_0 = 0.124$

5.6 Multiple Frequency Estimation Using Correlation Based Algorithm

The problem of frequency estimation of multiple sinusoids in noise is also of considerable concern in signal processing applications. Of various existing procedures, iterative filtering algorithm (IFA) can be considered as a consistent estimator, with fast convergent properties [7]. Because the sinusoids are localized in the frequency domain, bandpass filters can be employed to enhance the signal. In essence, one can use a single bandpass filter to sequentially estimate each frequency. But as shown in Section 5.5, the algorithm can not achieve CRB for the case of single sinusoid in AWGN. In this section, we propose the correlation based multiple tone frequency estimation, by extending the single tone correlation based frequency estimation to multiple frequency estimation. The DFT is used as a coarse initial estimate to achieve correct phase unwrapping. Note that the phase unwrapping is done as many times as the number of sinusoids existing in the signal.

A simulation example is shown in Figure 5.14 to demonstrate the algorithm. The signal in this example contains two exponentials of different frequencies. The noise is zero mean AWGN and $N = 32$. Figure 5.14 shows the MSE of frequency estimation for one exponential in the presence of a second exponential. The solid line shows the MSE of the single tone estimator in AWGN, and it achieves the CRB. The plots with symbols from top plot to bottom plot are for different frequency separations between the sinusoids. It can be seen that the correlation based frequency estimation can also be used in multiple tone frequency estimation and the separation of the frequencies determines the accuracy

of the estimator. When the frequency separation increases, the MSE becomes closer to CRB. At high SNR, it is the bias due to the second exponential that causes flattening of the plot. Figure 5.15 shows the performance of the multiple tone frequency estimation in multiplicative noise. The plot with square sign represents the estimator for single tone in multiplicative noise, where $f_1 = 0.124, f_d = 0.005$. The plot with star sign is the frequency estimator in the presence of another exponential in multiplicative noise, where $f_1 = 0.124, f_2 = 0.224, f_d = 0.005$. As a comparison, the frequency estimator in the presence of another exponential (without fading as in Figure 5.1.4) in AWGN is shown in circled line. It is seen that the multiplicative noise deteriorates the performance of the multiple tone frequency estimation, however, in the presence of multiplicative noise, the correlation based frequency estimator for single tone exponential has similar performance with the multiple tone estimator. On the other hand, the estimator in the presence of another exponential increases the threshold of the correlation based estimator due to the bias as shown in Figure 5.15.

Figure 5.14: The MSE of the correlation based multiple frequency estimation in AWGN

Figure 5.15: The MSE of the correlation based multiple frequency estimation in multiplicative noise

5.7 Comparison of the Correlation Based Frequency Estimation with Other Estimation Techniques

5.7.1 Periodogram Based Frequency Estimation

In the context of frequency estimation, MLE is a popular technique and for single tone frequency estimation, it is easily obtained using the Fourier transform of the data via the periodogram. In particular, the MLE of the frequency is the frequency where the periodogram attains its maximum [56]. This is usually implemented using an FFT [3]. In general, the fewer data points are used in the FFT, the worse the accuracy of the estimation. To reduce errors due to frequency quantization the data should be zero padded

so that a large number of frequency bins are computed. Alternatively, a coarse FFT search followed by a fine search can be implemented. To find the MLE for the parameters of multiple complex sinusoids in complex white noise, a highly nonlinear function of the unknown frequencies need to be maximized, and requires a search over multi-dimensional space which is complicated and impractical to implement. However, if the multiple frequencies are separated by much more than $1/N$, then the periodogram will yield discernable peaks even in multiplicative noise, if the Doppler spread is not very large. For closed spaced frequencies, to overcome the problems of periodogram, high resolution frequency estimators have been proposed in the literature. This will be discussed in the next section. The relation of the periodogram to the optimal window of the correlation based estimation has been discussed in Chapter 3 Section 3.3.1.

5.7.2 MUSIC Algorithm

A class of methods for estimating the frequency of a noise corrupted signal, which has received significant attention, is based on the eigen-decomposition of the sample covariance matrix. A representative class is the MUSIC (Multiple Signal Characterization) algorithm. The MUSIC algorithm employ the properties of the $M \times M$ theoretical autocorrelation matrix. The key property of the theoretical autocorrelation matrix are as follows:

1. The matrix is composed of a signal autocorrelation matrix and a noise autocorrelation matrix,
2. The signal autocorrelation matrix is not full rank,
3. The signal space is orthogonal to the noise space.

The MUSIC algorithm is based on the orthogonality of the the autocorrelation matrices.

