Stability of laminated composites with delaminations

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**STABILITY OF LAMINATED COMPOSITES
WITH DELAMINATIONS**

**YAP CHUN WEE
SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING
2010**

STABILITY OF LAMINATED COMPOSITES WITH DELAMINATIONS

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ABSTRACT

In this thesis, investigation on buckling problem of composite beam is carried out. A new formula which is denoted as CLT method 3 is derived to calculate the effective flexural stiffness of the thin composite beam. Two other existing equations indicated as CLT methods 1 and 2 that can also be used to evaluate the effective bending rigidity are presented. Furthermore, in order to examine the buckling phenomena of moderately thick composite beam, three different collections of expressions designated as FSDT methods 1, 2 and 3 are presented to compute three separate sets of values for the effective flexural stiffness, effective shear modulus and shear correction factor in the first-order shear deformation beam theory.

In conjunction with the above-mentioned equations, various analytical models are developed to analyze the thin and moderately thick composite beams with a single delamination, thin composite beam having semi-embedded delaminations with and without contact and thin composite beam with non-coinciding delaminations under combined axial and transverse loads including contact effect. These mathematical models are formulated via the suitable boundary conditions, kinematical compatibility equations and equilibrium of forces and moments. The contact effect is implemented appropriately into some analytical models to prevent the physically impermissible interpenetration of sub-laminates.

Various parametric studies are then conducted and they include boundary condition, type of composite materials, number of layers, ply orientation, stacking sequence, through-the-thickness location of the delamination, delamination length and along-the-span position of the delamination. For all the single-delaminated and multi-delaminated thin composite beams cases that are examined, the CLT methods 2 and 3 are able to form the desired upper and lower bounds, respectively, for the actual buckling loads. Analogously, for all the single-delaminated moderately thick composite beam cases, the FSDT methods 2 and 3 are competent of generating the required upper and lower limits, respectively.

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NOMENCLATURE

1–, 2– and 3– axes	Principal material coordinate system (see Figure 4.1)
a	Delamination length used in chapters 5 and 6 (see Figure 5.1)
a_1 and a_2	Delamination lengths used in chapters 7 and 8 (see Figures 7.1 and 8.1)
$A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}$ and H_{ij}	Various stiffness coefficients
c_1 and c_2	Constants in the third-order shear deformation theory
$\{C\}$	Column vector containing the unknowns
C_{51}	Coefficient required to evaluate the ${}^3(\overline{EI})$ (see equation 4.74)
C_{ij} for $i = 1..4$ and $j = 1,2$	Coefficients needed to compute the ${}^3(\overline{EI})$ (see equations c.1 to c.8)
d	Distance measuring from the center of the composite beam to the center of the delamination, which is used in chapters 5, 6 and 7 (see Figures 5.1 and 7.1)
d_1 and d_2	Distances measuring from the center of the composite beam to the center of the delaminations, which are used in chapter 8 (see Figure 8.1)
E	Young's modulus of isotropic materials
E_{11} and E_{22}	Young's moduli of fiber-reinforced composite materials along and transverse to the fiber, respectively
${}^j(\overline{EI})_i$ for $j = 1..3$	Three methods of calculating the effective flexural stiffness of fiber-reinforced composite beam for i^{th} beam (see equations 4.45, 4.55 and 4.73)
E_T	Through-thickness modulus of the fiber-reinforced composite material

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G	Shear modulus of isotropic materials
G_{12} , G_{23} and G_{13}	Shear moduli of fiber-reinforced composite materials in the 1-2, 2-3 and 1-3 planes, respectively
${}^j(\overline{Gh})_i$ for $j = 1..3$	Three methods of calculating the effective shear modulus of fiber-reinforced composite beam in first-order shear deformation theory for i^{th} beam (see equations 4.91, 4.96 and 4.105)
H	Total height of the delaminated fiber-reinforced composite beam
H_i	Height of i^{th} beam
$H[\mathbb{N}]$	Heaviside step function (see equation 7.9)
jK_i for $j = 1..3$	Three methods of calculating the shear correction factor of fiber-reinforced composite beam in first-order shear deformation theory for i^{th} beam (see section 4.3.3.1)
L	Total length of composite beam
$[M]$	Square matrix
$M_i(x)$	Moment in i^{th} beam
M_{xx} , M_{yy} and M_{xy}	Moments in the x -, y - and xy - directions, respectively (see Figure 4.3)
n and s	Local coordinates at the boundary (see Figure b.1)
n_x and n_y	x - and y - components of the unit normal vector $\vec{n}$
$\vec{n}$	Unit normal vector along the n local coordinate direction
N	Determination for the degree of polynomial for u_i (see equation 4.26)
N_{xx} , N_{yy} and N_{xy}	Forces in the x -, y - and xy - directions, respectively (see Figure 4.2)
p_c	Contact pressure
P	Axial compressive load on the laminated composite beam

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P_i	Axial compressive force in i^{th} beam
$P_{xx}, P_{yy}, P_{xy}, R_x$ and R_y	Higher order stress resultants in third-order shear deformation theory
q	Uniformly distributed transverse load
$\{Q\}$	Column vector consisting of constants
$Q_i(x)$	Shear force in i^{th} beam
Q_{ij} for $i, j = 1..6$	Reduced stiffnesses
$\overline{Q}_{ij}$ for $i, j = 1..6$	Transformed reduced stiffnesses
Q_x and Q_y	Shear forces in the xz and yz planes, respectively (see Figure 4.2)
S	Tensile strength of fiber or wire
t	Thickness per ply
u, v and w	Displacements in x -, y - and z - directions
u_i for $i = 1..3$	Three displacement fields in the x -, y - and z -directions (see equation 4.26)
$u_o, v_o, w_o, \phi_x, \phi_y, \theta_x, \theta_y, \lambda_x$ and λ_y	Various variables to describe the displacement fields in the classical laminate theory, first-order and third-order shear deformation theories
w_i	Deflection function of i^{th} beam
x, y and z	Cartesian coordinates
$\overline{Y}$	Distance from the top surface of the fiber-reinforced composite beam to the neutral axis of beam 1
$\overline{Y}_i$	Distance from the top surface of the fiber-reinforced composite beam to the neutral axis of i^{th} beam
z_{k-1} and z_k	Distance from middle surface to top and bottom of k^{th} layer, respectively (see Figure 4.4)

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α, β, γ and δ	Coefficients used in the calculation of $^2(\overline{EI})$ (see equations 4.56 to 4.59)
α_{ij}, β_{ij} and ϕ_{ij}	Coefficients required to compute the C_{ij} (see appendix C)
δU and δV	Variational operation on the virtual strain energy and virtual work done by applied forces, respectively
Δ_i^a and Δ_i^b	Change of length on the left and right side of i^{th} beam, respectively, due to bending
ΔP_i	Incremental force in i^{th} beam
$\varepsilon_{xx}^o, \varepsilon_{yy}^o$ and γ_{xy}^o	Membrane strains
ε_{11} (or ε_1), ε_{22} (or ε_2) and ε_{33} (or ε_3)	Normal strains in 1–, 2– and 3– directions, respectively
ε_{xx} (or ε_x), ε_{yy} (or ε_y) and ε_{zz} (or ε_z)	Normal strains in x –, y – and z – directions, respectively
γ_{12} (or $2\varepsilon_{12}$), γ_{23} (or $2\varepsilon_{23}$) and γ_{13} (or $2\varepsilon_{13}$)	Shear strains on the 1–2, 2–3 and 1–3 planes, respectively
γ_{xy} (or $2\varepsilon_{xy}$), γ_{yz} (or $2\varepsilon_{yz}$) and γ_{xz} (or $2\varepsilon_{xz}$)	Shear strains on the xy , yz and xz planes, respectively
κ_{xx}, κ_{yy} and κ_{xy}	Middle surface curvatures
λ_i	Defined as $\lambda_i = \sqrt{\frac{P_i}{(\overline{EI})_i}}$ for i^{th} beam
Ω_0	Domain that is occupied by the mid-plane of the plate
ϕ_i	Rotation function of i^{th} beam
ρ	Density of fiber or wire
$\sum F_x, \sum F_z$ and $\sum M_o$	Summation of forces in x – and z – directions, and moment about point O , respectively
τ_{11} (or σ_1), τ_{22} (or σ_2) and τ_{33} (or σ_3)	Normal stresses in 1–, 2– and 3– directions, respectively

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τ_{12}, τ_{23} and τ_{13}	Shear stresses on the 1–2 , 2–3 and 1–3 planes, respectively
τ_{xx} (or σ_x), τ_{yy} (or σ_y) and τ_{zz} (or σ_z)	Normal stresses in x –, y – and z – directions , respectively
τ_{xy}, τ_{yz} and τ_{xz}	Shear stresses on the xy , yz and xz planes, respectively
θ	Ply orientation
$^1\theta$ and $^2\theta$	Ply-angles for layers in general orientation composite beam
Γ	Boundary of the domain (see Figure b.1)
ν	Poisson’s ratio of isotropic materials
ν_{12} and ν_{21}	Poisson’s ratios of fiber-reinforced composite materials
$\mathcal{G}_i$	Defined as $\mathcal{G}_i = \left(\frac{\frac{P_i}{(EI)_i}}{1 - \frac{P_i}{(Gh)_i}(K_i)} \right)^{\frac{1}{2}}$ for i^{th} beam
$^i\xi_j$ for $i=1..3$ and $j=1..N$	Unknown functions in the polynomial for u_i (see equation 4.26)

CHAPTER ONE: INTRODUCTION

In this chapter, various kinds of composite materials will be introduced. This includes the fiber-reinforced composite which is selected as the composite matter to be examined in this thesis. The common terminologies to be employed in context of the fiber-reinforced composite materials are also addressed. In this chapter, it is pointed out that the main reasons for using the fiber-reinforced composite materials are their high strength-to-density and stiffness-to-density ratios, cost advantage and weight saving. In addition, their practical usages are presented. Then, the origins of delamination within the fiber-reinforced composite materials are described in detail. Finally, the objectives to be accomplished in the current research are listed.

1.1 Different types of composite materials

There are many different types of composites such as fibrous composite materials, laminated composite materials, particulate composite materials and combinations of composite materials. They are made up differently and have their own unique properties. Below are their descriptions.

1.1.1 Fibrous composite materials

This kind of composite materials are made up of fibers in a matrix. It is found that long fibers in different forms are inherently much stiffer and stronger than in bulk form. Some of the more popular fibers are graphite, carbon and boron fibers. These fibers are widely used because of their high strength-to-density and stiffness-to-density ratios. However, to form a structural element to carry load, a binder material known as a matrix must be used to bond these fibers together. However, the matrix not only serves as a support for the fibers, it also protects the fibers and provides stress transfer between broken fibers. The matrix usually has much lower density, stiffness and strength when compared with the fibers. Some of the materials used as matrix are polymers, metals, ceramics and carbon.

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Fiber or Wire	Density, ρ lb/in ³ (kN/m ³)	Tensile Strength, S 10 ³ lb/in ² (GN/m ²)	S/ρ 10 ⁵ in (km)	Tensile Stiffness, E 10 ⁶ lb/in ² (GN/m ²)	E/ρ 10 ⁷ in (Mm)
Aluminum	.097 (26.3)	90 (.62)	9 (24)	10.6 (73)	11 (2.8)
Titanium	.170 (46.1)	280 (1.9)	16 (41)	16.7 (115)	10 (2.5)
Steel	.282 (76.6)	600 (4.1)	21 (54)	30 (207)	11 (2.7)
E-Glass	.092 (25.0)	500 (3.4)	54 (136)	10.5 (72)	11 (2.9)
S-Glass	.090 (24.4)	700 (4.8)	78 (197)	12.5 (86)	14 (3.5)
Carbon	.051 (13.8)	250 (1.7)	49 (123)	27 (190)	53 (14)
Beryllium	.067 (18.2)	250 (1.7)	37 (93)	44 (300)	66 (16)
Boron	.093 (25.2)	500 (3.4)	54 (137)	60 (400)	65 (16)
Graphite	.051 (13.8)	250 (1.7)	49 (123)	37 (250)	72 (18)

Table 1.1: Properties of wires and fibers

Table 1.1 summarizes the specific strength (S/ρ) and specific stiffness (E/ρ) of the commonly used fibers and compares them with traditional metals.

1.1.2 Laminated composite materials

Layers of at least two different materials are bonded together to form the laminated composite materials. The reason of having this lamination is to produce a better material that has the best aspects of the layer and bonding materials. These desirable properties may include strength, stiffness, corrosion resistance, wear resistance, thermal insulation and many others. Such laminated composite materials include bimetals (for example thermostat), clad metals (for example copper-clad aluminum wire), laminated glass (for example safety glass), plastic-based laminates (for example Formica) and laminated fibrous composite.

1.1.3 Particulate composite materials

Particles of one or more materials suspended in a matrix of another material made up the particulate composite materials. Both the particles and matrix can be metallic or non-metallic. There are together four different groups of particulate composite materials. They are nonmetallic particulate in nonmetallic matrix composite materials (for example concrete), metallic particles in nonmetallic matrix composite materials (for example

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aluminum paint), metallic particles in metallic matrix composite materials and nonmetallic particles in metallic matrix composite materials (also known as cermets).

1.1.4 Combinations of composite materials

Many of the composite materials show a mixture of the characteristic of the different groups of composite materials discussed earlier. An obvious example will be the reinforced concrete. It is a particulate composite material as it is made up of gravel in a cement-paste binder. On the other hand, it is also a fibrous composite material as it contains steel reinforcement, which serves to maintain the concrete in compression. Another example will be the laminated fiber-reinforced composite materials, which are composed of fibrous composite materials laminated together. Therefore, they are both laminated and fibrous composite materials. In practical applications, fibrous composite materials with different fiber orientations are stacked together to provide strength and stiffness in various directions.

1.2 Mechanical properties of composite materials

1.2.1 Isotropic materials

For the isotropic materials, upon application of the normal force, there will only be extension of the material in the direction of the applied force and contraction in the directions perpendicular to it, there will not be any shear deformation. Similarly, shear force will not produce any extension or contraction of the material, only shear deformation. Hence, for isotropic materials, the stress-strain relations will simply appear as:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{C_{11}-C_{12}}{2} \right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{C_{11}-C_{12}}{2} \right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{C_{11}-C_{12}}{2} \right) \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} \quad (1.1)$$

where C_{ij} are the coefficient of the compliance matrix, which are functions of mechanical properties.

Only two material properties, Young's modulus and Poisson's ratio are required to evaluate the stiffness matrix. Shear modulus can also be used as a substitute for Young's modulus or Poisson's ratio in computing the stiffness matrix. It is because the three properties are related by the equation:

$$G = \frac{E}{2(1+\nu)} \quad (1.2)$$

1.2.2 Orthotropic materials

For orthotropic materials, similar to the isotropic materials, when a normal force is applied, the material will extend in the direction in which the force is acted, contract in the perpendicular directions and no shear deformation is produced. The difference is that for a given amount of force, the magnitude of extension in one principal direction will not be the same as that of the other principal direction. Therefore, more than one Young's moduli must be used to define the orthotropic materials. Furthermore, different pairs of principal material directions are associated with their individual Poisson's ratios. When there exists a shear force, there will only be shear deformation and no extension or contraction, and the deformation is entirely independent of the various Young's moduli and Poisson's ratios. Moreover, there is no interaction between shearing stresses and shearing strains in different planes. Hence, for orthotropic materials, the stress-strain relations will appear as:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} \quad (1.3)$$

There are nine independent constants of C_{ij} in this stiffness matrix.

1.2.3 Anisotropic materials

For materials that are anisotropic, when a normal force is applied, there will not only be extension of the material in the direction of the applied force and contraction in the directions perpendicular to it, there will also be shear deformation. Similarly, application of the shear force will not only cause shear deformations but also extension and contraction. This effect is known as the shear-extension coupling. Therefore, the stress-strain relations will look like this:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} \quad (1.4)$$

The stiffness matrix relating the stress and strain is symmetric, meaning that there will only be 21 independent constants in the matrix. This is the most general expression within the framework of linear elasticity.

1.3 Terminology used in this report

1.3.1 Laminae and laminate

Figure 1.1: Illustration of a (a) lamina and (b) laminate made up of several laminae

A lamina, as seen in Figure 1.1(a), consists of unidirectional fibers or woven fibers in a matrix. It is the fundamental building block for a laminate. The fibers in the lamina are the main component carrying the loads and are generally very strong and stiff. The roles of the matrix are to hold the fibers together, shelter the fibers and also to distribute the load to the fibers.

Laminae are stacked up to form laminate, as shown in Figure 1.1(b). This stacking process is called lamination. Laminae of different directions can be laminated to cater for the necessary strength and stiffness in various directions. Usually, the bonding material used will be the matrix itself. The resultant fiber-reinforced composite materials are generally regarded as linear elastic materials as the linear elastic fibers are the main strength and stiffness for the laminate.

1.4 Reasons for using fiber-reinforced composite materials

1.4.1 Strength and stiffness advantages

The fiber-reinforced composite materials have very high strength-to-density and stiffness-to-density ratios when compared with most metals, which made them as suitable materials for manufacturing weight-sensitive structures in aerospace industries. One example of fiber-reinforced composite materials with very high specific stiffness is graphite-epoxy, while glass-epoxy or Kelvar-epoxy can be used in applications requiring high specific strength. In addition, the laminae can be arranged in required directions to form a laminate that is capable of withstanding loads exerted in certain directions.

1.4.2 Cost advantage

The cost of some fibers, for example raw graphite and boron fibers has decreased over the years. This is due to the increased manufacturing experience accumulated over the years and more effective manufacturing technologies for mass production. The manufacturing processes required to work on the composite materials are usually simpler and hence cheaper than handling metal materials. The fastener count and bonding operations for composite structures are usually lower than that for the metal structures. Hence, the assembly cost will normally be lower for composite structures. This will also in turn reduce the other related costs such as labor cost required to assemble the parts. Furthermore, the operating cost for a composite structure is usually lower than that of a metallic structure.

In additional, for a factor,

$$MaterialsUtilizationFactor = \frac{rawmaterialweight}{finalpartweight}$$

, some parts that are made of metals may have very high values. This is a typical result of removing a significant amount of material from a metal block to obtain the final part. On the other hand, parts manufactured from the fiber-reinforced composite materials can

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have very low materials utilization factor. It is because composite structures are fabricated as close to the desired shape as possible, requiring minimum machining.

1.4.3 Weight advantages

Generally, weight saving has two advantages for vehicles that are manufactured from fiber-reinforced composite materials. One of them is with a lighter body, less fuel will be needed to travel the same distance. The other is the vehicle for example aircraft or truck can carry a heavier fuel and travel a longer range. This is especially true for space shuttles that will require a body as light as possible and that is able to carry more fuel to accomplish its mission.

1.5 Practical usage of fiber-reinforced composite materials

Fiber-reinforced composite materials have been used extensively in many areas especially the aerospace industries due to reasons mentioned earlier. Their main advantages over various metal alloys are their relatively higher strength-to-density and stiffness-to-density ratios. Their applications include:

1.5.1 Military aircrafts

Fiber-reinforced composite materials are used to make various parts of military aircrafts. The main reasons are weight and cost savings and eases of handling and design. Fiber-reinforced composite materials like the advanced composite materials, for example graphite-epoxy is occasionally used as substitutes for strategic metals such as titanium that are less readily available. For aircraft with forward-swept wings like Grumman X-29A, the inherent coupling effect of the composite laminate with laminae at various angles to the wing axis is used to counter the inevitable aerodynamic divergence when in flight. Some of the other significant milestones include Vought A-7 Speedbrake, Vought S-3A Spoiler, Boeing F-18, Boeing AV-8B Harrier, Grumman X-29A, Northrop Grumman B-2, Lockheed Martin F-22 and others. These aircraft utilized the benefits of using fiber-reinforced composite materials in one way or the other.

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1.5.2 Civil aircrafts

Parts of civil aircraft are manufactured from fiber-reinforced composite materials due to the same reasons as mentioned in previous section. However, there is a great leap in the use of composite materials when an aircraft is made entirely of composite materials. This plane is Rutan Voyager. Some other examples of civil aircrafts include Lockheed L-1011 Vertical Fin, Boeing 777, High-Speed Civil Transport and others.

1.5.3 Space applications

The main advantage that the fiber-reinforced composite materials can provide for space applications is weight saving. It is because the use of fiber-reinforced composite materials can mean an enormous saving in the cost of boosting every structure from earth into space. Some unique material properties are also made used of. For example, structures made up of graphite-epoxy can be constructed to obtain zero coefficient of thermal expansion. This special property is made used of in building large antennas that are subjected to large temperature changes in order to achieve dimensional stability for accuracy of pointing the signal.

1.5.4 Automotive applications

Making use of composite materials in the automotive industries has the advantages of low cost, weight saving and high-rate of production. Following the lower weight of automotives will have the obvious result of more efficient fuel consumption.

1.5.5 Other usages

Graphite-epoxy is utilized in already constructed columns of bridges to withstand earthquakes better. Other products made of fiber-reinforced composite materials include fishing rods, boats, tennis rackets, skis, surf boards golf clubs and many others.

1.6 Causes of delamination

Delamination in a composite laminate means cracking of the matrix (that holds the plies together), which normally exists between two plies. Once delamination occurs in the composite laminate, stresses within the composite laminate will be redistributed. This will then lower the compressive load carrying capacity of the composite laminate. The load carrying ability of the composite laminate will be even lower if the delamination is allowed to propagate. The causes for occurrence of delamination are:

1.6.1 Interlaminar stresses

Figure 1.2: Definitions for interlaminar stresses and edge delamination

Interlaminar stresses are σ_z , τ_{zx} and τ_{zy} as shown in Figure 1.2. Although these stresses are everywhere in the composite laminate, they are highest at the interplay surfaces. These interlaminar stresses that are very large at the free-edge are the culprits for free-edge delamination and subsequent delamination growth. The layer stacking sequence is the key to the values for these interlaminar stresses. These stresses are used at relatively large magnitude (to replace some of the stresses that cannot exist at the free edge) at the

free edge in order to obtain force and moment equilibriums. Moreover, conditions due to moisture and temperature can also cause interlaminar stress. The presence of these interlaminar stresses will mean that the laminated materials will delaminate at free edges like the edge of a plate or around a hole.

Figure 1.3: Ways to prevent free-edge delaminations

Various methods, as in Figure 1.3, are introduced to prevent this kind of delamination. One of them is the passive free-edge delamination-suppression concept. The idea of this concept is to decrease the delaminating effect of the interlaminar stresses by finding the best possible stacking sequence. The other type is the active delamination-suppression concepts. This family of ideas consists of edge reinforcement and edge modification. The methods in edge reinforcement include edge cap, stitching and interleaved layers and those in edge modification include ply termination, notching and tapering.

1.6.2 Impact

Impact by a foreign object onto the composite laminate can also cause delamination. Damage from impact can be grouped as Low Velocity Impact (LVI) and High Velocity Impact (HVI). LVI can cause internal delamination, which may not be obvious on the surface. However, significant degradation of overall stiffness and strength of the composite laminate may result. Therefore, the allowable design strain for composite structure is usually restricted to 0.4% of the ultimate design strain. HVI can result in severe surface damage that can be easily identified. At extreme cases, through penetration can occur. Moreover, there can be serious delaminations around the impact site.

1.6.3 Fabrication defects

Poor bonding between the plies can occur if there are errors in the manufacturing process. Some of the errors include insufficient matrix and air entrapment during the fabrication. Furthermore, introduction of foreign impurities and particles can also develop into delaminations.

1.7 Inadequacies of existing literatures

After the intensive literature review which will be presented in chapter two, it is found that there are still many aspects to examine in the research of the stability of laminated composites. It is detected that most of the existing literatures look into the buckling of perfect or delaminated composite plate or shell, the study on the stability of ideal or delaminated composite beam is minimal. The analysis on the buckling behavior of perfect or delaminated composite beam is crucial as such failure of this kind of structure is very common in many constructions. Moreover, from the literature review, it is discovered that there is a lack of research in the formulation of an appropriate equation to calculate the effective flexural stiffness of thin composite beam and to employ such equation in conjunction with the classical beam buckling theory to examine the buckling load of thin perfect composite beam. The next insufficient work in the literature is the inadequate use of the expressions that can be utilized to compute the effective flexural stiffness of thin composite beam in suitable mathematical model to examine the buckling load of thin delaminated composite beam. Moreover, another aspect that the literature review reveals is that there is a shortage of work in the development of suitable expressions to determine the effective flexural stiffness, effective shear modulus and shear correction factor and to utilize these expressions along with the first-order shear deformation theory beam buckling theory to investigate the critical load of moderately thick ideal composite beam. Furthermore, there is not enough research in employing these expressions into appropriate mathematical model to investigate the buckling load of moderately thick delaminated composite beam. In addition, it is spotted that most of the existing researchers constrain their studies to the buckling of the specially orthotropic,

symmetric cross-ply and symmetric angle-ply types of laminates. There is insufficient amount of analysis in laminates such as the anti-symmetric angle-ply and general orientated kinds of laminates. The use of these types of laminates can be found in many practical applications. Furthermore, although there are a handful of literatures which deal with mathematical models of thin composite beam with multiple delaminations, most of them do not include the contact effect. The involvement of the contact effect is vital in these mathematical models because if the contact effect is not reinforced, the physically impermissible interpenetration of sub-laminates in the delaminated regions will occur. In view of the above-mentioned inadequacy regarding research of the stability of laminated composites, the following objectives are proposed and relevant studies are carried out in this thesis.

1.8 Objectives

The fiber-reinforced composite materials are known to be very susceptible to interlaminar delamination. The delamination in turn can have a devastating negative influence on the compressive load carrying ability of the composite laminate. Hence, this thesis aims to investigate the effects of the delamination on load bearing capacity of the composite laminate by accomplishing the following objectives:

- 1) To develop a new formula (${}^3(\overline{EI})$) that can be utilized to compute the effective flexural rigidity of a thin composite beam through the use of the classical laminate theory and the Euler-Bernoulli beam theory. This novel expression can then be employed along with two other existing equations (${}^1(\overline{EI})$ and ${}^2(\overline{EI})$) that can also be applied to determine the effective bending stiffness of a thin composite beam to study the effect of effective flexural modulus estimates on the buckling load of the thin perfect composite beam. It should also be noted that the ${}^2(\overline{EI})$ will be applied for the first time to the thin delaminated composite beam cases which will be mentioned subsequently. Two out of the three formulae are identified and proposed to grant the wanted upper and lower buckling load bounds.

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- 2) To derive three different collections of expressions to evaluate three individual sets of values for the effective bending stiffness, effective shear modulus and shear correction factor (${}^1(\overline{EI})$, ${}^1(\overline{Gh})$, 1K ; ${}^2(\overline{EI})$, ${}^2(\overline{Gh})$, 2K and ${}^3(\overline{EI})$, ${}^3(\overline{Gh})$, 3K) of a moderately thick composite beam based on the first-order and third-order shear deformation theories. Analogously, these three separate groups of expressions can next be employed in conjunction with the first-order shear deformation beam theory to examine the critical load of the moderately thick composite beam. Then, recognize and suggest which two sets of equations offer the desired upper and lower limits for the critical load.
- 3) To establish a mathematical model to study the buckling of the thin composite beam with a single delamination. The three methods of computing the effective flexural stiffness (${}^j(\overline{EI})$, $j=1..3$) explained in the first point can then be utilized in cooperation with the analytical model to predict the critical load of the thin single-delaminated composite beam. Parameter studies are performed to comprehend the effects of ply orientation, layering pattern, number of plies, through-the-thickness position of the delamination, delamination span, composite materials and boundary condition on the critical load. Check to see whether the two recommended ${}^j(\overline{EI})$ expressions can still be used to acquire the longed upper and lower bounds for the buckling load of this thin single-delaminated composite beam.
- 4) To formulate a mathematical model to examine the buckling behavior of the moderately thick single-delaminated composite beam. The three approaches to compute the three unique groups of values for the effective bending stiffness, effective shear modulus and shear correction factor mentioned in the second point can then be used in conjunction with the analytical model to look into the buckling of the moderately thick composite beam with a single delamination. Parametric studies like the ones stated in point three are conducted. Analogously, inspection is done to determine whether the intended two groups of formulae to

- calculate the required ${}^j(\overline{EI})$, ${}^j(\overline{Gh})$ and jK are still capable of forming the wanted upper and lower bounds for the critical load of the moderately thick single-delaminated composite beam.
- 5) To construct two innovative analytical models to analyze the buckling phenomena of the thin composite beam inflicted by the semi-embedded delaminations with and without the contact effect at the delaminated zone. The ${}^2(\overline{EI})$ and ${}^3(\overline{EI})$ described in point one are then supplied to the two mathematical models to observe whether they are competent in providing the longed upper and lower limits, respectively, for the critical load of the thin composite beam containing the semi-embedded delaminations for both with and without contact analysis cases. Factors that may affect the buckling load such as the boundary condition, through-the-depth location of the delaminations, delaminations length, angle orientation and stacking sequence are investigated.
 - 6) To launch an original mathematical model to look into the stability problem of the thin composite beam consisting of the non-coinciding delaminations under combined axial and uniformly distributed transverse forces accounting for the contact analysis. Equivalently, the ${}^2(\overline{EI})$ and ${}^3(\overline{EI})$ are input into the analytical model to verify whether they are still proficient to grant the desired upper and lower bounds, respectively. Different aspects that may influence the buckling behavior like along-the-length position of the delaminations, delaminations span, through-the-depth location of the delaminations, class of laminate, ply-angle direction, type of composite materials and edge condition are also addressed.

1.9 Summary on major focuses of the thesis

In order to make clear the major focuses of this thesis, a summary on the key focuses of this thesis is included in this section. One of the major focuses of this thesis is to derive a novel closed form equation utilizing the classical laminate theory and Euler-Bernoulli beam theory to calculate the effective flexural stiffness ${}^3(\overline{EI})$ of a thin composite beam in

chapter 4. The next chief focus of this thesis is to use the ${}^3(\overline{EI})$ together with two other expressions ${}^1(\overline{EI})$ and ${}^2(\overline{EI})$ that can also be employed to compute effective bending rigidity of a thin composite beam to examine the buckling load of thin perfect composite beam in chapter 4. The other main focus of this thesis is to develop three original different collections of expressions to evaluate three individual sets of values for the effective bending stiffness, effective shear modulus and shear correction factor (${}^1(\overline{EI}), {}^1(\overline{Gh}), {}^1K$; ${}^2(\overline{EI}), {}^2(\overline{Gh}), {}^2K$ and ${}^3(\overline{EI}), {}^3(\overline{Gh}), {}^3K$) of a moderately thick composite beam based on the first-order and third-order shear deformation theories in chapter 4. The subsequent major focus of this thesis is to utilize the ${}^1(\overline{EI}), {}^1(\overline{Gh}), {}^1K$; ${}^2(\overline{EI}), {}^2(\overline{Gh}), {}^2K$ and ${}^3(\overline{EI}), {}^3(\overline{Gh}), {}^3K$ to investigate the critical load of moderately thick ideal composite beam in chapter 4. Another key focus which is presented in chapter 5 is to employ the ${}^1(\overline{EI}), {}^2(\overline{EI})$ and ${}^3(\overline{EI})$ to look into the buckling load of thin composite beam with a single delamination. The next chief focus of this thesis is to apply the ${}^1(\overline{EI}), {}^1(\overline{Gh}), {}^1K$; ${}^2(\overline{EI}), {}^2(\overline{Gh}), {}^2K$ and ${}^3(\overline{EI}), {}^3(\overline{Gh}), {}^3K$ to study the critical load of moderately thick single-delaminated composite beam in chapter 6. The other main focus of this thesis is to use the ${}^1(\overline{EI}), {}^2(\overline{EI})$ and ${}^3(\overline{EI})$ along with two newly developed analytical models to analyze the buckling behavior of thin composite beam having semi-embedded delaminations with and without contact in the delaminated zone in chapter 7. The subsequent major focus of this thesis is to employ the ${}^1(\overline{EI}), {}^2(\overline{EI})$ and ${}^3(\overline{EI})$ together with an original mathematical model to examine the buckling phenomena of thin composite beam with non-coinciding delaminations under combined axial and transverse loads including contact effect in chapter 8.

1.10 Thrusts of present findings

The thrusts of present findings are given in this section so as to inform the readers of the essences that are attained from the current findings first before proceeding to the later

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chapters. One of the thrusts of present findings is that the two equations ${}^2(\overline{EI})$ and ${}^3(\overline{EI})$ that can be used to calculate the effective flexural stiffness in the classical beam theory for a thin composite beam are able to provide the desired upper and lower bounds, respectively, for the actual buckling loads of the perfect, single-delaminated and multi-delaminated thin composite beams for all the cases that are considered. The other gist of present findings is that the two sets of expressions ${}^2(\overline{EI})$, ${}^2(\overline{Gh})$, 2K and ${}^3(\overline{EI})$, ${}^3(\overline{Gh})$, 3K that can be employed to determine the effective flexural stiffness, effective shear modulus and shear correction factor in the first-order shear deformation beam theory for a moderately thick composite beam are capable of supplying the longed upper and lower limits, respectively, for the actual critical loads of the ideal and single-delaminated moderately thick composite beam for all the cases that are examined.

CHAPTER TWO: LITERATURE REVIEW

The endless quest for lighter, more durable and excellent fatigue resistance construction materials has led the designers and engineers to the use of fiber-reinforced composite materials. By virtue of their high specific stiffness and strength, fiber-reinforced materials have been employed in many weight-sensitive industries and applications. Moreover, the production cost of the fiber-reinforced composite materials has declined due to the increase in manufacturing experience and improved fabrication technologies for mass production. However, composite laminates are susceptible to delaminations, which may occur due to high inter-laminar stresses at the free edge, impact and fabrication defects. Hence, throughout the years, researchers have been investigating on the effects of the delaminations on the composite laminates, with a great amount of attention turning to the study of the buckling behavior of delaminated composite laminates. These studies were done analytically, through the use of finite element method and experimentally. This chapter serves to gain insights of the research topic by reviewing the existing literature.

2.1 Analytical methods

Analytical methods such as thin film analysis, Bernoulli Euler beam theory, Rayleigh-Ritz energy method and plate theories are widely adopted to examine the delamination buckling, post-buckling and vibration behaviors of composite laminates. However, these techniques each have their own assumptions and limitations.

Bert [7] discussed about the deficiencies suffered by Bernoulli-Euler beam theory and classical thin plate theory (CPT). It was mentioned that the first attempt to take account of transverse shear deformation in plate theory was to apply the Bresse-Timoshenko beam theory to plates. Then, several so-called higher-order theories were developed by other researchers. Then, the classical thin plate theory (CPT) was extended to laminated plates. The dynamic and static methods of finding the suitable shear correction factors for composite laminates were also introduced. Moreover, higher-order laminated plate theories were derived. There were also further tries to consider each ply in a laminate as

an individual beam or plate so that the shear angle will be allowed to vary from layer to layer. The different theories were compared with the widely used closed-form solution for cylindrical bending of a simply supported laminate under a sinusoidal distributed normal pressure loading. The maximum deflection, bending stress distribution and transverse shear stress distribution for both homogenous orthotropic material and symmetric cross-ply laminate were used for the study. It was concluded that the Levinson-type theory was relatively more accurate and computationally less expensive than other theories.

Joshi and Suryanarayan [36] investigated the coupled flexural-torsional vibration of a narrow rectangular beam which was subjected to a static axial load and equal end moments. The static axial load could be tensile or compressive. For simply-supported edge conditions, closed form solution was achieved. The effects of the load and geometry of the narrow rectangular beam on the coupled frequencies and mode shapes were examined. Furthermore, an equivalent moment parameter was introduced. Moreover, an effective axial load parameter was defined to simplify the problem down to that of an equivalent beam-column. It was found from the results that lateral instability could still happen even if the beam was under tension if enough end moments were applied.

Simites [71] created a one-dimensional model to simulate a laminate plate with a through width delamination. This model was believed to be sufficient to describe the behavior of the laminate plate. In addition, delamination growth was considered. The Griffith fracture criterion was used to govern the delamination propagation. It was assumed that presence of the delamination separated the laminate plate in such a way that the Bernouli-Euler beam theory was applicable to the individual delaminated regions. Furthermore, it was assumed that delamination was already present in the laminate plate before the compressive load was applied. Both clamped and simply supported cases were considered. It was found that for composite laminates that had relatively long and thin delaminations, the buckling load was generally low while the chance for delamination propagation was high. On the other hand, for composite laminates with rather short and thick delaminations, the buckling load was a good estimate of the load-carrying capacity.

The author also emphasized that investigations ought to be carried out to examine the relationship between the delamination shapes and the stacking sequence of the composite laminate. Furthermore, the effect of the shape of the delamination on the buckling load and delamination propagation should also be considered.

Simites et al. [72] modeled the laminated plate in one dimension using the Euler-Bernoulli beam. Three assumptions were made, which included that the delamination existed before compressive load was applied and the direction of growth was always parallel to the horizontal axis, the properties of the plate were homogenous and the plate behaved in a linear elastic manner. Furthermore, the delamination was taken to be through-the-width. The influences of location, size and thickness of the delaminations on the buckling load were examined. Both simply supported and clamped boundary conditions were investigated. In addition, during the mathematical formulation, the boundary conditions, kinematic continuity conditions and equilibrium in moments and forces were used to solve the buckling problem. A new non-dimensionalised load, $\overline{P}_{dam}$, which provided the insight of the amount of load a fully delaminated configuration was introduced. It was found to give a good lower bound for the load carrying ability of a delaminated model. Furthermore, thin-film analysis was used for comparison. It was observed that the thin-film analysis held for all lengths of delamination when the delamination was thin, however, it was only applicable for longer delamination as the delaminations became thicker.

Williams and Stouffer [81] studied the critical buckling load and delamination growth of delaminated composite laminate. They investigated the problem analytically, using finite element analysis and experimentally. Excellent agreement was found when results obtained from the finite element analysis were compared with those from the analytical approach. It was shown that before the local buckling, the system energy release rate was zero. It was also observed that there would be a sudden and instant increase in the G value when local buckling occurred. Furthermore, from both the analytical method and experiment, global buckling of the whole laminate happened quite closely after the local buckling. It was pointed out that as the delamination length

increased, the system energy release rate would approach as asymptotic value. Once this value was less than the critical energy release rate, the delamination growth would be arrested. Otherwise, continuous delamination propagation may take place.

Yin et al. [82] studied about the buckling and post-buckling of a beam-plate with a through-the-width delamination symmetrically positioned at an arbitrary depth, undergoing axial compression. It was found in this paper that for a beam-plate with relatively slender delamination, the post-buckling axial load could be significantly higher than the buckling load. Furthermore, it was pointed out that in this case when the beam-plate had relatively slender delamination, delamination growth may not determine the failure of the plate. Different combinations of delamination lengths and axial load to calculate the energy-release rate related with the delamination growth. Curves were drawn to show the possibility and stability characteristics of delamination growth, which could be used to find the ultimate axial load capacity.

Kardomateas and Schmuesert [39] developed a formulation that accounted for transverse shear to determine the critical buckling load for the delaminated composites. The perturbation or small parameter method was used to handle the stability problem. Furthermore, an analytical equation for the initial post-buckling deflections of the delaminated composite was reported. The path-independent J-integral was modified to account for the transverse shear effects and examine the quasistatic delamination growth. It was found that the transverse shear effects would promote the onset of delamination growth. Three kinds of buckling modes, global mode, local mode and mixed mode were considered. The Kevlar/epoxy, graphite/epoxy and glass/epoxy composites were utilized in the numerical examples. The effects from the transverse shear for different composites, through-the-thickness location of the delamination and delamination length-to-delamination thickness ratios.

Mujumdar and Suryanarayan [55] examined how the delamination influenced the vibration characteristics of laminated beam kind of structures. A novel mathematical model was developed to investigate the full cycle linear analysis of vibration of beams

inflicted with through-width delaminations. For this mathematical model, it was assumed that the sub-laminates in the delaminated region were “constrained” and they shared the same transverse displacement. The formulation of this mathematical model was based on one-dimensional Euler beam analysis. Parametric studies were conducted by employing this “constrained mode model” to examine the effects of the boundary conditions, delamination span, lengthwise position and thicknesswise location of the delamination on the frequencies. Some examples on mode shapes were also addressed. Furthermore, experimental results on vibration of beams with midplane as well as off-midplane delamination were utilized to validate the proposed mathematical model. It was shown that the vibration results from the mathematical model were generally in good agreement with the experimental values.

Kapania and Raciti [37] wrote a review on the developments of shear effects, buckling, postbuckling and delamination buckling and growth of laminated beams and plates. Different shear deformation theories for plates and beams were presented. These theories were derived by assuming a variation of either the in-plane displacement components or the stress components or both in the thickness direction. The relevant kinematics relations, strain-displacement relations and equilibrium equations were discussed. Furthermore, the implementations of several high-order plate theories into finite element models were covered. The problem of local buckling and crippling of stiffeners were also presented. The closed-formed solutions or numerical differentiation were utilized to compute the energy release rates. Moreover, investigations on the multiple delaminations and on dynamic behavior of composite laminates under impact loads were covered.

Noor and Burton [57] did an assessment on the shear deformation theories for the multilayered composite plates. The aims of the article were to classify and review the different methods available for the two-dimensional shear deformation theories of multilayered plates, to examine the accuracy of the predictions from several two-dimensional shear deformation theories and potential directions for the improvement of shear deformation theories on the composite plates. Studies were carried out on the linear

static, free vibration and buckling problems. The four general approaches for developing the two-dimensional shear deformation theories for the composite plates were discussed, which included the method of hypotheses, method of expansion, asymptotic integration technique and iterative methods and methods of successive corrections. For the first technique, both theories based on replacing the laminated plate by an equivalent single-layered anisotropic plate and discrete layer theories based on piecewise approximation in the thickness direction were discussed. In the numerical studies, three factors were studied, which included the number of layers, the degree of orthotropy of the individual plies and the thickness ratio of the plate. The various shear deformation models were compared with the exact three-dimensional model. It was observed that the predictor-corrector approach was able to give accurate determination of the global as well as the detailed response characteristic of the plates. Furthermore, suggestions were made to improve the current predictive capability of the behavior and failure of the composite plates.

Shaw and Tsai [66] developed a method to find the strain energy release rate for mode I, mode II and mode III separately in their article. The classical plate theory was utilized to obtain the stress and moment fields at every point inside the compressed delaminated laminate. At the same time, the finite element method was used to study the region around the crack tip of the delaminated plate. The Irwin's virtual crack extension method was adopted to compute the strain energy release rate of the crack. The stress and moment resultants collected from the classical plate theory were then used as the boundary tractions for the finite element model. It was pointed out this methodology would require lesser computational time than using non-linear finite element method. Both cross-ply and angle-ply laminates were investigated in this paper. Investigations were carried out for various delamination lengths, compressive force and lay-ups of the laminate. It was discovered that these three parameters had a significance impact on the three different modes of strain release energy rates. The equilibrium of forces and moments, continuity of displacements and boundary conditions were used to determine the deflection of the individual sub-beams and the bending moments. Singular element was used to describe the behavior at the crack tip.

Sheinman et al. [67] investigated on the influence from the delamination and the bending-stretching coupling effect on the bifurcation load of the delaminated composite strip. Furthermore, the arbitrary axial and transverse loadings were covered. The composite strip was assumed to have a small thickness-to-length ratio, so that the shear deformation could be neglected. Both orthotropic and anisotropic strips were used. Both the prebuckling and buckling states were considered. A modified Newton's method together with the central finite-difference scheme were utilized to convert the nonlinear equations to an algebraic one.

Chen [10] used a variational energy principal method to formulate the stability problem of a delaminated composite laminate, taking account of transverse shear deformation. It was believed that this technique was more general and systematic in formulation. A specially orthotropic laminate was used and clamped boundary condition was considered. A Griffith-type fracture criterion was utilized to examine the delamination front propagation. With the variational energy principal, the governing expressions for static delamination buckling, postbuckling and delamination growth were derived. The general expressions of deflections and rotations of postbuckling solution were obtained. Furthermore, the energy release rate was derived analytically in terms of the axial forces, the bending moments, the transverse shear forces and the slopes at the delamination front. Effect from the shear deformation on the deflections, slopes and strain energy release rate was studied. The energy release rate was investigated for various delamination length, delamination height and shear deformation parameter.

Tessler [75] presented a higher-order beam theory, which was a one-dimensional version of the plate theory. In addition, a simple two-node beam element was created and the results generated for deep beam from this element were compared with the Timoshenko and exact elasticity solutions. It was emphasized that this element was able to account for the deformation of the mid-plane x-axis due to the transverse load, which was not considered in the Timoshenko theory. Both the desired equilibrium and suitable boundary equations were derived using the virtual work theorem. It was pointed out that accounted for both the transverse shear and transverse normal deformations, while the

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resulting equations were of same order as the Timoshenko theory. In this work, the longitudinal displacement was assumed to have a linear variation and the transverse displacement was taken to have a parabolic distribution. It was concluded that this beam element was able to compute the through-thickness distributions of all displacement, strain and strain components accurately.

Chen [11] created a one-dimensional thin-film delamination model to investigate an unsymmetric laminated structure. The formulation utilized the large deflection shear deformation theory that took into account both the bending-extension coupling and transverse shear deformation. Three kinds of in-plane loadings were considered. Then, closed-formed solutions for the critical buckling load and post-buckling deflections were derived. It was found that both the transverse shear deformation and bending-extension coupling would reduce the critical buckling load and increase the transverse deflection. These expressions were then used to study the thin-film strip delamination in a thick base laminate. The equation of the energy release rate was derived from the variational energy principle, which was believed to be a better approach than the path-independent J-integral counterpart, as the equation could then be expressed in terms of strains, curvatures, resultant forces and moments.

Figure 2.1: Presence of antisymmetric S-shaped buckling mode

Lim and Parsons [51] developed a numerical model in their paper to calculate the buckling load of a composite beam with multiple delaminations quickly. The Lagrange multipliers were utilized to enforce the boundary conditions and continuity requirements. Furthermore, an energy method, the Rayleigh-Ritz energy method was adopted in the computation. Parametric studies concerning the influences of the lengths and depths of the delaminations were studied. Finite element method was also employed to verify the results obtained from the model. An anti-symmetric S-shaped buckling mode was observed in this article. For example in Figure 2.1, a kink was observed when a delamination with a length 0.35 to 0.40 times the length of the laminate and was located at half the thickness of the laminate was inserted into the laminate. This suggested the presence of the s-shaped buckling mode. S-shaped buckling mode was also seen in double-delaminated beam, where two sub-laminates at the delaminated region had the same thicknesses. Critical loads calculated from the thin film theory were included for comparison. It was pointed out in this paper that in an unsymmetrical stacking sequence, the bending-stretching coupling existed, causing pre-buckling bending deformation and errors could occur when a classical linearized buckling analysis was utilized.

Wang et al. [78] used a continuous analysis to examine the buckling loads of delaminated beams and plates. The Stoke's transformation technique was used to investigate the buckling behavior of clamped and simply supported orthotropic beam-plates. This analysis was believed to be able to investigate composite laminates with multiple delaminations of different shapes, sizes and positions. In this method, the delaminated composite laminate was replaced by an intact body with forces at selected points in the original delaminated regions such that the net interfacial tractions disappear at these points. In addition to the Stoke's transformation technique, Fourier series was used to obtain the solutions for various boundary conditions. To show the use of this analysis, results associated with the influence of different length, shape and position of the delaminations, various boundary conditions, stiffness of adhesive layer and local buckling loads of the sub-laminates were presented. This continuous analysis method was believed to be simple but effective to solve problems of this nature. However, post-buckling and delamination growth were not considered in this paper.

Kutlu and Chang [42] examined laminated composite panels with multiple through-the-width delaminations, under compression analytically and experimentally. Both flat and cylindrical panels were investigated. In this paper, an analytical model was developed to examine the behavior of the laminated composite panels under compression, from the initial loading to the final failure. Therefore, this model would contain three components, which were the stress analysis, contact analysis and failure analysis. Large deformation theory was to develop the stress analysis, which was used to examine both the global and local deformations of the sub-laminates. For the contact analysis, Lagrange multiplier technique was used to model the surface conditions of the sub-laminates in contact. It was assumed that there was no friction between the sub-laminates. Furthermore, the delamination propagation was examined using the fracture mechanics. When predicting local fiber-matrix failure that would lead to global collapse, a set of failure criteria was used. Based on the model, a non-linear finite element code was produced. Numerical results from the non-linear finite element analysis were in good agreement with the existing analytical and experimental data. In addition, experiments were also carried out to verify the results from the model and numerical calculations.

Fraternali [21] created a one-dimensional numerical model to analyze the delamination behavior in the composite beam. The geometric nonlinearity arising from the moderately large rotations and shear strains is included in the stability analysis and interlaminar stress computation. Both energy release rate criterion and interface models were employed to examine the delamination growth. The classical Griffith fracture criterion was adopted in this article. Finally, the finite element method was used to complete the investigation on the buckling and propagation of delamination. The delamination considered was assumed to be through-the-width. Furthermore, the beam was assumed to be linearly elastic and orthotropic. Comparison was made between the predictions of the energy release rate from this new analytical model and thin film approximation. Several load vs. displacement curves were plotted by considering a wide range of critical (mode I) energy release rate and allowing the delamination to grow till 90% of the length of the beam.

Wang and Lin [79] determined the buckling load of a beam plate with single and multiple delaminations using an engineering analysis. Different supporting conditions were examined. Both the constrained and free mode models were considered. These two models represented two different buckling situations, where free mode referred to the case when the two sub-laminates did not touch each other during the buckling and constrained mode referred to the case where the two sub-laminates would be in contact during buckling. Furthermore, this paper intended to combine these two mode models to estimate the actual buckling load of a beam plate with multiple delaminations. The well-established beam-column model was used to represent the individual part of the delaminated beam plate, without containing any complicated details. Furthermore, this paper dealt with beam plates with single and multiple delaminations at different segments of the beam plates. This paper did not include the contact analysis. In order to treat each portion of the beam plate as a single dimensional model, the beam plate was assumed to be made up of orthotropic layers without coupling effects.

Figure 2.2: Effect of second delamination on the buckling load for a plate with $\bar{h}_1 = 0.125$ and $\bar{h}_2 = 0.25$

Figure 2.3: Normalized buckling load for a plate with $\bar{a}_1 = 0.5$, $\bar{h}_1 = 0.2$ for various $\bar{h}_2$ and $\bar{a}_2$

Huang and Kardomateas [29] employed suitable kinematical continuity conditions, equilibrium equations and boundary conditions with the linearized non-linear differential equation to derive a closed form solution for a plate with two delaminations. In this way, the buckling load of a plate with double delaminations of any sizes and locations could be computed easily. The clamped boundary condition was considered. Only half of the delaminated plate was modeled due to the symmetry of the structure. Investigation was carried out to examine the interactive effect of the two delaminations on the buckling load. In Figure 2.2, $\bar{a}_1$, $\bar{a}_2$, P_{cr} and P_{crs} were the length of delamination 1, length of the delamination 2, critical buckling load of the double delaminated laminate and the buckling load of the composite laminate with only one delamination respectively. For Figure 2.3, $\bar{h}_2$ was the distance from the top surface to the second delamination. It was found that when the delamination that was buried deeper in the plate was closer or longer than that near to the surface, it would have more effect on the buckling load. It was pointed out that if the two delaminations were too close, the middle sub-laminate would buckle first and contact analysis needed to be considered.

Gu and Chattopadhyay [26] solved the delamination buckling of the simply-supported orthotropic and laminated composite plates under cylindrical bending utilizing the elasticity approach. The resultant buckling expressions and the associated boundary conditions were derived via the variational principle, perturbation technique, linearization of the attained equations and the use of the exact constitutive equations. Furthermore, assumption of the uniform prebuckling stress state was also made. It was emphasized that the suggested technique will be able to act as a tool to assess the accuracy of the existing plate theories. It was explained that the proposed method was able to be used in cases with symmetric and cross-ply composites. As an illustration, a numerical example was done. In the numerical study, readings that were obtained from the classical lamination theory and a previously established higher-order shear deformation theory were compared with those that were acquired from the elasticity approach. The effects of the delamination span and position on the critical load were examined. It was found that the higher-order theory was able to provide very good buckling solutions for the thin and moderately thick delaminated composite plates.

Moradi and Taheri [52] applied the differential quadrature method (DQM), which could be used to solve linear and non-linear partial differential equations on the delamination buckling problem in composite laminates. This technique allowed one to transform the system of equilibrium expressions and boundary conditions to a list of linear algebraic equations that could be solved using standard eigensolver. The shear deformation was accounted for in the analysis. To verify this approach, results obtained from the DQM were compared with those reported by previous researchers and they were found to be in good agreement. Studies were also carried out on the effects of the shear deformation factor, dimension and location of the delamination

Shu [69] utilized the classical beam theorem to model a beam with double delaminations into five interconnected segments. Furthermore, two co-ordinate systems were used and the gradient was utilized as the unknown, which greatly decreased the complexity of the solution. It was found that different buckling modes could be obtained depending on the position of the most slender sub-laminate. The modes were

“constrained” mode and “free mode”. However, it was pointed out that the post-buckling behavior and delamination propagation were not included in this article. Both clamped and simply supported boundary conditions were studied. In addition, the upper and lower bounds solutions were formulated, which were believed to be applicable to multiple delaminations as well. The upper bound was obtained when the three sub-laminates buckled at the same time while the lower bound referred to when the three beams buckled independently. Parameters R and R_s were defined to study the different modes of buckling.

Figure 2.4: Influence of the number of sampling points on the critical load of a delaminated plate with a thin sub-laminate

Figure 2.5: Influence of the number of sampling points on the critical load of a delaminated plate with a thicker sub-laminate

Moradi and Taheri [53] used the differential quadrature method to examine the buckling behavior of the composite plate with through-the-width delamination. The differential quadrature method (Bellman et al) was used for their model. The Gauss method was employed by this differential quadrature method to find the derivative of a function. The composite plate consisted of one delamination positioned arbitrarily along the thickness, dividing the region into four segments. They also found that only 9 to 11 sampling points were required to discretize each segment to get converged and accurate results. Figure 2.4 showed the effect of the number of grid points on the prediction of the buckling load of a plate with thin delamination and the thickness of the thinner sublaminates was 10% of the perfect laminate. On the other hand, Figure 2.5 illustrated the influence of the sampling points on the critical load of a plate with thicker delamination. Furthermore, different grid spacing schemes were investigated to find the most efficient and accurate one. During the investigation on the effect of grid spacing, it was found that the use of unequal grid spacing and δ -technique resulted in faster convergence. After the introduction of suitable boundary conditions and application of the differential quadrature method to each segment, the problem was changed to a standard eigenvalue problem. In this paper, results from this method were used to demonstrate the efficiency and accuracy of this method in solving this type of buckling problem. Furthermore, with the use of this method, minimal computational and modeling efforts were required.

Figure 2.6: Influence of various material properties on the buckling load

Figure 2.7: Effect of the position of the delamination along the length on the critical load

Moradi and Taheri [54] used the differential quadrature method to analyze the buckling behavior of one-dimensional composite laminated beam-plates. In this article, the quadrature method was used with the Lagrange trial function and the selected grid sampling was based on the Chebyshev spacing. Influences of the shear deformation and bending-extension coupling were included when solving the problem using this differential quadrature method. After incorporating the necessary boundary conditions and kinematic constraints together with the use of the differential quadrature method, an eigenvalue problem was obtained. Then, the buckling loads and the corresponding mode shapes were obtained through an eigen-solver. Furthermore, various case studies were conducted to verify the accuracy and efficiency of this method. Then, investigations were carried out to examine the effects of different delamination lengths, thickness, ply stacking sequences and materials on the buckling load on the delaminated beam-plates. It was found that the buckling load could be reduced as much as 50% due to the bending-extension coupling. Slender composite laminates were observed to be more affected by the shear deformation than those with lower slenderness ratios. Effects of the material properties of the laminate composite and location of the delamination along the length of the laminate on the critical load, as shown in Figures 2.6 and 2.7 were investigated.

Gui and Li [28] used the Rayleigh-Ritz energy method to investigate the local buckling of a stitched laminate with an embedded elliptical delamination. Studies were carried out to examine the effects of the equivalent spring constant of the stitching thread, delamination size, stacking sequence and delamination orientation on the buckling strain. It was observed that stitching had a great impact on both buckling mode and buckling strain. When stitching thread of higher spring constant was used, the buckling strain was found to improve tremendously. On the other hand, the buckling strain decreased as the delamination size increased. Furthermore, both the delamination angle and stacking sequence affected the buckling strain. Multiple humps were seen when stitching thread of a critical equivalent spring constant (for certain delamination size) was used or a specific delamination size exists (for a particular value of the spring constant for the stitching thread).

Naik and Ramasimha [56] studied the buckling behavior of the typical woven fabric composites with a central delamination under axial compression. An analytical method had been developed to determine the compressive strength of such composites. Classical lamination theory, equilibrium of forces and finite element analysis were used in the approach. There were three steps in the analytical method, consisting of stress analysis of the laminate before buckling, eigenvalue study of the sublaminates and determination of the buckling strength from these results. Furthermore, experiments were carried out to verify this analytical method. Parametric studies examining the effects of the size and position across the thickness of the delamination on the buckling strength were carried out. Both the E-glass/epoxy and carbon/epoxy composites were used in this article. It was found that as the delamination size increased, the buckling strength of the composite decreased. On the other hand, as the thickness of the sublaminates increased, the damage tolerance decreased initially and then increased. It was concluded that the plain weave T-300/5208 carbon-epoxy composite WG02C had a higher tolerance to damage than the E-glass/epoxy composite GLE-12.

Daridon and Zidani [14] investigated the crack growth due to local buckling in a plate subjected to axial compression. The effect of the position of the delamination

through the thickness and the influence of existence of a damageable bridging component at the crack tip were examined. In this way, the combined influence of the fiber bridge and structural dimensions were investigated. Bernoulli beam was employed in this article. The non-linear fracture mechanism was utilized to govern the crack growth. In the model, the crack propagation was described by the fiber bridge formation and destruction. This study was carried out to examine the effect of fiber bridging on the delamination buckling of composite structures. Furthermore, simple damage law and calculation of released energy were utilized to determine the motion of the crack tip. In this article, the authors highlighted the relationship between the R-curve phenomenon and the reduction of the stabilization length.

