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VARIABLE STRUCTURE CONTROL WITH APPLICATIONS TO T-S FUZZY SYSTEMS

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Abstract

In this thesis, the problem of robust variable structure control for a class of complex uncertain nonlinear systems is investigated. A number of robustness and convergence results are presented for variable structure control systems with bounded unknown mismatched uncertainties. The finite time stabilization problems of uncertain nonlinear system and T-S fuzzy system are addressed. The major outcomes of the work described in this thesis are summarized as follows:

First, a simplified multiple-surface sliding control (MSSC) algorithm is proposed based on a recursive design approach that can significantly reduce the time complexity from $O(2^n)$ to $O(n)$, where n is the number of iterations. This methodology is then extended to the problem of adaptive fuzzy multiple-surface sliding mode control (AFMSMC) of a class of nonlinear systems with mismatched uncertainties. In this scheme, the Mamdani fuzzy logic-based approximator is first employed to learn the system uncertainties in Lyapunov sense, i.e., the weights are adaptively adjusted so that the output tracking error can asymptotically converge to zero according to the second method of Lyapunov stability. The iterative MSSC design is then carried out to ensure closed-loop stability. Strong robustness property and asymptotic error convergence are guaranteed.

Second, a systematic adaptive fast finite time control for a class of uncertain nonlinear systems is proposed. The upper bounds of the system uncertainties are unknown and not upper bounded by known positive constants (or smooth functions). Bernoulli-Jacobi

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equation like Lyapunov function candidate is used to construct the fast finite time stabilizer. Unlike the majority of existing results, global (rather than local) fast finite time stability of the closed-loop system is ensured, and the control input is bounded.

Third, a robust variable structure tracking control scheme is developed for a class of complex uncertain nonlinear systems with T-S fuzzy models. Here, a set of extreme matrices describing the upper bounds of all the interactions among the local fuzzy subsystems in the worst stability case are first derived. Using these extreme matrices as the bounding information, a variable structure control law is then designed to guarantee the asymptotic stability and robustness of the closed-loop system. In addition to ensuring strong robustness and asymptotic error convergence, it offers a simplified controller design.

Finally, a finite time variable structure control is designed for MIMO T-S fuzzy system. The concept of extreme fuzzy subsystems is used to design a terminal sliding mode (TSM) control and a set of TSM surfaces to guarantee finite time convergence of the closed-loop T-S fuzzy system. Unlike the conventional TSM control schemes, our design does not use the two-phase control method to guarantee bounded control input.

Symbols and acronyms

$\mathfrak{R}$	set of real numbers.
$\mathfrak{R}^n$	n—dimensional real Euclidean space.
$\mathfrak{R}^{n \times m}$	set of $n \times m$ real matrices.
I	identity matrix.
$I_{n \times n}$	identity matrix of dimension $n \times n$.
A^T	transpose of matrix A .
A^{-1}	inverse of matrix A .
A^{-T}	inverse-transpose of matrix A .
x^T	transpose of vector x .
$A^{1/2}$	a symmetric square root of matrix $A = A^T \geq 0$, i.e. $A^{1/2} . A^{1/2} = A$.
$diag\{A_1, A_2, ..., A_n\}$	block diagonal matrix with A_j being the diagonal entries.
$P \geq 0$	symmetric positive semidefinite matrix $P \in \mathfrak{R}^{n \times n}$.
$P > 0$	symmetric positive definite matrix $P \in \mathfrak{R}^{n \times n}$.
$P \geq Q$	$P - Q \geq 0$ for symmetric matrices $P, Q \in \mathfrak{R}^{n \times n}$.
$P > Q$	$P - Q > 0$ for symmetric matrices $P, Q \in \mathfrak{R}^{n \times n}$.
$\lambda(A)$	eigenvalues of matrix A .
$\det\{A\}$	determinant of A .

Symbols and Acronyms

$\ x\ _2$	l_2 norm of $x \in [0, \infty)$.
$trace(A)$	trace operation of matrix $A \in \Re^{n \times n}$.
VSC	variable structure control.
SMC	sliding mode control.
TSM	terminal sliding mode.
MSSC	multiple-surface sliding control.
AFMSMC	adaptive fuzzy multi-surface sliding mode control.

Chapter 1

INTRODUCTION

1.1. Motivations

The variable structure control technique was first used to solve control problems in former Soviet Union in the 1960s [1, 2] , and then developed and investigated by many researchers [3-7] in both theoretical and applied aspects.

A variable structure control system is characterized by a control structure, which is switched as the system states cross certain discontinuous surfaces in the state space. The intersection of these surfaces forms a sliding mode surface that is intended to constrain the dynamics of the system trajectories. When the sliding occurs, the trajectory is kept on the sliding mode surface resulting in the desired system dynamics that is insensitive to parameter variations, nonlinearities, and disturbances. It is due to the above advantages that the theory of variable structure systems has been widely used in the control of

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uncertain nonlinear systems. However, almost all the traditional sliding mode control designs are based on the restrictive assumption that the matching condition holds.

The first systematic variable structure control design method for SISO nonlinear system with mismatched uncertainties seems to be the work of Green and Hedrick [8], where multiple sliding mode surfaces are defined first, and virtual controllers are then designed to eliminate the effect of mismatched uncertainties by forcing the system states onto the sliding mode surfaces, sequentially. In the above variable structure control scheme, the upper bounds of the uncertainties are required in controller design. However, in practical situations, it is difficult to know the upper bounds of all the uncertainties.

In the recent work of Huang and Chen [9], great progress has been made for multiple-surface sliding mode control design. In this control scheme, function approximation technique is used to transform the uncertainties into a finite combination of orthonormal basis functions, and the control law is designed based on the approximated function. Theoretically, robustness and convergence can be obtained. However, still there are some cases where the selected basis functions are not rich enough for the estimation to converge to its true value.

In all the above variable structure control schemes, a linear sliding mode surface is first defined to describe the desired system dynamics, and a robust controller is designed to drive the switching plane variables to reach the sliding mode surface, and the asymptotic convergence of system state variables can then be obtained on the linear sliding mode surface. To get fast states convergence on the sliding mode surface, however, the sliding mode parameters must be chosen such that the poles of the sliding mode dynamics are far

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from the origin on the left half of the s -plane. This will, in turn, increase the gain of the controller. Considering the saturation property of control input signals in practical applications, a sliding mode controller with high gain is undesirable. On the other hand, a linear sliding mode technique can guarantee only the asymptotic convergence on the sliding mode surface, and therefore state variable vector cannot converge to zero in a finite time. To solve this problem, a finite time variable structure control, termed the Terminal Sliding Mode (TSM) control was developed by Man and Yu [10]. Using the TSM control, system state vector can converge to zero in a finite time, and the gain of the TSM controller can be significantly reduced with respect to the high gain of linear sliding mode controllers. However, the TSM control was initially designed for second order nonlinear system only.

Later, Yu and Man [11] present a new higher order recursive terminal sliding mode surface that can be used to design higher order TSM controller for n^{th} order SISO nonlinear system. This control scheme is effective but the robustness may be a problem if the dynamic systems contain mismatched uncertainties. In addition, because of the introduction of the recursive structure of the switching manifolds, each of which is characterized by a simple finite time dynamic equation, singularity problem may occur if the initial conditions are not selected carefully. This would result in an infinite control law. In order to overcome the possible singularity problem, a two-phase control strategy is proposed. The first phase is to induce the system state into a prescribed region in which singularity does not occur. The second phase is to switch to a TSM control that realizes finite time convergence. However, the major drawback of this strategy is that all trajectories have to start from a prescribed sector of state space in order to have

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nonsingular finite time convergence, which is restrictive in particular for practical applications.

The basis for control system design and stability analysis is a dynamic mathematical model that captures prominent features of the system under consideration. However, for complex uncertain nonlinear system, such a requirement is not feasible because the controlled system is time varying in nature, and has high nonlinearities and uncertain dynamics. Hence, simple linear or nonlinear differential equations cannot sufficiently represent the corresponding system, and therefore, the designed controller based on such a model cannot guarantee good performance such as stability and robustness.

During the last decade, fuzzy logic control has been suggested as an alternative to conventional control techniques for complex uncertain nonlinear system due to the fact that fuzzy logic combines human heuristic reasoning and expert experience to approximate a certain desired behaviour function [12, 13]. However, almost all the proposed fuzzy controllers only guarantee the asymptotic stability of the closed-loop system.

This thesis will further investigate variable structure control for nonlinear systems with mismatched uncertainties and present several new and improved robust variable structure control schemes. Also, the finite time stabilization problem of uncertain nonlinear systems and T-S fuzzy systems will be addressed.

1.2. Contributions of the Thesis

Major contributions of the thesis are summarized as follows:

- Simplified multiple-surface sliding control algorithm is proposed for a class of uncertain nonlinear systems. The time complexity is reduced from $O(2^n)$ to $O(n)$.
- Adaptive fuzzy multiple-surface sliding control is designed for SISO nonlinear system with mismatched uncertainties. The upper and lower bounds information of the system uncertainties is not required.
- Nonsingular fast finite time variable structure control is developed for a class of uncertain nonlinear systems. Conditions to ensure boundedness of control input are obtained.
- Sliding mode control is designed for a class of complex uncertain nonlinear systems with their T-S fuzzy models. The concept of extreme subsystems is used to simplify the sliding mode control design.
- The finite time stabilization problems of MIMO T-S fuzzy systems are addressed. Sufficient conditions are provided to guarantee finite time stability of the closed-loop T-S fuzzy system.

The main contents of this thesis are organized as follow:

Chapter 2 gives a brief survey for the basic variable structure theory. Some basic ideas such as sliding mode control designs, robustness analysis and controller design methods

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for linear or nonlinear systems are reviewed, and the recently developed terminal sliding mode control schemes are discussed.

Chapter 3 provides a background of fuzzy logic and fuzzy systems to be applied in the later chapters. Fuzzy sets, fuzzy set operations, and fuzzy linguistic representation such as linguistic variables and linguistic modifiers (hedged) will be briefly outlined. Fuzzy reasoning or approximate reasoning is considered from the engineering viewpoint with IF-THEN fuzzy implications using Mamdani's minimum inference. The basic concepts of Mamdani fuzzy systems and T-S fuzzy systems are summarized.

Chapter 4 points out that the time complexity of the main multiple-surface sliding control (MSSC) algorithm in [9] is $O(2^n)$. A simplified recursive-like MSSC algorithm with time complexity $O(n)$ is proposed, and, using mathematical induction, it is shown that this algorithm agrees with MSSC in [9].

Chapter 5 follows the line of chapter 4, in the sense that an adaptive fuzzy multiple-surface sliding mode control is proposed for SISO nonlinear system with mismatched uncertainties. It is shown that the Mamdani fuzzy logic-based function approximator is used to learn the system uncertainties first, and the iterative multiple-surface sliding mode design is then carried out. Using the concept of terminal sliding mode to design the baseline controller, not only strong robustness with respect to approximation errors and nonlinearities can be obtained, but the tracking error can also be proven to converge to a small neighborhood of the origin in finite time.

Chapter 6 concerns the adaptive fast finite time control of a class of uncertain nonlinear systems of which the upper bounds of the system uncertainties are unknown. By using

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the fast non-smooth control Lyapunov function and the method of so-called adding a power integrator merging with adaptive technique, a recursive design procedure is provided, which guarantees the fast finite time stability of the closed-loop system. It has been proven that the control input is bounded.

Chapter 7 considers a robust sliding mode control scheme for a class of complex uncertain nonlinear systems with T-S fuzzy models. It is shown that the concept of extreme matrices from H_∞ control technique is first used to determine the upper bound information of the interactions of the fuzzy subsystems, and a sliding mode control law is then designed to guarantee the asymptotic stability and robustness of the closed-loop system. It is the extreme matrices that simplify the sliding mode control design for the complex uncertain nonlinear systems with T-S fuzzy models.

Chapter 8 presents a terminal sliding mode control for MIMO T-S fuzzy systems. It is shown that the concept of extreme matrices is used to determine the upper bound information of the interactions of the fuzzy subsystems first, and a terminal sliding mode controller and a set of sliding variables are then designed to enable finite time reachability of the system origin.

Chapter 9 summarizes the results and draws conclusions, and the topics for future work are suggested.

Chapter 2

A SURVEY OF THE VARIABLE STRUCTURE CONTROL THEORY

2.1. Introduction

Variable structure control (VSC) [3, 4, 14-19] or sliding mode control (SMC) was first studied in former Soviet Union in the 1960's [18], and was introduced to western researchers by Utkin [5, 20-22]. The central feature of SMC is its sliding mode surfaces, in which the dynamic motion of the system is effectively constrained to lie within a certain subspace of the full state space. The sliding motion is then achieved by altering the system dynamics along sliding mode surfaces in the state space. On the sliding mode surface, the system is equivalent to an unforced system of lower order, termed the equivalent system, which is insensitive to system uncertainties.

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Variable structure control can also be considered to be an extension of conventional feedback control in the sense that the structure of a state feedback regulator is allowed to change as its states cross discontinuity surfaces, which results in discontinuous feedback control input on one or more manifolds in the state space. From the point of the conventional feedback control theory, a variable structure control system can be treated as a combination of subsystems. Each subsystem has a fixed structure and operates in a specified region of the state space. The combination of these subsystems according to some prescribed rules results in a new system, which is different from the individual subsystems and has the desired system response.

In this chapter, we will first review the basic variable structure control theory used to design robust control algorithms, we will then restrict our discussion to recent research results on the robust variable structure control for uncertain nonlinear systems.

This chapter is organized as follows: In Section 2.2, the basic variable structure control theory is briefly reviewed. The basic ideas and definitions, such as system model, the sliding mode surface, the condition for existence of sliding mode surface, robustness property, and an overview of variable structure controllers, are discussed. In Section 2.3, we address more complicated variable structure control for a class of nonlinear systems. A few useful variable structure control schemes for uncertain nonlinear systems are reviewed. In Section 2.4, we consider finite time variable structure control for a class of nonlinear systems with matched uncertainties. Finally, Section 2.5 gives conclusions.

2.2. Basic Variable Structure Control Theory

The main feature of a variable structure control system is the sliding motion. For the design of a variable structure controller, the first thing is to define a set of switching plane variables, which are a function of system states. The intersection of these switching planes forms a sliding mode surface. The purpose of the variable structure controller is to drive the system states into the sliding mode surface on which the sliding motion occurs and the motion of the system is thus formally equivalent to a system of low order, called as equivalent system. Actually, the sliding motion on the sliding mode surface is the convergence motion of the system states from arbitrary initial values to the origin. The convergence rate depends on the design of sliding mode parameters. It is due to this feature that the variable structure control is also called as sliding mode control.

Another feature of a variable structure system is that the transient response can be divided into two parts. First, the motion in which the variable structure controller drives the switching plane variables to reach the sliding mode surface. Second, the sliding motion in which the system states constrained on the sliding mode surface asymptotically converge to the origin. Usually, the sliding motion is determined only by the sliding mode parameters. However, the convergence of the switching plane variables are affected by the sliding mode parameters because the sliding mode parameters are involved in the controller gain matrices.

2.2.1. System Model and Sliding Mode Surface

Consider the following linear time invariant system:

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$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad (2.2.1)$$

where $\mathbf{x} \in \mathfrak{R}^n$ and $\mathbf{u} \in \mathfrak{R}^m$ represent the state variable vector and control signal vector, respectively, $\mathbf{A} \in \mathfrak{R}^{n \times n}$ and $\mathbf{B} \in \mathfrak{R}^{n \times m}$ are constant system matrices. It is assumed that $n > m$, $\mathbf{B}$ is of full rank m , and the pair is completely controllable.

Define a set of switching plane variables s_i ($i = 1, \dots, m$) passing through the state space origin:

$$s_i = \mathbf{c}_i^T \mathbf{x}, \quad i = 1, \dots, m \quad (2.2.2a)$$

or $\mathbf{s} = \mathbf{C}\mathbf{x}, \quad (2.2.2b)$

where $\mathbf{c}_i \in \mathfrak{R}^n$ is a constant vector and

$$\mathbf{C} = [\mathbf{c}_1^T \dots \mathbf{c}_m^T]^T, \quad (2.2.2c)$$

is an $m \times n$ constant matrix.

System (2.2.1) is said to attain a sliding mode surface when the state variable vector reaches and remains on the intersection ($\mathbf{s} = 0$) of the m switching plane variables

$$\mathbf{s}_0 = \bigcap_{i=1}^m s_i = \{\mathbf{x} : \mathbf{c}_i^T \mathbf{x} = 0, \ i = 1, \dots, m\}. \quad (2.2.3)$$

The control input vector $\mathbf{u}(t)$ in the variable structure control system usually has the following form [3]

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$$\mathbf{u}(t) = \mathbf{K}\mathbf{x} + \boldsymbol{\psi}\mathbf{x}, \quad (2.2.4)$$

where the first term in expression (2.2.4) is a linear feedback and the second term is a switching component with $\boldsymbol{\psi} = [\psi_{ij}]$ and

$$\psi_{ij} = \begin{cases} \alpha_{ij} & s_i x_j < 0, \\ \beta_{ij} & s_i x_j > 0. \end{cases} \quad (2.2.5)$$

The task of the control input $\mathbf{u}(t)$ in expression (2.2.4) is to drive the switching plane variables to reach the sliding mode surface in (2.2.3) by the suitable design of the controller gain matrices $\mathbf{K}$ and $\boldsymbol{\psi}$. Thereafter, the system performance will be determined by the sliding motion on the sliding mode surface. Generally, the sliding mode surface is designed such that the system response restricted on the sliding mode surface has a desired behavior such as asymptotic stability and prescribed transient response. Usually, the switching plane variables are designed as a linear function of the system states. Many researches have shown that it is convenient for the linear sliding mode surface to be used in the design and analysis for a variable structure control system.

The next important problem is how to design controller parameters to guarantee the switching plane variables to reach the sliding mode surface, and then remain on the sliding mode surface. The work in [3, 23] have shown that if the control input $\mathbf{u}(t)$ is designed such that the tangent vector or time derivative of the switching plane variables always pointing toward the sliding mode surfaces, then the switching plane variables $s_i (i=1,2,\dots,m)$ will asymptotically converge to zero, and the system states can remain on the sliding mode surface.

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In fact, the condition for the switching plane variables to reach the sliding mode surfaces is a convergence problem. Therefore, the second Lyapunov method can be used to provide a natural setting for the analysis [24, 25]. Generally, the following Lyapunov function is often used in the variable structure controller design

$$V = \frac{1}{2} \mathbf{s}^T \mathbf{s}. \quad (2.2.6)$$

In this case, the sufficient condition for the switching plane variables to reach the sliding mode surfaces can be expressed as follows

$$\mathbf{s}^T \dot{\mathbf{s}} < 0 \quad (2.2.7a)$$

$$\text{or} \quad s_i \dot{s}_i < 0, \quad i = 1, 2, \dots, m. \quad (2.2.7b)$$

It has been noted that most of the variable structure control algorithms are designed based on the sufficient condition in expression (2.2.7a) or expression (2.2.7b).

2.2.2. Equivalent Control

On the sliding mode surface, $s_i = 0$ and $\dot{s}_i = 0 (i = 1, \dots, m)$. By using expressions (2.2.1) and (2.2.2), we have

$$\begin{aligned} \dot{\mathbf{s}} &= \mathbf{C}\dot{\mathbf{x}} \\ &= \mathbf{C}\mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{B}\mathbf{u}_{eq} = 0, \end{aligned} \quad (2.2.8)$$

where $\mathbf{u}_{eq}$ is called as equivalent control.

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If $|CB| \neq 0$, the equivalent control u_{eq} can be written as

$$\begin{aligned} u_{eq} &= -(CB)^{-1} CAx \\ &= -Kx, \end{aligned} \tag{2.2.9a}$$

where $K = -(CB)^{-1} CA$. (2.2.9b)

The system response on the sliding mode surface can then be described by the following differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) - B(CB)^{-1} CAx(t) \\ &= [I - B(CB)^{-1} C] Ax(t). \end{aligned} \tag{2.2.10}$$

System (2.2.10) is called the equivalent system. The characteristics of the equivalent system (2.2.10) can be summarized as:

1. The dynamical behavior of the equivalent system is independent of the control input, but depends only on the choice of the matrix C in expression (2.2.2c). Therefore, the control input is just used to drive the system states into the sliding mode surface and thereafter to maintain it on the sliding mode surface. The determination of the matrix C may thus be completed with no prior knowledge of the form of the control input. The reason for the equivalent system to have an independent motion from the control input is due to the fact that the matrix CB is nonsingular. In fact, the condition $|CB| \neq 0$ means that the null space of C and the range space B are

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complementary subspaces. Thus, when the sliding motion occurs on the sliding mode surface or within $N(C)$, the behavior of the equivalent system is unaffected by the control input. If $|CB| = 0$ as shown in [3], the equivalent control is either not unique or does not exist. Therefore, sliding mode surface can not be reached.

2. Equivalent system (2.2.10) is an $(n-m)^{th}$ order system. The work in [26] has shown that for the matrix B with full rank m , there exist an orthogonal $n \times n$ transformation matrix T such that

$$TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad (2.2.11)$$

where B_2 is an $m \times m$ nonsingular matrix.

Define a transformed state variable vector $y = Tx$, state equation (2.2.1) becomes

$$\dot{y} = TAT^T y + TBu. \quad (2.2.12)$$

If y is partitioned as

$$y^T = \begin{bmatrix} y_1^T & y_2^T \end{bmatrix} \quad (2.2.13a)$$

and matrices TAT^T and CT^T are partitioned as

$$TAT^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (2.2.13b)$$

$$CT^T = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad (2.2.13c)$$

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then, the system (2.2.12) can be written in the following form

$$\dot{y}_1 = A_{11}y_1 + A_{12}y_2 \quad (2.2.14a)$$

$$\dot{y}_2 = A_{21}y_1 + A_{22}y_2 + B_2u. \quad (2.2.14b)$$

On the sliding mode surface, we have

$$C_1y_1 + C_2y_2 = 0 \quad (2.2.15a)$$

$$\text{or} \quad y_2 = -Fy_1, \quad (2.2.15b)$$

$$\text{where} \quad F = C_2^{-1}C_1. \quad (2.2.15c)$$

The equivalent system can then be written in the following form

$$\dot{y}_1 = (A_{11} - A_{12}F)y_1. \quad (2.2.16)$$

Therefore, we can see from expression (2.2.16) that the equivalent system is $(n-m)^{th}$ order system, i.e., the system dynamic is simplified on the sliding mode surface.

2.2.3. Robustness Property

Robustness property is an important feature of a variable structure control system. Suppose that system (2.2.1) has uncertainty in matrix A and an external disturbance, then the system state equation can be written in the following form

$$\dot{x}(t) = (A_0 + \Delta A)x(t) + Bu + Df, \quad (2.2.17)$$

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where A_0 is the nominal system matrix, ΔA is the uncertainty, $f \in \mathbb{R}^L$ is a bounded external disturbance vector, and matrix D is compatibly dimensioned. Without loss of generality, it can be assumed that matrices B and D are full rank and the uncertainty presented in the input distribution matrix B is incorporated in the system disturbance term. During the sliding motion, the state vector of the system satisfied the following equations

$$Cx = 0, \quad (2.2.18a)$$

$$C(A + \Delta A)x + CBu_{eq} + Cdf = 0. \quad (2.2.18b)$$

From expression (2.2.18b), the equivalent control can be achieved as

$$u_{eq} = -(CB)^{-1}C(Ax + \Delta Ax + Df), \quad (2.2.19)$$

and the equivalent system equation is then given by

$$\dot{x} = [I - B(CB)^{-1}C](Ax + \Delta Ax + Df), \quad (2.2.20a)$$

$$Cx = 0. \quad (2.2.20b)$$

Spurgeon has shown that the sliding mode system (2.2.20) is insensitive to parameter variations and the external disturbance if and only if the system uncertainty ΔA , matrices B and D satisfy the following rank relation [27]

$$\text{rank} [B : D] = \text{rank} [B : \Delta AT] = \text{rank} [B], \quad (2.2.21)$$

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where T is the matrix of the basis vectors of the reduced-order sliding subspace defined by expression (2.2.20).

Expression (2.2.21) is also called as the invariance condition. In addition, the robustness property of variable structure systems has been investigated by other researchers. For example, Gutman [28] and Bormish and Leitmann [6] have shown that if system uncertainties and disturbance satisfy the “matching condition”, then the system is completely insensitive on the sliding mode surface, and the effect of disturbance and parameter variations can be minimized by minimizing the time required to attain the sliding mode.

2.2.4. Two Methods of Sliding Mode Design

The quadratic minimization method for the sliding mode surface design was proposed by Utkin and Young [16]. In this method, the following cost function is defined

$$J(u) = \frac{1}{2} \int_{t_s}^{\infty} \mathbf{x}(t)^T \mathbf{Q} \mathbf{x}(t) dt, \quad (2.2.22)$$

where $\mathbf{Q}$ is a symmetric positive-definite matrix, and t_s denotes the time at which the sliding mode starts.

Partitioning the following matrix compatibly with y (see subsection 2.2.2)

$$T^T \mathbf{Q} T = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}, \quad (2.2.23)$$

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where matrix T is defined in expression (2.2.11), the cost function (2.2.22) can then be expressed in the following form

$$J(v) = \frac{1}{2} \int_s^{\infty} \{Y_1^T Q^* Y_1 + v^T Q_{22} v\} dt, \quad (2.2.24)$$

where $Q^* = Q_{11} - Q_{12} Q_{22}^{-1} Q_{21},$ (2.2.25a)

$$A^* = A_{11} - A_{12} Q_{22}^{-1} Q_{21}, \quad (2.2.25b)$$

$$v = y_2 + Q_{22}^{-1} Q_{21} y_1, \quad (2.2.25c)$$

$$\dot{y}_1 = A^* y_1 + A_{12} v(t). \quad (2.2.25d)$$

Expression (2.2.24) is the form of standard linear quadratic optimal regulator problem.

By minimizing expression (2.2.24), the optimal control $v(t)$ is given by

$$v(t) = -Q_{22}^{-1} A_{12}^T P y_1. \quad (2.2.25e)$$

Using expression (2.2.25e) in expression (2.2.25c), we have

$$y_2 = -Q_{22}^{-1} [Q_{21} + A_{12}^T P] y_1 = -F y_1, \quad (2.2.26)$$

where the matrix P satisfies the following Riccati equation

$$P Q^* + A^* P - P A_{12} Q_{22}^{-1} A_{12}^T P + Q^* = 0, \quad (2.2.27a)$$

and matrix

$$F = Q_{22}^{-1} [Q_{21} + A_{12}^T P], \quad (2.2.27b)$$

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can then be determined as required.

The eigenstructure assignment method is also very popular in the sliding mode designs and it was first used by Utkin and Young in 1978 [29].

Suppose that the sliding mode has commenced on $N(C)$. Then the equivalent system can be written as

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t), \quad (2.2.28)$$

where matrix $\mathbf{K}$ is given in expression (2.2.9b).

During the sliding motion, the state variables must remain in $N(C)$ so that

$$\mathbf{C}[\mathbf{A} - \mathbf{BK}] = 0 \Leftrightarrow R(\mathbf{A} - \mathbf{BK}) \subseteq N(C). \quad (2.2.29)$$

Let $\lambda_i (i=1, \dots, n)$ be the eigenvalues of $\mathbf{A} - \mathbf{BK}$ with corresponding eigenvectors $\mathbf{v}_i$, then, from expression (2.2.29), we have

$$\mathbf{C}[\mathbf{A} - \mathbf{BK}]\mathbf{v}_i = \lambda_i \mathbf{C}\mathbf{v}_i = 0. \quad (2.2.30)$$

Expression (2.2.30) shows that either λ_i is zero or $\mathbf{v}_i \in N(C)$. Since $\mathbf{A} - \mathbf{BK} = \mathbf{A}_{eq}$ has m zero-valued eigenvalues, we can set $\{\lambda_i : i=1, \dots, n-m\}$ be the nonzero eigenvalues and therefore, specifying the corresponding eigenvalues $\{\mathbf{v}_i : i=1, \dots, n-m\}$ fix the null space of $\mathbf{C}(\dim[N(C)] = n-m)$.

It is noted that $\mathbf{C}$ is not uniquely determined because the equation

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$$CV = 0, \quad V = [v_1 \dots v_{n-m}], \quad (2.2.31)$$

has m^2 degree of freedom, which may be easily seen. If we define

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = Tv, \quad (2.2.32)$$

where the partitioning of W is compatible with that of Y , then expression (2.2.31) becomes

$$0 = C^T Tv = [C_1 \quad C_2] \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = C_2 [F \quad I_m] \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}. \quad (2.2.33)$$

Therefore, F can be determined from the following equation

$$FW_1 = -W_2. \quad (2.2.34)$$

The work of Dorling and Zinober in [26] has shown that this approach has the drawback that the eigenvectors of matrix $A - BK$ are not freely assignable, and at most m elements of an eigenvector may be assigned arbitrarily, after which the remaining $n - m$ elements are fully determined by the assigned elements. Thus, one approach to assign eigenvector is to select m elements according to some scheme and accept the remaining elements as determined. This may allow a degree of adjustment to be carried out by inspection. Some other eigenvector assignment methods have also been proposed, and the details can be found in [30, 31].

2.2.5. Controller Designs

In most of the variable structure control schemes, the control law usually consists of a linear component u^L and a nonlinear component which are assumed to form the control input, u . The linear part is merely a state feedback

$$u^L = Kx, \quad (2.2.35)$$

while the nonlinear signal incorporates the discontinuous elements of the control. Some examples of possible types of nonlinearity are as given below.

(a) A nonlinear component with constant gains

$$u_i^{NL} = M_i \text{sign}(C_i x), \quad M_i > 0. \quad (2.2.36)$$

(b) A nonlinear component with state-dependent gains

$$u_i^{NL} = m_i(x) \text{sign}(C_i x), \quad m_i(\cdot) > 0. \quad (2.2.37)$$

(c) A linear feedback with switching gains

$$u_i^{NL} = \Psi x, \quad (2.2.38a)$$

where $\Psi = [\Psi_{ij}]$ and

$$\Psi_{ij} = \begin{cases} \alpha_{ij} & s_i x_j < 0, \\ \beta_{ij} & s_i x_j > 0. \end{cases} \quad (2.2.38b)$$

(d) A unit vector nonlinearity with scale factor

$$u^{NL} = \frac{N\mathbf{x}}{\|M\mathbf{x}\|}, \quad (2.2.39)$$

where the null spaces of N, M and C are coincident.

The nonlinear control component is discontinuous on the individual hyperplane in cases (a)-(c). This may result in wasted control effort as the system state pierces one hyperplane, and is forced into another surface. In case (d), the individual controls are continuous, except on the intersection of the switching plane variables where all the nonlinear control elements become discontinuous together.

2.3. Variable Structure Control of Nonlinear Systems

2.3.1. System Model

In section 2.2, we have briefly reviewed the basic variable structure control theory of linear systems. Most of these ideas can be extended to the variable structure control of nonlinear systems. However, the complexity of the analysis and the controller designs may be increased due to the nonlinearity in the nonlinear system model. From the engineering point of view, the following nonlinear system is often considered [15].

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}) + \mathbf{B}(t, \mathbf{x})\mathbf{u}(t), \quad (2.3.1)$$

where the state variable vector $\mathbf{x}(t) \in \mathcal{R}^n$, the control input vector $\mathbf{u}(t) \in \mathcal{R}^m$, $\mathbf{f}(t, \mathbf{x}) \in \mathcal{R}^n$, and $\mathbf{B}(t, \mathbf{x}) \in \mathcal{R}^{n \times m}$. Further, each entry in $\mathbf{f}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ is assumed to be continuous with continuous bounded derivative with respect to $\mathbf{x}$.

