Mathematical modelling of the digital tendon pulleys

Tan, Adin Ming

2008

Tan, A. M. (2008). Mathematical modelling of the digital tendon pulleys. Master's thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/41155>

<https://doi.org/10.32657/10356/41155>

Mathematical Modelling of the Digital Tendon Pulleys

Tan Ming Adin

School of Chemical and Biomedical Engineering

A thesis submitted to the Nanyang Technological University
in fulfilment of the requirement for the degree of
Master of Engineering

2008

Abstract

A mathematical model of the A2 pulley system will enable us to have a better understanding of the mechanics of the pulley-tendon system. The A2 pulley was modelled based on parallel pulley fibres attached to a phalanx with a tendon passing them. Mechanical properties of the pulleys such as stiffness, strength and friction were included in the model. A convergence test was done to ensure the accuracy of the test. The model managed to show the degree of flexion of the fingers affects the force distribution of the pulleys. High loads on flexed finger may lead to pulley ruptures. Further studies on the rupture mechanism showed that pulley ruptures are self propagating when a constant force is applied and the rate of rupture increases as less intact fibres are present to support the load. In addition to human applications, the model was applied to animals as well, and it proved the advantages of a curved phalanx as compared to a straight one. This is important in deciding the arboreality and terrestriality of primates and hominids. Further application includes explaining the tendon locking mechanism in bats, birds and some climbing rodents. The relationship between friction coefficient of the pulley-tendon interface and the residual force at the proximal tendon was developed for the tendon locking mechanism

Acknowledgements

The author wishes to express his most sincere appreciation and gratitude to the following:

1. Associate Professor Franz Konstantin Fuss, the supervisor for this project, for his invaluable advice and guidance throughout the first year of candidature;
2. Mr. Gunther Niegl for providing data essential for the success of the project;
3. Colleagues and friends at BMERC who one way or another provided me with the support throughout the course of this project;
4. My wife and family who supported me in their own quiet manner.

List of Figures

Figure 1: Complete flexion of the finger with pulleys coming into contact [9]	6
Figure 2: Sagittal and coronal depictions of a typical flexor tendon [10]	7
Figure 3: Different pulley layouts [11]	8
Figure 4: Differing nomenclature for the pulley system [12]	8
Figure 5: Finger pulleys with FDP and FDS The relative sizes of the pulleys is visible [13]	10
Figure 6: The A2 pulley is very thin and it covers both the FDP and FDS. It has to take the force exerted by both the tendons [13]	11
Figure 7: The FDS splits at the A2 pulley region, forming a groove for the FDP to pass through [13]	11
Figure 8: Attachment area of the A2 pulley. The striated nature of the pulley fibres can be clearly seen [13]	12
Figure 9: Flexor Digitorum Profundus (left), Flexor Digitorum Superficialis (middle), palmar Interossei (right) [14]	13
Figure 10: Flexor Pollicis Longus (left) and Flexor Pollicis Brevis (right) [14]	13
Figure 11: Maintenance of the tendon moment arm in the presence of pulleys [18] ..	16
Figure 12: Experiment set ups to investigate the mechanical properties of finger pulleys. Experiments by Lin et al [7] (left) and Tanaka et al [19] (right)	17
Figure 13: Increase in moment arm due to the absence of pulleys [18]	18
Figure 14: Methods of finding friction of the tendon-pulley interface. Proposals by Moro-oka [24] (left) and Zhao [25]	19
Figure 15: Comparison of gliding resistance before and after suture[25]	20

Figure 16: Six suture techniques: (a) Kessler, (b) Modified Kessler, (c) Tsuge, (d) Savage, (e) Lee, (f) Becker [23].....	20
Figure 17: Gliding resistance after suturing of the tendon by different suture techniques [23]. Massachusetts General Hospital (MGH) method is similar to the Becker's method.	20
Figure 18: Biomechanical model of forces acting on finger pulley system [30]	21
Figure 19: Sagittal MRI of fingers [31] shows bowstringing of the tendon due to the disruption of the A2 pulley indicates an intact pulley system of an uninjured hand.....	23
Figure 20: Taping over different positions of the A2 pulley to investigate their effectiveness in preventing pulley injuries.....	23
Figure 21: Experimental setup to explore the effectiveness of taping the A2 pulley ..	24
Figure 22: Different phalanx curvature obtained from different species [36] From top to bottom: <i>Archaeolemur</i> , <i>Indri</i> , <i>Babakotia</i> and <i>Palaeopropithecus</i>	26
Figure 23: Derivation of the included angle from the morphology of the phalanx [36]	27
Figure 24: Range of included angles of primates [37].....	27
Figure 25: Tendon locking mechanism of bats[39]	29
Figure 26: tendon locking mechanism of aves[38].....	30
Figure 27: Graphical explanation of the symbols used.....	36
Figure 28: Different interpulley stiffness obtained due to tension and compression ..	36
Figure 29: FBD for rigid pulley system with one fibre	37
Figure 30: FBD for rigid pulley system with two fibres.....	39
Figure 31: FBD for elastic pulley system with one fibre.....	41
Figure 32: FBD for elastic pulley system with two fibres	43

Figure 33: FBD of rigid pulleys with friction considered for only one fibre.....	46
Figure 34: FBD of rigid pulleys with friction considered for two fibres.....	47
Figure 35: Profile showing the force distribution in a pulley system where the force concentrations are at the marginal pulley fibres	64
Figure 36: Profile showing the force distribution in a pulley system where the force concentrations are at the central pulley fibres.....	64
Figure 37: Changes in forces of pulley fibres due to change in tendon angles.....	65
Figure 38: Comparison of forces on marginal and central fibres as tendon angle increases	66
Figure 39: Loading of the fingers when applying 2 different grips; open hand (left) and crimp (right) [42]. A lower tendon angle (left) and a higher tendon angle (right) ensue.	66
Figure 40: Schematic representation of phalanx at different curvatures.	67
Figure 41: Profile of a straight phalanx.	68
Figure 42: Changes in forces of pulley fibres due to change in tendon angles for a straight phalanx	69
Figure 43: Comparison of forces on marginal and central fibres as tendon angle increases for a straight phalanx.....	70
Figure 44: Non-symmetrical force distribution of pulleys due to the effects of friction	71
Figure 45: Effects of friction on the force distribution of pulley system.....	72
Figure 46: Comparison of the characteristics of the marginal and central fibres in a 'n' shaped curve.....	75
Figure 47: Curve fitting of the forces of the central fibres for 'n' shaped curves	76
Figure 48: Curve fitting of the forces of the marginal fibres for 'n' shaped curves	76

Figure 49: Comparison of the characteristics of the marginal and central fibres in a ‘u’ shaped curve.....	78
Figure 50: Curve fitting of the forces of the marginal fibres for ‘u’ shaped curves	78
Figure 51: Curve fitting of the forces of the central fibres for ‘u’ shaped curves	79
Figure 52: Variation in the shapes of the force distribution curves as the tendon angle varies	82
Figure 53: Variations of the coefficients A, B and C with respect to the Tendon Angle. Linear curve fit is applied to find the equation to relate the two variables.....	83
Figure 54: Positions in which the fibres with maximum (red line) and minimum forces (blue line) are found.....	85
Figure 55: Two kinds of ‘m’ shaped curves: Predominantly ‘n’ shaped (left) and predominantly ‘u’ shaped (right).	85
Figure 56: Depiction of a perfect ‘m’ where the marginal fibre force is equivalent to the central fibre force	86
Figure 57: Graph showing the perfect ‘m’ shaped curve having the most even force distribution	86
Figure 58: The difference between the maximum force and the minimum force is the smallest for a perfect ‘m’ shaped force distribution curve.....	88
Figure 59: Transition between ‘n’ shaped force distribution curves and ‘u’ shaped force distribution curves	89
Figure 60: Stepwise simulation of the rupture mechanism of the A2 pulley.....	91
Figure 61: Sequential rupture of the pulley fibres	92
Figure 62: Graph showing the sequence of pulley ruptures	93
Figure 63: Force distribution of the pulley system at an included angle of 25°	95

Figure 64: Marginal and Central pulley force ratio at varying tendon angles for different included angles.....	96
Figure 65: Marginal and Central pulley force ratio with respect to the marginal force at different tendon angles	97
Figure 66: Curve fitting of μ for different number of fibres.....	99
Figure 67: Profile of the pulley system in the presence of high friction coefficient..	100
Figure 68: Force distribution of pulleys with high friction coefficient.....	101
Figure 69: Relationship between friction coefficient and residual force in the proximal tendon.....	102
Figure 70: Application of superposition on creep.....	111

List of Tables

Table 1: Average axial dimensions of pulleys in mm for different fingers[6]	9
Table 2: Maximum breaking load (N) [7].....	16
Table 3: Stiffness of pulleys (N/mm) [7]	17
Table 4: Gliding resistance (N) of the tendon-pulley interface at different load [25] .	19
Table 5: Friction coefficient obtained by different researchers	21
Table 6: Grading system of finger pulley injuries [3].....	22
Table 7: comparison of injuries in rock climbers [3].....	25
Table 8: Statistics on locations of pulley injuries[3].....	25
Table 9: Values obtained from the curve-fitting of a 'n' shaped curve	77
Table 10: Comparison of the results obtained from each analysis and the corresponding asymptotic value for a 'n' shaped curve	77
Table 11: Values obtained from the curve-fitting of a 'u' shaped curve	79
Table 12: Comparison of the results obtained from each analysis and the corresponding asymptotic value for a 'u' shaped curve	79
Table 13: Values of the coefficients A, B and C with respect to the Tendon Angles .	83
Table 14: Comparisons of different shaped force distribution curves.....	89
Table 15: Values obtained from curve -fitting.....	99
Table 16: Degree of accuracy for varying number of fibres.....	100

List of Symbols

n	Number of pulley fibres used to model the A2 pulley
a_1, b_1	x - and y -coordinates of the start point of the finger tendon
a_2, b_2	x - and y -coordinates of the end point of the finger tendon
c_i, d_i	x - and y -coordinates of the attachment point of the finger pulley i^{th} pulley fibre on the phalanx
x_i, y_i	x - and y -coordinates of the contact point between the i^{th} pulley fibre and the tendon
F_t	Applied force of finger tendon
F_{p_i}	Force experienced by the i^{th} pulley fibre
F	Friction force (Tangential)
N	Force normal to the pulleys
θ	angle between the tendon and the horizontal at the pulley-tendon junction
C	Stiffness of A2 pulley
k	Stiffness of individual fibre
l	Length of pulley fibre from the sagittal view
μ	Friction coefficient at the pulley-tendon interface
q	Interpulley stiffness
φ	Included angle

Table of Contents

Abstract	i
Acknowledgements	ii
List of Figures	iii
List of Tables	viii
List of Symbols	ix
Table of Contents	x
1 INTRODUCTION	1
1.1 Background	1
1.2 Objectives	4
1.3 Scope	4
2 LITERATURE REVIEW	6
2.1 Morphology of the Pulley Tendon System	6
2.1.1 Flexor Tendon Sheath	6
2.1.2 Description of the Pulley-Tendon System	10
2.1.3 Muscles Involved with the Flexion of Digits	12
2.2 Mechanics of the Pulley-Tendon System	15
2.3 Pulley Injuries	22
2.4 Curvatures of Phalanges in Primates	26
2.5 Tendon Locking Mechanism	28
3 METHODOLOGY	31
3.1 Software Utilised	31
3.2 Mathematical Representation of the A2 Pulley	32
3.3 Derivation of Equations	34

3.3.1	Rigid Pulleys	37
3.3.2	Elastic Pulleys	41
3.3.3	Rigid Pulleys with Friction	46
3.3.4	Elastic Pulleys with Friction	52
3.3.5	Elastic Pulleys with Elastic Inter-Pulley Connections.....	53
3.4	Explanation of Program Syntax	57
4	FACTORS AFFECTING PULLEY FORCES	63
4.1	Tendon Angle.....	63
4.2	Phalanx Curvature.....	66
4.3	Friction	70
5	CONVERGENCE TEST	74
6	APPLICATIONS	81
6.1	Force Distribution and Tendon Angles – Grip at Different Finger Flexion Angles	81
6.2	Pulley Rupture Mechanism.....	90
6.3	Advantages of Curved Phalanges in Primates	94
6.4	Tendon Locking Mechanism	98
7	DISCUSSIONS.....	103
8	CONCLUSIONS.....	107
9	FUTURE WORKS.....	109
9.1	Experiments	109
9.2	Inclusion of Creep into Mathematical Model	110
9.3	Model of All Pulleys in Entire Finger.....	110
	REFERENCES	112
	APPENDICES	

1 INTRODUCTION

1.1 Background

The hand is one of the most versatile human effectors. It is used for the execution of precise motion where fine motor control is necessary. Hand function requires the cooperation of five digits to achieve its purpose. Consequently, the various structures in each finger play essential roles in its mechanics.

The structure of the human finger is described as 4 open chain links with 3 mobile links. It has 3 degrees of freedom (DOFs) for flexion and extension, activated by 3 separate muscles. Generally, the function of the finger is dependent on the bones, tendon sheaths, pulleys, tendons and muscles. The bones, consisting of the proximal phalanx, middle phalanx and distal phalanx, makes up the main support structure of the finger. These are the areas on the palmar side of the phalanges on which the tendon sheaths, pulleys and tendons are attached. The pulleys and tendons work in tandem to flex and extend the fingers. The forces needed to flex the finger come from the respective muscles via the tendons. The pulleys, on the other hand, provide leverage for the flexion to occur. The tendon sheath functions as a container for synovial fluid, which is essential to reduce the friction during tendon gliding [1].

Like all finely tuned mechanisms, a small defect in the fingers will render the whole cascade to malfunction. One common finger injury is that of pulley failure. This includes partial pulley rupture, complete pulley rupture and multiple pulley ruptures. The latter often leads to bowstringing [2] and this affects the function of the finger severely. In this case, excessive muscle flexion is needed to produce the same amount

of flexion of the finger in an uninjured hand. Ramification of bowstringing is a weakened grip, incomplete flexion and stiff joints.

Pulley injuries are well studied, and this is particularly true for rock climbers. Statistics have shown that pulley injuries in rock climbers are more prevalent than any other injuries [3]. This is because rock climbers place extreme forces on their upper extremities. It is not uncommon for the entire body weight to be supported by as little as one finger, leading to unnaturally huge amount of stresses on the fingers. This together with certain finger positions in rock climbing techniques, such as the crimp grip [4], leads to several injuries relating to the finger pulleys.

Another common finger injury is that of flexor tendon lacerations and this usually requires suturing of the flexor tendon. However, suturing introduces additional friction at the pulley-tendon interface. This affects finger flexion efficiency and impede tendon recovery. Several suture techniques had been studied to minimize the effects of increased friction. The amount of friction at the pulley-tendon interface is of paramount importance in the proper function of the finger and it seems like minimal friction is favourable. On the other hand, Schweizer [5] hinted that friction may be needed to enable rock climbers to resist immense forces by the finger flexors. Thus optimal fiction may not necessarily be minimum friction.

Although the effects of pulley injuries are thoroughly investigated, knowledge of its mechanism is scarce. Doyle [6] has done detailed studies on the morphology of the finger pulleys while Lin et al [7] has investigated on its mechanics. These provide critical information regarding the function and mechanism of the pulley-tendon

system. However, they are not sufficient in explaining the rupture mechanism of finger pulleys.

In relation to the function of the digits, the curvature of the phalanx seems to play an important role. Arboreal primates are noticed to have phalanges which are more curved than humans. Hypothesis had been made regarding the advantages of the curved phalanx but no mechanical explanation had been given [8]. As the attachment points of the pulleys are on the phalanx, the curvature is bound to affect the performance of the phalanx.

Another interesting application of the pulley-tendon system is found in bats, birds and some climbing rodents. Certain parts of the flexor tendons are covered with roughened patches to increase the friction coefficient. The pulley and tendon seem to have the ability to interlock, enabling the limbs to stay in a certain position without executing any muscle force. This ability is peculiar but rarely found in animals.

The above paragraphs covered some hypothesis and other unexplained phenomenon in relation to the pulley-tendon system. Some of these cannot be proven due to lack of materials, such as anthropological studies of primates, whereas others may just be too difficult to carry out, such as the rupture mechanism of the finger pulleys. Nevertheless, these problems may be overcome by the development of a pulley-tendon model.

A pulley-tendon model gives a mechanical perspective to hypotheses such as phalanx curvature and locking mechanisms. These hypotheses may not be proven or disproved

by the model, but at least it has the ability to strengthen their claims. In addition, complex experiments such as effects of fiction or pulley rupture mechanism may never be carried out. A simple model may not explain all phenomenon, but it may shed light on some of these problems. Moreover, a pulley-tendon model may explain why it is hazardous to execute certain rock climbing moves and prevent further injuries.

Hence, development of a new pulley-tendon model will be important in understanding the mechanics of the finger. The accuracy of the model can be validated by convergence tests and values obtained from previous studies.

1.2 Objectives

The objective of this study is to come up with a mathematical model of the digital tendon pulleys.

This model will be applied to explain behaviour of the pulleys under different conditions.

1.3 Scope

The current study will focus on the development of the A2 pulley system model. There are two tendons which pass through the A2 pulley. Both are summed up and considered as one single force.

The pulley fibres are modelled as elements in parallel connected to the phalanx.

The parameters considered are:

- 1) Number of pulleys
- 2) Tendon angle
- 3) Phalanx curvature
- 4) Friction at pulley-tendon interface
- 5) Inter-pulley forces

Detailed explanation of the behaviour of the model will be provided.

Comparisons between past experiments and results obtained from the model will be made.

Applications of the model include:

- 1) Influence of tendon angle on force distribution of pulleys and its application on sport climbing
- 2) Curvature of primate phalanges and force distribution in different grip positions
- 3) Explanation of finger pulley rupture mechanism
- 4) Investigation of locking mechanism in bats, birds and climbing rodents

2 LITERATURE REVIEW

2.1 Morphology of the Pulley Tendon System

2.1.1 Flexor Tendon Sheath

The flexor tendon sheath is a fibrous sheath consisting of retinacular tissues and synovial membrane, which encompasses a closed synovial system. The sheath begins at the metacarpals and continues distally till the end of the distal interphalangeal joint for the index, middle and ring finger. The retinacular structures are pulleys which hold the flexor tendon in close proximity to the phalanges of a finger. The pulley structures are tissue condensations arranged in annular, cruciform and transverse pattern over the synovial membrane. In areas where mechanical strength is important, annular pulleys arches of the tendon causing that section to be stiff [1]. In contrast, cruciform pulleys are found in areas which allow flexion of the finger, thus allowing the sheath to fold. In complete flexion, all the annular and cruciate pulleys come into contact as seen in Figure 1.

Figure 1: Complete flexion of the finger with pulleys coming into contact [9]

The sheath enables the containment of the synovial fluid, which bathes the tendon, providing it with nutrition and acts as a lubricant to reduce friction. In addition, the sheath with its retinacular structures, gives the tendon leverage allowing ease of finger flexion.

