Extended numerical manifold method for engineering failure analysis

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2010

An, X. (2010). Extended numerical manifold method for engineering failure analysis.
Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/42227>

<https://doi.org/10.32657/10356/42227>

**NANYANG
TECHNOLOGICAL
UNIVERSITY**

EXTENDED NUMERICAL MANIFOLD METHOD
FOR ENGINEERING FAILURE ANALYSIS

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SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING

2010

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School of Civil and Environmental Engineering

A thesis submitted to the Nanyang Technological University
in fulfillment of the requirement for the degree of
Doctor of Philosophy

2010

ACKNOWLEDGEMENT

First and foremost, I would like to express my sincere gratitude to my supervisor, Professor Ma Guowei for his warm encouragement, patient guidance and stimulus throughout this research. His unwavering enthusiasm and interest in scientific research especially on the development of the numerical manifold method are much appreciated and unforgettable. It is my honor to work with him.

I would also like to thank Professor Li Luxian from Xi'an Jiaotong University, China for his advices and suggestions to my research work. He has enhanced me a lot in my knowledge in the theoretical part.

I extend my gratitude to Dr Zhang Huihua from Xi'an Jiaotong University, China and Dr Ning Youjun for their great helps in my research, especially in the programming.

I am thankful to our whole NMM group, including Prof Ma Guowei, Dr Ning Youjun, Mr Fu Guoyang, Mr He Lei, Ms Jiao Liqing. We had a regular meeting each week. We discussed the problems we met and tried to figure out ways to solve the problems together. I benefited a lot from the discussions with them.

I thank my current and former colleagues in the UTRE project. We spent four years together. They made my life at NTU full of joys.

The scholarship provided by Nanyang Technological University is gratefully acknowledged.

Last but not the least, I would like to dedicate this work to my beloved husband, Gu Jiong, it is the most right decision I have ever made to marry him, to my parents for their love, they came a long way to Singapore to help taking care my newborn baby and then I can focus on my research work, and to my baby girl, Gu Kaixin, for all the joys she brings to my life.

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SUMMARY

In this thesis, the numerical manifold method (NMM) has been extended for engineering failure analysis.

The NMM has been coupled with the fracture mechanics to simulate the complex cracks and their growth. A new concept ‘singular physical cover’ is introduced. Asymptotic crack tip functions extracted from the analytical solution are chosen as the local approximation spaces of the singular physical covers. The domain form of the interaction integral and the maximum circumferential stress criterion are employed to evaluate the stress intensity factors at the crack tips and to predict the crack growth direction, respectively. Several typical crack problems are simulated. The numerical results show that the extended NMM can resolve the stress singularities around the crack tips well, and it is efficient and robust for complex crack problems. In addition, the extended NMM is obviously superior to the conventional finite element method (CFEM) and its various modifications, such as the extended finite element method (XFEM), the generalized finite element method (GFEM), etc.

The NMM has also been extended to account for the practical rock failure process. The Mohr-Coulomb criterion with a tensile cutoff is employed to predict the crack initiation and propagation. A cover-division strategy is adopted to fulfill the crack initiation and propagation. The cover-division strategy can avoid the mesh dependency to some extent because of the non-local nature of the stress. Algorithms are implemented to treat the manifold elements, the physical covers and the loops during the fracturing process. The developed program has been applied to simulate the progressive failure of rock slopes with non-persistent joints. Numerical results indicate that it is able to capture the fracturing in intact rock bridge and finally allow the kinematic release. The developed program has also been applied to investigate the potential failure mechanisms of footwall slopes in surface coal mining. It can reproduce the major failure mechanisms observed in footwall slopes.

The present study showed that the NMM is promising for engineering failure analysis and deserves to be further improved for more complex applications in the future.

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LIST OF SYMBOLS

M_I	=	I th mathematical cover
$\varphi_I(\mathbf{x})$	=	Partition of unity function for mathematical cover M_I
P_I^j	=	j th physical cover generated from mathematical cover M_I
δ_I^j	=	a modifier, equal to 1 within physical cover P_I^j , and 0 elsewhere
P_i	=	i th physical cover, reallocated with a single index i
$\mathbf{u}_i(\mathbf{x})$	=	local approximation function defined on physical cover P_i
$\mathbf{u}^h(\mathbf{x})$	=	global approximation on the displacement field
$\boldsymbol{\sigma}$	=	stress vector
$\boldsymbol{\varepsilon}$	=	strain vector
$\mathbf{D}$	=	elastic matrix
$\mathbf{L}$	=	differential operator matrix
$\mathbf{b}$	=	body loading per unit volume
$\ddot{\mathbf{u}}$	=	acceleration vector
∇_s	=	symmetric gradient tensor
$\bar{\mathbf{u}}$	=	displacement prescribed on the corresponding boundary Γ_u
$\bar{\mathbf{t}}$	=	traction prescribed on the corresponding boundary Γ_t
ρ	=	density of material
E	=	Young's modulus
ν	=	Poisson's ratio
$\mathbf{N}$	=	NMM shape function
$\mathbf{B}$	=	strain matrix, $\mathbf{B} = \mathbf{LN}$
$\mathbf{S}$	=	stress matrix, $\mathbf{S} = \mathbf{DLN}$
$\mathbf{K}$	=	stiffness matrix
$\mathbf{M}$	=	mass matrix
$\mathbf{F}$	=	loading vector
$\boldsymbol{\lambda}$	=	Lagrange multiplier vector
p	=	given positive penalty parameter
τ	=	shear strength

c	=	joint cohesion
ϕ	=	friction angle
α	=	dilation angle (or roughness angle)
$H(\mathbf{x})$	=	generalized Heaviside function
$J(\mathbf{x})$	=	junction function
$H_\alpha(\mathbf{x})$	=	discontinuous function (equal to 1 within aggregate R_α and 0 elsewhere)
$[[\cdot]]$	=	jump of a function
r, θ	=	polar coordinates in the local crack tip coordinate system
μ	=	shear modulus
κ	=	Kolosov constant
Φ_α	=	α th asymptotic crack tip function, $\alpha = 1 \sim 4$
K_I	=	mode I stress intensity factor
K_{II}	=	mode II stress intensity factor
J	=	J -integral
θ_c	=	crack propagation direction
K_I^M	=	mode I stress intensity factor at crack tip M
K_{II}^M	=	mode II stress intensity factor at crack tip M
F_I^M	=	normalized mode I stress intensity factor at crack tip M
F_{II}^M	=	normalized mode II stress intensity factor at crack tip M
a	=	crack length
h	=	cover size, diameter of the circumscribed circled of mathematical cover
∂R_α	=	boundary of region R_α
n_T	=	a set of nodes whose support completely cut by crack surfaces
n_Λ	=	a set of nodes whose support partially cut by crack surfaces
ω_i	=	support of node i
T	=	tensile strength
σ_1, σ_3	=	major and minor principal stresses
θ_1	=	dip angle of a discontinuity plane
θ_2	=	angle from bedding plane to joint set anticlockwise

LIST OF ABBREVIATION

CFEM	=	Conventional finite element method
DDA	=	Discontinuous deformation analysis
DEM	=	Discrete element method
EFG	=	Element-free Galerkin method
FCM	=	Finite cover method
FEM	=	Finite element method
GFEM	=	Generalized finite element method
NMM	=	Numerical manifold method
PFEM	=	Parasitic finite element method
PU	=	Partition of unity
XFEM	=	Extended finite element method

CHAPTER 1 INTRODUCTION

1.1 Background

Natural geological materials such as rock mass are highly discontinuous. In order to characterize such mechanical discontinuities in a numerical model, either explicitly or implicitly, various efforts have been dedicated by researchers.

The discontinuities are sometimes not taken into account explicitly when the engineering problems take material in quantities large enough so that the material can be described by the averaged material properties, which are assumed to be continuous. The continuum hypothesis is valid as long as the characteristic length of the engineering problem is much larger than its representative volume. The characteristic length is defined by either the smallest dimension of the problem, or the smallest dimension of a part of the problem. The representative volume is the smallest volume over which a measurement can be made that will yield a value representative of the whole. The continuum hypothesis enables us to define material properties as continuous functions of volume. These physical properties are often termed as constitutive law. The constitutive law together with various conservation laws (i.e. conservation of mass, conservation of energy, preservation of momentum balance, etc) results in a set of governing equations. Governing equations are usually given as a set of partial differential equations (i.e. the strong form) or integral equations (i.e. the weak form). Governing equations combined with boundary conditions and initial conditions form a boundary value problem. Analytical solution to a particular boundary value problem is usually absent, thus approximate numerical methods are often adopted. During the last a few decades, various continuum-based numerical methods have been developed, such as the finite difference method (FDM), the finite element method (FEM), the boundary element method (BEM), and various meshless (or meshfree) methods. Among them, the FEM is the most widely used one due to its flexibility in handling material heterogeneity and anisotropy, complex constitutive

models, complex boundary conditions, and dynamic problems.

However, for problems where the representative volume is either much larger than or of a similar order of the engineering problem, the continuum hypothesis is violated and an explicit representation of the discontinuities becomes essential. Representation of fractures in the FEM has been motivated since the late 1960s (Jing, 2003). Various joint element or interface element models, such as the Goodman joint element (Goodman, 1968), the six-node fracture element (Zienkiewicz et al., 1970), the joint element based on the theory of plasticity (Ghaboussi et al., 1973), the thin-layer element (Desai et al., 1984), the interface element in contact mechanics (Katona, 1983), etc., have been developed. Despite these efforts, the treatment of fractures and fracture growth remains limited in the FEM. The finite element mesh needs to coincide with fractures, which often complicates the meshing task. When fracture evolution needs to be modeled, remeshing is inevitable. Although having achieved great success, the meshing and the subsequent remeshing process is absolutely not an easy task. Moreover, additional inaccuracy may arise due to the mapping of various state variables involved in such processes.

In order to overcome the inconveniences in meshing and remeshing processes, a variety of modifications have been made to the conventional FEM based on the partition of unity (PU) (Melenk and Babuska, 1996; Babuska and Melenk, 1997). One example is the extended finite element method (Belytschko and Black, 1999; Moes et al., 1999; Daux et al., 2000; Sukumar et al., 2000; Belytschko et al., 2001; Dolbow et al., 2001; Sukumar et al., 2001; Sukumar and Prevost, 2003; Huang et al., 2003), which incorporates discontinuous enrichment functions into the standard finite element approximation space to account for the cracks and allows the cracks arbitrarily align with the finite element mesh and completely avoids remeshing for discontinuity evolution. Similar methods have been developed by Wells and Sluys (2001), de Borst et al. (2004), Gasser and Holzapfel (2005), Fries and Belytschko (2006). Hansbo and Hansbo (2004) introduced double nodes in the conventional FEM framework to describe the weak and strong discontinuities. Mergheim et al. (2005) called it ‘Hansbo method’ and applied it to the analysis of cohesive cracks. Another example is the generalized finite element method (GFEM) (Strouboulis et al., 2000a;

2000b; 2001; 2003; Duarte et al., 2000; Babuska et al., 2000) developed based on partition of unity. The mesh in the GFEM can be totally independent of the problem domain. A regular finite element mesh is always applicable. The GFEM augments the finite element space with high-order terms or handbook functions of boundary value problems to tackle some typical problems with multiple reentrant corners, voids, and/or cracks.

Other continuum-based numerical methods for fracturing analysis include the BEM and the meshless methods, such as the element-free Galerkin method (EFG) as a representative. However, in general, the BEM is not as efficient as the FEM in dealing with material heterogeneity and non-linearity, thus is suitable only for solving problems of fracturing in homogeneous and linearly elastic bodies. Meshless methods look promising, but are not sufficient to replace the FEM because of their difficulties in numerical integration of the weak form and imposition of essential boundary conditions.

Though great efforts have been made to the continuum-based numerical methods, block rotations, complete detachment and large-scale opening still cannot be properly treated. The number of discontinuities which can be dealt with is also limited.

To solve the aforementioned problems, another class of numerical techniques called the discrete element method (DEM) has been developed. The key concept of the DEM is that the problem domain is treated as an assemblage of rigid or deformable blocks and the contacts between them need to be identified and continuously updated during the entire deformation/motion process, and represented by proper constitutive models (Jing, 2003). The DEM is especially suitable for simulation of large-scale displacements of individual blocks, block rotations, and complete detachment. Typical examples of the DEM include the distinct element method (DEM) originated by Cundall (1971) and the discontinuous deformation analysis (DDA) pioneered by Shi (1988).

It is often the case that individual discrete blocks may also fracture or fragment, which is in essence a process of transition from continua to discontinua. Typical examples include practical rock failure, which involves the initial elastic deformation, crack initiation, propagation of both newly initiated cracks and pre-existing cracks,

crack coalescence, formation of a complete failure surface, and large movement of a discrete block system, rock blasting operation, where an explosive charge is detonated in a borehole to break the rock, rock crushing, where the rock is broken by rock crushing machine to achieve a desired size distribution, structural demolition, in which tall buildings are demolished by introducing carefully placed explosive charges, etc. Aforementioned problems are actually a combination of continua and discontinua, and are therefore termed as combined continua-discontinua problems. The deformation and fracturing of individual block is well represented by a continuum-based model. Interaction among individual blocks is best represented by a discontinuum-based model. Both the conventional continuum-based methods and discontinuum-based methods can only simulate the combined continua-discontinua problems to some extent. Combined continua-discontinua problems can only be best simulated by a combined continuum and discontinuum method.

The recently developed numerical manifold method (NMM) (Shi, 1991; 1992; 1995; 1996a; 1997) is such a hybrid method. The NMM combines the widely used finite element method (FEM) and the discontinuous deformation analysis (DDA) method, and is a natural bridge between the continuum and discrete representations. Using truncated discontinuous shape functions, the continuous block, fractured block and assemblage of discrete blocks are modeled in a unified framework.

The following distinct features make the NMM extremely suitable for discontinuity modeling: (1) its mathematical covers do not need to conform to neither the external boundaries nor the internal discontinuities, which makes the meshing task very convenient and remeshing be completely avoided for discontinuity evolution; (2) the NMM models the discontinuities in a straightforward manner by splitting each mathematical cover which is completely cut by the crack surfaces into several physical covers; each physical cover is assigned an independent local function; by such a way, complex cracks with arbitrary number of branches can be modeled in an exactly same way as the modeling of a single crack; (3) its simplex integration method makes the integration of the weak form over elements intersected by internal discontinuities and/or external boundaries be evaluated conveniently, without the need of element partitioning which is usually adopted by the extended finite element

method (XFEM); (4) its contact logic makes the frictional contact boundary conditions between the two sides of a crack surface or between two discrete blocks be satisfied accurately.

However, the conventional NMM also suffers from some limitations. The conventional NMM adopts polynomials as the local approximation spaces for each physical cover. Polynomials can approximate smooth functions well, but cannot represent well the stress singularities around the crack tips. In addition, the crack tips can be only allowed to stop at the sides of manifold elements. Although a continuous block, a block with pre-existing discontinuities and an assemblage of discrete blocks can be well represented, a fracturing algorithm is essential to transform a continua to a discontinua. This study aims to solve the aforementioned problems encountered by the conventional NMM.

1.2 Objectives of research

The purpose of this research is to extend the NMM for engineering failure analysis. The specific targets are outlined as follows:

- To extend the NMM for complex cracks and their growth:
 - To evaluate the applicability of the conventional NMM for such problems;
 - To incorporate the asymptotic crack tip functions into the local approximation spaces of singular physical covers based on partition of unity to accurately describe the stress singularities around the crack tips and to allow the crack tips stopping at arbitrary positions within a manifold element;
- To compare the NMM and the XFEM/GFEM in crack modeling and to incorporate the advantageous modeling strategy of the NMM into the XFEM/GFEM to improve their capabilities for complex crack problems;
- To extend the NMM for practical rock failure analysis:
 - To implement a fracturing algorithm based on the Mohr-Coulomb criterion with a tensile cutoff to realize the transition from continua to

discontinua;

- To apply the developed program to simulate the progressive failure in rock slopes with non-persistent joints;
- To apply the developed program to investigate the potential failure mechanisms of footwall slopes in surface coal mining with the emphasis on the parametric studies regarding to orientation of bedding plane, orientation of joint set, profile of bedding plane, joint persistence, etc.

1.3 Organization of thesis

The thesis is organized into 9 chapters.

Chapter 1 introduces the background and objectives of this research and the organization of this report.

Chapter 2 presents a review on the numerical manifold method (NMM). The origin and the basic theories of the NMM are briefly introduced. The recent developments to the NMM are reviewed. The comparison between the NMM and various other numerical methods is carried out. The similarities between the NMM and other numerical methods are pointed out. The advantageous features of the NMM compared with other numerical methods are emphasized. The limitations of the conventional NMM are also briefly discussed.

Chapter 3 introduces the numerical manifold method in a systemic manner, including construction of mathematical covers and their associated partition of unity functions, generation of physical covers and their local functions, generation of manifold elements and construction of global approximations over each manifold element, formulations of discrete system of equations, integration scheme, imposition of essential boundary conditions, contact detection and modeling. Several examples are simulated to demonstrate the capability of the NMM in accurately describing the stress/displacement field within a continuous body and in realistically modeling the interactions among multiple discrete blocks.

The NMM is extended to simulate complex cracks and their growth in Chapter 4. A new concept “singular physical cover” is introduced to represent the physical cover

which is partially cut by the crack surface and contains the crack tip singularity. Asymptotic crack tip functions are incorporated into the local approximation spaces of singular physical covers based on partition of unity. The domain form of interaction integral is implemented to evaluate the stress intensity factors at the crack tips. The maximum circumferential stress criterion is incorporated to determine the crack propagation direction. Several typical examples are simulated to verify the developed program.

The NMM is compared with the XFEM/GFEM in the aspect of crack modeling in Chapter 5. How the concept of covers makes the NMM extremely suitable for crack problems is discussed. The advantageous features of the NMM are implemented into the XFEM/GFEM to improve their capability and efficiency in modeling complex crack problems. The modified XFEM/GFEM is termed as the parasitic finite element method (PFEM). The validity of the PFEM is calibrated through two typical examples.

Chapter 6 extends the NMM for practical rock failure analysis. A fracturing algorithm based on the Mohr-Coulomb criterion with a tensile cutoff is implemented into the NMM. The computer implementation details, including treatment of manifold elements, treatment of physical covers, updating physical covers for each manifold element, updating loops for contact detection and modeling during the fracturing process, and the evaluation of the rock debris size using the simplex integration method are fully described.

The developed NMM enabling fracturing is calibrated through several crack initiation and propagation examples in Chapter 7. Then, through a simple crack propagation example, the NMM enabling fracturing is compared with the DDA coupled with a sub-block algorithm (or sub-block DDA in short) in fracturing modeling. The reason to choose the NMM rather than the DDA for rock failure analysis is discussed. Lastly, the NMM enabling fracturing is applied to investigate the progressive failure in rock with non-persistent joints.

Chapter 8 applies the developed program to simulate the potential failure mechanisms of footwall slopes in surface coal mining. Parametric studies regarding to orientation of bedding, orientation of joint set, footwall slope profile (with or without

CHAPTER 1 INTRODUCTION

bedding roll), joint persistence, etc, are carried out. Their influences on the footwall slope failure mechanisms are discussed.

Chapter 9 draws the conclusions and gives the recommendation for future study.

CHAPTER 2 LITERATURE REVIEW

2.1 Origin of the NMM

The numerical manifold method (NMM) is a newly developed numerical approach. It first appeared in a series of conference papers by Shi (1991; 1992; 1995; 1996a; 1997).

The term ‘manifold’ comes from the topological manifold and the differential manifold, which is the main subject of differential geometry, algebraic topology, differential topology and modern algebra of mathematics. The difference of the ‘numerical manifold’ and the traditional differential manifold lies in that the global functions of the differential manifold are highly differentiable and defined irrelevant to the covers, while the global functions of the numerical manifold here are defined based on covers and only piecewise differentiable.

In the NMM, the manifolds connect many overlapped small patches together to cover the entire problem domain. Each small patch is called a cover. A local function is defined on each cover. The global behavior is then determined by the weighted average of local functions defined on each physical cover.

Based on the finite covers, the NMM combines the well developed analytical methods, widely used finite element method (FEM) and the block-oriented discontinuous deformation analysis (DDA) in a unified form.

2.2 Basic concepts of the NMM

The NMM approximation is based on three basic concepts, i.e. the mathematical cover, the physical cover and the manifold element. With reference to an example in Fig. 2.1, these three basic concepts are introduced.

The mathematical covers are user-defined small patches. They may or may not overlap each other, but their union must be large enough to cover the entire problem

domain. In this context, the mathematical covers are consecutively numbered as M_l . The size of the mathematical covers determines the fine or rough approximations of the unknown field.

Fig. 2.1 An example to illustrate the basic concepts of the NMM: (a) problem domain; (b) mathematical covers; (c) physical covers; (d) manifold elements

One distinct feature of the NMM is that its mathematical covers do not need to

conform to neither the external boundaries nor the internal discontinuities, thus can always be regular. The meshing task in the NMM is very convenient. The remeshing is totally avoided for discontinuity evolution. In the example in Fig. 2.1, two mathematical covers, M_1 and M_2 are adopted to cover the problem domain, shown in Fig. 2.1b.

On each mathematical cover M_I , a partition of unity function $\varphi_I(\mathbf{x})$ is defined, which satisfies

$$\begin{aligned}\varphi_I(\mathbf{x}) &\in C^0(M_I) \\ \varphi_I(\mathbf{x}) &= 0, \mathbf{x} \notin M_I\end{aligned}\tag{2.1a}$$

and

$$\sum_{\substack{K \\ \text{if } \mathbf{x} \in M_K}} \varphi_K(\mathbf{x}) = 1\tag{2.1b}$$

Eq. (2.1a) indicates that the partition of unity function has non-zero value only on its corresponding mathematical cover, but zero elsewhere, whereas Eq. (2.1b) is just the partition of unity property to assure a conforming approximation.

In the original NMM, Shi used the term “weight function” for ‘the partition of unity function’. In the recent literature, the term ‘weight function’ has very different meaning. To remove a potential source of confusion, the notation ‘the partition of unity function’ is adopted, same as suggested by Lin (2003).