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Each entry $u_i(t)$ of the control input vector has the following form

$$u_i(t, \mathbf{x}) = \begin{cases} u_i^+(t, \mathbf{x}) & \text{with } s_i(\mathbf{x}) > 0, \\ u_i^-(t, \mathbf{x}) & \text{with } s_i(\mathbf{x}) < 0, \end{cases} \quad i = 1, \dots, m, \quad (2.3.2)$$

where $s_i(\mathbf{x})$ is the i^{th} switching surface associated with the $(n-m)$ dimensional switching surfaces

$$\mathbf{s}(\mathbf{x}) = [s_1(\mathbf{x}), \dots, s_m(\mathbf{x})]. \quad (2.3.3)$$

2.3.2. Sliding Mode Surface and Equivalent Control

Following the sliding mode surface design for linear systems in section 2.2.4, the method of equivalent control is a way to determine the system motion restricted to the sliding mode surface $\mathbf{s}(\mathbf{x}) = 0$. Suppose that there exists a time $t_0 > 0$, and the state of the system reaches the sliding mode surface after $t \geq t_0$. On the sliding mode surface, the following two equations are satisfied

$$\mathbf{s}(\mathbf{x}(t)) = 0 \quad t \geq t_0, \quad (2.3.4a)$$

$$\dot{\mathbf{s}}(\mathbf{x}(t)) = 0 \quad t \geq t_0. \quad (2.3.4 \text{ b})$$

Using system equation (2.3.1), expression (2.3.4.b) can be expressed as follows

$$\frac{\partial \mathbf{s}(\mathbf{x})}{\partial \mathbf{x}} [\mathbf{f}(t, \mathbf{x}) + \mathbf{B}(t, \mathbf{x}) \mathbf{u}_{eq}] = 0, \quad (2.3.5)$$

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where $\mathbf{u}_{eq}$ is the so-called equivalent control which can be obtained from expression (2.3.5) as follows

$$\mathbf{u}_{eq} = - \left[\frac{\partial s(\mathbf{x})}{\partial \mathbf{x}} \mathbf{B}(t, \mathbf{x}) \right]^{-1} \frac{\partial s(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}). \quad (2.3.6)$$

Using expression (2.3.6) in system model (2.3.1), the dynamics of the closed loop system on the sliding mode surface is given by

$$\dot{\mathbf{x}} = \left[\mathbf{I} - \mathbf{B}(t, \mathbf{x}) \left(\frac{\partial s(\mathbf{x})}{\partial \mathbf{x}} \mathbf{B}(t, \mathbf{x}) \right)^{-1} \frac{\partial s(\mathbf{x})}{\partial \mathbf{x}} \right] \mathbf{f}(t, \mathbf{x}). \quad (2.3.7)$$

Therefore, the problem of the sliding mode surface design is to choose sliding mode parameters such that the equivalent system (2.3.7) is stable. In most of the variable structure control schemes for nonlinear systems, the linear sliding mode surfaces are often used. Therefore, some methods of sliding mode design in section 2.2.4 can also be used.

2.3.3. Controller Designs

A. Diagonalisation Method

In general, for nonlinear system equation (2.3.1), the control input is a m dimensional vector and each entry has the structure of the form

$$u_i = \begin{cases} u_i^+(t, \mathbf{x}) & \text{for } s_i(\mathbf{x}) > 0, \\ u_i^-(t, \mathbf{x}) & \text{for } s_i(\mathbf{x}) < 0. \end{cases} \quad (2.3.8)$$

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To determine the switched feedback gains in control law (2.3.8), the following diagonalisation method is often used [15].

First, a new control vector is considered in terms of a nonsingular transformation

$$\mathbf{u}^*(t) = \mathbf{Q}^{-1}(t, \mathbf{x}) \left[\frac{\partial \mathbf{s}(\mathbf{x})}{\partial \mathbf{x}} \right] \mathbf{B}(t, \mathbf{x}) \mathbf{u}(t), \quad (2.3.9)$$

where $\mathbf{Q}^{-1}(t, \mathbf{x}) \left(\frac{\partial \mathbf{s}}{\partial \mathbf{x}} \right) \mathbf{B}(t, \mathbf{x})$ is a nonsingular transformation, and $\mathbf{Q}(t, \mathbf{x})$ is an arbitrary $m \times m$ diagonal matrix with elements $q_i(t, \mathbf{x})$, $i = 1, \dots, m$ such that $\inf |q_i(t, \mathbf{x})| > 0$.

Using expression (2.3.9) into expression (2.3.1), the system dynamics becomes

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) + \mathbf{B}(t, \mathbf{x}) \left[\frac{\partial \mathbf{s}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{B}(t, \mathbf{x}) \right]^{-1} \mathbf{Q}(t, \mathbf{x}) \mathbf{u}^*(t). \quad (2.3.10)$$

If $\mathbf{u}^*(t)$ is selected such that

$$q_i(t, \mathbf{x}) u_i^{*+} < -\nabla s_i(\mathbf{x}) \mathbf{f}(t, \mathbf{x}) \quad s_i(\mathbf{x}) > 0, \quad (2.3.11a)$$

$$q_i(t, \mathbf{x}) u_i^{*-} > -\nabla s_i(\mathbf{x}) \mathbf{f}(t, \mathbf{x}) \quad s_i(\mathbf{x}) < 0, \quad (2.3.11b)$$

then, $\mathbf{s}^T(\mathbf{x}) \dot{\mathbf{s}}(\mathbf{x}) < 0. \quad (2.3.12)$

Expression (2.3.12) is the reaching condition for the system states to reach the sliding mode surface $\mathbf{s}(\mathbf{x}) = 0$. On the sliding mode surface, the desired system dynamics can be obtained. Also, the control input $\mathbf{u}(t)$ can be obtained from equation (2.3.9).

B. Reaching Law Method

In addition to the above diagonalization method, the reaching law method [32] has been developed and proved to be efficient in controller design. The reaching law is a differential equation, which specifies the dynamics of a switching function $s(\mathbf{x})$. The differential equation of an asymptotically stable $s(\mathbf{x})$ is itself a reaching condition, i.e., $\mathbf{s}^T(\mathbf{x})\dot{\mathbf{s}}(\mathbf{x}) < 0$. In addition, by choice of the parameters in the differential equation, the dynamic quality of VSC system in the reaching mode can be controlled. A practical general form of the reaching law is

$$\dot{\mathbf{s}} = -\mathbf{Q}\text{sign}(\mathbf{s}) - \mathbf{K}\mathbf{h}(\mathbf{s}), \quad (2.3.13a)$$

where

$$\mathbf{Q} = \text{diag}[q_1, \dots, q_m], \quad q_i > 0,$$

$$\text{sign}(\mathbf{s}) = [\text{sign}(s_1), \dots, \text{sign}(s_m)]^T,$$

$$\mathbf{K} = \text{diag}[k_1, \dots, k_m],$$

$$\mathbf{h}(\mathbf{s}) = [h_1(s_1), \dots, h_m(s_m)]^T,$$

$$s_i h_i(s_i) > 0, \quad h_i(0) = 0. \quad (2.3.13b)$$

Three practical special cases of (2.3.13) are given below.

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A. Constant rate reaching

$$\dot{s} = Q \text{sign}(s). \quad (2.3.14)$$

This law forces the switching variable, $s(\mathbf{x})$ to reach the sliding mode surfaces $s(\mathbf{x}) = 0$ at a constant rate $|\dot{s}_i| = -q_i$. The merit of this reaching law is its simplicity. However, if q_i is too small, the reaching time will be too long. On the other hand, large q_i will cause severe chattering.

B. Constant plus proportional rate reaching

$$\dot{s} = -Q \text{sign}(s) - Ks. \quad (2.3.15)$$

Clearly, by adding the proportional rate term $-Ks$, the system state is forced to approach the sliding mode surfaces $s(\mathbf{x}) = 0$ faster when s is large. It can be shown that the reaching time for $\mathbf{x}$ to move from an initial state $\mathbf{x}_0$ to the sliding mode surfaces is finite, and is given by

$$T_i = \frac{1}{k_i} \ln \left(\frac{k_i |s_{i0}| + q_i}{q_i} \right). \quad (2.3.16)$$

C. Power rate reaching

$$\dot{s}_i = -k_i |s_i|^\alpha \text{sign}(s_i), \quad 0 < \alpha < 1, \quad i = 1, \dots, m. \quad (2.3.17)$$

This power rate reaching law increases the reaching speed when the state is far away from the sliding mode surfaces $s(\mathbf{x}) = 0$, but reduced the rate when the state is near the

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surfaces. The result is a fast reaching and low chattering reaching mode. Integrating (2.3.17) from $s_i = s_{i0}$ to $s_i = 0$ yields

$$T_i = \frac{1}{(1-\alpha)k_i} |s_{i0}|^{1-\alpha}, \quad i = 1, \dots, m, \quad (2.3.18)$$

showing that the reaching time is finite.

The control law can be determined by the time derivative of s and the reaching law (2.3.13), i.e.,

$$u = - \left[\frac{\partial s}{\partial x} B(t, x) \right]^{-1} \left[\frac{\partial s}{\partial x} f(t, x) + Q \text{sign}(s) + Kh(s) \right], \quad (2.3.19)$$

where the matrix $\left[\frac{\partial s}{\partial x} B(t, x) \right]$ is nonsingular.

2.3.4. Robust Control of Nonlinear Systems

In practical situations, the system dynamics of a nonlinear system is different from its nominal system model due to parameter uncertainties. To represent parameter uncertainties in the plant, the following state equation is considered [15]:

$$\dot{x} = [f(t, x) + \Delta f(t, x, r(t))] + [B(t, x) + \Delta B(t, x, r(t))]u(t), \quad (2.3.20a)$$

where $r(t)$ is a vector function of uncertain parameters. In most research works [2, 33], the plant uncertainties Δf and ΔB are assumed to lie in the image of $B(t, x)$ for all variables t and x which is called as the “matching condition”. Then the dynamic equation (2.3.20a) can be expressed as follows

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$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) + \mathbf{B}(t, \mathbf{x})\mathbf{u} + \mathbf{B}(t, \mathbf{x})\mathbf{e}(t, \mathbf{x}, \mathbf{r}, \mathbf{u}), \quad (2.3.20b)$$

where $\mathbf{e}(t, \mathbf{x}, \mathbf{r}, \mathbf{u})$ represents system uncertainties.

De Carlo [15] shows that if $\mathbf{e}(t, \mathbf{x}, \mathbf{r}, \mathbf{u})$ is bounded by a positive function $\rho(t)$

$$\|\mathbf{e}(t, \mathbf{x}, \mathbf{r}, \mathbf{u})\|_2 \leq \rho(t), \quad (2.3.21)$$

and control input has the following form

$$\mathbf{u} = \mathbf{u}_{eq} + \mathbf{u}_n, \quad (2.3.22a)$$

where

$$\mathbf{u}_{eq} = - \left[\frac{\partial \mathbf{s}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{B}(t, \mathbf{x}) \right]^{-1} \left[\frac{\partial \mathbf{s}(\mathbf{x})}{\partial t} + \frac{\partial \mathbf{s}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}) \right], \quad (2.3.22b)$$

$$\mathbf{u}_n = - \frac{\mathbf{B}^T(t, \mathbf{x}) \nabla_{\mathbf{x}} v(t, \mathbf{x})}{\|\mathbf{B}^T(t, \mathbf{x}) \nabla_{\mathbf{x}} v(t, \mathbf{x})\|^2} \hat{\rho}(t, \mathbf{x}), \quad (2.3.22c)$$

$$\hat{\rho}(t, \mathbf{x}) = \alpha + \rho(t, \mathbf{x}), \quad (2.3.22d)$$

$$\nabla_{\mathbf{x}} v(t, \mathbf{x}) = \left[\frac{\partial \mathbf{s}(\mathbf{x})}{\partial \mathbf{x}} \right]^T \mathbf{s}(t, \mathbf{x}), \quad (2.3.22e)$$

then system state can reach the sliding mode surfaces $\mathbf{s}(\mathbf{x}) = 0$, and the desired system dynamics can be obtained by the suitable choice of the sliding mode parameters.

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The results discussed in this section forms the foundation of the variable structure control theory for nonlinear systems. Although there are many classes of nonlinear systems, robustness and convergence of variable structure control systems may be established based on the results in this section.

2.4. Finite Time Variable Structure Control

In section 2.2 and 2.3, we have studied SMC design where linear sliding mode surface is used to describe the desired performance of the closed loop systems, that is, the system state variables will reach the system origin asymptotically on the linear sliding mode surface. Although the parameters of the linear sliding mode surface can be adjusted such that the convergence rate may be arbitrarily fast, the system states on the sliding mode surface cannot converge to zero in a finite time.

Recently, a so-called terminal sliding mode (TSM) control has been developed in [10, 11, 34-38] to achieve finite time convergence of the system dynamics on the TSM surface. Compare with linear sliding mode control designs, TSM offers some superior properties such as fast and finite time convergence. This control technique is particularly useful for high precision control as it speeds up the rate of convergence near an equilibrium point.

2.4.1. Terminal Sliding Mode Control of Second-order Uncertain Nonlinear System

Consider the second-order nonlinear system

$$\dot{x}_1 = x_2$$

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$$\dot{x}_2 = f(x) + g(x) + b(x)u, \quad (2.4.1)$$

where $x = [x_1 \ x_2]^T$ is the system state vector, $f(x)$ and $b(x) \neq 0$ are smooth nonlinear functions of x , and $g(x)$ represents the uncertainties and disturbances satisfying $\|g(x)\| \leq l_g$, where $l_g > 0$, and u is the scalar control input.

In order to obtain the terminal convergence of the state variables, the following first-order terminal sliding variable is defined:

$$s = x_2 + \beta x_1^{q/p}, \quad (2.4.2)$$

where $\beta > 0$ is a design constant, and p and q are positive odd integers which satisfy the following condition:

$$p > q. \quad (2.4.3)$$

Using a sliding mode controller of the form

$$u(x) = -b^{-1}(x)(f(x) + \beta \frac{q}{p} x_1^{q/p-1} x_2 + (l_g + \eta) \text{sign}(s)), \quad (2.4.4)$$

the terminal sliding variable s can be driven to the terminal sliding mode surface $s = 0$ in finite time. On the TSM surface, the system dynamics are determined by the following nonlinear differential equation:

$$\dot{x}_1 = -\beta x_1^{q/p}. \quad (2.4.5)$$

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It has been shown in [39, 40] that $x_1 = 0$ is the terminal attractor of the system (2.4.5).

Indeed, if the initial value of x_1 at $t = 0$ is $x_1(0)(\neq 0)$ and the two odd integers p and q are selected to satisfy (2.4.3), the finite time t_s that is taken to travel from $x_1(0)(\neq 0)$ to $x_1(0+t_s) = 0$ is given by

$$t_s = \frac{p}{\beta(p-q)} |x_1(0)|^{1-q/p}. \quad (2.4.6)$$

This means that, on the TSM surface, both the system states x_1 and x_2 converge to zero in finite time.

It is seen in TSM control (2.4.4) that the term $x_1^{q/p-1}x_2$ may cause singularity if $x_2 \neq 0$ when $x_1 = 0$. Note that this situation does not occur on the TSM surface because when $s = 0$, $x_2 = -\beta x_1^{q/p}$ and hence as long as $q < p < 2q$, i.e. $1 < q/p < 2$, the term $x_1^{q/p-1}x_2 = x_1^{(2q-p)/p}$ is nonsingular. The singularity problem may occur in the reaching phase when there is insufficient control to ensure $x_2 \neq 0$ while $x_1 = 0$.

2.4.2. Nonsingular Terminal Sliding Mode Control

In order to overcome the singularity problem for TSM systems, a new nonsingular terminal sliding mode (NTSM) control was proposed in [41], which is able to avoid this problem completely. The new nonsingular terminal sliding variable is defined as follows:

$$s = x_1 + \frac{1}{\beta} x_2^{p/q}, \quad (2.4.7)$$

where β , p , and q $x_1 = 0$ have been defined in (2.4.2). It is worth pointing out that

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when $s = 0$, the NTSM surface (2.4.7) is equivalent to (2.4.2) and hence the time taken to reach the equilibrium point $x_1 = 0$ on the NTSM surface is the same as in (2.4.6).

Using the NTSM controller of the form

$$u = -b^{-1}(x)(f(x) + \beta \frac{q}{p} x_2^{2-p/q} + (l_g + \eta) \text{sign}(s)), \quad (2.4.8)$$

where $1 < q/p < 2$, $\eta > 0$, the NTSM surface $s = 0$ can be reached in finite time. Then, the system states x_1 and x_2 will converge to zero in finite time. Since $2 - p/q > 0$, unlike the conventional TSM controller (2.4.4), the new NTSM controller (2.4.8) will not cause any singularity if $x_2 \neq 0$.

2.4.3. High-order Terminal Sliding Mode Control

The methodology discussed in subsections (2.4.1) and (2.4.2) has been extended to the control of the following high-order SISO linear system:

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i = 1, 2, \dots, n-1, \\ \dot{x}_n &= \sum_{j=1}^n a_j x_j + u(t), \end{aligned} \quad (2.4.9)$$

where a_j ($j = 1, \dots, n$) are known.

The following recursive TSM structure has been used for the control of high-order SISO linear system (2.4.9):

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$$s_0 = x_1$$

$$s_1 = \dot{s}_0 + \beta_1 s_0^{q_1/p_1}$$

$$s_2 = \dot{s}_1 + \beta_2 s_1^{q_2/p_2}$$

$$\vdots$$

$$s_{n-1} = \dot{s}_{n-2} + \beta_{n-1} s_{n-2}^{q_{n-1}/p_{n-1}}, \quad (2.4.10)$$

where $p_i, q_i (i=1, \dots, n-1)$ are positive odd integers, $p_i > q_i$, and $\beta_i > 0$. If a TSM controller is designed such that $s_{n-1} \dot{s}_{n-1} < 0$ is satisfied, the terminal sliding variables $s_{n-1}, \dots, s_0$ (that is x_1) will converge sequentially to zero in finite time.

2.5. Concluding Remarks

The theory of variable structure control systems has been briefly surveyed in sections 2.2, 2.3, and 2.4 of this chapter. Because the variable structure theory has many good features, it can be easily used to control linear or nonlinear systems. Although the robustness can be achieved without the exact knowledge of the control system, the system performance and control quality depend very much on the choice of sliding mode parameters and the estimate of bounding functions of the unknown parts. In practice, excessive control input and severe control chattering which may excite unmodelled high frequency dynamics are highly undesired.

Chapter 3

FUZZY THEORY AND FUZZY SYSTEMS

3.1. Introduction

In 1965 control scientist Lofti Zadeh [42-44] published his first paper on a novel way of characterizing non-probabilistic uncertainties, which he called as “fuzzy sets”. Fuzzy logic and fuzzy set theory has now evolved into a fruitful area containing various disciplines, such as calculus of fuzzy if-then rules, fuzzy graphs, fuzzy interpolation, fuzzy topology, fuzzy reasoning, fuzzy inference system, and fuzzy modeling. The applications, which are multi-disciplinary in nature, include automatic control, consumer electronics, signal processing, time-series prediction, information retrieval, database management, computer vision, data classification, decision making, and so on. The purpose of this section is to briefly summarize the basic concepts that are necessary in understanding fuzzy logic and fuzzy inference from a practical viewpoint.

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This chapter is organized as follows. In Section 3.2, we give the definitions of fuzzy sets, fuzzy set operations, fuzzy relation and compositions, and linguistic representations. In Section 3.3, description of fuzzy logic and fuzzy reasoning are provided. In Section 3.4, the T-S fuzzy system and Mamdani fuzzy system are discussed. Finally, Section 3.5 gives concluding remarks.

3.2. Fuzzy Set Theory

3.2.1. Fuzzy Sets

A *universe of discourse* U is the set of allowable values for a variable, denoted generically by $\{x\}$ which could be discrete or continuous, where x represents the *generic element* of U .

A *crisp set* A in a universe of discourse U can be defined as $A = \{x \mid \mu_A(x)\}$, where $\mu_A(x)$ is, in general, a condition by which $x \in A$. If we introduce a zero-one characteristic function such that $A \Rightarrow (\mu_A(x)) = 1$ if $x \in A$ and $\mu_A(x) = 0$ if $x \notin A$ then $\mu_A(x)$ is called as a membership function for A .

A *fuzzy set* F in a universe of discourse U is characterized by a membership function $\mu_A(x)$, which takes on values in the interval $[0,1]$. A fuzzy set may be viewed as a generalization of the concept of an ordinary (i.e. crisp) set whose membership function only takes two values $\{0,1\}$. A membership function for a fuzzy set F provides a *measure of the degree of similarity* of an element in U to F . A fuzzy set F in U may be

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represented as a set of ordered pairs of a generic element x and its grade of membership function: $F = \{(x, \mu_F(x)) | x \in U\}$. When U is continuous, F can be written concisely as $F = \int_U \mu_F(x) / x$, where the integral sign denotes the collection of all points $x \in U$ with associated membership function $\mu_F(x)$. When U is discrete, F is represented as $F = \sum_{x_i \in U} \mu_F(x_i) / x_i$, where the summation sign stands for the union of $(x, \mu_F(x))$ pairs. Similarly, the slash “/” in these expressions is only a marker and does not imply division. Note that in fuzzy sets, an element can reside in more than one set to different degrees of similarity. This cannot occur in a crisp set theory.

The *support* of a fuzzy set F is the crisp set of all points $x \in U$ such that $\mu_F(x) > 0$. The element $x \in U$ at which $\mu_F(x) > 0.5$ is called the *crossover point*. A fuzzy set whose support is a single point x in U with $\mu_F(x) = 1$ is referred to as *fuzzy singleton*. In a singleton output space, fuzzy set membership functions are represented as single vertical points.

3.2.2. Set Theory Operators

Let fuzzy sets A and B in U be described by their membership functions $\mu_A(x)$ and $\mu_B(x)$. The set theoretic operations of union, intersection and complement for fuzzy sets are defined as the following.

Union (disjunction): The membership function $\mu_{A \cup B}$ of the union $A \cup B$ is pointwise defined for all $x \in U$ by

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$$\mu_{A \cup B}(x) = \max \{ \mu_A(x), \mu_B(x) \}. \quad (3.2.1)$$

Intersection (conjunction): The membership function $\mu_{A \cap B}$ of the intersection $A \cap B$ is pointwise defined for all $x \in U$ by

$$\mu_{A \cap B}(x) = \min \{ \mu_A(x), \mu_B(x) \}. \quad (3.2.2)$$

Complement (negation): The membership function $\mu_{\bar{A}}$ of the complement of a fuzzy set A , $\bar{A}$ is pointwise defined for all $x \in U$ by

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x). \quad (3.2.3)$$

Other consistent definitions for fuzzy AND and OR have been proposed in the literature under the names “T-norm” and “T-conorm” operators, respectively.

T-norm: a t-norm, denoted by $*$, is a two-place function from $[0,1] \times [0,1]$ to $[0,1]$, which includes fuzzy intersection, algebraic product, bounded product, and drastic product,

$$\text{defined as } x * y = \begin{cases} \min \{x, y\} & \text{fuzzy intersection} \\ xy, & \text{algebraic product} \\ \max \{0, x + y - 1\}, & \text{bounded product} \\ \left. \begin{array}{l} x \text{ if } y = 1 \\ y \text{ if } x = 1 \\ 0 \text{ if } x, y < 1 \end{array} \right\}, & \text{drastic product} \end{cases}$$

where $x, y \in [0,1]$.

T-conorm: A t-conorm, denoted by $\oplus$, is a two-place function from $[0,1] \times [0,1]$ to $[0,1]$, which includes fuzzy union, algebraic sum, bounded sum, and drastic sum, defined as

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$$x \oplus y = \begin{cases} \max\{x, y\} & \text{fuzzy union} \\ x + y - xy, & \text{algebraic sum} \\ \min\{1, x + y\}, & \text{bounded sum} \\ \left. \begin{array}{l} x \text{ if } y = 0 \\ y \text{ if } x = 0 \\ 1 \text{ if } x, y > 0 \end{array} \right\}, & \text{drastic sum} \end{cases}$$

where $x, y \in [0, 1]$.

3.2.3. The Extension Principle

The extension principle is a tool for generalizing crisp mathematical concepts to fuzzy sets. It has been extensively used in the fuzzy literature.

The extension principle: Let U and V be two universes of discourse and f be a mapping from U to V . For a fuzzy set A in U , the extension principle defines a fuzzy set B in V by

$$\mu_B(y) = \sup_{x \in f^{-1}(y)} [\mu_A(x)]. \quad (3.2.4)$$

That is, $\mu_B(y)$ is the superium of $\mu_A(y)$ for all $x \in U$ such that $f(x) = y$, where $y \in V$ and we assume that $f^{-1}(y)$ is not empty. If $f^{-1}(y)$ is empty for some $y \in V$, define $\mu_B(y) = 0$.

3.2.4. Fuzzy Relations and Their Compositions

Cartesian product: If $A_1, \dots, A_n$ are fuzzy sets in $U_1, \dots, U_n$, respectively, the Cartesian product of $A_1, \dots, A_n$ is a fuzzy set in the product space $U_1 \times \dots \times U_n$ with the membership function

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$$\mu_{A_1 \times \dots \times A_n}(x_1, \dots, x_n) = \min \{ \mu_{A_1}(x_1), \dots, \mu_{A_n}(x_n) \}, \quad (4.2.5a)$$

or

$$\mu_{A_1 \times \dots \times A_n}(x_1, \dots, x_n) = \mu_{A_1}(x_1), \dots, \mu_{A_n}(x_n). \quad (4.2.5b)$$

Fuzzy relations: Fuzzy relations represent a degree of presence or absence of association, or interconnection between the elements of two or more fuzzy sets. An n - array fuzzy relation is a fuzzy set in $U_1 \times \dots \times U_n$ and is expressed as

$$R_{u_1 \times \dots \times u_n} = \{ ((x_1, \dots, x_n), \mu_R(x_1, \dots, x_n)) \mid (x_1, \dots, x_n) \in U_1 \times \dots \times U_n \}. \quad (3.2.6)$$

Compositions: Because fuzzy relations are fuzzy sets in the product space, set theoretic and algebraic operations can be defined using operators for fuzzy union, intersection, and complement. Let R and S be fuzzy relations in $U \times V$. The intersection and union of R and S , which are *compositions* of the two relations, are defined as

$$\mu_{R \cap S}(x, y) = \mu_R(x, y) * \mu_S(x, y), \quad (3.2.7a)$$

and

$$\mu_{R \cup S}(x, y) = \mu_R(x, y) \oplus \mu_S(x, y), \quad (3.2.7b)$$

respectively, where $*$ is any t-norm, and $\oplus$ is any t-conorm.

Next, consider the composition of fuzzy relations from different product space that share a common set. If R and S are two fuzzy relations in $U \times V$ and $V \times W$, respectively, the

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sup-star composition of $R(U \times V)$ and $S(V \times W)$ is a fuzzy relation denoted by $R \circ S(U \times V)$ and is defined by

$$R \circ S = \left\{ (x, y), \sup_{y \in V} [\mu_R(x, y) * \mu_S(y, z)], x \in U, y \in V, z \in W \right\}, \quad (3.2.8)$$

where $*$ could be any operator in the class of t-norm (the sup-product is most commonly used). When U , V , and W are discrete universe of discourse, the sup operation is the maximum.

3.2.5. Linguistic Representation

The use of fuzzy sets provides a basis for a systematic way for the management of vague and imprecise concepts. It is natural to explore the use of what might be called as the *Linguistic variable*, i.e., variables whose values are not numbers but words or sentence in a natural or artificial language. Fuzzy sets can be employed to represent linguistic variable. Consider a real line as a continuous universe of discourse U . A *fuzzy number* F in U is a fuzzy sets which is normal and convex, i.e.,

$$\max_{x \in U} \mu_F(x) = 1, \quad (3.2.9a)$$

$$\mu_F(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_F(x_1), \mu_F(x_2)), x_1, x_2 \in U, \lambda \in [0, 1]. \quad (3.2.9b)$$

A linguistic variable can be regarded as a variable whose value is a fuzzy number or whose values are defined in linguistic terms or labels. A linguistic variable is characterized by its name u ; the term set of u , i.e., the set of term $T(u)$ of linguistic values of u with each value being a fuzzy number defined on U ; a syntactic rule for generating

the names of values of u ; and a semantic rule for associating with each value of its meaning. For example, let SPEED (u) of a car be linguistic variable. It can be decomposed into the following set of terms $T(u) = \{\text{slow, medium, fast}\}$, where each term is characterized by a fuzzy set in the universe of discourse $U = [0 \text{ km/h, } 100\text{km/h}]$. “Slow” speed might be interpreted as “a speed below about 40 km/h”, “medium” as “a speed close to 60 km/h”, and “fast” as “a speed above 80 km/h”.

3.3. Fuzzy Logic and Fuzzy Reasoning

Fuzzy logic is a logical system that is much closer in spirit to human thinking and natural language than traditional logic systems. As a calculus of compatibility, fuzzy logic is quite different from probability, which is based on frequency distributions in a random population. Fuzzy logic describes the characteristic properties of continuously varying values by associating partitions of these values with a semantic label [45]. Fuzziness is different from probability. The laws of contraction and excluded middle are broken in fuzzy logic but are not broken in probability. Conditional probability, which must be defined in probability theory, can be derived from first principles using fuzzy logic [46]. Much of the description power of fuzzy logic comes from the fact that these semantic partitions can overlap corresponding to the transition from one term to the next.

Fuzzy set serves as the basis for fuzzy logic. Fuzzy set theory starts with some basic concepts coming from crisp set theory, fuzzy logic also begins by borrowing some crisp logic notions. In crisp logic, a *proposition* is an ordinary statement involving terms

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already defined, and must be meaningful to call it “true” or “false”. A *tautology* is a proposition formed by combining other propositions $p, q, r, \dots$.

Let us start with the basic primitive of fuzzy logic: fuzzy propositions. A fuzzy proposition represents a statement like “speed (u) is SLOW”, where SLOW is a linguistic label, defined by a fuzzy set on the universe of discourse of speed. Fuzzy propositions connect variables with linguistic labels defined for those variables. As in classical logic, fuzzy propositions can be combined by using logical connectives *and* and *or*, which are implemented by t-norms and t-conorms, respectively.

Fuzzy logic provides a means of modeling human decision making within the conceptual framework of fuzzy logic and approximate reasoning. In fact, reasoning with fuzzy logic is based on fuzzy rules which are expressed as logical implications, i.e. in the forms of IF-THEN statements. In fuzzy logic, the definition of a fuzzy implication may be expressed as a fuzzy implication function. Various implication functions can be classified into three main categories: *fuzzy conjunction*, *fuzzy disjunction*, and *fuzzy implication*. A fuzzy rule “IF u is A , THEN v is B ” is represented by a fuzzy implication and is denoted by $A \rightarrow B$, where A and B are fuzzy sets in the universe U and V with membership functions μ_A and μ_B , respectively. Membership function of the fuzzy implication, denoted by $\mu_{A \rightarrow B}(x, y)$, measures the degree of truth of the implication between $x \in U$ and $y \in V$.

Fuzzy conjunction: The fuzzy conjunction is defined for all $x \in U$ and $y \in V$ by

$$A \rightarrow B = A * B, \quad (3.3.1)$$

where $*$ is an operator representing a t-norm.

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Fuzzy disjunction: The fuzzy disjunction is defined for all $x \in U$ and $y \in V$ by

$$A \rightarrow B = A \oplus B, \tag{3.3.2}$$

where $\oplus$ is an operator representing a t-conorm.

Fuzzy implication: An extension is made by determining the fuzzy versions of some tautologies of $P \rightarrow q$ in crisp logic, e.g. $\sim p \vee p$, $\sim p \vee pq$, and $(\sim p \wedge \sim q) \vee q$. Thus, the fuzzy implication is associated with the corresponding families of fuzzy implication functions:

Material implication:

$$A \rightarrow B = \text{not} A \oplus B. \tag{3.3.3}$$

Propositional calculus:

$$A \rightarrow B = \text{not} A \oplus (A * B). \tag{3.3.4}$$

Extended propositional calculus:

$$A \rightarrow B = (\text{not} A * \text{not} B) \oplus B. \tag{3.3.5}$$

In traditional propositional logic, there are two very important inference rules, *Modus Ponens* and *Modus Tollens*. Modus ponens is associated with the implication $[A \rightarrow B]$:

Premises	Consequence
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Premise 1: x is A	y is B
Premise2: IF x is A THEN y is B	

In term of propositions p and q , modus ponens is expressed as $(p \wedge (p \rightarrow q)) \rightarrow q$.

Modus tollens is associated with the implication $[(\text{not } A \rightarrow \text{not } B)]$:

Premises	Consequence
Premise 1: y is B	x is not A
Premise2: IF x is A THEN y is B	

In terms of proposition p and q , modus tollens is expressed as $(\sim q \wedge (p \rightarrow q)) \rightarrow \sim p$. In

fuzzy logic, modus ponens is extended to *Generalized Modus Ponens* (GMP):

Premises	Consequence
Premise 1: x is A^*	y is B^*
Premise2: IF x is A THEN y is B	

and modus tollens to *Generalized Modus Tollens* (GMT):

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Premises	Consequence
Premise 1: y is B^* Premise2: IF x is A THEN y is B	y is A^*

where A, A^*, B, B^* are fuzzy sets introduced via linguistic variables x, y . Two more families of fuzzy implication function obtained from GMP and GMT are:

Generalization of modus ponens:

$$A \rightarrow B = \sup \{c \in [0,1], A^* c \leq B\} . \tag{3.3.6}$$

Generalization of modus tollens:

$$A \rightarrow B = \inf \{t \in [0,1], B \oplus t \leq A\} . \tag{3.3.7}$$

Based on the definition (3.3.1-3.3.7), many membership functions for fuzzy implication may be generated by employing appropriate triangular norms and conorms.