Figure 2: Sagittal and coronal depictions of a typical flexor tendon [10]

A standard nomenclature had been introduced for the annular and cruciform pulleys. The usual pattern is shown in Figure 2. However, it is not always easy to differentiate the pulleys. Hahn and Lanz [11] have done a detailed study on the structures and found varying types of pulley systems, depicted in Figure 3. In fact, nomenclature of these pulleys differed between researches as reflected in Figure 4. However, for this study, we would use the nomenclature as proposed by Doyle [6] which is shown in Figure 2.

Figure 3: Different pulley layouts [11]

Figure 4: Differing nomenclature for the pulley system [12]

From Figure 2, there are a total of 5 annular and 3 cruciform pulleys [1].

- A1 Pulley: arises from the palmer plate and found at the level of the MCP joint and the proximal end of the proximal phalanx. It may be represented by 2 or 3 annular bands occasionally.
- A2 Pulley: overlies the middle third to proximal two-thirds of the proximal phalanx. It originates from the ridges of the proximal and lateral areas of the proximal phalanx. Regarded to be the strongest pulley
- A3 Pulley: a narrow pulley located at the proximal interphalangeal joint and attaches to the palmar plate.
- A4 Pulley: overlies the middle third of the middle phalanx.
- A5 Pulley: found at the proximal end of the distal phalanx at the distal interphalangeal joint, just proximal to the termination of the tendon sheath.
- C1 Pulley: lies between the A2 and A3 pulleys
- C2 Pulley: located in the space between the A3 and A4 pulley
- C3 Pulley: found between the A4 and A5 pulleys. It is a prominent extension of the oblique fibres overlying A4 and is not always a separate structure.

Average dimensions of the pulleys from 61 fingers of fresh human cadavers were used in a study by Doyle [6]. The results obtained are shown in Table 1

Finger	A1	A2	A3	A4	A5	C1	C2	C3
Index	9.2	15.9	2.8	6.1	3.9	4.2	2.4	2.2
Middle	9.4	20.5	2.7	7.7	4.3	5.1	2.7	2.6
Ring	7.5	18.9	2.9	7.0	3.9	3.9	3.7	3.0
Little	5.3	11.7	2.7	5.9	4.3	3.8	2.6	2.4

Table 1: Average axial dimensions of pulleys in mm for different fingers[6]

2.1.2 Description of the Pulley-Tendon System

The ability of the pulley system to execute flexion of the digits without buckling of the entire system lies in the distribution of the annular and cruciform pulleys. The broad A2 and A4 with their distinct strength are located between joints while the narrow A1 and A3 pulleys are positioned over joints. The narrow and thin cruciform pulleys are located near joints where they are able to accommodate flexion more easily than the larger annular pulleys.

Fundamentally, both retinacular and synovial characteristics of the flexor tendon sheath are critical in the efficiency of digits flexion; one to maintain optimal glide path and the other to reduce friction during gliding so as to minimise attrition of the pulleys. The following figures (Figure 5 to Figure 8) show the details of the pulley-tendon system [13]. They give a clear illustration of the relative positions of the components of the pulley-tendon system.

Figure 5: Finger pulleys with FDP and FDS The relative sizes of the pulleys is visible [13]

Figure 6: The A2 pulley is very thin and it covers both the FDP and FDS. It has to take the force exerted by both the tendons [13]

Figure 7: The FDS splits at the A2 pulley region, forming a groove for the FDP to pass through [13]

Figure 8: Attachment area of the A2 pulley. The striated nature of the pulley fibres can be clearly seen [13]

2.1.3 Muscles Involved with the Flexion of Digits

There are five sets muscles that are related to the pulley tendon system; three for the fingers, as they have 3 degrees of freedom (DOFs), and two for the thumb which has 2 DOFs. The three muscles that flex the fingers are the flexor digitorum profundus (FDP), the flexor digitorum superficialis (FDS) and the palmar interossei muscles (PIO). The muscle that flexes the thumb is the flexor pollicis longus (FPL) and the flexor pollicis brevis (FPB) [1].

Figure 9: Flexor Digitorum Profundus (left), Flexor Digitorum Superficialis (middle), palmar Interossei (right) [14]

Figure 10: Flexor Pollicis Longus (left) and Flexor Pollicis Brevis (right) [14]

Flexor Digitorum Profundus

The FDP is a long muscle that originates from the proximal three fourths of the anterior and medial surfaces of the ulna and interosseous membrane as shown in Figure 9. As the single muscle mass travels distally in the forearm, it separates into an ulnar and radial bundle. The radial bundle forms the profundus tendon of the index finger. The ulnar bundle splits further to form the individual profundus tendons of the middle, ring and little fingers. Each tendon then proceeds between the pulleys and phalanges before passing through the gaps created by the divided FDS tendons, before inserting into the palmar bases of the bases of the distal phalanges. Each tendon from the FDP is designed to flex the proximal interphalangeal (PIP) joint, the distal interphalangeal (DIP) joint and the metacarpophalangeal joint (MCP).

Flexor Digitorum Superficialis

The FDS is a large and fleshy muscle that lies superficial to the FDP in the anterior portion of the forearm as shown in Figure 10. The muscle has two heads of origin, the humeroulnar and radial. The humeroulnar head arises from the medial epicondyle of the humerus and the medial ligament of the elbow joint. The radial head arises from the anterior border of the radius. The muscle then separates into two strata of muscle fibres, one superficial and the other deep. The superficial stratum divides into two tendons for the middle and ring finger. The deep stratum divides into two tendons for the index and little finger. In the region of the proximal phalanx, each tendon splits into two halves to form a channel for the FDP tendons to pass through, before inserting into the middle phalanges of the finger. The primary functions of the FDS tendons are to flex the PIP and MCP joints. They are similar in function to the FDP

tendons except that they cannot flex the distal phalanges since they are inserted at the middle phalanx.

Palmar Interossei

There are a total of 3 PIOs. Each arises from the entire length of the metacarpal bone of one finger, and is inserted into the side of the base of the proximal phalanx. It is used to flex the metacarpophalangeal joint while extending interphalangeal joints.

Flexor Pollicis Longus

The FPL is lateral to the FDP. It originates from the midanterior section of the radius and adjacent to the interosseus membrane as shown in Figure 9. This muscle gives rise to only a single flat tendon which passes the pulleys before attaching to the palmar surface of the base of the distal phalanx of the thumb.

This tendon is intended to flex the interphalangeal and metacarpophalangeal joints of the thumb.

Flexor Pollicis Brevis

The FPB arises from the flexor retinaculum and the lower part of the ridge on the greater multangular bone. It passes along the radial side of the FPL tendon and is inserted into the radial side of the base of the proximal phalanx of the thumb.

It is used to flex the metacarpophalangeal joint of the thumb.

2.2 Mechanics of the Pulley-Tendon System

The main purpose of pulleys is to ensure that the flexor tendons of the hand remain close to the joint axes so as to ascertain economy and efficiency in finger flexion [15].

The whole pulley system consists of transverse carpal ligament, the palmar aponeurosis pulley and the digital flexor pulley system. The pulleys attempt to maintain a constant tendon moment relationship between the flexor tendons and the joint axis as shown in Figure 11. Nevertheless, there is a slight increase in moment arm during flexion [16]. Whenever the digits are flexed, the tendon will exert a force on the pulley system. The tendon, with a greater cross-sectional area as compared to the pulleys, has a higher strength [17]. Thus excessive tendon excursion may cause pulley ruptures. Mechanical properties of the human pulleys including strength and stiffness had been investigated by Lin et al [7] using the set up in Figure 12 . The results are reflected in Table 2 and Table 3.

Figure 11: Maintenance of the tendon moment arm in the presence of pulleys [18]

Finger	A1	A2	A3	A4	A5	C1	C2	C3
Index	336.23	431.96	57.47	202.32	39.48	69.97	41.43	43.23
Middle	402.00	465.67	50.77	210.48	30.87	105.69	40.72	63.90
Ring	315.24	431.64	52.20	252.83	29.93	69.71	45.51	47.95
Little	243.66	300.69	30.74	172.41	31.51	79.94	31.77	45.11
Mean	324.28	407.49	47.80	209.51	32.86	78.33	39.86	50.05

Table 2: Maximum breaking load (N) [7]

Finger	A1	A2	A3	A4	A5	C1	C2	C3
Index	82.48	129.59	32.72	129.39	33.35	26.49	15.01	27.31
Middle	79.95	161.78	17.45	176.25	27.71	33.91	10.05	31.59
Ring	81.07	129.87	21.16	142.97	20.23	30.61	10.44	23.05
Little	57.82	117.81	15.75	203.48	29.70	31.84	17.10	34.13
Mean	74.82	133.34	22.00	167.99	22.71	30.71	13.58	27.94

Table 3: Stiffness of pulleys (N/mm) [7]

Figure 12: Experiment set ups to investigate the mechanical properties of finger pulleys. Experiments by Lin et al [7] (left) and Tanaka et al [19] (right)

The loss of any pulley will severely affect the efficiency of flexion in the digits. An absent pulley leads to an increased moment arm and thus an increased tendon excursion is required to produce the same arc of motion as seen in Figure 13. Contraction of muscles is limited to only 60%, beyond which no force is generated [20]. As muscle excursion is limited and proportional to muscle fibre length, the effectiveness of the tendon excursion relies heavily on the maintenance of the constant moment arm relationship.

Figure 13: Increase in moment arm due to the absence of pulleys [18]

During tendon excursion, the pulley-tendon interface experiences friction. It is commonly thought that repeated exposure to such friction and attrition could be detrimental to the pulley system in the long run. Thus, there had been several methods developed to quantify the magnitude of friction present (Figure 14) at the pulley-tendon interface and numerous ways to reduce the friction present.

The friction pulley-tendon interface is commonly referred to as gliding resistance [21-23]. It is a measure of the difference in tension on the distal and proximal end of the tendon. This is convenient as the raw data can be used as comparisons for relative resistance. Uchiyama *et al.* [21] made a study on the change in gliding resistance when the applied load on the flexor digitorum profundus tendon was varied. The results are shown in Table 4.

Figure 14: Methods of finding friction of the tendon-pulley interface. Proposals by Moro-oka [24] (left) and Zhao [25]

Specimen	0.98 N	2.45 N	4.9 N	9.8 N	14.7 N
1	0.12	0.17	0.24	0.38	0.46
2	0.03	0.06	0.10	0.18	0.26
3	0.04	0.06	0.11	0.18	0.23
4	0.09	0.12	0.19	0.23	0.35
5	0.13	0.23	0.34	0.55	0.75
6	0.03	0.05	0.06	0.07	0.10
7	0.06	0.08	0.13	0.14	0.22
Mean	0.07	0.11	0.17	0.25	0.34

Table 4: Gliding resistance (N) of the tendon-pulley interface at different load [25]

Detailed studies had been done to investigate the magnitude of gliding resistance after tendon repair and it had been discovered that there is a drastic increase before and after suturing of the tendon. Gliding resistance increased as much as 6 times as shown in Figure 15. In fact, different kinds of sutures (Figure 16) result in different gliding resistance [22, 23, 25] as can be seen in Figure 17. In addition, there had been research on the effectiveness of surface modification of the tendon by chemical means so as to reduce friction present [26-28].

Figure 15: Comparison of gliding resistance before and after suture[25]

Figure 16: Six suture techniques: (a) Kessler, (b) Modified Kessler, (c) Tsuge, (d) Savage, (e) Lee, (f) Becker [23]

Figure 17: Gliding resistance after suturing of the tendon by different suture techniques [23]. Massachusetts General Hospital (MGH) method is similar to the Becker's method.

In addition, methods had been developed for obtaining friction coefficients as well [5, 24, 29]. Table 5 shows the difference in values obtained by different researches.

Schweizer, contrary to other researches, claimed that there is an advantage in the presence of friction. His investigation showed that rock climbers could exert a higher load on the pulley-tendon system as a result of friction.

Author	Year	Friction Coefficient
Schweizer et al. [5]	2002	0.075
Moro-oka et al. [24]	1999	0.027
Uchiyama et al. [29]	1995	0.040

Table 5: Friction coefficient obtained by different researchers

A biomechanical model for the determination of the forces acting on the finger pulley system had been developed [30]. However, the pulley forces are derived from the frontal plane as can be seen in Figure 18. Moreover, it only considered the average forces on the pulley system and no friction was taken into account. Thus pulley failures due to non-uniform loading of the pulleys cannot be detected. In addition, it will not be able to explain any phenomenon involving friction at the pulley-tendon interface.

Figure 18: Biomechanical model of forces acting on finger pulley system [30]

2.3 Pulley Injuries

Excessive loading of the finger causes overstraining of the flexor pulleys. This may cause disruption of the pulley mechanism and result in pain. The magnitude of the damage is graded and reflected in Table 6. The loss of the pulley system no longer restricts the tendon from gliding juxtaposed to the phalanges. Bowstringing of the tendon occurs and pulls the tendon away from the bone. An increase in distance between the tendon and the bone can be seen in Figure 19. Bowstringing increases the tendon moment arm of the tendon at that point. An increase in the magnitude of contraction is necessary to permit the same amount of flexion. This is due to the need for a greater excursion of the tendon to execute the same amount of flexion. Bowstringing leads to weakening of the grip, incomplete flexion and stiff joints.

	<i>Grade 1</i>	<i>Grade 2</i>	<i>Grade 3</i>	<i>Grade 4</i>
Injury	Pulley strain	Complete rupture of A4 or partial rupture of A2 or A3	Complete rupture A2 or A3	Multiple ruptures, as A2/A3, A2/A3/A4 or single rupture (A2 or A3) combined with lumbricalis muscle or ligament damage
Therapy	Conservative	Conservative	Conservative	Surgical repair
Immobilisation	None	10 days	10–14 days	Postoperative 14 days
Functional therapy	2–4 wk	2–4 wk	4 wk	4 wk
Pulley protection	Tape	Tape	Thermoplastic or soft-cast ring	Thermoplastic or soft-cast ring
Easy sport-specific activities	After 4 wk	After 4 wk	After 6–8 wk	4 mo
Full sport-specific activities	6 wk	6–8 wk	3 mo	6 mo
Taping through climbing	3 mo	3 mo	6 mo	>12 mo

Table 6: Grading system of finger pulley injuries [3]

Figure 19: Sagittal MRI of fingers [31] shows bowstringing of the tendon due to the disruption of the A2 pulley indicates an intact pulley system of an uninjured hand

Attempts had been made in preventing pulley rupture by external mechanical means such as taping the proximal phalanx as shown in Figure 20. However, Schweizer [32] and Warne [33] have suggested that this method is probably ineffective in preventing ruptures. The experimental setup of Warne can be seen in Figure 21.

Figure 20: Taping over different positions of the A2 pulley to investigate their effectiveness in preventing pulley injuries

Figure 21: Experimental setup to explore the effectiveness of taping the A2 pulley

The occurrence of pulley injuries is extremely high in rock climbers. Conversely, a huge portion of rock climbing injuries is related to pulleys as shown in Table 7. Rock climbing places extreme forces on the fingers. It is not uncommon for the entire body weight to be supported by a single digit. Such manoeuvres place acute stress on the pulleys.

Pulley rupture	74
Pulley strain	48
Tendovaginitis	42
Joint capsular damage	37
Arthritis (acute)	13
Ganglion	11
Flexor tendon strain	7
Fracture	7
Arthritis (chronic)	7
Dupuytren contracture	5
Soft tissue injury, contusion	5
Flexor tendon partial tear	4
Collateral ligament injury	3
Osseous tear, fibrocartilago palmaris	2
Epiphyseal fracture	2
Lumbrical shift syndrome	2
Abscess	1
Finger amputation	1

Table 7: comparison of injuries in rock climbers [3]

	No.	M	F	Right	Left	D2	D3	D4	D5	A2	A3	A4	A2/A3	A2/A3/A4
Pulley strain	48	43	5	28	20	0	21	27	0	31	3	14	0	0
Pulley rupture	74	67	7	41	33	0	26	46	2	50	3	14	4	3

Table 8: Statistics on locations of pulley injuries[3]

Amongst the pulleys, the A2 has the highest probability of failure as seen in Table 8, although it had been established by Lin et al. [7] that the A2 is the strongest pulley. This was explained by Marco et al. [34]. The weaker A3 did not fail first as it is more compliant and deforms, thus transferring the stiffer A2 pulley. In addition, he reported that isolated rupture of a pulley is unlikely and obvious bowstringing is noticed only after the rupture of the A2, A3 and A4 pulleys. Thus multiple pulley failures are likely if bowstringing is noticeable.

2.4 Curvatures of Phalanges in Primates

The extremities of primates come in contact with their environment and form the biomechanical link in which forces are transmitted between the environment and the animal. As such, the designs of these extremities should be optimized and reflect adaptation in structure due to external stimuli. Curved phalanges of primates are often associated with their gripping function and climbing ability. Degree of curvature varies greatly from species to species as reflected in Figure 22. However, functional explanations for the degree of phalangeal curvature are hypothetical [8, 35]. It is currently not proven that the increase in curvature of the phalanx improves the gripping capability of the primates.

Figure 22: Different phalanx curvature obtained from different species [36] From top to bottom: *Archaeolemur*, *Indri*, *Babakotia* and *Palaeopropithecus*

Nevertheless, phalangeal curvature is easily and reliably quantified. The most widely used method is that of the included angle. The included angle requires three input variables which can be obtained from phalanx morphology. These input variables can also be easily measured from samples if they are available. From Figure 23, calculation of the included angle, theta, can be easily done with the input parameters.

Figure 23: Derivation of the included angle from the morphology of the phalanx [36]

Variations in included angle of different primates had been documented and Figure 24 shows some common examples.

Figure 24: Range of included angles of primates [37]

2.5 Tendon Locking Mechanism

It is often an advantage in mammals for the friction between the tendon and the sheath to be a minimum so as to maximize flexion efficiency. However, tendon locking mechanism is found in bats, birds and some climbing rodents [38-40]. The function of this mechanism is to lessen the amount of muscular activities used in sustained grasping.

Although there are variations in the tendon locking mechanisms, one basic principal remains the same. That is to increase the amount of friction by the presence of roughened tendon and sheath. The increased friction reduces the force needed for sustained flexion of the distal phalanx at that position as the phalanx is held in place by the friction. This allows continual grasping when required.

Due to the above effect, there is a difference in tendon forces at the proximal and distal ends of the pulleys. Thus the muscle at the proximal end is required to generate the residual force need to hold the phalanx in place, leading to a conservation of muscle force. Figure 25 and Figure 26 shows the locking mechanisms in bats and birds respectively. For birds, a transverse force needs to be applied to the tendon sheath to activate the mechanism.

Figure 25: Tendon locking mechanism of bats[39]

Explanation of abbreviations for Figure 25:

DEL: Dorsal Elastic Ligament

FS: Flexor Tendon Sheath

TD: Dorsal Tendon Tubercles

VEL: Ventral Elastic Ligament

UP: Ungual Phalanx

P: Proximal Phalanx

MC: Metacarpal

VT: Ventral Tendon Tubercles

Pr: Flexor Process

Figure 26: Tendon locking mechanism of aves[38]

3 METHODOLOGY

3.1 Software Utilised

The reliability of the software used in the modeling is of paramount importance. The precision of the results obtained, and subsequent analysis is dependent on it. Thus much consideration had to be made in the selection of software.