The physical covers are the subdivision of the mathematical covers by the physical features such as the external boundaries and the internal discontinuities. For example, if a mathematical cover M_I is divided by physical features into several pieces, and m_I of them are within the problem domain, then each piece is termed as a physical cover and totally m_I physical covers are formed and denoted as $P_I^j, j=1 \sim m_I$. In the example in Fig. 2.1, the mathematical cover M_1 is completely cut into three isolated pieces by the physical features and two of them are within the problem domain, so two physical covers, i.e. P_1^1 and P_1^2 are formed (Fig. 2.1c). Similarly, the mathematical cover M_2 also forms two physical covers, i.e. P_2^1 and

P_2^2 (Fig. 2.1c).

Each physical cover inherits the partition of unity function from its associated mathematical cover as

$$\varphi_I^j(\mathbf{x}) = \delta_I^j \cdot \varphi_I(\mathbf{x}) \quad (2.2)$$

where δ_I^j is a modifier, with its value equals to 1 within the physical cover P_I^j and 0 elsewhere.

In addition, each physical cover P_I^j is provided a local function $\mathbf{u}_I^j(\mathbf{x})$, which is usually chosen as polynomial functions. Polynomial functions can approximate smooth functions well. However, for some particular problems where the solution is non-smooth, such as problems with singularities at corners or edges, a prior knowledge about the solution can be used to make a better choice of the local approximation space.

Here, each physical cover has two indices, i.e. I and j . To simplify an implementation, we reallocate a single index to each physical cover, expressed as $P_i \triangleq P_I^j$ with i calculated by

$$i(I, j) = \sum_{l=1}^{I-1} m_l + j \quad (2.3)$$

Similarly, $\varphi_I^j(\mathbf{x})$ can also be reallocated a single index as $\varphi_i(\mathbf{x}) \triangleq \varphi_I^j(\mathbf{x})$, $\mathbf{u}_I^j(\mathbf{x})$ can also be reallocated a single index as $\mathbf{u}_i(\mathbf{x}) \triangleq \mathbf{u}_I^j(\mathbf{x})$, corresponding to the physical cover P_i , with the single index i determined by Eq. (2.3).

The manifold element is defined as the common region of several physical covers. For example, the common region of P_1^1 and P_2^1 forms a manifold element denoted as $E_1(P_1^1, P_2^1)$. The four physical covers in Fig. 2.1c form five manifold elements $E_1(P_1^1, P_2^1)$, $E_2(P_2^1)$, $E_3(P_1^2)$, $E_4(P_1^2, P_2^2)$, and $E_5(P_2^2)$ with their associated physical covers are given in the brackets (Fig. 2.1d).

On each manifold element, we use the partition of unity functions to paste all the local functions together to give a global approximation as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{\substack{i \\ \text{if } e \in P_i}} \varphi_i(\mathbf{x}) \cdot \mathbf{u}_i(\mathbf{x}) \quad (2.4)$$

In the NMM, the global approximation is first constructed in each manifold element. Element stiffness matrices are constructed and then assembled into a global stiffness matrix. In this respect, the manifold element is similar to the finite element in the FEM. Just due to this similarity, the NMM retains all the advantageous features of the system equations in the FEM such as the symmetry and sparsity.

2.3 Recent developments of the NMM

Since the initiation of the NMM in 1991 (Shi, 1991), various developments and applications have been achieved during the last two decades. Most of the publications are included in the series of proceedings of the International Conference on Analysis of Discontinuous Deformation (ICADD) symposia (Li et al., 1995; Salami and Banks, 1996; Ohnishi, 1997; Amadei, 1999; Bicanic, 2001; Hatzor, 2002; Lu, 2003; Maclaughlin and Sitar, 2005; Ju et al., 2007; Ma and Zhou, 2009). The main developments and applications of the NMM are summarized by categorizing them into several groups.

2.3.1 Improvement of the approximation accuracy

The original NMM approximation is based on triangular finite element covers. Shyu and Salami (1995) implemented quadrilateral isoparametric elements into the NMM. Generally speaking, the isoparametric elements are more efficient than the traditional triangular elements. However, for some particular problems such as the bending problem, the precision and efficiency of the quadrilateral isoparametric elements are not adequate either. Cheng et al. (2002) incorporated Wilson non-conforming elements into the NMM, derived the matrices of equilibrium equations and verified its efficiency and precision for a cantilever slab bending problem.

The original NMM approximation chooses constants as the local approximation space for each physical cover. Chen et al. (1998) introduced high-order polynomials

into the local approximation space for each physical cover. They provided detailed formulations on the high-order NMM with high-order local functions and verified its necessity for the complicated deformation problems through several illustration examples. In order to overcome the difficulties in deriving explicit expressions of element matrices for high-order NMM, a simple method is proposed by Su et al. (2003) to automatically produce the expressions and writing the subroutines with the software Mathematica. Lin et al. (2005) developed the formulations of three-dimensional NMM with high-order local functions, and proposed a fast simplex integration method based on special matrix operations.

2.3.2 Treatment of linear dependence problem involved in high-order NMM

Linear dependence problem, which leads to a singular stiffness matrix, widely exists in various partition of unity-based methods, such as the partition of unity method (Babuska and Melenk, 1997), the partition of unity finite element method (Melenk and Babuska, 1996), *hp*-cloud method (Duarte, 1996), the cloud-based *hp* finite element method (Oden et al., 1998), the partition of unity-based hierarchical finite element method (Taylor et al., 1998), and the generalized finite element method (GFEM) (Strouboulis et al., 2000a; 2000b; 2001; 2003; Duarte et al., 2000; 2001).

The NMM adopts a partition of unity to construct its global approximation, thus can also be viewed as a partition of unity-based method. Similar to various other partition of unity-based methods aforementioned, the NMM using finite element covers and polynomial local approximation functions also suffers from the linear dependence problem.

An et al. (2010) has investigated the linear dependence problem of the NMM approximation space using finite element covers and polynomial local functions at both element level and a global level. It was found that the reason of the linear dependence problem lies in the existence of multiple ways to reproduce a polynomial term. Two new methods were proposed to determine the rank deficiency of the NMM approximation space at a global level. These two new methods are much more convenient than the traditional method by counting the number of zero-eigenvalues of

the global stiffness matrix. A theorem about the relation among the number of elements, number of sides, the number of nodes was introduced and adopted to prove the equivalency of these two new methods.

An and Ma (2010) proposed a way to avoid the linear dependence problems. The original finite element covers and their associated partition of unity functions were adjusted. Taking the triangular finite element mesh for an example, the new mathematical cover takes a triangular element as a star, and the union of all the triangular finite elements immediately adjacent to this element as its domain. The new mathematical cover is actually the union of three traditional mathematical covers starred at the three nodes of its element star. The partition of unity function of a new mathematical cover is the summation of three partition of unity functions subordinate to the three traditional mathematical covers. The NMM approximation based on the modified finite element covers is proved linearly independent.

2.3.3 Extension of the NMM for strong discontinuity problems

Tsay et al. (1999) applied the NMM together with a local mesh refinement and auto-remeshing schemes to predict the crack growth. The crack opening displacement is adopted to evaluate the stress intensity factors. Zhang et al. (1999a; 1999b; 2003; 2007; 2010) coupled the NMM with the boundary element method (BEM) to simulate the crack propagation problems. Zhou et al. (1999) coupled the NMM with the element free method for crack propagation. Chiou et al. (2002) proposed the NMM combined with the virtual crack extension method to study the mixed-mode fracture propagation. Wang et al. (2003) extended the NMM to model the strong discontinuities resulting from shear failure. Materials under consideration were assumed to be elastic-brittle, i.e. if the shear stress in material reaches the shear strength, local shear failure would be induced. Interface elements were incorporated along the discontinuities on which Mohr-Coulomb law was applied. Li et al. (2005) and Gao and Cheng (2010) developed enriched meshless manifold method for two-dimensional crack modeling. Influence domains of the nodes form the mathematical covers. Partition of unity functions form the weight functions for each

mathematical cover. The physical covers are formed by the subdivisions of mathematical covers by the problem domain. The meshless manifold method was proved to be more efficient than the traditional meshless methods.

2.3.4 Extension of the NMM for weak discontinuity problems

Terada et al. (2003) introduced the finite cover method (FCM) as an alias of the NMM, presented the formulation for the static equilibrium state of a structure with arbitrary physical boundaries including material interfaces, and extended the FCM to analyze heterogeneous solids and structures involving the discontinuities in strains and discontinuities in displacement caused by debonding or rupture of material interfaces. The displacement compatibility conditions on the material interface are enforced by the Lagrange multiplier method. The FCM may result in generalized elements that are partially within physical domains. Terada and Kurumatani (2004) assessed the performance of such elements and concluded that the performance of the generalized elements in the FCM is equivalent to and is not inferior to that of the FEM. Kurumatani and Terada (2005) extended the FCM to elastoplasticity problems, in which the interface elements (or mortar elements) are introduced to impose the essential boundary conditions, displacement compatibility conditions on material interfaces. Terada et al. (2007a) extended the FCM for analyses of progressive failure processes involving cohesive zone fracture, starting from an interface debonding and evolving toward one of the constituents of heterogeneous solids and structures, in which constituents fail according to the maximum principal stress, while debonding is judged by multipliers of interface elements. Kurumatani and Terada (2009) extended the FCM to crack simulations for quasi-brittle heterogeneous solids by introducing a modeling method called multi-cover-layer modeling, in order to capture both weak and strong discontinuities.

2.3.5 Development of three-dimensional NMM

Terada and Kurumatani (2005) introduced an integrated procedure for three-dimensional structural analysis using the FCM. Formulations of the FCM with

interface elements for the static equilibrium state of a structure and detailed procedure of the 3D finite cover modeling, including the geometry modeling with 3D-CAD and the identification of the geometry covered by a regular patterned unit of mathematical covers for numerical integration are provided. Cheng and Zhang (2008) proposed a three-dimensional numerical manifold method with tetrahedron and hexahedron elements. Jiang et al. (2009) proposed a three-dimensional numerical manifold method based on tetrahedral meshes and derived the matrices of corresponding equilibrium equations from the minimum potential energy principle. He and Ma (2010) also proposed a three-dimensional NMM based on tetrahedron elements. Generation of mathematical covers, formulation of discrete equations, generating discrete blocks from a given fracture pattern were discussed in details.

2.3.6 Development of Eulerian NMM

Su and Xie (2005) proposed a NMM with fixed mathematical covers to compute large displacements of structures. After each time step, deformed boundaries of the structure intersect the fixed mathematical covers to generate new physical covers and new manifold elements. Several approaches are presented to compute initial stresses. Terada et al. (2007b) proposed an Eulerian FCM to perform Eulerian numerical analyses of hyperelastic bodies undergoing large deformation. In order to realize the implicit solution scheme for quasi-static equilibrium problems, they used both the Lagrangian and the Eulerian mathematical meshes simultaneously, and proposed a numerical algorithm to repeatedly perform re-covering and mapping over generalized elements. Okazawa et al. (2010) developed an Eulerian FCM for solid dynamics, which considers the kinematic and dynamic boundary conditions in the Eulerian framework by incorporating the approximation strategy of the FCM into the existing Eulerian explicit FEM.

2.3.7 Other developments and applications of the NMM

Lin (2003) examined the NMM from a partition of unity's perspective, and concluded that the framework of the NMM is similar to meshless methods, and the

most significant differences of the NMM come from its unique way of modeling discontinuities and its adoption of analytical integration scheme. Li et al. (2005) derived the governing equations of the NMM from the method of weighted residual (MWR), which enriched the mathematical foundation of the NMM and extended it to problems such as head conduction and potential flow, where the governing equations cannot be obtained from the minimum potential energy principle or other variational principles. A coupled discontinuous deformation analysis and numerical manifold method (NMM-DDA) has been developed by Miki et al. (2010) to take both methods' advantages while avoiding their shortcomings. The NMM has been applied to simulate various problems, such as saturated-unsaturated unsteady groundwater flow analysis (Ohnishi et al., 1999), rock masses containing joints of two different scales (Lin et al., 1999), data compression (Fang et al., 2005), dynamic non-linear analysis of saturated porous media (Zhang and Zhou, 2006), the shear response of heterogeneous rock joints (Ma et al., 2007a), dynamic friction mechanism of blocky rock system (Ma et al., 2007b), fluid-solid interaction analysis (Su and Huang, 2007), plane micropolar elasticity (Zhao et al., 2010).

2.4 Comparison between the NMM and other numerical methods

In this section, the NMM is compared with various other numerical methods. The similarities of the NMM with other numerical methods are pointed out. The distinct features which differentiate the NMM from other numerical methods are emphasized.

2.4.1 NMM vs. FEM

In the FEM, the problem domain is divided into a collection of elements of smaller sizes and standard shapes with fixed number of nodes at vertices and/or on the sides. Trial functions, usually polynomial, are used to approximate the unknown field at the element level with nodes serving as interpolation points for unknown variables and generate local algebraic equations. The elemental equations are then assembled into a global system of equations whose solution then produces unknowns at the nodes, after imposing the properly defined initial and boundary conditions.

On the other hand, the approximation in the NMM is established based on covers. Firstly, a set of mathematical covers are adopted to cover the entire problem domain. The mathematical covers are user-defined small patches. They do not need to conform to neither the external boundaries nor the internal physical features. Then, the intersection of each mathematical cover by the physical features forms physical covers. Each physical cover is assigned a local function. Common area of several physical covers forms a manifold element. A partition of unity is adopted to paste all the local functions together to give a global approximation over each manifold element. The elemental equations are constructed and assembled into a global system of equations. After imposing boundary conditions and initial conditions, the solutions to the global system of equations can be solved.

The NMM can convert to the standard FEM, if the following conditions are satisfied:

- A finite element mesh is adopted to generate its mathematical covers;
- Finite element shape functions are adopted as the partition of unity functions subordinate to the mathematical covers;
- Constants are adopted as the local approximation space for each physical cover;
- The finite element mesh coincides with the external boundaries and the internal discontinuities (i.e. the physical features cannot intersect the finite elements).

Under such circumstances, the mathematical cover in the NMM is equivalent to the nodal support in the FEM. The constant unknowns of each physical cover are equivalent to the nodal unknowns in the FEM. The manifold elements in the NMM are equivalent to the finite elements in the FEM.

The NMM combines the conventional FEM within its framework. The NMM exceeds the conventional FEM in the following aspects.

The NMM is more flexible than the FEM in discontinuity modeling. The conventional FEM requires the finite element mesh to be consistent with the internal discontinuities, which often complicates the meshing task. When discontinuity evolution needs to be modeled, remeshing is inevitable. Although having achieved

great success, the meshing and the subsequent remeshing process is absolutely not an easy task. Moreover, additional inaccuracy may arise due to the state variable mapping in such processes. In contrast, the mathematical covers in the NMM do not need to conform to neither the external boundaries nor the internal discontinuities, which makes the meshing task in the NMM very convenient and discontinuity evolution be modeled without remeshing.

The meshing task is very convenient in the NMM, which also makes the NMM more suitable than the FEM for complex domain problems with hundreds of inclusions, voids, and/or cracks.

The FEM is a continuum-based method, thus cannot well represent the block rotations, complete detachment and large-scale opening. In addition, the number of discontinuities which can be handled by the FEM is also limited. In contrast, because of its discrete root, the NMM can easily deal with the aforementioned problems.

Because of its partition of unity basis, the implementation of high-order approximations is greatly simplified in the NMM. The concept of edge and middle nodes, which is usually adopted in the conventional p -FEMs is not needed within the framework of the NMM. The implementation of p -adaptivity is simplified. Each physical cover can have a different polynomial order. The approximations can also be non-isotropic, i.e. different polynomial orders in different directions. The implementation of h -adaptivity is also simplified in the NMM. The implementation of h -adaptivity for high-order approximations is exactly same as for linear approximations.

In the conventional FEM, local approximation is always achieved by polynomials. The polynomials can approximate smooth functions well. However, for problems where the solution is rough (or highly oscillatory) (e.g. elasticity equation for laminated materials, material with stiffeners, etc), the usual piecewise polynomials spaces cannot resolve the essential features of the solution unless the mesh size h is very small or the polynomial degree p is very large, which makes the computational costs expensive or even impossible for today's computer (Melenk and Babuska, 1996). Same situation applies to problems with singularities at edges or corners and problems where the solution exhibits a boundary layer. In contrast, because of its partition of

unity basis, the NMM can use any prior known knowledge about the solution of a given boundary value problem to make a better choice of the local approximation space.

2.4.2 NMM vs. Other partition of unity-based methods

Motivated by the need to solve the problems where the classical FEM fails or is prohibitively expensive, various partition of unity (PU)-based methods have been developed during the last few decades. The common feature of these methods is the use of a partition of unity, which is a set of functions whose value sum to the unity at any point $\mathbf{x}$ in a domain Ω , to construct their approximations (Duarte et al., 2000).

The partition of unity method (PUM) first appeared in Melenk (1992), and subsequently in Babuska et al. (1994), Melenk (1995), and the papers that originated from the aforementioned works, e.g. Melenk and Babuska (1996), Babuska and Melenk (1997), Melenk and Babuska (1997), etc. The main features of the PUM lie in two aspects (Babuska and Melenk, 1997). Firstly, the PUM permits inclusion of a priori knowledge about the local behavior of the solution of a boundary value problem into its local spaces to develop a custom tailored approximation with optimal rate of convergence. Secondly, the PUM makes it easy for us to construct ansatz spaces of any desired regularity, and thus trial spaces to be used in variational formulations of higher-order differential equations (e.g. various plate and shell models) are available. Babuska and Melenk (Melenk, 1992; Melenk, 1995; Melenk and Babuska, 1996; Babuska and Melenk, 1997; Melenk and Babuska, 1997) illustrated the development of the PUM for the Laplacian, the elasticity and the Helmholtz problems, and a general class of elliptic equations, with the main focus on employing priori knowledge about the solution to construct their approximations.

Oden and Duarte (Duarte, 1996; Duarte and Oden, 1996a; Duarte and Oden, 1996b; Lyszka et al., 1996; Oden et al., 1998) presented a meshless construction of a PU-based method called the *hp* cloud method, which employs general partitions of unity called ‘clouds’, incorporates special functions which reflect the local character of the solution into the approximation, and develops both a priori and a posteriori

estimates.

Significant progress has been made in the last decade in the development of various so-called meshless (or meshfree) methods such as the smooth particle hydrodynamics (SPH) (Monaghan, 1988; Randles and Libersky, 1996), diffuse element method (Nayroles et al., 1992), element-free Galerkin method (EFG) (Belytschko et al., 1994), reproducing kernel particle methods (RKPM) (Liu et al., 1995; 1996; Chen et al., 1996), moving least squares reproducing kernel method (MLSRK) (Liu et al., 1997), method of finite spheres (De and Bathe, 2000), finite point method (Onate et al., 1996; Sulsky and Schreyer, 1996), natural element method (NEM) (Sukumar et al., 1998), etc. These methods differ from each other in terms of approximations and implementation processes. The PUM provides a leap in the understanding of various meshless methods for they have established ‘the partition of unity’ as a common theoretical thread (Lin, 2003). By basing an analysis on a set of nodes or particles, these methods reduce the demand for meshing, avoid the distortion of mesh when extreme large deformation is encountered (Chen et al., 1998), demonstrate great flexibility in modeling complex discontinuities (Belytschko et al., 1996; Oden and Duarte, 1997), and in addressing high gradient problems such as that occurred in strain localization (Li and Liu, 2000). However, the meshless methods also suffer from some drawbacks. The interpolation functions produced using the moving least-square techniques are non-polynomial functions, which makes the integration of the weak form more demanding compared with classical FEM. Another shortcoming of many meshless methods is the difficulty in enforcement of essential boundary conditions because the generated interpolation functions do not have the Kronecher delta property at nodes (Jing, 2003).

The PUM also impacted on mesh-based finite element methods and led to the development of the extended finite element method (Belytschko and Black, 1999; Moes et al., 1999; Daux et al., 2000; Sukumar et al., 2000; Belytschko et al., 2001; Dolbow et al., 2001; Sukumar et al., 2001; Sukumar and Prevost, 2003; Huang et al., 2003), which incorporates discontinuous enrichment functions into the standard finite element approximation space to account for the cracks and allows the cracks arbitrarily align within the finite element mesh and completely avoids remeshing for

discontinuity evolution, and the generalized finite element method (GFEM) (Strouboulis et al., 2000a; 2000b; 2001; 2003; Duarte et al., 2000; Babuska et al., 2000), which augments the finite element space with analytically or numerically generated solution (or handbook functions) for a given problem, allows finite element mesh constructed completely independent of the problem domain, and has been successfully used in typical problems with multiple reentrant corners, voids, inclusions and/or cracks.

The NMM adopts a partition of unity to construct its global approximation, thus can also be viewed as a partition of unity-based method.

The NMM is similar to various other partition of unity-based methods in the following aspects:

- the NMM also allows the incorporation of high-order terms or handbook functions into its local approximation space because of its partition of unity framework;
- similar to the XFEM/GFEM, the finite element mesh can be constructed totally independent of the presence of cracks, which simplifies the meshing task and makes the crack growth be modeled without remeshing;
- similar to the GFEM, a regular finite element mesh is always applicable.

The key features which differentiate the NMM from various other partition of unity-based methods lie in its unique way in discontinuity modeling:

- its mathematical covers do not need to conform to neither the external boundaries nor the internal cracks, which makes the meshing task very convenient and crack growth be modeled without remeshing;
- the NMM approximation is based on two sets of covers (i.e. mathematical cover and physical cover), which makes displacement discontinuities across complex cracks with arbitrary number of branches be modeled in a straightforward manner;
- its simplex integration scheme (Shi, 1996b) makes the integration of the weak form over manifold elements intersected by cracks be evaluated conveniently;
- its discrete root makes the modeling of interactions of multiple discrete blocks possible.

2.4.3 NMM vs. DDA

The discontinuous deformation analysis (DDA) (Shi, 1988) was developed originally to solve problems in which a rock mass is delimited into blocks by joints. Thus, in addition to the contact detection and frictional contact modeling, the DDA also presents algorithm to form blocks from joints. Since rock blocks generally undergo small deformation, the DDA simulates each discrete body as a simple deformable block with only six degrees of freedom (DOFs) (i.e. normal and shear strains of the block, rigid body translation of a specified point within the block, and the rotation angle of the block with the rotation center at the specified point within the block) in a two-dimensional setting. The NMM is developed based on the DDA. It preserves all the characteristics of the discrete element modeling such as the kinematics constraints, contact detection and modeling from the DDA. However, the NMM employs a number of covers to raise the DOFs to more accurately describe the displacement field and stress field in each block. So, we can say the NMM is the DDA with each block discretized into finite covers. In other words, if one discrete block is one manifold element with linear displacement field, then the NMM will be degraded to the DDA.

