Robust deconvolution and control of uncertain and time delay systems

Lin, Jun

2008

https://hdl.handle.net/10356/4111

https://doi.org/10.32657/10356/4111

Robust Deconvolution and Control of Uncertain and Time Delay Systems

Lin Jun

School of Electrical & Electronic Engineering

A thesis submitted to the Nanyang Technological University

in fulfillment of the requirements for the degree of

Doctor of Philosophy

2008

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

....

Date Lin Jun

To my family,
for their encouragement and love.

Acknowledgements

First of all and most importantly, I am deeply indebted to my supervisor, Professor Xie Lihua, for his professional guidance, constant support, encouragement and more. His insightful comments and thoughtful discussions have been an inspiration for my work. What I appreciated most were the many opportunities and the freedom he has offered that allowed me to find the research topics I am most interested in. Without his great enthusiastic help, I would not be able to complete this thesis.

I would also like to thank Prof. Zhang Huanshui for his helpful discussions and advice.

Appreciation also goes to the members of Instrumentation and System Lab and other friends who helped and supported me in these years. They brought me so much happiness and helped me in need.

I must also acknowledge the research scholarship I received from the Nanyang Technological University where the research work was carried out in the school of Electrical and Electronic Engineering.

Yet, in the end, I am most grateful to my parents for the love and encouragement provided through my entire life.

The Ph.D study is one of the most important periods for me, it means a lot and has changed my life.

Summary

This thesis presents some research results on control and deconvolution estimation for discrete-time linear systems with delay. The thesis focuses on two facets, the first is the robust deconvolution estimation of time-delay systems with stochastic parameter uncertainties and the other is the control of time-delay systems.

We begin with the topic of robust deconvolution for systems with input delay where we are concerned with the development of a polynomial approach to robust deconvolution filtering of linear discrete time-delay systems with random parameter uncertainties. The robust filtering problem is to find an estimator that minimizes the mean square estimation error with respect to the random parameter uncertainties and input and measurement noises. We discuss the problem for both the single-input single-output (SISO) and multi-input multi-output (MIMO) systems. The key to our solution is to quantify the effect of the random parameter uncertainties by introducing fictitious noises for which a simple way is given to calculate their covariances. The polynomial approach provides a lower computation cost than that of the state space approach.

We then look into control problems for systems with input delays. H_2 and H_∞ control problems for systems with time-varying but bounded delays in the input are first studied. We apply a state augmentation approach to obtain an H_2 state feedback controller. Sequential linear programming matrix method (SLPMM) is used to deal with non-convex bilinear matrix inequality (BMI). Although state augmentation has no computation advantage, it can be used when delay is relatively small. As for the H_∞ control problem, we obtain a sufficiency condition for the stability of the input-delay system by applying a Lyapunov-Krasovskii (L-K) functional. An H_∞ controller is further designed via a linear matrix inequality (LMI) approach.

More complicated problems such as the LQR, LQG, and H_∞ control for systems with known multiple input/ouput delays are studied. The key to our development of the LQR control is a duality between the LQR problem for systems with multiple input delays and a smoothing problem for an associated backward stochastic delay-free system. The duality allows us to obtain a simple solution to the LQR problem. To address the LQG problem, a separation principle is first established which converts the LQG problem into an LQR problem plus a Kalman filter for systems with multiple measurement delays. To address the latter, a reorganized innovation analysis is applied. We then extend the work to consider the H_∞ control problem. Similar to the LQR case, the H_∞ control problem is converted to a smoothing problem in Krein space. A sufficient condition is provided to check the existence of an H_∞ state feedback controller and a solution to the state feedback control is given in terms of Riccati difference equations (RDEs).

Finally, we look into the sampled-data LQR control for systems with multiple delayed inputs. A sampled state feedback controller with the zero-order hold is used. To solve the problem, the state is augmented with the zero-order hold inputs. The sampled-data LQR problem is then converted into a continuous-time LQR problem with the help of the Dirac delta function. By applying the LQR result for the continuous-time system and some simplification, we derive the optimal sampled-data controller.

Table of Contents

List of Figures

Symbols and Acronyms

Algebraic Operators

A^{-1}	Inverse of the matrix A
$A > 0(A \geq 0)$	A is positive definite (semi-positive definite)
$\delta(\cdot)$	Dirac delta function
adj	Adjoint matrix
det	Determinant
$Res(\cdot)$	Residue of a polynomial function
$sat(\cdot)$	saturation function
$Var(\cdot)$	variance
$tr(A)$	trace of matrix A
$\|y\|$	Euclidian norm of vector of y
$\|y\|_1$	$\sum_{i=1}^{n} \|y_i\|,\ y = [y_1, \cdots, y_n]^T$
A^T	Transpose of matrix A
I	Identity matrix

Sets and Set Operators

$\ell_2[0, N]$	Set of energy bounded discrete-time signals on $[0, N]$
$\ell_2[0, \infty)$	Set of energy bounded discrete-time signals on $[0, \infty)$
$\mathbb{N}$	Set of nature numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	Set of real vectors with n elements
$\mathbb{R}^{n\times m}$	Set of real matrices with n rows and m columns
$\mathbb{Z}$	Set of integer numbers

Acronyms

ABR	Available bit rate
ARE	Algebraic Riccati equation
arg	argument
ATM	Asynchronous transfer mode
BMI	Bilinear matrix inequality
CBR	Constant bit rate
ER	Explicit rate
GCC	Guaranteed cost control
HJB	Hamilton-Jacobi-Bellman
i.i.d	Independent and identical distribution
IIR	Infinite impulse response
L-K	Lyapunov-Krasovskii
LMI	Linear matrix inequality
LQG	Linear quadratic guassian
LQR	Linear quadratic reguation
LTI	Linear time invariant

ODE	Ordinary differential equation
Pr	Probability
QPSK	Quadrature phase shift keying
RDE	Riccati difference (differential) equation
RLS	Recursive least-squares
RM	Resource management
QoE	Quality of experience
QoS	Quality of service
SISO	Single input and Single output
SLMPP	Sequential linear programming matrix matrix method
SNR	Signal to noise ratio
MIMO	Multiple input and Multiple output
MMSE	Minimum mean square error
MSE	Mean square error
SD	Sampled-data
TDMA	Time division multiple access
VBR	Variable bit rate

Chapter 1

Introduction

1.1 Motivation

Control system design based on H_2 and H_∞ performance criteria has attracted a lot of interests in the past two decades. The H_2 concept can be traced back to the seminal work by Kalman and Bucy in the early 1960s [40] [41] in which state space was introduced to represent systems. On the other hand, the H_∞ control problem was firstly introduced by Zames [96] in 1981, initial developments in H_∞ control theory were based on frequency domain and operator theoretic methods. With the publication of the seminal work of [20], robust and optimal control/estimation entered a golden era, numerous works have been done in state space framework.

In the early 1990s, robust control/estimation [86, 93] emerged to conquer the difficulty when there is uncertainty in system model. Two kinds of parameter uncertainties are often discussed in literatures, one is bounded deterministic uncertainty [13, 70, 93], the other is stochastic (random) parameter uncertainty [89, 105]. The deterministic characterization of uncertainties assumes that the system parameters are within known lower and upper bounds which may not be realistic in applications such as mobile communications. In this situation, it may be more appropriate to characterize the modelling error in terms of random parameter uncertainties. Poor and Looze [70] consider robust estimation in state space where the process noise and measurement noise have uncertain second order statistics, but the system parameters are known exactly.

Deconvolution is concerned with the estimation of the input signal transmitted through a system (channel) based on some noise corrupted measurement. This problem has many applications including signal detection in communications, seismology, speech processing, equalization and numerical differentiation; see, e.g. [30, 54] and the references therein. Deconvolution problem is discussed in time domain [13] and frequency domain [15,19,45]. A polynomial approach (frequency domain) uses techniques such as inner-outer factorization and orthogonal principle and is often preferred in signal processing and communication communities. In the work of Chisci and Mosca [15], MMSE (minimum mean-square error) deconvolution problem is solved, Deng et al. [19] provide solution for optimal and self-tuning deconvolution estimation.

On the other hand, time delay exists in many engineering problems such as communications. Time delay cannot be ignored otherwise the closed loop system performance may not be guaranteed, even worse, stability may be jeopardized. With the development in engineering including networked control, congestion control in communication, chemical engineering, aircraft stabilization, time-delay systems have attracted a lot of interests [3, 79, 92].

Note that the stability of a linear time-invariant system can be judged by checking eigenvalue condition. However, checking eigenvalue condition is difficult for delay systems because there may be infinite number of poles in the system transfer function. The Lyapunov-Krasovskii (L-K) functional method [34] provides an effective way for establishing stability and has been extensively studied in the past decade.

However, the L-K approach generally only leads to a conservative sufficient stability condition.

Optimal control of time-delay systems has been an active research area since the late 1960s, first in the H_2 (LQG) and then in the H_∞ settings, see e.g. [16,85] for the discrete-time case and [7,44,60,61,83] for the continuous-time case. Time delay can appear in state and/or in input/output(I/O) of the system model. Many papers, see, e.g. [8,12,24] discuss systems with time delay in state. For discrete-time systems with delays, state augmentation approach can transform the system into delay-free one by augmenting the system state. However, it is computationally costly when the delay is large.

System with input/output (I/O) delay was investigated early by Smith [80]. For systems with input delay, 'memoryless' state feedback controller can be solved in terms of Lyapunov-Krasovskii functional and the result can be delay dependent or delay independent. However, the L-K approach is generally conservative as only sufficient conditions are known. There are mainly three approaches for control with I/O delay in the last two decades [60] and most of the results are for continuous-time systems. The first is a time-domain method. Operator interpolation and lifting originally derived for sampled-data control problem was used in control of time-delay systems [22,28,104]. Kojima and Ishijima [44] solve the generalized H_∞ preview and delayed control problem via an operator Riccati equation approach. Preview H_∞ control and estimation are studied in [84, 85] by Tadmor and Mirkin where game theoretic approach is used in the proof. In the full-information control, a standard H_2 ARE and a nonstandard H_∞-like ARE are used. Mirkin [55, 56] studies the H_∞ fixed-lag smoothing and control problems and provides a link between performance and delay. Meinsma and Mirkin [52] decompose the multiple I/O delays control problem into a series of nested elementary delay problems. Moelja and Meinsma [61] provide an interesting solution by converting the H_2 control problem to an equivalent

LQR problem.

The second approach is based on J-spectral factorization. The existence of J-spectral factor is then shown to be equivalent to the existence of nonnegative definite, stabilizing solutions to two indefinite algebraic Riccati equations, allowing a state-space formula for a linear fractional representation of all controllers. Meinsma and Zwart [53] use the J-spectral factorization approach to solve the H_∞ control for time delay system in frequency domain where Smith predictor is used in order to get a causal controller.

The last one is the Krein space approach. The key to this approach is the duality between the LQR problem for systems with multiple input delays and a smoothing problem for a backward stochastic delay free system, which extends the well known duality between the LQR of delay free systems and the optimal filtering. Zhang and his collaborators [97–100, 102] solve the linear quadratic regulation and the full-information H_∞ problems for multiple time delay systems using this approach.

In addition to those methods, there are some other methods for time-delay systems, e.g. Basin et al. [6, 7] provided finite time horizon LQR control for systems with time delay in state and input where Hamilton-Jacobi-Bellman (HJB) equation is used.

On the other hand, sampled-data (SD) control problems have been extensively discussed in the past decade [4, 9, 29, 42, 51, 71, 82, 88]. The analysis for SD systems is complicated because the continuous-time behavior is periodically time-varying and the dynamics are hybrid (continuous/discrete-time). SD systems are mostly considered as periodic systems and the controller is expressed in terms of periodic Riccati equations. There are mainly three approaches to the SD problems in recent years.

The first one is a direct method which solves SD problems in finite-dimensional hybrid state space. Kabamba [39] gave a solution for the H_∞ control in a finite-dimensional hybrid state space. Khargonekar et al. [42] synthesize the H_2 optimal control for SD systems and Yamamoto [95] also gives a solution to the H_2 control problem.

The second one is a frequency domain method based on parametric transfer function. Rosenwasser et al. [74] provides a solution to the H_2 time-delayed SD problem in frequency-domain, however, spectral factorization is hard to solve especially for systems with multiple inputs.

The third and commonly used one is the lifting method which relies on the property that all norms are preserved under lifting and so is the feedback interconnection structure. This method was first introduced by Yamamoto [95]. Toivonen and Sagfors [75] study the relationship between the lifting method and the two-Riccati equation solution in the H_∞ case. Trentelman and Stooroogel [88] deal with the H_2 problem for SD systems with the lifting method. Bamieh et al. [4] provide solutions to the H_∞ and H_2 problems where inter-sample behavior is discussed. Chen and Francis [10] work on the H_∞ SD control and evaluate the SD system according to the length of the sampling period. Mirkin et al. [58, 59] provide a rather comprehensive and insightful synthesis for the H_2 and H_∞ control of SD systems. For time delay systems, Chen and Francis [9] discusses their H_2 control problem. Park et al. [69] present a general framework of H_2 controllers for SD systems with single input delay.

There are some other results besides those mentioned above. Kabamba [38] investigate the generalized hold SD control problem. Furthermore, Sun et al. [81] and Shi [78] obtain necessary and sufficient conditions for the H_∞ control and filtering problems with generalized hold. Shergei et al. [77] discuss SD nonlinear H_∞ estimation and control where Dirac delta function is brought into discussion to transform the hybrid (continuous/discrete-time) system into a continuous-time one.

1.2 Objectives

The objectives of the present research are to develop methods for robust deconvolution estimation of time-delay systems with random parameter uncertainties and control of systems with time delay. In particular, we shall study the following two problems: (1) robust deconvolution for SISO/MIMO discrete-time systems with input delay and (2) delay dependent robust and optimal (H_2 and H_∞) control for systems with single or multiple input delays.

1.3 Major Contributions

The main contributions of the thesis are listed as follows:

(1) We propose a polynomial approach to robust deconvolution for discrete time-delay systems with random parameter uncertainties. Both the SISO and MIMO cases are investigated.

(2) H_2 and H_∞ problems for systems with unknown time-variant delay and known time-invariant delay are discussed and delay-dependent conditions are obtained.

(3) A solution to the LQG control of systems with multiple time-invariant I/O delays is provided based on a separation principle established. We also generalize the duality between LQR control with multiple delays and a smoothing for a backward stochastic system obtained in [97, 98] to more general case.

(4) LQR control for sampled-data systems with multiple input delays is studied. We transform the hybrid system into a continuous-time form by introducing Dirac delta function. This approach allows us to give an explicit solution to the SD problem using the LQR result for continuous-time systems.

1.4 Organization of the Thesis

The rest of the thesis is organized into two parts.

Part I includes Chapter 2 and Chapter 3 and it focuses on robust deconvolution for discrete-time systems with input delay. Chapter 2 and Chapter 3 use polynomial approach to solve robust deconvolution problem in SISO and MOMO systems, respectively. The details of the two chapters are as follows.

In Chapter 2, we develop a polynomial approach to robust deconvolution filtering of linear discrete-time systems with random parameter uncertainties and input delay. The uncertainties exist in both numerator and denominator of the system model. We aim to minimize the mean square estimation error with respect to the random bounded covariance uncertainties of the random parameters as well as input noises. The key to our solution is to quantify the effect of the random parameter uncertainties by introducing two fictitious noises for which a simple way is given to calculate their covariances. Some examples including application in wireless network are provided.

We extend our study in Chapter 3 to the MIMO case. A polynomial approach is adopted too. Covariance matrices of fictitious noises are calculated after some basic algebraic manipulations and they can be simplified in some case. The deconvolution estimator is expressed in terms of solution of a Diophantine equation and a spectral factorization. An example is used to show the efficiency of our method.

Part II includes chapters 4-8 and it focuses on control for systems with I/O delays. First we consider the H_2 and H_∞ control for system with time-variant input delay in Chapter 4 and Chapter 5. In Chapter 6 and Chapter 7, H_2 and H_∞ control for systems with multiple time-invariant delays are discussed. Sampled-data LQR control for systems with multiple input delays is solved in Chapter 8. The details of each chapter are as follows.

In Chapter 4, state feedback H_2 control for systems with time-variant but bounded delay in input is discussed. State augmentation is used to transform the system with input delay into a linear time varying system without delay and the H_2 controller is given in terms of BMIs. As an important application, ATM network congestion control with explicit rate feedback is provided, saturation in source rate and queue buffer is discussed as well.

In Chapter 5, the H_∞ state feedback control for systems with time-variant but bounded delay in input is studied. A sufficient condition for the asymptotic stability of the system is provided using a Lyapunov-Krasovskii functional. The state feedback controller is given in terms of LMIs. Application to congestion control in ATM networks is provided as well.

In Chapter 6, we revisit the classic linear quadratic Gaussian (LQG) problem for discrete-time systems with multiple input/output delays in finite horizon. A separation principle is established in order to convert the output feedback control problem into an LQR control problem in conjunction with the Kalman filtering. The LQR controller is obtained using a duality between the LQR for multiple-input delay systems and a smoothing problem for an associated backward stochastic delay free system. The Kalman filtering with multiple output delays is solved using a reorganized innovation analysis. One forward and one backward Riccati equations with the same dimension as the original system are to be solved. This approach has the advantage in computation as compared with the system augmentation approach.

In Chapter 7, we investigate the finite time horizon H_∞ control problem for discrete time systems with multiple input delays. We extend the existing work [97] by allowing a more general form in the controlled output. A linkage between the H_∞ control problem and a smoothing problem is first established which allows us to give a sufficient solvability condition for the H_∞ control problem by classical estimation theory. Our solution is given in terms of one Riccati difference equation of the same dimension of the plant (ignoring the delays). As a special case, a complete solution to the H_2 state feedback control of systems with multiple input delays is derived. The H_∞ control result is then applied to ATM congestion control. Simulations show that the proposed control technique can achieve desired control performance very efficiently and has certain robustness with respect to the varying round trip delay of the ATM network.

In Chapter 8, we consider the problem of sampled-data LQR for systems with multiple input delays. State feedback control with the zero-order hold is adopted and we apply the Dirac delta function to transform the hybrid (continuous and discrete) problem into a continuous-time LQR problem and solve it using the result of [98]. The sampled-data optimal controller is then derived via some algebraic manipulations.

In Chapter 9, we draw some conclusions and recommend some works for future research.

Part I

Robust Deconvolution for Systems with Input Delay

Chapter 2

Robust Deconvolution of Uncertain Discrete SISO Systems with Input Delay

2.1 Introduction

Deconvolution deals with the estimation of the input signal transmitted through a system (channel) based on some noise corrupted measurement [30] [1]. Much attention has been paid to this problem based on the Kalman filtering formulation [13] [43] or the polynomial approach [45] [1]. When system parameters are not known precisely or the parameters are time varying because of the perturbations of transmission medium, robust deconvolution can provide reliable estimation against these parameter uncertainties. In [13], the minimax robust deconvolution estimation is discussed in a state-space framework where Kalman filtering is used. For systems with random parameter uncertainty, a polynomial approach is adopted in [68] for dealing with the robust filtering problem. However, the uncertainties are assumed to exist only in the numerator of the system transfer function which may not be realistic. On the other hand, in most of the existing works, the parameters of the system and the statistics of the noises are assumed to be known exactly, which may not be the case in practice. For example, in mobile communications, channels are affected by the speeds of mobile terminals and transmission medium and are usually time-varying. It is impossible to determine the exact models of the channels. On the other hand, ignoring modeling uncertainties in the design of estimators may result in poor system performance [13]. Delay in input deteriorates the performance of deconvolution estimator in terms of mean square estimation error.

This chapter aims to present a polynomial approach to the robust deconvolution filtering problem for a general case where both the system with input delay and signal models contain random time-varying parameter uncertainties and the second order statistics of input and measurement noises are assumed to be within certain bounds. A robust filter is designed based on a minimax approach in the sense that the filter will minimize the upper bound of the estimator error covariance with respect to all the system uncertainties and noises. Unlike the work in [13], we adopt a polynomial approach which is usually preferred for signal processing applications and is computationally more attractive. To derive the robust optimal estimator, we first convert the random parameter uncertainties into two fictitious noises and give a simple way to calculate their covariances. Then, the estimator is designed following the standard polynomial approach. Our solution is given in terms of one spectral factorization and one polynomial equation.

The following notations are used throughout Chapter 2 and Chapter 3. q^{-1} is the backward shift operator. $X_*(q^{-1})$ is the conjugate polynomial of $X(q^{-1})$, i.e., if $X(q^{-1}) = x_0 + x_1q^{-1} + \ldots + x_{nx}q^{-nx}$, then $X_*(q^{-1}) = x_0 + x_1q + \cdots + x_{nx}q^{nx}$. For the sake of simplicity, the backward shift operator q^{-1} will be dropped wherever no confusion may be caused.

2.2 Problem Statement

We consider the source signal $u(k)$ and the noise corrupted measurement $y(k)$ that are generated by uncertain models [13, 15]:

$$\left[D(q^{-1}) + \Delta D(q^{-1})\right] u(k) = \left[C(q^{-1}) + \Delta C(q^{-1})\right] e(k), \tag{2.1}$$

$$\left[A(q^{-1}) + \Delta A(q^{-1})\right] y(k) = \left[B(q^{-1}) + \Delta B(q^{-1})\right] u(k-d) + \left[P(q^{-1}) + \Delta P(q^{-1})\right] v(k) \tag{2.2}$$

where d is the delay in input, $D(q^{-1})$, $C(q^{-1})$, $A(q^{-1})$, $B(q^{-1})$ and $P(q^{-1})$ are known polynomials which have the form

$$X(q^{-1}) = x_0 + x_1 q^{-1} + \ldots + x_{nx} q^{-nx}. \tag{2.3}$$

The polynomials $\Delta D(q^{-1})$, $\Delta C(q^{-1})$, $\Delta A(q^{-1})$, $\Delta B(q^{-1})$ and $\Delta P(q^{-1})$ represent parameter uncertainties of the models and they have the form

$$\Delta X(q^{-1}) = x_0(k) + x_1(k) q^{-1} + \cdots x_{nx}(k) q^{-nx} \tag{2.4}$$

with $x_i(k),\ i = 0, 1, \cdots, nx$ being time-varying random variables. In practice we may always assume [15, 30] that $d_0 = a_0 = 1$, $d_0(k) = 0$ and $a_0(k) = 0$.

For the sake of convenience in discussion, we adopt the notations:

$$\mathcal{X}(k) \triangleq \left[x_{nx_0}(k), x_{nx_0+1}(k), \cdots, x_{nx}(k)\right]^T, \tag{2.5}$$

$$\mathcal{Q}_x \triangleq \left[q^{-nx_0}, q^{-(nx_0+1)}, \cdots, q^{-nx}\right]^T, \tag{2.6}$$

where $nd_0 = na_0 = 1$ and $nc_0 = nb_0 = np_0 = 0$.

Remark 2.2.1. *By using the above notations, we have that* $\Delta X(q^{-1}) = \mathcal{Q}_x^T \mathcal{X}(k)$.

For example, $\Delta D(q^{-1}) = \mathcal{Q}_d^T\mathcal{D}(k)$ *with* $\mathcal{Q}_d = \left[q^{-1}, \cdots, q^{-nd}\right]^T$ *and* $\mathcal{D}(k) = \left[d_1(k), \cdots, d_{nd}(k)\right]^T$*, and* $\Delta C(q^{-1}) = \mathcal{Q}_c^T\mathcal{C}(k)$ *with* $\mathcal{Q}_c = \left[1, q^{-1}, \cdots, q^{-nc}\right]^T$ *and* $\mathcal{C}(k) = \left[c_0(k), \cdots, c_{nc}(k)\right]^T$*.*

In addition, we denote

$$\mathcal{D}_c(k) \triangleq \begin{bmatrix} -\mathcal{D}^T(k) & \mathcal{C}^T(k) \end{bmatrix}^T, \tag{2.7}$$

$$\mathcal{A}_b(k) \triangleq \begin{bmatrix} -\mathcal{A}^T(k) & \mathcal{B}^T(k) \end{bmatrix}^T, \tag{2.8}$$

$$\mathcal{A}_p(k) \triangleq \begin{bmatrix} -\mathcal{A}^T(k) & \mathcal{P}^T(k) \end{bmatrix}^T, \tag{2.9}$$

$$\mathcal{A}_{bp}(k) \triangleq \begin{bmatrix} -\mathcal{A}^T(k) & \mathcal{B}^T(k) & \mathcal{P}^T(k) \end{bmatrix}^T, \tag{2.10}$$

where $\mathcal{A}(k),\ \mathcal{B}(k),\ \mathcal{P}(k),\ \mathcal{D}(k)$ and $\mathcal{C}(k)$ are defined in (2.5).

The following assumptions are made throughout this chapter.

Assumption 2.2.1. *The input noises* $e(k)$ *and* $v(k)$ *and the random parameter uncertainties* $\mathcal{A}_{bp}(k)$ *and* $\mathcal{D}_c(k)$ *are mutually independent white noises with zero means and covariances:*

$$\bar{E}[e^2(k)] = \sigma_e^2, \quad \bar{E}[v^2(k)] = \sigma_v^2, \tag{2.11}$$

$$\tilde{E}\left[\mathcal{A}_{bp}(k)\mathcal{A}_{bp}^T(k)\right] = \mathcal{R}_{abp}, \quad \tilde{E}\left[\mathcal{D}_c(k)\mathcal{D}_c^T(k)\right] = \mathcal{R}_{dc}, \tag{2.12}$$

where $\bar{E}$ *and* $\tilde{E}$ *are respectively the mathematical expectations with respect to the input noises and the random parameter uncertainties in the models.*

In this chapter, the covariances $\sigma_e^2, \sigma_v^2, \mathcal{R}_{abp}$ and $\mathcal{R}_{dc}$ are not known exactly, but are from known ranges given below.

Assumption 2.2.2. *The unknown covariances* $\sigma_e^2,\ \sigma_v^2,\ \mathcal{R}_{abp}$ *and* $\mathcal{R}_{dc}$ *have the* *known upper and lower bounds, i.e.*

$$\underline{\sigma}_e^2 \le \sigma_e^2 \le \overline{\sigma}_e^2, \quad \underline{\sigma}_v^2 \le \sigma_v^2 \le \overline{\sigma}_v^2, \tag{2.13}$$

$$\underline{\mathcal{R}}_{abp} \le \mathcal{R}_{abp} \le \overline{\mathcal{R}}_{abp}, \tag{2.14}$$

$$\underline{\mathcal{R}}_{dc} \le \mathcal{R}_{dc} \le \overline{\mathcal{R}}_{dc}. \tag{2.15}$$

Denote

$$\begin{aligned} \mathcal{R}_a &\triangleq \tilde{E}\left[\mathcal{A}(k)\mathcal{A}^T(k)\right], \quad \mathcal{R}_{ab} \triangleq \tilde{E}\left[\mathcal{A}_b(k)\mathcal{A}_b^T(k)\right], \\ \mathcal{R}_{ap} &\triangleq \tilde{E}\left[\mathcal{A}_p(k)\mathcal{A}_p^T(k)\right], \quad \mathcal{R}_d \triangleq \tilde{E}\left[\mathcal{D}(k)\mathcal{D}^T(k)\right]. \end{aligned} \tag{2.16}$$

In view of (2.7)-(2.10), $\mathcal{R}_a$, $\mathcal{R}_{ab}$ and $\mathcal{R}_{ap}$ can be easily obtained from the covariance matrix $\mathcal{R}_{abp}$ and $\mathcal{R}_d$ from $\mathcal{R}_{dc}$. For example, $\mathcal{R}_a$ is the first $n_a \times n_a$ sub-block of $\mathcal{R}_{abp}$.

Furthermore, from (2.14)-(2.15), it is easy to know that

$$\underline{\mathcal{R}}_a \le \mathcal{R}_a \le \overline{\mathcal{R}}_a, \quad \underline{\mathcal{R}}_{ab} \le \mathcal{R}_{ab} \le \overline{\mathcal{R}}_{ab}, \quad \underline{\mathcal{R}}_{ap} \le \mathcal{R}_{ap} \le \overline{\mathcal{R}}_{ap}, \tag{2.17}$$

$$\underline{\mathcal{R}}_d \le \mathcal{R}_d \le \overline{\mathcal{R}}_d, \tag{2.18}$$

where $\underline{\mathcal{R}}_a$ ($\overline{\mathcal{R}}_a$), $\underline{\mathcal{R}}_{ab}$ ($\overline{\mathcal{R}}_{ab}$) and $\underline{\mathcal{R}}_{ap}$ ($\overline{\mathcal{R}}_{ap}$) can be obtained from the corresponding lower (upper) bounds $\underline{\mathcal{R}}_{abp}$ ($\overline{\mathcal{R}}_{abp}$), and $\underline{\mathcal{R}}_d$ ($\overline{\mathcal{R}}_d$) from $\underline{\mathcal{R}}_{dc}$ ($\overline{\mathcal{R}}_{dc}$).

Remark 2.2.2. *It is worth noting that the above model can be used to represent a system whose model may be subject to some variations from its nominal model and can find applications in, for example, mobile fading channels in communications [72], signal processing, control [17], nuclear fission and heat transfer, and population models [63]. Similar models have been used in [13, 89] where a state-space approach has been adopted. In fact, our model (2.1)-(2.2) will reduce to that of [13] when*

$\Delta D(q^{-1}) = \Delta C(q^{-1}) = 0$, $A(q^{-1}) = P(q^{-1})$ and $\Delta A(q^{-1}) = \Delta P(q^{-1})$, i.e., when there is no uncertainty in the model of the source signal and the measurement noise is white. On the other hand, a state-space model with its coefficient matrices containing random parameter uncertainties has been considered in [89] where a Kalman filtering problem is investigated.

Remark 2.2.3. *Assumption 2.2.2 on the second-order statistics of the uncertainties is standard; see, e.g. [70]. It should be noted that the system parameters in [70] are assumed to be known exactly, only the noise statistics contain uncertainty.*

Assumption 2.2.3. *The polynomials $A(q^{-1})$ and $D(q^{-1})$ are stable, i.e., all zeros of $A(q^{-1})$, $D(q^{-1})$ are in $|q| < 1$.*

The robust optimal deconvolution estimation problem to be addressed in this chapter is stated as follows:

Given the measurement $y(s)$, $0 \leq s \leq k-m$, where m is an integer, find a time-invariant estimator $\hat{u}(k \mid k-m)$ to minimize the following maximum mean square error:

$$\min_{\{\hat{u}\}} \max_{\{\mathcal{R}_{dc}, \mathcal{R}_{abp}, \sigma_e^2, \sigma_v^2\}} \tilde{E}\bar{E}\left[u(k) - \hat{u}(k \mid k-m)\right]^2, \tag{2.19}$$

where $\bar{E}$ is the mathematical expectation with respect to $e(k)$ and $v(k)$, and $\tilde{E}$ with respect to $\mathcal{D}_c(k)$ and $\mathcal{A}_{bp}(k)$.

The resultant estimator is called *robust optimal deconvolution estimator.*

Remark 2.2.4. *Since the white noises represented by $e(k)$ and $v(k)$ are independent of $\mathcal{D}_c(k)$ and $\mathcal{A}_{bp}(k)$, the objective function defined by (2.19) is obviously equivalent to*

$$E\left[u(k) - \hat{u}(k \mid k-m)\right]^2, \tag{2.20}$$

where E is the mathematical expectation over $e(k)$, $v(k)$, $\mathcal{D}_c(k)$ and $\mathcal{A}_{bp}(k)$.

2.3 Design of Optimal Deconvolution Estimator

In this section, we derive a robust optimal deconvolution estimator by using projection formulae and an innovation analysis approach. It is shown that the robust optimal deconvolution estimation involves computing covariance matrices of two fictitious noises, one spectral factorization and one polynomial equation which is slightly more complicated than the standard optimal design for systems without random parameter uncertainties.

2.3.1 Fictitious Noises

First, in view of (2.4), (2.1) and (2.2) are re-written as

$$D(q^{-1})u(k) = C(q^{-1})e(k) + e_0(k), \tag{2.21}$$

$$A(q^{-1})y(k) = B(q^{-1})u(k-d) + P(q^{-1})v(k) + v_0(k), \tag{2.22}$$

where

$$e_0(k) = -\Delta D(q^{-1})u(k) + \Delta C(q^{-1})e(k), \tag{2.23}$$

$$v_0(k) = -\Delta A(q^{-1})y(k) + \Delta B(q^{-1})u(k-d) + \Delta P(q^{-1})v(k). \tag{2.24}$$

$e_0(k)$ and $v_0(k)$ are termed as *fictitious noises*. Substituting (2.21) into (2.22) yields

$$ADy(k) = DPv(k) + BCe(k-d) + Be_0(k-d) + Dv_0(k), \tag{2.25}$$

where the operator q^{-1} has been omitted in the polynomials $A,\ B,\ P,\ C$ and D.

Assumption 2.3.1. *The upper bounds $\overline{\mathcal{R}}_d$ and $\overline{\mathcal{R}}_a$ of the covariance matrices $\mathcal{R}_d$ and $\mathcal{R}_a$ satisfy*

$$\frac{1}{2\pi i}\oint_{|z|=1}(DD_*)^{-1}\mathcal{Q}_d^T\overline{\mathcal{R}}_d[\mathcal{Q}_d^T]_*\frac{dz}{z} \quad < \quad 1, \tag{2.26}$$

$$\frac{1}{2\pi i}\oint_{|z|=1}(AA_*)^{-1}\mathcal{Q}_a^T\overline{\mathcal{R}}_a[\mathcal{Q}_a^T]_*\frac{dz}{z} \quad < \quad 1, \tag{2.27}$$

where $\mathcal{Q}_d = \left[q^{-1}, \cdots, q^{-nd}\right]^T$ *and* $\mathcal{Q}_a = \left[q^{-1}, \cdots, q^{-na}\right]^T$.

Remark 2.3.1. *Assumption 2.3.1 basically implies that the random uncertainties ΔD and ΔA should be 'smaller' than their nominal values D and A, respectively.*

Now, we have the following results for the fictitious noises, which will play an important role in the design of robust optimal deconvolution estimator.

Theorem 2.3.1.

(a). $e(k)$, $v(k)$, $e_0(k)$ and $v_0(k)$ satisfying Assumption 2.2.1 are mutually uncorrelated.

(b). $e_0(k)$ and $v_0(k)$ are white noises with zero means and the following covariances:

$$\sigma_{e_0}^2 \triangleq E[e_0^2(k)] \quad = \quad \gamma_1(1-\gamma_0)^{-1}, \tag{2.28}$$

$$\sigma_{v_0}^2 \triangleq E[v_0^2(k)] \quad = \quad (\sigma_{e_0}^2\lambda_1+\lambda_2)(1-\lambda_0)^{-1}, \tag{2.29}$$

where

$$\gamma_0 = \frac{1}{2\pi i}\oint_{|z|=1} (DD_*)^{-1}\mathcal{Q}_d^T \mathcal{R}_d [\mathcal{Q}_d^T]_* \frac{dz}{z}, \tag{2.30}$$

$$\gamma_1 = \frac{\sigma_e^2}{2\pi i}\oint_{|z|=1} (DD_*)^{-1}\left[C\mathcal{Q}_d^T \quad D\mathcal{Q}_c^T\right]\mathcal{R}_{dc}\left[C\mathcal{Q}_d^T \quad D\mathcal{Q}_c^T\right]_* \frac{dz}{z}, \tag{2.31}$$

$$\lambda_0 = \frac{1}{2\pi i}\oint_{|z|=1} (AA_*)^{-1}\mathcal{Q}_a^T \mathcal{R}_a [\mathcal{Q}_a^T]_* \frac{dz}{z}, \tag{2.32}$$

$$\lambda_1 = \frac{1}{2\pi i}\oint_{|z|=1} (AA_*DD_*)^{-1}\left[B\mathcal{Q}_a^T \quad A\mathcal{Q}_b^T\right]\mathcal{R}_{ab}\left[B\mathcal{Q}_a^T \quad A\mathcal{Q}_b^T\right]_* \frac{dz}{z}, \tag{2.33}$$

$$\begin{aligned}\lambda_2 = {} & \frac{1}{2\pi i}\oint_{|z|=1} (AA_*DD_*)^{-1}\times\Big\{\left[B\mathcal{Q}_a^T \quad A\mathcal{Q}_b^T\right]\mathcal{R}_{ab}\left[B\mathcal{Q}_a^T \quad A\mathcal{Q}_b^T\right]_* CC_*\sigma_e^2 \\ & +\left[P\mathcal{Q}_a^T \quad A\mathcal{Q}_p^T\right]\mathcal{R}_{ap}\left[P\mathcal{Q}_a^T \quad A\mathcal{Q}_p^T\right]_* \sigma_v^2\Big\}\frac{dz}{z},\end{aligned} \tag{2.34}$$

and $\mathcal{Q}_d$, $\mathcal{Q}_c$, $\mathcal{Q}_a$, $\mathcal{Q}_p$ *and* $\mathcal{Q}_b$ *are as defined in (2.6).*

Proof: See Appendix A.

2.3.2 Robust Optimal Deconvolution Estimator

Given the observation $y(s)$, $0 \le s \le k-m$, a stable time-invariant linear deconvolution estimator $\hat{u}(k \mid k-m)$ can be given as

$$\hat{u}(k \mid k-m) = \mathcal{F}(q^{-1})y(k-m), \tag{2.35}$$

where $\mathcal{F}$ is a causal and stable transfer function. From (2.21) and (2.23), it follows that

$$\begin{aligned} z(k) &\triangleq u(k) - \mathcal{F}y(k-m) \\ &= \left[\frac{C}{D} - q^{-m-d}\frac{\mathcal{F}BC}{AD}\right]e(k) - q^{-m}\frac{\mathcal{F}P}{A}v(k) + \left[\frac{1}{D} - q^{-m-d}\frac{\mathcal{F}B}{AD}\right]e_0(k) \\ &\quad -q^{-m}\frac{\mathcal{F}}{A}v_0(k). \end{aligned} \tag{2.36}$$

Applying (2.36) and Theorem 2.3.1, the estimation error is calculated as

$$\begin{aligned} Ez^2(k) &= \frac{1}{2\pi i}\oint\left\{\sigma_e^2\left[\frac{C}{D}-q^{-m-d}\mathcal{F}\frac{BC}{AD}\right][\digamma]_*+\sigma_v^2\mathcal{F}\mathcal{F}_*\frac{PP_*}{AA_*}\right. \\ &\left.+\sigma_{e_0}^2\left[\frac{1}{D}-q^{-m-d}\frac{\mathcal{F}B}{AD}\right][\daleth]_*+\sigma_{v_0}^2\frac{\mathcal{F}\mathcal{F}_*}{AA_*}\right\}\frac{1}{z}dz, \end{aligned} \tag{2.37}$$

where

$$\digamma=\frac{C}{D}-q^{-m-d}\mathcal{F}\frac{BC}{AD},\qquad \daleth=\frac{1}{D}-q^{-m-d}\mathcal{F}\frac{B}{AD}.$$

Lemma 2.3.1. *Under Assumption 2.2.1 and Assumption 2.3.1, we have*

$$arg\max_{\{\mathcal{R}_{dc},\mathcal{R}_{abp},\sigma_e^2,\sigma_v^2\}} Ez(k)^2 = \{\overline{\mathcal{R}}_{dc},\ \overline{\mathcal{R}}_{abp},\ \overline{\sigma}_e^2,\ \overline{\sigma}_v^2\}, \tag{2.38}$$

$$\max_{\{\mathcal{R}_{dc},\mathcal{R}_{abp},\sigma_e^2,\sigma_v^2\}} Ez(k)^2 = \overline{\sigma}_e^2k_1+\overline{\sigma}_v^2k_2+\overline{\sigma}_{e_0}^2k_3+\overline{\sigma}_{v_0}^2k_4, \tag{2.39}$$

where

$$\overline{\sigma}_{e_0}^2=\sigma_{e0\{\mathcal{R}_{dc}=\overline{\mathcal{R}}_{dc}\}}^2,\qquad \overline{\sigma}_{v_0}^2=\sigma_{v0\{\mathcal{R}_{abp}=\overline{\mathcal{R}}_{abp}\}}^2, \tag{2.40}$$

and

$$\begin{aligned} k_1 &= \frac{1}{2\pi i}\oint\left[\frac{C}{D}-q^{-m-d}\mathcal{F}\frac{BC}{AD}\right]\left[\frac{C}{D}-q^{-m-d}\mathcal{F}\frac{BC}{AD}\right]_*\frac{1}{z}dz, \\ k_2 &= \frac{1}{2\pi i}\oint\mathcal{F}\mathcal{F}_*\frac{PP_*}{AA_*}\frac{1}{z}dz, \\ k_3 &= \frac{1}{2\pi i}\oint\left[\frac{1}{D}-q^{-m-d}\mathcal{F}\frac{B}{AD}\right]\left[\frac{1}{D}-q^{-m-d}\mathcal{F}\frac{B}{AD}\right]_*\frac{1}{z}dz, \\ k_4 &= \frac{1}{2\pi i}\oint\mathcal{F}\mathcal{F}_*\frac{1}{AA_*}\frac{1}{z}dz. \end{aligned} \tag{2.41}$$

Proof: Note from (2.37) and (2.41) that

$$Ez^2(k) = \sigma_e^2k_1+\sigma_v^2k_2+\sigma_{e_0}^2k_3+\sigma_{v_0}^2k_4, \tag{2.42}$$

where $\sigma_{e_0}^2$ and $\sigma_{v_0}^2$ are computed by (2.28)-(2.29), i.e.,

$$\sigma_{e_0}^2 = \gamma_1(1-\gamma_0)^{-1}, \quad \sigma_{v_0}^2 = (\sigma_{e_0}^2\lambda_1 + \lambda_2)(1-\lambda_0)^{-1}, \tag{2.43}$$

with γ_i and λ_i being given by (2.30)-(2.34). It is not difficult to know that $\mathcal{R}_{abp} = \overline{\mathcal{R}}_{abp}$ implies $\mathcal{R}_{ab} = \overline{\mathcal{R}}_{ab}$, $\mathcal{R}_{ap} = \overline{\mathcal{R}}_{ap}$ and $\mathcal{R}_a = \overline{\mathcal{R}}_a$, and $\mathcal{R}_{dc} = \overline{\mathcal{R}}_{dc}$ implies $\mathcal{R}_d = \overline{\mathcal{R}}_d$. By applying Theorem 3.1, (2.38) and (2.39) follow from (2.42) directly.