Maple 9.5

Developed by Maplesoft, Waterloo, Ontario, Canada (2004). Maplesoft is a leading developer of advanced mathematical and analytical software for the past 25 years. Maple 9.5 has the ability to solve complicated math on a common PC platform.

Matlab 6.5

Developed by Mathworks, Natick, Massachusetts, USA (2002). Mathworks is the leading global provider of software for technical computing. Matlab 6.5 has a powerful tool box which allows curve fitting of complicated equations

Grapher 4

Developed by Golden Software, Golden, Colorado, USA (2002). Golden Software is the developer of premium scientific graphics software. Grapher 4 is user friendly and has good control over the graphing parameters.

AutoCAD 2004

Developed by Autodesk, San Rafael, California, USA. Autodesk is the developer of practical and innovative drafting software. AutoCAD is highly versatile and produces superior graphics.

3.2 Mathematical Representation of the A2 Pulley

Mathematical modelling of the A2 pulley system will enable us to have an in depth view of the pulley-tendon mechanism. The model is similar to that of Finite Element Method but with the advantage of control over all parameters of investigation. Factors affecting pulley function and performance have to be carefully considered to achieve reliable results.

Configuration and shape of the A2 pulley is considerably more consistent than the other pulleys. Thus it is the choice of pulley for our investigation. The A2 pulley consists of arcuate fibres that arches palmarly over the flexor digitorum profundus (FDP) and the flexor digitorum superficialis (FDS) tendons and attaches itself to the proximal and lateral areas of the proximal phalanx. The behaviour of the fibres is analogous to a series of flexible bucket handles.

In order to obtain a functional A2 finger pulley system, the pulley fibres were defined according to the following non-linear static conditions:

- The pulley is modelled as i equal fibres attached to the phalanx.

- The deflection point of the pulley at the free end of the i^{th} pulley fibre is defined by the point x_i, y_i . This is the point where the finger tendon exerts a force on the finger pulleys.
- The attachment point of the fibre at the bone (phalanx) is defined as c_i, d_i . The tendon passes through all the pulley fibres and is linked to the adjacent deflection points of the subsequent and preceding pulley fibre by x_{i+1}, y_{i+1} and x_{i-1}, y_{i-1} .
- The pulley fibre is represented by an elastic rope of length l_i and spring constant k .
- The pulley fibre reaction force Fp_i and the tendon forces Ft on each side of the fibre must be in equilibrium in x-and y-directions. Thus we obtain two force equilibrium equations.
- The endpoint x_i, y_i of the fibre is situated at the circumference of a circle of centre c_i, d_i and radius $l_i + Fp_i/k$. This will enable us to develop a kinematic equation.
- These 3 equations will form the basis of the A2 finger pulley system for i number of fibres.
- There are n fibres, leading to $3n$ equations and thus we obtain 2^n solutions. The boundary condition is that the pulley can only be stretched and the Fp

obtained must be positive. In addition, the pulleys are in the first quadrant and only positive coordinates are attainable, resulting in only one set of solution.

These fundamental equations will facilitate us in the development of the entire A2 pulley. A 2D approach is justified as all forces act in one plane. Detailed explanations of the development of the equations are found in the subsequent segment.

3.3 Derivation of Equations

The symbols used in this chapter and their respective descriptions:

n	Number of pulley fibres used to model the A2 pulley
a_1, b_1	x - and y -coordinates of the start point of the finger tendon Set at an arbitrary point to simulate the angle of the tendon at the metacarpo-phalangeal joint (MCP) as shown in Figure 27
a_2, b_2	x - and y -coordinates of the end point of the finger tendon Set at an arbitrary point to simulate the angle of the tendon at the proximal interphalangeal joint (PIP) as shown in Figure 27
c_i, d_i	x - and y -coordinates of the attachment point of the finger pulley i^{th} pulley fibre on the phalanx counting from the proximal end of the proximal phalanx as shown in Figure 27
x_i, y_i	x - and y -coordinates of the contact point between the i^{th} pulley fibre and the tendon as shown in Figure 27
F_t	Applied force of finger tendon
F_{p_i}	Force experienced by the i^{th} pulley fibre
F	Friction force (Tangential)

N	Force normal to the pulleys
θ	Tendon angle. Defined as the angle between the tendon and the horizontal at the pulley-tendon junction as shown in Figure 27
C	Stiffness of A2 pulley
k	Stiffness of individual fibre as shown in Figure 27
l	Length of pulley fibre from the sagittal view as shown in Figure 27
μ	Friction coefficient at the pulley-tendon interface
q	Interpulley stiffness. Defined as the elasticity between pulleys as seen in Figure 27. However, there is a difference in stiffness for tension and compression. When in tension, the stiffness is a result of the deformation of the pulley fibres and connective tissues (q_t). On the other hand, the stiffness obtained from compression is a result only of the deformation of the pulley fibres (q_c). This is shown in Figure 28.
φ	Included angle. Used to define the curvature of the phalanx as seen in Figure 27

Data needed to simulate the A2 pulley model:

Curvature of phalanx:	Included angle between 18° and 32° (Human) [37]
Axial width of attachment area:	15.9 mm to 20.5 mm [6]
Pulley Length:	14 mm [41]
Pulley stiffness:	117.81 N/mm to 161.87 N/mm [7]
Tendon force:	85 N to 256 N [34]
Friction coefficient:	0.027 to 0.075 [5, 24, 29]

Figure 27: Graphical explanation of the symbols used

Figure 28: Different interpulley stiffness obtained due to tension and compression

3.3.1 Rigid Pulleys

Figure 29: FBD for rigid pulley system with one fibre

From Figure 29:

$$Fp_{1x} = \frac{(c_1 - x_1)}{l} Fp_1$$

$$Fp_{1y} = \frac{(d_1 - y_1)}{l} Fp_1$$

$$Ft_{1x} = \left(\frac{(a_1 - x_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft$$

$$Ft_{1y} = \left(\frac{(b_1 - y_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft$$

$$Ft_{2x} = \left(\frac{(a_2 - x_1)}{\sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \cdot Ft$$

$$Ft_{2y} = \left(\frac{(b_2 - y_1)}{\sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \cdot Ft$$

Σ of forces in the x -direction

$$F_{t_{1x}} + F_{t_{2x}} + F_{p_{1x}} = 0$$

$$\left(\frac{(a_1 - x_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(a_2 - x_1)}{\sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \cdot Ft + \frac{(c_1 - x_1)}{l} Fp_1 = 0 \quad (1)$$

Σ of forces in the y -direction

$$F_{t_{1y}} + F_{t_{2y}} + F_{p_{1y}} = 0$$

$$\left(\frac{(b_1 - y_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(b_2 - y_1)}{\sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \cdot Ft + \frac{(d_1 - y_1)}{l} Fp_1 = 0 \quad (2)$$

For pulley kinematics, the pulley fibre can only be on a circular path with radius l and centre (c_l, d_l) . Thus we obtain,

$$(c_1 - x_1)^2 + (d_1 - y_1)^2 = (l)^2 \quad (3)$$

From Equations (1), (2) and (3), there are 3 equations with 3 unknowns, x_l , y_l and Fp_l .

They can be solved simultaneously to obtain the 3 unknowns which give us the position and applied force on the pulley fibre.

Extending the above to the modelling of 2 pulley fibres, we get:

Figure 30: FBD for rigid pulley system with two fibres

For pulley fibre 1 in Figure 30,

Σ of forces in the x -direction

$$\left(\frac{(a_1 - x_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \right) \cdot Ft + \frac{(c_1 - x_1)}{l} Fp_1 = 0 \quad (4)$$

Σ of forces in the y -direction

$$\left(\frac{(b_1 - y_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \right) \cdot Ft + \frac{(d_1 - y_1)}{l} Fp_1 = 0 \quad (5)$$

The path of pulley fibre 1 will be

$$(c_1 - x_1)^2 + (d_1 - y_1)^2 = (l)^2 \quad (6)$$

For pulley fibre 2,

Σ of forces in the x -direction

$$\left(\frac{(x_1 - x_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}} \right) \cdot Ft + \left(\frac{(a_2 - x_2)}{\sqrt{(a_2 - x_2)^2 + (b_2 - y_2)^2}} \right) \cdot Ft + \frac{(c_2 - x_2)}{l} Fp_2 = 0 \quad (7)$$

Σ of forces in the y -direction

$$\left(\frac{(y_1 - y_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}} \right) \cdot Ft + \left(\frac{(b_2 - y_2)}{\sqrt{(a_2 - x_2)^2 + (b_2 - y_2)^2}} \right) \cdot Ft + \frac{(d_2 - y_2)}{l} Fp_2 = 0 \quad (8)$$

The path of pulley fibre 2 will be

$$(c_2 - x_2)^2 + (d_2 - y_2)^2 = (l)^2 \quad (9)$$

From Equations (4) to (9), there are 6 equations with 6 unknowns, x_1 , x_2 , y_1 , y_2 , Fp_1 and Fp_2 . They can be solved simultaneously to obtain the 6 unknowns which give us the positions and applied forces on pulley fibre 1 and pulley fibre 2.

With the above analyses, the number of rigid pulley fibres can be increased indefinitely with the following general equations

$$\left(\frac{(x_{i-1} - x_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot Ft + \left(\frac{(x_{i+1} - x_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot Ft + \frac{(c_i - x_i) \cdot Fp_i}{l} = 0 \quad (10)$$

$$\left(\frac{(y_{i-1} - y_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot Ft + \left(\frac{(y_{i+1} - y_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot Ft + \frac{(d_i - y_i) \cdot Fp_i}{l} = 0 \quad (11)$$

$$(c_i - x_i)^2 + (d_i - y_i)^2 = (l_i)^2 \quad (12)$$

where i represents the i^{th} fibre.

Thus with n pulley fibres, we will obtain $3n$ equations with $3n$ unknowns. However, due to the numerical possibility of positive and negative answers, we obtain 2^n solutions. Fortunately, from the physical system, we know that that x , y and Fp can only take positive solutions as x , y and Fp are in the first quadrant of the Cartesian Coordinates and Fp is always under tension as it can only be pulled. Hence we obtain only a single set of solutions with $3n$ unknowns.

3.3.2 Elastic Pulleys

Figure 31: FBD for elastic pulley system with one fibre

From Figure 31, the red portion of the finger pulley represents the extension of the finger pulley when it is elastic. This extension is given as $\frac{Fp_1}{k}$ where k is the elasticity of the pulley fibre.

For the case of only a single elastic pulley fibre:

$$Fp_{1x} = \frac{(c_1 - x_1)}{l + \frac{Fp_1}{k}} Fp_1$$

$$Fp_{1y} = \frac{(d_1 - y_1)}{l + \frac{Fp_1}{k}} Fp_1$$

In contrast, Ft_{1x} , Ft_{1y} , Ft_{2x} and Ft_{2y} remains unchanged as they are only dependent on the coordinates of a_1 , b_1 , a_2 , b_2 and c_1 , d_1 , and not on the length of the pulley fibre.

$$Ft_{1x} = \left(\frac{(a_1 - x_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft$$

$$Ft_{1y} = \left(\frac{(b_1 - y_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft$$

$$Ft_{2x} = \left(\frac{(a_2 - x_1)}{\sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \cdot Ft$$

$$Ft_{2y} = \left(\frac{(b_2 - y_1)}{\sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \cdot Ft$$

Σ of forces in the x-direction

$$Ft_{1x} + Ft_{2x} + Fp_{1x} = 0$$

$$\left(\frac{(a_1 - x_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(a_2 - x_1)}{\sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \cdot Ft + \frac{(c_1 - x_1)}{l + \frac{Fp_1}{k}} Fp_1 = 0 \quad (13)$$

Σ of forces in the y -direction

$$F_{t_{1y}} + F_{t_{2y}} + F_{p_{1y}} = 0$$

$$\left(\frac{(b_1 - y_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(b_2 - y_1)}{\sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \cdot Ft + \frac{(d_1 - y_1)}{l + \frac{Fp_1}{k}} Fp_1 = 0 \quad (14)$$

For pulley kinematics, the elastic pulley fibre can only be on a circular path with

extended radius $l + \frac{Fp_1}{k}$ and centre (c_l, d_l) . Thus we obtain,

$$(c_l - x_1)^2 + (d_l - y_1)^2 = \left(l + \frac{Fp_1}{k} \right)^2 \quad (15)$$

From Equations (13), (14) and (15), there are 3 equations with 3 unknowns, x_l , y_l and

Fp_l .

Extending the above to the modelling of 2 pulley fibres, we get:

Figure 32: FBD for elastic pulley system with two fibres

For pulley fibre 1 of Figure 32,

Σ of forces in the x-direction

$$\left(\frac{(a_1 - x_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \right) \cdot Ft + \frac{(c_1 - x_1)}{l + \frac{Fp_1}{k}} Fp_1 = 0 \quad (16)$$

Σ of forces in the y-direction

$$\left(\frac{(b_1 - y_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \right) \cdot Ft + \frac{(d_1 - y_1)}{l + \frac{Fp_1}{k}} Fp_1 = 0 \quad (17)$$

The path of pulley fibre 1 will be

$$(c_1 - x_1)^2 + (d_1 - y_1)^2 = \left(l + \frac{Fp_1}{k} \right)^2 \quad (18)$$

For pulley fibre 2,

Σ of forces in the x-direction

$$\left(\frac{(x_1 - x_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}} \right) \cdot Ft + \left(\frac{(a_2 - x_2)}{\sqrt{(a_2 - x_2)^2 + (b_2 - y_2)^2}} \right) \cdot Ft + \frac{(c_2 - x_2)}{l + \frac{Fp_1}{k}} Fp_2 = 0 \quad (19)$$

Σ of forces in the y-direction

$$\left(\frac{(y_1 - y_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}} \right) \cdot Ft + \left(\frac{(b_2 - y_2)}{\sqrt{(a_2 - x_2)^2 + (b_2 - y_2)^2}} \right) \cdot Ft + \frac{(d_2 - y_2)}{l + \frac{Fp_1}{k}} Fp_2 = 0 \quad (20)$$

The path of pulley fibre 2 will be

$$(c_2 - x_2)^2 + (d_2 - y_2)^2 = \left(l + \frac{Fp_1}{k} \right)^2 \quad (21)$$

From Equations (16) to (21), there are 6 equations with 6 unknowns, x_1, x_2, y_1, y_2, Fp_1 and Fp_2 . They can be solved simultaneously to obtain the 6 unknowns which give us the positions and applied forces on pulley fibre 1 and pulley fibre 2.

With the above analyses, the number of elastic pulley fibres can be increased indefinitely with the general equations

$$\left(\frac{(x_{i-1} - x_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot Ft + \left(\frac{(x_{i+1} - x_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot Ft + \frac{(c_i - x_i) \cdot Fp_i}{l + \frac{Fp_1}{k}} = 0 \quad (22)$$

$$\left(\frac{(y_{i-1} - y_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot Ft + \left(\frac{(y_{i+1} - y_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot Ft + \frac{(d_i - y_i) \cdot Fp_i}{l + \frac{Fp_1}{k}} = 0 \quad (23)$$

$$(c_i - x_i)^2 + (d_i - y_i)^2 = \left(l + \frac{Fp_1}{k} \right)^2 \quad (24)$$

where i represents the i^{th} fibre.

Thus with n pulley fibres, obtain $3n$ equations with $3n$ unknowns. However, due to the numerical possibility of positive and negative answers, we obtain 2^n solutions. Fortunately, from the physical system, we know that that x, y and Fp can only take positive solutions as x, y are in the first quadrant of the Cartesian Coordinates and Fp is always under tension as it can only be pulled. Thus we obtain only a single set of solutions with $3n$ unknowns.

3.3.3 Rigid Pulleys with Friction

Figure 33: FBD of rigid pulleys with friction considered for only one fibre

When friction is present at the pulley-tendon interface, Ft_1 is not equal to Ft_2 . For this case, we assume that the system follows belt friction which has the equation as follows:

$$\frac{Ft_1}{Ft_2} = e^{\mu\alpha}$$

where μ is the friction coefficient and α is the belt contact angle.

The above equation is only true for $Ft_1 > Ft_2 > 0$.

Σ of forces in the x -direction

$$\left(\frac{(a_1 - x_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(a_2 - x_1)}{\sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \cdot \left(\frac{Ft}{e^{\mu\alpha}} \right) + \frac{(c_1 - x_1) \cdot Fp_1}{l_1} = 0 \quad (25)$$

Σ of forces in the y -direction

$$\left(\frac{(b_1 - y_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(b_2 - y_1)}{\sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \cdot \left(\frac{Ft}{e^{\mu\alpha}} \right) + \frac{(d_1 - y_1) \cdot Fp_1}{l_1} = 0 \quad (26)$$

and,

$$(c_1 - x_1)^2 + (d_1 - y_1)^2 = (l_1)^2 \quad (27)$$

There are 3 equations with 4 unknowns, α , x , y and Fp_1 . Thus we have to add another equation to solve for α . Since α is the belt contact angle, we get:

$$\alpha = \pi - \cos^{-1} \left(\frac{(a_1 - x_1)^2 + (b_1 - y_1)^2 + (a_2 - x_1)^2 + (b_2 - y_1)^2 - (a_1 - a_2)^2 - (b_1 - b_2)^2}{2\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2} \cdot \sqrt{(a_2 - x_1)^2 + (b_2 - y_1)^2}} \right) \quad (28)$$

Since α is known, Equations (25), (26) and (27) can be solved.

Extending the above to the modelling of 2 pulley fibres, we get

Figure 34: FBD of rigid pulleys with friction considered for two fibres

For pulley fibre 1,

Σ of forces in the x-direction

$$\left(\frac{(a_1 - x_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \right) \cdot \left(\frac{Ft}{e^{\mu\alpha 1}} \right) + \frac{(c_1 - x_1) \cdot Fp_1}{l_1} = 0 \quad (29)$$

Σ of forces in the y -direction

$$\left(\frac{(b_1 - y_1)}{\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2}} \right) \cdot Ft + \left(\frac{(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \right) \cdot \left(\frac{Ft}{e^{\mu\alpha_1}} \right) + \frac{(d_1 - y_1) \cdot Fp_1}{l_1} = 0 \quad (30)$$

and

$$(c_1 - x_1)^2 + (d_1 - y_1)^2 = (l_1)^2 \quad (31)$$

For pulley fibre 2,

Σ of forces in the x -direction

$$\left(\frac{(x_1 - x_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}} \right) \cdot Ft + \left(\frac{(a_2 - x_2)}{\sqrt{(a_2 - x_2)^2 + (b_2 - y_2)^2}} \right) \cdot \left(\frac{Ft}{e^{\mu\alpha_1} e^{\mu\alpha_2}} \right) + \frac{(c_1 - x_1) \cdot Fp_1}{l_1} = 0 \quad (32)$$

Σ of forces in the y -direction

$$\left(\frac{(y_1 - y_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}} \right) \cdot Ft + \left(\frac{(b_2 - y_2)}{\sqrt{(a_2 - x_2)^2 + (b_2 - y_2)^2}} \right) \cdot \left(\frac{Ft}{e^{\mu\alpha_1} e^{\mu\alpha_2}} \right) + \frac{(d_1 - y_1) \cdot Fp_1}{l_1} = 0 \quad (33)$$

and

$$(c_1 - x_1)^2 + (d_1 - y_1)^2 = (l_1)^2 \quad (34)$$

Note that

$$\alpha_1 = \pi - \cos^{-1} \left(\frac{(a_1 - x_1)^2 + (b_1 - y_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 - (a_1 - x_2)^2 - (b_1 - y_2)^2}{2\sqrt{(a_1 - x_1)^2 + (b_1 - y_1)^2} \cdot \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \right) \quad (35)$$

$$\alpha_2 = \pi - \cos^{-1} \left(\frac{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (a_2 - x_2)^2 + (b_2 - y_2)^2 - (x_1 - a_2)^2 - (y_1 - b_2)^2}{2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \cdot \sqrt{(a_2 - x_2)^2 + (b_2 - y_2)^2}} \right) \quad (36)$$

From Equations (29) to (34), there are 6 equations with 6 unknowns, x_1, x_2, y_1, y_2, Fp_1 and Fp_2 . They can be solved simultaneously to obtain the 6 unknowns which give us the positions and applied forces on pulley fibre 1 and pulley fibre 2.