2.4.4 NMM vs. UDEC

There is another class of numerical methods called discrete element method (DEM). The DEM treats the problem domain as an assemblage of distinct, intersecting bodies or blocks which are subjected to external loads and are expected to undergo significant motion with time. Joints are viewed as interfaces between the blocks and are treated as a boundary condition rather than a special element in the model. The DEM is especially suitable for simulation of large-scale displacements of individual blocks, block rotations, and complete detachment. Typical examples of the DEM include the distinct element code UDEC (Itasca, 2004) and the aforementioned discontinuous deformation analysis (DDA) pioneered by Shi (1988).

The UDEC represents a problem domain using discrete blocks. The blocks can be rigid or deformable by discretizing each block into a finite difference mesh. Various

constitutive models can be adopted to analyze the movements along discontinuities and deformation of blocks. Movements along discontinuities are represented by spring-slider system. The UDEC has been widely used in various engineering applications.

UDEC is an explicit version of the DEM, while the DDA is an implicit version of the DEM.

The comparison between the NMM and the DDA has been presented in Section 2.4.3. Thus, this sub-section will only focus on the comparison between the NMM and the UDEC.

The NMM differs from the UDEC in the following aspects:

- The UDEC is an explicit method. It calculates the state of a system at current step from the states of previous steps. It requires small time step Δt to keep the error in the result bounded. The NMM is an implicit method. It finds a solution by solving an equation involving both the previous step and the current step. Much larger time step Δt can be used. To achieve given accuracy, the NMM takes much less computational time than the UDEC, even taking into account that the NMM needs to solve the equilibrium equation;
- The UDEC uses numerical integration techniques (e.g. Gauss quadrature) while the NMM adopts closed-form integrations (i.e. simplex integration method) to evaluate the weak form;
- The joints in the UDEC are always assumed as persistent, failure occurs along pre-existing persistent joints. However, it is highly unlikely that such a network of fully persistent discontinuities exist in nature. The NMM allows non-persistent joints. In the NMM, cracks propagate and coalesce with each other to form a continuous failure path, thus seems more realistic;
- The UDEC uses a finite difference mesh while the NMM uses a cover system to resolve the stress / strain vibration within each block;
- The UDEC sometimes uses artificial joints to connect all the blocks together to represent an intact material, and then fracturing in intact material is realized by changing artificial joints to real joints. The fracturing follows pre-defined

artificial joints, thus the results are sensitive to the block configuration. The NMM allows the cracks arbitrarily align with the elements, thus mesh dependency is avoided to some extent.

- The UDEC is a relatively mature method. Various material models (e.g. elastic, Mohr-Coulomb plasticity, double-yield, strain-softening, etc) are available in the code. In addition, relative motion along the discontinuities can be linear or non-linear. The UDEC has been applied to various engineering applications. The NMM is a relatively new method. Only linear-elastic material model is available, and only linear motion along the discontinuities can be accounted for. Its application is also limited. Further developments are needed.

2.5 Advantageous features of the NMM

A detailed comparison between the NMM and various other numerical methods has been carried out in the last section. The distinct features which differentiate the NMM from other numerical methods are summarized as follows:

- A finite element mesh is usually adopted to generate the mathematical covers in the NMM. The so-called finite element mesh however does not need to conform to neither the external boundaries nor the internal discontinuities such as cracks and material interfaces, thus a regular finite element mesh is always applicable. The meshing task is very convenient. The NMM seems extremely suitable for complex domain problems with hundreds of inclusion, voids, and/or cracks.
- The NMM adopts a set of mathematical covers to cover the entire problem domain. The mathematical cover which is intersected by the internal discontinuities is partitioned into several physical covers. By adopting two sets of covers, the displacement jump across crack surface can be modeled in a straightforward manner. Complex cracks with arbitrary number of branches can be modeled in an exactly same way as a single crack.
- The NMM adopts a partition of unity to construct its approximation space.

Because of its partition of unity framework, the implementation of high-order approximations is greatly simplified in the NMM. The concept of edge and middle nodes, which is used in the conventional FEM is not needed. The approximation can be non-uniform, i.e. each physical cover can have a different polynomial order. The approximations can also be non-isotropic, i.e. different polynomial orders in different directions

- Because of its partition of unity framework, any prior known knowledge about the local behavior of the solution of a given boundary value problem can be incorporate into the local approximation spaces for each physical cover to improve the precision and efficiency of the approximations;
- Because of its simplex integration method, the integration of weak form over manifold elements which are intersected by external boundaries and/or internal discontinuities can be evaluated directly, without the need of element partitioning usually adopted in the XFEM;
- Its contact detection and modeling algorithm make the intersections of multiple discrete blocks be modeled in a realistic manner.

The above excellent features make the NMM promising for discontinuity modeling, especially for practical rock failure analysis.

2.6 Limitations of the conventional NMM

When the conventional NMM is applied to simulate the discontinuity problems, it suffers from two apparent limitations.

Firstly, the conventional NMM adopts polynomials as the local approximation space for each physical cover. Polynomials can approximate smooth functions well, but cannot represent well the singularities around the crack tips. In addition, the crack tips can only be allowed to stop at the sides of manifold elements.

Secondly, although a continuous block, a block with pre-existing discontinuities and an assemblage of discrete blocks can be well represented in the conventional NMM, a fracturing algorithm is essential to fulfill the transition from continua to discontinua.

CHAPTER 3 THE NUMERICAL MANIFOLD METHOD AND ITS CALIBRATION

3.1 Introduction

This chapter firstly introduces the numerical manifold method in a systemic manner, including construction of mathematical covers and their associated partition of unity functions, generation of physical covers and their local functions, generation of manifold elements and construction of global approximations over each manifold element, formulations of discrete system of equations, integration scheme, imposition of essential boundary conditions, and contact detection and modeling algorithms. Then, several examples are simulated to demonstrate the capability of the NMM in accurately resolving the stress/displacement field within a continuous body and in realistically modeling the interactions among multiple discrete blocks.

3.2 Fundamentals of the NMM

3.2.1 NMM approximation

As presented in Section 2.2, the NMM approximation is based on three basic concepts, i.e. the mathematical cover, the physical cover and the manifold element. The mathematical covers are a set of small patches used to cover the entire problem domain. On each mathematical cover, a partition of unity function is defined. The intersections of mathematical covers with the problem domain form physical covers. On each physical cover, a local function is defined. The common area of several physical covers forms a manifold element. The NMM uses the partition of unity functions to paste all the local functions together to give a global approximation over each manifold element. The elemental equations are constructed and assembled into a global system of equations. After imposing the boundary conditions and initial

conditions, the solutions to the equations are obtained.

Theoretically any shape of covers can be used in the NMM. However, construction of partition of unity functions and integration of the weak form are related to the cover shape, thus a reasonable choice of cover shape is very important (Chen et al., 1998; Li et al., 2005). To this end, the finite element mesh is usually adopted to generate the covers and their associated functions.

In the following three subsections, construction of finite element covers and their associated functions are presented.

Construction of mathematical covers and their partition of unity functions

The finite element mesh can be adopted to generate the mathematical covers. The so-called finite element mesh here does not need to conform to neither the external boundaries nor the internal discontinuities, thus actually different from the one used in the FEM. A regular finite element mesh is always applicable. This distinct feature makes the meshing task in the NMM very convenient.

Each node in the finite element mesh is regarded as a star. The union of all the finite elements sharing a common star forms a mathematical cover. Finite element shape functions naturally form a partition of unity subordinate to the mathematical covers.

For a two-dimensional problem, the mathematical covers can be generated from a regularly-patterned triangular finite element mesh or a regularly-patterned rectangular finite element mesh.

As shown in Fig. 3.1, we can use a finite element mesh composed of equilateral triangles to construct the mathematical covers. Each node in the triangular finite element mesh is termed as a star. The union of six triangles sharing a common star forms a hexagonal mathematical cover. A star and its associated mathematical cover are illustrated in Fig. 3.1.

Each triangular element is associated with three mathematical covers starred at its three nodes. Denote the three mathematical covers as $M_{(i)}$, $i = 1 \sim 3$, where i denotes the local index, the subscript $()$ transfers the local index into a global index.

Fig. 3.1 Generation of mathematical covers from a finite element mesh composed of equilateral triangles

On each triangular element, the three partition of unity functions of the three mathematical covers take the form as

$$\varphi_{(i)}(x, y) = S_{(i)} / S \quad (i = 1, 2, 3) \quad (3.1)$$

where $S_{(i)}$, $i = 1, 2, 3$ are the areas of the sub-triangles shown in Fig. 3.2, and S is the area of the whole triangle, with their expressions given as

$$S = \frac{1}{2} \begin{vmatrix} 1 & x_{(1)} & y_{(1)} \\ 1 & x_{(2)} & y_{(2)} \\ 1 & x_{(3)} & y_{(3)} \end{vmatrix} = \frac{1}{2} \left[(x_{(2)} - x_{(1)})(y_{(3)} - y_{(2)}) - (x_{(3)} - x_{(2)})(y_{(2)} - y_{(1)}) \right] \quad (3.2a)$$

$$S_{(1)} = \frac{1}{2} \begin{vmatrix} 1 & x & y \\ 1 & x_{(2)} & y_{(2)} \\ 1 & x_{(3)} & y_{(3)} \end{vmatrix} = \frac{1}{2} \left[(x_{(2)} - x)(y_{(3)} - y_{(2)}) - (x_{(3)} - x_{(2)})(y_{(2)} - y) \right] \quad (3.2b)$$

$$S_{(2)} = \frac{1}{2} \begin{vmatrix} 1 & x_{(1)} & y_{(1)} \\ 1 & x & y \\ 1 & x_{(3)} & y_{(3)} \end{vmatrix} = \frac{1}{2} \left[(x - x_{(1)})(y_{(3)} - y_{(1)}) - (x_{(3)} - x_{(1)})(y - y_{(1)}) \right] \quad (3.2c)$$

$$S_{(3)} = \frac{1}{2} \begin{vmatrix} 1 & x_{(1)} & y_{(1)} \\ 1 & x_{(2)} & y_{(2)} \\ 1 & x & y \end{vmatrix} = \frac{1}{2} \left[(x_{(2)} - x_{(1)})(y - y_{(2)}) - (x - x_{(2)})(y_{(2)} - y_{(1)}) \right] \quad (3.2d)$$

Fig. 3.2 Area coordinates in a triangular element

The partition of unity function of one hexagonal mathematical cover is depicted in Fig. 3.3. The function equals to the unity at the center point, and linearly declines to zero at all the six sides of the mathematical cover.

Fig. 3.3 Illustration of the partition of unity function of a hexagonal mathematical cover

It can be easily verified that the functions given in Eq. (3.1) satisfies the partition of unity condition in Eq. (2.1b) since the following identity holds

$$S = S_1 + S_2 + S_3 \quad (3.3)$$

We can also take the finite element mesh composed of right-angled triangles to construct the mathematical covers, as shown in Fig. 3.4. The union of six right-angled triangles sharing a common star forms an obliquely hexagonal mathematical cover. A star and its associated mathematical cover are also illustrated in Fig. 3.4.

Fig. 3.4 Generation of mathematical covers from a finite element mesh composed of right-angled triangles

Each right-angled triangle is associated with three mathematical covers. On each triangle, the three partition of unity functions of the three mathematical covers starred at its three nodes can also be obtained from Eqs. (3.1) and (3.2).

Similarly, we can also take the regularly-patterned rectangular finite element mesh to construct the mathematical covers, as shown in Fig. 3.5. The union of all the rectangular elements sharing a star forms a mathematical cover. A star and its associated mathematical cover are also given in Fig. 3.5.

Each finite element is associated with four mathematical covers starring at its four nodes. Denote the four mathematical covers as $M_{(i)}$, $i = 1 \sim 4$, where i denotes the local index, again the subscript $()$ transfers the local index into a global index.

On each rectangular element, the four partition of unity functions of the four mathematical covers take the form as

$$\varphi_{(1)}(x, y) = \frac{x - x_{(4)}}{x_{(1)} - x_{(4)}} \frac{y - y_{(2)}}{y_{(1)} - y_{(2)}} \quad (3.4a)$$

$$\varphi_{(2)}(x, y) = \frac{x - x_{(3)}}{x_{(2)} - x_{(3)}} \frac{y - y_{(1)}}{y_{(2)} - y_{(1)}} \quad (3.4b)$$

$$\varphi_{(3)}(x, y) = \frac{x - x_{(2)}}{x_{(3)} - x_{(2)}} \frac{y - y_{(4)}}{y_{(2)} - y_{(4)}} \quad (3.4c)$$

$$\varphi_{(4)}(x, y) = \frac{x - x_{(1)}}{x_{(4)} - x_{(1)}} \frac{y - y_{(3)}}{y_{(4)} - y_{(3)}} \quad (3.4d)$$

Fig. 3.5 Generation of mathematical covers from a regularly-patterned rectangular finite element mesh

The partition of unity function of a rectangular mathematical cover is depicted in Fig. 3.6. The function equals to 1 at its star (i.e. the center point) and bi-linearly declines to zero at its four sides.

Fig. 3.6 Illustration of the partition of unity function of a rectangular mathematical cover

It can be proved that the functions given in Eq. (3.4) satisfy the partition of unity condition in Eq. (2.1b).

Construction of physical covers and their local functions

The intersections of the mathematical covers and the problem domain form the physical covers.

In the following part, we will take one example to illustrate the generation of physical covers.

Fig. 3.7 shows a problem domain containing a joint and a tunnel.

Fig. 3.7 Schematic of a problem with a joint and a tunnel

Then, we use a regularly-patterned rectangular finite element mesh to cover the entire problem domain, as shown in Fig. 3.8.

Fig. 3.8 NMM model for the problem with a joint and a tunnel

Taking node 1 as a star, only rectangular element 1 contains this star, thus element 1 forms the mathematical cover M_1 . The M_1 is completely cut by the joint into two parts, and both parts are within the problem domain, thus two physical covers P_1^1 and P_1^2 are formed. As discussed in Section 2.2, for easy implementation, we reallocate a single index to each physical cover, as $P_1 \triangleq P_1^1$, $P_2 \triangleq P_1^2$. Taking node 2 as a star, the rectangular elements 1 and 2 shares this star, thus the union of elements 1 and 2 form the mathematical cover M_2 . The M_2 is completely cut by the external boundary and the joint into three parts, and two of them are within the problem domain, thus two physical covers P_2^1 and P_2^2 are formed. We reallocate a single index to each physical cover, as $P_3 \triangleq P_2^1$, $P_4 \triangleq P_2^2$. Other mathematical covers and

the physical covers can be generated in a similar way. Generation of physical covers are illustrated in Fig. 3.9, where the mathematical cover appears at the left side of the arrow, the physical covers generated from this mathematical cover are given at the right side of the arrow, and the number above the arrows indicates how many physical covers are generated from this mathematical cover.

CHAPTER 3 THE NUMERICAL MANIFOLD METHOD AND ITS CALIBRATION

Fig. 3.9 Generation of physical covers for the NMM model in Fig. 3.8

Each physical cover is assigned a local function. In the NMM, the local functions are usually taken as the polynomials as

$$\mathbf{u}_i(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \cdot \mathbf{a}_i \quad (3.5)$$

where $\mathbf{a}_i$ is an array of constant coefficients, and $\mathbf{p}^T(\mathbf{x})$ is the matrix of polynomial bases as

$$\mathbf{p}^T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 & \cdots & y^p & 0 \\ 0 & 1 & 0 & x & 0 & y & \cdots & 0 & y^p \end{bmatrix} \quad (3.6)$$

Polynomials can describe smooth functions well. For some particular problems such as problems containing singularities at edges or corners, any prior known knowledge about the solution can be adopted to make a better choice for the local functions of each physical cover.

Construction of manifold elements and the global approximation

The common area of several physical covers forms the manifold element.

Consider the NMM model in Fig. 3.8 for an example. The common area of four physical covers $P_1^I, P_2^I, P_6^I, P_7^I$ forms one manifold element, denoted as $E_1(P_1^I, P_2^I, P_6^I, P_7^I)$, where the subscript represents the number of this manifold element, while the associated physical covers are given in the bracket. The common area of four physical covers $P_1^2, P_2^2, P_6^2, P_7^2$ forms one manifold element, denoted as $E_2(P_1^2, P_2^2, P_6^2, P_7^2)$. Other manifold elements and the associated physical covers can be obtained in a similar way, as illustrated in Fig. 3.10.

Fig. 3.10 Generation of manifold elements for the NMM model in Fig. 3.8

The NMM adopts the partition of unity functions to paste all the local functions together to give the global approximation with the expression given in Eq. (2.4).

3.2.2 Formulation of discrete equations

Strong form of governing equations

We here present the governing equation for the displacement field for the dynamic equilibrium of an elastic body as shown in Fig. 3.11, where Ω is problem domain, and bounded by the boundary Γ , which is composed of natural (or traction) boundary Γ_t and the essential (or displacement) boundary Γ_u , i.e. $\Gamma = \Gamma_t \cup \Gamma_u$.

Fig. 3.11 A two-dimensional elastic problem

The governing equation for this problem is

$$\nabla \cdot \sigma + b = \rho \ddot{u} \quad \text{in } \Omega \quad (3.7)$$

where ∇ is the gradient operator, σ is the Cauchy stress tensor, b is the body force per unit volume, ρ is the density of the material, and $\ddot{u}$ is the acceleration tensor.

For an elastic material, the constitutive relationship between the stress and strain follows the generalized Hooke's law as

$$\sigma = C : \epsilon \quad (3.8)$$

where C is the elasticity (material) moduli tensor, and ϵ is the infinitesimal strain

tensor.

Under the assumption of small deformation, the strain displacement relation is

$$\boldsymbol{\varepsilon} = \nabla_s \mathbf{u} \quad (3.9)$$

where ∇_s is the symmetric gradient tensor, and $\mathbf{u}$ is the displacement vector.

The natural boundary condition is

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \Gamma_t \quad (3.10)$$

and the essential boundary condition is

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u \quad (3.11)$$

in which, $\bar{\mathbf{u}}$ and $\bar{\mathbf{t}}$ are respectively the values of displacement and traction prescribed on the corresponding boundaries.

So far, the problem is clearly stated in the tensor form. For the convenience of numerical implementation, the matrix form is often used. Accordingly, the equilibrium equation in Eq. (3.7) is rewritten in a matrix form as

$$\mathbf{L}\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}} \quad \text{in } \Omega \quad (3.12)$$

where $\boldsymbol{\sigma} = \{\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}\}^T$ is the stress vector, $\mathbf{b} = \{b_1 \quad b_2\}^T$ is the body loading per unit volume, $\ddot{\mathbf{u}}$ is the acceleration vector, $\mathbf{L}$ is a differential operator matrix, defined as

$$\mathbf{L} = \begin{bmatrix} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_1 \end{bmatrix} \quad (3.13)$$

The constitutive relationship in Eq. (3.8) is rewritten as

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \quad (3.14)$$

where $\boldsymbol{\varepsilon}^T = \{\varepsilon_{11} \quad \varepsilon_{22} \quad \gamma_{12} = 2\varepsilon_{12}\}$ is the strain vector, $\mathbf{D}$ is the elastic matrix, expressed as

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (3.15)$$

for a plane stress problem, where E and ν are respectively the Young's modulus and Poisson's ratio, and are respectively replaced by $E/(1-\nu^2)$ and $\nu/(1-\nu)$ for a plane strain problem.

The relationship between strain and displacement in Eq. (3.9) is rewritten as

$$\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u} \quad (3.16)$$

where $\mathbf{u} = \{u_1 \ u_2\}^T$ is the displacement vector.

Weak form of governing equations

In the numerical implementation, the governing equation is relaxed to be satisfied in the sense of integration over a certain domain. It makes the equilibrium equation weaker than those in the sense of differentiation. In this subsection, the principle of virtual work is employed to derive the weak form of the governing equations.

The principle of virtual work, which states that the virtual work done by external forces is equal to the virtual strain energy of the system, leads to the weak form as

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV - \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} dV - \int_{\Gamma_t} \delta \mathbf{u}^T \bar{\mathbf{t}} d\Gamma + \int_{\Omega} \delta \mathbf{u}^T \rho \ddot{\mathbf{u}} dV = 0 \quad (3.17)$$

In the dynamic analysis, an increment approach is often adopted. In this approach, the total time is divided into a finite number of time steps, and the variables such as displacement, velocity and acceleration are known at time t_n , the beginning of the current step, and the increments of various variables during the current time step $t_n \sim t_{n+1}$ are to be solved. So the incremental form of the principle of virtual work must be derived. To this end, the incremental form of Eq. (3.17) is obtained

$$\int_{\Omega} \delta \Delta \boldsymbol{\varepsilon}^T \Delta \boldsymbol{\sigma} dV - \int_{\Omega} \delta \Delta \mathbf{u}^T \Delta \mathbf{b} dV - \int_{\Gamma_t} \delta \Delta \mathbf{u}^T \Delta \bar{\mathbf{t}} d\Gamma + \int_{\Omega} \delta \Delta \mathbf{u}^T \rho \Delta \ddot{\mathbf{u}} dV = 0 \quad (3.18)$$

NMM interpolations

In each manifold element, the NMM interpolation is written as

$$\mathbf{u}^h = \mathbf{N}\mathbf{d} \quad (3.19)$$

where $\mathbf{N}$ is the NMM shape function, $\mathbf{d}$ is the unknown vector.

The strain is expressed as

$$\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{N}\mathbf{d} = \mathbf{B}\mathbf{d} \quad (3.20)$$

where $\mathbf{B} = \mathbf{L}\mathbf{N}$ is the strain matrix.

The constitutive relation is

$$\boldsymbol{\sigma} = \mathbf{D}\mathbf{L}\mathbf{N}\mathbf{d} = \mathbf{S}\mathbf{d} \quad (3.21)$$

where $\mathbf{S} = \mathbf{D}\mathbf{L}\mathbf{N}$ is the stress matrix.