Now the problem that is to be addressed in this chapter is converted to that of finding an estimator $\hat{u}(k \mid k-m)$ such that the following is minimized

$$Ez(k)^2_{|\{\mathcal{R}_{dc},\mathcal{R}_{abp},\sigma_e^2,\sigma_v^2\}=\{\overline{\mathcal{R}}_{dc},\ \overline{\mathcal{R}}_{abp},\ \overline{\sigma}_e^2,\ \overline{\sigma}_v^2\}}. \tag{2.44}$$

Theorem 2.3.2. *Consider the system defined by (2.1) and (2.2) and satisfying Assumptions 2.2.1-2.2.3 and 2.3.1. The robust optimal deconvolution estimator is given by (2.35) where the transfer function $\mathcal{F}$ has the form:*

$$\mathcal{F} = \frac{Q_1 A}{\overline{\beta}} \tag{2.45}$$

while the unknown polynomial Q_1 with an order of $max\{nc-m-d, nd-1\}$, together with the polynomial L_* of the order $\partial L = max(nb+nc+m+d, n\beta)-1$, is the unique solution to the equation

$$Q_1\overline{\beta}_*\overline{\sigma}_\varepsilon^2 + qL_*D = q^{m+d}\left[CC_*B_*\overline{\sigma}_e^2 + B_*\overline{\sigma}_{e_0}^2\right]. \tag{2.46}$$

In addition, the unknown polynomial $\overline{\beta}$ and the covariance $\overline{\sigma}_\varepsilon^2$ are the solution to the following spectral factorization:

$$\overline{\beta\beta}_*\overline{\sigma}_\varepsilon^2 = BB_*CC_*\overline{\sigma}_e^2 + DD_*PP_*\overline{\sigma}_v^2 + DD_*\overline{\sigma}_{v_0}^2 + BB_*\overline{\sigma}_{e_0}^2. \tag{2.47}$$

The minimal upper bound of the estimation error covariance is given by

$$Ez^2(k)_{min} = \frac{1}{2\pi i}\oint\left(\frac{\overline{\sigma}_e^2 CC_* + \overline{\sigma}_{e_0}^2}{DD_*} - \frac{BB_*(\overline{\sigma}_e^2 CC_* + \overline{\sigma}_{e_0}^2)(\overline{\sigma}_e^2 CC_* + \overline{\sigma}_{e_0}^2)}{\overline{\beta\beta}_* DD_* \overline{\sigma}_\varepsilon^2}\right)\frac{1}{z}dz + \frac{1}{2\pi i}\oint\frac{LL_*}{\overline{\sigma}_\varepsilon^2 \overline{\beta\beta}_*}\frac{1}{z}dz. \tag{2.48}$$

Proof: The result can be established by a similar argument as in [101].

Remark 2.3.2. *(2.46) is a bilateral polynomial equation. Note that D and $\overline{\beta}$ are stable. Thus, D and $\overline{\beta}_*$ have no common factors. This implies that the invariant polynomials of D are coprime with those of $\overline{\beta}_*$. Hence, a solution of (2.46) always exists and is unique [45]. Furthermore, a polynomial equation (2.46) can be easily solved by a linear system of equations $\mathcal{A}X = \mathcal{B}$, where $\mathcal{A}$ is a Sylvester matrix containing the coefficients of the polynomials in (2.46).*

Remark 2.3.3. *Since the polynomial $\overline{\beta}$, the spectral factorization factor, is always stable, the estimator given by (2.45) is always stable. Note that the calculation of the spectral factor $\overline{\beta}$ and $\overline{\sigma}_\varepsilon^2$ is very standard, a number of effective algorithms can be found in literature [37, 45].*

Remark 2.3.4. *As mentioned earlier, systems with random parameter uncertainties have been considered in [13, 89]. In [89], a state-space model with its coefficient matrices containing random parameter uncertainties has been studied for robust Kalman filtering using a linear matrix inequality (LMI) approach. Note that in [89] all the second-order statistics are assumed to be known exactly, only the initial values are uncertain and a finite horizon robust estimator is considered. In [13], the polynomial model is converted to a state-space model and a robust Kalman filter is then designed based on a Riccati equation approach; see also [65]- [66]. In the present work, we provide a robust filtering solution via a polynomial approach.*

To compare our work with that of [13], we consider the system tackled in [13] by setting $\Delta D(q^{-1}) = \Delta C(q^{-1}) = 0$ and $P(q^{-1}) = A(q^{-1})$, $nd \geq nc$, $na \geq nb$ and

$m = d = 0$ in our model. In this case, by [13], at least one Riccati equation of order $na + nd$ and one Lyapunov equation of order $na + nd$ are to be solved. In our approach, the calculation includes solving the spectral factorization (2.47) and the polynomial equation (2.46) (with order $\max\{nc, nd - 1\} + na + nd - 1$). Note that the calculation of λ_0, λ_1 and λ_2 is a simple application of residual theorem and the polynomial equation is solved by a linear system of equations which is much simpler than solving the Lyapunov equation in [13], especially when na or nd is large. Furthermore, it has been shown in [45] that the spectral factorization of (2.47) is computationally much more attractive than solving the Riccati equation in [13], especially when the order of the Riccati equation is high.

Remark 2.3.5. *Delay d in input affects the order of Q_1 in (2.46), furthermore the order of transfer function $\mathcal{F}$. When d increases, it is harder to predict the input signal through the information of output, i.e. the minimal upper bound of the estimation error covariance increases.*

2.4 Examples

In this section we shall apply the presented polynomial approach to the example in [13] and an application in wireless communication [48].

Example 2.4.1. *Consider the example in [13] where the models of the signal and measurement system are described by (2.1)-(2.2) with*

$$D(q^{-1}) = 1 - 0.8q^{-1}, \quad \Delta D(q^{-1}) = 0,$$

$$C(q^{-1}) = q^{-1}, \quad \Delta C(q^{-1}) = 0,$$

$$A(q^{-1}) = 1 - 0.2q^{-1}, \quad \Delta A(q^{-1}) = a(k)q^{-1},$$

$$B(q^{-1}) = 0.4q^{-1}, \quad \Delta B(q^{-1}) = b(k)q^{-1},$$

$$P(q^{-1}) = A(q^{-1}), \quad \Delta P(q^{-1}) = \Delta A(q^{-1}), \quad d = 0.$$

The model uncertainties $a(k)$, $b(k)$, the system noise, $e(k)$, and the observation noise, $v(k)$, are assumed to be mutually independent white noises with zero means and variances of $\sigma_a^2 \in [0.06, 0.26]$, $\sigma_b^2 \in [0.04, 0.24]$, $\sigma_e^2 \in [0.5, 1]$ and $\sigma_v^2 \in [0.5, 1]$, respectively.

Applying Theorem 2.3.1, we have $\bar{\sigma}_{e_0}^2 = 0$ and $\bar{\sigma}_{v_0}^2 = 1.1299$. Then, the spectral factor satisfies

$$\sigma_\varepsilon^2(1 + \overline{\beta}q^{-1})(1 + \overline{\beta}q) = -8.7956q^{-1} - 8.7956q + 18.2276,$$

which gives

$$\beta(q^{-1}) = 1 - 0.7619q^{-1} + 0.0628q^{-2}, \qquad \sigma_\varepsilon^2 = 2.5491.$$

The robust deconvolution estimator for filtering ($m = 0$) is obtained from Theorem 2.3.2 as

$$\hat{u}(k \mid k) = (1 - 0.7619q^{-1} + 0.0628q^{-2})^{-1}(1 - 0.8q^{-1})L(q^{-1})y(k),$$

where the polynomial $L(q^{-1})$ is the solution to (2.46) and is given by

$$L(q^{-1}) = 0.2915 - 0.0466q^{-1}.$$

It is not difficult to verify that the above estimator is equivalent to the state space solution given in [13].

Example 2.4.2. *In IS-136 (Second generation of the digital standard TDMA (Time division multiple access) wireless technology) system [64], the symbol time T will be set to $41.15\mu s$, carrier frequency is 1900 MHz. Among every $N = 162$ differ-*

ential QPSK (Quadrature phase shift keying)-modulated symbols, there are $N_{tr} = 14$ *leading training symbols. Here, Doppler rate* $\Omega_D = 0.04$ *or Doppler frequency* $f_D = 1600Hz$ *when mobile speed is* $90km/h$ *[48].*

A three tap Rayleigh fading symbol-spaced baseband channel with independently fading taps is simulated

$$y_k = h_{0,k}u_k + h_{1,k}u_{k-1} + h_{2,k}u_{k-2} + v_k = \Phi_k^* h_k + v_k$$

where $\Phi_k = [u_k \; u_{k-1} \; u_{k-2}]^T$ *and* $h_k = [h_{0,k} \; h_{1,k} \; h_{2,k}]^T$, Φ_k^* *is defined as the complex conjugate transpose of a column vector* Φ_k*. The symbols* $\{u_k\}$ *are assumed to have zero mean and constant modulus. It is assumed stationary with a known nonsingular autocorrelation matrix* $\mathbf{R} = E\Phi_k\Phi_k^* = I$*. The observation noise* v_k *has zero mean and variance* σ_v^2*.*

As the channel is time-varying, the taps are also time-varying. However, with a short period of time, we can consider that the taps consist of constants and random variations described by white Gaussian noises.

$$h_{i,k} = \bar{h}_{i,k} + \Delta h_{i,k}, \qquad i = 0, 1, 2.$$

Define

$$r_i = \frac{\bar{h}_{i,k}}{\sigma_{\Delta h_{i,k}}} \qquad i = 0, 1, 2,$$

as the ratio of nominal value against uncertainties. $\sigma_{\Delta h_{i,k}}$ *is the deviation of tap uncertainty. In this case, we assume* $r_0 = r_1 = r_2 = r$*. When channel is fast fading,* r *is small; when channel is slow fading,* r *is large.*

From the information of leading training period, we can use the RLS (recursive least-squares) [49] algorithm to obtain the nominal part of the taps. Now, we can apply robust estimator to estimate the transmitted signal in the following short time (e.g.

$N - N_{tr} = 148$ *symbol time).*

$\bar{h}_{0,k} = 0.53$; $\bar{h}_{1,k} = 0.02$; $\bar{h}_{2,k} = 0.06$ *are nominal values of the channel at a certain time. Robust estimator and nominal estimator are derived in the way presented in this chapter.*

The simulation result (Figure 2.1) shows that the robust estimator can achieve much better performance than nominal estimator when uncertainties are big (i.e. r is small).

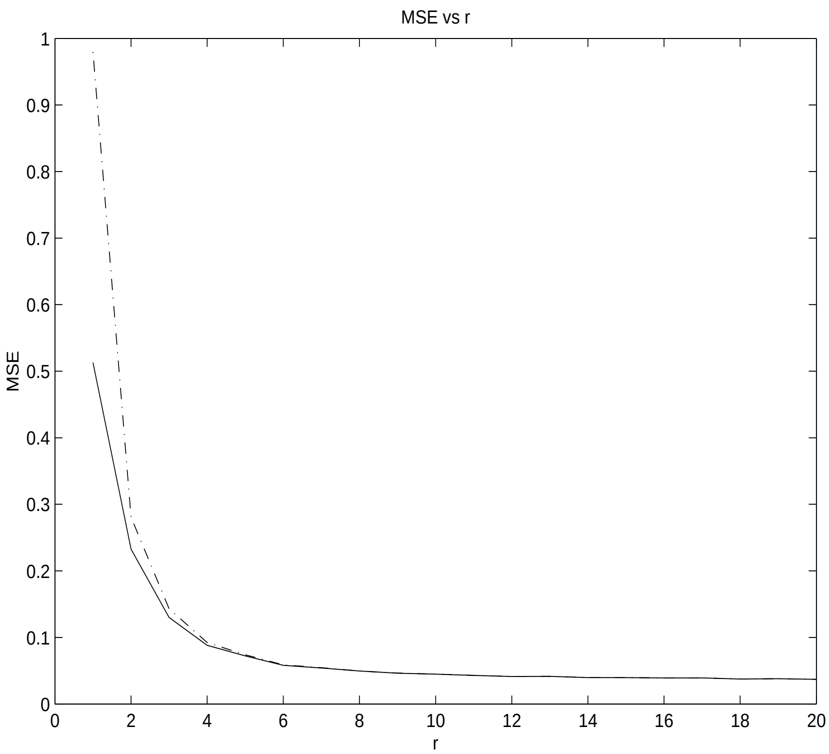

Figure 2.1: MSE performance of robust estimator (solid line) and nominal estimator (dash-dot line).

Now we discuss the situation when the SNR (defined as square of variance of e_0 versus variance of v_0) and r change. From Figure 2.2, we can find that the robust estimator outperforms the nominal estimator consistently, especially when r is small and/or SNR is large.

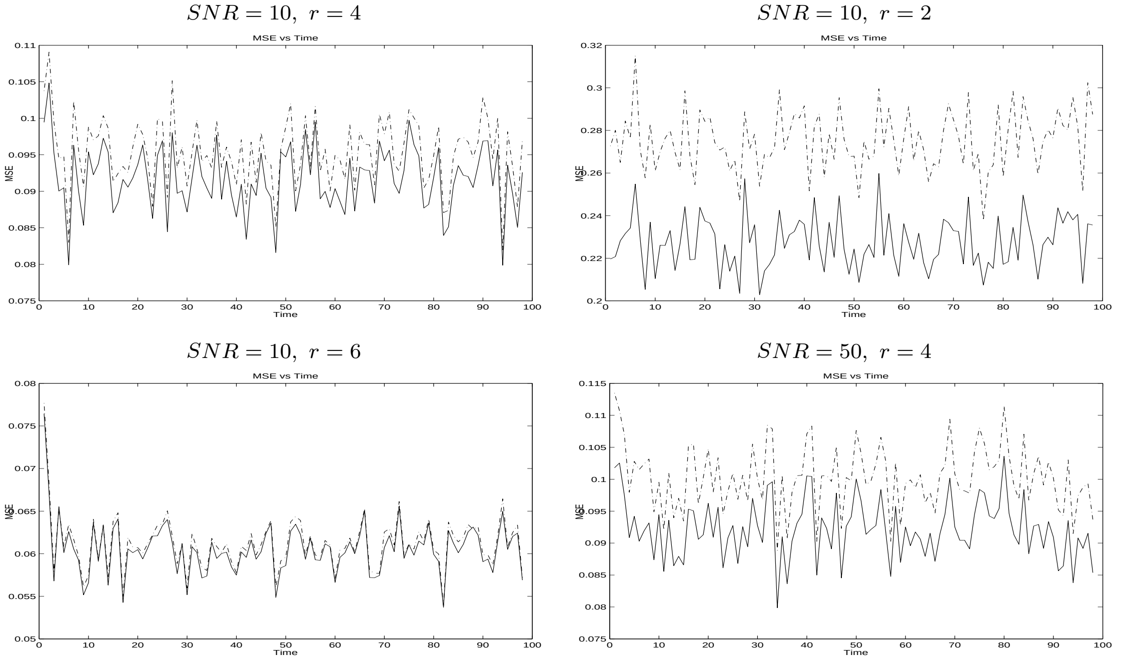

Figure 2.2: MSE Performances of robust estimator (solid line) and nominal estimator (dash-dot line).

Furthermore, we investigate the MSE performance when the nominal values of the channel change. Figure 2.3 and Figure 2.4 show the MSE performances under system parameters $\bar{h}_{0,k} = 0.1076$; $\bar{h}_{1,k} = 0.0739$; $\bar{h}_{2,k} = 0.0173$ *and* $\bar{h}_{0,k} = 1.0982$; $\bar{h}_{1,k} = 0.0263$; $\bar{h}_{2,k} = 0.0235$, *respectively. We conclude that MSE is smaller when the nominal parameters (especially dominant parameter* $\bar{h}_{0,k}$*) are larger. The reason is that when* $\bar{h}_{0,k}$ *is larger, the effect of observation noise* v_k *is smaller, so the estimation performance is better.*

Now we consider a more general situation, the input is modelled as colored noise and the parameter uncertainties also exist in the input signal model.

$$u_k = \frac{C(q^{-1}) + \Delta C(q^{-1})}{D(q^{-1}) + \Delta D(q^{-1})} e_k \tag{2.1}$$

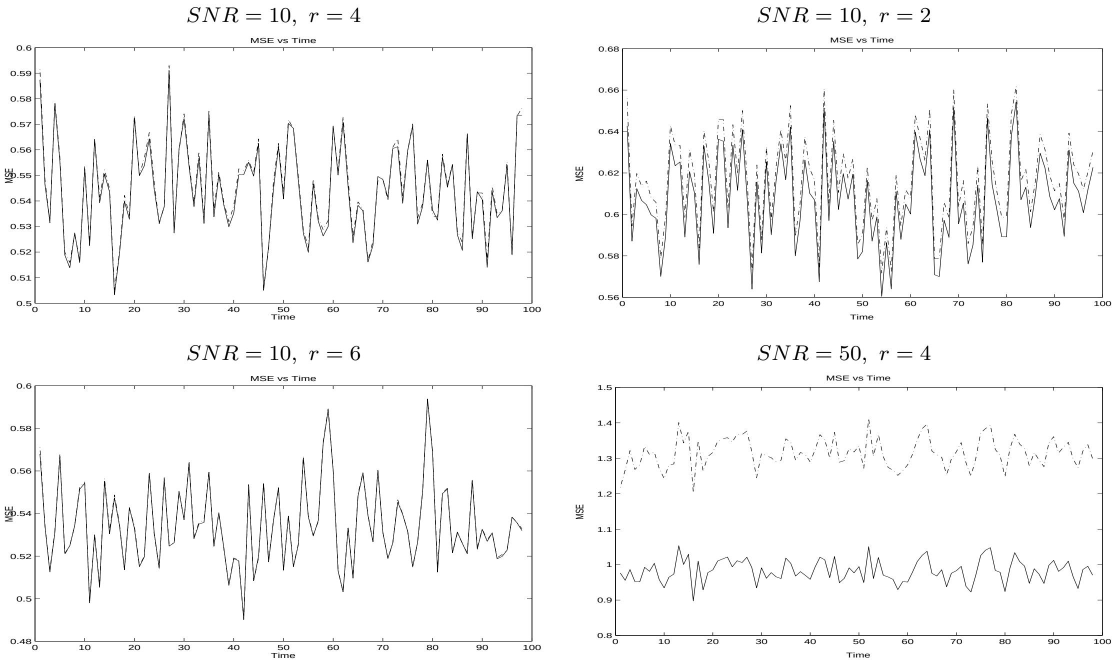

Figure 2.3: MSE Performances of robust estimator (solid line) and nominal estimator (dash-dot line).

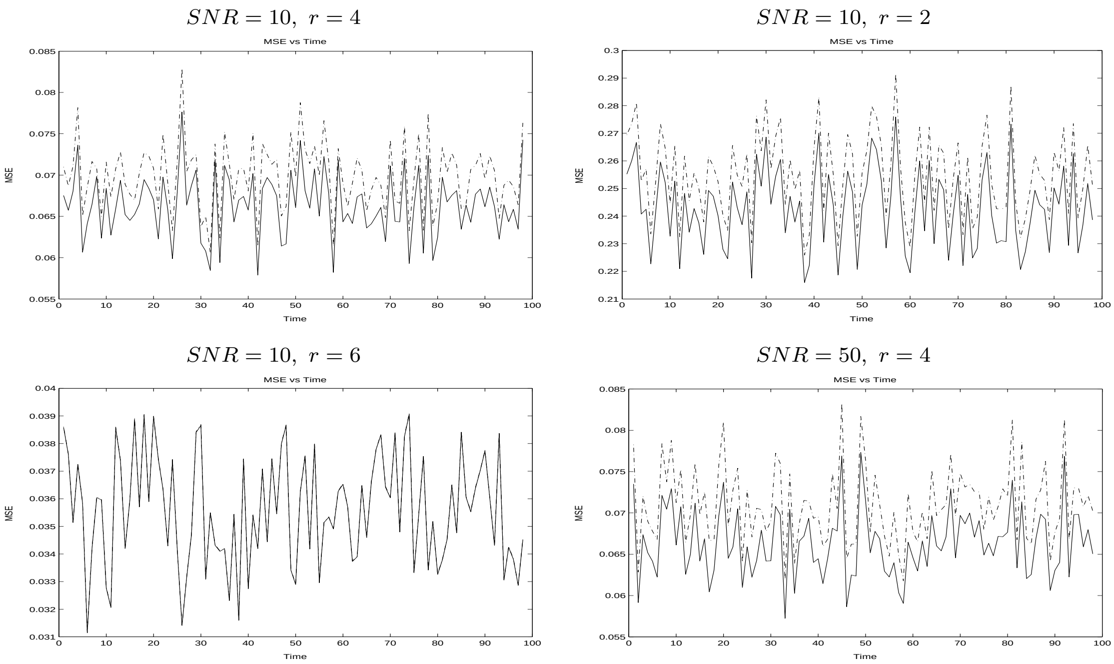

Figure 2.4: MSE Performances of MSE robust estimator (solid line) and nominal estimator (dash-dot line).

where e_k *and* u_k *are a white Gaussian noise and the transmitted signal, respectively.*

$$\begin{aligned} C(q^{-1}) &= 0.6q^{-1} \\ D(q^{-1}) &= 1+0.5q^{-1} \\ \Delta C(q^{-1}) &= \Delta c_1 q^{-1} \\ \Delta D(q^{-1}) &= \Delta d_1 q^{-1} \end{aligned} \tag{2.2}$$

The uncertainties Δc_1 *and* Δd_1 *are assumed to vary with* 25% *of their nominal values. Other conditions remain the same as in the first example (*$\bar{h}_{0,k} = 0.53$; $\bar{h}_{1,k} = 0.02$; $\bar{h}_{2,k} = 0.06$*) and* $r = 4$, $SNR = 50$*. The result is shown in Figure 2.5, the solid line is the one when we consider the uncertainty in the input signal model, whereas, the dash-dot line is the one when the uncertainty of the input signal model is not taken into consideration in the design of estimator. It is clear that the one which takes into account the input signal uncertainties gives better performance.*

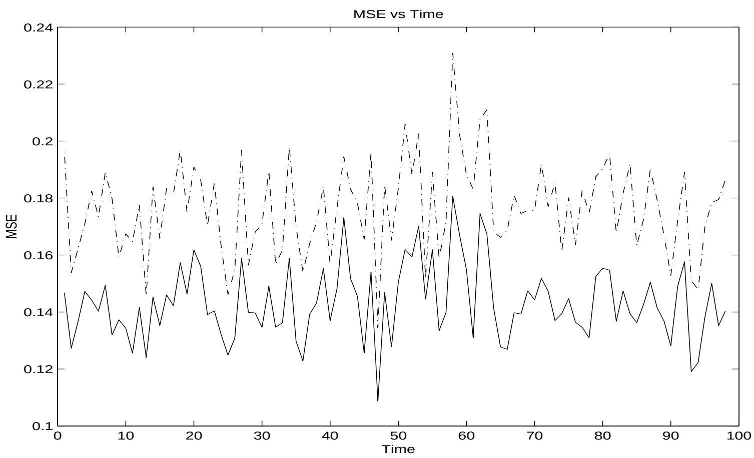

Figure 2.5: MSE Performances of two robust estimators: one considers the input signal model uncertainties (solid line) and the other does not consider the input signal model uncertainties (dash-dot line).

2.5 Conclusion

In this chapter, we have presented a polynomial approach to the robust deconvolution filtering for systems with random parameter uncertainties and input delay. The uncertainties appear not only in both the numerator and denominator of the system model but also in the input signal model. The covariances of the input and measurement noises are not exactly known but with known lower and upper bounds. We introduced two fictitious noises to quantify the effects of the random parameter uncertainties and presented simple formulae to compute the covariances of the fictitious noises. The optimal robust estimator is given in terms of one spectral factorization and one polynomial equation which can be solved efficiently.

The presented polynomial equation method is an alternative solution of [13] and is computationally more attractive than the latter [45]. Further, the model of the system under consideration is more general than the one in [13] where the input signal is required to be known exactly.

Chapter 3

Robust Deconvolution Estimation for Uncertain Discrete MIMO Systems with Input Delay

3.1 Introduction

Multiple-input multiple-output (MIMO) systems appear in communications where information from different users couples in channels. Different from an SISO system, polynomial matrices instead of polynomial functions in frequency domain are used to describe the channels and it is more difficult to solve the deconvolution estimation problem for MIMO systems.

There are some existing works on deconvolution estimation of MIMO systems, e.g., Ohrn et al. [68] discuss the robust MIMO deconvolution in frequency domain in which parameter uncertainties appear only in the numerators of the transfer functions, which, however, is restrictive in reality because there may exist uncertainties in denominators as well. In this chapter, we consider an MIMO system with input delay whose transfer function matrix contains random time-varying parameter uncertainties. The random parameter uncertainties are assumed to be of zero means and known covariances. A filter is designed that minimizes the estimation error covariance with respect to all the system uncertainties and noises. Unlike the work in [13], we adopt a polynomial approach which is often preferred for signal processing and communication applications. To derive the optimal estimator, we firstly convert the random parameter uncertainties into two fictitious noise vectors and calculate their covariance matrices. Under some conditions, the covariance calculation can be simplified. Similar to the SISO case, the estimator is then designed through spectral factorization and projection. The solution is given in terms of one spectral factorization and one polynomial matrix equation.

3.2 Problem Statement

We restate the system with its signal and channel models given by [30]. In order to highlight the main idea, we simplify the problem by discussing uncertainties only in the system model where the input signal model is uncertainty free.

$$\begin{aligned} [A(q^{-1}) + \Delta A(q^{-1})]y(k) &= [B(q^{-1}) + \Delta B(q^{-1})]u(k-d) + [P(q^{-1}) + \Delta P(q^{-1})]v(k), \\ u(k) &= D^{-1}(q^{-1})C(q^{-1})e(k) \end{aligned} \tag{3.1}$$

where d is the delay in input, $u(k) \in \mathbb{R}^m$ is the input signal, $y(k) \in \mathbb{R}^\iota$ is the output measurement, $e(k) \in \mathbb{R}^\varsigma$ and $v(k) \in \mathbb{R}^\varrho$ are zero-mean white noises with known covariance matrices Q_e and Q_v and they are independent of each other. Note that here we are considering systems with multiple inputs and multiple outputs.

The polynomial matrices $A(q^{-1})$, $B(q^{-1})$, $C(q^{-1})$, $D(q^{-1})$, and $P(q^{-1})$ are given, and they have the form

$$X(q^{-1}) \quad = \quad X_0 + X_1 q^{-1} + \cdots + X_{n_x} q^{-n_x}. \tag{3.2}$$

The polynomial matrices $\Delta A(q^{-1})$, $\Delta B(q^{-1})$ and $\Delta P(q^{-1})$ represent the time-varying uncertainties which have the form

$$\Delta X(q^{-1}) = X_0(k) + X_1(k) q^{-1} + \cdots + X_{n_x}(k) q^{-n_x}. \tag{3.3}$$

The time-varying uncertainties are characterized as random processes with zero mean and known covariances. They together with $e(k)$ and $v(k)$ are independent of each other. In practice, we may always assume that $D_0 = A_0 = I$ and $A_0(k) = 0$. Assume that the input signal has known statistics but the channel and measurement models have uncertainties. In fact, our study can be easily extended to the case where there exist random uncertainties in the signal model as well.

The problem under investigation is:

Find an estimator which minimizes the following averaged mean square error:

$$\tilde{E}\bar{E}[u(k) - \hat{u}(k|k-m)]^T [u(k) - \hat{u}(k|k-m)]$$

where $\bar{E}$ is the mathematical expectation over the external noise inputs $e(k)$ and $v(k)$, and $\tilde{E}$ over the random modeling uncertainties $\Delta A(q^{-1})$, $\Delta B(q^{-1})$ and $\Delta P(q^{-1})$. The integer m may be positive, zero or negative. We note that when $m = 0$, the above is a filtering problem; when $m < 0$, it represents a fixed lag smoothing problem and for $m > 0$, it is a prediction problem.

We make the following assumption concerning the system.

Assumption 3.2.1. *The polynomial matrices $A(q^{-1})$ and $D(q^{-1})$ are stable, i.e.,* *all zeros of $A(q^{-1})$ and $D(q^{-1})$ are inside the unit disk.*

Remark 3.2.1. *In the SISO case, the above model may be converted to a state-space form. In this situation, the deconvolution problem can be approached using the result of [13]. For the MIMO case, it is not easy to convert the system (3.1) into a state-space form due to the presence of uncertainties. Further, even if the system can be converted into a state-space one, the state-space techniques will require a higher computational cost. In the following, we shall address the above problem using a polynomial approach.*

3.3 Design of Optimal Deconvolution Estimator

In this section, we shall present a polynomial approach to the design of optimal estimator.

For the convenience of discussion we shall denote $X(q^{-1})$ by X and $\Delta X(q^{-1})$ by ΔX. It follows from (3.1) that

$$Ay(k) = Bu(k-d) + Pv(k) + \eta(k) \tag{3.4}$$

where $\eta(k) = -\Delta Ay(k) + \Delta Bu(k-d) + \Delta Pv(k)$.

Denote

$$\begin{aligned}
\mathcal{A}(k) &\triangleq [A_1(k) \;\; A_2(k) \;\; \cdots \;\; A_{n_a}(k)]^T, \\
\mathcal{B}(k) &\triangleq \left[B_0(k) \;\; B_1(k) \;\; \cdots \;\; B_{n_b}(k)\right]^T, \\
\mathcal{P}(k) &\triangleq \left[P_0(k) \;\; P_1(k) \;\; \cdots \;\; P_{n_p}(k)\right]^T, \qquad (3.5) \\
\mathcal{A}_{bp}(k) &\triangleq \left[-\mathcal{A}^T(k) \;\; \mathcal{B}^T(k) \;\; \mathcal{P}^T(k)\right]^T. \qquad (3.6)
\end{aligned}$$

Note that $\eta(k)$ can be expressed as

$$\eta(k) \;=\; \mathcal{A}_{bp}^T(k)\mathcal{Y}_u(k), \tag{3.7}$$

where

$$\mathcal{Y}_u(k) = \Big[y^T(k-1) \quad \cdots \quad y^T(k-n_a); \;\; u^T(k-d) \quad \cdots \quad u^T(k-d-n_b); \;\; v^T(k) \quad \cdots \quad v^T(k-n_p)\Big]^T.$$

Thus, it follows from (3.1) and (3.4) that

$$y(k) = A^{-1}BD^{-1}Ce(k-d) + A^{-1}Pv(k) + A^{-1}\eta(k) \tag{3.8}$$

and

$$\mathcal{Y}_u(k) \;=\; diag\{\mathcal{Q}_a, \mathcal{Q}_b, \mathcal{Q}_p\} \times \left\{ \begin{bmatrix} A^{-1}BD^{-1} \\ D^{-1} \\ 0 \end{bmatrix} Ce(k-d) + \begin{bmatrix} A^{-1}P \\ 0 \\ I \end{bmatrix} v(k) + \begin{bmatrix} A^{-1} \\ 0 \\ 0 \end{bmatrix} \eta(k) \right\}, \tag{3.9}$$

where

$$\mathcal{Q}_a \;=\; \Big[q^{-1}I \quad q^{-2}I \quad \cdots \quad q^{-n_a}I\Big]^T, \tag{3.10}$$

$$\mathcal{Q}_b \;=\; \Big[I \quad q^{-1}I \quad \cdots \quad q^{-n_b}I\Big]^T, \tag{3.11}$$

$$\mathcal{Q}_p \;=\; \Big[I \quad q^{-1}I \quad \cdots \quad q^{-n_p}I\Big]^T. \tag{3.12}$$

3.3.1 Calculation of the Covariance Matrix of η

Since $\Delta A,\ \Delta B,\ \Delta P$ are random matrices with zero means, it is easy to show that $e(k),\ v(k)$ and $\eta(k)$ are mutually uncorrelated white noise vectors. By using the Parseval's formula we obtain the following:

$$\begin{aligned}\mathcal{R}_{y_u} &= \tilde{E}\bar{E}[\mathcal{Y}_u\mathcal{Y}_u^T] \\ &= \frac{1}{2\pi i}\oint_{|z|=1} diag\{\mathcal{Q}_a, \mathcal{Q}_b, \mathcal{Q}_p\} \times \\ &\quad \{X_1CQ_eC_*X_{1*} + X_2Q_vX_{2*} + X_3Q_\eta X_{3*}\} \\ &\quad \times diag\{\mathcal{Q}_a, \mathcal{Q}_b, \mathcal{Q}_p\}_*\frac{dz}{z}, \end{aligned} \tag{3.13}$$

where $i = \sqrt{-1}$ and

$$X_1 = \begin{bmatrix} A^{-1}BD^{-1} \\ D^{-1} \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} A^{-1}P \\ 0 \\ I \end{bmatrix}, \quad X_3 = \begin{bmatrix} A^{-1} \\ 0 \\ 0 \end{bmatrix}. \tag{3.14}$$

Next, by multiplying $\mathcal{A}_{bp}^T(t)$ from the left hand side of (3.13) and $\mathcal{A}_{bp}(t)$ from the right side, and taking the mathematical expectation $\tilde{E}$, we obtain

$$\begin{aligned} Q_\eta &= \tilde{E}\bar{E}[\eta\eta^T] \\ &= \tilde{E}\bar{E}[\mathcal{A}_{bp}^T\mathcal{Y}_u\mathcal{Y}_u^T\mathcal{A}_{bp}] \\ &= \frac{1}{2\pi i}\oint_{|z|=1} \tilde{E}\begin{bmatrix} -\Delta A & \Delta B & \Delta P \end{bmatrix}\{X_1CQ_eC_*X_{1*} + X_2Q_vX_{2*} + X_3Q_\eta X_{3*}\}\begin{bmatrix} -\Delta A_* \\ \Delta B_* \\ \Delta P_* \end{bmatrix}\frac{dz}{z} \\ &= \frac{1}{2\pi i}\oint_{|z|=1}\Big\{\tilde{E}\left(-\Delta AA^{-1}BD^{-1} + \Delta BD^{-1}\right)CQ_eC_*\left(-\Delta AA^{-1}BD^{-1} + \Delta BD^{-1}\right)_* \\ &\quad + \tilde{E}\left(-\Delta AA^{-1}P + \Delta P\right)Q_v\left(-\Delta AA^{-1}P + \Delta P\right)_* + \tilde{E}(\Delta AA^{-1})Q_\eta(\Delta AA^{-1})_*\Big\}\frac{dz}{z} \end{aligned}$$

$$
\begin{aligned}
&= \Phi_0 + \frac{1}{2\pi i}\oint_{|z|=1} \tilde{E}(\Delta A A^{-1} Q_\eta A_*^{-1} \Delta A_*) \frac{dz}{z} \\
&= \Phi_0 + \frac{1}{2\pi i}\oint_{|z|=1} \tilde{E}[\Delta A (adj A) Q_\eta (adj A_*) \Delta A_*] \frac{1}{det A det A_*} \frac{dz}{z} \qquad (3.15)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_0 &= \frac{1}{2\pi i}\oint_{|z|=1} \tilde{E}\left\{\left(-\Delta A A^{-1} B D^{-1} + \Delta B D^{-1}\right) C Q_e C_* \left(-\Delta A A^{-1} B D^{-1} + \Delta B D^{-1}\right)_* \right. \\
&\left. + \left(-\Delta A A^{-1} P + \Delta P\right) Q_v \left(-\Delta A A^{-1} P + \Delta P\right)_* \right\} \frac{dz}{z}, \qquad (3.16)
\end{aligned}
$$

and the operators *adj* and *det* are used to calculate the adjoint matrix and determinant of a certain matrix, respectively.

Remark 3.3.1. *If in addition there are uncertainties in the input signal model, then the expression of Q_η needs to be modified. In fact, assume that*

$$
[D(q^{-1}) + \Delta D(q^{-1})]u(k) = [C(q^{-1}) + \Delta C(q^{-1})]e(k).
$$

Define $\tilde{e}(k) = -\Delta D(q^{-1})u(k) + \Delta C(q^{-1})e(k)$, then (3.15) is modified as

$$
\begin{aligned}
Q_{\tilde{e}} &= \frac{1}{2\pi i}\oint_{|z|=1} \tilde{E}\left\{(-\Delta D D^{-1} C + \Delta C) Q_e (-\Delta D D^{-1} C + \Delta C) \right. \\
&\left. + \tilde{E}(\Delta D D^{-1} Q_{\tilde{e}} D_*^{-1} \Delta D_*)\right\} \frac{dz}{z} \\
Q_\eta &= \frac{1}{2\pi i}\oint_{|z|=1} \left\{\tilde{E}(-\Delta A A^{-1} B + \Delta B D^{-1}) D^{-1} C Q_e C_* D_*^{-1} (-\Delta A A^{-1} B + \Delta B D^{-1})_* \right. \\
&+ \tilde{E}(-\Delta A A^{-1} B + \Delta B) D^{-1} Q_{\tilde{e}} D_*^{-1} (-\Delta A A^{-1} B + \Delta B)_* \\
&\left. + \tilde{E}(-\Delta A A^{-1} P + \Delta P) Q_v (-\Delta A A^{-1} P + \Delta P)_* + \tilde{E} \Delta A A^{-1} Q_\eta A_*^{-1} \Delta A_* \right\} \frac{dz}{z}.
\end{aligned}
$$

To solve the optimal estimation, it is crucial to evaluate the covariance matrix Q_η. In the following, we shall address this problem by presenting a number of technical lemmas.

Lemma 3.3.1. *Let P and Q be given complex matrices of dimensions $r \times n$ and $n \times n$ respectively and with Q diagonal. Define $\Omega = [\omega_{lj}] = \tilde{E}[\Gamma P Q P_* \Gamma_*]$, where Γ is a random complex matrix of dimension $\kappa \times r$. Then*

$$\begin{pmatrix} \omega_{11} \\ \vdots \\ \omega_{1\kappa} \\ \omega_{21} \\ \vdots \\ \omega_{\kappa\kappa} \end{pmatrix} = \begin{pmatrix} \tilde{E}(\gamma_{11}\gamma_{11*}) & \dots & \tilde{E}(\gamma_{11}\gamma_{1r*}) & \dots & \tilde{E}(\gamma_{1r}\gamma_{1r*}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{E}(\gamma_{11}\gamma_{\kappa 1*}) & \dots & \tilde{E}(\gamma_{11}\gamma_{\kappa r*}) & \dots & \tilde{E}(\gamma_{1r}\gamma_{\kappa r*}) \\ \tilde{E}(\gamma_{21}\gamma_{11*}) & \dots & \tilde{E}(\gamma_{21}\gamma_{1r*}) & \dots & \tilde{E}(\gamma_{2r}\gamma_{1r*}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{E}(\gamma_{\kappa 1}\gamma_{\kappa 1*}) & \dots & \tilde{E}(\gamma_{\kappa 1}\gamma_{\kappa r*}) & \dots & \tilde{E}(\gamma_{\kappa r}\gamma_{\kappa r*}) \end{pmatrix} \begin{pmatrix} p_{11}p_{11*} & \dots & p_{1n}p_{1n*} \\ \vdots & \ddots & \vdots \\ p_{11}p_{r1*} & \dots & p_{1n}p_{rn*} \\ \vdots & \ddots & \vdots \\ p_{r1}p_{r1*} & \dots & p_{rn}p_{rn*} \end{pmatrix} \begin{pmatrix} q_{11} \\ \vdots \\ q_{nn} \end{pmatrix} \tag{3.17}$$

where γ_{lj}, p_{lj} and q_{lj} are elements of Γ, P and Q respectively.

Proof: Note that

$$\begin{aligned} \omega_{lj} &= \sum_t \sum_\tau \tilde{E}\gamma_{lt}[PQP_*]_{t\tau}\gamma_{j\tau*} \\ &= \sum_t \sum_\tau [\tilde{E}\gamma_{lt}\gamma_{j\tau*}][PQP_*]_{t\tau} \\ &= \begin{pmatrix} \tilde{E}(\gamma_{l1}\gamma_{j1}) & \dots & \tilde{E}(\gamma_{l1}\gamma_{jr*}) & \dots & \tilde{E}(\gamma_{lr}\gamma_{jr*}) \end{pmatrix} \begin{pmatrix} [PQP_*]_{11} \\ \vdots \\ [PQP_*]_{1r} \\ \vdots \\ [PQP_*]_{rr} \end{pmatrix} \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} [PQP_*]_{t\tau} &= \sum_x p_{tx} q_{xx} p_{\tau x*} \\ &= \begin{pmatrix} p_{t1}p_{\tau 1*} & \dots & p_{tn}p_{\tau n*} \end{pmatrix} \begin{pmatrix} q_{11} \\ \vdots \\ q_{nn} \end{pmatrix}. \end{aligned} \quad (3.19)$$

By substituting (3.19) into (3.18), we get (3.17).

Lemma 3.3.2. *In Lemma 3.3.1, if all the elements of* Γ *are random variables with zero means and uncorrelated with each other, then* Ω *becomes a diagonal matrix with*

$$\begin{pmatrix} \omega_{11} \\ \omega_{22} \\ \vdots \\ \omega_{\kappa\kappa} \end{pmatrix} = \begin{pmatrix} \tilde{E}(\gamma_{11}\gamma_{11*}) & \dots & \tilde{E}(\gamma_{1r}\gamma_{1r*}) \\ \tilde{E}(\gamma_{21}\gamma_{21*}) & \dots & \tilde{E}(\gamma_{2r}\gamma_{2r*}) \\ \vdots & \ddots & \vdots \\ \tilde{E}(\gamma_{\kappa 1}\gamma_{\kappa 1*}) & \dots & \tilde{E}(\gamma_{\kappa r}\gamma_{\kappa r*}) \end{pmatrix} \begin{pmatrix} p_{11}p_{11*} & \dots & p_{1n}p_{1n*} \\ \vdots & \ddots & \vdots \\ p_{r1}p_{r1*} & \dots & p_{rn}p_{rn*} \end{pmatrix} \begin{pmatrix} q_{11} \\ \vdots \\ q_{nn} \end{pmatrix}. \quad (3.20)$$

Proof: Observe that

$$\tilde{E}\gamma_{lt}\gamma_{j\tau*} = \delta_{lj}\delta_{t\tau}\tilde{E}\gamma_{lt}\gamma_{lt*}, \quad (3.21)$$

where $\delta_{ij} \triangleq \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

Hence,

$$\begin{aligned} \omega_{lj} &= \sum_t \sum_\tau \delta_{lj}\delta_{t\tau}\tilde{E}\gamma_{lt}[PQP_*]_{t\tau}\gamma_{j\tau*} \\ &= \sum_t [\tilde{E}\gamma_{lt}\gamma_{lt*}][PQP_*]_{tt} \end{aligned} \quad (3.22)$$

which leads to (3.20) directly.