Ghugal and Shimpi [25] presented a detailed review on the theories for homogenous, isotropic plates. Then, different displacement and stress based refined theories for shear deformable isotropic and anisotropic laminated plates were addressed. The theories for homogenous and isotropic plates covered included classical plate theory, first-order shear deformation theories, second-order shear deformation theories, various higher-order shear deformation theories and trigonometric shear deformation theory. These plate theories were then extended to the laminated anisotropic plates, forming the two classes of so-called equivalent single layer theories and discrete layer theories or layerwise theories. Furthermore, available exact elasticity solutions for the plate problem were cited. It was pointed out that more research work should be done on unsymmetric laminated plates. There was also room for investigation for the refined theories that were based on trigonometric functions. Moreover, it was realized that refined deformation theories for laminated plate to obtain meaningful transverse shear stresses, which were single valued at layer interfaces should be derived.

Lee et al. [48] utilized the classical laminated beam theory to examine the free-vibration of through-thickness multi-delaminated composite beams analytically. The multi-delaminated beams were divided into many individual sub-beams at the front of the delamination tip. The kinematics continuity conditions were developed from the assumption of constant curvature at the multi-delamination tip. Furthermore, a free mode

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was assumed and the separate beams segments were allowed to vibrate freely without touching each other. In order to simplify the formulation, the effect of bending-extension coupling induced from the presence of delamination and transverse shear deformation was neglected. To validate the mathematical model, experiments were conducted for the single-delaminated case while the finite element method was employed to look into the multi-delaminated case. The glass/epoxy delaminated beams made by semi-autoclave were used in the experimental study. Composite beams with arbitrary lateral, longitudinal, and both multi-delamination cases were analyzed to study the effects of size and location of multiple delaminations on the natural frequencies.

Shan and Pelegri [65] used an approximate method to investigate the buckling behavior, which could be used to serve as the first step to study of the contact zone at the crack tip. Only single delamination was examined. Local buckling in this paper referred to the buckling of the thinner sublaminate before the overall buckling of the composite, while global buckling referred to the overall buckling of the composite. Furthermore, experiments were carried out to verify the prediction critical axial displacement at which local and global buckling occurred by this approximation method. However, there existed a distinct difference between the results obtained from this approximate method and the data from experiment. It was mentioned that this variation may be partly due to geometrical imperfections and shearing loading effects. Moreover, the author recommended a correction factor for the approximate method to predict the critical load more accurately. It was observed from the experiments that as the axial load increased, transitions occurred from local buckling to base bending and finally to global buckling. Furthermore, delamination propagation was observed. For both the local and global buckling, this approximate method could be used as an easy and efficient way to evaluate the forces, moments and energy. It was also pointed out that other existing methods were only able to examine the influence of the thickness ratio, the ratio of the thickness of the thinner sublaminate to that of the whole composite on the total strain energy at discrete points, while this approximate method was able to investigate in a continuous range.

Wang and Lu [80] considered laminated plates with embedded rectangular and triangular delaminations near the surface. Parametric studies were also carried out to examine the effect of the delamination's orientation. An energy method was used to examine the buckling behavior of such plates under mechanical and thermal loadings. The influence of tension-shear coupling and bending-torsion coupling were taken into account. The delamination was placed in such a way that the originally symmetrical laminate was divided into two asymmetrical sub-laminates at the delaminated region. Furthermore, experiments were carried out to investigate the buckling failure mechanism of the laminate plates with a single rectangular and triangular delamination under mechanical loading only. These results were found to compare well with the analytical predictions. The stacking sequence used in this paper was $[0/\pm 45/90]_{2s}$. The delamination was located just below the two layers.

Parlapalli and Shu [58] examined the buckling behavior of a beam that was made of two different materials with an asymmetry delamination was examined. New non-dimensionalized parameters, axial and bending stiffnesses and effective-slenderness ratio were introduced. Parametric studies were carried out to investigate the effects of the delamination length and the location of the delamination on the critical load. Furthermore, both clamped and simply supported boundary conditions were considered. Three types of buckling shapes were identified, namely the local, mixed and global mode shapes and they were found to be greatly influenced by the effectiveness-slenderness ratio. The delaminated beam was divided into four segments and each part was simulated by a Bernoulli beam. It was assumed in this article that each layer of the beam was made of isotropic linearly elastic and homogenous material and there was already a delamination in the composite before the load was applied. Contact analysis was not covered in this paper. It was observed that for a given delamination length, the buckling load decreased with increasing normalized bending stiffness. Moreover, for a specific value of normalized bending stiffness, the critical load increased with the normalized axial stiffness. It was found that the buckling load decreased with increasing delamination length. Furthermore, the effective-slenderness ratio was observed to give an insight of the buckling mode.

Shu and Della [70] formulated a mathematical model to look into the free vibration of composite beams with two non-overlapping delaminations. The analytical model was developed based on the Euler-Bernoulli beam theory. In the formulation of the analytical model, the boundary conditions, kinematical continuities and equilibrium of forces and moments were enforced. In the analysis, both the ‘constrained model’ and ‘free model’ were utilized. In addition, the influences from the differential stretching and bending-extension coupling were included. Before the actual study, the analytical results were compared with the experimental and analytical data in the literatures and good agreement was found between them. From the parametric studies, it was found that the primary frequency and respective mode shape of the composite beam were greatly affected by the sizes and locations of the delaminations. It was also observed that when the mode shape from the ‘free model’ hardly displayed any opening, vibration results from both ‘constrained model’ and ‘free model’ were very close. However, if there was big opening, their vibration results would be very different.

Andrews et al. [4] created a model to investigate the elastic interaction of multiple delaminations in plates that are subjected to static out of plane loading, the effects of this interaction on the fracture behavior and the structural response of the laminate. The model used the classical beam theory, which neglects the shear deformation to determine the response of the delaminated structure. Furthermore, the model accounted for the non-frictional contact along the delamination surfaces employing three approximations, two of which permitted closed form solution of the problem. The approaches that were established for beams and plates with a single delamination were applied to compute the energy release rate and stress intensity factors at every delamination crack tip. The cantilever plate containing multiple delaminations deforming in cylindrical bending was utilized as a case study. Shielding and amplification of the energy release rates of the cracks, modification of the mode ratio and interaction effects on the macro structural response of the plate were examined. When delamination growth is considered, a discontinuity in the energy release rate and mode ratio was observed when the delaminations reached the same length. This led to behaviors like the local snap-back,

snap-through instabilities, hyperstrength, crack pull-along and crack arrest. Furthermore, finite element method was employed to verify the model.

Della and Shu [17] examined the free vibration of delaminated bimaterial beam. Two parameters which were known as the axial stiffness and bending stiffness were established to look into the vibration problem and mode shape. Both the ‘free mode’ and ‘constrained model’ were employed in the investigation. It was found that there was a monotonic relation between the natural frequencies and stiffness ratios. Furthermore, an innovative slenderness ratio that could be used in vibration analysis was proposed. It was found that this slender ratio parameter would affect the resultant vibration mode shape. Three different types of mode shapes were observed and they were known as global, mixed and local. From the analysis, it was discovered that as the axial stiffness and bending stiffness ratios rose and the delamination span increased, the effect of the ratios on the natural frequencies dropped. It was also observed that increase in the delamination length would not reduce the natural frequencies by a lot for higher axial stiffness and bending stiffness ratios.

2.2 Finite element analysis

With the advance in the ability of computer to deal with numerical values, finite element method has become increasingly popular. While analytical solutions are often subjected to some idealizations, finite element method does not have such restrictions, hence, it can reflect the actual situation. Furthermore, finite element analysis must be used when the problem gets complicated and includes complexities such as non-linear effects, plasticity and dynamic analysis.

Figure 2.8: Double-delaminated beam-plate with five neutral axes

Figure 2.9: Nodal forces and moments for each segment

Figure 2.10: Resultant nodal forces and moments after transformation

Kapania and Wolfe [38] examined the buckling behavior of a beam plate with two delaminations of equal length, one above the other. Moreover, the large laminate with double delaminations was divided into five regions. A beam plate element with 6 degree of freedom was used. The element was positioned along the neutral axes of each of these segments to simulate these five separate parts. This was illustrated in Figure 2.8. The resultant finite element model appeared in Figure 2.9 or 2.10. The delaminations were taken to be through-the-width. Both clamped and simply supported boundary conditions were considered. Furthermore, the energy release rate was computed using the numerical differentiation. For beam-plate with single delamination, when the results calculated from numerical differentiation were compared with the incremental solutions, they were found to be in excellent agreement. The column beam was assumed to be homogenous and linearly elastic. It was found that for a beam-plate with double delaminations, if one of the sub-laminate was much thinner than the other two, the buckling load would be much

determined by this sub-laminate. However, if all the three sub-laminates had almost the same thickness, the second delamination would have a significant influence on the buckling load.

Lee et al. [47] investigated the buckling behavior of the beam plate with multiple delaminations under compression. Using the layer-wise theory, a finite element method was formed to solve the problem. Then, results were obtained stating the influences of the lengths of the delaminations, number of delaminations, their positions through the thickness and their locations along the length of the composite laminates. Furthermore, the corresponding mode shapes were also recorded. This approach was proposed in hope to cut down the computing time compared to use of plane elasticity finite element method. It was observed that the ratio of the thickness to length of the composite laminate affected the buckling load. For a rather thick composite laminate with a single delamination at the central plane, global symmetric, local symmetric and antisymmetric buckling determined the buckling load for different ranges of delamination length. On the other hand, for the composite laminate that was relatively thin, global symmetric buckling dominated while antisymmetric buckling governed the buckling load for a short range of delamination length. The buckling load and mode depended greatly on where the delaminations were located along the length of the composite laminate. Generally, when the delamination was off-centre, global buckling was observed. Depending on the delamination length, local and antisymmetric bucklings were seen when the delamination approached the centre.

Gaudenzi [22] employed a classical non-linear finite element method utilizing the linearized buckling analysis to examine the delamination buckling in practical applications. Both two- or three-dimensional solid elastic finite elements were used to model the delaminated plate. Results obtained by using the two-dimensional solid elastic finite elements were compared with those from another article. Then, a square plate with an embedded circular delamination was investigated using the three-dimensional solid elastic finite elements. However, the contact constraints were not considered in the numerical simulation and growth of the delamination was not accounted. The effects of

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the delamination size and location through the thickness are investigated. Three buckling modes were identified, which included the global, local and mixed critical shapes. It was found that as the delamination became larger and approached the free surface, the buckling load of the delaminated plate decreased.

Figure 2.11: Deflections of the upper and lower sub-laminates for different delamination lengths

Figure 2.12: Moment reactions for different delamination lengths

Kim and Hong [41] studied both cases of composite laminate with either a through-the-width or embedded delamination. In addition, both buckling and post-buckling were examined. A geometrically non-linear finite-element analysis was used in the post-buckling state of the laminate to deal with the geometric complexity and large deformation. Furthermore, reasonable assumptions that a delamination already existed before loading and crack propagation was ignored were made. These valid assumptions were made to simplify the problem. Both the unidirectional and cross-ply laminates were examined in this paper. For the through-the-width and embedded delaminations, the influences of the delamination size and boundary conditions on the buckling and post-buckling phenomena were investigated. Two constraints were imposed on the finite-element model to transform a three-dimensional theory to a two-dimensional one. One of them was straight lines that are perpendicular to the middle plane will remain straight but not perpendicular after bending. The other constraint was that the transverse stress normal to the middle plane was ignored. In this paper, it was found that a delamination size existed below which the buckling load and postbuckling behavior would not be affected. It was also observed that presence of a delamination would cause the extension-bending coupling, which would in turn affect the buckling load and post-buckling behavior greatly. Both the boundary conditions and delamination size determined the buckling mode. The buckling mode would in turn affect the buckling load and postbuckling behavior. Comparing Figure 2.11 and Figure 2.12, it was interesting to notice that the reaction moment increased sharply when the buckling load was reached. Furthermore, it was pointed out that the reaction moment could also be used to judge the direction for the deflection of the sub-laminate.

Figure 2.13: Finite element modeling of delaminated area using virtual beam element

Kyoung et al. [44] examined the buckling and post-buckling behaviors of laminates with multiple through-the-width delaminations in their paper. Furthermore, these delaminations were modeled in a more realistic configuration. Contact node pair using the virtual beam element with the assumption of node-to-node contact was employed to prevent overlapping in the delaminated region. It was assumed that there was no curvature in the width direction. The 8-node degenerated shell element that had 5 degrees of freedom per node was used to model both the sound and delaminated portions of the laminate. Figure 2.13 showed how the virtual beam element was utilized in the finite element modeling of the delaminated area. The updated Lagrangian formulation and the improved arc-length method were utilized in the non-linear finite element formulation. Then, this modeling technique was used to simulate laminates with delaminations of uniform lengths, laminates with delaminations of different lengths and cross-ply laminates with double delaminations. It was observed that the buckling and post-buckling phenomenon were greatly affected by the arrangement of the delaminations. The article concluded that multiple delaminations ought to be considered in practical applications to gain a more realistic insight.

Lee et al. [49] wrote a non-linear finite element code using an eight-node hexahedral element to investigate the buckling behavior of a composite laminate with multiple delaminations under an axial compression. Furthermore, experiments were conducted to verify the data that were obtained from the analytical model. The results from the experiment and simulation were generally in good agreement. Three important bifurcation points, which were the initial buckle, initial delaminations and collapse, were compared. The parameters that were studied in this paper included different numbers, sizes, positions of delaminations and layer orientations. In addition, the strain energy release rate $G_I/G_{II}/G_{III}$ and the represented mode I/II/III fracture mode were used to examine the crack growth. The DCB/ENF methods were used to determine the values for G_{IC} and G_{IIC} . The Irwin's crack closure integral was adopted. The three-dimensional Hashine failure criterion was used to investigate the final collapse behavior of the composite laminate.

Chen and Sun [9] created the numerical analysis models and methods to investigate the buckling behavior such as the initial buckling, postbuckling, contact response of the delamination front, fiber and matrix failure. In order to compute the stress field of the delaminated plates, the finite element method based on the first-order shear deformation and Von Karman nonlinear plate theory was utilized. Furthermore, for the same values of x and y in the delaminated region, virtual truss element is used to connect the nodes of the upper and lower sub-laminates to prevent overlapping. A modified crack closure technique was employed to work on the delamination initiation and propagation. For the local compressive damage response, the Tsai-Hill failure criteria coupled with the corresponding stiffness degradable rules were used. It was emphasized in this article whether buckling or compressive strength failure was the first failure mode before the actual analysis of the compressive load abilities of the delaminated plate. It was found that stiffness degradation had effects on the buckling loads, postbuckling deflection and distribution of strain energy release rate at the delamination front.

Kyoung et al. [45] studied the buckling and post-buckling of the composite laminates with multiple delaminations under compression. Both one-dimensional composite

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laminates with multiple through-the-width delaminations and two-dimensional cross-ply composite laminates with circular delaminations were investigated. The updated Lagrangian description and the modified arc-length method were employed in the finite element formulation. Furthermore, the eight-node degenerated shell and rigid beam elements were used to model the composite laminates. The eight-node degenerated shell element had five degrees of freedom (3 displacement components corresponding to the global coordinates and two rotations about local unit vectors of the element). First order shear deformation theory was implemented. Mindlin plate theory was used for the boundary conditions on the delamination front. The rigid beam elements were used to connect the sound sublaminates to the sublaminae in the delaminated region. Furthermore, constraints were imposed at the delamination front to satisfy the compatibility conditions. Virtual beam element was used to form the contact node pair to prevent overlapping. Results from the finite element analyses showed that presence of the multiple delaminations in the composite laminates would not only lower the buckling load, but would also affect the post-buckling behavior. It was also found that the projected area of the delaminations would influence the buckling load of the composite laminates. Furthermore, it was observed that dimensions of the delaminations would affect the post-buckling behavior of the composite laminates. Configurations of the delaminations, which would lead to local reduction of bending stiffness and the contact between the surfaces of the layers at the delaminated region, were observed to affect the load carrying capacity of the composite laminate.

Tay et al. [74] observed that the mesh refinement did not really affect the values of the strain energy release rate (SERR) predicted by the virtual crack closure technique (VCCT). However, it would influence the results obtained using direct crack closure technique (CCT). It was also shown that 2-D analysis was different from 3-D analysis. A number of 2-D models with various mesh designs and mesh refinements were constructed to study the delaminated plates under plane strain conditions. It was observed that the location of the delamination along the thickness of the plate had a great impact on the amount of mode I and mode II components of the SERR. Furthermore, rectangular laminate with a circular embedded delamination was investigated. Discussion and

comparison were made between the values obtained from VCCT and CCT. It was observed that although VCCT underestimated the values for GI and GII, it could still be used to predict the crack growth direction. The influence of the meshing along the delamination front was also assessed.

Ali Kouchakzadeh and Sekine [1] used the Mindlin plate elements in a finite element method, including the shear effect, to study the buckling behavior of rectangular composite laminates with multiple embedded delaminations. After which an eigenproblem was solved to get the buckling mode and load. Individual mid-plane was allocated to each sub-laminate. Overlapping between sub-laminates was avoided by enforcing constraints through the use of the penalty function method. In addition, results from the model with this constraint were compared with the model without constraint (ie. Overlapping is allowed). The influences of the size, number and shape of the delaminations on the buckling load were then examined. The shapes of the embedded delaminations included circular and elliptical. Furthermore, comparison was made between a composite laminate with multiple delaminations and a composite laminate with a hole. It was found that presence of multiple delaminations decreased the buckling load of a composite laminate and this reduction was more significant as the number and size of the delaminations increased. In this paper, extension of the delamination was not covered. It was emphasized that the shape and orientation of the delamination would also affect the buckling load.

Riccio et al. [64] created a numerical tool to investigate the buckling behavior of the composite panel with either through-the-width or embedded delaminations. Both the modified virtual crack closure and penalty method were utilized to compute the delamination propagation and avoid penetration between the sub-laminates, respectively. For the 2-dimensional analyses, with through-the-width delaminations, only half of the laminate was considered due to the symmetry of geometries and forces. Furthermore, the size and location of the delamination through the thickness were varied and the response of the delamination growth was noted. It was observed in this article that contact analysis

must be included in order to obtain the actual compressive behavior of panels in 3-D analyses.

Gaudenzi et al. [23] tried to modify the numerical procedure in their previous work in an attempt to increase the efficiency of the numerical analysis especially for the prediction of the multiple unstable delamination. The improvement was done continuation method by Riks, which was a particular form of predictor equation. The modified virtual crack closure technique was utilized to determine the delamination extension. Both through-the-width and embedded delaminations were studied. The modified incremental continuation Riks formulation was tested against available experimental data and the effectiveness of the proposed method was successfully demonstrated.

Figure 2.14: Half model undergoing mixed buckling mode shape

Figure 2.15: Half model undergoing local buckling mode shape

Hwang and Liu [31] investigated the effects of the multiple delaminations on the buckling load and buckling mode. Various configurations of multiple delaminations were considered. One of the configurations resembled that caused by low velocity impact. Finite element method employing non-linear analysis was used to predict the buckling behavior. Furthermore, contact elements were used to prevent the overlapping situation and the crack tip was meshed with singular elements to include the singularity behavior. Due to the symmetry of the three possible buckling shapes considered in this article, only half of the specimen was modeled. The half model for the mixed buckling mode shape and local buckling mode shape were shown in Figure 2.14 and Figure 2.15, respectively. It was observed that if a long delamination existed just below the free surface, the presence of shorter delaminations beneath it would not affect the buckling load and buckling shape. However, when the shorter delaminations existed above the longer delamination, their existence would have an influence on the buckling load and buckling mode. It was also discovered that as the longer delamination moved towards the mid-plane, the shorter delaminations above it would have more impact on the buckling load.

de Morais [18] conducted the 3D finite element (FE) analyses of carbon fibre multidirectional specimens with $45^\circ/-45^\circ$, $0^\circ/45^\circ$ and $0^\circ/90^\circ$ delaminating interfaces. The G_{Ic} of multidirectional specimens was expressed as a function of the G_{Ic} of the unidirectional laminate through the use of a fracture criterion based on interlaminar stresses. The ratio between the G_{Ic} of a multidirectional specimen and that of a unidirectional laminate was found to be between 1.1 and 1.4. The corrected beam theory (CBT) and the classical laminate theory (CLT) were used to check the validity of the finite element model. For the multidirectional specimens, the curvature of the delamination front, effect of the residual stresses on the measured G_{Ic} and contribution of each fracture mode to the cracking between layers of different were analyzed. It was found that these features did not have significant effect on the measured G_{Ic} if the stacking sequence was carefully chosen.

Figure 2.16: Connection of shell and gap elements for the study of global buckling

Figure 2.17: Connection of shell and gap elements for the study of local buckling

Tafreshi and Oswald [73] studied the influences of the delamination size, shape and alignment on the critical buckling load of a delaminated composite plate in their paper. Furthermore, the local crack propagation was also examined. The displacement equations Mindlin/Kirchhoff plate theory was adopted to connect the corresponding nodes of the upper and lower sublaminates. Eight-noded shell and gap elements were used to model the laminated plate. Figure 2.16 and 2.17 showed how the shell and gap elements were

connected to examine the effects of global and local delamination buckling in this article, respectively. Experiments were also conducted and compared with the finite element analysis. Experimental results were found to be in good agreement with those from the finite element analysis for cases with embedded delaminations, different readings were obtained for situations when the delaminations were located at the edge of the plate. When investigating the crack extension, due to symmetry, only one quarter of the actual full model was utilized so as to have more elements around the crack front. Circular, elliptic aligned longitudinal and elliptic aligned transverse delamination shapes are employed to investigate the growth potential at the delamination border. The strain energy release rate was computed using the modified crack closure technique. It was observed that for high strain loading level, the potential in which the crack would grow strongly depend on the delamination shape and local buckling mode, which was in turn highly affected by the stacking sequence.

Figure 2.18: Effects of fiber orientation, stacking sequence and delamination width on the buckling load

Zor [84] used the carbon/epoxy woven-fiber system to make the composite plate. The shape of the composite plate was square, with single through-the-width delamination and

the boundary condition used was simply supported. The delamination was kept in the middle along the thickness and length of the plate for all cases. Effects of different fiber angles and delamination widths were investigated. 3D finite elements method was used to perform the linear buckling analysis. Two groups of volumes with two layers each, one on top and the other below were used in the modeling. Furthermore, an important parameter, critical delamination length was determined for all cases. It was also found that in this paper, the influences of the contact elements on the results are unimportant and could be neglected. It was observed that composite plate with larger fiber angle had a greater buckling load than that with a smaller fiber angle. However, when the buckling loads of the delaminated composite plates with different fiber-orientations were normalized with the respective buckling loads of the perfect composite plates, the composite plate with larger fiber angle would have a lower normalized load. This was illustrated in Figure 2.18.

Zor [85] considered the simply supported boundary condition in his paper and the composite used was woven steel-reinforced thermoplastic (LDFE, F.2.12). Furthermore, the effects of different ply orientations and delamination width were assessed. The two kinds of single delamination were included for investigation. One of them was the familiar through-the-width delamination (vertical delamination), which appeared perpendicular to the axial load and the other was through-the-length delamination (horizontal delamination) existing parallel to the axial force. Critical delamination width (CDW) was also used in this paper to represent the maximum delamination size that a structure could tolerate before its compressive strength started to decrease. It was found that composite laminates with horizontal through-the-length delamination had lower critical delamination width than that with vertical through-the-width delamination. It was noticed that for high values of orientation angles, the values and changing ratios (ratio of the buckling load of the composite laminate with delamination to that of a perfect composite laminate) were higher than those of the low angles. However, for each kind of delamination, the orientation angle did not affect the value of critical delamination width. Contact elements were included in the 3-D finite element models. However, it was found that the effect of these contact elements was negligible and could be neglected.

de Morais and de Moura [20] used the progressive based finite element models of the DCB and ENF specimens to evaluate the accuracy of the NL and 5%-Max criteria that were used in interlaminar fracture tests of composites. Interface elements had been developed to carry out the analysis and simulation of the delamination growth. Furthermore, for the DCB and ENF tests, the Beam Theory based data reduction schemes were used. Both carbon fibre reinforced plastics and glass fibre reinforced plastics were utilized in the simulation. Error generated by friction was also investigated and was found to be fairly small in all the cases. It was found that the 5%-Max criterion showed good performance, while the NL criterion was not good enough for the ENF test.

Figure 2.19: Two types of multiple delaminations

Hwang and Huang [30] employed the non-linear buckling analysis utilizing the finite element method to study the interaction a long delamination and a short delamination in the post-buckling stage. Two configurations were proposed and they were shown in Figure 2.19. Furthermore, the possible fracture mode of the delamination growth was discussed. The two modes were opening and shearing modes. Only half of

the specimen was modeled due to symmetry. The elements used were two-dimensional quadratic eight-node plane strain element, singular element and contact elements. The modified crack closure integral was used to compute the components of the strain energy release rate. It was found that the way in which the two delaminations buckled had an effect on the fracture mode of the delamination propagation.

Pradhan and Panda [62] examined the interlaminar fracture pattern of a multiple quasi-isotropic laminate with two symmetrically located embedded elliptical delaminations with respect to mid-plane symmetry of the total laminate. The lay-up sequence of the laminate was $[(0^\circ/\pm 45^\circ/90^\circ)]_{2s}$. Four laminates were studied by changing the through-the-thickness position of the delaminations. The effect of ply-orientation and the interaction between thermal residual stresses and mechanical loading on the delamination growth were investigated. The modified crack closure method was utilized to calculate the energy release rate for the individual fracture mode. The 3-D eight-node layered brick element and 3-D node-to-node contact element were used in the finite element analysis. This paper suggested that the effect of the residual thermal stresses and lamina orientation propagation behavior in the fiber-reinforced laminates were quite strong and should be taken into account in the damage development studies.

Aslan and Sahin [5] studied the influence of the delaminations size on the buckling load and compressive failure load of the E-glass composite laminates containing several large delaminations. These delaminations were induced by the low velocity impact on the laminates. The clamped $(0^\circ/90^\circ/0^\circ/90^\circ)_s$ laminated composite plates with and without multiple large delaminations were examined both experimentally and numerically. The experimental and simulation results were found to be in good agreement. From the analysis, it was observed that the near-surface delamination had a great impact on the buckling load of the composite laminates while the beneath delaminations offered an insignificant effect. Symmetrical and anti-symmetrical delaminated composite laminates with different fiber orientations were also studied numerically. It was found that composite laminates with different fiber orientations presented different critical loads for

the same near-surface delamination. Furthermore, the compressive failure load of the composite laminates that are inflicted with large delaminations was investigated utilizing the anti-buckling compression test fixture. It was discovered that the beneath delaminations had a significant influence on the compression failure load of the composite laminates.

Zor [86] created the 3-D models for the low-density polyethylene thermoplastic laminated composite plates reinforced by woven steel fibers, which contained the lateral vertical or horizontal strip delaminations that existed among all the four layers. It was discovered that there was no need to include the contact elements in the finite element model. The composite plates were simply-supported at all the edges and buckling loads were obtained for various stacking sequences. Both the symmetric and antisymmetric laminates were analyzed for fiber angles 0° , 15° , 30° and 45° . It was found that the larger the fiber angle the composite laminate had, the higher the buckling load. The critical delamination width ratio (CDWR) was introduced as an index for quality control to represent the maximum delamination size that a structure could withstand without lowering its compressive strength. The CDWR was attained for all the cases that were investigated. It was observed that composite plates that had the vertical delaminations had a CDWR of 10% while those that had the horizontal delaminations had a CDWR of 5%.

2.3 Experiment

Not many attempts have been made in performing experiments to investigate the buckling and post-buckling behavior of composite laminate due to the relatively large amount of time and effort that must be put in. Nevertheless, there are still some active researchers who are willing to conduct experiments and obtain valuable data.

Davidson and Ferrie [15] combined the results from the theoretical and experimental examination to study the influence of stretching-shearing coupling on the delamination buckling and propagation. Honeycomb sandwich laminates containing near-the-surface delaminations under compression were used. Cross-ply laminate that did not

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have the stretching-shearing coupling and another configuration that was chosen to maximize the stretching-shearing coupling (A_{16}) were used in the study. Linear mixed-mode delamination growth law and crack-tip elements were used to predict the delamination growth strains. The $[33_2/10]_s$ laminate was observed to have a significantly higher delamination buckling strain than that of $[0_2/90/0_2]$. However, the $[33_2/10]_s$ laminate was found to have much lower critical strain in which the delamination started to grow than the $[0_2/90/0_2]$ lay-up. Investigation was also carried out to study the critical compressive strain in the parent laminate when failure occurred. Hence, a rather simple closed form method was developed to predict the delamination buckling and growth easily.

Kutlu and Chang [43] carried out experiments to examine the buckling behavior of composite laminates with multiple delaminations. The results were used to verify the analytical and finite element analyses that were developed by the same authors previously. The material used in the experiments was T300/976 graphite/epoxy. Panels were made from plies with different orientations manufactured into cylindrical and flat composite panels. Either one or two through-the-width delaminations were introduced. Delaminations were introduced to the composite laminate by inserting very thin Telfon film strips. Moreover, before any testing, an X-radiograph was taken for every specimen to see if there is any internal damage due to improper manufacturing or cutting. Furthermore, right from the application of the initial load to the final failure, strain gauges were used to record the strain as the compressive load increased. Finite element method was also used and its results were compared with the data obtained from the experiment. Good agreement was found between them. Moreover, finite element analysis was utilized to determine the effects of the size, position and quantity of the delaminations on the buckling load of the composite laminates. It was pointed out that presence of the thermal residual stress could greatly affect the postbuckling behavior of the composite laminate. It was concluded that delamination growth could greatly affect the post-buckling behavior of the composite laminate and composite laminates with

multiple delaminations could behave very differently when compared with those with only one delamination.

Lifshitz and Gildin [50] conducted experiments to investigate how the carbon/epoxy composite beam behaved under cyclic compressive loading. The composite beam was made of AS4/3502 carbon/epoxy and the stacking sequence of $[(45/90/-45/0)_6]_S$ was used. Firstly, the effect of a delamination on the fatigue life was studied. Then, the respond of the composite beam from the initial cyclic compressive loading to failure was examined. In addition to finding the critical stress of the delaminated beam, effort was also put in to determine the average stress in the sub-laminate when it buckled. The delaminated beam was subjected to cyclic loadings that were 70%, 75% and 80% of the static strength and the number of cycles to the formation of the first thin sub-laminate that buckled was noted. It was observed that for a laminate under cyclic loading, delaminations could be created, causing the laminate to fail below its static failure strength. It was also found that for laminate under cyclic loading, in many cases, the delamination growth was sporadic instead of constant at certain rate, which was assumed in many papers. When the distance between the outer surface and the pre-existing crack was smaller than a critical value, the laminate would fail sooner.

Lachaud et al. [46] utilized both thermoset and thermoplastic carbon-fibre composites in the experimental and numerical studies of delamination caused by local buckling. The carbon-fibre composite materials used were T300/914 (thermoset) and AS4/Peek (thermoplastic). It was found that growth of delamination was detected when macrocracks appeared within the buckling piles. It was found that propagation of the delamination appeared much later in AS4/PEEK than in T300/914. Before the delamination growth, acoustic emission (Felicity ratio) was used to determine the damage accumulation. Then, two numerical methods, the global restitution energy rate (virtual crack extension method) and its partition into modes I, II and III (modified crack-closure integral) were used to predict the start of delamination growth. A stacking configuration and two kinds of materials were used with a mechanical failure criterion. After that, an interfacial stresses criterion determined by the finite-element method was compared with

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this criterion. The acoustic parameter was introduced in this paper, which seems to be able to track the macrocracking caused by local buckling.

Figure 2.20: Setup of mechanical equipment

Figure 2.21: Close-up of the setup

Gu and Chattopadhyay [27] carried out detailed experiments to examine the behavior of delaminations buckling and postbuckling in composites. Figure 2.20 and Figure 2.21 showed the setup for the experiment. The parameters that they included in their study consisted of ply stacking sequence, delaminations positions and delamination length. It was observed that the resultant buckling shape was strongly dependent on the length and location through the thickness of the delaminations. The results from this experimental investigation were also used to validate the previous new higher-order theory for delaminations buckling and postbuckling analysis developed by the same authors. In order to capture the transverse shear effects in the delaminated composite, in this theory, higher-order through-thickness displacement field was used to define the transverse shear deformations. Moreover, the classical laminate theory was also used in the validation. The results obtained from the higher-order theory were found to be in very good agreement with those from the experiments. On the other hand, when the results calculated from the classical laminate theory were compared to those from both the higher-order theory and experiments, noticeable difference could be seemed. For the initial postbuckling behavior, data from the experiments and higher-order theory was compared and they correlated quite well.

Figure 2.22: Critical buckling load of the interply hybrid composite for $\frac{t}{H} = \frac{1}{2}$

Figure 2.23: Critical buckling load of the interply hybrid composite for $\frac{t}{H} = \frac{3}{8}$

Figure 2.24: Critical buckling load of the interply hybrid composite for $\frac{t}{H} = \frac{1}{8}$

Hwang and Mao [33] examined both the single-fibre system and interply hybrid composites in their paper. Delaminations were introduced into these composites and their buckling characteristics were investigated. Although composites that were made up of boron or carbon fibres had high modulus/weight ratios, they generally had weaker impact resistance compared to composites made up of steel alloys or glass fibres. Therefore, different quantities of carbon and glass plies were used in various configurations to form the interply hybrid composites with desired material properties. Experiments and finite element method considering the contact elements were employed to investigate the buckling behavior of these composite laminates. Furthermore, singular elements were used around the crack tip to model the singularity behavior and also to cut down the computing time. It was observed that both the experiments and the non-linear analysis displayed similar results. For both the single-fibre system and interply hybrid composites, it was found that their buckling loads decreased as the delamination appeared closer to the surface and the delamination became longer. These trends could be observed from Figure 2.22 to Figure 2.24. In these graphs, the terms $\frac{a}{L}$ and $\frac{t}{H}$ were ratio of delamination length to the total length of the laminate and ratio of the thickness of the thinner sub-laminate to the thickness of the entire composite, respectively. In addition, the amount of glass layers used and how these glass layers were stacked with the carbon layers were observed to affect the buckling load. The critical delamination length that was defined as the delamination length below which the buckling load of the composite would not be affected was also examined.

de Morais et al. [19] conducted DCB experimental tests on the carbon/epoxy $[0^\circ/90^\circ]_{12}$ specimens. Furthermore, finite element method was used to investigate the local crack propagation observed in the tests. It was found that intralaminar damage occurred to the 90° mid-layer. It was also observed that the measured G_{Ic} values were significantly higher than those of the unidirectional $[0^\circ]_{24}$ laminates especially the final propagation values. It was observed that the interlaminar G is in good agreement with the corrected beam theory and did not depend on the delaminating $[0^\circ/90^\circ]$ interface.

However, the intralaminar G_{Ic} was observed to be significantly smaller than the interlaminar G_{Ic} and transverse cracking maybe unavoidable in the DCB multidirectional specimens.

Hwang and Liu [32] studied composite laminates with multiple delaminations, under uniaxial load, for their buckling and post-buckling phenomenon through experiments. In addition, the critical loads in which the delaminations would start to propagate were also examined. Impact damage was assumed to have caused these delaminations. Furthermore, comparisons were made between the results obtained from experiments and those from a non-linear buckling analysis from the finite element method. The composite laminate was meshed with two-dimensional quadratic eight-node plane strain elements and the crack tips were meshed with singular elements in order to simulate the singularity effect. Furthermore, contact elements were introduced at the delaminated region to avoid overlapping. The difference on the buckling and post-buckling behaviors between laminates with single and multiple delaminations were investigated. It was found that for a composite laminate with multiple delaminations, if the longest delamination was located close to the free surface, then, the shorter delaminations below it would not have significant effect on the buckling load and these delaminations could be replaced by a single delamination with its length and position the same as the longest delamination. However, it would be too unconservative to use the critical delamination propagation load of single-delaminated laminate to predict the critical delamination growth load of a composite laminate with multiple delaminations.

Ge et al. [24] used the digital speckle correlation method to get the inplane displacement. The composite panel investigated contained an embedded elliptical delamination. In the experiments, different ply layering sequences and orientation of the elliptical damage were considered. The finite element analysis was used to obtain the strain energy release rate. Furthermore, the Mindlin plate theory and virtual crack closure technique were utilized in this study. It was observed that the energy release rate from the mode III fracture was important and should be included in the total energy release rate.

Finally, it was also pointed out that the stacking sequence had effects on the delamination growth.

Pereira et al. [59] carried out ENF tests on multi-directional carbon/epoxy laminates, which stacking sequences were studied previously using the finite element analysis. In the later stage, an average interlaminar stress based criterion was developed to predict the effect of the delaminating interface on the measured G_{IIC} . Two types of compliance calibrations were utilized and they were denoted as CC1 and CC2. Both compliance calibration schemes fitted well with the finite element results. It was found that the presence of residual stress did not affect the widthwise average of the G_{II} . Moreover, when the friction was introduced, the error was very small when the compliance calibration schemes were used. The non-linearity (NL) and the 5% offset or maximum load criteria (5/M) were used to define the exact instant of crack initiation.

Zhang and Taheri [83] used carbon/epoxy and E-glass epoxy in their investigation. The dynamic pulse-buckling behavior of the composite beam with a lay-up sequence of $[(\pm 67.5)_n]_s$ was examined through experiments. In additional, the post-impact behavior was studied. It was observed that beams with slenderness ratios of $\frac{L}{r} = 227$, only one crest was developed when the beams buckled. On the other hand, for beams with slenderness ratio of $\frac{L}{r} = 320$, two or more crests were formed. It was found that beams with lay-up sequence of $[(\pm 67.5)_3]_s$ had a lower axial and bending stiffness than beams with stacking sequences of $[0_{12}]$, $[(\pm 22.5)_n]_s$ and $[(\pm 45)_3]_s$. Furthermore, after the impact, the beams with $[(\pm 67.5)_3]_s$ lay-lose their elasticity and were unable to return to their original form. The deflection of the beam varied linearly with the impact energy. It was observed that the behavior of the E-glass/epoxy laminate beams exhibited the same trend.

CHAPTER THREE: FINITE ELEMENT ANALYSIS AND EXPERIMENTAL STUDY

The finite element software ANSYS 8.1 is used in the research. With proper engineering knowledge and sufficient experience with the software, one can obtain fast and highly reliable findings from ANSYS 8.1. The software is capable of performing various kinds of analyses such as structural, fluid and thermal studies. The software is utilized to solve for the buckling problem, which is a type of structural investigation. The standard procedures of applying the ANSYS 8.1 are modeling, solving and post-processing of the results. One can write the source codes for these procedures in text form and input into the ANSYS 8.1 to execute the steps. This grants tremendous ease when one is trying to conduct a large number of parametric studies. The user simply needs to alter the source codes in the text to achieve the desired configuration for the simulation.

In addition, experiment is carried out on the composite beam samples. The main stages of the experiment comprise of producing of the composite beam specimens, the actual buckling test and post-processing. In the manufacturing phase, after the prepreg is layered to what is desired, the stacked prepreg is sent for curing. The curing table and the HeatCon 9200 temperature controller are the main devices that are employed in the curing process. The cured composite beam specimen can then be mounted onto the Instron 5565 test machine to execute the actual buckling test. Lastly, the raw data that is generated during the test is processed and analyzed.

3.1 Finite element method

In the following section, the essential steps of employing the ANSYS 8.1 software to accomplish the buckling analysis will be discussed in detail. These vital stages consist of the constructing phase, solving phase and post-processing stage.

3.1.1 Constructing phase

The entire constructing process consists of pointing to the desired working directory, providing the proper file name, entering the preprocessor, giving suitable title for the

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analysis to be performed, assigning variables, selecting the types of elements to be used, defining the material properties, identifying the real constants, building the physical model and meshing it. A sample source code for these operations is available in Appendix A.

3.1.1.1 Working directory

In the source code, the working directory is pointed to 'newanalysis2'.

3.1.1.2 Filename

The filename associated with the files that are created along with the simulation is given as 'antisymmetric15degree'

3.1.1.3 Preprocessor

The command '/prep7' is issued to enter the preprocessor in ANSYS 8.1.

3.1.1.4 Title

The numerical analysis is named as 'Carbon-epoxy 8 plies C-C antisymmetric 15 degree H2/H=1/2 a/L=0.50'

3.1.1.5 Variables

After the title is supplied, variables that will be used subsequently are defined. The left hand sides of the equations are the variables while the right hand sides are their respective values.

3.1.1.6 Elements

In the source code in Appendix A, the element SHELL91 is employed. SHELL91 is an 8-noded layered shell element that is suitable of modeling fiber-reinforced laminated composites. The element has six degrees of freedom at each node. Furthermore, the element has a node offset option that can be used to construct the delaminated composite beam. The solid element counterpart for SHELL91 is SOLID46. SOLID 46 is an 8-noded

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layered structural solid element with three degrees of freedom at each node. The other details on these elements can be found in the ANSYS 8.1 help file.

3.1.1.7 Material properties

For the simulation in Appendix A, the carbon/epoxy composite material utilized has the following material properties:

E_{11} (GPa)	E_{22} (GPa)	G_{12} (GPa)	ν_{12}
150	9	4.8	0.28

Table 3.1: Mechanical properties of carbon/epoxy composite matter

3.1.1.8 Real constants

In the source code in Appendix A, the real constants are written for the SHELL91 such that it has the material properties that are described in the previous section, thickness per layer of t ($t = 0.125\text{mm}$) and individual ply-orientation of either angle1 ($angle1 = 15^\circ$) or angle2 ($angle2 = -15^\circ$). These values can also be input in a similar fashion when the SOLID46 element is utilized.

3.1.1.9 Modeling and meshing

In this section, the five steps of creating and meshing the model will be explained. These five phases are reflected in the source code in Appendix A. Therefore, the five steps turn out to be:

Carbon-epoxy 8 plies C-C antisymmetric 15 degree $H_2/H=1/2$ $a/L=0.50$

Figure 3.1: Creating areas A1, A2 and A4

- 1) Firstly, three areas are constructed as displayed in Figure 3.1. The two areas at the two sides A1 and A2 are used to simulate the intact regions. In contrast, area A4 is employed to represent the delaminated zone.

Carbon-epoxy 8 plies C-C antisymmetric 15 degree $H_2/H=1/2$ $a/L=0.50$

Figure 3.2: Meshing of areas A1, A2 and A4

- 2) Then, the SHELL91 element is utilized to mesh the areas A1, A2 and A4. As illustrated in the diagram above, the first set of 4 plies out of the total 8 plies is

produced after meshing. When meshing the three areas, the node position option is set such that the nodes are located at the top surface of the element. This means that the nodes will reside at the top surface of the element with the first set of 4 plies below the top surface as displayed in Figure 3.2.

Carbon-epoxy 8 plies C-C antisymmetric 15 degree H2/H=1/2 a/L=0.50

Figure 3.3: Generating areas A3, A5 and A7

- 3) As depicted in Figure 3.3, another set of areas, A3, A5 and A7 are constructed at the same location as areas A1, A2 and A4, respectively. Analogously, the areas A3 and A5 are used to form the perfect sectors while area A7 is utilized to realize the delaminated sector.

Figure 3.4: Meshing of areas A3, A5 and A7

- 4) Once again, as shown in the illustration above, the SHELL91 element is applied to mesh areas A3, A5 and A7 to generate the desired second set of 4 plies. However, in this case, the node location option is adjusted in such a way that the nodes will now appear at the bottom surface of the element. The second set of 4 plies will emerge above the bottom surface as demonstrated in Figure 3.4.

Figure 3.5: Merging of selected coincident nodes

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- 5) Letting the nodes associated with the first set of 4 plies to be at the top surface of the element in step 2 and placing the nodes corresponding to the second set of 4 plies to be at the bottom surface of the element in step 4 make it possible for all these nodes to be on the same plane. As areas A1, A2, A3 and A5 are employed to build the ideal portions, the nodes of the SHELL91 element of areas A1 and A3 and the nodes of the SHELL91 element of areas A2 and A5 must be merged, as portrayed in Figure 3.5. This is done in the ANSYS 8.1 by issuing the command ‘nummrg,node’ to merge these coincident nodes to obtain the required intact parts. As areas A4 and A7 are left untouched, their nodes will not be merged, thus forming the wanted delaminated region.

After the model is created and meshed, it is saved and the command ‘finish’ is input to exit the pre-processor. This simulation can also be done using the SOLID46 element in the similar manner. The next procedure after the modeling and meshing stage will be the solving and post-processing stage.

3.1.2 Solving phase

The complete solving phase includes entering the solving tab, specifying the appropriate boundary condition, activating the prestress effect and actual carrying out of the buckling analysis. The source code for this solving phase can be found in Appendix A.

3.1.2.1 Solving tab

The command ‘/solu’ is delivered to go into the solving tab.

3.1.2.2 Boundary conditions

In the source code in Appendix A, the clamped boundary condition is then imposed onto the meshed model. At one end of the meshed model where the global x is 0, all the nodes are not allowed to translate along or rotate about the three global Cartesian co-ordinate directions. This prescribes the necessary clamped boundary condition at this edge. At the other end, the same set of restrictions is enforced on the nodes except for translation in

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the x – direction. This exception is to permit the application of the in-plane compressive force in the next section.

3.1.2.3 Prestress effects

Before proceeding to the actual buckling analysis, the prestress effects must first be applied onto the elements. This is accomplished via the command ‘pstress,on’, applying 1N of compressive force onto every node at the end that is still able to move in the x – direction and performing the static analysis. These operations are reflected clearly in the source code in Appendix A. Then, the command ‘finish’ is keyed in to leave the solving tab.

3.1.2.4 Buckling analysis

It is important to note that one ought to exit the solving tab after the static analysis for the prestress effects and reenter the solving tab for the actual buckling analysis. Before solving the buckling analysis, one has to specify the extraction method to be used and the number of modes to be extracted. In the source code in the Appendix A, the command ‘bucopt,subsp,1’ indicates the use of the subspace iteration and to extract only the first mode. As one is usually just interested in the lowest critical load, attaining just the first mode will be sufficient. The command ‘mxpand,1’ is input to write this first buckled mode shape to the result file for subsequent post-processing.

3.1.3 Post-processing stage

The post-processing stage involves displaying the buckled mode shape and obtaining the desired critical load. The source code for this step is also addressed in the Appendix A.

3.1.3.1 Post-processor

The command ‘/post1’ is input to enter the post-processor.

3.1.3.2 Reviewing results

Figure 3.6: Buckled composite beam with a delamination

Figure 3.6 demonstrates the buckled mode shape of the composite beam. The marked number that appears at the top left-hand corner of the diagram above is attained via the buckling analysis and is known as the load factor. Since a unit load is specified on every node at the end of the model that is still competent to travel in the x -direction, multiplying the resultant load factor with the total number of nodes at the same end will grant the wanted buckling load.

3.1.4 Overview on finite element method

Figure 3.7: Summary on the whole finite element method

The flow chart above depicts a complete overview on the finite element method.

3.2 Experimental study

Aside from finite element method, experiment is also conducted to gain practical insights into the buckling problem. In the experiment, the carbon/epoxy composite material Fibredux 913C-HTA (12K)-5-35% is utilized to fabricate the composite beam specimens.

This notation can be interpreted as:

913	:	120°C curing epoxy matrix
C	:	carbon fiber
HTA	:	fiber type (standard modulus)
12K	:	number of filaments (12,000)
5	:	cured prepreg thickness at 60% V_f (5 thousandths of an inch = 0.125mm)
34%	:	resin content of prepreg, nominal % by weight

All the specimens are composed of 16 specially orthotropic plies with cured effective length, width and thickness of 100mm, 20mm and 2mm, respectively. Both the clamped and simply-supported boundary conditions are examined. While the length of the simply-supported composite beam specimen is 100mm, the span for the clamped one must be 140mm. The extra 40mm is to facilitate the gripping of the two ends (ie. 20mm at each side) of the composite beam specimen to form the desired clamped edge condition. Furthermore, both the perfect and delaminated cases are explored. For the delaminated cases, the delamination is located at the mid-plane and at the center along the spanwise direction. The delamination lengths that are considered include $\frac{a}{L} = 0.2, 0.4, 0.6$ and 0.8 ,

where a is the delamination span and L is the effective length of the composite beam specimen.

The whole experiment mainly involves making of the composite beam specimen, performing the buckling test and gathering of the useful information such as the critical load and buckled mode shape. After completing the experiment, the experimental results will be compared with the readings that are acquired from the finite element method in the later section.

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3.2.1 Fabrication of composite beam specimen

In the following text, a detail description of the entire process of specimen production will be presented as individual steps as:

- 1) Prepreg is supplied in a large roll and it is preserved in the refrigerator at temperature as low as -20°C . Hence, the prepreg roll must be defrosted at ambient temperature before it can be used.
- 2) Utilizing a set square, pen knife and ruler, cut the prepreg needed dimensions. They are $140\text{mm} \times 20\text{mm}$ for clamped specimen and $100\text{mm} \times 20\text{mm}$ for simply-supported specimen.

Figure 3.8: Different elements for a piece of prepreg

- 3) The various constituents for a layer of unidirectional prepreg is illustrated in Figure 3.8. When manufacturing the specimen, the blue polythene protective film is first peeled off from the first layer and the brown release paper is removed from the second layer. Then, the bottom surface of the second layer is attached carefully onto the top surface of the first layer.

Figure 3.9: Prepreg roller

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- 4) A roller which is displayed in Figure 3.9 is utilized to apply pressure onto the two layers to ensure complete cohesion of the two layers and guarantee that the adhesive layers stick to one another. The roller also acts to reduce any air bubbles in the specimen which will become voids after curing and affect the specimen's structural integrity.
- 5) For the perfect composite beam specimen, the stacking process is stopped when 16 layers are bonded together. On the other hand, for the delaminated composite beam specimen, a very thin piece of the non-porous Telfon sheet has to be inserted between the 8th and 9th layers to simulate the longed delamination of different lengths at the center lengthwise position of the specimen. For both the ideal and delaminated composite beam specimens, the brown release paper that is still glued to the first layer and the blue polythene protective film that is still attached to the 16th layer will only be stripped off just prior to the curing process. Three specimens are constructed for the separate perfect and different delaminated cases.

Figure 3.10: Fabricated jig to assist curing process

- 6) Figure 3.10 depicts a jig that is constructed to aid the curing process. Both the aluminum base plate and the aluminum guides are wrapped with the non-porous Telfon sheet. The guides are attached to the base plate via the heat resistant

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adhesive blue tape. The composite beam specimens can then be placed onto the base plate, between the guides as shown in Figure 3.10. The base plate is to prevent the melted resin matrix from reaching and staining the heating table. The guides are included to help achieve a more rectangular cross-section and straighter side edges.

Figure 3.11: Primary layering on the specimens and jig

- 7) After laying the composite beam specimens onto the jig, cover them with a film of porous Telfon sheet, followed by a 'breather' cloth and finally a piece of non-porous Telfon sheet. A typical arrangement is illustrated in Figure 3.11. Transfer this assembly onto the curing table and place a thermocouple near the setup to monitor the temperature.

Figure 3.12: Curing table

Figure 3.13: HeatCon 9200 temperature controller

Figure 3.14: Thermocouple

- 8) The two key devices in the curing process are the curing table and the HeatCon 9200 temperature controller as displayed in Figures 3.12 and 3.13. The curing table consists of the metal surface that supplies heat to melt the resin matrix and the vacuum suction that creates the necessary pressure within the table. In addition, the rubber cover avoids the pressure from escaping during the curing course. The HeatCon 9200 temperature controller manages and monitors the temperature of the metal surface of the curing table. It can also be programmed to produce the desired curing cycle. The thermocouple as shown in Figure 3.14 measures the temperature on the metal surface and feedbacks the value to the HeatCon 9200 temperature controller.

- 9) After the assembly that is mentioned in step 7 is rested on the metal surface of the curing table, before commencing the actual curing process, a big piece of ‘breather’ cloth is covered over the arrangement. This ‘breather’ cloth will envelop the entire metal surface of the curing table to ensure that heat is retained to offer a smooth curing cycle. Then, the rubber cover of the curing table is closed. Another big sheet of ‘breather’ cloth is coated over the rubber cover to speed up the curing course.

Figure 3.15: Curing cycle

- 10) Figure 3.15 demonstrates the curing cycle. The initial portion of the curing cycle is known as the ‘ramping up’, when the temperature increases slowly from the ambient temperature to 120°C. At this stage, a chemical reaction occurs as the resin matrix viscosity decreases rapidly. A noticeable short dwell of 10 minutes is observed in this primary phase of the curing cycle. After the first phase, there is a dwell of approximately one hour at 120°C. The function of this secondary period is to control the rate of chemical reaction. At the end of the second step, the resin viscosity is at a minimum and the pressure squeezes out the excess resin. The application of the constant pressure throughout the cycle prevents the consolidation of the plies till the resin in the specimen reaches its glass state at the

end of the cooling down or ramping down phase. The full curing process takes about 180 minutes to finish.

Figure 3.16: Cured and trimmed (a) perfect and (b) delaminated composite beam samples

- 11) After the complete curing cycle, the composite beam samples are taken out of the curing table and allowed to cool to room temperature. Then, the spiky edges are filed for easy handling. Figure 3.16 demonstrates the final look of the ideal and delaminated composite beam samples. The specimens are now ready to go through the buckling experiment.

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3.2.2 Compression testing

In this section, the procedure for the buckling test on the composite beam specimen will be discussed:

Figure 3.17: Instron 5565 test machine

Figure 3.18: Instron 5565 Control panel

- 1) The compression test on the composite beam specimen is accomplished via the Instron 5565 machine which is shown in Figure 3.17. Figure 3.18 illustrates the control panel that is connected to the machine to facilitate the buckling experiment.

- 2) Login to the 'MERLIN AUTOMATIC MATERIAL TESTING SYSTEM (MERLIN AMTS)' and key in the correct parameters. In addition, input the dimensions of the specimen and set the velocity of the crosshead to be 1.5mm/minute.

Figure 3.19: Specimens that are under (a) clamped and (b) simply-supported edge conditions

- 3) Fixtures that can be secured to the Instron 5565 machine are custom-made to realize the longed clamped and simply-supported boundary conditions. The Figure 3.19 (a) depicts the clamped end of the specimen while Figure 3.19 (b) demonstrates the simply-supported edge of the sample. These fixtures that are fastened to the lower grip of the Instron 5565 machine are stationary. These fixtures are also attached to the upper movable grip to give the necessary edge condition. After the composite beam specimen is first secured to the fixture at the lower end of the Instron 5565 machine, the upper grip can be adjusted in the lateral direction using the 'JOG UP/DOWN' button to reach a suitable height before mounting the specimen to the upper fixture. Furthermore, one can utilize the 'FINE POSITION' button to obtain a more precise vertical location of the upper grip after acquiring a rough one via the 'JOG UP/DOWN' button.
- 4) After putting the specimen in place, one can begin the buckling test by pressing the 'START TEST' button. The upper grip will then start to travel downwards and exert an increasing compressive force onto the specimen. Simultaneously, the MERLIN AMTS will record down the displacement of the upper grip (upper edge movement of the specimen) and the load that is applied onto the specimen from the machine. Later, the data can be utilized for further processing.

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- 5) When the composite beam specimen is observed to buckle, the ‘STOP TEST’ button is pushed to halt the upper grip. At this point, one can take photographs of the buckled specimen for analysis afterward. Then, the ‘RETURN’ button is hit to bring the upper grip back to the original position prior to pressing the ‘START TEST’ button in step 4. Subsequently, remove the tested sample from the Instron 5565 machine.

Figure 3.20: A graph of compressive load on a specimen versus end displacement of the upper grip

Figure 3.21: A Southwell plot

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- 6) The readings that are stored in the MERLIN AMTS can be extracted out and utilized to plot a graph of compressive load (P) on the specimen versus end displacement (δ) of the upper grip. An example of such a diagram for one of the specimens is shown in Figure 3.20 for illustration. This specimen is simply-supported, has a delamination of $\frac{a}{L} = 0.6$ and its other details are same as those describe previously in section 3.2. Then, the famous Southwell plot will be employed to determine the buckling load. The equation of the Southwell plot appears as:

$$\delta = \frac{\delta}{P} P_{cr} + C \quad (3.1)$$

In the above expression, P_{cr} is the buckling load while C is a constant. One can identify that equation (3.1) is actually an equation of a straight line. From equation (3.1), it is clear that one should plot δ against $\frac{\delta}{P}$ and the gradient of the straight line will give the desired buckling load. Therefore, the data in Figure 3.20 are used to plot δ against $\frac{\delta}{P}$ to create the corresponding Southwell plot and this plot is presented in Figure 3.21. The straight line equation in Figure 3.21 corresponds to the line beside it. Hence, by comparing this straight line equation with equation (3.1), one can conclude that the desired buckling load is 1561N.

- 7) The six steps above can be repeated to investigate the next specimen.

Since three composite beam specimens are fabricated for each individual case, a total of 30 samples are manufactured. The findings from these specimens will be presented in the subsequent section.

3.2.3 Summary of experiment

Figure 3.22: Overall outline of the experimental study

The summary above provides an overview of the entire experimental procedure.

3.3 Results and discussion

In this section, convergence test will be conducted on the simulation model that is described in section 3.2 to find out the proper number of mesh to be used. In addition, the readings that are attained from the experiment in section 3.3 will be compared with the values that are generated from the finite element method.

3.3.1 Convergence test

Figure 3.23: SHELL91 element convergence test

The above diagram illustrates the convergence test on the SHELL91 element. In the test, the SHELL91 element attempts to simulate a 16 specially orthotropic plies Fibredux 913C-HTA (12K)-5-35% carbon/epoxy delaminated composite beam, with effective length, overall width and total height of 100mm, 20mm and 2mm, respectively. The $\frac{a}{L} = 0.6$ delamination resides at the mid-plane and the center position along the lengthwise direction of the composite beam. Furthermore, both ends of the composite beam are clamped. From Figure 3.23, it can be noted that one will need around 300 to 400 elements to achieve a converged result.

3.3.2 Comparison between simulation and experimental readings

Figure 3.24: Critical load of 16 layers specially orthotropic composite beam for different delamination lengths under simply-supported boundary condition

Figure 3.25: Normalized buckling load of 16 plies specially orthotropic composite beam for various delamination lengths under both simply-supported and clamped end conditions

Figure 3.24 shows the buckling load of simply-supported composite beam that is composed of 16 specially orthotropic plies for dissimilar delamination spans. The three triangular points for each $\frac{a}{L}$ in the graph denote the three results that are acquired from

the three specimens in the experiment for the individual $\frac{a}{L}$ cases. For every $\frac{a}{L}$ scenario, the three buckling load values appear to be consistent. Moreover, the experimental

readings turn out to tally very well with the data that is gathered from the finite element method. The discrepancies may be due to the experimental errors. One of the experimental errors may be due to the inevitable presence of air voids and other impurities in the final composite beam specimen. Furthermore, it is found to be impossible to obtain a specimen which has an exact and uniform thickness of 2mm. Nevertheless, great effort is put in to minimize the potential experimental errors.