Assume the presence of a single frequency component in white noise, in this case, we have

$$x(n) = A_1 \exp(j\omega_1 n) + z(n) \quad (5.7.11)$$

where A_1 is the amplitude of the sinusoid, and $z(n)$ is the zero mean white noise with variance σ_z^2 . Recall that the autocorrelation of the signal can be expressed as

$$r(k) = A_1^2 \exp(j\omega_1 k) + \sigma_z^2 \delta(k) \quad (5.7.12)$$

Writing the above equation in matrix form, we see that

$$\mathbf{R}_x = \mathbf{R}_s + \mathbf{R}_n \quad (5.7.13)$$

$$\mathbf{R}_s = \begin{bmatrix} A_1^2 & A_1^2 \exp(-j\omega_1) & \cdots & A_1^2 \exp(-j\omega_1(M-1)) \\ A_1^2 \exp(j\omega_1) & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & A_1^2 \exp(-j\omega_1) \\ A_1^2 \exp(j\omega_1(M-1)) & A_1^2 \exp(j\omega_1) & A_1^2 \exp(-j\omega_1) & A_1^2 \end{bmatrix} \quad (5.7.14)$$

where M is the size of the correlation matrix and $\mathbf{R}_n = \sigma_z^2 \mathbf{I}$.

Define $\mathbf{e}_1 = [1 \ \exp(j\omega_1) \ \dots \ \exp(-j\omega_1(M-1))]^T$, then we have $\mathbf{R}_s = A_1^2 \mathbf{e}_1 \mathbf{e}_1^H$. By inspection, $\text{rank}(\mathbf{R}_s) = 1$, so there is only one nonzero eigenvalue $\lambda \neq 0$, and moreover,

$$\mathbf{R}_s \mathbf{e}_1 = A_1^2 \mathbf{e}_1 \mathbf{e}_1^H \mathbf{e}_1 = M A_1^2 \mathbf{e}_1 \quad (5.7.15)$$

Specifically, for $M = 2$, using (5.7.15), we can easily show that frequency estimation from MUSIC is given by $-j \arg(r(1))$. This is also the popular pulse pair frequency estimator used in radar and sonar applications [69]. The statistical performance of the MUSIC is analyzed and it is shown that MUSIC is a large samples approximation of MLE if and only if the signals are uncorrelated [22]. In Section 5.7.4, We will compare the MUSIC algorithm in the presence of multiplicative noise and colored noise for single tone and multiple tone frequency estimation. It will be shown that MUSIC has the best resolution for multiple tone frequencies estimation, but in term of achieving the CRB it has poorer performance compared to correlation based method.

5.7.3 Iterative Filtering Algorithm

The iterative filtering algorithm (IFA) is first proposed as an approximation to MLE [6]. At first, the received signal is filtered with an AR filter and then an AR estimation method is applied to the output data of the filter. The Burg algorithm has been found to work well since it constrains the poles to be within the unit circle, rendering a stable AR filter. In summery, the IFA is as follows [6]:

1. Filter $x(n)$ with a first order AR filter to yield $y^k(n)$.
2. Use $y^k(n)$ to estimate and update the AR parameters
3. Repeat steps 1-2 until convergent to the true frequency.

where $x(n)$ is the noise corrupted signal. The initial AR parameter can be set as zero. Convergence is obtained when the zeros of the filter are close to the unit circle. The

iterative filtering algorithm can also be applied to the application of multiple frequency estimation in additive white noise [7–9].

5.7.4 Comparison Examples

In this section, we compare the performance of various frequency estimators in different noise environments. A comparison is made between the following estimators:

1. MLE, when the frequencies are separated by much more than $1/N$, the MLE simplified as resolving the peak of the periodogram.
2. Correlation based frequency estimator, we use DFT as the initial estimate to resolve the phase unwrapping ambiguity, followed with the multiple lags of correlation based estimator.
3. Iterative filtering algorithm. In the simulations, for each SNR, 32 iterations are computed.
4. MUSIC algorithm. FFT of length 1024 are used to calculate the pseudo spectrum.

The data chosen for the simulation consist of one complex sinusoids in complex white noise, colored noise and multiplicative noise, with $f_0 = 0.124$ and $N = 32$. The colored noise is modeled as an MA process with coefficients $[1, -2.8, 1.6, -1.4]$, and the multiplicative noise is a complex circular gaussian process with the Doppler spread of $f_d = 0.003$.

Figure 5.16 summarizes the frequency estimation accuracy of the algorithms described in the presence of additive white noise. As seen for Figure 5.16, the MLE and correlation based frequency estimators have the best performance, attaining CRB down to a SNR