To apply the above lemmas for the calculation of Q_η, we denote the second term of

(3.15) by Π, i.e. $\Pi = \frac{1}{2\pi i}\oint_{|z|=1}\tilde{E}(\Delta A A^{-1} Q_\eta A_*^{-1}\Delta A_*)\frac{dz}{z}$. Further, denote $\check{A} = adjA$ and $\check{A}_* = adjA_*$.

If all the elements of $\Delta A = [\Delta a_{lj}]$ are random variables of zero means and uncorrelated with each other, then from Lemma 3.3.2, we know that Π is a diagonal matrix, and the diagonal elements are given by

$$
\begin{aligned}
&\begin{bmatrix}\pi_{11} & \pi_{22} & \dots & \pi_{NN}\end{bmatrix}^T \\
= \;& \frac{1}{2\pi i}\oint_{|z|=1}\begin{pmatrix}\tilde{E}\Delta a_{11}\Delta a_{11} & \dots & \tilde{E}\Delta a_{1N}\Delta a_{1N*} \\ \vdots & \ddots & \vdots \\ \tilde{E}\Delta a_{N1}\Delta a_{N1} & \dots & \tilde{E}\Delta a_{NN}\Delta a_{NN*}\end{pmatrix}\begin{pmatrix}\check{a}_{11}\check{a}_{11*} & \dots & \check{a}_{1N}\check{a}_{1N*} \\ \vdots & \ddots & \vdots \\ \check{a}_{N1}\check{a}_{N1*} & \dots & \check{a}_{NN}\check{a}_{NN*}\end{pmatrix} \\
& \times \begin{pmatrix} q_{\eta_{11}} \\ \vdots \\ q_{\eta_{NN}}\end{pmatrix}\frac{1}{detAdetA_*}\frac{dz}{z} \\
= \;& [Res(f_{ij})]\begin{pmatrix} q_{\eta_{11}} \\ \vdots \\ q_{\eta_{NN}}\end{pmatrix}
\end{aligned}
\tag{3.23}
$$

where

$$
F = [f_{ij}] = \begin{pmatrix}\tilde{E}\Delta a_{11}\Delta a_{11} & \dots & \tilde{E}\Delta a_{1N}\Delta a_{1N*} \\ \vdots & \ddots & \vdots \\ \tilde{E}\Delta a_{N1}\Delta a_{N1} & \dots & \tilde{E}\Delta a_{NN}\Delta a_{NN*}\end{pmatrix}\begin{pmatrix}\check{a}_{11}\check{a}_{11*} & \dots & \check{a}_{1N}\check{a}_{1N*} \\ \vdots & \ddots & \vdots \\ \check{a}_{N1}\check{a}_{N1*} & \dots & \check{a}_{NN}\check{a}_{NN*}\end{pmatrix}\frac{1}{zdetAdetA_*}.
$$

Similar calculations can be applied to Φ_0 of (3.15). Hence, the matrix equation (3.15) becomes N linear equations, from which we can compute Q_η easily.

Lemma 3.3.3. *Let Λ be a given $n \times n$ matrix and Γ a random $n \times n$ matrix whose* *elements are random variables satisfying*

$$E[\gamma_{lj}\gamma_{t\tau}] = \begin{cases} \gamma, & l = t, j = \tau, \\ 0, & otherwise. \end{cases} \tag{3.24}$$

Then,

$$\tilde{E}(\Gamma\Lambda\Gamma^T) = \gamma tr(\Lambda) I_n \tag{3.25}$$

where I_n is the identity matrix of $n \times n$.

Proof: Assume $\Omega = [\omega_{lj}] = \tilde{E}(\Gamma\Lambda\Gamma^T)$. Then,

$$\omega_{lj} = \sum_t \sum_\tau \tilde{E}\gamma_{lt}\lambda_{t\tau}\gamma_{j\tau}.$$

If $l \neq j$, $\omega_{lj} = 0$. Otherwise,

$$\begin{aligned} \omega_{ll} &= \sum_t \sum_\tau \tilde{E}[\gamma_{lt}\lambda_{t\tau}\gamma_{l\tau}] \\ &= \sum_t \tilde{E}[\gamma_{lt}\lambda_{tt}\gamma_{lt}] \\ &= \gamma tr(\Lambda). \end{aligned}$$

Hence, (3.25) follows.

Corollary 3.3.1. *Let* $\Delta A = \sum_{j=1}^{n_a} A_j(k) z^{-j}$. *If all the elements of* $A_j(k)$, $j = 1, \cdots, n_a$ *are random variables of zero means and are uncorrelated with each other and* $\tilde{E}A_j(k)A_j^T(k) = \gamma_j I_N$ $(j = 1, 2, \ldots, n_a)$, *then*

$$\frac{1}{2\pi i}\oint_{|z|=1} \Delta A A^{-1} Q_\eta A_*^{-1} \Delta A_* \frac{dz}{z} = \sum_{j=1}^{n_a} \gamma_j \frac{1}{2\pi i}\oint_{|z|=1} (tr(A^{-1}Q_\eta A_*^{-1}))\frac{dz}{z} I_N. \tag{3.26}$$

Proof:

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{|z|=1} \Delta A A^{-1} Q_\eta A_*^{-1} \Delta A_* \frac{dz}{z} &= \frac{1}{2\pi i}\oint_{|z|=1} [\sum_{j=1}^{n_a} A_j(k) z^{-j}](A^{-1} Q_\eta A_*^{-1})[\sum_{j=1}^{n_a} A_j^T(k) z^j]\frac{dz}{z} \\
&= \frac{1}{2\pi i}\oint_{|z|=1} [\sum_j \sum_l A_j(k) z^{-j}(A^{-1} Q_\eta A_*^{-1}) A_l^T(k) z^l]\frac{dz}{z} \\
&= \frac{1}{2\pi i}\oint_{|z|=1} \sum_{j=1}^{n_a} \gamma_j tr(A^{-1} Q_\eta A_*^{-1}) I_N \frac{dz}{z} \\
&= \sum_{j=1}^{n_a} \gamma_j \frac{1}{2\pi i}\oint_{|z|=1} [tr(A^{-1} Q_\eta A_*^{-1})]\frac{dz}{z} I_N.
\end{aligned}$$

The third equality is due to the assumption that $\tilde{E}(A_j(k)A_l^T(k)) = 0$, if $j \neq l$.

Furthermore, consider $\Delta B = \sum_{j=0}^{n_b} B_j(k) z^{-j}$ and $\Delta P = \sum_{l=0}^{n_p} P_l(k) z^{-l}$. If all the elements of $B_j(k)$ and $P_l(k)$, $j = 0, 1, \cdots, n_b$, $l = 0, 1, \cdots, n_p$ are random variables of zero means and satisfying

$$\tilde{E} B_j(k) B_l^T(k) = \delta_{jl}\gamma_j I, \quad \tilde{E} P_l(k) P_j^T(k) = \delta_{lj}\gamma_l I,$$

then using Corollary 3.1, (3.15) can be rewritten as

$$Q_\eta - \sum_{j=1}^{n_a} \gamma_j \frac{1}{2\pi i}\oint_{|z|=1} (tr(A^{-1} Q_\eta A_*^{-1})) I_N = \Phi_0 = \phi_0 I_N,$$

$$\begin{aligned}
\phi_0 &= \sum_{j=1}^{n_a} \gamma_i \frac{1}{2\pi i}\oint_{|z|=1} tr(A^{-1} B D^{-1} C Q_e C_* D_*^{-1} B_* A_*^{-1} + A^{-1} P Q_v P_* A_*^{-1})\frac{dz}{z} \\
&\quad + \sum_{j=0}^{n_b} \gamma_{bi} \frac{1}{2\pi i}\oint_{|z|=1} tr(D^{-1} C Q_e C_* D_*^{-1})\frac{dz}{z} + \sum_{j=0}^{n_p} \gamma_{pi} tr(Q_v).
\end{aligned}$$

From the equation above, we can see that Q_η can be expressed as a diagonal matrix

$diag\{q_{\eta 11} \dots q_{\eta NN}\} = \gamma_\eta I_N$, where γ_η is a constant given by

$$\gamma_\eta = \frac{\phi_0}{1 - \sum_{j=1}^{n_a} \gamma_j \frac{1}{2\pi i} \oint_{|z|=1} [tr(A^{-1} A_*^{-1})] \frac{dz}{z}}. \tag{3.27}$$

3.3.2 Calculation of Optimal Estimator

With the computed covariance matrix Q_η, we now discuss the computation of the optimal estimator. To this end, let $(\tilde{B}, \tilde{D})$ be a left-coprime pair [30] of $B(q^{-1})D^{-1}(q^{-1})$, i.e.

$$B(q^{-1})D^{-1}(q^{-1}) = \tilde{D}^{-1}(q^{-1})\tilde{B}(q^{-1}). \tag{3.28}$$

From (3.28) and (3.1), (3.4) can be expressed as

$$\begin{aligned} Ay(k) &= BD^{-1}Ce(k-d) + Pv(k) + \eta(k) \\ &= \tilde{D}^{-1}\tilde{B}Ce(k-d) + Pv(k) + \eta(k). \end{aligned} \tag{3.29}$$

Since $e(k), v(k)$ and $\eta(k)$ are mutually independent, it can be easily observed that the spectral density of the output $y(k)$ is of the form

$$(\tilde{D}A)^{-1}\mathcal{W}(z, z^{-1})(\tilde{D}A)_*^{-1}$$

where $\mathcal{W}(z, z^{-1}) = \tilde{B}CQ_eC_*\tilde{B}_* + \tilde{D}PQ_vP_*\tilde{D}_* + \tilde{D}Q_\eta\tilde{D}_*$.

We make the following standard assumption on the above spectral $W(z, z^{-1})$.

Assumption 3.3.1. *The spectral $W(z, z^{-1})$ is positive definite on $|z| = 1$.*

Under Assumption 3.3.1, a unique stable spectral factor β exists, with an order of

$$n_\beta = max\{n_{\tilde{b}} + n_c,\ n_{\tilde{d}} + n_p\}$$

and satisfies the following

$$\beta Q_\epsilon \beta_* = \mathcal{W}.$$

Or, equivalently, we have

$$\beta\epsilon(k) = \tilde{B}Ce(k-d) + \tilde{D}Pv(k) + \tilde{D}\eta(k). \tag{3.30}$$

Note that the spectral factor β and Q_ϵ can be computed using the Riccati equation approach [18] or by rewriting the original equation $\mathcal{W}$ in terms of a reduced Sylvester matrix [19].

$$\tilde{D}Ay(k) = \beta\epsilon(k)$$

Note that β is an $N \times N$ matrix and $\epsilon(k)$ is a vector.

Theorem 3.3.1. *Consider the system (3.1) satisfying Assumptions 3.2.1 and 3.3.1. The robust deconvolution estimator* $\hat{u}(k|k-m)$ *is given by*

$$\hat{u}(k|k-m) = D^{-1}(q^{-1})L(q^{-1})\beta^{-1}\tilde{D}(q^{-1})A(q^{-1})y(k-m) \tag{3.31}$$

where the polynomial $L(q^{-1})$ *has the form*

$$L(q^{-1}) = R(q^{-1}) - S(q^{-1})$$

and

$$R(q^{-1}) = R_m + R_{m-1}q^{-1} + \ldots + R_{n_c-k}q^{-(n_c-m)},$$

$$S(q^{-1}) = S_0 + S_1q^{-1} + \ldots + S_{n_d-1}q^{-(n_d-1)}$$

with the coefficient matrices R_i and S_i given by

$$\begin{aligned} R_i &= E\{Ce(k)\epsilon^T(k-i)\}Q_\epsilon^{-1}, \\ S_i &= \sum_{j=i}^{n_d} D_j E[u(k-j)\epsilon^T(k-m-i)]Q_\epsilon^{-1}. \end{aligned}$$

Proof: By taking the projection of each term of (3.1) onto the linear space generated by $\{\epsilon(t-m), \epsilon(t-m-1), ...\}$ [30] [37], it follows that

$$\sum_{i=0}^{n_d} D_i \hat{u}(t-i|t-m) = Proj\{Ce(t)|\epsilon(t-m), \epsilon(t-m-1), ...\}. \quad (3.32)$$

Note that

$$\hat{u}(t-i|t-m) = \hat{u}(t-i|t-i-m) + \sum_{j=0}^{i-1} E[u(t-i)\epsilon^T(t-j-m)]Q_\epsilon^{-1}\epsilon(t-j-m), \quad (3.33)$$

$$Proj\{Ce(t)|\epsilon(t-m), \epsilon(t-m-1), ...\} = \sum_{\infty}^{m} E[Ce(t)\epsilon^*(t-i)]Q_\epsilon^{-1}\epsilon(t-i). \quad (3.34)$$

So,

$$\begin{aligned}
\sum_{i=0}^{n_d} D_i\hat{u}(t-i|t-m) &= \sum_{i=0}^{n_d} D_i\hat{u}(t-i|t-i-m) \\
&\quad + \sum_{i=0}^{n_d} D_i \sum_{j=0}^{i-1} E[u(t-i)\epsilon^*(t-j-m)]Q_\epsilon^{-1}\epsilon(t-j-m) \\
&= \sum_{i=0}^{n_d} D_i q^{-i}\hat{u}(t|t-m) \\
&\quad + \sum_{i=0}^{n_d}\sum_{j=0}^{i-1} D_i E[u(t-i)\epsilon^*(t-j-m)]Q_\epsilon^{-1}\epsilon(t-j-m)] \\
&= D(q^{-1})\hat{u}(t|t-m) \\
&\quad + \sum_{i=0}^{n_d-1}[\sum_{j=i}^{n_d} D_j E[u(t-j)\epsilon(t-m-i)]Q_\epsilon^{-1}]q^{-i}\epsilon(t-m) \\
&= D(q^{-1})\hat{u}(t|t-m) + S(q^{-1})\epsilon(t-m). \qquad (3.35)
\end{aligned}$$

From equation (3.30), it is easy to show that

$$E[e(t)\epsilon(t-i)] = \begin{cases} 0, & i > 0 \\ \neq 0, & i \leq 0. \end{cases} \qquad (3.36)$$

This implies that

$$E[C(q^{-1})e(t)\epsilon^T(t-i)] = \begin{cases} 0, & i > n_c \\ \neq 0, & i \leq n_c. \end{cases} \qquad (3.37)$$

Using (3.37), (3.34) yields

$$\begin{aligned}
Proj\{Ce(t)|\epsilon(t-m), \epsilon(t-m-1), ...\} &= \sum_{n_c}^{m} R_i\epsilon(t-i) \qquad (3.38) \\
&= R(q^{-1})\epsilon(t-m). \qquad (3.39)
\end{aligned}$$

Substitute (3.35) and (3.39) into (3.32)

$$\begin{aligned} R(q^{-1})\epsilon(t-m) &= S(q^{-1})\epsilon(t-m) + D(q^{-1})\hat{u}(t|t-m) \\ L(q^{-1}) &= R(q^{-1}) - S(q^{-1}). \end{aligned}$$

From the above we can obtain

$$\begin{aligned} L(q^{-1})\epsilon(t-m) &= D(q^{-1})\hat{u}(t|t-m), \\ \hat{u}(t|t-m) &= D^{-1}(q^{-1})L(q^{-1})\beta^{-1}(q^{-1})\tilde{D}(q^{-1})A(q^{-1})y(t-m). \qquad \square \end{aligned}$$

In practice, we can calculate $L(q^{-1})$ directly without calculating $R(q^{-1})$ and $S(q^{-1})$, as shown in the theorem below.

Theorem 3.3.2. *Consider the system defined by (3.1) which satisfies Assumptions 3.2.1 and 3.3.1. The robust deconvolution estimator is given by (3.31) with $L(q^{-1})$ satisfying the Diophantine equation:*

$$LQ_\epsilon\beta_* + zDM_* = z^{m+d}CQ_eC_*\tilde{B}_* \tag{3.40}$$

where M_* is a polynomial matrix, L and M_* are the unique solution of the Diophantine equation. The minimal estimation error is given by

$$\begin{aligned} Ez^T(k)z(k)_{min} &= tr[\frac{1}{2\pi i}\oint_{|z|=1}[D^{-1}(I - L\beta^{-1}q^{-m-d}\tilde{B})CQ_eC_*(I - L\beta^{-1}q^{-m-d}\tilde{B})_*D_*^{-1}] \\ &+D^{-1}L\beta^{-1}\tilde{D}(PQ_vP_* + Q_v^0)\tilde{D}_*\beta_*^{-1}L_*D_*^{-1}\frac{dz}{z}]. \end{aligned} \tag{3.41}$$

Proof:

$$\begin{aligned} z(k) &= u(k) - \hat{u}(k|k-m) \\ &= u(k) - D^{-1}L\beta^{-1}\tilde{D}Ay(k-m) \\ &= D^{-1}Ce(k) - D^{-1}L\beta^{-1}\tilde{D}Aq^{-m}A^{-1}(BD^{-1}Ce(k-d) + Pv(k) + v_0(k)) \\ &= D^{-1}Ce(k) - D^{-1}L\beta^{-1}\tilde{D}q^{-m}(\tilde{D}^{-1}\tilde{B}Ce(k-d) + Pv(k) + v_0(k)) \\ &= D^{-1}Ce(k) - D^{-1}L\beta^{-1}q^{-m}\tilde{B}Ce(k-d) - D^{-1}L\beta^{-1}q^{-m}\tilde{D}(Pv(k) + v_0(k)). \end{aligned}$$

Assume that $\xi(k-m)$ is an arbitrary signal generated from a linear combination of measurement, $y(k-m)$, which can be expressed as

$$\begin{aligned} \xi(k-m) &= \mathcal{M}(q^{-1})y(k-m) \\ &= \mathcal{M}(q^{-1})q^{-m}A^{-1}(BD^{-1}Ce(k-d) + Pv(k) + \eta(k)) \\ &= \mathcal{M}(q^{-1})q^{-m}A^{-1}(\tilde{D}^{-1}\tilde{B}Ce(k-d) + Pv(k) + \eta(k)). \end{aligned}$$

Based on the projection theory, $\hat{u}(k|k-m)$ is an MMSE estimate, given the observation $\{y(k-m),\ y(k-m-1)\ldots\}$, iff

$$E[z(k)\xi^T(k-m)] = 0.$$

On the other hand,

$$\begin{aligned}
& Ez(k)\xi^T(k-m) \\
= & \frac{1}{2\pi i}\oint_{|z|=1}\{D^{-1}CQ_eC_*D_*^{-1}B_*A_*^{-1}z^{m+d}\mathcal{M}_* - [D^{-1}L\beta^{-1}\tilde{B}CQ_eC_*D_*^{-1}B_*A_*^{-1}\mathcal{M}_* \\
& +D^{-1}L\beta^{-1}\tilde{D}PQ_vP_*A_*^{-1}\mathcal{M}_* + D^{-1}L\beta^{-1}\tilde{D}Q_\eta A_*^{-1}\mathcal{M}_*]\}\frac{dz}{z} \\
= & \frac{1}{2\pi i}\oint_{|z|=1}\{D^{-1}CQ_eC_*D_*^{-1}B_*A_*^{-1}z^{m+d}\mathcal{M}_* \\
& -D^{-1}L\beta^{-1}(\tilde{B}CQ_eC_*\tilde{B}_* + \tilde{D}PQ_vP_*\tilde{D}_* + \tilde{D}Q_v^0\tilde{D}_*)\tilde{D}_*^{-1}A_*^{-1}\mathcal{M}_*\}\frac{dz}{z} \\
= & \frac{1}{2\pi i}\oint_{|z|=1} D^{-1}\{z^{m+d}CQ_eC_*\tilde{B}_* - LQ_\epsilon\beta_*\}\tilde{D}_*^{-1}A_*^{-1}\mathcal{M}_*\frac{dz}{z}. \qquad (3.42)
\end{aligned}$$

Because there are no poles in $|z| = 1$ in (3.42) and the polynomial matrix A is stable, the part in the flower brace must satisfy

$$z^{m+d}CQ_eC_*\tilde{B}_* - LQ_\epsilon\beta_* = zDM_*.$$

The equation above is called Diophantine equation, from which we can get $L(q^{-1})$. The unknown polynomial matrix L with an order of $\max\{n_c-m-d, n_d-1\}$ together with M_* can be solved, where M_* is a polynomial matrix with an order of

$$\partial M = max(n_b + n_c + m + d,\ n_\beta) - 1.$$

The minimal estimation error is given by

$$\begin{aligned}
Ez^T(k)z(k)_{min} \;=\; & tr[\frac{1}{2\pi i}\oint_{|z|=1}[D^{-1}(I - L\beta^{-1}z^{-m-d}\tilde{B})CQ_eC_*(I - L\beta^{-1}z^{-m-d}\tilde{B})_*D_*^{-1}] \\
& +D^{-1}L\beta^{-1}\tilde{D}(PQ_vP_* + Q_v^0)\tilde{D}_*\beta_*^{-1}L_*D_*^{-1}\frac{dz}{z}]. \quad \square \qquad (3.43)
\end{aligned}$$

Remark 3.3.2. *Theorem 3.3.2 presents a design method for the optimal deconvolution estimation of MIMO systems with random parameter uncertainties in the system transfer function matrix using a polynomial approach. It should be pointed out that most of the existing works on deconvolution of systems with random parameter uncertainty using a polynomial approach are for SISO systems and they allow the random parameter uncertainties to appear only in the numerator of transfer function.*

3.4 Example

We consider a two-transmitter-and-two-receiver case. For the system (3.1) with

$$A(q^{-1}) = \begin{pmatrix} 1 - 0.1q^{-2} & -q^{-1} \\ 0.39q^{-1} & 1 - 0.9q^{-2} \end{pmatrix},$$

$$B(q^{-1}) = \begin{pmatrix} 1 - 0.3q^{-1} & 0.5 \\ 0.5 & 1 + 0.3q^{-1} \end{pmatrix},$$

$$C(q^{-1}) = \begin{pmatrix} 1 - 0.4q^{-1} & 0 \\ 0 & 1 - 0.4q^{-1} \end{pmatrix},$$

$$D(q^{-1}) = \begin{pmatrix} 1 + 0.7q^{-1} & 0 \\ 0 & 1 + 0.4q^{-1} \end{pmatrix},$$

$$P(q^{-1}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\Delta A(q^{-1}) = \begin{pmatrix} a_1 q^{-2} & a_2 q^{-2} \\ a_3 q^{-2} & a_4 q^{-2} \end{pmatrix},$$

$$\Delta B(q^{-1}) = \begin{pmatrix} b_1 q^{-1} & b_2 q^{-1} \\ b_3 q^{-1} & b_4 q^{-1} \end{pmatrix},$$

$$\Delta P(q^{-1}) = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix},$$

$$d = 0,$$

where a_i, b_i and P_i, $i = 1,2,3,4$, are uncorrelated with each other. a_i and b_i have the same variance of $\sigma_1^2 = 0.003$, and p_i, $i = 1,2,3,4$ have the same variance of $\sigma_2^2 = 0.004$. Also

$$\sigma_\xi^2 = 0.003, \quad \sigma_\zeta^2 = 0.004,$$

$$Q_e = I_2, \quad Q_v = 0.3 I_2.$$

So,

$$\begin{aligned} A^{-1}(q^{-1}) &= \frac{1}{(1+0.6q^{-1})(1-0.6q^{-1})(1-0.5q^{-1})(1+0.5q^{-1})} \begin{pmatrix} 1-0.9q^{-2} & q^{-1} \\ -0.39q^{-1} & 1-0.1q^{-2} \end{pmatrix}, \\ D^{-1}(q^{-1}) &= \frac{1}{(1+0.4q^{-1})(1+0.7q^{-1})} \begin{pmatrix} 1+0.4q^{-1} & 0 \\ 0 & 1+0.7q^{-1} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} & Q_v^0 - \sum_{j=0}^{n_A} \gamma_j \bar{E}(tr(A^{-1} Q_v^0 A_*^{-1})) I_N \\ = & \sum \gamma_i \bar{E} tr(A^{-1} B D^{-1} C Q_e C_* D_*^{-1} B_* A_*^{-1} + A^{-1} P Q_v P_* A_*^{-1}) I_N \\ & + \sum \gamma_{bi} \bar{E} tr(D^{-1} C Q_e C_* D_*^{-1}) I_N + \sum \gamma_{pi} tr Q_v I_N, \end{aligned}$$

where

$$\bar{E} tr(A^{-1} B D^{-1} C Q_e C_* D_*^{-1} B_* A_*^{-1}) = 17.3867,$$

$$\bar{E} tr(A^{-1} P Q_v P_* A_*^{-1}) = 0.3 * 6.8185 = 2.0456,$$

$$\bar{E}tr(D^{-1}CQ_eC_*D_*^{-1}) = 2.2348,$$

$$\bar{E}(tr(A^{-1}A_*^{-1})) = 4.1878.$$

We hence obtain

$$(1-0.003*4.1878)q_v^0 = 0.003*17.3867+0.003*2.0456+0.003*2.2348+0.004*0.6,$$

i.e.

$$Q_v^0 = q_v^0 I = 0.0683I.$$

From (3.28), we can get

$$\begin{aligned}\tilde{B} &= \begin{pmatrix} (1-0.3q^{-1})(1+0.4q^{-1}) & 0.5(1+0.7q^{-1}) \\ 0.5(1+0.4q^{-1}) & (1+0.3q^{-1})(1+0.7q^{-1}) \end{pmatrix}, \\ \tilde{D} &= \begin{pmatrix} (1+0.7q^{-1})(1+0.4q^{-1}) & 0 \\ 0 & (1+0.7q^{-1})(1+0.4q^{-1}) \end{pmatrix}.\end{aligned}$$

Furthermore,

$$\mathcal{W} = \tilde{B}CQ_eC_*\tilde{B}_* + \tilde{D}PQ_vP_*\tilde{D}_* + \tilde{D}Q_v^0\tilde{D}_*$$

with

$$\begin{aligned}\mathcal{W}(1,1) &= 0.05q^{-3} - 0.06q^{-2} + 0.74q^{-1} + 2.94 + 0.74q - 0.06q^2 + 0.05q^3, \\ \mathcal{W}(1,2) &= 0.024q^{-3} - 0.05^{-2} + 0.76q^{-1} + 2.50 + 1.15q - 0.02q^2 - 0.042q^3, \\ \mathcal{W}(2,1) &= -0.042q^{-3} - 0.02q^{-2} + 1.15q^{-1} + 2.50 + 0.76q - 0.05q^2 + 0.024q^3, \\ \mathcal{W}(2,2) &= -0.084q^{-3} - 0.09q^{-2} + 1.44q^{-1} + 3.19 + 1.44q - 0.09q^2 - 0.084q^3.\end{aligned}$$

Apply the spectral factorization using [30]

$$\beta(q^{-1})\beta^T(q) = \mathcal{W}(q, q^{-1}) = \mathcal{M}(q^{-1})\Omega\mathcal{M}^T(q),$$

where

$$\begin{aligned}
\mathcal{M} &= M_0 + M_1 q^{-1} + M_2 q^{-2} + M_3 q^{-3}, \\
\Omega &= \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \\
M_0 &= \begin{pmatrix} 1 & 0 & 1.47 & 2.5 \\ 0 & 1 & 0 & 1.60 \end{pmatrix}, \\
M_1 &= \begin{pmatrix} 0 & 0 & 0.74 & 0.76 \\ 0 & 0 & 1.15 & 1.44 \end{pmatrix}, \\
M_2 &= \begin{pmatrix} 0 & 0 & -0.06 & -0.05 \\ 0 & 0 & -0.02 & -0.09 \end{pmatrix}, \\
M_3 &= \begin{pmatrix} 0 & 0 & 0.05 & 0.024 \\ 0 & 0 & -0.042 & -0.084 \end{pmatrix}.
\end{aligned}$$

After calculation, we obtain that

$$\begin{aligned}
\beta_{11}(q^{-1}) &= 0.30q^{-1} - 0.03q^{-2} + 0.03q^{-3} + 1.46, \\
\beta_{12}(q^{-1}) &= 0.45q^{-1} - 0.03q^{-2} - 0.001q^{-3} + 0.72, \\
\beta_{21}(q^{-1}) &= 0.32q^{-1} + 0.02q^{-2} + 0.0045q^{-3} + 0.72, \\
\beta_{22}(q^{-1}) &= 0.95q^{-1} - 0.03q^{-2} - 0.07q^{-3} + 1.29, \\
Q_\epsilon &= I_2.
\end{aligned}$$

The Diophantine equation of (3.40) is given by

$$\begin{aligned}
& L(q^{-1})\begin{pmatrix} 0.30q-0.03q^2+0.03q^3+1.46 & 0.32q+0.02q^2+0.0045q^3+0.72 \\ 0.45q-0.03q^2-0.001q^3+0.72 & 0.95q-0.03q^2-0.07q^3+1.29 \end{pmatrix} \\
& +q\begin{pmatrix} 1+0.7q^{-1} & 0 \\ 0 & 1+0.4q^{-1} \end{pmatrix} M_*(q) \\
= & \begin{pmatrix} 1-0.4q^{-1} & 0 \\ 0 & 1-0.4q^{-1} \end{pmatrix}\begin{pmatrix} 1-0.4q & 0 \\ 0 & 1-0.4q \end{pmatrix} \\
& \times\begin{pmatrix} (1-0.3q)(1+0.4q) & 0.5(1+0.4q) \\ 0.5(1+0.7q) & (1+0.3q)(1+0.7q) \end{pmatrix} \\
= & \left(\begin{matrix} 1.12-0.2360q-0.1792q^2+0.0480q^3-0.4q^{-1} \\ 0.44+0.206q-0.14q^2-0.2q^{-1} \end{matrix}\right. \\
& \left.\begin{matrix} 0.5+0.032q-0.08q^2-0.2q^{-1} \\ 0.76+0.676q-0.1564q^2-0.084q^3-0.4q^{-1} \end{matrix}\right).
\end{aligned}$$

Solving, we can get

$$\begin{aligned}
L(q^{-1}) &= \begin{pmatrix} 1.0383-0.2725q^{-1} & 0.0025-0.0029q^{-1} \\ 0.0247+0.0220q^{-1} & 0.8298-0.3223q^{-1} \end{pmatrix}, \\
M_*(q) &= \begin{pmatrix} -0.4494-0.1517q+0.0169q^2 & -0.2297-0.0965q-0.0045q^2 \\ -0.1377-0.1154q+0.0001q^2 & -0.0726-0.1443q-0.0260q^2 \end{pmatrix}.
\end{aligned}$$

At last, we obtain the estimator$(m=0)$

$$\begin{aligned}
\hat{u}(k|k) &= D^{-1}(q^{-1})L(q^{-1})\beta^{-1}(q^{-1})\tilde{D}(q^{-1})A(q^{-1})y(k) \\
&= \frac{1}{\mathcal{S}_0}\begin{pmatrix} \mathcal{S}_1 & \mathcal{S}_2 \\ \mathcal{S}_3 & \mathcal{S}_4 \end{pmatrix} y(k)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}_0 &= 1.3650 + 1.2196q^{-1} + 0.0657q^{-2} - 0.1029q^{-3} + 0.0073q^{-4}, \\
\mathcal{S}_1 &= 1.3376 + 0.8811q^{-1} - 0.3916q^{-2} - 0.2805q^{-3} + 0.0179q^{-4}, \\
\mathcal{S}_2 &= -0.7439 - 1.9097q^{-1} - 0.4586q^{-2} + 0.6037q^{-3} + 0.1382q^{-4}, \\
\mathcal{S}_3 &= -0.5656 + 0.0880q^{-1} + 0.4051q^{-2} - 0.0061q^{-3} - 0.0338q^{-4}, \\
\mathcal{S}_4 &= 1.1937 + 1.1526q^{-1} - 1.0015q^{-2} - 0.7042q^{-3} + 0.2143q^{-4}.
\end{aligned}$$

Simulation result

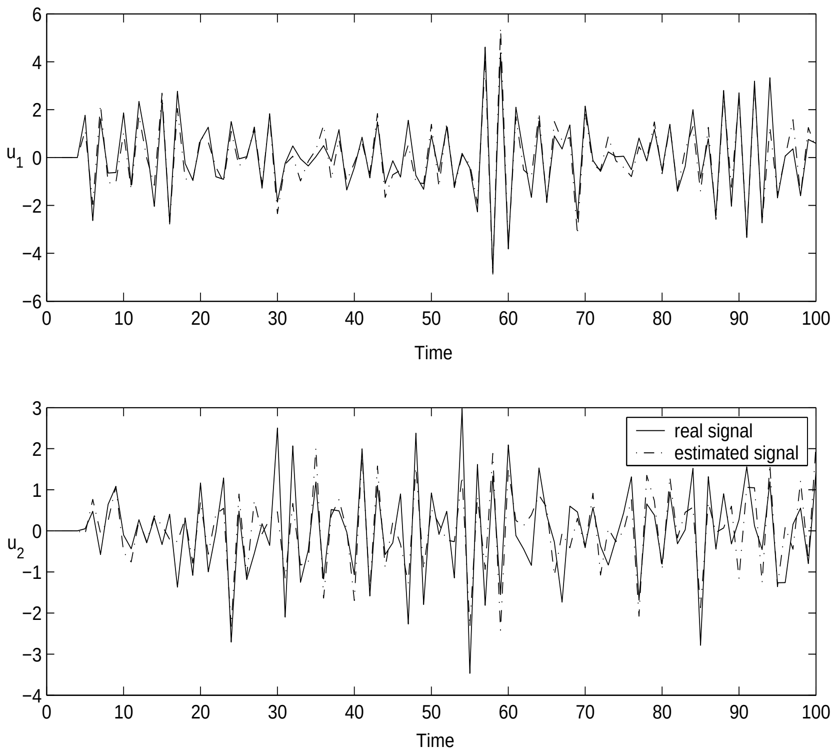

Figure 3.1: Performance of MIMO system, solid line denotes the true signal, dash-dot line denotes its estimate by the robust estimation.

From Figure 3.1, we can find robust deconvolution estimate tracks the real actual signal well. In order to observe the average performance of the robust deconvolution estimator, we calculate the MSE (100 tests were used) of u_1 and u_2 using the robust and standard MSE without considering parameter uncertainty, respectively.

$$\frac{robust\ MSE\ of\ u_1}{standard\ MSE\ of\ u_1} = 0.7037,$$
$$\frac{robust\ MSE\ of\ u_2}{standard\ MSE\ of\ u_2} = 0.8475.$$

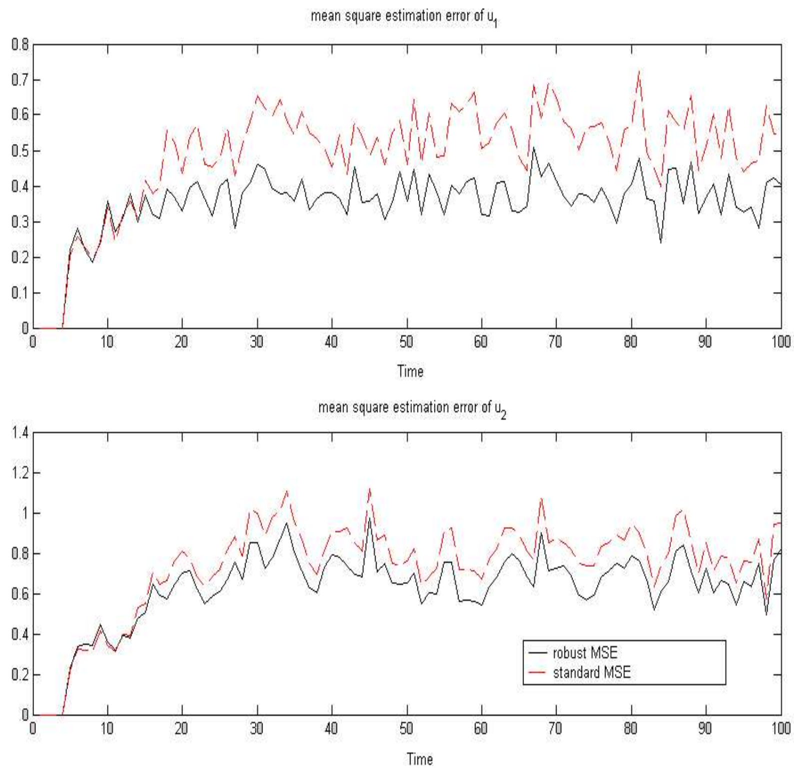

Figure 3.2: Performance of MIMO system, solid line denotes the mean square estimation error of the robust estimator, dash line denotes the mean square estimation error of the standard estimator.

From Figure 3.2 (100 tests were used), we can see that performance of the robust estimator is better than that of the nominal estimator. When we design an estimator, the factor of model uncertainties cannot be ignored if the uncertainties are larger.

3.5 Conclusion

In this chapter, an MIMO robust deconvolution estimator design for systems with input delay has been presented using a polynomial approach. The approach has the advantage, compared to other existing works, that the denominator uncertainties are allowed in the transfer function matrix, which is more general than the one allowing uncertainties only in the numerators of the system transfer functions. The technique of projection and spectral factorization are used in the design. We have presented a simplified method to compute the covariance of the fictitious noise. The estimator in polynomial form is relatively simple and has an advantage in computational cost compared to the state space counterpart. Simulation results have shown that the robust estimator performs better than the standard MSE. When the uncertainties are larger, the advantage of the robust estimator over non-robust one becomes more obvious.

Part II

Control for Systems with I/O Delays

Chapter 4

H_2 Control for Systems with Time-variant Input Delay

4.1 Introduction

Starting from this chapter, we will focus on the study of control problems for systems with time delay in input/output. Time delay exists in many engineering systems and it is a factor that affects the performance of a system or even the stability of the system. There have been quite a few approaches to control problems for systems with input delay, see review papers [60, 73, 90]. Gouaisbaout et al. [31, 32] provide sliding mode control for linear time delay systems where an upper bound of delay is assumed known. Xia et al. [91] provide a 'memoryless' state feedback sliding mode controller in terms of LMIs where no upper-bound of delay is needed. 'Memoryless' state feedback control has a simple structure and can be applied to address performance control in addition to stability.

In this chapter, a BMI approach to state feedback H_2 control for systems with time-variant input delay is discussed. Firstly, we augment the state to transform the time delay system into a delay-free system and then we apply an BMI approach for the delay-free system. The delay is assumed to be time-varying but bounded. We present a robust control design that guarantees the stability and H_2 performance for the system for all admissible delays.

The application of the proposed robust control in congestion control is also investigated. In congestion control, we are concerned with the best average performance of the network over a long period of time. Hence, the H_2 performance measure would be an appropriate candidate. Our objective is to design a congestion control that would give rise to a guaranteed H_2 performance regardless of the time-varying delay on the return path. Saturation in source rate and queue buffer is also taken into consideration in congestion control.

4.2 Problem Statement

Introduce a discrete time system with time delay input as

$$x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k - d_k), \tag{4.1}$$

$$z(k) = Cx(k) + Du(k - d_k), \tag{4.2}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $w(k) \in \mathbb{R}^p$ and $z(k) \in \mathbb{R}^r$ represent the state, the control input, the exogenous input (noise) and the controlled output, respectively. $d_k \in \{1, 2, \cdots, \bar{d}\}$ is time-variant delay in input at time k with $\bar{d}$ known and $w(k)$ is a Gaussian white noise.

Our objective in this chapter is:

Find a suitable state feedback controller such that the closed-loop system is asymp-

totically stable and the cost

$$J = \lim_{N\to\infty} \frac{1}{N} E\{\sum_{k=1}^{N} z^T(k)z(k)\} \tag{4.3}$$

is minimized, where $E(\cdot)$ *denotes the mathematical expectation.*

4.3 Design of Robust State-feedback H_2 Controller

We consider a state feedback control $u(k) = Fx(k)$ for the system (4.1)-(4.2). The closed-loop system is given by

$$\begin{cases} x(k+1) & = Ax(k) + B_2Fx(k-d_k) + B_1w(k), \\ z(k) & = Cx(k) + DFx(k-d_k). \end{cases} \tag{4.4}$$

By considering that $d_k \in \{1, 2, \cdots, \bar{d}\}$, we transform the system (4.4) into an equivalent system as follows by state augmentation:

$$\begin{aligned} (\Sigma): \quad \xi(k+1) &= \bar{A}_k\xi(k) + \bar{B}w(k) \\ z(k) &= \bar{C}_k\xi(k) \end{aligned}$$

where

$$\xi(k) = \begin{pmatrix} x(k) \\ \xi_1(k) \\ \vdots \\ \xi_{\bar{d}}(k) \end{pmatrix}, \ \bar{A}_k = \begin{pmatrix} A & 0 & \cdots & \overbrace{B_2}^{(d_k+1)-th\ block} & \cdots & 0 & 0 \\ F & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & I_m & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & I_m & 0 \end{pmatrix}, \ \bar{B} = \begin{pmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\bar{C}_k = \begin{pmatrix} C & 0 & \cdots & 0 & \overbrace{D}^{(d_k+1)-th\ block} & 0 & \cdots & 0 \end{pmatrix}.$$

Here, B_2 and D are at the $(d_k+1)-th$ block column of $\bar{A}_k$ and $\bar{C}_k$, respectively, and the dimensions of $\xi(k)$, $\bar{A}_k$, $\bar{B}$, $\bar{C}_k$ are $(n+\bar{d}m)\times 1$, $(n+\bar{d}m)\times(n+\bar{d}m)$, $(n+\bar{d}m)\times p$ and $r\times(n+\bar{d}m)$, respectively.

Under the state feedback $u(k) = Fx(k)$, for a given constant time delay $d_k = d$, it is well known that the cost of (4.3) is in fact the square of the H_2 norm of the system (Σ). Hence, the time delay H_2 control problem becomes the problem of designing a state feedback control gain F such that the closed-loop system (Σ) is stable and its H_2 norm is minimized.