Figure 3.25 illustrates the normalized critical load of the specially orthotropic composite beam for different delamination spans. The composite beam is comprised of 16 plies and both the simply-supported and clamped boundary conditions are studied. In Figure 3.25, in addition of the experimental and finite element method findings, the relevant published results from Simites et al. [72] are included for comparison. Each normalized experimental buckling load in the Figure 3.25 is the outcome of the average of the three specimens results for an individual case. A value in the Figure 3.25 from the finite element method, experiment or Simites et al. [72] is attained via normalizing its actual buckling load with the critical load of its own perfect composite beam. From the diagram, one can deduce that the readings from the finite element method and Simites et al. [72] are in excellent agreement. This infers that the finite element method that is addressed in section 3.2 is capable of providing results that are very close to those predicted by the kind of analytical model that is formulated by Simites et al. [72]. Hence, the finite element method in section 3.2 will be employed to validate the mathematical models that will be developed in the following chapters. The experimental values are also in good agreement with those from the finite element method and Simites et al. [72]. The difference may again be due to the experimental errors.

CHAPTER FOUR: THEORY

In the present chapter, the geometrical relations for small deformations and the constitutive relations for a lamina in a chosen coordinate system are addressed. Then, various laminated plate theories are discussed. They include the classical lamination theory, first-order shear deformation theory and third-order shear deformation. An innovative expression (${}^3(\overline{EI})$) which can be used to work out the flexural stiffness of a thin laminated beam is derived. It is utilized along with two other bending rigidity formulae (${}^1(\overline{EI})$ and ${}^2(\overline{EI})$) to look into the buckling problem of a thin perfect composite beam in the results and discussion section. Three distinct ways of computing the effective flexural stiffness, effective shear modulus and shear correction factor are established via the first-order shear deformation theory and third-order shear deformation theories. These developed equations are used to investigate the buckling behavior of the moderately thick ideal composite beam towards the end of the chapter.

4.1 Geometrical relations

For small strains and rotations of transverse normal, Reddy [63] addressed that the strain-displacement relations become linear as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad (4.1)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \quad (4.2)$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \quad (4.3)$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (4.4)$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad (4.5)$$

$$\gamma_{yz} = 2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad (4.6)$$

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The x , y and z are the coordinates present in the orthogonal Cartesian system. The u , v and w are the displacements in the x -, y - and z -directions, respectively. The normal strains in the x -, y - and z -directions are denoted as ε_{xx} , ε_{yy} and ε_{zz} , respectively. Furthermore, the $\gamma_{xy}(=2\varepsilon_{xy})$, $\gamma_{xz}(=2\varepsilon_{xz})$ and $\gamma_{yz}(=2\varepsilon_{yz})$ are the shear strains in the xy -, xz - and yz -planes, respectively.

4.2 Constitutive relations

Figure 4.1: Principal material coordinate system and problem coordinate directions

The 1- and 2-axes in Figure 4.1 are known as the material coordinate system. The x - and y -axes are the chosen coordinate system for the problem formulation. Both the 3- and z -axes are pointing out of the page. In addition, Figure 4.1 demonstrates θ , which is the positive rotation of the principal material axes from the x - y axes.

Reddy [63] mentions that for a unidirectionally reinforced lamina, which is a very thin component, most laminate theories neglect the transverse normal stress, τ_{33} . Therefore, as pointed out by Decolon [16] and Reddy [63], the stress-strain relations can be reduced to:

$$\begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} \quad (4.7)$$

$$\begin{Bmatrix} \tau_{23} \\ \tau_{13} \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix} \quad (4.8)$$

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The τ_{11} and τ_{22} are the normal stresses in the 1– and 2– directions, respectively. On the other hand, the τ_{12} , τ_{23} and τ_{13} are the shear stresses on the 1–2, 2–3 and 1–3 planes, respectively. The ε_{11} and ε_{22} are the normal strains in the principal material directions. The γ_{12} , γ_{23} and γ_{13} are the shear strains on the 1–2, 2–3 and 1–3 planes, respectively. The Q_{ij} (for $i, j = 1..6$) in equations (4.7) and (4.8) are known as the reduced stiffnesses. They can be evaluated as:

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \quad (4.9)$$

$$Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \quad (4.10)$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \quad (4.11)$$

$$Q_{66} = G_{12} \quad (4.12)$$

$$Q_{44} = G_{23} \quad (4.13)$$

$$Q_{55} = G_{13} \quad (4.14)$$

Hence, the reduced stiffnesses will involve six independent engineering constants. The E_1 and E_2 are the Young's moduli along and transverse to the fiber, respectively. G_{12} , G_{23} and G_{13} are the shear moduli in the 1-2, 2-3 and 1-3 plane, respectively. Finally, ν_{12} and ν_{21} are the Poisson's ratios.

It should be noted that the stresses and strains defined in equations (4.7) and (4.8) are referred to a coordinate system that coincides with the principal material coordinates for an orthotropic material. However, in general, the principal directions of orthotropy do not coincide with the selected coordinate system for the problem formulation. Thus, it is necessary to establish transformation relations among stresses and strains in one coordinate system to the corresponding ones in another coordinate system. These relations can then be used to transform constitutive equations from the principal material coordinate system to the system used in the problem description. After proper stress and strain transformations, the stress-strain relations for a lamina, defined in the selected coordinate system for the problem formulations turn out to be:

$$\begin{Bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad (4.15)$$

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad (4.16)$$

The τ_{xx} and τ_{yy} are the normal stresses in the x - and y -directions, respectively.

The τ_{xy} , τ_{yz} and τ_{xz} are the shear stresses in the xy -, yz - and xz - planes,

respectively. The $\bar{Q}_{ij}$ (for $i, j = 1..6$) in equations (4.15) and (4.16) are called the transformed reduced stiffnesses and can be computed by:

$$\bar{Q}_{11} = Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta \quad (4.17)$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12} (\sin^4 \theta + \cos^4 \theta) \quad (4.18)$$

$$\bar{Q}_{22} = Q_{11} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^4 \theta \quad (4.19)$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \theta \cos \theta \quad (4.20)$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \theta \cos \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin \theta \cos^3 \theta \quad (4.21)$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66} (\sin^4 \theta + \cos^4 \theta) \quad (4.22)$$

$$\bar{Q}_{44} = Q_{44} \cos^2 \theta + Q_{55} \sin^2 \theta \quad (4.23)$$

$$\bar{Q}_{45} = (Q_{55} - Q_{44}) \cos \theta \sin \theta \quad (4.24)$$

$$\bar{Q}_{55} = Q_{55} \cos^2 \theta + Q_{44} \sin^2 \theta \quad (4.25)$$

Then, the equations (4.17) to (4.25) can be utilized to determine the laminate coefficients in the stiffness matrices.

4.3 Laminate theories

Different laminated plate theories are established by taking the displacement field to be a linear combination of unknown functions and the thickness coordinate (z -axis):

$$u_i(x, y, z) = \sum_{j=0}^N z^j *^i \xi_j(x, y), \text{ for } i = 1, 2, 3. \quad (4.26)$$

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The u_i are the three displacement fields in the x -, y - and z -directions. The value of N will determine the degree of polynomial to be used for the displacement field. The ${}^i\xi_j$ are the unknown functions in the polynomial for u_i .

In this section, the classical laminate theory (CLT), first-order shear deformation theory (FSDT) and third-order shear deformation theory (TSDT) on laminated plate will be discussed. Then, suitable assumptions are made to the CLT and FSDT on laminated plate to develop governing equations and effective flexural stiffnesses for both wide and narrow laminated beams. The TSDT on laminated plate will be utilized to derive the shear correction factor for the FSDT wide and narrow laminated beams.

4.3.1 Classical laminate theory

Reddy [40] indicated that in the CLT on laminated plate, the displacement field used is:

$$u(x, y, z) = u_o^C(x, y) - z \frac{\partial w_o^C}{\partial x} \quad (4. 27)$$

$$v(x, y, z) = v_o^C(x, y) - z \frac{\partial w_o^C}{\partial y} \quad (4. 28)$$

$$w(x, y, z) = w_o^C(x, y) \quad (4. 29)$$

The u_o^C , v_o^C and w_o^C are the displacements of the mid-plane in the x -, y - and z -directions, respectively. The superscript ‘C’ will be used to denote quantities in CLT. Therefore, substituting the above three expressions into equations (4.1) into (4.6) give the strain-displacement relations:

$$\varepsilon_{xx}^C = \frac{\partial}{\partial x} \left(u_o^C - z \frac{\partial w_o^C}{\partial x} \right) = \frac{\partial u_o^C}{\partial x} - z \frac{\partial^2 w_o^C}{\partial x^2} \quad (4. 30)$$

$$\varepsilon_{yy}^C = \frac{\partial}{\partial y} \left(v_o^C - z \frac{\partial w_o^C}{\partial y} \right) = \frac{\partial v_o^C}{\partial y} - z \frac{\partial^2 w_o^C}{\partial y^2} \quad (4. 31)$$

$$\varepsilon_{zz}^C = 0 \quad (4. 32)$$

$$\gamma_{xy}^C = 2\varepsilon_{xy}^C = \frac{\partial}{\partial y} \left(u_o^C - z \frac{\partial w_o^C}{\partial x} \right) + \frac{\partial}{\partial x} \left(v_o^C - z \frac{\partial w_o^C}{\partial y} \right) = \left(\frac{\partial u_o^C}{\partial y} + \frac{\partial v_o^C}{\partial x} \right) - 2z \frac{\partial^2 w_o^C}{\partial x \partial y} \quad (4. 33)$$

$$\gamma_{xz}^C = 2\varepsilon_{xz}^C = \frac{\partial}{\partial z} \left(u_o^C - z \frac{\partial w_o^C}{\partial x} \right) + \frac{\partial w_o^C}{\partial x} = -\frac{\partial w_o^C}{\partial x} + \frac{\partial w_o^C}{\partial x} = 0 \quad (4.34)$$

$$\gamma_{yz}^C = 2\varepsilon_{yz}^C = \frac{\partial}{\partial z} \left(v_o^C - z \frac{\partial w_o^C}{\partial y} \right) + \frac{\partial w_o^C}{\partial y} = -\frac{\partial w_o^C}{\partial y} + \frac{\partial w_o^C}{\partial y} = 0 \quad (4.35)$$

Hence, in the CLT, the transverse normals remain straight and perpendicular to the mid-plane after the deformation. Therefore, the CLT is more applicable for thin plate.

The plate governing equation developed in Appendix B using variational formulation for this theory turns out to be:

$$\frac{\partial^2 M_{xx}^C}{\partial x^2} + \frac{\partial^2 M_{yy}^C}{\partial y^2} + 2 \frac{\partial^2 M_{xy}^C}{\partial x \partial y} + N_{xx}^C \frac{\partial^2 w_o^C}{\partial x^2} + N_{yy}^C \frac{\partial^2 w_o^C}{\partial y^2} + 2N_{xy}^C \frac{\partial^2 w_o^C}{\partial x \partial y} = 0 \quad (4.36)$$

Figure 4.2: Force resultants on a flat laminate

Figure 4.3: Moment resultants on a flat laminate

The various force resultants per unit width and moment resultants per unit width present in equation (4.36) are shown in Figure 4.2 and 4.3, respectively. These force and moment resultants can be obtained by substituting equations (4.15) and (4.16) into (b.2) and (b.3) of Appendix B, considering equations (4.30) to (4.35). This involves integration of stresses through the laminate thickness that requires lamina-wise integration. Decolon [16], Jones [35] and Reddy [63] reveal that the force and moment resultants can be worked out to be:

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$$\begin{Bmatrix} N_{xx}^C \\ N_{yy}^C \\ N_{xy}^C \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^o \\ \varepsilon_{yy}^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \quad (4.37)$$

$$\begin{Bmatrix} M_{xx}^C \\ M_{yy}^C \\ M_{xy}^C \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^o \\ \varepsilon_{yy}^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \quad (4.38)$$

The membrane strains are defined as $\begin{Bmatrix} \varepsilon_{xx}^o \\ \varepsilon_{yy}^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_o^C}{\partial x} \\ \frac{\partial v_o^C}{\partial y} \\ \frac{\partial u_o^C}{\partial y} + \frac{\partial v_o^C}{\partial x} \end{Bmatrix}$ (4.39)

and curvatures as $\begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = - \begin{Bmatrix} \frac{\partial^2 w_o^C}{\partial x^2} \\ \frac{\partial^2 w_o^C}{\partial y^2} \\ 2 \frac{\partial^2 w_o^C}{\partial x \partial y} \end{Bmatrix}$. (4.40)

The A_{ij} are extensional stiffnesses, B_{ij} are the bending extension coupling stiffnesses and D_{ij} are the bending stiffnesses. They evaluated as:

$$A_{ij} = \sum_{k=1}^N (\overline{Q_{ij}})_k (z_k - z_{k-1}) \quad (4.41)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^N (\overline{Q_{ij}})_k (z_k^2 - z_{k-1}^2) \quad (4.42)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^N (\overline{Q_{ij}})_k (z_k^3 - z_{k-1}^3) \quad (4.43)$$

Figure 4.4: Geometry of N-layered laminate

Figure 4.4 illustrates how the z_{k-1} and z_k are defined. The $(\overline{Q}_{ij})_k$ in equations (4.41) to (4.43) are the transformed reduced stiffnesses of the k_{th} layer, which can be obtained from equations (4.17) to (4.25).

4.3.1.1 Beam effective flexural rigidity

The beam buckling governing equation utilizing classical theory can be found in many textbooks such as those written by Iyengar [34], Popov [61] and Vinson and Sierakowski [76] to be:

$$\frac{\partial^2 M_{xx}^c}{\partial x^2} + N_{xx}^c \frac{\partial^2 w_o^c}{\partial x^2} = 0 \quad (4.44)$$

This classical beam theory is more applicable for thin beams as it does not account for the transverse shear deformation. For an isotropic and homogenous beam, the flexural rigidity is merely EI , however, for a beam made of fiber-reinforced composite material, the computation for the flexural rigidity is not so straightforward due to the various mechanical coupling terms. The conventional formula to calculate the effective bending stiffness of a laminated beam is highlighted by Altenbach et al. [2] and Vinson and Sierakowski [76] as:

$$^1(\overline{EI}) = \frac{A_{11}D_{11} - B_{11}^2}{A_{11}} \quad (4.45)$$

The derivation and accompanying assumptions of expression (4.45) can be found in Appendix H. The pre-superscript 1 on the left side of the equation is to denote that it is method 1. Therefore, from Appendix H, equation (4.45) will appear in equation (4.44) as:

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$$^1\left(\overline{EI}\right)\frac{d^4w_o^C}{dx^4} + P\frac{d^2w_o^C}{dx^2} = 0 \quad (4.46)$$

In equation (4.46), P is the axial compressive force, $P = -N_{xx}^C$. The right side of equation (4.45) only consists of three coupling terms, which will not be enough to describe the flexural behavior of a general laminated beam. In this section, two other expressions that can be used to compute the flexural stiffness of a laminated beam will be introduced.

Method 2

In this approach, the derived effective bending stiffness consists of more coupling terms. Therefore, the effective bending stiffness obtained using this method is believed to be able to reflect the actual behavior of the laminated beam more accurately than method 1 discussed earlier. The assumptions used in this approach consist of:

- 1) The amount of deflection of a point on the composite beam is only dependent on the position of this point on the composite laminate in the x – direction.
- 2) In equation (4.37), only N_{xx}^C exists while N_{yy}^C and N_{xy}^C are taken to be zero.
- 3) M_y^C and M_{xy}^C in equation (4.38) are negligible and small and therefore neglected.

With the above assumptions, from equations (4.37) and (4.38), four expressions can be written:

$$N_{xx}^C = A_{11}\varepsilon_{xx}^o + A_{12}\varepsilon_{yy}^o + A_{16}\gamma_{xy}^o + B_{11}\kappa_{xx} \quad (4.47)$$

$$0 = A_{12}\varepsilon_{xx}^o + A_{22}\varepsilon_{yy}^o + A_{26}\gamma_{xy}^o + B_{12}\kappa_{xx} \quad (4.48)$$

$$0 = A_{16}\varepsilon_{xx}^o + A_{26}\varepsilon_{yy}^o + A_{66}\gamma_{xy}^o + B_{16}\kappa_{xx} \quad (4.49)$$

$$M_{xx}^C = B_{11}\varepsilon_{xx}^o + B_{12}\varepsilon_{yy}^o + B_{16}\gamma_{xy}^o + D_{11}\kappa_{xx} \quad (4.50)$$

Equations (4.48) and (4.49) are utilized to get ε_{yy}^o and γ_{xy}^o as functions of other terms and the results are:

$$\varepsilon_{yy}^o = \frac{(A_{16}A_{26} - A_{12}A_{66})}{(A_{22}A_{66} - A_{26}^2)}\varepsilon_{xx}^o + \frac{(A_{26}B_{16} - A_{66}B_{12})}{(A_{22}A_{66} - A_{26}^2)}\kappa_{xx} \quad (4.51)$$

$$\gamma_{xy}^o = \frac{(A_{12}A_{26} - A_{16}A_{22})}{(A_{22}A_{66} - A_{26}^2)}\varepsilon_{xx}^o + \frac{(A_{26}B_{12} - A_{22}B_{16})}{(A_{22}A_{66} - A_{26}^2)}\kappa_{xx} \quad (4.52)$$

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Then, equations (4.51) and (4.52) are substituted into equation (4.47). After that, equation (4.47) will consist of only ε_{xx}^o , κ_{xx} , N_{xx}^C and the different coupling stiffness. Later, from equation (4.47), ε_{xx}^o can be expressed as a function of κ_{xx} , N_{xx}^C and the various coupling terms and the expression turns out to be:

$$\varepsilon_{xx}^o = \frac{(A_{12}A_{26}B_{16} - A_{12}A_{66}B_{12} + A_{16}A_{26}B_{12} - A_{16}A_{22}B_{16} + A_{22}A_{66}B_{11} - A_{26}^2B_{11})}{(A_{11}A_{22}A_{66} - A_{11}A_{22}A_{26}^2 + A_{16}^2A_{22} + 2A_{12}A_{16}A_{26} - A_{12}^2A_{66})} \kappa_{xx} + \frac{(A_{22}A_{66} - A_{26}^2)}{(A_{11}A_{22}A_{66} - A_{11}A_{22}A_{26}^2 + A_{16}^2A_{22} + 2A_{12}A_{16}A_{26} - A_{12}^2A_{66})} N_{xx}^C \quad (4.53)$$

After obtaining equation (4.53), we can insert equations (4.51), (4.52) and (4.53) into equation (4.50) to obtain an expression for M_{xx}^C , which will be just in terms of κ_{xx} , N_{xx}^C

and the different coupling stiffnesses. Noting that $\kappa_{xx} = -\frac{d^2w_o^C}{dx^2}$ and N_{xx}^C is a constant along x -axis at the onset of buckling, when this expression is substituted into equation (4.44), the resulting governing differential equation becomes:

$${}^2(\overline{EI}) \frac{d^4w_o^C}{dx^4} + P \frac{d^2w_o^C}{dx^2} = 0 \quad (4.54)$$

The expression for ${}^2(\overline{EI})$ turns out to be:

$${}^2(\overline{EI}) = D_{11} - \beta B_{12} - \delta B_{16} - \frac{(B_{11} + \alpha B_{12} + \gamma B_{16})(B_{11} - \beta A_{12} - \delta A_{16})}{(A_{11} + \alpha A_{12} + \gamma A_{16})} \quad (4.55)$$

The coefficients α , β , γ and δ are given by:

$$\alpha = \frac{A_{16}A_{26} - A_{12}A_{66}}{A_{22}A_{66} - A_{26}^2} \quad (4.56)$$

$$\beta = \frac{A_{66}B_{12} - A_{26}B_{16}}{A_{22}A_{66} - A_{26}^2} \quad (4.57)$$

$$\gamma = \frac{A_{12}A_{26} - A_{22}A_{16}}{A_{22}A_{66} - A_{26}^2} \quad (4.58)$$

$$\delta = \frac{A_{22}B_{16} - A_{26}B_{12}}{A_{22}A_{66} - A_{26}^2} \quad (4.59)$$

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This effective flexural rigidity formula will be shown in later sections that it can be used with the classical beam buckling equation in equation (4.54) to give an upper bound of buckling load of a laminated beam.

A possible explanation for equation (4.55) to be able to form the upper bound lies in the assumptions made in its derivation. In the derivation, it is assumed that κ_y and κ_{xy} are equal to zero. This means that there will be no curvature in the y – and xy – directions, which is probably more appropriate for wide beams as mentioned by Kaw [40], where the beam is more affected by the end conditions. Nevertheless, in reality, no matter how much the end effect may be, there will always be κ_y and κ_{xy} . Hence, with such assumption, where the beam is not allowed to bend in y – direction and twist in xy – direction, the beam will turn out to be stiffer than one that has zero curvatures only at its ends, resulting in a higher effective flexural modulus value, thus giving the upper bound.

Method 3

In this section, a closed-form expression that can be used to determine the effective flexural modulus for a laminated composite beam is derived. From equations (4.37) and (4.38), the classical laminate theory in full form is:

$$N_{xx}^C = A_{11}\epsilon_{xx}^o + A_{12}\epsilon_{yy}^o + A_{16}\gamma_{xy}^o + B_{11}\kappa_{xx} + B_{12}\kappa_{yy} + B_{16}\kappa_{xy} \quad (4.60)$$

$$N_{yy}^C = A_{12}\epsilon_{xx}^o + A_{22}\epsilon_{yy}^o + A_{26}\gamma_{xy}^o + B_{12}\kappa_{xx} + B_{22}\kappa_{yy} + B_{26}\kappa_{xy} \quad (4.61)$$

$$N_{xy}^C = A_{16}\epsilon_{xx}^o + A_{26}\epsilon_{yy}^o + A_{66}\gamma_{xy}^o + B_{16}\kappa_{xx} + B_{26}\kappa_{yy} + B_{66}\kappa_{xy} \quad (4.62)$$

$$M_{xx}^C = B_{11}\epsilon_{xx}^o + B_{12}\epsilon_{yy}^o + B_{16}\gamma_{xy}^o + D_{11}\kappa_{xx} + D_{12}\kappa_{yy} + D_{16}\kappa_{xy} \quad (4.63)$$

$$M_{yy}^C = B_{12}\epsilon_{xx}^o + B_{22}\epsilon_{yy}^o + B_{26}\gamma_{xy}^o + D_{12}\kappa_{xx} + D_{22}\kappa_{yy} + D_{26}\kappa_{xy} \quad (4.64)$$

$$M_{xy}^C = B_{16}\epsilon_{xx}^o + B_{26}\epsilon_{yy}^o + B_{66}\gamma_{xy}^o + D_{16}\kappa_{xx} + D_{26}\kappa_{yy} + D_{66}\kappa_{xy} \quad (4.65)$$

Since the analysis is confined to the bending behavior of columns or beams subjected to loads applied only in only one plane, equations (4.61), (4.62), (4.64) and (4.65) are set equal to zero and solving these four simultaneous equations for the out-of-plane strain

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and curvature components in terms of required in-plane strain and curvature component yields:

$$\varepsilon_{yy}^o = C_{11}\varepsilon_{xx}^o + C_{12}\kappa_{xx} \quad (4.66)$$

$$\gamma_{xy}^o = C_{21}\varepsilon_{xx}^o + C_{22}\kappa_{xx} \quad (4.67)$$

$$\kappa_{yy} = C_{31}\varepsilon_{xx}^o + C_{32}\kappa_{xx} \quad (4.68)$$

$$\kappa_{xy} = C_{41}\varepsilon_{xx}^o + C_{42}\kappa_{xx} \quad (4.69)$$

The C_{ij} (for $i=1..4$ and $j=1,2$) are introduced for compactness and they are compiled in Appendix C. Substituting the above equations into equation (4.60) gives a form of solution for strain ε_{xx}^o as:

$$\varepsilon_{xx}^o = C_{51}\kappa_{xx} + C_{52}N_{xx}^C \quad (4.70)$$

The C_{51} and C_{52} are in terms of the C_{ij} defined in equations (4.66) to (4.69).

Finally, substituting equation (4.70) into equation (4.63) yields the conventional form of the moment equation:

$$M_{xx}^C = {}^3(\overline{EI})\kappa_{xx} + C_{61}N_{xx}^C = {}^3(\overline{EI})\frac{d^2w_o^C}{dx^2} + C_{61}N_{xx}^C \quad (4.71)$$

As at the point of buckling, N_{xx}^C is constant along the axial direction, therefore, when equation (4.71) is substituted into equation (4.44), the C_{61} term vanishes yielding:

$${}^3(\overline{EI})\frac{d^4w_o^C}{dx^4} + P\frac{d^2w_o^C}{dx^2} = 0 \quad (4.72)$$

Hence, the effective bending modulus turns out to be:

$${}^3(\overline{EI}) = D_{11} + (C_{31}C_{51} + C_{32})D_{12} + (C_{41}C_{51} + C_{42})D_{16} + B_{11}C_{51} \\ + (C_{11}C_{51} + C_{12})B_{12} + (C_{21}C_{51} + C_{22})B_{16} \quad (4.73)$$

The coefficient C_{51} is found to be:

$$C_{51} = -\frac{C_{12}A_{12} + C_{22}A_{16} + B_{11} + C_{32}B_{12} + C_{42}B_{16}}{A_{11} + C_{11}A_{12} + C_{21}A_{16} + C_{31}B_{12} + C_{41}B_{16}} \quad (4.74)$$

This innovative closed-form formula in equation (4.73) will be demonstrated in the later section that it can be used to compute the lower limit for the critical load of a laminated beam.

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During derivation, M_{yy}^C in equation (4.64) and M_{xy}^C in expression (4.65) are set to zero, while κ_{yy} and κ_{xy} are not put to zero. These conditions are more appropriate for narrow beams that are less affected by the end conditions, as addressed by Kaw [40], which have smaller values of resultant M_{yy}^C and M_{xy}^C to prevent bending and twisting in the y – and xy – directions. However, no matter how narrow a beam can be, the end conditions will still be able to provide certain amount of M_{yy}^C and M_{xy}^C to lower the curvatures. Hence, putting M_{yy}^C and M_{xy}^C to zero will yield greater κ_{yy} and κ_{xy} than actual value, resulting in lower effective flexural modulus than actual value, thus forming the lower limit.

4.3.2 First-order shear deformation theory

In the FSDT, the displacement field is taken as:

$$u(x, y, z) = u_o^F(x, y) + z\phi_x^F(x, y) \quad (4.75)$$

$$v(x, y, z) = v_o^F(x, y) + z\phi_y^F(x, y) \quad (4.76)$$

$$w(x, y, z) = w_o^F(x, y) \quad (4.77)$$

The superscript ‘ F ’ is used to denote variables in FSDT. The u_o^F , v_o^F and w_o^F are the displacements of the mid-plane in the x –, y – and z – directions, respectively. The ϕ_x^F and ϕ_y^F are the rotations of a transverse normal about the y – and x – axes, respectively.

Substituting equations (4.75) to (4.77) into equations (4.1) to (4.6), the strain-displacement relations turn out to be:

$$\varepsilon_{xx}^F = \frac{\partial}{\partial x}(u_o^F + z\phi_x^F) = \frac{\partial u_o^F}{\partial x} + z \frac{\partial \phi_x^F}{\partial x} \quad (4.78)$$

$$\varepsilon_{yy}^F = \frac{\partial}{\partial y}(v_o^F + z\phi_y^F) = \frac{\partial v_o^F}{\partial y} + z \frac{\partial \phi_y^F}{\partial y} \quad (4.79)$$

$$\varepsilon_{zz}^F = 0 \quad (4.80)$$

$$\gamma_{xy}^F = 2\varepsilon_{xy}^F = \frac{\partial}{\partial y}(u_o^F + z\phi_x^F) + \frac{\partial}{\partial x}(v_o^F + z\phi_y^F) = \left(\frac{\partial u_o^F}{\partial y} + \frac{\partial v_o^F}{\partial x} \right) + z \left(\frac{\partial \phi_x^F}{\partial y} + \frac{\partial \phi_y^F}{\partial x} \right) \quad (4.81)$$

$$\gamma_{xz}^F = 2\varepsilon_{xz}^F = \frac{\partial}{\partial z}(u_o^F + z\phi_x^F) + \frac{\partial w_o^F}{\partial x} = \frac{\partial w_o^F}{\partial x} + \phi_x^F \quad (4.82)$$

$$\gamma_{yz}^F = 2\varepsilon_{yz}^F = \frac{\partial}{\partial z}(v_o^F + z\phi_y^F) + \frac{\partial w_o^F}{\partial y} = \frac{\partial w_o^F}{\partial y} + \phi_y^F \quad (4.83)$$

The FSDT plate governing equations can be found in Appendix D. Figure 4.2 and 4.3 illustrates the various force resultants per unit width ($N_{xx}^F, N_{yy}^F, N_{xy}^F, Q_x^F$ and Q_y^F) and moment resultants per unit width (M_{xx}^F, M_{yy}^F and M_{xy}^F) existing in the governing equations, respectively. Noting equations (4.78) to (4.81), substitution of equation (4.15) into equations (b.2) and (b.3) of Appendix B results in the three in-plane forces resultants and the three moment resultants having the same form as equations (4.37) and (4.38) (except the superscript 'C' on the variables should be changed to 'F'). However, for FSDT, while the definition of the membrane strains are kept as equation (4.39) (except the superscript 'C' should be changed to 'F'), the curvatures are now written as:

$$\begin{Bmatrix} K_{xx} \\ K_{yy} \\ K_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x^F}{\partial x} \\ \frac{\partial \phi_y^F}{\partial y} \\ \frac{\partial \phi_x^F}{\partial y} + \frac{\partial \phi_y^F}{\partial x} \end{Bmatrix} \quad (4.84)$$

In addition, unlike the classical theory, the transverse shear strains γ_{xz}^F and γ_{yz}^F in FSDT are not zero but constant through the thickness. As FSDT accounts for the transverse shear deformation, it can be used to model moderately thick plate. Utilizing equation (4.16) together with equations (4.82) and (4.83) into equation (d.2) in Appendix D yields:

$$\begin{Bmatrix} Q_y^F \\ Q_x^F \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^F \\ \gamma_{xz}^F \end{Bmatrix} \quad (4.85)$$

The Q_x^F and Q_y^F are known as the transverse force resultants and K is called the shear correction factor. The terms in the square matrix can be computed using equation (4.41) with i and j equal to 4 or 5.

4.3.2.1 Beam effective bending stiffness

The FSDT beam buckling governing equations can also be gotten from textbooks like those written by Altenbach et al. [2] and Vinson and Sierakowski [76] as:

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$$\frac{dN_{xx}^F}{dx} = 0 \quad (4.86)$$

$$\frac{dM_{xx}^F}{dx} - Q_x^F = 0 \quad (4.87)$$

$$\frac{dQ_x^F}{dx} - P \frac{d^2 w_o^F}{dx^2} = 0 \quad (4.88)$$

As the FSDT caters for transverse shear strain, it can be used to study moderately thick laminated beam. From Altenbach et al. [2] and Vinson and Sierakowski [76], the conventional FSDT buckling equations for laminated beam in terms of the displacement are:

$${}^1(\overline{EI}) \frac{d^2 \phi_x^F}{dx^2} - {}^1(\overline{Gh}) ({}^1K) \left(\phi_x^F + \frac{dw_o^F}{dx} \right) = 0 \quad (4.89)$$

$${}^1(\overline{Gh}) ({}^1K) \left(\frac{d\phi_x^F}{dx} + \frac{d^2 w_o^F}{dx^2} \right) - P \frac{d^2 w_o^F}{dx^2} = 0 \quad (4.90)$$

The traditional formula to evaluate ${}^1(\overline{EI})$ in equation (4.89) turns out to be equation (4.45). Again, the pre-superscripts in 1K and ${}^1(\overline{Gh})$ denote that these values are used in method 1. The 1K expression will be derived in the later section. Conventionally, the ${}^1(\overline{Gh})$ is calculated as:

$${}^1(\overline{Gh}) = A_{55} \quad (4.91)$$

Method 2

Analogous to method 2 in section 4.3.1.1, it is assumed that there is no bending in the y -direction or twisting. The transverse normal is inextensible. Therefore, w_o^F and ϕ_x^F are only functions of x and ϕ_y^F is taken as zero. Furthermore, for a beam undergoing axial force only in the x -direction, N_y^F and N_{xy}^F are set as zero. Similar to the method 2 in section 4.3.1.1, with these assumptions, one can develop the effective flexural rigidity and effective shear stiffness that can be used with the FSDT beam to compute the upper bound for the buckling load of a laminated composite beam. Again, with these assumptions, one can arrive at equations (4.47) to (4.50) as the relations between the in-

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plane force resultants and moment resultants with the membrane strains and curvatures are the same for both the CLT and FSDT (except the superscript ‘ C ’ on the variables should be changed to ‘ F ’). Following the same procedure as method 2 in section 4.3.1.1, one can obtain an expression for M_{xx}^F which is just in terms of κ_{xx} , N_{xx}^F and the different coupling stiffnesses. Substituting this expression into equation (4.87), noting that

$$\kappa_{xx} = \frac{d\phi_x^F}{dx} \text{ instead yields:}$$

$${}^2(\overline{EI}) \frac{d^2 \phi_x^F}{dx^2} - Q_x^F = 0 \quad (4.92)$$

The ${}^2(\overline{EI})$ in the above equation can be calculated using equation (4.55).

Furthermore, with the above assumptions, one can let the transverse shear strain

$$\gamma_{yz}^F = \phi_y^F + \frac{dw_o^F}{dy} \text{ to be 0. Hence, from equation (4.85), one can have:}$$

$$Q_x^F = {}^2KA_{55}\gamma_{xz}^F = {}^2KA_{55}\left(\phi_x^F + \frac{dw_o^F}{dx}\right) \quad (4.93)$$

By substituting the above equation in equations (4.92) and (4.88), the FSDT laminated beam governing equations will become:

$${}^2(\overline{EI}) \frac{d^2 \phi_x^F}{dx^2} - {}^2(\overline{Gh})({}^2K)\left(\phi_x^F + \frac{dw_o^F}{dx}\right) = 0 \quad (4.94)$$

$${}^2(\overline{Gh})({}^2K)\left(\frac{d\phi_x^F}{dx} + \frac{d^2 w_o^F}{dx^2}\right) - P \frac{d^2 w_o^F}{dx^2} = 0 \quad (4.95)$$

The shear correction factor, 2K to be used in the above two equations will be developed in the later section. The effective shear modulus for this method is found to be:

$${}^2(\overline{Gh}) = A_{55} \quad (4.96)$$

It will be demonstrated in later section that the ${}^2(\overline{EI})$ and ${}^2(\overline{Gh})$ presented here can be utilized to give an upper limit for the buckling load of a moderately thick laminated beam.

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Method 3

Parallel to method 3 in section 4.3.1.1, suitable formula will be derived from FSDT on fiber-reinforced composite, which can be utilized to obtain a lower bound for the buckling load of a moderately thick laminated beam. Once again, the CLT and FSDT will have the same expanded form of equations for the in-plane force resultants and moment resultants, as in equations (4.60) to (4.65) (except the superscript ‘ C ’ on the variables should be changed to ‘ F ’). Following the steps in method 3 of section 4.3.1.1, the relation below can be found as in equation (4.71):

$$M_{xx}^F = {}^3(\overline{EI})\kappa_{xx} + C_{61}N_{xx}^F \quad (4.97)$$

The formula for ${}^3(\overline{EI})$ in the above equation is the same as equation (4.73). However, for FSDT, $\kappa_{xx} = \frac{d\phi_x^F}{dx}$. The C_{61} term disappears when equation (4.97) is substituted into equation (4.87) as N_{xx}^F is constant at the point of buckling leaving:

$${}^3(\overline{EI})\frac{d^2\phi_x^F}{dx^2} - Q_x^F = 0 \quad (4.98)$$

The expanded form of equation (4.85) is:

$$Q_y^F = {}^3KA_{44}\gamma_{yz}^F + {}^3KA_{45}\gamma_{xz}^F \quad (4.99)$$

$$Q_x^F = {}^3KA_{45}\gamma_{yz}^F + {}^3KA_{55}\gamma_{xz}^F \quad (4.100)$$

Likewise, the shear correction factor 3K for this method 3 will be established in the later section. As the study is restricted to the bending behavior of columns or beams subjected to loads applied only in only one plane, equation (4.99) is set as zero. Then, solving this equation for γ_{yz}^F results in:

$$\gamma_{yz}^F = -\frac{A_{45}}{A_{44}}\gamma_{xz}^F \quad (4.101)$$

Inserting the above equation into equation (4.100), noting equations (4.82) and (4.83) gives:

$$Q_x^F = {}^3KA_{45}\left(-\frac{A_{45}}{A_{44}}\gamma_{xz}^F\right) + {}^3KA_{55}\gamma_{xz}^F = {}^3K\left(A_{55} - \frac{A_{45}^2}{A_{44}}\right)\left(\phi_x^F + \frac{dw_o^F}{dx}\right) \quad (4.102)$$

Then equation (4.102) is substituted into equations (4.98) and (4.88) to yield:

$${}^3(\overline{EI}) \frac{d^2 \phi_x^F}{dx^2} - {}^3(\overline{Gh}) ({}^3K) \left(\phi_x^F + \frac{dw_o^F}{dx} \right) = 0 \quad (4.103)$$

$${}^3(\overline{Gh}) ({}^3K) \left(\frac{d\phi_x^F}{dx} + \frac{d^2 w_o^F}{dx^2} \right) - P \frac{d^2 w_o^F}{dx^2} = 0 \quad (4.104)$$

The effective shear modulus in this case is now defined as:

$${}^3(\overline{Gh}) = A_{55} - \frac{A_{45}^2}{A_{44}} \quad (4.105)$$

Similarly, in the later section, it will be illustrated that the ${}^3(\overline{EI})$ and ${}^3(\overline{Gh})$ can be used with the FSDT beam buckling equations to compute the lower bound for the critical load of a moderately thick laminated beam.

4.3.3 Third-order shear deformation theory

In the TSDT, the displacement field is given as:

$$u(x, y, z) = u_o^T(x, y) + z\phi_x^T(x, y) + z^2\theta_x^T(x, y) + z^3\lambda_x^T(x, y) \quad (4.106)$$

$$v(x, y, z) = v_o^T(x, y) + z\phi_y^T(x, y) + z^2\theta_y^T(x, y) + z^3\lambda_y^T(x, y) \quad (4.107)$$

$$w(x, y, z) = w_o^T(x, y) \quad (4.108)$$

The superscript ‘ T ’ will be used to indicate terms that are employed in TSDT. The definitions for u_o^T , v_o^T , w_o^T , ϕ_x^T and ϕ_y^T are similar to those in FSDT (except the superscript ‘ F ’ should be changed to ‘ T ’). The θ_x^T , θ_y^T , λ_x^T and λ_y^T are further unknown functions to be determined in this theory. However, the number of unknowns can be decreased by imposing zero shear stresses on the top and bottom of the laminate:

$$\sigma_{xz} \left(x, y, z = \pm \frac{h}{2} \right) = \sigma_{yz} \left(x, y, z = \pm \frac{h}{2} \right) = 0 \quad (4.109)$$

The h is total height of the laminate.

Utilizing the stress-strain relations in equation (4.16) give the following two equations:

$$\overline{Q}_{55} \gamma_{xz}^T \left(x, y, z = \pm \frac{h}{2} \right) + \overline{Q}_{45} \gamma_{yz}^T \left(x, y, z = \pm \frac{h}{2} \right) = 0 \quad (4.110)$$

$$\overline{Q}_{45} \gamma_{xz}^T \left(x, y, z = \pm \frac{h}{2} \right) + \overline{Q}_{44} \gamma_{yz}^T \left(x, y, z = \pm \frac{h}{2} \right) = 0 \quad (4.111)$$

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The equation (4.110) can be rearranged to have:

$$\gamma_{xz}^T \left(x, y, z = \pm \frac{h}{2} \right) = -\frac{\bar{Q}_{45}}{\bar{Q}_{55}} \times \gamma_{yz}^T \left(x, y, z = \pm \frac{h}{2} \right) \quad (4.112)$$

Substituting equation (4.112) into (4.111) and rearranging yields:

$$\left(\bar{Q}_{44} - \frac{\bar{Q}_{45}^2}{\bar{Q}_{55}} \right) \times \gamma_{yz}^T \left(x, y, z = \pm \frac{h}{2} \right) = 0 \quad (4.113)$$

As the term inside the first bracket cannot be zero, it can be concluded that:

$$\gamma_{yz}^T \left(x, y, z = \pm \frac{h}{2} \right) = 0 \quad (4.114)$$

Using this result into equation (4.112) results in:

$$\gamma_{xz}^T \left(x, y, z = \pm \frac{h}{2} \right) = 0 \quad (4.115)$$

Using the strain-displacement relations defined in equations (4.5) and (4.6) and the assumed displacement field from equations (4.106) to (4.108) into equations (4.114) and (4.115) gives the following expressions:

$$\gamma_{xz}^T \left(x, y, z = \pm \frac{h}{2} \right) = \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right]_{z=\pm \frac{h}{2}} = \left[\phi_x^T + 2z\theta_x^T + 3z^2\lambda_x^T + \frac{\partial w_o^T}{\partial x} \right]_{z=\pm \frac{h}{2}} = 0 \quad (4.116)$$

$$\gamma_{yz}^T \left(x, y, z = \pm \frac{h}{2} \right) = \left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right]_{z=\pm \frac{h}{2}} = \left[\phi_y^T + 2z\theta_y^T + 3z^2\lambda_y^T + \frac{\partial w_o^T}{\partial y} \right]_{z=\pm \frac{h}{2}} = 0 \quad (4.117)$$

Substituting the $z = \pm \frac{h}{2}$ into the two equations above produces four equations:

$$\gamma_{xz}^T \left(x, y, z = -\frac{h}{2} \right) = \phi_x^T - h\theta_x^T + \frac{3h^2}{4}\lambda_x^T + \frac{\partial w_o^T}{\partial x} = 0 \quad (4.118)$$

$$\gamma_{xz}^T \left(x, y, z = +\frac{h}{2} \right) = \phi_x^T + h\theta_x^T + \frac{3h^2}{4}\lambda_x^T + \frac{\partial w_o^T}{\partial x} = 0 \quad (4.119)$$

$$\gamma_{yz}^T \left(x, y, z = -\frac{h}{2} \right) = \phi_y^T - h\theta_y^T + \frac{3h^2}{4}\lambda_y^T + \frac{\partial w_o^T}{\partial y} = 0 \quad (4.120)$$

$$\gamma_{yz}^T \left(x, y, z = +\frac{h}{2} \right) = \phi_y^T + h\theta_y^T + \frac{3h^2}{4}\lambda_y^T + \frac{\partial w_o^T}{\partial y} = 0 \quad (4.121)$$

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Then, these equations can be solved simultaneously to find θ_x^T , θ_y^T , λ_x^T and λ_y^T as:

$$\theta_x^T = 0 \quad (4.122)$$

$$\theta_y^T = 0 \quad (4.123)$$

$$\lambda_x^T = -\frac{4}{3h^2} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \quad (4.124)$$

$$\lambda_y^T = -\frac{4}{3h^2} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) \quad (4.125)$$

Hence, equations (4.106) to (4.108) can be simplified to contain less unknown functions as:

$$u(x, y, z) = u_o^T + z\phi_x^T - \frac{4z^3}{3h^2} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \quad (4.126)$$

$$v(x, y, z) = v_o^T + z\phi_y^T - \frac{4z^3}{3h^2} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) \quad (4.127)$$

$$w(x, y, z) = w_o^T \quad (4.128)$$

From equations (4.126) to (4.128) into equations (4.1) to (4.6), the strains can be found to be:

$$\varepsilon_{xx}^T = \frac{\partial}{\partial x} \left(u_o^T + z\phi_x^T - \frac{4z^3}{3h^2} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right) = \frac{\partial u_o^T}{\partial x} + z \frac{\partial \phi_x^T}{\partial x} - c_1 z^3 \left(\frac{\partial \phi_x^T}{\partial x} + \frac{\partial^2 w_o^T}{\partial x^2} \right) \quad (4.129)$$

$$\varepsilon_{yy}^T = \frac{\partial}{\partial y} \left(v_o^T + z\phi_y^T - \frac{4z^3}{3h^2} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) \right) = \frac{\partial v_o^T}{\partial y} + z \frac{\partial \phi_y^T}{\partial y} - c_1 z^3 \left(\frac{\partial \phi_y^T}{\partial y} + \frac{\partial^2 w_o^T}{\partial y^2} \right) \quad (4.130)$$

$$\varepsilon_{zz}^T = 0 \quad (4.131)$$

$$\begin{aligned} \gamma_{xy}^T &= 2\varepsilon_{xy}^T = \frac{\partial}{\partial y} \left(u_o^T + z\phi_x^T - \frac{4z^3}{3h^2} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right) + \frac{\partial}{\partial x} \left(v_o^T + z\phi_y^T - \frac{4z^3}{3h^2} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) \right) \\ &= \left(\frac{\partial u_o^T}{\partial y} + \frac{\partial v_o^T}{\partial x} \right) + z \left(\frac{\partial \phi_x^T}{\partial y} + \frac{\partial \phi_y^T}{\partial x} \right) - c_1 z^3 \left(\frac{\partial \phi_x^T}{\partial y} + \frac{\partial \phi_y^T}{\partial x} + 2 \frac{\partial^2 w_o^T}{\partial x \partial y} \right) \end{aligned} \quad (4.132)$$

$$\gamma_{xz}^T = 2\varepsilon_{xz}^T = \frac{\partial}{\partial z} \left(u_o^T + z\phi_x^T - \frac{4z^3}{3h^2} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right) + \frac{\partial w_o^T}{\partial x} = \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) - c_2 z^2 \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \quad (4.133)$$

$$\gamma_{yz}^T = 2\varepsilon_{yz}^T = \frac{\partial}{\partial z} \left(v_o^T + z\phi_y^T - \frac{4z^3}{3h^2} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) \right) + \frac{\partial w_o^T}{\partial y} = \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) - c_2 z^2 \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) \quad (4.134)$$

The constants c_1 and c_2 are $\frac{4}{3h^2}$ and $\frac{4}{h^2}$, respectively.

Similar to CLT and FSDT, the various force and moment resultants for this theory can be obtained by substituting equations (4.15) and (4.16) into equations (b.2) and (b.3) from Appendix B, expression (d.2) of Appendix D and relations (e.2) and (e.3) in Appendix E, considering equations (4.129) to (4.134). Carrying out the lamina-wise integration gives:

$$\begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \\ N_{xy}^T \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_o^T}{\partial x} \\ \frac{\partial v_o^T}{\partial y} \\ \frac{\partial u_o^T}{\partial y} + \frac{\partial v_o^T}{\partial x} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x^T}{\partial x} \\ \frac{\partial \phi_y^T}{\partial y} \\ \frac{\partial \phi_x^T}{\partial y} + \frac{\partial \phi_y^T}{\partial x} \end{Bmatrix} + \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \begin{Bmatrix} -c_1 \left(\frac{\partial \phi_x^T}{\partial x} + \frac{\partial^2 w_o^T}{\partial x^2} \right) \\ -c_1 \left(\frac{\partial \phi_y^T}{\partial y} + \frac{\partial^2 w_o^T}{\partial y^2} \right) \\ -c_1 \left(\frac{\partial \phi_x^T}{\partial y} + \frac{\partial \phi_y^T}{\partial x} + 2 \frac{\partial^2 w_o^T}{\partial x \partial y} \right) \end{Bmatrix} \quad (4.135)$$

$$\begin{Bmatrix} M_{xx}^T \\ M_{yy}^T \\ M_{xy}^T \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_o^T}{\partial x} \\ \frac{\partial v_o^T}{\partial y} \\ \frac{\partial u_o^T}{\partial y} + \frac{\partial v_o^T}{\partial x} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x^T}{\partial x} \\ \frac{\partial \phi_y^T}{\partial y} \\ \frac{\partial \phi_x^T}{\partial y} + \frac{\partial \phi_y^T}{\partial x} \end{Bmatrix} + \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{Bmatrix} -c_1 \left(\frac{\partial \phi_x^T}{\partial x} + \frac{\partial^2 w_o^T}{\partial x^2} \right) \\ -c_1 \left(\frac{\partial \phi_y^T}{\partial y} + \frac{\partial^2 w_o^T}{\partial y^2} \right) \\ -c_1 \left(\frac{\partial \phi_x^T}{\partial y} + \frac{\partial \phi_y^T}{\partial x} + 2 \frac{\partial^2 w_o^T}{\partial x \partial y} \right) \end{Bmatrix} \quad (4.136)$$

$$\begin{Bmatrix} P_{xx}^T \\ P_{yy}^T \\ P_{xy}^T \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_o^T}{\partial x} \\ \frac{\partial v_o^T}{\partial y} \\ \frac{\partial u_o^T}{\partial y} + \frac{\partial v_o^T}{\partial x} \end{Bmatrix} + \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x^T}{\partial x} \\ \frac{\partial \phi_y^T}{\partial y} \\ \frac{\partial \phi_x^T}{\partial y} + \frac{\partial \phi_y^T}{\partial x} \end{Bmatrix} + \begin{bmatrix} H_{11} & H_{12} & H_{16} \\ H_{12} & H_{22} & H_{26} \\ H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{Bmatrix} -c_1 \left(\frac{\partial \phi_x^T}{\partial x} + \frac{\partial^2 w_o^T}{\partial x^2} \right) \\ -c_1 \left(\frac{\partial \phi_y^T}{\partial y} + \frac{\partial^2 w_o^T}{\partial y^2} \right) \\ -c_1 \left(\frac{\partial \phi_x^T}{\partial y} + \frac{\partial \phi_y^T}{\partial x} + 2 \frac{\partial^2 w_o^T}{\partial x \partial y} \right) \end{Bmatrix} \quad (4.137)$$

$$\begin{Bmatrix} Q_y^T \\ Q_x^T \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \phi_y^T + \frac{\partial w_o^T}{\partial y} \\ \phi_x^T + \frac{\partial w_o^T}{\partial x} \end{Bmatrix} + \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} \begin{Bmatrix} -c_2 \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) \\ -c_2 \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \end{Bmatrix} \quad (4.138)$$

$$\begin{Bmatrix} R_y^T \\ R_x^T \end{Bmatrix} = \begin{bmatrix} D_{44} & D_{45} \\ D_{45} & D_{55} \end{bmatrix} \begin{Bmatrix} \phi_y^T + \frac{\partial w_o^T}{\partial y} \\ \phi_x^T + \frac{\partial w_o^T}{\partial x} \end{Bmatrix} + \begin{bmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{bmatrix} \begin{Bmatrix} -c_2 \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) \\ -c_2 \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \end{Bmatrix} \quad (4.139)$$

The A_{ij} , B_{ij} and D_{ij} in equations (4.135) to (4.139) can be evaluated using equations (4.41) to (4.43). The additional stiffnesses from this theory, E_{ij} , F_{ij} and H_{ij} can be computed using:

$$E_{ij} = \frac{1}{4} \sum_{k=1}^N (\overline{Q_{ij}})_k (z_k^4 - z_{k-1}^4) \quad (4.140)$$

$$F_{ij} = \frac{1}{5} \sum_{k=1}^N (\overline{Q_{ij}})_k (z_k^5 - z_{k-1}^5) \quad (4.141)$$

$$H_{ij} = \frac{1}{7} \sum_{k=1}^N (\overline{Q_{ij}})_k (z_k^7 - z_{k-1}^7) \quad (4.142)$$

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4.3.3.1 Determination of shear correction factor

Both Reddy [63] and Wang et al. [77] remark that unlike the FSDT, TSDT does not require shear correction factor, K . It is because the displacement field used accommodates quadratic variation of transverse shear strains and vanishing of transverse shear stresses on the top and bottom of a general laminate. Wang et al. [77] also stated that it is possible to obtain the K needed for FSDT from the TSDT. Therefore, in this section, an attempt will be made to derive two expressions for K to be used in equations (4.93) and (4.102) for method 2 and 3 in section 4.3.2.1, respectively. This is done by utilizing the assumptions used in these methods, some simplification of the TSDT and relating FSDT with the TSDT. It is mentioned by Reddy [63] that a simplified third-order shear deformation theory (STSDT) can be deduced from the full TSDT by letting c_1 to be 0 but not c_2 . Hence, while the governing equations (e.11) and (e.12) in Appendix E remains, equations (e.13) to (e.15) are reduced to:

$$\frac{\partial M_{xx}^T}{\partial x} + \frac{\partial M_{xy}^T}{\partial y} - \bar{Q}_x^T = 0 \quad (4.143)$$

$$\frac{\partial M_{xy}^T}{\partial x} + \frac{\partial M_{yy}^T}{\partial y} - \bar{Q}_y^T = 0 \quad (4.144)$$

$$\begin{aligned} & \frac{\partial \bar{Q}_x^T}{\partial x} + \frac{\partial \bar{Q}_y^T}{\partial y} + \frac{\partial}{\partial x} \left(N_{xx}^T \frac{\partial w_o^T}{\partial x} \right) + \frac{\partial}{\partial y} \left(N_{yy}^T \frac{\partial w_o^T}{\partial y} \right) \\ & + \frac{\partial}{\partial y} \left(N_{xy}^T \frac{\partial w_o^T}{\partial x} \right) + \frac{\partial}{\partial x} \left(N_{xy}^T \frac{\partial w_o^T}{\partial y} \right) = 0 \end{aligned} \quad (4.145)$$

Hence, one can recover the governing equations of the FSDT by setting c_1 to be zero, as pointed out by both Reddy [63] and Wang et al. [77]. By comparing the above equations with equations (d.7) to (d.9) in Appendix D for FSDT, one can try to relate the effective transverse shear force $\bar{Q}_x^T$ and $\bar{Q}_y^T$ with Q_x^F and Q_y^F in the FSDT. In doing this, one can attain the K value that is required in the FSDT.

Shear correction factor for method 2, 2K

In this section, an attempt will be made to develop the K value that can be used in method 2 of section 4.3.2.1, 2K . This 2K can then be utilized in equations (4.94) and

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(4.95) to solve the buckling problem. Parallel to method 2 in section 4.3.2.1, in this STSDT, it is assumed that there is no bending in the y –direction or twisting. Furthermore, the transverse normal is inextensible. Therefore, w_o^T and ϕ_x^T are only functions of x and ϕ_y^T is taken as zero. In this way, using equations (4.138) and (4.139), the effective shear force $\bar{Q}_x^T$ can be written as:

$$\begin{aligned}\bar{Q}_x^T &= Q_x^T - c_2 R_x^T \\ &= A_{55} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) - c_2 D_{55} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) - c_2 \left(D_{55} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) - c_2 F_{55} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right) \\ &= (A_{55} - c_2 D_{55} - c_2 (D_{55} - c_2 F_{55})) \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right)\end{aligned}\quad (4.146)$$

A couple of assumptions are used to relate the FSDT beam to the STSDT beam. They are:

- 1) Both beams are able to calculate the transverse shear force of the actual beam at any arbitrary x location accurately so that $Q_x^F = \bar{Q}_x^T$.
- 2) The two beams are able to model the transverse shear strain of the actual beam at any arbitrary x position accurately such that $\gamma_{xz}^F = \gamma_{xz}^T$ in an average sense, Gu and Chattopadhyay [26].

Therefore, from assumption 2,

$$\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} dz \right]^F = \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} dz \right]^T \quad (4.147)$$

$$LHS = \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} dz \right]^F \quad (4.148)$$

$$\Rightarrow LHS = \int_{-\frac{h}{2}}^{\frac{h}{2}} \phi_x^F + \frac{\partial w_o^F}{\partial x} dz \quad (4.149)$$

$$\Rightarrow LHS = \left[\left(\phi_x^F + \frac{\partial w_o^F}{\partial x} \right) z \right]_{-\frac{h}{2}}^{\frac{h}{2}} \quad (4.150)$$

$$\Rightarrow LHS = \left[\left(\phi_x^F + \frac{\partial w_o^F}{\partial x} \right) \frac{h}{2} - \left(\phi_x^F + \frac{\partial w_o^F}{\partial x} \right) \left(\frac{-h}{2} \right) \right] \quad (4.151)$$

$$\therefore LHS = h \left(\phi_x^F + \frac{\partial w_o^F}{\partial x} \right) \quad (4.152)$$

$$RHS = \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} dz \right]^T \quad (4.153)$$

$$\Rightarrow RHS = \int_{-\frac{h}{2}}^{\frac{h}{2}} \phi_x^T - \frac{4}{h^2} z^2 \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) + \frac{\partial w_o^T}{\partial x} dz \quad (4.154)$$

$$\Rightarrow RHS = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) - \frac{4}{h^2} z^2 \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) dz \quad (4.155)$$

$$\Rightarrow RHS = \left[z \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) - \frac{4}{3h^2} z^3 \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right]_{-\frac{h}{2}}^{\frac{h}{2}} \quad (4.156)$$

$$\Rightarrow RHS = \left\{ \begin{aligned} & \left[\frac{h}{2} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) - \frac{4}{3h^2} \left(\frac{h}{2} \right)^3 \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right] \\ & - \left[-\frac{h}{2} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) - \frac{4}{3h^2} \left(\frac{-h}{2} \right)^3 \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right] \end{aligned} \right\} \quad (4.157)$$

$$\therefore RHS = \frac{2h}{3} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \quad (4.158)$$

Hence, the relation between $\left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right)$ and $\left(\phi_x^R + \frac{\partial w_o^R}{\partial x} \right)$, from assumption 2 turns out

to be:

$$\left(\phi_x^F + \frac{\partial w_o^F}{\partial x} \right) = \frac{2}{3} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \quad (4.159)$$

From assumption 1, utilizing equations (4.93), (4.146) and (4.159), one can write:

$${}^2KA_{55} \left(\phi_x^F + \frac{dw_o^F}{dx} \right) = \left(A_{55} - c_2 D_{55} - c_2 (D_{55} - c_2 F_{55}) \right) \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \quad (4.160)$$

$$\Rightarrow {}^2KA_{55} \left(\frac{2}{3} \right) \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) = \left(A_{55} - c_2 D_{55} - c_2 (D_{55} - c_2 F_{55}) \right) \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \quad (4.161)$$

$$\therefore {}^2K = \frac{3(A_{55} - c_2 D_{55} - c_2 (D_{55} - c_2 F_{55}))}{2A_{55}} \quad (4.162)$$

It should be noted that the shear correction factor to be used in the conventional method 1 in section 4.3.2.1, 1K , will be the same as 2K . It is because in the FSDT, this traditional method exclude the γ_{yz}^F for Q_x^F and the expression for Q_x^F will turn out to be the same as equation (4.93) in method 2. If one extend the conventional method to the STSDT and exclude the γ_{yz}^T for the effective transverse shear force $\bar{Q}_x^T$, the $\bar{Q}_x^T$ expression for the method 1 will turn out to be the same as equation (4.146). Hence, following the procedure above will yield the 1K which will be the same as 2K .