With the above analyses, the number of rigid pulley fibres can be increased indefinitely with the following general equations

$$\left(\frac{(x_{i-1} - x_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot \left(\frac{Ft}{\prod_{j=1}^{i-1} e^{\mu\theta_j}} \right) + \left(\frac{(x_{i+1} - x_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot \left(\frac{Ft}{\prod_{j=1}^i e^{\mu\theta_j}} \right) + \frac{(c_i - x_i) \cdot Fp_i}{l_i} = 0 \quad (37)$$

$$\left(\frac{(y_{i-1} - y_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot \left(\frac{Ft}{\prod_{j=1}^{i-1} e^{\mu\theta_j}} \right) + \left(\frac{(y_{i+1} - y_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot \left(\frac{Ft}{\prod_{j=1}^i e^{\mu\theta_j}} \right) + \frac{(d_i - y_i) \cdot Fp_i}{l_i} = 0 \quad (38)$$

$$(c_i - x_i)^2 + (d_i - y_i)^2 = (l_i)^2 \quad (39)$$

where

$$\alpha_j = \pi - \cos^{-1} \left(\frac{(x_{j-1} - x_j)^2 + (y_{j-1} - y_j)^2 + (x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2 - (x_{j-1} - x_{j+1})^2 - (y_{j-1} - y_{j+1})^2}{2\sqrt{(x_{j-1} - x_j)^2 + (y_{j-1} - y_j)^2} \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2}} \right) \quad (40)$$

i represents the i^{th} fibre.

Thus with n pulley fibres, we will obtain $3n$ equations with $3n$ unknowns. However, due to the numerical possibility of positive and negative answers, we obtain 2^n solutions. Fortunately, from the physical system, we know that that x, y and Fp can only take positive solutions as x, y are in the first quadrant of the Cartesian

Coordinates and Fp is always under tension as it can only be pulled. Hence we obtain only a single set of solutions with $3n$ unknowns.

Nonetheless, the first two equations (Equations (37) and (38)) are non-algebraic, transcendent equations, due to exponential functions as well as trigonometric functions (Equation (40) to be substituted into Equations (37) and (38)). These equations, after having converted them into Maple syntax, they cannot be solved any more. Hence it is required to convert them into pure algebraic, 2nd order equations by getting rid of the transcendent elements.

Based on the above belt friction equation, the ratio R of the 1 shanks of a belt drive, or tendon in finger pulleys, is

$$R = \frac{T_2}{T_1}$$

where $T_1 > T_2$

Solving for μ delivers

$$\mu = \frac{\left(\ln \frac{1}{R} \right)}{\theta}$$

When replacing the transcendent elements in Equations (1) and (2) by the ratio R , they then become algebraic and μ can be solved afterwards from R and μ .

Upon substitution, we get

$$\left(\frac{(x_{i-1} - x_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot (Ft \cdot R^{i-1}) + \left(\frac{(x_{i+1} - x_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot (Ft \cdot R^i) + \frac{(c_i - x_i) \cdot Fp_i}{l_i} = 0 \quad (41)$$

$$\left(\frac{(y_{i-1} - y_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot (Ft \cdot R^{i-1}) + \left(\frac{(y_{i+1} - y_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot (Ft \cdot R^i) + \frac{(d_i - y_i) \cdot Fp_i}{l_i} = 0 \quad (42)$$

This method offers 3 advantages and one disadvantage.

Advantage 1: all 3 governing equations are algebraic and can be solved by Maple

Advantage 2: the ratio R of the 2 tendon shanks can be controlled directly, which accounts for the mechanical advantage of a friction pulley system, i.e., the ratio of output force over input force; the input force is provided by the muscles.

Advantage 3: in a system consisting of n pulleys the total ratio R_{TOT} of the proximal tendon shank (connected to the muscle) and the distal one (connected to the phalanx) can be pre-set, as the ratio R applies n -times.

Consequently,

$$R_{TOT} = R^n$$

and

$$R = \sqrt[n]{R_{TOT}}$$

This is decisive for investigating the tendon locking mechanism of bats, which suggests that the digital flexor muscles are not required to produce a force for upside-down static hanging. Due to the nature of the ratio R as T_2/T_1 , T_2 can never become zero, as a zero T_2 depends on infinite μ . Yet, muscles still produce a passive muscle tension such that T_2 can become very small. In a system consisting of 19 pulley fibres,

and T_2 be 1% of T_1 , R_{TOT} be 0.01, and R becomes 0.785 for each individual pulley fibre.

The disadvantage of 3 algebraic equations is that the friction coefficient μ cannot be controlled a priori. μ has to be calculated afterwards for each single fibre, based on the ratio R and the included tendon angle θ . As the angle θ changes slightly between the pulley fibres, we can expect slight variation in μ . This, however, means that μ is no longer a constant. Yet, different authors (Table 5) provided different friction coefficients for human finger pulley systems, and moreover, we have no proof of a uniform μ over the entire pulley length. The μ provided by the authors mentioned in Table 5 hence is a mean value over the entire pulley length.

3.3.4 Elastic Pulleys with Friction

The analysis for elastic pulleys with friction is similar to the development of equations in Sections 3.3.2 and 3.3.3.

The general equations obtained are:

$$\left(\frac{(x_{i-1} - x_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot (Ft \cdot R^{i-1}) + \left(\frac{(x_{i+1} - x_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot (Ft \cdot R) + \frac{(c_i - x_i) \cdot Fp_i}{l_i + \frac{Fp_i}{k}} = 0 \quad (43)$$

$$\left(\frac{(y_{i-1} - y_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot (Ft \cdot R^{i-1}) + \left(\frac{(y_{i+1} - y_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot (Ft \cdot R) + \frac{(d_i - y_i) \cdot Fp_i}{l_i + \frac{Fp_i}{k}} = 0 \quad (44)$$

$$(c_i - x_i)^2 + (d_i - y_i)^2 = \left(l_i + \frac{Fp_i}{k} \right)^2 \quad (45)$$

Where

$$\alpha_j = \pi - \cos^{-1} \left(\frac{(x_{j-1} - x_j)^2 + (y_{j-1} - y_j)^2 + (x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2 - (x_{j-1} - x_{j+1})^2 - (y_{j-1} - y_{j+1})^2}{2\sqrt{(x_{j-1} - x_j)^2 + (y_{j-1} - y_j)^2} \cdot \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2}} \right) \quad (46)$$

3.3.5 Elastic Pulleys with Elastic Inter-Pulley Connections

In order to investigate the effects of inter-pulley stiffness, at least 2 fibres have to be considered. We have to define the inter-fibre length s and spring constant q , the latter as compressive spring constant q_C and tensile constant q_T (due to elastic inequality of compression and tension)

In addition to the equations derived for the elastic pulley fibres, both Equations (22) and (23) are augmented by TWO additional forces each, one from i to $i-1$, and the other from i to $i+1$.

These 2 forces can be tensile or compressive. By convention, tensile forces on the $i-1$ side of the pulley are negative, and on the $i+1$ side are positive. Hence, the sign has to be changed for compressive forces.

However, we have to decide the nature of the inter-pulley forces; compression or tension of inter-pulley connections, based on length s and length change Δs :

$$\Delta s = \sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2} - s \quad (47)$$

or

$$\Delta s = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - s \quad (48)$$

Case1: root $> s$, then $\Delta s > 0 \rightarrow$ tension

Case2: root $< s$, then $\Delta s < 0 \rightarrow$ compression

However, we have a problem with the values of q_C and q_T :

If Δs is positive (tension), then the value of q_T should be used

If Δs is negative (compression), then the value of q_C should be used

Thus we need an equation to do the decision making:

$$q = \frac{q_r[\text{sgn}(\Delta s)+1]}{2} - \frac{q_r[\text{sgn}(\Delta s)-1]}{2} \quad (49)$$

The Signum (sgn) function above returns ± 1 .

For example:

Case1: Δs is positive

Meaning that the inter-pulley connection is under tension and thus $q_C = 0$ and $q = q_T$

$$q = \frac{q_r[+1+1]}{2} - \frac{q_r[+1-1]}{2}$$

$$q = \frac{q_r[+2]}{2} - \frac{q_r[0]}{2}$$

$$q = +q_r$$

Case2: Δs is negative

Meaning that the inter-pulley connection is under compression and thus $q_T = 0$ and q

$= q_C$

$$q = \frac{q_r[-1+1]}{2} - \frac{q_r[-1-1]}{2}$$

$$q = \frac{q_r[0]}{2} - \frac{q_r[-2]}{2}$$

$$q = -q_c - 1$$

$$q = +q_c$$

The equation we need to add to the equations derived for elastic pulley fibres for inter-pulley analysis is:

$$\left(\frac{(x_{i-1} - x_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot Fs + \left(\frac{(x_{i+1} - x_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot Fs$$

where Fs = force of inter-pulley connections

Yet, $Fs = \Delta s \cdot q$

Substitute Equations (47) and (48) for Δs , and Equation (49) for q and the additional equations is this:

$$\begin{aligned} & \left(\frac{(x_{i-1} - x_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot \left(\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2} - s \right) \cdot \\ & \left(\frac{q_r}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2} - s \right) + 1 \right) - \frac{q_c}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2} - s \right) - 1 \right) \right) + \\ & \left(\frac{(x_{i+1} - x_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot \left(\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - s \right) \cdot \\ & \left(\frac{q_r}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - s \right) + 1 \right) - \frac{q_c}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - s \right) - 1 \right) \right) \end{aligned}$$

Thus the final 3 equations for inter-pulley analysis are as follows:

$$\begin{aligned}
 & \left(\frac{(x_{i-1} - x_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot Ft + \left(\frac{(x_{i+1} - x_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot Ft + \frac{(c_i - x_i) \cdot Fp_i}{l_i + \frac{Fp_i}{k}} + \\
 & \left(\frac{(x_{i-1} - x_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot \left(\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2} - s_i \right) \cdot \\
 & \left(\frac{q_T}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2} - s_i \right) + 1 \right) - \frac{q_C}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2} - s_i \right) - 1 \right) \right) + \\
 & \left(\frac{(x_{i+1} - x_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot \left(\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - s_i \right) \cdot \\
 & \left(\frac{q_T}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - s_i \right) + 1 \right) - \frac{q_C}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - s_i \right) - 1 \right) \right) = 0
 \end{aligned} \tag{50}$$

$$\begin{aligned}
 & \left(\frac{(y_{i-1} - y_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot Ft + \left(\frac{(y_{i+1} - y_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot Ft + \frac{(d_i - y_i) \cdot Fp_i}{l_i + \frac{Fp_i}{k}} + \\
 & \left(\frac{(y_{i-1} - y_i)}{\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2}} \right) \cdot \left(\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2} - s_i \right) \cdot \\
 & \left(\frac{q_T}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2} - s_i \right) + 1 \right) - \frac{q_C}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2} - s_i \right) - 1 \right) \right) + \\
 & \left(\frac{(y_{i+1} - y_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \right) \cdot \left(\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - s_i \right) \cdot \\
 & \left(\frac{q_T}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - s_i \right) + 1 \right) - \frac{q_C}{2} \left(\operatorname{sgn} \left(\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - s_i \right) - 1 \right) \right) = 0
 \end{aligned} \tag{51}$$

$$(c_i - x_i)^2 + (d_i - y_i)^2 = \left(l_i + \frac{Fp_i}{k} \right)^2 \tag{52}$$

3.4 Explanation of Program Syntax

Having derived the equations needed for analysis, we now have to write a program in Maple 9.5 in order to analyse different input parameters. The syntax is as follows with explanation for ease of understanding.

Explanation of syntax	Maple Syntax
Clears all variables in buffer	> restart;
Input of variables:	> n:=19; Ft:=1; l:=0.45; a[1]:=3; b[1]:=3.8; a[2]:=7; b[2]:=3.8; C:=10; phi:=25;
Calculation of Individual fibre stiffness	> k:=C/n;
Conversion of included angle from degrees to radians	> phi_rad:=phi/180*3.141592654;
Calculation of length of phalanx	> r:=0.8/sin(phi_rad);
Calculation of inter-pulley space	> s:=(r-l)*2*phi_rad/(n-1);

Generation of n number of pulley equations	<pre> > for i from 1 to n do > eqn[1+3*(i-1)]:=((x[i-1]-x[i])/sqrt((x[i-1]-x[i])^2+(y[i-1]-y[i])^2))* Ft+((x[i+1]-x[i])/sqrt((x[i+1]-x[i])^2+(y[i+1]-y[i])^2))*Ft+(c[i]- x[i])*Fp[i]/(1+Fp[i]/k)+((x[i-1]-x[i])/sqrt((x[i-1]-x[i])^2+(y[i-1]- y[i])^2))*(sqrt((x[i-1]-x[i])^2+(y[i-1]-y[i])^2)-s)*qc[1]+((x[i+1]- x[i])/sqrt((x[i+1]-x[i])^2+(y[i+1]-y[i])^2))*(sqrt((x[i+1]-x[i])^2+(y[i+1]- y[i])^2)-s)*qc[2]=0; > eqn[2+3*(i-1)]:=((y[i-1]-y[i])/sqrt((x[i-1]-x[i])^2+(y[i-1]-y[i])^2))* Ft+((y[i+1]-y[i])/sqrt((x[i+1]-x[i])^2+(y[i+1]-y[i])^2))*Ft+(d[i]- y[i])*Fp[i]/(1+Fp[i]/k)+((y[i-1]-y[i])/sqrt((x[i-1]-x[i])^2+(y[i-1]- y[i])^2))*(sqrt((x[i-1]-x[i])^2+(y[i-1]-y[i])^2)-s)*qc[1]+((y[i+1]- y[i])/sqrt((x[i+1]-x[i])^2+(y[i+1]-y[i])^2))*(sqrt((x[i+1]-x[i])^2+(y[i+1]- y[i])^2)-s)*qc[2]=0; > eqn[3+3*(i-1)]:=(c[i]-x[i])^2+(d[i]-y[i])^2=(1+Fp[i]/k)^2; > od: > eqn[1]:=subs({x[0]=a[1],y[0]=b[1],qc[1]=0,qt[1]=0},eqn[1]); > eqn[2]:=subs({x[0]=a[1],y[0]=b[1],qc[1]=0,qt[1]=0},eqn[2]); > eqn[3*n-2]:=subs({x[n+1]=a[2],y[n+1]=b[2],qc[2]=0,qt[2]=0},eqn[3*n-2]) > eqn[3*n-1]:=subs({x[n+1]=a[2],y[n+1]=b[2],qc[2]=0,qt[2]=0},eqn[3*n- 1]): </pre>
Definition of inter-pulley stiffness	<pre> > qc[1]:=1: qc[2]:=1: </pre>
Calculation of pulley attachment points and substituting into pulley equations	<pre> > for i from 1 to n do > e[i]=5-r*cos((3.141592654/2-phi_rad)+2*phi_rad*(i-1)/(n-1)); > assign(%); > f[i]=5-r+r*sin((3.141592654/2-phi_rad)+2*phi_rad*(i-1)/(n-1)); > assign(%); > od: > for i from 1 to n do > eqn[1+3*(i-1)]:=subs(c[i]=e[i],d[i]=f[i],eqn[1+3*(i-1)]); > eqn[2+3*(i-1)]:=subs(c[i]=e[i],d[i]=f[i],eqn[2+3*(i-1)]); > eqn[3+3*(i-1)]:=subs(c[i]=e[i],d[i]=f[i],eqn[3+3*(i-1)]); > od: </pre>

Program output:

$$\begin{aligned} eqn_1 &:= \frac{3 - x_1}{\sqrt{23.44 - 6x_1 + x_1^2 - 7.6y_1 + y_1^2}} + \frac{x_2 - x_1}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} \\ &+ \frac{(4.200000000 - x_1)Fp_1}{0.45 + \frac{19}{10}Fp_1} + \frac{(x_2 - x_1)\mathfrak{e}\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2} - 0.06995673632\ddot{o}}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} = 0 \\ eqn_2 &:= \frac{3.8 - y_1}{\sqrt{23.44 - 6x_1 + x_1^2 - 7.6y_1 + y_1^2}} + \frac{y_2 - y_1}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} \\ &+ \frac{(4.822644270 - y_1)Fp_1}{0.45 + \frac{19}{10}Fp_1} + \frac{(y_2 - y_1)\mathfrak{e}\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2} - 0.06995673632\ddot{o}}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} = 0 \\ eqn_3 &:= (4.200000000 - x_1)^2 + (4.822644270 - y_1)^2 = \mathfrak{e}0.45 + \frac{19}{10}Fp_1\ddot{o}^2 \end{aligned}$$

Note: $\mathfrak{e}$ is open bracket whereas $\ddot{o}$ is close. This is due to errors during file export.

Only the first 3 equations are shown here. The rest can be obtained in Appendix A.