If delay d_k is a constant, the system (Σ) will be time-invariant ($\bar{A}_k = \bar{A}$, $\bar{C}_k = \bar{C}$), the H_2 norm square of the system can be computed as [103]

$$||G(z)||_2^2 = tr(\bar{B}^T L_o \bar{B}) = tr(\bar{C} L_c \bar{C}^T) \tag{4.5}$$

where L_c and L_o are the reachability and observability Gramians

$$\bar{A}^T L_o \bar{A} - L_o + \bar{C}^T \bar{C} = 0, \tag{4.6}$$

$$\bar{A} L_c \bar{A}^T - L_c + \bar{B}\bar{B}^T = 0. \tag{4.7}$$

When d_k is time-variant, $\bar{A}_k$ and $\bar{C}_k$ will be time-variant. In this case, we have the following result.

Theorem 4.3.1. *Given the system (Σ) with time-varying $\bar{A}_k$ and $\bar{C}_k$, the following hold [5]:*

(a) If the system (Σ) is exponentially stable, then

$$||\Sigma||_2^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} tr[\bar{C}_k L_c(k) \bar{C}_k^T] \tag{4.8}$$

where L_c satisfies the following difference Lyapunov equation

$$L_c(k+1) = \bar{A}_k L_c(k) \bar{A}_k^T + \bar{B}\bar{B}^T,\ L_c(0) = 0. \tag{4.9}$$

(b) If there exist bounded matrices P, Q, W and a scalar γ such that for $i =$ $1, 2,\ \cdots, \bar{d}$

$$\begin{bmatrix} P - \bar{B}\bar{B}^T & \tilde{A}_i \\ \tilde{A}_i^T & Q \end{bmatrix} > 0, \tag{4.10}$$

$$\begin{bmatrix} W & \tilde{C}_i \\ \tilde{C}_i^T & Q \end{bmatrix} > 0, \tag{4.11}$$

$$tr(W) < \gamma^2, \tag{4.12}$$

$$PQ = I, \tag{4.13}$$

where

$$\tilde{A}_i = \begin{pmatrix} A & 0 & \cdots & \overbrace{B_2}^{(i+1)-th\ \ block} & \cdots & 0 & 0 \\ F & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & I_m & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & I_m & 0 \end{pmatrix}, \tag{4.14}$$

$$\tilde{C}_i = \begin{pmatrix} C & 0 & \cdots & 0 & \overbrace{D}^{(i+1)-th\ \ block} & 0 & \cdots & 0 \end{pmatrix}, \tag{4.15}$$

then the system (Σ) is exponentially stable and

$$||\Sigma||_2 < \gamma. \tag{4.16}$$

In this situation, a suitable state feedback controller is

$$u(k) = Fx(k). \tag{4.17}$$

Proof: (a) In view of [5], define $L_c(k) \triangleq E[\xi(k)\xi^T(k)]$.

$$\begin{aligned} L_c(k+1) &= E[\xi(k+1)\xi^T(k+1)] \\ &= E\{[\bar{A}_k\xi(k) + \bar{B}w(k)][\bar{A}_k\xi(k) + \bar{B}w(k)]^T\} \\ &= E[\bar{A}_k\xi(k)\xi^T(k)\bar{A}_k^T] + \bar{B}\bar{B}^T \\ &= \bar{A}_k L_c(k)\bar{A}_k^T + \bar{B}\bar{B}^T. \end{aligned} \tag{4.18}$$

From the definition of the H_2 norm,

$$\begin{aligned} ||\Sigma||_2^2 &= \lim_{N\to\infty} E\left(\frac{1}{N}\sum_{k=1}^{N} z^T(k)z(k)\right) \\ &= \lim_{N\to\infty} E\frac{1}{N}\sum_{k=1}^{N} tr(z(k)z^T(k)) \\ &= \lim_{N\to\infty} E\frac{1}{N}\sum_{k=1}^{N} tr(\bar{C}_k\xi(k)\xi^T(k)\bar{C}_k^T) \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} tr(\bar{C}_k L_c(k)\bar{C}_k^T). \end{aligned} \tag{4.19}$$

(b) It can be easily known that if there exist constant matrices P and W such that for $k = 1, 2, \cdots$

$$\begin{pmatrix} P - \bar{B}\bar{B}^T & \bar{A}_k P \\ P\bar{A}_k^T & P \end{pmatrix} > 0, \tag{4.20}$$

$$\begin{pmatrix} W & \bar{C}_k P \\ P\bar{C}_k^T & P \end{pmatrix} > 0, \tag{4.21}$$

then

$$
\begin{aligned}
P &> \bar{A}_k P \bar{A}_k^T + \bar{B}\bar{B}^T, \\
W &> \bar{C}_k P \bar{C}_k^T.
\end{aligned}
$$

Since $L_c(0) = 0$ and P is positive definite, we have that $L_c(k) < P$ and

$$
||\Sigma||_2^2 < \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} tr(W). \tag{4.22}
$$

Moreover, $\bar{A}_k \in \{\tilde{A}_i\},\ \bar{C}_k \in \{\tilde{C}_i\},\ i = 1, 2, \cdots, \bar{d}$. From (4.10) and (4.11), we have that for all k

$$
\begin{pmatrix} P - \bar{B}\bar{B}^T & \bar{A}_k P \\ P\bar{A}_k^T & P \end{pmatrix} > 0, \tag{4.23}
$$

$$
\begin{pmatrix} W & \bar{C}_k P \\ P\bar{C}_k^T & P \end{pmatrix} > 0. \tag{4.24}
$$

Then

$$
\begin{aligned}
||\Sigma||_2^2 &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} tr\left(\bar{C}_k L_c(k) \bar{C}_k^T\right) \\
&\leq \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} tr\left(\bar{C}_k P \bar{C}_k^T\right) \\
&\leq tr(W). \qquad \Box
\end{aligned}
$$

Remark 4.3.1. *The state-feedback control problem can be extended to static output feedback control problem. In this case, F in $\tilde{A}_i$ will be replaced by FH where H is the output matrix, i.e. $y(k) = Hx(k)$.*

Note, however, that (4.10)-(4.13) do not constitute a convex optimization due to the equality constraint (4.13) which is bilinear in P and Q. Although no global solution is available for the bilinear problem, some useful algorithms have been proposed; see e.g. [27, 46]. In [27], the above constrained non-convex optimization is formulated as:

$$\min tr(QP) \tag{4.25}$$

$$subject\ to\ LMIs\ (4.10)-(4.12)\ and \tag{4.26}$$

$$\begin{pmatrix} P & I \\ I & Q \end{pmatrix} \geq 0. \tag{4.27}$$

The above optimization solves the bilinear problem (4.10)-(4.13) if and only if the minimum solution of $tr(QP) = n + \bar{d}m$. We should also observe that F is embedded in $\tilde{A}_i$ and $\tilde{C}_i$.

To find an optimal robust H_2 controller, minimization of γ^2 is to be carried out. From the description above, we need to minimize both $tr(W)$ and $tr(QP)$. By extending the sequential linear programming matrix method (SLPMM) in [46], we propose the following procedure for the optimization.

Algorithm 4.3.1. *1. Given a positive scalar γ, solve the semi-definite programming problem (SDP) (4.10)-(4.12) and (4.27) for an initial (P^0, Q^0, W^0, F^0) and set $k = 0$. Note that if the SDP does not admit a solution, increase γ until a feasible solution subject to LMIs (4.10)-(4.12) exists.*

2. Solve the following LMI problem for the variables P, Q, W, F :

$$\min \ tr(P^k Q + PQ^k)$$

$$subject\ to\ LMIs(4.10)-(4.12).$$

Set $P^{k+1} = P$, $Q^{k+1} = Q$.

3. If a stopping criterion is satisfied, say $k = k_{max}$, here k_{max} is the largest iteration we choose, then exit. Otherwise, set $k = k + 1$ and go to step 2.

On the other hand, smaller γ means better H_2 performance, so we are interested in the following minimization problem:

$$\min_{(P,\ Q,\ W,\ F)} \gamma,\ \textit{subject to}\ (4.10) - (4.13). \tag{4.28}$$

We can define the following multi-objective programming problem:

$$\min_{(P,\ Q,\ W,\ F)} tr(PQ) + tr(W),\ \textit{subject to}\ (4.10),\ (4.11)\ \textit{and}\ (4.27).$$

Algorithm 4.3.2. *(SLPMM [46])*

1. Find (P^0, Q^0, W^0, F^0) that satisfy (4.10), (4.11) and (4.27).

For $k = 1, 2, ...,$ do

2. Determine (U^k, V^k, Z^k, H^k) as the solution of

$$\min_{(P,\ Q,\ W,\ F)} tr(PQ^k + P^kQ) + tr(W),\ \textit{subject to}\ (4.10),\ (4.11)\ \textit{and}\ (4.27).$$

3. If $tr(U^kQ^k + P^kV^k) + tr(Z^k) = 2\ tr(P^kQ^k) + tr(W^k)$, then stop.

4. Compute $\beta \in [0, 1]$ by solving

$$\min_{\beta\in[0,1]} tr[(P^k + \beta(U^k - P^k))(Q^k + \beta(V^k - Q^k)) + (W^k + \beta(Z^k - W^k))].$$

5. Set $P^{k+1} = (1-\beta)P^k + \beta U^k,\ Q^{k+1} = (1-\beta)Q^k + \beta V^k,\ W^{k+1} = (1-\beta)W^k + \beta Z^k,\ F^{k+1} = H^k$, go to Step 2.

Theorem 4.3.2. *If the delay d_k is known and there exist matrices $P_i,\ Q_i,\ W_i\ (i =$*

$1, 2, \cdots, \bar{d}$) *and scalar* γ *such that*

$$\begin{bmatrix} P_j - \bar{B}\bar{B}^T & \tilde{A}_i \\ \tilde{A}_i^T & Q_i \end{bmatrix} > 0 \quad (4.29)$$

$$\begin{bmatrix} W_i & \tilde{C}_i \\ \tilde{C}_i^T & Q_i \end{bmatrix} > 0 \quad (4.30)$$

$$tr(W_i) < \gamma^2 \quad (4.31)$$

$$P_i Q_i = I, \; i, j = 1, 2, \; \cdots, \bar{d}, \quad (4.32)$$

where $\tilde{A}_i$, $\tilde{C}_i$ *are defined in (4.14) and (4.15), then the system* (Σ) *is exponentially stable and*

$$||\Sigma||_2 < \gamma. \quad (4.33)$$

In this situation, a suitable controller can be

$$u(k) = Fx(k). \quad (4.34)$$

Proof: In view of (4.32), it follows from (4.29)-(4.30) that

$$\begin{pmatrix} P_j - \bar{B}\bar{B}^T & \tilde{A}_i P_i \\ P_i \tilde{A}_i^T & P_i \end{pmatrix} > 0 \quad (4.35)$$

$$\begin{pmatrix} W_i & \tilde{C}_i P_i \\ P_i \tilde{C}_i^T & P_i \end{pmatrix} > 0, \; i = 1, 2, \cdots, \bar{d}. \quad (4.36)$$

Hence, for any k, there exist bounded matrix sequences $P(k), W(k)$ such that

$$\begin{bmatrix} P(k+1) - \bar{B}\bar{B}^T & \bar{A}_k P(k) \\ P(k)\bar{A}_k^T & P(k) \end{bmatrix} > 0, \tag{4.37}$$

$$\begin{bmatrix} W(k) & \bar{C}_k P(k) \\ P(k)\bar{C}_k^T & P(k) \end{bmatrix} > 0, \tag{4.38}$$

where $\bar{A}_k = \tilde{A}_{d_k} \in \{\tilde{A}_i\}$, $\bar{C}_k = \tilde{C}_{d_k} \in \{\tilde{C}_i\}$ and $P(k) = P_{d_k}$, $d_k \in \{1, 2, \cdots, \bar{d}\}$. According to [5], we know that the system (Σ) is exponentially stable and

$$||\Sigma||_2^2 < \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} tr[W(k)] < \gamma^2.$$

Remark 4.3.2. *Theorem 4.3.2 is obviously less conservative than Theorem 4.3.1 as when setting $P_i = P$ and $W_i = W$, $i = 1, 2, \cdots, \bar{d}$, the former reduces to the latter.*

4.4 Application in Congestion Control in ATM Networks

4.4.1 ATM Network Congestion Control Model

A mathematical model of congestion control in ATM (asynchronous transfer mode) is taken from [2]. The ABR source is the only traffic class which responds to feedback information for the node for rate adjustment to prevent network congestion and to maintain quality of service (QoS) to all connections. The feedback information is the available transmission capacity (bandwidth) and queue level at the bottleneck node. Since the available node capacity for the ABR source changes over time in an unpredictable way due to the higher priority sources, the CBR (constant bit rate) and VBR (variable bit rate) source rates are represented as interferences. In this section, without causing confusion, we adopt similar notations of ATM network model as in [2] where the subscript represents time instant. Let ζ_k denote the higher priority source (interference) which is modelled as a stable ARMA process [3]. Such a formulation allows for long-range correlated traffic. Let q_k be the queue length at the bottleneck and μ_k the effective service rate available for the traffic of the given source in that link at the beginning of the kth time slot. Let r_k denote the effective source rate measured at the congestion switch. Without loss of generality, we consider the case of single connection. Therefore, the queue length equation is given by

$$q_{k+1} = q_k + r_k - \mu_k. \tag{4.39}$$

The effective service rate is modelled as

$$\mu_k = \mu + \zeta_k, \tag{4.40}$$

$$\zeta_{k+1} = \sum_{i=1}^{p_1} l_i \zeta_{k+1-i} + \rho w_k, \tag{4.41}$$

where μ is the constant nominal service rate and $\{l_i\}_{i=1,2,\cdots,p_1}$ are known parameters. $\{w_k\}_{k\geq 1}$ is a zero-mean i.i.d. Gaussian sequence with unit variance and ρ is a known constant. We assume that there is no cell loss and let u_k denote explicit cell rate (ER) calculated by switch. The delay between u_k and r_k is d_k which is the round trip delay. d_k consists of two path delays, one is return path delay and the other is forward path delay. On the return path RM cells travel from the switch to the source. On the forward path the user data travels from the source through the congested switch. In ATM network, packet delays, transmission delays, processing delays and queueing delays exist in transmission on the both paths and the queuing delay is dominant [79]. Actually the two paths are one single communication link and hence the round trip delay d_k can be considered. The relationship between u_k and r_k can be expressed as

$$r_k = u_{k-d_k} \tag{4.42}$$

where d_k is known to be bounded with upper bound $\bar{d}$.

In the case when d_k is known, i.e. the explicit ER is time-stamped, we formulate the congestion control problem as an LQG stochastic control problem [3], that is, we seek a feedback control $u_k = u_k(q_k, \mu_k)$ that minimizes the cost

$$J = \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{k=1}^{N}[(q_k - q_d)^2 + \lambda^2(r_k - \mu_k)^2]\right\} \tag{4.43}$$

where q_d is the target queue length and λ is a weighting factor. It is clear that the objective is to make the queue buffer close to the desired level while the difference between the source rate and the service rate should not be too large.

The above criterion combines the performance of queue length and accumulation of the difference between switch input and output. We should note that round trip delay in transmission is one reason of the disagreement between the switch input and output.

In the case when the delay d_k is not known, we shall be concerned with designing a controller that minimizes an upper bound of (4.43) for all possible time-varying uncertainties within the bound of $\bar{d}$.

Observe that the above criterion does not consider the saturation in queue buffer and service rate. We shall address this issue later.

Remark 4.4.1. *In [36], an H_∞ control approach is adopted where w_k is assumed to an energy bounded deterministic signal. While guaranteeing the worst-case performance, the H_∞ approach is generally conservative.*

Remark 4.4.2. *In the multi-source case, the control objective becomes finding inputs $r_{i,k}$ so as to minimize*

$$J = \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{k=1}^{N}[(q_k - q_d)^2 + \sum_{i=1}^{M}\lambda_i^2(r_{i,k} - a_i\mu_k)^2]\right\} \tag{4.44}$$

where M, λ_i, $r_{i,k}$, a_i *are respectively the number of sources, weighting, effective source rate for source* i *and share of input bandwidth of the* i*-th source* $(\sum_{i=1}^{M} a_i = 1)$*. If we consider fair sharing of available bandwidth, we can simply define* $a_i = 1/M,\ i = 1, 2, \cdots, M$*. There will be jitter in queue length when new sources add in the congested switch. The reason is that RM cells come back to different sources with different delays. See the examples in Section 4.4.2. In spite of the jitter, the fair sharing of bandwidth is still a good scheme for its simplicity in application.*

Denote

$$\begin{aligned} x(k) &= \begin{pmatrix} q_k - q_d \\ \zeta_{k+1-p_1} \\ \vdots \\ \zeta_k \end{pmatrix}, \\ u(k-d_k) &= u_{k-d_k} - \mu = r_k - \mu, \\ z(k) &= \begin{bmatrix} q_k - q_d \\ \lambda(r_k - \mu_k) \end{bmatrix}. \end{aligned}$$

Then, the system (4.39)-(4.42) can be rewritten as (4.1)-(4.2), where $0 < d_k \leq \bar{d}$,

$w(k) = w_k$ is a white noise with unit variance, and

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & l_{p_1} & l_{p_1-1} & l_{p_1-2} & \cdots & l_1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0 & 0 & \cdots & 0 & \rho \end{bmatrix}^T, \\ B_2 &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T, \\ C &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -\lambda \end{bmatrix}, \\ D &= \begin{bmatrix} 0 & \lambda \end{bmatrix}^T. \end{aligned}$$

Obviously, (4.43) can be written as (4.3).

We assume that each ABR source is both compliant and greedy, which implies that a command received from the congested switch is immediately executed.

In Theorems 4.3.1 and 4.3.2, we present an approach that achieves a guaranteed H_2 performance. So, we can use the result of the previous section to solve the congestion control problem. And a model of effective service rate is used, which can give rise to better performance of congestion control. It is worth highlighting that the sampling period of controller is larger than the RM cell spacing, there is plenty of time to perform the required computation [79].

Remark 4.4.3. *In the multi-source case, one switch needs to allocate bandwidths for a lot of users and information may experience different delays for each user. It is a big burden to design different state feedback controller for each user, to make* *it even worse, the number of users is changing. Here, one single fixed controller is designed to deal with this problem. We apply the same state feedback for each user, namely,* $u_j(k) = f_j F x(k),\ j = 1, 2, \cdots, M$, *here* M *is the number of inputs,* F *is the same as in Theorem 4.3.1 and the system is*

$$x(k+1) = Ax(k) + B_2 \sum_{j=1}^{M} u_j(k - d_k^j) + B_1 w(k), \tag{4.45}$$

$$z(k) = Cx(k) + D \sum_{j=1}^{M} u_j(k - d_k^j) \tag{4.46}$$

where $\sum_{j=1}^{M} f_j = 1,\ f_j \geq 0, j = 1, 2, \cdots, M,\ f_j$ *is the bandwidth share of each user and* d_k^j *is the delay of input* j. *Substitute the state feedback control in the system*

$$x(k+1) = Ax(k) + B_2 \sum_{j=1}^{M} f_j F x(k - d_k^j) + B_1 w(k), \tag{4.47}$$

$$z(k) = Cx(k) + D \sum_{j=1}^{M} f_j F x(k - d_k^j). \tag{4.48}$$

After state augmentation, we get

$$\begin{bmatrix} P - \bar{B}\bar{B}^T & \bar{A}_k \\ \bar{A}_k^T & Q \end{bmatrix} = \sum_{j=1}^{M} f_j \begin{bmatrix} P - \bar{B}\bar{B}^T & \tilde{A}_k^j \\ \tilde{A}_k^{jT} & Q \end{bmatrix} > 0, \tag{4.49}$$

$$\begin{bmatrix} W & \bar{C}_k \\ \bar{C}_k^T & Q \end{bmatrix} = \sum_{j=1}^{M} f_j \begin{bmatrix} W & \tilde{C}_k^j \\ \tilde{C}_k^{jT} & Q \end{bmatrix} > 0. \tag{4.50}$$

(4.51)

From the description above, we know in the multi-input case, state feedback controller $u_j(k) = f_j F x(k),\ j = 1, 2, \cdots, M$ *can achieve the required* H_2 *performance for the system. Note that* $f_j, j = 1, 2, \cdots, M$ *can be treated as weights of the source, in the later congestion control simulation, we define* $f_j = 1/M, j = 1, 2, \cdots, M$.

On the other hand, saturation exists in sources and switch because of limited available bandwidth and queue buffer. Saturation can affect the efficiency of congestion control, so it should be taken into consideration. However, since the system we are dealing with is stochastic with white Gaussian noise inputs, it makes sense to discuss the saturation problem in terms of probability. It is well known that for a Gaussian random variable η with zero mean and variance σ^2,

$$Pr(\eta < 3\sigma) = 0.985$$

where Pr means probability.

Assume that the upper bound of source rate is $\bar{r}$, that is

$$r_k = sat(u_{k-d_k}), \tag{4.52}$$

$$sat(u_{k-d_k}) = \begin{cases} 0, \ if \ u_{k-d_k} < 0 \\ u_{k-d_k}, \ if \ 0 \le u_{k-d_k} \le \bar{r} \\ \bar{r}, \ if \ \bar{r} < u_{k-d_k}. \end{cases} \tag{4.53}$$

Observe that

$$u(k) = u_k - \mu, \tag{4.54}$$

$$u(k) = Fx(k) = F \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} \xi(k), \tag{4.55}$$

where $\xi(k)$ is a stochastic process with covariance $L_c(k) = E[\xi(k)\xi^T(k)]$ and $L_c(k) < P$. Under the condition that the system is stable, if

$$F \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} P \begin{bmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} F^T \le (\frac{\bar{r} - \mu}{3})^2 \tag{4.56}$$

then the probability that

$$u_{k-d_k} \leq \bar{r} \tag{4.57}$$

is 0.985.

Note that (4.56) can be also expressed as

$$\begin{pmatrix} (\bar{r}-\mu)^2/9 & F\begin{bmatrix} \mathbf{I}_m & 0 & \cdots & 0 \end{bmatrix} \\ \begin{bmatrix} \mathbf{I}_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} F^T & Q \end{pmatrix} \geq 0 \tag{4.58}$$

where $PQ = I$.

Similarly, assume that the upper bound of queue buffer is $\bar{q}$. We have that

$$\begin{aligned} q_k - q_d &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x(k) & (4.59) \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} \xi(k) & (4.60) \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \xi(k). & (4.61) \end{aligned}$$

Then, if

$$\begin{pmatrix} (\bar{q}-q_d)^2/9 & \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & Q \end{pmatrix} \geq 0, \tag{4.62}$$

then the probability that

$$q_k < \bar{q} \tag{4.63}$$

is 0.985.

4.4.2 Simulation Studies

Firstly, we consider a congested switch with one source. We adopt the similar parameters as given by [79].

The bandwidth available for ABR traffic $b_0 = 1500$ cells/s.
The maximum rate $R_{1,max} = 2b_0 = 3000$ cells/s.
The buffer length $\bar{q} = 10000$ cells/s.
The buffer set point $q_0 = (1/2)y_{max} = 5000$ cells.
The controller cycle time $T = 1$ ms.
The maximum delay on the path $m = 10$ms (including delay in the forward and backward paths).

Here, the delay is unknown and time-varying. We assume that the link capacity is a 2^{nd} order auto-regressive (AR) process with parameters $l_1 = l_2 = 0.4$ and the driving zero-mean Gaussian white noise process has a variance equal to 1 [79]. By using the LMI toolbox of the MATLAB package and Algorithm 4.3.1, the following feedback controller is obtained with a chosen $\gamma = 3.25$:

$$F = [-0.0114 \;\; 0.1210 \;\; 0.1379].$$

Figure 4.1 shows the buffer occupancy trajectory for time-variant return path delays between 0 and 9 ms, while keeping VBR delays in the forward path fixed at 1 ms. The desired equilibrium at $q_d = 5000$ cells is clearly maintained. Figure 4.2 shows the control signal $u(k)$ calculated by the switch.

Further, we will consider a more realistic example, with 100 sources feeding into the same congested switch and the system starts at the equilibrium. We use the same

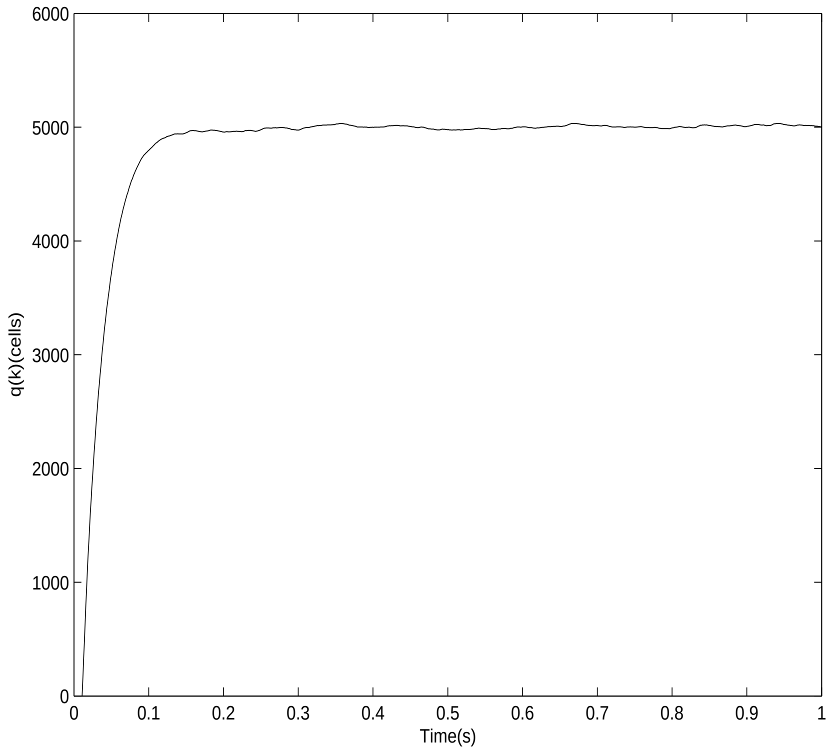

Figure 4.1: Buffer-level vs time.

parameters as in the previous example, and fix the delay in the forward path to be 1 ms.

Since the same controller that stabilizes one source stabilizes multiple sources, the same controller, $F = [-0.0114 \;\; 0.1210 \;\; 0.1379]$ will be used (see Remark 4.4.3). Ten additional sources join in the network sequentially at 200-ms intervals. The results of the simulation are shown in Figure 4.3 and Figure 4.4. Figure 4.3 shows the result under the condition of constant bandwidth. As each source joins the switch, a small glitch can be observed. This is the effect of delays in updating the weights. When a new source is connected, the switch will compute new weights and send them to the sources. However, the updated weights and the reaction of the sources to those updated weights are delayed and therefore, for a brief period of time, the sum of the weights may not be equal to one. After all switches are updated, the control scheme kicks in and brings the buffer to the desired set point. After considering the

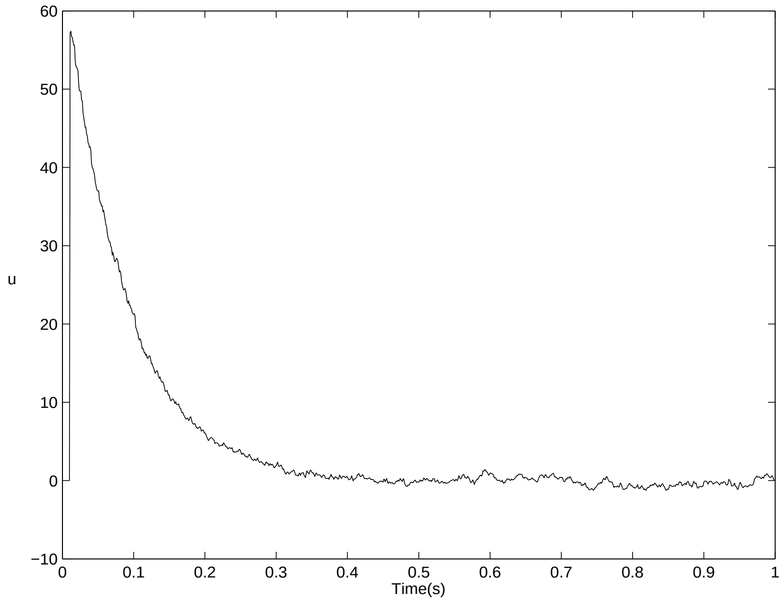

Figure 4.2: Control Signal.

random nature of the bandwidth available, that is

$$\zeta_{k+1} = 0.4\zeta_k + 0.4\zeta_{k-1} + w_k$$

where w_k is a white gaussian noise, the result is shown in Figure 4.4.

When buffer length is near to the buffer set point, saturation may happen during transmission. We assume the buffer length $\bar{q} = 5050\ cells/s$ and we want the probability of the queue length saturation is below 0.015. By incorporating (4.62) in the optimization, the optimal state feedback gain is

$$F = \begin{bmatrix} -0.0117 & 0.1198 & 0.1382 \end{bmatrix}.$$

We carry out 50 simulations and compute the total number of instants where queue buffer is saturated. Figure 4.5 is the result of one simulation, where dash line is the result of the design where the upper bound of queue length is taken into consideration and solid line is the result without considering the upper bound of

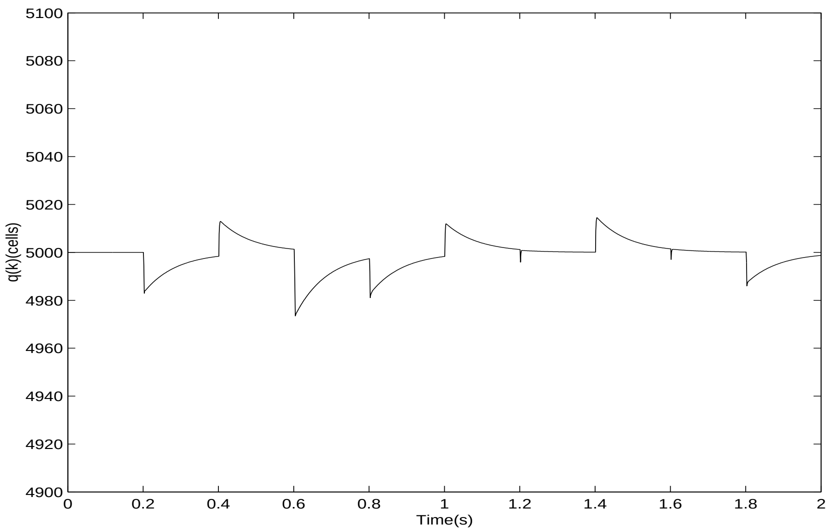

Figure 4.3: Buffer-level vs time.

queue length. Figure 4.6 shows the control signal $u(k)$ of one source calculated by switch, the curve is step-wise. The result shows that the total number of saturation points for the case when buffer saturation is not considered in congestion control design is 1926 in the 50 simulations. The number is reduced to 1758 when the buffer saturation limit is considered and the percentage of the number of the unsaturated times over total number of times

$$\frac{number\ of\ unsaturated\ times}{total\ sampling\ points} = 98.242\%$$

is very near to the theoretical value of 98.5%.

4.5 Conclusion

This chapter studied the H_2 control problem for systems with time-delay in input and its applications in congestion control of ATM network. State augmentation was applied to convert the delay problem into a delay-free control problem for systems with parameter uncertainties. An optimization approach for solving bilinear ma-

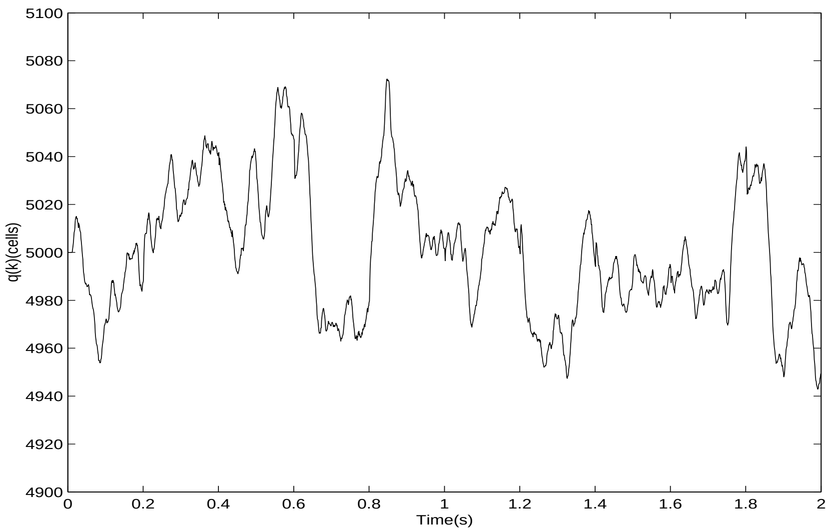

Figure 4.4: Buffer-level vs time.

trix inequalities was proposed and a controller with a guaranteed H_2 performance was derived. The congestion control of ATM network has been formulated as an H_2 control problem for linear systems with delays and solved using the proposed H_2 control design. An example was given to demonstrate the effectiveness of the controller.

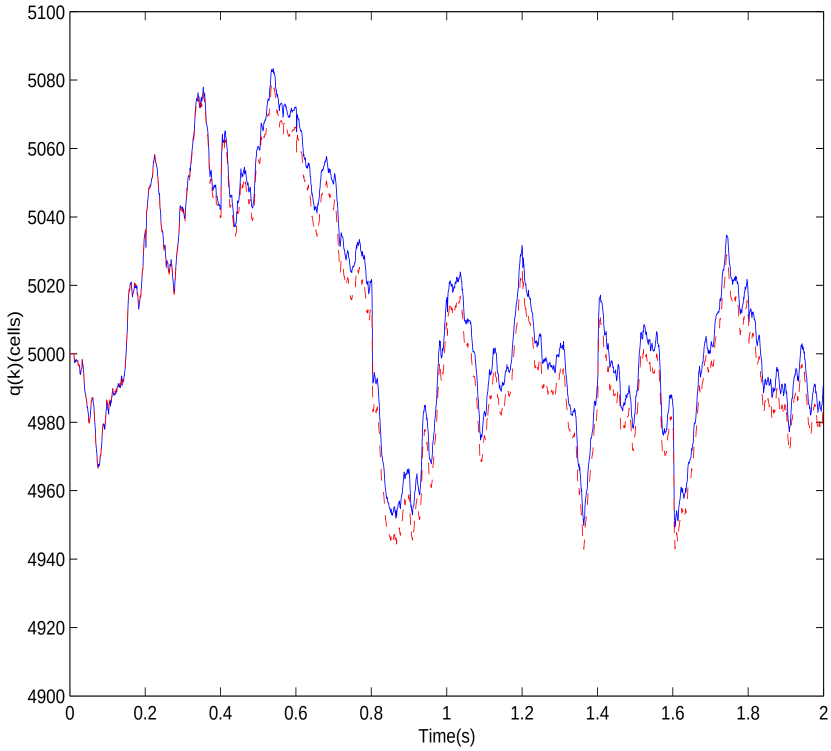

Figure 4.5: Buffer-level vs time.

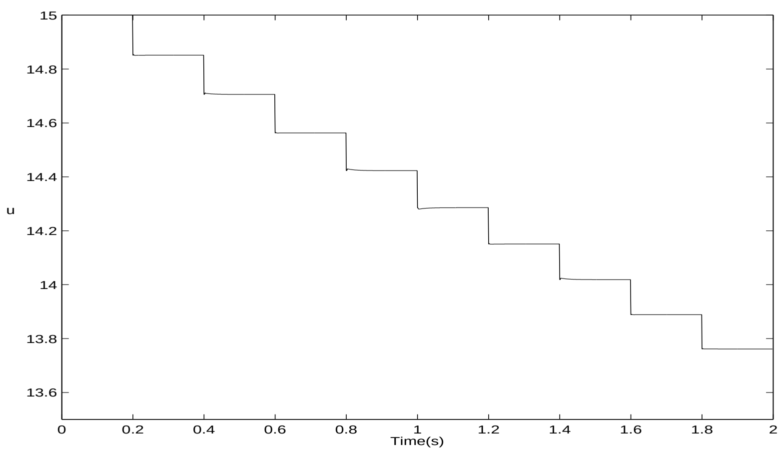

Figure 4.6: Control Signal.

Chapter 5

H_∞ Control for Systems with Time-variant Input Delay

5.1 Introduction

In the last chapter, we have discussed the state feedback H_2 control problem for systems with time delay in input. In this chapter, we will continue to study the state feedback H_∞ control problem for the same system.

There have been extensive studies on stability of time-delay systems in the past decade [21, 94]. Existing results based on Lyapunov-Krasovskii functionals can be classified into delay independent [21] and delay dependent [8, 12, 24, 47]. Delay independent results imply that the stability of the system is guaranteed regardless the delay. Delay dependent results have been given for both unknown constant delay [14, 47, 94] and time-varying bounded delay [8, 12, 24]. In addition to stability, H_2 and H_∞ performances have also been studied. To the best of our knowledge, most of the works are discussed for continuous-time systems. In this chapter, we will discuss the H_∞ control problem for discrete-time systems.

In this chapter, we discuss systems with unknown time-variant but bounded delay. A state feedback delay-dependent H_∞ controller is derived by a proper choice of Lyapunov-Krasovskii functional. A controller with guaranteed H_∞ performance is given in terms of LMIs.

We also apply our approach to congestion control in ATM networks. The mathematical model of the problem description and assumptions are adopted from [2] with the following modifications: (i) the time invariant delays in [2, 3] are now time-variant; (ii) the zero-mean i.i.d noise introduced in the AR model for the description of stochastic available bandwidth is now replaced with an energy bounded noise. These modifications allows wider class of modelling errors that are more realistic. Based on the assumption that w_k in the higher priority source's IIR model is an energy bounded deterministic signal in ATM networks, we can model the congestion control problem as an H_∞ control problem.

5.2 Problem Statement

Consider the input delay system

$$x(k+1) = Ax(k) + B_2u(k-d_k) + B_1w(k) \tag{5.1}$$

$$z(k) = C_1x(k) + D_2u(k-d_k) + D_1w(k) \tag{5.2}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $w(k) \in \mathbb{R}^p$ are system state, control input and exogenous output(noise), respectively. d_k is an unknown time-variant delay satisfying $0 < d_k \leq \bar{d}$ with known upper bound $\bar{d}$.

Consider a state feedback controller $u(k) = Fx(k)$. Then, the closed-loop system is

$$\begin{cases} x(k+1) &= Ax(k) + B_2Fx(k-d_k) + B_1w(k), \\ z(k) &= C_1x(k) + D_2Fx(k-d_k) + D_1w(k). \end{cases} \tag{5.3}$$

The H_∞ control problem can be described as:

Find an appropriate controller $u(k) = \mathcal{F}(x(k))$, so that the closed-loop system is asymptotically stable and satisfies

$$||z||_2 < \gamma||w||_2 \tag{5.4}$$

for any non-zero $w \in \ell_2[0,\infty)$ under zero initial condition ($x(k) = 0,\ -d_0 \le k \le 0$), where $\gamma > 0$ is a given scalar.

5.3 Design of State-feedback H_∞ Controller

Before dealing with the above H_∞ control problem, we first consider the stability of the unforced system

$$x(k+1) = Ax(k) + A_1x(k-d_k) \tag{5.5}$$

where $0 < d_k \le \bar{d}$. The following gives a sufficient condition for the stability of (5.5).

Lemma 5.3.1. *The system (5.5) is asymptotically stable for any $0 < d_k \le \bar{d}$ if there exist matrices $P > 0$, $Q > 0$, S, R and Z such that*

$$\pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ * & \pi_{22} \end{bmatrix} < 0, \quad \begin{bmatrix} S & R \\ R^T & Z \end{bmatrix} \ge 0 \tag{5.6}$$

where

$$\begin{aligned}
\pi_{11} &= A^T PA - P + \bar{d}S + R + R^T + \bar{d}Q + \bar{d}(A-I)^T Z(A-I), \\
\pi_{12} &= A^T PA_1 + \bar{d}(A-I)^T ZA_1 - R, \\
\pi_{22} &= A_1^T (P + \bar{d}Z)A_1 - Q.
\end{aligned}$$

Proof: Define a Lyapunov functional candidate as

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{5.7}$$

where

$$\begin{aligned}
V_1(k) &= x^T(k)Px(k), \\
V_2(k) &= \sum_{j=-\bar{d}}^{-1} \sum_{i=k+j+1}^{k} \Delta x^T(i) Z \Delta x(i), \\
V_3(k) &= \sum_{i=k-d_k}^{k-1} x^T(i)Qx(i), \\
V_4(k) &= \sum_{\theta=-\bar{d}+1}^{-1} \sum_{l=k+\theta}^{k-1} x^T(l)Qx(l)
\end{aligned}$$

and $\Delta x(i) = x(i) - x(i-1)$.

Observe that

$$x(k) - x(k-d_k) = \sum_{i=k-d_k+1}^{k} \Delta x(i).$$

Then,

$$x(k+1) = (A + A_1)x(k) - A_1 \sum_{i=k-d_k+1}^{k} \Delta x(i) = (A + A_1)x(k) - A_1 g_x(k)$$

where $g_x(k) = \sum_{i=k-d_k+1}^{k} \Delta x(i)$.