Shear correction factor for method 3, 3K

In this section, an attempt will also be made to establish the shear correction factor from the STSDT for the FSDT beam in method 3 of section 4.3.2.1, 3K . This 3K can then be utilized in equations (4.103) and (4.104) to solve the buckling problem. For STSDT, if the study is confined to bending behavior of a beam in the xz -plane as in method 3 of section 4.3.2, one can let $\bar{Q}_y^T$ to be zero:

$$\bar{Q}_y^T = Q_y^T - c_2 R_y^T = 0 \quad (4.163)$$

Using equations (4.138) and (4.139), the above equation can be written as:

$$\begin{aligned} & A_{44} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) + A_{45} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) - c_2 \left(D_{44} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) + D_{45} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right) \\ & - c_2 \left(D_{44} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) + D_{45} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right) \\ & - c_2 \left(F_{44} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) + F_{45} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right) \right) = 0 \end{aligned} \quad (4.164)$$

Solving the above equation for $\left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right)$ in terms of $\left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right)$ results in:

$$\left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) = - \frac{(A_{45} + c_2^2 F_{45} - 2c_2 D_{45})}{(A_{44} + c_2^2 F_{44} - 2c_2 D_{44})} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \quad (4.165)$$

From equations (4.138) and (4.139), the expression for $\bar{Q}_x^T$ is:

$$\begin{aligned}\bar{Q}_x^T &= Q_x^T - c_2 R_x^T = A_{45} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) + A_{55} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \\ &- c_2 \left(D_{45} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) + D_{55} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right) \\ &- c_2 \left(D_{45} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) + D_{55} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right. \\ &\quad \left. - c_2 \left(F_{45} \left(\phi_y^T + \frac{\partial w_o^T}{\partial y} \right) + F_{55} \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \right) \right)\end{aligned}\quad (4.166)$$

Substituting equation (4.165) into the above equation and simplifying yields:

$$\bar{Q}_x^T = \left(\begin{aligned} &A_{55} - 2c_2 D_{55} + c_2^2 F_{55} + \frac{4c_2 A_{45} D_{45} - A_{45}^2 - 4c_2^2 D_{45}^2}{A_{44} - 2c_2 D_{44} + c_2^2 F_{44}} \\ &+ \frac{2c_2^2 (2c_2 D_{45} - A_{45}) F_{45}}{A_{44} - 2c_2 D_{44} + c_2^2 F_{44}} - \frac{c_2^4 F_{45}^2}{A_{44} - 2c_2 D_{44} + c_2^2 F_{44}} \end{aligned} \right) \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right) \quad (4.167)$$

Therefore, using assumption 1 in the previous section, equations (4.102), (4.159) and (4.167), one can obtain:

$$\begin{aligned}&{}^3K \left(A_{55} - \frac{A_{45}^2}{A_{44}} \right) \left(\phi_x^F + \frac{dw_o^F}{dx} \right) \\ &= \left(\begin{aligned} &A_{55} - 2c_2 D_{55} + c_2^2 F_{55} + \frac{4c_2 A_{45} D_{45} - A_{45}^2 - 4c_2^2 D_{45}^2}{A_{44} - 2c_2 D_{44} + c_2^2 F_{44}} \\ &+ \frac{2c_2^2 (2c_2 D_{45} - A_{45}) F_{45}}{A_{44} - 2c_2 D_{44} + c_2^2 F_{44}} - \frac{c_2^4 F_{45}^2}{A_{44} - 2c_2 D_{44} + c_2^2 F_{44}} \end{aligned} \right) \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right)\end{aligned}\quad (4.168)$$

$$\begin{aligned}&\Rightarrow {}^3K \left(A_{55} - \frac{A_{45}^2}{A_{44}} \right) \left(\frac{2}{3} \right) \left(\phi_x^T + \frac{dw_o^T}{dx} \right) \\ &= \left(\begin{aligned} &A_{55} - 2c_2 D_{55} + c_2^2 F_{55} + \frac{4c_2 A_{45} D_{45} - A_{45}^2 - 4c_2^2 D_{45}^2}{A_{44} - 2c_2 D_{44} + c_2^2 F_{44}} \\ &+ \frac{2c_2^2 (2c_2 D_{45} - A_{45}) F_{45}}{A_{44} - 2c_2 D_{44} + c_2^2 F_{44}} - \frac{c_2^4 F_{45}^2}{A_{44} - 2c_2 D_{44} + c_2^2 F_{44}} \end{aligned} \right) \left(\phi_x^T + \frac{\partial w_o^T}{\partial x} \right)\end{aligned}\quad (4.169)$$

$$\therefore {}^3K = \frac{3A_{44} \left(\begin{aligned} &A_{55} - 2c_2D_{55} + c_2^2F_{55} + \frac{4c_2A_{45}D_{45} - A_{45}^2 - 4c_2^2D_{45}^2}{A_{44} - 2c_2D_{44} + c_2^2F_{44}} \\ &+ \frac{2c_2^2(2c_2D_{45} - A_{45})F_{45}}{A_{44} - 2c_2D_{44} + c_2^2F_{44}} - \frac{c_2^4F_{45}^2}{A_{44} - 2c_2D_{44} + c_2^2F_{44}} \end{aligned} \right)}{2(A_{44}A_{55} - A_{45}^2)} \quad (4.168)$$

Therefore, shear correction factors that can be used for the three methods develop utilizing FSDT in section 4.3.2.1 have been established. These K values contain the higher-order stiffness coefficients from the STSDT and it is believed that they can be used in the FSDT to give a better prediction for the buckling solution. For a specially orthotropic laminated beam, it is found that ${}^1K = {}^2K = {}^3K = \frac{4}{5}$, which is slightly lower than the conventional value of $\frac{5}{6}$.

4.4 Results and discussion

In this section, the buckling loads of both the thin and moderately thick perfect laminated beams will be studied using the methods developed previously. The dimension of the laminated beam is taken to be $20\text{mm} \times 100\text{mm}$ (width $\times$ length). The thickness per ply is 0.125mm . The fiber-reinforced material used here is the carbon/epoxy fiber-reinforced composite material and it has a fiber volume fraction of 0.65 (or 65%). Anti-symmetric angle-ply laminates are utilized in this investigation. Both the clamped and simply-supported cases are studied.

4.4.1 Thin ideal laminated beam

In this section, the buckling behaviors of thin ideal laminated beams will be examined.

(a)

(b)

(c)

(d)

Figure 4.5: Comparison of buckling loads of anti-symmetric angle-ply laminated thin perfect beams, which composed of (a) 2 plies, (b) 4 plies, (c) 8 plies and (d) 16 plies for different ply-angles, under clamped boundary condition.

Figures 4.5(a) to (d) show the buckling loads of anti-symmetric laminated beams for 2, 4, 8 and 16 plies, respectively, which are clamped at both ends. The three methods indicated in these diagrams refer to the three different approaches established in section 4.3.1.1. The methods 1, 2 and 3 utilize ${}^1(\overline{EI})$, ${}^2(\overline{EI})$ and ${}^3(\overline{EI})$, respectively, in conjunction with the classical beam theory to compute the desired critical loads of these thin ideal laminated beams. For all the cases considered in Figures 4.5(a) to (d), it is observed that all the readings obtained from the finite element method are confined between those evaluated using methods 2 and 3, where values determined using method 2 serves as the upper bound and results computed with method 3 acts as the lower limit. It is also found that solutions calculate utilizing method 1 always appear above those worked out employing method 2. These findings support the earlier suggestion that readings attained using methods 2 and 3 can be utilized to form a pair of upper and lower bounds within which the actual buckling loads lie.

(a)

(b)

(c)

(d)

Figure 4.6: Comparison of buckling loads of anti-symmetric angle-ply laminated thin perfect beams, which composed of (a) 2 plies, (b) 4 plies, (c) 8 plies and (d) 16 plies for different ply-angles, under simply-supported boundary condition.

Figures 4.6 (a) to (d) illustrate the critical loads for the same class of laminates that are composed of 2, 4, 8 and 16 layers, respectively, under simply-supported boundary condition. Similar to the clamped cases, all the readings acquired from the finite element method are encompassed by the results calculated employing methods 2 and 3 formulated in section 4.3.1.1. Hence, once again, it is proven that the methods 2 and 3 in section 4.3.1.1 can be applied together with the classical beam theory to enclose a region in which the real critical loads will be. Furthermore, for these simply-supported situations, it is noticed that the results gained from the finite element method tends to be closer to the solutions gotten from method 3 than those obtained from method 2.

4.4.2 Moderately thick perfect laminated beam

In this section, the buckling phenomena of moderately thick perfect laminated beam will be looked into.

(a)

(b)

Figure 4.7: Comparison of buckling loads of anti-symmetric angle-ply laminated moderately thick perfect beams, which composed of (a) 40 plies and (b) 48 plies for different ply-angles, under clamped boundary condition.

Figures 4.7 (a) and (b) demonstrate the critical loads of clamped anti-symmetric angle-ply laminated beams that are composed of 40 and 48 layers, respectively. The three methods mentioned in these plots are referring to the three techniques developed in section 4.3.2.1. They include method 1 that used ${}^1(\overline{EI})$, ${}^1(\overline{Gh})$ and 1K , method 2 that employed ${}^2(\overline{EI})$, ${}^2(\overline{Gh})$ and 2K and method 3 that utilized ${}^3(\overline{EI})$, ${}^3(\overline{Gh})$ and 3K , which can be used with the FSDT beam theory to work out the buckling force of a moderately thick laminated beam. From the two illustrations, careful observation reveals that for both the 40 and 48 plies laminated beams, all the buckling loads obtained from the finite

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element method fall between the upper and lower bounds computed from method 2 and 3, respectively. Although the results gained from method 1 are very close to those attained from method 2, they still have bigger values and are thus further away from the true buckling loads. Therefore, these observations show that methods 2 and 3 in section 4.3.2.1 can be utilized to form a pair of upper and lower limits for the actual critical loads of a moderately thick laminated beam.

(a)

(b)

Figure 4.8: Comparison of buckling loads of anti-symmetric angle-ply laminated moderately thick perfect beams, which composed of (a) 40 plies and (b) 48 plies for different ply-angles, under simply-supported boundary condition.

Figures 4.8 (a) and (b) illustrate the buckling behaviors of 40 and 48 plies anti-symmetric laminated beams, respectively. Both beams are simply-supported at the two ends. Once more, the buckling loads gathered from the finite element method are within the upper

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and lower bounds worked out via methods 2 and 3 in section 4.3.2.1. Like the clamped cases, the solutions calculated employing method 1 in section 4.3.2.1 are very near to the values evaluated using method 2 in section 4.3.2.1. However, a close look tells that the results determined utilizing method 2 in section 4.3.2.1 are closer to the actual buckling loads acquired from the finite element method than the readings attained via method 1 in section 4.3.2.1. Together with the similar findings from the clamped cases, it can be implied that method 2 in section 4.3.2.1 is a better mean of getting the desired upper limit for the real critical loads of a moderately thick laminated beam over method 1 in section 4.3.2.1. It is also noted that for these simply-supported cases, the data obtained from the finite element method are much closer to the lower bound than to the upper limit. This observation agrees with the previous study on the simply-supported thin perfect laminated beam.

CHAPTER FIVE: BUCKLING ANALYSIS OF THIN SINGLE-DELAMINATED FIBER-REINFORCED COMPOSITE BEAM

The development of the analytical model to study the buckling of the thin fiber-reinforced composite beam with a delamination is presented. The mathematical model is formulated using the balance of forces and moments, compatibility conditions and edge conditions. Then, the three methods of calculating the effective flexural stiffness of the composite beam in the section 4.3.1.1, including the novel method $3^3(\overline{EI})$, are applied to the mathematical model to obtain the required critical load. In this chapter as well as others, the composite beam is kept to have full length of 100mm, width of 20mm and thickness per layer of 0.125mm for consistency. The 8 plies anti-symmetric angle-ply and 16 layers generally orientated carbon/epoxy beams are examined. Parametric studies are carried out on factors such as delamination length, through-the-depth location of the delamination and edge conditions such as clamped and simply-supported ends. Furthermore, the boron/epoxy and E-glass epoxy composite materials participate in the analysis.

5.1 Problem description

Figure 5.1: Composite beam with a single delamination

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This chapter is dedicated to the study of buckling of thin delaminated fiber-reinforced composite beam. As only thin structures are considered in this chapter, only materials covered in section 4.3.1 in chapter four (CLT) will be used to solve the buckling problem. The composite beam in Figure 5.1 has a total length of L and overall height of H . It contains a delamination of length a and the distance from the center of the composite laminate to the center of the delamination is denoted by d . Furthermore, the delamination divides the perfect beam into four segments, beam 1 on the left of the delamination, beam 2 below the delamination, beam 3 above the delamination and beam 4 to the right of the delamination, as illustrated in Figure 5.1. The heights of beam 2 and beam 3 are given as H_2 and H_3 . This delaminated beam is subjected to a compressive force, P at the two ends. The x co-ordinate axis is used to measure the distance of any point on the composite along the length of the beam. On the other hand, the w co-ordinate axis is utilized to indicate the deflection of the beam.

5.2 Fundamental expressions

From equation (4.44), the beam buckling governing equation employing CLT appears as:

$$\frac{\partial^2 M_{xx}^c}{\partial x^2} + N_{xx}^c \frac{\partial^2 w_o^c}{\partial x^2} = 0 \quad (5.1)$$

In terms of displacement, the above equation can be rewritten as:

$${}^j(\overline{EI})_i \frac{d^4(w_o^c)_i}{dx^4} + P_i \frac{d^2(w_o^c)_i}{dx^2} = 0 \text{ for } (i = 1..4 \text{ and } j = 1..3) \quad (5.2)$$

The subscript i in the above equation is employed to indicate the individual beam segment in Figure 5.1. The pre-superscript j is used to denote the three methods that can be used to compute the desired effective flexural stiffness of a laminated beam. These three formulae can be found in section (4.3.1.1) as equations (4.45), (4.55) and (4.73). It is desirable to reduce the terms in equation (5.2) to a simpler form for ease of use in the later sections. Keeping in mind that only CLT is utilized in this chapter and the nought on the variable w is only used to refer to the middle surface, both the superscript C and subscript o will be dropped from the variable w . Also, remembering that the effective bending rigidity of the individual beam can be evaluated in three different ways, the pre-

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superscript j will also be removed from the term $(\overline{EI})$ for the rest of this chapter. In this way, equation (5.2) can be expressed in a more compact manner as:

$$(\overline{EI})_i \frac{d^4 w_i}{dx^4} + P_i \frac{d^2 w_i}{dx^2} = 0 \quad (5.3)$$

The general solution for equation (5.3) has the form:

$$\frac{d^2 w_i}{dx^2} = \alpha \cos\left(\sqrt{\frac{P_i}{(\overline{EI})_i}} x\right) + \beta \sin\left(\sqrt{\frac{P_i}{(\overline{EI})_i}} x\right) \quad (5.4)$$

The α and β are unknown coefficients.

By letting $\lambda_i = \sqrt{\frac{P_i}{(\overline{EI})_i}}$, the general solution is simplified to:

$$\frac{d^2 w_i}{dx^2} = \alpha \cos(\lambda_i x) + \beta \sin(\lambda_i x) \quad (5.5)$$

$$\text{For beam 1, } \lambda_1 = \sqrt{\frac{P_1}{(\overline{EI})_1}} \quad (5.6)$$

$$\text{For beam 2, } \lambda_2 = \sqrt{\frac{P_2}{(\overline{EI})_2}} = \sqrt{\frac{H_2 P_1}{H (\overline{EI})_2}} = \sqrt{\frac{(\overline{EI})_1}{(\overline{EI})_2}} \sqrt{\frac{H_2}{H}} \sqrt{\frac{P_1}{(\overline{EI})_1}} = \lambda_1 \sqrt{\frac{(\overline{EI})_1 H_2}{(\overline{EI})_2 H}} \quad (5.7)$$

$$\text{As for beam 3, } \lambda_3 = \sqrt{\frac{P_3}{(\overline{EI})_3}} = \sqrt{\frac{H_3 P_1}{H (\overline{EI})_3}} = \sqrt{\frac{(\overline{EI})_1}{(\overline{EI})_3}} \sqrt{\frac{H_3}{H}} \sqrt{\frac{P_1}{(\overline{EI})_1}} = \lambda_1 \sqrt{\frac{(\overline{EI})_1 H_3}{(\overline{EI})_3 H}} \quad (5.8)$$

$$\text{Beam 4 will have } \lambda_4 = \sqrt{\frac{P_4}{(\overline{EI})_4}} = \sqrt{\frac{H P_1}{H (\overline{EI})_4}} = \sqrt{\frac{(\overline{EI})_1}{(\overline{EI})_4}} \sqrt{\frac{P_1}{(\overline{EI})_1}} = \sqrt{\frac{P_1}{(\overline{EI})_1}} = \lambda_1 \quad (5.9)$$

Therefore from equation (5.5), the deflection, slope, moment and shear force for the individual beam can be written as:

$$w_i(x) = A_i \cos(\lambda_i x) + B_i \sin(\lambda_i x) + C_i x + D_i \quad (5.10)$$

$$\frac{dw_i(x)}{dx} = -\lambda_i A_i \sin(\lambda_i x) + \lambda_i B_i \cos(\lambda_i x) + C_i \quad (5.11)$$

$$\frac{d^2 w_i(x)}{dx^2} = -\lambda_i^2 A_i \cos(\lambda_i x) - \lambda_i^2 B_i \sin(\lambda_i x) \quad (5.12)$$

$$\frac{d^3 w_i(x)}{dx^3} = \lambda_i^3 A_i \sin(\lambda_i x) - \lambda_i^3 B_i \cos(\lambda_i x) \quad (5.13)$$

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The displacement w and its derivatives in the above equations will be applied in the subsequent section for the analytical formulation.

5.3 Formulation

In this section, the necessary system of equations to solve the buckling problem of the thin fiber-reinforced composite beam containing a delamination will be presented. These equations arise from the boundary conditions at two ends of the delaminated composite beam, kinematical conditions and equilibrium in internal forces and moments. The internal shear force and moment for the i th beam can be written in term of displacement w as $Q_i(x) = -(\overline{EI})_i \frac{d^3 w_i(x)}{dx^3}$ and $M_i(x) = -(\overline{EI})_i \frac{d^2 w_i(x)}{dx^2}$, respectively. All the x values (position along the beam) in this section can be found in Figure 5.1.

5.3.1 Boundary conditions

Both the clamped and simply supported boundary conditions are addressed in the subsequent table.

<i>Clamped</i> (5.××a)	<i>Simply-supported</i> (5.××b)	
$w_1(x=0)=0$	$w_1(x=0)=0$	(5.14)
$\frac{dw_1(x=0)}{dx}=0$	$M_1(x=0)=0$ $\therefore -(\overline{EI})_1 \frac{d^2 w_1(x=0)}{dx^2}=0$	(5.15)
$w_4(x=L)=0$	$w_4(x=L)=0$	(5.16)
$\frac{dw_4(x=L)}{dx}=0$	$M_4(x=L)=0$ $\therefore -(\overline{EI})_4 \frac{d^2 w_4(x=L)}{dx^2}=0$	(5.17)

5.3.2 Kinematical compatibility equations

Figure 5.2: Kinematical continuity conditions at $x = \frac{L}{2} - \frac{a}{2} + d$

In the classical beam theory, the transverse normal is inextensible. Furthermore, planes that are originally straight and perpendicular to the neutral axis will remain straight and normal to the neutral axis even after bending. These are illustrated in Figure 5.2. Moreover, as the two sub-laminates (Beams 2 and 3) remain attached to the base laminate (Beam 1) after bending, the following kinematical continuity equations can be deduced:

$$w_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = w_2\left(x = \frac{L}{2} - \frac{a}{2} + d\right)$$

$$\therefore w_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) - w_2\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = 0 \quad (5.18)$$

$$w_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = w_3\left(x = \frac{L}{2} - \frac{a}{2} + d\right)$$

$$\therefore w_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) - w_3\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = 0 \quad (5.19)$$

$$\frac{dw_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right)}{dx} = \frac{dw_2\left(x = \frac{L}{2} - \frac{a}{2} + d\right)}{dx}$$

$$\therefore \frac{dw_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right)}{dx} - \frac{dw_2\left(x = \frac{L}{2} - \frac{a}{2} + d\right)}{dx} = 0 \quad (5.20)$$

$$\begin{aligned} \frac{dw_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right)}{dx} &= \frac{dw_3\left(x = \frac{L}{2} - \frac{a}{2} + d\right)}{dx} \\ \therefore \frac{dw_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right)}{dx} - \frac{dw_3\left(x = \frac{L}{2} - \frac{a}{2} + d\right)}{dx} &= 0 \end{aligned} \quad (5.21)$$

Figure 5.3: Kinematical continuity conditions at $x = \frac{L}{2} + \frac{a}{2} + d$

Similarly, at the right crack-tip (as shown in Figure 5.3), the following kinematical continuity equations can be written:

$$\begin{aligned} w_2\left(x = \frac{L}{2} + \frac{a}{2} + d\right) &= w_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right) \\ \therefore w_2\left(x = \frac{L}{2} + \frac{a}{2} + d\right) - w_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right) &= 0 \end{aligned} \quad (5.22)$$

$$w_3\left(x = \frac{L}{2} + \frac{a}{2} + d\right) = w_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right)$$

$$\therefore w_3\left(x = \frac{L}{2} + \frac{a}{2} + d\right) - w_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right) = 0 \quad (5.23)$$

$$\begin{aligned} \frac{dw_2\left(x = \frac{L}{2} + \frac{a}{2} + d\right)}{dx} &= \frac{dw_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right)}{dx} \\ \therefore \frac{dw_2\left(x = \frac{L}{2} + \frac{a}{2} + d\right)}{dx} - \frac{dw_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right)}{dx} &= 0 \end{aligned} \quad (5.24)$$

$$\begin{aligned} \frac{dw_3\left(x = \frac{L}{2} + \frac{a}{2} + d\right)}{dx} &= \frac{dw_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right)}{dx} \\ \therefore \frac{dw_3\left(x = \frac{L}{2} + \frac{a}{2} + d\right)}{dx} - \frac{dw_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right)}{dx} &= 0 \end{aligned} \quad (5.25)$$

Figure 5.4: Deformation caused by incremental forces (ΔP_2 and ΔP_3)

Figure 5.4 demonstrates the deformed state of the thin single-delaminated composite beam after buckling. The curves that are made up of short and long dashes denote the neutral axes of the individual beams. On the other hand, the thick solid lines represent the transverse normals located at the delamination tips. Δ_2^a plus Δ_2^b gives the total change in length of beam 2 due to bending. The incremental force in beam 2 associated with this

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alter in length is indicated as ΔP_2 . Similarly, adding up Δ_3^a and Δ_3^b provides the overall change in span of beam 3 and the corresponding incremental force in beam 3 is ΔP_3 . Furthermore, $\bar{Y}$, $\bar{Y}_2$ and $\bar{Y}_3$ are the distances measuring from the top surface to the neutral axes of beam 1 (or beam 4), beam 2 and beam 3, respectively. Therefore, $\bar{Y}_2 > \bar{Y} > \bar{Y}_3$. It is assumed that right after the buckling, the axial deformation formula can be used to calculate the amount of shortening or extension in the beams. Hence, for beam 2, the resultant equation is:

$$\frac{\Delta P_2 a}{H_2 E_2} = \Delta_2^a + \Delta_2^b = (\bar{Y}_2 - \bar{Y}) \left(\frac{dw_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} - \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) \quad (5.26)$$

From Figure 5.4, one can deduce that a negative sign must be placed in front of the term

$$\frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \text{ in the above equation. It is because the } \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \text{ will be}$$

a negative value due to how the w -axis is defined and a minus sign must be multiplied in

$$\text{front of it so that it can be added to the term } \frac{dw_1 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \text{ (which is a positive}$$

number) in equation (5.26). As $(\bar{Y}_2 - \bar{Y})$ is a positive value, the product at the right hand side of equation (5.26) will yield a positive figure. From Figure 5.4, one can notice that beam 2 experiences tension, which means that $\Delta P_2 > 0$. Hence, there will be no need to include any minus sign in the left hand side of equation (5.26) to make both sides to be equal. It should be noted that equation (5.26) remains true regardless of how beam 2 bends. Rearranging the equation gives:

$$\frac{\Delta P_2 a}{H_2 E_2} - (\bar{Y}_2 - \bar{Y}) \left(\frac{dw_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} - \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) = 0 \quad (5.27)$$

On the other hand, for beam 3,

$$-\frac{\Delta P_3 a}{H_3 E_3} = \Delta_3^a + \Delta_3^b = (\bar{Y} - \bar{Y}_3) \left(\frac{dw_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} - \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) \quad (5.28)$$

As $(\bar{Y} - \bar{Y}_3)$ is positive, similar to equation (5.26), the right hand side of the above equation will be positive. However, according to Figure 5.4, ΔP_3 is a compressive force and hence a negative value. Therefore, a minus sign must be incorporated in the left hand side of equation (5.28) to make it a valid expression. Also, in spite of how beam 3 bends, equation (5.28) remains applicable. Equation (5.28) can be rearranged to have:

$$\frac{\Delta P_3 a}{H_3 E_3} + (\bar{Y} - \bar{Y}_3) \left(\frac{dw_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} - \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) = 0 \quad (5.29)$$

Therefore, by equating equations (5.27) and (5.29), one can obtain:

$$\begin{aligned} & \frac{\Delta P_2 a}{H_2 E_2} - (\bar{Y}_2 - \bar{Y}) \left(\frac{dw_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} - \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) \\ &= \frac{\Delta P_3 a}{H_3 E_3} + (\bar{Y} - \bar{Y}_3) \left(\frac{dw_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} - \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) \\ \therefore & -\frac{\Delta P_2 a}{H_2 E_2} + (\bar{Y}_2 - \bar{Y}) \left(\frac{dw_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} - \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) \\ &+ \frac{\Delta P_3 a}{H_3 E_3} + (\bar{Y} - \bar{Y}_3) \left(\frac{dw_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} - \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) = 0 \end{aligned} \quad (5.30)$$

5.3.3 Equilibrium of forces and moments

Figure 5.5: Internal forces and moments at $x = \frac{L}{2} - \frac{a}{2} + d$

Figure 5.5 illustrates the conventions used for the axial forces, shear forces and moments.

This portion is located at the left crack tip of the delamination, where $x = \frac{L}{2} - \frac{a}{2} + d$.

Referring to Figure 5.5, the essential equations can be set up utilizing the equilibrium of forces and moments.

From $\sum F_x = 0$, one will have:

$$\Delta P_2 + \Delta P_3 = 0 \quad (5.31)$$

Taking $\sum F_z = 0$, the resultant expression is:

$$\begin{aligned} -V_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) + V_2 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) + V_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) &= 0 \\ \therefore (\overline{EI})_1 \frac{d^3 w_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx^3} - (\overline{EI})_2 \frac{d^3 w_2 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx^3} \\ - (\overline{EI})_3 \frac{d^3 w_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx^3} &= 0 \end{aligned} \quad (5.32)$$

Employing $\sum M_o = 0$, taking clockwise to be positive, one will obtain:

$$\begin{aligned} M_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) - M_2 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) - M_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) \\ + (\overline{Y} - \overline{Y}_3) \Delta P_3 - (\overline{Y}_2 - \overline{Y}) \Delta P_2 &= 0 \end{aligned}$$

$$\begin{aligned} \therefore & -(\overline{EI})_1 \frac{d^2 w_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx^2} + (\overline{EI})_2 \frac{d^2 w_2 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx^2} \\ & + (\overline{EI})_3 \frac{d^2 w_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx^2} + (\overline{Y} - \overline{Y}_3) \Delta P_3 - (\overline{Y}_2 - \overline{Y}) \Delta P_2 = 0 \end{aligned} \quad (5.33)$$

Figure 5.6: Internal forces and moments at $x = \frac{L}{2} + \frac{a}{2} + d$

Similarly, Figure 5.6 demonstrates the forces and moments present at the right delamination tip. Once again, vital equations for the formulation can be attained from the balances of forces and moments.

Utilizing $\sum F_z = 0$, one can get:

$$\begin{aligned} & V_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) - V_2 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) - V_3 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) = 0 \\ \therefore & -(\overline{EI})_4 \frac{d^3 w_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx^3} + (\overline{EI})_2 \frac{d^3 w_2 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx^3} \\ & + (\overline{EI})_3 \frac{d^3 w_3 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx^3} = 0 \end{aligned} \quad (5.34)$$

Furthermore, by having $\sum M_o = 0$, taking clockwise to be positive, one can obtain:

$$\begin{aligned}
& M_2 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) + M_3 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) - M_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) \\
& - (\bar{Y} - \bar{Y}_3) \Delta P_3 + (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0 \\
& \therefore -(\bar{EI})_2 \frac{d^2 w_2 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx^2} - (\bar{EI})_3 \frac{d^2 w_3 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx^2} \\
& + (\bar{EI})_4 \frac{d^2 w_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx^2} - (\bar{Y} - \bar{Y}_3) \Delta P_3 + (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0
\end{aligned} \tag{5.35}$$

The equations (5.14) to (5.17) (equations (5.××a) for clamped cases and expressions (5.××b) for simply-supported scenarios) together with expressions (5.18) till (5.25) and equations (5.30) to (5.35) are utilized to solve the buckling problem. Then, equations (5.10) to (5.13) are substituted into the above-mentioned equations to yield 18 simultaneous expressions, which can then be written in a matrix form as:

$$[M]\{C\} = \{0\} \tag{5.36}$$

The matrix $[M]$ is a 18×18 square matrix while $\{C\}$ is a column vector containing all the 16 undetermined unknowns (A_i, B_i, C_i and D_i , for $i = 1..4$) together with ΔP_2 and ΔP_3 . In order for the terms in the $\{C\}$ column vector to have valid solutions, the matrix $[M]$ must vanish and the lowest eigenvalue will give the desired buckling load.

5.4 Results and discussion

Materials	E_{11} (GPa)	E_{22} (GPa)	G_{12} (GPa)	E_{11}/E_{22}	G_{12}/E_{22}	ν_{12}
Carbon/epoxy	150	9	4.8	16.667	0.533	0.28
Boron/epoxy	210	21	6.3	10.000	0.300	0.21
E-Glass/epoxy	42	10	4	4.200	0.400	0.28

Table 5.1: Material properties for different composite materials

In this section, the three methods of calculating the effective flexural stiffness of a composite beam addressed in section 4.3.1.1 will be incorporated into the mathematical model presented in section 5.3 to study the buckling load of the single-delaminated composite beam. These three techniques are denoted as $^1(\bar{EI})$, $^2(\bar{EI})$ and $^3(\bar{EI})$ and can be

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evaluated using equations (4.45), (4.54) and (4.73), respectively. The finite element method will be employed to verify the analytical results. It should be noted that the readings obtained via the finite element method in the current chapter and chapter six to eight are all converged results. The fiber-reinforced composite materials used in this investigation include carbon/epoxy, boron/epoxy and E-glass/epoxy. The material properties for these composites are listed in Table 5.1. The composite beam is configured to be 100mm length, 20mm width and the thickness per ply is 0.125mm. The thin delaminated composite beams analyzed comprise of either 8 or 16 plies. Both the clamped and simply-supported boundary conditions are examined. Furthermore, parametric studies on ply orientation, stacking sequence, through-the-thickness location of the delamination and delamination length are conducted.

(a/L)	$(H_3/H) = 0.200$		$(H_3/H) = 0.300$		$(H_3/H) = 0.400$	
	Present	Ref. [72]	Present	Ref. [72]	Present	Ref. [72]
0.200	0.9266	0.9264	0.9925	0.9924	0.9952	0.9950
0.400	0.2471	0.2470	0.5315	0.5314	0.7884	0.7883
0.600	0.1104	0.1103	0.2436	0.2435	0.4124	0.4124
0.800	0.0623	0.0623	0.1390	0.1390	0.2428	0.2428

(a)

(a/L)	$(H_3/H) = 0.200$		$(H_3/H) = 0.300$		$(H_3/H) = 0.400$	
	Present	Ref. [72]	Present	Ref. [72]	Present	Ref. [72]
0.200	0.9994	0.9996	0.9997	0.9997	0.9997	0.9997
0.400	0.9017	0.9016	0.9852	0.9827	0.9902	0.9902
0.600	0.4331	0.4331	0.8149	0.8149	0.9198	0.9198
0.800	0.2450	0.2450	0.5118	0.5118	0.7264	0.7264

(b)

Table 5.2: Comparisons of present results with those from Simites et al. [72] for delaminated homogenous beam with (a) clamped and (b) simply-supported ends

Prior to the parametric studies, the well-known results from Simites et al. [72] for homogenous delaminated beams are used to validate the current mathematical model. The comparison is shown in Tables 5.2 (a) and (b). Each reading in the two tables is attained by nondimensionalizing the actual buckling load of the delaminated beam with respect to the critical load of the perfect beam that can be determined utilizing the conventional Euler's formula for the clamped or simply-supported end. It is found that

data generated via the present mathematical model are observed to be in good agreement with those reported by Simitses et al. [72].

5.4.1 Carbon/epoxy composite material

A stacking-sequence terminology is introduced to describe the overall stacking sequence of a delaminated composite beam in a simple and concise fashion. If a delaminated composite beam has a stacking-sequence of $[0^\circ_2 // -45^\circ / 45^\circ / (90^\circ / 30^\circ / -30^\circ)_2]$ will be a 10 layered delaminated composite beam with an expanded sequence of $[0^\circ / 0^\circ // -45^\circ / 45^\circ / 90^\circ / 30^\circ / -30^\circ / 90^\circ / 30^\circ / -30^\circ]$. The first angle in the list will be referring to the angle of the ply nearest to the top surface. The square bracket is used to enclose the entire stacking sequence. The single slash “/” is employ to separate the various angles while the double slash “//” is utilized to symbolize the through-the-thickness location of the delamination. Hence, in this case, the composite beam will contain a delamination two plies below the top surface. The subscript after an angle will indicate the number of times the angle is repeated before going to the next angle in the piling list. Similarly, the subscript behind a curve bracket will specify the number of times the angles inside the bracket are recurred before proceeding to the subsequent angle.

In this section, the carbon/epoxy will be used as the composite material to investigate the buckling behavior of a thin single-delaminated composite beam. Its material properties are presented in Table 5.1. Both the anti-symmetric angle-ply and general orientated laminated beams will be looked into.

5.4.1.1 Thin single-delaminated anti-symmetric angle-ply beam

All the delaminated anti-symmetric angle-ply beams addressed in this section are made up of 8 layers.

(a)

(b)

(c)

Figure 5.7: Comparison of buckling loads of $[(-\theta/\theta)_2 // (-\theta/\theta)_2]$ composite beam with a delamination of length (a) 10mm, (b) 50mm and (c) 80mm under clamped boundary condition

(a)

(b)

(c)

Figure 5.8: Comparison of buckling loads of $[(-\theta/\theta)_2 // (-\theta/\theta)_2]$ composite beam with a delamination of length (a) 10mm, (b) 50mm and (c) 80mm under simply-supported boundary condition

(a)

(b)

(c)

Figure 5.9: Comparison of buckling loads of $[-\theta/\theta//(-\theta/\theta)_3]$ composite beam with a delamination of length (a) 10mm, (b) 50mm and (c) 80mm under clamped boundary condition

Figure 5.10: Buckling mode shapes of clamped $[(-\theta/\theta)_2 // (-\theta/\theta)_2]$ composite beam with a delamination span of (a) 10mm, (b) 50mm and (c) 80mm

Figure 5.11: Buckling mode shapes of clamped $[-\theta/\theta//(-\theta/\theta)_3]$ composite beam with a delamination span of (a) 10mm, (b) 50mm and (c) 80mm

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The three methods stated in the diagrams above point to the three different means of computing the effective flexural rigidity of a composite beam. These techniques are denoted as $^1(\overline{EI})$, $^2(\overline{EI})$ and $^3(\overline{EI})$ in section 4.3.1.1, where the per-superscript indicates the type of method used. Thus, the three analytical solutions in each of the graphs above are obtained by applying these three different methods into the mathematical model formulated in section 5.3 and solving the eigenproblem. Such labels will be utilized in other graphs for the rest of the sections in this chapter. In Figures 5.7 to 5.9, the 10mm delamination is used to represent a short delamination, the 50mm delamination is employed to characterize a delamination of intermediate length and the 80mm delamination is used to stand for a long delamination. It is apparent from these figures that increases in delamination length and ply-angle have a negative effect on the load carrying capacity of the delaminated beam.

The plots in Figures 5.7 (a) to (c) demonstrate the variation of the critical load of $[(-\theta/\theta)_2 // (-\theta/\theta)_2]$ delaminated composite beam that is clamped at both ends for various ply-angles for three delamination lengths. The angle θ covers from 0° to 90° . From these diagrams, it can be seen that the values acquired from the finite element method are enveloped by the analytical results evaluated employing methods 2 and 3. The upper bound is formed by employing method 2 in cooperation with the mathematical model while the lower limit is generated by exercising method 2 in conjunction with the analytical model. This finding matches with the previous observation in Figures 4.5 (a) to (d) for thin perfect anti-symmetric laminated beam under clamped boundary condition. Hence, one can conclude that the effective flexural stiffnesses calculated using methods 2 and 3 are not only able to determine the desired upper and lower bounds of the buckling load of an ideal composite beam, they can also be utilized to work out the wanted higher and lower limits of a delaminated composite beam. The solution produced via method 1 in conjunction with the mathematical model is always higher than the data computed using method 2 together with the analytical model. Analogous to the perfect laminated beam cases, this implies that the method 2 is a better mean of working with the mathematical model than method 1 to obtain the desired upper bound. Nevertheless, the buckling load attained applying method 1 is preserved in this chapter to serve as a reference.

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In Figures 5.8 (a) to (c), the change of the buckling force of the simply-supported $[(-\theta/\theta)_2 // (-\theta/\theta)_2]$ delaminated composite beam for different ply-angles is examined for three delamination lengths. Once more, the finite element method readings are enclosed by the upper and lower bounds attained via method 2 and 3 together with the analytical model, respectively. Furthermore, it is interesting to observe that for all these simply supported cases, the points acquired from the finite element method are much closer to the lower bound than the upper limit. Such finding is also seen for thin ideal anti-symmetric angle-ply composite beam in Figures 4.6 (a) to (d).

The graphs in Figures 5.9 (a) to (c) illustrate alter of the critical load of the $[-\theta/\theta // (-\theta/\theta)_3]$ composite beam that is clamped for various θ values for three delamination lengths. As anticipated, the method 2 and 3 are still able to work with the analytical model to create the longed upper and lower limits for the results determined from the finite element method.

Lastly, the buckling mode shapes for the clamped $[(-\theta/\theta)_2 // (-\theta/\theta)_2]$ and $[-\theta/\theta // (-\theta/\theta)_3]$ cases are displayed in Figures 5.10 and 5.11, respectively, for the three different delamination lengths. The simply-supported counterparts can be found in the Appendix F.

5.4.1.2 Thin general orientation composite beam with a delamination

The delaminated composite beams to be examined in this section are composed of plies with general orientations. They are all made up of 16 plies and clamped at the two ends.

(a)

(b)

(c)

Figure 5.12: Comparison of critical loads of $[(\theta/0^\circ/90^\circ/-\theta)_2/(\theta/0^\circ/90^\circ/-\theta)_2]$ delaminated composite beam for θ equals to (a) 22.5° , (b) 30° and (c) 45° for various delamination lengths

(a)

(b)

(c)

Figure 5.13: Comparison of critical loads of $[(0^\circ/{}^1\theta/-{}^1\theta/90^\circ/0^\circ/{}^2\theta/-{}^2\theta/90^\circ)]/[(0^\circ/{}^1\theta/-{}^1\theta/90^\circ/0^\circ/{}^2\theta/-{}^2\theta/90^\circ)]$ delaminated composite beam for (a) ${}^1\theta = 30^\circ$ and ${}^2\theta = 22.5^\circ$; (b) ${}^1\theta = 45^\circ$ and ${}^2\theta = 22.5^\circ$; and (c) ${}^1\theta = 45^\circ$ and ${}^2\theta = 30^\circ$ for various delamination lengths

(a)

(b)

(c)

Figure 5.14: Comparison of critical loads of $[(\theta/90^\circ/-\theta/0^\circ)_2/(\theta/90^\circ/-\theta/0^\circ)_2]$ delaminated composite beam for θ equals to (a) 22.5° , (b) 30° and (c) 45° for various delamination lengths

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Figures 5.12 (a) to (c) show the buckling loads of $[(\theta/0^\circ/90^\circ/-\theta)_2 // (\theta/0^\circ/90^\circ/-\theta)_2]$ delaminated composite beam for different delamination lengths. The angle θ adopts the values of 22.5° , 30° and 45° . Hence, the thin delaminated composite beam is compiled of angle-ply with dissimilar angle orientations. Once more, the analytical answers from methods 2 and 3 are able to provide the required upper and lower bounds for the finite element method results, respectively. Furthermore, the mathematical curve attained from method 2 offers a better upper limit than the analytical solution from method 1.

Figures 5.13 (a) to (c) illustrate the critical loads of $[(0^\circ/^1\theta/-^1\theta/90^\circ/0^\circ/^2\theta/-^2\theta/90^\circ) // (0^\circ/^1\theta/-^1\theta/90^\circ/0^\circ/^2\theta/-^2\theta/90^\circ)]$ delaminated composite beam for various delamination spans. In this piling arrangement, the angles $^1\theta$ and $^2\theta$ can take on values of 22.5° , 30° and 45° . Thus, this composite beam will consists of a mixture of not only 0° and 90° but also plies that have two other angular directions.

Figures 5.14 (a) to (c) depict the buckling loads of $[(\theta/90^\circ/-\theta/0^\circ)_2 // (\theta/90^\circ/-\theta/0^\circ)_2]$ for a range of delamination lengths. Likewise, the angle θ undertakes inputs as 22.5° , 30° and 45° . Again, the mathematical data computed by methods 2 and 3 is able to supply the required upper and lower limits for the finite element method readings for these two kinds of delaminated composite beams for cases in both Figures 5.13 and 5.14.

All the figures demonstrate that the buckling load of a thin delaminated composite beam decreases as the delamination becomes longer.

5.4.2 Other composite materials

In addition of carbon/epoxy composite, boron/epoxy and E-glass/epoxy composite materials are used in this study. Their material properties are listed in Table 5.1. These two composite materials are introduced into the analysis to test whether the analytical solutions extracted from methods 2 and 3 can also work on other types of composite materials. All the delaminated composite beams examined in this section are compiled of 8 layers. They have a stacking arrangement of $[(-\theta/\theta)_2 // (-\theta/\theta)_2]$ and the angle θ stretches from 0° to 90° . Parallel with section 5.4.1.1, the delamination lengths are selected to be 10mm, 50mm and 80mm to correspond to short, intermediate and long

delaminations, respectively. Both clamped and simply-supported boundary conditions will be examined.

(a)

(b)

(c)

Figure 5.15: Buckling loads of delaminated boron/epoxy composite beam, in which the delamination is (a) 10mm, (b) 50mm and (c) 80mm long, clamped at both ends

(a)

(b)

(c)

Figure 5.16: Buckling loads of delaminated boron/epoxy composite beam, in which the delamination is (a) 10mm, (b) 50mm and (c) 80mm long, simply-supported at both ends

(a)

(b)

(c)

Figure 5.17: Buckling loads of delaminated E-glass/epoxy composite beam, in which the delamination is (a) 10mm, (b) 50mm and (c) 80mm long, clamped at both ends

(a)

(b)

(c)

Figure 5.18: Buckling loads of delaminated E-glass/epoxy composite beam, in which the delamination is (a) 10mm, (b) 50mm and (c) 80mm long, simply-supported at both ends

Figures 5.15 (a) to (c) and 5.16 (a) to (c) illustrate the critical loads of thin delaminated boron/epoxy composite beams that are clamped and simply-supported at both sides, respectively, for three different delamination lengths. On the other hand, Figures 5.17 (a) to (c) and 5.18 (a) to (c) show the buckling loads of thin delaminated E-glass/epoxy composite beams that are clamped and simply-supported at both ends, respectively, for three unlike delamination lengths. From Table 5.1, one can observe that carbon/epoxy, boron/epoxy and E-glass/epoxy can be used to represent composite materials with high, intermediate and low E_{11} to E_{22} ratios, respectively. Hence, Figures 5.15 to 5.18 can be utilized to examine whether the upper and lower limits proposed are capable of capturing the actual buckling loads of these composite materials with dissimilar material characteristics. From Figures 5.15 to 5.18, the plots show that all the data generated via finite element method is encompassed within the upper and lower bounds acquired from the mathematical solution determined utilizing methods 2 and 3, respectively. Thus, once again, the results support the suggestion that the approach is capable of identifying a region where the actual critical load of a delaminated composite beam will lay.

Other observations include decline of the buckling load as the ply-angle and delamination length increases. Moreover, for both composite materials, the readings obtained using the finite element method are much closer to the lower bound than the higher limit for the simply-supported cases.

5.5 Concluding remarks

In this chapter, the buckling performance of the thin single-delaminated composite beam is studied. An analytical model is formulated via the boundary conditions, kinematical compatibility and the balance of inner forces and moments. Then, the three approaches of calculating the effective bending rigidity of the composite beam are used in cooperation with the mathematical model to predict the critical load of the thin composite beam with a delamination. It is found that the values that are obtained from the analytical model are in excellent agreement with those that are published by Simitses et al. [72].

The 8 layers anti-symmetric angle-ply carbon/epoxy delaminated beam is examined. The delamination resides either at the mid-plane or 2 plies beneath the free surface. Three delamination lengths are studied and they are $\frac{a}{L} = 0.1, 0.5$ and 0.8 . In addition, both the clamped and simply-supported boundary conditions are investigated. It is observed that as the delamination length and ply-angle rise, the buckling load of the composite beam drops. It is also noticed that the critical load of the thin single-delaminated beam falls as the delamination moves deeper into the composite beam. For all the cases that are analyzed, the methods 2 and 3 are able to cooperate with the analytical model to build the longed upper and lower bounds for the actual buckling data that is gathered via the finite element method. It is noted that method 2 always gives a better upper limit than method 1 as it is always able to provide buckling values that are closer to the actual results from the finite element method. Furthermore, for the simply-supported scenario, it is found that the readings that are acquired from the finite element method tend to be much closer to the lower bound than the upper limit.

The clamped 16 plies generally orientated carbon/epoxy is also studied for several delamination lengths. There are three kinds of layouts and they are $[(\theta/0^\circ/90^\circ/-\theta)_2 // (\theta/0^\circ/90^\circ/-\theta)_2]$, $[(0^\circ/1\theta/-1\theta/90^\circ/0^\circ/2\theta/-2\theta/90^\circ) // (0^\circ/1\theta/-1\theta/90^\circ/0^\circ/2\theta/-2\theta/90^\circ)]$ and $[(\theta/90^\circ/-\theta/0^\circ)_2 // (\theta/90^\circ/-\theta/0^\circ)_2]$. The unknown angles take on values including 22.5° , 30° and 45° . Once again, from this investigation, it is observed that increase in the delamination span has a negative effect on the critical load of the composite beam and the method 2 always supply a more superior upper bound than method 1. Most

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importantly, once more, the results demonstrate that the methods 2 and 3 are still able to yield the desired upper and lower limits for the generally orientated composite beam, respectively.

The boron/epoxy and E-glass/epoxy composite materials are also involved in the analysis. The single-delaminated composite beam has a layering sequence of $[(-\theta/\theta)_2 // (-\theta/\theta)_2]$ and the angle θ covers from 0° to 90° . Both the clamped and simply-supported end conditions are studied. The delamination lengths that are examined include $\frac{a}{L} = 0.1, 0.5$ and 0.8 . Similar to the findings from the examination on the carbon/epoxy composite material, it is observed that increases in both the delamination length and the ply-angle have a harmful influence on the critical load of the composite beam. It is also noted that other than the carbon/epoxy composite material, the methods 2 and 3 are still capable of producing the desired upper and lower bounds, respectively, for other categories of composite materials like the boron/epoxy and E-glass/epoxy.

CHAPTER SIX:

STABILITY PHENOMENA OF MODERATELY THICK FIBER-REINFORCED COMPOSITE BEAM WITH A DELAMINATION

Unlike the previous chapter which studies the thin composite beam, in this chapter, the buckling behavior of the moderately thick composite beam with a single delamination is analyzed. A mathematical model is established via the suitable boundary conditions, kinematical compatibility and equilibrium of forces and moments. The three unique techniques of evaluating the $(\overline{EI})$, $(\overline{Gh})$ and K that are addressed in chapter four will then be used in conjunction with the analytical model to compute the longed critical load of the 48 plies composite beam with a delamination. Studies are conducted on parameters that may affect the buckling load of the composite beam and they include the layer orientation, piling arrangement, delamination length, through-the thickness position of the delamination, boundary conditions and the kind of composite matters.

6.1 Problem explanation

In this chapter, the buckling behavior of the moderately thick single-delaminated composite beam will be examined. When looking into buckling of moderately thick composite beam, the transverse shear deformation will be significant and must be taken into consideration. Hence, the materials addressed in section 4.3.2 in chapter four employing the FSDT will be applied in this chapter to work out the stability problem. Keeping in mind that the composite beam in the current chapter will compose of more layers than the previous chapter, the different variables in Figure 5.1 will still be preserved to describe the configuration of the moderately thick single-delaminated composite beam in this chapter.

6.2 Primary equations

The necessary expressions to solve for the FSDT beam buckling problem can be extracted from equations (4.87) and (4.88) as:

$$\frac{dM_{xx}^F}{dx} - Q_x^F = 0 \quad (6.1)$$

$$\frac{dQ_x^F}{dx} - P \frac{d^2 w_o^F}{dx^2} = 0 \quad (6.2)$$

Equations (6.1) and (6.2) can be written in terms of the displacements as:

$$^j(\overline{EI})_i \frac{d^2(\phi_x^F)_i}{dx^2} - ^j(\overline{Gh})_i \left(^j K_i \left((\phi_x^F)_i + \frac{d(w_o^F)_i}{dx} \right) \right) = 0 \quad (6.3)$$

$$^j(\overline{Gh})_i \left(^j K_i \left(\frac{d(\phi_x^F)_i}{dx} + \frac{d^2(w_o^F)_i}{dx^2} \right) \right) - P_i \frac{d^2(w_o^F)_i}{dx^2} = 0 \quad (6.4)$$

Similar to chapter five, the subscript i is utilized to symbolize the separate beams in Figure 5.1. The pre-superscript j is employed to refer to the three methods, which can be used to compute three sets of different values for the effective bending stiffness $(\overline{EI})$, the effective shear modulus $(\overline{Gh})$ and shear correction factor K of a composite beam. For method 1, the expressions needed to calculate $^1(\overline{EI})$, $^1(\overline{Gh})$ and 1K can be found in equations (4.45), (4.91) and (4.162), respectively. On the other hand, equations (4.55), (4.96) and (4.162) are applied to work out $^2(\overline{EI})$, $^2(\overline{Gh})$ and 2K in method 2. Lastly, expressions (4.73), (4.105) and (4.170) may be used to determine $^3(\overline{EI})$, $^3(\overline{Gh})$ and 3K in method 3. Here, an attempt will be made to reduce both equations (6.3) and (6.4) to a more compact form. Analogous to chapter five, noting that only FSDT is employed to resolve the buckling problem in this chapter and the nought is included on w to refer to the deflection of the mid-plane, both the superscript F and subscript o will be removed from the displacement terms. Similarly, bearing in mind that there are three techniques of evaluating $(\overline{EI})$, $(\overline{Gh})$ and K , the pre-superscript j will also be discarded. In addition, remembering that only rotation of the transverse normal in the x -direction is considered, the subscript x will be eliminated from the displacement term ϕ . In this manner, equations (6.3) and (6.4) can be transformed into a more reduced form:

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$$(\overline{EI})_i \frac{d^2 \phi_i}{dx^2} - (\overline{Gh})_i (K_i) \left(\phi_i + \frac{dw_i}{dx} \right) = 0 \quad (6.5)$$

$$(\overline{Gh})_i (K_i) \left(\frac{d\phi_i}{dx} + \frac{d^2 w_i}{dx^2} \right) - P_i \frac{d^2 w_i}{dx^2} = 0 \quad (6.6)$$

The subsequent steps will provide the mean of attaining the general solutions for the two differential equations above.

Firstly, differentiating equation (6.5) once yields:

$$(\overline{EI})_i \frac{d^3 \phi_i}{dx^3} - (\overline{Gh})_i (K_i) \left(\frac{d\phi_i}{dx} + \frac{d^2 w_i}{dx^2} \right) = 0 \quad (6.7)$$

$$\Rightarrow \frac{d\phi_i}{dx} + \frac{d^2 w_i}{dx^2} = \left(\frac{(\overline{EI})_i}{(\overline{Gh})_i (K_i)} \right) \frac{d^3 \phi_i}{dx^3} \quad (6.8)$$

Substituting (6.8) into (6.6) gives:

$$(\overline{EI})_i \frac{d^3 \phi_i}{dx^3} - P_i \frac{d^2 w_i}{dx^2} = 0 \quad (6.9)$$

Rearranging equation (6.6) results in:

$$\frac{d\phi_i}{dx} = - \left(1 - \frac{P_i}{(\overline{Gh})_i (K_i)} \right) \frac{d^2 w_i}{dx^2} \quad (6.10)$$

$$\Rightarrow \frac{d^3 \phi_i}{dx^3} = - \left(1 - \frac{P_i}{(\overline{Gh})_i (K_i)} \right) \frac{d^4 w_i}{dx^4} \quad (6.11)$$

Inputting equation (6.11) into (6.9) produces:

$$- (\overline{EI})_i \left(1 - \frac{P_i}{(\overline{Gh})_i (K_i)} \right) \frac{d^4 w_i}{dx^4} - P_i \frac{d^2 w_i}{dx^2} = 0 \quad (6.12)$$

$$\Rightarrow \frac{d^4 w_i}{dx^4} + \left(\frac{\frac{(\overline{EI})_i}{1 - \frac{P_i}{(\overline{Gh})_i (K_i)}}}{P_i} \right) \frac{d^2 w_i}{dx^2} = 0 \quad (6.13)$$

The general solution for equation (6.13) turns out to be:

$$w_i = A_i \sin \left[\left(\frac{\frac{P_i}{(\overline{EI})_i}}{1 - \frac{P_i}{(\overline{Gh})_i(K_i)}} \right)^{\frac{1}{2}} x \right] + B_i \cos \left[\left(\frac{\frac{P_i}{(\overline{EI})_i}}{1 - \frac{P_i}{(\overline{Gh})_i(K_i)}} \right)^{\frac{1}{2}} x \right] + C_i x + D_i \quad (6.14)$$

From equation (6.10), making w_i the subject, one can achieve the following relation:

$$\frac{d^2 w_i}{dx^2} = - \frac{1}{\left(1 - \frac{P_i}{(\overline{Gh})_i(K_i)} \right)} \frac{d\phi_i}{dx} \quad (6.15)$$

Inserting the above expression into equation (6.7) yields:

$$(\overline{EI})_i \frac{d^3 \phi_i}{dx^3} - (\overline{Gh})_i(K_i) \left(\frac{d\phi_i}{dx} - \frac{1}{\left(1 - \frac{P_i}{(\overline{Gh})_i(K_i)} \right)} \frac{d\phi_i}{dx} \right) = 0 \quad (6.16)$$

$$\Rightarrow (\overline{EI})_i \frac{d^3 \phi_i}{dx^3} + \left(-(\overline{Gh})_i(K_i) + \frac{(\overline{Gh})_i(K_i)}{\left(1 - \frac{P_i}{(\overline{Gh})_i(K_i)} \right)} \right) \frac{d\phi_i}{dx} = 0 \quad (6.17)$$

$$\Rightarrow (\overline{EI})_i \frac{d^3 \phi_i}{dx^3} + \left(\frac{-(\overline{Gh})_i(K_i) + P_i + (\overline{Gh})_i(K_i)}{\left(1 - \frac{P_i}{(\overline{Gh})_i(K_i)} \right)} \right) \frac{d\phi_i}{dx} = 0 \quad (6.18)$$

$$\therefore \frac{d^3 \phi_i}{dx^3} + \left(\frac{\frac{P_i}{(\overline{EI})_i}}{\left(1 - \frac{P_i}{(\overline{Gh})_i(K_i)} \right)} \right) \frac{d\phi_i}{dx} = 0 \quad (6.19)$$

The general solution for the differential equation above can be written as:

$$\begin{aligned}
\phi_i = & -A_i \left(\frac{P_i}{(\overline{EI})_i} \right) \left(\frac{\frac{P_i}{(\overline{EI})_i}}{1 - \frac{P_i}{(\overline{Gh})_i(K_i)}} \right)^{-\frac{1}{2}} \cos \left[\left(\frac{\frac{P_i}{(\overline{EI})_i}}{1 - \frac{P_i}{(\overline{Gh})_i(K_i)}} \right)^{\frac{1}{2}} x + B_i \left(\frac{P_i}{(\overline{EI})_i} \right) \right. \\
& \times \left. \left(\frac{\frac{P_i}{(\overline{EI})_i}}{1 - \frac{P_i}{(\overline{Gh})_i(K_i)}} \right)^{-\frac{1}{2}} \sin \left[\left(\frac{\frac{P_i}{(\overline{EI})_i}}{1 - \frac{P_i}{(\overline{Gh})_i(K_i)}} \right)^{\frac{1}{2}} x - C_i x \right] \right.
\end{aligned} \tag{6.20}$$

Here, for the individual i th beam, a parameter will be introduced as:

$$\mathcal{G}_i = \left(\frac{\frac{P_i}{(\overline{EI})_i}}{1 - \frac{P_i}{(\overline{Gh})_i(K_i)}} \right)^{\frac{1}{2}} \tag{6.21}$$

Inserting the above parameter in equations (6.14) and (6.20) condenses these bulky general solutions and their derivatives into:

$$w_i(x) = A_i \sin(\mathcal{G}_i x) + B_i \cos(\mathcal{G}_i x) + C_i x + D_i \tag{6.22}$$

$$\frac{dw_i(x)}{dx} = A_i \mathcal{G}_i \cos(\mathcal{G}_i x) - B_i \mathcal{G}_i \sin(\mathcal{G}_i x) + C_i \tag{6.23}$$

$$\phi_i(x) = -A_i \left(\frac{P_i}{(\overline{EI})_i \mathcal{G}_i} \right) \cos(\mathcal{G}_i x) + B_i \left(\frac{P_i}{(\overline{EI})_i \mathcal{G}_i} \right) \sin(\mathcal{G}_i x) - C_i x \tag{6.24}$$

$$\frac{d\phi_i(x)}{dx} = -A_i \left(\frac{P_i}{(\overline{EI})_i} \right) \sin(\mathcal{G}_i x) + B_i \left(\frac{P_i}{(\overline{EI})_i} \right) \cos(\mathcal{G}_i x) - C_i \tag{6.25}$$

Equations (6.22) to (6.25) will be applied to form the desired equations in the formulation.