threshold of -5dB. The MUSIC algorithm has acceptable performance, around 4dB above the CRB. The MUSIC algorithm employs $M = 8$ since for smaller values of M the spectral peaks are not resolvable. Increasing M will increase the accuracy of MUSIC, but the computational complexity will increase dramatically. The IFA has similar SNR threshold, but due to the inherent bias, the performance does not improve at high SNR as the bias persists. Figure 5.17 shows the performance of the algorithms in the presence of colored noise. As expected, the SNR threshold is larger than that of white noise. The correlation based algorithm and the MLE still has the best performance, and the IFA has better SNR threshold than MUSIC. Figure 5.18 shows the performance of the various estimators in the presence of multiplicative noise, and Figure 5.19 shows the performance comparison in the presence of multiplicative noise and additive colored noise. From these simulations, we conclude that the correlation based estimator has the best performance for single tone frequency estimation in different noise environments. However, since we use DFT as the initial estimate to resolve phase unwrapping, the accuracy of the estimator is determined by the DFT coarse search. In the presence of multiple tone frequencies, MUSIC as a high resolution frequency estimation method is of advantage among this algorithms. Figure 5.20 is the performance comparison of MUSIC and correlation based estimator using DFT phase unwrapping for multiple frequency estimation in AWGN. The accuracy of correlation based estimator and MUSIC in the presence of colored noise and multiplicative is similar with Figure 5.18 and Figure 5.19. In Figure 5.20, the circled lines show the correlation based estimation and the plus lines represent the MUSIC based estimation algorithm. From the top plot to the bottom plot of circled lines, the frequencies are selected as $f_1 = 0.124$, $f_2 = 0.154$, $f_1 = 0.124$, $f_2 = 0.254$, $f_1 = 0.124$, $f_2 = 0.354$. It

is seen from Figure 5.20 that when the two sinusoids are spaced closer than $1/N$, i. e. , the Fourier resolution, the DFT peak search can not locate the frequency accurately, thus the performance of the correlation based estimator deteriorates, but MUSIC provides estimation with better resolution. When the spacing of the two sinusoids increases, the variance of the correlation based estimator using DFT phase unwrapping is closer to CRB. As discussed in section 5.6, the correlation based frequency estimator use DFT as an initial estimate to resolve phase unwrapping, as such, the resolution is within $1/N$ for multiple tone frequency estimation.

5.7.5 Computational Complexity of Various Frequency Estimators

In comparing various estimation approaches, the computational complexity of algorithms should be considered. Generally, the eigenanalysis techniques are computationally very intensive, especially for multiple tone frequency estimation because it involve large eigenvalue decomposition of matrices. The complexity of the algorithm is around $O(L^3)$, where L is the size of the matrix.

Of the methods considered, the IFA requires least amount of computations, with the total computational complexity of $O(N \log N)$ [9], which is comparable to that of an N point fast Fourier transform. However, the performance of the IFA is not satisfying and the bias persist at high SNR.

The periodogram can be efficiently implemented using the FFT, which requires about $6N \log_2 N$ real-valued computations. In the presence of white noise, the MLE can also be

implemented efficiently using FFT, but in order to achieve the CRB, the computational load is heavy since a significant amount of zero padding is required.

As mentioned in the introduction, correlation-based methods are an alternative to the periodogram based algorithm to reduce the complexity. The total number of multiplications for the correlation based estimator using DFT phase unwrapping technique is around $2N \log N + N - 1$. Although the correlation based algorithm has a marginally higher computational complexity than IFA, it is much more computational efficient than MLE and eigenvalue based algorithm. Using the DFT initial estimate as phase unwrapping, the SNR threshold is quite low and approaches that of the MLE. The correlation based algorithms can also be applied to the multiple frequency estimation. In this case, if the frequency separation is larger than $1/N$, it can achieve the best performance while maintaining a low computational complexity.

Figure 5.16: Comparison of several single frequency estimators in AWGN

Figure 5.17: Comparison of several single frequency estimators in additive colored noise

Figure 5.18: Comparison of several single frequency estimators in multiplicative noise and AWGN

Figure 5.19: Comparison of several single frequency estimators in the presence of multiplicative noise and additive colored noise

Figure 5.20: Comparison of correlation based estimator using DFT phase unwrapping and MUSIC algorithm in the presence of multiple frequency

5.8 Conclusions

This chapter has discussed the frequency estimation in the presence of both additive colored noise and multiplicative noise. By prior estimation of the Doppler spread and the order of the colored noise, the performance of the correlation based estimator can be improved. It is illustrated via simulation that the correlation based frequency estimation is suitable for various multiplicative and additive noise conditions. We have also discussed different phase unwrapping techniques, and simulation shows that in multiplicative noise, DFT based phase unwrapping has a better SNR threshold. Finally, we have discussed MLE, MUSIC, IFA and correlation based frequency estimator in different noise environments. The multiple frequency estimation is also discussed. In comparing computational complexity and mean square error performances, it is shown that, overall, the correlation based method provides the most suitable approach for frequency estimation.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

The work in this thesis comprehensively investigated the correlation based frequency estimation in AWGN, additive colored noise and multiplicative noise (fading) environments. In this section, a summary of the work accomplished in this thesis is presented. Concluding remarks for the frequency estimation methods in different fading channels are also highlighted.

The frequency estimation in AWGN for damped or undamped complex exponential signals has been extensively studied in the literature. However, it turns out that in some applications, the additive noise model is not fully satisfactory for describing all the aspects of the received signal. In most cases, a signal having a time varying amplitude is received due to the fluctuating channel conditions and additive returned echoes. This is commonly the case in the applications noted in this thesis. In addition, the case of additive colored

noise is also taken into account and the performance is evaluated considering complete channel conditions.