Solve for Fp, x and y of individual fibre	>fsolve({seq(eqn[m],m=1..3*n)},{seq(x[j],j=1..n),seq(y[k],k=1..n),seq(Fp[l],l=1..n)},0..infinity);
---	--

Program output:

{Fp[3] = 0.3704208142e-1, y[7] = 4.452855243, y[10] = 4.466524506, x[2] = 4.451789504, Fp[4] = 0.3907967113e-1, Fp[13] = 0.4282580609e-1, y[5] = 4.428797875, x[17] = 5.481146640, y[17] = 4.393443993, y[1] = 4.348092172, x[13] = 5.207587441, y[13] = 4.452855243, Fp[12] = 0.4345013553e-1, Fp[6] = 0.4191593313e-1, x[12] = 5.138488247, y[12] = 4.460438362, x[6] = 4.723500421, Fp[7] = 0.4282580605e-1, x[5] = 4.654860649, y[6] = 4.442288715, x[3] = 4.518853360, y[3] = 4.393443993, Fp[19] = 0.3134772367e-1, x[15] = 5.345139351, y[15] = 4.428797875, Fp[5] = 0.4068357335e-1, x[7] = 4.792412560, x[16] = 5.413399996, y[16] = 4.412472493, x[19] = 5.614386456, y[19] = 4.348092172, Fp[16] = 0.3907967108e-1, x[8] = 4.861511753, y[8] = 4.460438362, x[18] = 5.548210496, y[18] = 4.371898749, x[10] = 5.000000000, Fp[8] = 0.4345013555e-1, y[2] = 4.371898749, y[4] = 4.412472493, Fp[10] = 0.4393447037e-1, Fp[17] = 0.3704208151e-1, Fp[18] = 0.3449477284e-1, Fp[2] = 0.3449477273e-1, Fp[15] = 0.4068357331e-1, x[4] = 4.586600004, Fp[9] = 0.4381463839e-1, x[9] = 4.930727802, y[9] = 4.465001403, Fp[11] = 0.4381463841e-1, Fp[1] = 0.3134772374e-1, x[14] = 5.276499579, y[14] = 4.442288715, x[11] = 5.069272199, y[11] = 4.465001403, Fp[14] = 0.4191593310e-1, x[1] = 4.385613545}

Sorting of solutions obtained	> assign(%); > seq(x[j],j=1..n); > seq(y[k],k=1..n); > seq(Fp[l],l=1..n);
-------------------------------	--

Program output:

4.385613545, 4.451789504, 4.518853360, 4.586600004, 4.654860649, 4.723500421,
4.792412560, 4.861511753, 4.930727802, 5.000000000, 5.069272199, 5.138488247,
5.207587441, 5.276499579, 5.345139351, 5.413399996, 5.481146640, 5.548210496,
5.614386456
4.348092172, 4.371898749, 4.393443993, 4.412472493, 4.428797875, 4.442288715,
4.452855243, 4.460438362, 4.465001403, 4.466524506, 4.465001403, 4.460438362,
4.452855243, 4.442288715, 4.428797875, 4.412472493, 4.393443993, 4.371898749,
4.348092172
0.3134772374e-1, 0.3449477273e-1, 0.3704208142e-1, 0.3907967113e-1, 0.4068357335e-1,
0.4191593313e-1, 0.4282580605e-1, 0.4345013555e-1, 0.4381463839e-1, 0.4393447037e-1,
0.4381463841e-1, 0.4345013553e-1, 0.4282580609e-1, 0.4191593310e-1, 0.4068357331e-1,
0.3907967108e-1, 0.3704208151e-1, 0.3449477284e-1, 0.3134772367e-1

Pulley fibre under maximum load	> m1:=max(%);
Pulley fibre under minimum load	> m2:=min(%%);
Assigning values to individual fibres with respect to the maximum and minimum force so as to display plots in multiple colours in accordance to the magnitude of force applied	<pre> > for i from 1 to n do > p[i]:=1-(m1-Fp[i])/(m1-m2) > od; > for i from 1 to n do > if p[i]<0.5 then j[i]:=0 > elif p[i]<0.75 then j[i]:=4*(p[i]-0.5) > else j[i]:=1 > end if; > od; > for i from 1 to n do > if p[i]<0.25 then k[i]:=4*p[i] > elif p[i]<0.75 then k[i]:=1 > else k[i]:=-4*(p[i]-1) > end if; > od; > for i from 1 to n do > if p[i]<0.25 then l[i]:=1 > elif p[i]<0.5 then l[i]:=-4*(p[i]-0.5) > else l[i]:=0 > end if; > od; </pre>
Display of plots with colours for individual fibres	<pre> > with(plots): > for i from 1 to n do > L[i]:=[[e[i],f[i]],x[i],y[i]]]: > R[i]:=plot(L[i],color=COLOR(RGB,j[i],k[i],l[i]),view=[3..7,3..7],thickness= [floor(120/n)]): > od; > M:=[seq([e[k],f[k]],k=1..n)]: > p2:=plot(M,colour=[black],view=[3..7,3..7],thickness=[5]): </pre>

```
> N:=[seq([x[k],y[k]],k=1..n)]:
> p3:=plot(N,colour=[tan],view=[3..7,3..7],thickness=[4]):
> P:=[a[1],b[1]],x[1],y[1]]:
> p4:=plot(P,colour=[tan],view=[3..7,3..7],thickness=[4]):
> Q:=[x[n],y[n]],a[2],b[2]]:
> p5:=plot(Q,colour=[tan],view=[3..7,3..7],thickness=[4],title="Profile of
Phalanx"):
> display(seq(R[i],i=1..n),R[5],p2,p3,p4,p5);
```

Program output:

Profile of Phalanx

Fp vs pulley fibre number	> plot([seq([k,Fp[k]],k=1..n)]);
---------------------------	----------------------------------

Program output:

Calculation of inter-pulley strain	> for i from 1 to n-1 do > strain[i]=(sqrt((x[i+1]-x[i])^2+(y[i+1]-y[i])^2)-(1.260356024/(n-1)))/(1.260356024/(n-1)); > od;
Calculation of inter-pulley force	> for i from 1 to n-1 do > IPF[i]=(sqrt((x[i+1]-x[i])^2+(y[i+1]-y[i])^2)-(1.260356024/(n-1)))*qc[1]; > assign(%); > od;
Display of inter-pulley force with respect to pulley fibre number	> plot([seq([u,IPF[u]],u=1..n)]);
Calculation of tendon angle	> alpha[radians]:=arctan((y[1]-b[1])/(x[1]-a[1])); > alpha[degrees]:=(arctan((y[1]-b[1])/(x[1]-a[1])))/3.142*180; > beta[radians]:=arctan((y[n]-b[2])/(a[2]-x[n])); > beta[degrees]:=(arctan((y[n]-b[2])/(a[2]-x[n])))/3.142*180;

4 FACTORS AFFECTING PULLEY FORCES

4.1 Tendon Angle

The flexor tendon goes through a wide range of motion during our daily life. The angle formed between the pulley system and the flexor tendon varies consequently. We would like to investigate how different degrees of flexion affect the force distribution on the pulley system. To track the different magnitudes of flexion, we have to consider the tendon angle, as defined earlier. From Figure 35, two of such angles are possible (one from the proximal end and the other from the distal end) and these angles formed at the MCP joint and the PIP joint are usually different. However, for ease of calculation and better understanding of the force distribution, we assumed that the two joints are symmetrical and the tendon angles are the same. Using equations developed in Section 3.3.2, we modelled the A2 pulley with 9 pulley fibres.

Typical sets of results are depicted in Figure 37 and Figure 36. As can be seen from the coloured fibres, the force distribution is symmetrical about the central fibre. The colours describe the magnitude of the force on the pulley. They are rainbow colours from red (maximum pulley force) to blue (minimum pulley force). For this particular case, the force concentrations are at the marginal pulleys. We then proceeded to explore the change in tendon angle as the finger flexes. Figure 38 summarises the results obtained from a change in tendon angle from 20° to 70° , with 10° steps.

Figure 35: Profile showing the force distribution in a pulley system where the force concentrations are at the marginal pulley fibres

Figure 36: Profile showing the force distribution in a pulley system where the force concentrations are at the central pulley fibres

From Figure 37 it can be seen that the curves changes from 'n' shape to 'u' shaped curves as the tendon angle increases. It is a gradual change and it shows the change in load distribution at different amount of flexion of the fingers. The force is initially concentrated in the central fibre for low tendon angles. This force concentration slowly moves towards the marginal fibres as the tendon angle increases. In addition, the maximum force of each curve increases with an increase in tendon angle and the force distribution becomes less uniform. The gradients at the sides of the curve are steeper when the tendon angle is increased. This adds up to a combine effect of high loads at the marginal fibres at high tendon angles.

From Figure 38, a comparison between the marginal pulley fibres and the central pulley fibre is made. The marginal fibre is noticed to be undertaking a higher force in comparison to the central fibre except for tendon angles below 29° . However, at low tendon angles, the force concentration at the central fibres is low compared to forces

at higher tendon angles. Thus the area of concern should be the marginal fibres where high force concentration will be experienced.

Figure 37: Changes in forces of pulley fibres due to change in tendon angles

Discussion:

A higher tendon angle results in higher forces on the pulley system. In addition, the higher forces are usually experienced by the marginal fibres. Extensive flexion of the fingers in addition to applying a high force is highly undesirable for the pulley system. Figure 39 shows a comparison between the kinds of holds. This may explain why ruptures of the A2 pulley usually occur from the marginal end.

Figure 38: Comparison of forces on marginal and central fibres as tendon angle increases

Figure 39: Loading of the fingers when applying 2 different grips; open hand (left) and crimp (right) [42]. A lower tendon angle (left) and a higher tendon angle (right) ensue.

4.2 Phalanx Curvature

The phalanx is slightly curved for the human finger. We would like to examine the advantage of this curved phalanx as compared to a straight one. It is known that non-

human primates have more curved phalanges and this is often associated to cheiridial use in an arboreal setting. However, details of this association is not developed thus it should be investigated in the perspective of force loading in the pulley system.

As an initial study, only nine pulley fibres are used and the phalanx is modelled as shown in Figure 41. The equations utilized are those developed in Section 3.3.2. However, the attachment points, c_i d_i , are different from those used to model a curved phalanx. These points can be found by dividing the length of the attachment area, on a straight line, by the number of fibres.

Detailed studies of the variation of phalanx curvature in primates are covered in Section 6.3. Although no primates have a straight phalanx configuration, this study is done to give us an idea how the force distribution of the pulleys vary due to a difference in phalanx curvature. Figure 42 shows a typical set of results obtained for a profile with straight phalanx.

Figure 40: Schematic representation of phalanx at different curvatures.

Figure 41: Profile of a straight phalanx.

From Figure 42, it can be seen that 'u' shaped curves are formed for a wide range of motion. In fact, the range of tendon angles used (from 20° to 70°) resulted in all 'u' shaped curves. This shows that a phalanx with a straight profile will lead to force concentrations at the marginal fibres.

Comparing Figure 37 and Figure 42, the pulley force at the marginal fibres for the straight phalanx is higher than that for the curved phalanx for the same tendon angle. In addition, the gradient between the marginal fibre and the adjacent fibre is steeper than in a curved phalanx, accounting for a higher stress concentration at the marginal fibres. This is true for the entire range of motion under investigation. This goes to show that the curvature of the phalanx affects the force distribution in the finger pulley system.

Figure 42: Changes in forces of pulley fibres due to change in tendon angles for a straight phalanx

It can be seen from Figure 43 that the marginal fibres always have a higher force as compared to the central fibres. Upon extrapolation, the two lines will intersect at 0° . This shows that there are always force concentrations at the marginal fibres for flexion of the fingers for a phalanx with straight configuration.

Discussion:

In avoiding finger pulley injuries, stress concentrations have to be avoided and the phalanx design is optimized with a curve attachment area for the finger pulleys, to provide a more even force distribution. For a straight phalanx, the stress

concentrations are always experienced at the marginal fibres. In addition, the forces experienced by these fibres are always higher. Thus there is a higher chance for pulley failure initiating at the marginal fibres for a phalanx with straight configuration.

Figure 43: Comparison of forces on marginal and central fibres as tendon angle increases for a straight phalanx

4.3 Friction

Friction is assumed to be kept a minimum so as to allow smooth gliding of the tendon over the pulley. Although the model developed is a static one, we would like to simulate friction at the pulley-tendon interface and to find out the effects of static friction. This may explain the claim made by Schweizer [5], who suggested that friction may be essential so as to allow rock climbers to exert high loads on the fingers.

The equations utilized are those developed in Section 3.3.4. A total of 9 fibres are used to model the effects of friction. Only one tendon angle is used and the friction coefficient at the pulley-tendon interface varied.

A typical set of results obtained is depicted in Figure 44. It can be seen that the force distribution is not symmetrical. The marginal pulley force on one side is much higher than that for the other.

Figure 44: Non-symmetrical force distribution of pulleys due to the effects of friction

Figure 45 shows the effects of friction on the pulley system. An increase in friction coefficient causes the graph to be more asymmetrical. The marginal pulley fibre force at the end the pulley where the tendon force is applied experienced a much higher force than that of the other marginal fibre. Moreover, it was observed that an increase in friction caused a decrease in forces on the pulley system.

Figure 45: Effects of friction on the force distribution of pulley system

Discussion:

The friction coefficient between tendon and pulley range from 0.027 to 0.075 for normal functional pulleys. The friction coefficient value of 0.05 was plotted on Figure 45. It can be seen that the shift in symmetry is not significant. However, there is a distinct drop in the forces experienced by the pulley fibres. This may support Schweizer’s stand that the presence of friction may increase the force bearing capacity of the pulley fibres.

The range of friction coefficient plotted was between 0 and 0.4. This is to magnify the effects of friction on the pulley-tendon system. In addition, it hints that there is a possibility for the force on the pulley fibre to be zero at higher friction coefficient. This is explored in greater detail in Section 6.4.

5 CONVERGENCE TEST

As our method of analysis is similar to that of the Finite Element Method (FEM), there is a need for verification of the results obtained. The number of elements in FEM is critical. Likewise for our modelling, there is a need for us to ensure that the results obtained converge to a certain value. Thus this section is carried out to find out the number of fibres required in the analysis to achieve a certain confidence level.

For the purpose of a more reliable convergence test, the equations derived from Section 3.3.5 were used. These equations included the elasticity of the pulley fibres and the inter-pulley stiffness. The numbers of pulley fibres were increased for each analysis with all other factors remaining. However, the number of pulley fibres in the system affects its stiffness. Thus the overall stiffness of the pulley system has to be divided by the number of fibres to maintain its overall stiffness.

Nevertheless, this does not solve the problem completely. As the number of fibres increased, each fibre has a lower stiffness. As the forces are not loaded uniformly, some will displace more than others and this will affect the tendon angle. For example, a model with only 5 fibres will have stiffer fibres than those of a model with 21 fibres. Thus the marginal fibres of the 5 fibre model will extend less and this will affect the tendon angle. As known from the analysis in Section 4.1, the tendon angle will affect the force distribution of the pulley fibres. Even so, we could get past the problem by varying the tendon input angle and ensure that the output tendon angles remained the same for all the analysis.

We decided to start off the analysis with 5 pulley fibres and a tendon angle of 23° and from there, the number of fibres increased gradually. The results obtained are depicted in Figure 46. This analysis is for 'n' shaped curve as the forces of the central fibre is always higher than that of the marginal fibre. As can be seen from the figure, forces on both central and marginal will reach asymptotic values. Thus curve fitting with a hyperbolic function may assist us in obtaining the asymptotic values.

Figure 46: Comparison of the characteristics of the marginal and central fibres in a 'n' shaped curve

Matlab was used to curve-fit the results. The generic hyperbolic function, $Y=A/(X+B)+C$ was used, where the value of C would be the asymptotic value. The results obtained are shown in Figure 47, Figure 48 and Table 9.

Figure 47: Curve fitting of the forces of the central fibres for 'n' shaped curves

Figure 48: Curve fitting of the forces of the marginal fibres for 'n' shaped curves

Entity	Values obtained for Marginal Fibre	Values obtained for Central Fibre
A	-2.066	0.4611
B	15.17	-0.9309
C	0.757	0.8351
R- Square Value	0.9976	1.000

Table 9: Values obtained from the curve-fitting of a 'n' shaped curve

No. of Fibres	% of asymptotic value of marginal fibre	% of asymptotic value of central fibre
5	86.5	113.6
7	87.5	109.1
9	88.6	106.8
11	89.6	105.5
15	91.0	103.9
19	92.1	103.0
23	92.9	102.5
27	93.6	101.8
31	93.9	101.1

Table 10: Comparison of the results obtained from each analysis and the corresponding asymptotic value for a 'n' shaped curve

From Table 10, the values of the central fibres converge faster than that of the marginal fibres. Therefore, the parameter of concern should be that of the marginal fibre as it takes more fibres to converge to its asymptotic value. From the results obtained, a minimum of 15 fibres is needed to achieve more than 90% accuracy.

Nevertheless, the results obtained are for 'n' shaped curves. There is a need to do a convergence test for 'u' shaped curves as well, to ensure that they converge in the same manner. Thus a higher tendon angle was selected and the angle used was 32.5°. The results obtained are depicted in Figure 49 . Likewise, a hyperbolic curve fit was carried out and the results are shown in Figure 50, Figure 51 and Table 11.

Figure 49: Comparison of the characteristics of the marginal and central fibres in a 'u' shaped curve

Figure 50: Curve fitting of the forces of the marginal fibres for 'u' shaped curves

Figure 51: Curve fitting of the forces of the central fibres for 'u' shaped curves

Entity	Values obtained for Marginal Fibre	Values obtained for Central Fibre
A	-2.583	0.3858
B	3.684	-1.297
C	1.428	1.005
R- Square Value	0.9999	0.9999

Table 11: Values obtained from the curve-fitting of a 'u' shaped curve

No. of Fibres	% of asymptotic value of marginal fibre	% of asymptotic value of central fibre
5	79.2	110.4
7	83.0	106.8
9	85.7	105.0
13	89.1	103.2
15	90.1	102.8
17	91.3	102.4
21	92.6	101.9
25	93.7	101.6

Table 12: Comparison of the results obtained from each analysis and the corresponding asymptotic value for a 'u' shaped curve

From Table 10, the values of the central fibres converge faster than that of the marginal fibres. Therefore, the parameter of concern should be that of the marginal fibre as it takes more fibres to converge to its asymptotic value. From the results obtained, a minimum of 15 fibres is needed to achieve more than 90% accuracy. Moreover, the number of pulley fibres required can be calculated for certain accuracy.

For example: Accuracy of 95% for the marginal fibres of an 'u' shaped curve.

$$C*(0.95) = A/(X+B)+C$$

$$-0.05C = A/(X+B)$$

$$-0.05(1.428) = -2.583/(X+3.684)$$

$$X=32.5$$

$$X \approx 33$$

For an accuracy of 95% we will need 33 fibres. This will take approximately 6000s for Maple to process using a Pentium 4, 3 GHz, 512 MB RAM, PC. However, for 19 fibres, it takes only 2000s. Thus 19 fibres were used for all future analysis unless the equations get too complex.

6 APPLICATIONS

6.1 Force Distribution and Tendon Angles – Grip at Different Finger Flexion Angles

It had been noticed that at certain tendon angles, neither an ‘n’ nor a ‘u’ force distribution curve is produced. Instead, an awkward shaped curve, something between ‘n’ and ‘u’ shape is produced. We decided to name it the ‘m’ shaped curve. This is the force distribution curve produced for the transition between ‘n’ shaped curves and ‘u’ shaped curves as the tendon angle increases. It is peculiar that a curve like this is produced and thus we decided to study it in greater detail for an included angle of 25° for the phalanx.

Figure 52 is a multiple plot of force distribution curves with different tendon angles. As can be seen, the ‘n’ shaped curves transform to ‘m’ shaped curves before ending up as ‘u’ shaped curves as the tendon angle increases. It is only a narrow transition zone but we would like to find the commencement of the development of the ‘m’ shaped curve to the termination of its existence.