Eq. (3.19) can be rewritten in an incremental form as

$$\Delta\mathbf{u} = \mathbf{N}\Delta\mathbf{d} \quad (3.22)$$

The incremental strain is expressed as

$$\Delta\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{N}\Delta\mathbf{d} = \mathbf{B}\Delta\mathbf{d} \quad (3.23)$$

The constitutive relation in the incremental form as

$$\Delta\boldsymbol{\sigma} = \mathbf{D}\mathbf{L}\mathbf{N}\Delta\mathbf{d} = \mathbf{S}\Delta\mathbf{d} \quad (3.24)$$

Discrete system of equations

With Eqs. (3.22)-(3.24), by applying Eq. (3.18) on element level and considering the arbitrariness of variation $\delta\{\Delta\mathbf{d}\}$, we finally obtain the discrete equations for a dynamic analysis

$$\mathbf{K}\Delta\mathbf{d} + \mathbf{M}\Delta\ddot{\mathbf{d}} = \Delta\mathbf{F} \quad (3.25)$$

where the stiffness matrix $\mathbf{K}$, mass matrix $\mathbf{M}$ and the incremental load at the current time step $\Delta\mathbf{F}$ are given as

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} dV \quad (3.26)$$

$$\mathbf{M} = \int_{\Omega} \rho \mathbf{N}^T \mathbf{N} dV \quad (3.27)$$

and

$$\Delta \mathbf{F} = \int_{\Omega} \mathbf{N}^T \Delta \mathbf{b} dV + \int_{\Gamma_t} \mathbf{N}^T \Delta \bar{\mathbf{t}} d\Gamma \quad (3.28)$$

respectively.

Note that the stiffness and mass matrices are symmetric from Eqs. (3.26) and (3.27), and in fact they are also banded and sparse.

In the dynamic analysis, the acceleration and the velocity (if any) can also be expressed by using the Newmark scheme, expressed as

$$\begin{aligned} \mathbf{d}_{n+1} &= \mathbf{d}_n + \Delta t \dot{\mathbf{d}}_n + \frac{(\Delta t)^2}{2} ((1-2\beta)\ddot{\mathbf{d}}_n + 2\beta\ddot{\mathbf{d}}_{n+1}) \\ \dot{\mathbf{d}}_{n+1} &= \dot{\mathbf{d}}_n + \Delta t ((1-\gamma)\ddot{\mathbf{d}}_n + \gamma\ddot{\mathbf{d}}_{n+1}) \end{aligned} \quad (3.29)$$

where γ and β are two constants.

Then,

$$\Delta \ddot{\mathbf{d}} = \frac{1}{\beta(\Delta t)^2} \left[\Delta \mathbf{d} - \Delta t \dot{\mathbf{d}}_n - \frac{(\Delta t)^2}{2} \ddot{\mathbf{d}}_n \right] \quad (3.30)$$

If we use the scheme of the average acceleration, i.e. $\beta = 1/4$ and $\gamma = 1/2$, Eq. (3.30) becomes

$$\Delta \ddot{\mathbf{d}} = \frac{4}{(\Delta t)^2} \Delta \mathbf{d} - \frac{4}{\Delta t} \dot{\mathbf{d}}_n - 2\ddot{\mathbf{d}}_n \quad (3.31)$$

Substituting Eq. (3.31) into Eq. (3.25), we have

$$\bar{\mathbf{K}} \Delta \mathbf{d} = \Delta \bar{\mathbf{F}} \quad (3.32)$$

where $\bar{\mathbf{K}}$ and $\Delta \bar{\mathbf{F}}$ are respectively the equivalent stiffness matrix and the equivalent loading vector, expressed as

$$\bar{\mathbf{K}} = \mathbf{K} + \frac{4}{(\Delta t)^2} \mathbf{M} \quad (3.33)$$

and

$$\Delta \bar{\mathbf{F}} = \Delta \mathbf{F} + \frac{4}{\Delta t} \mathbf{M} \dot{\mathbf{d}}_n + 2 \mathbf{M} \ddot{\mathbf{d}}_n \quad (3.34)$$

The element matrices will be assembled to form the system matrix, and then the system equation is solved for the unknowns.

In this subsection, the discrete equations are derived based on a dynamic analysis. For a static analysis, discrete equations can be derived similarly by simply removing the inertia item (i.e. the right-hand term in Eq. (3.12)) from the governing equation.

3.2.3 Integration scheme

Although the mathematical covers are usually formed by a regularly-patterned triangular or rectangular finite element mesh, the manifold elements may have arbitrary shapes because of the intersection with the external boundaries and/or the internal discontinuities. Thus, direction integration scheme over the whole element like that in the conventional FEM is not applicable. In the NMM, the simplex integration method is adopted to evaluate the integrations over an arbitrary manifold element.

Assume an arbitrary manifold element with ordered vertices $V_1, V_2, \dots, V_6$, $V_7 = V_1$. The integration over this manifold element is expressed as

$$\iint_A f(x, y) dx dy \quad (3.35)$$

where $f(x, y)$ is an arbitrary polynomial function, A represents the domain occupied by manifold element $V_1 V_2 V_3 \dots V_7$.

Simplex integration takes simplex as its integration domain. Simplex has orientations, i.e. positive and negative. Simplex in a two-dimensional setting is an oriented triangle. In order to adopt the simplex integration to evaluate the integration of the weak form over the arbitrary manifold element $V_1 V_2 V_3 \dots V_7$, we need to partition this element into a set of simplex, i.e. oriented triangles. Triangulation of this arbitrary manifold element $V_1 V_2 V_3 \dots V_7$ is illustrated in Fig. 3.12. A set of oriented

triangles are obtained as $\Delta_{V_0V_1V_2}$, $\Delta_{V_0V_2V_3}$, ..., $\Delta_{V_0V_6V_7}$, where V_0 is an arbitrary point. In Fig. 3.12, the orientation of each triangle is represented by an oriented circle. Each triangle with three ordered vertices is a two-dimensional simplex. The sign '+' means the simplex area of the oriented triangle (or simplex) is positive, while '-' means simplex area of the oriented triangle (or simplex) is negative. '+' or '-' of an orientated triangle can be determined by the sign of its Jacobi determinant J , where the expression of J will be provided later in Eq. (3.40). In Fig. 3.12, simplex areas of $\Delta_{V_0V_1V_2}$, $\Delta_{V_0V_4V_5}$, $\Delta_{V_0V_5V_6}$, $\Delta_{V_0V_6V_7}$ are positive, while simplex areas of $\Delta_{V_0V_2V_3}$ and $\Delta_{V_0V_3V_4}$ are negative. The algebraic sum of all these simplex areas is exactly the area of polygonal manifold element $V_1V_2V_3 \cdots V_7$. In ordinary integration, the integration has no orientations; therefore the integration has no algebraic addition. By introducing orientation to each triangle, algebraic addition becomes applicable.

Fig. 3.12 Triangulation of an arbitrary manifold element for simplex integration

The integration over an arbitrary manifold element $V_1V_2V_3 \cdots V_{n+1}$, where $V_{n+1} = V_1$, becomes a sum of simplex integration over each two-dimensional simplex

$$\iint_A f(x, y) dx dy = \sum_{k=1}^n \int_{V_0V_kV_{k+1}} f(x, y) D(x, y) \quad (3.36)$$

where n is the number of edge of the polygonal manifold element, $D(x,y)$ represents two-dimensional simplex integration, defined as normal integration times the sign of Jacobi determinant J , where the expression of J will be provided later in Eq. (3.40).

Integration over a two-dimensional simplex can be evaluated analytically. Its derivation follows two steps. First, a basic integration in terms of area coordinates, u_0 , u_1 and u_2 over a coordinate simplex with vertices $U_0 (0,0)$, $U_1 (1,0)$, and $U_2 (0,1)$, is obtained as

$$\iint_{U_0 U_1 U_2} u_0^{i_0} u_1^{i_1} u_2^{i_2} du_1 du_2 = \frac{i_0! i_1! i_2!}{(i_0 + i_1 + i_2 + 2)!} \quad (3.37)$$

Second, a general integration in normal coordinate over a simplex with vertices of $V_0 (x_0, y_0)$, $V_k (x_k, y_k)$, and $V_{k+1} (x_{k+1}, y_{k+1})$ is derived by the following coordinate transformation

$$\begin{aligned} 1 &= u_0 + u_k + u_{k+1} \\ x &= x_0 u_0 + x_k u_k + x_{k+1} u_{k+1} \\ y &= y_0 u_0 + y_k u_k + y_{k+1} u_{k+1} \end{aligned} \quad (3.38)$$

such that

$$\int_{V_0 V_k V_{k+1}} x^{m_1} y^{m_2} D(x, y) = \text{sign}(J) \iint_{V_0 V_k V_{k+1}} x^{m_1} y^{m_2} dx dy = J \iint_{U_0 U_1 U_2} x^{m_1} y^{m_2} du_1 du_2 \quad (3.39)$$

where J is the Jacobi determinant, expressed as

$$J = \begin{vmatrix} \frac{\partial x}{\partial u_k} & \frac{\partial x}{\partial u_{k+1}} \\ \frac{\partial y}{\partial u_k} & \frac{\partial y}{\partial u_{k+1}} \end{vmatrix} = \begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_k & y_k \\ 1 & x_{k+1} & y_{k+1} \end{vmatrix} \quad (3.40)$$

The simplex integrations of monomial polynomials up to degree 1 over the arbitrary manifold element $V_1 V_2 V_3 \cdots V_n V_{n+1}$ is derived for illustration purpose as following

$$\begin{aligned}
 \iint_A 1 dx dy &= \sum_{k=1}^n \int_{V_0 V_k V_{k+1}} 1 D(x, y) = \sum_{k=1}^n J \iint_{U_0 U_1 U_2} du_1 du_2 \\
 &= \sum_{k=1}^n J \frac{0!0!0!}{(0+0+0+2)!} \\
 &= \sum_{k=1}^n \frac{1}{2} J
 \end{aligned} \tag{3.41}$$

$$\begin{aligned}
 \iint_A x dx dy &= \sum_{k=1}^n \int_{V_0 V_k V_{k+1}} x D(x, y) = \sum_{k=1}^n J \iint_{U_0 U_1 U_2} (x_0 u_0 + x_k u_k + x_{k+1} u_{k+1}) du_1 du_2 \\
 &= \sum_{k=1}^n J \frac{1!}{(1+2)!} (x_0 + x_k + x_{k+1}) \\
 &= \sum_{k=1}^n \frac{1}{6} J (x_0 + x_k + x_{k+1})
 \end{aligned} \tag{3.42}$$

$$\begin{aligned}
 \iint_A y dx dy &= \sum_{k=1}^n \int_{V_0 V_k V_{k+1}} y D(x, y) = \sum_{k=1}^n J \iint_{U_0 U_1 U_2} (y_0 u_0 + y_k u_k + y_{k+1} u_{k+1}) du_1 du_2 \\
 &= \sum_{k=1}^n J \frac{1!}{(1+2)!} (y_0 + y_k + y_{k+1}) \\
 &= \sum_{k=1}^n \frac{1}{6} J (y_0 + y_k + y_{k+1})
 \end{aligned} \tag{3.43}$$

The simplex integrations of higher-order monomial polynomials can be derived in a similar manner.

Then, the simplex integration of arbitrary polynomial functions will be the linear combination of simplex integrations of monomial polynomials. For example,

$$\iint_A (a_0 + a_1 x + a_2 y) dx dy = \sum_{k=1}^n \left(\frac{1}{2} a_0 J + \frac{1}{6} a_1 J (x_k + x_{k+1}) + \frac{1}{6} a_2 J (y_k + y_{k+1}) \right) \tag{3.44}$$

where a_0 , a_1 and a_2 are constant coefficients.

Details on the simplex integration method can be found in Shi (1996b).

Once the coordinates of all the vertices of a manifold element are known, the simplex integration over this manifold element can be easily evaluated no matter whatever shape it is. Element partitioning involved in the PU-based XFEM is totally unnecessary. However, from the above derivation, we can see that the simplex integration scheme is limited to the cases where polynomials are chosen as local

approximation spaces. For the case where non-polynomial functions are adopted, the simplex integration is not applicable any more. In such circumstances, we turn to the numerical integration method. Details will be provided in Section 4.2.5.

3.2.4 Imposition of boundary conditions

In general, the boundary conditions are categorized into two distinct types, i.e. the natural boundary condition and the essential boundary condition.

In structural analysis, the natural boundary conditions are for tractions at the boundaries, as expressed in Eq. (3.10). The natural boundary conditions are satisfied by including it in the weak form given in Eq. (3.17).

So, here the focus will be on the imposition of essential boundary conditions, i.e. the constraints on displacements.

For a general problem, the essential boundary conditions are given in Eq. (3.11).

Another kind of boundary condition needed to be treated is the displacement compatibility condition along the material interface.

Fig. 3.13 shows a problem domain with two-phase materials. The whole domain is divided into two distinct sub-domains, i.e. $\Omega^{[1]}$ and $\Omega^{[2]}$, corresponding to one material and to the other, respectively. The interface between $\Omega^{[1]}$ and $\Omega^{[2]}$ called material interface, is denoted by $\Gamma^{[1-2]}$. If we recognize that the material interface $\Gamma^{[1-2]}$ consists of two coincident surfaces, i.e. $\Gamma^{[1]}$ and $\Gamma^{[2]}$, which are associated with the sub-domains $\Omega^{[1]}$ and $\Omega^{[2]}$, respectively, then we can define outward unit normal vectors $\mathbf{n}^{[1]}$ and $\mathbf{n}^{[2]}$, associated with these two distinct surfaces. The interface compatibility condition along the material interface $\Gamma^{[1-2]}$ can be expressed as

$$\begin{aligned}\mathbf{u}^{[1]} &= \mathbf{u}^{[2]} \\ \mathbf{t}^{[1]} &= -\mathbf{t}^{[2]}\end{aligned}\tag{3.45}$$

In the NMM, the mathematical covers are constructed totally independent of the external boundaries and material interfaces. The essential boundary conditions thus cannot be accomplished by directly enforcing the degrees of freedom (DOFs) like in

the FEM, but is usually prescribed by the Lagrange multiplier method, the penalty method or the augmented Lagrange multiplier method in the weak form of governing equations. These three imposition techniques will be discussed in the next three subsections, respectively.

Fig. 3.13 The interface compatibility between two phases of materials

The Lagrange multiplier method

The Lagrange multiplier method can be employed to impose the two essential boundary conditions in the weak sense method (Terada et al., 2003; Terada and Kurumatani, 2005; 2009).

To this end, Eq. (3.17) is modified to

$$\begin{aligned} & \int_{\Omega} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV - \int_{\Gamma^{1-2}} \delta (\mathbf{u}^{[1]} - \mathbf{u}^{[2]})^T \cdot \boldsymbol{\lambda} d\Gamma - \int_{\Gamma_u} \delta (\mathbf{u} - \bar{\mathbf{u}})^T \cdot \boldsymbol{\lambda} d\Gamma \\ & = \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} dV + \int_{\Gamma_t} \delta \mathbf{u}^T \bar{\mathbf{t}} d\Gamma - \int_{\Omega} \delta \mathbf{u}^T \rho \ddot{\mathbf{u}} dV \end{aligned} \quad (3.46a)$$

$$\int_{\Gamma^{1-2}} \delta \boldsymbol{\lambda}^T \cdot (\mathbf{u}^{[1]} - \mathbf{u}^{[2]}) d\Gamma + \int_{\Gamma_u} \delta \boldsymbol{\lambda}^T \cdot (\mathbf{u} - \bar{\mathbf{u}}) d\Gamma = 0 \quad (3.46b)$$

where $\boldsymbol{\lambda}$ is the vector of Lagrange multiplier on the boundaries.

Note that the Lagrange multiplier vector $\boldsymbol{\lambda}$ can be identified to be the traction vector on the corresponding boundaries. Thus, we have

$$\boldsymbol{\lambda} = \mathbf{t} \quad \text{on } \Gamma_u \quad (3.47a)$$

$$\boldsymbol{\lambda} = \mathbf{t}^{[1]} = -\mathbf{t}^{[2]} \quad \text{on } \Gamma^{[1-2]} \quad (3.47b)$$

So, the single displacement field problem in Eq. (3.17) turns into a two-field, i.e. the displacement field and the traction field, hybrid problem in Eq. (3.46). There are several approaches to treat the Lagrange multipliers, of which the most rigorous approach is regarding the Lagrange multipliers as unknowns of the interface (or mortar) element, which are independent of the displacement field and then to be determined. The discrete equations can be derived accordingly. Details can be referred to Terada et al. (2003).

The penalty method

The Lagrange multiplier method comes at the cost of additional variables and an indefinite structure of resulting matrices. These difficulties motivate other considerations.

As a matter of fact, the simplest and widely used way to deal with the essential boundary conditions is the use of the penalty method. In this method, the Lagrange multipliers are approximated to be the corresponding displacement multiplied by a penalty parameter such that

$$\boldsymbol{\lambda} = p(\mathbf{u} - \bar{\mathbf{u}}) \quad \text{on } \Gamma_u \quad (3.48a)$$

$$\boldsymbol{\lambda} = p(\mathbf{u}^{[1]} - \mathbf{u}^{[2]}) \quad \text{on } \Gamma^{[1-2]} \quad (3.48b)$$

where p is a given positive penalty parameter, which must be considerably large enough to enforce Eq. (3.46), but not too large to induce the numerical instability. Substituting Eq. (3.48) into Eq. (3.46) gives the penalized weak form. Then the corresponding discrete equations can be derived, referring to Terada et al. (2003).

Augmented Lagrange multiplier method

In the analysis of problem involving multiple materials, debonding of material interfaces is judged by the traction on the material interfaces, i.e. the Lagrange

multipliers. So, it is required to obtain the Lagrange multipliers accurately. To this end, the augmented Lagrange multiplier method with the Uzawa update, a well-known iterative procedure, is proved effective. The augmented Lagrange multiplier method can be regarded as a combination of the Lagrange multiplier method and the penalty method. In the augmented Lagrange multiplier method, it is assumed that

$$\lambda = \bar{\lambda} + p[\mathbf{u}] \quad (3.49)$$

where $\bar{\lambda}$ is a given algorithmic multiplier, and p is the penalty parameter with a approximate order, $[\mathbf{u}]$ is the error of the essential boundary conditions, given by

$$[\mathbf{u}] = \mathbf{u} - \bar{\mathbf{u}} \quad \text{on } \Gamma_u \quad (3.50a)$$

$$[\mathbf{u}] = \mathbf{u}^{[1]} - \mathbf{u}^{[2]} \quad \text{on } \Gamma^{[1-2]} \quad (3.50b)$$

Substituting Eq. (3.49) into Eq. (3.46a) yields the corresponding weak form, see Terada et al. (2007).

With the augmented Lagrange multiplier method, the iterative procedure is followed by setting the multipliers at step $k+1$ as

$$\bar{\lambda}^{k+1} = \bar{\lambda}^k + p[\mathbf{u}]^k \quad (3.51)$$

So far, $\bar{\lambda}^{k+1}$ is used to judge the debonding of the material interfaces.

3.2.5 Contact detection and modeling

Origin of contact problems in the NMM

The NMM aims at the discontinuous problems, even with movements. When intersected with physical features like cracks and material interfaces, each mathematical cover forms several independent physical covers associated with different local functions. So, the adjacent manifold elements formed by these physical covers are independent on each other in the framework of NMM.

However, the fact is that for the case involving internal discontinuities, there are often some relations between the adjacent manifold elements across discontinuous

entities. For example, for a problem domain containing a crack, the displacement across the crack surface is discontinuous, however, the two sides, i.e. upper side and lower side, of a crack surface cannot penetrate each other in geometry even under a complex stress state. Another example is the problem with discrete bodies. The displacement across each body boundary is discontinuous; however, one body cannot penetrate into another body. Such constraints are normally termed as non-penetration or unilateral condition, and attributed to a contact problem in physics.

Treatment of contact problems in the NMM

Since the frictional contact problems are inherently nonlinear and irreversible, for the sake of generality, an incremental approach is adopted in the NMM. We assume that: (1) the contact state at $t = t_n$, the beginning of the current time step, is known and the contact state at $t = t_{n+1}$, the end of the current step, after time interval $\Delta t = t_{n+1} - t_n$ is to be solved; (2) the time incremental for each time step is chosen small enough so that the displacements of all the points within the problem domain are less than a predefined maximum displacement limit ρ .

Contact areas are not known a priori and can change within a time step considerably. A surface point of a body may contact any portion of the surface of another body, and such a point can even come into contact with a part of the surface of its own body. So, the algorithms for detecting contact are of utmost importance. In the NMM, detecting contact at the beginning of each time step is split into two steps: a global search is first done to find the bodies which might possible come into contact by representing the bodies by bounding boxes, i.e. if the distance between the bounding boxes of any two bodies is less than 2ρ , they might contact within this time step; a local search is then conducted to figure out all the vertex-edge contact pairs.

There are generally three types of contacts: angle-to-edge, angle-to-angle, edge-to-edge, as shown in Fig. 3.14. Among them, edge-to-edge contact can be treated as two angle-to-edge contacts.

The contact will be of angle-to-angle if both of the following conditions are satisfied: (1) minimum distance of the two angle vertices is less than 2ρ , (2) the

maximum overlapping angle of the two angles is less than 2δ when one angle vertex translates to the vertex of the other angle without rotation.

Fig. 3.14 Three types of contacts: (a) angle-to-angle; (b) angle-to-edge; (c) edge-to-edge

The contact will be of angle-to-edge if both of the following conditions are satisfied: (1) minimum distance between the vertex and the edge is less than 2ρ ; (2) maximum overlapping angle of the angle and the edge is less than 2δ when the angle vertex translates to the edge without rotation.

An angle-to-angle contact has two entrance lines and determined as followings:

(1) If both angles are less than 180° , the two entrance lines are determined according to Table 3.1. Illustrations of the first, second and forth cases are depicted in Figs. 3.15 to 3.17, respectively. The third case is similar to the second one, thus its illustration is omitted to save pages. An angle-to-angle contact takes place when both the two entrance lines are passed respectively by the vertex of the other angle;

(2) If one of the two associated angles is greater than 180° , the two edges of the angle greater than 180° will be the entrance lines (as illustrated in Fig. 3.18); angle-to-angle contact takes place if one of the two entrance lines is passed by the vertex of the other angle.