Hence,

$$\begin{aligned}\Delta V_1(k) &= V_1(k+1) - V_1(k)\\ &= [x^T(k)(A+A_1)^T - g_x^T(k)A_1^T]P[(A+A_1)x(k) - A_1 g_x(k)] - x^T(k)Px(k)\\ &= x^T(k)[(A+A_1)^T P(A+A_1) - P]x(k) - 2x^T(k)(A+A_1)^T PA_1 g_x(k)\\ &\quad + g_x^T(k)A_1^T PA_1 g_x(k). \end{aligned} \tag{5.8}$$

By taking into account (5.6),

$$\begin{aligned}&-2x^T(k)(A+A_1)^T PA_1 g_x(k)\\ \le\ &\sum_{i=k-d_k+1}^{k} \begin{pmatrix} x(k)\\ \Delta x(i)\end{pmatrix}^T \begin{pmatrix} S & R\\ R^T & Z\end{pmatrix} \begin{pmatrix} x(k)\\ \Delta x(i)\end{pmatrix} - x^T(k)(A+A_1)^T PA_1 \sum_{i=k-d_k+1}^{k} \Delta x(k)\\ =\ &\sum_{i=k-d_k+1}^{k} [x^T(k)Sx(k) + 2x^T(k)R\Delta x(i) + \Delta x^T(i)Z\Delta x(i)]\\ &-x^T(k)(A+A_1)^T PA_1 \sum_{i=k-d_k+1}^{k} \Delta x(k)\\ =\ & d_k x^T(k)Sx(k) + 2x^T(k)[R - (A+A_1)^T PA_1] \sum_{i=k-d_k+1}^{k} \Delta x(i)\\ &+ \sum_{i=k-d_k+1}^{k} \Delta x^T(i)Z\Delta x(i)\\ \le\ & \bar{d}x^T(k)Sx(k) + 2x^T(k)[R - (A+A_1)^T PA_1] \sum_{i=k-d_k+1}^{k} \Delta x(i)\\ &+ \sum_{i=k-\bar{d}+1}^{k} \Delta x^T(i)Z\Delta x(i).\end{aligned}$$

Also,

$$\begin{aligned}
\Delta V_2(k) &= V_2(k+1) - V_2(k) \\
&= \bar{d}\Delta x^T(k+1)Z\Delta x(k+1) - \sum_{i=k-\bar{d}+1}^{k} \Delta x^T(i)Z\Delta x(i) \\
&= \bar{d}[(A-\mathbf{I})x(k) + A_1 x(k-d_k)]^T Z[(A-\mathbf{I})x(k) + A_1 x(k-d_k)] \\
&\quad - \sum_{i=k-\bar{d}+1}^{k} \Delta x^T(i)Z\Delta x(i), \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
\Delta V_3(k) &= V_3(k+1) - V_3(k) \\
&= [\sum_{i=k+1-d_{k+1}}^{k} - \sum_{i=k-d_k}^{k-1}]x^T(i)Qx(i) \\
&\leq x^T(k)Qx(k) - x^T(k-d_k)Qx(k-d_k) + \sum_{l=k+1-\bar{d}}^{k-1} x^T(l)Qx(l), \qquad (5.10)
\end{aligned}$$

and

$$\Delta V_4(k) = (\bar{d}-1)x^T(k)Qx(k) - \sum_{l=k-\bar{d}+1}^{k-1} x^T(l)Qx(l). \qquad (5.11)$$

It then follows from (5.8)-(5.11) that along the state trajectory of (5.5),

$$\begin{aligned}
\Delta V(k) &= V(k+1) - V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k) \\
&\leq \begin{bmatrix} x(k) \\ x(k-d_k) \end{bmatrix}^T \begin{bmatrix} \pi_{11} & \pi_{12} \\ * & \pi_{22} \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-d_k) \end{bmatrix}.
\end{aligned}$$

Therefore, the system is robustly stable for any $0 < d_k \leq \bar{d}$ if $\pi < 0$. This completes the proof. □

Next, we consider the H_∞ performance problem for the system

$$x(k+1) = Ax(k) + A_1 x(k-d_k) + B_1 w(k), \tag{5.12}$$

$$z(k) = C_1 x(k) + E_1 x(k-d_k) + D_1 w(k), \tag{5.13}$$

where $w \in \ell_2[0,\infty)$ and $1 \leq d_k \leq \bar{d}$. Given a scalar $\gamma > 0$, the system is said to have the H_∞ performance γ if the system is asymptotically stable and under zero initial condition $(x(k) = 0, \ -d_0 \leq k \leq 0)$,

$$||z||_2 < \gamma ||w||_2$$

for any non-zero w and $1 \leq d_k \leq \bar{d}$.

Theorem 5.3.1. *Given a scalar $\gamma > 0$, the system (5.12)-(5.13) has the H_∞ performance γ if there exist matrices P, Q, S, R and Z such that the following matrix inequalities hold:*

$$\Gamma = \begin{bmatrix} M & -R & 0 & A^T P & \bar{d}(A-I)^T Z & C_1^T \\ * & -Q & 0 & A_1^T P & \bar{d} A_1^T Z & D_1^T \\ * & * & -\gamma^2 I & B_1^T P & \bar{d} B_1^T Z & E_1^T \\ * & * & * & -P & 0 & 0 \\ * & * & * & * & -\bar{d} Z & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{5.14}$$

$$\begin{bmatrix} S & R \\ R^T & Z \end{bmatrix} \geq 0, \tag{5.15}$$

where

$$M = -P + \bar{d} S + R + R^T + \bar{d} Q. \tag{5.16}$$

Proof: In fact, it is easy to verify that along the state trajectory of (5.12)-(5.13),

$$\Delta V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \leq \begin{bmatrix} x(k) \\ x(k-d_k) \\ w(k) \end{bmatrix}^T \Gamma \begin{bmatrix} x(k) \\ x(k-d_k) \\ w(k) \end{bmatrix}$$

where

$$\Gamma_1 = \begin{bmatrix} \pi_{11} + C_1^T C_1 & \pi_{12} + C_1^T E_1 & A^T P B_1 + \bar{d}(A-I)^T Z B_1 + C_1^T D_1 \\ * & \pi_{22} + E_1^T E_1 & A_1^T P B_1 + \bar{d} A_1^T Z B_1 + E_1^T D_1 \\ * & * & B_1^T P B_1 + \bar{d} B_1^T Z B_1 + D_1^T D_1 - \gamma^2 I \end{bmatrix}$$

and π_{11}, π_{12} and π_{22} are as in (5.6). By the Schur complement, we know that $\Gamma_1 < 0$ if and only if (5.14) holds. Further, (5.6) is implied by (5.14)-(5.15). Thus, (5.14)-(5.15) implies the stability of the system and

$$\Delta V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) < 0, \ \forall (x(k), x(k-d_k), w(k)) \neq 0.$$

That is, $||z||_2 < \gamma ||w||_2$ by considering the zero initial condition. This completes the proof.

We now apply Theorem 5.3.1 to the system (5.3). Then, the system (5.3) has the

H_∞ performance γ if

$$\begin{bmatrix} M & -R & 0 & A^TP & \bar{d}(A-I)^TZ & C_1^T \\ * & -Q & 0 & F^TB_2^TP & \bar{d}F^TB_2^TZ & F^TD_2^T \\ * & * & -\gamma^2 I & B_1^TP & \bar{d}B_1^TZ & E_1^T \\ * & * & * & -P & 0 & 0 \\ * & * & * & * & -\bar{d}Z & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{5.17}$$

$$\begin{bmatrix} S & R \\ R^T & Z \end{bmatrix} \geq 0, \tag{5.18}$$

where

$$M = -P + \bar{d}S + R + R^T + \bar{d}Q.$$

Pre- and post-multiply (5.17) by $diag\{P^{-1}, P^{-1}, I, P^{-1}, I, I\}$ and $diag\{P^{-1}, P^{-1}, I, P^{-1}, I, I\}$ and let

$$\bar{S} = P^{-1}SP^{-1},\ \bar{R} = P^{-1}RP^{-1},\ Y = P^{-1},\ H = Z^{-1},\ \bar{Q} = P^{-1}QP^{-1}.$$

It results that

$$\begin{bmatrix} \bar{M} & -\bar{R} & 0 & YA^T & \bar{d}Y(A-I)^T & YC_1^T \\ * & -\bar{Q} & 0 & LB_2^T & \bar{d}LB_2^T & LD_2^T \\ * & * & -\gamma^2 I & B_2^T & \bar{d}B_2^T & D_1^T \\ * & * & * & -Y & 0 & 0 \\ * & * & * & * & -\bar{d}H & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{5.19}$$

where

$$L = YF^T, \ \ \bar{M} = -Y + \bar{d}\bar{S} + \bar{R} + \bar{R}^T + \bar{d}\bar{Q}.$$

Also, multiply (5.18) from the left and the right by $diag\{P^{-1}, P^{-1}\}$ and $diag\{P^{-1}, P^{-1}\}$, respectively. We get

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix} \begin{bmatrix} S & R \\ R^T & Z \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix} = \begin{bmatrix} \bar{S} & \bar{R} \\ \bar{R}^T & YH^{-1}Y \end{bmatrix} \geq 0. \tag{5.20}$$

Note that $(Y-H)H^{-1}(Y-H) \geq 0$, i.e. $YH^{-1}Y \geq 2Y - H$. Hence, (5.20) can be implied by

$$\begin{bmatrix} \bar{S} & \bar{R} \\ \bar{R}^T & 2Y - H \end{bmatrix} \geq 0. \tag{5.21}$$

Therefore, the system (5.3) has the H_∞ performance γ if the LMIs (5.19) and (5.21) admit a solution $(\bar{S}, \bar{R}, \bar{Q}, H, Y, L, Z)$. In this situation, a suitable state feedback gain is given by

$$F = L^T Y^{-1}. \tag{5.22}$$

In the following section, we shall apply this result to design H_∞ congestion control laws.

5.4 Simulation Studies for Congestion Control in ATM Networks

In this section, we revisit the ATM congestion control model in Section 4.4.1. Assume that w_k is an energy bounded deterministic signal instead of a white Gaussian noise with known statistics, then the congestion control problem can be formulated as an H_∞ control problem. The objective is to achieve both the system stability and the performance index

$$J_\infty = \sup_{w \neq 0} \frac{\sum_{k=1}^{\infty}[(q_k - q_d)^2 + \lambda^2(r_k - \mu_k)^2]}{\sum_{k=1}^{\infty} w_k^2} < \gamma^2 \tag{5.23}$$

for some pre-specified $\gamma > 0$, where q_d is the desired queue length, λ is a constant and $w \in \ell_2[0, \infty)$. The first term of the numerator represents a penalty for deviating from a desired queue length. The second term is a measure of the quality with which the input rate tracks the available link capacity, and λ is the weighting to balance the importance of these two terms.

Introduce

$$\begin{aligned} x(k) &= [(q_k - q_d), \zeta_{k+1-p_1}, \cdots, \zeta_k]^T, \\ u(k) &= u_k - \mu, \end{aligned} \tag{5.24}$$

and let $z_1(k) = q_k - q_d$, $z_2(k) = r_k - \mu_k = u_{k-d_k} - \zeta_k$. The congestion control system can be put into the form (5.1)-(5.2) with $0 < d_k \leq \bar{d}$ where $\bar{d}$ is known upper bound and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & l_{p_1} & l_{p_1-1} & l_{p_1-2} & \cdots & l_1 \end{bmatrix}, \quad (5.25)$$

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \quad (5.26)$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \quad (5.27)$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & -e \end{bmatrix}, \quad (5.28)$$

$$D_2 = \begin{bmatrix} 0 & e \end{bmatrix}^T, \quad (5.29)$$

$$D_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T. \quad (5.30)$$

5.4.1 Single Source Case

Using the parameters of the simulation of the last chapter except the assumption that in this chapter $w \in \ell_2[0, \infty)$ is energy bounded. By using the LMI toolbox of the MATLAB package, the following feedback controller is obtained:

$$\gamma = 500, \;\; F = \begin{bmatrix} -0.0396 & 0.1827 & 0.0608 \end{bmatrix}.$$

Let the sum of return and forward delay vary between 0 and 10 ms. The buffer level response is shown in Figure 5.1 and the corresponding control signal $u(k)$ calculated by switch is shown in Figure 5.2.

Clearly the system has reached and maintained the equilibrium $q_d = 5000$ cells

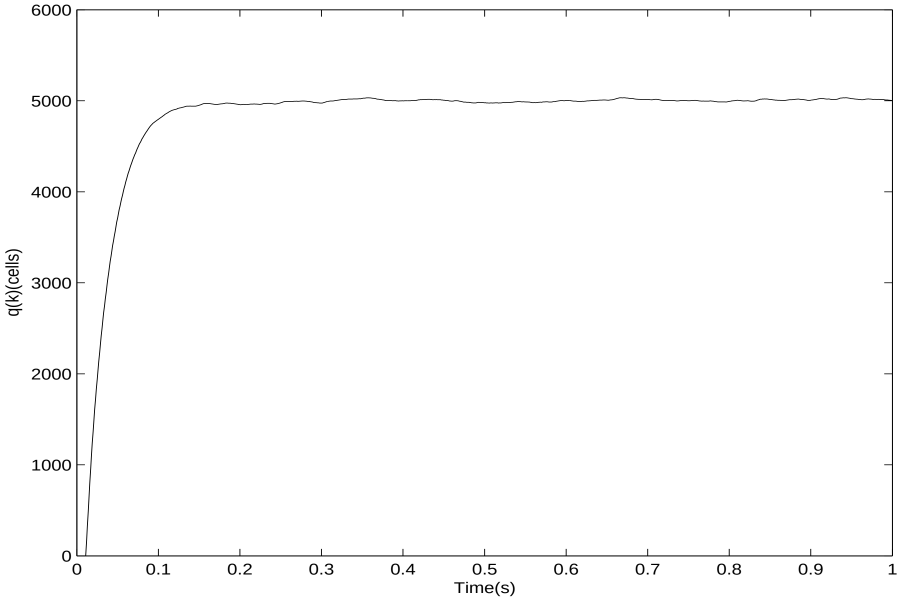

Figure 5.1: Buffer level $q(k)$ with single source connection time-variant delay randomly varying between 0 and 10 ms.

nicely under the random disturbance of other high priority traffic. The transient response result is quite similar to [79].

5.4.2 Multiple Sources Case

In this subsection, we will discuss the case when there are multiple sources in the ATM networks. The example is similar to the multiple source example used in [79]. We assume that there are 100 sources feeding into the same congested switch. We will use the same parameters as in the single source example. The maximum delay among all connections is 10 ms with each connection's delay varying randomly between 0 and 10 ms. Since the same controller that stabilizes one source stabilizes multiple sources [79] (see also Remark 4.4.3), the same controller gain $F = \begin{bmatrix} -0.0396 & 0.1827 & 0.0608 \end{bmatrix}$ will be used.

The system starts at equilibrium with 100 sources feeding into the congested switch. Then ten additional sources join in sequentially at 200-ms intervals. The results of

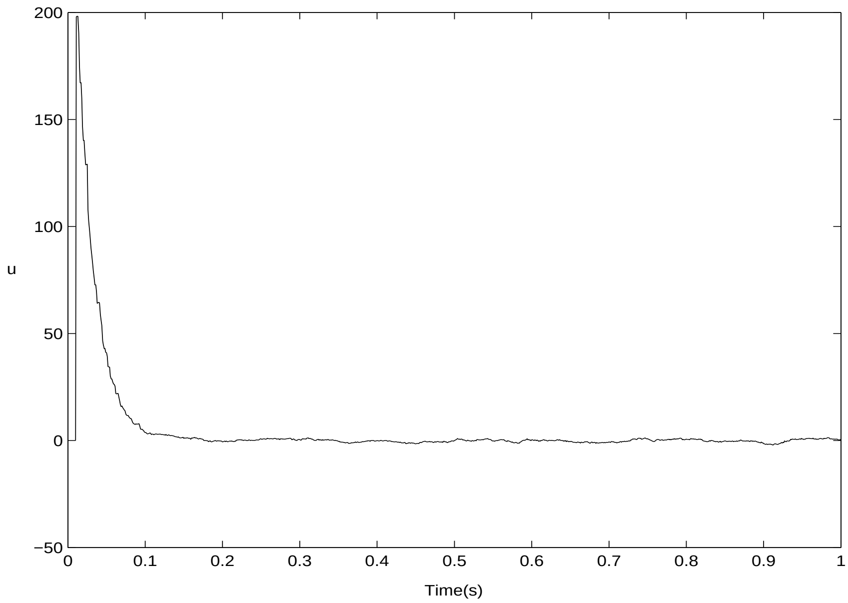

Figure 5.2: Control Signal.

this simulation are shown in Figure 5.3 and Figure 5.4. Figure 5.3 shows the result under the condition of constant bandwidth. There is a small glitch when a new source joins the switch. This is the effect of delays in updating the weight, which is very similar to the result in [79].

In the above simulations, we only consider the situation when $w_k = 0$ in the IIR model. Now we consider the random nature of the bandwidth available, that is, w_k is a white Gaussian noise and the simulation is shown in Figure 5.4. The buffer level clearly keeps tracking the equilibrium with small fluctuations to compensate for the random fluctuations in the bandwidth available. The control signal $u(k)$ of one source calculated by the switch is shown in Figure 5.5. The curve is step-wise and jumps happen at the moment when new source joins in the network. The simulations can demonstrate the promising robust performance of the controller.

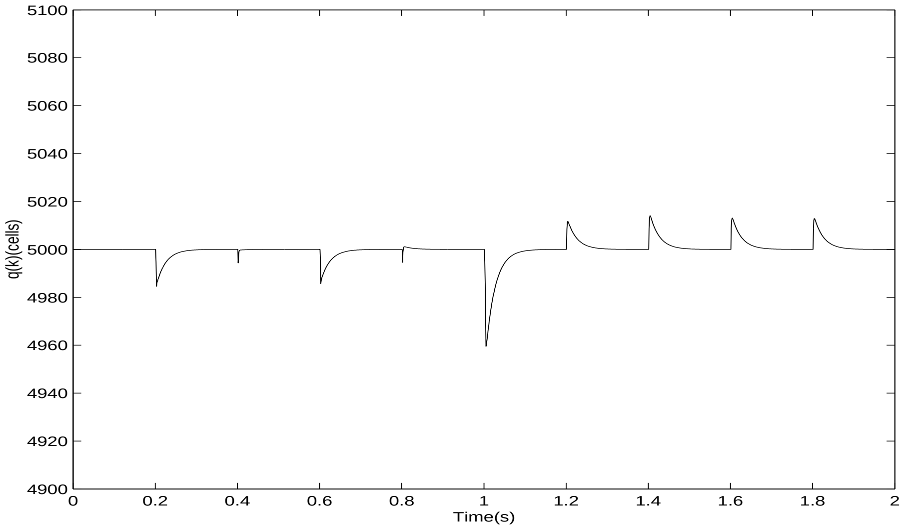

Figure 5.3: Buffer level $q(k)$ as ten additional sources join the congested switch sequentially at 200-ms intervals. The bandwidth available is a constant.

5.5 Conclusion

This chapter has discussed the delay-dependent H_∞ control problem for systems with time-variant input delay. State feedback control is considered and a Lyapunov Krasovskii functional is introduced to provide a sufficient condition for the stability as well as the H_∞ performance of the closed-loop system. As an important application, the issue of ATM network congestion control with explicit rate feedback has been discussed. The ATM network is modelled as a system with time delay in input and we study the performance control under an H_∞ criterion.

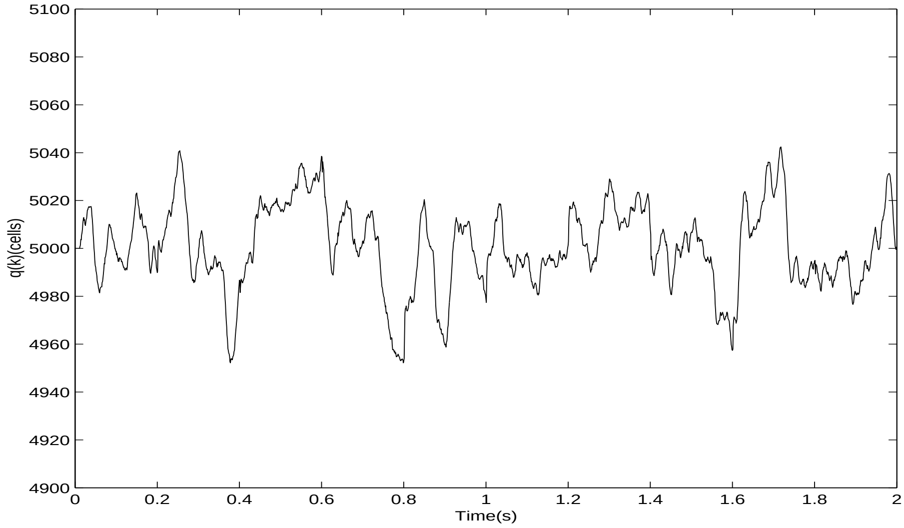

Figure 5.4: Buffer level $q(k)$ as ten additional sources join the congested switch sequentially at 200-ms intervals. The bandwidth available is a function of nominal value 1500 plus a 2^{nd} order AR disturbance process.

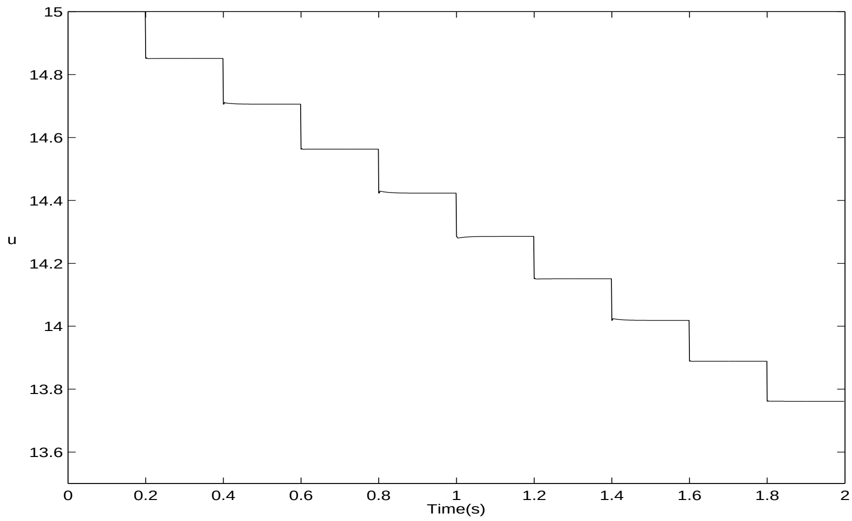

Figure 5.5: Control Signal.

Chapter 6

Linear Quadratic Gaussian Control for Linear Systems with Multiple I/O Delays

6.1 Introduction

In this chapter and Chapter 7, we shall discuss linear systems with multiple input delays. Different from the last two chapters, the study here is limited to systems with time-invariant and known delays.

Since the introduction of Smith predictor [80], there have been a lot of works on optimal control for systems with I/O delays [7, 33, 83]. In recent years, multiple time-delay issues have attracted significant research interest.

The LQG (H_2) control problem for systems with multiple input delays can be discussed in frequency domain and time domain. Most of the works are for continuous-time systems. Grimble and Hearns [33], Moelja, Meinsma and Kuipers [62] solve the LQG problem in frequency domain. Mirkin et al. [55] [57] solve the H_2 control by extracting Smith predictor and observer-predictor from the system model. Moelja and Meinsma [61] provide a solution by converting the H_2 optimal problem to an equivalent H_2 regulator one and tackling the regulator problem in time domain.

Compared to state feedback control, output feedback control needs state estimation, when there are delays in the observers, the difficulty aggregates. When the observations are multiple delay dependent, Kalman filter cannot be applied directly. Nagpal and Ravi [67] solve the continuous-time H_∞ control and estimation problem with single delay in measurement. In the work of Shaked and de Souza [76], H_∞ control for systems with single delayed measurement is discussed, however, only a sufficient stability condition is provided. Moelja and Meinsma [61] give a solution for the H_2 optimal control for continuous-time systems with multiple i/o delays, nevertheless, the solution is not explicit in the multiple delay case and the result is derived under some extra assumptions. In [102], the H_∞ filtering problem for continuous-time systems with single measurement delay is discussed.

In this chapter, we focus on the finite horizon LQG (H_2) problem for discrete-time systems with multiple input/output delays based on the LQR result in Zhang et al. [97]. We adopt the approach of reorganized innovation analysis [50]. A separation principle is established which converts the problem into an LQR control and an H_2 filtering problems. We address the LQR problem using the duality between the LQR problem for systems with multiple input delays and a smoothing problem for a backward stochastic delay-free system combining the LQR result with the Kalman filter for systems with measurement delays, a solution to the LQG problem is given. Compared with the state augmentation approach, our result is economic in computation where only two Riccati equations are needed and they have the same dimension with the original system (ignoring the delays).

6.2 Problem Statement

We consider the following discrete linear system with multiple input delays

$$\begin{aligned} x(k+1) &= Ax(k) + \sum_{i=0}^{l} B_i u_i(k-h_i) + w(k), \;\; l \geq 1, && (6.1) \\ y_{(i)}(k) &= C_{(i)} x(k-d_i) + v_{(i)}(k), \quad i = 0, \cdots, l, && (6.2) \end{aligned}$$

where $x(k) \in \mathbb{R}^n$, $u_i(k) \in \mathbb{R}^{m_i}$, $y_i(k) \in \mathbb{R}^{\iota_i}$, $w(k) \in \mathbb{R}^p$, $v_i(k) \in \mathbb{R}^{\varrho_i}$, and

$$\left\langle \begin{bmatrix} x(0) \\ w(k) \\ v_{(i)}(k) \end{bmatrix}, \begin{bmatrix} x(0) \\ w(s) \\ v_{(i)}(s) \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_w \delta_{k,s} & 0 \\ 0 & 0 & Q_{v_{(i)}} \delta_{k,s} \end{bmatrix}, \qquad (6.3)$$

where $\delta_{k,s} = \begin{cases} 1, & k = s, \\ 0, & k \neq s. \end{cases}$

Without loss of generality, we assume that $0 = h_0 < h_1 < \cdots < h_l$ and $0 = d_0 < d_1 < \cdots < d_l$.

Let $y(k)$ be the observation of system (6.2) at time k, then $y(k)$ is given by

$$y(k) \triangleq \begin{cases} \begin{bmatrix} y_{(0)}(k) \\ \vdots \\ y_{(i-1)}(k) \end{bmatrix}, & d_{i-1} \leq k < d_i, \\ \begin{bmatrix} y_{(0)}(k) \\ \vdots \\ y_{(l)}(k) \end{bmatrix}, & k \geq d_l. \end{cases} \qquad (6.4)$$

The measurement-feedback control problem can now be stated as follows.

Find the input sequences $\{u_i(k) = \mathcal{F}_i(y(0), \cdots, y(k)\}$ *(*$i = 0, \cdots, l$*) such that the* *cost function*

$$J_N = E[x_{N+1}^T P_{N+1} x_{N+1} + \sum_{i=0}^{l} \sum_{k=0}^{N-h_i} u_i^T(k) R_i u_i(k) + \sum_{k=0}^{N} x^T(k) Q x(k)] \quad (6.5)$$

is minimized.

In the above, E is the mathematical expectation with respect to the random noises w and v, $N > h_d$ is an integer, x_{N+1} is the terminal state, i.e. $x_{N+1} = x(N+1)$, $P_{N+1} = P_{N+1}^T \geq 0$ reflects the penalty on the terminal state, the matrices as R_i, $i = 0, 1, \cdots, l$, are positive definite and the matrix Q is non-negative definite.

6.3 Solution to the LQG control

We first present the Kalman filtering formulation for the system (6.1)-(6.2). Suppose that $\hat{x}(k+1 \mid k+1)$ is the optimal estimation of state $x(k+1)$ given the measurements of $y(0), \cdots, y(k+1)$. Then, by projection formula, it follows that

$$\begin{aligned} \hat{x}(k+1 \mid k+1) &= A\hat{x}(k \mid k) + \sum_{i=0}^{l} B_i u_i(k-h_i) \\ &\quad + \langle x(k+1), \ \bar{w}(k+1) \rangle Q_{\bar{w}}^{-1}(k+1) \bar{w}(k+1), \end{aligned} \quad (6.6)$$

where $\bar{w}(k+1)$ is the innovation of the measurement $y(k+1)$, i.e.,

$$\bar{w}(k+1) = y(k+1) - \hat{y}(k+1 \mid k),$$

while $\hat{y}(k+1 \mid k)$ is the optimal linear estimation of $y(k+1)$ based on the measurements $y(0), \cdots, y(k)$ and $Q_{\bar{w}}(k+1)$ is the covariance matrix of the innovation

$\bar{w}(k+1)$. For the convenience of discussion, denote

$$K(k) = \langle x(k+1),\ \bar{w}(k+1)\rangle\, Q_{\bar{w}}^{-1}(k+1). \tag{6.7}$$

Then, it follows from (6.6) that

$$\hat{x}(k+1 \mid k+1) \quad = \quad A\hat{x}(k \mid k) + \sum_{i=0}^{l} B_i u_i(k-h_i) + K(k)\bar{w}(k+1). \tag{6.8}$$

Note that $\bar{w}(k)$ is a white noise with zero mean and covariance matrix of $Q_{\bar{w}}(k)$.

Remark 6.3.1. *(6.8) has a similar form as the standard Kalman filtering formulation, however, since the measurements are with multiple delays, there is no direct way to calculate the gain matrix of $K(k)$ as in the standard Kalman filtering where the gain matrix is computed by performing one Riccati equation. We shall present the approach of reorganized innovation analysis to solve the H_2 filtering problem.*

6.3.1 Optimal Measurement Feedback Controller

In this subsection, we will derive the measurement feedback controller. Firstly, let

$$\tilde{x}(k \mid k) = x(k) - \hat{x}(k \mid k), \tag{6.9}$$

where $\hat{x}(k \mid k)$ is the optimal filter of (6.8). It is obvious that $\tilde{x}(k \mid k)$, the filtering error, is uncorrelated with $\hat{x}(k \mid k)$. By some simple algebraic evaluations, we obtain that

$$J_N = J_N^0 + J_N^1, \tag{6.10}$$

where

$$\begin{aligned} J_N^0 &= E\left(\hat{x}_{N+1|N+1}^T P_{N+1} \hat{x}_{N+1|N+1} + \sum_{i=0}^{l} \sum_{k=0}^{N-h_i} u_i^T(k) R_i u_i(k) \right. \\ &\quad \left. + \sum_{k=0}^{N} \hat{x}^T(k \mid k) Q \hat{x}(k \mid k) \right), \\ J_N^1 &= E\left(\tilde{x}_{N+1}^T P_{N+1} \tilde{x}_{N+1} + \sum_{k=0}^{N} \tilde{x}^T(k) Q \tilde{x}(k)\right). \end{aligned} \tag{6.11}$$

Note that J_N^1 does not contain any control input u, the problem of minimizing J_N is converted into one that seeks the control input u such that J_N^0 is minimized, with the constraint of (6.8). On the other hand, the optimal solution to the minimization of J_N^1 is the optimal estimate $\hat{x}(k|k)$ of $x(k)$. Thus, the LQG problem is equivalent to an LQR control in conjunction with an optimal filtering problem, i.e., we establish a separation principle.

In the following, we first assume that $\hat{x}(k|k)$ is given and concerned with the LQR problem associated with J_N^0. The solution follows from [97] which will be given below. Our aim is to find the optimal controller of $u_i(\tau)$ ($i = 0, 1, \cdots, l,\ 0 \le \tau \le N$), in terms of the current estimated state $x(\tau|\tau)$. The problem can be addressed by shifting the time interval from $[0, h_l]$ to $[\tau, \tau + h_l]$ ([97]). To this end, we introduce the backward RDE equation

$$P_j^\tau = A^T P_{j+1}^\tau A + Q - K_j^\tau M_j^\tau (K_j^\tau)^T, \tag{6.12}$$

$$P_{N-\tau+1}^\tau = P_{N+1}, \ \tau > 0, \tag{6.13}$$

where

$$\bar{A}_j^\tau = A^T - K_j^\tau (B_j^\tau)^T, \tag{6.14}$$

$$K_j^\tau = A^T P_{j+1}^\tau B_j^\tau (M_j^\tau)^{-1}, \tag{6.15}$$

$$M_j^\tau = R_j^\tau + (B_j^\tau)^T P_{j+1}^\tau B_j^\tau, \tag{6.16}$$

and

$$B_j^\tau = \begin{cases} [B_0, \cdots, B_i], & h_i \le j < h_{i+1}, \\ [B_0, \cdots, B_l], & j \ge h_l, \end{cases} \tag{6.17}$$

$$R_j^\tau = \begin{cases} diag\{R_0, \cdots, R_i\}, & h_i \le j < h_{i+1}, \\ diag\{R_0, \cdots, R_l\}, & j \ge h_l. \end{cases} \tag{6.18}$$

Denote

$$\bar{A}_{j,j}^\tau \triangleq I, \tag{6.19}$$

$$\bar{A}_{j,m}^\tau \triangleq \bar{A}_j^\tau \cdots \bar{A}_{m-1}^\tau, \; m \ge j. \tag{6.20}$$

Then, the optimal LQR solution is given by [97]

$$u_i^*(k) = -\overbrace{[0 \; \cdots \; 0 \; I_m]}^{i+1 \text{ blocks}} \times \left(\left[\mathcal{F}_0^k(h_i)\right]^T \hat{x}(k \mid k) + \sum_{j=1}^{h_i} \left[\mathcal{F}_j^k(h_i)\right]^T \tilde{u}^{k*}(j-1) + \sum_{j=h_i+1}^{h_l} \left[\mathcal{S}_j^k(h_i)\right]^T \tilde{u}^{k*}(j-1) \right), \tag{6.21}$$

where

$$\tilde{u}^\tau(k) \triangleq \begin{cases} \sum_{j=i+1}^{l} B_j u_j(k+\tau-h_j), & h_i \le k < h_{i+1}, k+\tau \le N \\ 0, & k \ge h_l, \end{cases} \tag{6.22}$$

$$\mathcal{S}_j^\tau(0) = P_j^\tau(\bar{A}_{1,j}^\tau)^T B_0^\tau(M_0^\tau)^{-1}, \tag{6.23}$$

$$\mathcal{S}_j^\tau(k) = P_j^\tau[(\bar{A}_{k+1,j}^\tau)^T B_k^\tau(M_k^\tau)^{-1} - (\bar{A}_{k,j}^\tau)^T G^\tau(k) K_k^\tau], 0 < k \leq j-1, \tag{6.24}$$

$$\mathcal{F}_j^\tau(k) = [I_n - P_j^\tau G^\tau(j)]\bar{A}_{j,k}^\tau K_k^\tau, \; j \leq k \leq N, \tag{6.25}$$

and

$$G^\tau(k) = \sum_{j=1}^{k} (\bar{A}_{j,k}^\tau)^T B_{j-1}^\tau (M_{j-1}^\tau)^{-1} (B_{j-1}^\tau)^T \bar{A}_{j,k}^\tau. \tag{6.26}$$

What remains is to derive the optimal estimate $\hat{x}(k|k)$ which is given in the next subsection.

6.3.2 Calculation of the Kalman Filter $\hat{x}(k \mid k)$

It is easy to know that $\hat{x}(k \mid k)$ is not a standard Kalman filter as the measurements $y(0), \cdots, y(k+1)$ are with multiple time delays. In order to calculate $\hat{x}(k \mid k)$, we shall adopt the reorganized innovation analysis approach [50]. Note that the measurement sequence $\{y(s), \; s = 0, 1, \cdots, k\}$ contains the same information about the system as the reorganized sequence:

$$\{y_{l+1}(0), \; \cdots, y_{l+1}(k_l); \; \cdots ; \; y_i(k_i+1), \cdots, y_i(k_{i-1}); \; \cdots ; \\ y_1(k_1+1), \; \cdots, \; y_1(k)\}, \tag{6.27}$$

where $k_i = k - d_i$,

$$y_i(s) = \begin{bmatrix} y_{(0)}(s) \\ \vdots \\ y_{(i-1)}(s + d_{i-1}) \end{bmatrix}. \tag{6.28}$$

It is clear that

$$y_i(k) = C_i x(k) + v_i(k), \quad i = 1, \cdots, l+1, \tag{6.29}$$

where

$$C_i = \begin{bmatrix} C_{(0)} \\ \vdots \\ C_{(i-1)} \end{bmatrix}, \quad v_i(k) = \begin{bmatrix} v_{(0)}(k) \\ \vdots \\ v_{(i-1)}(k+d_{i-1}) \end{bmatrix}, \tag{6.30}$$

and $v_i(k)$ are white noises of zero means and covariances

$$Q_{v_i} \ = \ diag\{Q_{v_{(0)}}, \cdots, Q_{v_{(i-1)}}\}, i = 1, \ \cdots, l+1. \tag{6.31}$$

Now we have the following result.

Theorem 6.3.1. *Consider the system (6.1)-(6.2). The optimal filter $\hat{x}(k \mid k)$ is given by*

$$\hat{x}(k \mid k) = \left[I_n - P_1(k)C_1^T Q_{\bar{w}}^{-1}(k,1)C_1\right]\hat{x}(k,1) + P_1(k)C_1^T Q_{\bar{w}}^{-1}(k,1)y_1(k), \tag{6.32}$$

where $Q_{\bar{w}}(k,1) = C_1 P_1(k) C_1^T + Q_{v_1}$, while the estimator $\hat{x}(k,1)$ and the matrix $P_1(k)$ are computed by

$$\begin{aligned} \hat{x}(k_{i-1}+1, i) \ &= \ A_i(k_{i-1}+1, k_i+1)\hat{x}(k_i+1, i+1) \\ &\quad + \sum_{s=k_i+1}^{k_{i-1}} A_i(k_{i-1}+1, s+1)\left[K_i(s)y_i(s) + \tilde{u}(s)\right], \ i = l, l-1, \cdots, 1, \end{aligned} \tag{6.33}$$

where $A_i(m,m) = I_n$,

$$A_i(\zeta, m) = A_i(\zeta - 1)\cdots A_i(m), \quad \zeta \geq m, \tag{6.34}$$

$$\begin{aligned} \tilde{u}(s) &= \sum_{i=0}^{l} B_i u_i(s - h_i) && (6.35)\\ A_i(s) &= A - K_i(s)C_i, && (6.36)\\ K_i(s) &= AP_i(s)C_i^T Q_{\bar{w}}^{-1}(s,i), && (6.37)\\ Q_{\bar{w}}(s,i) &= C_i P_i(s) C_i^T + Q_{v_i}, && (6.38) \end{aligned}$$

and $P_i(s)$ *is the solution to the following Riccati equation,*

$$\begin{aligned} P_i(s+1) &= AP_i(s)A^T - AP_i(s)C_i^T Q_{\bar{w}}^{-1}(s,i)C_iP_i(s)A^T + Q_w,\\ P_i(k_i+1) &= P_{i+1}(k_i+1), \quad i = l, \cdots, 1. \end{aligned} \tag{6.39}$$

In the above the initial values $\hat{x}(k_l+1, l+1)$ *and* $P_{l+1}(k_l+1)$ *are the terminal values of the following standard Kalman filter*

$$\begin{aligned} \hat{x}(s+1, l+1) &= A_{l+1}(s)\hat{x}(s, l+1) + \tilde{u}(s) + K_{l+1}(s)y_{l+1}(s),\\ \hat{x}(0, l+1) &= 0, \end{aligned} \tag{6.40}$$

where $A_{l+1}(s) = A - K_{l+1}(s)C_{l+1}$, $K_{l+1}(s) = AP_{l+1}(s)C_{l+1}^T Q_{\bar{w}}^{-1}(s, l+1)$ *and* $P_{l+1}(s)$ *is computed by*

$$\begin{aligned} P_{l+1}(s+1) &= AP_{l+1}(s)A^T - AP_{l+1}(s)C_{l+1}^T Q_{\bar{w}}^{-1}(s,l+1)C_{l+1}P_{l+1}(s)A^T + Q_w,\\ P_{l+1}(0) &= \langle x(0), x(0)\rangle. \end{aligned} \tag{6.41}$$

Proof: We begin the proof by introducing the following definitions:

1) The estimator $\hat{\xi}(s,i)$ for $s > k_i + 1$ denotes the optimal estimation of $\xi(s)$ given the observation

$$\{y_{l+1}(0), \cdots, y_{l+1}(k_l); \quad \cdots; \quad y_i(k_i+1), \cdots, y_i(s-1)\}. \tag{6.42}$$

2) For $s = k_i + 1$, $\hat{\xi}(s,i)$ is the optimal estimation of $\xi(s)$ given the observation

$$\{y_{l+1}(0), \cdots, y_{l+1}(k_l); \quad \cdots; \quad y_{i+1}(k_{i+1}+1), \cdots, y_{i+1}(k_i)\}. \tag{6.43}$$

Similarly, $\hat{y}_i(\cdot, i)$ and $\hat{x}(\cdot, i)$ are defined. For $i = l+1$, it is clear that $\bar{w}(k, l+1)$ is the standard Kalman filtering innovation sequence for the system (6.1) and (6.29). Define

$$\bar{w}_i(s,i) \triangleq y(s) - \hat{y}_i(s,i). \tag{6.44}$$

In view of (6.29), it follows

$$\bar{w}_i(s,i) \quad = \quad C_i\tilde{x}(s,i) + v_i(s), \;\; i = 1, \cdots, l+1, \tag{6.45}$$

where

$$\tilde{x}(s,i) \quad = \quad x(s) - \hat{x}(s,i), \quad i = 1, \cdots, l+1, \tag{6.46}$$

is the one step ahead prediction error of the state $x(s)$ based on the observations (6.42) or (6.43). Thus, $\hat{x}(k \mid k)$ is the projection of the state $x(k)$ onto the Hilbert space spanned by the innovation sequence $\{\bar{w}_{l+1}(0, l+1), \cdots, \bar{w}_{l+1}(k_l, l+1); \cdots; \bar{w}_i(k_i+1, i), \cdots, \bar{w}_i(k_{i-1}, i); \cdots; \bar{w}_1(k_1+1, 1), \cdots, \bar{w}_1(k, 1)\}$. Since $\bar{w}$ is a white noise, the filter $\hat{x}(k \mid k)$ is calculated by using the projection formula as

$$\begin{aligned}
& \hat{x}(k \mid k) \\
= \;& Proj\{x(k) \mid \bar{w}_{l+1}(0, l+1), \cdots, \bar{w}_{l+1}(k_l, l+1); \cdots; \bar{w}_1(k_1+1, 1), \cdots, \bar{w}_1(k_1, 1)\} \\
& + Proj\{x(k) \mid \bar{w}_1(k, 1)\}. \\
= \;& \hat{x}(k, 1) + E\left[x(k)\bar{w}_1^T(k, 1)\right] Q_{\bar{w}}^{-1}(k, 1)\bar{w}_1(k, 1) \\
= \;& \hat{x}(k, 1) + P_1(k)C_1^T Q_{\bar{w}}^{-1}(k, 1)\left[y_1(k) - C_1\hat{x}(k, 1)\right] \\
= \;& \left[I_n - P_1(k)C_1^T Q_{\bar{w}}^{-1}(k, 1)C_1\right]\hat{x}(k, 1) + P_1(k)C_1^T Q_{\bar{w}}^{-1}(k, 1)y_1(k), \qquad (6.47)
\end{aligned}$$

which is (6.32). Similarly, $\hat{x}(s+1, i)$ is the projection of the state $x(s+1)$ onto the linear space spanned by the innovation $\{\bar{w}_{l+1}(0, l+1), \cdots, \bar{w}_{l+1}(k_l, l+1); \cdots; \bar{w}_i(k_i+1, i), \cdots, \bar{w}_i(s, i)\}$, it follows from the projection formula that

$$\begin{aligned}
& \hat{x}(s+1, i) \\
= \;& Proj\{x(s+1) \mid \bar{w}_{l+1}(0, l+1), \cdots, \bar{w}_{l+1}(k_l, l+1); \cdots; \bar{w}_i(k_i+1, i), \cdots, \bar{w}_i(s, i)\} \\
= \;& Proj\{x(s+1) \mid \bar{w}_{l+1}(0, l+1), \cdots, \bar{w}_{l+1}(k_l, l+1); \cdots; \bar{w}_i(k_i+1, i), \cdots, \bar{w}_i(s-1, i)\} \\
& + Proj\{x(s+1) \mid \bar{w}_i(s, i)\} \\
= \;& A\hat{x}(s, i) + \tilde{u}(s) + Proj\{w(s) \mid \bar{w}_{l+1}(0, l+1), \cdots, \bar{w}_{l+1}(k_l, l+1); \\
& \cdots; \bar{w}_i(k_i+1, i), \cdots, \bar{w}_i(s-1, i)\} + Proj\{x(s+1) \mid \bar{w}_i(s, i)\}. \qquad (6.48)
\end{aligned}$$

Noting that $w(s)$ is uncorrelated with the innovation $\{\bar{w}_{l+1}(0, l+1), \cdots, \bar{w}_{l+1}(k_l, l+1), \cdots; \bar{w}_i(k_i+1, i), \cdots, \bar{w}_i(s, i)\}$, we have

$$\begin{aligned}
\hat{x}(s+1, i) &= A\hat{x}(s, i) + A \cdot E\left[x(s)\bar{w}_i^T(s, i)\right] Q_{\bar{w}}^{-1}(s, i)\bar{w}_i(s) + \tilde{u}(s) \\
&= A\hat{x}(s, i) + AP_i(s)C_i^T Q_{\bar{w}}^{-1}(s, i)\left[y_i(s) - C_i\hat{x}(s, i)\right] + \tilde{u}(s), \qquad (6.49)
\end{aligned}$$

which can be rewritten as

$$\hat{x}(s+1,i) = A_i(s)\hat{x}(s,i) + K_i(s)y_i(s) + \tilde{u}(s), \tag{6.50}$$

with the initial value as $\hat{x}(k_i+1,i) = \hat{x}(k_i+1,i+1)$. Note that for each i, $k_i + 1 \le s \le k_i + d_i - d_{i-1} = k_{i-1}$. From (6.50), it follows that

$$\begin{aligned}
\hat{x}(k_{i-1}+1,i) &= A_i(k_{i-1})\hat{x}(k_{i-1},i) + K_i(k_{i-1})y_i(k_{i-1}) + \tilde{u}(k_{i-1}) \\
&= A_i(k_{i-1})A_i(k_{i-1}-1)\hat{x}(k_{i-1}-1,i) \\
&\quad + A_i(k_{i-1})K_i(k_{i-1}-1)y_i(k_{i-1}-1) + K_i(k_{i-1})y_i(k_{i-1}) \\
&\quad + A_i(k_{i-1})\tilde{u}(k_{i-1}-1) + \tilde{u}(k_{i-1}) \qquad (6.51) \\
&= \cdots \\
&= A_i(k_{i-1})A_i(k_{i-1}-1)\cdots A_i(k_i+1)\hat{x}(k_i+1,i) + \\
&\quad \sum_{j=k_i+1}^{k_{i-1}} A_i(k_{i-1})A_i(k_{i-1}-1)\cdots A_i(j+1)K_i(j)y_i(j) + \\
&\quad \sum_{j=k_i+1}^{k_{i-1}} A_i(k_{i-1})A_i(k_{i-1}-1)\cdots A_i(j+1)\tilde{u}(j) \\
&= A_i(k_{i-1}+1,k_i+1)\hat{x}(k_i+1,i) + \\
&\quad \sum_{j=k_i+1}^{k_{i-1}} A_i(k_{i-1}+1,j+1)K_i(j)y_i(j) + \\
&\quad \sum_{j=k_i+1}^{k_{i-1}} A_i(k_{i-1}+1,j+1)\tilde{u}(j), \qquad (6.52)
\end{aligned}$$

which is (6.33). Thus the proof is completed.