6.3 Formulation

In this section, the essential set of expressions to solve for the buckling load of a moderately thick single-delaminated composite beam is addressed. These equations are established from the boundary conditions, kinematical conditions and balance of internal

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forces and moments. The internal shear forces and moments can be expressed in terms of the displacements as $Q_i = (\overline{Gh})_i (K_i) \left(\phi_i(x) + \frac{dw_i(x)}{dx} \right)$ and $M_i = (\overline{EI})_i \frac{d\phi_i(x)}{dx}$.

6.3.1 End conditions

The two kinds of end conditions to be presented in this section include the simply-supported and clamped boundary conditions.

Clamped ($6.\times\times a$)	Simply-supported ($6.\times\times b$)
$w_1(x=0)=0$	$w_1(x=0)=0$

(6.26)

$\phi_1(x=0)=0$	$M_1(x=0)=0$ $\therefore -(\overline{EI})_1 \frac{d\phi_1(x=0)}{dx} = 0$
-----------------	---

(6.27)

$w_4(x=L)=0$	$w_4(x=L)=0$
--------------	--------------

(6.28)

$\phi_4(x=L)=0$	$M_4(x=L)=0$ $\therefore -(\overline{EI})_4 \frac{d\phi_4(x=L)}{dx} = 0$
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(6.29)

6.3.2 Kinematical continuity expressions

Figure 6. 1: Kinematical compatibility at $x = \frac{L}{2} - \frac{a}{2} + d$

In FSDT, the transverse normal that is initially straight and at right angle to the neutral axis remains straight but not perpendicular to the neutral axis after deformation. It is a

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relaxation of the classical theory. This amounts to including transverse shear strain in the theory. Figure 6.1 depicts rotation of the transverse normal after bending at the left delamination tip where $x = \frac{L}{2} - \frac{a}{2} + d$. Moreover, as the displacement w_i does not depend on the z -direction, the transverse normal is inextensible. With these, the following compatibility equations can be arrived:

$$w_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = w_2\left(x = \frac{L}{2} - \frac{a}{2} + d\right)$$

$$\therefore w_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) - w_2\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = 0 \quad (6.30)$$

$$w_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = w_3\left(x = \frac{L}{2} - \frac{a}{2} + d\right)$$

$$\therefore w_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) - w_3\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = 0 \quad (6.31)$$

$$\phi_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = \phi_2\left(x = \frac{L}{2} - \frac{a}{2} + d\right)$$

$$\therefore \phi_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) - \phi_2\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = 0 \quad (6.32)$$

$$\phi_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = \phi_3\left(x = \frac{L}{2} - \frac{a}{2} + d\right)$$

$$\therefore \phi_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right) - \phi_3\left(x = \frac{L}{2} - \frac{a}{2} + d\right) = 0 \quad (6.33)$$

Figure 6.2: Kinematical compatibility at $x = \frac{L}{2} + \frac{a}{2} + d$

Analogously, from Figure 6.2, the following continuity expressions at $x = \frac{L}{2} + \frac{a}{2} + d$ can

be realized:

$$w_2\left(x = \frac{L}{2} + \frac{a}{2} + d\right) = w_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right)$$

$$\therefore w_2\left(x = \frac{L}{2} + \frac{a}{2} + d\right) - w_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right) = 0 \quad (6.34)$$

$$w_3\left(x = \frac{L}{2} + \frac{a}{2} + d\right) = w_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right)$$

$$\therefore w_3\left(x = \frac{L}{2} + \frac{a}{2} + d\right) - w_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right) = 0 \quad (6.35)$$

$$\phi_2\left(x = \frac{L}{2} + \frac{a}{2} + d\right) = \phi_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right)$$

$$\therefore \phi_2\left(x = \frac{L}{2} + \frac{a}{2} + d\right) - \phi_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right) = 0 \quad (6.36)$$

$$\phi_3\left(x = \frac{L}{2} + \frac{a}{2} + d\right) = \phi_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right)$$

$$\therefore \phi_3\left(x = \frac{L}{2} + \frac{a}{2} + d\right) - \phi_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right) = 0 \quad (6.37)$$

Figure 6. 3: Deformations due to bending of beams 2 and 3

Figure 6.3 depicts a buckled moderately thick single-delaminated composite beam. Basically, the lines and variables in Figure 6.3 have the same meaning as those in Figure 5.3, except for the angle of rotation of the transverse normals at the delamination tips.

Here, the transverse normals turn by ϕ_i instead of $\frac{dw_i}{dx}$. Again, it is taken that the axial

deformation formula can be utilized to determine the change in the length of the beams 2 and 3 due to bending. Hence the following expression can be written for beam 2:

$$\frac{\Delta P_2 a}{H_2 E_2} = \Delta_2^a + \Delta_2^b \quad (6.38)$$

$$\therefore \frac{\Delta P_2 a}{H_2 E_2} = (\bar{Y}_2 - \bar{Y}) \left(-\phi_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) \right) + (\bar{Y}_2 - \bar{Y}) \left(\phi_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) \right) \quad (6.39)$$

From Figure 6.3, it is clear that ΔP_2 has a positive value since beam 2 is under tension. Thus, the left hand side of the above equation will give the absolute value for the amount of change in beam 2's length. On the right side of equation (6.39), the term $(\bar{Y}_2 - \bar{Y})$ will definitely be positive as the neutral axis of beam 2 is further away from the top surface

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than beam 1. Hence, $\bar{Y}_2 \succ \bar{Y}$. The positive convention for the rotation of the transverse normal ϕ_i will follow the right-hand-rule (curling from z - to x - axis) and happens to be counter-clockwise. It is due to how the x - and z - axes are defined and the displacement functions selected. Therefore, from Figure 6.3, one can conclude that $\phi_1\left(x = \frac{L}{2} - \frac{a}{2} + d\right)$ will be a negative number while $\phi_4\left(x = \frac{L}{2} + \frac{a}{2} + d\right)$ is a positive value.

Hence, a minus sign must be added to the first product on the right hand side of equation (6.39) to produce a positive figure. In contrast, there is no need to include a minus sign to the second product on the right side of the same expression as the product is already positive. Then, these two positive resultant products can be added to attain the desired total absolute change in span of beam 2 and equate to the absolute solution found via the axial deformation formula to achieve equation (6.39). Rearranging equation (6.39) yields:

$$\frac{\Delta P_2 a}{H_2 E_2} + (\bar{Y}_2 - \bar{Y}) \left(\phi_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) - \phi_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) \right) = 0 \quad (6.40)$$

Similarly, for beam 3, the following compatibility equation can be derived:

$$\frac{-\Delta P_3 a}{H_3 E_3} = \Delta_3^a + \Delta_3^b \quad (6.41)$$

$$\therefore \frac{-\Delta P_3 a}{H_3 E_3} = (\bar{Y} - \bar{Y}_3) \left(-\phi_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) \right) + (\bar{Y} - \bar{Y}_3) \left(\phi_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) \right) \quad (6.42)$$

From Figure 6.3, it can be observed that beam 3 is under compression and so ΔP_3 will be negative. Hence, a minus sign must be incorporated to the first term of equation (6.42) so as to generate a positive value for the left hand side of expression (6.42). The term $(\bar{Y} - \bar{Y}_3)$ on the right hand side of equation (6.42) provides the distance between the neutral axes of beams 1 and 3, and turns out to be positive since $\bar{Y} \succ \bar{Y}_3$. The reasons for including a negative sign in front of ϕ_1 but not ϕ_4 are identical to those for beam 2. With these in mind and equating the result from the axial deformation formula to the total change in span of beam 3 yields (6.42). Reorganizing equation (6.42) produces:

$$\frac{\Delta P_3 a}{H_3 E_3} - (\bar{Y} - \bar{Y}_3) \left(\phi_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) - \phi_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) \right) = 0 \quad (6.43)$$

Then, by equating expressions (6.40) and (6.43) one can acquire:

$$\begin{aligned}
 & \frac{\Delta P_2 a}{H_2 E_2} + (\bar{Y}_2 - \bar{Y}) \left(\phi_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) - \phi_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) \right) \\
 &= \frac{\Delta P_3 a}{H_3 E_3} - (\bar{Y} - \bar{Y}_3) \left(\phi_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) - \phi_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) \right) \\
 &\therefore \frac{\Delta P_2 a}{H_2 E_2} + (\bar{Y}_2 - \bar{Y}) \left(\phi_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) - \phi_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) \right) \\
 &- \frac{\Delta P_3 a}{H_3 E_3} + (\bar{Y} - \bar{Y}_3) \left(\phi_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) - \phi_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) \right) = 0
 \end{aligned} \tag{6.44}$$

6.3.3 Balance of forces and moments

Referring to Figure 5.5, one can obtain the subsequent important equilibrium equations at

$$x = \frac{L}{2} - \frac{a}{2} + d.$$

Employing equilibrium of forces in the x -direction at $x = \frac{L}{2} - \frac{a}{2} + d$, one can get:

$$\Delta P_2 + \Delta P_3 = 0 \tag{6.45}$$

Applying $\sum F_z = 0$ yields:

$$\begin{aligned}
 & -V_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) + V_2 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) + V_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) = 0 \\
 & \therefore (\overline{Gh})_1 (K_1) \left(\phi_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) + \frac{dw_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} \right) \\
 & - (\overline{Gh})_2 (K_2) \left(\phi_2 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) + \frac{dw_2 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} \right) \\
 & - (\overline{Gh})_3 (K_3) \left(\phi_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) + \frac{dw_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} \right) \\
 & - \left(\begin{aligned} & P_1 \frac{dw_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} - P_2 \frac{dw_2 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} \\ & - P_3 \frac{dw_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} \end{aligned} \right) = 0
 \end{aligned} \tag{6.46}$$

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Moreover, utilizing $\sum M_o = 0$, considering clockwise to be positive, brings the equation below:

$$\begin{aligned}
 & M_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) - M_2 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) - M_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right) \\
 & + (\bar{Y} - \bar{Y}_3) \Delta P_3 - (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0 \\
 & \therefore (\overline{EI})_1 \frac{d\phi_1 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} - (\overline{EI})_2 \frac{d\phi_2 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} \\
 & - (\overline{EI})_3 \frac{d\phi_3 \left(x = \frac{L}{2} - \frac{a}{2} + d \right)}{dx} + (\bar{Y} - \bar{Y}_3) \Delta P_3 - (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0
 \end{aligned} \tag{6.47}$$

Similarly, the essential expressions from $x = \frac{L}{2} + \frac{a}{2} + d$ can be acquired via Figure 5.6.

From $\sum F_z = 0$ at $x = \frac{L}{2} + \frac{a}{2} + d$, one can deduce:

$$\begin{aligned}
 & V_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) - V_2 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) - V_3 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) = 0 \\
 & \therefore (\overline{Gh})_4 (K_4) \left(\phi_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) + \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) \\
 & - (\overline{Gh})_2 (K_2) \left(\phi_2 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) + \frac{dw_2 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) \\
 & - (\overline{Gh})_3 (K_3) \left(\phi_3 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) + \frac{dw_3 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \right) \\
 & - \left(\begin{aligned} & P_4 \frac{dw_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} - P_2 \frac{dw_2 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \\ & - P_3 \frac{dw_3 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \end{aligned} \right) = 0
 \end{aligned} \tag{6.48}$$

Also, performing moment equilibrium at $x = \frac{L}{2} + \frac{a}{2} + d$ yields:

$$\begin{aligned}
 & M_2 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) + M_3 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) - M_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right) \\
 & - (\bar{Y} - \bar{Y}_3) \Delta P_3 + (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0 \\
 & \therefore (\bar{EI})_2 \frac{d\phi_2 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} + (\bar{EI})_3 \frac{d\phi_3 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} \\
 & - (\bar{EI})_4 \frac{d\phi_4 \left(x = \frac{L}{2} + \frac{a}{2} + d \right)}{dx} - (\bar{Y} - \bar{Y}_3) \Delta P_3 + (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0
 \end{aligned} \tag{6.49}$$

One can apply equations (6.26) to (6.29) (equations (6.××a) for clamped cases and expressions (6.××b) for simply-supported scenarios) in conjunction with expressions (6.30) through (6.37) and equations (6.44) to (6.49) to attain the desired critical load of the moderately thick single-delaminated composite beam. Then, equations (6.22) to (6.25) can be input into the aforementioned expressions for individual boundary conditions and create a matrix:

$$[M]\{C\} = \{0\} \tag{6.50}$$

The $[M]$ is a 18×18 square matrix and $\{C\}$ is a 18×1 vector. The vector $\{C\}$ is composed of the unknowns A_i, B_i, C_i and D_i for $i = 1..4$ and the incremental forces ΔP_2 and ΔP_3 . In order to achieve non-trivial solutions, the matrix $[M]$ must vanish and the lowest eigenvalue provides the longest critical load.

6.4 Results and discussion

In this section, the three different methods of evaluating individual sets of ${}^j(\bar{EI})$, ${}^j(\bar{Gh})$ and jK for $j = 1..3$ presented in chapter four will be applied in conjunction with the mathematical model developed in the previous section to predict the buckling load of a moderately thick delaminated composite beam. Again, the finite element method will be utilized to verify the analytical solutions. The composite materials that are used in the investigation together with their material properties are listed in Table 5.1.

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The delaminated composite beam's length, width and thickness per layer are kept as 100mm, 20mm and 0.125mm, respectively. However, unlike the former chapter, the moderately thick delaminated composite beam in this chapter is composed of 48 plies and thus much thicker. The boundary conditions on the moderately thick delaminated beam include clamped and simply-supported edges. Moreover, several other factors that may affect the critical load such as through-the-depth position of the delamination, delamination span, layer orientation and stacking sequence are examined.

(a/L)	$(H_3/H) = 0.10$		$(H_3/H) = 0.30$		$(H_3/H) = 0.50$	
	Present	Refs. [10] and [52]	Present	Refs. [10] and [52]	Present	Refs. [10] and [52]
0.025	0.83333	0.83333	0.83334	0.83333	0.83333	0.83333
0.10	0.81934	0.81934	0.83321	0.83322	0.83324	0.83324
0.20	0.23767	0.23767	0.82804	0.82804	0.83025	0.83025
0.30	0.10853	0.10853	0.73249	0.73249	0.80806	0.80806
0.50	0.03964	0.03968	0.32440	0.32439	0.60599	0.60599
0.70	0.02033	0.02032	0.17408	0.17408	0.39677	0.39677
0.90	0.01234	0.01231	0.10810	0.10811	0.27705	0.27705

Table 6.1: Comparison of current results with previously published data from Chen [10] and Moradi and Taheri [52] for various through-the-thickness position and length of delamination

Figure 6.4: Comparison of present readings with published ones from Moradi and Taheri [52] for a composite beam with a delamination at the mid-plane for different delamination length and along-the-span location of the delamination

Before going to the actual analysis, it is important to verify the analytical model formulated in the previous section. To do this, the readings gathered from the current mathematical model are compared with those in the existing literature. The articles that are written by Chen [10] and Moradi and Taheri [52] are chosen for this validation. In these papers, a single-delaminated specially orthotropic beam axially loaded between two clamped edges is studied. The data generated via the present analytical model is addressed in Table 6.1 and Figure 6.4. These values are obtained by normalizing the buckling load computed from the mathematical model with the critical load calculated utilizing the standard Euler's formula for a clamped column. It is found that the results attained from the current analytical model are in very good agreement with the published ones, hence validating the present analytical model.

6.4.1 Carbon/epoxy composite material

In the following sections 6.4.1.1 and 6.4.1.2, moderately thick single-delaminated anti-symmetric angle-ply and general orientated composite beams will be studied, respectively. Both classes of beams will be made of carbon/epoxy composite material. Its material properties are shown in Table 5.1.

6.4.1.1 Anti-symmetric angle-ply composite beam with a delamination

In this section, 48 layers anti-symmetric angle-ply single delaminated composite beam is examined. Parametric studies are conducted to investigate how factors like the ply orientation, delamination length, through-the-thickness location of the delamination and edge conditions influence the buckling load.

(a)

(b)

(c)

Figure 6.5: Critical loads of moderately thick $[(-\theta/\theta)_{12} // (-\theta/\theta)_{12}]$ composite beam with a delamination of (a) 10mm, (b) 50mm and (c) 80mm long held via clamped edges

(a)

(b)

(c)

Figure 6.6: Critical loads of moderately thick $\left[(-\theta/\theta)_{12} // (-\theta/\theta)_{12}\right]$ composite beam with a delamination of (a) 10mm, (b) 50mm and (c) 80mm long held via simply-supported edges

(a)

(b)

(c)

Figure 6.7: Critical loads of moderately thick $\left[(-\theta/\theta)_6/(-\theta/\theta)_{18}\right]$ composite beam with a delamination of (a) 10mm, (b) 50mm and (c) 80mm long held via clamped edges

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The three methods in Figures 6.5 till 6.7 refer to the different techniques of computing the three individual sets of values for $(\overline{EI})$, $(\overline{Gh})$ and K . The relevant equations to calculate $(\overline{EI})$, $(\overline{Gh})$ and K for each method are addressed earlier in section 6.2. Therefore, the analytical solution plots in the above diagrams are obtained by incorporating these three separate groups of numbers for $(\overline{EI})$, $(\overline{Gh})$ and K into the mathematical model established in section 6.3. The delamination length chosen for the investigation include 10mm, 50mm and 80mm. From the graphs in Figures 6.5 to 6.7, it can be deduced that the critical load of the moderately thick delaminated composite beam drops as the delamination length and ply-angle increase.

Figure 6.5 (a) to (c) depicts the buckling load of $[(-\theta/\theta)_{12} // (-\theta/\theta)_{12}]$ anti-symmetric angle-ply delaminated composite beam with clamped edges. This composite beam composes of 48 plies and the delamination exists at the mid-plane. From Figure 6.5 (a) to (c), it can be seen that the mathematical results acquired from methods 2 and 3 are able to give the wanted upper and lower bounds for the finite element method readings, respectively. This observation tallies with the previous inspection in section 4.4.2 on moderately thick perfect laminated beam. From Figure 6.5, it may appear that the results attained via methods 1 and 2 are the same. However, a comparison of their solutions with the finite element method data reveals that method 2 yields values that are closer to the actual buckling load than method 1. Hence, one can conclude that method 2 is a better mean of obtaining the desired upper limit over method 1. Nonetheless, the results determined from method 1 will be kept for the rest of this chapter to act as a reference.

Figure 6.6 (a) to (c) illustrates the variation of buckling load of a single-delaminated moderately thick laminated beam with ply-angle under simply-supported boundary condition. Similarly, in this case, all the finite element method values are enclosed by the higher and lower limits that are created from methods 2 and 3, respectively. In addition, it is found that the finite element method points tend to fall much closer to the lower bound than the upper limit. This kind of trend is also observed on the moderately thick ideal composite beam in section 4.4.2.

Figure 6.7 (a) to (c) demonstrates the critical load of $[(-\theta/\theta)_6 // (-\theta/\theta)_{18}]$ composite beam with clamped edges. Once more, it is noticed that the method 2 and 3 are capable of

working with the mathematical model to form the required upper and lower bounds to encompass the finite element results.

6.4.1.2 Single-delaminated composite beam with general ply orientation

Again, carbon/epoxy composite material will be utilized for the study of buckling in this section. However, in this section, the moderately thick delaminated composite beam will consist of layers with general ply orientation. All of these composite beams are taken to be clamped at both edges.

(a)

(b)

(c)

Figure 6.8: Buckling results of $[(0^\circ/\theta_1/0^\circ/-\theta_2/0^\circ/2\theta_1/0^\circ/-2\theta_2)_3]_s / [(0^\circ/\theta_1/0^\circ/-\theta_2/0^\circ/2\theta_1/0^\circ/-2\theta_2)_3]$ composite beam with (a) $\theta_1 = 22.5^\circ$ and $\theta_2 = 30^\circ$; (b) $\theta_1 = 22.5^\circ$ and $\theta_2 = 45^\circ$; and (c) $\theta_1 = 30^\circ$ and $\theta_2 = 45^\circ$ for different delamination spans

(a)

(b)

(c)

Figure 6.9: Buckling results of $[(\theta/0^\circ/90^\circ/-\theta)_6 // (\theta/0^\circ/90^\circ/-\theta)_6]$ composite beam with angle θ equals to (a) 22.5° , (b) 30° and (c) 45° for different delamination spans

(a)

(b)

(c)

Figure 6.10: Buckling of $\left[(0^\circ/{}^1\theta/-{}^1\theta/90^\circ/0^\circ/{}^2\theta/-{}^2\theta/90^\circ)_3\right]//\left[(0^\circ/{}^1\theta/-{}^1\theta/90^\circ/0^\circ/{}^2\theta/-{}^2\theta/90^\circ)_3\right]$ composite beam with (a) ${}^1\theta = 30^\circ$ and ${}^2\theta = 22.5^\circ$; (b) ${}^1\theta = 45^\circ$ and ${}^2\theta = 22.5^\circ$; and (c) ${}^1\theta = 45^\circ$ and ${}^2\theta = 30^\circ$ for different delamination spans

Figure 6.8 (a) to (c) shows the buckling behavior of the moderately thick single-delaminated composite beam with a piling sequence of $\left[(0^\circ/{}^1\theta/0^\circ/-{}^1\theta/0^\circ/{}^2\theta/0^\circ/-{}^2\theta)_3\right]//\left[(0^\circ/{}^1\theta/0^\circ/-{}^1\theta/0^\circ/{}^2\theta/0^\circ/-{}^2\theta)_3\right]$. The angles ${}^1\theta$ and ${}^2\theta$ can have values of 22.5° , 30° and 45° . Thus, this composite beam will not only include the 0° ply but also layers with two other fiber directions. It is examined that even if the moderately thick delaminated composite beam is composed of plies with dissimilar ply orientations, the methods 2 and 3 are still competent of generating the longest upper and

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lower limits for the finite element method data, respectively. A careful examination tells that method 2 produces a better upper bound than method 1 as its results are nearer to those gotten via finite element method.

The graphs in Figures 6.9 (a) till (c) and 6.10 (a) through (c) depict the variation of the critical load of $[(\theta/0^\circ/90^\circ/-\theta)_6 // (\theta/0^\circ/90^\circ/-\theta)_6]$ and $[(0^\circ/^1\theta/-^1\theta/90^\circ/0^\circ/^2\theta/-^2\theta/90^\circ)_3 // (0^\circ/^1\theta/-^1\theta/90^\circ/0^\circ/^2\theta/-^2\theta/90^\circ)_3]$ composite beams. Similarly, the unknown angles in the stacking lists can take up values of 22.5° , 30° and 45° . Therefore, these two types of moderately thick delaminated composite beam will consist of plies with four different angle orientations. The plots in Figure 6.9 and 6.10 reflect that the proposed methods 2 and 3 can still work out the wanted higher and lower limits for the buckling load predicted by the finite element method. The other observation is that for all the cases considered in this section, the buckling load of the composite beam decreases as the delamination span increases.

6.4.2 Analysis on dissimilar composite materials

Besides carbon/epoxy, other composite materials such as boron/epoxy and E-glass/epoxy are utilized in present study. Their mechanical properties are addressed in Table 5.1. The moderately thick delaminated composite beam to be inspected in this section has a piling sequence of $[(-\theta/\theta)_{12} // (-\theta/\theta)_{12}]$, where the angle θ adopts a value of 0° to 90° . Hence, the composite beam will compose of 48 layers. The delamination length is chosen to be 10mm, 50mm and 80mm long to characterize short, intermediate and long delaminations, respectively. The composite beam is tested with both clamped and simply-supported boundary conditions.

(a)

(b)

(c)

Figure 6.11: Critical loads of a clamped moderately thick $[(-\theta/\theta)_{12}/(-\theta/\theta)_{12}]$ boron/epoxy composite beam with a (a) 10mm, (b) 50mm and (c) 80mm delamination

(a)

(b)

(c)

Figure 6.12: Critical loads of a simply-supported moderately thick $[(-\theta/\theta)_{12}/(-\theta/\theta)_{12}]$ boron/epoxy composite beam with a (a) 10mm, (b) 50mm and (c) 80mm delamination

(a)

(b)

(c)

Figure 6.13: Critical loads of a clamped moderately thick $[(-\theta/\theta)_{12} // (-\theta/\theta)_{12}]$ E-glass/epoxy composite beam with a (a) 10mm, (b) 50mm and (c) 80mm delamination

(a)

(b)

(c)

Figure 6.14: Critical loads of a simply-supported moderately thick $[(-\theta/\theta)_{12} // (-\theta/\theta)_{12}]$ E-glass/epoxy composite beam with a (a) 10mm, (b) 50mm and (c) 80mm delamination

Figure 6.11 (a) to (c) and 6.12 (a) to (c) illustrate the buckling phenomena of moderately thick single-delaminated boron/epoxy composite beam under clamped and simply-supported boundary conditions, respectively. Conversely, Figure 6.13 (a) to (c) and 6.14 (a) to (c) demonstrate the critical load of moderately thick delaminated composite beam that is made of E-glass composite materials with its edges clamped and simply-supported, respectively. From Table 5.1, it is noted that the carbon/epoxy, boron/epoxy and E-glass/epoxy have wildly different E_{11}/E_{22} relations. Consequently, this put up a stringent test to see whether the methods 2 and 3 developed will still be able to construct the desired bounds. For all the cases that are examined in this section, it is found that all the finite element method results are encompassed by the upper and lower bounds produced from methods 2 and 3, respectively. This observation gives further evident that the methods 2 and 3 proposed are able to enclose a region to contain the exact buckling load of a moderately thick single-delaminated composite beam.

Out of the three composite materials analyzed, boron/epoxy appears to be the most superior composite material while E-glass/epoxy seems to be the most inferior in terms of buckling load. It is found that increase in the delamination span has a negative influence on the load carrying capacity of the composite beam. Moreover, it is observed that the finite element method points seem to locate much closer to the lower bound than the upper limit for the simply-supported cases.

6.5 Concluding remarks

In the current chapter, the buckling behavior of the moderately thick composite beam with a single delamination is examined. The transverse shear deformation is accounted for in the analysis. An analytical model is formulated to look into the buckling problem. After that, the three approaches of calculating the $(\overline{EI})$, $(\overline{Gh})$ and K can work together with the mathematical model to determine for the moderately thick single-delaminated beam. Before proceeding to further investigation, the analytical model developed is validated by comparing its results with those that are published earlier by Chen [10] and Moradi and Taheri [52]. All the moderately thick single-delaminated composite beams in the present study have the total length, overall width and thickness per ply of 100mm,

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20mm and 0.125mm, respectively. Moreover, the composite beams are all made up of 48 layers.

The anti-symmetric angle-ply carbon/epoxy composite beam is analyzed for delamination spans of 10mm, 50mm and 80mm. The delamination occurs at either the mid-plane or 12 layers below the free surface. Furthermore, both the clamped and simply-supported edge conditions are considered. It is observed that as the delamination resides deeper into the composite beam or increase in length, the buckling load of the composite beam falls. Bigger ply-angle also has a negative effect on the critical load. For all the cases that are investigated, it is found that the methods 2 and 3 are able to work in cooperation with the analytical model to form the needed upper and lower bounds for the buckling readings that are attained from the finite element method. It is also noticed that the method 2 will always yield a better upper limit than the method 1. For the simply-supported case, the results from the finite element method tend to be much nearer to the lower limit than the upper bound.

The composite beam with layers of general ply-orientation is also investigated for several delamination spans. The composite beam is manufactured from the carbon/epoxy composite material and is clamped at both ends. The delamination exists at the mid-plane. The three different forms of composite beam that are included in the study consist of $[(0^\circ/1\theta/0^\circ/-1\theta/0^\circ/2\theta/0^\circ/-2\theta)_3 // (0^\circ/1\theta/0^\circ/-1\theta/0^\circ/2\theta/0^\circ/-2\theta)_3]$, $[(\theta/0^\circ/90^\circ/-\theta)_6 // (\theta/0^\circ/90^\circ/-\theta)_6]$ and $[(0^\circ/1\theta/-1\theta/90^\circ/0^\circ/2\theta/-2\theta/90^\circ)_3 // (0^\circ/1\theta/-1\theta/90^\circ/0^\circ/2\theta/-2\theta/90^\circ)_3]$. The unknown angles adopt values of 22.5° , 30° and 45° . It is noted that the methods 2 and 3 are still competent in forming the required upper and lower bounds, respectively, for these generally orientated composite beams. Once again, it is shown that the method 2 is a better mean of obtaining the upper limit than the method 1. It is also observed that as the length of the delamination increases, the critical load of the composite beam decreases.

Other composite matters like the boron/epoxy and E-glass/epoxy are also examined. For these composite materials, the clamped and simply-supported $[(-\theta/\theta)_{12} // (-\theta/\theta)_{12}]$ composite beam is analyzed. The angle θ varies from 0° to 90° and the delamination spans are $\frac{a}{L} = 0.1, 0.5$ and 0.8 . For all the cases that are studied, in spite of their wildly

dissimilar material properties, the methods 2 and 3 are still capable in providing the desired upper and lower limits, respectively. Moreover, out of the three composite materials that are tested in the present chapter, the boron/epoxy seems to yield the highest buckling load while the E-glass/epoxy offers the lowest critical load. Analogous to the observation from the carbon/epoxy composite beam, for the simply-supported cases from the other two composite matters, the readings from the finite element method appear to fall much closer to the lower bound than the upper limit.

CHAPTER SEVEN:**BUCKLING BEHAVIOR OF THIN COMPOSITE BEAM
HAVING SEMI-EMBEDDED DELAMINATIONS WITH
AND WITHOUT CONTACT IN DELAMINATED ZONE**

In the present chapter, two original mathematical models are derived to investigate the buckling performance of the thin composite beam containing the semi-embedded delamination with and without the contact effect in the delaminated region. The analytical model that does not account for the contact effect is known as the contactless model while the mathematical model that caters for the contact analysis is called the contact-integrated model. The two approaches of computing the effective bending rigidity of the composite beam ${}^2(\overline{EI})$ and the novel ${}^3(\overline{EI})$ in section 4.3.1.1 can then be used with the two models to work out the critical loads for cases that include or exclude interpenetration of sub-laminates in the delaminated sector. The $[-\theta/\theta/-\theta/(\theta/-\theta)_2/\theta/(-\theta/\theta)_4]$, $[(-\theta/\theta)_2/-\theta//\theta/-\theta/\theta/(-\theta/\theta)_4]$ and $[(-\theta/\theta)_3//-\theta/\theta/(-\theta/\theta)_4]$ beams to be examined are referred to as configurations 1, 2 and 3, respectively. 16 plies of specially orthotropic and anti-symmetric angle ply carbon/epoxy composite beams are analyzed. Other studies include boundary conditions, delaminations lengths and through-the-depth position of the delaminations.

7.1 Configuration depiction

Figure 7.1: Composite beam having semi-embedded delaminations

Similar to chapter five, the current chapter will be devoted to investigation of no other than thin delaminated fiber-reinforced composite beam. Hence, only the CLT covered in section 4.3.1 of chapter four will be applied to look into the buckling phenomena in this chapter. The above diagram illustrates a thin fiber-reinforced composite beam with semi-embedded delaminations. This delaminated composite beam is compressed by a pair of axial loads P . Its total height and span are denoted by H and L , respectively. Within the composite beam, the lower delamination has a length of a_1 while the upper one has a span of a_2 . The symbol d designates the distance measuring from the center of the composite beam to the mid-span of the lower delamination. The two delaminations split the composite beam into six portions. These individual segments and their respective heights are shown in Figure 7.1. Analogous to chapter five, the x - and w - axes indicate the spanwise position along the composite beam and transverse deflection of the composite beam, respectively.

7.2 Essential equations

In this chapter, an analytical model is established to examine the buckling load of a thin composite beam with semi-embedded delaminations. In addition, a novel approach is developed to incorporate a pressure-contact model that will be explained shortly, into the analytical formulation of the thin composite beam with semi-embedded delaminations to

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prevent interpenetration in the delaminated regions and determine the buckling load exactly. Therefore, two mathematical models will be set up in this chapter. From this point onwards, the model that does not include the pressure-contact model will be labeled as “contactless model” while the analytical model that comprises the pressure-contact model will be identified as “contact-integrated model”.

When dealing with the contactless model, the governing differential equations for the separate beams in Figure 7.1 will appear as equation (5.3). The general solution for differential equation (5.3) emerges as expression (5.10), thus, equations (5.10) till (5.13) can be applied to the contactless model to compute the desired deflection, slope, moment and shear force in the formulation in the later section. Parallel to chapter five, the

parameter λ_i occurring in the above-mentioned equations is defined as $\lambda_i = \sqrt{\frac{P_i}{(EI)_i}}$ in this chapter, for $i = 1..6$.

$$\text{Hence, for beam 1, one can have } \lambda_1 = \sqrt{\frac{P_1}{(EI)_1}} \quad (7.1)$$

$$\text{For beam 2, } \lambda_2 = \sqrt{\frac{P_2}{(EI)_2}} = \lambda_1 \sqrt{\frac{(EI)_1 (E_2 H_2)}{(EI)_2 (E_2 H_2 + E_3 H_3 + E_4 H_4)}} \quad (7.2)$$

$$\text{For beam 3, } \lambda_3 = \sqrt{\frac{P_3}{(EI)_3}} = \lambda_1 \sqrt{\frac{(EI)_1 (E_3 H_3)}{(EI)_3 (E_2 H_2 + E_3 H_3 + E_4 H_4)}} \quad (7.3)$$

$$\text{For beam 4, } \lambda_4 = \sqrt{\frac{P_4}{(EI)_4}} = \lambda_1 \sqrt{\frac{(EI)_1 (E_4 H_4)}{(EI)_4 (E_2 H_2 + E_3 H_3 + E_4 H_4)}} \quad (7.4)$$

$$\text{For beam 5, } \lambda_5 = \sqrt{\frac{P_5}{(EI)_5}} = \lambda_1 \sqrt{\frac{(EI)_1 (E_5 H_5)}{(EI)_5 (E_2 H_2 + E_5 H_5)}} \quad (7.5)$$

$$\text{For beam 6, } \lambda_6 = \sqrt{\frac{P_6}{(EI)_6}} = \lambda_1 \quad (7.6)$$

Figure 7.2: Buckling mode shapes of 16 plies 100mm long clamped (a) configuration 1, (b) configuration 2 and (c) configuration 3 composite beams with delamination lengths of $a_1 = 50\text{mm}$ and $a_2 = 30\text{mm}$, not taking account of the contact effect

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In this chapter, three different configurations of thin composite beam with semi-embedded delaminations are investigated. These stacking sequences $[-\theta/\theta/-\theta//(\theta/-\theta)_2/\theta//(-\theta/\theta)_4]$, $[(-\theta/\theta)_2/-\theta//\theta/-\theta/\theta//(-\theta/\theta)_4]$ and $[(-\theta/\theta)_3//-\theta/\theta//(-\theta/\theta)_4]$ will be known as configurations 1, 2 and 3 throughout this chapter, respectively. Before carrying out the buckling analysis on the thin composite beam with semi-embedded delaminations utilizing the contact-integrated model, one must first acquire the buckling mode shape of the same composite beam employing the contactless model. It is because from this buckling mode shape, one can then know whether any interpenetration of sub-laminates in the delaminated region occurs and recognize which sub-beam penetrates which other. Whenever interpenetration is found to arise, the contact-integrated model should be employed to prevent this physically impermissible event and obtain the more realistic critical load. Figure 7.2 (a) to (c) demonstrates the buckling mode shapes of configurations 1, 2 and 3 attained via the contactless model, respectively. Clearly, from Figure 7.2 (a), it can be seen that there is no interpenetration of the sub-beams in the delaminated zone as beam 4 buckles the most, followed by beam 3 and then beam 2 for configuration 1. On the other hand, for configurations 2 and 3 (Figure 7.2 (b) and (c), respectively), it is observed that the beam 3 which is sandwiched between beams 2 and 4 penetrates beam 4. It is also noticed that configuration 3 suffers a more severe sub-laminates interpenetration than configuration 2. The reason may be that the beam 3 is much thinner than beam 4 in configuration 3 as compared with configuration 2. The consequence is that the beam 3 in configuration 3 will deflect much more than beam 4, resulting in a more severe interpenetration than configuration 2. The same observations can also be seen in another set of buckling mode shapes depicted in Appendix G. Thus, one can conclude that the configuration 1 should use the contactless model to solve its buckling problem. In contrast, both configurations 2 and 3 will need the contact-integrated model to work out their critical loads. In addition, employing the contactless model, one can tell that configuration 3 bears a more serious interpenetration than configuration 2.

Figure 7.3: Pressure on beams 3 and 4 as a result of contact between the two beams

As an outcome of resisting interpenetration between beams 3 and 4 in configurations 2 and 3, pressure due to contact will be set up on the top and bottom surfaces of beams 3 and 4, respectively. This pressure is denoted as p_c and it is illustrated in Figure 7.3. For both the contactless and contact-integrated model, the buckling governing differential equations for beams 3 and 4 can be written in a general form as:

$$\begin{aligned} & (\overline{EI})_3 \frac{d^4 w_3}{dx^4} + P_3 \frac{d^2 w_3}{dx^2} = p_c \\ \Rightarrow & (\overline{EI})_3 \frac{d^4 w_3}{dx^4} + P_3 \frac{d^2 w_3}{dx^2} = H[\mathfrak{N} = (w_4 - w_3)] \times \frac{2E_T}{H_3 + H_4} (w_4 - w_3) \end{aligned} \quad (7.7)$$

$$\begin{aligned} & (\overline{EI})_4 \frac{d^4 w_4}{dx^4} + P_4 \frac{d^2 w_4}{dx^2} = -p_c \\ \Rightarrow & (\overline{EI})_4 \frac{d^4 w_4}{dx^4} + P_4 \frac{d^2 w_4}{dx^2} = -H[\mathfrak{N} = (w_4 - w_3)] \times \frac{2E_T}{H_3 + H_4} (w_4 - w_3) \end{aligned} \quad (7.8)$$

Analogous to the work by Andrews et al. [4], the last term in equations (7.7) and (7.8) is used to portray the pressure (p_c) due to contact between beams 3 and 4 and it is referred to as the pressure-contact model. In this term, the E_T is the through-thickness modulus of the material and $H[\mathfrak{N}]$ is the Heaviside step function. The Heaviside step function is introduced to ensure that the pressure-contact model will resist only interpenetration but

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not opening between the beam segments. The Heaviside step function has the following property:

$$H[\mathfrak{N}] = \begin{cases} 0, \mathfrak{N} < 0 \\ 1, \mathfrak{N} > 0 \end{cases} \quad (7.9)$$

Thus, for configuration 1 that does not have any interpenetration, equations (7.7) and (7.8) will have $H[\mathfrak{N}] = 0$ (contactless model). As the solving methodology for the contactless model has been discussed earlier in the second paragraph of the current section, the focus now will be shifted to configurations 2 and 3. For these two configurations, the $H[\mathfrak{N}]$ in equations (7.7) and (7.8) will be equal to 1 since they contain interpenetration and must employ the contact-integrated model to resolve their buckling problem. Hence, for configurations 2 and 3, the governing differential equations for beams 3 and 4 will turn out to be:

$$(\overline{EI})_3 \frac{d^4 w_3}{dx^4} + P_3 \frac{d^2 w_3}{dx^2} = \frac{2E_T}{H_3 + H_4} (w_4 - w_3) \quad (7.10)$$

$$\therefore \frac{d^4 w_3}{dx^4} + \lambda_3^2 \frac{d^2 w_3}{dx^2} = \frac{2E_T}{(\overline{EI})_3 (H_3 + H_4)} (w_4 - w_3) \quad (7.11)$$

$$(\overline{EI})_4 \frac{d^4 w_4}{dx^4} + P_4 \frac{d^2 w_4}{dx^2} = -\frac{2E_T}{H_3 + H_4} (w_4 - w_3) \quad (7.12)$$

$$\therefore \frac{d^4 w_4}{dx^4} + \lambda_4^2 \frac{d^2 w_4}{dx^2} = -\frac{2E_T}{(\overline{EI})_4 (H_3 + H_4)} (w_4 - w_3) \quad (7.13)$$

Rearranging equation (7.11), making w_4 the subject gives:

$$w_4 = \frac{(\overline{EI})_3 (H_3 + H_4)}{2E_T} \frac{d^4 w_3}{dx^4} + \frac{\lambda_3^2 (\overline{EI})_3 (H_3 + H_4)}{2E_T} \frac{d^2 w_3}{dx^2} + w_3 \quad (7.14)$$

The second and fourth derivatives of w_4 are:

$$\frac{d^2 w_4}{dx^2} = \frac{(\overline{EI})_3 (H_3 + H_4)}{2E_T} \frac{d^6 w_3}{dx^4} + \frac{\lambda_3^2 (\overline{EI})_3 (H_3 + H_4)}{2E_T} \frac{d^4 w_3}{dx^2} + \frac{d^2 w_3}{dx^2} \quad (7.15)$$

$$\frac{d^4 w_4}{dx^4} = \frac{(\overline{EI})_3 (H_3 + H_4)}{2E_T} \frac{d^8 w_3}{dx^8} + \frac{\lambda_3^2 (\overline{EI})_3 (H_3 + H_4)}{2E_T} \frac{d^6 w_3}{dx^6} + \frac{d^4 w_3}{dx^4} \quad (7.16)$$

Inputting expressions (7.14) till (7.16) above into equation (7.13) yields:

$$\begin{aligned} & \left(\frac{(\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^8 w_3}{dx^8} + \frac{\lambda_3^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^6 w_3}{dx^6} + \frac{d^4 w_3}{dx^4} \right) \\ & + \lambda_4^2 \left(\frac{(\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^6 w_3}{dx^4} + \frac{\lambda_3^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^4 w_3}{dx^2} + \frac{d^2 w_3}{dx^2} \right) \end{aligned} \quad (7.17)$$

$$\begin{aligned} & = - \frac{2E_T}{(\overline{EI})_4(H_3 + H_4)} \left(\frac{(\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^4 w_3}{dx^4} + \frac{\lambda_3^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^2 w_3}{dx^2} + w_3 - w_3 \right) \\ & \Rightarrow \frac{(\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^8 w_3}{dx^8} + \frac{\lambda_3^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^6 w_3}{dx^6} + \frac{d^4 w_3}{dx^4} \\ & + \frac{\lambda_4^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^6 w_3}{dx^4} + \frac{\lambda_3^2 \lambda_4^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^4 w_3}{dx^2} + \lambda_4^2 \frac{d^2 w_3}{dx^2} \\ & = - \frac{(\overline{EI})_3}{(\overline{EI})_4} \frac{d^4 w_3}{dx^4} - \frac{\lambda_3^2 (\overline{EI})_3}{(\overline{EI})_4} \frac{d^2 w_3}{dx^2} \end{aligned} \quad (7.18)$$

$$\begin{aligned} & \therefore \frac{(\overline{EI})_3(H_3 + H_4)}{2E_T} \frac{d^8 w_3}{dx^8} + \left(\frac{\lambda_3^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} + \frac{\lambda_4^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} \right) \frac{d^6 w_3}{dx^6} \\ & + \left(1 + \frac{\lambda_3^2 \lambda_4^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} + \frac{(\overline{EI})_3}{(\overline{EI})_4} \right) \frac{d^4 w_3}{dx^4} + \left(\lambda_4^2 + \frac{\lambda_3^2 (\overline{EI})_3}{(\overline{EI})_4} \right) \frac{d^2 w_3}{dx^2} = 0 \end{aligned} \quad (7.19)$$

The characteristic equation of differential equation (7.19) is:

$$\begin{aligned} & \therefore \frac{(\overline{EI})_3(H_3 + H_4)}{2E_T} m^8 + \left(\frac{\lambda_3^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} + \frac{\lambda_4^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} \right) m^6 \\ & + \left(1 + \frac{\lambda_3^2 \lambda_4^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} + \frac{(\overline{EI})_3}{(\overline{EI})_4} \right) m^4 + \left(\lambda_4^2 + \frac{\lambda_3^2 (\overline{EI})_3}{(\overline{EI})_4} \right) m^2 = 0 \end{aligned} \quad (7.20)$$

From the Maple mathematical software, the roots for the above expression are found to be $0, 0, \pm ni, p \pm qi$ and $s \pm ti$. It should be noted that in equation (7.20), the parameters $(\overline{EI})_3, (\overline{EI})_4, E_T, H_3$ and H_4 are all known while λ_3 and λ_4 can be stated in terms of λ_1 . Thus, the n, p, q, s and t appearing in the above-mentioned solution roots will contain λ_1 as the only unknown. Therefore, the general solution for differential equation (7.19) can be deduced as:

$$w_3(x) = A_{3;4} \cos(nx) + B_{3;4} \sin(nx) + e^{px} (C_{3;4} \cos(qx) + D_{3;4} \sin(qx)) + e^{sx} (E_{3;4} \cos(tx) + F_{3;4} \sin(tx)) + G_{3;4}x + H_{3;4} \quad (7.21)$$

The subscript “3;4” is used on the unknown coefficients of the above expression because equation (7.21) is the general solution to the differential equation (7.19), which is in turn produced from the two coupled governing differential equations for beams 3 and 4 (equations 7.11) and (7.13)). Hence, one can have:

$$\begin{aligned} \frac{dw_3(x)}{dx} &= -A_{3;4}n \sin(nx) + B_{3;4}n \cos(nx) \\ &+ e^{px} ((pC_{3;4} + qD_{3;4}) \cos(qx) + (pD_{3;4} - qC_{3;4}) \sin(qx)) \\ &+ e^{sx} ((sE_{3;4} + tF_{3;4}) \cos(tx) + (sF_{3;4} - tE_{3;4}) \sin(tx)) + G_{3;4} \end{aligned} \quad (7.22)$$

$$\begin{aligned} \frac{d^2w_3(x)}{dx^2} &= -A_{3;4}n^2 \cos(nx) + B_{3;4}n^2 \sin(nx) \\ &+ e^{px} ((p^2C_{3;4} + 2pqD_{3;4} - q^2C_{3;4}) \cos(qx) + (p^2D_{3;4} - 2pqC_{3;4} - q^2D_{3;4}) \sin(qx)) \\ &+ e^{sx} ((s^2E_{3;4} + 2stF_{3;4} - t^2E_{3;4}) \cos(tx) + (s^2F_{3;4} - 2stE_{3;4} - t^2F_{3;4}) \sin(tx)) \end{aligned} \quad (7.23)$$

$$\begin{aligned} \frac{d^3w_3(x)}{dx^3} &= A_{3;4}n^3 \sin(nx) - B_{3;4}n^3 \cos(nx) \\ &+ e^{px} \left((p^3C_{3;4} + 3p^2qD_{3;4} - 3pq^2C_{3;4} - q^3D_{3;4}) \cos(qx) \right. \\ &\quad \left. + (p^3D_{3;4} - 3p^2qC_{3;4} - 3pq^2D_{3;4} + q^3C_{3;4}) \sin(qx) \right) \\ &+ e^{sx} \left((s^3E_{3;4} + 3s^2tF_{3;4} - 3st^2E_{3;4} - t^3F_{3;4}) \cos(tx) \right. \\ &\quad \left. + (s^3F_{3;4} - 3s^2tE_{3;4} - 3st^2F_{3;4} + t^3E_{3;4}) \sin(tx) \right) \end{aligned} \quad (7.24)$$

Equations (7.21) through (7.24) can then be applied to the analytical model that will be formulated in the next section to describe the deflection, slope, shear force or moment of beam 3, giving rise to the contact-integrated model. Then, the appropriate solution for w_4 can be acquired by inputting the expression for w_3 (equation (7.21)) together with its second and fourth derivatives into equation (7.14):

$$\begin{aligned}
w_4(x) = & \frac{(\overline{EI})_3(H_3 + H_4)}{2E_T} \times \\
& \left[\begin{aligned}
& A_{3;4} n^4 \cos(nx) + B_{3;4} n^4 \sin(nx) \\
& + e^{px} \left((p^4 C_{3;4} + 4p^3 q D_{3;4} - 6p^2 q^2 C_{3;4} - 4pq^3 D_{3;4} + q^4 C_{3;4}) \cos(qx) \right. \\
& \quad \left. + (p^4 D_{3;4} - 4p^3 q C_{3;4} - 6p^2 q^2 D_{3;4} + 4pq^3 C_{3;4} + q^4 D_{3;4}) \sin(qx) \right) \\
& + e^{sx} \left((s^4 E_{3;4} + 4s^3 t F_{3;4} - 6s^2 t^2 E_{3;4} - 4st^3 F_{3;4} + t^4 E_{3;4}) \cos(tx) \right. \\
& \quad \left. + (s^4 F_{3;4} - 4s^3 t E_{3;4} - 6s^2 t^2 F_{3;4} + 4st^3 E_{3;4} + t^4 F_{3;4}) \sin(tx) \right) \\
& + \frac{\lambda_3^2 (\overline{EI})_3(H_3 + H_4)}{2E_T} \times \\
& \left[\begin{aligned}
& -A_{3;4} n^2 \cos(nx) - B_{3;4} n^2 \sin(nx) \\
& + e^{px} \left((p^2 C_{3;4} + 2pq D_{3;4} - q^2 C_{3;4}) \cos(qx) + (p^2 D_{3;4} - 2pq C_{3;4} - q^2 D_{3;4}) \sin(qx) \right) \\
& + e^{sx} \left((s^2 E_{3;4} + 2st F_{3;4} - t^2 E_{3;4}) \cos(tx) + (s^2 F_{3;4} - 2st E_{3;4} - t^2 F_{3;4}) \sin(tx) \right) \\
& + A_{3;4} \cos(nx) + B_{3;4} \sin(nx) + e^{px} (C_{3;4} \cos(qx) + D_{3;4} \sin(qx)) \\
& + e^{sx} (E_{3;4} \cos(tx) + F_{3;4} \sin(tx)) + G_{3;4} x + H_{3;4}
\end{aligned} \right]
\end{aligned} \right] \quad (7.25)
\end{aligned}$$

Similarly, the above equation can be differentiated to attain the necessary deflection, slope, shear force and moment of beam 4 to be used in the contact-integrated model. For beams 1, 2, 5 and 6, as no interpenetration is spotted, equations (5.10) till (5.13) will still be employed to obtain the essential deflection, slope, shear force and moment of these beams.

In summary, this section introduces both the contactless and contact-integrated models. As the contactless model does not involve any contact model, the buckling governing equations for all the sub-beams will take in the form of equation (5.3). Thus, for all the sub-beams equations (5.10) to (5.13) can be utilized in conjunction with the formulation (in section 7.3) for this contactless model to achieve the longed critical load. The use of contactless model serves two main purposes. One of them is to solve the buckling problem of the thin composite beam with semi-embedded delaminations that does not have any interpenetration between the sub-laminates, such as configuration 1. The other function is to identify which sub-beam penetrates which other for cases that exhibits interpenetration, like configurations 2 and 3. For configurations 2 and 3, the contact-integrated model must be employed since the pressure-contact model is applied on their beams 3 and 4 to resist interpenetration between the two beams. Consequently, for

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configurations 2 and 3, equations (7.21) to (7.25) together with the first, second and third derivatives of w_4 must be utilized in the formulation (in section 7.3) to gain the required deflection, slope, shear force and moments of beams 3 and 4. However, the other beams (beams 1, 2, 5 and 6) in these two configurations will still use equations (5.10) till (5.13) in the formulation (in section 7.3) to compute their respective deflection, slope, shear force and moments.

7.3 Formulation

In this section, the vital equations in a mathematical model for a thin composite beam with semi-enclosed delaminations will be delivered. These expressions are extracted from the boundary conditions, proper kinematical compatibilities and equilibrium of forces and moments.

7.3.1 Edge conditions

In this chapter, both the clamped and simply-supported end constrains will be examined.

The equations arising from the two boundary conditions are summarized as:

<i>Clamped</i> (7.××a)	<i>Simply-supported</i> (7.××b)	
$w_1(x=0)=0$	$w_1(x=0)=0$	(7.26)
$\frac{dw_1(x=0)}{dx}=0$	$M_1(x=0)=0$ $\therefore -(\overline{EI})_1 \frac{d^2 w_1(x=0)}{dx^2}=0$	(7.27)
$w_6(x=L)=0$	$w_6(x=L)=0$	(7.28)
$\frac{dw_6(x=L)}{dx}=0$	$M_6(x=L)=0$ $\therefore -(\overline{EI})_6 \frac{d^2 w_6(x=L)}{dx^2}=0$	(7.29)

7.3.2 Kinematical constancy equations

Figure 7.4: Kinematical constancy at $x = \frac{L}{2} - \frac{a_1}{2} + d$

Figure 7.4 demonstrates the leftmost delamination fronts of a thin composite beam with semi-enclosed delaminations. As the transverse normals of the beams are inextensible and stay perpendicular to the neutral axis even after bending, the following kinematical compatibility equations can be inferred:

$$w_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right) = w_2\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)$$

$$\therefore w_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right) - w_2\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right) = 0 \quad (7.30)$$

$$w_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right) = w_3\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)$$

$$\therefore w_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right) - w_3\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right) = 0 \quad (7.31)$$

$$w_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right) = w_4\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)$$

$$\therefore w_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right) - w_4\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right) = 0 \quad (7.32)$$

$$\frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx} = \frac{dw_2\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}$$

$$\therefore \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx} - \frac{dw_2\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx} = 0 \quad (7.33)$$

$$\frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx} = \frac{dw_3\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}$$

$$\therefore \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx} - \frac{dw_3\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx} = 0 \quad (7.34)$$

$$\frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx} = \frac{dw_4\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}$$

$$\therefore \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx} - \frac{dw_4\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx} = 0 \quad (7.35)$$

Figure 7.5: Kinematical constancy at $x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d$

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Analogously, subsequent continuity equations can be deduced from Figure 7.5:

$$w_3\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right) = w_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)$$

$$\therefore w_3\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right) - w_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right) = 0 \quad (7.36)$$

$$w_4\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right) = w_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)$$

$$\therefore w_4\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right) - w_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right) = 0 \quad (7.37)$$

$$\frac{dw_3\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx} = \frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}$$

$$\therefore \frac{dw_3\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx} - \frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx} = 0 \quad (7.38)$$

$$\frac{dw_4\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx} = \frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}$$

$$\therefore \frac{dw_4\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx} - \frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx} = 0 \quad (7.39)$$

Figure 7.6: Kinematical constancy at $x = \frac{L}{2} + \frac{a_1}{2} + d$

Figure 7.6 depicts the rightmost delamination tip. From this diagram, one can realize the following kinematical compatibility expressions:

$$w_2\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right) = w_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)$$

$$\therefore w_2\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right) - w_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right) = 0 \quad (7.40)$$

$$w_5\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right) = w_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)$$

$$\therefore w_5\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right) - w_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right) = 0 \quad (7.41)$$

$$\frac{dw_2\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx} = \frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx}$$

$$\therefore \frac{dw_2\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx} - \frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx} = 0 \quad (7.42)$$

$$\frac{dw_5\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx} = \frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx}$$

$$\therefore \frac{dw_5\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx} - \frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx} = 0 \quad (7.43)$$

Figure 7.7: Shortening and extension of the sub-laminates as a result of bending

Figure 7.7 illustrates the buckled configuration of a thin composite beam having semi-embedded delaminations. The incremental forces within beams 2, 3, 4 and 5 as a consequence of bending are designated by ΔP_2 , ΔP_3 , ΔP_4 and ΔP_5 , respectively. Firstly, focusing on beam 3, one can derive the following relation from Figure 7.7:

$$-\frac{\Delta P_3 a_2}{H_3 E_3} = \Delta_3^a - \Delta_3^b \quad (7.44)$$

$$\therefore -\frac{\Delta P_3 a_2}{H_3 E_3} = \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx} (\bar{Y} - \bar{Y}_3) + \frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx} (\bar{Y}_3 - \bar{Y}_5) \quad (7.45)$$

Here, it is assumed that beam 3 is experiencing a compression. Hence, the incremental force in beam 3 ΔP_3 will be negative. Therefore, one has to add a minus sign to the left hand side of both equations (7.44) and (7.45) to attain a positive value for the amount of contraction that is suffered by beam 3. On the right hand side of equation (7.44), this

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shortening is computed via subtracting the extension at the right end of beam 3 Δ_3^b from the reduction in length at the left edge of beam 3 Δ_3^a (see Figure 7.7). As $\bar{Y}, \bar{Y}_2, \bar{Y}_3, \bar{Y}_4$ and $\bar{Y}_5$ are the distances that are measured along the transverse normals from the top surface of the composite beam to the neutral axes of beams 1 (and beam 6), 2, 3, 4 and 5, respectively, one can conclude from Figure 7.7 that $\bar{Y}_2 > \bar{Y} > \bar{Y}_3 > \bar{Y}_5 > \bar{Y}_4$. Thus, the two terms $(\bar{Y} - \bar{Y}_3)$ and $(\bar{Y}_3 - \bar{Y}_5)$ on the right hand side of expression (7.45) will both be

positive. On top of that, the term $\frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}$ is positive due to how the w -axis

is defined. Therefore, the first product on the right hand side of equation (7.45),

$\frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_3)$, will grant the desired positive value for the contraction at

the left end of beam 3. Conversely, as $\frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}$ is negative, the second

product on the right hand side of equation (7.45), $\frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}(\bar{Y}_3 - \bar{Y}_5)$, will

turn out to be negative. This explains why a plus is placed in front of this

$\frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}(\bar{Y}_3 - \bar{Y}_5)$ instead of a minus to get $-\Delta_3^b$ in equation (7.44).