As observed in Figure 52, the maximum forces of some ‘m’ shaped curves occur somewhere between the central and marginal fibres, unlike ‘n’ shaped curves where the maximum forces always occur at the central fibre, and for ‘u’ shaped curves, the marginal fibres. Thus from the observation above, it can be deduced that the ‘m’ shaped curves can be found in regions where the fibres with the maximum load are found between the marginal and central fibres. As Maple is not able to solve for

continuous function of tendon angles (we can only define the input angle and not the output angle), an alternative approach is needed.

Figure 52: Variation in the shapes of the force distribution curves as the tendon angle varies

As all the curves produced have a maximum of three maximum points, a forth order polynomial fit will be suitable for finding equations of individual force distribution curve. Moreover, the curves are symmetrical about the central fibre and thus odd-

powered functions can be eliminated. In other words, the equation, $Y = A \cdot X^4 + B \cdot X^2 + C$ can be used. The curve fit results are reflected in Table 13 and Figure 53.

Tendon Angle	A	B	C
21.578932	-0.00301	-0.00957	0.043928
24.364867	-0.00157	-0.0041	0.046597
25.703821	-0.00087	-0.00145	0.047861
26.489912	-0.00047	0.000115	0.048599
26.749071	-0.00034	0.000633	0.048841
27.006799	-0.0002	0.001148	0.049082
31.866080	0.002193	0.011019	0.053544
36.190379	0.004082	0.020117	0.057392

Table 13: Values of the coefficients A, B and C with respect to the Tendon Angles

Figure 53: Variations of the coefficients A, B and C with respect to the Tendon Angle. Linear curve fit is applied to find the equation to relate the two variables

The characteristics of A, B and C can be analysed by deriving an appropriate equation with respect to the tendon angle. Upon a linear fit of the derived data, the relationships of the 3 coefficients with respect to the tendon angles (θ) are as follows:

$$A = 0.000485584943 \cdot \theta - 0.01337348747$$

$$B = 0.002034314976 \cdot \theta - 0.05369219704$$

$$C = 0.0009194323434 \cdot \theta + 0.02420193146$$

$$\text{Overall Equation: } Y = (0.000485584943 * \theta - 0.01337348747) * X^4 + (0.002034314976 * \theta - 0.05369219704) * X^2 + 0.0009194323434 * \theta + 0.02420193146$$

In order to determine this region where the ‘m’ shaped curves occur, the graph in Figure 54 is produced. The red lines represent the location where the maximum forces are found with respect to the tendon angles. As can be seen, the region in which ‘m’ shaped force distribution curves occur is between 26.4° and 26.8°. However, the fibres with the minimum forces, represented by the blue lines, behave in an entirely different manner. The changes are discrete and not continuous like that of the fibres with maximum force. The fibres with minimum forces can only be found in the central or marginal fibres and this occurs at 26.6°.

This results in two types of ‘m’ shaped curves, one predominantly ‘n’ shaped and the other, predominantly ‘u’ shaped as depicted in Figure 55. Moreover, at around 26.6°, the force in the marginal fibre is equivalent to the force in the central fibre. This results in a perfect ‘m’ as shown in Figure 56. Hence for the ‘m’ shaped curves, we name the predominantly ‘n’ shaped curves as ‘m_n’, the predominantly ‘u’ shaped curve as ‘m_u’ and the perfect ‘m’ as ‘m_p’. The perfect ‘m’ is the point at which we achieve the most even force distribution in the pulley system. Thus we can use this as a bench mark in comparison for other force distributions in the pulley system. For example, if the elasticity of the pulleys changes, the perfect ‘m’ will not exist at a tendon angle of 26.6°. It will appear in another tendon angle. Thus we can use this benchmark to gauge the effects in the change in elasticity of the pulley fibres. This is true for other parameters as well, such as phalanx curvature and inter-pulley stiffness.

Figure 54: Positions in which the fibres with maximum (red line) and minimum forces (blue line) are found

Figure 55: Two kinds of 'm' shaped curves: Predominantly 'n' shaped (left) and predominantly 'u' shaped (right).

Figure 56: Depiction of a perfect ‘m’ where the marginal fibre force is equivalent to the central fibre force

Figure 57: Graph showing the perfect ‘m’ shaped curve having the most even force distribution

Figure 57 shows the peak force, force of marginal and central fibre within the 'm' region. Typically for 'm', the peak force (red line) is higher than marginal fibre force (blue) and central fibre force (green).

The point where green and blue lines cross, reflects equal marginal and central forces, and a ratio of marginal fibre force over central fibre force of 1. This point separates the 'm'-region into 'm_n' and 'm_u'.

'm_p' shows the most even force distribution throughout the entire pulley length. This is reflected in the turquoise line, representing the difference between the peak force and the minimal force, the latter may be the marginal or central fibre force, whichever is smaller. The smaller the difference in force between maximal (peak) and minimal force, the more even is the force distribution. The minimal difference is found exactly at the point where marginal and central forces are equal.

Figure 58: The difference between the maximum force and the minimum force is the smallest for a perfect 'm' shaped force distribution curve

From the above study, we know that change in tendon angles will result in changes in different force distribution curves. Figure 59 shows the transition between 'n' shaped curves and 'u' shaped curves as the tendon angles increase. 'm' shaped curves can be distinctly seen in the figure as well.

Figure 59: Transition between ‘n’ shaped force distribution curves and ‘u’ shaped force distribution curves

Explanation of Figure 59 is as follows:

- Blue section: ‘n’ shaped curves
- Green section: ‘m’ shaped curves which are predominantly ‘n’ shaped
- Yellow section: ‘m’ shaped curves which are predominantly ‘u’ shaped
- Red section: ‘u’ shaped curves

Shape of curve	Force Distribution	Magnitude of force	Risk of Rupture
‘u’	Uneven	High	High
‘m’	Even	Medium	Low
‘n’	Uneven	Low	Low

Table 14: Comparisons of different shaped force distribution curves

From the results, it is clear that flexed fingers under high load results in a higher risk of pulley injury. This is especially true for rock climbers who apply unnaturally amount of force on the fingers and often in a much flexed finger position. This results in the high number of pulley-related injuries.

6.2 Pulley Rupture Mechanism

This section of the study aims to assess the rupture mechanism of the finger pulley system. We would like to determine if a change in tendon angle due to pulley rupture be self contained and prevent propagation of the rupture; or the reduction in pulley fibre numbers be of a greater effect, rendering the rupture self-propagating, thus only depending on the feedback mechanism of pain to compel one to reduce applied force so as to prevent complete pulley rupture.

19 pulley fibres were modelled as suggested by the above convergence test. In addition, another pulley fibre is included at the distal end of the A2 pulley system. This additional fibre is used to represent a functional A3 pulley, which does not rupture so as to maintain the angle of approach towards the A2 pulleys.

After running the first simulation, whereby none of the pulleys are ruptured yet, it was found that the marginal fibre adjacent to the simulated A4 pulley experienced a marginal force of 0.0572. In order to simulate the onset of rupture, it was assumed that all fibres will fail at this value. Next, the failed pulley is removed and the simulation was repeated taking 0.0572 as the rupture force. This is just an arbitrary value, needed to initiate pulley rupture. Subsequently, the process was repeated until the complete rupture of the A2 pulley.

Figure 60: Stepwise simulation of the rupture mechanism of the A2 pulley

From Figure 60, the coloured lines are explained as follows:

- Red line: represents the failure of the marginal pulley at a value of 1.
- Yellow line: one fibre is removed and the simulation repeated. This time round, three fibres exceeded the predetermined failure force of the fibre
- Green line: three pulley fibres were removed leading to the failure of six pulleys
- Blue line: six fibres were removed resulting in the failure of six pulleys
- Purple line: complete rupture of all the pulleys

Figure 61: Sequential rupture of the pulley fibres

The sequential rupture of the A2 pulley is depicted in Figure 61. Rupture occurs when the marginal pulley experiences a force greater than the rupture point. When this occurs, the adjacent fibre experiences an even higher force and so forth. The force on the remaining pulley fibres increase as the number of fibres is gradually reduced.

Thus the rupture will propagate upon application of a constant force. Consequently, to summarise the effects of the rupture, Figure 62 is plotted.

Figure 62: Graph showing the sequence of pulley ruptures

From Figure 62, it can be seen that the rate of propagation varies at different phases of the rupture. The initial propagation is much slower and this accelerates as more fibres are ruptured. However, at the final phase of the rupture, there is a decrease in the rupture rate. This is due to the lack of pulleys as all the pulleys had ruptured.

In addition, the common occurrence of partial pulley failures [3] can be explained. When the pulley starts to rupture, it requires a certain time interval before the pain reflex sets in. Fortunately, the rupture rate is slow initially and this gives the victim ample time to reduce the applied force on the fingers before complete rupture occurs.

6.3 Advantages of Curved Phalanges in Primates

The curvature of the phalanx is largely used to associate the arboreality and bipedality of primates. It is generally thought that a phalanx with a higher curvature relates to a more arboreal species, as there is a greater need for grasping and climbing. However there is no evidence to show the advantage of a phalanx with a higher curvature compared to that of a flatter one. This section aims to show that it is indeed an advantage to have a more curved phalanx in terms of force distribution in the pulley system and how a the pulley system can take a higher force before the onset of rupture.

Figure 63 shows the force distribution over the A2 pulley for different tendon angles. This is for the specific case where the included angle is 25° . All other included angle between 10° and 70° can be found in Appendix B. It can be seen that at around a tendon angle of 27° , stress concentration points start to form at the marginal fibres as the hand is flexed more.

Figure 63: Force distribution of the pulley system at an included angle of 25°

Figure 64: Marginal and Central pulley force ratio at varying tendon angles for different included angles

Figure 65: Marginal and Central pulley force ratio with respect to the marginal force at different tendon angles

Assuming a uniformly distributed force on the pulley system, the ratio between the marginal fibre and the central fibre should be 1. This is to allow a basis for comparison between the different included angles. Figure 64 shows the comparison for all the included angles. As can be seen, stress concentration points only start to appear at a tendon angle of 76° for an included angle of 70°, as compared to a tendon angle of 11° for an included angle of 10°. This allows the hands of primates to flex more without causing additional stress on the pulley system.

Furthermore, from Figure 65, a higher marginal fibre force is recorded at a higher included angle.

6.4 Tendon Locking Mechanism

Bats have the most peculiar habit of hanging upside down when sleeping. They have a tendon locking mechanism that provides a friction mechanism between the sheath and the flexor tendon, which lessens the amount of muscle force needed to actively gripping a substrate. we would like to find out what would be the percentage tendon force reduction due to this friction mechanism and its corresponding friction coefficient.

Based on the equations derived in Section 3.3.4, it is known that only the difference between the tendon force at the proximal end and tendon force at the distal end of the pulley system is needed. In addition to do this, we have to find out the minimum number of pulley fibres needed in our model such that the average fiction coefficient remains the same. Thus we fix the ratio between the proximal tendon and distal tendon, and vary the number of pulley fibres in the model. Matlab was used to curve-fit the results. The generic hyperbolic function, $Y=A/(X+B)+C$ was used, where the value of C would be the asymptotic value. The results obtained are shown in Figure 66 and Table 15.

Figure 66: Curve fitting of μ for different number of fibres

Entity	Values obtained from curve fit
A	-0.6096
B	1.753
C	2.805
R-Square Value	0.9999

Table 15: Values obtained from curve -fitting

From the asymptotic value obtained from Table 15, we can find the degree of accuracy at different fibre numbers. This is done in Table 16. Figure 67 shows the profile of the pulley fibres in the presence of high friction coefficient. All the fibres are pulled toward the distal end and the red fibre shows that the distal fibre experiences the highest load.

Number of Fibres	Average μ	Percentage of Asymptotic Value
5	2.714	96.8
7	2.735	97.6
9	2.748	98.0
11	2.757	98.4
13	2.763	98.6
15	2.768	98.8

Table 16: Degree of accuracy for varying number of fibres

Figure 67: Profile of the pulley system in the presence of high friction coefficient

For friction analysis, even with only 5 fibres, a high degree of accuracy is achieved. The friction coefficient is not sensitive to the changes in μ . Hence we plotted the pulley fibre force with respect to the fibre number in . As expected, the force distribution is not symmetrical. The most distal pulley fibre experiences the highest force whereas the proximal pulley fibre the lowest. For high friction coefficient, the force on the distal fibre is the highest. However, the force decreases rapidly as it proceeds to the proximal end. On the other hand, for lower friction coefficient, the force on the distal fibre is the lowest but the decrease in force to the proximal end is much slower.

Figure 68: Force distribution of pulleys with high friction coefficient

Figure 69: Relationship between friction coefficient and residual force in the proximal tendon

Figure 69 shows the relationship between the friction coefficient and the amount of force experienced by the tendon at the proximal end. It can be seen that the tendon force will not be able to reach absolute zero, but achieving a very small force in the tendon is possible. To achieve 10% residual force in the tendon, a friction coefficient of 1.76

7 DISCUSSIONS

A biomechanical model of the A2 pulley has given us much insight to its characteristics and functionality. The development of such a model has opened up several windows of opportunities to understand the mechanics of the pulley-tendon system better. The model has allowed us to understand the effects of finger flexion at high loads. This is important to rock climbers as they have to execute a variety of finger positions in order to utilize all handholds present. Very often, these positions have detrimental effects on the fingers and from this study, we understand why there is a need to avoid certain finger flexion positions when a high load is applied. In addition, the model allowed us to investigate the rupture mechanism of the pulley system. It explains the nature of pulley rupture from initiation to complete failure. Other than human applications, this model can be applied to animals as well. The phalanx curvatures of primates and hominids are often related to their arboreality and terrestriality. These were observed by researchers based on data collected. Nonetheless, to date this correlation does not have any mechanical proof. Using the model developed, we have the ability to show the advantages of a curved phalanx with respect to the force distribution on the pulley system. Besides the curvature of the phalanx, another phenomenon found in bats, birds and some climbing rodents was explored. These animals have a tendon locking mechanism which allows them to have a sustained grip on a substrate without prolonged muscle activity. The basic principle behind this mechanism is to increase friction at the pulley-tendon interface. This model correlates the friction present at the interface and the residual tendon force. Thus the muscle force saving capacity of the tendon locking mechanism can be investigated.

Convergence tests are critical in deciding the accuracy and reliability of the results obtained from the model. These tests show the model converges as the number of elements is increased. In addition, it reflects on how close the results obtained are with respect to the asymptotic values. These tests had been carried out on different scenarios related to the pulley-tendon system. From the study, it had been found out that a minimum of 15 fibres is needed in order to achieve 90% accuracy. A higher level of accuracy requires a higher number of fibres. However, this is limited to the processing capability of the software and time taken to solve all the equations. An increase of one fibre requires 3 additional equations and thus this renders an exponential increase in the processing time.

The phalanx angles reflect on the degree of flexion of the fingers. Thus this study is important as it shows the force distribution in the pulley system due to different flexion angles of the fingers. It had been found that for extended fingers, the force distribution curve for the pulleys is 'n' shaped. The average force is low and the force concentration is in the middle of the pulley system. Thus failures are rare if the fingers are extended, even if a high force is applied. However, when the fingers are highly flexed, the force distribution curve returned is 'u' shaped. The average force is higher and the force concentrations are at the marginal fibres. This causes pulley failures if a huge load is applied and they always occur from the marginal end of the pulleys. This is particularly important to rock climbers, where the fingers experience extreme forces. Hence it is good to take note that huge loads on flexed fingers can be very detrimental and should be avoided to prevent pulley injuries.

The process of pulley fibre failure had never been investigated. When pulley fibres start to fail, there are two factors affecting the forces in the pulley fibres. The first factor is the number of fibres, where a decrease in fibre number leads to an increase in stress. The other factor is the tendon angle, where any changes in the tendon angle lead to different force distribution. This study takes into account both factors and shows that the rate of pulley failure is not uniform throughout the entire rupture process when a constant force is applied. From the study, it is shown that the rupture is self-propagating and the reduction of pulley fibres plays a greater role in the rupture mechanism. Initially, the failure process is slow due to the presence of high percentage of intact pulley fibres. However, the rate increases as there are less intact fibres left. The initial slow rupture process is very important as it prevents total pulley rupture. Pain reflex sets in at the onset of pulley failure and this gives the victim a certain amount of time to reduce the applied force and this prevents total pulley rupture. Partial rupture has a much higher chance of recovery without a need for surgery. Moreover, the recovery period is much shorter and the presence of a partial pulley system prevents bowstringing of the tendon.

In order to minimise finger pulley injuries, stress concentrations have to be avoided and the human phalanx is optimized with a curve attachment area for the finger pulley, to provide an even force distribution. This avoids stress concentration, in contrast to a straight phalanx profile. Moreover, the effects of phalanx curvature can be extended to explain the arboreality and terrestriality of primates and early hominids. It had been stated that the sensitivity of phalangeal curvature to functional use in extant primates suggests that it faithfully reflects arboreal use in early hominids. Yet, despite of correlation between phalanx morphology and arboreality, there is no available

biomechanical proof, which indicates that a curved phalanx has a biomechanical advantage in powerful grips with increased finger flexion. If the phalanx is curved, then the finger pulleys have to follow the phalangeal curvature. The result obtained for this specific pulley study is that a higher curvature of the finger pulley results in a more even force distribution at higher finger flexion and tendon angles. This shows that the hypothesis made by Richmond [43] was correct from a biomechanics point of view. A phalanx with a greater curvature prevents injuries to the finger pulleys even at high flexion and load.

Bats have the ability to sleep hanging from branches of trees due to the presence of tendon locking mechanism in their pulley-tendon system. The mechanism reduces the amount of force needed by the muscle to sustain prolonged gripping. From our study, it can be seen that zero force on the tendon at proximal end is not possible as this requires the friction at the pulley-tendon interface to be infinity. In order to maintain only 1% force, a friction coefficient of 3.36 is required and a friction coefficient of 1.76 for 10% residual force. Thus from this study, we have developed the relationship between the residual muscle force required and the coefficient of friction needed to achieve that level of force conservation.

8 CONCLUSIONS

This project was initiated with the aim of developing a mathematical model of the A2 pulley system to explain its mechanics. Several parameters had been explored and some interesting results had been achieved.

Convergence test is done and it is discovered that a minimum of 15 fibres is required to achieve 90% accuracy. Higher level accuracy can be achieved but limited to the software solving capabilities.

Tendon angles reflect on the degree of flexion of the finger. The force distribution of the pulley changes from 'n' to 'm' to 'u' as the finger changes extension to flexion. High loads at high flexion angles cause stress concentrations to be developed at the marginal fibres and this is detrimental to the finger.

Rupture mechanism had been investigated and it had been discovered that the rate of rupture of the pulley is not uniform throughout the entire process. It is slow initially but accelerates as the number of intact fibres reduces. In addition, pulley rupture is self-propagating when a constant force is applied.

Studies in the variations of curvatures of phalanges explain the biomechanical advantages of a curved phalanx over a straight one. A curved phalanx allows a more even force distribution and increase the threshold for the applied tendon force without the formation of stress concentration points.

Some animals depend on the tendon locking mechanism to reduce the amount of force needed for sustained gripping. The relationship between the friction due to the tendon locking mechanism and the residual tendon force had been developed in this study.

9 FUTURE WORKS

9.1 Experiments

There are several points in the study which requires additional information. So far, data obtained from previous studies are not completely appropriate to validate the model. Experiments should be conducted authenticate the results obtained.