The main result is summarized as

Theorem 6.3.2. *(LQG Control)*

Consider the state-space model (6.1)-(6.2). Suppose the controller $u_i(k)$ is allowed *to be a causal linear function of the measurements* $y(0), \cdots, y(k)$, *i.e.,*

$$u_i(k) = \mathcal{F}_i(y(0), \cdots, y(k)). \tag{6.53}$$

Then the optimal solution that minimizes (6.5) is given by (6.21), where $\hat{x}(k \mid k)$ *is the Kalman filter calculated by Theorem 6.3.1.*

6.4 Examples

6.4.1 Simple Numerical Example

Consider the system (6.1) and (6.2) with $l = 2,\ A = 1,\ h_0 = 0,\ h_1 = 1,\ h_2 = 2,\ d_0 = 0,\ d_1 = 2,\ d_2 = 3,\ N = 100,\ B_i = 1,\ C_{(i)} = 2,\ i = 0, 1, 2$. The parameters of (6.3) and (6.5) are $P_{N+1} = 2,\ R_i = 1,\ i = 0, 1, 2,\ Q = 2$ and

$$\left\langle \begin{bmatrix} x(0) \\ w(k) \\ v_{(i)}(k) \end{bmatrix}, \begin{bmatrix} x(0) \\ w(s) \\ v_{(i)}(s) \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta_{k,s} & 0 \\ 0 & 0 & \delta_{k,s} \end{bmatrix}.$$

The initial state $x(0) = 50$. By applying Theorem 6.3.1, we obtain the controller:

$$\begin{aligned} u_i^*(k) &= -\overbrace{[0 \ \cdots \ 0 \ I_m]}^{i+1 \text{ blocks}} \times \\ &\quad \left(\left[\mathcal{F}_0^k(h_i)\right]^T \hat{x}(k \mid k) + \sum_{j=1}^{h_i} \left[\mathcal{F}_j^k(h_i)\right]^T \tilde{u}^{k*}(j-1) \right. \\ &\quad \left. + \sum_{j=h_i+1}^{h_l} \left[\mathcal{S}_j^k(h_i)\right]^T \tilde{u}^{k*}(j-1) \right), \end{aligned}$$

The trajectories of the actual and estimated states are in Figure 6.1.

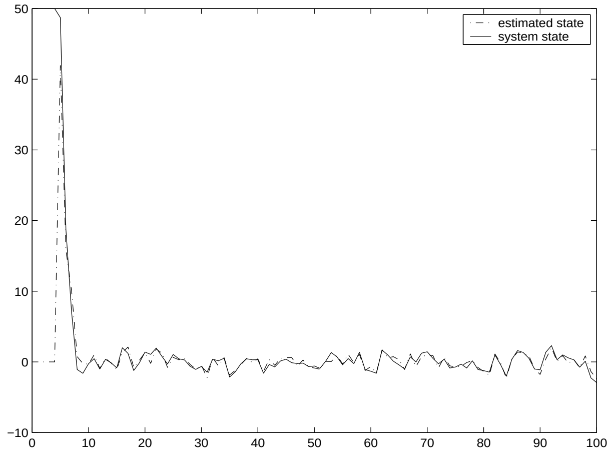

Figure 6.1: Actual system state and estimated state.

6.4.2 Application in Unilateral Delay Systems

Introduce a three-layer discrete-time unilateral delay system (see Figure 6.2) which is similar to the continuous-time case [44]:

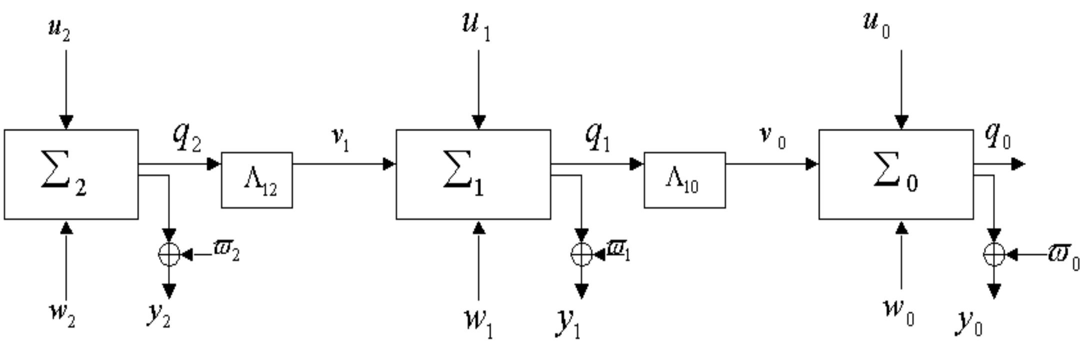

Figure 6.2: Unilateral delay system.

$$\begin{aligned}\check{x}_i(k+1) &= \check{A}_i\check{x}_i(k) + \check{D}_i\check{w}_i(k) + \check{B}_i u_i(k) + \check{E}_i v_i(k)\\ q_i(k) &= \check{C}_i\check{x}_i(k)\\ v_0(k) &= q_1(k-h),\ v_1(k) = q_2(k-h),\ v_2(k) = 0\\ \check{y}_i(k) &= \check{F}_i\check{x}_i(k) + \check{\varpi}_i(k),\ i = 0,1,2\end{aligned}$$

where $\check{x}_i(k),\ q_i(k),\ \check{y}_i(k),\ v_i(k),\ \check{w}_i(k),\ \check{\varpi}_i(k),\ i = 0,1,2,$ are system states, layer outputs, layer observed outputs, layer inputs, layer exogenous noises, layer observer noises, respectively. $\check{w}_i(k),\ \check{\varpi}_i(k),\ i = 0,1,2$ are uncorrelated Gaussian white noises with zero means and variances of 1.

In order to transform the unilateral delay system into the system with multiple input delays, we define

$$x(k) = \begin{pmatrix} x_0(k)\\ x_1(k)\\ x_2(k)\end{pmatrix} = \begin{pmatrix}\check{x}_0(k)\\ \check{x}_1(k-h)\\ \check{x}_2(k-2h)\end{pmatrix},\quad w(k) = \begin{pmatrix}\check{w}_0(k)\\ \check{w}_1(k-h)\\ \check{w}_2(k-2h)\end{pmatrix},$$

$$y(k) = \begin{pmatrix}\check{y}_0(k)\\ \check{y}_1(k-h)\\ \check{y}_2(k-2h)\end{pmatrix},\quad \varpi(k) = \begin{pmatrix}\check{\varpi}_0(k)\\ \check{\varpi}(k-h)\\ \check{\varpi}(k-2h)\end{pmatrix},$$

and $h_0 = 0,\ h_1 = h,\ h_2 = 2h.$

We obtain the system with multiple input delays (see Figure 6.3)

$$\begin{aligned}x(k+1) &= Ax(k) + Dw(k) + \sum_{i=0}^{d} B_i u_i(k-h_i)\\ y(k) &= Fx(k) + \varpi(k)\end{aligned}$$

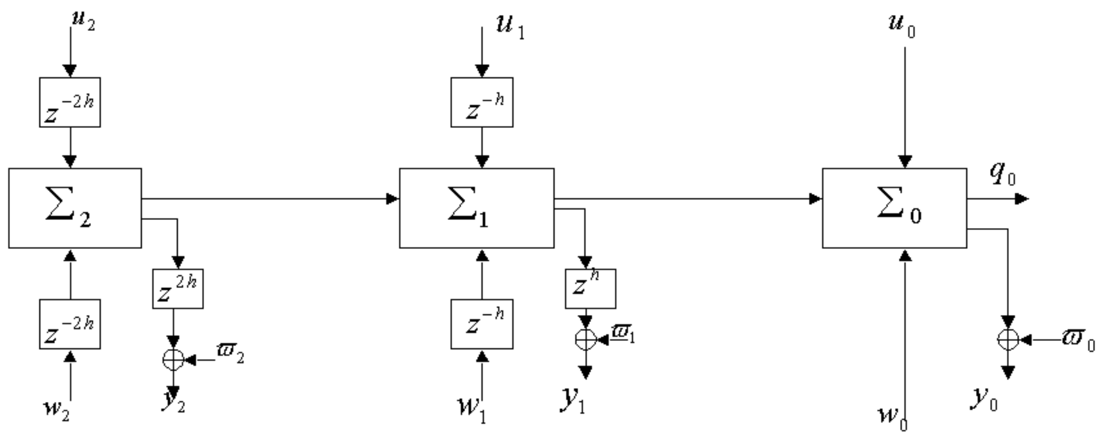

Figure 6.3: Auxiliary form.

where

$$A = \begin{pmatrix} \check{A}_0 & \check{E}_0\check{C}_1 & 0 \\ 0 & \check{A}_1 & \check{E}_1\check{C}_2 \\ 0 & 0 & \check{A}_2 \end{pmatrix}, \quad D = \begin{pmatrix} \check{D}_0 & 0 & 0 \\ 0 & \check{D}_1 & 0 \\ 0 & 0 & \check{D}_2 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} \check{B}_0 \\ 0 \\ 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ \check{B}_1 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ \check{B}_2 \end{pmatrix},$$

$$F = \begin{pmatrix} \check{F}_0 & 0 & 0 \\ 0 & \check{F}_1 & 0 \\ 0 & 0 & \check{F}_2 \end{pmatrix}.$$

The performance index of the unilateral delay system is given by

$$J_N = E\sum_{k=0}^{N}[q_0^T(k)q_0(k) + \sum_{i=0}^{2} u_i^T(k-h_i)u_i(k-h_i)]. \tag{6.54}$$

Assume that initial conditions $u_i(t) = 0$, $t < 0$, $i = 0, 1, 2$, and $var[x(0)] = diag\{1, 1, 1\}$, where $var(\cdot)$ denotes variance. Then, (6.54) has an equivalent expression as (6.5) where $P_{N+1} = 0,\ R_i = 1,\ i = 0, 1, 2$ and

$$Q = \begin{pmatrix} \check{C}_0^T \check{C}_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In order to simplify the computation, we set the parameters $\check{A}_i = \check{C}_i = \check{D}_i = \check{B}_i = \check{F}_i = 1,\ i = 0, 1, 2,\ \check{E}_0 = -0.5,\ \check{E}_1 = -0.7,\ h = 1$, and $N = 100$. So we can solve the LQG control problem for unilateral delay system with Theorem 6.3.2.

Figure 6.4 shows the performances of actual system state and estimated state where $x(0) = diag\{20, 20, 20\}$ and Figure 6.5 shows the control inputs. The simulation shows that our approach in this chapter can achieve good performance for the unilateral delay system.

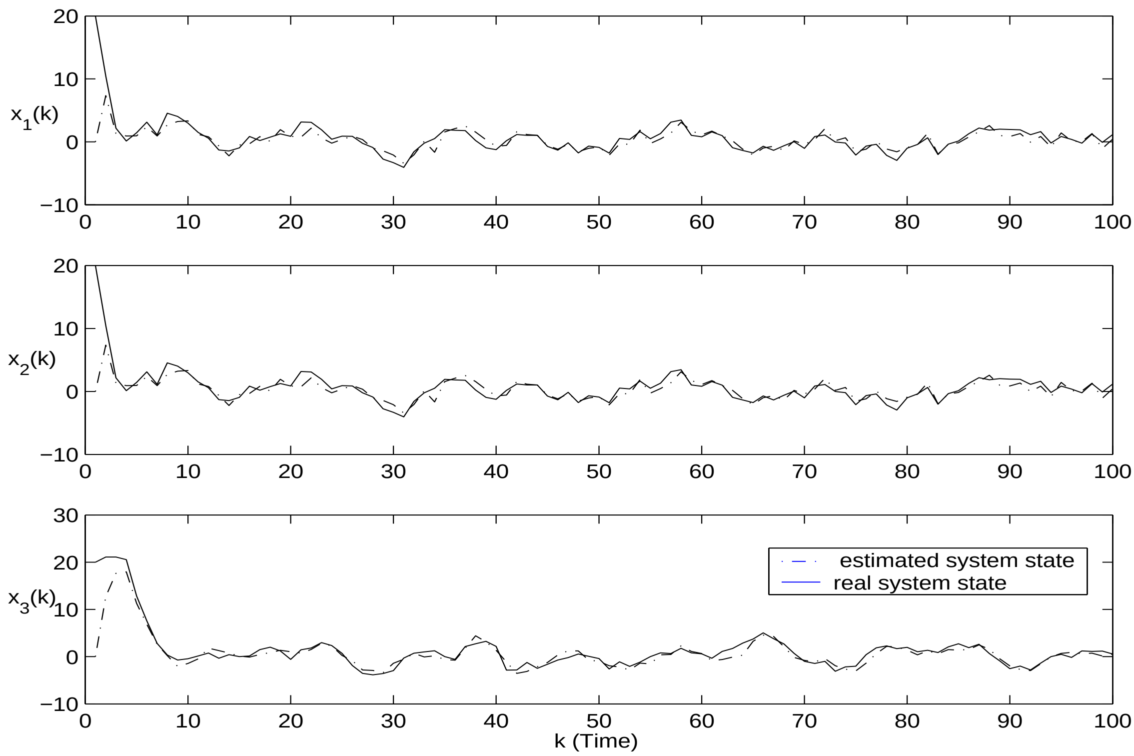

Figure 6.4: System state and estimated state.

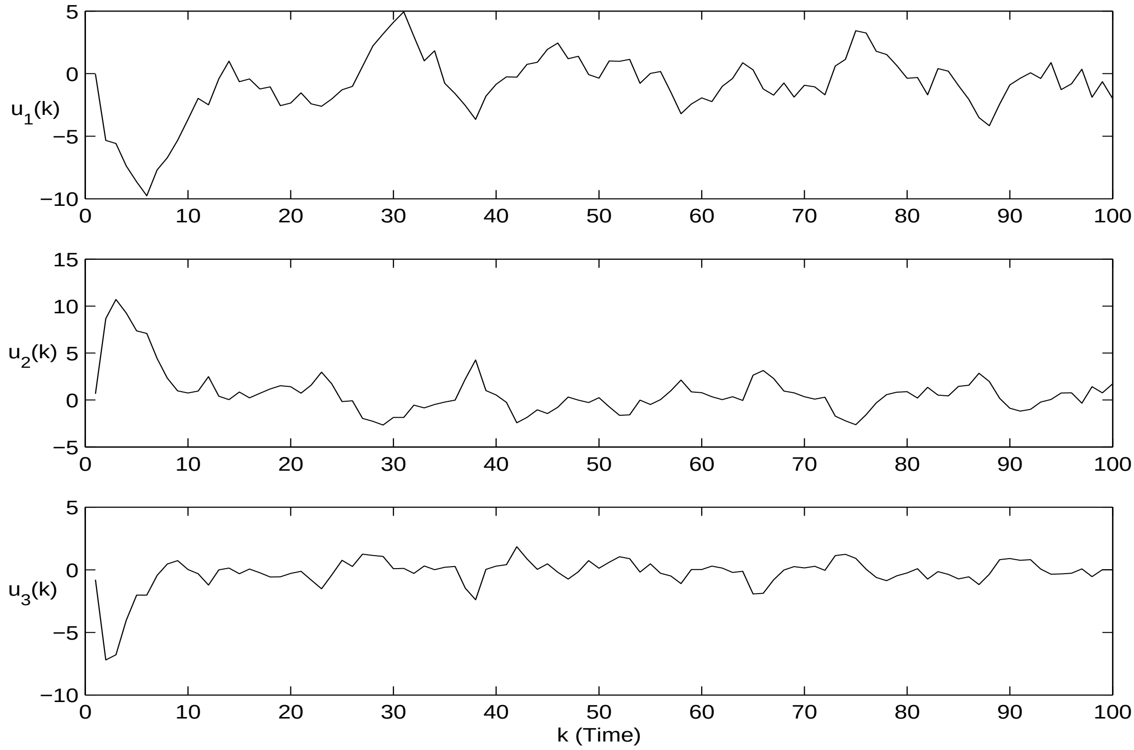

Figure 6.5: Control inputs.

The minimization result of J_N is related to the time delay h. When h is larger, J_N becomes larger too, e.g. $J_N = 620.9082$ when $h = 1$, however, $J_N = 868.6360$ when $h = 2$. This fact demonstrates that time delays make the performance of output feedback control worse.

Remark 6.4.1. *In [26], delays in the unilateral system are approximated by a finite dimensional model. Here, we provide a direct solution to the LQG problem for unilateral delay systems without introducing any approximation.*

6.5 Conclusion

We have investigated the discrete-time LQG problem in finite horizon for systems with multiple input/output delays. A separation principle has been obtained which divides the LQG problem into the LQR control for the multiple input delays system and the H_2 filtering problem with multiple output delays. For the filtering problem, a reorganized innovation analysis approach has been proposed. Our approach has an advantage in computation compared with the state augmentation approach. Further research on continuous-time counterpart can be carried out.

Chapter 7

H_∞ Control for Linear Systems with Multiple Input Delays

7.1 Introduction

In this chapter, we shall discuss the H_∞ control for systems with multiple input delays. In the last few years, some important progress has been made for the H_∞ control of systems with I/O delays. Tadmor [83] obtains a necessary and sufficient condition for the H_∞ control of continuous-time systems with single input delay and an H_∞ controller is given in terms of solutions of two algebraic Riccati equations and one differential Riccati equation over the delay interval. Mirkin [55] treats the time delay controllers as constrained ones and extracts time delay controllers from the parametrization of all delay-free controllers. Meinsma and Zwart [53] use a special controller transformation to reduce a standard H_∞ problem to an equivalent finite dimensional one. Meinsma and Mirkin [52] treat the multiple delay operator as a special series of nested elementary delay operators. Kojima and Ishijima [44] discuss the problem in function space and transform the resulting operator Riccati equations into algebraic Riccati equations. Zhang et al. [99, 102] give an elegant solution for discrete-time systems where a duality between the LQR problem with multiple input delays and a smoothing problem for a backward stochastic delay free system is provided.

In this chapter, we discuss the discrete-time H_∞ control problem for systems with multiple input delays. Our work is based on [99] where the H_∞ full information control and H_∞ control with single input delay are considered. We consider a more general case where cross terms of state and control inputs exist in the cost function. An H_∞ controller is in terms of an RDE. Moreover, the H_2 optimal controller is obtained when the H_∞ performance γ is set to infinity. At last, we apply our result to the congestion control in ATM network by transforming the congestion network model to a linear system with multiple input delays.

7.2 Problem Statement

We consider a time-variant linear system with multiple delayed inputs

$$x(k+1) = A_k x(k) + \sum_{i=1}^{d} B_{i,k} u_i(k-h_i) + \bar{B}_k w(k), \tag{7.1}$$

$$z(k) = C_k x(k) + \sum_{i=1}^{d} D_{i,k} u_i(k-h_i), \tag{7.2}$$

where $x(k) \in \mathbb{R}^n$, $u_i(k) \in \mathbb{R}^{m_i}, w(k) \in \mathbb{R}^p$ and $z(k) \in \mathbb{R}^r$ represent the state, the control input, the exogenous input, and the controlled output, respectively. We assume that $u_i(k) = 0$ when $k < 0$. $A_k, B_{i,k}, \bar{B}_k, C_k$ and $D_{i,k}$ are bounded time-varying matrices. It is assumed that the exogenous input is from $\ell_2[0,N]$ where N is the time-horizon of the control problem under investigation. Without loss of generality, we assume that the delays are in a strictly increasing order: $0 < h_1 < \cdots < h_d$.

For a given positive scalar γ, the finite-horizon H_∞ state feedback control problem is stated as:

Find a control strategy

$$u_i(k) = \mathcal{F}_i(x(k), u_j(\tau)|1 \leq j \leq d,\ 0 \leq \tau < k),$$
$$i = 1, 2, \cdots, d \tag{7.3}$$

such that for any non-zero $w \in \ell_2[0, N]$*,*

$$\frac{||z||^2_{[0,N]}}{||w||^2_{[0,N]}} < \gamma^2 \tag{7.4}$$

under zero initial condition $x(k) = 0, \ \ -h_d \leq k \leq 0.$

7.3 Design of H_∞ Controller

Define $J_N = ||z||^2_{[0,N]} - \gamma^2||w||^2_{[0,N]}$, $h_0 = 0$, $u_0(k) \triangleq w(k)$ and

$$u(k) \triangleq \begin{cases} \begin{bmatrix} u_0(k-h_0) \\ \vdots \\ u_i(k-h_i) \end{bmatrix}, & \begin{array}{c} h_i \leq k < h_{i+1}, \\ i = 0, \cdots, d-1, \end{array} \\ \\ \begin{bmatrix} u_0(k-h_0) \\ \vdots \\ u_d(k-h_d) \end{bmatrix}, & k \geq h_d. \end{cases}$$

Then, the cost function J_N can be rewritten as:

$$\begin{aligned} J_N &= \sum_{k=0}^{N} x^T(k)Q_k x(k) + u^T(k)R_k u(k) + u^T(k)L_k x(k) \\ &\quad + x^T(k)L_k^T u(k) \end{aligned} \tag{7.5}$$

where

$$Q_k = C_k^T C_k, \tag{7.6}$$

$$\bar{D}_{i,k} = [D_{1,k} \;\; D_{2,k} \;\; \cdots \;\; D_{i,k}], \; i = 1, 2, \cdots, d \tag{7.7}$$

$$R_k = \begin{cases} \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & \bar{D}_{i,k}^T \bar{D}_{i,k} \end{pmatrix}, & \begin{array}{c} h_i \le k < h_{i+1} \\ i = 1, 2, \cdots, d-1 \end{array} \\ \\ \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & \bar{D}_{d,k}^T \bar{D}_{d,k} \end{pmatrix}, & k > h_d \end{cases} \tag{7.8}$$

$$L_k = \begin{cases} \begin{pmatrix} 0 \\ \bar{D}_{i,k}^T C_k \end{pmatrix}, & h_i \le k < h_{i+1} \\ \\ \begin{pmatrix} 0 \\ \bar{D}_{d,k}^T C_k \end{pmatrix}, & k > h_d. \end{cases} \tag{7.9}$$

Also, note that the system (7.1) can be written as

$$x(k+1) = A_k x(k) + \sum_{i=0}^{d} B_{i,k} u_i(k - h_i) \tag{7.10}$$

where $B_{0,k} = \bar{B}_k$.

Denote

$$\tilde{u}(k) \triangleq \begin{cases} \sum_{j=i+1}^{d} B_{j,k} u_j(k-h_j), & h_i \le k < h_{i+1}, \\ 0, & k \ge h_d, \end{cases} \tag{7.11}$$

$$B_k = \begin{cases} [B_{0,k} \ \cdots \ B_{i,k}], & h_i \le k < h_{i+1}, \\ [B_{0,k} \ \cdots \ \Gamma_{d,k}], & k \ge h_d, \end{cases} \tag{7.12}$$

$$\xi \triangleq \begin{pmatrix} x(0) \\ \tilde{u}(0) \\ \vdots \\ \tilde{u}(h_d-1) \end{pmatrix}, \ u \triangleq \begin{pmatrix} u(0) \\ \vdots \\ u(N) \end{pmatrix}. \tag{7.13}$$

Using the above notations and taking into account that $u_i(k) = 0, \ i = 0, 1, \cdots, d$, for $k < 0$, system (7.10) can be rewritten as

$$x(k+1) = \begin{cases} A_k x(k) + B_k u(k) + \tilde{u}(k), & h_i \le k < h_{i+1}, \\ A_k x(k) + B_k u(k), & k \ge h_d. \end{cases} \tag{7.14}$$

Associated with system (7.14) and the cost (7.5), we introduce the following stochastic system:

$$\begin{cases} \mathbf{x}(k) = A_k^T \mathbf{x}(k+1) + \mathbf{u}(k), \\ \mathbf{y}(k) = B_k^T \mathbf{x}(k+1) + \mathbf{v}(k), \end{cases} \tag{7.15}$$

where $\mathbf{x}(N+1) = 0$ and

$$\begin{aligned} < \mathbf{u}(k), \mathbf{u}(s) > &= Q_k \delta_{k,s}, \\ < \mathbf{v}(k), \mathbf{v}(s) > &= R_k \delta_{k,s}, \\ < \mathbf{u}(k), \mathbf{v}(s) > &= L_k \delta_{k,s}, \end{aligned}$$

where $\delta_{k,s} = \begin{cases} 1, & k = s, \\ 0, & k \neq s. \end{cases}$

Remark 7.3.1. *In (7.15), the covariance R_k of the measurement noise is indefinite as seen from (7.8). Therefore, the system (7.15) should be studied in Krein space ([35]) rather than Hilbert space.*

Denote

$$\mathbf{y^c} \triangleq \begin{pmatrix} \mathbf{y}(0) \\ \vdots \\ \mathbf{y}(N) \end{pmatrix}, \quad \mathbf{x_0} \triangleq \begin{pmatrix} \mathbf{x}(0) \\ \vdots \\ \mathbf{x}(h_d) \end{pmatrix}.$$

We introduce the following lemma to bridge the deterministic H_∞ problem and a stochastic optimization problem.

Lemma 7.3.1. *The cost function J_N can be given by*

$$J_N = \begin{pmatrix} \xi \\ u \end{pmatrix}^T \Pi \begin{pmatrix} \xi \\ u \end{pmatrix} \tag{7.16}$$

where ξ and u are defined in (7.13), $\Pi = \begin{pmatrix} \mathbf{R_{x_0}} & \mathbf{R_{x_0 y^c}} \\ \mathbf{R_{y^c x_0}} & \mathbf{R_{y^c}} \end{pmatrix}$, *and* $\mathbf{R_{xy}} =< \mathbf{x}, \mathbf{y} >$.

Proof: The proof follows a similar line of arguments as in [97].

From the above lemma, when $\mathbf{R_{y^c}}$ is invertible, the minimizing solution over u for J_N is known to be ([35])

$$\begin{aligned} \hat{u} &= -\mathbf{R}_{\mathbf{y^c}}^{-1}\mathbf{R}_{\mathbf{y^c x_0}}\xi \\ &= -\mathbf{R}_{\mathbf{y^c}}^{-1}\mathbf{R}_{\mathbf{y^c x}(0)}x(0) - \textstyle\sum_{k=1}^{h_d} \mathbf{R}_{\mathbf{y^c}}^{-1}\mathbf{R}_{\mathbf{y^c x}(k)}\tilde{u}(k-1). \end{aligned} \tag{7.17}$$

Remark 7.3.2. *Observe that $\mathbf{R}_{\mathbf{y^c}}^{-1}\mathbf{R}_{\mathbf{y^c x}(k)}$ in (7.17) is in fact the transpose of the* *filtering ($k=0$) or smoothing gain ($k=1,2,\cdots,h_d$) of the stochastic backward system (7.15) in Krein space. Thus, the standard estimation theory can be applied to obtain the gain.*

The following result is then given. The detail of the derivation is similar to [99] and is omitted.

Theorem 7.3.1. *Consider the system (7.1)-(7.2) and the performance (7.4). For a given $\gamma>0$ and $0\le\tau\le N$, assume that the RDE*

$$\begin{aligned} P_j^\tau &= A_{\tau+j}^T P_{j+1}^\tau A_{\tau+j} + Q_{\tau+j} - (A_{\tau+j}^T P_{j+1}^\tau B_j^\tau + (L_j^\tau)^T) \\ &\quad \times (R_j^\tau + (B_j^\tau)^T P_{j+1}^\tau B_j^\tau)^{-1}((B_j^\tau)^T P_{j+1}^\tau A_{\tau+j} + L_j^\tau) \end{aligned}$$

with initial condition $P_{N-\tau+1}^\tau = 0$,

$$B_j^\tau = \begin{cases} [B_{0,j+\tau}, \ \cdots, \ B_{i,j+\tau}], & h_i \le j < h_{i+1} \\ [B_{0,j+\tau}, \ \cdots, \ B_{d,j+\tau}], & j \ge h_d, \end{cases}$$

$$L_j^\tau = \begin{cases} \begin{pmatrix} 0 \\ \bar{D}_{i,j+\tau}^T C_{j+\tau} \end{pmatrix}, & h_i \le j < h_{i+1} \\ \\ \begin{pmatrix} 0 \\ \bar{D}_{d,j+\tau}^T C_{j+\tau} \end{pmatrix}, & j > h_d, \end{cases}$$

and

$$R_j^\tau = \begin{cases} \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & \bar{D}_{i,j+\tau}^T \bar{D}_{i,j+\tau} \end{pmatrix}, & \begin{matrix} h_i \le j < h_{i+1} \\ i = 1,2,\cdots,d-1 \end{matrix} \\ \\ \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & \bar{D}_{d,j+\tau}^T \bar{D}_{d,j+\tau} \end{pmatrix}, & j > h_d, \end{cases}$$

admits a bounded solution $P_j^\tau,\ j = 0, 1, \cdots, min\{h_d, N - \tau + 1\}$*. Then a controller that solves the* H_∞ *state feedback control problem exists if and only if*

$$\bar{M}_{1,1}(k) < 0, \tag{7.18}$$

where $\bar{M}_{1,1}(k)$ *is the* $(1,1)$ *block of* $\bar{M}_k$*:*

$$\bar{M}_k = \begin{cases} diag\{\bar{B}_k^T,\ B_{1,k}^T, \cdots, B_{d,k}^T\}[\bar{P}_{k+1}(i,j)]_{(d+1)\times(d+1)} \\ \quad \times diag\{\bar{B}_k,\ B_{1,k}, \cdots B_{d,k}\} \\ \quad + diag\{-\gamma^2 I_p,\ \bar{D}_{d,k}^T \bar{D}_{d,k}\}, \\ \qquad\qquad\qquad\qquad k \le N - h_d; \\ \\ diag\{\bar{B}_k^T,\ B_{1,k}^T, \cdots, B_{l,k}^T\}[\bar{P}_{k+1}(i,j)]_{(l+1)\times(l+1)} \\ \quad \times diag\{\bar{B}_k,\ B_{1,k}, \cdots B_{l,k}\} \\ \quad + diag\{-\gamma^2 I_p,\ \bar{D}_{l,k}^T \bar{D}_{l,k}\}, \\ \qquad\qquad\qquad\qquad N - h_{l+1} < k \le N - h_l \end{cases}$$

with $\bar{P}_\tau(i,j)$*:*

$$\begin{aligned} \bar{P}_\tau(i,j) = {} & P_{h_i}^\tau (\bar{A}_{h_i-1}^\tau)^T \cdots (\bar{A}_{h_j}^\tau)^T - \sum_{s=0}^{h_j-1} P_{h_i}^\tau (\bar{A}_{h_i-1}^\tau)^T \cdots (\bar{A}_{s+1}^\tau)^T B_s^\tau \\ & \times (M_s^\tau)^{-1} (B_s^\tau)^T \{P_{h_j}^\tau (\bar{A}_{h_j-1}^\tau)^T \cdots (\bar{A}_{s+1}^\tau)^T\}^T,\ i \ge j. \end{aligned}$$

In this case, a suitable H_∞ controller (central controller) $u_i^*(\tau)$ is given by

$$
\begin{aligned}
u_i^*(\tau) &= -[\overbrace{0 \ \ \cdots \ \ 0 \ \ I_m}^{i+1 \ blocks}]\{[\mathcal{F}_0^\tau(h_i)]^T x(\tau) \\
&+ \sum_{l=1}^{h_i}[\mathcal{F}_l^\tau(h_i - l)]^T \sum_{j=i+1}^{d} B_{j,l+\tau-1} u_j(l+\tau-h_j-1) \\
&+ \sum_{l=h_i+1}^{h_d}[\mathcal{S}_l^\tau(h_i)]^T \sum_{j=i+1}^{d} B_{j,l+\tau-1} u_j(l+\tau-h_j-1)\}, \\
& \qquad i = 1, 2, \cdots, d,
\end{aligned}
\tag{7.19}
$$

where

$$
\begin{aligned}
\mathcal{F}_l^\tau(j) &= \{\bar{A}_l^\tau \cdots \bar{A}_{l+j-1}^\tau \\
&\quad - \sum_{s=1}^{l} \mathcal{A}_l^\tau(s)(B_{s-1}^\tau)^T \bar{A}_s^\tau \cdots \bar{A}_{j+l-1}^\tau\} K_{l+j}^\tau, 1 \le j \le l-1, \\
\mathcal{S}_l^\tau(j) &= \mathcal{A}_l^\tau(j+1) \\
&\quad -\{\sum_{s=1}^{j} \mathcal{A}_l^\tau(s)(B_{s-1}^\tau)^T \bar{A}_s^\tau \cdots \bar{A}_{j-1}^\tau\} K_j^\tau, 1 \le j \le l-1, \\
\mathcal{S}_l^\tau(0) &= \mathcal{A}_l^\tau(1), \\
\mathcal{A}_l^\tau(s) &= \begin{cases} 0, \ \ l < s, \\ P_l^\tau B_{l-1}^\tau (M_{l-1}^\tau)^{-1}, \ \ l = s, \\ P_l^\tau (\bar{A}_{l-1}^\tau)^T \cdots (\bar{A}_s^\tau)^T B_{s-1}^\tau (M_{s-1}^\tau)^{-1}, \ \ l > s, \end{cases}
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{A}_j^\tau &= A_{\tau+j}^T - K_j^\tau (B_j^\tau)^T, \\
K_j^\tau &= A_{\tau+j}^T P_{j+1}^\tau B_j^\tau (M_j^\tau)^{-1}, \\
M_j^\tau &= R_j^\tau + (B_j^\tau)^T P_{j+1}^\tau B_j^\tau.
\end{aligned}
$$

Remark 7.3.3. *Note that in [99], the H_∞ full-information control problem is in-*

vestigated. Here, we consider the state feedback H_∞ control for system with multiple input delays.

Remark 7.3.4. *It is well known that the optimal H_2 control solution can be obtained by setting $\gamma \to \infty$ in the corresponding H_∞ control solution. Thus, consider the system (7.1)-(7.2), where w is a white noise with unit variance. The H_2 state feedback control problem is to find an optimal state feedback law that minimizes $E(\|z\|^2)$, where $E(\cdot)$ stands for the mathematical expectation.*

Denote

$$\begin{aligned}
\check{B}_k^\tau &= \begin{cases} [B_{1,\tau+k} \ \cdots \ B_{i,\tau+k}], & h_i \le k < h_{i+1}, \\ [B_{1,\tau+k} \ \cdots \ B_{d,\tau+k}], & k \ge h_d, \end{cases} \\
\check{L}_k^\tau &= \begin{cases} (\bar{D}_{i,k}^\tau)^T C_k, & h_i \le k < h_{i+1}, \\ (\bar{D}_{d,k}^\tau)^T C_k, & k > h_d, \end{cases} \\
\bar{D}_{i,k}^\tau &= [D_{1,\tau+k-h_1} \ \ D_{2,\tau+k-h_2} \ \cdots \ D_{i,\tau+k-h_i}], \\
& \quad\quad i = 1, 2, \cdots, d,
\end{aligned}$$

and

$$\check{R}_k^\tau = \begin{cases} \bar{D}_{i,k}^T \bar{D}_{i,k}, & h_i \le k < h_{i+1}, \\ \bar{D}_{d,k}^T \bar{D}_{d,k}, & k > h_d. \end{cases}$$

Introduce the Riccati equation

$$\begin{aligned}
P_j^\tau &= A_{\tau+j}^T P_{j+1}^\tau A_{\tau+j} + Q_{\tau+j} - (A_{\tau+j}^T P_{j+1}^\tau \check{B}_j^\tau + (\check{L}_j^\tau)^T) \\
& \quad \times (\check{R}_j^\tau + (\check{B}_j^\tau)^T P_{j+1}^\tau \check{B}_j^\tau)^{-1} ((\check{B}_j^\tau)^T P_{j+1}^\tau A_{\tau+j} + \check{L}_j^\tau), \\
& \quad\quad\quad\quad P_{N-\tau+1}^\tau = 0.
\end{aligned} \tag{7.20}$$

The optimal H_2 controller can be given by

$$
\begin{aligned}
u_i^*(\tau) = -[\overbrace{0 \quad \cdots \quad 0}^{i \text{ blocks}} \quad I_m]\{[\mathcal{F}_0^\tau(h_i)]^T x(\tau) \\
+\sum_{l=1}^{h_i}[\mathcal{F}_l^\tau(h_i-l)]^T \sum_{j=i+1}^{d} B_{j,l+\tau-1}u_j(l+\tau-h_j-1) \\
+\sum_{l=h_i+1}^{h_d}[\mathcal{S}_l^\tau(h_i)]^T \sum_{j=i+1}^{d} B_{j,l+\tau-1}u_j(l+\tau-h_j-1)\}, \\
i = 0,1,2,\cdots,d,
\end{aligned}
\tag{7.21}
$$

where

$$
\begin{aligned}
\mathcal{F}_l^\tau(j) &= \{\bar{A}_l^\tau \cdots \bar{A}_{l+j-1}^\tau \\
&\quad -\sum_{s=1}^{l}\mathcal{A}_l^\tau(s)(\check{B}_{s-1}^\tau)^T\bar{A}_s^\tau\cdots\bar{A}_{j+l-1}^\tau\}K_{l+j}^\tau, \qquad (7.22)\\
\mathcal{S}_l^\tau(j) &= \mathcal{A}_l^\tau(j+1) \\
&\quad -\{\sum_{s=1}^{j}\mathcal{A}_l^\tau(s)(\check{B}_{s-1}^\tau)^T\bar{A}_s^\tau\cdots\bar{A}_{j-1}^\tau\}K_j^\tau, \\
&\quad 1 \le j \le l-1, \qquad (7.23)\\
\mathcal{S}_l^\tau(0) &= \mathcal{A}_l^\tau(1), \\
\mathcal{A}_l^\tau(s) &= \begin{cases} 0, \ l < s, \\ P_l^\tau\check{B}_{l-1}^\tau(M_{l-1}^\tau)^{-1}, \ l = s, \\ P_l^\tau(\bar{A}_{l-1}^\tau)^T\cdots(\bar{A}_s^\tau)^T\check{B}_{s-1}^\tau(M_{s-1}^\tau)^{-1}, \ l > s, \end{cases} \qquad (7.24)
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{A}_j^\tau &= A_{\tau+j}^T - K_j^\tau(\check{B}_j^\tau)^T, \\
K_j^\tau &= A_{\tau+j}^T P_{j+1}^\tau \check{B}_j^\tau (M_j^\tau)^{-1}, \\
M_j^\tau &= \check{R}_j^\tau + (\check{B}_j^\tau)^T P_{j+1}^\tau \check{B}_j^\tau,
\end{aligned}
$$

and $\breve{B}_0^{\tau} \triangleq 0$.