To evaluate the performance of the estimators, the CRB on the estimate variances is given in Chapter 2. Here we provide a general framework to evaluate the CRB both in time domain and frequency domain. It has shown that under high SNR conditions, and in certain multiplicative noise models (e.g. Jake's model), the covariance matrix involved in CRB evaluation tends to be singular which makes the evaluation of the CRB numerically unstable. In an effort to improve the numerical stability, the CMT technique is proposed when the covariance matrix of the signal is singular. This technique evaluates the CRB via time domain processing and is robust and accurate when the number of data samples is small.

For CRB evaluation with a large number of samples, for computational simplicity, we also propose the use of DFT-based frequency domain CRB evaluation in conjunction with the CMT technique. Using this approach, the asymptotic CRB have been derived using the periodogram of the data in frequency domain. It is noted that this technique is accurate under the condition that the channel is a fast fading channel. For a finite duration sequence, the mean value of the periodogram contains a bias for the true power spectral density and, consequently, the periodogram-based CRB obtained in frequency domain is not accurate for finite data samples. We propose an improved periodogram-based CRB based on the Blackman-Tukey power spectral estimate. As an improved version of periodogram-based CRB, the method shows a faster convergence to the exact CRB with the increasing number of data samples. Finally, a closed form expression for

the large sample CRB is given and a general expression for an asymptotic CRB in the presence of multiplicative noise and colored additive noise is provided.

Chapter 3 investigated the performance of the correlation-based frequency estimator in the presence of additive colored noise. In this chapter, we have modelled the broadband colored noise as an MA process in order to obtain an optimal estimator. It is shown that the variance of the proposed estimator using multiple lags of correlations asymptotically approaches CRB for large data samples. The estimator has also shown to be unbiased for low and medium SNR, whereas the estimation using the DFT-based peak search is biased and has a high SNR threshold level. For practical implementation, based on the error analysis of the estimator, frequency estimation from multiple lags of correlations with different phase unwrapping techniques were compared. The optimal linear weights that were derived can be used to achieve CRB at any length of data samples. Several alternative windows were also compared with the optimal window for reduced computational complexity. In this chapter, we have demonstrated that better estimation to DFT fine peak search is possible with use of an iterative filtering based initial phase un-wrapping technique.

In Chapter 3, we have also shown by using all the lags of the correlations, the frequency estimator can be related to the MLE, and the performance is the same as that of MLE when the DFT phase unwrapping is used. However, this is not true in the presence of multiplicative noise. The multiplicative noise seriously degrades the performance of the correlation based frequency estimator, although it is shown that the efficiency (with respect to the CRB) of the estimator is not deteriorating.

The performance of the correlation based frequency estimator for different fading conditions is presented in Chapter 4. Based on the performance evaluation in multiplicative noise, direct insights into the selection of the optimal lag value for estimation is provided. The Doppler spread of the fading channel is also estimated because the optimal frequency estimator requires the spectral bandwidth information of the signal. It is shown that for all kind of fading channels (either Rician or Rayleigh fading), by proper selection of the optimal lag value used in the correlation estimation, the variance of the frequency estimator can be made closer to the CRB. Using the Doppler spread information, it is shown how an estimator based on multiple lags of correlations can be used to improve the performance at low signal to noise ratios.

Finally, a general model including both additive colored noise and multiplicative noise is provided in Chapter 5 and the correlation based method is also used in the multiple frequency estimation. Furthermore, different multiple frequency estimation techniques are compared, and we conclude that the correlation based method is the most suitable approach for frequency estimation.

6.2 Future Work

As noted previously, most practical applications encounter the presence of a multiplicative noise model. The thesis has mainly focused on the frequency estimation in a flat fading channel, which has widespread applications in mobile and personal radio communications. Identifying the time varying multipath channels becomes increasingly important since the

parameter estimation techniques depend on the channel conditions. The multipath effect causes the time delay (delay spread) of the transmitted signal while the fading effect causes the Doppler shift (Doppler spread).

The assumption of flat fading implies that the signal bandwidth is considerably smaller than the reciprocal of multipath delay spread. For wideband signal transmissions, more generally, the wireless channel can be described as a frequency selective channel. Using a tap delay line representation for wideband communications [58], the work presented here can be extended for a frequency selective fading channel, which could be a challenging problem.

Based on the model representations of the wireless channel, the parameter estimation in the presence of multiplicative noise can be further extended to cover amplitude estimation, phase estimation and time delay estimation. For example, various such parameter estimation techniques can be used in practical situations, specifically in signal processing for communication applications under the model of multiplicative noise. These applications include: estimation channel matrix for space-time processing (this is amplitude estimation) carrier offset estimation/Doppler compensation (frequency estimation) carrier synchronization (phase estimation) asynchronous DS-CDMA code timing extraction (usually using sub-space techniques).