For beam 5, the subsequent expression can be developed:

$$-\frac{\Delta P_5(a_1 - a_2)}{H_5 E_5} = \Delta_5 \quad (7.46)$$

$$-\frac{\Delta P_5(a_1 - a_2)}{H_5 E_5} = -\frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_5) \quad (7.47)$$

Similarly, beam 5 is taken to be in compression. Thus, a negative sign must be included on the left hand side of equation (7.47) to obtain the wanted total contraction of beam 5 in

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positive number. On the right hand side of equation (7.47), the term $(\bar{Y} - \bar{Y}_5)$ is positive

since $\bar{Y} > \bar{Y}_5$. However, the term $\frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx}$ on the right side of equation

(7.46) is negative. As a result, a minus sign must be placed in front of the product

$\frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_5)$ in order to acquire a positive value for the shortening of

beam 5 (Δ_5) which can then be equated to the left hand side which is also a positive number.

Adding up equations (7.45) and (7.47) yields the total amount of contraction received by both beams 3 and 5:

$$\begin{aligned} & -\frac{\Delta P_3 a_2}{H_3 E_3} - \frac{\Delta P_5 (a_1 - a_2)}{H_5 E_5} = \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_3) \\ & + \frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}(\bar{Y}_3 - \bar{Y}_5) - \frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_5) \\ & \therefore \frac{\Delta P_3 a_2}{H_3 E_3} + \frac{\Delta P_5 (a_1 - a_2)}{H_5 E_5} + \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_3) \\ & + \frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}(\bar{Y}_3 - \bar{Y}_5) - \frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_5) = 0 \end{aligned} \quad (7.48)$$

One can also establish the following expression for beam 4:

$$-\frac{\Delta P_4 a_2}{H_4 E_4} = \Delta_4^a + \Delta_4^b \quad (7.49)$$

$$-\frac{\Delta P_4 a_2}{H_4 E_4} = \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_4) - \frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}(\bar{Y}_5 - \bar{Y}_4) \quad (7.50)$$

For beam 4, one can also presume that the beam is under compression. Likewise, the incremental force in beam 4 ΔP_4 will be negative and a minus sign must be incorporated into the left hand side of the above two equations to secure the required overall

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shortening of beam 4 in positive value. On the right hand side of equation (7.50), both terms $(\bar{Y} - \bar{Y}_4)$ and $(\bar{Y}_5 - \bar{Y}_4)$ are positive as $\bar{Y} \succ \bar{Y}_4$ and $\bar{Y}_5 \succ \bar{Y}_4$. In this way, the first

product on the right hand side of equation (7.50), $\frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_4)$, will

contribute the desired amount of contraction at the left end of beam 4 (Δ_4^a) in positive

number. In contrast, the term $\frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}$ will produce a negative value.

Hence, a minus sign must be included to the second product on the right hand side of

equation (7.50), $\frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}(\bar{Y}_5 - \bar{Y}_4)$, in attempt to compute the longed

amount of shortening occurring at the right tip of beam 4 (Δ_4^b) in positive value.

Summing up the $\frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_4)$ and $-\frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}(\bar{Y}_5 - \bar{Y}_4)$

will then provide the wanted complete amount of contraction undertaken by beam 4.

Adding equations (7.47) and (7.50) together presents the entire shortening withstood by both beams 4 and 5:

$$\begin{aligned}
 & -\frac{\Delta P_4 a_2}{H_4 E_4} - \frac{\Delta P_5 (a_1 - a_2)}{H_5 E_5} = \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_4) \\
 & - \frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}(\bar{Y}_5 - \bar{Y}_4) - \frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_5) \\
 & \therefore \frac{\Delta P_4 a_2}{H_4 E_4} + \frac{\Delta P_5 (a_1 - a_2)}{H_5 E_5} + \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_4) \\
 & - \frac{dw_5\left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d\right)}{dx}(\bar{Y}_5 - \bar{Y}_4) - \frac{dw_6\left(x = \frac{L}{2} + \frac{a_1}{2} + d\right)}{dx}(\bar{Y} - \bar{Y}_5) = 0
 \end{aligned} \tag{7.51}$$

7.3.3 Equilibrium of forces and moments

Figure 7.8: Individual forces and moments at $x = \frac{L}{2} - \frac{a_1}{2} + d$

Figure 7.8 shows the interior forces and moments that are available at position $x = \frac{L}{2} - \frac{a_1}{2} + d$. From this drawing, one can achieve the necessary equations for the formulation through the balance of forces and moments.

Employing $\sum F_x = 0$, one can arrive at:

$$\Delta P_2 + \Delta P_3 + \Delta P_4 = 0 \quad (7.52)$$

Utilizing $\sum F_z = 0$ yields:

$$\begin{aligned} & -V_1 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right) + V_2 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right) + V_3 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right) \\ & + V_4 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right) = 0 \\ \therefore & (\overline{EI})_1 \frac{d^3 w_1 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right)}{dx^3} - (\overline{EI})_2 \frac{d^3 w_2 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right)}{dx^3} \\ & - (\overline{EI})_3 \frac{d^3 w_3 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right)}{dx^3} - (\overline{EI})_4 \frac{d^3 w_4 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right)}{dx^3} = 0 \end{aligned} \quad (7.53)$$

Moreover, from $\sum M_o = 0$, taking clockwise to be positive, one can expect:

$$\begin{aligned}
 & M_1 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right) - M_2 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right) - M_3 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right) \\
 & - M_4 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right) + (\bar{Y} - \bar{Y}_3) \Delta P_3 + (\bar{Y} - \bar{Y}_4) \Delta P_4 - (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0 \\
 & \therefore -(\bar{EI})_1 \frac{d^2 w_1 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right)}{dx^2} + (\bar{EI})_2 \frac{d^2 w_2 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right)}{dx^2} \\
 & + (\bar{EI})_3 \frac{d^2 w_3 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right)}{dx^2} + (\bar{EI})_4 \frac{d^2 w_4 \left(x = \frac{L}{2} - \frac{a_1}{2} + d \right)}{dx^2} \\
 & + (\bar{Y} - \bar{Y}_3) \Delta P_3 + (\bar{Y} - \bar{Y}_4) \Delta P_4 - (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0
 \end{aligned} \tag{7.54}$$

Figure 7.9: Internal forces and moments at $x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d$

Figure 7.9 depicts the right tip of the upper delamination. Similarly, crucial expressions for the formulation are derived from the diagram.

Carrying out force balance in the vertical direction gives:

$$\begin{aligned}
 & -V_3 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right) - V_4 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right) \\
 & + V_5 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right) = 0
 \end{aligned}$$

$$\begin{aligned} \therefore (\overline{EI})_3 \frac{d^3 w_3 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right)}{dx^3} + (\overline{EI})_4 \frac{d^3 w_4 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right)}{dx^3} \\ - (\overline{EI})_5 \frac{d^3 w_5 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right)}{dx^3} = 0 \end{aligned} \quad (7.55)$$

Taking moment equilibrium about point O produces:

$$\begin{aligned} M_3 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right) + M_4 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right) \\ - M_5 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right) + (\overline{Y}_3 - \overline{Y}_5) \Delta P_3 - (\overline{Y}_5 - \overline{Y}_4) \Delta P_4 = 0 \\ \therefore -(\overline{EI})_3 \frac{d^2 w_3 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right)}{dx^2} - (\overline{EI})_4 \frac{d^2 w_4 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right)}{dx^2} \\ + (\overline{EI})_5 \frac{d^2 w_5 \left(x = \frac{L}{2} - \frac{a_1}{2} + a_2 + d \right)}{dx^2} + (\overline{Y}_3 - \overline{Y}_5) \Delta P_3 - (\overline{Y}_5 - \overline{Y}_4) \Delta P_4 = 0 \end{aligned} \quad (7.56)$$

Figure 7.10: Inner forces and moments at $x = \frac{L}{2} + a_1 + d$

Analogously, one can attain the vital set of equation from the figure above.

Applying $\sum F_x = 0$, one can get:

$$\Delta P_2 + \Delta P_5 = 0 \quad (7.57)$$

Use of $\sum F_z = 0$ results in:

$$\begin{aligned}
& -V_2\left(x = \frac{L}{2} + a_1 + d\right) - V_5\left(x = \frac{L}{2} + a_1 + d\right) + V_6\left(x = \frac{L}{2} + a_1 + d\right) = 0 \\
& \therefore (\overline{EI})_2 \frac{d^3 w_2\left(x = \frac{L}{2} + a_1 + d\right)}{dx^3} + (\overline{EI})_5 \frac{d^3 w_5\left(x = \frac{L}{2} + a_1 + d\right)}{dx^3} \\
& - (\overline{EI})_6 \frac{d^3 w_6\left(x = \frac{L}{2} + a_1 + d\right)}{dx^3} = 0
\end{aligned} \tag{7.58}$$

Furthermore, the following expression can be obtained via $\sum M_o = 0$:

$$\begin{aligned}
& M_2\left(x = \frac{L}{2} + a_1 + d\right) + M_5\left(x = \frac{L}{2} + a_1 + d\right) - M_6\left(x = \frac{L}{2} + a_1 + d\right) \\
& + (\overline{Y}_2 - \overline{Y})\Delta P_2 - (\overline{Y} - \overline{Y}_5)\Delta P_5 = 0 \\
& \therefore -(\overline{EI})_2 \frac{d^2 w_2\left(x = \frac{L}{2} + a_1 + d\right)}{dx^2} - (\overline{EI})_5 \frac{d^2 w_5\left(x = \frac{L}{2} + a_1 + d\right)}{dx^2} \\
& + (\overline{EI})_6 \frac{d^2 w_6\left(x = \frac{L}{2} + a_1 + d\right)}{dx^2} + (\overline{Y}_2 - \overline{Y})\Delta P_2 - (\overline{Y} - \overline{Y}_5)\Delta P_5 = 0
\end{aligned} \tag{7.59}$$

Thus, equations (7.26) to (7.29) (equations (7.××a) for clamped cases and expressions (7.××b) for simply-supported scenarios), (7.30) till (7.43), (7.48) and (7.51) through (7.59) are employed in the formulation to work out the buckling solution of a thin composite beam that contains semi-embedded delaminations. As previously addressed in the last paragraph in section 7.2, all the beams in the contactless model should use expressions (5.10) to (5.13) for the above-mentioned equations to resolve buckling problems like configuration 1. On the other hand, for configurations 2 and 3 which utilize the contact-integrated model, only beams 1, 2, 5 and 6 should apply equations (5.10) till (5.13) for the aforementioned equations. Beam 3 ought to use expressions (7.21) through (7.23) while beam 4 should employ equation (7.25) and its derivatives. After applying the suitable equations for either the contactless or contact-integrated model, the 28 equations stated in the first sentence of this paragraph can be arranged in a compact matrix form as:

$$[M]\{C\} = \{0\} \quad (7.60)$$

The $[M]$ is a 28×28 square matrix that is composed of only λ_1 . The $\{C\}$ is a column vector that consists of the undetermined constants (A_i, B_i, C_i and D_i , for $i = 1..6$) and the incremental forces (ΔP_i , for $i = 2..5$). In order for the unknowns in $\{C\}$ to have valid solutions, the determinant of $[M]$ must be 0. Then, the lowest result that is obtained from resolving of the determinant of $[M]$ will provide the wanted critical load.

7.4 Findings and discussion

In this section, the buckling behavior of a thin composite beam with semi-embedded delaminations is examined. The investigation is conducted with and without the contact analysis. In the current chapter, two approaches that are covered in section 4.3.1.1 will be applied to the individual sub-laminates to compute their effective flexural stiffnesses. These two methods are designated as $^2(\overline{EI})$ and $^3(\overline{EI})$ and can be calculated employing expressions (4.54) and (4.73), respectively. Once again, the finite element method will be used as a tool to validate the analytical results. For convenience, for the rest of this chapter, “contactless simulation” will be used to imply finite element method that does not incorporate contact and “contact-integrated simulation” to stand for finite element method that includes the contact effect between the sub-laminates. The carbon/epoxy is chosen as the composite material in this analysis and its mechanical properties are presented in Table 5.1. The dimensions of the composite beam remain as 100mm, 20mm and 0.125mm for the overall composite beam span, width and thickness per layer, respectively. In this chapter, all the thin delaminated composite beams are composed of 16 plies. Both the clamped and simply-supported end conditions are studied. In addition, both specially orthotropic and anti-symmetric angle-ply type of composite beams will be explored. Further parameters to be considered include delamination lengths and through-the-thickness location of the delaminations.

7.4.1 Specially orthotropic thin composite beam with semi-embedded delaminations

In this portion, the specially orthotropic thin composite beam containing semi-embedded delaminations will be analyzed. All the three configurations that are discussed earlier will

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be studied here. It should be noted that for all the cases considered, a_1 is fixed to be 90mm while a_2 is set as a variable. Both clamped and simply-supported boundary conditions will be examined. Only ${}^3(\overline{EI})$ will be employed to acquire the effective bending modulus of the sub-laminates since ${}^2(\overline{EI})$ and ${}^3(\overline{EI})$ yield almost the same value for specially orthotropic circumstance.

(a)

(b)

Figure 7.11: Buckling load of specially orthotropic configuration 1 composite beam, taking a_1 to be 90mm and a_2 as a variable under (a) clamped and (b) simply-supported boundary conditions, without contact analysis

(a)

(b)

Figure 7.12: Buckling load of specially orthotropic configuration 2 composite beam, taking a_1 to be 90mm and a_2 as a variable under (a) clamped and (b) simply-supported boundary conditions, with and without contact analysis

Figure 7.13: Percentage difference between analytical and simulation data for specially orthotropic configuration 2 composite beam, picking a_1 to be 90mm and a_2 as a variable under clamped and simply-supported end conditions, accounting of contact

Figure 7.14: Percentage deviation between readings that are obtained by accounting and not accounting of contact from both analytical and simulation model for specially orthotropic configuration 2 composite beam, picking a_1 to be 90mm and a_2 as a variable under clamped and simply-supported end conditions

(a)

(b)

Figure 7.15: Buckling load of specially orthotropic configuration 3 composite beam, taking a_1 to be 90mm and a_2 as a variable under (a) clamped and (b) simply-supported boundary conditions, with and without contact analysis

Figure 7.16: Percentage difference between analytical and simulation data for specially orthotropic configuration 3 composite beam, picking a_1 to be 90mm and a_2 as a variable under clamped and simply-supported end conditions, accounting of contact

Figure 7.17: Percentage deviation between readings that are obtained by accounting and not accounting of contact from both analytical and simulation model for specially orthotropic configuration 3 composite beam, picking a_1 to be 90mm and a_2 as a variable under clamped and simply-supported end conditions

Figure 7.18: Buckling mode shape of simply-supported configuration 1 composite beam adopting a_1 to be 90mm and a_2 as 60mm obtained from the contactless model (see Figure 7.11 (b))

Figure 7.19: Buckling mode shapes of simply-supported configuration 2 composite beam adopting a_1 to be 90mm and a_2 as 60mm obtained from the (a) contactless and (b) contact-integrated models (see Figure 7.12 (b))

Figure 7.20: Buckling mode shapes of clamped configuration 3 composite beam adopting a_1 to be 90mm and a_2 as 60mm obtained from the (a) contactless and (b) contact-integrated models (see Figure 7.15 (b))

Figure 7.11 (a) and (b) demonstrate the critical load for the specially orthotropic configuration 1 under clamped and simply-supported edge conditions, respectively, for different $\frac{a_2}{L}$. No interpenetration is detected from the contactless simulation. This observation tallies with the earlier one that is attained via the contactless model in section 7.2. The reason for this occurrence is also given in section 7.2, in the paragraph right after equation (7.6). From Figure 7.11, it can be seen that the results obtained from the contactless model are in excellent agreement with the solution gathered from the contactless simulation.

Figure 7.12 (a) and (b) illustrate the buckling load of specially orthotropic configuration 2 for various $\frac{a_2}{L}$. The composite beam is either clamped or simply-supported at both ends.

The investigation is carried out both analytically and numerically, with and without contact effect. Both contactless model and contactless simulation indicate that there will be interpenetration between beams 3 and 4 if contact effect is not included. Likewise, the explanation for this incident is provided in the paragraph just after expression (7.6) in section 7.2. Hence, there will be a need to use the contact-integrated model and contact-integrated simulation for configuration 2 type of composite beam to attain the more

realistic critical load. Figure 7.12 (a) and (b) show that the values determined from the mathematical model are all in good agreement with the readings generated from the simulation for both with and without contact scenarios. Although interpenetration occurs in configuration 2 composite beam, the readings that are obtained from the contactless model and contactless simulation are preserved in this section to correspond to most of the related works in the existing literature where no contact effect is considered. The percentage difference between the simulation and analytical results for the case in which contact effect is included, is reflected in Figure 7.13 for both clamped and simply-supported boundary conditions. It can be seen that for all the cases, the percentage differences are all below 3.5%. Thus, one can conclude that the proposed contact-integrated model is capable of solving the buckling problem of a thin composite beam with semi-embedded delaminations and contact effect correctly. Figure 7.14 depicts the percentage deviation between critical loads that are acquired while accounting and not accounting contact effect. Both analytical model and simulation under clamped and simply-supported end conditions are looked into. It is observed that the values calculated from the analytical model and simulation are very close together for both boundary conditions. From Figure 7.14, it can also be found that the percentage difference in buckling load for cases that include and exclude contact effect becomes larger as the delamination span $\frac{a_2}{L}$ increases for both edge conditions.

Figure 7.15 demonstrates the buckling load of specially orthotropic configuration 3 composite beam with and without the contact effect for a range of a_2 . Once again, both the clamped and simply-supported edge conditions will be studied. Analogous to configuration 2, both the analytical and numerical studies suggest that interpenetration between beams 3 and 4 will arise if contact effect is not implemented. In addition, it is noted that the interpenetration in configuration 3 is more severe than in configuration 2. The rationales behind these happenings are given in detail in the paragraph just after equation (7.6) in section 7.2. From Figure 7.15, for both clamped and simply-supported ends, the results obtained from the contactless model match very well with those produced from the contactless simulation, while the readings computed via the contact-integrated model are also in very good agreement with those gained from the contact-

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integrated simulation. Once more, this proves that the contactless and contact-integrated mathematical models developed are very competent in representing the thin composite beam with semi-embedded delaminations without and with contact, respectively. From Figure 7.16, it is revealed that the percentage difference between the data acquired from the contact-integrated model and the solution attained from the contact-integrated simulation is always less than 3.4% for both clamped and simply-supported boundary conditions. Figure 7.17 illustrates the percentage deviation between readings that are extracted from accounting and not accounting contact effect for both analytical model and simulation under clamped and simply-supported situations. Parallel to configuration 2, the values determined from the analytical model and simulation are very near to each other and the percentage deviation tends to increase as $\frac{a_2}{L}$ increases. Another important information can be unfolded when one compare Figure 7.17 with Figure 7.14. A careful inspection tells that the percentage difference between with and without contact cases is significantly higher in Figure 7.17 (for configuration 3) than in Figure 7.14 (for configuration 2). Therefore, as the addition of the contact effect in configuration 3 will raise the critical load by so much, one ought to incorporate the contact effect when working with configuration 3 to prevent underestimating the actual buckling load seriously.

Nevertheless, all the three configurations share a common trait that the buckling load decreases as the $\frac{a_2}{L}$ increases regardless of boundary conditions and whether the contact effect is accounted or not. Finally, buckling mode shape of the simply-supported configuration 1 having $a_1 = 90\text{mm}$ and $a_2 = 60\text{mm}$ is acquired via the contactless model and demonstrated in Figure 7.18. The buckling mode shapes of both the simply-supported configurations 2 and 3 with the same delaminations are plotted in Figures 7.19 and 7.20, respectively, utilizing both the contactless and contact-integrated models. Their clamped counterparts are exhibited in Appendix G. A point to note when looking at Figure 7.19 (b) is that both beams 3 and 4 may appear to be directing at the same single line. However, by visualizing two lines (that are very close together) instead of one, the diagram is actually trying to portray that the beam 4 is pressing on top of beam 3 and they

are in contact. The same idea applies to Figures 7.20 (b) above and g.3 (b) and g.4 (b) in Appendix G.

7.4.2 Anti-symmetric angle-ply composite beam containing semi-embedded delaminations

In this section, the anti-symmetric angle-ply configurations 1, 2 and 3 composite beams will be studied. Since it is examined from the previous section that configuration 1 does not suffer any interpenetration when no contact effect is applied, only the contactless model and contactless simulation will be utilized to look into this configuration. On the other hand, as it is observed in the preceding section that both configurations 2 and 3 require contact effect to prevent interpenetration of beams 3 and 4, only contact-integrated model and contact-integrated simulation will be employed to investigate these two configurations in this section. Furthermore, both the clamped and simply-supported end conditions will be discussed.

7.4.2.1 Clamped support condition

(a)

(b)

(c)

Figure 7.21: Critical load of configuration 1 composite beam with delaminations (a) $a_1 = 50\text{mm}$ and $a_2 = 30\text{mm}$; (b) $a_1 = 70\text{mm}$ and $a_2 = 45\text{mm}$; and (c) $a_1 = 90\text{mm}$ and $a_2 = 60\text{mm}$ for various ply-angles under clamped ends

(a)

(b)

(c)

Figure 7.22: Critical load of configuration 2 composite beam with delaminations (a) $a_1 = 50\text{mm}$ and $a_2 = 30\text{mm}$; (b) $a_1 = 70\text{mm}$ and $a_2 = 45\text{mm}$; and (c) $a_1 = 90\text{mm}$ and $a_2 = 60\text{mm}$ for various ply-angles under clamped ends

(a)

(b)

(c)

Figure 7.23: Critical load of configuration 3 composite beam with delaminations (a) $a_1 = 50\text{mm}$ and $a_2 = 30\text{mm}$; (b) $a_1 = 70\text{mm}$ and $a_2 = 45\text{mm}$; and (c) $a_1 = 90\text{mm}$ and $a_2 = 60\text{mm}$ for various ply-angles under clamped ends

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In the above diagrams for all the three anti-symmetric angle-ply configurations, the case (a) is utilized to represent composite beam with shorter pair of delaminations, the case (b) is employed to characterize composite beam with a pair of delaminations of moderate length and the case (c) is used to portray composite beam with a pair of long delaminations. The ply-angle θ covers from 0° to 90° . The methods 2 and 3 appearing in the drawings above refer to the two ways of calculating the bending rigidity which are ${}^2(\overline{EI})$ and ${}^3(\overline{EI})$, respectively. This notation will be employed throughout the rest of this chapter. These flexural stiffnesses can then be input into the contactless or contact-integrated model to obtain the desired buckling load. Conversely, the method 1 ${}^1(\overline{EI})$ is excluded from this chapter as it is detected in chapter five that this method will always yield a more inferior upper bound as compared to method 2.

Figure 7.21 illustrates the buckling load of anti-symmetric angle-ply configuration 1 composite beam for three different pairs of delamination spans (a_1 and a_2) under clamped boundary condition. Only contactless model and contactless simulation will be used to treat configuration 1 composite beam as it does not have any interpenetration trouble. From Figure 7.21, one can realize that the methods 2 and 3 can be used in conjunction with the contactless model to construct the longed upper and lower bounds for the actual critical load, respectively. On the other side, Figures 7.22 and 7.23 show the critical force of anti-symmetric angle-ply configurations 2 and 3 composite beams for three sets of dissimilar semi-embedded delaminations under clamped ends. Both contact-integrated model and contact-integrated simulation are employed to analyze these two configurations to prevent any interpenetration between their beams 3 and 4. From Figures 7.22 and 7.23, one can conclude that the proposed methods 2 and 3 are still able to work with the contact-integrated model to create the required upper and lower limits for the buckling load readings that are acquired from the contact-integrated simulation, respectively. For all the three configurations, increase in the delamination lengths a_1 and a_2 will have a negative impact on the load bearing capacity of the composite beam. Moreover, it is observed that for all the cases considered in this section, the critical load tends to decline as the ply-angle θ becomes larger.

7.4.2.2 Simply-supported boundary condition

(a)

(b)

(c)

Figure 7.24: Critical load of configuration 1 composite beam with delaminations (a) $a_1 = 50\text{mm}$ and $a_2 = 30\text{mm}$; (b) $a_1 = 70\text{mm}$ and $a_2 = 45\text{mm}$; and (c) $a_1 = 90\text{mm}$ and $a_2 = 60\text{mm}$ for various ply-angles under simply-supported ends

(a)

(b)

(c)

Figure 7.25: Critical load of configuration 2 composite beam with delaminations (a) $a_1 = 50\text{mm}$ and $a_2 = 30\text{mm}$; (b) $a_1 = 70\text{mm}$ and $a_2 = 45\text{mm}$; and (c) $a_1 = 90\text{mm}$ and $a_2 = 60\text{mm}$ for various ply-angles under simply-supported ends

(a)

(b)

(c)

Figure 7.26: Critical load of configuration 3 composite beam with delaminations (a) $a_1 = 50\text{mm}$ and $a_2 = 30\text{mm}$; (b) $a_1 = 70\text{mm}$ and $a_2 = 45\text{mm}$; and (c) $a_1 = 90\text{mm}$ and $a_2 = 60\text{mm}$ for various ply-angles under simply-supported ends

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In this section, the simply-supported anti-symmetric angle-ply thin composite beam with semi-embedded delaminations will be discussed. Similarly, three different pairs of a_1 and a_2 will be used to stand for short, intermediate and long sets of delaminations. Parallel with the previous section, the two approaches of attaining the flexural stiffness ${}^2(\overline{EI})$ and ${}^3(\overline{EI})$ will be utilized in conjunction with the contactless or contact-integrated model to get the wanted buckling load.

Figure 7.24 depicts the critical load of configuration 1 composite beam for various combinations of a_1 and a_2 . The contactless model and contactless simulation will be employed to resolve the stability problem of configuration 1 as this configuration does not suffer any interpenetration. From Figure 7.24, one can see that the method 2 is able to provide the desired upper bound and the method 3 is capable of supplying the longed lower limit for the points that are acquired from the contactless simulation. Figures 7.25 and 7.26 show the buckling load of configurations 2 and 3 for three groups of a_1 and a_2 . Both the contact-integrated model and contact-integrated simulation are applied on configurations 2 and 3 since their beams 3 and 4 undergo interpenetration. From both Figures 7.25 and 7.26, it can be noticed that all the data generated from the contact-integrated simulation is enclosed by the upper and lower bounds that are constructed via methods 2 and 3, respectively. Hence, once again, it is proven that the suggested methods 2 and 3 are able to form a region that can encompass the actual buckling load. From all the graphs in this section, it can also be noted that both the increase in delamination lengths a_1 and a_2 and ply-angle θ have a harmful effect on the load carrying capacity of the composite beam.

7.5 Concluding remarks

In this chapter, both the contactless and contact-integrated analytical models are formulated to analyze the buckling load of a thin composite beam with semi-embedded delaminations. The contactless model does not take into account of contact. It can be utilized for two purposes. One of them is to get the mode shape of a buckled composite beam when examining the composite beam for the first time. In this way, one can then identify whether there is any interpenetration in the delaminated zone and distinguish

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which sub-laminate penetrates which other. The second use is to employ it to solve cases where there is no interpenetration. In contrast, the contact-integrated model considers the contact effect by incorporating the pressure-contact model into the analytical formulation. This model may be used to resolve cases where interpenetration crop up. The details on the formulation of both the contactless and contact-integrated models are explained in sections 7.2 and 7.3. In this chapter, three classes of thin composite beam with semi-embedded delaminations are studied. They are $[-\theta/\theta/-\theta//(\theta/-\theta)_2/\theta//(-\theta/\theta)_4]$, $[(-\theta/\theta)_2/-\theta//\theta/-\theta/\theta//(-\theta/\theta)_4]$ and $[(-\theta/\theta)_3//-\theta/\theta//(-\theta/\theta)_4]$ which are designated as configurations 1, 2 and 3, respectively.

In order to put a test on the analytical models proposed, the 16 plies specially orthotropic configurations 1, 2 and 3 carbon-epoxy beams are examined. The contactless simulation and contact-integrated simulation are employed to verify results that are computed from the contactless model and contact-integrated model, respectively. Both the clamped and simply-supported boundary conditions are involved. In this investigation, the a_1 is set at 90mm while a_2 is varied from 45mm to 85mm. For configuration 1, no interpenetration is detected and the readings calculated from the contactless model are in excellent agreement with the data that is gathered from the contactless simulation. Interpenetration is spotted in both configurations 2 and 3. It is also found that configuration 3 suffers a more serious interpenetration than configuration 2. Similarly, the solution that is computed from the contactless model is in great agreement with the buckling load that is generated from the contactless simulation and the data that is evaluated from the contact-integrated model is also very close to the results that are produced from the contact-integrated simulation. Furthermore, it is noticed that the buckling load of the composite beam decreases as the a_2 increases.

In addition, the anti-symmetric angle-ply configurations 1, 2 and 3 composite beams are analyzed. The composite beams are also made up of carbon/epoxy and they are comprised of 16 layers. Both clamped and simply-supported end conditions are engaged in the study. Three different pairs of delamination lengths a_1 and a_2 are used in the analysis. They consist of $a_1 = 50\text{mm}$ and $a_2 = 30\text{mm}$; $a_1 = 70\text{mm}$ and $a_2 = 45\text{mm}$; and $a_1 = 90\text{mm}$ and $a_2 = 60\text{mm}$. The ply-angle θ ranges from 0° to 90° . The contactless

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model and contactless simulation are employed to examine the configuration 1 composite beam while the contact-integrated model and contact-integrated simulation are utilized to investigate the configurations 2 and 3 composite beams in this research. For configuration 1, it is found that the two methods of determining the bending stiffness ${}^2(\overline{EI})$ and ${}^3(\overline{EI})$ are able to work together with the contactless model to produce the upper and lower limits for the points that are produced from the contactless simulation for both kinds of boundary conditions, respectively. Moreover, it is observed that the ${}^2(\overline{EI})$ and ${}^3(\overline{EI})$ can be utilized in conjunction with the contact-integrated model to build the desired upper and lower bounds for the buckling load that is gathered from the contact-integrated simulation for both types of end conditions, respectively. It is also seen that increase in the delamination lengths a_1 and a_2 and ply-angle θ has a negative effect on the load carrying capacity of the composite beam.

CHAPTER EIGHT:

STABILITY STUDY ON THIN COMPOSITE BEAM WITH NON-COINCIDING DELAMINATIONS UNDER COMBINED AXIAL AND TRANSVERSE LOADS INCLUDING CONTACT EFFECT

In this chapter, an original analytical model has been formulated to investigate the stability phenomena of the thin composite beam with the non-coinciding delaminations under the combined axial and transverse forces, taking account of the contact effect to prevent the interpenetration at the delaminated zone. The mathematical model developed in the current chapter is known as the contact-accommodated model. The two ways of evaluating the effective bending stiffness discussed in section 4.3.1.1, the ${}^2(\overline{EI})$ and the innovative ${}^3(\overline{EI})$ can then be employed in conjunction with the contact-accommodated model to determine the required buckling load. The carbon/epoxy, boron/epoxy and E-glass/epoxy composite materials are utilized in the analysis. Furthermore, 16 plies anti-symmetric angle-ply composite beam is explored for both clamped and simply-supported edges. The other influencing factors that are inspected include the delaminations spans, through-the-depth location of the delaminations, along-the-length position of the delaminations.

8.1 Delaminated beam layout

Figure 8.1: Composite beam containing non-coinciding delaminations subjected to both axial and vertical forces

Analogous to chapters five and seven, the analysis in this chapter will only be restricted to thin fiber-reinforced composite beam. Hence, just the hypothesis that is addressed in section 4.3.1 of chapter four (CLT) will be employed in the current chapter. Figure 8.1 shows a thin composite beam inflicted with non-coinciding delaminations. In addition, this thin delaminated composite beam experiences both axial force P and uniformly distributed transverse load q . The composite beam has a total depth of H and a full length of L . The delamination closer to the left end has a length of a_1 while the delamination nearer to the right edge has a span of a_2 . The distance measuring from the center of the whole beam to the middle of the delamination closer to the left end is denoted by d_1 . On the other hand, the range covering from the center of the entire beam to the midpoint of the delamination near to the right edge is indicated by d_2 . The two delaminations divide the ideal composite beam into seven pieces. These distinct parts and their corresponding heights are illustrated in Figure 8.1. The x -axis is utilized to specify any location along the span of the composite beam. The w -axis is used to designate the vertical displacement of the composite beam.

8.2 Vital expressions

In this chapter, an innovative analytical model that involves the contact effect will be developed to study the buckling behavior of the thin composite beam with non-coinciding delaminations under combined axial and uniformly distributed transverse loads. In order to distinguish this mathematical model from the contact-integrated model that is introduced in chapter seven, the analytical model that is established in this chapter will be called the “contact-accommodated model” for the rest of this chapter. As the uniformly distributed transverse load is imposed onto the composite beam, the axial force is also applied. The axial force is increased steadily until buckling occurs. As a result of the pressure from the uniformly distributed transverse load, the beam 3 will penetrate beam 2 and the beam 6 will go through beam 5 if no contact effect is implemented. Hence, the pressure-contact model must be enforced at the delaminated regions in the formulation to prevent this physically impermissible event. The presence of the uniformly distributed load has two effects on the composite beam. The first is to prescribe an initial imperfection to the composite beam while the second is to force the composite beam to buckle to a global mode shape where all the individual beams buckle in the same direction. This type of buckling mode shape is also seen by Hwang and Liu [31].

Before proceeding further to solve the governing differential equations for the individual beams, a parameter λ_i will now be brought in to reduce the general solutions for these equations to a more compact form. The λ_i will be defined for the remaining of this chapter as:

$$\lambda_i = \sqrt{\frac{P_i}{(EI)_i}} \quad (8.1)$$

$$\text{Thus, parallel with chapter five, for beam 1, } \lambda_1 = \sqrt{\frac{P_1}{(EI)_1}} \quad (8.2)$$

$$\text{For beam 2, } \lambda_2 = \sqrt{\frac{P_2}{(EI)_2}} = \sqrt{\frac{H_2 P_1}{H (EI)_2}} = \sqrt{\frac{(EI)_1}{(EI)_2}} \sqrt{\frac{H_2}{H}} \sqrt{\frac{P_1}{(EI)_1}} = \lambda_1 \sqrt{\frac{(EI)_1 H_2}{(EI)_2 H}} \quad (8.3)$$

$$\text{For beam 3, } \lambda_3 = \sqrt{\frac{P_3}{(EI)_3}} = \sqrt{\frac{H_3 P_1}{H (EI)_3}} = \sqrt{\frac{(EI)_1}{(EI)_3}} \sqrt{\frac{H_3}{H}} \sqrt{\frac{P_1}{(EI)_1}} = \lambda_1 \sqrt{\frac{(EI)_1 H_3}{(EI)_3 H}} \quad (8.4)$$

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$$\text{For beam 4, } \lambda_4 = \sqrt{\frac{P_4}{(EI)_4}} = \sqrt{\frac{HP_1}{H(EI)_4}} = \sqrt{\frac{(EI)_1}{(EI)_4}} \sqrt{\frac{P_1}{(EI)_1}} = \sqrt{\frac{P_1}{(EI)_1}} = \lambda_1 \quad (8.5)$$

$$\text{For beam 5, } \lambda_5 = \sqrt{\frac{P_5}{(EI)_5}} = \sqrt{\frac{H_5 P_1}{H(EI)_5}} = \sqrt{\frac{(EI)_1}{(EI)_5}} \sqrt{\frac{H_5}{H}} \sqrt{\frac{P_1}{(EI)_1}} = \lambda_1 \sqrt{\frac{(EI)_1 H_5}{(EI)_5 H}} \quad (8.6)$$

$$\text{For beam 6, } \lambda_6 = \sqrt{\frac{P_6}{(EI)_6}} = \sqrt{\frac{H_6 P_1}{H(EI)_6}} = \sqrt{\frac{(EI)_1}{(EI)_6}} \sqrt{\frac{H_6}{H}} \sqrt{\frac{P_1}{(EI)_1}} = \lambda_1 \sqrt{\frac{(EI)_1 H_6}{(EI)_6 H}} \quad (8.7)$$

$$\text{Finally, for beam 7, } \lambda_7 = \sqrt{\frac{P_7}{(EI)_7}} = \sqrt{\frac{HP_1}{H(EI)_7}} = \sqrt{\frac{(EI)_1}{(EI)_7}} \sqrt{\frac{P_1}{(EI)_1}} = \sqrt{\frac{P_1}{(EI)_1}} = \lambda_1 \quad (8.8)$$

From Figure 8.1, focusing on just beams 1, 4 and 7, one can form the following governing differential equations for them:

$$(\overline{EI})_i \frac{d^4 w_i}{dx^4} + P_i \frac{d^2 w_i}{dx^2} = q \quad (8.9)$$

The solution for the above differential expression can be worked out to be:

$$w_i(x) = A_i \cos(\lambda_i x) + B_i \sin(\lambda_i x) + \frac{q}{2\lambda_i^2 (\overline{EI})_i} x^2 + C_i x + D_i \quad (8.10)$$

Then, expression (8.10) can be differentiated to obtain:

$$\frac{dw_i(x)}{dx} = -\lambda_i A_i \sin(\lambda_i x) + \lambda_i B_i \cos(\lambda_i x) + \frac{q}{\lambda_i^2 (\overline{EI})_i} x + C_i \quad (8.11)$$

$$\frac{d^2 w_i(x)}{dx^2} = -\lambda_i^2 A_i \cos(\lambda_i x) - \lambda_i^2 B_i \sin(\lambda_i x) + \frac{q}{\lambda_i^2 (\overline{EI})_i} \quad (8.12)$$

$$\frac{d^3 w_i(x)}{dx^3} = \lambda_i^3 A_i \sin(\lambda_i x) - \lambda_i^3 B_i \cos(\lambda_i x) \quad (8.13)$$

Equations (8.10) till (8.13) can then be applied on beams 1, 4 and 7 in the analytical model that will be formulated in the next section.

The attention shall now be shifted to the beams above and below the two delaminations. The beams 2 and 3 shall be covered first. The forces that are exerted on beam 2 consist of the horizontal load P_2 and the pressure p_c which is created as the beam 3 pushes the beam 2 downwards. Hence, the same pressure p_c will also act on the beam 3 as the beam 2 pushes the beam 3 upwards. Furthermore, the axial force P_3 and the uniformly

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distributed downward load q are applied on the beam 3. Therefore, one can write down the governing differential equations for these two beams as:

$$\left(\overline{EI}\right)_2 \frac{d^4 w_2}{dx^4} + P_2 \frac{d^2 w_2}{dx^2} = p_c \quad (8.14)$$

$$\therefore \left(\overline{EI}\right)_2 \frac{d^4 w_2}{dx^4} + P_2 \frac{d^2 w_2}{dx^2} = \frac{2E_T}{H_2 + H_3} (w_3 - w_2) \quad (8.15)$$

$$\left(\overline{EI}\right)_3 \frac{d^4 w_3}{dx^4} + P_3 \frac{d^2 w_3}{dx^2} = -p_c + q \quad (8.16)$$

$$\therefore \left(\overline{EI}\right)_3 \frac{d^4 w_3}{dx^4} + P_3 \frac{d^2 w_3}{dx^2} = -\frac{2E_T}{H_2 + H_3} (w_3 - w_2) + q \quad (8.17)$$

Equation (8.15) can be reshuffled to yield:

$$w_3 = \frac{\left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^4 w_2}{dx^4} + \frac{\lambda_2^2 \left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^2 w_2}{dx^2} + w_2 \quad (8.18)$$

The second and fourth derivatives of w_3 can be evaluated as:

$$\frac{d^2 w_3}{dx^2} = \frac{\left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^6 w_2}{dx^6} + \frac{\lambda_2^2 \left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^4 w_2}{dx^4} + \frac{d^2 w_2}{dx^2} \quad (8.19)$$

$$\frac{d^4 w_3}{dx^4} = \frac{\left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^8 w_2}{dx^8} + \frac{\lambda_2^2 \left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^6 w_2}{dx^6} + \frac{d^4 w_2}{dx^4} \quad (8.20)$$

Substituting expressions (8.18) through (8.20) into equation (8.17) produces:

$$\begin{aligned} & \left(\frac{\left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^8 w_2}{dx^8} + \frac{\lambda_2^2 \left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^6 w_2}{dx^6} + \frac{d^4 w_2}{dx^4} \right) \\ & + \lambda_3^2 \left(\frac{\left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^6 w_2}{dx^6} + \frac{\lambda_2^2 \left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^4 w_2}{dx^4} + \frac{d^2 w_2}{dx^2} \right) \\ & = -\frac{2E_T}{\left(\overline{EI}\right)_3 (H_2 + H_3)} \left(\frac{\left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^4 w_2}{dx^4} + \frac{\lambda_2^2 \left(\overline{EI}\right)_2 (H_2 + H_3)}{2E_T} \frac{d^2 w_2}{dx^2} + w_2 - w_2 \right) + \frac{q}{\left(\overline{EI}\right)_3} \end{aligned} \quad (8.21)$$

$$\begin{aligned} \Rightarrow & \frac{(\overline{EI})_2(H_2 + H_3)}{2E_T} \frac{d^8 w_2}{dx^8} + \frac{\lambda_2^2(\overline{EI})_2(H_2 + H_3)}{2E_T} \frac{d^6 w_2}{dx^6} + \frac{d^4 w_2}{dx^4} \\ & + \frac{\lambda_3^2(\overline{EI})_2(H_2 + H_3)}{2E_T} \frac{d^6 w_2}{dx^6} + \frac{\lambda_2^2 \lambda_3^2(\overline{EI})_2(H_2 + H_3)}{2E_T} \frac{d^4 w_2}{dx^4} + \lambda_3^2 \frac{d^2 w_2}{dx^2} \end{aligned} \quad (8.22)$$

$$\begin{aligned} = & -\frac{(\overline{EI})_2}{(\overline{EI})_3} \frac{d^4 w_2}{dx^4} - \frac{\lambda_2^2(\overline{EI})_2}{(\overline{EI})_3} \frac{d^2 w_2}{dx^2} + \frac{q}{(\overline{EI})_3} \\ \therefore & \frac{(\overline{EI})_2(H_2 + H_3)}{2E_T} \frac{d^8 w_2}{dx^8} + \left(\frac{\lambda_2^2(\overline{EI})_2(H_2 + H_3)}{2E_T} + \frac{\lambda_3^2(\overline{EI})_2(H_2 + H_3)}{2E_T} \right) \frac{d^6 w_2}{dx^6} \\ & \left(1 + \frac{\lambda_2^2 \lambda_3^2(\overline{EI})_2(H_2 + H_3)}{2E_T} + \frac{(\overline{EI})_2}{(\overline{EI})_3} \right) \frac{d^4 w_2}{dx^4} + \left(\lambda_3^2 + \frac{\lambda_2^2(\overline{EI})_2}{(\overline{EI})_3} \right) \frac{d^2 w_2}{dx^2} = \frac{q}{(\overline{EI})_3} \end{aligned} \quad (8.23)$$

By equating the left hand side of equation (8.23) to zero, one can write down the homogeneous differential equation for the above expression as:

$$\begin{aligned} \therefore & \frac{(\overline{EI})_2(H_2 + H_3)}{2E_T} \frac{d^8 (w_2)_h}{dx^8} + \left(\frac{\lambda_2^2(\overline{EI})_2(H_2 + H_3)}{2E_T} + \frac{\lambda_3^2(\overline{EI})_2(H_2 + H_3)}{2E_T} \right) \frac{d^6 (w_2)_h}{dx^6} \\ & \left(1 + \frac{\lambda_2^2 \lambda_3^2(\overline{EI})_2(H_2 + H_3)}{2E_T} + \frac{(\overline{EI})_2}{(\overline{EI})_3} \right) \frac{d^4 (w_2)_h}{dx^4} + \left(\lambda_3^2 + \frac{\lambda_2^2(\overline{EI})_2}{(\overline{EI})_3} \right) \frac{d^2 (w_2)_h}{dx^2} = 0 \end{aligned} \quad (8.24)$$

The characteristic equation for the above homogeneous differential expression appears to be:

$$\begin{aligned} \therefore & \frac{(\overline{EI})_2(H_2 + H_3)}{2E_T} b^8 + \left(\frac{\lambda_2^2(\overline{EI})_2(H_2 + H_3)}{2E_T} + \frac{\lambda_3^2(\overline{EI})_2(H_2 + H_3)}{2E_T} \right) b^6 \\ & \left(1 + \frac{\lambda_2^2 \lambda_3^2(\overline{EI})_2(H_2 + H_3)}{2E_T} + \frac{(\overline{EI})_2}{(\overline{EI})_3} \right) b^4 + \left(\lambda_3^2 + \frac{\lambda_2^2(\overline{EI})_2}{(\overline{EI})_3} \right) b^2 = 0 \end{aligned} \quad (8.25)$$

The solutions for b in the above equation can be attained via the Maple mathematical software to be $0, 0, c \pm di, f \pm gi$ and $\pm hi$. It should be noted that the c, d, f, g and h existing in the roots are only functions of $H_2, H_3, E_T, (\overline{EI})_2, (\overline{EI})_3, \lambda_2$ and λ_3 only. While $H_2, H_3, E_T, (\overline{EI})_2$ and $(\overline{EI})_3$ are known, λ_2 and λ_3 can be expressed in terms of λ_1 utilizing equations (8.3) and (8.4). Therefore, λ_1 will be the only unknown variable within c, d, f, g and h . Hence, the homogenous solution will emerge as:

$$\begin{aligned} (w_2(x))_h = & A_{2;3} \cos(hx) + B_{2;3} \sin(hx) + e^{cx} (C_{2;3} \cos(dx) + D_{2;3} \sin(dx)) \\ & + e^{fx} (E_{2;3} \cos(gx) + F_{2;3} \sin(gx)) + G_{2;3}^\diamond x + H_{2;3}^\diamond \end{aligned} \quad (8.26)$$

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All the capital letters in the preceding equations with a subscript “2;3” are unknown constants to be resolved. Next, noticing that there are two real roots for equation (8.25), the subsequent particular solution can be realized using the method of undetermined coefficients:

$$(w_2(x))_p = P_1 x^2 + P_2 x + P_3 \quad (8.27)$$

The unidentified P_1 , P_2 and P_3 in the former equation can be found by inputting equation (8.27) into (8.23) resulting in:

$$\begin{aligned} \therefore \frac{(\overline{EI})_2(H_2 + H_3)}{2E_T}(0) + \left(\frac{\lambda_2^2(\overline{EI})_2(H_2 + H_3)}{2E_T} + \frac{\lambda_3^2(\overline{EI})_2(H_2 + H_3)}{2E_T} \right)(0) \\ \left(1 + \frac{\lambda_2^2 \lambda_3^2(\overline{EI})_2(H_2 + H_3)}{2E_T} + \frac{(\overline{EI})_2}{(\overline{EI})_3} \right)(0) + \left(\lambda_3^2 + \frac{\lambda_2^2(\overline{EI})_2}{(\overline{EI})_3} \right)(2P_1) = \frac{q}{(\overline{EI})_3} \end{aligned} \quad (8.28)$$

$$\therefore P_1 = \frac{q}{2(\lambda_2^2(\overline{EI})_2 + \lambda_3^2(\overline{EI})_3)} \quad (8.29)$$

Thus, the desired particular solution turns out to be:

$$(w_2(x))_p = \frac{q}{2(\lambda_2^2(\overline{EI})_2 + \lambda_3^2(\overline{EI})_3)} x^2 + P_2 x + P_3 \quad (8.30)$$

The complete solution for the differential equation (8.23) addition of the homogeneous and particular solutions from expressions (8.26) and (8.30), respectively:

$$w_2(x) = (w_2(x))_h + (w_2(x))_p \quad (8.31)$$

$$\begin{aligned} \Rightarrow w_2(x) = A_{2;3} \cos(hx) + B_{2;3} \sin(hx) + e^{cx} (C_{2;3} \cos(dx) + D_{2;3} \sin(dx)) \\ + e^{fx} (E_{2;3} \cos(gx) + F_{2;3} \sin(gx)) \end{aligned} \quad (8.32)$$

$$\begin{aligned} + \frac{q}{2(\lambda_2^2(\overline{EI})_2 + \lambda_3^2(\overline{EI})_3)} x^2 + (G_{2;3}^\diamond + P_2)x + (H_{2;3}^\diamond + P_3) \\ \therefore w_2(x) = A_{2;3} \cos(hx) + B_{2;3} \sin(hx) + e^{cx} (C_{2;3} \cos(dx) + D_{2;3} \sin(dx)) \\ + e^{fx} (E_{2;3} \cos(gx) + F_{2;3} \sin(gx)) + \frac{q}{2(\lambda_2^2(\overline{EI})_2 + \lambda_3^2(\overline{EI})_3)} x^2 + G_{2;3}x + H_{2;3} \end{aligned} \quad (8.33)$$

After the expression for $w_2(x)$ has been identified, the equation for $w_3(x)$ can be determined utilizing equation (8.18):

$$\begin{aligned}
w_3(x) = & \frac{(\overline{EI})_2(H_2 + H_3)}{2E_T} \times \\
& \left[\begin{aligned} & A_{2,3}h^2(h^2 - \lambda_2^2)\cos(hx) + B_{2,3}h^2(h^2 - \lambda_2^2)\sin(hx) \\ & + e^{cx} \left(\begin{aligned} & \left(c^4 C_{2,3} + 4c^3 d D_{2,3} - 6c^2 d^2 C_{2,3} - 4cd^3 D_{2,3} + d^4 C_{2,3} \right) \cos(dx) \\ & + \lambda_2^2 (c^2 C_{2,3} + 2cd D_{2,3} - d^2 C_{2,3}) \end{aligned} \right) \\ & + \left(\begin{aligned} & c^4 D_{2,3} - 4c^3 d C_{2,3} - 6c^2 d^2 D_{2,3} + 4cd^3 C_{2,3} + d^4 D_{2,3} \end{aligned} \right) \sin(dx) \\ & + \lambda_2^2 (c^2 D_{2,3} - 2cd C_{2,3} - d^2 D_{2,3}) \end{aligned} \right) \\ & + e^{fx} \left(\begin{aligned} & \left(f^4 E_{2,3} + 4f^3 g F_{2,3} - 6f^2 g^2 E_{2,3} - 4fg^3 F_{2,3} + g^4 E_{2,3} \right) \cos(gx) \\ & + \lambda_2^2 (f^2 E_{2,3} + 2fg F_{2,3} - g^2 E_{2,3}) \end{aligned} \right) \\ & + \left(\begin{aligned} & f^4 F_{2,3} - 4f^3 g E_{2,3} - 6f^2 g^2 F_{2,3} + 4fg^3 E_{2,3} + g^4 F_{2,3} \end{aligned} \right) \sin(gx) \\ & + \lambda_2^2 (f^2 F_{2,3} - 2fg E_{2,3} - g^2 F_{2,3}) \end{aligned} \right) \end{aligned} \right] \\
& + \frac{q\lambda_2^2(\overline{EI})_2(H_2 + H_3)}{2E_T(\lambda_2^2(\overline{EI})_2 + \lambda_3^2(\overline{EI})_3)} + A_{2,3}\cos(hx) + B_{2,3}\sin(hx) + e^{cx}(C_{2,3}\cos(dx) + D_{2,3}\sin(dx)) \\
& + e^{fx}(E_{2,3}\cos(gx) + F_{2,3}\sin(gx)) + \frac{q}{2(\lambda_2^2(\overline{EI})_2 + \lambda_3^2(\overline{EI})_3)}x^2 + G_{2,3}x + H_{2,3} \tag{8.34}
\end{aligned}$$

Equations (8.33) and (8.34) can then be used to work on beams 2 and 3, respectively, in the analytical model. Following the procedure to attain $w_2(x)$ and $w_3(x)$, the $w_5(x)$ and $w_6(x)$ for beams 5 and 6, respectively, can be determined to be:

$$\begin{aligned}
w_5(x) = & A_{5,6}\cos(\bar{h}x) + B_{5,6}\sin(\bar{h}x) + e^{\bar{c}x}(C_{5,6}\cos(\bar{d}x) + D_{5,6}\sin(\bar{d}x)) \\
& + e^{\bar{f}x}(E_{5,6}\cos(\bar{g}x) + F_{5,6}\sin(\bar{g}x)) + \frac{q}{2(\lambda_5^2(\overline{EI})_5 + \lambda_6^2(\overline{EI})_6)}x^2 + G_{5,6}x + H_{5,6} \tag{8.35}
\end{aligned}$$

$$\begin{aligned}
w_6(x) = & \frac{(\overline{EI})_5(H_5 + H_6)}{2E_T} \times \\
& \left[\begin{aligned} & A_{5,6} \bar{h}^2 (\bar{h}^2 - \lambda_2^2) \cos(\bar{h}x) + B_{5,6} \bar{h}^2 (\bar{h}^2 - \lambda_2^2) \sin(\bar{h}x) \\ & + e^{\bar{c}x} \left(\begin{aligned} & \left(\bar{c}^4 C_{5,6} + 4\bar{c}^3 \bar{d} D_{5,6} - 6\bar{c}^2 \bar{d}^2 C_{5,6} - 4\bar{c} \bar{d}^3 D_{5,6} + \bar{d}^4 C_{5,6} \right) \cos(\bar{d}x) \\ & + \lambda_5^2 \left(\bar{c}^2 C_{5,6} + 2\bar{c} \bar{d} D_{5,6} - \bar{d}^2 C_{5,6} \right) \end{aligned} \right) \\ & + \left(\begin{aligned} & \bar{c}^4 D_{5,6} - 4\bar{c}^3 \bar{d} C_{5,6} - 6\bar{c}^2 \bar{d}^2 D_{5,6} + 4\bar{c} \bar{d}^3 C_{5,6} + \bar{d}^4 D_{5,6} \\ & + \lambda_5^2 \left(\bar{c}^2 D_{5,6} - 2\bar{c} \bar{d} C_{5,6} - \bar{d}^2 D_{5,6} \right) \end{aligned} \right) \sin(\bar{d}x) \end{aligned} \right) \\ & + e^{\bar{f}x} \left(\begin{aligned} & \left(\bar{f}^4 E_{5,6} + 4\bar{f}^3 \bar{g} F_{5,6} - 6\bar{f}^2 \bar{g}^2 E_{5,6} - 4\bar{f} \bar{g}^3 F_{5,6} + \bar{g}^4 E_{5,6} \right) \cos(\bar{g}x) \\ & + \lambda_5^2 \left(\bar{f}^2 E_{5,6} + 2\bar{f} \bar{g} F_{5,6} - \bar{g}^2 E_{5,6} \right) \end{aligned} \right) \\ & + \left(\begin{aligned} & \bar{f}^4 F_{5,6} - 4\bar{f}^3 \bar{g} E_{5,6} - 6\bar{f}^2 \bar{g}^2 F_{5,6} + 4\bar{f} \bar{g}^3 E_{5,6} + \bar{g}^4 F_{5,6} \\ & + \lambda_5^2 \left(\bar{f}^2 F_{5,6} - 2\bar{f} \bar{g} E_{5,6} - \bar{g}^2 F_{5,6} \right) \end{aligned} \right) \sin(\bar{g}x) \end{aligned} \right) \\ & + \frac{q \lambda_5^2 (\overline{EI})_5 (H_5 + H_6)}{2E_T (\lambda_5^2 (\overline{EI})_5 + \lambda_6^2 (\overline{EI})_6)} + A_{5,6} \cos(\bar{h}x) + B_{5,6} \sin(\bar{h}x) + e^{\bar{c}x} (C_{5,6} \cos(\bar{d}x) + D_{5,6} \sin(\bar{d}x)) \\ & + e^{\bar{f}x} (E_{5,6} \cos(\bar{g}x) + F_{5,6} \sin(\bar{g}x)) + \frac{q}{2(\lambda_5^2 (\overline{EI})_5 + \lambda_6^2 (\overline{EI})_6)} x^2 + G_{5,6} x + H_{5,6} \end{aligned} \right] \quad (8.36)
\end{aligned}$$

Similarly, the equations (8.35) and (8.36) can be applied on beams 5 and 6, respectively, into the mathematical model that will be formulated in the next section.

In summary, beams 1, 4 and 7 utilizes expressions (8.10) to (8.13) in the analytical model to describe their deflection, slope, moment and shear force, respectively. On the other hand, the beams 2, 3, 5 and 6 will require equations (8.33) till (8.36), respectively, together with their corresponding derivatives to work out their deflection, slope, moment and shear force in the mathematical model that will be established in the following section.

8.3 Formulation

In this section, the crucial set of equations will be established to formulate the desired analytical model. These expressions are developed via the edge conditions, kinematical continuity and balance of internal forces and moment.