7.4 Example

In this section, we revisit the ATM congestion control problem. The ATM system model and symbols are the same as in Section 4.4.1 except that we consider multiple input rates here. The queue length equation is given by

$$\begin{aligned} q_{k+1} &= q_k + \sum_{i=1}^{d} r_{i,k} - \mu_k, \\ &\equiv q_k + \sum_{i=1}^{d} \tilde{u}_{i,k-h_i} - \mu_k, \qquad (7.25) \\ \mu_k &= \mu + \zeta_k \qquad (7.26) \\ \zeta_k &= \sum_{i=1}^{p_1} l_i \zeta_{k-i} + w_{k-1}, \qquad (7.27) \end{aligned}$$

where μ is the constant nominal service rate, $r_{i,k}$ and $\tilde{u}_{i,k-h_i}$ are respectively the input rate of i-th source and the calculated i-th source rate at switch, $\{l_i\}_{i=1,2,\cdots,p_1}$ are known parameters. w_k is a zero-mean i.i.d. Gaussian sequence with variance ρ^2. h_i is the round trip delay of the i-th source.

The H_∞ congestion control problem is given below.

Find a source rate $\tilde{u}$ *such that*

$$\sup_{\{q_0, \tilde{u}_{i,t-h_i} | 0 \le i \le d, 0 \le k \le N-h_i\}} J(q_0, \tilde{u}_{i,k-h_i}, \mu_k) < \gamma^2, \qquad (7.28)$$

for a prescribed $\gamma > 0$, *where*

$$J(q_0, \tilde{u}_{i,k-h_i}, \mu_k) = \frac{\sum_{k=1}^{N} [(q_k - q)^2 + \sum_{i=1}^{d} \lambda^2 (r_{i,k} - a_i \mu_k)^2]}{\sum_{t=1}^{N} w_k^2}, \qquad (7.29)$$

where q is the target queue length, λ is a weighting factor and $0 \neq w \in \ell_2[0, N]$.

It is clear that the objective is to make the queue buffer close to the desired level while the difference between the source rate and the service rate should not be too large. The above criterion combines the performance of queue length and accumulation of the difference between switch input and output.

We assume that link capacity is a second order AR process, i.e. $p_1 = 2$ [3]. To formulate the above congestion control as the H_∞ control problem for systems with multiple inputs studied in the previous section, define

$$
\begin{aligned}
x(k) &\triangleq \begin{pmatrix} q_k - q \\ \zeta_{k-1} \\ \zeta_k \end{pmatrix}, \\
u_i(k - h_i) &\triangleq \tilde{u}_{i,k-h_i} - a_i \mu, \\
z(k) \triangleq \begin{pmatrix} q_k - q \\ \lambda(r_{1,k} - a_1 \mu_k) \\ \vdots \\ \lambda(r_{d,k} - a_d \mu_k) \end{pmatrix} &= \begin{pmatrix} q_k - q \\ \lambda(\bar{u}_1(k - h_1) - a_1 \zeta_k) \\ \vdots \\ \lambda(\bar{v}_d(k - h_d) - a_d \zeta_k) \end{pmatrix}, \\
w(k) &\triangleq w_k.
\end{aligned}
$$

Here a_i is the weight for different source rates and $\sum_{i=1}^{d} a_i = 1$.

The system (7.25)-(7.27) can be expressed as the form of (7.1)-(7.2) with

$$
A_k = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & l_2 & l_1 \end{pmatrix}, \quad \bar{B}_k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad B_{i,t} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
$$

From the cost function (7.29), it is easy to know that Q_k, R_k and L_k of (7.5) are given by

$$Q_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sum_{i=1}^{\vartheta(k)} a_i^2 \end{pmatrix}, \quad R_k = \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & \lambda^2 I \end{pmatrix},$$

$$L_k^T = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & -a_1 & \cdots & -a_{\vartheta(k)} \end{pmatrix},$$

where $\vartheta(k) = \begin{cases} i, & h_i \leq k < h_{i+1}, i = 1, 2, \cdots, d-1 \\ d, & k > h_d \end{cases}$ and I is an identity matrix of appropriate dimension.

An H_∞ controller $u_i(\tau)$ can be obtained using Theorem 7.3.1.

Remark 7.4.1. *Note that in congestion control $\mathcal{F}_l^\tau(j), \mathcal{S}_l^\tau(j), \bar{M}_{1,1}(k)$ and $\Gamma_{j,l+\tau-1}$ are known or can be calculated off-line, so the source rates (controller) can be computed real time.*

We consider that the congestion control model has the same parameters as that of Chapter 4, except that $w \in \ell_2[0, N]$. We further assume that there are 4 sources with round trip delay from 1 to 4, respectively. Time length is 100 and we set $\gamma = 15$. The weighting between the queue length and the transmission rate is $\lambda = 1$ and $a_i = 1/d(i = 1, 2, \cdots, d)$ are the source sharing. Simulation result is in Figure 7.1 and 7.2. The vertical axis in Figure 7.1 is queue length q_k and the controllers $u_i(k)$ in Figure 7.2 are defined as $u_i(k) = \tilde{u}_{i,k} - a_i\mu$ where $\tilde{u}_{i,k}$ are calculated i-th source rate at switch and μ is the constant nominal service rate. The initial queue length of the congested switch is set to be 5100. From Figure 7.1 we can see that the queue length quickly converges to the target queue length.

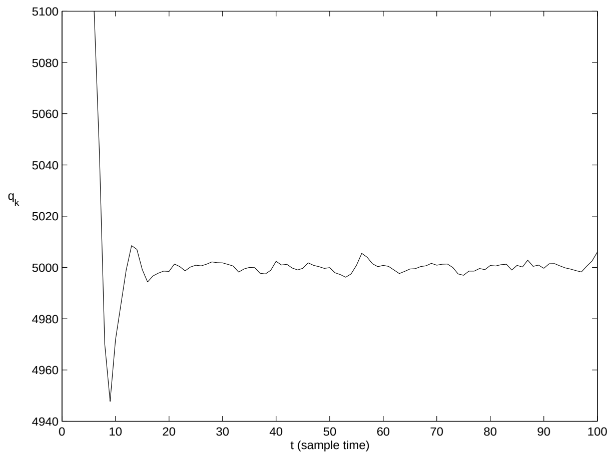

Figure 7.1: Queue length response.

7.5 Conclusion

This chapter has addressed the H_∞ control problem for systems with multiple input delays. We extended the existing work [99] where the H_∞ full-information control is studied, by allowing cross terms of state and delayed inputs in the quadratic criterion. The key to solve the H_∞ problem is the duality between the H_∞ control problem and a smoothing estimation problem for an associated system without delays. We also applied the H_∞ technique to multi-user-one-switch congestion control in ATM networks. Compared with the existing results of ATM congestion control, our approach has the advantage of lower computational cost as no state augmentation is needed. The simulation demonstrated the effectiveness of the proposed approach.

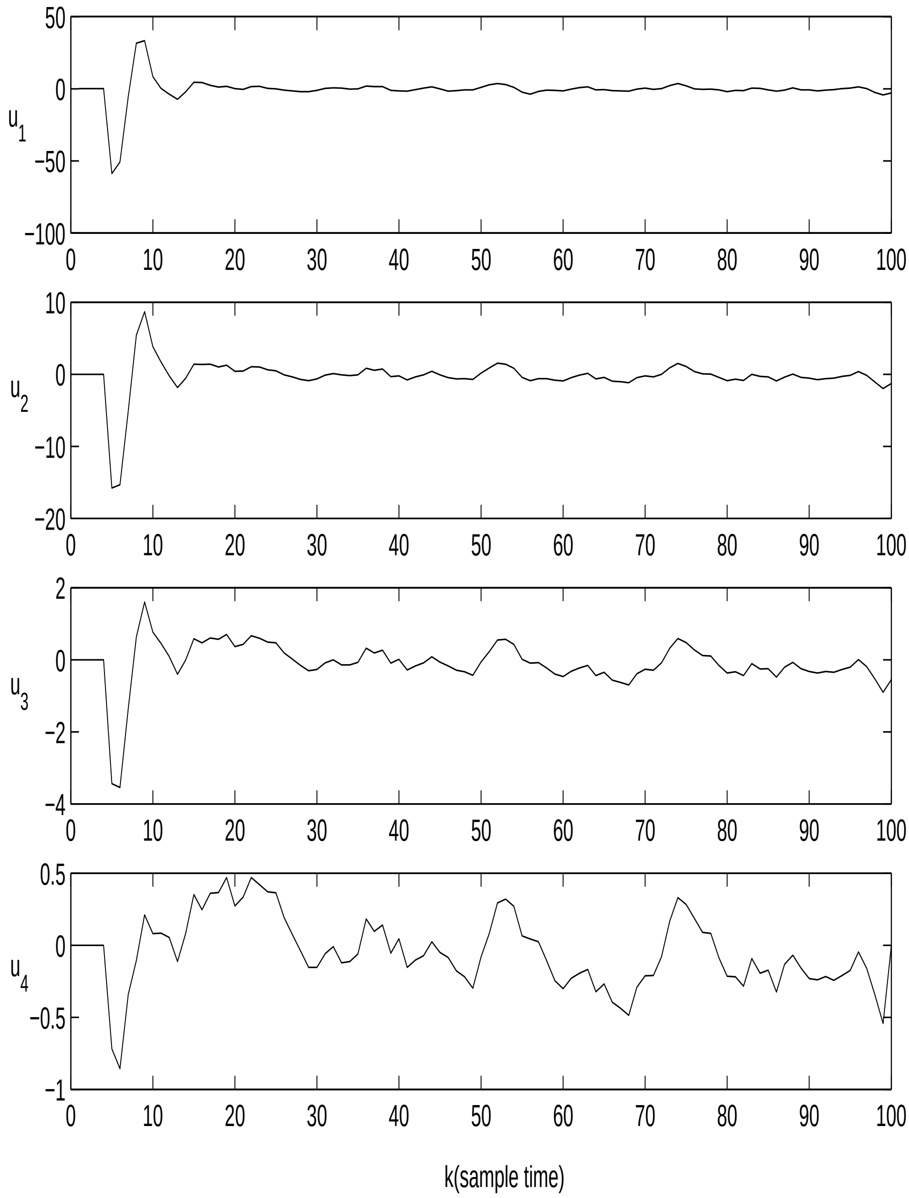

Figure 7.2: Control Signal.

Chapter 8

Sampled-data LQR Control for Systems with Multiple Input Delays

8.1 Introduction

Sampled-data control is commonly used in digital control and communication and has been extensively studied in the early 1990's [4, 9, 11, 38, 81, 88]. There are several methods to deal with SD systems: direct method [95], frequency domain method [74] and lifting method [87].

There are some works about SD systems with I/O delays, e.g. [23,25,69]. Lyapunov-Krasovskii functional is used to get sufficient stability conditions for the H_∞ control problem in [23, 25] where the controller relies on the bound of I/O delay and it is a suboptimal result. A lifting method is used in [69] to deal with single time delay H_2 problem. To the best of our knowledge, SD control for systems with multiple input delays has not been solved yet. The work in this chapter is primarily motivated by Zhang et al. [97, 98] where the LQR problem for both the discrete-time and continuous-time systems with multiple input delays is studied.

In this chapter, we study SD systems with zero-order hold (ZOH) controllers. We first convert the sampled-data LQR problem into the LQR for a continuous-time system by introducing Dirac delta function. The 'continuous-time' system has multiple input delays, for which the LQR solution can be obtained using the result of [98]. After some basic algebraic manipulations, we derive the LQR sampled-data controller with ZOH.

8.2 Problem Statement

In this section, we formulate the sampled-data LQR problem for systems with the zero-order hold.

Consider the system with multiple input delays described by:

$$\dot{x}(t) = A_t x(t) + \sum_{i=1}^{d} B_{i,t} u_i(t - h_i) \tag{8.1}$$

where $x(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^{m_i}$ are the system state and the i_{th} control input, respectively, h_i, $i = 1, 2, \cdots, d$, is the i_{th} input delay which is assumed to be known. Without loss of generality, we assume that $0 < h_1 < h_2 < \cdots < h_d$ and $m_i = m$, $i = 1, 2, \cdots, d$. The initial state is

$$x(t) = \phi(t), \quad -h_d \leq t \leq 0. \tag{8.2}$$

The cost function is given by

$$\begin{aligned} J_{t_f} &= x_{t_f}^T P_{t_f} x_{t_f} + \int_0^{t_f} x^T(t) Q x(t) dt + \sum_{i=1}^{d} \int_0^{t_f - h_i} u_i^T(t) R_{1,i} u_i(t) dt + \\ &+ \sum_{i=1}^{d} \sum_{j | jp \in [0, t_f - h_i], j \in \mathbb{Z}} (u_i^T(jp) - u_i^T(jp^-)) R_{2,i} (u_i(jp) - u_i(jp^-)) \end{aligned} \quad (8.3)$$

where $u_i(jp^-) = u_i(t)|(t < jp \text{ and } t - jp \to 0)$, p is the sampling period, t_f is the time horizon under consideration, and the matrices P_{t_f}, $R_{1,i}$, $R_{2,i}$ and Q are weighting matrices with $R_{2,i}$ positive definite and $R_{1,i}$ and Q positive semi-definite. Here, we do not require $R_{1,i}$ to be positive definite since the zero-order hold will be used and we are concerned with input at sampling instants. In order to simplify the discussion, we assume that h_i and t_f are multiple of p, i.e. $h_i = \kappa_i p$ and $t_f = Np$ where $\kappa_i, N \in \mathbb{N}$. We also assume that $u_i(t) = 0$ for $t < 0$.

Remark 8.2.1. *Note that in the standard H_2 and H_∞ sampled-data control (see [9] and [81], respectively), the cost function only involves the first control input term in (8.3) which can be considered as the total energy cost. The second control input term in (8.3) can be considered as the penalty on input jump at sampling instant and it is introduced also for the reason that in our approach to be introduced later, we need to use this term for controller design [77]. Observe that since we shall only consider the zero-order hold, the cost function of (8.3) is also meaningful if we set $R_{1,i}$ to be zero.*

Since $u_i(t)$ is constant for $kp \leq t < (k+1)p$, we can write $u_i(t)$ in the form:

$$u_i(t) = \int_0^t \tilde{u}_i(\tau) d\tau, \ \tilde{u}_i(\tau) = \sum_{j=0}^{N-\kappa_i} \mu_{i,j} \delta(\tau - jp) \quad (8.4)$$

where $\delta(t)$ is the Dirac delta function.

The problem of sampled-data LQR with the zero-order hold is then stated as:

Find the control inputs $u_i(t),\ 0 \le t \le t_f - h_i$ *or equivalently* $\mu_{i,j}$ *of (8.4),* $j = 0, 1, \cdots, N - \kappa_i,\ i = 1, 2, \cdots, d$ *based on the sampled state* $x(kp),\ kp \le t,$ *that minimize the cost (8.3).*

8.3 Design of Sampled-Data LQR Controller

In this section, we shall present a solution to the sampled-data LQR problem. We adopt the approach in [77] where the sampled-data H_∞ control problem for systems without delays is tackled by augmenting the system state with control input.

First, denote

$$\tilde{x}(t) \triangleq \begin{pmatrix} x(t) \\ u_1(t-h_1) \\ \vdots \\ u_d(t-h_d) \end{pmatrix}, \tag{8.5}$$

$$\tilde{u}(t) \triangleq \begin{cases} 0, & 0 \le t < h_1, \\ \begin{pmatrix} \tilde{u}_1(t-h_1) \\ \vdots \\ \tilde{u}_i(t-h_i) \end{pmatrix}, & h_i \le t < h_{i+1}, \\ \begin{pmatrix} \tilde{u}_1(t-h_1) \\ \vdots \\ \tilde{u}_d(t-h_d) \end{pmatrix}, & t \ge h_d, \end{cases} \tag{8.6}$$

$$\check{u}(t) \triangleq \begin{cases} \begin{pmatrix} 0 \\ \tilde{u}_1(t-h_1) \\ \vdots \\ \tilde{u}_d(t-h_d) \end{pmatrix}, & 0 \leq t < h_1, \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{u}_{i+1}(t-h_{i+1}) \\ \vdots \\ \tilde{u}_d(t-h_d) \end{pmatrix}, & h_i \leq t < h_{i+1}, \\ 0, & t \geq h_d. \end{cases} \tag{8.7}$$

In view of the assumption that $u_i(t) = 0$ for $t < 0$, (8.1) can be expressed as the following equivalent augmented system:

$$\begin{aligned} \dot{\tilde{x}}(t) &= \begin{cases} \bar{A}_t\tilde{x}(t) + \sum_{l=1}^{i} \bar{B}_{l,t}\tilde{u}_l(t-h_l) + \check{u}(t), & h_i \leq t < h_{i+1} \\ \bar{A}_t\tilde{x}(t) + \sum_{l=1}^{d} \bar{B}_{l,t}\tilde{u}_l(t-h_l), & t \geq h_d \end{cases} && (8.8) \\ &= \begin{cases} \bar{A}_t\tilde{x}(t) + \bar{B}_t\tilde{u}(t) + \check{u}(t), & h_i \leq t < h_{i+1} \\ \bar{A}_t\tilde{x}(t) + \bar{B}_t\tilde{u}(t), & t \geq h_d, \end{cases} && (8.9) \end{aligned}$$

where

$$\bar{A}_t = \begin{pmatrix} A_t & B_{1,t} & \cdots & B_{d,t} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{8.10}$$

$$\bar{B}_{l,t} = \begin{cases} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, & t \geq h_l, \\ 0, & t < h_l, \end{cases} \tag{8.11}$$

$$\bar{B}_t = \begin{cases} 0, & 0 \leq t < h_1, \\ [\bar{B}_{1,t}, \cdots, \bar{B}_{i,t}], & h_i \leq t < h_{i+1}, \\ [\bar{B}_{1,t}, \cdots, \bar{B}_{d,t}], & t \geq h_d, \end{cases} \tag{8.12}$$

and the identity matrix I in $\bar{B}_{l,t}$ is in the $(l+1)-th$ row block.

By taking into consideration of (8.4), it is easy to know that the cost function (8.3) can be rewritten as

$$\begin{aligned} J_{t_f} &= x_{t_f}^T P_{t_f} x_{t_f} + \sum_{i=1}^{d} \int_0^{t_f} u_i^T(t-h_i) R_{1,i} u_i(t-h_i) dt + \int_0^{t_f} x^T(t) Q x(t) dt \\ &\quad + \sum_{i=1}^{d} \int_0^{t_f - h_i} \tilde{u}_i^T(t) \left(R_{2,i} \frac{1}{\sum_{j=0}^{N-1} \delta(t-jp)} \right) \tilde{u}_i(t) dt \qquad (8.13) \\ &= \tilde{x}_{t_f}^T \tilde{P}_{t_f} \tilde{x}_{t_f} + \int_0^{t_f} \tilde{x}^T(t) \tilde{Q}_t \tilde{x}(t) dt + \int_0^{t_f} \tilde{u}^T(t) \tilde{R}_t \tilde{u}(t) dt, \qquad (8.14) \end{aligned}$$

where

$$\begin{aligned}
\tilde{x}_{t_f} &= \tilde{x}(t_f), & (8.15)\\
\tilde{P}_{t_f} &= diag\{P_{t_f}, 0, \cdots, 0\}, & (8.16)\\
\tilde{Q}_t &= diag\{Q, R_{1,1}, \cdots, R_{1,d}\}, & (8.17)\\
\tilde{R}_t &= \frac{1}{\sum_{j=0}^{N-1} \delta(t - jp)} \bar{R}_t, & (8.18)\\
\bar{R}_t &= \begin{cases} 0, & 0 \le t < h_1, \\ diag\{R_{2,1}, \cdots, R_{2,i}\}, & h_i \le t < h_{i+1}, \\ diag\{R_{2,1}, \cdots, R_{2,d}\}, & t \ge h_d. \end{cases} & (8.19)
\end{aligned}$$

We observe that the cost function (8.14) is now in the form of standard quadratic cost function for continuous-time systems. Note that the LQR problem for continuous-time systems with input delays has been addressed in [98] where the analogue state $x(t)$ is available for feedback.

In the following, we shall apply the result of [98] and demonstrate how the optimal control based on the sampled state $x(jp)$ can be obtained.

To this end, we introduce the following RDE:

$$-\frac{dP(t)}{dt} = \bar{A}_t^T P(t) + P(t)\bar{A}_t + \tilde{Q}_t - K_t \tilde{R}_t K_t^T, \quad P(t_f) = \tilde{P}_{t_f}, \tag{8.20}$$

associated with the system (8.9) and the cost (8.14), where

$$K_t = P(t)\bar{B}_t(\tilde{R}_t)^{-1} \tag{8.21}$$

and $\bar{B}_t$ is defined in (8.12).

Note that

$$\begin{aligned} K_t\tilde{R}_tK_t^T &= P(t)\bar{B}_t\tilde{R}_t^{-1}\tilde{R}_t\tilde{R}_t^{-1}\bar{B}_t^TP(t) \\ &= P(t)\bar{B}_t\sum_{j=0}^{N-1}\delta(t-jp)\bar{R}_t^{-1}\bar{B}_t^TP(t), \end{aligned} \tag{8.22}$$

which implies that $K_t\tilde{R}_tK_t^T = 0$ when $t \in [(j-1)p, jp)$ and $K_t\tilde{R}_tK_t^T \to \infty$ for $t = jp$.

Therefore, for $t \in [(j-1)p, jp)$, the RDE (8.20) becomes

$$-\frac{dP(t)}{dt} = \bar{A}_t^TP(t) + P(t)\bar{A}_t + \tilde{Q}_t, \tag{8.23}$$

whereas at sampling instant $t = jp$, (8.20) can be rewritten as

$$-\dot{P}(t) = -K_t\tilde{R}_tK_t^T. \tag{8.24}$$

By taking into consideration of (8.21), we have

$$-\dot{P}(t) = -P(t)\bar{B}_t\tilde{R}_t^{-1}\bar{B}_t^TP(t) = -P(t)\bar{B}_t\sum_{j=0}^{N-1}\delta(t-jp)\bar{R}_t^{-1}\bar{B}_t^TP(t) \tag{8.25}$$

or yet

$$\frac{d(P^{-1}(t))}{dt} = -\bar{B}_t\sum_{j=0}^{N-1}\delta(t-jp)\bar{R}_t^{-1}\bar{B}_t^T. \tag{8.26}$$

We now approximate $\delta(t)$ by $\frac{1}{\Delta}[\delta_{-1}(t) - \delta_{-1}(t-\Delta)]$ where δ_{-1} is the unit Heaviside step function at $t = 0$ [77]. By taking the integration of (8.26) from jp^- to jp, we obtain

$$-P^{-1}(jp) + P^{-1}(jp^-) = \bar{B}_{jp}\bar{R}_{jp}^{-1}\bar{B}_{jp}^T,$$

i.e.,

$$P(jp^-) = P(jp)[I + \bar{B}_{jp}\bar{R}_{jp}^{-1}\bar{B}_{jp}^T P(jp)]^{-1}. \quad (8.27)$$

From (8.20) and (8.27), we know that $P(t)$ will be continuous inter-samples but there is a jump at each sampling instant.

With the solution of the RDE (8.23) and (8.27), the optimal control is given by [98]:

$$\begin{aligned}\tilde{u}_i^*(jp) &= -\bar{K}(jp+h_i)\tilde{x}(0) - \int_0^{h_i^-} \bar{K}_1(jp+h_i,s)\check{u}^*(s)ds - \int_{h_i}^{h_d} \bar{K}_2(jp+h_i,s)\check{u}^*(s)ds \\ &= -\bar{K}_{10}(jp+h_i)x(0) - \bar{K}_{11}(jp+h_i)u_1(-h_1) - \cdots - \bar{K}_{1d}(jp+h_i)u_d(-h_d) \\ &\quad - \int_0^{h_i^-} \bar{K}_1(jp+h_i,s)\check{u}^*(s)ds - \int_{h_i}^{h_d} \bar{K}_2(jp+h_i,s)\check{u}^*(s)ds \quad (8.28)\end{aligned}$$

where h_i^- is defined as $h_i^- < h_i$ and $(h_i^- - h_i) \to 0$,

$$\begin{aligned}\bar{K}(jp+h_i) &= R_{2,i}^{-1}\sum_{k=0}^{N-1}\delta(jp-kp)\bar{B}_{i,jp+h_i}^T P(jp+h_i)[\bar{\Psi}(0,jp+h_i)]^T, \quad (8.29) \\ \bar{K}_1(jp+h_i,s) &= R_{2,i}^{-1}\sum_{k=0}^{N-1}\delta(jp-kp)\bar{B}_{i,jp+h_i}^T P(jp+h_i) \\ &\quad \times[\bar{\Psi}(s,jp+h_i)]^T[I_n - G(s)P(s)], \quad (8.30) \\ \bar{K}_2(jp+h_i,s) &= R_{2,i}^{-1}\sum_{k=0}^{N-1}\delta(jp-kp)\bar{B}_{i,jp+h_i}^T[I_n - P(jp+h_i)G(jp+h_i)] \\ &\quad \times\bar{\Psi}(jp+h_i,s)P(s), \quad (8.31) \\ \bar{K}(\cdot) &= [\bar{K}_{10}(\cdot)\ \bar{K}_{11}(\cdot)\ \cdots\ \bar{K}_{1d}(\cdot)], \quad (8.32)\end{aligned}$$

and

$$\begin{aligned}G(s) &= \int_0^s [\bar{\Psi}(r,s)]^T\bar{B}_r\tilde{R}_r^{-1}\bar{B}_r^T\bar{\Psi}(r,s)dr \\ &= \sum_{j|0\le jp\le s}[\bar{\Psi}(jp,s)]^T\bar{B}_{jp}\bar{R}_{jp}^{-1}\bar{B}_{jp}^T\bar{\Psi}(jp,s), \quad (8.33)\end{aligned}$$

while $\bar{\Psi}(s,\cdot)$ is the transition matrix of

$$-\tilde{A}_s = K_s\bar{B}_s^T - \bar{A}_s^T. \tag{8.34}$$

Recall the definition

$$\tilde{u}_i(\tau) = \sum_{k=0}^{N-\kappa_i} \delta(\tau - kp)\mu_{i,k}.$$

Then, by taking integration of (8.28) from jp^- to jp and taking into consideration (8.29)-(8.32), we get

$$\begin{aligned}\mu_{i,j}^* &= -K_{10}(jp+h_i)x(0) - K_{11}(jp+h_i)u_1(-h_1) - \cdots - K_{1d}(jp+h_i)u_d(-h_d)\\ &\quad - \int_0^{h_i^-} K_1(jp+h_i,s)\check{u}^*(s)ds - \int_{h_i}^{h_d} K_2(jp+h_i,s)\check{u}^*(s)ds \end{aligned}\tag{8.35}$$

where

$$\begin{aligned}K(jp+h_i) &= R_{2,i}^{-1}\bar{B}_{i,jp+h_i}^T P(jp+h_i)[\bar{\Psi}(0,jp+h_i)]^T,\\ K_1(jp+h_i,s) &= R_{2,i}^{-1}\bar{B}_{i,jp+h_i}^T P(jp+h_i)[\bar{\Psi}(s,jp+h_i)]^T[I_n - G(s)P(s)],\\ K_2(jp+h_i,s) &= R_{2,i}^{-1}\bar{B}_{i,jp+h_i}^T [I_n - P(jp+h_i)G(jp+h_i)]\bar{\Psi}(jp+h_i,s)P(s),\\ K(\cdot) &= [K_{10}(\cdot)\ K_{11}(\cdot)\ \cdots\ K_{1d}(\cdot)].\end{aligned}$$

We further note that

$$\begin{aligned}
&\int_0^{h_i^-} K_1(jp+h_i,s)\check{u}^*(s)ds \\
&= \sum_{k=0}^{i-1}\int_{h_k}^{h_{k+1}^-} K_1(jp+h_i,s)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{l=0}^{N-1}\mu_{k,l}\delta(s-h_k-lp) \\ \vdots \\ \sum_{l=0}^{N-1}\mu_{d,l}\delta(s-h_d-lp)\end{pmatrix} ds \\
&= 0,
\end{aligned} \tag{8.36}$$

where $h_0 \triangleq 0$.

Similarly,

$$\begin{aligned}
&\int_{h_i}^{h_d} K_2(jp+h_i,s)\check{u}^*(s)ds \\
&= \sum_{k=i+1}^{d}\int_{h_{k-1}}^{h_k} K_2(jp+h_i,s)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{l=0}^{N-1}\mu_{k,l}\delta(s-h_k-lp) \\ \vdots \\ \sum_{l=0}^{N-1}\mu_{d,l}\delta(s-h_d-lp)\end{pmatrix} ds \\
&= 0.
\end{aligned} \tag{8.37}$$

Moreover, note that $u_i(\tau)=0$ if $\tau<0$, the optimal control is given by

$$u_i^*(jp) = -K_{10}(jp+h_i)x(0). \tag{8.38}$$

However, the above optimal control is given in terms of the initial state $x(0)$ rather than the current state $x(jp)$. This problem can be addressed by shifting the time interval from $[0, h_d]$ to $[\tau, \tau + h_d]$. Here, we introduce the following notations. For any given $\tau \geq 0$, denote

$$
\begin{aligned}
\tilde{x}^\tau(t) &\triangleq \begin{pmatrix} x(\tau+t) \\ u_1(\tau+t-h_1) \\ \vdots \\ u_d(\tau+t-h_d) \end{pmatrix}, \\
\tilde{u}^\tau(t) &\triangleq \begin{cases} 0, & 0 \leq t < h_1 \\ \begin{pmatrix} \tilde{u}_1(\tau+t-h_1) \\ \vdots \\ \tilde{u}_i(\tau+t-h_i) \end{pmatrix}, & h_i \leq t < h_{i+1} \\ \begin{pmatrix} \tilde{u}_1(\tau+t-h_1) \\ \vdots \\ \tilde{u}_d(\tau+t-h_d) \end{pmatrix}, & t \geq h_d, \end{cases} \\
\breve{u}^\tau(t) &\triangleq \begin{cases} \begin{pmatrix} 0 \\ \tilde{u}_1(\tau+t-h_1) \\ \vdots \\ \tilde{u}_d(\tau+t-h_d) \end{pmatrix}, & 0 \leq t < h_1 \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{u}_l(\tau+t-h_l) \\ \vdots \\ \tilde{u}_d(\tau+t-h_d) \end{pmatrix}, & h_l \leq t < h_{l+1},\ t+\tau \leq t_f \\ 0, & t \geq h_d. \end{cases}
\end{aligned}
$$

Define the following RDE:

$$-\frac{dP^{\tau}(t)}{dt} = \bar{A}^T_{\tau+t}P^{\tau}(t) + P^{\tau}(t)\bar{A}_{\tau+t} + \tilde{Q}_{\tau+t} - K^{\tau}_t\tilde{R}^{\tau}_t(K^{\tau}_t)^T \tag{8.39}$$

with initial condition $P^{\tau}(t_f - \tau) = \tilde{P}_{t_f}$, where

$$\tilde{R}^{\tau}_t = \frac{1}{\sum_{j=0}^{N-1}\delta(t+\tau-jp)}\bar{R}^{\tau}_t, \tag{8.40}$$

$$\bar{R}^{\tau}_t = \begin{cases} 0, & 0 \le t < h_1 \\ diag\{R_{2,1}, \cdots, R_{2,i}\}, & h_i \le t < h_{i+1} \\ diag\{R_{2,1}, \cdots, R_{2,d}\}, & t \ge h_d, \end{cases} \tag{8.41}$$

$$\bar{B}^{\tau}_t = \begin{cases} 0, & 0 \le t < h_1 \\ [\bar{B}_{1,t+\tau}, \cdots, \bar{B}_{i,t+\tau}], & h_i \le t < h_{i+1} \\ [\bar{B}_{1,t+\tau}, \cdots, \bar{B}_{d,t+\tau}], & t \ge h_d, \end{cases} \tag{8.42}$$

$$K^{\tau}_t = P^{\tau}(t)\bar{B}^{\tau}_t(\tilde{R}^{\tau}_t)^{-1}. \tag{8.43}$$

Furthermore, the RDE (8.39) is equivalent to

$$\begin{aligned} -\frac{dP^{\tau}(t)}{dt} &= \bar{A}^T_{\tau+t}P^{\tau}(t) + P^{\tau}(t)\bar{A}_{\tau+t} + \tilde{Q}_{\tau+t}, \ \tau \ne jp, t \in [0,p), j = 0, 1, \cdots, \\ P^{jp}(0^-) &= P^{jp}(0)[I + \bar{B}^{jp}_0(\bar{R}^{jp}_0)^{-1}(\bar{B}^{jp}_0)^TP^{jp}(0)]^{-1}. \end{aligned} \tag{8.44}$$

Similar to the case of $\tau = 0$, the optimal controller can be expressed as

$$\begin{aligned} \mu^*_{i,j} &= -K^{jp}(h_i)\tilde{x}^{jp}(0) - \int_0^{h_i^-} K^{jp}_1(h_i,s)\breve{u}^{jp*}(s)ds - \int_{h_i}^{h_d} K^{jp}_2(h_i,s)\breve{u}^{jp*}(s)ds \\ &= -K^{jp}_{10}(h_i)x(jp) - K^{jp}_{11}(h_i)u_1(jp-h_1) - \cdots - K^{jp}_{1d}(h_i)u_d(jp-h_d) \\ &\quad - \int_0^{h_i^-} K^{jp}_1(h_i,s)\breve{u}^{jp*}(s)ds - \int_{h_i}^{h_d} K^{jp}_2(h_i,s)\breve{u}^{jp*}(s)ds, \end{aligned} \tag{8.45}$$

where

$$
\begin{aligned}
K^{jp}(h_i) &= R_{2,i}^{-1}\bar{B}_{i,jp+h_i}^{T}P^{jp}(h_i)[\bar{\Psi}^{jp}(0,h_i)]^{T}, && (8.46)\\
K_1^{jp}(h_i,s) &= R_{2,i}^{-1}\bar{B}_{i,jp+h_i}^{T}P^{jp}(h_i)[\bar{\Psi}^{jp}(s,h_i)]^{T}[I_n - G^{jp}(s)P^{jp}(s)], && (8.47)\\
K_2^{jp}(h_i,s) &= R_{2,i}^{-1}\bar{B}_{i,jp+h_i}^{T}[I_n - P^{jp}(h_i)G^{jp}(h_i)]\bar{\Psi}^{jp}(h_i,s)P^{jp}(s), && (8.48)\\
G^{jp}(s) &= \int_0^s [\bar{\Psi}^{jp}(r,s)]^T\bar{B}_r^{jp}(\tilde{R}_r^{jp})^{-1}(\bar{B}_r^{jp})^T\bar{\Psi}^{jp}(r,s)dr\\
&= \sum_{k|0\le kp\le s}[\bar{\Psi}^{jp}(kp,s)]^T\bar{B}_{kp}^{jp}(\bar{R}_{kp})^{-1}(\bar{B}_{kp}^{jp})^T\bar{\Psi}^{jp}(kp,s), && (8.49)\\
K^{jp}(h_i) &= [K_{10}^{jp}(h_i)\ K_{11}^{jp}(h_i)\ \cdots\ K_{1d}^{jp}(h_i)], && (8.50)
\end{aligned}
$$

and $\bar{\Psi}^{\tau}(s,\centerdot)$ is the transition matrix of

$$
-\tilde{A}_s^{\tau} = K_s^{\tau}(\bar{B}_s^{\tau})^T - (\bar{A}_s^{\tau})^T. \tag{8.51}
$$

Then,

$$
\begin{aligned}
\int_0^{h_i^-} K_1^{jp}(h_i,s)\breve{u}^{jp*}(s)ds &= \sum_{k=1}^{i}\int_{h_{k-1}}^{h_k^-}K_1^{jp}(h_i,s)\begin{pmatrix}0\\ \vdots\\ 0\\ \sum_{l=0}^{N-1}\mu_{k,l}\delta(s-h_k-lp+jp)\\ \vdots\\ \sum_{l=0}^{N-1}\mu_{d,l}\delta(s-h_d-lp+jp)\end{pmatrix}ds\\
&= \sum_{k=1}^{i}\sum_{\zeta=h_{k-1}/p}^{h_k/p-1}K_1^{jp}(h_i,\zeta p)\begin{pmatrix}0\\ \vdots\\ 0\\ \mu_{k,j+\zeta-h_k/p}\\ \vdots\\ \mu_{d,j+\zeta-h_d/p}\end{pmatrix}.
\end{aligned}
$$

Similarly,

$$\int_{h_i}^{h_d} K_2^{jp}(h_i, s)\check{u}^{jp*}(s)ds = \sum_{k=i+1}^{d} \sum_{\zeta=h_{k-1}/p}^{h_d/p} K_2^{jp}(h_i, \zeta p) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu_{k,j+\zeta-h_k/p} \\ \vdots \\ \mu_{d,j+\zeta-h_d/p} \end{pmatrix}.$$

So the controller (8.45) is simplified into

$$\begin{aligned} \mu_{i,j}^* &= -K_{10}^{jp}(h_i)x(jp) - K_{11}^{jp}(h_i)u_1(jp-h_1) - \cdots - K_{1d}^{jp}(h_i)u_d(jp-h_d) \\ &- \sum_{k=1}^{i} \sum_{\zeta=h_{k-1}/p}^{h_k/p-1} K_1^{jp}(h_i, \zeta p) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu_{k,j+\zeta-h_k/p} \\ \vdots \\ \mu_{d,j+\zeta-h_d/p} \end{pmatrix} - \sum_{k=i+1}^{d} \sum_{\zeta=h_{k-1}/p}^{h_d/p} K_2^{jp}(h_i, \zeta p) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu_{k,j+\zeta-h_k/p} \\ \vdots \\ \mu_{d,j+\zeta-h_d/p} \end{pmatrix}, \\ &= -K_{10}^{jp}(h_i)x(jp) - K_{11}^{jp}(h_i)u_1(jp-h_1) - \cdots - K_{1d}^{jp}(h_i)u_d(jp-h_d) \\ &- \sum_{k=1}^{i} \sum_{\zeta=\kappa_{k-1}}^{\kappa_k-1} K_1^{jp}(h_i, \zeta p) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu_{k,j+\zeta-\kappa_k} \\ \vdots \\ \mu_{d,j+\zeta-\kappa_d} \end{pmatrix} - \sum_{k=i+1}^{d} \sum_{\zeta=\kappa_{k-1}}^{\kappa_d} K_2^{jp}(h_i, \zeta p) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu_{k,j+\zeta-\kappa_k} \\ \vdots \\ \mu_{d,j+\zeta-\kappa_d} \end{pmatrix} \end{aligned} \quad (8.52)$$

where $\mu_{\cdot,j} = 0$ when $j < 0$. The last equality is derived according to the definition of h_k and $\kappa_0 \triangleq 0$.

Theorem 8.3.1. *Given linear time-varying system (8.1) with multiple input delays and the LQR cost function (8.3), the optimal state feedback sampled-data controller with ZOH is given by*

$$u_i(t) = \sum_{j=0}^{k} \mu_{i,j}, \quad kp \le t < (k+1)p, i = 1, 2, \cdots, d, \tag{8.53}$$

where $u_{i,j}$ is defined as (8.52).

8.4 Example

Here we present a simple example to show the efficiency of our approach.

Introduce a system

$$\dot{x}(t) = x(t) + u(t - h_1)$$

with quadratic performance

$$\begin{aligned} J_{t_f} &= x_{t_f}^T P_{t_f} x_{t_f} + \int_0^{t_f} x^T(t) Q x(t) dt + \int_0^{t_f - h_1} u^T(t) R_1 u(t) dt \\ &+ \sum_{j | jp \in [0, t_f - h_1]} (u(jp) - u(jp^-))^T R_2 (u(jp) - u(jp^-)), \end{aligned}$$

where $p = 0.1,\ h_1 = 1,\ Q = 2,\ R_1 = R_2 = 0.5,\ t_f = 10,\ P_{t_f} = 3$. We further assume that $x(0) = 1$ and $u(t) = 0,\ t < 0$.

According to Section 8.3, we have

$$\begin{aligned}\bar{A}_t &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \\ \bar{B}_t &= \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & t \geq 1, \\ 0, & t < 1. \end{cases}\end{aligned}$$

So the RDEs (Riccati differential equations) are

$$-\frac{dP^{\tau}(t)}{dt} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} P^{\tau}(t) + P^{\tau}(t) \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}, \tau = 0, 0.1, \cdots, 10, t \in [0, 0.1) \tag{8.54}$$

$$\begin{aligned}P^{jp}(0^-) &= P^{jp}(0) \left[I + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} P^{jp}(0) \right]^{-1}, \\ P^{10}(0) &= P(10) = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \; j = 0, 1, \cdots, 10.\end{aligned}$$

Now, the RDE (8.54) leads to

$$\begin{aligned}-\dot{P}^{\tau}_{11} &= 2P^{\tau}_{11} + 2, \\ -\dot{P}^{\tau}_{12} &= P^{\tau}_{11} + P^{\tau}_{12}, \\ P^{\tau}_{21} &= P^{\tau}_{12}, \\ -\dot{P}^{\tau}_{22} &= 2P^{\tau}_{12} + 0.5.\end{aligned}$$

Hence, for $t, t_0 \in [0, 0.1)$,

$$\begin{aligned}
P_{11}^{\tau}(t) &= e^{-2(t-t_0)}P_{11}^{\tau}(t_0) + \int_{t_0}^{t} e^{-2(t-\sigma)}(-2)d\sigma \\
&= e^{2t_0-2t}(P_{11}^{\tau}(t_0)+1) - 1, \\
P_{12}^{\tau}(t) &= e^{-(t-t_0)}P_{12}^{\tau}(t_0) + \int_{t_0}^{t} e^{-(t-\sigma)}(-P_{11}^{\tau}(\sigma))d\sigma \\
&= e^{-(t-t_0)}P_{12}^{\tau}(t_0) + (P_{11}^{\tau}(t_0)+1)(e^{2t_0-2t} - e^{t_0-t}) + (1 - e^{-t+t_0}), \\
P_{22}^{\tau}(t) &= P_{22}^{\tau}(t_0) - \int_{t_0}^{t}(2P_{12}^{\tau}(\sigma) + 0.5)d\sigma \\
&= P_{22}^{\tau}(t_0) - 2\int_{t_0}^{t} P_{12}^{\tau}(\sigma)d\sigma - 0.5(t-t_0) \\
&= P_{22}^{\tau}(t_0) - 0.5(t-t_0) - 2[-e^{t_0-t}P_{12}^{\tau}(t_0) + P_{12}^{\tau}(t_0) \\
&\quad +(P_{11}^{\tau}(t_0)+1)(-\frac{1}{2}e^{2t_0-2t} + \frac{1}{2} + e^{t_0-t} - 1) + (t-t_0) + e^{t_0-t} - 1].
\end{aligned}$$

The solution of $P(t)$ is shown in Figure 8.1, we can see that $P(jp), j = 0, 1, \cdots, 99$, backward converges to a constant matrix and there is a jump at each sampling time $jp, \ j = 1, \cdots, 99$.