3.4. Fuzzy Systems

In practice, most physical systems are nonlinear which are too complex to be modeled by rigorous mathematical models. One well-known solution for this modeling problem is the fuzzy rule-based modeling. The fuzzy rule-based models are generally classified into two major types: the Mamdani fuzzy systems and the Takagi-Sugeno (T-S) fuzzy systems. In

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this section, the basic concepts of Mamdani fuzzy systems and T-S fuzzy systems are summarized.

3.4.1. Mamdani Fuzzy Systems

To model a complex system using Mamdani fuzzy model, three steps are usually enclosed. First, the whole state-space of the complex system is decomposed into a few subspaces. Second, within each subspace, the complex system is approximated using a fuzzy set. Finally, the global Mamdani fuzzy model of the complex system is constructed by aggregating all the fuzzy sets of the subsystems using the weight average fuzzy inference method.

A general Mamdani fuzzy system is described as follows:

$$R_i : \text{IF } z_1(t) \text{ is } F_1^i \text{ AND } z_2(t) \text{ is } F_2^i \text{ AND } \dots \text{ AND } z_n(t) \text{ is } F_n^i$$

THEN

$$y \text{ is } B^i$$

$$\text{for } i = 1, \dots, m, \quad (3.4.1)$$

where R_i represents the i^{th} fuzzy inference rule, m is the number of inference rules,

$F_j (j = 1, \dots, n)$ is the fuzzy sets of the premise part. $z(t) = (z_1(t), \dots, z_n(t))^T$ is the vector of

some measurable input variables. y is the output linguistic variable, and B^i is the fuzzy set of the consequent part for the i^{th} fuzzy inference rule.

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Denote $\mu_B(y)$ as the aggregates output membership function, and S as the support of $\mu_B(y)$. By using *Centroid of Area* defuzzification, the crisp output of the Mamdani system could be represented as

$$y^* = \frac{\int_S y \cdot \mu_B(y) dy}{\int_S \mu_B(y) dy}. \quad (3.4.2)$$

In the next subsection, we will present the T-S fuzzy systems, which are more compact and computationally efficient in representation than the Mamdani fuzzy systems.

3.4.2. T-S Fuzzy Systems

Three steps are often enclosed to model a complex system using T-S modeling technique. First, the whole state-space is decomposed into a few subspaces. Second, within each subspace, the complex system is approximated using a linear time invariant model. Finally, the global fuzzy model of the complex system is constructed using the weighted average fuzzy inference method to aggregate all the subsystem matrices.

The general T-S fuzzy system model can be represented as:

$$R^i : \text{IF} \quad z_1(t) \text{ is } F_1^i \text{ AND } \dots z_n(t) \text{ is } F_n^i,$$

THEN

$$\dot{\mathbf{x}}(t) = \mathbf{A}^i \mathbf{x}(t) + \mathbf{B}^i \mathbf{u}(t),$$

$$\mathbf{y}(t) = \mathbf{C}_i \mathbf{x}(t),$$

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$$\text{for } i = 1, \dots, m, \quad (3.4.3)$$

where R^i represents the i^{th} fuzzy rule, m the number of inference rules, F_j ($j = 1, \dots, n$) the fuzzy sets. $\mathbf{x}(t) \in \mathcal{R}^{n \times 1}$ the state variable vector, $\mathbf{u}(t) \in \mathcal{R}^{q \times 1}$ the system input vector, and $\mathbf{y}(t) \in \mathcal{R}^{p \times 1}$ the system output. The matrices $\mathbf{A}^i \in \mathcal{R}^{n \times n}$, $\mathbf{B}^i \in \mathcal{R}^{n \times q}$, and $\mathbf{C}_i \in \mathcal{R}^{p \times n}$ are parameter matrices of the i^{th} subsystem. $\mathbf{z}(t) = (z_1(t), \dots, z_n(t))^T$ represents the vector of some measurable system variables. Denote $\mu^i(\mathbf{z}(t))$ be the normalized fuzzy membership function of the inferred fuzzy set F^i where

$$F^i = \prod_{j=1}^n F_j^i, \quad (3.4.4)$$

$$\mu^i = \frac{F^i}{\sum_{i=1}^m F^i}, \quad (3.4.5)$$

and

$$\sum_{i=1}^m \mu^i = 1. \quad (3.4.6)$$

Using the weighted average fuzzy inferences, we obtain the following global T-S fuzzy state space model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t), \end{aligned} \quad (3.4.7)$$

where

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$$\begin{aligned}
A(t) &= \sum_{i=1}^m \mu_i(t) A_i, \quad B(t) = \sum_{i=1}^m \mu_i(t) B_i, \\
C(t) &= \sum_{i=1}^m \mu_i(t) C_i.
\end{aligned} \tag{3.4.8}$$

T-S fuzzy systems are more compact and computationally efficient in representation than Mamdani fuzzy systems, therefore they have been more widely applied to fuzzy modeling and control of engineering systems.

3.5. Concluding Remarks

As an introduction, this chapter provides an overview of fuzzy logic and fuzzy systems. Fuzzy sets, fuzzy sets operations, and fuzzy linguistic representation such as linguistic variables have been briefly outlined. Fuzzy logic and fuzzy reasoning is considered from the engineering viewpoint with IF-THEN fuzzy implications. Based on the construction of the rule consequent, the fuzzy systems are generally classified into two major types: the Mamdani fuzzy systems and the T-S fuzzy systems. However, the stability properties of T-S fuzzy systems have not been fully explored. In the following chapters of this thesis, an improved robust variable structure control scheme for complex uncertain nonlinear systems will be proposed by combining conventional methods and recently developed fuzzy extreme subsystems concept to improve the system performance and enhance the control quality.

Chapter 4

MULTIPLE-SURFACE SLIDING CONTROL OF A CLASS OF UNCERTAIN NONLINEAR SYSTEMS

4.1. Introduction

Sliding mode control has been widely developed and extensively implemented in recent years [47-50]. It is seen in Chapter 2 that, on the sliding mode surface, the system is equivalent to an unforced system of lower order, termed the equivalent system, which is insensitive to matched uncertainties.

In order to control nonlinear systems with mismatched uncertainties, multiple-surface sliding control (MSSC) was developed in [9, 47]. However, the mathematical proof of the

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main MSSC algorithm in [9] is not given. Further, the time complexity of the algorithm is in $O(2^n)$.

In this chapter, we will first propose a simplified recursive-like MSSC algorithm that reduces the time complexity from $O(2^n)$ to $O(n)$. Then, using mathematical induction, we show that the developed simplified algorithm agrees with MSSC in [9].

The chapter is organized as follows: In Section 4.2, the basic principle of MSSC design will be discussed. Then, in Section 4.3, the simplified MSSC algorithm, which is the main result of this chapter, will be proposed and analyzed. Finally, in Section 4.4, we conclude this chapter with some remarks.

4.2. Problem Formulation

In this chapter, we focus on the design of multiple-surface sliding control for a class of nonlinear systems. Consider a family of uncertain nonlinear systems of the form

$$\begin{aligned}\dot{x}_i &= x_{i+1} + d_i(\mathbf{x}, t), \quad 1 \leq i \leq n-1, \\ \dot{x}_n &= f(\mathbf{x}, t) + m(\mathbf{x}, t)u + d_n(\mathbf{x}, t), \\ y &= x_1,\end{aligned}\tag{4.2.1}$$

where $\mathbf{x} = [x_1 \quad \cdots \quad x_n]^T \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are input and output signals, respectively, the functions $d_i(\mathbf{x}, t) \in \mathbb{R}$, $i = 1, 2, \dots, n$, $f(\mathbf{x}, t)$ and $m(\mathbf{x}, t)$ are

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unknown, the function $m(x, t)$ does not vanish in the set of all feasible x and for all $t \geq 0$.

The n sliding mode surfaces are defined as

$$s_i = x_i - x_{id}, \quad i = 1, \dots, n, \quad (4.2.2)$$

where x_{id} represents the desired value of state x_i .

Let $x_{1d} = y_d$, $\bar{d}_1 = d_1$, and

$$x_{2d} = \dot{y}_d - \hat{\bar{d}}_1 - c_1 \frac{s_1}{\phi_1}, \quad (4.2.3)$$

where $\hat{\bar{d}}_1$ is an estimate of $\bar{d}_1$, $c_1 > 0$ is a design parameter, and $\phi_1 > 0$ is the thickness of the boundary layer of the sliding mode surface $s_1 = 0$. Then for s_1 , its time derivative along the system dynamics (4.2.1) is

$$\dot{s}_1 = s_2 + (\bar{d}_1 - \hat{\bar{d}}_1) - c_1 \frac{s_1}{\phi_1}. \quad (4.2.4)$$

Taking the time derivative of the 2nd sliding surface, s_2 , we have

$$\dot{s}_2 = (s_3 + x_{3d}) + d_2 - \dot{x}_{2d}. \quad (4.2.5)$$

Taking the time derivative of x_{2d} in (4.2.3), we have

$$\dot{x}_{2d} = \ddot{y}_d - \dot{\hat{\bar{d}}}_1 - \frac{c_1}{\phi_1} (x_2 + d_1 - \dot{y}_d). \quad (4.2.6)$$

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Separate $\dot{x}_{2d}$ into two parts as $\dot{x}_{2d} = \dot{x}'_{2d} + \bar{x}_{2d}$, where $\dot{x}'_{2d} = \ddot{y}_d - \frac{c_1}{\phi_1}(x_2 - \dot{y}_d)$ is composed of known values and $\bar{x}_{2d} = -\hat{\bar{d}}_1 - \frac{c_1}{\phi_1}(d_1)$ is the unknown portion of $\dot{x}_{2d}$. Denote $\bar{d}_2 = d_2 - \bar{x}_{2d}$ as a lumped uncertainty, and then (4.2.5) can be rewritten as

$$\dot{s}_2 = s_3 + x_{3d} + \bar{d}_2 - \dot{x}'_{2d}. \quad (4.2.7)$$

Choosing

$$x_{3d} = \dot{x}'_{2d} - \hat{\bar{d}}_2 - c_2 \frac{s_2}{\phi_2}, \quad (4.2.8)$$

where $\dot{x}'_{2d}$ is a known value, $\hat{\bar{d}}_2$ is an estimate of $\bar{d}_2$, and $\phi_2 > 0$ is the thickness of the boundary layer of $s_2 = 0$, (4.2.7) can be written as

$$\dot{s}_2 = s_3 + (\bar{d}_2 - \hat{\bar{d}}_2) - c_2 \frac{s_2}{\phi_2}. \quad (4.2.9)$$

Proceeding along the same lines as before, for $i = 1, 2, \dots, n-1$, Huang and Chen [9] had obtained the following MSSC algorithm:

$$\dot{s}_i = s_{i+1} + (\bar{d}_i - \hat{\bar{d}}_i) - c_i \frac{s_i}{\phi_i}, \quad (4.2.10)$$

$$x_{(i+1)d} = \dot{x}'_{id} - \hat{\bar{d}}_i - c_i \frac{s_i}{\phi_i}, \quad (4.2.11)$$

$$\bar{d}_i = d_i - \bar{x}_{id}, \quad (4.2.12)$$

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$$\dot{\bar{x}}_{(i+1)d} = \dot{x}'_{(i+1)d} + \bar{\dot{x}}_{(i+1)d}, \quad (4.2.13)$$

$$\dot{x}'_{(i+1)d} = y_d^{(i+1)} - \sum_{j=1}^i \frac{c_j}{\phi_j} (x_{i+1} - \alpha'_{id_j}), \quad (4.2.14)$$

$$\bar{\dot{x}}_{(i+1)d} = -\dot{\bar{d}}_i - \sum_{j=1}^i \frac{c_j}{\phi_j} (d_i + \bar{\alpha}_{id_j}), \quad (4.2.15)$$

$$\alpha'_{id_j} = \begin{cases} y_d^{(i)} & \text{for } j=1, \\ y_d^{(i)} - \sum_{k=1}^{j-1} \frac{c_k}{\phi_k} (x_i - \alpha'_{(i-1)d_k}) & \text{for } 2 \leq j \leq i-1, \\ \dot{x}'_{id} & \text{for } j=i, \end{cases} \quad (4.2.16)$$

$$\bar{\alpha}_{id_j} = \begin{cases} 0 & \text{for } j=1, \\ \sum_{k=1}^{j-1} \frac{c_k}{\phi_k} (d_{i-1} + \bar{\alpha}_{(i-1)d_k}) & \text{for } 2 \leq j \leq i-1, \\ \bar{d}_i - d_i & \text{for } j=i. \end{cases} \quad (4.2.17)$$

Remark 4.1: Let $T(n)$ be the number of function calls, i.e., α'_{id_j} , $\bar{\alpha}_{id_j}$, $\alpha'_{(i-1)d_k}$, and $\bar{\alpha}_{(i-1)d_k}$ in (4.2.14), (4.2.15), (4.2.16), and (4.2.17), respectively, we can express the number of functions called by $\dot{x}'_{(n)d}$ and $\bar{\dot{x}}_{(n)d}$ as

$$T(n) = \begin{cases} 2(2n-3) & \text{if } 2 \leq n \leq 3, \\ 2 \left(\sum_{j=0}^{n-2} 2^j \right) & \text{if } n > 3. \end{cases} \quad (4.2.18)$$

Equation (4.2.18) shows that, the MSSC algorithm in [9] performs $T(n) = 2(2^{n-1}) = 2^n$ number of function calls in order to obtain $\dot{x}'_{(n)d}$ and $\bar{\dot{x}}_{(n)d}$ for $n > 3$. It follows that

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$T(n) = O(2^n)$. Since the operation of function calls is a basic operation [51] of equations (4.2.14)-(4.2.17), we conclude that the time complexity is $O(2^n)$.

4.3. Simplified MSSC Algorithm

Based on equations (4.2.14)-(4.2.17), we propose the following simplified MSSC algorithm

$$\dot{x}'_{(i+1)d} = \varphi_{i+1}(i+1) + h_{i+1}(i+1), \quad (4.3.1)$$

$$\bar{\dot{x}}_{(i+1)d} = \begin{cases} -\dot{d}_1 - \frac{c_1}{\phi_1} d_1 & \text{for } i=1, \\ -\dot{d}_i + \mathfrak{z}_{i+1}(i+1) + \eta_{i+1}(i+1) & \text{for } i \geq 2, \end{cases} \quad (4.3.2)$$

$$\varphi_j(k) = \begin{cases} y_d^k & \text{for } j=1, \\ \varphi_{j-1}(k) - \frac{c_{j-2}}{\phi_{j-2}} [x_k - \varphi_{j-1}(k-1)] & \text{for } j \geq 2, \end{cases} \quad (4.3.3a)$$

$$h_j(k) = -\frac{c_{j-1}}{\phi_{j-1}} [x_k - \dot{x}'_{(k-1)d}], \quad (4.3.3b)$$

$$\mathfrak{z}_j(k) = \begin{cases} 0 & \text{for } j \leq 2, \\ -\frac{c_1}{\phi_1} [d_{k-1}] & \text{for } j=3, \\ \mathfrak{z}_{j-1}(k) - \frac{c_{j-2}}{\phi_{j-2}} [d_{k-1} - \mathfrak{z}_{j-1}(k-1)] & \text{for } j \geq 4, \end{cases} \quad (4.3.4a)$$

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$$\eta_j(k) = -\frac{c_{j-1}}{\phi_{j-1}}[d_{k-1} - \bar{x}_{(k-1)d}]. \quad (4.3.4b)$$

Remark 4.2: For the proposed MSSC algorithm (4.3.1)-(4.3.2), let $C(n)$ be the number of function calls, i.e., $\varphi_{j-1}(k)$ and $\mathfrak{z}_{j-1}(k)$ in (4.3.3a) and (4.3.4a), respectively. We can then use the recurrence to express the number of functions called by $\dot{x}'_{(n)d}$ and $\bar{x}_{(n)d}$ as

$$C(n) = \begin{cases} 2 & \text{if } n=1, \\ C(n-1) + 2 & \text{if } n \geq 2. \end{cases} \quad (4.3.5)$$

Equation (4.3.5) means that $C(n) = 2n$ and the time complexity of the proposed simplified algorithm in (4.3.1)~(4.3.4b) is $O(n)$.

Remark 4.3: Next, we will use mathematical induction to show that the simplified MSSC algorithm agrees with MSSC law in [9]. Following the design steps in [9], for steps $i = 1, 2$, and 3 , we have

$$\begin{aligned} \dot{x}_{2d} &= y_d^{(2)} - \frac{c_1}{\phi_1}[x_2 - y_d^{(1)}] - [\dot{\bar{d}}_1 + \frac{c_1}{\phi_1}d_1] \\ &= \varphi_2(2) + h_2(2) + \bar{x}_{(2)d} \\ &= \dot{x}'_{(2)d} + \bar{x}_{(2)d}, \end{aligned} \quad (4.3.6)$$

$$\dot{x}_{3d} = y_d^{(3)} - \frac{c_1}{\phi_1}[x_3 - y_d^{(2)}] - \frac{c_2}{\phi_2}[x_3 - \dot{x}'_{2d}]$$

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$$\begin{aligned}
 & -\frac{\dot{\bar{d}}_2}{\phi_1} - \frac{c_1}{\phi_1}[d_2] - \frac{c_2}{\phi_2}[d_2 - \bar{x}_{2d}] \\
 & = \varphi_3(3) + h_3(3) - \frac{\dot{\bar{d}}_2}{\phi_1} + \mathfrak{z}_3(3) + \eta_3(3) \\
 & = \dot{x}'_{(3)d} + \bar{\dot{x}}_{(3)d}, \tag{4.3.7}
 \end{aligned}$$

$$\begin{aligned}
 \dot{x}_{4d} &= y_d^{(4)} - \frac{c_1}{\phi_1}[x_4 - y_d^{(3)}] - \frac{c_2}{\phi_2}[x_4 - [y_d^{(3)} - \frac{c_1}{\phi_1}[x_3 - y_d^{(2)}]]] \\
 & - \frac{c_3}{\phi_3}[x_4 - \dot{x}'_{3d}] - \frac{\dot{\bar{d}}_3}{\phi_1} - \frac{c_1}{\phi_1}[d_3] - \frac{c_2}{\phi_2}[d_3 - [-\frac{c_1}{\phi_1}[d_2]]] \\
 & - \frac{c_3}{\phi_3}[d_3 - \bar{x}_{3d}] \\
 & = \varphi_4(4) + h_4(4) - \frac{\dot{\bar{d}}_3}{\phi_1} + \mathfrak{z}_4(4) + \eta_4(4) \\
 & = \dot{x}'_{(4)d} + \bar{\dot{x}}_{(4)d}. \tag{4.3.8}
 \end{aligned}$$

It is seen that this algorithm is true for steps 1, 2, and 3.

For further analysis, we assume that the simplified MSSC algorithm is valid for step i , $\forall i \leq p-1$. In the following we will prove that it is also true for step p , where $p > 3$, to complete the induction process.

If $\dot{x}'_{(p)d}$, $\bar{\dot{x}}_{(p)d}$, $\varphi_p(p)$, $h_p(p)$, $\mathfrak{z}_p(p)$, and $\eta_p(p)$ are chosen as defined in (4.3.1), (4.3.2), (4.3.3a), (4.3.3b), (4.3.4a), and (4.3.4b), respectively, then, for step p

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$$\begin{aligned}
 \dot{x}_{(p)d} &= \ddot{x}'_{(p-1)d} - \frac{\dot{\bar{d}}}{\bar{d}_{p-1}} - \frac{c_{p-1}}{\phi_{p-1}} [\dot{s}_{p-1}] \\
 &= \varphi_p(p) + h_p(p) - \frac{\dot{\bar{d}}}{\bar{d}_{p-1}} + \mathfrak{z}_p(p) + \eta_p(p) \\
 &= \dot{x}'_{(p)d} + \bar{\ddot{x}}_{(p)d}.
 \end{aligned} \tag{4.3.9}$$

Substituting the index p for $i+1$ and differentiating (4.2.11) with respect to time, we have

$$\dot{x}_{(p)d} = \ddot{x}'_{(p-1)d} - \frac{\dot{\bar{d}}}{\bar{d}_{p-1}} - \frac{c_{p-1}}{\phi_{p-1}} [\dot{s}_{p-1}]. \tag{4.3.10}$$

Using (4.2.1) and (4.2.13) in (4.3.10), we get

$$\begin{aligned}
 \dot{x}_{(p)d} &= \ddot{x}'_{(p-1)d} - \frac{\dot{\bar{d}}}{\bar{d}_{p-1}} - \frac{c_{p-1}}{\phi_{p-1}} [x_p + d_{p-1} - \dot{x}'_{(p-1)d} - \bar{\ddot{x}}_{(p-1)d}] \\
 &= \ddot{x}'_{(p-1)d} - \frac{c_{p-1}}{\phi_{p-1}} [x_p - \dot{x}'_{(p-1)d}] - \frac{\dot{\bar{d}}}{\bar{d}_{p-1}} \\
 &\quad - \frac{c_{p-1}}{\phi_{p-1}} [d_{p-1} - \bar{\ddot{x}}_{(p-1)d}].
 \end{aligned} \tag{4.3.11}$$

Substituting the index $p-1$ for $i+1$ in (4.3.1), and, using (4.3.3) in (4.3.11), we get

$$\dot{x}_{(p)d} = \dot{\phi}_{p-2}(p-1) - \frac{c_{p-3}}{\phi_{p-3}} [\dot{x}_{p-1} - \dot{\phi}_{p-2}(p-2)]$$

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$$\begin{aligned}
 & -\frac{c_{p-2}}{\phi_{p-2}}[\dot{x}_{p-1} - \ddot{x}'_{(p-2)d}] - \frac{c_{p-1}}{\phi_{p-1}}[x_p - \dot{x}'_{(p-1)d}] \\
 & -\dot{\bar{d}}_{p-1} - \frac{c_{p-1}}{\phi_{p-1}}[d_{p-1} - \bar{x}_{(p-1)d}].
 \end{aligned} \tag{4.3.12}$$

Using (4.2.1) again, we get

$$\begin{aligned}
 \dot{x}_{(p)d} &= \dot{\phi}_{p-2}(p-1) - \frac{c_{p-3}}{\phi_{p-3}}[x_p + d_{p-1} - \dot{\phi}_{p-2}(p-2)] \\
 & -\frac{c_{p-2}}{\phi_{p-2}}[x_p + d_{p-1} - \ddot{x}'_{(p-2)d}] - \frac{c_{p-1}}{\phi_{p-1}}[x_p - \dot{x}'_{(p-1)d}] \\
 & -\dot{\bar{d}}_{p-1} - \frac{c_{p-1}}{\phi_{p-1}}[d_{p-1} - \bar{x}_{(p-1)d}].
 \end{aligned} \tag{4.3.13}$$

To proceed, we introduce two additional propositions.

Proposition 4.1: For $l \leq p-1$,

$$\dot{\phi}_{l+1}(k) = \phi_{l+1}(k+1) + \mathfrak{z}_{l+1}(k+1). \tag{4.3.14}$$

Proof: Without loss of generality, we assume that the proposed MSSC algorithm is true.

Then,

$$\begin{aligned}
 \dot{\phi}_2(k) &= y_d^{k+1} + 0 \\
 &= \phi_2(k+1) + \mathfrak{z}_2(k+1),
 \end{aligned} \tag{4.3.15}$$

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$$\begin{aligned}\dot{\varphi}_3(k) &= \varphi_2(k+1) - \frac{c_1}{\phi_1}[x_{k+1} - \varphi_2(k)] - \frac{c_1}{\phi_1}[d_k] \\ &= \varphi_3(k+1) + \mathfrak{z}_3(k+1),\end{aligned}\tag{4.3.16}$$

$$\begin{aligned}\dot{\varphi}_4(k) &= \varphi_2(k+1) - \frac{c_1}{\phi_1}[x_{k+1} - \varphi_2(k)] - \frac{c_2}{\phi_2}[x_{k+1} - \varphi_3(k)] \\ &\quad - \frac{c_1}{\phi_1}[d_k] - \frac{c_2}{\phi_2}[d_k + \frac{c_1}{\phi_1}[d_{(k-1)}]] \\ &= \varphi_4(k+1) + \mathfrak{z}_4(k+1),\end{aligned}\tag{4.3.17}$$

By assuming that Proposition 4.1 is valid for step $l-1$ and differentiating (4.3.3a) with respect to time, for $l > 3$, we have

$$\begin{aligned}\dot{\varphi}_l(k) &= \dot{\varphi}_{l-1}(k) - \frac{c_{l-2}}{\phi_{l-2}}[\dot{x}_k - \dot{\varphi}_{l-1}(k-1)] \\ &= \varphi_{l-1}(k+1) + \mathfrak{z}_{l-1}(k+1) \\ &\quad - \frac{c_{l-2}}{\phi_{l-2}}[\dot{x}_k - \varphi_{l-1}(k) - \mathfrak{z}_{l-1}(k)].\end{aligned}\tag{4.3.18}$$

Using (4.2.1) in (4.3.18), it is easy to get

$$\begin{aligned}\dot{\varphi}_l(k) &= \varphi_{l-1}(k+1) + \mathfrak{z}_{l-1}(k+1) \\ &\quad - \frac{c_{l-2}}{\phi_{l-2}}[x_{k+1} + d_k - \varphi_{l-1}(k) - \mathfrak{z}_{l-1}(k)]\end{aligned}$$

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$$\begin{aligned}
 &= \varphi_{l-1}(k+1) - \frac{c_{l-2}}{\phi_{l-2}}[x_{k+1} - \varphi_{l-1}(k)] \\
 &\quad + \mathfrak{z}_{l-1}(k+1) - \frac{c_{l-2}}{\phi_{l-2}}[d_k - \mathfrak{z}_{l-1}(k)] \\
 &= \varphi_l(k+1) + \mathfrak{z}_l(k+1).
 \end{aligned} \tag{4.3.19}$$

This completes the proof of the inductive step.

Proposition 4.2: For $1 < l \leq p-1$,

$$\ddot{x}'_{(l)d} = \varphi_{l+1}(l+1) + \mathfrak{z}_{l+1}(l+1). \tag{4.3.20}$$

Proof: Without loss of generality, we assume that the proposed MSSC algorithm is true.

Substituting the index $l+1$ for p in (4.3.10), one arrives at

$$\begin{aligned}
 \dot{x}_{(l+1)d} &= \ddot{x}'_{(l)d} - \frac{\dot{\hat{d}}}{\phi_l} - \frac{c_l}{\phi_l}[\dot{s}_l] \\
 &= \ddot{x}'_{(l)d} - \frac{\dot{\hat{d}}}{\phi_l} - \frac{c_l}{\phi_l}[x_{l+1} + d_l - \dot{x}'_{(l)d} - \bar{x}_{(l)d}].
 \end{aligned} \tag{4.3.21}$$

Rearranging (4.3.21) yields

$$\ddot{x}'_{(l)d} = \dot{x}_{(l+1)d} + \frac{c_l}{\phi_l}[x_{l+1} - \dot{x}'_{(l)d}] + \frac{\dot{\hat{d}}}{\phi_l} + \frac{c_l}{\phi_l}[d_l - \bar{x}_{(l)d}]. \tag{4.3.22}$$

Using $\dot{x}_{(l+1)d} = \dot{x}'_{(l+1)d} + \bar{x}_{(l+1)d}$ in (4.3.22), we have

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$$\begin{aligned}\ddot{x}'_{(l)d} &= \dot{x}'_{(l+1)d} + \frac{c_l}{\phi_l} [x_{l+1} - \dot{x}'_{(l)d}] + \bar{x}_{(l+1)d} \\ &\quad + \dot{\bar{d}}_l + \frac{c_l}{\phi_l} [d_l - \bar{x}_{(l)d}].\end{aligned}\tag{4.3.23}$$

Using (4.3.1) and (4.3.2) in (4.3.23), we get

$$\begin{aligned}\ddot{x}'_{(l)d} &= \varphi_{l+1}(l+1) + h_{l+1}(l+1) + \frac{c_l}{\phi_l} [x_{l+1} - \dot{x}'_{(l)d}] \\ &\quad + [-\dot{\bar{d}}_l + \mathfrak{z}_{l+1}(l+1) + \eta_{l+1}(l+1)] + \dot{\bar{d}}_l + \frac{c_l}{\phi_l} [d_l - \bar{x}_{(l)d}] \\ &= \varphi_{l+1}(l+1) + \mathfrak{z}_{l+1}(l+1),\end{aligned}\tag{4.3.24}$$

and the proof is thus completed.

Using Proposition 4.1 and Proposition 4.2, it follows that

$$\begin{aligned}\dot{x}_{(p)d} &= \varphi_{p-2}(p) + \mathfrak{z}_{p-2}(p) \\ &\quad - \frac{c_{p-3}}{\phi_{p-3}} [x_p + d_{p-1} - \varphi_{p-2}(p-1) - \mathfrak{z}_{p-2}(p-1)] \\ &\quad - \frac{c_{p-2}}{\phi_{p-2}} [x_p + d_{p-1} - \varphi_{p-1}(p-1) - \mathfrak{z}_{p-1}(p-1)] \\ &\quad - \frac{c_{p-1}}{\phi_{p-1}} [x_p - \dot{x}'_{(p-1)d}] - \dot{\bar{d}}_{p-1} - \frac{c_{p-1}}{\phi_{p-1}} [d_{p-1} - \bar{x}_{(p-1)d}]\end{aligned}$$

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$$\begin{aligned}
&= \varphi_{p-2}(p) - \frac{c_{p-3}}{\phi_{p-3}}[x_p - \varphi_{p-2}(p-1)] \\
&\quad - \frac{c_{p-2}}{\phi_{p-2}}[x_p - \varphi_{p-1}(p-1)] - \frac{c_{p-1}}{\phi_{p-1}}[x_p - \dot{x}'_{(p-1)d}] \\
&\quad - \frac{\dot{\bar{d}}}{\phi_{p-1}} + \mathfrak{z}_{p-2}(p) - \frac{c_{p-3}}{\phi_{p-3}}[d_{p-1} - \mathfrak{z}_{p-2}(p-1)] \\
&\quad - \frac{c_{p-2}}{\phi_{p-2}}[d_{p-1} - \mathfrak{z}_{p-1}(p-1)] - \frac{c_{p-1}}{\phi_{p-1}}[d_{p-1} - \bar{x}_{(p-1)d}]. \tag{4.3.25}
\end{aligned}$$

Using (4.3.3a) and (4.3.4a) in (4.3.25), one has

$$\begin{aligned}
\dot{x}_{(p)d} &= \varphi_{p-1}(p) - \frac{c_{p-2}}{\phi_{p-2}}[x_p - \varphi_{p-1}(p-1)] \\
&\quad - \frac{c_{p-1}}{\phi_{p-1}}[x_p - \dot{x}'_{(p-1)d}] - \frac{\dot{\bar{d}}}{\phi_{p-1}} + \mathfrak{z}_{p-1}(p) \\
&\quad - \frac{c_{p-2}}{\phi_{p-2}}[d_{p-1} - \mathfrak{z}_{p-1}(p-1)] - \frac{c_{p-1}}{\phi_{p-1}}[d_{p-1} - \bar{x}_{(p-1)d}]. \tag{4.3.26}
\end{aligned}$$

Equation (4.3.26) can also be written as

$$\dot{x}_{(p)d} = \varphi_p(p) + h_p(p) - \frac{\dot{\bar{d}}}{\phi_{p-1}} + \mathfrak{z}_p(p) + \eta_p(p), \tag{4.3.27}$$

if $\varphi_p(p)$, $h_p(p)$, $\mathfrak{z}_p(p)$, and $\eta_p(p)$ are chosen as defined in (4.3.3a), (4.3.3b), (4.3.4a), and (4.3.4b), respectively. Clearly, (4.3.27) can further be written as

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$$\dot{x}_{(p)d} = \dot{x}'_{(p)d} + \bar{\dot{x}}_{(p)d}, \quad (4.3.28)$$

if $\dot{x}'_{(p)d}$ and $\bar{\dot{x}}_{(p)d}$ are chosen as in (4.3.1) and (4.3.2), respectively, we obtain (4.3.9),

which completes the induction process.

4.4. Conclusions

Using the recursive design, a simplified MSSC algorithm has been proposed to obtain the MSSC law. It is proven that the proposed algorithm reduces the time complexity of the conventional MSSC algorithm in [9].