Another challenging application is to estimate the Doppler shift in underwater acoustic communication systems. In these systems, carrier tracking and symbol synchronization are severely affected by Doppler shift induced by relative motion of the transmitting and receiving structures which will induce a multiplicative noise. It is noted that, the under-

water acoustic channel has limitations for high data rate transfers due to high frequency attenuation. Estimating the Doppler shift associated with wideband signals in the presence of multiplicative noise presents a major challenge to the modem designers. The multiple frequency estimation in the presence of multiplicative noise in array signal processing is another research topic. Due to the interference of co-channels and signals, the performance evaluation of the MUSIC algorithm in the presence of multiple frequencies and multiplicative noise could be of interest for the future research.

Author's Publications

- [1] Z. Wang and S. S. Abeysekera, "Frequency estimation via multiple lags of correlations in the presence of broadband colored noise," *Signal Processing*, vol. 86, pp. 3371-3387, 2006.
- [2] Z. Wang and S. S. Abeysekera, "Performance of correlation based frequency estimation methods in fading channels," *IEEE Transactions on Vehicular Technology*, vol. 55, no. 4, pp. 1281-1290, 2006.
- [3] Z. Wang and S. S. Abeysekera, "Asymptotic Bounds for Frequency Estimation in the Presence of Multiplicative Noise," *Journal of Applied Signal Processing*, accepted for publication.
- [4] Z. Wang and S. S. Abeysekera, "Frequency estimation from multiple lags of correlations in the presence of MA colored noise," in *IEEE International Conference on Acoustic, Speech and Signal Processing*, Philadelphia, PA, USA, 2005, pp. 693-696.
- [5] Z. Wang and S. S. Abeysekera, "Frequency estimation using the pulse-pair method in different fading environments," in *IEEE International Conference on Acoustics, Speech, and Signal Processing*, Hong Kong, Apr. 2003, pp. 517-520.
- [6] Z. Wang and S. S. Abeysekera, "Bounds on frequency estimation under additive and multiplicative noise," in *The Fourth IEEE International Conference on Information, Communications and Signal Processing, and the Fourth Pacific Rim Conference on Multimedia*, Singapore, Dec. 2003, pp. 15-18.
- [7] S. S. Abeysekera and Z. Wang, "Performance of the pulse pair method with an optimal lag value for frequency estimation in fading channels," in *IEEE International Symposium on Circuits and Systems*, Japan, May 2005, pp. 1734-1737.

Bibliography

- [1] B. Parat, *Digital Processing of Random Signals: Theory and Methods*. New Jersey: Prentice Hall international editions, 1994.
- [2] K. S. Miller and M. M. Rochwarger, "A covariance approach to spectral moment estimation," *IEEE Transactions on Information Theory*, vol. IT-18, no. 5, pp. 588–596, 1972.
- [3] D. C. Rife and R. R. Boorstyn, "Single-tone parameter estimation from discrete-time observations," *IEEE Transactions on Information Theory*, vol. 20, no. 5, pp. 591–598, 1974.
- [4] —, "Multiple tone parameter estimation from discrete-time observations," *Bell Syst. Tech. J.*, vol. 55-9, pp. 1389–1410, 1976.
- [5] S. M. Kay, *Modern Spectral Estimation Theory and Application*. Prentice Hall: PTR Prentice Hall, pp. 540, 1988.
- [6] —, "Accurate frequency estimation at low signal to noise ratio," *IEEE Transactions on Signal Processing*, vol. 32, no. 3, pp. 540–547, 1984.
- [7] T.-H. Li and B. Kedem, "Strong consistency of the contraction mapping method for frequency estimation," *IEEE Transactions on Information Theory*, vol. 39, no. 3, pp. 989–997, 1993.
- [8] —, "Iterative filtering for multiple frequency estimation," *IEEE Transactions on Signal Processing*, vol. 42, no. 5, pp. 1120–1132, 1994.