We also have

$$\begin{aligned}
-\tilde{A}_s^{\tau} &= K_s^{\tau}(\bar{B}_s^{\tau})^T - (\bar{A}_s^{\tau})^T \\
&= P^{\tau}(s)\bar{B}_s^{\tau}(\tilde{R}_s^{\tau})^{-1}(\bar{B}_s^{\tau})^T - (\bar{A}_s^{\tau})^T \\
&= -\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},
\end{aligned}$$

when $0 \le s < h_1$. The last equality is derived according to $\bar{B}_s^{\tau} = 0$, when $0 \le s < h_1$.

The transition matrix can then be achieved

$$\bar{\Psi}^{jp}(s, h_1) = \begin{pmatrix} e^{h_1-s} & 0 \\ -1 + e^{h_1-s} & 1 \end{pmatrix}, \ 0 \le s < h_1. \tag{8.55}$$

Then

$$\mu_{1,j}^* = -K^{jp}(h_1)\tilde{x}(jp) - \sum_{\zeta=0}^{h_1/p-1} K_1^{jp}(h_1, \zeta p) \begin{pmatrix} 0 \\ \mu_{1,j+\zeta-h_1/p} \end{pmatrix}, \tag{8.56}$$

where

$$\begin{aligned} K^{jp}(h_1) &= R_2^{-1}\bar{B}_{jp+h_1}^T P^{jp}(h_1)[\bar{\Psi}^{jp}(0,h_1)]^T, \\ K_1^{jp}(h_1,s) &= R_2^{-1}\bar{B}_{jp+h_1}^T P^{jp}(h_1)[\bar{\Psi}^{jp}(s,h_1)]^T[I_2 - G^{jp}(s)P^{jp}(s)] \end{aligned}$$

and $G^{jp}(s) = 0$ when $0 \leq s < h_1$, which is derived according to equation (8.49).

Furthermore, $u(t) = \sum_{j=0}^{k} \mu_{1,j},\ kp \leq t < (k+1)p$.

The control signal $u(t)$ and the trajectory of the closed-loop system state $x(t)$ are shown in Figure 8.2 and Figure 8.3, respectively. We can see that $x(t)$ converges to zero quickly. There is a glitch in the Figure 8.3 because we assume initial condition $u(t) = 0,\ t < 0$, and the input of the system is zero (i.e. open-loop) when $t \leq h_1$. The control begins to affect the performance of $x(t)$ after time h_1 and $x(t)$ is almost equal to 0 when $t > 5$.

8.5 Conclusion

We have studied the finite-horizon LQR problem for sampled-data systems with multiple input delays. Dirac delta function is used to transform the SD system into a 'continuous' LQR problem. The LQR SD controller is then derived from the continuous LQR controller by some algebraic manipulations. Compared with other approaches to sampled-data problems, our approach is easy to understand and has a simple expression. A simple example is provided to show that the proposed method is effective.

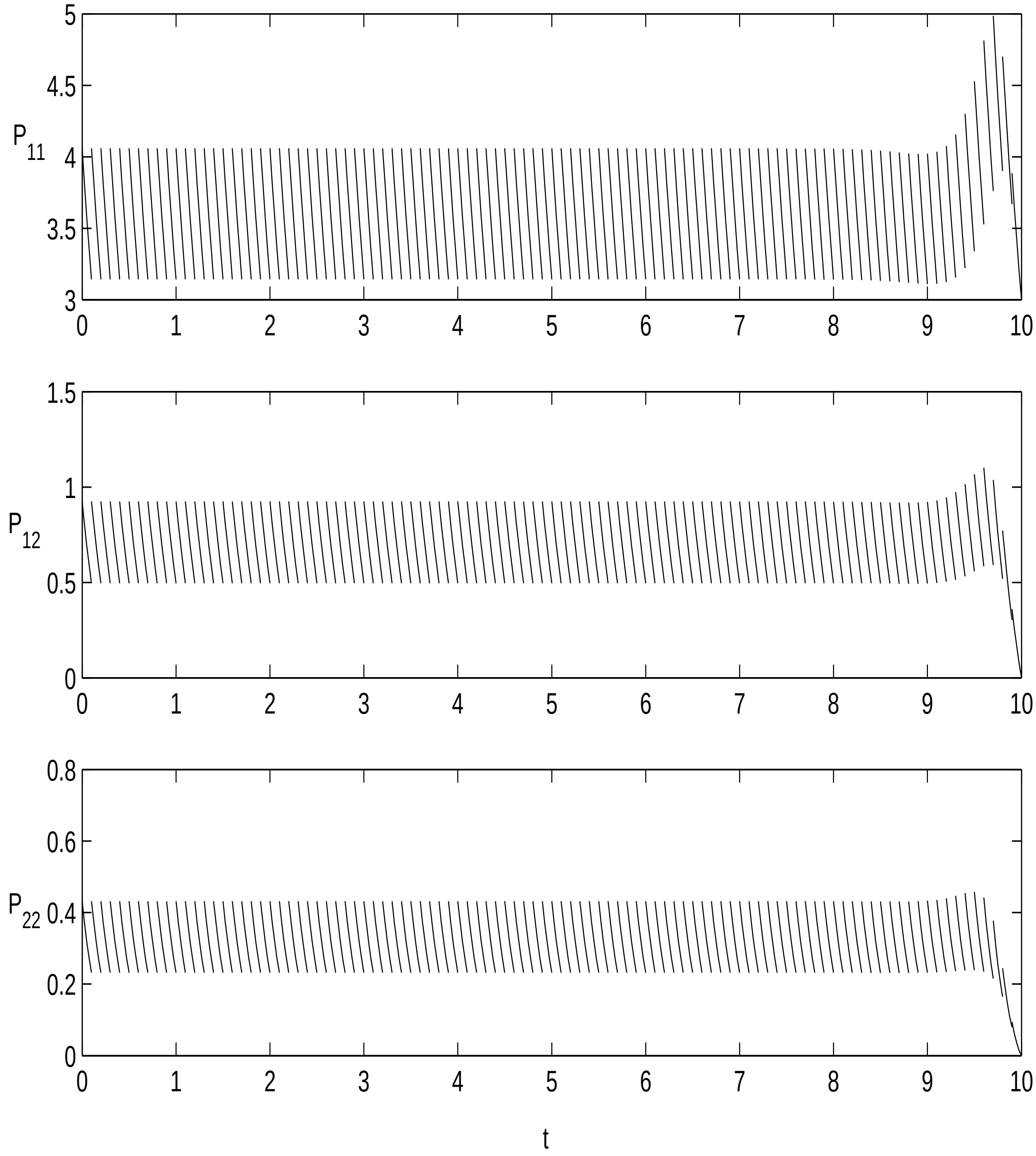

Figure 8.1: The solution of P(t).

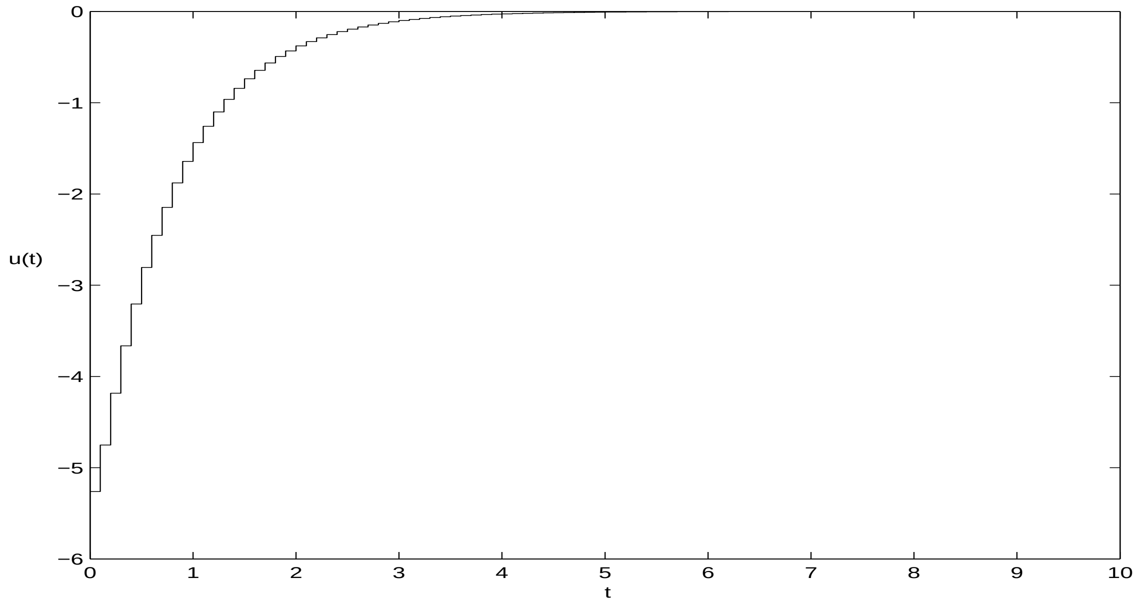

Figure 8.2: Control signal $u(t)$.

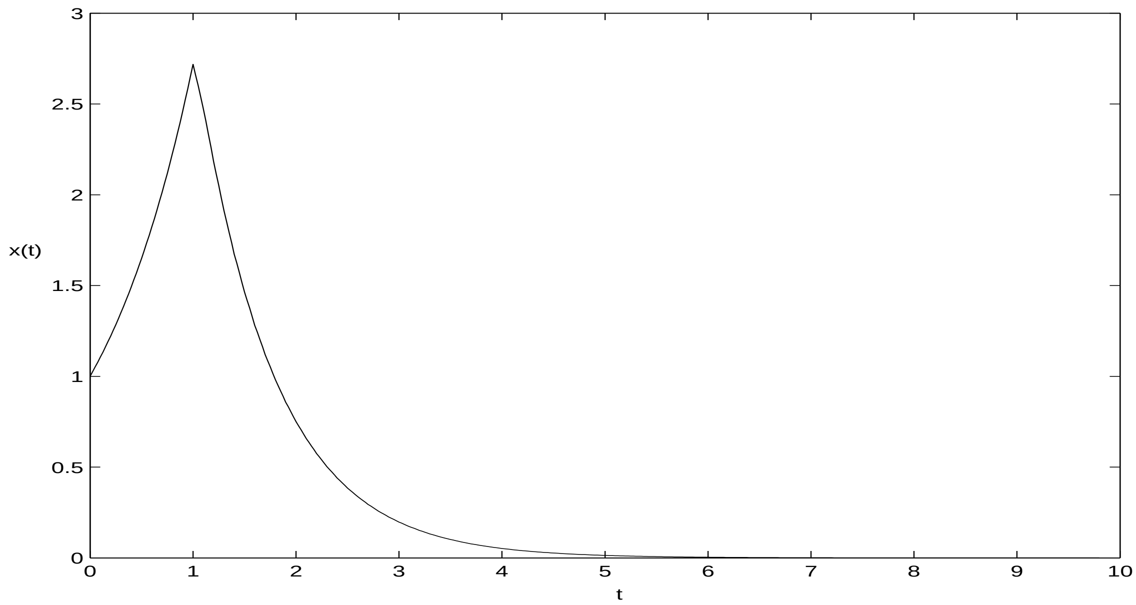

Figure 8.3: State trajectory.

Chapter 9

Conclusion and Future Work

9.1 Conclusions

In this thesis, we have discussed some issues of robust deconvolution and optimal and robust control of systems with input/output delays. We summarize the contributions of the thesis in the following aspects.

We investigated robust deconvolution for SISO and MIMO time-delay systems with a polynomial approach. The uncertainties under consideration appear in both the numerator and denominator of the system model and input signal model. Fictitious noises are introduced to evaluate the effect of the random parameter uncertainties and the robust deconvolution estimator is given in terms of a Diophantine equation and a spectral factorization. For the SISO case, the deconvolution estimator minimizes the mean square estimation error with respect to the uncertainties and noises whose covariances are of known lower and upper bounds. For the MIMO case, the covariances of uncertainties and noises are known and the covariances of the fictitious noises are simplified with some algebraic manipulations.

We provided a solution to the H_2 optimal state feedback control for systems with time delay in the input. The system with time delay is transformed to a delay-free time-varying system by state augmentation and the H_2 control design is then formulated as a bilinear matrix inequality (BMIs) problem for which a sequential linear programming method is applied.

We also addressed the H_∞ state feedback control for systems with time delay in input. A Lyapunov-Krasovskii functional method is applied to obtain a sufficient condition for stability analysis and an H_∞ controller is designed in terms of linear matrix inequalities.

We proposed solutions for the H_2 and H_∞ control for systems with multiple input delays. The key to our development is the duality between the LQR control with multiple input delays and a smoothing problem for an associated backward stochastic delay free system. We also tackled the LQG problem for systems with multiple I/O delays. A separation principle has been developed which converts the LQG problem into a LQR problem plus the Kalman filtering. The Kalman filtering with multiple output delays is derived using a re-organized innovation analysis. For the H_∞ case, we considered more general case than that in an existing literature by incorporating cross terms of state and exogenous inputs in the quadratic cost function.

Finally, we looked into the sampled-data LQR control with multiple input delays. Dirac delta function has been introduced to transform the hybrid (continuous/discrete-time) problem into a continuous-time form. State feedback controllers are expressed in terms of the solution of the 'continuous' LQR problem. The sampled data controller with zero-order hold is then obtained with some algebraic manipulations.

9.2 Future work

In Chapters 2 and 3, we have studied the robust deconvolution estimation for time-delay systems with random parameter uncertainties under the minimal mean square error criterion where partial information about the statistics of input noise is assumed. In the case when no information on the input noise statistics is available, a robust deconvolution under an H_∞ performance criterion would be desirable. Thus, it seems natural to extend the work in Chapters 2 and 3 to the robust H_∞ deconvolution estimation for systems with random parameter uncertainty.

In Chapters 6 and 7, we have investigated the finite horizon LQR, LQG and H_∞ control problems for systems with multiple input delays. It remains challenging to establish parallel results for the infinite horizon case where stability analysis for the closed-loop system has to be carried out. Furthermore, the state feedback control in Chapter 7 may be extended to the dynamic output feedback case.

We have investigated the optimal H_2 and H_∞ control problems for systems with multiple input delays. It would be interesting to study systems with uncertainty. In this case, the question of whether a similar duality between an optimal control problem and a smoothing estimation problem holds remains to be investigated. Another problem worthy of investigating is the output feedback H_∞ control problem for systems with multiple I/O delays.

In Chapter 8, we have investigated sampled-data LQR control problem for system with multiple input delays. It is worth investigating the H_2 and H_∞ sampled-data control problems for systems with input delay.

Appendix A Proof of Theorem 2.4.1

Proof: (a). Note from (2.23)-(2.24) that $e_0(k)$ and $v_0(k)$ can be expressed as

$$e_0(k) = \mathcal{D}_c^T(k)\mathcal{U}_e(k), \quad (A.1)$$

$$v_0(k) = \mathcal{A}_{bp}^T(k)\mathcal{Y}_u(k), \quad (A.2)$$

where $\mathcal{D}_c(k)$ and $\mathcal{A}_{bp}(k)$ are as defined in (2.7) and (2.10), respectively, and $\mathcal{U}_e(k)$ and $\mathcal{Y}_u(k)$ are as follows

$$\mathcal{U}_e(k) = \Big[u(k-1) \quad \cdots \quad u(k-nd); \quad e(k) \quad \cdots \quad e(k-nc)\Big]^T, \quad (A.3)$$

$$\mathcal{Y}_u(k) = \Big[y(k-1) \quad \cdots \quad y(k-na); \quad u(k-d) \quad \cdots \quad u(k-d-nb); \quad v(k) \quad \cdots \quad v(k-np)\Big]^T. \quad (A.4)$$

Observe that $\mathcal{D}_c(k)$ is independent of $\mathcal{U}_e(k)$ and $\mathcal{A}_{bp}(k)$ is independent of $\mathcal{Y}_u(k)$. From Assumption 2.2.1, we know that $e(k)$, $v(k)$, $e_0(k)$ and $v_0(k)$ are mutually uncorrelated. Actually, it is easy to verify that $E\left[e(k)v(k)\right] = E\left[e(k)e_0(k)\right] = E\left[e(k)v_0(k)\right] = E\left[v(k)e_0(k)\right] = E\left[v(k)v_0(k)\right] = E\left[e_0(k)v_0(k)\right] = 0$. Thus, $e(k)$, $v(k)$, $e_0(k)$ and $v_0(k)$ are mutually uncorrelated.

(b). Under Assumption 2.2.1, for any $k \neq s$, say $k > s$, $\mathcal{D}_c(k)$ is independent of

$\mathcal{D}_c(s)$, $\mathcal{U}_e(s)$ and $\mathcal{U}_e(k)$. Then, we have

$$\begin{aligned} E[e_0(k)e_0(s)] &= E\left[\mathcal{D}_c^T(k)\right] E\left[\mathcal{U}_e(k)\mathcal{U}_e^T(s)\mathcal{D}_c(s)\right] = 0, \\ E[e_0(k)] &= E\left[\mathcal{D}_c^T(k)\right] E\left[\mathcal{U}_e(k)\right] = 0, \end{aligned} \tag{A.5}$$

where $E\left[\mathcal{D}_c^T(k)\right] = 0$ has been applied. Hence, $e_0(k)$ is a white noise with zero mean. Similarly, we can show that $v_0(k)$ is a white noise with zero mean.

Next, we compute the covariances of $e_0(k)$ and $v_0(k)$. Define

$$\mathcal{R}_{ue} \triangleq E\left[\mathcal{U}_e(k)\mathcal{U}_e^T(k)\right], \quad \mathcal{R}_{yu} \triangleq E\left[\mathcal{Y}_u(k)\mathcal{Y}_u^T(k)\right]. \tag{A.6}$$

From (A.1), since $\mathcal{D}_c(k)$ is independent of $\mathcal{U}_e(k)$, it follows that

$$\sigma_e^0 = E\left[e_0^2(k)\right] = \tilde{E}\left[\mathcal{D}_c^T(k)E\{\mathcal{U}_e(k)\mathcal{U}_e^T(k)\}\mathcal{D}_c(k)\right] = \tilde{E}\left[\mathcal{D}_c^T(k)\mathcal{R}_{ue}\mathcal{D}_c(k)\right] \tag{A.7}$$

Similarly,

$$\sigma_v^0 = E\left[v_0^2(k)\right] = \tilde{E}\left[\mathcal{A}_{bp}^T(k)\mathcal{R}_{yu}\mathcal{A}_{bp}(k)\right]. \tag{A.8}$$

On the other hand, from (6.6) we have

$$\begin{aligned} \mathcal{U}_e(k) &= \begin{bmatrix} D^{-1}\mathcal{Q}_d\left\{Ce(k) + e_0(k)\right\} \\ \mathcal{Q}_c e(k) \end{bmatrix} \\ &= D^{-1}diag\{\mathcal{Q}_d, \mathcal{Q}_c\}\left\{\begin{bmatrix} C \\ D \end{bmatrix} e(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_0(k)\right\}, \end{aligned} \tag{A.9}$$

where $\mathcal{U}_e(k)$ is as defined in (A.3) and

$$\mathcal{Q}_d = \begin{bmatrix} q^{-1} & q^{-2} & \cdots & q^{-nd} \end{bmatrix}^T, \tag{A.10}$$

$$\mathcal{Q}_c = \begin{bmatrix} 1 & q^{-1} & \cdots & q^{-nc} \end{bmatrix}^T. \tag{A.11}$$

Since $e(k)$ and $e_0(k)$ are mutually uncorrelated white noises, using Parseval's formula yields the following

$$\begin{aligned} \mathcal{R}_{ue} &= E[\mathcal{U}_e(k)\mathcal{U}_e^T(k)] \\ &= \frac{1}{2\pi i}\oint_{|z|=1}(DD_*)^{-1}diag\{\mathcal{Q}_d, \mathcal{Q}_c\}\left\{\begin{bmatrix} C \\ D \end{bmatrix}\begin{bmatrix} C_* & D_* \end{bmatrix}\sigma_e + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\sigma_e^0\right\} \times \\ &\quad diag\{\mathcal{Q}_{d*}, \mathcal{Q}_{c*}\}\frac{dz}{z}. \end{aligned} \tag{A.12}$$

We multiply the left side of (A.12) by $\mathcal{D}_c^T(k)$ and the right side by $\mathcal{D}_c(k)$, and take the mathematical expectation $\tilde{E}$, it follows that

$$\begin{aligned} \sigma_e^0 &= \tilde{E}[\mathcal{D}_c^T(k)\mathcal{U}_e(k)\mathcal{U}_e^T(k)\mathcal{D}_c(k)] \\ &= \tilde{E}\frac{1}{2\pi i}\oint_{|z|=1}(DD_*)^{-1}\begin{bmatrix} -\Delta D & \Delta C \end{bmatrix}\left\{\begin{bmatrix} C \\ D \end{bmatrix}\begin{bmatrix} C_* & D_* \end{bmatrix}\sigma_e + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\sigma_e^0\right\} \\ &\quad \times \begin{bmatrix} -\Delta D_* \\ \Delta C_* \end{bmatrix}\frac{dz}{z}. \end{aligned} \tag{A.13}$$

Thus,

$$\sigma_e^0 = \sigma_e^0\gamma_0 + \gamma_1, \tag{A.14}$$

where

$$\gamma_0 = \frac{1}{2\pi i}\oint_{|z|=1}(DD_*)^{-1}\tilde{E}\left[\Delta D\Delta D_*\right]\frac{dz}{z}, \tag{A.15}$$

$$\gamma_1 = \frac{\sigma_e}{2\pi i}\oint_{|z|=1}(DD_*)^{-1}\tilde{E}\left[(-C\Delta D+D\Delta C)(-C\Delta D+D\Delta C)_*\right]\frac{dz}{z}, \tag{A.16}$$

By applying some simple algebra and noting that $\gamma_0 < 1$, γ_0 and γ_1 are obtained as (2.30) and (2.31), thus σ_e^0 of (2.28) is obtained from (A.14). By applying a similar discussion, we obtain (2.29). This completes the proof.

Author's Publications

1. H. Zhang, D. Zhang, L. Xie and J. Lin, "Robust filtering under stochastic parametric uncertainties," *Automatica*, vol.40, no.9, pp. 1583-1589, 2004.

2. J. Lin, L. Xie and H. Zhang, "An LMI approach to robust congestion control of ATM networks," *International Journal of Control, Automation, and Systems*, vol. 4, no. 1, pp. 53-62, Feb. 2006.

3. H. Hu, J. Lin and L. Xie, "Robust congestion control for high speed data networks with uncertain time-variant delays: an LMI control approach," *IEEE Conference on Local Computer Networks 30th Anniversary*, pp.474-475, Nov. 2005.

4. J. Lin, L. Xie and H. Zhang, "H_∞ control of linear systems with multiple input delays with application to ATM network congestion control," *The 6th World Congress on Intelligent Control and Automation*, Dalian, China, June 2006.

5. J. Lin, L. Xie and H. Zhang, "Output feedback H_2 Control of multiple input delay systems with application to congestion control," *The 9th International Conference on Control, Automation, Robotics and Vision*, Singapore, Dec. 2006.

6. J. Lin and L. Xie, "Sampled-data LQR control with multiple input delays," *in preparation.*

Bibliography

[1] A. Ahlen and M. Sternad. Optimal deconvolution based on polynomial methods. *IEEE Trans. Acoustics, Speech and Signal Processing*, 37(2):217–226, Feb. 1989.

[2] E. Altman and T. Basar. Optimal rate control for high speed telecommunication networks. In *proceedings of 34th IEEE Conference on Decision and Control*, pages 1389–1394, New Orleans, LA., 1995.

[3] E. Altman, T. Basar, and R. Srikant. Congestion control as a stochastic control problem with action delays. *Automatica*, 35:1937–1950, 1999.

[4] B. Bamieh, J. Pearson, B. Francis, and A. Tannenbaum. A lifting technique for linear periodic systems with applications to sampled-data control. *Systme & Control Letters*, 17:70–88, 1991.

[5] K. A. Barbosa, C. E. de Souza, and A. Trofino. Robust H_2 filtering for discrete-time uncertain linear systems using parameter-dependent Lyapunov functions. In *Proceeding of American Control Conference*, pages 3224–3229, 2002.

[6] M. Basin and J.Rodriguez-Gonzalez. Optimal control for linear systems with multiple time delays in control input. *IEEE Trans. Automatic Contr.*, 51(1), 2006.

[7] M. Basin and J. Rodriguez-Gonzalez. Optimal control for linear systems with time delay in control input based on the duality principle. In *Proc. American Control Conf.*, pages 2144–2148, Denver, 2003.

[8] Y. Cao, Y. Sun, and J. Lam. Delay-dependent robust H_∞ control for uncertain system with time-varying delays. *IEE Proc.-Control Theory Appl.*, 145(3):338–344, May 1998.

[9] T. Chen and B. Francis. H_2-optimal sampled-data control. *IEEE Trans. Automatic Control*, 36:387–397, 1991.

[10] T. Chen and B. Francis. H_∞-optimal sampled-data control: computataion and design. In *Proc. of American Control Conference*, pages 2767–1771, Baltimore, Maryland, 1994.

[11] T. Chen and B. Francis. *Optimal sampled-data control systems.* Springer, 1995.

[12] W. Chen, Z. Guan, and P. Yu. Delay-dependent stability and H_∞ control of uncertain discrete-time markovian jump systems with mode-dependent time delays. *System and Control Letters*, 52(5):361–376, 2004.

[13] Y. Chen and B. Chen. Minimax robust deconvolution filters under stochastic parametric and noise uncertainties. *IEEE Trans. Signal Processing*, (1):33–45, 1994.

[14] E. Cheres, S. Gutman, and Z. Palmor. Stabilization of uncertain dynamic systems includeing state delay. *IEEE Trans. Automatic Control*, 34:1199–1203, 1989.

[15] L. Chisci and E. Mosca. MMSE deconvolution via polynomial methods and its dual LQG regulation. *Automatica*, (30):1197–1201, 1994.

[16] D. Chyung. Discrete systems with delays in control. *IEEE Trans. on Automat. Contr.*, 14:196–197, 1969.

[17] O. Costa and C. Kubrusly. State-feedback h_∞ control for discrete-time infinite -dimensional stochastic bilinear systems. *Journal of Mathematical Systems, Estimation and Control*, (6):1–32, 1996.

[18] H. S. Dabis and T. J. Moir. A unified approach to optimal estimation using diophantine equations. *Int. J. Control.*, 57(3):577–598, 1993.

[19] Z. Deng, H. Zhang, S. Liu, and L. Zhou. Optimal and self-tuning white noise estimators with applications to deconvolution and filtering problems. *Automatica*, 32:199–216, 1996.

[20] J. Doyle, K. Glover, P. Khargonekar, and B. Francis. State space solutions to standard H_2 and H_∞ control problems. *IEEE Trans. on Automatic Control*, 34:831–847, 1989.

[21] L. Dugrard and E. Verriest. Stability and control of time-delay systems. In *Lecture notes in control and information sciences*, volume 228. Berlin: Springer, 1997.

[22] C. Foias, A. Tannenbaum, and G. Zames. Weighted sensitivity minimization for delay system. *IEEE Trans. Automatic Control*, 31:763–766, 1986.

[23] E. Fridman, A. Seuret, and J. Richard. Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, 40:1441–1446, 2004.

[24] E. Fridman and U. Shaked. H_∞-control of linear state-delay descriptor systems: an LMI approach. *Linear Algebra and its Applications*, 351-352:271–302, 2002.

[25] E. Fridman, U. Shaked, and V. Suplin. Input/output delay approach to robust sampled-data H_∞ control. *Systems & Control Letters*, 54:271–282, 2005.

[26] T. Fujinaka and M. Araki. Discrete-time optimal control of systems with unilateral time-delays. *Automatica*, 23(6):763–765, 1987.

[27] L. Ghaoui, F. Oustry, and M. Rami. A cone complementary linearization algorithms for static output-deedback and related problems. *IEEE Trans. Automatic Control*, 42:1171–1176, 1997.

[28] K. Glover and J. Partington. Robust stabilization of delay systems by approximation of co-prime factors. *System and Control Letters*, 14:325–331, 1990.

[29] O. Gonzalez, H. Herencia-Zapanna, and W. Gray. Stochastic stability of sampled data systems with a jump linear controller. In *Decision and Control, 2004. CDC. 43rd IEEE Conference*, 2004.

[30] G. Goodwin and K. Sin. *Adaptive filtering, prediction and control*. Prentice-Hall, In., 1984.

[31] F. Gouaisbaut, M. Dambrine, and J. Richard. Robust control of delay systems: a sliding mode control design via LMI. *Systems & Control Letters*, 46:219–230, 2002.

[32] F. Gouaisbaut, W. Perruquetti, and J. Richaprd. A sliding mode control for linear systems with input and state delays. In *Proceeding of the 38th Conf. Decision & Control*, pages 4234–4239, Phoenix, Arizona, USA, December 1999.

[33] M. Grimble and G. Hearns. LQG controllers for state-space systems with pure transport delays: Application to hot strip mills. *Automatica*, 34(10):1169–1184, 1998.

[34] J. Hale and S. Verduyn-Lunel. Introduction to functional differential equations. *in Applied Mathematical Sciences, New York: Springer*, 1993.

[35] B. Hassibi, A. Sayed, and T. Kailath. *Indefinite Quadratic Estimation and Control: A Unified Approach to H_2 and H_∞ Theories.* SIAM Studies in Applied Mathematics series, 1998.

[36] J. Hu, J. Lin, and L. Xie. Robust congestion control for high speed data networks with uncertain time-variant delays: An LMI control approach. In *Proc. the 30th IEEE Conf. on Local Computer Networks*, Sydney, Australia, Nov. 2005.

[37] J. Jezek and V. Kucera. Efficient algorithm for matrix spectral factorization. *Automatica*, 21:663–669, 1985.

[38] P. Kabamba. Control of linear systems using generalized sampled-data hold functions. *IEEE trans. Automatic control*, 32(9):772–783, 1987.

[39] P. Kabamba and S. Hara. Worst-case analysis and design of sampled-data control systems. *IEEE trans. Automatic Control*, 38(9):1337–1357, 1993.

[40] R. Kalman. A new approach to linear filteringand prediction problems. *Trans. of the ASME-Journal of Basic engineering*, 1960.

[41] R. Kalman and R. Bucy. New results in linear filtering and prediction. *Trans. of the ASME-Journal of Basic Engineering*, pages 95–107, 1961.

[42] P. Khargonekar and N. Sivashankar. H_2 optimal control for sampled-data systmes. *System & Control Letters*, 17:435–436, 1991.

[43] H. Kim, A. Jalali, C. Sims, and Y. Kim. Prediction, filtering, smoothing and devonvolution in a discrete H_∞ setting: a game theory approach. *Int. J. Control*, 70(6):841–857, 1998.

[44] A. Kojima and S. Ishijima. H_∞ control for preview and delayed strategies. In *Proc. of the 40th IEEE Conf. Decision & Control*, pages 991–996, Orlando, Florida USA, Dec. 2001.

[45] V. Kučera. *Discrete Linear Control. The Polynomial Equation Approach.* Chienester: Wiley, 1979.

[46] F. Leibfritz. An LMI-based algorithm for designing suboptimal static H_2/H_∞ output feedback controllers. *SIAM J. Control Optim.*, 39(6):1711–1735, 2001.

[47] X. Li and C. D. Souza. Robust stabiliztion and H_∞ control of uncertain linear time-delay systems. In *Proc. 13th IFAC World Congress, San Francisco, USA*, volume H, pages 113–118, 1996.

[48] L. Lindbom, M. Sternad, and A. Ahlen. Tracking of time-varying mobile radio channels-part II: A case study. *IEEE Trans. Communications*, 50:156–167, 2002.

[49] L. Ljung. *System identification: theory for the user.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1987.

[50] X. Lu and H. Zhang. Kalman filtering for multiple time delay measurements. *Automatica*, 41(7), 2005.

[51] Y. Lu, A. Xue, and Y. Sun. Guaranteed cost control for smapled-data systems. In *Proc. of American Contr. Conf.*, pages 185–189, Chicago, Illinois, 2000.

[52] G. Meinsma and L. Mirkin. H_∞ control of systems with multiple I/O delays via decomposition to adobe problems. *IEEE Trans. on Automatic Control*, 50(2):199–211, 2005.

[53] G. Meinsma and H. Zwart. On H_∞ control for dead-time system. *IEEE Trans. Automatic Control*, 45(2):272–285, 2000.

[54] J. M. Mendel. *Optimal Seismic Deconvolution: An Estimation-Based Approach.* New York: Academic, 1983.

[55] L. Mirkin. On the extraction of dead-time controllers and estimators from delay-free parametrizations. *IEEE Trans. Automatic Control*, 48(4):543–553, April 2003.

[56] L. Mirkin. On the H_∞ fixed-lag smoothing: how to exploit the information preview. *Automatica*, 39:1495–1504, 2003.

[57] L. Mirkin and N. Raskin. Every stabilzing dead-time controller has an observer-predictor-based structure. *Automatica*, 39:1747–1754, 2003.

[58] L. Mirkin, H. Rotstein, and Z. Palmor. H_2 and H_∞ design of sampled-data systems using lifting. Part I: general framework and solutions. *SIAM J. Control Optim.*, 38(1):175–196, 1999.

[59] L. Mirkin, H. Rotstein, and Z. Palmor. H_2 and H_∞ design of sampled-data systems using lifting. Part II: properties of systems in the lifted domain. *SIAM J. Control Optim.*, 38(1):197–218, 1999.

[60] L. Mirkin and G. Tadmour. H_∞ control of system with I/O delay: A review of some problem-oriented methods. *IMA J. Math. Control and Information*, 19:185–199, 2002.

[61] A. Moelja and G. Meinsma. H_2-optimal control of systems with multiple i/o delays: time domain approach. *Automatica*, 41(7):1229–1238, 2005.

[62] A. Moelja, G. Meinsma, and J. Kuipers. On h_2 control of systems with multiple i/o delays. *IEEE Trans. Automatic Control*, 51(8):1347–1354, 2006.

[63] M. Mohler and W. Kolodziej. An overview of stochastic bilinear control processes. *IEEE Trans. Systems Man Cybernetics*, 10, 1980.

[64] A. Molisch. *Wideband wireless digital communications.* Prentice Hall PTR, 2001.

[65] J. Morris. The Kalman filter: A robust estimator for some classes of linear quadratic problem. *IEEE Trans. Inform. Theory*, IT-22, 1976.

[66] G. Mouxstakides and S. Kassam. Minimax equalization for random signals,.

[67] K. Nagpal and R. Ravi. H_∞ control and estimation problems with delayed measurements: state-space solutions. *SIAM J. Control Optim.*, 35(4):1217–1243, 1997.

[68] K. Ohrn, A. Ahlen, and M. Sternad. A probabilitstic approach to multivariable robust filtering and open-loop control. *IEEE Trans. Automatic Control*, 40(3):405–418, March 1995.

[69] K. Park, J. Park, Y. Choi, Z. Li, and N. Kim. Design of H_2 controllers for sampled-data systems with input time delays. *Real time systems*, 26:231–260, 2004.

[70] H. Poor and D. Looze. Minimax state estimation for linear stochastic systems with noise uncertainty. *IEEE Trans. Automatic Control*, AC-26:902–906, 1981.

[71] L. Qiu and K. Tan. Direct state space solution of multirate sampled-data H_2 optimal control. *Automatica*, 34(11):1431–1437, 1998.

[72] T. Rappaport. *Wireless Communications: Principles and Practice.* Prentice Hall, 2002.

[73] J. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39:1667–1694, 2003.

[74] Y. Rosenwasser, K. Polyakov, and B. Lampe. Frequency-domain method for H_2 optimization of time-delayed sampled-data systems. *Automatica*, 33(7):1387–1392, 1997.

[75] M. Sagfors and H. Toivonen. The sampled-data H_∞ problem: The equivalence of discretization-based methods and a Riccati equation solution. In *Proceeding of Conf. on Decision and Control*, pages 428–433, Kobe, Japan, 1996.

[76] U. Shaked and C. de Souza. H_∞ control of linear systems with delayed measurements. In *Int. Symposium on Computer Aided Control System Design*, pages 351–356, Hawaii, 1999.

[77] M. Shergei and U. Shaked. Sampled-data non-linear H_∞ estimation and control. *Int. J. Control*, 73(8):704–719, 2000.

[78] P. Shi. *Issues in robust filtering and control of sampled-data systems.* Ph.d thesis, ECE Dept.,Univeristy of Newcastle, New South Wales, 1994.

[79] M. Sichitiu, P. Bauer, and K. Premaratne. The effect of uncertain time-variant delays in ATM networks with explicit rate feedback: a control theoretic approach. *IEEE/ACM Transactions on Networking*, 11(4):628–637, 2003.

[80] O. Smith. A controller to overcome dead time. *ISA Journal of Insturment Society of America*, 6:28–33, 1959.

[81] W. Sun, K. Nagpal, and P. Khargonekar. H_∞ control and filtering for sampled-data systems. *IEEE Trans. Automatic Control*, 38:1162–1175, 1993.

[82] W. Sun, K. Nagpal, P. Khargonekar, and K. Poolla. Digital control systems: H_∞ controller design with a zero-order hold function. In *Proceeding of Conf. Decision and Control*, pages 475–480, Tucson, Artzone, 1992.

[83] G. Tadmor. The standard H_∞ problem in systems with a single input delay. *IEEE Trans. Automatic Control*, 45:382–397, 2000.

[84] G. Tadmor and L. Mirkin. H_∞ control and estimation with preview- Part I: matrix ARE solutions in continuous-time. *IEEE Trans. on Automatic Control*, 50(1):19–28, 2005.

[85] G. Tadmor and L. Mirkin. H_∞ control and estimation with preview- Part II: matrix ARE solutions in discrete-time. *IEEE Trans. on Automatic Control*, 50(1):29–39, 2005.

[86] Y. Theodor, U. Shaked, and C. de Souza. A game theory approach to robust discrete-time H_∞ estimation. *IEEE Trans. Signal Processing*, 42(6):1486–1495, June 1994.

[87] H. Toivonen and M. Sagfors. The sampled-data H_∞ problem: a unified framework for discretization-based methods and riccati equation solution. *Int. J. Control*, 66(2):289–309, 1997.

[88] H. Trentelman and A. Stoorvogel. Sampled-data and discrete-time H_2 optimal control. In *Proceeding of Conf. Decision and Control*, pages 331–336, San Antonio, Texas, 1993.

[89] F. Wang and V. Balakrishnan. Robust Kalman filters for linear time-varying systems with stochastic parametric uncertainties. *IEEE Trans. Signal Processing*, 50:803–813, 2002.

[90] K. Watanabe, E. Nobuyanma, and A. Kojima. Recent advances in control of time delay systems-a tutorial review. In *Proc. Conf. 35th Decision and Control, Kobe, Japan*, pages 2083–2089, 1996.

[91] Y. Xia, J. Han, and Y. Jia. A sliding mode control for linear systems with input and state delays. In *Proceeding of the 41st IEEE Conf. Decision & Control*, pages 3332–3337, 2002.

[92] L. Xiao, A. Hassibi, and J. P. How. Control with random communication delays via a discrete-time jump system approach. In *Proc. American Control Conf.*, Chicago, 2000.

[93] L. Xie and C. de Souza. Robust H_∞ control for linear systems with norm-bounded tme-varying uncertainty. *IEEE Trans. Automatic Control*, 37(8):1188–1191, 1992.

[94] L. Xie and C. D. Souza. Robust stablization and disturbance attenuation for uncertain delay systems. In *Proc. 1993 European Control conference, Groningen, the Netherlands*, pages 667–672, 1993.

[95] Y. Yamamoto. A function space approach to sampled data control systems and tracking problems. *IEEE trans. Automatic Control*, 39(4):703–713, 1994.

[96] G. Zames. Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Trans. Automatic Control*, 26:301–320, 1981.

[97] H. Zhang, G. Duan, and L. Xie. Linear quadratic regulation for linear time-varying systems with multiple input delays Part I: discrete-time case. In *Proc. 5th International Conf. on Control and Automation*, Budapest, June 2005.

[98] H. Zhang, G. Duan, and L. Xie. Linear quadratic regulation for linear time-varying systems with multiple input delays Part II: continuous-time case. In *Proc. 5th International Conf. on Control and Automation*, Budapest, June 2005.

[99] H. Zhang, L. Xie, and G. Duan. H_∞ control of discrete-time systems with multiple input delays. In *Proc. 44th IEEE Conf. Decision and Control and European Control Control*, Seville, Spain, Dec. 2005.

[100] H. Zhang, L. Xie, and Y. Soh. A unified appraoch to linear estimation for discrete-time systems-Part I: H_2 estiamtion. In *Proc. IEEE conf. Decision Control*, USA, 2001.

[101] H. Zhang, L. Xie, and Y. C. Soh. Optimal and self-tuning deconvolution in time domain. *IEEE Trans. Signal Processing*, 47(8):2253–2261, 1999.

[102] H. Zhang, D. Zhang, and L. Xie. An innovation approach to H_∞ prediction with application to systems with time delayed measurements. *Automatica*, 40:1253–1261, 2004.

[103] K. Zhou, J. Doyle, and K. Golver. *Robust and Optimal Control.* Prentice-Hall, 1995.

[104] K. Zhou and P. Khargonekar. On the weighted sensitivity minimization problem for delay systems. *System and Control Letters*, 8:307–312, 87.

[105] X. Zhu, Y. Soh, and L. Xie. Design and analysis of discrete-time robust Kalman filters. *Automatica*, 38:1069–1077, 2002.