An angle-to-edge contact has only one entrance line, which is the edge (illustrated in Fig. 3.19), and angle-to-edge contact takes place if the entrance line is passed by the angle vertex.

Table 3.1 Definition of two entrance lines when both angles less than 180°

Associated angles		Entrance lines	
$\alpha \leq 180^\circ$	$\beta \leq 180^\circ$	OE_3	OE_2
$\alpha \leq 180^\circ$	$\beta > 180^\circ$	OE_3	OE_4
$\alpha > 180^\circ$	$\beta \leq 180^\circ$	OE_1	OE_2
$\alpha > 180^\circ$	$\beta > 180^\circ$	OE_1	OE_4

Note: OE_1 and OE_2 are two sides of angle 1;

OE_3 and OE_4 are two sides of angle 2;

α is the angle from OE_1 to OE_3 ;

β is the angle from OE_2 to OE_4

Fig. 3.15 Two entrance lines (marked by thick lines) of an angle-to-angle contact

$$(\angle E_1OE_2 < 180^\circ, \angle E_3OE_4 < 180^\circ, \alpha \leq 180^\circ, \beta \leq 180^\circ)$$

Fig. 3.16 Two entrance lines (marked by thick lines) of an angle-to-angle contact

$$(\angle E_1OE_2 < 180^\circ, \angle E_3OE_4 < 180^\circ, \alpha \leq 180^\circ, \beta > 180^\circ)$$

Fig. 3.17 Two entrance lines (marked by thick lines) of an angle-to-angle contact
 $(\angle E_1OE_2 < 180^\circ, \angle E_3OE_4 < 180^\circ, \alpha > 180^\circ, \beta > 180^\circ)$

Fig. 3.18 Two entrance lines (marked by thick lines) of a contact between a convex angle and a concave angle

Fig. 3.19 The entrance line (marked by thick line) for an angle-to-edge contact

To evaluate the penetration, an entrance distance d_n is introduced, which is defined as the distance from the vertex P_1 of the angle to the entrance line P_2P_3 at $t = t_{n+1}$.

Contacts are pre-determined according to the entrance distance. On this basis, the stiff springs are applied to push the vertex P_1 away from P_2P_3 . For the case that both the two angles are less than 180° , a stiff spring is attached between the vertex and its entrance line which has a smaller entrance distance d_n ; for the case that one of the two angles is greater than 180° , a stiff spring is applied if only one of the two entrance lines is passed by the corresponding vertex of the other angle, and two stiff springs are applied if both the two entrance lines are passed; for an angle-to-edge contact, one stiff spring is applied.

For each vertex-entrance line contact pair (contact pair in short), there are three possible contact modes: open, sliding and sticking. At the beginning of each time step, the contact modes for all contact pairs are assumed as sticking except those contact pairs inherited from the last time step. For a sticking contact pair, a normal spring is applied to push the vertex away from the entrance line in the normal direction and another shear spring is applied to avoid the tangential displacement between the vertex and the entrance line.

Assume P_1 is the vertex, P_2P_3 is the entrance line and (x_k, y_k) and (u_k, v_k) are the coordinates and displacement of P_k ($k=1,2,3$), respectively; point P_1 is in element i , and points P_2, P_3 are in element j , stiffness of the normal spring is k_n . If three points P_1, P_2, P_3 rotate anticlockwise, then the normal distance d_n from vertex P_1 to P_2P_3 is

$$d_n = \frac{1}{l} \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix} \quad (3.52)$$

where l is the distance between points P_2 and P_3 .

Fig. 3.20 Illustration of entrance distance d_n between a vertex and its entrance line

Ignoring the second-order infinite small terms in Eq. (3.52), and rearranging the terms give

$$d_n = \mathbf{H}^T \mathbf{D}_i + \mathbf{G}^T \mathbf{D}_j + \frac{S_0}{l} \quad (3.53)$$

where

$$S_0 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \quad (3.54)$$

$$\mathbf{H} = \frac{1}{l} \mathbf{N}_i(x_1, y_1) \begin{Bmatrix} y_2 - y_3 \\ x_3 - x_2 \end{Bmatrix} \quad (3.55)$$

$$\mathbf{G} = \frac{1}{l} \mathbf{N}_j(x_2, y_2) \begin{Bmatrix} y_3 - y_1 \\ x_1 - x_3 \end{Bmatrix} + \frac{1}{l} \mathbf{N}_j(x_3, y_3) \begin{Bmatrix} y_1 - y_2 \\ x_2 - x_1 \end{Bmatrix} \quad (3.56)$$

The potential energy due to the normal spring is

$$\begin{aligned} \Pi_n &= \frac{k_n}{2} d_n^2 = \frac{k_n}{2} \left(\mathbf{H}^T \mathbf{D}_i + \mathbf{G}^T \mathbf{D}_j + \frac{S_0}{l} \right)^2 \\ &= \frac{k_n}{2} \left(\mathbf{D}_i \mathbf{H} \mathbf{H}^T \mathbf{D}_i + \mathbf{D}_j \mathbf{G} \mathbf{G}^T \mathbf{D}_j + 2 \mathbf{D}_i \mathbf{H} \mathbf{G}^T \mathbf{D}_j + 2 \left(\frac{S_0}{l} \right) \mathbf{D}_i \mathbf{H} + 2 \left(\frac{S_0}{l} \right) \mathbf{D}_j \mathbf{G} + \left(\frac{S_0}{l} \right)^2 \right) \end{aligned} \quad (3.57)$$

Thus, the sub-matrices due to the normal spring can be obtained and assembled into global stiffness matrix and global loading vector as

$$\begin{aligned}
 k_n \mathbf{H}_{i(r)} \mathbf{H}_{i(s)}^T &\rightarrow \mathbf{K}_{i(r)i(s)}, \quad (r, s = 1, 2, 3) \\
 k_n \mathbf{H}_{i(r)} \mathbf{G}_{j(s)}^T &\rightarrow \mathbf{K}_{i(r)j(s)}, \quad (r, s = 1, 2, 3) \\
 k_n \mathbf{G}_{j(r)} \mathbf{H}_{i(s)}^T &\rightarrow \mathbf{K}_{j(r)i(s)}, \quad (r, s = 1, 2, 3) \\
 k_n \mathbf{G}_{j(r)} \mathbf{G}_{j(s)}^T &\rightarrow \mathbf{K}_{j(r)j(s)}, \quad (r, s = 1, 2, 3) \\
 -k_n (S_0/l) \mathbf{H}_{i(r)} &\rightarrow \mathbf{F}_{i(r)}, \quad (r, s = 1, 2, 3) \\
 -k_n (S_0/l) \mathbf{G}_{j(r)} &\rightarrow \mathbf{F}_{j(r)}, \quad (r, s = 1, 2, 3)
 \end{aligned} \tag{3.58}$$

Assume $P_0(x_0, y_0)$ is the projection point of vertex P_1 on entrance line P_2P_3 (see Fig. 3.20), with its coordinates expressed as

$$\begin{aligned}
 x_0 &= (1-t_0)x_2 + t_0x_3 \\
 y_0 &= (1-t_0)y_2 + t_0y_3
 \end{aligned} \tag{3.59}$$

The shear displacement of point P_1 within the current step is

$$d_s = \frac{1}{l} \overrightarrow{P_0P_1} \cdot \overrightarrow{P_2P_3} = \frac{1}{l} (x_1 + u_1 - x_0 - u_0 \quad y_1 + v_1 - y_0 - v_0) \begin{Bmatrix} x_3 + u_3 - x_2 - u_2 \\ y_3 + v_3 - y_2 - v_2 \end{Bmatrix} \tag{3.60}$$

Ignoring the second-order infinite small terms and rearranging Eq. (3.60) give

$$d_s = \tilde{\mathbf{H}}^T \mathbf{D}_i + \tilde{\mathbf{G}}^T \mathbf{D}_j + \frac{\tilde{S}_0}{l} \tag{3.61}$$

where

$$\tilde{S}_0 = (x_1 - x_0 \quad y_1 - y_0) \begin{Bmatrix} x_3 - x_2 \\ y_3 - y_2 \end{Bmatrix} \tag{3.62}$$

$$\tilde{\mathbf{H}} = \frac{1}{l} \mathbf{N}_i(x_1, y_1)^T \begin{Bmatrix} x_3 - x_2 \\ y_3 - y_2 \end{Bmatrix} \tag{3.63}$$

$$\mathbf{G} = \frac{1}{l} \mathbf{N}_j(x_2, y_2)^T \begin{Bmatrix} -x_1 + 2(1-t_0)x_2 - (1-2t_0)x_3 \\ -y_1 + 2(1-t_0)y_2 - (1-2t_0)y_3 \end{Bmatrix} + \frac{1}{l} \mathbf{N}_j(x_3, y_3)^T \begin{Bmatrix} x_1 + (2t_0-1)x_2 - 2t_0x_3 \\ y_1 + (2t_0-1)y_2 - 2t_0y_3 \end{Bmatrix} \tag{3.64}$$

The potential energy induced by the shear spring is

$$\begin{aligned}\Pi_s &= \frac{k_s}{2} d_s^2 = \frac{k_s}{2} \left(\tilde{\mathbf{H}}^T \mathbf{D}_i + \tilde{\mathbf{G}}^T \mathbf{D}_j + \frac{\tilde{S}_0}{l} \right)^2 \\ &= \frac{k_s}{2} \left(\mathbf{D}_i \tilde{\mathbf{H}} \tilde{\mathbf{H}}^T \mathbf{D}_i + \mathbf{D}_j \tilde{\mathbf{G}} \tilde{\mathbf{G}}^T \mathbf{D}_j + 2 \mathbf{D}_i \tilde{\mathbf{H}} \tilde{\mathbf{G}}^T \mathbf{D}_j + 2 \left(\frac{\tilde{S}_0}{l} \right) \mathbf{D}_i \tilde{\mathbf{H}} + 2 \left(\frac{\tilde{S}_0}{l} \right) \mathbf{D}_j \tilde{\mathbf{G}} \left(\frac{\tilde{S}_0}{l} \right)^2 \right)\end{aligned}\quad (3.65)$$

Lastly, the sub-matrices due to the shear spring are obtained and assembled to the global stiffness matrix and global loading vector as

$$\begin{aligned}k_s \tilde{\mathbf{H}}_{i(r)} \tilde{\mathbf{H}}_{i(s)}^T &\rightarrow \mathbf{K}_{i(r)i(s)}, \quad (r, s = 1, 2, 3) \\ k_s \tilde{\mathbf{H}}_{i(r)} \tilde{\mathbf{G}}_{j(s)}^T &\rightarrow \mathbf{K}_{i(r)j(s)}, \quad (r, s = 1, 2, 3) \\ k_s \tilde{\mathbf{G}}_{j(r)} \tilde{\mathbf{H}}_{i(s)}^T &\rightarrow \mathbf{K}_{j(r)i(s)}, \quad (r, s = 1, 2, 3) \\ k_s \tilde{\mathbf{G}}_{j(r)} \tilde{\mathbf{G}}_{j(s)}^T &\rightarrow \mathbf{K}_{j(r)j(s)}, \quad (r, s = 1, 2, 3) \\ -k_s (S_0 / l) \tilde{\mathbf{H}}_{i(r)} &\rightarrow \mathbf{F}_{i(r)}, \quad (r, s = 1, 2, 3) \\ -k_s (S_0 / l) \tilde{\mathbf{G}}_{j(r)} &\rightarrow \mathbf{F}_{j(r)}, \quad (r, s = 1, 2, 3)\end{aligned}\quad (3.66)$$

For the sliding mode, except the normal spring, a pair of frictional forces instead of a shear spring will be added. Based on the Coulomb's friction law, the frictional force is calculated as

$$F = \text{sign} \cdot k_n d_n \mu \quad (3.67)$$

where 'sign' is assigned as '+' or '-' according to the direction of relative sliding, $k_n d_n$ is the normal force, μ is the friction coefficient.

Then, the potential energy due to frictional force F on an element i is

$$\Pi_{f_i} = \frac{f}{l} (u_1 \quad v_1) \begin{Bmatrix} x_3 - x_2 \\ y_3 - y_2 \end{Bmatrix} - \frac{f}{l} (u_0 \quad v_0) \begin{Bmatrix} x_3 - x_2 \\ y_3 - y_2 \end{Bmatrix} = F \mathbf{D}_i^T \bar{\mathbf{H}} + F \mathbf{D}_j^T \bar{\mathbf{G}} \quad (3.68)$$

where

$$\bar{\mathbf{H}} = \frac{F}{l} \mathbf{N}_i(x_1, y_1) \begin{Bmatrix} x_3 - x_2 \\ y_3 - y_2 \end{Bmatrix} \quad (3.69)$$

$$\bar{\mathbf{G}} = \frac{F}{l} \mathbf{N}_j(x_0, y_0) \begin{Bmatrix} x_3 - x_2 \\ y_3 - y_2 \end{Bmatrix} \quad (3.70)$$

Lastly, the sub-matrix due to the frictional force are obtained and assembled to the global loading vector as

$$\begin{aligned} -F\bar{\mathbf{H}}_{i(r)} &\rightarrow \mathbf{F}_{i(r)}, \quad (r=1,2,3) \\ -F\bar{\mathbf{G}}_{j(r)} &\rightarrow \mathbf{F}_{j(r)}, \quad (r=1,2,3) \end{aligned} \quad (3.71)$$

After the load increment of the current time step is applied, the equilibrium equation is solved and the state quantities such as displacements, stresses, etc, are obtained. The entrance distance between the vertex and the entrance line for each possible contact pair is updated. If d_n of a contact pair is positive, its contact mode is changed to open and all the springs and frictional forces are removed if any. If d_n of a contact pair is negative, the numerical penetration will violate the physical contact behavior occurring and a normal spring is applied to push the vertex away from the entrance. If the tangential resultant contact traction exceeds its maximum value, i.e. $F \geq N\mu + c$, where F is the tangential resultant contact traction, N is the normal resultant contact traction, μ is the friction coefficient, c is the cohesion, the vertex will slide along the entrance line, sliding mode is assigned, a pair of frictional force is added. In contrast, if the tangential resultant contact traction is smaller than its maximum value, i.e. $F < N\mu + c$, sticking mode is assigned, a shear spring is added to avoid the relative tangential displacement between the vertex and the edge. The equilibrium equation is updated accordingly and solved again. The above open-close iteration procedure is repeated until all the possible vertex-edge contact pairs satisfy the contact conditions, the computation then proceeds to the next time step.

Detailed contact detection and modeling can be referred to Shi (1991; 1992; 1995; 1996a; 1997).

3.3 Calibration of the NMM

This section presents several examples to calibrate the NMM program in continua modeling and discontinua modeling. Among them, the first two are designed to verify the capability of the NMM in accurately describing the displacement field and stress field with the refinement of the mathematical covers. The third example is to check

the ability of the NMM in accurately dealing with the frictional contact between two blocks. The forth example is designed to demonstrate the convenience of meshing in the NMM for discontinuity modeling. The last two examples are presented to illustrate the efficiency and robustness of the NMM in treating the interaction among multiple discrete blocks.

3.3.1 Beam bending

Under study is a simply supported beam subjected a concentrated load at its center, as shown in Fig. 3.21. The dimension of the beam is 10 m long, 1 m deep and 1m wide. The point loading is $P = 200$ N. The material of the beam is assumed elastic without damage with properties of $E = 1 \times 10^5$ MPa and $\nu = 0.24$. There are totally 11 measure points shown in Fig. 3.21.

Fig. 3.21 Beam bending problem under central point loading

To simulate this problem, three meshes are adopted. The left bottom end point of the beam is fixed while the right end is laid on another fixed rigid block so as to allow the horizontal deformation of the beam. Fig. 3.22 shows the final configurations of the beam with different meshes.

(a)

Fig. 3.22 Configuration of deformed beam with different meshes

The theoretical solutions for simply supported beam under central point loading can be expressed as (Timoshenko and Goodier, 1970; Chen et al., 1998):

$$v(x) = -\frac{P}{12EI}x^3 + \frac{Pl^2}{16EI}x \quad (0 \leq x \leq \frac{l}{2}) \quad (3.72)$$

where E is the Young's modulus, I is the moment of inertia and l is the length of the beam.

Fig. 3.23 Comparison of numerical results and theoretical solution for beam bending problem

The numerical results of the three meshes and the theoretical solutions are compared in Fig. 3.23. It can be found that the numerical result converges to the

theoretical solutions by refining the meshes.

3.3.2 Infinite plate with a circular hole under uniaxial tension

An infinite plate with a traction free circular hole subjected to a far-field uniaxial tension is considered here. The problem geometry is depicted in Fig. 3.24. The diameter of the hole is 0.2 m. The dimension of the plate is 2 m × 2 m.

Fig. 3.24 Numerical model for an infinite plate with a traction free circular hole

The analytical solution for this problem was provided by Kirsch (1898) as follows:

$$\begin{aligned}\sigma_r &= \frac{\sigma}{2} \left(1 - \frac{a^2}{r^2} \right) + \frac{\sigma}{2} \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2} \right) \cos 2\theta \\ \sigma_\theta &= \frac{\sigma}{2} \left(1 + \frac{a^2}{r^2} \right) - \frac{\sigma}{2} \left(1 + \frac{3a^4}{r^4} \right) \cos 2\theta \\ \sigma_{r\theta} &= -\frac{\sigma}{2} \left(1 - \frac{3a^4}{r^4} + 2\frac{a^2}{r^2} \right) \sin 2\theta\end{aligned}\tag{3.73}$$

where σ is the far-field uniaxial tensile stress applied to the plate, a is the radius of

the circular hole, r and θ are polar coordinates, as defined in Fig. 3.24.

As can be derived from Eq. (3.73), the stress σ_θ reaches its maximum value $(\sigma_\theta)_{\max} = 3\sigma$ at the periphery of the hole ($r = a$), at a diametric position transverse to the loading direction ($\theta = \pi/2$). The stress concentration factor (SCF) for this problem is therefore 3. The x -direction stress falls to zero at the position $\theta = \pi/2$, $r = a$, as it must satisfy the stress-free boundary condition at the periphery of the hole.

Parameters used in the calculation are: Young's Modulus $E = 20$ GPa, Poisson's ratio $\nu = 0.3$, far-field uniaxial tension $\sigma = 1$ Pa. The NMM model for a quarter of the problem is shown in Fig. 3.25. A regular cover system generated from a regularly-patterned triangular mesh is adopted. Because of symmetry, only one quarter of the model is shown here.

This discretization gives satisfactory results for the present analysis. Fig. 3.26 gives a plot of computed σ_y versus the analytical solution. A good agreement has been observed.

Fig. 3.25 NMM model for a quarter of the infinite plate problem

Fig. 3.26 Comparison of numerical result and theoretical solution for infinite plate problem

3.3.3 Frictional sliding of a block along a slope

A sliding problem is studied here using NMM. The numerical model is shown in Fig. 3.27.

Fig. 3.27 Numerical model for a block sliding along a slope

The slope is an isosceles left-angled triangle of the size of $1\text{ m} \times 1\text{ m}$ with the bottom fixed. The block with the size of $0.05\text{ m} \times 0.05\text{ m}$ is sliding along the slope by

the gravity. The vertical displacement time history can be derived based on Newton's second law as:

$$h = \frac{1}{4} \cdot (1 - \tan \phi) \cdot g \cdot t^2 \quad (3.74)$$

where h is the vertical displacement of the block, g is the acceleration of gravity, ϕ is the friction angle, t is the time.

In this case, we use an over-sized triangle to cover the whole problem domain. Only two manifold elements are generated. As mentioned in Section 2.4.3, under such a circumstance, NMM is equivalent to DDA.

Three cases with various friction angles of 0° , 10° , and 20° are simulated. The vertical displacement time histories are obtained and compared with theoretical solutions in Fig. 3.28. Good agreements have been found.

Fig. 3.28 Comparison of numerical results and theoretical solutions for block sliding problem

3.3.4 Direct shear test

Here, a numerical investigation on the mechanical behavior of initially matched artificial joints with saw-tooth shaped asperities is carried out to demonstrate the convenience of the meshing for the discontinuity problem with complex discontinuity surface.

Fig. 3.29a shows a schematic view of the NMM model for the direct shear test.

The joint surface (marked by the dash line box) consists of 20 asperities. The upper half (60 cm $\times$ 10 cm in dimension) specimen is smaller than the lower half (75 cm $\times$ 20 cm in dimension). The lower half is fixed at the bottom, while the upper half is sheared to the right by the velocity boundary applied on the left side. A compressive stress is uniformly applied to the top of the specimen to simulate the lateral pressure effect on the joint. A velocity is uniformly applied to the left edge of the upper specimen to realize the shearing process. In all simulations, the compressive stress and the boundary velocity are applied slowly to insure obtaining quasi-static response. The normal stress is first applied at a rate of 0.25 MPa/s until a preset average normal stress is reached, and then the normal stress is held to be a constant. A velocity boundary is then uniformly applied to the left edge of the upper specimen so that the upper specimen moves from its initial position at a rate of 0.75 mm/s toward the right side to a preset maximum shear displacement. A total number of 81 measure points are uniformly distributed at the left edge of the upper specimen. The stress and the displacement of each measurement point are recorded during the shearing process. Then, the average shear stress and the shear displacement of the upper specimen are obtained by taking the average of the corresponding values of all the measure points.

In the NMM, a finite element mesh is adopted to generate the mathematical covers. The so-called finite element mesh can be constructed totally independent of the problem domain. Thus, a regular mesh is always applicable. The meshing task is very convenient. It allows us to adopt the exactly same mesh, with various joint surfaces arbitrarily aligning within the mesh. This distinct feature makes the NMM extremely suitable for the direct shear test modeling. The NMM mesh of one case is shown in Fig. 3.29b.

The rock material is assumed to be elastic without damage with the parameters: mass density = 2650 kg/m³; Young's Modulus = 80 GPa; Poisson's Ratio = 0.3. The joint obeys Coulomb slip criterion with the parameters: friction angle = 30°; tensile strength = 0 MPa; cohesion = 0 MPa.