- [9] T.-H. Li and K.-S. Song, "Asymptotic analysis of a fast algorithm for efficient multiple frequency estimation," *IEEE Transactions on Information Theory*, vol. 48, no. 10, pp. 2709–2720, 2002.
- [10] B. G. Quinn, "Estimating frequency by interpolation using Fourier coefficients," *IEEE Transactions on Signal Processing*, vol. 42, no. 5, pp. 1264–1268, 1994.
- [11] . "Estimation of frequency, amplitude, and phase from the DFT of a time series," *IEEE Transactions on Signal Processing*, vol. 45, no. 3, pp. 814–817, 1997.
- [12] S. N. Crozier and K. W. Moreland, "Performance of a simple delay multiply average technique for frequency estimation," in *Canadian Conf. Elect. Comput. Eng.*, 1992, paper WM-10.3.
- [13] S. W. Lang and B. R. Musicus, "Frequency estimation from phase differences," in *IEEE International Conference on Acoustic, Speech and Signal Processing*, vol. 4, 1989, pp. 2141–2143.
- [14] S. M. Kay, "A fast and accurate single frequency estimation by linear prediction," *IEEE Transactions on Signal Processing*, vol. 37, no. 7, pp. 1987–1990, 1989.
- [15] B. Völcker and P. Händel, "Frequency estimation from proper sets of correlations," *IEEE Transactions on Signal Processing*, vol. 50, no. 4, pp. 791–802, 2002.
- [16] B. Lovell and R. Williamson, "The statistical performance of some instantaneous frequency estimators," *IEEE Transactions on Signal Processing*, vol. 40, no. 7, pp. 1708–1723, 1992.
- [17] Z. Wang and S. S. Abeysekera, "Efficient frequency estimation using the pulse-pair method at different lags," in *IEEE International Conference on Acoustics, Speech, and Signal Processing*, Hong Kong, Apr. 2003, pp. 517–520.
- [18] M. S. Puckette and J. C. Brown, "Accuracy of frequency estimates using the phase vocoder," *IEEE Transactions on Speech and Audio Processing*, vol. 6, no. 2, pp. 166–176, 1998.

- [19] J. A. K. P. Padhi, S. S. Abeysekera and S. George, "Scaling of audio signals using frequency domain techniques," in *IEEE 34th Annual Asilomar Conference on Signals, Systems and Computers*, Pacific Grove, California, USA, 2000, pp. 1484–1488.
- [20] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramer-Rao bound," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 5, pp. 720–741, 1989.
- [21] M. P. Fitz, "Further results in the fast estimation of a single frequency," *IEEE Transactions on Communications*, vol. 42, no. 2/3/4, pp. 862–864, 1994.
- [22] P. Händel, A. Eriksson, and T. Wigren, "Performance analysis of a correlation based single tone frequency estimator," *Signal Processing*, vol. 44, pp. 223–231, 1995.
- [23] J. M. Francos and B. Friedlander, "Bounds for estimation of complex exponential in unknown colored noise," *IEEE Transactions on Signal Processing*, vol. 43, no. 9, pp. 2176–2185, 1995.
- [24] P. Stoica, "Cisoid parameter estimation in the colored noise case: Asymptotic Cramer-Rao bound, maximum likelihood, and nonlinear least-squares," *IEEE Transactions on Signal Processing*, vol. 45, no. 8, pp. 2048–2059, 1997.
- [25] B. Friedlander and A. J. Weiss, "Direction finding using noise covariance modeling," *IEEE Transactions on Signal Processing*, vol. 43, no. 7, pp. 1557–1567, 1995.
- [26] J. P. L. Cadre, "Parametric methods for spatial signal processing in the presence of unknown colored noise fields," *IEEE Transactions on Acoustic, Speech and Signal Processing*, vol. 37, no. 7, pp. 965–983, 1989.
- [27] F. Gini and M. Luise, "Cramer Rao bounds in the parametric estimation of fading radiotransmission channels," *IEEE Transactions on Communications*, vol. 46, no. 10, pp. 1390–1398, 1998.
- [28] M. Ghogho, A. Swami, and B. Garel, "Performance analysis of cyclic statistics for the estimation of harmonics in multiplicative and additive noise," *IEEE Transactions on Signal Processing*, vol. 47, no. 12, pp. 3235–3249, 1999.

- [29] M. Ghogho, A. Swami, and T. S. Durrani, "Frequency estimation in the presence of Doppler spread: Performance analysis," *IEEE Transactions on Signal Processing*, vol. 49, no. 4, pp. 777–789, 2001.
- [30] S. S. Abeysekera, "Performance of pulse-pair method of Doppler estimation," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 34, no. 2, pp. 520–531, 1998.
- [31] V. M. Baronkin, Y. V. Zakharov, and T. C. Tozer, "Frequency estimation in slowly fading multipath channels," *IEEE Transactions on Communications*, vol. 50, no. 11, pp. 1848–1859, 2002.
- [32] P. Stoica, O. Besson, and A. B. Gershman, "Direction of arrival estimation of an amplitude-distorted wavefront," *IEEE Transactions on Signal Processing*, vol. 49, no. 2, pp. 269–275, 2002.
- [33] O. Besson, F. Vincent, P. Stoica, and A. B. Gershman, "Approximate maximum likelihood estimators for array processing in multiplicative noise environments," *IEEE Transactions on Signal Processing*, vol. 48, no. 9, pp. 2506–2518, 2000.
- [34] J. Ringelstein, A. B. Gershman, and J. F. Bohme, "Direction finding in random inhomogeneous media in the presence of multiplicative noise," *IEEE Signal Processing Letters*, vol. 7, no. 10, pp. 269–272, 2000.
- [35] F. Gini, G. B. Giannakis, M. Greco, and G. T. Zhou, "Time-averaged subspace methods for radar clutter texture retrieval," *IEEE Transactions on Signal Processing*, vol. 49, no. 9, pp. 1886–1898, 2001.
- [36] W. Y. Kuo and M. P. Fitz, "Frequency offset compensation of pilot symbol assisted modulation in frequency flat fading," *IEEE Transactions on Communications*, vol. 45, no. 11, pp. 1412–1416, 1997.
- [37] S. S. Abeysekera and Z. Wang, "Performance of the pulse pair method with an optimal lag value for frequency estimation in fading channels," in *IEEE International Symposium on Circuits and Systems*, Japan, May 2005, pp. 1734–1737.