Firstly, we would like to obtain the total force in the pulley due to a difference in tendon angle. To date, no experiments regarding forces on the pulley system due to a change in tendon angle had been conducted. It would be ideal to detect the forces on individual forces so as to compare the force distribution from the results in the model. However, for an initial study, we would only like to find the total force on the pulley system and compare it to the total force obtained form the model. This is experimentally more manageable at the moment as force transducers for independent fibres may be difficult to produce.

Secondly, we would like to find the values of inter-pulley stiffness for compression and tension. No studies had been done on this. Investigation on the inter-pulley stiffness will make the model more complete and accurate. This will actually reflect on the significance of the inter-pulley stiffness in the model. If it does not have any significance, it can be removed from the equations and render the processing time to be reduced.

Lastly, the friction in the tendon locking mechanism is of interest. We would like to find out the friction coefficient between the sheath and the tendon, and in the process,

calculate the force saving capability of the tendon locking mechanism. Attempts had been made to obtain some bats for testing. Unfortunately, Agri-Food and Veterinary Authority (AVA) of Singapore refused to allow us to import any from Malaysia due to the presence of Nipah virus. Thus we will attempt to get this data from other researches or bats from alternative sources

9.2 Inclusion of Creep into Mathematical Model

We would like to investigate the effects of creep on the model. Creep causes the fibre length to change and this affects the force distribution in the pulleys. This problem can be solved by applying the superposition method developed by Findley et al [44]. The application of superposition can be seen in Figure 70.

9.3 Model of All Pulleys in Entire Finger

Only the A2 pulley had been modelled in this study. Detailed studies on the loading behaviour of the pulley systems in the entire pulley can be studied simultaneously if all the pulleys are modelled. This will give an overall view of the force distribution in the finger and identification of injuries risk due to different factors such as tendon angle, phalanx curvature and friction

Figure 70: Application of superposition on creep

REFERENCES

1. Williams PL: **Gray's Anatomy**. New York: Churchill Livingstone; 1995.
2. Tang JB, Xie RG: **Effect of A3 pulley and adjacent sheath integrity on tendon excursion and bowstringing**. *The Journal of Hand Surgery* 2001, **26**(5):855-861.
3. Schoffl V, Hochholzer T, Winkelmann HP, Strecker W: **[Therapy of injuries of the pulley system in sport climbers]**. *Handchir Mikrochir Plast Chir* 2004, **36**(4):231-236. (Article in German)
4. Schweizer A: **Biomechanical properties of the crimp grip position in rock climbers**. *Journal of Biomechanics* 2001, **34**(2):217-223.
5. Schweizer A, Frank O, Ochsner PE, Jacob HAC: **Friction between human finger flexor tendons and pulleys at high loads**. *Journal of Biomechanics* 2003, **36**(1):63-71.
6. Doyle JR: **Anatomy and function of the palmar aponeurosis pulley**. *J Hand Surg [Am]* 1990, **15**(1):78-82.
7. Lin G-T, Cooney WP, Amadio PC, An K-N: **Mechanical properties of human pulleys**. *The Journal of Hand Surgery: Journal of the British Society for Surgery of the Hand* 1990, **15**(4):429-434.
8. Stern JT, Jr., Jungers WL, Susman RL: **Quantifying phalangeal curvature: an empirical comparison of alternative methods**. *Am J Phys Anthropol* 1995, **97**(1):1-10.
9. Strauch B, de Moura W: **Digital flexor tendon sheath: an anatomic study**. *J Hand Surg [Am]* 1985, **10**(6 Pt 1):785-789.

10. Hauger O, Chung CB, Lektrakul N, Botte MJ, Trudell D, Boutin RD, Resnick D: **Pulley system in the fingers: normal anatomy and simulated lesions in cadavers at MR imaging, CT, and US with and without contrast material distention of the tendon sheath.** *Radiology* 2000, **217**(1):201-212.
11. Hahn P, Lanz U: **[Flexor pulleys of the fingers. Anatomy, biomechanics, reconstruction].** *Handchir Mikrochir Plast Chir* 1996, **28**(5):265-270. (Article in German)
12. Wilhelmi BJ, Snyder Nt, Verbesey JE, Ganchi PA, Lee WP: **Trigger finger release with hand surface landmark ratios: an anatomic and clinical study.** *Plast Reconstr Surg* 2001, **108**(4):908-915.
13. **Turn Till Burn** [<http://www.turntillburn.ch/ttbs/en/med/ringband.htm>]
14. **Upper Extremity Muscle Atlas** [<http://www.rad.washington.edu/atlas/>]
15. Doyle JR: **Palmar and digital flexor tendon pulleys.** *Clin Orthop Relat Res* 2001(383):84-96.
16. Mester S, Schmidt B, Derczy K, Nyarady J, Biro V: **Biomechanics of the human flexor tendon sheath investigated by tenography.** *J Hand Surg [Br]* 1995, **20**(4):500-504.
17. Boyer MI, Meunier MJ, Lescheid J, Burns ME, Gelberman RH, Silva MJ: **The influence of cross-sectional area on the tensile properties of flexor tendons.** *The Journal of Hand Surgery* 2001, **26**(5):828-832.
18. Moutet F: **[Flexor tendon pulley system: anatomy, pathology, treatment].** *Chir Main* 2003, **22**(1):1-12. (Article in French)
19. Tanaka T, Amadio PC, Zhao C, Zobitz ME, An K-N: **The effect of partial A2 pulley excision on gliding resistance and pulley strength in vitro.** *The Journal of Hand Surgery* 2004, **29**(5):877-883.

20. Marieb EN: **Human Anatomy and Physiology**, 3 edn. California: The Benjamin/Cummings Publishing Company; 1995.
21. Uchiyama S, Amadio PC, Coert JH, Berglund LJ, An KN: **Gliding resistance of extrasynovial and intrasynovial tendons through the A2 pulley**. *J Bone Joint Surg Am* 1997, **79**(2):219-224.
22. Paillard PJ, Amadio PC, Zhao C, Zobitz ME, An KN: **Gliding resistance after FDP and FDS tendon repair in zone II: an in vitro study**. *Acta Orthop Scand* 2002, **73**(4):465-470.
23. Zhao C, Amadio PC, Zobitz ME, Momose T, Couvreur P, An KN: **Gliding resistance after repair of partially lacerated human flexor digitorum profundus tendon in vitro**. *Clin Biomech (Bristol, Avon)* 2001, **16**(8):696-701.
24. Moro-oka T, Miura H, Higaki H, Arimura S, Mawatari T, Murakami T, Iwamoto Y: **A new friction tester of the flexor tendon**. *Journal of Biomechanics* 1999, **32**(10):1131-1134.
25. Zhao C, Amadio PC, Zobitz ME, An KN: **Gliding characteristics of tendon repair in canine flexor digitorum profundus tendons**. *J Orthop Res* 2001, **19**(4):580-586.
26. Momose T, Amadio PC, Sun YL, Zhao C, Zobitz ME, Harrington JR, An KN: **Surface modification of extrasynovial tendon by chemically modified hyaluronic acid coating**. *J Biomed Mater Res* 2002, **59**(2):219-224.
27. K. N. An. MEZ, L. Berglund, C. F. Zhao, S. Uchiyama: **Biomechanics of Tendon Gliding**. In: *American Society of Biomechanics: 2001; University of California, San Diego*; 2001: 2.

28. Uchiyama S, Amadio PC, Ishikawa J, An KN: **Boundary lubrication between the tendon and the pulley in the finger.** *J Bone Joint Surg Am* 1997, **79**(2):213-218.
29. Uchiyama S, Coert JH, Berglund L, Amadio PC, An KN: **Method for the measurement of friction between tendon and pulley.** *J Orthop Res* 1995, **13**(1):83-89.
30. Roloff I, Schoffl VR, Vigouroux L, Quaine F: **Biomechanical model for the determination of the forces acting on the finger pulley system.** *Journal of Biomechanics* 2005.
31. Gabl M, Rangger C, Lutz M, Fink C, Rudisch A, Pechlaner S: **Disruption of the finger flexor pulley system in elite rock climbers.** *Am J Sports Med* 1998, **26**(5):651-655.
32. Schweizer A: **Biomechanical effectiveness of taping the A2 pulley in rock climbers.** *J Hand Surg [Br]* 2000, **25**(1):102-107.
33. Warme WJ, Brooks D: **The effect of circumferential taping on flexor tendon pulley failure in rock climbers.** *Am J Sports Med* 2000, **28**(5):674-678.
34. Marco RA, Sharkey NA, Smith TS, Zissimos AG: **Pathomechanics of closed rupture of the flexor tendon pulleys in rock climbers.** *J Bone Joint Surg Am* 1998, **80**(7):1012-1019.
35. Susman RL, Stern JT, Jr., Jungers WL: **Arboreality and bipedality in the Hadar hominids.** *Folia Primatol (Basel)* 1984, **43**(2-3):113-156.
36. Jungers WL, Godfrey LR, Simons EL, Chatrath PS: **Phalangeal curvature and positional behavior in extinct sloth lemurs (Primates, Palaeopropithecidae).** *Proc Natl Acad Sci U S A* 1997, **94**(22):11998-12001.

37. Susman RL: **Oreopithecus bambolii: an unlikely case of hominid-like grip capability in a Miocene ape.** *J Hum Evol* 2004, **46**(1):105-117.
38. Quinn TH, Baumel JJ: **The digital tendon locking mechanism of the avian foot (Aves).** *Zoomorphology* 1990, **109**:281-293.
39. Quinn TH, Baumel JJ: **Chiropteran tendon locking mechanism.** *J Morphol* 1993, **216**(2):197-208.
40. Haffner M: **A tendon-locking mechanism in two climbing rodents, Muscardinus avellanarius and Micromys minutus (Mammalia, Rodentia).** *J Morphol* 1996, **229**(2):219-227.
41. Niegl G: **Length of pulleys.** In. Vienna; 2005. (Personal communication)
42. **Individual Sports for Health Promotion** [<http://www2.eou.edu/peh494/>]
43. Richmond BG: **Ontogeny of Phalangeal Curvature and Locomotor Behavior in lar gibbons.** *American Journal of Physical Anthropology* 1997, **24**:197.
44. Findley WN, Lai JS, Onran K: **Creep and relaxation of nonlinear viscoelastic materials.** New York: Dover Publications; 1989.

APPENDIX A

Please Note: Maple has a bug and the symbol $\frac{\partial}{\partial}$ is open bracket and the symbol $\frac{\partial}{\partial}$ is the close bracket

$$\begin{aligned} eqn_1 := & \frac{3 - x_1}{\sqrt{23.44 - 6x_1 + x_1^2 - 7.6y_1 + y_1^2}} + \frac{x_2 - x_1}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} \\ & + \frac{(4.200000000 - x_1) Fp_1}{0.45 + \frac{19}{10} Fp_1} + \frac{(x_2 - x_1) \frac{\partial}{\partial} \sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2} - 0.06995673632 \frac{\partial}{\partial}}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} = 0 \end{aligned}$$

$$\begin{aligned} eqn_2 := & \frac{3.8 - y_1}{\sqrt{23.44 - 6x_1 + x_1^2 - 7.6y_1 + y_1^2}} + \frac{y_2 - y_1}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} \\ & + \frac{(4.822644270 - y_1) Fp_1}{0.45 + \frac{19}{10} Fp_1} + \frac{(y_2 - y_1) \frac{\partial}{\partial} \sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2} - 0.06995673632 \frac{\partial}{\partial}}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} = 0 \end{aligned}$$

$$eqn_3 := (4.200000000 - x_1)^2 + (4.822644270 - y_1)^2 = \frac{\partial}{\partial} 0.45 + \frac{19}{10} Fp_1 \frac{\partial}{\partial}^2$$

$$\begin{aligned} eqn_4 := & \frac{x_1 - x_2}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} + \frac{x_3 - x_2}{\sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2}} \\ & + \frac{(4.284082318 - x_2) Fp_2}{0.45 + \frac{19}{10} Fp_2} + \frac{(x_1 - x_2) \frac{\partial}{\partial} \sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2} - 0.06995673632 \frac{\partial}{\partial}}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} \\ & + \frac{(x_3 - x_2) \frac{\partial}{\partial} \sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2} - 0.06995673632 \frac{\partial}{\partial}}{\sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2}} = 0 \end{aligned}$$

$$\begin{aligned} eqn_5 := & \frac{y_1 - y_2}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} + \frac{y_3 - y_2}{\sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2}} \\ & + \frac{(4.859398351 - y_2) Fp_2}{0.45 + \frac{19}{10} Fp_2} + \frac{(y_1 - y_2) \frac{\partial}{\partial} \sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2} - 0.06995673632 \frac{\partial}{\partial}}{\sqrt{x_2^2 - 2x_2x_1 + x_1^2 + y_2^2 - 2y_2y_1 + y_1^2}} \\ & + \frac{(y_3 - y_2) \frac{\partial}{\partial} \sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2} - 0.06995673632 \frac{\partial}{\partial}}{\sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2}} = 0 \end{aligned}$$

APPENDIX A

$$eqn_6 := (4.284082318 - x_2)^2 + (4.859398351 - y_2)^2 = \frac{\mathfrak{A}}{\mathfrak{C}} 0.45 + \frac{19}{10} Fp_2 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}$$

$$\begin{aligned} eqn_7 := & \frac{x_2 - x_3}{\sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2}} + \frac{x_4 - x_3}{\sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2}} \\ & + \frac{(4.369847029 - x_3) Fp_3}{0.45 + \frac{19}{10} Fp_3} + \frac{(x_2 - x_3) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2} - 0.06995673632 \ddot{\mathfrak{O}}}{\sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2}} \\ & + \frac{(x_4 - x_3) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2} - 0.06995673632 \ddot{\mathfrak{O}}}{\sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2}} = 0 \end{aligned}$$

$$\begin{aligned} eqn_8 := & \frac{y_2 - y_3}{\sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2}} + \frac{y_4 - y_3}{\sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2}} \\ & + \frac{(4.892034417 - y_3) Fp_3}{0.45 + \frac{19}{10} Fp_3} + \frac{(y_2 - y_3) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2} - 0.06995673632 \ddot{\mathfrak{O}}}{\sqrt{x_3^2 - 2x_3x_2 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2}} \\ & + \frac{(y_4 - y_3) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2} - 0.06995673632 \ddot{\mathfrak{O}}}{\sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2}} = 0 \end{aligned}$$

$$eqn_9 := (4.369847029 - x_3)^2 + (4.892034417 - y_3)^2 = \frac{\mathfrak{A}}{\mathfrak{C}} 0.45 + \frac{19}{10} Fp_3 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}$$

$$\begin{aligned} eqn_{10} := & \frac{x_3 - x_4}{\sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2}} + \frac{x_5 - x_4}{\sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2}} \\ & + \frac{(4.457092589 - x_4) Fp_4}{0.45 + \frac{19}{10} Fp_4} + \frac{(x_3 - x_4) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2} - 0.06995673632 \ddot{\mathfrak{O}}}{\sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2}} \\ & + \frac{(x_5 - x_4) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2} - 0.06995673632 \ddot{\mathfrak{O}}}{\sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2}} = 0 \end{aligned}$$

APPENDIX A

$$\begin{aligned}
 eqn_{11} := & \frac{y_3 - y_4}{\sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2}} + \frac{y_5 - y_4}{\sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2}} \\
 & + \frac{(4.920475774 - y_4) Fp_4}{0.45 + \frac{19}{10} Fp_4} + \frac{(y_3 - y_4) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2} - 0.06995673632 \frac{\ddot{\mathcal{O}}}{\emptyset}}{\sqrt{x_4^2 - 2x_4x_3 + x_3^2 + y_4^2 - 2y_4y_3 + y_3^2}} \\
 & + \frac{(y_5 - y_4) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2} - 0.06995673632 \frac{\ddot{\mathcal{O}}}{\emptyset}}{\sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2}} = 0
 \end{aligned}$$

$$eqn_{12} := (4.457092589 - x_4)^2 + (4.920475774 - y_4)^2 = \frac{\mathfrak{A}}{\mathfrak{C}} 0.45 + \frac{19}{10} Fp_4 \frac{\ddot{\mathcal{O}}}{\emptyset}^2$$

$$\begin{aligned}
 eqn_{13} := & \frac{x_4 - x_5}{\sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2}} + \frac{x_6 - x_5}{\sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2}} \\
 & + \frac{(4.545613972 - x_5) Fp_5}{0.45 + \frac{19}{10} Fp_5} + \frac{(x_4 - x_5) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2} - 0.06995673632 \frac{\ddot{\mathcal{O}}}{\emptyset}}{\sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2}} \\
 & + \frac{(x_6 - x_5) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2} - 0.06995673632 \frac{\ddot{\mathcal{O}}}{\emptyset}}{\sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2}} = 0
 \end{aligned}$$

$$\begin{aligned}
 eqn_{14} := & \frac{y_4 - y_5}{\sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2}} + \frac{y_6 - y_5}{\sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2}} \\
 & + \frac{(4.944655585 - y_5) Fp_5}{0.45 + \frac{19}{10} Fp_5} + \frac{(y_4 - y_5) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2} - 0.06995673632 \frac{\ddot{\mathcal{O}}}{\emptyset}}{\sqrt{x_5^2 - 2x_5x_4 + x_4^2 + y_5^2 - 2y_5y_4 + y_4^2}} \\
 & + \frac{(y_6 - y_5) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2} - 0.06995673632 \frac{\ddot{\mathcal{O}}}{\emptyset}}{\sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2}} = 0
 \end{aligned}$$

$$eqn_{15} := (4.545613972 - x_5)^2 + (4.944655585 - y_5)^2 = \frac{\mathfrak{A}}{\mathfrak{C}} 0.45 + \frac{19}{10} Fp_5 \frac{\ddot{\mathcal{O}}}{\emptyset}^2$$

APPENDIX A

$$\begin{aligned}
eqn_{16} := & \frac{x_5 - x_6}{\sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2}} + \frac{x_7 - x_6}{\sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2}} \\
& + \frac{(4.635203156 - x_6) Fp_6}{0.45 + \frac{19}{10} Fp_6} + \frac{(x_5 - x_6) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2}} \\
& + \frac{(x_7 - x_6) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2}} = 0
\end{aligned}$$

$$\begin{aligned}
eqn_{17} := & \frac{y_5 - y_6}{\sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2}} + \frac{y_7 - y_6}{\sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2}} \\
& + \frac{(4.964517029 - y_6) Fp_6}{0.45 + \frac{19}{10} Fp_6} + \frac{(y_5 - y_6) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_6^2 - 2x_6x_5 + x_5^2 + y_6^2 - 2y_6y_5 + y_5^2}} \\
& + \frac{(y_7 - y_6) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2}} = 0
\end{aligned}$$

$$eqn_{18} := (4.635203156 - x_6)^2 + (4.964517029 - y_6)^2 = \frac{\mathfrak{A}}{\mathfrak{C}} 0.45 + \frac{19}{10} Fp_6 \frac{\ddot{\mathfrak{O}}}{\varnothing}^2$$

$$\begin{aligned}
eqn_{19} := & \frac{x_6 - x_7}{\sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2}} + \frac{x_8 - x_7}{\sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2}} \\
& + \frac{(4.725649605 - x_7) Fp_7}{0.45 + \frac{19}{10} Fp_7} + \frac{(x_6 - x_7) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2}} \\
& + \frac{(x_8 - x_7) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2}} = 0
\end{aligned}$$