Chapter 5

ADAPTIVE FUZZY MULTI-SURFACE SLIDING MODE CONTROL OF A CLASS OF NONLINEAR SYSTEMS WITH MISMATCHED UNCERTAINTIES

5.1. Introduction

Sliding mode control [3, 17] has been widely applied to various systems since 1960s. As pointed out by Drazenovic [2], the salient feature of the sliding mode control is that it is completely robust to matching uncertainties that lie in the range space of the input matrix, after the system dynamics reaches the sliding mode surface. Although many systems can

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be classified under this category, there are still many systems with mismatched uncertainties, and the complex for the design of the sliding mode controllers is increasing.

Multi-surface sliding mode control (MSMC) has been developed in [8] to control nonlinear systems with mismatched uncertainties. The upper bound information of the uncertainties is required to design the MSMC. However, this upper bound information may not always be available. Therefore, in [9], MSMC and the function approximation technique were combined to control a more general class of nonlinear systems with mismatched uncertainties of which the upper bound information of the uncertainties is not available. It is seen that the finite combination of orthonormal Fourier basis functions were used to design this function approximator. Since these basis functions are obtained by trial and error approach, the functions selected may not be rich enough to approximate the uncertainties.

Among many function approximation techniques, neural networks and fuzzy systems are widely used [52-55]. For instance, neural networks were used to approximate a highly nonlinear robot function in [52, 53], fuzzy systems were used to approximate system uncertainties and unknown part of the nonlinear system in [54, 55]. Further, it has been proven in [56] that the Mamdani type fuzzy systems are universal approximators.

In this chapter, we combine the Mamdani fuzzy logic-based function approximator and the MSMC introduced in Chapter 4 to design an adaptive fuzzy multi-surface sliding mode control (AFMSMC). The Mamdani fuzzy logic-based approximator is first employed to learn the system uncertainties in Lyapunov sense, i.e., the weights are adaptively adjusted so that the output tracking error can asymptotically converge to zero

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according to the second method of Lyapunov stability. The iterative MSMC design is then carried out. We note here that the time bound for the convergence of the tracking error can be reduced significantly by employing the concept of terminal sliding mode (TSM) [10] to design the baseline controller.

This chapter is organized as follows: Section 5.2 presents the design procedure of MSMC. Section 5.3 presents the fuzzy systems used for function approximation. Section 5.4 presents the mathematical proof on the closed-loop stability in detail. Section 5.5 presents simulation results of a benchmark problem. Section 5.6 gives concluding remarks.

5.2. Multi-surface Sliding Mode Control

Consider the n th order SISO nonlinear system:

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \Delta_i(\mathbf{x}, t), \quad 1 \leq i \leq n-1, \\ \dot{x}_n &= b(\mathbf{x}, t)u(t) + f(\mathbf{x}, t) + \Delta_n(\mathbf{x}, t), \\ y &= x_1,\end{aligned}\tag{5.2.1}$$

where $\mathbf{x} = [x_1, \dots, x_n]^T \in \mathfrak{R}^n$ is a vector of measurable states, $u(t) \in \mathfrak{R}$ and $y \in \mathfrak{R}$ represent the input and output signals, respectively. The functions $\Delta_i(\mathbf{x}, t) \in \mathfrak{R}$, $i = 1, \dots, n$ and $b(\mathbf{x}, t)$ are unknown, the function $b(\mathbf{x}, t)$ does not vanish for all feasible $\mathbf{x}$ and for all $t \geq 0$.

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The goal of this chapter is to design an adaptive control law so that the output y can converge to the desired reference signal y_d with a guaranteed transient performance and tracking accuracy. The following steps formulate the design procedure for MSMC.

Step 1

Define n sliding mode surfaces as

$$s_i = x_i - x_{id}, \quad i = 1, \dots, n, \quad (5.2.2)$$

where x_{id} represents the desired value of state x_i .

Let $x_{1d} = y_d$, and differentiating s_1 with respect to time t , we have

$$\begin{aligned} \dot{s}_1 &= x_2 + \Delta_1(\mathbf{x}, t) - \dot{y}_d \\ &= s_2 + x_{2d} + \Delta_1(\mathbf{x}, t) - \dot{y}_d. \end{aligned} \quad (5.2.3)$$

Let $\bar{\Delta}_1(\mathbf{x}, t) = \Delta_1(\mathbf{x}, t)$ and view x_{2d} as a synthetic control. We can then design a control law for s_1 such that x_1 tracks its desired trajectory x_{1d} .

The control law for the synthetic control is chosen as,

$$\begin{aligned} x_{2d} &= \dot{y}_d - \hat{\bar{\Delta}}_1(\mathbf{x}, \Theta_{\Delta_1}) - \alpha_1 s_1 - k_1 s_1 \\ &= \dot{y}_d - \hat{\bar{\Delta}}_1(\mathbf{x}, \Theta_{\Delta_1}) - w_1 s_1, \end{aligned} \quad (5.2.4)$$

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where $w_1 = \alpha_1 + k_1$, α_1 , and k_1 are the design parameters to be determined later, $\hat{\Delta}_1(\mathbf{x}, \Theta_{\Delta_1})$ is the estimate of $\bar{\Delta}_1(\mathbf{x}, t)$, and Θ_{Δ_1} is the collection of the tuning parameters c , φ , and ω .

Substituting (5.2.4) into (5.2.3), we write the time derivative of the first sliding mode surface as follows

$$\dot{s}_1 = s_2 + \left(\bar{\Delta}_1(\mathbf{x}, t) - \hat{\Delta}_1(\mathbf{x}, \Theta_{\Delta_1}) \right) - w_1 s_1. \quad (5.2.5)$$

Step 2

Differentiating x_{2d} with respect to time, we have

$$\dot{x}_{2d} = y_d^2 - \dot{\Delta}_1(\mathbf{x}, \Theta_{\Delta_1}) - w_1 (x_2 + \Delta_1(\mathbf{x}, t) - y_d), \quad (5.2.6)$$

where $y_d^2 = \frac{d^2 y}{dt^2}$.

For clarity and simplicity, we separate (5.2.6) into the certain and the uncertain portions.

By defining the certain terms as $\tilde{x}_{2d}$ and the uncertain terms as $\hat{x}_{2d}$, $\dot{x}_{2d}$ is expressed as

$$\dot{x}_{2d} = \tilde{x}_{2d} + \hat{x}_{2d}, \quad (5.2.7)$$

$$\text{where} \quad \tilde{x}_{2d} = y_d^2 - w_1 (x_2 - y_d) \quad (5.2.8a)$$

$$\text{and} \quad \hat{x}_{2d} = -\dot{\Delta}_1(\mathbf{x}, \Theta_{\Delta_1}) - w_1 (\Delta_1(\mathbf{x}, t)). \quad (5.2.8b)$$

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Differentiating s_2 with respect to time, we have

$$\begin{aligned}\dot{s}_2 &= \dot{x}_3 + \Delta_2(\mathbf{x}, t) - \dot{\tilde{x}}_{2d} \\ &= s_3 + x_{3d} + \bar{\Delta}_2(\mathbf{x}, t) - \tilde{\dot{x}}_{2d},\end{aligned}\tag{5.2.9}$$

where $\bar{\Delta}_2(\mathbf{x}, t) = \Delta_2(\mathbf{x}, t) - \hat{\dot{x}}_{2d}$. (5.2.10)

Also, the synthetic control input x_{3d} is chosen as

$$\begin{aligned}x_{3d} &= \tilde{\dot{x}}_{2d} - \hat{\bar{\Delta}}_2(\mathbf{x}, \Theta_{\Delta_2}) - \alpha_2 s_2 - k_2 s_2 \\ &= \tilde{\dot{x}}_{2d} - \hat{\bar{\Delta}}_2(\mathbf{x}, \Theta_{\Delta_2}) - w_2 s_2,\end{aligned}\tag{5.2.11}$$

where $\tilde{\dot{x}}_{2d}$ is the known function from (5.2.7), $\hat{\bar{\Delta}}_2(\mathbf{x}, \Theta_{\Delta_2})$ is an estimate of $\bar{\Delta}_2(\mathbf{x}, t)$, and $w_2 = \alpha_2 + k_2$, $\alpha_2, k_2 > 0$ are the design parameters. Substituting (5.2.11) into (5.2.9), we obtain the second sliding mode surface as follows

$$\dot{s}_2 = s_3 + \left(\bar{\Delta}_2(\mathbf{x}, t) - \hat{\bar{\Delta}}_2(\mathbf{x}, \Theta_{\Delta_2}) \right) - w_2 s_2.\tag{5.2.12}$$

Step i

Similarly, the synthetic control input $x_{(i+1)d}$, for $i = 1, 2, \dots, n-1$, can be chosen as [57]:

$$x_{(i+1)d} = \tilde{\dot{x}}_{id} - \hat{\bar{\Delta}}_i(\mathbf{x}, \Theta_{\Delta_i}) - w_i s_i,\tag{5.2.13a}$$

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where $w_i = \alpha_i + k_i$, $\alpha_i > 0$, $k_i \geq \frac{(E_{i+1} + \delta_{\Delta_i})^2}{4e_i}$, $E_i = \sqrt{\frac{e_i}{2\alpha_i}}$, and $e_i > 0$, represent the design parameters, $\delta_{\Delta_i} \geq 0$ being the upper bound of the fuzzy approximation error in (5.3.10a), then

$$\dot{s}_i = s_{i+1} + \left(\bar{\Delta}_i(\mathbf{x}, t) - \hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i}) \right) - w_i s_i, \quad (5.2.13b)$$

$$\bar{\Delta}_i(\mathbf{x}, t) = \Delta_i(\mathbf{x}, t) - \hat{x}_{id}, \quad (5.2.13c)$$

$$\dot{\hat{x}}_{(i+1)d} = \tilde{\hat{x}}_{(i+1)d} + \hat{\dot{x}}_{(i+1)d}, \quad (5.2.13d)$$

$$\tilde{\hat{x}}_{(i+1)d} = y_d^{(i+1)} - \sum_{j=1}^i w_j (x_{i+1} - \tilde{\Phi}_{ij}), \quad (5.2.13e)$$

$$\hat{\dot{x}}_{(i+1)d} = -\hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i}) - \sum_{j=1}^i w_j (\Delta_i(\mathbf{x}, t) + \bar{\Phi}_{ij}), \quad (5.2.13f)$$

$$\tilde{\Phi}_{ij} = \begin{cases} y_d^j & \text{for } j = 1, \\ y_d^j - \sum_{k=1}^{j-1} w_k (x_i - \tilde{\Phi}_{(i-1)k}) & \text{for } 2 \leq j \leq i-1, \\ \tilde{\hat{x}}_{id} & \text{for } j = i, \end{cases} \quad (5.2.13g)$$

$$\bar{\Phi}_{ij} = \begin{cases} 0 & \text{for } j = 1, \\ \sum_{k=1}^{j-1} w_k (\Delta_{i-1}(\mathbf{x}, t) + \bar{\Phi}_{(i-1)k}) & \text{for } 2 \leq j \leq i-1, \\ \bar{\Delta}_i(\mathbf{x}, t) - \Delta_i(\mathbf{x}, t) & \text{for } j = i. \end{cases} \quad (5.2.13h)$$

Step n

Differentiating the final sliding mode surface s_n with respect to time, we have

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$$\begin{aligned}\dot{s}_n &= f(\mathbf{x}, t) + b(\mathbf{x}, t)u(t) + \Delta_n(\mathbf{x}, t) - \dot{\hat{x}}_{nd} \\ &= b(\mathbf{x}, t)u(t) + \bar{\Delta}_n(\mathbf{x}, t) - \tilde{f}(\mathbf{x}, t),\end{aligned}\quad (5.2.14)$$

where

$$\tilde{x}_{nd} = y_d^n - \sum_{j=1}^{n-1} w_j \left(x_n - \tilde{\Phi}_{(n-1)j} \right), \quad (5.2.15a)$$

$$\hat{\dot{x}}_{nd} = -\hat{\Delta}_{n-1}(\mathbf{x}, \Theta_{\Delta_{n-1}}) - \sum_{j=1}^{n-1} w_j \left(\Delta_{n-1}(\mathbf{x}, t) + \bar{\Phi}_{(n-1)j} \right), \quad (5.2.15b)$$

$$\tilde{f}(\mathbf{x}, t) = \tilde{x}_{nd} - f(\mathbf{x}, t), \quad (5.2.15c)$$

$$\text{and} \quad \bar{\Delta}_n(\mathbf{x}, t) = \Delta_n(\mathbf{x}, t) - \hat{\dot{x}}_{nd}. \quad (5.2.15d)$$

The baseline control law is chosen as

$$u(t) = \frac{1}{\hat{b}(\mathbf{x}, \Theta_b)} \left[\tilde{f}(\mathbf{x}, t) - \hat{\Delta}_n(\mathbf{x}, \Theta_{\Delta_n}) - \alpha_n s_n - \beta |s_n|^\gamma \text{sign}(s_n) \right], \quad (5.2.16)$$

where $\alpha_n, \beta > 0, 0 < \gamma < 1$, are the design parameters, $\hat{b}(\mathbf{x}, \Theta_b)$ and $\hat{\Delta}_n(\mathbf{x}, \Theta_{\Delta_n})$ are the estimates of $b(\mathbf{x}, t)$ and $\bar{\Delta}_n(\mathbf{x}, t)$, respectively.

5.3. Fuzzy Approximator

5.3.1. Mamdani Fuzzy Logic System

The Mamdani fuzzy logic system with m inputs, $I_1, \dots, I_m$ and the center of average defuzzifier is described as [55]

$$F = \frac{\sum_{k=1}^g \prod_{j=1}^m \mu_{P_j^k}(I_j) \omega_k}{\sum_{k=1}^g \prod_{j=1}^m \mu_{P_j^k}(I_j)}, \quad (5.3.1)$$

where g is the number of fuzzy rules, P_j^k is the j th fuzzy set corresponding to the k th fuzzy rule, $\mu_{P_j^k}$ is the membership function of the fuzzy set P_j^k , and ω_k is the centroid of the k th fuzzy set corresponding to the controller's output.

Assumption 5.1: The membership function used is a C^1 function.

To proceed, we represent (5.3.1) as

$$F = \sum_{k=1}^g \left(\frac{\prod_{j=1}^m \mu_{P_j^k}(I_j)}{\sum_{k=1}^g \prod_{j=1}^m \mu_{P_j^k}(I_j)} \right) \omega_k. \quad (5.3.2)$$

Let $\omega = [\omega_1 \quad \omega_2 \quad \dots \quad \omega_g]^T$,

and $M = [M_1 \quad M_2 \quad \dots \quad M_g]^T$

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$$= \left[\frac{\prod_{j=1}^m \mu_{p_j^1}(I_j)}{\sum_{k=1}^g \prod_{j=1}^m \mu_{p_j^k}(I_j)} \quad \dots \quad \frac{\prod_{j=1}^m \mu_{p_j^g}(I_j)}{\sum_{k=1}^g \prod_{j=1}^m \mu_{p_j^k}(I_j)} \right]^T.$$

Then, we write (5.3.2) as

$$F = \omega^T M. \quad (5.3.3)$$

In this chapter, we use the following Gaussian membership function for P_j^k

$$\mu_{A_j^k} = \exp \left(-\frac{(I_j - c_{p_j^k})^2}{2\varphi_{p_j^k}^2} \right), \quad (5.3.4)$$

where $c_{p_j^k}$ and $\varphi_{p_j^k}$ represent the centre and the gradient of the Gaussian membership function, respectively.

5.3.2. Modeling Uncertainties

In this subsection, we approximate the unknown functions, $\bar{\Delta}_i(\mathbf{x}, t)$ and $b(\mathbf{x}, t)$ using the Mamdani fuzzy logic systems $\hat{\bar{\Delta}}_i(\mathbf{x}, \Theta_{\Delta_i})$ and $\hat{b}(\mathbf{x}, \Theta_b)$, respectively, where:

$$\hat{\bar{\Delta}}_i(\mathbf{x}, \Theta_{\Delta_i}) = \omega_{\Delta_i}^T M_{\Delta_i}, \quad \hat{b}(\mathbf{x}, \Theta_b) = \omega_b^T M_b, \quad (5.3.5)$$

$$\text{with} \quad \Theta_{\Delta_i} = \{c_{p_j^k}^{\Delta_i}, \varphi_{p_j^k}^{\Delta_i}, \omega_k^{\Delta_i}\} \quad (5.3.6a)$$

$$\text{and} \quad \Theta_b = \{c_{p_j^k}^b, \varphi_{p_j^k}^b, \omega_k^b\} \quad \text{for } i = 1, \dots, n. \quad (5.3.6b)$$

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In the following, we use $\Theta_{\Delta_i}^*$ and Θ_b^* to represent the optimal fuzzy parameters such that

$$\Theta_{\Delta_i}^* = \arg \min_{\Theta_{\Delta_i} \in \Phi} \sup_{x,t \in \Omega} \left| \bar{\Delta}_i(x,t) - \hat{\Delta}_i(x, \Theta_{\Delta_i}) \right|, \quad (5.3.7a)$$

$$\Theta_b^* = \arg \min_{\Theta_b \in \Phi} \sup_{x,t \in \Omega} \left| b(x,t) - \hat{b}(x, \Theta_b) \right|, \quad (5.3.7b)$$

where Φ is a closed constraint set for Θ_{Δ_i} and Θ_b , $\Omega \subseteq \mathcal{R}^{n+1}$ is a region where the state x and time t are constrained to reside.

For further analysis, we have the following assumptions:

Assumption 5.2: For all $x, t \in \Omega$, the minimum approximation errors satisfy the following bounded conditions:

$$\left| \bar{\Delta}(x,t) - \hat{\Delta}(x, \Theta_{\Delta_i}^*) \right| \leq d_{\Delta_i} \text{ and } \left| b(x,t) - \hat{b}(x, \Theta_b^*) \right| \leq d_b \quad (5.3.8)$$

with $d_{\Delta_i}, d_b \geq 0$, for $i=1, \dots, n$.

Assumption 5.3: The optimal fuzzy approximators, $\hat{\Delta}_i(x, \Theta_{\Delta_i}^*)$ and $\hat{b}(x, \Theta_b^*)$ are bounded:

$$\hat{\Delta}_i(x, \Theta_{\Delta_i}^{\min}) \leq \hat{\Delta}_i(x, \Theta_{\Delta_i}^*) \leq \hat{\Delta}_i(x, \Theta_{\Delta_i}^{\max}), \quad (5.3.9a)$$

$$0 < \hat{b}(x, \Theta_b^{\min}) \leq \hat{b}(x, \Theta_b^*) \leq \hat{b}(x, \Theta_b^{\max}). \quad (5.3.9b)$$

Furthermore,

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$$\left| \bar{\Delta}_i(\mathbf{x}, t) - \hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i}^{\min}) \right| \leq \delta_{\Delta_i}, \quad (5.3.10a)$$

$$\left| b(\mathbf{x}, t) - \hat{b}(\mathbf{x}, \Theta_b^{\min}) \right| \leq \delta_b, \quad (5.3.10b)$$

and

$$\left| \delta_{\Delta_n} + \delta_b u \right| \leq h_n, \quad (5.3.10c)$$

where δ_{Δ_i} , δ_b , and $h_n > 0$, for $i = 1, \dots, n$.

The above assumptions will be used in the stability analysis of the closed-loop system in the following section.

5.4. The Controller Design

In this section, we first discuss the concept of TSM surface, then consider the design of the AFMSMC law and the proof of the closed-loop stability.

Definition 5.1: The fast TSM surface can be described by the following first order nonlinear differential equation [58]:

$$\dot{s} + \alpha s + \beta |s|^\gamma \text{sign}(s) = 0, \quad (5.4.1)$$

where $\alpha, \beta > 0$, $0 < \gamma < 1$, respectively.

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Remark 5.1: For any given initial condition s_0 , s converges to $s = 0$ in finite time, and the finite convergence time T can be computed as follows:

$$T = \frac{1}{\alpha(1-\gamma)} \ln \left(\frac{\alpha |s_0|^{1-\gamma} + \beta}{\beta} \right). \quad (5.4.2)$$

In order to obtain the upper bound of s_n , we need to minimize the following cost function:

$$J = \left(\tau \frac{h_n}{\alpha_n} - \left(\frac{(1-\tau)h_n}{\beta_n} \right)^{1/\gamma} \right)^2. \quad (5.4.3)$$

The optimal value of τ is denoted as τ^* where

$$\tau^* = \arg \min_{\tau} \{J\}, \quad (5.4.4)$$

with $0 \leq \tau \leq 1$.

Now, we can approximate the fuzzy systems $\hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i})$ and $\hat{b}(\mathbf{x}, \Theta_b)$ by using the truncated Taylor series. Taking the Taylor expansion of both $\hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i})$ and $\hat{b}(\mathbf{x}, \Theta_b)$, we obtain

$$\hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i}^*) - \hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i}) = (\Theta_{\Delta_i}^* - \Theta_{\Delta_i}) \left(\frac{d\hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i})}{d\Theta_{\Delta_i}} \right) + H \quad (5.4.5a)$$

$$\hat{b}(\mathbf{x}, \Theta_b^*) - \hat{b}(\mathbf{x}, \Theta_b) = (\Theta_b^* - \Theta_b) \left(\frac{d\hat{b}(\mathbf{x}, \Theta_b)}{d\Theta_b} \right) + H, \quad (5.4.5b)$$

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where H represents the higher order terms, which are small enough to be ignored.

Theorem 5.1: Consider the n th-order nonlinear system with mismatched uncertainties (5.2.1) under Assumptions 5.1 and 5.2. If we apply the control law (5.2.16) for the final sliding mode surface s_n , and the control law (5.2.13a) for the i^{th} sliding mode surface s_i where $i = 1, 2, \dots, n-1$, with the following update laws:

$$\begin{aligned} \dot{c}_{p_j^k}^{\Delta_i} &= \frac{s_i}{\rho_{jk}^{\Delta_i}} \left(\frac{d\hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i})}{dc_{p_j^k}^{\Delta_i}} \right), \quad \dot{\phi}_{p_j^k}^{\Delta_i} = \frac{s_i}{\sigma_{jk}^{\Delta_i}} \left(\frac{d\hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i})}{d\phi_{p_j^k}^{\Delta_i}} \right), \\ \dot{\omega}_k^{\Delta_i} &= \frac{s_i}{\eta_k^{\Delta_i}} \left(\frac{d\hat{\Delta}_i(\mathbf{x}, \Theta_{\Delta_i})}{d\omega_k^{\Delta_i}} \right), \end{aligned} \quad (5.4.6a)$$

$$\begin{aligned} \dot{c}_{p_j^k}^b &= \frac{s_n}{\rho_{jk}^b} \left(\frac{d\hat{b}(\mathbf{x}, \Theta_b)}{dc_{p_j^k}^b} \right), \quad \dot{\phi}_{p_j^k}^b = \frac{s_n}{\sigma_{jk}^b} \left(\frac{d\hat{b}(\mathbf{x}, \Theta_b)}{d\phi_{p_j^k}^b} \right), \\ \dot{\omega}_k^b &= \frac{s_n}{\eta_k^b} \left(\frac{d\hat{b}(\mathbf{x}, \Theta_b)}{d\omega_k^b} \right), \end{aligned} \quad (5.4.6b)$$

where $\rho_{jk}^{\Delta_i}$, $\sigma_{jk}^{\Delta_i}$, $\eta_k^{\Delta_i}$, ρ_{jk}^b , σ_{jk}^b , $\eta_k^b > 0$ represent the positive weights, then the following properties can be guaranteed under any finite initial conditions: (i) the n^{th} sliding mode surface S_n will first converge to the neighborhood of $s_n = 0$ as

$$|s_n| \leq \varepsilon_n \quad (5.4.7)$$

in finite time $T \leq T_n$, where

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$$\varepsilon_n = \begin{cases} \max \left(\frac{h_n}{\tau^* \alpha_n}, \left(\frac{(\tau^* - 1) h_n}{\tau^* \beta_n} \right)^{1/\gamma} \right) & J \neq 0, \\ \frac{h_n}{\tau^* \alpha_n} = \left(\frac{(\tau^* - 1) h_n}{\tau^* \beta_n} \right)^{1/\gamma} & J = 0, \end{cases} \quad (5.4.8)$$

$$T_n = \frac{1}{2\alpha_n(1-\zeta)} \ln \left(\frac{2\alpha_n |V_{n0}| + 2^\zeta \beta}{2^\zeta \beta} \right), \quad (5.4.9)$$

$$\zeta = \frac{1+\gamma}{2},$$

and next, (ii) the i^{th} sliding mode surface s_i will converge sequentially to the neighborhood of $s_i = 0$ as

$$|s_i| \leq \varepsilon_i \quad (5.4.10)$$

in finite time $T = T_i$, with

$$T_i = \frac{\ln \left(\frac{2V_{i0}}{\varepsilon_i^2 - 2E_i^2} \right)}{2\alpha_i}, \quad (5.4.11)$$

where $\varepsilon_i^2 > 2E_i^2$.

Proof:

Case 1: Consider the following Lyapunov function candidate:

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$$\begin{aligned}
 V_n = & \frac{1}{2}s_n^2 + \frac{1}{2} \left(\sum_{j=1}^m \sum_{k=1}^g \rho_{jk}^{\Delta_n} \left(\tilde{c}_{p_j^k}^{\Delta_n} \right)^2 + \sum_{j=1}^m \sum_{k=1}^g \sigma_{jk}^{\Delta_n} \left(\tilde{\varphi}_{p_j^k}^{\Delta_n} \right)^2 \right. \\
 & + \sum_{k=1}^g \eta_k^{\Delta_n} \left(\tilde{\omega}_k^{\Delta_n} \right)^2 + \sum_{j=1}^m \sum_{k=1}^g \rho_{jk}^b \left(\tilde{c}_{p_j^k}^b \right)^2 \\
 & \left. + \sum_{j=1}^m \sum_{k=1}^g \sigma_{jk}^b \left(\tilde{\varphi}_{p_j^k}^b \right)^2 + \sum_{k=1}^g \eta_k^b \left(\tilde{\omega}_k^b \right)^2 \right), \quad (5.4.12)
 \end{aligned}$$

where

$$\begin{aligned}
 \tilde{c}_{p_j^k}^{\Delta_n} &= c_{p_j^k}^{*\Delta_n} - c_{p_j^k}^{\Delta_n}, \quad \tilde{c}_{p_j^k}^b = c_{p_j^k}^{*b} - c_{p_j^k}^b, \quad \tilde{\varphi}_{p_j^k}^{\Delta_n} = \varphi_{p_j^k}^{*\Delta_n} - \varphi_{p_j^k}^{\Delta_n}, \\
 \tilde{\varphi}_{p_j^k}^b &= \varphi_{p_j^k}^{*b} - \varphi_{p_j^k}^b, \quad \tilde{\omega}_k^{\Delta_n} = \omega_k^{*\Delta_n} - \omega_k^{\Delta_n}, \quad \tilde{\omega}_k^b = \omega_k^{*b} - \omega_k^b.
 \end{aligned}$$

Differentiating V_n with respect to time, we have

$$\begin{aligned}
 \dot{V}_n = & s_n \dot{s}_n + \sum_{j=1}^m \sum_{k=1}^g \rho_{jk}^{\Delta_n} \tilde{c}_{p_j^k}^{\Delta_n} \dot{\tilde{c}}_{p_j^k}^{\Delta_n} + \sum_{j=1}^m \sum_{k=1}^g \sigma_{jk}^{\Delta_n} \tilde{\varphi}_{p_j^k}^{\Delta_n} \dot{\tilde{\varphi}}_{p_j^k}^{\Delta_n} \\
 & + \sum_{k=1}^g \eta_k^{\Delta_n} \tilde{\omega}_k^{\Delta_n} \dot{\tilde{\omega}}_k^{\Delta_n} + \sum_{j=1}^m \sum_{k=1}^g \rho_{jk}^b \tilde{c}_{p_j^k}^b \dot{\tilde{c}}_{p_j^k}^b \\
 & + \sum_{j=1}^m \sum_{k=1}^g \sigma_{jk}^b \tilde{\varphi}_{p_j^k}^b \dot{\tilde{\varphi}}_{p_j^k}^b + \sum_{k=1}^g \eta_k^b \tilde{\omega}_k^b \dot{\tilde{\omega}}_k^b. \quad (5.4.13)
 \end{aligned}$$

Substituting (5.2.14) and (5.2.15) into (5.4.13), we have

$$\begin{aligned}
 \dot{V}_n = & s_n \left(\left(\hat{b}(\mathbf{x}, \Theta_b^*) - \hat{b}(\mathbf{x}, \Theta_b) \right) u(t) \right. \\
 & \left. + \left(\hat{\Delta}_n(\mathbf{x}, \Theta_{\Delta_n}^*) - \hat{\Delta}_n(\mathbf{x}, \Theta_{\Delta_n}) \right) - \alpha_n s_n - \beta |s_n|^\gamma \text{sign}(s_n) \right)
 \end{aligned}$$

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$$\begin{aligned}
 & + \sum_{j=1}^m \sum_{k=1}^g \rho_{jk}^{\Delta_n} \tilde{c}_{p_j^k}^{\Delta_n} \dot{\tilde{c}}_{p_j^k}^{\Delta_n} + \sum_{j=1}^m \sum_{k=1}^g \sigma_{jk}^{\Delta_n} \tilde{\phi}_{p_j^k}^{\Delta_n} \dot{\tilde{\phi}}_{p_j^k}^{\Delta_n} \\
 & + \sum_{k=1}^g \eta_k^{\Delta_n} \tilde{\omega}_k^{\Delta_n} \dot{\tilde{\omega}}_k^{\Delta_n} + \sum_{j=1}^m \sum_{k=1}^g \rho_{jk}^b \tilde{c}_{p_j^k}^b \dot{\tilde{c}}_{p_j^k}^b \\
 & + \sum_{j=1}^m \sum_{k=1}^g \sigma_{jk}^b \tilde{\phi}_{p_j^k}^b \dot{\tilde{\phi}}_{p_j^k}^b + \sum_{k=1}^g \eta_k^b \tilde{\omega}_k^b \dot{\tilde{\omega}}_k^b.
 \end{aligned} \tag{5.4.14}$$

Substituting (5.4.5) in (5.4.14) and subsequently, using (5.4.6) and Assumption 5.2, we have

$$\begin{aligned}
 \dot{V}_n & \leq s_n \left(h_n - \alpha_n s_n - \beta |s_n|^\gamma \text{sign}(s_n) \right), \\
 & = s_n \left(\tau h_n - \alpha_n s_n + (1 - \tau) h_n - \beta |s_n|^\gamma \text{sign}(s_n) \right).
 \end{aligned} \tag{5.4.15}$$

It is seen that, if $h_n = 0$,

$$\begin{aligned}
 \dot{V}_n & = s_n \left(-\alpha_n s_n - \beta |s_n|^\gamma \text{sign}(s_n) \right) \\
 & \leq -2\alpha_n V_n - 2^\zeta \beta |V_n|^\zeta.
 \end{aligned} \tag{5.4.16}$$

From Definition 5.1 and Remark 5.1, the Lyapunov function candidate (5.4.12) converges to zero in finite time

$$T_n = \frac{1}{2\alpha_n(1-\zeta)} \ln \left(\frac{2\alpha_n |V_{n0}|^{(1-\zeta)} + 2^\zeta \beta}{2^\zeta \beta} \right). \tag{5.4.17}$$

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It is noted that, if $h_n \neq 0$, and $\dot{s}_n > 0$, for τ^* obtained with $J = 0$, s_n can reach the following region in finite time

$$|s_n| \leq \frac{\tau^* h_n}{\alpha_n} = \left(\frac{(1 - \tau^*) h_n}{\beta_n} \right)^{1/\gamma}. \quad (5.4.18)$$

However, for τ^* obtained with $J \neq 0$, s_n can reach the following region in finite time

$$|s_n| \leq \max \left(\frac{\tau^* h_n}{\alpha_n}, \left(\frac{(1 - \tau^*) h_n}{\beta_n} \right)^{1/\gamma} \right). \quad (5.4.19)$$

Case 2: Consider the following Lyapunov function candidate

$$\begin{aligned} V_i = & \frac{1}{2} s_i^2 + \frac{1}{2} \left(\sum_{j=1}^m \sum_{k=1}^g \rho_{jk}^{\Delta_i} \left(\tilde{c}_{A_j^k}^{\Delta_i} \right)^2 \right. \\ & \left. + \sum_{j=1}^m \sum_{k=1}^g \sigma_{jk}^{\Delta_i} \left(\tilde{\varphi}_{A_j^k}^{\Delta_i} \right)^2 + \sum_{k=1}^g \eta_k^{\Delta_i} \left(\tilde{\omega}_k^{\Delta_i} \right)^2 \right). \end{aligned} \quad (5.4.20)$$

Differentiating V_i with respect to time, we have

$$\begin{aligned} \dot{V}_i = & s_i \dot{s}_i + \sum_{j=1}^m \sum_{k=1}^g \rho_{jk}^{\Delta_i} \tilde{c}_{A_j^k}^{\Delta_i} \dot{\tilde{c}}_{A_j^k}^{\Delta_i} \\ & + \sum_{j=1}^m \sum_{k=1}^g \sigma_{jk}^{\Delta_i} \tilde{\varphi}_{A_j^k}^{\Delta_i} \dot{\tilde{\varphi}}_{A_j^k}^{\Delta_i} + \sum_{k=1}^g \eta_k^{\Delta_i} \tilde{\omega}_k^{\Delta_i} \dot{\tilde{\omega}}_k^{\Delta_i}. \end{aligned} \quad (5.4.21)$$

Using (5.2.13a), (5.4.5a), and (5.4.6a), similar to the prove of the Case 1, we obtain

$$\dot{V}_i \leq s_i \left((E_{i+1} + \delta_{\Delta_i}) - \alpha_i s_i - k_i s_i \right)$$

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$$\begin{aligned}
 &\leq -2\alpha_i V_i - \left(\frac{-(E_{i+1} + \delta_{\Delta_i})}{2\sqrt{k_i}} + \sqrt{k_i} s_i \right)^2 + \left(\frac{(E_{i+1} + \delta_{\Delta_i})}{2\sqrt{k_i}} \right)^2 \\
 &\leq -2\alpha_i V_i + \left(\frac{(E_{i+1} + \delta_{\Delta_i})}{2\sqrt{k_i}} \right)^2 \\
 &\leq -2\alpha_i V_i + e_i.
 \end{aligned} \tag{5.4.22}$$

By solving the linear differential equation (5.4.22), we get

$$V_i \leq \exp(-2\alpha_i t) V_{i0} + E_i^2. \tag{5.4.23}$$

Hence, the Lyapunov function candidate (5.4.20) converges to $V_i \leq \frac{\varepsilon_i^2}{2}$, which implies that the i^{th} sliding mode surface s_i converges to $|s_i| \leq \varepsilon_i$ in finite time (5.4.11).