- [38] R. Dabora, J. Goldberg, and H. Messer, "Inherent limitations in data-aided time synchronization of continuous phase-modulation signals over time-selective fading channels," *IEEE Transactions on Signal Processing*, vol. 50, no. 6, pp. 1470–1482, 2002.
- [39] T. McWhorter and L. Scharf, "Cramer-Rao bounds for deterministic model analysis," *IEEE Transactions on Signal Processing*, vol. 41, no. 5, pp. 1847–1865, 1993.
- [40] S. M. Kay and V. Nagesha, "Maximum likelihood estimation of signals in autoregressive noise," *IEEE Transactions on Signal Processing*, vol. 42, no. 1, pp. 88–101, 1994.
- [41] D. Sengupta and S. M. Kay, "Parameter estimation and GLRT detection in colored non-Gaussian autoregressive processes," *IEEE Transactions on Acoustic, Speech and Signal Processing*, vol. 38, no. 10, pp. 1661–1676, 1990.
- [42] Y. Zhao, "Frequency-domain maximum likelihood estimation for automatic speech recognition in additive and convolutive noises," *IEEE Transactions on Speech and Audio Processing*, vol. 8, no. 3, pp. 255–266, 2000.
- [43] M. Bertocco, C. Offelli, and D. Petri, "Analysis of damped sinusoidal signals via a frequency domain interpolation algorithm," *IEEE Transactions on Instrumentation and Measurement*, vol. 43, no. 2, pp. 245–250, 1994.
- [44] J. Lin, J. G. Proakis, and F. Ling, "Optimal tracking of time varying channels: A frequency domain approach for known and new algorithms," *IEEE Transactions on Selected Areas in Communications*, vol. 13, no. 1, pp. 141–153, 1995.
- [45] J. M. B. Dias and J. M. B. Leito, "Nonparametric estimation of mean Doppler and spectral width," *IEEE Transactions on Geosciences and Remote Sensing*, vol. 38, no. 1, pp. 271–282, 2000.
- [46] B. Friedlander and B. Porat, "The exact Cramer-Rao bound for Gaussian autoregressive process," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 25, no. 1, pp. 1456–1458, 1989.

- [47] B. Porat and B. Friedlander, "Computation of the exact information matrix of Gaussian time series with stationary random components," *IEEE Transactions on Acoustic, Speech and Signal Processing*, vol. 34, no. 1, pp. 118–130, 1986.
- [48] A. Zeira and A. Nehorai, "Frequency domain Cramer Rao bound for Gaussian processes," *IEEE Transactions on Signal Processing*, vol. 38, no. 6, pp. 1063–1066, 1990.
- [49] P. Whittle, "The analysis of multiple stationary time series," *J. Roy. Stat. Soc.*, vol. 15, pp. 125–139, 1953.
- [50] C. W. Therrien, *Discrete Random Signals and Statistical Signal Processing*. NJ: Prentice-hall, Englewood Cliffs, 2000.
- [51] B. Friedlander, "On the Cramer-Rao bound for time delay and Doppler estimation," *IEEE Transactions on Information Theory*, vol. IT-30, no. 3, pp. 575–580, 1984.
- [52] R. Frehlich, "Cramer Rao bound for Gaussian random processes and applications to radar processing of atmosphere signals," *IEEE Transactions on Geosciences and Remote Sensing*, vol. 31, no. 6, pp. 1123–1131, 1993.
- [53] D. N. Swingler, "Approximate bounds on frequency estimates for short cissoids in colored noise," *IEEE Transactions on Signal Processing*, vol. 46, no. 5, pp. 1456–1458, 1998.
- [54] , "Frequency estimation for closely spaced sinusoids: Simple approximations to the Cramer-Rao lower bound," *IEEE Transactions on Signal Processing*, vol. 41, no. 1, pp. 489–495, 1993.
- [55] Y. Yao and T.-S. Ng, "Feed-forward carrier frequency and timing synchronization for MSK modulation," in *ICICS-PCM*, Singapore, 2003.
- [56] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*. Englewood Cliffs, NJ: PTR Prentice Hall, pp. 540, 1993.
- [57] B. Picindono, "On circularity," *IEEE Transactions on Signal Processing*, vol. 42, no. 12, pp. 3473–3482, 1994.