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8.3.1 Boundary state

In this section, both the clamped and simply-supported cases will be explored. The expressions that emerge from these two classes of boundary conditions can be grouped as:

Clamped ($8.\times\times a$)	Simply-supported ($8.\times\times b$)
$w_1(x=0)=0$	$w_1(x=0)=0$

(8.37)

$\frac{dw_1(x=0)}{dx}=0$	$M_1(x=0)=0$ $\therefore -(EI)_1 \frac{d^2 w_1(x=0)}{dx^2}=0$
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(8.38)

$w_7(x=L)=0$	$w_7(x=L)=0$
--------------	--------------

(8.39)

$\frac{dw_7(x=L)}{dx}=0$	$M_7(x=L)=0$ $\therefore -(EI)_7 \frac{d^2 w_7(x=L)}{dx^2}=0$
--------------------------	--

(8.40)

8.3.2 Kinematical consistency expressions

Figure 8.2: Kinematical continuity regarding the left delamination

From Figure 8.1 and the drawing above, one can deduce the subsequent essential compatibility equations:

$$\begin{aligned}
w_1\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right) &= w_2\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right) \\
\therefore w_1\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right) - w_2\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right) &= 0
\end{aligned} \tag{8.41}$$

$$\begin{aligned}
w_1\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right) &= w_3\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right) \\
\therefore w_1\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right) - w_3\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right) &= 0
\end{aligned} \tag{8.42}$$

$$\begin{aligned}
\frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right)}{dx} &= \frac{dw_2\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right)}{dx} \\
\therefore \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right)}{dx} - \frac{dw_2\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right)}{dx} &= 0
\end{aligned} \tag{8.43}$$

$$\begin{aligned}
\frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right)}{dx} &= \frac{dw_3\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right)}{dx} \\
\therefore \frac{dw_1\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right)}{dx} - \frac{dw_3\left(x = \frac{L}{2} - \frac{a_1}{2} - d_1\right)}{dx} &= 0
\end{aligned} \tag{8.44}$$

$$\begin{aligned}
w_2\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right) &= w_4\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right) \\
\therefore w_2\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right) - w_4\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right) &= 0
\end{aligned} \tag{8.45}$$

$$\begin{aligned}
w_3\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right) &= w_4\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right) \\
\therefore w_3\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right) - w_4\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right) &= 0
\end{aligned} \tag{8.46}$$

$$\frac{dw_2\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right)}{dx} = \frac{dw_4\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right)}{dx}$$

$$\therefore \frac{dw_2\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right)}{dx} - \frac{dw_4\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right)}{dx} = 0 \quad (8.47)$$

$$\frac{dw_3\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right)}{dx} = \frac{dw_4\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right)}{dx}$$

$$\therefore \frac{dw_3\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right)}{dx} - \frac{dw_4\left(x = \frac{L}{2} + \frac{a_1}{2} - d_1\right)}{dx} = 0 \quad (8.49)$$

Figure 8.3: Kinematical continuity related to right delamination

Similarly, the following valuable constancy expressions may be extracted from the diagram above:

$$w_4\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right) = w_5\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right)$$

$$\therefore w_4\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right) - w_5\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right) = 0 \quad (8.50)$$

$$w_4\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right) = w_6\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right)$$

$$\therefore w_4\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right) - w_6\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right) = 0 \quad (8.51)$$

$$\frac{dw_4\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right)}{dx} = \frac{dw_5\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right)}{dx}$$

$$\therefore \frac{dw_4\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right)}{dx} - \frac{dw_5\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right)}{dx} = 0 \quad (8.52)$$

$$\frac{dw_4\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right)}{dx} = \frac{dw_6\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right)}{dx}$$

$$\therefore \frac{dw_4\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right)}{dx} - \frac{dw_6\left(x = \frac{L}{2} - \frac{a_2}{2} + d_2\right)}{dx} = 0 \quad (8.53)$$

$$w_5\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right) = w_7\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right)$$

$$\therefore w_5\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right) - w_7\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right) = 0 \quad (8.54)$$

$$w_6\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right) = w_7\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right)$$

$$\therefore w_6\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right) - w_7\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right) = 0 \quad (8.55)$$

$$\frac{dw_5\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right)}{dx} = \frac{dw_7\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right)}{dx}$$

$$\therefore \frac{dw_5\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right)}{dx} - \frac{dw_7\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right)}{dx} = 0 \quad (8.56)$$

$$\frac{dw_6\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right)}{dx} = \frac{dw_7\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right)}{dx}$$

$$\therefore \frac{dw_6\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right)}{dx} - \frac{dw_7\left(x = \frac{L}{2} + \frac{a_2}{2} + d_2\right)}{dx} = 0 \quad (8.57)$$

Figure 8.4: Contraction and lengthening of the beams due to bending

Figure 8.4 displays the shape of the thin composite beam with non-coinciding delaminations under both axial and uniformly distributed loads after bending. The incremental forces experienced by beams 2, 3, 5 and 6 due to the bending are symbolized as ΔP_2 , ΔP_3 , ΔP_5 and ΔP_6 , respectively. Once again, in this chapter, it is believed that the axial deformation formula is able to evaluate the shortening and extension at the two

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end of a beam. Firstly, concentrating on beam 3 in Figure 8.4, one can derive the following equation:

$$-\frac{\Delta P_3 a_1}{H_3 E_3} = \Delta_3^a - \Delta_3^b \quad (8.58)$$

$$\therefore -\frac{\Delta P_3 a_1}{H_3 E_3} = (\bar{Y} - \bar{Y}_3) \frac{dw_1 \left(x = \frac{L}{2} - \frac{a_1}{2} - \frac{d_1}{2} \right)}{dx} - (\bar{Y} - \bar{Y}_3) \frac{dw_3 \left(x = \frac{L}{2} + \frac{a_1}{2} - \frac{d_1}{2} \right)}{dx} \quad (8.59)$$

In this case, it is first assumed that the beam 3 is suffering from compression. Hence, the ΔP_3 will turn out to be negative and a minus sign must be arranged in front of it to make the left hand side of both equations (8.58) and (8.59) to be positive. Upon doing so, the left hand side of expressions (8.58) and (8.59) will grant the desired positive contraction value that is experienced by the beam 3. This same amount of shortening can also be acquired from Figure 8.4 by subtracting the extension Δ_3^b at the right end of the beam 3 from the contraction Δ_3^a at the left edge of beam 3. The $\bar{Y}, \bar{Y}_2, \bar{Y}_3, \bar{Y}_5$ and $\bar{Y}_6$ are the distances covering from the top surface of the composite beam to the neutral axes of beams 1 (also beams 4 and 7), 2, 3, 5 and 6, respectively. From Figure 8.4, one can observe that $\bar{Y} > \bar{Y}_3$. This implies that the term $(\bar{Y} - \bar{Y}_3)$ appearing on the right hand side

of equation (8.59) will be positive. Furthermore, the term $\frac{dw_1 \left(x = \frac{L}{2} - \frac{a_1}{2} - \frac{d_1}{2} \right)}{dx}$

emerging in the first product on the right hand side of expression (8.59) is positive due to

how the w -axis is defined. Thus, the first product $(\bar{Y} - \bar{Y}_3) \frac{dw_1 \left(x = \frac{L}{2} - \frac{a_1}{2} - \frac{d_1}{2} \right)}{dx}$ on the

right side of equation (8.59) will provide the wanted positive valued contraction Δ_3^a

happening at the left end of beam 3. The $\frac{dw_3 \left(x = \frac{L}{2} + \frac{a_1}{2} - \frac{d_1}{2} \right)}{dx}$ occurring in the second

product on the right hand side of expression (8.59) is also positive. Hence, the second

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product $(\bar{Y} - \bar{Y}_3) \frac{dw_3 \left(x = \frac{L}{2} + \frac{a_1}{2} - \frac{d_1}{2} \right)}{dx}$ on the right hand side of equation (8.59) will offer

the necessary positive valued elongation Δ_3^b taking place at the right edge of beam 3.

Subtracting this product from the first product $(\bar{Y} - \bar{Y}_3) \frac{dw_1 \left(x = \frac{L}{2} - \frac{a_1}{2} - \frac{d_1}{2} \right)}{dx}$ will then

yield the desired effective contraction withstood by the beam 3 in positive number. It should be noted that expression (8.59) is still valid even if beam 3 is suffering from tension instead of compression. Reshuffling equation (8.59) results in:

$$\frac{\Delta P_3 a_1}{H_3 E_3} + (\bar{Y} - \bar{Y}_3) \left(\frac{dw_1 \left(x = \frac{L}{2} - \frac{a_1}{2} - \frac{d_1}{2} \right)}{dx} - \frac{dw_3 \left(x = \frac{L}{2} + \frac{a_1}{2} - \frac{d_1}{2} \right)}{dx} \right) = 0 \quad (8.60)$$

Next, looking at beam 6 in Figure 8.4, one can state the subsequent expression:

$$-\frac{\Delta P_6 a_2}{H_6 E_6} = -\Delta_6^a + \Delta_6^b \quad (8.61)$$

$$\therefore -\frac{\Delta P_6 a_2}{H_6 E_6} = (\bar{Y} - \bar{Y}_6) \frac{dw_6 \left(x = \frac{L}{2} - \frac{a_2}{2} + \frac{d_2}{2} \right)}{dx} - (\bar{Y} - \bar{Y}_6) \frac{dw_7 \left(x = \frac{L}{2} + \frac{a_2}{2} + \frac{d_2}{2} \right)}{dx} \quad (8.62)$$

Similarly, beam 6 is taken to be under compression. Therefore, ΔP_6 will be a negative number. Hence, a minus sign must be included at the left hand side of both equations (8.61) and (8.62) to obtain the positive value for the contraction endured by beam 6. From Figure 8.4, it can be seen that this shortening can be attained by subtracting the extension at the left end of beam 6 Δ_6^a from the reduction of length at the right edge of beam 6 Δ_6^b . This deduction is reflected on the right hand side of expression (8.61). From Figure 8.4, since $\bar{Y} > \bar{Y}_6$, the term $(\bar{Y} - \bar{Y}_6)$ on the right hand side of equation (8.62) will be positive. Conversely, as a result of the way the w -axis is defined, both the terms

$\frac{dw_6 \left(x = \frac{L}{2} - \frac{a_2}{2} + \frac{d_2}{2} \right)}{dx}$ and $\frac{dw_7 \left(x = \frac{L}{2} + \frac{a_2}{2} + \frac{d_2}{2} \right)}{dx}$ on the right side of equation (8.62)

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will be negative. Thus, the first product on the right hand side of equation (8.62)

$(\bar{Y} - \bar{Y}_6) \frac{dw_6 \left(x = \frac{L}{2} - \frac{a_2}{2} + \frac{d_2}{2} \right)}{dx}$ will turn out to be negative, corresponding to the term

$-\Delta_6^a$ on the right hand side of equation (8.61). Similarly, the second product

$(\bar{Y} - \bar{Y}_6) \frac{dw_7 \left(x = \frac{L}{2} + \frac{a_2}{2} + \frac{d_2}{2} \right)}{dx}$ on the right hand side of equation (8.62) will also be

negative. Therefore, a minus sign must be placed in front of this product to produce the longed positive value for the amount of shortening occurring at the right end of beam 6. It should be reminded that equation (8.62) remains true even if beam 6 is bearing tension rather than compression. Reorganizing expression (8.62) yields the following equation:

$$\frac{\Delta P_6 a_2}{H_6 E_6} + (\bar{Y} - \bar{Y}_6) \left(\frac{dw_6 \left(x = \frac{L}{2} - \frac{a_2}{2} + \frac{d_2}{2} \right)}{dx} - \frac{dw_7 \left(x = \frac{L}{2} + \frac{a_2}{2} + \frac{d_2}{2} \right)}{dx} \right) = 0 \quad (8.63)$$

8.3.3 Balance of forces and moments

Looking at Figure 5.5 and realizing that the delamination tip is now at $x = \frac{L}{2} - \frac{a_1}{2} - d_1$,

one can obtain the subsequent equilibrium expressions.

Utilizing the $\sum F_x = 0$, one can attain:

$$\Delta P_2 + \Delta P_3 = 0 \quad (8.64)$$

The balance of forces in the vertical direction yields:

$$\begin{aligned} -V_1 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right) + V_2 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right) + V_3 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right) &= 0 \\ \therefore (\bar{EI})_1 \frac{d^3 w_1 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right)}{dx^3} - (\bar{EI})_2 \frac{d^3 w_2 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right)}{dx^3} \\ - (\bar{EI})_3 \frac{d^3 w_3 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right)}{dx^3} &= 0 \end{aligned} \quad (8.65)$$

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The next equation can be obtained via $\sum M_o = 0$ as:

$$\begin{aligned}
 & M_1 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right) - M_2 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right) - M_3 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right) \\
 & + (\bar{Y} - \bar{Y}_3) \Delta P_3 - (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0 \\
 & \therefore -(\overline{EI})_1 \frac{d^2 w_1 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right)}{dx^2} + (\overline{EI})_2 \frac{d^2 w_2 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right)}{dx^2} \\
 & + (\overline{EI})_3 \frac{d^2 w_3 \left(x = \frac{L}{2} - \frac{a_1}{2} - d_1 \right)}{dx^2} + (\bar{Y} - \bar{Y}_3) \Delta P_3 - (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0
 \end{aligned} \tag{8.66}$$

Referring to Figure 5.6 and taking note that the delamination front is now at

$x = \frac{L}{2} + \frac{a_1}{2} - d_1$, one can acquire the following balance equations.

Employing $\sum F_z = 0$ results in:

$$\begin{aligned}
 & V_4 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right) - V_2 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right) - V_3 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right) = 0 \\
 & \therefore -(\overline{EI})_4 \frac{d^3 w_4 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right)}{dx^3} + (\overline{EI})_2 \frac{d^3 w_2 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right)}{dx^3} \\
 & + (\overline{EI})_3 \frac{d^3 w_3 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right)}{dx^3} = 0
 \end{aligned} \tag{8.67}$$

One can get the subsequent equation by applying moment equilibrium about the point O :

$$\begin{aligned}
 & M_2 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right) + M_3 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right) - M_4 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right) \\
 & - (\bar{Y} - \bar{Y}_3) \Delta P_3 + (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0 \\
 & \therefore -(\overline{EI})_2 \frac{d^2 w_2 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right)}{dx^2} - (\overline{EI})_3 \frac{d^2 w_3 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right)}{dx^2} \\
 & + (\overline{EI})_4 \frac{d^2 w_4 \left(x = \frac{L}{2} + \frac{a_1}{2} - d_1 \right)}{dx^2} - (\bar{Y} - \bar{Y}_3) \Delta P_3 + (\bar{Y}_2 - \bar{Y}) \Delta P_2 = 0
 \end{aligned} \tag{8.68}$$

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Similarly, crucial expressions can be attained via equilibrium consistency at

$$x = \frac{L}{2} - \frac{a_2}{2} + d_2.$$

Summation of the forces in the horizontal direction results in:

$$\Delta P_5 + \Delta P_6 = 0 \quad (8.69)$$

From the $\sum F_z = 0$, the following equation can be achieved:

$$\begin{aligned} & -V_4 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right) + V_5 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right) + V_6 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right) = 0 \\ \therefore & (\overline{EI})_4 \frac{d^3 w_4 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right)}{dx^3} - (\overline{EI})_5 \frac{d^3 w_5 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right)}{dx^3} \\ & - (\overline{EI})_6 \frac{d^3 w_6 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right)}{dx^3} = 0 \end{aligned} \quad (8.70)$$

By taking $\sum M_o = 0$, one can acquire:

$$\begin{aligned} & M_4 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right) - M_5 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right) - M_6 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right) \\ & + (\overline{Y} - \overline{Y}_6) \Delta P_6 - (\overline{Y}_5 - \overline{Y}) \Delta P_5 = 0 \\ \therefore & -(\overline{EI})_4 \frac{d^2 w_4 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right)}{dx^2} + (\overline{EI})_5 \frac{d^2 w_5 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right)}{dx^2} \\ & + (\overline{EI})_6 \frac{d^2 w_6 \left(x = \frac{L}{2} - \frac{a_2}{2} + d_2 \right)}{dx^2} + (\overline{Y} - \overline{Y}_6) \Delta P_6 - (\overline{Y}_5 - \overline{Y}) \Delta P_5 = 0 \end{aligned} \quad (8.71)$$

Analogously, the subsequent expressions can be achieved by employing the balance of forces and moments at $x = \frac{L}{2} + \frac{a_2}{2} + d_2$.

Through $\sum F_z = 0$ the following equation can be established:

$$V_7 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right) - V_5 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right) - V_6 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right) = 0$$

$$\begin{aligned} \therefore -(\overline{EI})_7 \frac{d^3 w_7 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right)}{dx^3} + (\overline{EI})_5 \frac{d^3 w_5 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right)}{dx^3} \\ + (\overline{EI})_6 \frac{d^3 w_6 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right)}{dx^3} = 0 \end{aligned} \quad (8.72)$$

By having moment balance about the point O , one can develop the expression below:

$$\begin{aligned} M_5 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right) + M_6 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right) - M_7 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right) \\ - (\overline{Y} - \overline{Y}_6) \Delta P_6 + (\overline{Y}_5 - \overline{Y}) \Delta P_5 = 0 \\ \therefore -(\overline{EI})_5 \frac{d^2 w_5 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right)}{dx^2} - (\overline{EI})_6 \frac{d^2 w_6 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right)}{dx^2} \\ + (\overline{EI})_7 \frac{d^2 w_7 \left(x = \frac{L}{2} + \frac{a_2}{2} + d_2 \right)}{dx^2} - (\overline{Y} - \overline{Y}_6) \Delta P_6 + (\overline{Y}_5 - \overline{Y}) \Delta P_5 = 0 \end{aligned} \quad (8.73)$$

Figure 8.5: Axial force (P_1) versus deflection plot

In this section, a mathematical model which comprises of equations (8.37) to (8.40) (equations (8.××a) for clamped cases and expressions (8.××b) for simply-supported scenarios), equations (8.41) through (8.57), expressions (8.60) and (8.63) and equations (8.64) till (8.73) is developed. This analytical model can work in conjunction with the deflection equations that are derived in the previous section for the individual beams to

look into the stability problem of a thin composite beam with non-coinciding delaminations under combined axial and uniformly distributed transverse loads. This analytical model considers the contact effect at the delaminated region and is named as the “contact-accommodated model” as mentioned in the preceding section. It should be reminded that beams 1, 4 and 7 ought to employ the expressions (8.10) through (8.13). Conversely, beams 2, 3, 5 and 6 will need expressions (8.33) till (8.36), respectively, alongside with their derivatives. Upon substituting the appropriate expressions for the individual beams into the 32 equations present in the mathematical model, the resultant 32 expressions can be written in a matrix structure as:

$$[M]\{C\} = \{Q\} \quad (8.74)$$

The $[M]$ is a 32×32 square matrix that is composed of only λ_1 . The column vector $\{C\}$ is comprised of the 28 unknown coefficients from the deflection equations for the separate beams and 4 unidentified incremental forces ($\Delta P_2, \Delta P_3, \Delta P_5$ and ΔP_6). On the other hand, $\{Q\}$ is a column vector that consists of all the constants that are brought to the right hand side of all the 32 equations. By first assuming a low value for λ_1 , the $[M]$ becomes a defined matrix. With both the $[M]$ and $\{Q\}$ recognized, all the 32 unknowns residing in the column vector $\{C\}$ can be solved from equation (8.74). With all the unknown coefficients in the deflection equations for the divided beams resolved, the entire deformed profile for the thin composite beam with non-coinciding delaminations is revealed. One can then take note of the amount of mid-point deflection that is endured by the composite beam at this level of λ_1 . This process can be repeated by increasing the value for λ_1 and recording the mid-point deflection as the λ_1 changes. The readings for the axial load P_1 (which can be calculated using $P_1 = \lambda_1^2 (\overline{EI})_1$) and the mid-point deflection can then be graphed into a curve like in Figure 8.5. The required buckling load can then be acquired from the plot to be the load when the graph becomes horizontal.

8.4 Observations and analysis

In this section, parametric studies are conducted on the thin composite beam with non-coinciding delaminations. Both the axial and uniformly distributed transverse loads are

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exerted onto the composite beam and contact effect is also considered to avoid interpenetration. The two methods of determining the effective flexural rigidity of a composite beam that are derived in section 4.3.1.1 will be used together with the mathematical model established here to evaluate the longed critical load for the composite beam wreaked with non-coinciding delaminations. These two approaches end up to be ${}^2(\overline{EI})$ and ${}^3(\overline{EI})$ in equations (4.54) and (4.73), respectively. Parallel with the mathematical model, the finite element method that includes the contact effect will be employed to verify the analytical results. This finite element method will be described as the “contact-accommodated simulation” for the remaining of this chapter. The composite materials that are utilized in the study consist of carbon/epoxy, boron/epoxy and glass epoxy. The mechanical properties for these composite materials are tabulated in Table 5.1. The composite beam with non-coinciding delaminations to be examined in this section has an overall length of 100mm, total width of 20mm and the thickness per layer is 0.125mm. For all the cases that are studied in this section, the thin composite beams are all comprised of 16 plies. Both the clamped and simply-supported edge conditions are investigated. Furthermore, anti-symmetric angle-ply composite beam is analyzed. The other aspects that may influence the buckling load of the thin composite beam with non-coinciding delaminations include ply-orientation, along-the-span position of the delaminations, delamination length and through-the-thickness location of the delaminations.

8.4.1 Carbon/epoxy composite material

The clamped and simply-supported anti-symmetric angle-ply composite beam with non-coinciding delaminations addressed in this section is manufactured from carbon/epoxy composite matter.

8.4.1.1 Clamped ends

(a)

(b)

(c)

Figure 8.6: Buckling load of $[(-\theta/\theta)_3 // (-\theta/\theta)_2 // (-\theta/\theta)_3]$ composite beam with non-coinciding delaminations in which $d_1 = 0$, (a) $a_1 = 20\text{mm}$, $a_2 = 5\text{mm}$ and $d_2 = 30\text{mm}$; (b) $a_1 = 40\text{mm}$, $a_2 = 5\text{mm}$ and $d_2 = 35\text{mm}$ and (c) $a_1 = 40\text{mm}$, $a_2 = 25\text{mm}$ and $d_2 = 35\text{mm}$ under combined axial and transverse loads, accounting for contact effect

(a)

(b)

(c)

Figure 8.7: Buckling load of $[(-\theta/\theta)_4/(-\theta/\theta)_4]$ composite beam with non-coinciding delaminations in which $d_1 = 0$, (a) $a_1 = 20\text{mm}$, $a_2 = 25\text{mm}$ and $d_2 = 30\text{mm}$; (b) $a_1 = 40\text{mm}$, $a_2 = 25\text{mm}$ and $d_2 = 35\text{mm}$ and (c) $a_1 = 50\text{mm}$, $a_2 = 5\text{mm}$ and $d_2 = 37.5\text{mm}$ under combined axial and transverse loads, accounting for contact effect

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In all the diagrams above, the methods 2 and 3 refer to the two unique techniques of computing the effective bending rigidity of a composite beam. The method 2 points to ${}^2(\overline{EI})$ which can be calculated employing equation (4.54) while method 3 implies ${}^3(\overline{EI})$ which can be determined via expression (4.73). Such designation will be used for the rest of this chapter. Analogous to chapter seven, the method 1 ${}^1(\overline{EI})$ will not be included in this chapter.

Figures 8.6 (a) till (c) demonstrates the critical load of $[(-\theta/\theta)_3 // (-\theta/\theta)_2 // (-\theta/\theta)_3]$ composite beam with non-coinciding delaminations for various a_1, a_2 and d_2 but letting $d_1 = 0$. In the analysis, the contact effect is considered and the composite beam is subjected to both axial and uniformly distributed transverse loads. Furthermore, the angle θ stretches from 0° to 90° . In all the three plots, the methods 2 and 3 are able to work in conjunction with the contact-accommodated model to produce the desired upper and lower bounds, respectively, embracing the readings that are gathered from the contact-accommodated simulation. By comparing Figures 8.6 (a) and (b), one can spot that rise in the a_1 value lowers that buckling load of the composite beam. Moreover, looking at Figures 8.6 (b) and (c) suggests that increase in a_2 reduces the load carrying capacity of the thin composite beam with non-coinciding delaminations.

On the other hand, Figures 8.7 (a) through (c) depicts the buckling load for the $[(-\theta/\theta)_4 // (-\theta/\theta)_4]$ composite beam under both axial and uniformly distributed vertical forces. Thus, for this composite beam, both the non-coinciding delaminations are located at the mid-plane. However, the length and spanwise position of the two delaminations are not the same. Moreover, the contact effect is included to avoid interpenetration at the delaminated zones. Similarly, from Figure 8.7 (a) to (c), one can observe that the exact buckling loads that are generated by the contact-accommodated simulation are still encompassed by the upper and lower limits that are constructed utilizing the methods 2 and 3, respectively. This further proves that the methods 2 and 3 are capable of working in association with the contact-accommodated model to construct the longed upper and lower curves. Comparing Figures 8.7 (a) and (b) confirms that increase in the a_1 will decrease the load bearing ability of the composite beam. Moreover, looking at Figures 8.6

(c) and 8.7 (b) tells that the critical load of the composite beam falls as the non-coinciding delaminations get deeper into the composite beam, approaching the mid-plane. Additional finding involves drop of the buckling load of the thin composite beam with non-coinciding delaminations as the ply-angle θ becomes larger.

8.4.1.2 Simply-supported edges

(a)

(b)

(c)

Figure 8.8: Critical load of $[(-\theta/\theta)_2 // (-\theta/\theta)_3 // (-\theta/\theta)_3]$ composite beam with non-coinciding delaminations in which $d_1 = d_2 = 25\text{mm}$, (a) $a_1 = a_2 = 20\text{mm}$; (b) $a_1 = 40\text{mm}$, $a_2 = 10\text{mm}$ and (c) $a_1 = a_2 = 40\text{mm}$ under combined axial and transverse loads, considering contact effect

Figure 8.9: Critical load of $[(-\theta/\theta)_3 // -\theta/\theta // (-\theta/\theta)_4]$ composite beam with non-coinciding delaminations in which $a_1 = a_2 = 40\text{mm}$ and $d_1 = d_2 = 25\text{mm}$ under combined axial and transverse loads, considering contact effect

This section examines the buckling load of simply supported composite beam with non-coinciding delaminations at $d_1 = d_2 = 25\text{mm}$ and contact effect under combined horizontal and uniformly distributed transverse loads. Figure 8.8 (a) till (c) demonstrates the results for the $[(-\theta/\theta)_2 // (-\theta/\theta)_3 // (-\theta/\theta)_3]$ composite beam for several

delamination lengths while Figure 8.9 exhibits the outcome for the $[(-\theta/\theta)_3//-\theta/\theta/(-\theta/\theta)_4]$ composite beam with $a_1 = a_2 = 40\text{mm}$. For all the four cases that are analyzed here, all the points that are collected via the contact-accommodated simulation are still captured by the upper and lower bounds erected utilizing the methods 2 and 3, respectively, together with the contact-accommodated model. Among the four scenarios covered in this section, the case addressed in Figure 8.8 (a) has the highest critical load. Comparing Figures 8.8 (b) and (c) reveals that higher a_2 has a negative influence on the load carrying capacity of the composite beam. Looking at Figure 8.8 (c) and 8.9, one can realize that the buckling load of the composite drops as the non-coinciding delaminations drift further away from the top and bottom free surfaces into the composite beam. The other observation is that the critical load of the composite beam decreases as the angle θ becomes larger.

8.4.2 Other composite materials

In addition of the carbon/epoxy composite matter, the thin boron/epoxy and E-glass/epoxy composite beams with non-coinciding delaminations, withstanding combined axial and uniformly distributed transverse loads are studied. Furthermore, the contact effect is implemented to prevent interpenetration at the delaminated regions and both the clamped and simply-supported boundary conditions are discussed.

8.4.2.1 Clamped boundary condition

(a)

(b)

Figure 8.10: Buckling load of $[(-\theta/\theta)_3 // (-\theta/\theta)_2 // (-\theta/\theta)_3]$ boron/epoxy beam with non-coinciding delaminations in which $d_1 = 0$, (a) $a_1 = 20\text{mm}$, $a_2 = 5\text{mm}$ and $d_2 = 30\text{mm}$ and (b) $a_1 = 40\text{mm}$, $a_2 = 5\text{mm}$ and $d_2 = 35\text{mm}$ under combined axial and transverse loads, accounting for contact effect

(a)

(b)

(c)

Figure 8.11: Buckling load of $[(-\theta/\theta)_4//(-\theta/\theta)_4]$ boron/epoxy beam with non-coinciding delaminations in which $d_1 = 0$, (a) $a_1 = 20\text{mm}$, $a_2 = 25\text{mm}$ and $d_2 = 30\text{mm}$; (b) $a_1 = 40\text{mm}$, $a_2 = 25\text{mm}$ and $d_2 = 35\text{mm}$ and (c) $a_1 = 50\text{mm}$, $a_2 = 5\text{mm}$ and $d_2 = 37.5\text{mm}$ under combined axial and transverse loads, accounting for contact effect

(a)

(b)

Figure 8.12: Buckling load of $[(-\theta/\theta)_3 // (-\theta/\theta)_2 // (-\theta/\theta)_3]$ E-glass/epoxy beam with non-coinciding delaminations in which $d_1 = 0$, (a) $a_1 = 20\text{mm}$, $a_2 = 5\text{mm}$ and $d_2 = 30\text{mm}$ and (b) $a_1 = 40\text{mm}$, $a_2 = 5\text{mm}$ and $d_2 = 35\text{mm}$ under combined axial and transverse loads, accounting for contact effect

(a)

(b)

(c)

Figure 8.13: Buckling load of $[(-\theta/\theta)_4/(-\theta/\theta)_4]$ E-glass/epoxy beam with non-coinciding delaminations in which $d_1 = 0$, (a) $a_1 = 20\text{mm}$, $a_2 = 25\text{mm}$ and $d_2 = 30\text{mm}$; (b) $a_1 = 40\text{mm}$, $a_2 = 25\text{mm}$ and $d_2 = 35\text{mm}$ and (c) $a_1 = 50\text{mm}$, $a_2 = 5\text{mm}$ and $d_2 = 37.5\text{mm}$ under combined axial and transverse loads, accounting for contact effect

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A glance at the Table 5.1 discloses that carbon/epoxy, boron/epoxy and E-glass/epoxy have significantly different material properties. Therefore, this section can be used to see whether the conclusions that are drawn on the carbon/epoxy composite beam in section 8.4.1.1 are also valid for other dissimilar composite matters. From the comparison of the graphs above for the individual boron/epoxy and E-glass/epoxy composite materials, it is again verified that increases in a_1 and a_2 and moving of the delaminations towards to the mid-plane lower the buckling load of the composite beam. Likewise, increase in the ply-angle θ will also have a harmful effect on the load bearing capacity of the composite beam. Furthermore, for all the cases considered, the proposed methods 2 and 3 are still able to operate together with the contact-accommodated model to provide the necessary higher and lower limits for the true buckling loads that are gained from the contact-accommodated simulation. This means that the suggested methods 2 and 3 apply not only to carbon/epoxy but also to boron/epoxy and E-glass/epoxy composite materials as well.

8.4.2.2 Simply-supported boundary condition

(a)

(b)

(c)

Figure 8.14: Critical load of $[(-\theta/\theta)_2 // (-\theta/\theta)_3 // (-\theta/\theta)_3]$ boron/epoxy beam with non-coinciding delaminations in which $d_1 = d_2 = 25\text{mm}$, (a) $a_1 = a_2 = 20\text{mm}$; (b) $a_1 = 40\text{mm}$, $a_2 = 10\text{mm}$ and (c) $a_1 = a_2 = 40\text{mm}$ under combined axial and transverse loads, considering contact effect

Figure 8.15: Critical load of $[(-\theta/\theta)_3 // -\theta/\theta // (-\theta/\theta)_4]$ boron/epoxy beam with non-coinciding delaminations in which $a_1 = a_2 = 40\text{mm}$ and $d_1 = d_2 = 25\text{mm}$ under combined axial and transverse loads, considering contact effect

(a)

(b)

(c)

Figure 8.16: Critical load of $[(-\theta/\theta)_2 // (-\theta/\theta)_3 // (-\theta/\theta)_3]$ E-glass/epoxy beam with non-coinciding delaminations in which $d_1 = d_2 = 25\text{mm}$, (a) $a_1 = a_2 = 20\text{mm}$; (b) $a_1 = 40\text{mm}$, $a_2 = 10\text{mm}$ and (c) $a_1 = a_2 = 40\text{mm}$ under combined axial and transverse loads, considering contact effect

Figure 8.17: Critical load of $[(-\theta/\theta)_3 // -\theta/\theta // (-\theta/\theta)_4]$ E-glass/epoxy beam with non-coinciding delaminations in which $a_1 = a_2 = 40\text{mm}$ and $d_1 = d_2 = 25\text{mm}$ under combined axial and transverse loads, considering contact effect

In this section, the simply-supported thin boron/epoxy and E-glass epoxy beams with non-coinciding delaminations under combined axial and uniformly distributed transverse loads are explored with contact effect. Analogous to the arguments in section 8.4.1.2, the results in this section indicates that increase in the delamination length and angle θ and

progress of the delaminations towards the mid-plane have a detrimental effect on the buckling load of the composite beam. Moreover, all the plots in this section illustrate that the data that is attained via the contact-accommodated simulation is all successfully enveloped by the upper and lower bounds that are assembled employing the methods 2 and 3, respectively. The other conclusion that can be drawn from this chapter is that the boron/epoxy appears to have the most superior load carrying capacity followed by the carbon/epoxy, while the E-glass/epoxy has the worst load bearing ability.

8.5 Concluding remarks

In this chapter, the novel contact-accommodated mathematical model is developed to study the behavior of the thin composite beam with non-coinciding delaminations under both the axial force and uniformly distributed load. The contact effect is accounted in the contact-accommodated model to prevent any interpenetration in the delaminated regions. The different expressions for the individual beams in Figure 8.1 are derived in section 8.2. The beams 1, 4 and 7 should use equation (8.10). On the other hand, the beams 2, 3, 5 and 6 ought to employ equations (8.33), (8.34), (8.35) and (8.36) for their deflection profiles. The 32 essential equations that are involved in the contact-accommodated model are addressed in section 8.3. Furthermore, the entire phase of solving for the desired buckling load is explained in the last portion of section 8.3.

Various parametric tests are performed on the thin carbon/epoxy beam with non-coinciding delaminations withstanding the combined axial and uniformly distributed transverse loads, considering contact effect. The clamped $[(-\theta/\theta)_3 // (-\theta/\theta)_2 // (-\theta/\theta)_3]$ and $[(-\theta/\theta)_4 // (-\theta/\theta)_4]$ composite beams are examined for various ply-angles, lengthwise positions of the delaminations, through-the thickness locations of the delaminations and delaminations lengths. In addition, the simply-supported $[(-\theta/\theta)_2 // (-\theta/\theta)_3 // (-\theta/\theta)_3]$ and $[(-\theta/\theta)_3 // -\theta/\theta // (-\theta/\theta)_4]$ composite beams are studied. It is found that larger ply-angle and longer delaminations have a harmful effect on the buckling load of the composite beam. Moreover, moving the delaminations deeper into the composite beam will lower the critical load. For all the investigations that are done on the carbon/epoxy beam, the suggested $^2(\overline{EI})$ and $^3(\overline{EI})$ are able to work together

with the contact-accommodated model to form the desired upper and lower limits for the actual buckling load that is attained from the contact-accommodated simulation, respectively.

Aside from carbon/epoxy, boron/epoxy and E-glass/epoxy composite materials are also utilized in the analysis. It is also noticed that boron/epoxy has the best load bearing ability, followed by the carbon/epoxy, while E-glass/epoxy has the poorest load carrying capacity.

CHAPTER NINE: CONCLUSIONS AND FUTURE WORK

9.1 Conclusions

Three different methods of computing the effective flexural stiffness of the thin laminated composite beam are addressed. These three dissimilar methods are derived from the CLT through different sets of assumptions. They are method 1 $^1(\overline{EI})$, method 2 $^2(\overline{EI})$ and the newly developed method 3 $^3(\overline{EI})$, which can be found in expressions (4.45), (4.55) and (4.73), respectively. These equations are applied to the conventional Euler's buckling formula and the results are compared with those that are attained via the finite element method. The thin perfect composite beam in the investigation has a full length, overall width and thickness per ply of 100mm, 20mm and 0.125mm, respectively. This configuration is kept the same throughout all the studies for consistency. The thin ideal composite beam is composed of the carbon/epoxy anti-symmetric laminates and both the clamped and simply-supported boundary conditions are examined. The total number of layers used includes 2, 4, 8 and 16. It is observed that the data generated by the finite element method can be successfully encompassed by the upper bound created utilizing the method 2 $^2(\overline{EI})$ and the lower limit constructed via the novel method 3 $^3(\overline{EI})$. The method 2 $^2(\overline{EI})$ is found to produce solution that is closer to the actual buckling load computed from the finite element method than method 1 $^1(\overline{EI})$. This implies that method 2 $^2(\overline{EI})$ is a better mean of attaining the upper bound than method 1 $^1(\overline{EI})$. Analogously, three different techniques are also established from the FSDT to look into moderately thick composite beam. They consist of method 1 $(^1(\overline{EI}), ^1(\overline{Gh}) \text{ and } ^1K)$; method 2 $(^2(\overline{EI}), ^2(\overline{Gh}) \text{ and } ^2K)$ and method 3 $(^3(\overline{EI}), ^3(\overline{Gh}) \text{ and } ^3K)$, which can be evaluated employing equations (4.45), (4.91) and (4.162); (4.55), (4.96) and (4.162) and (4.73), (4.105) and (4.170), respectively. This time the composite beam is much thicker, comprising of either 40 or 48 plies. It is noticed that the suggested method 2 $(^2(\overline{EI}), ^2(\overline{Gh}) \text{ and } ^2K)$ and 3 $(^3(\overline{EI}), ^3(\overline{Gh}) \text{ and } ^3K)$ are capable of forming the desired upper and lower bounds for the actual buckling load. It is observed that method 2

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$\left({}^2(\overline{EI}), {}^2(\overline{Gh}) \text{ and } {}^2K \right)$ always grant a better upper bound than method 1 $\left({}^1(\overline{EI}), {}^1(\overline{Gh}) \text{ and } {}^1K \right)$ as it is able to produce results that are nearer to the readings attained via the finite element method all the time.

In chapter five, the buckling of the thin composite beam with single delamination is investigated. As only thin composite beam is considered, just the three approaches that are established from the CLT are applied in chapter five. An analytical model is formulated based on the classical beam theory through the boundary conditions, kinematical compatibility and balance of the interior forces and moments. The three different ways of calculating the effective bending rigidity of the composite beam can then be used in conjunction with the mathematical model to yield the longed theoretical buckling load predictions. From the investigation on the 8 layered single-delaminated carbon/epoxy anti-symmetric composite beam, it is detected that increases in the delamination length and ply-angle have a negative effect on the load carrying ability of the composite beam for both clamped and simply-supported end conditions. It is also seen that the critical load falls as the delamination resides deeper into the beam. The 16 plied clamped generally orientated carbon/epoxy composite beam with a delamination at the mid-plane is also examined for various delamination spans. Furthermore, the study is extended to other fiber-reinforced composite materials such as the boron/epoxy and E-glass/epoxy. It is noticed that out of the three composite materials that are employed in the analysis, the boron/epoxy gives the highest critical load while the E-glass/epoxy yields the lowest buckling load. For all the cases considered, it is discovered that the method 2 ${}^2(\overline{EI})$ and method 3 ${}^3(\overline{EI})$ are still capable of building the wanted upper and lower bounds for the exact critical load that is acquired from the finite element method. Similar to the thin perfect composite beam that is discussed in the previous paragraph, the method 2 ${}^2(\overline{EI})$ always deliver a more superior upper bound than method 1 ${}^1(\overline{EI})$ for the delaminated case.

In chapter six, the stability behavior of the moderately thick single-delaminated composite beam is studied. A mathematical model is developed employing the first order shear deformation beam theory that accounts for the transverse shear deformation. Then,

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the three methods of evaluating the $(\overline{EI})$, $(\overline{Gh})$ and K (that are derived via FSDT) are used together with the analytical model to determine the buckling load for the moderately thick composite beam with a delamination. Analogous to chapter five, influencing parameters like the type of composite materials utilized, boundary conditions, ply orientation, layering sequence, delamination length and through-the thickness position of the delamination are studied. However, in chapter six, the delaminated composite beam is made up of 48 plies and thus much thicker than that in chapter five. Nevertheless, the above-mentioned factors affect the buckling load of the moderately thick composite beam with a delamination in a similar fashion as the critical load of the thin single-delaminated composite beam in chapter five. Furthermore, it is noticed that the buckling solutions from the method 2 $(^2(\overline{EI}), ^2(\overline{Gh}) \text{ and } ^2K)$ and method 3 $(^3(\overline{EI}), ^3(\overline{Gh}) \text{ and } ^3K)$ are able to offer the desired upper and lower limits for the actual critical load that is obtained via the finite element method. Parallel to the moderately thick ideal composite beam, the method 2 $(^2(\overline{EI}), ^2(\overline{Gh}) \text{ and } ^2K)$ will present a better upper limit than method 1 $(^1(\overline{EI}), ^1(\overline{Gh}) \text{ and } ^1K)$ all the time.

The chapter seven is dedicated to the study of the thin composite beam containing the semi-embedded delaminations, with and without the contact analysis. Three different piling arrangements, consisting of $[-\theta/\theta/-\theta/(\theta-\theta)_2/\theta/(-\theta/\theta)_4]$, $[(-\theta/\theta)_2/-\theta/\theta/-\theta/\theta/(-\theta/\theta)_4]$ and $[(-\theta/\theta)_3/-\theta/\theta/(-\theta/\theta)_4]$ are used in the investigation and they are denoted as the configurations 1, 2 and 3, respectively. The mathematical model that does not take into account of the contact analysis is known as the contactless model while the analytical model that considers the contact analysis is called the contact-integrated model. Once again, the method 2 $(^2(\overline{EI}))$ and method 3 $(^3(\overline{EI}))$ are applied to the analytical models to investigate the buckling problem. All the delaminated composite beams that are analyzed in chapter seven are manufactured from 16 plies of carbon/epoxy composite material. In the study on the specially orthotropic thin composite beam, the a_1 is set to be 90mm while the a_2 is allowed to vary. Both the clamped and simply-supported boundary conditions are examined. It is found that there is no interpenetration of the sub-laminates for configuration 1 and the contactless model

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should be used. On the other hand, the configurations 2 and 3 should employ the contact-integrated model as interpenetration of the sub-laminates occurs in these two cases. It is discovered that the presence of the contact effect becomes more crucial as the delamination a_2 gets longer. It is also noted that the configuration 3 has a more serious interpenetration than the configuration 2. Generally, the readings from the mathematical models are in great agreement with the values that are attained via the finite element method. Moreover, the anti-symmetric angle-ply configurations 1, 2 and 3 are studied for three different pairs of a_1 and a_2 . Once more, both the clamped and simply-supported edge conditions are investigated. Applying the contactless model on configuration 1 and utilizing the contact-integrated model on configurations 2 and 3, the method $2^2(\overline{EI})$ and method $3^3(\overline{EI})$ are once again able to erect the desired upper and lower limits, respectively, for the buckling results obtained from their respective finite element analyses. As expected, increase in the a_1 and a_2 and rise in the ply-angle have a detrimental effect on the load bearing capacity of the thin composite beam with semi-embedded delaminations.

In chapter eight, the stability performance of the thin composite beam inflicted with the non-coinciding delaminations under combined axial and uniformly distributed transverse loads is examined. A mathematical model that incorporates the contact effect is formulated to look into the problem and it is named as the contact-accommodated model. The contact effect is implemented to avoid the physically impermissible interpenetration of the sub-laminates in the delaminated zones. The edge conditions, kinematical continuity and equilibrium of internal forces and moments are also involved in the formulation of the analytical model. Then, the method $2^2(\overline{EI})$ and method $3^3(\overline{EI})$ are applied in conjunction with the mathematical model to determine the required buckling load. The carbon/epoxy, boron/epoxy and E-glass/epoxy composite materials are utilized in the study. The 16 layers anti-symmetric angle-ply delaminated composite beam is employed in the analysis. Further aspects that are considered in the investigation include the delaminations length, along-the-length position of the delaminations, through-the-thickness location of the delaminations and the ply-orientation. Moreover, both the clamped and simply-supported end conditions are explored. The conclusions that can be

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drawn are as the delaminations length and ply-angle rise, the critical load of the composite beam drops. It is also observed that the buckling load of the composite beam falls as the delaminations get deeper into the composite beam. Out of the three composite materials, it is noted that the boron/epoxy has the highest load carrying ability, followed by the carbon/epoxy while the E-glass/epoxy has the most inferior loading bearing capacity. The other important finding is that for all the cases that are considered, the suggested method 2 $^2(\overline{EI})$ and method 3 $^3(\overline{EI})$ are competent of working with the developed analytical model to produce the wanted upper bound and lower limit for the buckling readings that are gathered from the contact-accommodated simulation.

Some unusual or surprise results can be observed from the work in this thesis. One of these unusual or surprise results are the Figures 7.15 (a) and (b), and 7.17 in chapter seven where it is found that the buckling load results for the cases where contact effect is not accounted can be very different from those that take contact effect into consideration. The other unusual or surprise results comes from Figures 5.17 (a) to (c), 5.18 (a) to (c), 6.13 (a) to (c), 6.14 (a) to (c), 8.12 (a) and (b), 8.13 (a) to (c), 8.16 (a) to (c) and 8.17. From these diagrams which are the buckling load results for E-glass/epoxy composite material, it is unusual or surprising to detect that the loop that is formed by the upper and lower buckling load bounds appears to be rather smaller than those of the carbon/epoxy and boron/epoxy composite materials.

The current research in this thesis can be applied to investigate practical fiber-reinforced composite structures with or without delamination. One of the practical applications of the present research in this thesis is on the wing of some aircrafts. It is stated by Cutler [13] that the skin of the wing of certain aircrafts is composed of fiber-reinforced composite materials. When such an aircraft is in flight, the upward force on the wing will cause the wing to bend upward. This will in turn exert a compressive force on the fiber-reinforced composite upper skin of the wing, which may lead to buckling. The next practical application of the current research in this thesis is regarding the chimney. Bank [6] and Peters [60] mention that fiber-reinforced polymers (FRP) composite material strip or tape is applied along the length of the chimney in practice. This is to help in resisting

bending of the chimney as a result of the wind load. Hence, the present research in this thesis can be employed to monitor or analyze whether the strip that is applied on the side of the chimney undergoing compression will suffer buckling under certain wind load. The present research in this thesis can also be extended to study sandwich beam. Altenbach et al. [2] and Cheremisinoff [12] point out that fiber-reinforced composite materials can be employed as facing materials for the sandwich beam. Shenoi and Wellicome [68] remarks that it is possible for sandwich beam to experience a failure known as wrinkling when it is under compression. It is explained by Bitzer [8] that wrinkling is a kind of local buckling when the compressive facing buckles into the core or buckles outward and fails the facing-to-core adhesive bond. Hence, the current research in this thesis can be employed to study this local buckling of the fiber-reinforced composite facing.

9.2 Future work

In this section, recommendations for the future work are listed:

- 1) The three techniques of evaluating the effective flexural stiffness $\left({}^j\overline{EI} \right)$, for $j = 1..3$ can be applied to look into bending and free vibration problems of the thin perfect and delaminated composite beams.
- 2) The three unique groups of equations to determine the three separate sets of values for the effective flexural stiffness, effective shear modulus and shear correction factor $\left({}^1\overline{EI}, {}^1\overline{Gh}, {}^1K; {}^2\overline{EI}, {}^2\overline{Gh}, {}^2K \text{ and } {}^3\overline{EI}, {}^3\overline{Gh}, {}^3K \right)$ in the first-order shear deformation beam theory can be employed to study the buckling behavior of the moderately thick laminated beam with double delaminations (instead of only single delamination).
- 3) The present research can be extended to analyze hybrid thin or moderately thick and ideal or delaminated composite beam that is made up of two or more kinds of fiber-reinforced composite materials with different piling sequence and ply-orientation.

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- 4) The influence of the coupling effects from the delaminated composite beam with an arbitrary stacking sequence on the fracture mechanics at the delamination front during the buckling and post-buckling phases can also be examined.
- 5) Work can be done to improve the accuracies of both the upper and lower bounds so that they are closer to the desired actual buckling load for the thin or moderately thick and perfect or delaminated composite beam cases.
- 6) The research into the limits of when the interpenetration of layers will occur can also be explored.

LIST OF PUBLICATIONS

1. Yap, C.W. and Chai, G.B., 2007, "Analytical and numerical studies on the buckling of delaminated composite beams", Journal of Composite Structures, Vol. 80, pp. 307-319.
2. Yap, C.W., Chai, G.B. and Parlapalli, M.S.R., 2008, "Effect of flexural stiffness estimates on the buckling load of delaminated composite beams", Proceedings of the Institution of Mechanical Engineers, Part L: Journal of Materials: Design and Applications, Vol. 222, No. 2, pp. 91-102.
3. Chai, G.B. and Yap, C.W., 2008, "Coupling effects in bending, buckling and free vibration of generally laminated composite beams", Journal of Composite Science and Technology, Vol. 68, pp. 1664-1670.

REFERENCES

1. Ali Kouchakzadeh, M. and Sekine, H., 2000, "Compressive buckling analysis of rectangular composite laminates containing multiple delaminations", *Journal of Composite Structures*, Vol. 50, pp. 249-255.
2. Altenbach, H., Altenbach, J. and Kissing, W., 2004, "Mechanics of Composite Structural elements", Springer-Verlag.
3. Anderson, T.L., 2005, "Fracture mechanics: Fundamentals and applications, 3rd Edition", CRC Press Taylor & Francis Group.
4. Andrews, M.G., Massabo, R. and Cox, B.N., 2006, "Elastic interaction of multiple delaminations in plates subject to cylindrical bending", *International Journal of Solids and Structures*, Vol. 43, pp.855-886.
5. Aslan, Z. and Sahin, M., 2009, "Buckling behavior and compressive failure of composite laminates containing multiple large delaminations", *Journal of Composite Structures*, Vol. 89, pp. 382-390.
6. Bank, L.C., 2006, "Composites for construction: Structural design with FRP materials", John Wiley & Sons.
7. Bert, C.W., 1984, "A critical evaluation of new plate theories applied to laminated composites", *Journal of Composite Structures*, Vol. 2, pp. 329-347.
8. Bitzer, T., 1997, "Honeycomb technology: Materials, design, manufacturing, applications and testing", Chapman & Hall.
9. Chen, H. and Sun, X., 1999, "Residual compressive strength of laminated plates with delamination", *Journal of Composite Structures*, Vol. 47, pp. 711-717.
10. Chen, Hsin-Piao, 1991, "Shear deformation theory for compressive delamination buckling and growth", *Journal of AIAA*, Vol. 29, No. 5, pp. 813-819.
11. Chen, Hsin-Piao, 1993, "Transverse shear effects on buckling and postbuckling of laminated and delaminated plates, *Journal of AIAA*, Vol. 31, No. 1, pp. 163-169.
12. Cheremisinoff, N.P., 1990, "Handbook of ceramics and composites: Synthesis and properties", CRC Press.

NANYANG TECHNOLOGICAL UNIVERSITY

13. Cutler, J., 1999, "Understanding aircraft structures, 3rd edition", Blackwell Science Ltd.
14. Daridon, L. and Zidani, K., 2002, "The stabilizing effects of fiber bridges on delamination cracks", *Journal of Composite Science and Technology*, Vol. 62, pp. 83-90.
15. Davidson, B.D. and Ferrie, C.H., 1994, "Effect of stretching-shearing coupling on instability-related delamination growth", *Journal of Composite Structures*, Vol. 29, pp. 383-392.
16. Decolon, C., 2004, "Analysis of composite structures", Kogan Page Science.
17. Della, C.N. and Shu, D., 2007, "Free vibration analysis of delaminated biomaterial beams", *Journal of Composite Structures*, Vol. 80, pp. 212-220.
18. de Morais, A.B., 2003, "Double cantilever beam testing of multidirectional laminates", *Journal of Composites: Part A*, Vol. 34, no. 12, pp. 1135-1142.
19. de Morais, A.B., de Moura, M.F, Marques, A.T. and de Castro, P.T., 2002, "Mode-I interlaminar fracture of carbon/epoxy cross-ply composites", *Journal of Composites Science and Technology*, Vol. 62, pp. 679-686.
20. de Morais, A.B. and de Moura, M.F.S.F., 2005, "Assessment of initiation criteria used in interlaminar fracture tests of composites", *Journal of Engineering Fracture Mechanics*, Vol. 72, pp. 2615-2627.
21. Fraternali, F., 1996, "Energy release rates for delamination of composite beams", *Journal of Theoretical and Applied Fracture Mechanics*, Vol. 25, pp. 225-232.
22. Gaudenzi, P., 1997, "On delamination buckling of composite laminates under compressive loading", *Journal of Composite Structures*, Vol. 39, No. 1-2, pp. 21-30.
23. Gaudenzi, P., Perugini, P. and Riccio, A., 2001, "Post-buckling behavior of composite panels in the presence of unstable delaminations", *Journal of Composite Structures*, Vol. 51, pp. 301-309.
24. Ge, D., Liu, F. and Lu, M., 2004, "Effect of mode III fracture on composite laminates", *Journal of Composite Structures*, Vol. 63, pp. 469-480.

NANYANG TECHNOLOGICAL UNIVERSITY

25. Ghugal, Y.M. and Shimpi, R.P., 2002, "A review of refined shear deformation theories of isotropic and anisotropic laminated plates", *Journal of Reinforced Plastics and Composites*, Vol. 21, No. 9, pp. 775-812.
26. Gu, H and Chattopadhyay, A., 1998, "Elasticity approach for delamination buckling of composite beam plates", *American Institute of Aeronautics and Astronautics*, Vol. 36, pp.1529-1534.
27. Gu, H and Chattopadhyay, A., 1999, "An experimental investigation of delamination buckling and post-buckling of composite laminates", *Journal of Composites Science and Technology*, Vol. 59, pp. 903-910.
28. Gui, L. and Li, Z., 2001, "Delamination buckling of stitched laminates", *Journal of Composite Science and Technology*, Vol. 61, pp. 629-636.
29. Huang, H. and Kardomateas, G.A., 1997, "Buckling of orthotropic beam-plates with multiple central delaminations", *International Journal of Solid Structures*, Vol. 35, No. 13 pp. 1355-1362.
30. Hwang, S.-F. and Huang, S.-M., 2005, "Postbuckling behavior of composite laminates with two delaminations under uniaxial compression", *Journal of Composite Structures*, Vol. 68, pp. 157-165.
31. Hwang, S.-F. and Liu, G.-H., 2001, "Buckling behavior of composite laminates with multiple delaminations under axial compression", *Journal of Composite Structures*, Vol. 53, pp.235-243.
32. Hwang, S.-F. and Liu, G.-H., 2002, "Experimental study for buckling and post-buckling behaviors of composite laminates with multiple delaminations", *Journal of Reinforced Plastics and Composites*, Vol. 21, No. 4, pp. 333-349.
33. Hwang, S.-F. and Mao, C.-P., 1999, "The delamination buckling of single-fibre system and interply hybrid composites", *Journal of Composite Structures*, Vol. 46, pp. 279-287.
34. Iyengar, N.G.R., 1988, "Structural stability of columns and plate", *Ellis Horwood*.
35. Jones, R.M., 1999, "Mechanics of composite materials", *Taylor and Francis Group*.

NANYANG TECHNOLOGICAL UNIVERSITY

36. Joshi, A. and Suryanarayan, S., 1984, "Coupled flexural-torsional vibration of beams in the presence of static axial loads and end moments", *Journal of Sound and Vibration*, Vol. 92, pp. 583-589.
37. Kapania, Rakesh K. and Raciti, Stefano, 1989, "Recent advances in analysis of laminated beams and plates, part I: shear effects and buckling" *Journal of AIAA*, Vol. 27, No.7, pp.923-934.
38. Kapania, R.K. and Wolfe, D.R., 1989, "Buckling of axially loaded beam plate with multiple delaminations", *Journal of Pressure Vessel Technology*, Vol. 111, pp. 151-158.
39. Kardomateas, G.A. and Schmuesert, D.W., 1988, "Buckling and Postbuckling of delaminated composites under compressive loads including transverse shear effects", *Journal of AIAA*, Vol. 26, No. 3, pp. 337-343.
40. Kaw, A.K., 2006, "Mechanics of composite materials, 2nd edition", CRC Press Taylor and Francis Group.
41. Kim, H.-J. and Hong, C.-S., 1997, "Buckling and post-buckling behavior of composite laminates with a delamination", *Journal of Composites Science and Technology*, Vol. 57, pp. 557-564.
42. Kutlu, Z and Chang, F.-K., 1995, "Composite panels containing multiple through-the-width delaminations and subjected to compression. Part I: analysis.", *Composite Structures*, Vol. 31, pp. 273-296.
43. Kutlu, Z and Chang, F.-K., 1995, "Composite panels containing multiple through-the-width delaminations and subjected to compression. Part II: experiments and verifications", *Composite Structures*, Vol. 31, pp. 297-314.
44. Kyoung, W.-M., Kim, C.-G. and Hong, C.-S., 1998, "Modeling of composite laminates with multiple delaminations under compressive loading", *Journal of Composite Materials*, Vol. 32, No. 10, pp. 951-968.
45. Kyoung, W.-M., Kim, C.-G. and Hong, C.-S., 1999, "Buckling and postbuckling behavior of composite cross-ply laminates with multiple delaminations", *Composite Structures*, Vol. 43, pp. 257-274.
46. Lachaud, F., Lorrain, B., Michel, L. and Barriol, R., 1998, "Experimental and numerical study of delamination caused by local buckling of thermoplastic and

NANYANG TECHNOLOGICAL UNIVERSITY

- thermoset composites”, *Composites Science and Technology*, Vol. 58, pp. 727-733.
47. Lee, Jaehong, Gurdal, Zafer and Griffin Jr., O.Hayden, 1993, “Layer-wise approach for the bifurcation problem in laminated composites with delaminations”, *American Institute of Aeronautics and Astronautics*, Vol. 31, No. 2, pp. 331-338.
48. Lee, S., Park, T. and Voyiadjis, G.Z., 2003, “Vibration analysis of multi-delaminated beams”, *Composite Part B: Engineering*, Vol. 34, pp. 647-659.
49. Lee, Y.J., Lee, C.H. and Fu, W.S., 1998, “Study on the compressive strength of laminated composite with through-the-width delamination”, *Journal of Composite Structures*, Vol. 41, pp. 229-241.
50. Lifshitz, J.M. and Gildin, D., 1997, “Failure of delaminated carbon/epoxy composite beams under cyclic compression”, *Journal of Composite Structures*, Vol. 39, Nos. 3-4, pp. 289-296.
51. Lim, Y.B. and Parsons, I.D., 1993, “The linearized buckling analysis of a composite beam with multiple delaminations”, *International Journal of Solids Structures*, Vol. 30, pp. 3085-3099.
52. Moradi, S. and Taheri, F., 1998, “Differential quadrature approach for delamination buckling analysis of composites with shear deformation”, *Journal of AIAA*, Vol. 36, No. 10, pp.1869-1873.
53. Moradi, S. and Taheri, F., 1999, “Application of differential quadrature method to the delamination buckling of composite plates”, *Computers and Structures*, Vol. 70, pp. 615-623.
54. Moradi, S. and Taheri, F., 1999, “Delamination buckling analysis of general laminated composite beams by differential quadrature method”, *Composites: Part B*, Vol. 30, pp. 503-511.
55. Mujumdar, P.M. and Suryanarayan, S., 1988, “Flexural vibrations of beams with delaminations”, *Journal of Sound and Vibration*, Vol. 125, pp. 441-461.
56. Naik, N.K. and Ramasimha, R., 2001, “Estimation of compressive strength of delaminated composites”, *Journal of composite Structures*, Vol. 52, pp. 199-204.