Many studies on joint shear strength under static loading have been theoretically and experimentally performed. Several criteria are commonly used to estimate the

joint shear strengths.

(a)

(b)

Fig. 3.29 Numerical model of a direct shear test: (a) geometric model; (b) NMM model

The shear strength of a smooth, clean and dry joint can be described by simple Coulomb linear slip law:

$$\tau = c + \sigma \tan \phi \quad (3.75)$$

where ϕ is the frictional angle, σ is the normal stress, c is the joint cohesion, which is very small and often neglected, then, Eq. (3.75) becomes

$$\tau = \sigma \tan \phi \quad (3.76)$$

Natural joints are far from smooth. The shear behavior does not follow a linear criterion. For a rough clean joint, the peak strength at constant normal stress is reached after a small shear displacement. The shear resistance falls within further

shear displacement until the residual strength is eventually reached.

At low normal stresses, shearing occurs by riding over asperities. The shear strength increases with normal stress linearly as described by the dilatant model (Patton, 1966)

$$\tau = \sigma \tan(\phi + \alpha) \quad (3.77)$$

where α is the dilation angle (or asperity inclination angle).

At higher normal stresses, the asperities start to be sheared off and the dilation angle finally entirely disappears by shearing. So, a bilinear shear strength criterion has often been used by combining the dilatant model for low normal stresses and the Coulomb linear criterion for high normal stresses.

In the current study, the NMM is adopted to simulate the shear behavior of artificial rock joint with the aim to demonstrate the convenience of the NMM pre-processing for arbitrary discontinuity modeling. In our NMM model, the upper specimen only rides over the asperities, but the shearing of the asperities is not accounted for, thus, the dilatant model given in Eq. (3.77) can be used as the analytical solution.

The parametric studies regarding joint roughness and confining stress are carried out.

Joint roughness is a measure of the inherent joint surface waviness and unevenness relative to the mean plane. It is a very important component of shear strength, especially for undislocated and interlocked joints. The joint roughness is represented by the asperity inclination angle in this study. Six cases with various asperity inclination angles of 0° , 5° , 10° , 15° , 20° , and 25° are simulated. The corresponding joint surfaces for six cases are depicted in Fig. 3.30.

(a) $\alpha = 0^\circ$

(b) $\alpha = 5^\circ$

Fig. 3.30 Uniform joint surface with various joint roughnesses

During the shearing process, the upper specimen will move to the right side until a preset shearing displacement of 3 mm. The shearing process of the case with $\alpha = 25^\circ$ is shown in Fig. 3.31. It can be seen that for the geometrically uniform rock joint, the upper half slides along the surface of each asperity of the lower half during the shearing process.

Fig. 3.31 Shearing process of uniform rock joint with roughness of $\alpha = 25^\circ$

The shear strengths from NMM modeling are compared with theoretical solutions (Eq. (3.77)) in Fig. 3.32. Good agreement has been observed. The results indicate that the joint roughness has a significant influence on the shear strength. It is obvious that larger joint roughness will lead to higher shear strength.

Fig. 3.32 Comparison of NMM results and theoretical solutions in shear strength with different joint roughnesses

The influence of normal stresses is also investigated. Seven cases with various normal stresses of 0.5 MPa, 1.0 MPa, 2.0 MPa, 3.0 MPa, 4.0 MPa, 5.0 MPa, and 6.0 MPa are simulated to investigate the influences of normal stress on the joint shear strength. The numerical result again matches well with the theoretical solutions (shown in Fig. 3.33). The shear strength increases linearly with the normal stresses.

Fig. 3.33 Comparison of NMM result with theoretical solutions in shear strength under different normal stresses

The geometrical non-uniformity effect is also investigated with each asperity configured with different inclination angles following the Weibull distribution.

The two-parameter Weibull distribution can be expressed as (Weibull, 1951; Liu et al., 2002; Tang, 1997)

$$f(T) = \frac{m}{\mu} \left(\frac{T}{\mu} \right)^{m-1} \exp \left(- \left(\frac{T}{\mu} \right)^m \right) \quad (3.78)$$

where T is the rock joint parameter, μ is the mean value of T , m is the shape parameter describing the scattering of T .

The cumulative distribution function of Weibull distribution is derived as

$$Q(T) = \int_0^T f(T) dT = 1 - \exp \left(- \left(\frac{T}{\mu} \right)^m \right) \quad (3.79)$$

Then, the disorder of the probability distribution for joint asperity inclination angles can be achieved by the Monte Carlo method (Liu et al., 2002). One simple way is to generate a series of random data which have a uniform distribution between 0 and 1. Then the desired inclination angle for each asperity can be obtained by the following expression:

$$T = \mu \left(\ln(U(0,1)) \right)^{1/m} \quad (3.80)$$

where $U(0,1)$ is the random data between 0 and 1 generated by Monte Carlo method. T is the desired random parameters which can be assigned to each asperity.

In the current study, the mean inclination angle is set to 20° . Various non-uniformity indices of 1.4, 1.5, 2, 3, 5, 10 are considered.

The shear mechanism becomes significantly different for the geometrically non-uniform rock joint. An illustration example is plotted in Fig. 3.34 by using the case with $\alpha = 20^\circ$, and $m = 2$. It is observed that after a small shearing distance, the upper half specimen will slide along the surface of the asperity with the largest roughness while most of the rock joint surface will be open. Similar phenomenon has been observed for other cases.

(a) Step 0

(b) Step 150

(c) Step 300

(d) Step 500

Fig. 3.34 Shearing process of rock joint with non-uniform asperities, $\alpha = 20^\circ$, $m = 2$

3.3.5 Domino run

A typical domino run problem shown in Fig. 3.35 is numerically investigated here.

Fig. 3.35 Configuration of a domino run problem

The numerical model consists totally 41 rectangular blocks with the size of 20 mm $\times$ 150 mm and the spacing of 30 mm on a horizontal surface. The material

parameters for the blocks are: Young's modulus $E = 200$ GPa, Poisson's ratio $\nu = 0.3$. The friction coefficient μ between the blocks and the horizontal surface is 0.3. An initial pulse with the magnitude of $6.7 \text{ N}\cdot\text{s}$ (triangular pulse loading, magnitude of the peak force is 44.72 kN , duration is 0.3 ms) and the direction given in Fig. 3.35 is applied to the first block to make it topple to the second block, which induces a toppling process shown in Fig. 3.36.

Similar to all the examples in the previous sections, a regular cover system generated from a regularly-patterned triangular mesh is adopted. In the NMM model for domino run problem, there are totally 96 manifold elements.

The numerically obtained result is consistent with the experimentally observed phenomenon (Bates, 2004) and DDA modeling results (Scheele and Bates, 2005).

Fig. 3.36 Domino run process modeled by the NMM

3.3.6 Vibrating screen

Vibrating screen is widely used in mining, construction material, transportation, energy and chemical industry for physical separation of particles. The efficiency of the screen is determined by the size and the shape of the screen openings, the amplitude and the frequency of the screen motion (Cleary, 2004). Here, the mineral separation process using the vibrating screen is numerically investigated to illustrate the efficiency and robustness of the NMM in modeling interaction of discrete block system.

The simplified model is shown in Fig. 3.37, which includes a set of coal blocks, a vibrating screen and a container. The coal blocks are generated by cutting a whole block with two set of joints. The screen moves in the horizontal direction with a frequency of 1 Hz and amplitude of 5.5 cm.

Fig. 3.37 Configuration of the simplified model for the vibrating screen modeling

In the current study, an oversize triangle is selected to cover the whole problem domain, shown in Fig. 3.38a, which treats each block as a single manifold element with constant stress field and linear displacement field. This assumption is reasonable because the fact that the coal blocks generally undergo small deformation during the sieving process and this simple mesh will save a lot of computational time. The generated manifold elements are depicted in Fig. 3.38b, where the colors are randomly assigned to distinguish different elements while without any physical meanings.

(a) meshing

(b) generated manifold elements

(c) an enlarged concave block marked by circle in (b)

Fig. 3.38 NMM model for the vibrating screen modeling

During the sieving process, the contact between individual blocks is complex because of the large block size range, and the arbitrary block shape (both convex and concave blocks are available).

The size separation process with the vibrating screen is shown in Fig. 3.39. During the whole size separation process, no penetration has been found, which demonstrates the efficiency of the contact detection and modeling algorithm in the NMM.

Fig. 3.39 Size separation process with the vibrating screen modeled by the NMM

3.4 Summary

This chapter presents a review on the fundamentals of the NMM, including the NMM approximation, derivation of discrete system of equations, simplex integration scheme, imposition of boundary conditions, and contact detection and modeling, which builds up the foundation of the thesis work to apply the NMM for engineering failure analysis.

Then, through several examples, we see that the NMM can simulate both the continua and discontinua well. To be more specific, the NMM is not only capable of accurately describing the displacement and stress/strain fields within a continuous body but also able to represent the interactions of discrete block system realistically. In addition, the meshing task in the NMM is very convenient. Once a proper fracturing algorithm is implemented, the NMM is promising for the practical rock failure analysis which involves continua modeling, transition from continua to discontinua, and discontinua modeling.

CHAPTER 4 NMM COUPLED WITH FRACTURE MECHANICS FOR CRACK PROBLEMS

4.1 Introduction

Modeling cracks and their growth is highly important for researchers and engineers to quantitatively predict the life span of cracked structures under service conditions. For this purpose, a number of numerical approaches have been proposed in the past several decades.

Through embedding the singular elements at crack tips, the finite element method (FEM) was successfully used to compute the stress intensity factors (SIFs) (Barsoum, 1977; Kwon and Akin, 1989). However, the conventional FEM requires the finite element mesh to coincide with the cracks, which often complicates the meshing task. When crack growth is involved, remeshing is inevitable, which makes the simulation tedious and time-consuming. Moreover, the variables such as displacements, strains and stresses are required to map onto a new set of nodes and quadrature points at each time step. Additional inaccuracies may arise during such processes. In order to overcome the aforementioned inconveniences, a variety of modifications to the conventional FEM have been made within the framework of partition of unity.

Belytschko and Black (1999) enriched the nodes of the finite elements near the crack tips and along the crack surfaces with the asymptotic crack tip functions. This method allows the cracks to arbitrarily align within the finite element mesh. Only minimal remeshing is required even for severely curved cracks.

Moes et al. (1999) adopted the generalized Heaviside function $H(\mathbf{x})$ ($H(\mathbf{x})=1$ when $\mathbf{x}$ at one side of crack surface, and -1 when $\mathbf{x}$ at the other side of the crack surface) instead of asymptotic crack tip functions to introduce a discontinuous field across the crack surface away from the crack tips. The finite element mesh is allowed to be independent of the cracks and remeshing is completely avoided for crack growth

problems.

The generalized Heaviside function $H(\mathbf{x})$ can only describe the displacement discontinuity across a single crack surface. In order to account for branched and intersecting cracks, Daux et al. (2000) introduced another discontinuous function termed as junction function $J(\mathbf{x})$ and named their method the extended finite element method (XFEM).

The XFEM has been successfully applied to two-dimensional static and quasi-static crack growth problems (Moes et al., 1999; Dolbow et al., 2000a; 2000b; Moes and Belytschko, 2002; Dolbow et al., 2001), three-dimensional crack problems (Sukumar et al., 2000; Sukumar et al., 2003), with its extension to modeling voids (Daux et al., 2000; Sukumar et al., 2001), material interfaces (Sukumar et al., 2001; Belytschko et al., 2001), tangential discontinuities (Belytschko et al., 2001), etc.

Another example of modifications to the conventional FEM is the generalized finite element method (GFEM) developed based on partition of unity. The mesh in the GFEM is allowed to be completely independent of the problem domain and can always be regular. The GFEM augments the standard finite element approximation space with high-order terms or handbook functions of boundary value problems to tackle some typical problems with multiple reentrant corners, voids, cracks, etc. Recently, Simone et al. (2006) applied the GFEM for polycrystals with discontinuous grain boundaries by incorporating discontinuous functions $H_\alpha(\mathbf{x})$ ($H_\alpha(\mathbf{x})=1$ if $\mathbf{x}$ within aggregate R_α , $H_\alpha(\mathbf{x})=0$ elsewhere). Later, Duarte et al. (2007) extended the method for branched crack problems. The so-called GFEM in Simone et al. (2006) and Duarte et al. (2007) is in essence same to the XFEM but adopts different enrichment functions. Compared with the XFEM, the GFEM can construct the enrichment functions for arbitrarily branched cracks much more easily. However, the enrichment to the nodes in the GFEM needs to be properly designed since its enrichment functions $H_\alpha(\mathbf{x})$ are not linearly independent. In addition, the GFEM can only allow the crack tips stopping at the finite element sides because its enrichment functions are defined based on aggregate topology.

Some earlier researchers (Tsay et al., 1999; Chiou et al., 2002) implemented local

mesh refinement and auto-remeshing schemes into the NMM to model crack problems. However, such methods suffer from the same limitations as the conventional FEM, thus tedious and time-consuming. Moreover, the crack tips still cannot stop inside manifold elements.

In order to overcome the limitations of the conventional NMM, this chapter introduces a new concept ‘singular physical cover’ (in contrast to other conventional physical cover) to represent the physical cover which is partially cut by the crack surface and contains the crack tip singularity. Then, the asymptotic crack tip functions extracted based on the fracture mechanics are incorporated into the local approximation spaces of singular physical covers based on partition of unity with the aims to capture the crack tip singularities and to enable the crack tips stopping at arbitrary positions within a manifold element. Detailed modeling strategies of the NMM for crack problems are presented in Section 4.2. Several typical examples are simulated to verify the program and to demonstrate the efficiency and robustness of the NMM for complex crack problems and their growth in Section 4.3. Section 4.4 gives the summary of this chapter.

4.2 NMM coupled with fracture mechanics for crack problems

4.2.1 Generating covers

The NMM approximation is based on covers. The covers are user-defined small patches covering the problem domain. As presented in Chapter 3, a finite element mesh is usually adopted to generate the mathematical covers. The finite element mesh is not required to coincide with neither the external boundaries nor the internal cracks, thus can always be regular.

Taking the regularly-patterned triangular finite element mesh in Fig. 3.1 for an example, each node in the triangular finite element mesh is a star. The union of six triangles sharing a common star forms a hexagonal mathematical cover.

If intersected with the internal cracks, each hexagonal mathematical cover will be partitioned into several physical covers. For example, a single crack surface may

completely cut the mathematical cover M_I into two physical covers, i.e. P_I^1 and P_I^2 (Fig. 4.1a); a branched crack may completely cut the mathematical cover M_J into three physical covers, i.e. P_J^1 , P_J^2 , and P_J^3 (Fig. 4.1b); an intersecting crack may completely cut the mathematical cover M_K into four physical covers, i.e. P_K^1 , P_K^2 , P_K^3 , and P_K^4 (Fig. 4.1c). In general, if the crack surfaces completely cut a mathematical cover into several isolated regions and m of them are within the problem domain, m physical covers will be formed. However, for the mathematical cover partially cut by the crack surface, only one physical cover will be formed (Fig. 4.1d). This kind of physical covers contain the crack tip singularity, thus are termed as singular physical covers in this context, in contrast to other conventional physical covers.

Fig. 4.1 Generating physical covers for crack problems: (a) M_I cut by a single crack, forms P_I^1 and P_I^2 ; (b) M_J cut by a branched crack, forms P_J^1 , P_J^2 , and P_J^3 ; (c) M_K cut by an intersecting crack, forms P_K^1 , P_K^2 , P_K^3 , and P_K^4 ; (d) M_L partially cut by a crack, forms P_L^1

4.2.2 Applicability of the conventional NMM for crack problems

Fig. 4.2a shows a NMM model for an internal crack. In this case, a regularly-patterned rectangular finite element mesh is adopted to generate the mathematical covers. As aforementioned, the finite element mesh does not need to conform to neither the external boundary nor the internal cracks, thus can always be regular. Each node in the finite element mesh is regarded as a star. The union of all the rectangular finite elements sharing a star forms a mathematical cover. Each mathematical cover with a squared star is completely cut by the crack into two isolated pieces, thus forms two physical covers, while each mathematical cover with a circled star is partially cut by the crack, thus forms one singular physical cover. The internal crack will lead to displacement jumps across the crack surface in the rectangular finite elements 1 to 7 marked in Fig. 4.2a.

(a)

(b)

Fig. 4.2 Modeling an internal crack with the NMM: (a) NMM model; (b) local indices of four mathematical covers of a finite element (MC-mathematical cover; PC-physical cover)

We first investigate the displacement field in finite element 2. There are four mathematical covers associated with element 2, denoted as $M_{(i)}$, $i=1 \sim 4$, where i , $i=1 \sim 4$ are the local indices of four mathematical covers defined in Fig. 4.2b, and $()$ transforms the local index i into a global index. Mathematical cover $M_{(1)}$ forms one singular physical cover, denoted as $P_{(1)}^1$. Mathematical cover $M_{(2)}$ forms two physical covers, denoted as $P_{(2)}^1$ and $P_{(2)}^2$. Mathematical cover $M_{(3)}$ forms two physical covers, denoted as $P_{(3)}^1$ and $P_{(3)}^2$. Mathematical cover $M_{(4)}$ forms one singular physical cover, denoted as $P_{(4)}^1$.

From Eq. (2.4), the displacement jump across the crack surface in finite element 2 is obtained as

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = \sum_{i \in \{2,3\}} \varphi_{(i)} \left(\delta_{(i)}^1(\mathbf{x}) \mathbf{u}_{(i)}^1 - \delta_{(i)}^2(\mathbf{x}) \mathbf{u}_{(i)}^2 \right) \quad (4.1)$$

where $\llbracket \cdot \rrbracket$ represents the jump of a function, $\mathbf{u}_{(i)}^\alpha$ ($i=2,3$; $\alpha=1 \sim 2$) are the local functions corresponding to physical covers $P_{(i)}^\alpha$ ($i=2,3$; $\alpha=1 \sim 2$). Considering $\delta_{(i)}^1=1$, $\delta_{(i)}^2=0$ at the upper crack surface, $\delta_{(i)}^1=0$, $\delta_{(i)}^2=1$ at the lower crack surface, Eq. (4.1) becomes

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = \sum_{i \in \{2,3\}} \varphi_{(i)} \left(\mathbf{u}_{(i)}^1 - \mathbf{u}_{(i)}^2 \right) \quad (4.2)$$

With the knowledge that $\mathbf{u}_{(i)}^1$ ($i=2,3$) and $\mathbf{u}_{(i)}^2$ ($i=2,3$) are individually defined local functions and contain independent degrees of freedom (DOFs), the displacement discontinuity across the crack surface in finite element 2 is obviously manifested.

Take finite element 3 for another example. There are also four mathematical covers associated with element 3, denoted as $M_{(i)}$, $i=1 \sim 4$, where i , $i=1 \sim 4$ are the local indices of four mathematical covers defined in Fig. 4.2b, and $()$ transforms the local index i into a global index. Mathematical cover $M_{(1)}$ forms two physical

cover, denoted as $P_{(1)}^1$ and $P_{(1)}^2$. Mathematical cover $M_{(2)}$ forms two physical covers, denoted as $P_{(2)}^1$ and $P_{(2)}^2$. Mathematical cover $M_{(3)}$ forms two physical covers, denoted as $P_{(3)}^1$ and $P_{(3)}^2$. Mathematical cover $M_{(4)}$ forms one singular physical cover, denoted as $P_{(4)}^1$. From Eq. (2.4), the displacement jump across the crack surface in finite element 3 is obtained as

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = \sum_{i \in \{1,2,3\}} \varphi_{(i)} \left(\delta_{(i)}^1(\mathbf{x}) \mathbf{u}_{(i)}^1 - \delta_{(i)}^2(\mathbf{x}) \mathbf{u}_{(i)}^2 \right) \quad (4.3)$$

where $\llbracket \cdot \rrbracket$ represents the jump of a function, $\mathbf{u}_{(i)}^\alpha$ ($i=1,2,3$; $\alpha=1 \sim 2$) are the local functions corresponding to physical covers $P_{(i)}^\alpha$ ($i=1,2,3$; $\alpha=1 \sim 2$). Considering $\delta_{(i)}^1=1$, $\delta_{(i)}^2=0$ at the upper crack surface, $\delta_{(i)}^1=0$, $\delta_{(i)}^2=1$ at the lower crack surface, Eq. (4.3) becomes

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = \sum_{i \in \{1,2,3\}} \varphi_{(i)} \left(\mathbf{u}_{(i)}^1 - \mathbf{u}_{(i)}^2 \right) \quad (4.4)$$

With the knowledge that $\mathbf{u}_{(i)}^1$ ($i=1,2,3$) and $\mathbf{u}_{(i)}^2$ ($i=1,2,3$) are individually defined local functions and contain independent degrees of freedom (DOFs), the displacement discontinuity across the crack surface in finite element 3 is obviously described.

Following an exactly same way, the displacement jump across the crack surface in rectangular finite elements 4 to 6 can be proved.

However, things are different in finite elements 1 and 7. Take element 1 for an example. All the four mathematical covers associated with element 1, i.e. $M_{(i)}$, $i=1 \sim 4$ are only partially cut by the crack surface, and thus each forms one singular physical cover, denoted as $P_{(1)}^1$, $P_{(2)}^1$, $P_{(3)}^1$, and $P_{(4)}^1$, respectively. If we take the usually adopted polynomials as local approximation spaces for these physical covers, the displacement filed in finite element 1 will be continuous. In other words, the displacement jump across the crack surface in element 1 cannot be described. Same problem applies to finite element 7.

To sum up, in the conventional NMM, the displacement jump across crack surfaces in fully cracked finite elements can be well described by partitioning each mathematical cover which is completely cut by the cracks into several physical covers. However, by adopting polynomial local approximation space, the displacement jump across the crack surfaces in partially cracked elements cannot be described. In addition, the stress singularity near the crack tips cannot be captured. Therefore, modifications to the conventional NMM are needed.

4.2.3 Extensions to the conventional NMM

As discussed in the last section, the conventional NMM adopts the polynomial as its local approximation spaces. Polynomials can approximate smooth functions well. However, for crack problems, the NMM adopting polynomial local approximation spaces (i.e. the conventional NMM) suffers from two limitations: (1) the singularities near the crack tips cannot be well captured; (2) the crack tips are only allowed to stop at the sides of the elements. In this section, we will use prior known knowledge about the solution to make a better choice for its local approximation spaces based on partition of unity.