- [58] J. G. Proakis, *Digital Communications*. Singapore: McGraw-Hill, Inc., 1995.
- [59] J. K. Cavers, "An analysis of pilot symbol assisted modulation for Rayleigh fading channels," *IEEE Transactions on Vehicular Technology*, vol. 40, no. 4, pp. 686–693, 1991.
- [60] J. M. Wozencraft and I. M. Jacobs, *Principles of Communication Engineering*. USA: Waveland Press, 1990.
- [61] M. Morelli and U. Mengali, "An improved frequency offset estimator for OFDM applications," *IEEE Communications Letters*, vol. 3, no. 3, pp. 75–77, 1999.
- [62] K. E. Baddour and N. C. Beaulieu, "Autoregressive models for fading channel simulation," in *The IEEE Global Telecommunications Conference*, San Antonio, Nov. 2001, pp. 1187–1192.
- [63] L. Rugini, P. Banelli, and S. Cacopardi, "Regularized MMSE multiuser detection using covariance matrix tapering," in *IEEE International Conference on Communication*, Anchorage, AK, May 2003, pp. 2460–2464.
- [64] P. Stoica, "Parameter estimation problems with singular information matrices," *IEEE Transactions on Signal Processing*, vol. 49, no. 1, pp. 87–90, 2001.
- [65] J. R. Guerci, "Theory and application of covariance matrix tapers for robust adaptive beamforming," *IEEE Transactions on Signal Processing*, vol. 47, no. 4, pp. 977–985, 1999.
- [66] J. G. Proakis, *Digital Signal Processing Principles, Algorithms, and Applications*. NJ: Prentice hall international editions, 1996.
- [67] B. E. Hajek, "On the strong information singularity of certain stationary processes," *IEEE Transactions on Information Theory*, vol. 25, no. 5, pp. 605–609, 1979.
- [68] S. D. B. Geoffrey W.K. Colman and N. C. Beaulieu, "An ARMA multipath fading simulator," in *The 7th Annual Virginia Tech Symposium on Wireless Personal Communications*, Virginia.

- [69] F. C. Benham, H. L. Groginsky, A. S. Soltes, and G. Works, "Pulse pair estimation of Doppler spectrum parameters," Raytheon Co., Wayland, MA USA, Tech. Rep. F-19628-71-C-0126, Feb 1972.
- [70] G. W. Lank, I. S. Reed, and G. E. Pollon, "A semicoherent detection and Doppler estimation statistic," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 9, no. 2, pp. 151–165, 1973.
- [71] S. S. Abeysekera, "Efficient frequency estimation using the pulse-pair method at different lags," in *IEEE International Symposium on Circuits and Systems*, Bangkok, Thailand, May 2003.
- [72] S. M. Kay and V. Nagesha, "Maximum likelihood estimation of signals in autoregressive noise," *IEEE Transactions on Signal Processing*, vol. 42, no. 1, pp. 88–101, 1994.
- [73] V. Clarkson, P. J. Kootsookos, and B. G. Quinn, "Analysis of the variance threshold of Kay's weighted linear predictor frequency estimator," *IEEE Transactions on Signal Processing*, vol. 42, no. 9, pp. 2370–2379, 1994.
- [74] T. Brown and M. M. Wang, "An iterative algorithm for single frequency estimation," *IEEE Transactions on Signal Processing*, vol. 50, no. 11, pp. 2671–2682, 2002.
- [75] D. W. Tufts and P. D. Fiore, "Simple, effective estimation of frequency based on prony's method," in *IEEE International Conference on Acoustics, Speech and Signal Processing*, vol. 5, Atlanta, GA, USA, May, 1996, pp. 2801–2804.
- [76] Q. Liu, "Frequency synchronization in global satellite communications systems," *IEEE Transactions on Communications*, vol. 51, no. 3, pp. 359–365, 2003.
- [77] Z. Wang and S. S. Abeysekera, "Frequency estimation from multiple lags of correlations in the presence of MA colored noise," in *IEEE International Conference on Acoustic, Speech and Signal Processing*, Philadelphia, PA, USA, 2005, pp. 693–696.
- [78] L.-P. Lafrance, M.-A. Cantin, Y. Savaria, S. Sung, and P. Lavoie, "Architecture and performance characterization of hardware and software implementations of the

- crozier frequency estimation algorithm,” in *IEEE International Symposium on Circuits and Systems*, USA, May 2002, pp. 823–826.
- [79] T. S. Rappaport, *Wireless Communications Principles and Practice*. USA: Prentice Hall, 1996.
- [80] H. Akaike, “Fitting autoregressive models for prediction,” *Ann. Inst. of Stat. Math.*, vol. 21, pp. 243–347, 1969.
- [81] ———, “Information theory and extension of the maximum likelihood principle.” Tsahkadsor, Armenian, USSR, 1971.
- [82] G. Liang, D. M. Wilkes, and J. A. Cadzow, “ARMA model order estimation based on the eigenvalues of the covariance matrix,” *IEEE Transactions on Signal Processing*, vol. 41, no. 10, pp. 3003–3009, 1993.