APPENDIX A

$$\begin{aligned}
eqn_{20} := & \frac{y_6 - y_7}{\sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2}} + \frac{y_8 - y_7}{\sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2}} \\
& + \frac{(4.980013431 - y_7) Fp_7}{0.45 + \frac{19}{10} Fp_7} + \frac{(y_6 - y_7) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_7^2 - 2x_7x_6 + x_6^2 + y_7^2 - 2y_7y_6 + y_6^2}} \\
& + \frac{(y_8 - y_7) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2}} = 0
\end{aligned}$$

$$eqn_{21} := (4.725649605 - x_7)^2 + (4.980013431 - y_7)^2 = \frac{\mathfrak{E}}{\mathfrak{E}} 0.45 + \frac{19}{10} Fp_7 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}^2$$

$$\begin{aligned}
eqn_{22} := & \frac{x_7 - x_8}{\sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2}} + \frac{x_9 - x_8}{\sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2}} \\
& + \frac{(4.816740772 - x_8) Fp_8}{0.45 + \frac{19}{10} Fp_8} + \frac{(x_7 - x_8) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2}} \\
& + \frac{(x_9 - x_8) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2}} = 0
\end{aligned}$$

$$\begin{aligned}
eqn_{23} := & \frac{y_7 - y_8}{\sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2}} + \frac{y_9 - y_8}{\sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2}} \\
& + \frac{(4.991108374 - y_8) Fp_8}{0.45 + \frac{19}{10} Fp_8} + \frac{(y_7 - y_8) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_8^2 - 2x_8x_7 + x_7^2 + y_8^2 - 2y_8y_7 + y_7^2}} \\
& + \frac{(y_9 - y_8) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2}} = 0
\end{aligned}$$

$$eqn_{24} := (4.816740772 - x_8)^2 + (4.991108374 - y_8)^2 = \frac{\mathfrak{E}}{\mathfrak{E}} 0.45 + \frac{19}{10} Fp_8 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}^2$$

APPENDIX A

$$\begin{aligned}
eqn_{25} := & \frac{x_8 - x_9}{\sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2}} + \frac{x_{10} - x_9}{\sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2}} \\
& + \frac{(4.908262596 - x_9) Fp_9}{0.45 + \frac{19}{10} Fp_9} + \frac{(x_8 - x_9) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2}} \\
& + \frac{(x_{10} - x_9) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2}} = 0
\end{aligned}$$

$$\begin{aligned}
eqn_{26} := & \frac{y_8 - y_9}{\sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2}} + \frac{y_{10} - y_9}{\sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2}} \\
& + \frac{(4.997775787 - y_9) Fp_9}{0.45 + \frac{19}{10} Fp_9} + \frac{(y_8 - y_9) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_9^2 - 2x_9x_8 + x_8^2 + y_9^2 - 2y_9y_8 + y_8^2}} \\
& + \frac{(y_{10} - y_9) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2}} = 0
\end{aligned}$$

$$eqn_{27} := (4.908262596 - x_9)^2 + (4.997775787 - y_9)^2 = \frac{\mathfrak{E}}{\mathfrak{E}} 0.45 + \frac{19}{10} Fp_9 \frac{\ddot{\mathfrak{O}}}{\emptyset}^2$$

$$\begin{aligned}
eqn_{28} := & \frac{x_9 - x_{10}}{\sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2}} + \frac{x_{11} - x_{10}}{\sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2}} \\
& + \frac{(5.000000000 - x_{10}) Fp_{10}}{0.45 + \frac{19}{10} Fp_{10}} \\
& + \frac{(x_9 - x_{10}) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2}} \\
& + \frac{(x_{11} - x_{10}) \frac{\mathfrak{E}}{\mathfrak{E}} \sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2}} = 0
\end{aligned}$$

APPENDIX A

$$\begin{aligned}
eqn_{29} := & \frac{y_9 - y_{10}}{\sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2}} + \frac{y_{11} - y_{10}}{\sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2}} \\
& + \frac{(5.000000000 - y_{10}) Fp_{10}}{0.45 + \frac{19}{10} Fp_{10}} \\
& + \frac{(y_9 - y_{10}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_{10}^2 - 2x_{10}x_9 + x_9^2 + y_{10}^2 - 2y_{10}y_9 + y_9^2}} \\
& + \frac{(y_{11} - y_{10}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2}} = 0
\end{aligned}$$

$$eqn_{30} := (5.000000000 - x_{10})^2 + (5.000000000 - y_{10})^2 = \frac{\mathfrak{A}}{\mathfrak{E}} 0.45 + \frac{19}{10} Fp_{10} \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}^2$$

$$\begin{aligned}
eqn_{31} := & \frac{x_{10} - x_{11}}{\sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2}} + \frac{x_{12} - x_{11}}{\sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2}} \\
& + \frac{(5.091737405 - x_{11}) Fp_{11}}{0.45 + \frac{19}{10} Fp_{11}} \\
& + \frac{(x_{10} - x_{11}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2}} \\
& + \frac{(x_{12} - x_{11}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2}} = 0
\end{aligned}$$

$$\begin{aligned}
eqn_{32} := & \frac{y_{10} - y_{11}}{\sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2}} + \frac{y_{12} - y_{11}}{\sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2}} \\
& + \frac{(4.997775787 - y_{11}) Fp_{11}}{0.45 + \frac{19}{10} Fp_{11}}
\end{aligned}$$

APPENDIX A

$$\begin{aligned}
& + \frac{(y_{10} - y_{11}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_{11}^2 - 2x_{11}x_{10} + x_{10}^2 + y_{11}^2 - 2y_{11}y_{10} + y_{10}^2}} \\
& + \frac{(y_{12} - y_{11}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2}} = 0
\end{aligned}$$

$$eqn_{33} := (5.091737405 - x_{11})^2 + (4.997775787 - y_{11})^2 = \frac{\mathfrak{A}}{\mathfrak{E}} 0.45 + \frac{19}{10} Fp_{11} \frac{\ddot{\mathfrak{O}}}{\emptyset}^2$$

$$\begin{aligned}
eqn_{34} := & \frac{x_{11} - x_{12}}{\sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2}} + \frac{x_{13} - x_{12}}{\sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2}} \\
& + \frac{(5.183259228 - x_{12}) Fp_{12}}{0.45 + \frac{19}{10} Fp_{12}} \\
& + \frac{(x_{11} - x_{12}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2}} \\
& + \frac{(x_{13} - x_{12}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2}} = 0
\end{aligned}$$

$$\begin{aligned}
eqn_{35} := & \frac{y_{11} - y_{12}}{\sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2}} + \frac{y_{13} - y_{12}}{\sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2}} \\
& + \frac{(4.991108374 - y_{12}) Fp_{12}}{0.45 + \frac{19}{10} Fp_{12}} \\
& + \frac{(y_{11} - y_{12}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_{12}^2 - 2x_{12}x_{11} + x_{11}^2 + y_{12}^2 - 2y_{12}y_{11} + y_{11}^2}} \\
& + \frac{(y_{13} - y_{12}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\emptyset}}{\sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2}} = 0
\end{aligned}$$

$$eqn_{36} := (5.183259228 - x_{12})^2 + (4.991108374 - y_{12})^2 = \frac{\mathfrak{A}}{\mathfrak{E}} 0.45 + \frac{19}{10} Fp_{12} \frac{\ddot{\mathfrak{O}}}{\emptyset}^2$$

APPENDIX A

$$\begin{aligned}
 eqn_{37} := & \frac{x_{12} - x_{13}}{\sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2}} + \frac{x_{14} - x_{13}}{\sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2}} \\
 & + \frac{(5.274350396 - x_{13}) Fp_{13}}{0.45 + \frac{19}{10} Fp_{13}} \\
 & + \frac{(x_{12} - x_{13}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2} - 0.06995673632 \frac{\ddot{\mathbf{O}}}{\mathfrak{O}}}{\sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2}} \\
 & + \frac{(x_{14} - x_{13}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2} - 0.06995673632 \frac{\ddot{\mathbf{O}}}{\mathfrak{O}}}{\sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2}} = 0
 \end{aligned}$$

$$\begin{aligned}
 eqn_{38} := & \frac{y_{12} - y_{13}}{\sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2}} + \frac{y_{14} - y_{13}}{\sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2}} \\
 & + \frac{(4.980013431 - y_{13}) Fp_{13}}{0.45 + \frac{19}{10} Fp_{13}} \\
 & + \frac{(y_{12} - y_{13}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2} - 0.06995673632 \frac{\ddot{\mathbf{O}}}{\mathfrak{O}}}{\sqrt{x_{13}^2 - 2x_{13}x_{12} + x_{12}^2 + y_{13}^2 - 2y_{13}y_{12} + y_{12}^2}} \\
 & + \frac{(y_{14} - y_{13}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2} - 0.06995673632 \frac{\ddot{\mathbf{O}}}{\mathfrak{O}}}{\sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2}} = 0
 \end{aligned}$$

$$eqn_{39} := (5.274350396 - x_{13})^2 + (4.980013431 - y_{13})^2 = \frac{\mathfrak{A}}{\mathfrak{E}} 0.45 + \frac{19}{10} Fp_{13} \frac{\ddot{\mathbf{O}}}{\mathfrak{O}}$$

$$\begin{aligned}
 eqn_{40} := & \frac{x_{13} - x_{14}}{\sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2}} + \frac{x_{15} - x_{14}}{\sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2}} \\
 & + \frac{(5.364796844 - x_{14}) Fp_{14}}{0.45 + \frac{19}{10} Fp_{14}}
 \end{aligned}$$

APPENDIX A

$$\begin{aligned}
& + \frac{(x_{13} - x_{14}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2}} \\
& + \frac{(x_{15} - x_{14}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2}} = 0 \\
\\
eqn_{41} := & \frac{y_{13} - y_{14}}{\sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2}} + \frac{y_{15} - y_{14}}{\sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2}} \\
& + \frac{(4.964517029 - y_{14}) Fp_{14}}{0.45 + \frac{19}{10} Fp_{14}} \\
& + \frac{(y_{13} - y_{14}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{14}^2 - 2x_{14}x_{13} + x_{13}^2 + y_{14}^2 - 2y_{14}y_{13} + y_{13}^2}} \\
& + \frac{(y_{15} - y_{14}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2}} = 0 \\
\\
eqn_{42} := & (5.364796844 - x_{14})^2 + (4.964517029 - y_{14})^2 = \frac{\mathfrak{A}}{\mathfrak{E}} 0.45 + \frac{19}{10} Fp_{14} \frac{\ddot{\mathfrak{O}}}{\varnothing}^2 \\
\\
eqn_{43} := & \frac{x_{14} - x_{15}}{\sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2}} + \frac{x_{16} - x_{15}}{\sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2}} \\
& + \frac{(5.454386028 - x_{15}) Fp_{15}}{0.45 + \frac{19}{10} Fp_{15}} \\
& + \frac{(x_{14} - x_{15}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2}} \\
& + \frac{(x_{16} - x_{15}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2}} = 0
\end{aligned}$$

APPENDIX A

$$\begin{aligned}
eqn_{44} := & \frac{y_{14} - y_{15}}{\sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2}} + \frac{y_{16} - y_{15}}{\sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2}} \\
& + \frac{(4.944655585 - y_{15}) Fp_{15}}{0.45 + \frac{19}{10} Fp_{15}} \\
& + \frac{(y_{14} - y_{15}) \frac{\mathfrak{E}}{\mathfrak{e}} \sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_{15}^2 - 2x_{15}x_{14} + x_{14}^2 + y_{15}^2 - 2y_{15}y_{14} + y_{14}^2}} \\
& + \frac{(y_{16} - y_{15}) \frac{\mathfrak{E}}{\mathfrak{e}} \sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2}} = 0
\end{aligned}$$

$$eqn_{45} := (5.454386028 - x_{15})^2 + (4.944655585 - y_{15})^2 = \frac{\mathfrak{E}}{\mathfrak{e}} 0.45 + \frac{19}{10} Fp_{15} \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}$$

$$\begin{aligned}
eqn_{46} := & \frac{x_{15} - x_{16}}{\sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2}} + \frac{x_{17} - x_{16}}{\sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2}} \\
& + \frac{(5.542907411 - x_{16}) Fp_{16}}{0.45 + \frac{19}{10} Fp_{16}} \\
& + \frac{(x_{15} - x_{16}) \frac{\mathfrak{E}}{\mathfrak{e}} \sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2}} \\
& + \frac{(x_{17} - x_{16}) \frac{\mathfrak{E}}{\mathfrak{e}} \sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\mathfrak{O}}}{\sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2}} = 0
\end{aligned}$$

$$\begin{aligned}
eqn_{47} := & \frac{y_{15} - y_{16}}{\sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2}} + \frac{y_{17} - y_{16}}{\sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2}} \\
& + \frac{(4.920475774 - y_{16}) Fp_{16}}{0.45 + \frac{19}{10} Fp_{16}}
\end{aligned}$$

APPENDIX A

$$\begin{aligned}
& + \frac{(y_{15} - y_{16}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{16}^2 - 2x_{16}x_{15} + x_{15}^2 + y_{16}^2 - 2y_{16}y_{15} + y_{15}^2}} \\
& + \frac{(y_{17} - y_{16}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2}} = 0
\end{aligned}$$

$$eqn_{48} := (5.542907411 - x_{16})^2 + (4.920475774 - y_{16})^2 = \frac{\mathfrak{A}}{\mathfrak{E}} 0.45 + \frac{19}{10} Fp_{16} \frac{\ddot{\mathfrak{O}}}{\varnothing}^2$$

$$\begin{aligned}
eqn_{49} := & \frac{x_{16} - x_{17}}{\sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2}} + \frac{x_{18} - x_{17}}{\sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2}} \\
& + \frac{(5.630152972 - x_{17}) Fp_{17}}{0.45 + \frac{19}{10} Fp_{17}} \\
& + \frac{(x_{16} - x_{17}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2}} \\
& + \frac{(x_{18} - x_{17}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2}} = 0
\end{aligned}$$

$$\begin{aligned}
eqn_{50} := & \frac{y_{16} - y_{17}}{\sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2}} + \frac{y_{18} - y_{17}}{\sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2}} \\
& + \frac{(4.892034417 - y_{17}) Fp_{17}}{0.45 + \frac{19}{10} Fp_{17}} \\
& + \frac{(y_{16} - y_{17}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{17}^2 - 2x_{17}x_{16} + x_{16}^2 + y_{17}^2 - 2y_{17}y_{16} + y_{16}^2}} \\
& + \frac{(y_{18} - y_{17}) \frac{\mathfrak{A}}{\mathfrak{E}} \sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2} - 0.06995673632 \frac{\ddot{\mathfrak{O}}}{\varnothing}}{\sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2}} = 0
\end{aligned}$$

$$eqn_{51} := (5.630152972 - x_{17})^2 + (4.892034417 - y_{17})^2 = \frac{\mathfrak{A}}{\mathfrak{E}} 0.45 + \frac{19}{10} Fp_{17} \frac{\ddot{\mathfrak{O}}}{\varnothing}^2$$

APPENDIX A

$$\begin{aligned}
eqn_{52} := & \frac{x_{17} - x_{18}}{\sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2}} + \frac{x_{19} - x_{18}}{\sqrt{x_{19}^2 - 2x_{19}x_{18} + x_{18}^2 + y_{19}^2 - 2y_{19}y_{18} + y_{18}^2}} \\
& + \frac{(5.715917683 - x_{18}) Fp_{18}}{0.45 + \frac{19}{10} Fp_{18}} \\
& + \frac{(x_{17} - x_{18}) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2} - 0.06995673632 \frac{\ddot{\mathbf{O}}}{\varnothing}}{\sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2}} \\
& + \frac{(x_{19} - x_{18}) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_{19}^2 - 2x_{19}x_{18} + x_{18}^2 + y_{19}^2 - 2y_{19}y_{18} + y_{18}^2} - 0.06995673632 \frac{\ddot{\mathbf{O}}}{\varnothing}}{\sqrt{x_{19}^2 - 2x_{19}x_{18} + x_{18}^2 + y_{19}^2 - 2y_{19}y_{18} + y_{18}^2}} = 0
\end{aligned}$$

$$\begin{aligned}
eqn_{53} := & \frac{y_{17} - y_{18}}{\sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2}} + \frac{y_{19} - y_{18}}{\sqrt{x_{19}^2 - 2x_{19}x_{18} + x_{18}^2 + y_{19}^2 - 2y_{19}y_{18} + y_{18}^2}} \\
& + \frac{(4.859398351 - y_{18}) Fp_{18}}{0.45 + \frac{19}{10} Fp_{18}} \\
& + \frac{(y_{17} - y_{18}) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2} - 0.06995673632 \frac{\ddot{\mathbf{O}}}{\varnothing}}{\sqrt{x_{18}^2 - 2x_{18}x_{17} + x_{17}^2 + y_{18}^2 - 2y_{18}y_{17} + y_{17}^2}} \\
& + \frac{(y_{19} - y_{18}) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_{19}^2 - 2x_{19}x_{18} + x_{18}^2 + y_{19}^2 - 2y_{19}y_{18} + y_{18}^2} - 0.06995673632 \frac{\ddot{\mathbf{O}}}{\varnothing}}{\sqrt{x_{19}^2 - 2x_{19}x_{18} + x_{18}^2 + y_{19}^2 - 2y_{19}y_{18} + y_{18}^2}} = 0
\end{aligned}$$

$$eqn_{54} := (5.715917683 - x_{18})^2 + (4.859398351 - y_{18})^2 = \frac{\mathfrak{A}}{\mathfrak{C}} 0.45 + \frac{19}{10} Fp_{18} \frac{\ddot{\mathbf{O}}}{\varnothing}^2$$

$$\begin{aligned}
eqn_{55} := & \frac{x_{18} - x_{19}}{\sqrt{x_{19}^2 - 2x_{19}x_{18} + x_{18}^2 + y_{19}^2 - 2y_{19}y_{18} + y_{18}^2}} + \frac{7 - x_{19}}{\sqrt{63.44 - 14x_{19} + x_{19}^2 - 7.6y_{19} + y_{19}^2}} \\
& + \frac{(5.800000000 - x_{19}) Fp_{19}}{0.45 + \frac{19}{10} Fp_{19}} \\
& + \frac{(x_{18} - x_{19}) \frac{\mathfrak{A}}{\mathfrak{C}} \sqrt{x_{19}^2 - 2x_{19}x_{18} + x_{18}^2 + y_{19}^2 - 2y_{19}y_{18} + y_{18}^2} - 0.06995673632 \frac{\ddot{\mathbf{O}}}{\varnothing}}{\sqrt{x_{19}^2 - 2x_{19}x_{18} + x_{18}^2 + y_{19}^2 - 2y_{19}y_{18} + y_{18}^2}} = 0
\end{aligned}$$

APPENDIX B

APPENDIX B

APPENDIX B

APPENDIX B

APPENDIX B