5.5. Numerical Simulation

The proposed AFMSMC controller is tested on the benchmark system in Freeman and Kokotovic [59] and Won and Hedrick [47]. The dynamic equation is as follows:

$$\begin{aligned}
 \dot{x}_1 &= x_2 + \theta x_1^3, \\
 \dot{x}_2 &= (1 + \mu)u, \\
 y &= x_1,
 \end{aligned} \tag{5.5.1}$$

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where θ and μ are unknown parameters. In the actual simulation, θ is chosen as 0.95, and μ as 0.09. The bound of the unknown functions are assumed to be unknown to the fuzzy approximator. The desired output y_d is chosen as $y_d = 1 + \sin(2\pi t)$. The initial values of the system states are chosen as $x_1(0) = x_2(0) = 0$.

The sliding mode surfaces are $s_1 = x_1 - y_d$ and $s_2 = x_2 - x_{2d}$. Here, we choose $\alpha_1 = 20$, $\alpha_2 = 10$, $\beta = 5$, and $\gamma = 0.5$, respectively. System states, x_1 and x_2 are chosen as the input variables to the fuzzy approximator to model the unknown function, θx_1^3 . Three membership functions are assigned to each fuzzy input variable. However, for unknown functions, $(1 + \mu)$, s_1 and s_2 , are chosen as the input variables to the fuzzy approximator and similarly, three membership functions are assigned to each fuzzy input variable.

For simplicity, we assigned the initial values to the fuzzy parameters according to human experience where

$$c^\Delta = [0.3 \quad 1.4 \quad 2.2 \quad -8.9 \quad 1.3 \quad 11.4],$$

$$\varphi^\Delta = [0.7 \quad 0.6 \quad 1.2 \quad 4.9 \quad 5.1 \quad 4.9],$$

and
$$\omega^\Delta = [0.5 \quad -5 \quad -1 \quad -3 \quad -14 \quad 0.1 \quad 1 \quad 27 \quad 7].$$

Fig. 5.1 and Fig. 5.2 show the tracking performance and the control input of the AFMSMC, respectively. It is seen that the superior performance of fast and high precision tracking performance with the proposed AFMSMC scheme is obtained.

5.6. Conclusions

An adaptive fuzzy controller has been developed such that the knowledge and boundary conditions of the uncertainties are not required. The proposed adaptive control scheme has been proven to be able to guarantee the tracking error to converge to a small neighborhood of the origin in finite time. The controller is simple to design, yet gives good tracking performance for a class of nonlinear systems with mismatched uncertainties. Simulation results have been provided to show the effectiveness of the proposed control scheme.

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Fig. 5.1. The output tracking.

Fig. 5.2. The control input signal.

Chapter 6

ADAPTIVE FAST FINITE TIME CONTROL OF A CLASS OF UNCERTAIN NONLINEAR SYSTEMS

6.1. Introduction

A finite time stabilization problem was initially studied in the literature of optimal control [60]. The finite time control laws [36, 58, 61-65] are usually time-varying, discontinuous, or even depending directly on the initial conditions of considered systems [66]. In [67], a Lyapunov stability theorem has been used to test the finite time stability of continuous nonlinear system. In [68, 69], the idea of non-smooth control Lyapunov method was used to design finite time controllers for a class of nonlinear systems.

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The non-smooth control Lyapunov function that is commonly used is in the form $\dot{V} + \beta V^\gamma \leq 0$. In comparison with the linear control Lyapunov function $\dot{V} + \alpha V \leq 0$, the non-smooth control Lyapunov function may deliver poor convergence performance when the initial condition, $V(0)$, is far away from the origin, albeit its finite time convergence lies in its exponentially growing convergence rate when $V(t)$ is near the origin. We hope to know: Can we combine the advantages of the non-smooth control Lyapunov function and the linear control Lyapunov function to design a fast finite time control law that guarantees faster finite time stability of the closed-loop system?

For this problem, the Bernoulli-Jacobi equation [70] like fast non-smooth control Lyapunov function has been used in [58] to design a continuous terminal sliding mode control law. This fast non-smooth control Lyapunov function can deliver faster finite time stability of the closed loop system that cannot be realized by using either control Lyapunov function alone.

It is observed that in most of the existing works mentioned, the upper bounds of the system uncertainties are assumed known or unknown but upper bounded by known positive constants (or smooth functions). Thus, naturally an interesting question arises: Without these assumptions, how to design a robust finite time controller? This question motivates us to investigate the problem of adaptive control design that can guarantee the finite time stability of the closed-loop system by means of discontinuous feedback.

In this chapter, an adaptive fast finite time control scheme is proposed and studied to solve the problem stated above. We first define the appropriate unknown parameters, and then successfully solve the mentioned problem using the fast control Lyapunov function

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and the adding a power integrator approach [63, 71, 72], merging with adaptive technique.

Finally, we will prove that the control input is bounded.

6.2. Problem Formulation

Consider a family of uncertain nonlinear systems of the form

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \Delta_i(\mathbf{x}, t, u), \quad 1 \leq i \leq n-1, \\ \dot{x}_n &= u(t) + f(\mathbf{x}, t) + \Delta_n(\mathbf{x}, t, u) + d_n(\mathbf{x}, t),\end{aligned}\tag{6.2.1}$$

where $\mathbf{x} = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}$ represents the input signal,

$\Delta_i(\mathbf{x}, t) \in \mathbb{R}$, $i = 1, \dots, n$ are C^1 uncertain functions with $\Delta_i(0, t) = 0$, $\forall t$, and

$d_n(\mathbf{x}, t) \in \mathbb{R}$ is unknown function with $d_n(\mathbf{x}, t) \neq 0$, $\forall t, \forall \mathbf{x}$.

To proceed, we have the following assumptions.

Assumption 6.1: For $1 \leq i \leq n$, there are some unknown C^1 functions $\sigma_i(x_1, \dots, x_i) \geq 0$, such that

$$|\Delta_i(\mathbf{x}, t, u)| \leq (|x_1| + \dots + |x_i|) \sigma_i(x_1, \dots, x_i).\tag{6.2.2}$$

Assumption 6.2: There exists a known continuous function $d_{\max}(\mathbf{x}, t)$ such that

$$|d_n(\mathbf{x}, t)| \leq d_{\max}(\mathbf{x}, t).\tag{6.2.3}$$

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Theorem 6.1: Consider the non-Lipschitz continuous nonlinear system

$$\dot{x} = f(x), \text{ with } f(0) = 0. \quad (6.2.4)$$

Suppose there are C^1 Lyapunov function $V(x)$, and real numbers $\alpha > 0$, $\beta > 0$ and $0 < \gamma < 1$, such that

(1) $V(x)$ is positive definite;

$$(2) \dot{V}(x) + \alpha V(x) + \beta V^\gamma(x) \leq 0.$$

Then, the origin of system (6.2.4) is globally finite time stable and the settling time, depending on the initial state $x(0) = x_0$ is given by

$$T(x_0) \leq \frac{1}{\alpha(1-\gamma)} \ln\left(\frac{\alpha V^{1-\gamma}(x_0) + \beta}{\beta}\right). \quad (6.2.5)$$

Lemma 6.1: (Lin and Qian [73]): Let c, d be positive real numbers and $\phi(x)$, $\varphi(x)$, and $\mathcal{G}(x)$ be real-valued continuous functions. For any continuous function $\tau(x) > 0$, there is a continuous function $\rho(x) \geq 0$ such that

$$|\phi(x)| |\varphi(x)|^c |\mathcal{G}(x)|^d \leq \tau(x) |\varphi(x)|^{c+d} + |\mathcal{G}(x)|^{c+d} \rho(x). \quad (6.2.6)$$

Lemma 6.2: (Huang, Lin, and Yang [63]): For any real numbers x_i , $i = 1, 2, \dots, n$ and $0 < b \leq 1$, the following inequality holds:

$$(|x_1| + \dots + |x_n|)^b \leq |x_1|^b + \dots + |x_n|^b. \quad (6.2.7)$$

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When $b = p/q \leq 1$, where $p > 0$ and $q > 0$ are odd integers,

$$|x^b - y^b| \leq 2^{1-b} |x - y|^b. \quad (6.2.8)$$

Proof: The proof of inequality (6.2.8) can be found in [74]. For inequality (6.2.7), let $q = 1/b$. Then, by the Taylor expansion formula with the integration remainder, we have

$$\left(|x_1|^b + \dots + |x_n|^b\right)^q \geq \left(|x_1| + \dots + |x_n|\right)^b, \quad (6.2.9)$$

that imply

$$\left(|x_1|^b + \dots + |x_n|^b\right) \geq \left(|x_1| + \dots + |x_n|\right)^b, \quad (6.2.10)$$

which is identical to (6.2.7).

In this section, we propose a multiple-surface sliding mode [8, 9, 47, 57] like procedure to construct adaptive fast finite time controllers of system (6.2.1). The controllers are designed in n steps:

Step 1: Define n sliding mode surfaces $s_i = x_i^{\frac{q_i}{p_i}} - x_{id}^{\frac{q_i}{p_i}}$ for $i = 1, \dots, n$, where x_{id} represents the desired value of state x_i , q_i and p_i are odd positive integers, $q_i \geq p_i$, $0 < p_n/q_n < \dots < p_1/q_1 \leq 1$. Let $Q_i = p_i/q_i$, $x_{id} = 0$, $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$, where $\hat{\theta}_i$ is the estimate of θ_i . Let us first consider subsystem

$$\dot{x}_1 = x_2 + \Delta_1(x, t, u) \quad (6.2.11)$$

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where $\Delta_1(x, t, u) \leq (|x_1|)\sigma_1(x_1)$. In the case that parameter $\sigma_1(x_1)$ is unknown, we propose a robust adaptive virtual control law to guarantee the subsystem finite time stable as detailed in Lemma 6.3.

Lemma 6.3: Consider subsystem (6.2.11), if $s_2 = 0$ and the virtual controller x_{2d} is chosen as

$$\begin{aligned} x_{2d} &= -s_1^{(1+Q_1)\gamma-1} \left(n + 2\beta_1 + 2\alpha_1 s_1^{(1+Q_1)(1-\gamma)} + \sqrt{1 + \hat{\theta}_1^2} \right) \\ &= -s_1^{(1+Q_1)\gamma-1} \kappa_1(x_1, \hat{\theta}_1), \end{aligned} \quad (6.2.12)$$

where $\kappa_1(x_1, \hat{\theta}_1) > 0$, $\beta_1 > 0$, $\alpha_1 > 0$, $0 < \gamma < 1$, and the parameter $\hat{\theta}_1$ is updated by

$$\dot{\hat{\theta}}_1 = s_1^{(1+Q_1)\gamma}, \quad (6.2.13)$$

then, for any finite initial condition $x_1(0)$ and $\hat{\theta}_1(0)$, the system state x_1 will converge to zero in finite time

$$T_1 \leq \frac{1}{\alpha_1(1-\gamma)} \ln \left(\frac{\alpha V_1^{1-\gamma}(x_1(0)) + \beta_1}{\beta_1} \right), \quad (6.2.14)$$

where $V_1(x_1(0))$ is the Lyapunov function candidate (6.2.15).

Proof: To show that x_1 converges to zero in finite time (6.2.14), based on Theorem 6.1,

we first find a Lyapunov function candidate $V_1(x_1) > 0$ such that

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$\dot{V}_1(x_1) + \alpha_1 V_1(x_1) + \beta_1 V_1^\gamma(x_1) \leq 0, \forall x_1$. Let us consider the following Lyapunov function candidate:

$$V_1(x_1) = \frac{Q_1}{1+Q_1} s_1^{1+Q_1}. \quad (6.2.15)$$

Since $\frac{1}{2} \tilde{\theta}_1^2 \geq 0$, we have

$$V_1(x_1) \leq \frac{Q_1}{1+Q_1} s_1^{1+Q_1} + \frac{1}{2} \tilde{\theta}_1^2. \quad (6.2.16)$$

Differentiating (6.2.16) with respect to time, we have

$$\begin{aligned} \dot{V}_1(x_1) &\leq s_1 \left(x_2 + \Delta_1(x, t, u) \right) - \tilde{\theta}_1 \dot{\hat{\theta}}_1 \\ &\leq s_1 \left(x_2 + (|x_1|) \sigma_1(x_1) \right) - \tilde{\theta}_1 \dot{\hat{\theta}}_1 \\ &= s_1(x_2 - x_{2d}) + s_1 x_{2d} + s_1^{(1+Q_1)\gamma} \left(|s_1|^{(1+Q_1)(1-\gamma)} \sigma_1(x_1) \right) - \tilde{\theta}_1 \dot{\hat{\theta}}_1. \end{aligned} \quad (6.2.17)$$

Letting $\theta = |s_1|^{(1+Q_1)(1-\gamma)} \sigma_1(x_1)$, we get

$$\begin{aligned} \dot{V}_1(x_1) &\leq s_1(x_2 - x_{2d}) + s_1 x_{2d} + s_1^{(1+Q_1)\gamma} \hat{\theta}_1 + \tilde{\theta}_1 \left(s_1^{(1+Q_1)\gamma} - \hat{\theta}_1 \right) \\ &\leq 2s_1 |s_2|^{Q_2} + s_1 x_{2d} + s_1^{(1+Q_1)\gamma} \hat{\theta}_1 + \tilde{\theta}_1 \left(s_1^{(1+Q_1)\gamma} - \hat{\theta}_1 \right). \end{aligned} \quad (6.2.18)$$

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Substituting the virtual control (6.2.12) and updating law (6.2.13) into (6.2.18), if $s_2 = 0$, we have

$$\begin{aligned}\dot{V}_1(x_1) &\leq -2\beta_1 s_1^{(1+Q_1)\gamma} - 2\alpha_1 s_1^{(1+Q_1)} \\ &\leq -\beta_1 V_1^\gamma - \alpha_1 V_1,\end{aligned}\tag{6.2.19}$$

which means that x_1 converges to zero in finite time (6.2.14).

Step i ($2 \leq i \leq n-1$): Similar procedures are taken for each steps when $i = 2, \dots, n-1$ as in Step 1. Let us consider subsystems

$$\dot{x}_i = x_{i+1} + \Delta_i(x_1, \dots, x_i, t, u).\tag{6.2.20}$$

The sufficient conditions of the existence of adaptive virtual control laws to render (6.2.20) finite time stable are detailed in Corollary 6.1. Following the design steps in Step 1 and using Lemmas 6.1~6.2, we are able to prove this corollary.

Corollary 6.1: Consider i subsystems (6.2.20) where $i = 2, \dots, n-1$, if $s_{i+1} = 0$ and there exist a set of C^1 Lyapunov functions $V_i(x_1, \dots, x_i, \hat{\theta}_1 \dots \hat{\theta}_{i-1}) > 0$, and a set of C^0 virtual controller $x_{3d}, \dots, x_{i+1d}$ defined by

$$\begin{aligned}x_{3d} &= -s_2^{(1+Q_2)\gamma-1} \kappa_2(x_1, \dots, x_2, \hat{\theta}_2), \\ \dot{\hat{\theta}}_2 &= s_2^{(1+Q_2)\gamma}, \\ &\vdots\end{aligned}\tag{6.2.21a}$$

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$$\begin{aligned} x_{(i+1)d} &= -s_i^{(1+Q_i)\gamma-1} \kappa_i(x_1, \dots, x_i, \hat{\theta}_i), \\ \dot{\hat{\theta}}_i &= s_i^{(1+Q_i)\gamma}, \end{aligned} \quad (6.2.21b)$$

with, $\kappa_2(x_1, \dots, x_2, \hat{\theta}_2) > 0, \dots, \kappa_i(x_1, \dots, x_i, \hat{\theta}_i) > 0$ being C^1 , such that

$$\begin{aligned} \dot{V}_i(.) &\leq -\alpha_i V_i(.) - \beta_i V_i^\gamma(.) - (n-i+1)(s_1^{(1+Q_1)\gamma} + \dots \\ &\quad + s_i^{(1+Q_i)\gamma}) + s_i(x_{i+1} - x_{(i+1)d}) \end{aligned} \quad (6.2.22a)$$

$$\leq -\alpha_i V_i(.) - \beta_i V_i^\gamma(.), \quad (6.2.22b)$$

where $\alpha_1 > \dots > \alpha_i > 0$, $\beta_1 > \dots > \beta_i > 0$, then for any finite initial condition $x_i(0)$,

$\hat{\theta}_{i-1}(0)$, the system state x_i will converge to zero in finite time

$$T_i \leq \frac{1}{\alpha_i(1-\gamma)} \ln \left(\frac{\alpha_i V_i^{1-\gamma}(x_i(0)) + \beta_i}{\beta_i} \right). \quad (6.2.23)$$

Proof: We prove this corollary by induction. Consider the Lyapunov function candidate

$$V_i(.) = V_{i-1}(.) + W_i(x_1, \dots, x_i, \hat{\theta}_1, \dots, \hat{\theta}_{i-1}), \quad (6.2.24)$$

where
$$W_i(x_1, \dots, x_i, \hat{\theta}_1, \dots, \hat{\theta}_{i-1}) = \int_{x_{id}}^{x_i} (\chi_i^{\frac{1}{Q_i}} - x_i^{\frac{1}{Q_i}}) d\chi. \quad (6.2.25)$$

The Lyapunov function $V_i(.)$ thus defined has several useful properties collected in the following two propositions.

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Proposition 6.1: $W_i(x_1, \dots, x_i, \hat{\theta}_1, \dots, \hat{\theta}_{i-1})$ is C^1 . Moreover,

$$\begin{aligned} \frac{dW_i}{dx_i} &= x_i^{\frac{1}{Q_i}} - x_{id}^{\frac{1}{Q_i}} = s_i, \\ \frac{dW_i}{dx_l} &= (x_i - x_{id}) \frac{d(-x_{id}^{\frac{1}{Q_i}})}{dx_l}, \\ \frac{dW_i}{d\hat{\theta}_l} &= (x_i - x_{id}) \frac{d(-x_{id}^{\frac{1}{Q_i}})}{d\hat{\theta}_l}, \quad \forall l = 1, \dots, i-1. \end{aligned} \quad (6.2.26)$$

Proof: The proof is straightforward and therefore is omitted.

Proposition 6.2: $V_i(x_1, \dots, x_i, \hat{\theta}_1, \dots, \hat{\theta}_{i-1})$ is C^1 , positive definite and proper satisfying

$$V_i(x_1, \dots, x_i, \hat{\theta}_1, \dots, \hat{\theta}_{i-1}) \leq 2(s_1^{1+Q_1} + \dots + s_i^{1+Q_i}). \quad (6.2.27)$$

Proof: The proof is straightforward and therefore is omitted.

Without loss of generality, suppose at step $i-1$, there is a C^1 Lyapunov function such that

$$\begin{aligned} V_{i-1}(\cdot) &\leq - \left\{ (n-i+2) \left(\sum_{l=1}^{i-1} s_l^{(1+Q_l)\gamma} \right) + 2 \left(\sum_{l=1}^{i-1} \beta_l s_l^{(1+Q_l)\gamma} \right) \right. \\ &\quad \left. + 2 \left(\sum_{l=1}^{i-1} \alpha_l s_l^{(1+Q_l)} \right) \right\} + s_{i-1} (x_i - x_{(i)d}). \end{aligned} \quad (6.2.28)$$

Using Proposition 6.1, it is deduced from (6.2.28) that

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$$\begin{aligned}
 \dot{V}_i(.) &\leq -\left\{ (n-i+2) \left(\sum_{l=1}^{i-1} s_l^{(1+Q_l)\gamma} \right) + 2 \left(\sum_{l=1}^{i-1} \beta_l s_l^{(1+Q_l)\gamma} \right) \right. \\
 &\quad \left. + 2 \left(\sum_{l=1}^{i-1} \alpha_l s_l^{(1+Q_l)} \right) \right\} + s_{i-1} (x_i - x_{(i)d}) \\
 &\quad + s_i (x_{i+1} + \Delta_i(.)) + \sum_{l=1}^{i-1} \frac{dW_i}{dx_l} \dot{x}_l + \sum_{l=1}^{i-1} \frac{dW_i}{d\hat{\theta}_l} \dot{\hat{\theta}}_l - \tilde{\theta}_i \dot{\hat{\theta}}_i. \tag{6.2.29}
 \end{aligned}$$

Here, we estimate each term on the right side of (6.2.29). Using Lemmas 6.1~6.2, we have

$$|s_{i-1} (x_i - x_{id})| \leq \frac{|s_{i-1}|^{(1+Q_{i-1})\gamma}}{4} + 4 |s_i|^{(1+Q_i)\gamma} \hat{s}_i, \tag{6.2.30}$$

where $\hat{s}_i = |s_i|^{2Q_i - (1+Q_i)\gamma} |s_{i-1}|^{2 - (1+Q_{i-1})\gamma}$.

To continue the proof, we introduce three additional propositions.

Proposition 6.3: For $j = 1, \dots, n$, there are C^1 functions $\tilde{\sigma}_j(s_1, \dots, s_j) \geq 0$ such that

$$|\Delta_j(\mathbf{x}, t, u)| \leq (|s_1|^{(1+Q_1)\gamma-1} + \dots + |s_j|^{(1+Q_j)\gamma-1}) \tilde{\sigma}_j(.). \tag{6.2.31}$$

Proof: By Lemma 2, for $l = 2, \dots, j$,

$$|x_l| \leq \left| s_l + x_{ld}^{\frac{1}{Q_l}} \right|^{Q_l} \leq |s_l|^{Q_l} + |s_{l-1}|^{(1+Q_{l-1})\gamma-1} \kappa_{l-1}(.). \tag{6.2.32}$$

Using Assumption 6.1, we have

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$$\begin{aligned} |\Delta_j(\mathbf{x}, t, \mathbf{u})| &\leq \left(\sum_{l=1}^j |s_l|^{(1+Q_l)\gamma-1} |s_l|^{(1+Q_l)(1-\gamma)} + \sum_{l=2}^j |s_{l-1}|^{(1+Q_{l-1})\gamma-1} \kappa_{l-1}(\cdot) \right) \sigma_j(\cdot) \\ &\leq (|s_1|^{(1+Q_1)\gamma-1} + \dots + |s_j|^{(1+Q_j)\gamma-1}) \tilde{\sigma}_j(\cdot), \end{aligned} \quad (6.2.33)$$

where $\sigma_j(\cdot) \geq 0$ is a C^1 function.

Proposition 6.4: There is a C^1 function $\tau_i(\cdot) \geq 0$ such that

$$\left| \sum_{l=1}^{i-1} \frac{dW_l}{dx_l} \dot{x}_l \right| \leq \frac{1}{4} \left\{ |s_1|^{(1+Q_1)\gamma} + \dots + |s_{i-1}|^{(1+Q_{i-1})\gamma} \right\} + |s_i|^{(1+Q_i)\gamma} \tau_i(\cdot). \quad (6.2.34)$$

Proof: Using Propostion 6.3 and Lemma 6.2, for $l = 2, \dots, i-1$,

$$\begin{aligned} |\dot{x}_l| &\leq s_{l+1}^{Q_{l+1}} + |x_{(l+1)d}| + \left(\sum_{j=1}^l |s_j|^{(1+Q_j)\gamma-1} \right) \tilde{\sigma}_j(\cdot) \\ &\leq \left(\sum_{j=1}^{l+1} |s_j|^{(1+Q_j)\gamma-1} \right) \bar{\sigma}_l(\cdot), \end{aligned} \quad (6.2.35)$$

where $\bar{\sigma}_l(\cdot) \geq 0$ is a C^1 function.

With the form of x_{id} in mind, we can prove that

$$\left| \frac{d(-x_{id}^{\frac{1}{Q_i}})}{dx_l} \right| \leq |s_{i-1}|^{\frac{(1+Q_{i-1})\gamma-1}{Q_i}} C_{i-1}(\cdot). \quad (6.2.36)$$

Putting (6.2.35) and (6.2.36) together and using Lemma 6.1, we have

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$$\begin{aligned}
 \left| \frac{d(-x_{id}^{\frac{1}{Q_i}})}{dx_i} \dot{x}_i \right| &\leq |s_{i-1}|^{\frac{(1+Q_{i-1})\gamma-1}{Q_i}} C_{i-1}(\cdot) \left\{ \left(\sum_{j=1}^{l+1} |s_j|^{(1+Q_j)\gamma-1} \right) \bar{\sigma}_l(\cdot) \right\} \\
 &\leq \sum_{j=1}^{l+1} \left(|s_j|^{(1+Q_j)\gamma-Q_i} |s_j|^{\frac{(1+Q_{i-1})\gamma-1}{Q_i}+Q_i-2} + |s_{i-1}|^{\frac{(1+Q_{i-1})\gamma-1}{Q_i}+(1+Q_j)\gamma-2} \tilde{\tau}_j(\cdot) \right) \\
 &\leq \left(\sum_{j=1}^i |s_j|^{(1+Q_j)\gamma-Q_i} \right) \bar{\tau}_l(\cdot), \tag{6.2.37}
 \end{aligned}$$

where $\tilde{\tau}_j(\cdot) \geq 0$, $\bar{\tau}_l(\cdot) \geq 0$ are C^1 functions.

Now, we are ready to prove Proposition 6.4. Using (6.2.37) and proposition 6.1,

$$\begin{aligned}
 \left| \sum_{l=1}^{i-1} \frac{dW_i}{dx_l} \dot{x}_l \right| &\leq \sum_{l=1}^{i-1} |x_i - x_{id}| \left| \frac{d(-x_{id}^{\frac{1}{Q_i}})}{dx_i} \dot{x}_i \right| \\
 &\leq \left(2 |s_i|^{Q_i} \sum_{j=1}^i |s_j|^{(1+Q_j)\gamma-Q_i} \right) \sum_{l=1}^{i-1} \bar{\tau}_l(\cdot). \tag{6.2.38}
 \end{aligned}$$

Using Lemma 6.1 again, we arrive at (6.2.34), which conclude the proof.

Proposition 6.5: There is C^1 function $\rho_i(\cdot) \geq 0$ such that

$$\left| \sum_{l=1}^{i-1} \frac{dW_i}{d\hat{\theta}_l} \dot{\hat{\theta}}_l \right| \leq \frac{1}{4} \left\{ |s_1|^{(1+Q_1)\gamma} + \dots + |s_{i-1}|^{(1+Q_{i-1})\gamma} \right\} + |s_i|^{(1+Q_i)\gamma} \rho_i(\cdot). \tag{6.2.39}$$

Proof: With the form of $\hat{\theta}_l$ in mind and using (6.2.36), we have

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$$\begin{aligned} \left| \frac{d(-x_{id}^{\frac{1}{Q_i}})}{d\hat{\theta}_l} \dot{\hat{\theta}}_l \right| &\leq |s_{i-1}|^{\frac{(1+Q_{i-1})\gamma-1}{Q_i}} C_{i-1}(\cdot) \{s_i^{(1+Q_i)\gamma}\} \\ &\leq \left\{ |s_i|^{(1+Q_i)\gamma-Q_i} + |s_{i-1}|^{(1+Q_i)\gamma+\frac{(1+Q_{i-1})\gamma-1}{Q_i}} \right\} \bar{\rho}_l(\cdot), \end{aligned} \quad (6.2.40)$$

where $\bar{\rho}_l(\cdot) \geq 0$ is a C^1 function.

Using (6.2.40), Lemma 6.1, and Proposition 6.1,

$$\begin{aligned} \left| \sum_{l=1}^{i-1} \frac{dW_l}{d\hat{\theta}_l} \dot{\hat{\theta}}_l \right| &\leq 2|s_i|^{Q_i} \left\{ \sum_{l=1}^{i-1} (|s_l|^{(1+Q_l)\gamma-Q_l} + |s_{l-1}|^{(1+Q_l)\gamma+\frac{(1+Q_{l-1})\gamma-1}{Q_l}}) \bar{\rho}_l(\cdot) \right\} \\ &\leq \sum_{l=1}^{i-1} \left(\frac{1}{4} |s_l|^{(1+Q_l)\gamma} + |s_l|^{(1+Q_l)\gamma} C_l(\cdot) \right) + \sum_{l=1}^{i-1} |s_{l-1}|^{(1+Q_l)\gamma+\frac{(1+Q_{l-1})\gamma-1}{Q_l}} \bar{\rho}_l(\cdot) \\ &\quad \times |s_i|^{(1+Q_i)\gamma} |s_i|^{Q_i-(1+Q_i)\gamma}. \end{aligned} \quad (6.2.41)$$

Since $(1+Q_l) \geq (1+Q_i)$, we have inequality (6.2.39).

Using Proposition 6.3 and Lemma 6.1, we get

$$\begin{aligned} |s_i \Delta_i(\cdot)| &\leq |s_i| (|s_1|^{(1+Q_1)\gamma-1} + \dots + |s_i|^{(1+Q_i)\gamma-1}) \tilde{\sigma}_i(\cdot) \\ &\leq \sum_{l=1}^{i-1} \left(\frac{1}{4} |s_l|^{(1+Q_l)\gamma} + |s_l|^{(1+Q_l)\gamma} C_l(\cdot) \right) + |s_i|^{(1+Q_i)\gamma} \bar{\omega}_i(\cdot). \end{aligned} \quad (6.2.42)$$

Since $(1+Q_l) \geq (1+Q_i)$, we have

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$$|s_i \Delta_i(\cdot)| \leq \frac{1}{4} \left\{ |s_1|^{(1+Q_1)\gamma} + \dots + |s_{i-1}|^{(1+Q_{i-1})\gamma} \right\} + |s_i|^{(1+Q_i)\gamma} \bar{\varpi}_i(\cdot), \quad (6.2.43)$$

where $\bar{\varpi}_i(\cdot) \geq 0$, $\varpi_i(\cdot) \geq 0$ are C^1 functions.