The near tip displacement fields for combined Modes I and II loading are (Belytschko and Black, 1999)

$$\begin{aligned} u(x, y) &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \cos\left(\frac{\theta}{2}\right) \left[\kappa - 1 + 2 \sin^2\left(\frac{\theta}{2}\right) \right] + \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left[\kappa + 1 + 2 \cos^2\left(\frac{\theta}{2}\right) \right] \\ v(x, y) &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left[\kappa + 1 - 2 \sin^2\left(\frac{\theta}{2}\right) \right] - \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \cos\left(\frac{\theta}{2}\right) \left[\kappa - 1 - 2 \sin^2\left(\frac{\theta}{2}\right) \right] \end{aligned} \quad (4.5)$$

where r and θ are polar coordinates in the local crack tip coordinate system, shown in Fig. 4.3, μ is the shear modulus, κ is Kolosov constant defined as

$$\kappa = \begin{cases} 3 - 4\nu, & \text{plane strain} \\ (3 - \nu)/(1 + \nu), & \text{plane stress} \end{cases} \quad (4.6)$$

The displacement field in Eq. (4.5) is contained in the span of the following four

functions:

$$[\Phi_1 \quad \Phi_2 \quad \Phi_3 \quad \Phi_4] = \left[\sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2} \right] \quad (4.7)$$

Fig. 4.3 Illustration of local crack tip coordinate system

Then, the asymptotic crack tip functions in Eq. (4.7) are proposed to be incorporated into the local approximation spaces of singular physical covers. The four terms in Eq. (4.7) can reproduce any displacement field near crack tips, thus the singularities near crack tips can be well represented. In addition, the first term of Eq. (4.7) is discontinuous across the crack surface, i.e. $\theta = \pm\pi$, as illustrated in Fig. 4.4, thus, the displacement jumps across the crack surfaces in partially cracked elements can be modeled, which will be explained in the following part.

Fig. 4.4 2D view of asymptotic crack tip functions

The NMM approximation in Eq. (2.4) is revised to

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i=1}^{n_M} \varphi_i(\mathbf{x}) \left(\sum_{j=1}^{m_i} \delta_i^j \mathbf{u}_i^j + \underbrace{\sum_{\alpha=1}^4 \Phi_\alpha \mathbf{a}_i^\alpha}_{i \in n_\Lambda} \right) \quad (4.8)$$

where $\mathbf{a}_i^\alpha$ is the unknown associated with Φ_α , Φ_α is the matrix of enrichment bases as

$$\Phi_\alpha = \begin{bmatrix} \Phi_\alpha & 0 \\ 0 & \Phi_\alpha \end{bmatrix}, \quad \alpha = 1 \sim 4 \quad (4.9)$$

Take finite element 7 in Fig. 4.2 for an example. All the four mathematical covers associated with element 7, i.e. $M_{(i)}, i=1 \sim 4$ are only partially cut by the crack surface, and thus each forms one singular physical cover, denoted as $P_{(1)}^1, P_{(2)}^1, P_{(3)}^1$, and $P_{(4)}^1$, respectively. Singular physical covers are enriched with the asymptotic crack tip functions in Eq. (4.7).

Then, from Eq. (4.8), the displacement jump across the crack surface in element 7 can be obtained as

$$\begin{aligned} \llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket &= \sum_{i \in \{1,2,3,4\}} \varphi_{(i)} \sqrt{r} \begin{pmatrix} \sin \frac{\theta}{2} \Big|_{\theta=\pi} - \sin \frac{\theta}{2} \Big|_{\theta=-\pi} & 0 \\ 0 & \sin \frac{\theta}{2} \Big|_{\theta=\pi} - \sin \frac{\theta}{2} \Big|_{\theta=-\pi} \end{pmatrix} \mathbf{a}_i^1 \\ &= \sum_{i \in \{1,2,3,4\}} \varphi_{(i)} \sqrt{r} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{a}_i^1 \end{aligned} \quad (4.10)$$

The displacement jump across the element 7 is obviously manifested in Eq. (4.10). Similarly, the displacement jump across the crack surface in element 1 can be proved.

To sum up, by adopting two sets of covers and by incorporating asymptotic crack tip functions in Eq. (4.7) into the local approximation spaces of singular physical covers, the entire crack be modeled in the extended (or modified) NMM. Moreover, asymptotic crack tip functions in Eq. (4.7) can well describe the stress singularities around the crack tips.

4.2.4 Formulation of discrete equations

Let $\mathbf{u} \in V$ be the displacement trial function with $V = H^1(\Omega)$, and $\delta\mathbf{u} \in V$ be a test function. A weak form of the discrete problem on a manifold element e is to find $\mathbf{u}^h$ in the finite dimensional subspace $V^h \subset V$, $\forall \delta\mathbf{u}^h \in V^h$, such that

$$\int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\delta\mathbf{u}^h) dV + \lambda \int_{\Gamma_u} (\mathbf{u}^h - \bar{\mathbf{u}}) \cdot \delta\mathbf{u}^h d\Gamma = \int_{\Gamma_t} \bar{\mathbf{t}} \cdot \delta\mathbf{u}^h d\Gamma + \int_{\Omega} \mathbf{b} \cdot \delta\mathbf{u}^h dV \quad (4.11)$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are stress tensor and strain tensor, Ω_e is the domain occupied by the cover-based ME e subjected to the body force $\mathbf{b}$. The boundary conditions consist of the essential boundary condition $\mathbf{u} = \bar{\mathbf{u}}$ on Γ_u and traction boundary condition $\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}}$ on Γ_t . In the NMM, the mathematical covers are constructed totally independent of the external boundaries, thus the essential boundary condition cannot be accomplished by directly enforcing the degrees of freedom like in the FEM, but is usually prescribed by additional technique in the weak form of governing equations. The second-term of Eq. (4.11) is to enforce the essential boundary conditions by using the penalty method with a real penalty number λ .

On the other hand, from Eq. (4.8), the trial function $\mathbf{u}^h$ and the test function $\delta\mathbf{u}^h$ are expressed as

$$\mathbf{u}^h(\mathbf{x}) = \sum \varphi_i(\mathbf{x}) \cdot (\mathbf{p}^T(\mathbf{x}) \cdot \mathbf{a}_i + \Phi \mathbf{c}_i) \quad (4.12a)$$

$$\delta\mathbf{u}^h(\mathbf{x}) = \sum \varphi_i(\mathbf{x}) \cdot (\mathbf{p}^T(\mathbf{x}) \cdot \delta\mathbf{a}_i + \Phi \delta\mathbf{c}_i) \quad (4.12b)$$

By substituting the above two equations into Eq. (4.11), and considering the arbitrariness of the test functions, we formally obtain

$$\mathbf{K}\mathbf{q} = \mathbf{F} \quad (4.13)$$

where $\mathbf{q}$ is the vector of unknowns, $\mathbf{K}$ and $\mathbf{F}$ are the equivalent stiffness matrix and load vector, respectively. The element contribution to $\mathbf{K}$ and $\mathbf{F}$ are as follows

$$\mathbf{k}_{ij}^e = \begin{bmatrix} \mathbf{k}^{a_i a_j} & \mathbf{k}^{a_i c_j} \\ \mathbf{k}^{c_i a_j} & \mathbf{k}^{c_i c_j} \end{bmatrix}, \quad (4.14)$$

$$\mathbf{f}_i^e = \left\{ \mathbf{f}^{a_i} \quad \mathbf{f}^{c_i} \right\}^T \quad (4.15)$$

with their entries being

$$\mathbf{k}^{r_i s_j} = \int_{\Omega^e} \left(\mathbf{B}^{r_i} \right)^T \mathbf{D} \mathbf{B}^{s_j} d\Omega + \lambda \int_{\Gamma_u^e} \left(\varphi_i \mathbf{P} \right)^T \cdot \left(\varphi_j \mathbf{P} \right) d\Gamma \quad (r_i, s_i = \mathbf{a}_i, \mathbf{c}_i) \quad (4.16)$$

$$\mathbf{f}^{a_i} = \int_{\Gamma_t^e} \varphi_i \mathbf{P} \bar{\mathbf{t}} d\Gamma + \int_{\Omega^e} \varphi_i \mathbf{P} \mathbf{b} d\Omega + \lambda \int_{\Gamma_u^e} \varphi_i \mathbf{P} \bar{\mathbf{u}} d\Gamma \quad (4.17a)$$

$$\mathbf{f}^{c_i} = \int_{\Gamma_t^e} \varphi_i \Phi \bar{\mathbf{t}} d\Gamma + \int_{\Omega^e} \varphi_i \Phi \mathbf{b} d\Omega + \lambda \int_{\Gamma_u^e} \varphi_i \Phi \bar{\mathbf{u}} d\Gamma \quad (4.17b)$$

where $\mathbf{D}$ is the elasticity matrix. The entries of the strain-displacement matrix $\mathbf{B}$ are

$$\mathbf{B}^{a_i} = \begin{bmatrix} (\varphi_i \mathbf{P})_{,x} & 0 \\ 0 & (\varphi_i \mathbf{P})_{,y} \\ (\varphi_i \mathbf{P})_{,y} & (\varphi_i \mathbf{P})_{,x} \end{bmatrix}, \quad (4.18a)$$

$$\mathbf{B}^{c_i} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \mathbf{B}_3 \quad \mathbf{B}_4], \quad (4.18b)$$

$$\mathbf{B}_\alpha = \begin{bmatrix} (\varphi_i \Phi_\alpha)_{,x} & 0 \\ 0 & (\varphi_i \Phi_\alpha)_{,y} \\ (\varphi_i \Phi_\alpha)_{,y} & (\varphi_i \Phi_\alpha)_{,x} \end{bmatrix} \quad (\alpha = 1 \sim 4). \quad (4.18c)$$

4.2.5 Integration scheme

Although the mathematical covers are usually formed by a regularly-patterned triangular or rectangular finite element mesh, the manifold elements may have general shape because of the intersection with the physical features. Thus, direction integration scheme over the whole element like that in the FEM is not applicable any more.

In the conventional NMM, simplex integration scheme is adopted to evaluate the integration of weak form, as introduced in Section 3.2.3.

However, the simplex integration scheme is limited to the cases where

polynomial functions are adopted as local functions. For the manifold elements covered by singular physical covers enriched with asymptotic crack tip functions, which are non-polynomial functions, the simplex integration is not applicable any more. In such circumstances, we turn to the widely used Gaussian quadrature.

The manifold element of arbitrary shape is first partitioned into several sub-triangles, as illustrated in Fig. 4.5.

Fig. 4.5 Partitioning manifold elements into sub-triangles by connecting each vertex to the centroid: (a) an element completely cut by a crack surface; (b) an element partially cut by the crack surface

Then, Gaussian quadrature is carried out on each sub-triangle, and finally summation is made to obtain the integration over the whole manifold element.

For a conventional element, 1-point Gaussian quadrature is used over each sub-triangle, while for a singular element, 13-point Gaussian quadrature is used over each sub-triangle. For a blending element that has both conventional and singular physical covers, 7-point Gaussian quadrature is used over each sub-triangle.

4.2.6 Evaluation of stress intensity factors

A number of approaches have been developed to evaluate the stress intensity factors (SIFs) at the crack tips, e.g. the crack opening displacement method (Paris and Sih, 1965; Tsay et al., 1999), virtual crack extension method (Hellen, 1975; Xie et al., 1995; Chiou et al., 2002), and the domain form of the interaction integral method (Yau et al., 1980; Shih and Asaro, 1988; Moes et al., 1999; Belytschko and Black,

1999), etc. In this study, domain form of the interaction integral is employed to evaluate the SIFs at the crack tips. For completeness, this method is briefly introduced in this section.

The coordinates are taken to be the local crack tip coordinates with the x_1 -axis parallel to the crack faces (see Fig. 4.3). For a general mixed-mode problem we have the following relationship between the J -integral and the SIFs (Rice, 1968)

$$J = \frac{K_I^2}{E^*} + \frac{K_{II}^2}{E^*} \quad (4.19)$$

where $E^* = E$ for a plane stress problem, and $E^* = E/(1-\nu^2)$ for a plane strain problem, E is the Young's modulus of the material, ν is the Poisson's ratio, .

Two states of cracked body are considered. State 1, $(\sigma_{ij}^{(1)}, \varepsilon_{ij}^{(1)}, u_i^{(1)})$ corresponding to the present state and state 2, $(\sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)}, u_i^{(2)})$, is an auxiliary state which will be chosen as the asymptotic fields for mode I or mode II. The J -integral for the sum of the two states is

$$J^{(1+2)} = \int_{\Gamma} \left(\frac{1}{2} (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) (\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}) \delta_{1j} - (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) \frac{\partial (u_i^{(1)} + u_i^{(2)})}{\partial x_1} \right) n_j d\Gamma \quad (4.20)$$

Expanding and rearranging terms give

$$J^{(1+2)} = J^{(1)} + J^{(2)} + I^{(1,2)} \quad (4.21)$$

where $I^{(1,2)}$ is called the interaction integral for states 1 and 2

$$I^{(1,2)} = \int_{\Gamma} \left(W^{(1,2)} \delta_{1j} - \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} - \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} \right) n_j d\Gamma \quad (4.22)$$

where $W^{(1,2)}$ is the interaction strain energy

$$W^{(1,2)} = \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} = \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)} \quad (4.23)$$

Writing Eq. (4.20) for the combined states and rearranging terms give

$$J^{(1+2)} = J^{(1)} + J^{(2)} + \frac{2}{E^*} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)}) \quad (4.24)$$

Eqs. (4.21) and (4.24) lead to

$$I^{(1,2)} = \frac{2}{E^*} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)}) \quad (4.25)$$

Choosing state 2 as the pure mode I asymptotic field with $K_I^{(2)} = 1$, $K_{II}^{(2)} = 0$ gives the mode I SIF for state 1 as

$$K_I^{(1)} = \frac{2}{E^*} I^{(1, \text{Mode I})} \quad (4.26)$$

Similarly, we can obtain the mode II SIF for state 1 as

$$K_{II}^{(1)} = \frac{2}{E^*} I^{(1, \text{Mode II})} \quad (4.27)$$

For the convenience of calculation, the contour integral in Eq. (4.22) is transformed to an equivalent form by multiplying the integrand by a sufficiently smooth weighting function $q(\mathbf{x})$ which takes a value of unity on an open set containing the crack tip and vanishes on an outer prescribed contour C_0 . Then for each contour Γ in Fig. 4.6, assuming the crack faces are straight in the region between Γ and C_0 , the interaction integral may be written as

$$I^{(1,2)} = \int_C \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right) q m_j dC \quad (4.28)$$

where contour $C = \Gamma + C_+ + C_- + C_0$ and $\mathbf{m}$ is the unit outward normal. In deriving the above equation, we have used the relations $m_j = -n_j$ on Γ , $m_j = n_j$ on C_0 , C_+ , C_- , and $m_1 = 0$ on C_+ and C_- .

Using the divergence theorem on the closed integral over the contour C and passing to the limit as the contour Γ is shrunk to the crack tip, we arrive at the following equation for the interaction integral in domain form

$$I^{(1,2)} = \int_A \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} - \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} dA \quad (4.29)$$

In deriving the above equation, we have used the knowledge that

$\sigma_{i2}^{(2)} m_2$ ($i=1,2$) vanishes on the crack faces as the auxiliary fields satisfy traction-free crack faces. We also note the importance of including the integral over C_+ and C_- in the case of contact between crack faces. While the normal traction which arises on the crack faces during the contact does not contribute to the J -integral, they do contribute to the interaction integral. The actual normal forces do work in conjunction with the auxiliary displacements.

Fig. 4.6 Convention at crack tip, domain A is enclosed by Γ , C_+ , C_- and C_0 , unit normal $m_j = n_j$ on C_+ , C_- and C_0 , and $m_j = -n_j$ on Γ

4.2.7 Modeling crack growth

Crack representation

In preparation of the analysis, the initial crack geometry, and the parameters of the crack growth criterion should be given. For a two-dimensional problem, similar to the XFEM (Sukumar and Prevost, 2003), cracks are represented by continuous piecewise linear segments, i.e., $\Gamma_c = \cup_1^n \mathbf{I}_i$, where n is the number of crack segments, $\mathbf{I}_i = \{(x_i, y_i), (x_{i+1}, y_{i+1})\}$ for $i=1, 2, \dots, n$, with the crack tip being a point, which is

either (x_i, y_i) or (x_{n+1}, y_{n+1}) , or both. The crack growth criterion will be discussed in the next sub-section.

Crack growth criterion

For a given crack problem, a criterion is needed to judge whether the crack will propagation or not at the beginning of each time step, and then the crack growth direction and increment if the propagation starts.

Several crack growth criterion have been proposed, including the maximum energy release rate criterion (Nuismer, 1975), the minimum strain energy density criterion (Erdogan and Sih, 1963), and the maximum circumferential stress criterion (Sih, 1974).

In the present study, the maximum circumferential stress criterion is chosen. This criterion states that a crack will propagate when the circumferential stress reaches its critical value, and propagate from its tip in a direction θ_c so that the circumferential stress σ_θ is maximum.

Under general mixed-mode loadings, the asymptotic near-tip circumferential and shear stresses take the following form (Moes et al, 1999):

$$\begin{aligned}\sigma_\theta &= \frac{K_I}{\sqrt{2\pi r}} \frac{1}{4} \left(3 \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \frac{1}{4} \left(-3 \sin \frac{\theta}{2} - 3 \sin \frac{3\theta}{2} \right) \\ \sigma_{r\theta} &= \frac{K_I}{\sqrt{2\pi r}} \frac{1}{4} \left(\sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \frac{1}{4} \left(\cos \frac{\theta}{2} + 3 \cos \frac{3\theta}{2} \right)\end{aligned}\quad (4.30)$$

Therefore, the critical angle θ_c defining the radial direction of propagation can be determined by

$$\frac{\partial \sigma_\theta}{\partial \theta} = 0 \quad (4.31)$$

After a few manipulations, the following expression is obtained

$$K_I \sin \theta_c + K_{II} (3 \cos \theta_c - 1) = 0 \quad (4.32)$$

Solving this equation gives

$$\theta_c = 2 \arctan \left(\frac{1}{4} \left(\frac{K_I}{K_{II}} \pm \sqrt{\left(\frac{K_I}{K_{II}} \right)^2 + 8} \right) \right) \quad (4.33)$$

We know that if $K_{II} = 0$, then $\theta_c = 0$ (pure mode I), and if $K_{II} > 0$, the crack growth angle $\theta_c < 0$, and if $K_{II} < 0$, then $\theta_c > 0$. Then, Eq. (4.33) is re-summarized as

$$\theta_c = \begin{cases} 2 \arctan \left(\frac{1}{4} \left(\frac{K_I}{K_{II}} + \sqrt{\left(\frac{K_I}{K_{II}} \right)^2 + 8} \right) \right) & K_{II} < 0 \\ 0 & K_{II} = 0 \\ 2 \arctan \left(\frac{1}{4} \left(\frac{K_I}{K_{II}} - \sqrt{\left(\frac{K_I}{K_{II}} \right)^2 + 8} \right) \right) & K_{II} > 0 \end{cases} \quad (4.34)$$

Submitting θ_c into Eq. (4.30) leads to the maximum circumferential stress at a radius $r = r_0$ as

$$(\sigma_\theta)_{\max} = \frac{1}{\sqrt{2\pi r_0}} \cos \frac{\theta_c}{2} \left(K_I \cos^2 \frac{\theta_c}{2} - \frac{3}{2} K_{II} \sin \theta_c \right) \quad (4.35)$$

Then, the crack propagation criterion reads

$$(\sigma_\theta)_{\max} = (\sigma_\theta)_c \quad (4.36)$$

where $(\sigma_\theta)_c$ is the critical value for the maximum circumferential stress, which can be determined from the mode I crack toughness K_{IC} . For a crack under mode I loading, $\theta_c = 0^\circ$, $K_{II} = 0$, $K_I = K_{IC}$. Substituting the above values into Eq. (4.35), we obtain

$$(\sigma_\theta)_c = \frac{K_{IC}}{\sqrt{2\pi r_0}} \quad (4.37)$$

Substituting Eqs. (4.35) and (4.37) into Eq. (4.36) leads to the crack propagation criterion

$$K_{eq} = K_{IC} \quad (4.38)$$

where K_{eq} is the equivalent mode I SIF, expressed as

$$K_{eq} = \cos \frac{\theta_c}{2} \left(K_I \cos^2 \frac{\theta_c}{2} - \frac{3}{2} K_{II} \sin \theta_c \right) \quad (4.39)$$

Fig. 4.7 Illustration of crack growth direction and increment at time step n

A sketch for crack growth is illustrated in Fig. 4.7, in which the crack growth direction angle θ_c is determined from Eq. (4.34), while the crack growth increment da is pre-defined before calculation, which is adopted by many other researchers (Bittencourt et al., 1996; Moes et al., 1999; Ventura et al., 2002).

Treatment of manifold elements during fracturing process

If the crack growth criterion is satisfied, the crack will grow from its crack tip to a predefined distance and divide some elements. In the present study, we add elements to achieve the element division. If element i is divided into two sub-elements, one of them will be numbered as i , and the other one numbered as $i+1$. All the elements numbered after this element will be reallocated element numbers from $i+2$. The two sub-elements will be assigned with physical covers temporally same with those of the original element before division. The update of the physical covers for each element will be presented in the following section.

Treatment of physical covers during fracturing process

During the crack growth, some physical covers are completely cut by the crack and need to be partitioned and some physical covers are partially cut by the crack surfaces and become singular physical covers. If the original physical cover P_i is partitioned into two isolated parts P_i^1 and P_i^2 because of the growing crack, we add one cover in the same place to represent the partition, which will be numbered from the total number of covers (say N) in the present state as P_{N+1} . When constant local functions are adopted for each conventional physical cover, one more physical cover will introduce two more degrees of freedom into the global discrete equations. If a physical cover P_j is partially cut by the growing crack, it will become a singular physical cover and be enriched with asymptotic crack tip functions. During the crack growing, some of the original singular physical covers will become conventional physical covers and the enrichment functions will be removed from such physical covers.