57. Noor, A.K. and Burton, W.S., 1989, "Assessment of shear deformation theories for multilayered composite plates", *Journal of Applied Mechanics Reviews*, Vol. 42, No. 1, pp. 1-13.
58. Parlapalli, M. and Shu, D., 2004, "Buckling analysis of two-layer beams with an asymmetric delamination", *Journal of Engineering Structures*, Vol. 26, pp. 651-658.
59. Pereira, A.B., de Morais, A.B., Marques, A.T. and de Castro, P.T., 2004, "Mode II interlaminar fracture of carbon/epoxy multidirectional laminates", *Journal of Composites Science and Technology*, Vol. 64, pp. 1653-1659.
60. Peters, S.T., 1998, "Handbook of composites, 2nd edition", Chapman & Hall.
61. Popov, E.P., 1999, "Engineering mechanics of solids, 2nd Edition", Prentice Hall.
62. Pradhan, B. and Panda, S.K., 2005, "The influence of ply sequence and thermoelastic stress field on asymmetric delamination crack growth behavior of embedded elliptical delaminations in laminated FRP composites", *Journal of Composites Science and Technology*, Vol. 66, pp. 417-426.
63. Reddy, J.N., 2004, "Mechanics of laminated composite plates and shells: theory and analysis, 2nd edition", CRC Press LLC.
64. Riccio, A., Perugini, P. and Scaramuzzino, F., 2000, "Modeling compressive behavior of delaminated composite panels", *Journal of Computers and Structures*, Vol. 78, pp. 73-81.
65. Shan, B. and Pelegri, A.A., 2003, "Approximate analysis of the buckling behavior of composites with delamination", *Journal of Composite Materials*, Vol. 37, No. 8, pp. 673-685.
66. Shaw, D. and Tsai, M.Y., 1989, "Analysis of delamination in compressively loaded laminates", *Journal of Composites Science and Technology*, Vol. 34, pp. 1-17.
67. Sheinman, I., Bass, M. and Ishai, O., 1989, "Effect of delamination on stability of laminated composite strip", *Journal of Composite Structures*, Vol. 11, pp. 227-242.
68. Shenoi, R.A. and Wellicome, J.F., "Composite materials in maritime structures: Fundamental aspects", Cambridge University Press.

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69. Shu, D., 1996, "Buckling of multiple delaminated beams", International Journal of Solids Structures, Vol. 35, No. 13, pp. 1451-1465.
70. Shu, D. and Della, C.N., 2004, "Free vibration analysis of composite beams with two non-overlapping delaminations", International Journal of Mechanical Sciences, Vol. 46, pp. 509-526.
71. Simiteses, G.J., 1985, "Delamination buckling of flat laminates", Journal of Composite Structures, Vol. 4, no. 4, pp. 361-381.
72. Simiteses, G.J., Sallam, S and Yin, W.L., 1985, "Effect of delamination of axially loaded homogenous laminated plates", America Institute of Aeronautics and Astronautics, pp.1437-1444.
73. Tafreshi, A. and Oswald, T., 2003, "Global buckling behavior and local damage propagation in composite plates with embedded delaminations", International Journal of Pressure Vessels and Pipping, Vol. 80, pp. 9-20.
74. Tay, T.E., Shen, F., Lee, K.H., Scaglione, A. and Di Sciuva, M., 1999, "Mesh design in finite element analysis of post-buckled delamination in composite laminates", Journal of Composite Structures, Vol. 47, pp. 603-611.
75. Tessler, Alexander, 1991, "A two-node beam element including transverse shear and transverse normal deformation", Journal of Numerical Methods in Engineering, Vol. 32, No. 5, pp.1027-1039.
76. Vinson, J.R. and Sierakowski, R.L., 2002, "The behavior of structures composed of composite materials, 2nd edition", Kluwer Academic Publishers.
77. Wang, C.M., Reddy, J.N. and Lee, K.H., 2000, "Shear deformable beams and plates: Relationships with classical solutions", Elsevier.
78. Wang, James Ting-Shun, Cheng, Shou-Hsiung and Lin, Chien-Chang, 1994, "Local buckling of delaminated beams and plates using continuous analysis", Journal of Composite Materials, Vol. 29, No. 10, pp. 1374-1402.
79. Wang, James Ting-Shun and Lin, Chien-Chang, 1996, "Engineering analysis of buckling of delaminated beam plates", Journal of Composite Structures, Vol. 34, pp. 397-407.

80. Wang, X. and Lu, G., 2003, "Local buckling of composite laminar plates with various delaminated shapes", *Journal of Thin-walled Structures*, Vol. 41, No. 6, pp. 493-506.
81. Williams, J.F., Stouffer, D.C., Ilic, S. and Jones, R., 1986, "An analysis of delamination behavior", *Journal of Composite Structures*, Vol. 5, pp. 203-216.
82. Yin, Wan-Lee, Sallam, Sayed N. and Simites, George J., 1986, "Ultimate axial load capacity of a delaminated beam-plate", *American Institute of Aeronautics and Astronautics*, Vol. 24, No. 1, pp. 123-128.
83. Zhang, Z. and Taheri, F., 2004, "Dynamic pulse-buckling behavior of 'quasi-ductile' carbon/epoxy and E-glass/epoxy laminated composite beams", *Journal of Composite Structures*, Vol. 64, pp. 269-274.
84. Zor, M., 2003, "Delamination width effect on buckling loads of simply supported woven-fabric laminated composite plates of carbon/epoxy", *Journal of Reinforced Plastics and Composites*, Vol. 22, No. 17, pp. 1535-1546.
85. Zor, M., 2004, "Three-dimensional buckling analysis of thermoplastic composite laminated plates with single vertical or horizontal strip delamination", *Journal of Thermoplastic Composite Materials*, Vol. 17, pp. 557-568.
86. Zor, M., 2009, "Determination of critical width of multiple delamination in laminated composite plate under buckling load", *Journal of Reinforced Plastics and Composites*, Vol. 28, No. 3, pp. 305-316.

APPENDIX A

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```

/cwd,D:\aw\chunwee\newanalysis2      !Pointing to the desired working directory
/filename,antisymmetric15degree,1    !Giving filename
/prep7                                !Entering per-processor
/title,Carbon-epoxy 8 plies C-C antisymmetric 15 degree H2/H=1/2 a/L=0.50    !Giving title
nl1=4                                !Assigning variables (Number of layers in first set)
nl2=4                                !Assigning variables (Number of layers in second set)
t=0.1250e-03                         !Assigning variables (Thickness per ply)
al=1.000e-01                         !Assigning variables (Total length of composite beam)
bl=2.000e-02                         !Assigning variables (Width of composite beam)
abar=0.500                           !Assigning variables (Delamination span to beam length ratio)
b=20                                 !Assigning variables (Number of element divisions for delamination span)
t1=nl1*t                             !Assigning variables (Total thickness in first set)
t2=nl2*t                             !Assigning variables (Total thickness in second set)
tlam=t1+t2                           !Assigning variables (Total thickness of composite beam)
angle1=15                            !Assigning variables (First ply orientation specification)
angle2=-15                           !Assigning variables (Second ply orientation specification)
et,1,91,4,1                          !Selecting elements (Shell 91)
keyopt,1,11,2                        !Selecting elements (Shell 91)
et,2,91,4,1                          !Selecting elements (Shell 91)
keyopt,2,11,1                        !Selecting elements (Shell 91)
mp,ex,1,150.000e09                   !Defining material 1 properties
mp,ey,1,9.000e09                     !Defining material 1 properties
mp,ez,1,9.000e09                     !Defining material 1 properties
mp,prxy,1,0.28                       !Defining material 1 properties
mp,prxz,1,0.28                       !Defining material 1 properties
mp,pryz,1,0.28                       !Defining material 1 properties
mp,gxy,1,4.800e09                    !Defining material 1 properties
mp,gxz,1,4.800e09                    !Defining material 1 properties
mp,gyz,1,4.000e09                    !Defining material 1 properties
mp,dens,1,1230                       !Defining material 1 properties
mp,ex,2,150.000e09                   !Defining material 2 properties
mp,ey,2,9.000e09                     !Defining material 2 properties
mp,ez,2,9.000e09                     !Defining material 2 properties
mp,prxy,2,0.28                       !Defining material 2 properties
mp,prxz,2,0.28                       !Defining material 2 properties
mp,pryz,2,0.28                       !Defining material 2 properties
mp,gxy,2,4.800e09                    !Defining material 2 properties
mp,gxz,2,4.800e09                    !Defining material 2 properties
mp,gyz,2,4.000e09                    !Defining material 2 properties
mp,density,2,1230                    !Defining material 2 properties
r,1,nl1                              !Assigning real constants 1
rmore                                !Assigning real constants 1
rmore,1,angle1,t                     !Assigning real constants 1
rmore,1,angle2,t                     !Assigning real constants 1
rmore,1,angle1,t                     !Assigning real constants 1
rmore,1,angle2,t                     !Assigning real constants 1
r,2,nl2                              !Assigning real constants 2
rmore                                !Assigning real constants 2
rmore,2,angle1,t                     !Assigning real constants 2

```


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rmore,2,angle2,t,	<i>!Assigning real constants 2</i>
rmore,2,angle1,t,	<i>!Assigning real constants 2</i>
rmore,2,angle2,t,	<i>!Assigning real constants 2</i>
wpoffs,,,t1	<i>!Building model (Step 1)</i>
csys,4	<i>!Building model (Step 1)</i>
k,1	<i>!Building model (Step 1)</i>
k,2,al	<i>!Building model (Step 1)</i>
k,3,al,bl	<i>!Building model (Step 1)</i>
k,4,,bl	<i>!Building model (Step 1)</i>
ls=4	<i>!Building model (Step 1) (Specifies number of divisions on a line)</i>
1,1,4,ls	<i>!Building model (Step 1)</i>
1,2,3,ls	<i>!Building model (Step 1)</i>
kbtw,2,1,5,ratio,0.5*(1+abar)	<i>!Building model (Step 1)</i>
kbtw,2,1,6,ratio,0.5*(1-abar)	<i>!Building model (Step 1)</i>
kbtw,3,4,7,ratio,0.5*(1+abar)	<i>!Building model (Step 1)</i>
kbtw,3,4,8,ratio,0.5*(1-abar)	<i>!Building model (Step 1)</i>
1,5,7,ls	<i>!Building model (Step 1)</i>
1,6,8,ls	<i>!Building model (Step 1)</i>
ls=(b/(2*abar))-b/2	<i>!Building model (Step 1)</i>
1,1,5,ls	<i>!Building model (Step 1)</i>
1,4,7,ls	<i>!Building model (Step 1)</i>
1,6,2,ls	<i>!Building model (Step 1)</i>
1,8,3,ls	<i>!Building model (Step 1)</i>
ls=b	<i>!Building model (Step 1)</i>
1,5,6,ls	<i>!Building model (Step 1)</i>
1,7,8,ls	<i>!Building model (Step 1)</i>
a,1,2,3,4	<i>!Building model (Step 1)</i>
asbl,1,3	<i>!Building model (Step 1)</i>
asbl,3,4	<i>!Building model (Step 1)</i>
type,1	<i>!Meshing (Step 2)</i>
mat,1	<i>!Meshing (Step 2)</i>
real,1	<i>!Meshing (Step 2)</i>
local,11	<i>!Meshing (Step 2)</i>
esys,11	<i>!Meshing (Step 2)</i>
amesh,1	<i>!Meshing (Step 2)</i>
amesh,2	<i>!Meshing (Step 2)</i>
amesh,4	<i>!Meshing (Step 2)</i>
csys,4	<i>!Building model (Step 3)</i>
k,9	<i>!Building model (Step 3)</i>
k,10,al	<i>!Building model (Step 3)</i>
k,11,al,bl	<i>!Building model (Step 3)</i>
k,12,,bl	<i>!Building model (Step 3)</i>
ls=4	<i>!Building model (Step 3)</i>
1,9,12,ls	<i>!Building model (Step 3)</i>
1,10,11,ls	<i>!Building model (Step 3)</i>
kbtw,10,9,13,ratio,0.5*(1+abar)	<i>!Building model (Step 3)</i>
kbtw,10,9,14,ratio,0.5*(1-abar)	<i>!Building model (Step 3)</i>
kbtw,11,12,15,ratio,0.5*(1+abar)	<i>!Building model (Step 3)</i>
kbtw,11,12,16,ratio,0.5*(1-abar)	<i>!Building model (Step 3)</i>
1,13,15,ls	<i>!Building model (Step 3)</i>
1,14,16,ls	<i>!Building model (Step 3)</i>

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ls=(b/(2*abar))-b/2	<i>!Building model (Step 3)</i>
l,9,13,ls	<i>!Building model (Step 3)</i>
l,12,15,ls	<i>!Building model (Step 3)</i>
l,14,10,ls	<i>!Building model (Step 3)</i>
l,16,11,ls	<i>!Building model (Step 3)</i>
ls=b	<i>!Building model (Step 3)</i>
l,13,14,ls	<i>!Building model (Step 3)</i>
l,15,16,ls	<i>!Building model (Step 3)</i>
a,9,10,11,12	<i>!Building model (Step 3)</i>
asbl,3,13	<i>!Building model (Step 3)</i>
asbl,6,14	<i>!Building model (Step 3)</i>
type,2	<i>!Meshing (Step 4)</i>
mat,2	<i>!Meshing (Step 4)</i>
real,2	<i>!Meshing (Step 4)</i>
local,11	<i>!Meshing (Step 4)</i>
esys,11	<i>!Meshing (Step 4)</i>
amesh,5	<i>!Meshing (Step 4)</i>
amesh,3	<i>!Meshing (Step 4)</i>
amesh,7	<i>!Meshing (Step 4)</i>
asel,s,,1	<i>!Meshing (Step 5)</i>
asel,a,,,3	<i>!Meshing (Step 5)</i>
nsla,s,1	<i>!Meshing (Step 5)</i>
nummrg,node	<i>!Meshing (Step 5)</i>
nsel,all	<i>!Meshing (Step 5)</i>
asel,s,,2	<i>!Meshing (Step 5)</i>
asel,a,,,5	<i>!Meshing (Step 5)</i>
nsla,s,1	<i>!Meshing (Step 5)</i>
nummrg,node	<i>!Meshing (Step 5)</i>
save	<i>!Save file</i>
finish	<i>!Exit pre-processor</i>

Solving phase

/solu	<i>!Going into solving tab</i>
csys,0	<i>!Specifying boundary conditions</i>
nsel,all	<i>!Specifying boundary conditions</i>
nsel,x,0	<i>!Specifying boundary conditions</i>
d,all,all	<i>!Specifying boundary conditions</i>
nsel,x,al	<i>!Specifying boundary conditions</i>
d,all,uy,0	<i>!Specifying boundary conditions</i>
d,all,uz,0	<i>!Specifying boundary conditions</i>
d,all,rotz,0	<i>!Specifying boundary conditions</i>
d,all,rotx,0	<i>!Specifying boundary conditions</i>
d,all,roty,0	<i>!Specifying boundary conditions</i>
nsel,all	<i>!Enforcing prestress effects</i>
pstress,on	<i>!Enforcing prestress effects</i>
nsel,x,al	<i>!Enforcing prestress effects</i>
f,all,fx,-1	<i>!Enforcing prestress effects</i>
nsel,all	<i>!Enforcing prestress effects</i>

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solve	<i>!Enforcing prestress effects</i>
finish	<i>!Exiting solving tab</i>
/solu	<i>!Going into solving tab</i>
antype,buckle	<i>!Working out buckling analysis</i>
bucopt,subsp,1	<i>!Working out buckling analysis</i>
mxpand,1	<i>!Working out buckling analysis</i>
solve	<i>!Working out buckling analysis</i>
finish	<i>!Exiting solving tab</i>

Post-processing stage

/post1	<i>!Getting into post-processor</i>
set,last	<i>!Assessing results</i>
pldisp,0	<i>!Assessing results</i>

APPENDIX B

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For the CLT, the total virtual strain energy can be written as:

$$\begin{aligned}
 \delta U &= \int_{\Omega_0} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} (\tau_{xx} \delta \varepsilon_{xx} + \tau_{yy} \delta \varepsilon_{yy} + 2\tau_{xy} \delta \varepsilon_{xy}) dz \right] dxdy \\
 &= \int_{\Omega_0} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\tau_{xx} \frac{\partial \delta u_0}{\partial x} - \tau_{xx} z \frac{\partial^2 \delta w_0}{\partial x^2} + \tau_{yy} \frac{\partial \delta v_0}{\partial y} - \tau_{yy} z \frac{\partial^2 \delta w_0}{\partial y^2} \right. \right. \\
 &\quad \left. \left. + \tau_{xy} \left(\frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} \right) - 2\tau_{xy} z \frac{\partial^2 \delta w_0}{\partial x \partial y} \right) dz \right] dxdy \quad (b.1) \\
 &= \int_{\Omega_0} \left[N_{xx} \frac{\partial \delta u_0}{\partial x} - M_{xx} \frac{\partial^2 \delta w_0}{\partial x^2} + N_{yy} \frac{\partial \delta v_0}{\partial y} - M_{yy} \frac{\partial^2 \delta w_0}{\partial y^2} \right. \\
 &\quad \left. + N_{xy} \left(\frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} \right) - 2M_{xy} \frac{\partial^2 \delta w_0}{\partial x \partial y} \right] dxdy
 \end{aligned}$$

The delta symbol δ is the variational operator. The Ω_0 is the domain that is occupied by the mid-plane of the plate. The h is the thickness of the plate. The τ_{xx} and τ_{yy} are the normal stresses in the x - and y -directions, respectively. The τ_{xy} is the shear stress in the xy -plane. The quantities N_{xx} , N_{yy} and N_{xy} are the in-plane force resultants while M_{xx} , M_{yy} and M_{xy} are the moment resultants. Figures 4.2 and 4.3 show how these force and moment resultants are defined. They are evaluated as:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{Bmatrix} dz \quad \text{and} \quad (b.2)$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{Bmatrix} z dz \quad (b.3)$$

Performing integration by parts to relieve the virtual displacements $(\delta u_0, \delta v_0, \delta w_0)$ in the Ω_0 integral of any differentiation gives:

$$\delta U = \int_{\Omega_0} \left[- \left(\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_0 - \left(\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \delta v_0 - \left(\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) \delta w_0 \right] dxdy + \oint_{\Gamma} \left[(N_{xx}n_x + N_{xy}n_y) \delta u_0 + (N_{xy}n_x + N_{yy}n_y) \delta v_0 - (M_{xx}n_x + M_{xy}n_y) \frac{\partial \delta w_0}{\partial x} - (M_{xy}n_x + M_{yy}n_y) \frac{\partial \delta w_0}{\partial y} + \left(\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) n_x + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} \right) n_y \right) \delta w_0 \right] ds \quad (b.4)$$

Figure b.1: Boundary Γ and its coordinate system (n , s and z)

The Γ is the boundary of the domain Ω_0 as shown in Figure b.1. Moreover, s is the coordinate measured along Γ . The n_x and n_y are the x - and y -components of the unit normal vector $\vec{n}$. They are given as $n_x = \cos \theta$ and $n_y = \sin \theta$. The θ is the angle between the $\vec{n}$ and the x -axis.

The virtual work done by the in-plane forces is:

$$\delta V = \frac{1}{2} \int_{\Omega_0} \left[N_{xx} \cdot 2 \frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial x} + N_{yy} \cdot 2 \frac{\partial w_0}{\partial y} \frac{\partial \delta w_0}{\partial y} + 2 N_{xy} \left(\frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial y} + \frac{\partial \delta w_0}{\partial x} \frac{\partial w_0}{\partial y} \right) \right] dxdy \quad (b.5)$$

Performing integration by parts to relieve the virtual displacements of any differentiation gives:

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$$\begin{aligned} \delta V = & \int_{\Omega_0} \left[- \left(\frac{\partial}{\partial x} \left(N_{xx} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial y} \left(N_{yy} \frac{\partial w_0}{\partial y} \right) \right) \right. \\ & \left. + \frac{\partial}{\partial y} \left(N_{xy} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial x} \left(N_{xy} \frac{\partial w_0}{\partial y} \right) \right] \delta w_0 dx dy \\ & + \oint_{\Gamma} \left[\left(\left(N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x + \left(N_{yy} \frac{\partial w_0}{\partial y} + N_{xy} \frac{\partial w_0}{\partial x} \right) n_y \right) \delta w_0 \right] ds \end{aligned} \quad (b.6)$$

From the principle of virtual displacements, one can have:

$$\begin{aligned} \delta U + \delta V \equiv \delta W = 0 \\ \therefore \int_{\Omega_0} \left[- \left(\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_0 - \left(\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \delta v_0 \right. \\ \left. - \left(\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial}{\partial x} \left(N_{xx} \frac{\partial w_0}{\partial x} \right) \right. \right. \\ \left. \left. + \frac{\partial}{\partial y} \left(N_{yy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_{xy} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial x} \left(N_{xy} \frac{\partial w_0}{\partial y} \right) \right) \delta w_0 \right] dx dy \\ + \oint_{\Gamma} \left[\begin{aligned} & (N_{xx} n_x + N_{xy} n_y) \delta u_0 + (N_{xy} n_x + N_{yy} n_y) \delta v_0 \\ & - (M_{xx} n_x + M_{xy} n_y) \frac{\partial \delta w_0}{\partial x} \\ & - (M_{xy} n_x + M_{yy} n_y) \frac{\partial \delta w_0}{\partial y} \\ & + \left(\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x \right. \\ & \left. + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + N_{yy} \frac{\partial w_0}{\partial y} + N_{xy} \frac{\partial w_0}{\partial x} \right) n_y \right) \delta w_0 \end{aligned} \right] ds = 0 \end{aligned} \quad (b.7)$$

The δW is the total internal and external virtual work.

The Euler-Lagrange equations are then obtained by setting the coefficients of the δu_0 ,

δv_0 and δw_0 in Ω_0 to be zero:

$$\delta u_0 : \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (b.8)$$

$$\delta v_0 : \quad \frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \quad (b.9)$$

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$$\delta w_0 : \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial}{\partial x} \left(N_{xx} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial y} \left(N_{yy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_{xy} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial x} \left(N_{xy} \frac{\partial w_0}{\partial y} \right) = 0 \quad (\text{b.10})$$

Utilizing equations (b.8) and (b.9), equation (b.10) can be simplified to become:

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + N_{xx} \frac{\partial^2 w_0}{\partial x^2} + N_{yy} \frac{\partial^2 w_0}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w_0}{\partial x \partial y} = 0 \quad (\text{b.11})$$

From the boundary integrals in equation (b.7), the form of boundary conditions can be determined to be:

$$\text{either } \delta u_0 = 0 \text{ or } N_{xx} n_x + N_{xy} n_y = 0, \quad (\text{b.12})$$

$$\text{either } \delta v_0 = 0 \text{ or } N_{xy} n_x + N_{yy} n_y = 0, \quad (\text{b.13})$$

$$\text{either } \frac{\partial \delta w_0}{\partial x} = 0 \text{ or } M_{xx} n_x + M_{xy} n_y = 0, \quad (\text{b.14})$$

$$\text{either } \frac{\partial \delta w_0}{\partial y} = 0 \text{ or } M_{xy} n_x + M_{yy} n_y = 0 \text{ and} \quad (\text{b.15})$$

$$\text{either } \delta w_0 = 0 \text{ or } \left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + N_{xy} \frac{\partial w_0}{\partial y} + N_{yy} \frac{\partial w_0}{\partial x} \right) n_y = 0 \quad (\text{b.16})$$

APPENDIX C

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$$C_{11} = \frac{\alpha_{11}D_{22} + \alpha_{12}D_{66} + \alpha_{13}D_{26} + (A_{66}A_{12} - A_{26}A_{16})(D_{22}D_{66} - D_{26}^2) + \alpha_{14}}{\beta_{11}D_{22} + \beta_{12}D_{66} + \beta_{13}D_{26} + (A_{26}^2 - A_{22}A_{66})(D_{22}D_{66} - D_{26}^2) + \beta_{14}} \quad (c.1)$$

$$C_{12} = \frac{\left\{ \begin{aligned} &\phi_{11}D_{22} + \phi_{12}D_{12} + \phi_{13}D_{66} + \phi_{14}D_{16} + \phi_{15}D_{26} \\ &+ \phi_{16}(D_{22}D_{16} - D_{12}D_{26}) + \phi_{17}(D_{12}D_{66} - D_{16}D_{26}) + \phi_{18}(D_{22}D_{66} - D_{26}^2) \end{aligned} \right\}}{\beta_{11}D_{22} + \beta_{12}D_{66} + \beta_{13}D_{26} + (A_{26}^2 - A_{22}A_{66})(D_{22}D_{66} - D_{26}^2) + \beta_{14}} \quad (c.2)$$

where $\alpha_{11} = -A_{12}B_{66}^2 - A_{66}B_{16}B_{26} + A_{16}B_{26}B_{66} + A_{26}B_{16}B_{66}$

$$\alpha_{12} = -A_{12}B_{26}^2 - A_{66}B_{12}B_{22} + A_{26}B_{26}B_{12} + A_{16}B_{22}B_{26}$$

$$\alpha_{13} = 2A_{12}B_{26}B_{66} + A_{66}(B_{22}B_{16} + B_{26}B_{12}) - A_{16}(B_{26}^2 + B_{22}B_{66}) - A_{26}(B_{16}B_{26} + B_{12}B_{66})$$

$$\alpha_{14} = B_{16}B_{26}^3 + B_{12}B_{22}B_{66}^2 - B_{12}B_{66}B_{26}^2 - B_{22}B_{66}B_{16}B_{26}$$

$$\beta_{11} = A_{22}B_{66}^2 + A_{66}B_{26}^2 - 2A_{26}B_{66}B_{26} \quad \beta_{12} = A_{22}B_{26}^2 + A_{66}B_{22}^2 - 2A_{26}B_{22}B_{26}$$

$$\beta_{13} = 2A_{26}B_{26}^2 + 2A_{26}B_{22}B_{66} - 2A_{22}B_{26}B_{66} - 2A_{66}B_{22}B_{26}$$

$$\beta_{14} = 2B_{26}^2B_{22}B_{66} - B_{22}^2B_{66}^2 - B_{26}^4$$

$$\phi_{11} = B_{16}B_{26}B_{66} - B_{12}B_{66}^2 \quad \phi_{12} = B_{22}B_{66}^2 - B_{66}B_{26}^2$$

$$\phi_{13} = B_{22}B_{16}B_{26} - B_{12}B_{26}^2 \quad \phi_{14} = B_{26}^3 - B_{22}B_{66}B_{26}$$

$$\phi_{15} = -B_{26}^2B_{16} + 2B_{12}B_{66}B_{26} - B_{22}B_{66}B_{16}$$

$$\phi_{16} = B_{66}A_{26} - B_{26}A_{66} \quad \phi_{17} = B_{26}A_{26} - B_{22}A_{66}$$

$$\phi_{18} = -B_{16}A_{26} + B_{12}A_{66}$$

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$$C_{21} = \frac{\alpha_{21}D_{22} + \alpha_{22}D_{66} + \alpha_{23}D_{26} + (A_{22}A_{16} - A_{26}A_{12})(D_{22}D_{66} - D_{26}^2) + \alpha_{24}}{\beta_{21}D_{22} + \beta_{22}D_{66} + \beta_{23}D_{26} + (A_{26}^2 - A_{22}A_{66})(D_{22}D_{66} - D_{26}^2) + \beta_{24}} \quad (\text{c.3})$$

$$C_{22} = \frac{\left\{ \begin{aligned} &\phi_{21}D_{22} + \phi_{22}D_{12} + \phi_{23}D_{66} + \phi_{24}D_{16} + \phi_{25}D_{26} \\ &+ \phi_{26}(D_{22}D_{16} - D_{12}D_{26}) + \phi_{27}(D_{12}D_{66} - D_{16}D_{26}) + \phi_{28}(D_{22}D_{66} - D_{26}^2) \end{aligned} \right\}}{\beta_{21}D_{22} + \beta_{22}D_{66} + \beta_{23}D_{26} + (A_{26}^2 - A_{22}A_{66})(D_{22}D_{66} - D_{26}^2) + \beta_{24}} \quad (\text{c.4})$$

where $\alpha_{21} = A_{12}B_{26}B_{66} - A_{22}B_{66}B_{16} - A_{16}B_{26}^2 + A_{26}B_{26}B_{16}$

$$\alpha_{22} = -A_{16}B_{22}^2 - A_{22}B_{12}B_{26} + A_{12}B_{22}B_{26} + A_{26}B_{12}B_{22}$$

$$\alpha_{23} = A_{22}(B_{66}B_{12} + B_{16}B_{26}) - A_{12}(B_{26}^2 + B_{22}B_{66}) - A_{26}(B_{16}B_{22} + B_{12}B_{26}) + 2A_{16}B_{22}B_{26}$$

$$\alpha_{24} = B_{12}B_{26}^3 - B_{22}B_{16}B_{26}^2 + B_{16}B_{66}B_{22}^2 - B_{22}B_{66}B_{12}B_{26}$$

$$\phi_{21} = -B_{16}B_{26}^2 + B_{12}B_{26}B_{66} \quad \phi_{22} = B_{26}^3 - B_{22}B_{66}B_{26}$$

$$\phi_{23} = -B_{16}B_{22}^2 + B_{22}B_{12}B_{26} \quad \phi_{24} = B_{66}B_{22}^2 - B_{22}B_{26}^2$$

$$\phi_{25} = 2B_{16}B_{26}B_{22} - B_{26}^2B_{12} - B_{22}B_{66}B_{12}$$

$$\phi_{26} = B_{26}A_{26} - B_{66}A_{22} \quad \phi_{27} = B_{22}A_{26} - B_{26}A_{22}$$

$$\phi_{28} = B_{16}A_{22} - B_{12}A_{26}$$

$$\beta_{21} = \beta_{11} \quad \beta_{22} = \beta_{12} \quad \beta_{23} = \beta_{13} \quad \beta_{24} = \beta_{14}$$

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$$C_{31} = \frac{\alpha_{31}D_{66} + \alpha_{32}D_{26} + \alpha_{33}}{\beta_{31}D_{22} + \beta_{32}D_{66} + \beta_{33}D_{26} + (A_{26}^2 - A_{22}A_{66})(D_{22}D_{66} - D_{26}^2) + \beta_{34}} \quad (c.5)$$

$$C_{32} = \frac{\phi_{31}D_{12} + \phi_{32}D_{66} + \phi_{33}D_{16} + \phi_{34}D_{26} + (A_{22}A_{66} - A_{26}^2)(D_{12}D_{66} - D_{26}D_{16}) + \phi_{35}}{\beta_{31}D_{22} + \beta_{32}D_{66} + \beta_{33}D_{26} + (A_{26}^2 - A_{22}A_{66})(D_{22}D_{66} - D_{26}^2) + \beta_{34}} \quad (c.6)$$

where

$$\begin{aligned} \alpha_{31} &= B_{22}(A_{26}A_{16} - A_{66}A_{12}) + B_{26}(A_{26}A_{12} - A_{22}A_{16}) + B_{12}(A_{22}A_{66} - A_{26}^2) \\ \alpha_{32} &= B_{66}(A_{22}A_{16} - A_{12}A_{26}) + B_{26}(A_{12}A_{66} - A_{26}A_{16}) - B_{16}(A_{22}A_{66} - A_{26}^2) \\ \alpha_{33} &= A_{22}(B_{66}B_{16}B_{26} - B_{12}B_{66}^2) + A_{66}(B_{22}B_{16}B_{26} - B_{12}B_{26}^2) + A_{12}(B_{22}B_{66}^2 - B_{66}B_{26}^2) \\ &\quad + A_{16}(B_{26}^3 - B_{22}B_{66}B_{26}) + A_{26}(2B_{66}B_{12}B_{26} - B_{16}B_{26}^2 - B_{22}B_{66}B_{16}) \\ \beta_{31} &= A_{66}B_{26}^2 + A_{22}B_{66}^2 - 2A_{26}B_{26}B_{66} \\ \beta_{32} &= A_{66}B_{22}^2 + A_{22}B_{26}^2 - 2A_{26}B_{22}B_{26} \\ \beta_{33} &= 2A_{26}(B_{26}^2 + B_{22}B_{66}) - 2A_{66}B_{22}B_{26} - 2A_{22}B_{26}B_{66} \\ \beta_{34} &= 2B_{22}B_{66}B_{26}^2 - B_{26}^4 - B_{22}^2B_{66}^2 \\ \phi_{31} &= 2A_{26}B_{66}B_{26} - A_{22}B_{66}^2 - A_{66}B_{26}^2 \\ \phi_{32} &= -A_{22}B_{26}B_{16} - A_{66}B_{22}B_{12} + A_{26}(B_{16}B_{22} + B_{12}B_{26}) \\ \phi_{33} &= A_{22}B_{66}B_{26} + A_{66}B_{22}B_{26} - A_{26}(B_{22}B_{66} + B_{26}^2) \\ \phi_{34} &= A_{22}B_{66}B_{16} + A_{66}B_{12}B_{26} - A_{26}(B_{26}B_{16} + B_{12}B_{66}) \\ \phi_{35} &= B_{16}B_{26}^3 + B_{22}B_{12}B_{66}^2 - B_{12}B_{66}B_{26}^2 - B_{22}B_{66}B_{16}B_{26} \end{aligned}$$

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$$C_{41} = \frac{\alpha_{41}D_{22} + \alpha_{42}D_{26} + \alpha_{43}}{\beta_{41}D_{22} + \beta_{42}D_{66} + \beta_{43}D_{26} + (A_{26}^2 - A_{22}A_{66})(D_{22}D_{66} - D_{26}^2) + \beta_{44}} \quad (c.7)$$

$$C_{42} = \frac{\phi_{41}D_{12} + \phi_{42}D_{22} + \phi_{43}D_{16} + \phi_{44}D_{26} + (A_{22}A_{66} - A_{26}^2)(D_{16}D_{22} - D_{26}D_{12}) + \phi_{45}}{\beta_{41}D_{22} + \beta_{42}D_{66} + \beta_{43}D_{26} + (A_{26}^2 - A_{22}A_{66})(D_{22}D_{66} - D_{26}^2) + \beta_{44}} \quad (c.8)$$

where

$$\begin{aligned} \alpha_{41} &= B_{66}(A_{12}A_{26} - A_{22}A_{16}) + B_{26}(A_{26}A_{16} - A_{12}A_{66}) + B_{16}(A_{22}A_{66} - A_{26}^2) \\ \alpha_{42} &= B_{22}(A_{12}A_{66} - A_{16}A_{26}) + B_{26}(A_{22}A_{16} - A_{12}A_{26}) - B_{12}(A_{22}A_{66} - A_{26}^2) \\ \alpha_{43} &= A_{22}(B_{66}B_{12}B_{26} - B_{16}B_{26}^2) + A_{66}(B_{22}B_{12}B_{26} - B_{16}B_{22}^2) + A_{12}(B_{26}^3 - B_{22}B_{66}B_{26}) \\ &\quad - A_{16}(B_{22}B_{26}^2 - B_{66}B_{22}^2) + A_{26}(2B_{22}B_{26}B_{16} - B_{22}B_{12}B_{66} - B_{12}B_{26}^2) \\ \phi_{41} &= A_{22}B_{66}B_{26} + A_{66}B_{22}B_{26} - A_{26}(B_{26}^2 + B_{22}B_{66}) \\ \phi_{42} &= -A_{22}B_{66}B_{16} - A_{66}B_{26}B_{12} + A_{26}(B_{16}B_{26} + B_{12}B_{66}) \\ \phi_{43} &= 2A_{26}B_{22}B_{26} - A_{66}B_{22}^2 - A_{22}B_{26}^2 \\ \phi_{44} &= A_{22}B_{26}B_{16} + A_{66}B_{22}B_{12} - A_{26}(B_{26}B_{12} + B_{22}B_{16}) \\ \phi_{45} &= B_{12}B_{26}^3 + B_{66}B_{16}B_{22}^2 - B_{22}B_{16}B_{26}^2 - B_{22}B_{66}B_{12}B_{26} \\ \beta_{41} &= \beta_{31} \quad \beta_{42} = \beta_{32} \quad \beta_{43} = \beta_{33} \quad \beta_{44} = \beta_{34} \end{aligned}$$

APPENDIX D

For the FSDT, the virtual strain energy can be worked out to be:

$$\begin{aligned}
 \delta U &= \int_{\Omega_0} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\tau_{xx} \delta \varepsilon_{xx} + \tau_{yy} \delta \varepsilon_{yy} + 2\tau_{xy} \delta \varepsilon_{xy} + 2\tau_{xz} \delta \varepsilon_{xz} + 2\tau_{yz} \delta \varepsilon_{yz} \right) dz \right] dxdy \\
 &= \int_{\Omega_0} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\tau_{xx} \frac{\partial \delta u_0}{\partial x} + \tau_{xx} z \frac{\partial \delta \phi_x}{\partial x} + \tau_{yy} \frac{\partial \delta v_0}{\partial y} + \tau_{yy} z \frac{\partial \delta \phi_y}{\partial y} + \tau_{xy} \left(\frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} \right) \right. \right. \\
 &\quad \left. \left. + \tau_{xy} z \left(\frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} \right) + \tau_{xz} \left(\frac{\partial \delta w_0}{\partial x} + \delta \phi_x \right) + \tau_{yz} \left(\frac{\partial \delta w_0}{\partial y} + \delta \phi_y \right) \right) dz \right] dxdy \quad (d.1) \\
 &= \int_{\Omega_0} \left[N_{xx} \frac{\partial \delta u_0}{\partial x} + M_{xx} \frac{\partial \delta \phi_x}{\partial x} + N_{yy} \frac{\partial \delta v_0}{\partial y} + M_{yy} \frac{\partial \delta \phi_y}{\partial y} + N_{xy} \left(\frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} \right) \right. \\
 &\quad \left. + M_{xy} \left(\frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} \right) + Q_x \left(\frac{\partial \delta w_0}{\partial x} + \delta \phi_x \right) + Q_y \left(\frac{\partial \delta w_0}{\partial y} + \delta \phi_y \right) \right] dxdy
 \end{aligned}$$

The τ_{xz} and τ_{yz} are the shear stresses in the xz – and yz –planes, respectively. The quantities Q_x and Q_y are the transverse force resultants. Figure 4.2 depicts how these force resultants are defined. These force resultants are calculated as:

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz \quad (d.2)$$

The parameter K is known as the shear correction factor.

Similarly, performing integration by parts to relieve the virtual displacements of any differentiation gives:

$$\begin{aligned}
 \delta U &= \int_{\Omega_0} \left[- \left(\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_0 - \left(\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \delta v_0 \right. \\
 &\quad \left. - \left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x \right) \delta \phi_x \right. \\
 &\quad \left. - \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y \right) \delta \phi_y - \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) \delta w_0 \right] dxdy \quad (d.3) \\
 &+ \oint_{\Gamma} \left[(N_{xx} n_x + N_{xy} n_y) \delta u_0 + (N_{xy} n_x + N_{yy} n_y) \delta v_0 + (M_{xx} n_x + M_{xy} n_y) \delta \phi_x \right. \\
 &\quad \left. + (M_{xy} n_x + M_{yy} n_y) \delta \phi_y + (Q_x n_x + Q_y n_y) \delta w_0 \right] ds
 \end{aligned}$$

From the principle of virtual displacements, utilizing equations (b.6) and (d.3), one can obtain:

$$\delta U + \delta V \equiv \delta W = 0$$

$$\begin{aligned}
& \therefore \int_{\Omega_0} \left[-\left(\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_0 - \left(\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \delta v_0 - \left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x \right) \delta \phi_x \right. \\
& \quad \left. - \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y \right) \delta \phi_y - \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x} \left(N_{xx} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial y} \left(N_{yy} \frac{\partial w_0}{\partial y} \right) \right) \delta w_0 \right] dx dy \\
& \quad + \oint_{\Gamma} \left[(N_{xx} n_x + N_{xy} n_y) \delta u_0 + (N_{xy} n_x + N_{yy} n_y) \delta v_0 + (M_{xx} n_x + M_{xy} n_y) \delta \phi_x \right. \\
& \quad \left. + (M_{xy} n_x + M_{yy} n_y) \delta \phi_y + \left(\left(Q_x + N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x \right) \delta w_0 \right. \\
& \quad \left. + \left(\left(Q_y + N_{yy} \frac{\partial w_0}{\partial y} + N_{xy} \frac{\partial w_0}{\partial x} \right) n_y \right) \delta w_0 \right] ds = 0
\end{aligned} \tag{d.4}$$

Similarly, the governing equations can be attained by setting the coefficients of the δu_0 , δv_0 , $\delta \phi_x$, $\delta \phi_y$ and δw_0 in Ω_0 to be zero:

$$\delta u_0 : \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{d.5}$$

$$\delta v_0 : \quad \frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \tag{d.6}$$

$$\delta \phi_x : \quad \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{d.7}$$

$$\delta \phi_y : \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = 0 \tag{d.8}$$

$$\begin{aligned}
\delta w_0 : \quad & \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x} \left(N_{xx} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial y} \left(N_{yy} \frac{\partial w_0}{\partial y} \right) \\
& + \frac{\partial}{\partial y} \left(N_{xy} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial x} \left(N_{xy} \frac{\partial w_0}{\partial y} \right) = 0
\end{aligned} \tag{d.9}$$

Then, the form of boundary conditions can be found from the boundary integrals in equation (d.4) as:

$$\text{either } \delta u_0 = 0 \text{ or } N_{xx} n_x + N_{xy} n_y = 0, \tag{d.10}$$

$$\text{either } \delta v_0 = 0 \text{ or } N_{xy} n_x + N_{yy} n_y = 0, \tag{d.11}$$

$$\text{either } \delta \phi_x = 0 \text{ or } M_{xx} n_x + M_{xy} n_y = 0, \tag{d.12}$$

$$\text{either } \delta \phi_y = 0 \text{ or } M_{xy} n_x + M_{yy} n_y = 0 \text{ and} \tag{d.13}$$

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$$\text{either } \delta w_0 = 0 \text{ or } \begin{aligned} & \left(Q_x + N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x \\ & + \left(Q_y + N_{yy} \frac{\partial w_0}{\partial y} + N_{xy} \frac{\partial w_0}{\partial x} \right) n_y = 0 \end{aligned} \quad (\text{d.14})$$

APPENDIX E

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For the TSDT, the total virtual strain energy will be:

$$\begin{aligned}
 \delta U &= \int_{\Omega_0} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\tau_{xx} \delta \varepsilon_{xx} + \tau_{yy} \delta \varepsilon_{yy} + 2\tau_{xy} \delta \varepsilon_{xy} + 2\tau_{xz} \delta \varepsilon_{xz} + 2\tau_{yz} \delta \varepsilon_{yz} \right) dz \right] dxdy \\
 &= \int_{\Omega_0} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\begin{aligned} &\tau_{xx} \frac{\partial \delta u_0}{\partial x} + \tau_{xx} z \frac{\partial \delta \phi_x}{\partial x} - c_1 \tau_{xx} z^3 \left(\frac{\partial \delta \phi_x}{\partial x} + \frac{\partial^2 \delta w_0}{\partial x^2} \right) + \tau_{yy} \frac{\partial \delta v_0}{\partial y} \\ &+ \tau_{yy} z \frac{\partial \delta \phi_y}{\partial y} - c_1 \tau_{yy} z^3 \left(\frac{\partial \delta \phi_y}{\partial y} + \frac{\partial^2 \delta w_0}{\partial y^2} \right) + \tau_{xy} \left(\frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} \right) \\ &+ \tau_{xy} z \left(\frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} \right) - c_1 \tau_{xy} z^3 \left(\frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} + 2 \frac{\partial^2 \delta w_0}{\partial x \partial y} \right) \\ &+ \tau_{xz} \left(\delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right) - c_2 \tau_{xz} z^2 \left(\delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right) \\ &+ \tau_{yz} \left(\delta \phi_y + \frac{\partial \delta w_0}{\partial y} \right) - c_2 \tau_{yz} z^2 \left(\delta \phi_y + \frac{\partial \delta w_0}{\partial y} \right) \end{aligned} \right) dz \right] dxdy \\
 &= \int_{\Omega_0} \left[\begin{aligned} &N_{xx} \frac{\partial \delta u_0}{\partial x} + M_{xx} \frac{\partial \delta \phi_x}{\partial x} - c_1 P_{xx} \left(\frac{\partial \delta \phi_x}{\partial x} + \frac{\partial^2 \delta w_0}{\partial x^2} \right) + N_{yy} \frac{\partial \delta v_0}{\partial y} \\ &+ M_{yy} z \frac{\partial \delta \phi_y}{\partial y} - c_1 P_{yy} \left(\frac{\partial \delta \phi_y}{\partial y} + \frac{\partial^2 \delta w_0}{\partial y^2} \right) + N_{xy} \left(\frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} \right) \\ &+ N_{xy} z \left(\frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} \right) - c_1 P_{xy} \left(\frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} + 2 \frac{\partial^2 \delta w_0}{\partial x \partial y} \right) \\ &+ Q_x \left(\delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right) - c_2 R_x \left(\delta \phi_x + \frac{\partial \delta w_0}{\partial x} \right) \\ &+ Q_y \left(\delta \phi_y + \frac{\partial \delta w_0}{\partial y} \right) - c_2 R_y \left(\delta \phi_y + \frac{\partial \delta w_0}{\partial y} \right) \end{aligned} \right] dxdy \quad (e.1)
 \end{aligned}$$

The P_{xx} , P_{yy} , P_{xy} , R_x and R_y are known as the higher order stress resultants and they are given as:

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{Bmatrix} z^3 dz \quad \text{and} \quad (e.2)$$

$$\begin{Bmatrix} R_x \\ R_y \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} z^2 dz \quad (e.3)$$

Upon integration by parts, the total virtual strain energy becomes:

$$\delta U = \int_{\Omega_0} \left[- \left(\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_0 - \left(\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \delta v_0 - \left(\frac{\partial \bar{M}_{xx}}{\partial x} + \frac{\partial \bar{M}_{xy}}{\partial y} - \bar{Q}_x \right) \delta \phi_x \right. \\ \left. - \left(\frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_{yy}}{\partial y} - \bar{Q}_y \right) \delta \phi_y \right. \\ \left. - \left(\frac{\partial \bar{Q}_x}{\partial x} + \frac{\partial \bar{Q}_y}{\partial y} + c_1 \left(\frac{\partial^2 P_{xx}}{\partial x^2} + \frac{\partial^2 P_{yy}}{\partial y^2} + 2 \frac{\partial^2 P_{xy}}{\partial x \partial y} \right) \right) \delta w_0 \right] dx dy \\ + \oint_{\Gamma} \left[\left(N_{xx} n_x + N_{xy} n_y \right) \delta u_0 + \left(N_{xy} n_x + N_{yy} n_y \right) \delta v_0 + \left(\bar{M}_{xx} n_x + \bar{M}_{xy} n_y \right) \delta \phi_x \right. \\ \left. + \left(\bar{M}_{xy} n_x + \bar{M}_{yy} n_y \right) \delta \phi_y \right. \\ \left. + \left(\left(\bar{Q}_x + c_1 \left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} \right) \right) n_x + \left(\bar{Q}_y + c_1 \left(\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} \right) \right) n_y \right) \delta w_0 \right. \\ \left. - c_1 \left(P_{xx} n_x + P_{xy} n_y \right) \frac{\partial \delta w_0}{\partial x} - c_1 \left(P_{xy} n_x + P_{yy} n_y \right) \frac{\partial \delta w_0}{\partial y} \right] ds \quad (e.4)$$

The new terms that are introduced for compactness are defined as:

$$\bar{M}_{xx} = M_{xx} - c_1 P_{xx}, \quad (e.5)$$

$$\bar{M}_{yy} = M_{yy} - c_1 P_{yy}, \quad (e.6)$$

$$\bar{M}_{xy} = M_{xy} - c_1 P_{xy}, \quad (e.7)$$

$$\bar{Q}_x = Q_x - c_2 R_x \text{ and} \quad (e.8)$$

$$\bar{Q}_y = Q_y - c_2 R_y. \quad (e.9)$$

From the principle of virtual displacements, using equations (b.6) and (e.4), one can get:

$$\delta U + \delta V \equiv \delta W = 0$$

$$\begin{aligned}
& \therefore \int_{\Omega_0} \left[\begin{aligned} & - \left(\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_0 - \left(\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \delta v_0 \\ & - \left(\frac{\partial \bar{M}_{xx}}{\partial x} + \frac{\partial \bar{M}_{xy}}{\partial y} - \bar{Q}_x \right) \delta \phi_x \\ & - \left(\frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_{yy}}{\partial y} - \bar{Q}_y \right) \delta \phi_y \\ & - \left(\frac{\partial \bar{Q}_x}{\partial x} + \frac{\partial \bar{Q}_y}{\partial y} + c_1 \left(\frac{\partial^2 P_{xx}}{\partial x^2} + \frac{\partial^2 P_{yy}}{\partial y^2} + 2 \frac{\partial^2 P_{xy}}{\partial x \partial y} \right) + \frac{\partial}{\partial x} \left(N_{xx} \frac{\partial w_0}{\partial x} \right) \right. \\ & \left. + \frac{\partial}{\partial y} \left(N_{yy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_{xy} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial x} \left(N_{xy} \frac{\partial w_0}{\partial y} \right) \right] \delta w_0 \Bigg] dx dy \\ & + \oint_{\Gamma} \left[\begin{aligned} & (N_{xx} n_x + N_{xy} n_y) \delta u_0 + (N_{xy} n_x + N_{yy} n_y) \delta v_0 + (\bar{M}_{xx} n_x + \bar{M}_{xy} n_y) \delta \phi_x \\ & + (\bar{M}_{xy} n_x + \bar{M}_{yy} n_y) \delta \phi_y \\ & + \left(\bar{Q}_x + c_1 \left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} \right) + N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x \\ & + \left(\bar{Q}_y + c_1 \left(\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} \right) + N_{yy} \frac{\partial w_0}{\partial y} + N_{xy} \frac{\partial w_0}{\partial x} \right) n_y \right] \delta w_0 \\ & - c_1 (P_{xx} n_x + P_{xy} n_y) \frac{\partial \delta w_0}{\partial x} - c_1 (P_{xy} n_x + P_{yy} n_y) \frac{\partial \delta w_0}{\partial y} \end{aligned} \right] ds = 0 \quad (e.10)
\end{aligned}$$

Then, the governing equations can be obtained by letting the coefficients of δu_0 , δv_0 , $\delta \phi_x$, $\delta \phi_y$ and δw_0 in Ω_0 to be zero:

$$\delta u_0: \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (e.11)$$

$$\delta v_0: \quad \frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \quad (e.12)$$

$$\delta \phi_x: \quad \frac{\partial \bar{M}_{xx}}{\partial x} + \frac{\partial \bar{M}_{xy}}{\partial y} - \bar{Q}_x = 0 \quad (e.13)$$

$$\delta \phi_y: \quad \frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_{yy}}{\partial y} - \bar{Q}_y = 0 \quad (e.14)$$

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$$\delta w_0: \frac{\partial \bar{Q}_x}{\partial x} + \frac{\partial \bar{Q}_y}{\partial y} + c_1 \left(\frac{\partial^2 P_{xx}}{\partial x^2} + \frac{\partial^2 P_{yy}}{\partial y^2} + 2 \frac{\partial^2 P_{xy}}{\partial x \partial y} \right) + \frac{\partial}{\partial x} \left(N_{xx} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial y} \left(N_{yy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_{xy} \frac{\partial w_0}{\partial x} \right) + \frac{\partial}{\partial x} \left(N_{xy} \frac{\partial w_0}{\partial y} \right) = 0 \quad (e.15)$$

The form of boundary conditions can be deduced from the terms in the boundary integral as:

$$\text{either } \delta u_0 = 0 \text{ or } N_{xx} n_x + N_{xy} n_y = 0, \quad (e.16)$$

$$\text{either } \delta v_0 = 0 \text{ or } N_{xy} n_x + N_{yy} n_y = 0, \quad (e.17)$$

$$\text{either } \delta \phi_x = 0 \text{ or } \bar{M}_{xx} n_x + \bar{M}_{xy} n_y = 0, \quad (e.18)$$

$$\text{either } \delta \phi_y = 0 \text{ or } \bar{M}_{xy} n_x + \bar{M}_{yy} n_y = 0, \quad (e.19)$$

$$\text{either } \delta w_0 = 0 \text{ or } \left(\bar{Q}_x + c_1 \left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} \right) + N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x + \left(\bar{Q}_y + c_1 \left(\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} \right) + N_{yy} \frac{\partial w_0}{\partial y} + N_{xy} \frac{\partial w_0}{\partial x} \right) n_y = 0, \quad (e.20)$$

$$\text{either } \frac{\partial \delta w_0}{\partial x} = 0 \text{ or } c_1 (P_{xx} n_x + P_{xy} n_y) = 0 \text{ and} \quad (e.21)$$

$$\text{either } \frac{\partial \delta w_0}{\partial y} = 0 \text{ or } c_1 (P_{xy} n_x + P_{yy} n_y) = 0 \quad (e.22)$$

APPENDIX F

Figure f.1: Buckling mode shapes of simply-supported $[(-\theta/\theta)_2 // (-\theta/\theta)_2]$ composite beam with a delamination span of (a) 10mm, (b) 50mm and (c) 80mm

APPENDIX G

Figure g.1: Buckling mode shapes of 16 plies 100mm long clamped (a) configuration 1, (b) configuration 2 and (c) configuration 3 composite beams with delamination lengths of $a_1 = 70\text{mm}$ and $a_2 = 45\text{mm}$, not taking account of the contact effect

Figure g.2: Buckling mode shape of clamped configuration 1 composite beam adopting a_1 to be 90mm and a_2 as 60mm obtained from the contactless model (see Figure 7.11 (a))

Figure g.3: Buckling mode shapes of clamped configuration 2 composite beam adopting a_1 to be 90mm and a_2 as 60mm obtained from the (a) contactless and (b) contact-integrated models (see Figure 7.12 (a))

Figure g.4: Buckling mode shapes of clamped configuration 3 composite beam adopting a_1 to be 90mm and a_2 as 60mm obtained from the (a) contactless and (b) contact-integrated models (see Figure 7.15 (a))

APPENDIX H

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For this traditional formula in method 1, the simplifying assumptions are:

- 1) The cross sectional dimensions of the beam are very small compared to the length of the beam.
- 2) The composite material is linearly elastic.
- 3) The deflections are small.
- 4) Plane sections through a beam normal to its axis remain plane after the beam is subjected to bending.
- 5) The assumptions made in CLT are also applicable.

Since it is assumed that the cross sectional dimensions of the beam are very small compared to the length (x – direction) of the beam, it can also be said that the variation in the response in the y – direction (width) can be ignored. Hence, equations (4.37) and (4.38) in chapter four will become:

$$N_{xx}^C = A_{11} \frac{du_o^C}{dx} - B_{11} \frac{d^2 w_o^C}{dx^2} \quad (h.1)$$

$$N_{yy}^C = A_{12} \frac{du_o^C}{dx} - B_{12} \frac{d^2 w_o^C}{dx^2} \quad (h.2)$$

$$N_{xy}^C = A_{16} \frac{du_o^C}{dx} - B_{16} \frac{d^2 w_o^C}{dx^2} \quad (h.3)$$

$$M_{xx}^C = B_{11} \frac{du_o^C}{dx} - D_{11} \frac{d^2 w_o^C}{dx^2} \quad (h.4)$$

$$M_{yy}^C = B_{12} \frac{du_o^C}{dx} - D_{12} \frac{d^2 w_o^C}{dx^2} \quad (h.5)$$

$$M_{xy}^C = B_{16} \frac{du_o^C}{dx} - D_{16} \frac{d^2 w_o^C}{dx^2} \quad (h.6)$$

Substituting equation (h.4) into equation (4.44) in chapter four, the following expression is formed:

$$\frac{d^2}{dx^2} \left(B_{11} \frac{du_o^C}{dx} - D_{11} \frac{d^2 w_o^C}{dx^2} \right) + N_{xx}^C \frac{d^2 w_o^C}{dx^2} = 0 \quad (h.7)$$

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From equation (h.1), expressing $\frac{du_o^C}{dx}$ with respect to other terms, the following equation

can be obtained:

$$\frac{du_o^C}{dx} = \frac{N_{xx}^C}{A_{11}} + \frac{B_{11}}{A_{11}} \frac{d^2 w_o^C}{dx^2} \quad (\text{h.8})$$

Inserting equation (h.8) into (h.7), the resultant equation is:

$$\frac{d^2}{dx^2} \left(B_{11} \left(\frac{N_{xx}^C}{A_{11}} + \frac{B_{11}}{A_{11}} \frac{d^2 w_o^C}{dx^2} \right) - D_{11} \frac{d^2 w_o^C}{dx^2} \right) + N_{xx}^C \frac{d^2 w_o^C}{dx^2} = 0 \quad (\text{h.9})$$

$$\Rightarrow \frac{d^2}{dx^2} \left(\frac{B_{11} N_{xx}^C}{A_{11}} + \frac{B_{11}^2}{A_{11}} \frac{d^2 w_o^C}{dx^2} - D_{11} \frac{d^2 w_o^C}{dx^2} \right) + N_{xx}^C \frac{d^2 w_o^C}{dx^2} = 0 \quad (\text{h.10})$$

$$\Rightarrow \frac{d^2}{dx^2} \left(\frac{B_{11} N_{xx}^C}{A_{11}} \right) + \left(\frac{B_{11}^2 - A_{11} D_{11}}{A_{11}} \right) \frac{d^4 w_o^C}{dx^4} + N_{xx}^C \frac{d^2 w_o^C}{dx^2} = 0 \quad (\text{h.11})$$

Assuming that at the point of buckling, N_{xx}^C is independent of x , hence,

$$\left(\frac{B_{11}^2 - A_{11} D_{11}}{A_{11}} \right) \frac{d^4 w_o^C}{dx^4} + N_{xx}^C \frac{d^2 w_o^C}{dx^2} = 0 \quad (\text{h.12})$$

By introducing $P = -N_{xx}^C$, where P is the axial compressive force, the equation above transforms into the following form:

$${}^1(\overline{EI}) \frac{d^4 w_o^C}{dx^4} + P \frac{d^2 w_o^C}{dx^2} = 0 \quad (\text{h.13})$$

Therefore, the effective flexural stiffness that can be derived employing the method 1 will be:

$${}^1(\overline{EI}) = \frac{A_{11} D_{11} - B_{11}^2}{A_{11}} \quad (\text{h.14})$$

The pre-superscript 1 on the left side of the equation is to denote that it is method 1.