Substituting (6.2.30), (6.2.34), (6.2.39), and (6.2.43) into (6.2.29), yields

$$\begin{aligned} \dot{V}_i(\cdot) \leq & - \left\{ (n-i+1) \left(\sum_{l=1}^{i-1} s_l^{(1+Q_l)\gamma} \right) + 2 \left(\sum_{l=1}^{i-1} \beta_l s_l^{(1+Q_l)\gamma} \right) \right. \\ & \left. + 2 \left(\sum_{l=1}^{i-1} \alpha_l s_l^{(1+Q_l)\gamma} \right) \right\} + s_i x_{i+1} + s_i^{(1+Q_i)\gamma} (4\hat{s}_i(\cdot) + \varpi_i(\cdot) \\ & + \tau_i(\cdot) + \rho_i(\cdot)) - \tilde{\theta}_i \hat{\theta}_i. \end{aligned} \quad (6.2.44)$$

Letting $\theta_i = 4\hat{s}_i(\cdot) + \varpi_i(\cdot) + \tau_i(\cdot) + \rho_i(\cdot)$, we get

$$\begin{aligned} \dot{V}_i(\cdot) \leq & - \left\{ (n-i+1) \left(\sum_{l=1}^{i-1} s_l^{(1+Q_l)\gamma} \right) + 2 \left(\sum_{l=1}^{i-1} \beta_l s_l^{(1+Q_l)\gamma} \right) + 2 \left(\sum_{l=1}^{i-1} \alpha_l s_l^{(1+Q_l)\gamma} \right) \right\} \\ & + s_i (x_{i+1} - x_{(i+1)d}) + s_i x_{(i+1)d} + s_i^{(1+Q_i)\gamma} \hat{\theta}_i + \tilde{\theta}_i (s_i^{(1+Q_i)\gamma} - \hat{\theta}_i). \end{aligned} \quad (6.2.45)$$

Obviously, the C^0 virtual controller

$$\begin{aligned} x_{(i+1)d} &= -s_i^{(1+Q_i)\gamma-1} \left((n-i+1) + 2\beta_i + 2\alpha_i s_i^{(Q_i+1)(1-\gamma)} + \sqrt{1 + \hat{\theta}_i^2} \right) \\ &:= -s_i^{(1+Q_i)\gamma-1} \kappa_i(x_1, \dots, x_i, \hat{\theta}_i) \end{aligned} \quad (6.2.46)$$

with $\kappa_i(x_1, \dots, x_i, \hat{\theta}_i) > 0$ and the updating law

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$$\hat{\theta}_i = s_i^{(1+Q_i)\gamma} \quad (6.2.47)$$

result in

$$\begin{aligned} \dot{V}_i(.) &\leq -\left\{ (n-i+1) \left(\sum_{l=1}^i s_l^{(1+Q_l)\gamma} \right) + 2 \left(\sum_{l=1}^i \beta_l s_l^{(1+Q_l)\gamma} \right) \right. \\ &\quad \left. + 2 \left(\sum_{l=1}^i \alpha_l s_l^{(1+Q_l)} \right) \right\} + s_i (x_{i+1} - x_{(i+1)d}) \\ &\leq -\alpha_i V_i^\gamma - \beta_i V_i + 2s_i |s_{i+1}|^{Q_i}. \end{aligned} \quad (6.2.48)$$

This completes the proof of the inductive step.

Step n : This is the final step and using the inductive argument above, we have

$$\begin{aligned} \dot{V}_n(.) &\leq -\left\{ \left(\sum_{l=1}^{n-1} s_l^{(1+Q_l)\gamma} \right) + 2 \left(\sum_{l=1}^{n-1} \beta_l s_l^{(1+Q_l)\gamma} \right) + 2 \left(\sum_{l=1}^{n-1} \alpha_l s_l^{(1+Q_l)} \right) \right\} \\ &\quad + s_{n-1} (x_n - x_{nd}) + s_n (u + f(.) + \Delta_n(.) + d_n(.)) \\ &\quad + \sum_{l=1}^{n-1} \frac{dW_n}{dx_l} \dot{x}_l + \sum_{l=1}^{n-1} \frac{dW_n}{d\hat{\theta}_l} \dot{\hat{\theta}}_l - \tilde{\theta}_n \dot{\hat{\theta}}_n \\ &\leq -\left\{ 2 \left(\sum_{l=1}^{n-1} \beta_l s_l^{(1+Q_l)\gamma} \right) + 2 \left(\sum_{l=1}^{n-1} \alpha_l s_l^{(1+Q_l)} \right) \right\} + s_n (u + f(.) \\ &\quad + d_n(.)) + s_n^{(1+Q_n)\gamma} (4\hat{s}_n(.) + \varpi_n(.) + \tau_n(.) + \rho_n(.)) - \tilde{\theta}_n \dot{\hat{\theta}}_n. \end{aligned} \quad (6.2.49)$$

The final controller

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$$u = -(|f(\cdot)| + d_{mx}(\cdot))\text{sign}(s_n) - s_n^{(1+Q_n)\gamma-1} (2\beta_n + 2\alpha_n s_n^{(1+Q_n)(1-\gamma)} + \sqrt{1 + \hat{\theta}_n^2}) \quad (6.2.50)$$

with updating law

$$\dot{\hat{\theta}}_n = s_n^{(1+Q_n)\gamma} \quad (6.2.51)$$

where $\theta_n = 4\hat{s}_n(\cdot) + \varpi_n(\cdot) + \tau_n(\cdot) + \rho_n(\cdot)$ results in

$$\dot{V}_n(\cdot) \leq -\alpha_n V_n^\gamma - \beta_n V_n. \quad (6.2.52)$$

Theorem 6.2: For system (6.2.1), if the virtual control laws and the actual control law are designed as (6.2.46) and (6.2.50), respectively, Q_n and γ are chosen such that the following inequalities hold

$$(1 + Q_n)\gamma > 1, \quad (6.2.53)$$

$$\frac{(1 + Q_{n-1})\gamma - 1}{Q_n} + Q_n > 2, \quad (6.2.54)$$

then the state x will reach zero in finite time $T \leq \sum_{l=1}^n T_l$ and the actual control signal $u(t)$

is bounded.

Proof: By Theorem 6.1, the sliding mode surfaces $s_i = 0$, $i = n, \dots, 1$, are reached sequentially in finite time T . We obtain $s_1 = x_1 = 0$, then $x_2 = \dot{x}_1 - \Delta_1(\cdot) = 0$. Similarly, $x_3 = \dot{x}_2 - \Delta_2(\cdot), \dots, x_n = \dot{x}_{n-1} - \Delta_{n-1}(\cdot) = 0$. We conclude that $x = 0$ in finite time T .

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From (6.2.21), since (6.2.53) holds and $Q_1 > \dots > Q_n$, then the virtual controllers $x_{2d}, \dots, x_{(n-1)d}$ are bounded, $u(t)$ is therefore bounded. From (6.2.37), since (6.2.54) holds,

then $\left| \frac{d(-x_{id}^{\frac{1}{Q_i}})}{dx_l} \dot{x}_l \right|$ for $l = 1, \dots, n-1$ are bounded, the virtual controllers $x_{2d}, \dots, x_{(n-1)d}$ are

bounded. Hence, we conclude that the actual control signal $u(t)$ is bounded.

6.3. Simulation Example

To illustrate the effectiveness of the proposed adaptive fast finite time control scheme, we consider the following SISO uncertain nonlinear system.

$$\dot{x}_1 = x_2 + \varepsilon_1 x_1$$

$$\dot{x}_2 = u + 0.2 \sin(t) + \varepsilon_2(x_1 + x_2) + \varepsilon_3 \cos(t),$$

where ε_1 , ε_2 , and ε_3 are unknown parameters. In the actually simulation, ε_1 , ε_2 , and ε_3 are chosen as 0.1, 0.1, and 0.2, respectively. The initial values of the states are chosen as $x_1(0) = 5$, $x_2(0) = -11$.

The sliding mode surfaces are designed as $s_1 = x_1^{19/17}$, $s_2 = x_2^{19/3} - x_{2d}^{19/3}$. Using (6.2.12),

(6.2.13) in Lemma 6.3, the virtual control law x_{2d} and the update law $\hat{\theta}_1$ can be designed

as

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$$x_{2d} = -s_1^{(1+17/19)(21/23)-1} (2 + 2(3 \times 10^{-6}) + 2(7 \times 10^{-7})s_1^{(1+17/19)(1-21/23)} + \sqrt{1 + \hat{\theta}_1^2}),$$

$$\dot{\hat{\theta}}_1 = s_1^{(1+17/19)(21/23)},$$

respectively. Using (6.2.50), we have

$$u = -(0.6 + |0.2 \sin(t)|) \text{sign}(s_2) - s_2^{(1+19/3)(21/23)-1} (2(2 \times 10^{-6})$$

$$+ 2(1 \times 10^{-7})s_2^{(1+3/19)(1-21/23)} + \sqrt{1 + \hat{\theta}_2^2}),$$

$$\dot{\hat{\theta}}_2 = s_2^{(1+3/19)(21/23)}.$$

Fig. 6.1 shows the system states $x_1(t)$ and $x_2(t)$, and Fig. 6.2 shows the phase plot of system states. It is seen that finite time convergence of system states has been achieved using the proposed control scheme.

6.4. Conclusions

In this chapter, using the fast non-smooth control Lyapunov function and the method of adding a power integrator merging with adaptive technique, we have successfully developed an adaptive fast finite time control scheme for a class of uncertain nonlinear systems. A simulation example is given in support of the proposed control scheme.

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Fig. 6.1. The system states.

Fig. 6.2. The phase plot of system states.

Chapter 7

SLIDING MODE CONTROL OF FUZZY DYNAMIC SYSTEMS

7.1. Introduction

Sliding mode control (SMC) systems have been studied extensively and received many applications [17, 23, 75-77]. For the design of a SMC system, a sliding mode surface is first defined to specify the desired system dynamics [3]. A sliding mode controller is then designed to drive the sliding variable to converge to zero. On the sliding mode surface, not only the desired system dynamics can be achieved, but the closed-loop system also exhibits invariant properties with respect to parameter variations and external disturbances.

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It is known that, in order to design a SMC system, only a rough mathematical model of the physical system is needed, i.e., the upper and lower bounds of the unknown parameters are required. Because of this advantage, SMC technique has been used to control complex nonlinear systems with their T-S fuzzy models [78, 79].

The characteristics of the T-S fuzzy modeling technique for complex nonlinear systems can be briefly described as follows: (1) The whole state-space is divided into subspaces, (2) a simple linear model is used to approximate the complex nonlinear system in each subspace, (3) the fuzzy inference rules are then used to fuzzify all the subsystem parameter matrices to obtain a global T-S fuzzy model [12, 13].

Based on the above characteristics of T-S fuzzy models, many approaches have been developed for the control of T-S fuzzy systems [10-18]. The control scheme proposed in [10] requires a common positive definite matrix for all subsystems. In fact, it is very difficult to obtain such a common positive definite matrix in practice. However, the control schemes developed in [11-18], using the linear matrix inequality (LMI) approaches, the fuzzy Lyapunov approaches, and the robust H_∞ approaches, do not require the determination of a common positive definite matrix as in [10]. Therefore, it is possible to use these schemes for practical applications.

In this chapter, we present a systematic SMC design methodology for a class of complex nonlinear systems with T-S fuzzy models. We first introduce the concept of the extreme fuzzy subsystems [80] where, in each subspace, a dominant membership function, together with its local subsystem, dominates the global T-S fuzzy system. A set of extreme matrices describing the upper bounds of all the interactions among the local

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fuzzy subsystems in the worst stability case are then derived. A sliding mode control law using these extreme matrices as the bounded information is then designed to guarantee the asymptotic stability and robustness of the closed-loop system. It is the extreme matrices that simplify the sliding mode control design for the complex nonlinear systems with T-S fuzzy models.

The chapter is organized as follows: Section 2 formulates a class of complex nonlinear systems with T-S fuzzy models. Section 3 discusses the concepts of the extreme fuzzy subsystems and the extreme matrices. Section 4 presents a robust SMC design for complex nonlinear systems with matched uncertainties. Section 5 presents numerical simulation examples to show the effectiveness of the control strategy. Section 6 gives conclusions.

7.2. Problem Formulation

Consider a class of complex nonlinear systems described by

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}) + \Delta\mathbf{A}(\mathbf{x}) + [\mathbf{B}(\mathbf{x}) + \Delta\mathbf{B}(\mathbf{x})]\mathbf{u}, \quad (7.2.1)$$

where $\mathbf{x} \in \mathfrak{R}^n$ is a measurable state variable vector, $\mathbf{u} \in \mathfrak{R}$ is a control input, $\mathbf{A}(\mathbf{x}) \in \mathfrak{R}^n$ and $\mathbf{B}(\mathbf{x}) \in \mathfrak{R}^n$ are known vectors whose elements are nonlinear functions of $\mathbf{x}$, $\Delta\mathbf{A}(\mathbf{x}) \in \mathfrak{R}^n$ and $\Delta\mathbf{B}(\mathbf{x}) \in \mathfrak{R}^n$ represent the system uncertainties.

Assumption 7.1: The uncertainties $\Delta\mathbf{A}(\mathbf{x})$ and $\Delta\mathbf{B}(\mathbf{x})$ satisfy the following matching conditions:

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$$\Delta A(x) = B(x)\Delta f_m(x) \quad \text{and} \quad \Delta B(x) = B(x)\Delta B_m(x) \quad (7.2.2)$$

$$\text{with} \quad \|\Delta f_m(x)\| \leq \rho_m \quad \text{and} \quad \|\Delta B_m(x)\| < \varepsilon_m < 1. \quad (7.2.3)$$

For further analysis, the nominal model of the complex nonlinear system (7.2.1) without uncertainties can be written as:

$$\dot{x} = A(x) + B(x)u. \quad (7.2.4)$$

It is well known that it is impossible to represent a complex nonlinear system by using a single linear model in the whole state-space. However, it is usually possible to approximate the nonlinear system by using linear models in a number of operating points. Then, we can obtain a global T-S fuzzy model, by using the fuzzy inference rule to fuzzify all local linear models in the whole state-space, to represent the complex nonlinear system [5].

Therefore, the T-S fuzzy model of the nominal complex nonlinear system (7.2.4) may be described by the following fuzzy inference rules [78, 80]:

$$\begin{aligned} R^i : \text{IF} \quad & z_1(t) \text{ is } F_1^i \text{ AND } \dots z_n(t) \text{ is } F_n^i \\ \text{THEN} \quad & \dot{x} = A^i x + B^i u, \quad \text{for } i = 1, \dots, m, \end{aligned} \quad (7.2.5)$$

where R^i represents the i^{th} fuzzy inference rule, m the number of inference rules, $F_j(z(t))$ ($j = 1, \dots, n$) the fuzzy sets, x the system state variable vector, u the system input. $A^i \in \mathbb{R}^{n \times n}$ and $B^i \in \mathbb{R}^{n \times 1}$ are i^{th} subsystem matrices, respectively. $z(t) = (z_1(t), \dots, z_n(t))^T$ represents some measurable system variables.

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Denote $\mu^i(z(t))$ as the normalized fuzzy membership function of the inferred fuzzy set F^i where

$$F^i(z(t)) = \prod_{j=1}^n F_j^i(z(t)), \quad (7.2.6)$$

$$\mu^i(z(t)) = \frac{F^i(z(t))}{\sum_{i=1}^m F^i(z(t))}, \quad (7.2.7)$$

and
$$\sum_{i=1}^m \mu^i(z(t)) = 1. \quad (7.2.8)$$

Using the fuzzy inference method with a centre-average defuzzifier, product inference, and singleton fuzzifier, we obtain a T-S fuzzy model of the nominal complex nonlinear system (7.2.4) as follows:

$$\dot{\mathbf{x}} = \mathbf{A}(\mu(z(t)))\mathbf{x} + \mathbf{B}(\mu(z(t)))u(t), \quad (7.2.9)$$

where
$$\mathbf{A}(\mu(z(t))) = \sum_{i=1}^m \mu^i(z(t))\mathbf{A}^i, \quad \mathbf{B}(\mu(z(t))) = \sum_{i=1}^m \mu^i(z(t))\mathbf{B}^i, \quad (7.2.10)$$

$$\mu(z(t)) = (\mu^1(z(t)), \mu^2(z(t)), \dots, \mu^m(z(t))). \quad (7.2.11)$$

For the further process, we have the following assumptions:

Assumption 7.2: Each local linear subsystem in (7.2.5) is controllable, i.e. the

controllability matrices $\mathbf{M}^i = [\mathbf{B}^i, \mathbf{A}^i\mathbf{B}^i, (\mathbf{A}^i)^2\mathbf{B}^i, \dots, (\mathbf{A}^i)^{n-1}\mathbf{B}^i]$, for $i = 1, \dots, m$, have full rank.

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Assumption 7.3: The global fuzzy model (6.1.9) is controllable in the state space, i.e. the

controllability matrix $M = [B(\mu(z(t))), A(\mu(z(t)))B(\mu(z(t))), \dots, A^{n-1}(\mu(z(t))) \times B(\mu(z(t)))]$ has full rank in the state space.

For convenience, we let the desired reference model share the same fuzzy sets with the fuzzy system in (7.2.5), where

$$\begin{aligned} R^i : IF \quad & z_1(t) \text{ is } F_1^i \text{ AND } \dots z_n(t) \text{ is } F_n^i \\ THEN \quad & \dot{x}_r = A_r^i x_r + B_r^i r, \quad \text{for } i = 1, 2, \dots, m, \end{aligned} \quad (7.2.12)$$

with $A_r^i \in \mathbb{R}^{n \times n}$, $\Re\{\lambda[A_r^i]\} < 0$, $B_r^i \in \mathbb{R}^{n \times 1}$, and $r \in \mathbb{R}$ is a known input of the desired reference model.

Equation (7.2.12) can also be expressed as the following T-S global model:

$$\dot{x}_r = A_r(\mu(z(t)))x_r + B_r(\mu(z(t)))r, \quad (7.2.13)$$

$$\text{where} \quad A_r(\mu(z(t))) = \sum_{i=1}^m \mu^i(z(t))A_r^i, \quad B_r(\mu(z(t))) = \sum_{i=1}^m \mu^i(z(t))B_r^i, \quad (7.2.14)$$

$$\mu(z(t)) = (\mu^1(z(t)), \mu^2(z(t)), \dots, \mu^m(z(t))). \quad (7.2.15)$$

For simplicity, let $\mu = \mu(z(t))$, $\mu^i = \mu^i(z(t))$, and $F^i = F^i(z(t))$.

In the following, we present the concepts of the extreme fuzzy subsystem and the extreme matrices, whereby the obtained extreme matrices are then used to design a

sliding mode controller to ensure that the output of a complex nonlinear system described by (7.2.1) can asymptotically converge to the output of the desired reference model (7.2.13).

7.3. Extreme Fuzzy Subsystems

Following [80], the state-space is decomposed into m subspaces according to the following rules:

$$\mathbb{S}_i = \{ \mathbf{x} \mid \mu^l \geq \mu^i, l = 1, 2, \dots, m, l \neq i \}, \quad \text{for } i = 1, 2, \dots, m, \quad (7.3.1)$$

where the characteristic function of $\mathbb{S}_i$ is defined by

$$\eta^i = \begin{cases} 1 & (\mathbf{x} \in \mathbb{S}_i), \\ 0 & (\mathbf{x} \notin \mathbb{S}_i). \end{cases} \quad (7.3.2)$$

Let G be the set of membership functions satisfying (7.2.8). Then, within each subspace $\mathbb{S}_i$, the T-S fuzzy system (7.2.9) can be denoted by

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}^i(\mu)\mathbf{x} + \tilde{\mathbf{B}}^i(\mu)u \quad (7.3.3a)$$

$$= [\mathbf{A}^i + \nabla \mathbf{A}^i(\mu)]\mathbf{x} + [\mathbf{B}^i + \nabla \mathbf{B}^i(\mu)]u, \quad (7.3.3b)$$

where $\nabla \mathbf{A}^i(\mu) = \sum_{l=1}^{m_l} \bar{\mu}^l \nabla \mathbf{A}^{il}, \quad \nabla \mathbf{B}^i(\mu) = \sum_{l=1}^{m_l} \bar{\mu}^l \nabla \mathbf{B}^{il}, \quad (7.3.3c)$

$$\nabla \mathbf{A}^{il} = \mathbf{A}^l - \mathbf{A}^i, \quad \nabla \mathbf{B}^{il} = \mathbf{B}^l - \mathbf{B}^i, \quad (7.3.3d)$$

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$$\bar{\mu}^l \in G, \quad l=1, \dots, m, \quad \bar{\mu}^l \neq \mu^i, \quad (7.3.3e)$$

$$\bar{\mu}^l \neq 0, \quad \forall x \in \mathbb{S}_i, \quad \text{for } i=1, 2, \dots, m. \quad (7.3.3f)$$

It should be noted that the i th subsystem in (7.3.3b) and the i th local subsystem in (7.2.5) are different. The i th local subsystem in (7.2.5) describes the linear approximation of the complex nonlinear system in the i th subspace decomposed according to the fuzzy inference rules. However, the i th subsystem in (7.3.3b) describes the dynamic behavior of the T-S fuzzy system (7.2.9) in the subspace $\mathbb{S}_i$ where $\mu^i \geq \mu^l$. It is easy to see from (7.3.3b) that the i th subsystem is dominated by parameter matrices A^i and B^i , and the interactions of the local fuzzy subsystems are represented by $\nabla A^i(\mu)$ and $\nabla B^i(\mu)$.

Using the characteristic function in (6.3.2), we denote the i^{th} subsystem (7.3.3b) as:

$$\dot{x}^i = [A^i + \nabla A^i(\mu)]x^i + [B^i + \nabla B^i(\mu)]w^i, \quad (7.3.4)$$

where $i=1, 2, \dots, m$, $x^i = \eta^i x$, and $w^i = \eta^i u$.

Similarly, using the rule (7.3.1), the characteristic function (7.3.2), and the decomposition principle (7.3.3), we denote the desired reference model in (7.2.13) as:

$$\dot{x}_r^i = [A_r^i + \nabla A_r^i(\mu)]x_r^i + [B_r^i + \nabla B_r^i(\mu)]v_r^i, \quad (7.3.5)$$

where $i=1, 2, \dots, m$, $x_r^i = \eta^i x_r$, and $v_r^i = \eta^i r$.

It is seen that $\nabla A^i(\mu)$, $\nabla B^i(\mu)$ and $\nabla A_r^i(\mu)$, $\nabla B_r^i(\mu)$ in (7.3.4) and (7.3.5), respectively, are time varying, and in order to use SMC method to design a controller, some upper

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bound information about $\nabla A^i(\mu)$, $\nabla B^i(\mu)$, $\nabla A_r^i(\mu)$, and $\nabla B_r^i(\mu)$ are needed to be known. For this purpose, we define the following upper bounds as in [81]:

$$\begin{aligned} [\nabla \bar{A}^i(\mu)]^T [\nabla \bar{A}^i(\mu)] &\leq (E^i)^T (E^i) = [E^{i1} \quad E^{i2}]^T [E^{i1} \quad E^{i2}], \\ \nabla \bar{A}^i(\mu) &= [\nabla A^i(\mu) \quad \nabla B^i(\mu)], \end{aligned} \quad (7.3.6)$$

$$\begin{aligned} [\nabla \bar{A}_r^i(\mu)]^T [\nabla \bar{A}_r^i(\mu)] &\leq (E_r^i)^T (E_r^i) = [E_r^{i1} \quad E_r^{i2}]^T [E_r^{i1} \quad E_r^{i2}], \\ \nabla \bar{A}_r^i(\mu) &= [\nabla A_r^i(\mu) \quad \nabla B_r^i(\mu)]. \end{aligned} \quad (7.3.7)$$

In (7.3.6) and (7.3.7), the upper bound matrices E^{i1} , E^{i2} , E_r^{i1} , and E_r^{i2} are called the extreme matrices. Because these upper bounds are defined by the inequalities (7.3.6) and (7.3.7), the choice of the extreme matrices is not unique [82-84].

If we replace $\nabla A^i(\mu)$ by E^{i1} , $\nabla B^i(\mu)$ by E^{i2} , $\nabla A_r^i(\mu)$ by E_r^{i1} , $\nabla B_r^i(\mu)$ by E_r^{i2} , we can then express (7.3.4) and (7.3.5) as follows:

$$\dot{x}^i = [A^i + E^{i1}] x^i + [B^i + E^{i2}] w^i, \quad \dot{x}_r^i = [A_r^i + E_r^{i1}] x_r^i + [B_r^i + E_r^{i2}] v_r^i. \quad (7.3.8)$$

In view of [16], we define the above dynamic equations as the extreme fuzzy subsystems of (7.3.4) and (7.3.5).

In the following section, we will use the extreme matrices E^{i1} , E^{i2} , E_r^{i1} , and E_r^{i2} in (7.3.6) and (7.3.7) to design a sliding mode controller to guarantee that the output tracking error can asymptotically converge to zero and strong robustness with respect to the matched uncertainties can be guaranteed.

7.4. Design of A Robust Sliding Mode Controller

In this section, we consider the SMC design for a class of complex nonlinear systems with matched uncertainties. Using (7.2.2) and (7.2.9) in (7.2.1), we obtain:

$$\dot{\mathbf{x}} = \mathbf{A}(\mu)\mathbf{x} + \mathbf{B}(\mu)\left[\left[I + \Delta\mathbf{B}_m(\mathbf{x})\right]u + \Delta\mathbf{f}_m(\mathbf{x})\right]. \quad (7.4.1)$$

Defining the output tracking error $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_r$, and an error vector

$$\tilde{\mathbf{x}} = \left[\tilde{x}, \dot{\tilde{x}}, \dots, \tilde{x}^{(n-1)}\right]^T, \quad (7.4.2)$$

where $\tilde{x}^{(k)} = x^{(k)} - x_r^{(k)}$, ($k = 0, 1, \dots, n-1$), and using (7.2.13), (7.4.1) and (7.4.2), we obtain the error dynamics as follows:

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}(\mu)\mathbf{x} + \mathbf{B}(\mu)\left[\left[1 + \Delta\mathbf{B}_m(\mathbf{x})\right]u + \Delta\mathbf{f}_m(\mathbf{x})\right] - \mathbf{A}_r(\mu)\mathbf{x}_r - \mathbf{B}_r(\mu)r. \quad (7.4.3)$$

For further analysis, a sliding variable $s(\tilde{\mathbf{x}})$ is defined as:

$$s(\tilde{\mathbf{x}}) = \mathbf{c}^T \tilde{\mathbf{x}}, \quad (7.4.4)$$

where $\mathbf{c} = [c_1, \dots, c_n]^T$ is chosen such that zeros of the polynomial $\mathbf{c}^T \tilde{\mathbf{x}}$ are in the left half of the complex plane. For convenience, we let $c_n > 0$.

Differentiating the sliding variable (7.4.4) with respect to time, we have

$$\dot{s}(\tilde{\mathbf{x}}) = \mathbf{c}^T \left\{ \mathbf{A}(\mu)\mathbf{x} + \mathbf{B}(\mu)\left[\left[1 + \Delta\mathbf{B}_m(\mathbf{x})\right]u + \Delta\mathbf{f}_m(\mathbf{x})\right] - \mathbf{A}_r(\mu)\mathbf{x}_r - \mathbf{B}_r(\mu)r \right\}. \quad (7.4.5)$$

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If we replace $A(\mu)$ by $A^i + \nabla A^i(\mu)$, $B(\mu)$ by $B^i + \nabla B^i(\mu)$, A_r by $A_r^i + \nabla A_r^i(\mu)$, B_r by $B_r^i + \nabla B_r^i(\mu)$, we can then express $\dot{s}(\tilde{x})$ in the subspace $\mathbb{S}_i$ as follows:

$$\begin{aligned} \dot{s}(\tilde{x}^i) = & c^T \left\{ \left[A^i + \nabla A^i(\mu) \right] x^i + \left[B^i + \nabla B^i(\mu) \right] \left[[1 + \Delta B_m(x)] w^i \right. \right. \\ & \left. \left. + \Delta f_m(x^i) \right] - \left[A_r^i + \nabla A_r^i(\mu) \right] x_r^i - \left[B_r^i + \nabla B_r^i(\mu) \right] r^i \right\}. \end{aligned} \quad (7.4.6)$$

The results on the design of the sliding mode controller and the analysis of the error convergence is summarized in the following theorem.

Theorem 7.1: Consider the nonlinear system (7.4.1). If the control input for the i^{th} subsystem in $\mathbb{S}_i$ is designed as:

$$w^i = k^i(x^i) - \frac{s(\tilde{x}^i)}{\|s(\tilde{x}^i)\| \|c^T B^i (1 - \varepsilon_m)\|} \{\psi_1^i + \psi_2^i\}, \quad (7.4.7)$$

where $k^i(x^i) = -(c^T B^i)^{-1} (c^T A^i x^i - c^T A_r^i x_r^i - c^T B_r^i r^i), \quad (7.4.8)$

$$\psi_1^i = \left\{ \|c^T E^{i1} x^i\| + \|c^T B^i \varepsilon_m k^i(x^i)\| \right\}, \quad (7.4.9)$$

$$\begin{aligned} \psi_2^i = & \left\{ \|c^T E^{i2} (1 + \varepsilon_m) k^i(x^i)\| + \|c^T B^i \rho_m\| \right. \\ & \left. + \|c^T E^{i2} \rho_m\| + \|c^T E_r^{i1} x_r^i\| + \|c^T E_r^{i2} r^i\| \right\}, \end{aligned} \quad (7.4.10)$$

then the output tracking error $\tilde{x} = x - x_r$ will asymptotically converge to zero.

Proof: Consider the following Lyapunov function for the i^{th} subsystem in $\mathbb{S}_i$:

$$V^i = 0.5 s^T(\tilde{\mathbf{x}}^i) s(\tilde{\mathbf{x}}^i) \quad (7.4.11)$$

and differentiating V^i with respect to time, we have

$$\begin{aligned} \dot{V}^i &= s^T(\tilde{\mathbf{x}}^i) \dot{s}(\tilde{\mathbf{x}}^i) \\ &= s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T [\dot{\mathbf{x}}^i - \dot{\mathbf{x}}_r^i] \\ &= s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \left\{ \left[\mathbf{A}^i + \nabla \mathbf{A}^i(\mu) \right] \mathbf{x}^i + \left[\mathbf{B}^i + \nabla \mathbf{B}^i(\mu) \right] \left[1 + \Delta B_m(\mathbf{x}^i) \right] \mathbf{w}^i \right. \\ &\quad \left. + \Delta f_m(\mathbf{x}^i) \right\} - \left[\mathbf{A}_r^i + \nabla \mathbf{A}_r^i(\mu) \right] \mathbf{x}_r^i - \left[\mathbf{B}_r^i + \nabla \mathbf{B}_r^i(\mu) \right] \mathbf{r}^i \} \\ &= s^T(\tilde{\mathbf{x}}^i) \left\{ \mathbf{c}^T \nabla \mathbf{A}^i(\mu) \mathbf{x}^i - \mathbf{c}^T \mathbf{B}^i k^i(\mathbf{x}^i) + \mathbf{c}^T \left[\mathbf{B}^i + \nabla \mathbf{B}^i(\mu) \right] \left[1 + \Delta B_m(\mathbf{x}^i) \right] \mathbf{w}^i \right. \\ &\quad \left. + \mathbf{c}^T \left[\mathbf{B}^i + \nabla \mathbf{B}^i(\mu) \right] \Delta f_m(\mathbf{x}^i) - \mathbf{c}^T \nabla \mathbf{A}_r^i(\mu) \mathbf{x}_r^i - \mathbf{c}^T \nabla \mathbf{B}_r^i(\mu) \mathbf{r}^i \right\} \\ &= s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \nabla \mathbf{A}^i(\mu) \mathbf{x}^i + s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \mathbf{B}^i \Delta B_m(\mathbf{x}^i) k^i(\mathbf{x}^i) \\ &\quad + s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \nabla \mathbf{B}^i(\mu) \left[1 + \Delta B_m(\mathbf{x}^i) \right] k^i(\mathbf{x}^i) \\ &\quad - \frac{\|s(\tilde{\mathbf{x}}^i)\| \mathbf{c}^T \left[\mathbf{B}^i + \nabla \mathbf{B}^i(\mu) \right] \left[1 + \Delta B_m(\mathbf{x}^i) \right]}{\|\mathbf{c}^T \mathbf{B}^i (1 - \varepsilon_m)\|} \{ \psi_1^i + \psi_2^i \} \\ &\quad + s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \left[\mathbf{B}^i + \nabla \mathbf{B}^i(\mu) \right] \Delta f_m(\mathbf{x}^i) \\ &\quad - s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \nabla \mathbf{A}_r^i(\mu) \mathbf{x}_r^i - s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \nabla \mathbf{B}_r^i(\mu) \mathbf{r}^i \end{aligned}$$

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$$\begin{aligned}
&\leq s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \mathbf{E}^i \mathbf{x}^i + s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \mathbf{B}^i \Delta B_m(\mathbf{x}^i) k^i(\mathbf{x}^i) \\
&\quad + s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \nabla \mathbf{B}^i(\mu) [1 + \Delta B_m(\mathbf{x}^i)] k^i(\mathbf{x}^i) \\
&\quad - \frac{\|s(\tilde{\mathbf{x}}^i)\| \mathbf{c}^T [\mathbf{B}^i + \nabla \mathbf{B}^i(\mu)] [1 + \Delta B_m(\mathbf{x}^i)]}{\|\mathbf{c}^T \mathbf{B}^i (1 - \varepsilon_m)\|} \{\psi_1^i + \psi_2^i\} \\
&\quad + s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T [\mathbf{B}^i + \nabla \mathbf{B}^i(\mu)] \Delta f_m(\mathbf{x}^i) \\
&\quad - s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \nabla A_r^i(\mu) \mathbf{x}_r^i - s^T(\tilde{\mathbf{x}}^i) \mathbf{c}^T \nabla \mathbf{B}_r^i(\mu) \mathbf{r}^i \\
&\leq \|s(\tilde{\mathbf{x}}^i)\| \|\mathbf{c}^T \mathbf{E}^i \mathbf{x}^i\| + \|s(\tilde{\mathbf{x}}^i)\| \|\mathbf{c}^T \mathbf{B}^i \varepsilon_m k^i(\mathbf{x}^i)\| - \|s(\tilde{\mathbf{x}}^i)\| \phi^i \{\psi_1^i\} \\
&\quad + \|s(\tilde{\mathbf{x}}^i)\| \|\mathbf{c}^T \nabla \mathbf{B}^i(\mu) (1 + \varepsilon_m) k^i(\mathbf{x}^i)\| + \|s(\tilde{\mathbf{x}}^i)\| \|\mathbf{c}^T [\mathbf{B}^i + \nabla \mathbf{B}^i(\mu)] \rho_m\| \\
&\quad + \|s(\tilde{\mathbf{x}}^i)\| \|\mathbf{c}^T \nabla A_r^i(\mu) \mathbf{x}_r^i\| + \|s(\tilde{\mathbf{x}}^i)\| \|\mathbf{c}^T \nabla \mathbf{B}_r^i(\mu) \mathbf{r}^i\| - \|s(\tilde{\mathbf{x}}^i)\| \phi^i \{\psi_2^i\} \\
&\leq \|s(\tilde{\mathbf{x}}^i)\| (1 - \phi^i) \left\{ \|\mathbf{c}^T \mathbf{E}^i \mathbf{x}^i\| + \|\mathbf{c}^T \mathbf{B}^i \varepsilon_m k^i(\mathbf{x}^i)\| \right\} \\
&< 0 \quad \text{for } s(\tilde{\mathbf{x}}^i) \neq 0, \tag{7.4.12}
\end{aligned}$$

where

$$\phi^i = \frac{(\|\mathbf{c}^T \mathbf{B}^i\| + \|\mathbf{c}^T \nabla \mathbf{B}^i(\mu)\|) (1 + \|\Delta B_m(\mathbf{x}^i)\|)}{\|\mathbf{c}^T \mathbf{B}^i (1 - \varepsilon_m)\|} > 1. \tag{7.4.13}$$

Equation (7.4.12) shows that the sliding mode controller (7.4.7) can drive the sliding variable $s(\tilde{\mathbf{x}}^i)$ to converge to zero, and then, on the sliding mode surface, $s(\tilde{\mathbf{x}}^i) = 0$, the output tracking error can asymptotically converge to zero.

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Remark 7.1: The control law (7.4.7) is discontinuous across the sliding mode surface $s(\tilde{\mathbf{x}}^i) = 0$, however, which may excite undesired high frequency dynamics [1]. To eliminate the chattering, we use the following boundary layer control law in place of the discontinuous control law in expression (7.4.7)

$$\mathbf{w}^i = \begin{cases} \mathbf{k}^i(\mathbf{x}^i) - \frac{s(\tilde{\mathbf{x}}^i)}{\|s(\tilde{\mathbf{x}}^i)\| \| \mathbf{c}^T \mathbf{B}^i (1 - \varepsilon_m) \|} \{ \psi_1^i + \psi_2^i \} & \text{if } \|s(\tilde{\mathbf{x}}^i)\| \geq \delta, \\ \mathbf{k}^i(\mathbf{x}^i) - \frac{s(\tilde{\mathbf{x}}^i)}{\delta^2 \| \mathbf{c}^T \mathbf{B}^i (1 - \varepsilon_m) \|} \{ \psi_1^i + \psi_2^i \} & \text{if } \|s(\tilde{\mathbf{x}}^i)\| < \delta, \end{cases} \quad (7.4.14)$$

where δ is a positive number [34, 58].

7.5. Simulation Examples

In this section, we illustrate the effectiveness of the proposed control scheme by two simulation examples.

Example 7.1: In this example, we consider the tracking control of a one-link robotic manipulator. The dynamic equation of the one-link robotic manipulator is given by [52, 85]

$$ml^2\ddot{\theta} + d\dot{\theta} + mgl \cos(\theta) + \Delta f_m = [1 + \Delta B_m]u, \quad (7.5.1)$$

where the link is of length l and mass m , θ is the angular position with initial values $\theta(0) = 17.2^\circ$ and $\dot{\theta}(0) = 0$, and Δf_m and ΔB_m represent the matched uncertainties. For

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simplicity, the parameters in (7.5.1) are chosen as $l = 1\text{m}$, $m = 1\text{kg}$, $d = 1\text{kgm}^2/\text{s}$, and $g = 9.8\text{m/s}^2$. To illustrate the robustness of the proposed control scheme, the matched uncertainties are assumed to be

$$\Delta f_m = \sin(t) + 0.5 \sin(200\pi t), \quad (7.5.2a)$$

$$\Delta B_m = 0.2 \cos(t) + 0.1 \sin(200\pi t). \quad (7.5.2b)$$

Since the matched uncertainties are chosen as in (7.5.2), it follows from (7.2.3) that ρ_m and ε_m can be set to $\rho_m = 2 \geq 1.5$ and $\varepsilon_m = 0.5 \geq 0.3$.

In this simulation, the local fuzzy models, which represent the nominal model of the dynamical equation (7.5.1), in the range of $[-90^\circ, 90^\circ]$, are obtained as

$$R^1: \text{IF } x_1 \text{ is about } -90^\circ$$

$$\text{THEN } \dot{x} = A^1 x + B^1(u - u^1), \quad (7.5.3a)$$

$$R^2: \text{IF } x_1 \text{ is about } -60^\circ$$

$$\text{THEN } \dot{x} = A^2 x + B^2(u - u^2), \quad (7.5.3b)$$

$$R^3: \text{IF } x_1 \text{ is about } 0^\circ$$

$$\text{THEN } \dot{x} = A^3 x + B^3(u - u^3), \quad (7.5.3c)$$

$$R^4: \text{IF } x_1 \text{ is about } 60^\circ$$

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$$\text{THEN } \dot{\mathbf{x}} = \mathbf{A}^4 \mathbf{x} + \mathbf{B}^4 (u - u^4), \quad (7.5.3d)$$

R^5 : IF x_1 is about 90°

$$\text{THEN } \dot{\mathbf{x}} = \mathbf{A}^5 \mathbf{x} + \mathbf{B}^5 (u - u^5), \quad (7.5.3e)$$

where $x_1 = \theta$, $x_2 = \dot{\theta}$, $\mathbf{x} = [x_1, x_2]^T$,

$$\mathbf{A}^1 = \begin{bmatrix} 0 & 1 \\ -9.81 & -1 \end{bmatrix}, \quad \mathbf{B}^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u^1 = 0,$$

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ -8.5 & -1 \end{bmatrix}, \quad \mathbf{B}^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u^2 = 4.9,$$

$$\mathbf{A}^3 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B}^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u^3 = 9.81,$$

$$\mathbf{A}^4 = \begin{bmatrix} 0 & 1 \\ 8.5 & -1 \end{bmatrix}, \quad \mathbf{B}^4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u^4 = 4.9,$$

$$\mathbf{A}^5 = \begin{bmatrix} 0 & 1 \\ 9.81 & -1 \end{bmatrix}, \quad \mathbf{B}^5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u^5 = 0.$$

And the fuzzy sets for x_1 are chosen as in Fig. 7.1.

The desired reference model is defined as

$$\dot{\mathbf{x}}_r = \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix} \mathbf{x}_r + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r,$$

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where $\mathbf{x}_r(0) = [0 \ 0]^T$ and r is a periodic rectangular signal as in Fig. 7.2.

In view of Fig. 1, it is obvious that, based on (6.3.1), the following state space regions

$$x_1 = [-90^\circ, -75^\circ], [-75^\circ, -30^\circ], [-30^\circ, 30^\circ], [30^\circ, 75^\circ], [75^\circ, 90^\circ],$$

can be designed as $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \mathbb{S}_4$, and $\mathbb{S}_5$, respectively.

Further, it can be shown that the following extreme matrices,

$$\mathbf{E}^{11} = 0.5|\mathbf{A}^2 - \mathbf{A}^1|, \quad \mathbf{E}^{21} = 0.5|\mathbf{A}^3 - \mathbf{A}^2|,$$

$$\mathbf{E}^{31} = 0.5|\mathbf{A}^4 - \mathbf{A}^3|, \quad \mathbf{E}^{41} = 0.5|\mathbf{A}^5 - \mathbf{A}^4|, \quad \mathbf{E}^{51} = 0.5|\mathbf{A}^5 - \mathbf{A}^4|,$$

satisfy the inequality (7.3.6). In this example, the sliding variable is defined as:

$$s(\tilde{\mathbf{x}}) = [10 \ 1]\tilde{\mathbf{x}},$$

Fig. 7.3(a) and Fig. 7.3(b) show the system output tracking and the control input signal. It is seen that the effects of system uncertainties are eliminated, and good tracking performance is achieved. To eliminate chattering, we implement the boundary layer control law (7.4.14) with $\delta = 0.1$. The good system performance is shown in Fig. 7.4, where the chattering is eliminated.

Example 7.2: In this example, we compare the proposed control scheme with the fuzzy H_∞ control scheme in [19] for the control of an inverted pendulum. The dynamic equation of the inverted pendulum is given by [25]

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$$\ddot{\theta} = \left[\frac{g \sin(\theta) - aml\dot{\theta}^2 \sin(2\theta)/2}{4l/3 - aml \cos^2(\theta)} \right] + \Delta f_m - \left[\frac{a \cos(\theta)}{4l/3 - aml \cos^2(\theta)} + \Delta B_m \right] u \quad (7.5.4)$$

where the pendulum is of length $2l$ and mass M , θ is the angular position with initial values $\theta(0) = 85^\circ$ and $\dot{\theta}(0) = 0$, $a = 1/(m + M)$, and Δf_m and ΔB_m represent the matched uncertainties. In this simulation, the parameters in (7.5.4) are chosen as $l = 0.5\text{m}$, $m = 2\text{kg}$, $M = 8\text{kg}$, and $g = 9.8\text{kgm/s}^2$. The system uncertainties are assumed to satisfy the following matching conditions:

$$\Delta f_m = 10 \sin(t) + 5 \sin(200\pi t), \quad (7.5.5a)$$

$$\Delta B_m = 0.2 \cos(t) + 0.1 \sin(200\pi t), \quad (7.5.5b)$$

with $\rho_m = 20$ and $\varepsilon_m = 0.5$.

In this simulation, the local fuzzy models, which represent the nominal model of the inverted pendulum (7.5.4), in the range of $[-90^\circ, 90^\circ]$, are obtained as

R^3 : IF x_1 is about -60°

$$\text{THEN } \dot{x} = A^1 x + B^1 u, \quad (7.5.6a)$$

R^3 : IF x_1 is about 0°

$$\text{THEN } \dot{x} = A^2 x + B^2 u, \quad (7.5.6b)$$

R^3 : IF x_1 is about 60°

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$$\text{THEN } \dot{\mathbf{x}} = \mathbf{A}^3 \mathbf{x} + \mathbf{B}^3 u, \quad (7.5.6c)$$

where $x_1 = \theta$, $x_2 = \dot{\theta}$, $\mathbf{x} = [x_1, x_2]^T$,

$$\mathbf{A}^1 = \begin{bmatrix} 0 & 1 \\ 5.8512 & 0 \end{bmatrix}, \quad \mathbf{B}^1 = \begin{bmatrix} 0 \\ -0.0779 \end{bmatrix},$$

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix}, \quad \mathbf{B}^2 = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix},$$

$$\mathbf{A}^3 = \begin{bmatrix} 0 & 1 \\ 5.8512 & 0 \end{bmatrix}, \quad \mathbf{B}^3 = \begin{bmatrix} 0 \\ -0.0779 \end{bmatrix}.$$

And the fuzzy sets for x_1 are chosen as in Fig. 5.

Similarly, in view of Fig. 7.5, based on (7.3.1), the following state space regions

$$x_1 = [-120, -30], [-30, 30], [30, 120],$$

can be designed as $\mathbb{S}_1$, $\mathbb{S}_2$, and $\mathbb{S}_3$, respectively.

It is easy to show that the following selected extreme matrices

$$\mathbf{E}^{11} = \mathbf{E}^{21} = \mathbf{E}^{31} = |\mathbf{A}^2 - \mathbf{A}^1|, \quad \mathbf{E}^{12} = \mathbf{E}^{22} = \mathbf{E}^{32} = |\mathbf{B}^2 - \mathbf{B}^1|,$$

satisfy the inequality in (7.3.6).

In this example, the sliding variable is defined as:

$$s(\mathbf{x}) = [25 \quad 1] \mathbf{x} = 0.$$

It is noted that, as in [19], the fuzzy H_∞ control law for the inverted pendulum can be designed as

$$u = -\mathbf{B}(\mu)^T P \mathbf{x}$$

$$= \sum_{i=1}^3 \mu^i K^i \mathbf{x}, \quad K^i = -(\mathbf{B}^i)^T P,$$

where $P = \begin{bmatrix} 1884800 & 128700 \\ 128700 & 10600 \end{bmatrix}$, is the solution of the Riccati equation in (18) of [19].

Fig. 7.6(a) and Fig. 7.6(b) show the comparisons of system outputs and the control input signals of the proposed control scheme and the fuzzy H_∞ control scheme. It is observed that the proposed control law not only provides faster converging rate but also reduces the control amplitude compared with the fuzzy H_∞ control law. Fig. 7.7(a) and Fig. 7.7(b) show the comparisons of system outputs and the control input signals of the proposed control scheme and the fuzzy H_∞ control scheme, where the boundary layer controller is used to eliminate the chattering ($\delta = 0.5$). It is seen that the chattering is eliminated and the amplitude of the proposed control input signal is greatly reduced.

7.6. Conclusions

A robust SMC design methodology has been proposed for a class of complex nonlinear systems with T-S fuzzy models. It has been shown that the concept of extreme matrices describing the bounded information of interactions has been used to design a sliding mode controller, and several advantages of the proposed control scheme have been seen

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from the theoretical analysis and the simulation results. These include strong robustness property with respect to system uncertainties and fast convergence of the output tracking error on the sliding mode surface. It is worth pointing out that the proposed control scheme not only simplifies the controller design, but also reduces the control amplitude compared with the fuzzy H_∞ control scheme.

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Fig. 7.1. Fuzzy sets of x_1 .

Fig. 7.2. The control input.

Fig. 7.3(a). The output tracking of the link.

Fig. 7.3(b). The control input.

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Fig. 7.4(a). The output tracking of the link.

Fig. 7.4(b). The control input.

Fig. 7.5. Fuzzy sets of x_1 .

Fig. 7.6(a). The output response.

Fig. 7.6(b). The control input.

Fig. 7.7(a). The output response.

Fig. 7.7(b). The control input.

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Chapter 8

TERMINAL SLIDING MODE CONTROL OF MIMO T-S FUZZY SYSTEMS

8.1. Introduction

Fuzzy logic control has been proven to be a successful control approach for a class of complex nonlinear systems, see [12, 86, 87] for example. In the conventional fuzzy logic control design, T-S fuzzy model is used to model the complex nonlinear system first, so that a local controller can be designed for each local fuzzy subsystem. The global fuzzy logic control law is then be obtained by aggregating all the local controllers using the fuzzy inference law.

Recently, many new stability analyses and controller design results for T-S fuzzy systems have appeared in the fuzzy control literature [88-90]. The system states can be proven to

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be asymptotically stable, that is, the convergence in such systems can at best be exponentially with infinite settling time. It is therefore interesting to design controller for T-S fuzzy systems with finite time convergence property. Comparing to the infinite time controller, due to the non-Lipschitz property, the finite time controller is able to not only provide faster convergence rate, but also perform better in the presence of system uncertainties and disturbances [91, 92].

Terminal sliding mode (TSM) control was developed in [11, 36, 37] to achieve finite time convergence of the system dynamics on the sliding mode surface. In [11, 37], the hierarchical terminal sliding mode structure was proposed for SISO nonlinear system such that the sliding variables can converge to zero sequentially and then, the system states can reach the system origin in finite time. A new TSM surface was proposed in [36] for MIMO linear system. It was shown that by suitably designing the parameter matrices of the sliding variable, on the sliding mode surface, the system states converge to zero in finite time.

In this chapter, we present a systematic TSM control design for MIMO T-S fuzzy system. Based on the concept of the extreme matrices [80, 93] introduced in Chapter 6, a TSM control law and a set of TSM surfaces using these extreme matrices as the bounded information are designed to guarantee finite time convergence of the closed-loop system. It is the extreme matrices that simplify the TSM control design for MIMO T-S fuzzy system.

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To our best knowledge, there is little work dealing with finite time control of T-S fuzzy systems in the literature at present stage. The proposed control scheme expands the class of nonlinear systems that can be handled using TSM technique.

The rest of the chapter is organized as follows: Section 8.2 introduces the MIMO T-S fuzzy system. Section 8.3 discusses the concept about the extreme matrices. Section 8.4 presents TSM control design for MIMO T-S fuzzy system. Section 8.5 presents a numerical simulation example in support of the developed control scheme. Section 8.6 gives conclusions.

8.2. Problem Formulation

Consider the MIMO T-S Fuzzy system with both fuzzy inference rules and local analytic linear models as follows:

$$R^i : IF \ x_1 \text{ is } F_1^i \text{ AND } \dots x_n \text{ is } F_n^i$$

$$THEN \ \dot{x}_1 = A_{11}^i x_1 + A_{12}^i x_2$$

$$\dot{x}_2 = A_{21}^i x_1 + A_{22}^i x_2 + B_2 u, \quad \text{for } i = 1, \dots, m, \quad (8.2.1)$$

where R^i represents the i^{th} fuzzy inference rule, m is the number of inference rules, F_j ($j = 1, \dots, n$) are fuzzy sets, $x_1 \in \mathbb{R}^{n-p}$ and $x_2 \in \mathbb{R}^p$ are the system state variable vector, $u \in \mathbb{R}^p$ is the control input, $A_{11}^i \in \mathbb{R}^{(n-p) \times (n-p)}$, $A_{12}^i \in \mathbb{R}^{(n-p) \times p}$, $A_{21}^i \in \mathbb{R}^{p \times (n-p)}$,

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$A_{22}^i \in \Re^{p \times p}$, and $B_2 \in \Re^{p \times p}$ are the system matrices, B_2 is non-singular, and $x = (x_1, \dots, x_n)^T$.

Denote μ^i as the normalized fuzzy membership function

$$F^i = \prod_{j=1}^n F_j^i, \mu^i = \frac{F^i}{\sum_{i=1}^m F^i}, \text{ and } \sum_{i=1}^m \mu^i = 1. \quad (8.2.2)$$

Using the fuzzy inference method with a center-average defuzzifier, product inference and singleton fuzzifier, the MIMO T-S fuzzy system (8.2.1) can be expressed by the following global model:

$$\begin{aligned} \dot{x}_1 &= A_{11}(\mu)x_1 + A_{12}(\mu)x_2 \\ \dot{x}_2 &= A_{21}(\mu)x_1 + A_{22}(\mu)x_2 + B_2 u, \end{aligned} \quad (8.2.3)$$

where

$$\begin{aligned} A_{11}(\mu) &= \sum_{i=1}^m \mu^i A_{11}^i, \quad A_{12}(\mu) = \sum_{i=1}^m \mu^i A_{12}^i, \\ A_{21}(\mu) &= \sum_{i=1}^m \mu^i A_{21}^i, \quad A_{22}(\mu) = \sum_{i=1}^m \mu^i A_{22}^i, \\ \mu &= (\mu^1, \mu^2, \dots, \mu^m). \end{aligned} \quad (8.2.4)$$

For the further analysis, we let

$$A^i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad (8.2.5a)$$

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$$A(\mu) = \begin{bmatrix} A_{11}(\mu) & A_{12}(\mu) \\ A_{21}(\mu) & A_{22}(\mu) \end{bmatrix}. \quad (8.2.5b)$$

Before we proceed, we have the following assumptions:

Assumption 8.1: Each linear subsystem in (8.2.5a) is controllable, i.e. the controllability

matrices $M^i = \begin{bmatrix} B, A^i B, (A^i)^2 B, \dots, (A^i)^{n-1} B \end{bmatrix}$ for $i = 1, \dots, m$ have full rank.

Assumption 8.2: The global fuzzy model (8.2.5b) is controllable in the state space, i.e.

the controllability matrix $M = \begin{bmatrix} B, A(\mu)B, \dots, A^{n-1}(\mu)B \end{bmatrix}$ has full rank in the state space [93].

8.3. Extreme Matrices

Following [80, 93], we decompose the state-space of the MIMO T-S fuzzy system (8.2.3)

into m subspaces as follows:

$$\mathbb{S}_i = \{x \mid \mu^l \geq \mu^i, l = 1, 2, \dots, m, l \neq i\}, \text{ for } i = 1, 2, \dots, m. \quad (8.3.1)$$

The characteristic function of $\mathbb{S}_i$ is defined by

$$\eta^i = \begin{cases} 1 & (x \in \mathbb{S}_i), \\ 0 & (x \notin \mathbb{S}_i). \end{cases} \quad (8.3.2)$$

Let G be the set of membership functions satisfying (8.2.2). Then, the global model of the

MIMO T-S fuzzy system can be expressed in each subspace as

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$$\dot{\mathbf{x}}_1 = [\mathbf{A}_{11}^i + \nabla \mathbf{A}_{11}^i(\mu)]\mathbf{x}_1 + [\mathbf{A}_{12}^i + \nabla \mathbf{A}_{12}^i(\mu)]\mathbf{x}_2$$

$$\dot{\mathbf{x}}_2 = [\mathbf{A}_{21}^i + \nabla \mathbf{A}_{21}^i(\mu)]\mathbf{x}_1 + [\mathbf{A}_{22}^i + \nabla \mathbf{A}_{22}^i(\mu)]\mathbf{x}_2 + \mathbf{B}_2 \mathbf{u},$$

where

$$\nabla \mathbf{A}_{jk}^i(\mu) = \sum_{l=1}^{m_i} \bar{\mu}^l \nabla \mathbf{A}_{jk}^{il}, \quad \nabla \mathbf{A}_{jk}^{il} = \mathbf{A}_{jk}^{il} - \mathbf{A}_{jk}^i,$$

$$\bar{\mu}^l \in G, \quad l=1, \dots, m, \quad \bar{\mu}^l \neq \mu^l,$$

$$\bar{\mu}^l \neq 0, \quad \forall x \in \mathbb{S}_i,$$

$$\text{for } i=1, 2, \dots, m, \quad j=1, 2, \text{ and } k=1, 2. \quad (8.3.3)$$

Obviously, the interactions of the fuzzy subsystems are represented by $\nabla \mathbf{A}_{jk}^i(\mu)$. Here the i^{th} subsystem (8.3.3) is different from the fuzzy dynamical local model in (8.2.3), because it considers all the interactions among the local fuzzy models.

Using the characteristic function of $\mathbb{S}_i$, (8.3.3) can be denoted by

$$\dot{\mathbf{x}}_1^i = [\mathbf{A}_{11}^i + \nabla \mathbf{A}_{11}^i(\mu)]\mathbf{x}_1^i + [\mathbf{A}_{12}^i + \nabla \mathbf{A}_{12}^i(\mu)]\mathbf{x}_2^i$$

$$\dot{\mathbf{x}}_2^i = [\mathbf{A}_{21}^i + \nabla \mathbf{A}_{21}^i(\mu)]\mathbf{x}_1^i + [\mathbf{A}_{22}^i + \nabla \mathbf{A}_{22}^i(\mu)]\mathbf{x}_2^i + \mathbf{B}_2 \mathbf{w}^i, \quad (8.3.4)$$

where $i=1, 2, \dots, m$, $\mathbf{x}^i = \eta^i \mathbf{x}$, and $\mathbf{w}^i = \eta^i \mathbf{u}$.

It can be seen that the i^{th} subsystem is time-varying. For the design of a TSM controller, we define the following upper bounds:

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$$(\nabla A_{jk}^i(\mu))^T \nabla A_{jk}^i(\mu) \leq (E_{jk}^i)^T E_{jk}^i, \quad (8.3.5)$$

for $i=1,2,\dots,m$, $j=1,2$, and $k=1,2$. Here we call the matrices, E_{jk}^i , the extreme matrices.

8.4. TSM for MIMO T-S Fuzzy Systems

In this section, we will develop a TSM controller for MIMO T-S fuzzy systems introduced in Section 8.2. In order to obtain the terminal convergence of the state variables, we have the following definitions.

Definition 8.1: The TSM surface can be described by the following first order nonlinear differential equations [58]

$$s = \dot{x} + \beta |x|^{q/Q} \text{sign}(x) = 0, \quad (8.4.1)$$

where $x \in \mathbb{R}$, $\alpha, \beta > 0$, q and Q are odd integers and $Q > q$.

Definition 8.2: The equilibrium point $x = 0$ of the differential equation (8.4.1) is globally finite-time stable, i.e., for any given initial condition $x(0) = x_0$, the system state converges to $x = 0$ in finite time [58]

$$T = \frac{1}{\beta(1-q/Q)} |x_0|^{1-Q}, \quad (8.4.2)$$

and stays there forever.

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Based on [36], the TSM surface for the MIMO T-S fuzzy system in the i^{th} subspace is defined as

$$s^i = c_1^i x_1 + c_2^i x_2 + c_3^i x_1^{q/Q} = 0, \quad (8.4.3)$$

where $c_1^i \in \mathbb{R}^{p \times (n-p)}$, $c_2^i \in \mathbb{R}^{p \times p}$, and $c_3^i \in \mathbb{R}^{p \times (n-p)}$ are the parameter matrices of the TSM surface.

Lemma 8.1: (Man and Yu [36]) If

$$2q > Q, \quad (8.4.4)$$

then the control input w^i is bounded, and hence u is bounded.

For further analysis, we have the following assumptions.

Assumption 8.3: The system matrices $[(A_{12}^i)^T A_{12}^i]$ and $[A_{12}^i (A_{12}^i)^T]$ are non-singular for $i = 1, \dots, m$.

Assumption 8.4: The interactions of the fuzzy subsystems $\nabla A_{12}^i(\mu)$ in the i^{th} subspace are bounded in Euclidean norm with $\|\nabla A_{12}^i(\mu)((A_{12}^i)^T A_{12}^i)^{-1}(A_{12}^i)^T\| < 1$.

Assumption 8.5: E_{12}^i are designed such that not only condition (8.3.5) is satisfied, but it is also bounded in Euclidean norm with $\|E_{12}^i((A_{12}^i)^T A_{12}^i)^{-1}(A_{12}^i)^T\| < 1$.

To obtain the sufficient condition of the existence of the TSM surface (8.4.3) for the MIMO T-S fuzzy system in the i^{th} subspace, we have the following theorem.

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Theorem 8.1: Consider the MIMO T-S fuzzy system in (8.2.3). If the control input for the i^{th} subsystem is designed as:

$$\mathbf{w}^i = -(\mathbf{c}_2^i \mathbf{B}_2)^{-1} \left\{ \mathbf{k}_1^i + \text{sign}(\mathbf{s}^i) \mathbf{k}_2^i + \beta^i \begin{bmatrix} s_1^{(q_s/Q_s)i} \\ \vdots \\ s_m^{(q_s/Q_s)i} \end{bmatrix} \right\}, \quad (8.4.5a)$$

where

$$\begin{aligned} \mathbf{k}_1^i &= \mathbf{c}_1^i \mathbf{A}_{11}^i \mathbf{x}_1 + \mathbf{c}_1^i \mathbf{A}_{12}^i \mathbf{x}_2 + \mathbf{c}_2^i \mathbf{A}_{21}^i \mathbf{x}_1 + \mathbf{c}_2^i \mathbf{A}_{22}^i \mathbf{x}_2 \\ &+ \mathbf{c}_3^i (q/Q) \text{diag}(x_i^{(q-Q)/Q}) (\mathbf{A}_{11}^i \mathbf{x}_1 + \mathbf{A}_{12}^i \mathbf{x}_2), \end{aligned} \quad (8.4.5b)$$

$$\begin{aligned} \mathbf{k}_2^i &= \|\mathbf{c}_1^i \mathbf{E}_{11}^i \mathbf{x}_1\| + \|\mathbf{c}_1^i \mathbf{E}_{12}^i \mathbf{x}_2\| + \|\mathbf{c}_2^i \mathbf{E}_{21}^i \mathbf{x}_1\| + \|\mathbf{c}_2^i \mathbf{E}_{22}^i \mathbf{x}_2\| \\ &+ \|\mathbf{c}_3^i (q/p) \text{diag}(x_i^{(q-Q)/Q}) (\mathbf{E}_{11}^i \mathbf{x}_1 + \mathbf{E}_{12}^i \mathbf{x}_2)\|, \end{aligned} \quad (8.4.5c)$$

$$\text{diag}(x_i^{(q-Q)/Q}) = \begin{bmatrix} x_1^{(q-Q)/Q} & & \\ & \ddots & \\ & & x_{n-p}^{(q-Q)/Q} \end{bmatrix}, \quad (8.4.5d)$$

then the terminal sliding variable vector $\mathbf{s}^i$ will reach the TSM surface $\mathbf{s}^i = 0$ in finite time

$$T_s \leq \sum_i T_s^i, \quad (8.4.5e)$$

where

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$$T_s^i = \frac{1}{\beta^i(1-q_s/Q_s)} \left| (s^i(0))^T s^i(0) \right|^{1-q_s/Q_s}. \quad (8.4.5f)$$

Proof: Defining a Lyapunov function

$$V^i = 1/2 (s^i)^T s^i \quad (8.4.6)$$

and differentiating V^i with respect to time, we have

$$\begin{aligned} \dot{V}^i &= (s^i)^T \dot{s}^i \\ &= (s^i)^T \left[c_1^i \dot{x}_1 + c_2^i \dot{x}_2 + c_3^i (q/Q) \text{diag}(x_1^{(q-Q)/Q}) \dot{x}_1 \right] \\ &= (s^i)^T \left[c_1^i [A_{11}^i + \nabla A_{11}^i(\mu)] x_1^i + c_1^i [A_{12}^i + \nabla A_{12}^i(\mu)] x_2^i \right. \\ &\quad \left. + c_2^i [A_{21}^i + \nabla A_{21}^i(\mu)] x_1^i + c_2^i [A_{22}^i + \nabla A_{22}^i(\mu)] x_2^i + c_2^i B_2 w^i \right. \\ &\quad \left. + c_3^i (q/Q) \text{diag}(x_1^{(q-Q)/Q}) [A_{11}^i + \nabla A_{11}^i(\mu)] x_1^i \right. \\ &\quad \left. + c_3^i (q/Q) \text{diag}(x_1^{(q-Q)/Q}) [A_{12}^i + \nabla A_{12}^i(\mu)] x_2^i \right] \\ &= (s^i)^T \left[k_1^i + c_1^i \nabla A_{11}^i(\mu) x_1^i + c_1^i \nabla A_{12}^i(\mu) x_2^i \right. \\ &\quad \left. + c_2^i \nabla A_{21}^i(\mu) x_1^i + c_2^i \nabla A_{22}^i(\mu) x_2^i + c_2^i B_2 w^i \right. \\ &\quad \left. + c_3^i (q/Q) \text{diag}(x_1^{(q-Q)/Q}) \nabla A_{11}^i(\mu) x_1^i \right. \\ &\quad \left. + c_3^i (q/Q) \text{diag}(x_1^{(q-Q)/Q}) \nabla A_{12}^i(\mu) x_2^i \right] \end{aligned}$$

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$$\begin{aligned}
 & \leq \|s^i\| \|c_1^i \nabla A_{11}^i(\mu) x_1^i\| + \|s^i\| \|c_1^i \nabla A_{12}^i(\mu) x_2^i\| + \|s^i\| \|c_2^i \nabla A_{21}^i(\mu) x_1^i\| \\
 & \quad + \|s^i\| \|c_2^i \nabla A_{22}^i(\mu) x_2^i\| - \|s^i\| k_2^i \\
 & \quad + \|s^i\| \|c_3^i (q/Q) \text{diag}(x_1^{(q-Q)/Q}) \nabla A_{11}^i(\mu) x_1^i + c_3^i (q/Q) \text{diag}(x_1^{(q-Q)/Q}) \\
 & \quad \times \nabla A_{12}^i(\mu) x_2^i\| - \beta^i (s^i)^T \begin{bmatrix} s_1^{(q_s/Q_s)i} & \dots & s_m^{(q_s/Q_s)i} \end{bmatrix}^T \\
 & \leq -2\beta^i (V^i)^{(q_s+Q_s)/2Q_s}.
 \end{aligned} \tag{8.4.7}$$

According to the finite time stability criteria (8.4.1), TSM surface (8.4.3) will be reached in the finite time (8.4.5e).

The design of the parameter matrices of the TSM variable in (8.4.3) and the convergence property of the MIMO T-S fuzzy system on the TSM surface are stated in the following theorem.

Theorem 8.2: Consider the MIMO T-S fuzzy system in (8.2.3) with the TSM controller in (8.4.5a) under Assumptions 3-5. If the TSM parameter matrices are designed such that

$$c_1^i = (A_{12}^i)^T (A_{12}^i (A_{12}^i)^T)^{-1} \|A_{11}^i + E_{11}^i\| I, \tag{8.4.8a}$$

$$c_2^i = \left(1 - \|E_{12}^i ((A_{12}^i)^T A_{12}^i)^{-1} (A_{12}^i)^T\|\right) I, \tag{8.4.8b}$$

$$c_3^i = (A_{12}^i)^T (A_{12}^i (A_{12}^i)^T)^{-1} \text{diag}(\beta_1^i), \tag{8.4.8c}$$

then the system states will converge to zero in finite time on the TSM surface.

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Proof: On the TSM surface (14), the state variable vector $\mathbf{x}_2$ can be written as

$$\mathbf{x}_2 = -\frac{(\mathbf{A}_{12}^i)^T (\mathbf{A}_{12}^i (\mathbf{A}_{12}^i)^T)^{-1}}{1 - \|\mathbf{E}_{12}^i ((\mathbf{A}_{12}^i)^T \mathbf{A}_{12}^i)^{-1} (\mathbf{A}_{12}^i)^T\|} \left[\|\mathbf{A}_{11}^i + \mathbf{E}_{11}^i\| \mathbf{I} \mathbf{x}_1 + \text{diag}(\beta_l^i) \mathbf{x}_1^{q/Q} \right]. \quad (8.4.9)$$

It is seen that $\mathbf{x}_2$ is the linear combinations of $\mathbf{x}_1$. Therefore to prove that the system states can converge to zero in finite time on the TSM surface, we only need to show that $\mathbf{x}_1$ can reach the system origin in finite time.

Consider the Lyapunov function

$$V = 1/2 \mathbf{x}_1^T \mathbf{x}_1. \quad (8.4.10)$$

Differentiating V with respect to time and using (8.3.5) and (8.4.9), we have

$$\begin{aligned} \dot{V} &= \mathbf{x}_1^T \dot{\mathbf{x}}_1 \\ &= \mathbf{x}_1^T \left[[\mathbf{A}_{11}^i + \nabla \mathbf{A}_{11}^i(\mu)] \mathbf{x}_1 \right. \\ &\quad \left. - [\mathbf{I} + \nabla \mathbf{A}_{12}^i(\mu) ((\mathbf{A}_{12}^i)^T \mathbf{A}_{12}^i)^{-1} (\mathbf{A}_{12}^i)^T] \right. \\ &\quad \left. \times \frac{\mathbf{A}_{12}^i (\mathbf{A}_{12}^i)^T (\mathbf{A}_{12}^i (\mathbf{A}_{12}^i)^T)^{-1}}{\|\mathbf{I} - \mathbf{E}_{12}^i ((\mathbf{A}_{12}^i)^T \mathbf{A}_{12}^i)^{-1} (\mathbf{A}_{12}^i)^T\|} \left[\|\mathbf{A}_{11}^i \right. \right. \\ &\quad \left. \left. + \mathbf{E}_{11}^i\| \mathbf{I} \mathbf{x}_1 + \text{diag}(\beta_l^i) \mathbf{x}_1^{q/Q} \right] \right] \\ &\leq \|\mathbf{A}_{11}^i + \nabla \mathbf{A}_{11}^i(\mu)\| \|\mathbf{x}_1\|^2 \end{aligned}$$

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$$\begin{aligned}
 & -\frac{1-\|\nabla A_{12}^i(\mu)((A_{12}^i)^T A_{12}^i)^{-1}(A_{12}^i)^T\|}{1-\|E_{12}^i((A_{12}^i)^T A_{12}^i)^{-1}(A_{12}^i)^T\|} \left[\|A_{11}^i \right. \\
 & \left. + E_{11}^i \left(\|\mathbf{x}_1\|^2 + \text{diag}(\beta_i) \mathbf{x}_1^T \mathbf{x}_1^{q/Q} \right) \right] \\
 & < -\text{diag}(\beta_i) \mathbf{x}_1^T \mathbf{x}_1^{q/Q} \\
 & = -2\beta^i V^{(q+Q)/2Q}. \tag{8.4.11}
 \end{aligned}$$

According to the finite time stability criteria (8.4.1), the system states $\mathbf{x}_1$ and $\mathbf{x}_2$ will reach the system origin in a finite time.

8.5. Simulation Example

To illustrate the finite time stabilization property of the proposed TSM controller, we consider the following MIMO T-S fuzzy system:

$$\begin{aligned}
 R^1 : IF \quad & x_1 \text{ is } F_1 \\
 THEN \quad & \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & -2 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{8.5.1a}
 \end{aligned}$$

$$R^2 : IF \quad x_1 \text{ is } F_2$$

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$$\text{THEN} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \quad (8.5.1b)$$

We use the following membership functions, F_1 and F_2 :

$$F_1 = \frac{1 - 1/(1 + \exp(-14(x_1 - \pi/8)))}{1 + \exp(-14(x_1 + \pi/8))},$$

$$F_2 = 1 - F_1.$$

In this simulation, we choose

$$A_{11}^1 = A_{11}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{12}^1 = A_{12}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_{21}^1 = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}, \quad A_{21}^2 = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix},$$

$$A_{22}^1 = \begin{bmatrix} 0 & 0 \\ -3 & -4 \end{bmatrix}, \quad A_{22}^2 = \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The extreme matrices are selected as:

$$E_{11}^1 = E_{11}^2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad E_{12}^1 = E_{12}^2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$E_{21}^1 = E_{21}^2 = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad E_{22}^1 = E_{22}^2 = \begin{bmatrix} 0 & 0.5 \\ 4 & 5 \end{bmatrix}.$$

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Simulation result shown in Fig. 8.1 illustrates the effectiveness of the TSM controller (8.4.5a). Figs. 8.2 and 8.3 show the control inputs u_1 and u_2 , respectively. It is seen that the good system performance has been achieved using the proposed control scheme. Finite time convergence property is guaranteed.

8.6. Conclusions

A terminal sliding mode control scheme has been developed in this chapter for MIMO T-S fuzzy system. The extreme matrices have been used to design not only the TSM controller, but also the TSM surface. It has been shown that the TSM controller is able to drive the system states to reach the system origin in a finite time. A simulation example has been given in support of the proposed control scheme.

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Fig. 8.1. The system states.

Fig. 8.2. The control input u_1 .

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Fig. 8.3. The control input u_2 .

CHAPTER 9 CONCLUSIONS

Chapter 9

CONCLUSIONS

9.1. Summary

Variable structure control technique is a powerful approach for the control of nonlinear systems. A number of factors, such as over estimation of system uncertainties and wrong selection of initial conditions and power rules, are known to not only affect the performance of the variable structure control systems, but also result in an infinite control law. Therefore, this thesis has been mainly concerned with the study and improvements of variable structure control designs by employing adaptive technique to realize finite time convergence with bounded control law.

CHAPTER 9 CONCLUSIONS

Chapter two of this thesis has provided a survey for the basic variable structure control theory and recent significant results on variable structure control. The limitations of these results have also been highlighted.

Chapter three has provided a background for fuzzy logic and fuzzy systems being applied in this thesis. Fuzzy sets, fuzzy set operations, and fuzzy linguistic representation such as linguistic variables, linguistic modifiers (hedged) have been outlined. The basic concept of T-S fuzzy system and Mamdani fuzzy system has been discussed.

In chapter four, a simplified multiple-surface sliding control (MSSC) algorithm has been proposed in order to reduce the time complexity, compared with the conventional MSSC algorithm.

In chapter five, an adaptive fuzzy multiple-surface sliding mode control has been developed such that the knowledge and boundary conditions of the uncertainties are not required. The proposed adaptive control scheme has been proven to be able to guarantee the tracking error to converge to a small neighborhood of the origin in finite time. The controller is simple to design, yet gives good tracking performance for SISO nonlinear system with mismatched uncertainties.

In chapter six, an adaptive fast finite time multiple-surface sliding mode control has been proposed for a class of uncertain nonlinear systems. Unlike the control scheme proposed in chapter five, the controller design in this chapter can guarantee fast finite time stability of the closed-loop system. Also, it is proven that the control input is bounded under some weak conditions.

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In chapter seven, a robust tracking control scheme has been developed for a class of complex nonlinear systems with T-S fuzzy model. This scheme is suitable for controlling a class of complex nonlinear systems whose global nonlinear dynamics cannot be described by a single or a set of linear differential equations. Here, the concept of extreme matrices from H_∞ control technique is used to determine the upper bound information of the interactions of the fuzzy subsystems, and a sliding mode control law is then designed to guarantee the asymptotic stability and robustness of the closed-loop system.

In chapter eight, a terminal sliding mode control scheme has been proposed for MIMO T-S fuzzy system. It is shown that the extreme matrices can be used to design not only the terminal sliding mode controller, but also the terminal sliding mode surface. Sufficient conditions are provided to guarantee finite time stability of the closed-loop T-S fuzzy system and boundedness of control law.

In summary, the thesis has provided several new and improved robust variable structure control schemes aimed at achieving robustness and finite time convergence against nonlinearities, parameter uncertainties, and external disturbances in the control of a class of nonlinear systems and T-S fuzzy systems. Simulation examples are given to illustrate the effectiveness of our theoretical results.

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9.2. Future Work

Some areas for future research are as follows:

- (1) Time delays often occur in many dynamic systems. The existence of time delays usually becomes the source of instability and deteriorates the performance of the systems. Motivated by the work in [94, 95], we believe that, the MSSC control schemes developed in Chapter 4, Chapter 5, and Chapter 6 can be extended to control a class of nonlinear systems with time delays. Using fuzzy approximation technique and by choosing appropriate Lyapunov-Krasovskii functions in the Lyapunov function candidate, the uncertainties from unknown time delays can be removed such that the design of the stabilizing MSSC control law is free from these uncertainties. In this way, the iterative MSSC design procedure can be carried out directly.
- (2) All the results provided in this thesis are for continuous time case. However, in practical situations, the controller is designed from the sampled data. Therefore, it is necessary to carry out research investigations for the multiple-surface sliding mode control and terminal sliding mode control for the discrete case.
- (3) In recent years, many effective control methods have been developed for chaotic systems [96-98] and T-S fuzzy modeling technique has been used to model chaotic systems. It is possible to use SMC control scheme developed in chapter 7 to control the chaotic systems with their T-S fuzzy models.

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(4) Bifurcation control is a relatively new area of active research [99].

Conventionally, bifurcation control is aimed at shifting the position of the bifurcation point or changing its type. Such a significant change of the system properties requires good knowledge of its model. In practice, however, in most cases, the system model has uncertainties and only part of the system state is available for measurement. It may be possible to design an adaptive robust fuzzy controller to eliminate the effect of uncertainties and then tunes a nonlinear uncertain dynamical system to its bifurcation point.

REFERENCES

REFERENCES

- [1] S. V. Emelyanov and A. I. Fedotova, "Design of a static tracking systems with variable structure.," *Automation and Remote Control* pp. 1223-1235, 1962.
- [2] B. Drazenovic, "The invariance conditions in variable structure systems," *Automatica*, vol. 5, pp. 287-295, 1969.
- [3] V. Utkin, "Variable structure systems with sliding modes," *IEEE Transactions on Automatic Control*, vol. 22, pp. 212-222, 1977.
- [4] K.-K. Young, P. Kokotovic, and V. Utkin, "A singular perturbation analysis of high-gain feedback systems," *IEEE Transactions on Automatic Control*, vol. 22, pp. 931-938, 1977.
- [5] V. Utkin, "Application of equivalent control method to the systems with large feedback gain," *IEEE Transactions on Automatic Control*, vol. 23, pp. 484-486, 1978.
- [6] M. C. B. Barmish, G. Leitmann, "A New Class of Stabilizing Controllers for Uncertain Dynamical Systems," *SIAM J Control and Optimization*, vol. 21, pp. 246-255, 1983.
- [7] J.-J. E. Slotine, Sastry, S. S., "Tracking control of nonlinear system using sliding surface with application to robotic manipulators," *International Journal of Control*, vol. 38, pp. 465-492, 1983.
- [8] J. H. Green, & Hedrick, J. K., "Nonlinear speed control for automotive engines," presented at American Control Conference, 1990.
- [9] A.-C. Huang and Y.-C. Chen, "Adaptive multiple-surface sliding control for non-autonomous systems with mismatched uncertainties," *Automatica*, vol. 40, pp. 1939-1945, 2004.

REFERENCES

- [10] M. Zhihong, A. P. Paplinski, and H. R. Wu, "A robust MIMO terminal sliding mode control scheme for rigid robotic manipulators," *IEEE Transactions on Automatic Control*, vol. 39, pp. 2464-2469, 1994.
- [11] X. Yu and Z. Man, "Model reference adaptive control systems with terminal sliding modes," *International Journal of Control*, vol. 64, pp. 1165 - 1176, 1996.
- [12] M. Sugeno and G. T. Kang, "Fuzzy modelling and control of multilayer incinerator," *Fuzzy Sets and Systems*, vol. 18, pp. 329-345, 1986.
- [13] M. Sugeno and G. T. Kang, "Structure identification of fuzzy model," *Fuzzy Sets and Systems*, vol. 28, pp. 15-33, 1988.
- [14] B. Paden and S. Sastry, "A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Transactions on Circuits and Systems*, vol. 34, pp. 73-82, 1987.
- [15] R. A. DeCarlo, S. H. Zak, and G. P. Matthews, "Variable structure control of nonlinear multivariable systems: a tutorial," *Proceedings of the IEEE*, vol. 76, pp. 212-232, 1988.
- [16] V. I. V. I. Utkin, *Sliding modes in control and optimization*: Berlin ; New York : Springer-Verlag, 1992.
- [17] J.-J. E. Slotine, *Applied nonlinear control*: Prentice Hall, 1991.
- [18] U. Itkis, *Control systems of variable structure*, 1976.
- [19] U. O. K.D. Young, *Variable Structure Systems, Sliding Mode and Nonlinear Control*: Springer, 1999.
- [20] V. I. Utkin, "Sliding mode control design principles and applications to electric drives," *IEEE Transactions on Industrial Electronics*, vol. 40, pp. 23-36, 1993.
- [21] V. I. Utkin, H.-C. Chang, and A. Keyhani, "Automobile climate control using sliding mode," presented at Electric Machines and Drives Conference, 2001. IEMDC 2001. IEEE International, 2001.
- [22] V. I. Utkin and A. Sabanovic, "Sliding modes applications in power electronics and motion control systems," presented at IEEE International Symposium on Industrial Electronics, 1999. ISIE '99, 1999.
- [23] K.-K. D. Young and H. G. Kwatny, "Variable structure servomechanism design and applications to overspeed protection control," *Automatica*, vol. 18, pp. 385-400, 1982.

REFERENCES

- [24] K. D. YOUNG and U. OZGUNER, "Sliding-mode Design for Robust Linear Optimal Control," *Automatica*, vol. 33, pp. 1313-1323, 1997.
- [25] A. Bacciotti, *Liapunov functions and stability in control theory*, 2nd ed: Springer, 2005.
- [26] C. M. Dorling, and Zinober, A.S.I., "Two approaches to hyperplane design in multivariable variable structure control systems," *International Journal of Control*, vol. 44, pp. 65-82, 1986.
- [27] S. K. Spurgeon, "Choice of discontinuous control component for robust sliding mode performance," *International Journal of Control*, vol. 53, pp. 163-179, 1991.
- [28] S. Gutman, "Uncertain dynamical systems--A Lyapunov min-max approach," *IEEE Transactions on Automatic Control*, vol. 24, pp. 437-443, 1979.
- [29] S. H. çZak, *Systems and control*: Oxford University Press, 2003.
- [30] B. Moore, "On the flexibility offered by state feedback in multivariable systems beyond closed loop eigenvalue assignment," *IEEE Transactions on Automatic Control*, vol. 21, pp. 689-692, 1976.
- [31] G. Klein and B. Moore, "Eigenvalue-generalized eigenvector assignment with state feedback," *IEEE Transactions on Automatic Control*, vol. 22, pp. 140-141, 1977.
- [32] W. Gao and J. C. Hung, "Variable structure control of nonlinear systems: a new approach," *IEEE Transactions on Industrial Electronics*, vol. 40, pp. 45-55, 1993.
- [33] I. Petersen, "Structural Stabilization of Uncertain Systems: Necessity of the Matching Condition," *SIAM J Control and Optimization*, vol. 23, pp. 286-296, 1985.
- [34] M. Zhihong and M. Palaniswami, "Robust tracking control for rigid robotic manipulators," *IEEE Transactions on Automatic Control*, vol. 39, pp. 154-159, 1994.
- [35] M. Zhihong and X. H. Yu, "Terminal sliding mode control of MIMO linear systems," presented at Decision and Control, 1996., Proceedings of the 35th IEEE, 1996.
- [36] M. Zhihong and X. H. Yu, "Terminal sliding mode control of MIMO linear systems," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 44, pp. 1065-1070, 1997.

REFERENCES

- [37] Y. Wu, X. Yu, and Z. Man, "Terminal sliding mode control design for uncertain dynamic systems," *Systems & Control Letters*, vol. 34, pp. 281-287, 1998.
- [38] X. Yu and M. Zhihong, "Multi-input uncertain linear systems with terminal sliding-mode control," *Automatica*, vol. 34, pp. 389-392, 1998.
- [39] M. Zak, "Terminal attractors for addressable memory in neural networks," *Physics Letters A*, vol. 133, pp. 18-22, 1988.
- [40] M. Zak, "Terminal attractors in neural networks," *Neural Networks*, vol. 2, pp. 259-274, 1989.
- [41] Y. Feng, X. Yu, and Z. Man, "Non-singular terminal sliding mode control of rigid manipulators," *Automatica*, vol. 38, pp. 2159-2167, 2002.
- [42] L. A. Zadeh, "Fuzzy sets versus probability," *Proceedings of the IEEE*, vol. 68, pp. 421, 1980.
- [43] L. A. Zadeh, "Fuzzy Sets," *Information and control*, vol. 8, pp. 338-353, 1965.
- [44] L. A. Zadeh, "Fuzzy logic," *Computer*, vol. 21, pp. 83-93, 1988.
- [45] E. Cox, *The fuzzy systems handbook : a practitioner's guide to building, using, and maintaining fuzzy systems*: AP Professional, 1994.
- [46] B. Kosko, *Neural networks and fuzzy systems : a dynamical systems approach to machine intelligence*: Prentice Hall, 1992.
- [47] M. Won, & Hedrick, J.K., "Multi-surface sliding control of a class of uncertain nonlinear systems," *International Journal of Control*, vol. 64, pp. 693-706, 1996.
- [48] R. Hirschorn, "Sliding-Mode Control Variations," *IEEE Transactions on Automatic Control*, vol. 52, pp. 468-480, 2007.
- [49] S. Laghrouche, F. Plestan, and A. Glumineau, "Higher order sliding mode control based on integral sliding mode," *Automatica*, vol. 43, pp. 531-537, 2007.
- [50] L. Xiangjun, W. Changhong, W. Qiyong, Y. Xinghuo, and W. Bin, "Digital Redesign of Sliding Mode Control of LTI Systems," presented at Intelligent Control and Automation, 2006. WCICA 2006. The Sixth World Congress on, 2006.
- [51] M. H. Alsuwaiyel, *Algorithms : design techniques and analysis*. Singapore ; New Jersey: World Scientific, 1999.

REFERENCES

- [52] M. Zhihong, H. R. Wu, and M. Palaniswami, "An adaptive tracking controller using neural networks for a class of nonlinear systems," *IEEE Transactions on Neural Networks*, vol. 9, pp. 947-955, 1998.
- [53] R. Fierro and F. L. Lewis, "Control of a nonholonomic mobile robot using neural networks," *IEEE Transactions on Neural Networks*, vol. 9, pp. 589-600, 1998.
- [54] W. Wang, J. Yi, D. Zhao, and X. Liu, "Design of cascade fuzzy sliding-mode controller," presented at American Control Conference, 2005. Proceedings of the 2005, 2005.
- [55] Y. Lee and S. H. Zak, "Uniformly ultimately bounded fuzzy adaptive tracking controllers for uncertain systems," *IEEE Transactions on Fuzzy Systems*, vol. 12, pp. 797-811, 2004.
- [56] D. Yongsheng, Y. Hao, and S. Shihuang, "Necessary conditions on minimal system configuration for general MISO Mamdani fuzzy systems as universal approximators," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 30, pp. 857-864, 2000.
- [57] S. Khoo, Z. Man, and S. Zhao, "Comments on "Adaptive multiple-surface sliding control for non-autonomous systems with mismatched uncertainties"," *Automatica*, *accepted*, 2007.
- [58] S. Yu, X. Yu, B. Shirinzadeh, and Z. Man, "Continuous finite-time control for robotic manipulators with terminal sliding mode," *Automatica*, vol. 41, pp. 1957-1964, 2005.
- [59] R. A. Freeman and P. V. Kokotovic, "Design of 'softer' robust nonlinear control laws," *Automatica*, vol. 29, pp. 1425-1437, 1993.
- [60] E. B. Lee and L. Markus, *Foundations of optimal control theory*. Malabar, Fla.: R.E. Krieger Pub. Co., 1986.
- [61] J.-M. Coron, "On the Stabilization in Finite Time of Locally Controllable Systems by Means of Continuous Time-Varying Feedback Law," *SIAM Journal on Control and Optimization*, vol. 33, pp. 804-833, 1995.
- [62] S. P. Bhat and D. S. Bernstein, "Finite-Time Stability of Continuous Autonomous Systems," *SIAM Journal on Control and Optimization*, vol. 38, pp. 751-766, 2000.
- [63] X. Huang, W. Lin, and B. Yang, "Global finite-time stabilization of a class of uncertain nonlinear systems," *Automatica*, vol. 41, pp. 881-888, 2005.

REFERENCES

- [64] G. Bartolim, A. Pisano, and E. Usai, "Global stabilization for nonlinear uncertain systems with unmodeled actuator dynamics," *IEEE Transactions on Automatic Control*, vol. 46, pp. 1826-1832, 2001.
- [65] A. Levant, "Homogeneity approach to high-order sliding mode design," *Automatica*, vol. 41, pp. 823-830, 2005.
- [66] Y. Hong, "Finite-time stabilization and stabilizability of a class of controllable systems," *Systems & Control Letters*, vol. 46, pp. 231-236, 2002.
- [67] S. P. Bhat and D. S. Bernstein, "Continuous, bounded, finite-time stabilization of the translational and rotational double integrators," *Proceedings of the IEEE International Conference on Control Applications*, 1996.
- [68] Y. Hong and Z. P. Jiang, "Finite-Time Stabilization of Nonlinear Systems With Parametric and Dynamic Uncertainties," *IEEE Transactions on Automatic Control*, vol. 51, pp. 1950-1956, 2006.
- [69] J.-M. Coron and L. Praly, "Adding an integrator for the stabilization problem," *Systems & Control Letters*, vol. 17, pp. 89-104, 1991.
- [70] E. L. E. L. Ince, 1891-1941., *Ordinary differential equations*: Dover Publications, 1956.
- [71] Q. Chunjiang and L. Wei, "A continuous feedback approach to global strong stabilization of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 46, pp. 1061-1079, 2001.
- [72] Z. Sun and Y. Liu, "Adaptive state-feedback stabilization for a class of high-order nonlinear uncertain systems," *Automatica*, vol. 43, pp. 1772-1783, 2007.
- [73] L. Wei and Q. Chunjiang, "Adaptive control of nonlinearly parameterized systems: a nonsmooth feedback framework," *IEEE Transactions on Automatic Control*, vol. 47, pp. 757-774, 2002.
- [74] C. Qian and W. Lin, "Non-Lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization," *Systems & Control Letters*, vol. 42, pp. 185-200, 2000.
- [75] M. C. M. Teixeira and S. H. Zak, "Stabilizing controller design for uncertain nonlinear systems using fuzzy models," *IEEE Transactions on Fuzzy Systems*, vol. 7, pp. 133-142, 1999.
- [76] S. H. Zak and S. Hui, "On variable structure output feedback controllers for uncertain dynamic systems," *IEEE Transactions on Automatic Control*, vol. 38, pp. 1509-1512, 1993.

REFERENCES

- [77] W. Chang, J. B. Park, Y. H. Joo, and G. Chen, "Design of robust fuzzy-model-based controller with sliding mode control for SISO nonlinear systems," *Fuzzy Sets and Systems*, vol. 125, pp. 1-22, 2002.
- [78] X. Yu, Z. Man, and B. Wu, "Design of fuzzy sliding-mode control systems," *Fuzzy Sets and Systems*, vol. 95, pp. 295-306, 1998.
- [79] C.-L. Hwang, "A novel Takagi-Sugeno-based robust adaptive fuzzy sliding-mode controller," *IEEE Transactions on Fuzzy Systems*, vol. 12, pp. 676-687, 2004.
- [80] S. G. Cao, N. W. Rees, and G. Feng, "Analysis and design for a class of complex control systems part II: Fuzzy controller design," *Automatica*, vol. 33, pp. 1029-1039, 1997.
- [81] G. Feng, "Stability analysis of discrete-time fuzzy dynamic systems based on piecewise Lyapunov functions," *IEEE Transactions on Fuzzy Systems*, vol. 12, pp. 22-28, 2004.
- [82] G. Feng, S. G. Cao, and N. W. Rees, "An approach to $H[\infty]$ control of a class of nonlinear systems," *Automatica*, vol. 32, pp. 1469-1474, 1996.
- [83] G. Feng, S. G. Cao, N. W. Rees, and C. M. Cheng, "Analysis and design of model based fuzzy control systems," *Proceedings of the Sixth IEEE International Conference on Fuzzy Systems*, 1997.
- [84] J. Ma and G. Feng, "An approach to $H[\infty]$ control of fuzzy dynamic systems," *Fuzzy Sets and Systems*, vol. 137, pp. 367-386, 2003.
- [85] F. Mei, Z. Man, and T. Nguyen, "Fuzzy modelling and tracking control of nonlinear systems," *Mathematical and Computer Modelling*, vol. 33, pp. 759-770, 2001.
- [86] G.-C. Hwang and S.-C. Lin, "A stability approach to fuzzy control design for nonlinear systems," *Fuzzy Sets and Systems*, vol. 48, pp. 279-287, 1992.
- [87] K. Tanaka and M. Sugeno, "Stability analysis and design of fuzzy control systems," *Fuzzy Sets and Systems*, vol. 45, pp. 135-156, 1992.
- [88] C. Lin, Q.-G. Wang, and T. H. Lee, "Stabilization of uncertain fuzzy time-delay systems via variable structure control approach," *IEEE Transactions on Fuzzy Systems*, vol. 13, pp. 787-798, 2005.
- [89] B.-J. Rhee and S. Won, "A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design," *Fuzzy Sets and Systems*, vol. 157, pp. 1211-1228, 2006.

REFERENCES

- [90] H. K. Lam and F. H. F. Leung, "Stability analysis of fuzzy control systems subject to uncertain grades of membership," *IEEE Transactions on Systems, Man and Cybernetics, Part B*, vol. 35, pp. 1322-1325, 2005.
- [91] S. P. Bhat and D. S. Bernstein, "Finite-time stability of homogeneous systems," in *Proceedings of the American Control Conference*, vol. 4, 1997, pp. 2513-2514
- [92] X. Yu, J.-X. Xu, Y. Hong, and S. Yu, "Analysis of a class of discrete-time systems with power rule," *Automatica*, vol. 43, pp. 562-566, 2007.
- [93] S. Khoo, Z. Man, and S. Zhao, "Sliding Mode Control of Fuzzy Dynamic Systems," in *Proceedings of the 9th IEEE International Conference on Control, Automation, Robotics and Vision*, 2006, pp. 711-716.
- [94] C. Lin, Q.-G. Wang, T. H. Lee, Y. He, and B. Chen, "Observer-based $H[\infty]$ fuzzy control design for T-S fuzzy systems with state delays," *Automatica*, vol. 44, pp. 868-874, 2008.
- [95] S. S. Ge, F. Hong, and T. H. Lee, "Robust adaptive control of nonlinear systems with unknown time delays," *Automatica*, vol. 41, pp. 1181-1190, 2005.
- [96] X. Yu, G. Chen, Y. Xia, Y. Song, and Z. Cao, "An invariant-manifold-based method for chaos control," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 48, pp. 930-937, 2001.
- [97] G. Feng and G. Chen, "Adaptive control of discrete-time chaotic systems: a fuzzy control approach," *Chaos, Solitons & Fractals*, vol. 23, pp. 459-467, 2005.
- [98] B. Polyak, "Stabilizing Chaos with Predictive Control," *Automation and Remote Control*, vol. 66, pp. 1791-1804, 2005.
- [99] D. V. Efimov and A. L. Fradkov, "Adaptive tuning to bifurcation for time-varying nonlinear systems," *Automatica*, vol. 42, pp. 417-425, 2006.

AUTHOR'S PUBLICATIONS

AUTHOR'S PUBLICATIONS

- [1] S. Khoo, Z. Man, and S. Zhao, "Sliding Mode Control for T-S Fuzzy Systems," *International Journal of Systems and Control*, vol. 1, pp. 22-30, 2007.
- [2] S. Khoo, Z. Man, and S. Zhao, "Comments on "Adaptive multiple-surface sliding control for non-autonomous systems with mismatched uncertainties", *Automatica*, *in press*, 2007.
- [3] S. Khoo, Z. Man, and S. Zhao, "Sliding Mode Control for A Class of Complex Nonlinear Systems with T-S Fuzzy models" *WSEAS Trans. on Systems*, *accepted*, 2007.
- [4] S. Khoo, Z. Man, and S. Zhao, "Sliding Mode Control of Fuzzy Dynamic Systems," in *Proceedings of the 9th IEEE International Conference on Control, Automation, Robotics and Vision*, 2006, pp. 711-716.
- [5] S. Khoo, Z. Man, and S. Zhao, "Adaptive Fuzzy Multi-surface Sliding Mode Control for a Class of Nonlinear Systems," in *Proceedings of the 2nd IEEE International Conference on Industrial Electronics and Applications*, 2007, pp. 1095-1100.
- [6] S. Khoo, Z. Man, and S. Zhao, "Terminal Sliding Mode Control for MIMO T-S Fuzzy Systems," in *Proceedings of the 6th IEEE International Conference on Information, Communications and Signal Processing*, 2007.
- [7] S. Khoo, Z. Man, and S. Zhao, "Adaptive Fast Finite Time Control of A Class of Nonlinear Uncertain Systems," in *Proceedings of the 3rd IEEE International Conference on Industrial Electronics and Applications*, 2007.
- [8] S. Zhao, Z. Man, and S. Khoo, "Modified LMS and NLMS Algorithm with a New Variable Step Size," in *Proceedings of the 9th IEEE International Conference on Control, Automation, Robotics and Vision*, 2006, pp. 43-48.

AUTHOR'S PUBLICATIONS

- [9] S. Zhao, Z. Man, and S. Khoo, "A Fast Variable Step-Size LMS Algorithm with System Identification," in *Proceedings of the 2nd IEEE International Conference on Industrial Electronics and Applications*, 2007, pp. 2340-2345.
- [10] S. Zhao, Z. Man, and S. Khoo, "A new variable step-size transform domain LMS algorithm with system identification," in *Proceedings of the 6th IEEE International Conference on Control and Automation*, 2007, pp. 2721-2724.
- [11] S. Zhao, Z. Man, and S. Khoo, "Conjugate Gradient Algorithm design with RLS Normal Equation," in *Proceedings of the 6th IEEE International Conference on Information, Communications and Signal Processing*, 2007.