Since some of the physical covers are partitioned during the crack growing, the three or four (three for a triangular finite element mesh, four for a rectangular finite element mesh) physical covers for each element should be updated. This process is achieved through two steps. Firstly, we search for all the elements associated with the split physical cover. Secondly, we allocate correct physical covers (P_i or P_{N+1}) to these elements. In the NMM code, the orientation test (Sukumar and Prevost, 2003; Shewchuk, 1997), which determines whether a point lies to the left of, to the right of, or on a line or plane defined by other points, is adopted as the computational algorithm to allocate the correct physical covers for each element. To do so, the centroid (x, y) of each element is calculated by the simplex integration method; triangles with vertices $((x_1, y_1), (x_2, y_2), (x, y))$ and $((x_1, y_1), (x_2, y_2), (x', y'))$ where (x_1, y_1) and (x_2, y_2) are the coordinates to define a crack segment, (x', y') is the coordinates of the star corresponding to the split physical cover. The determinants Δ and Δ' are evaluated

$$\Delta = (x_1 - x)(y_2 - y) - (x_2 - x)(y_1 - y) \quad (4.40a)$$

$$\Delta' = (x_1 - x')(y_2 - y') - (x_2 - x')(y_1 - y') \quad (4.40b)$$

when $\Delta \cdot \Delta' > \varepsilon$, element and the star on the same side of the crack segment, cover P_i is assigned to the element; while $\Delta \cdot \Delta' < -\varepsilon$, element and the star on the difference side of the crack segment, cover P_{N+1} is allocated to the element, where a tolerance $\varepsilon = 10^{-6}$ is used in the current study.

4.3 Illustration examples

In this section, eight examples are presented to verify the extended NMM and to exhibit its capability for complex crack problems and their growth. Five stationary crack problems, i.e. an edge crack in a finite plate under uniaxial tension, an inclined central crack in a square plate under uniaxial tension, a symmetrically branched crack problem, a star-shaped crack problem, and a tree-shaped crack are first considered to show the accuracy and convergence of the extended NMM in evaluation of stress intensity factors (SIFs), and then crack propagations are simulated for another three examples, i.e. edge crack growth in double cantilever beam, double cracks in a plate with two circular holes, growth of a branched crack.

4.3.1 An edge crack in a finite plate under uniaxial tension

As the first example, a problem with an edge crack in a finite plate under uniaxial tension shown in Fig. 4.8 is investigated. It is a typical mode I crack problem.

The reference solution for the mode I SIF was given by Ewalds and Wanhill (1989) as

$$K_I = F\sigma\sqrt{\pi a} \quad (4.41)$$

where F is the modification factor to reflect the side effect, and if $a/W \leq 0.6$, approximated by

$$F = 1.12 - 0.231\left(\frac{a}{W}\right) + 10.55\left(\frac{a}{W}\right)^2 - 21.72\left(\frac{a}{W}\right)^3 + 30.39\left(\frac{a}{W}\right)^4 \quad (4.42)$$

In calculation, the plate size is taken to be $W = 2.0$ m, $H = 3.0$ m, and the

uniaxial tension $\sigma = 1.0$ Pa. The uniform hexagonal mathematical covers are used. For easy interpretation of the resolution of the mathematical covers, we introduce a parameter called cover size h defined as the maximum diameter of the circumscribed circle of the mathematical cover.

Fig. 4.8 An edge crack in a finite plate under uniaxial tension

To study the convergence of the numerically obtained mode I stress intensity factor K_I , various mathematical covers with cover size h ranging from $0.5W$ to $0.05W$ are adopted, while the crack length a is fixed at 1.0 m. To save pages, only the mathematical covers with cover size $h = 0.5W$ is depicted in Fig. 4.9 for illustration purpose, in which there are totally 94 mathematical covers, 97 physical covers and 152 manifold elements.

The mode I SIF K_I is computed for the above various cases. Fig. 4.10 plots the mode I SIF K_I normalized by the reference solution in Eq. (4.41). It is easily seen that when the mathematical covers are gradually refined, the normalized K_I converges to the reference solution. For example, for the case with $h = 0.05W$, the

relative error is within 0.4%.

Next, we focus on the accuracy of SIF results for different crack lengths ranging from 0.5 to 1.0. A constant mathematical cover size of $h = 0.05W$ is used throughout the computation. The calculated mode I SIF for different crack lengths is plotted in Fig. 4.11, as compared with the reference solution. The agreement is satisfactory.

Fig. 4.9 A representative mathematical cover system for the edge crack problem,
 $h = 0.5W$

Fig. 4.10 Convergence of the normalized K_I with the refinement of cover size

Fig. 4.11 Variation of K_I with crack length a

4.3.2 An inclined central crack in a square plate under uniaxial tension

In this example, a typical mixed mode crack problem is investigated. As shown in Fig. 4.12, we consider a square plate with an inclined central crack subjected to uniform uniaxial tension.

Fig. 4.12 An inclined central crack in a square plate under uniaxial tension

In calculation, the plate size is set to be $W = 10.0$ with the half crack length of $a = 1.0$. The uniform uniaxial tension at the top and bottom is assumed to be $\sigma = 1.0$. When the length of the crack is sufficiently small, as compared with the plate size, the exact SIF results for the inclined crack in an infinite plate can be regarded as the reference solution, which can be expressed as (Karihaloo, 1973)

$$\begin{aligned} K_I &= \sigma \sqrt{\pi a} \cos^2 \beta \\ K_{II} &= \sigma \sqrt{\pi a} \sin \beta \cos \beta \end{aligned} \quad (4.43)$$

During the simulation, six cases with different crack inclination angles of 0° , 15° , 30° , 45° , 60° and 75° are examined, respectively. For all the cases, 1862 mathematical covers with cover size $h = 0.2W$ are used, but the number of physical covers and manifold elements slightly differ for different cases with different inclined angles, e.g. 1866 physical covers and 3560 manifold elements for the case with $\beta = 0^\circ$, and 1868 physical covers and 3562 manifold elements for the case with $\beta = 75^\circ$. The calculated SIFs are plotted in Fig. 4.13. It is obvious that the results by the present NMM agree well with the reference solution.

Fig. 4.13 Variation of K_I , K_{II} with the crack inclination angle β

4.3.3 A symmetrically branched crack in a finite plate

Our third numerical experiment is a finite plate with a symmetrically branched crack under uniform uniaxial tension as shown in Fig. 4.14.

Throughout the calculation, the plate dimensions are fixed to be $W = 5$ and $H = 4$, and material constants are the Young's modulus $E = 1000$ and the Poisson's ratio $\nu = 0.3$. The uniform uniaxial tension at the top and bottom is assumed to be $\sigma = 1.0$. The normalized stress intensity factors for crack tips A and B are defined as

$$\begin{aligned} F_I^A &= K_I^A / \sigma \sqrt{\pi c}, \\ F_I^B &= K_I^B / \sigma \sqrt{\pi c}, \\ F_{II}^B &= K_{II}^B / \sigma \sqrt{\pi c} \end{aligned} \quad (4.44)$$

with $c = (a + b \cos \theta) / 2$ (see Fig. 4.14).

Fig. 4.14 A symmetrically branched crack in a finite plate under tension

The uniform hexagonal mathematical covers generated from a uniform triangular finite element mesh is adopted, based on which physical covers and manifold elements are formed accordingly.

Fig. 4.15 A representative mathematical cover system for the branched crack problem,
cover size $h = 0.18W$

To study the convergence, a branched crack with $a/W = b/W = 0.2$ and $\theta = 45^\circ$ is investigated. The representative discretized domain with cover size $h = 0.18W$ is shown in Fig. 4.15, in which there are 493 mathematical covers, 503 physical covers and 913 manifold elements.

The SIFs for different cover size h are calculated and the normalized SIF results versus W/h are plotted in Fig. 4.16. It is evident that the SIF results converge to a certain value as the mathematical covers are consecutively refined.

(a)

(b)

(c)

Fig. 4.16 Variations of the normalized SIFs with the cover size for the branched crack problem: (a) F_I^A ; (b) F_I^B ; (c) F_{II}^B (dash line represents the converging value of each normalized SIF)

Next, the influence of the finiteness of plate on the SIFs is examined by varying a/W from 0.1 to 0.9 while keeping $a/b=1$, $\theta=45^\circ$ and $h=0.04W$. The SIFs are calculated and compared with the reference solutions from the XFEM (Daux et al., 2000) in Fig. 4.17. It is seen that the agreement between the present results and the reference solutions is satisfactory.

(a)

(b)

(c)

Fig. 4.17 Comparison between the normalized SIFs from the NMM and reference solutions for the branched crack problem

The influence of the crack geometry on the SIFs is also investigated by varying b/a and θ , while fixing $a/W = 0.05$. In calculation, h takes the value of $0.02W$ for relatively larger b/a (e.g. 0.5, 0.7 and 0.9), while $h = 0.01W$ for smaller b/a to assure that there is only one crack tip in a manifold element. Since the crack size is sufficiently small as compared with the plate size in our modeling, it is reasonable to choose the SIFs of a branched crack in an infinite plate as the reference solution (Chen and Hasebe, 1995). The SIF results from the extended NMM and the reference solution are listed in Table 4.1. It is seen that the relative error (RE) is less than 1.6% for all the cases.

Table 4.1 Normalized SIFs comparison for a branched crack

b/a	θ	30°			45°			60°		
		NMM	*	RE (%)	NMM	*	RE (%)	NMM	*	RE (%)
0.1	F_I^A	1.020	1.008	1.19	0.999	1.010	1.09	1.002	1.012	0.99
	F_I^B	0.669	0.679	1.47	0.553	0.560	1.25	0.386	0.390	1.03
	F_{II}^B	0.196	0.199	1.51	0.352	0.347	1.44	0.430	0.435	1.15
0.3	F_I^A	1.022	1.016	0.59	1.026	1.019	0.69	1.020	1.023	0.29
	F_I^B	0.664	0.661	0.45	0.509	0.512	0.59	0.307	0.309	0.65
	F_{II}^B	0.274	0.276	0.72	0.441	0.438	0.68	0.521	0.526	0.95
0.5	F_I^A	1.013	1.021	0.78	1.016	1.026	0.97	1.026	1.033	0.68
	F_I^B	0.652	0.658	0.91	0.496	0.500	0.80	0.290	0.288	0.69
	F_{II}^B	0.311	0.309	0.65	0.470	0.474	0.84	0.563	0.559	0.72
0.7	F_I^A	1.027	1.024	0.29	1.036	1.033	0.29	1.038	1.046	0.76
	F_I^B	0.658	0.657	0.15	0.499	0.496	0.60	0.284	0.282	0.71
	F_{II}^B	0.324	0.327	0.92	0.497	0.493	0.81	0.577	0.573	0.70
0.9	F_I^A	1.034	1.028	0.58	1.042	1.040	0.19	1.054	1.061	0.66
	F_I^B	0.655	0.658	0.46	0.499	0.495	0.81	0.282	0.281	0.36
	F_{II}^B	0.396	0.399	0.75	0.507	0.503	0.80	0.580	0.577	0.52

* Chen and Hasebe (1995)

4.3.4 A star-shaped crack in a square plate under biaxial tension

Next, we examine a star-shaped crack in a square plate subjected to biaxial tension as shown in Fig. 4.18. The normalized stress intensity factors at tips A and B

are defined as

$$\begin{aligned} F_I^A &= K_I^A / \sigma \sqrt{\pi a}, \\ F_I^B &= K_I^B / \sigma \sqrt{\pi a}, \\ F_{II}^B &= K_{II}^B / \sigma \sqrt{\pi a} \end{aligned} \quad (4.45)$$

Fig. 4.18 A star-shape crack in a square plate subjected to bi-axial tension

A representative mathematical cover system for the star-shaped crack problem is shown in Fig.4.19, with $a / W = 0.1$.

(a)

(b)

Fig. 4.19 A representative mathematical cover system for the star-shape crack problem

($a/W = 0.1$): (a) global view; (b) zoom in the central region

In the analysis, the plate size is taken to be $W = 2.0$, and the biaxial tension to be unity. The normalized SIFs are inspected. The results for different cover sizes are plotted in Fig. 4.20 for the case of $a/W = 0.2$, and the convergence can be apparently found as the cover size is consecutively decreased.

(a)

(b)

(c)

Fig. 4.20 Variation of the normalized SIFs with the cover size for a star-shaped crack problem: (a) F_I^A ; (b) F_I^B ; (c) F_{II}^B (dash line represents the converging value of each normalized SIF)

Similar to the previous example, the influence of the finiteness of plate on the SIFs is also investigated. This time, the cover size $h = 0.05W$ is used. In this case, the number of mathematical covers is fixed at 2895, while numbers of physical covers and manifold elements changes with the crack length (see Table 4.2).

Table 4.2 Number of physical covers and manifold elements for different a/W

a/W	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
PCs	2921	2951	2979	3009	3037	3065	3095	3123	3153
CEs	5612	5642	5670	5700	5728	5756	5786	5814	5844

It should be noted that the numbers of mathematical covers, physical covers and manifold elements in this study can be sharply decreased if adaptive finite element mesh is adopted. However, it is not the topic of the current study and will not be discussed here. The computed SIF results are summarized and compared with three reference solutions from Daux et al. (2000), Muravin and Turkel (2006), and Cheung et al. (1994) in Table 4.3. From this table, the excellent accuracy of the present NMM can also be easily seen.

Table 4.3 Normalized SIFs comparison for a start-shaped crack

a/W	F_I^A				F_I^B				F_{II}^B			
	NMM	R1	R2	R3	NMM	R1	R2	R3	NMM	R1	R2	R3
0.1	0.758	0.751	-	0.741	0.767	0.769	-	0.741	0	0	-	0
0.2	0.771	0.767	0.769	0.757	0.771	0.768	0.769	0.758	0	0.001	0.001	0.001
0.3	0.789	0.793	0.797	0.785	0.798	0.798	0.799	0.788	0.002	0.002	0.002	0.002
0.4	0.821	0.829	0.835	0.826	0.854	0.847	0.853	0.837	0.007	0.008	0.008	0.007
0.5	0.887	0.886	0.892	0.885	0.924	0.926	0.923	0.909	0.016	0.018	0.02	0.017
0.6	0.971	0.967	0.975	0.976	1.04	1.045	1.041	1.018	0.036	0.036	0.045	0.034
0.7	1.107	1.097	1.102	1.114	1.234	1.237	1.238	1.194	0.061	0.059	0.062	0.053
0.8	1.34	1.342	1.345	-	1.559	1.562	1.558	-	0.082	0.086	0.08	-
0.9	1.93	1.931	1.915	-	2.166	2.193	2.161	-	0.089	0.087	0.091	-

D: Daux et al. (2000)

MT: Muravin and Turkel (2006)

C: Cheung et al. (1984)

‘-’ means no corresponding solution

4.3.5 A tree-shaped crack in a finite plate

The fifth example intends to show the capability of the present NMM in solving very complex crack problems. As shown in Fig. 4.21, this is a coined problem with a tree-shaped crack in a finite plate under biaxial tension.

In computation, the dimensions are $W = H = 6.0$, $a = b = 2c = 1.0$ and $\alpha = 45^\circ$, $\beta = 60^\circ$. The external load is biaxial uniform tension with $\sigma = 1.0$. The material constants are $E = 1000$ for the Young's modulus and $\nu = 0.3$ for the Poisson's ratio.

To the authors' knowledge, the reference solution to this problem is not available. So, the current work focuses on the convergence test for such a problem, and the crack tips labeled by A, B, C and D in Fig. 4.21 are investigated. The normalized SIFs for these crack tips are expressed as

$$\begin{aligned} F_I^M &= K_I^M / \sigma \sqrt{\pi a}, \\ F_{II}^M &= K_{II}^M / \sigma \sqrt{\pi a} \quad (M = A, B, C, D) \end{aligned} \quad (4.46)$$

Fig. 4.21 A tree-shaped crack in a squared plate under bi-axial tension

(a)

(b)

Fig. 4.22 A representative mathematical cover system for the tree-shaped crack problem ($h = 0.073W$): (a) the discretized domain; (b) zoom in the central region

The calculation of SIFs for different cover size h is performed. For the case of $h = 0.073W$, 3636 overlapped hexagonal mathematical covers are used to cover the whole physical domain, and 3744 physical covers together with 7125 manifold elements are eventually formed, as shown in Fig. 4.22.

(a)

(b)

(c)

(d)

Fig. 4.23 Variation of the SIFs with the cover size for tree-shaped crack:(a) F_I^A ; (b) F_I^B ; (c) F_{II}^B ; (d) F_{II}^B ; (e) F_I^C ; (f) F_{II}^C ; (g) F_I^D ; (h) F_{II}^D (dash line represents the converging value of each normalized SIF)

The SIFs results for different value of W/h are plotted in Fig. 4.23. It is seen that, even for such a complex problem, the present NMM shows a consistent convergence as the mathematical covers are gradually refined.

4.3.6 Edge crack growth in double cantilever beam

In Fig. 4.24, a double cantilever beam with a pair of concentrated forces applied at the left end and the right end of the beam clamped is illustrated. If the initial crack is exactly located on the mid-plane of the sample, by symmetry, the crack is under

pure mode I loading, and it would propagate straight ahead. However, this straight path is unstable. A crack slightly off the mid-plane tends to curve away from the mid-plane.

In simulation, the beam dimensions are chosen as $W = 6.0$, $H = 1.0$, and the initial crack length is $a = 2.0$ with an initial offset $\delta = 0.05$ from the mid-plane. The fracture toughness of the beam is assumed to be $K_{IC} = 10.0$.

Fig. 4.24 Double cantilever beam with a slightly offset edge crack under concentrated loadings

The state of crack depends on the magnitude of the concentrated force F . Under a sufficiently small loading, the crack is stationary. When the loading is continuously increased until the crack growth criterion in Eq. (4.39) is satisfied, the crack begins to propagate. For a new growth step, the cracked beam is analyzed with same load F as that in the previous step, and then the load and corresponding responses, e. g. the deflection (i.e. vertical displacement) of the loading point, are scaled linearly to satisfy the crack growth criterion.

To test the convergence of crack growth path, analysis is performed for a fixed crack increment $da = 0.15$ as the mathematical cover size h decreases. The crack growth path and corresponding load-deflection curve are plotted in Fig. 4.25. As we can see from Fig. 4.25a, for relative coarse mathematical covers, e.g., when $h = 0.5$ the crack growth path is unsmooth with sudden change of growth direction. This is

because the SIF is not so accurately calculated that the accumulative error causes the sharp deviation of the growth direction. For finer cover sizes (e.g. $h = 0.30$ and $h = 0.20$), the path becomes smooth and convergent. The convergent results are comparable with those in Huang et al. (2003).

(a)

(b)

Fig. 4.25 Influence of the cover size on: (a) crack path; (b) load-deflection curve

In the above computation, the increment at each step was pre-defined. However, the choice of da is still suspended. Here, we try to present a guideline through studying the influence of da on the crack growth path.

Simulations for different increments with cover size fixed at $h = 0.16$ are carried out. The results are shown in Fig. 4.26.

(a)

(b)

Fig. 4.26 Influence of increment da on: (a) crack path; (b) load-deflection curve

From Fig. 4.26a, we see that when $0.08 \leq da \leq 0.12$ the crack paths agree well with each other. For greater increments, e.g. $da = 0.15$ and 0.20 , the crack paths are obviously different. It is reasonable. Since the real and smooth crack path is approximated by piecewise linear segment in theory, use of smaller increments may cause fewer accumulative errors, and therefore can lead to a smoother and stable growth path. For the load-deflection curves in Fig. 4.26b, we can also see the curves consistently converge for much smaller increments such as $0.08 \leq da \leq 0.12$. It is

noted that our present simulations are limited to the crack growth increments greater than $0.5h = 0.08$. This is because for much smaller crack increments, the crack tip may stop or kink in the same element in the subsequent growth steps, which causes difficulty in the partition of elements.

So far, we can suggest that it is better for the crack growth increment to take a value in the interval of $0.5h \leq da \leq 0.75h$. This will be further confirmed in the following two examples.

4.3.7 Double cracks in a plate with two circular holes

In this section, the propagation of multiple cracks in a rectangular plate with two circular holes is investigated. As shown in Fig. 4.27, the dimensions of the plate and cracks are designed so that the problem is symmetric with respect to the center of the plate in physics.

Fig. 4.27 Double cracks in a plate with two circular holes

The dimensions labeled in Fig. 4.27 are listed in Table 4.4.

Table 4.4 Dimensions of the problem geometry

L	H	L_1	H_1	H_2	R	a
20	10	1	2.85	1	2	1

On account of symmetry, displacement boundary conditions are symmetrically prescribed on the bottom and top edge of the plate with the values of $\Delta v = \pm 0.1$ in the vertical direction.

In this example, we assume that the cracks are not arrested until they reach the free boundary (including the hole boundary). For the symmetry reason, the same crack growth increments are chosen for both of the cracks.

For all growth steps, a fixed mathematical cover system with cover size $h = 0.80$ is used to trace the crack growth, and at the beginning, there are 1501 physical covers and 2774 manifold elements in total, which will be gradually increased as the cracks propagate. Throughout the computation, the crack growth increment is fixed at $da = 0.5h$.

The crack growth process is plotted in Fig. 4.28. We can see that at the earlier step such as step 5 (see Fig. 4.28a), the growth paths are almost symmetric. However, as the cracks further propagate (e.g. at the steps shown in Fig. 4.28b-d), the growth paths gradually lose the symmetry because of non-symmetry of the mathematical cover system, as well as the accumulative error of SIFs.

(a) Step 5

(b) Step 15

(c) Step 35

(d) Step 55

Fig. 4.28 Crack growth process of double cracks in a plate with two circular holes

To show the accuracy of the present method, a finer cover size of $h = 0.2$ is used, and the crack growth increment remains $da = 0.5h$. Repeating the procedure, the crack growth paths are obtained, and shown in Fig. 4.29. Compared with Fig. 4.28d with $h = 0.8$, we see that the two cracks propagate almost symmetrically at all the steps, and the results agree well with that from the FEM in Bouchard et al. (2003).

Fig. 4.29 Comparison of crack growth paths (solid line from the extended NMM; dash line from Bouchard et al. (2003))

4.3.8 Growth of a branched crack

As the last example, the propagation of a branched crack is studied to show the robustness of the NMM in simulating the crack growth in complex crack problems.

As shown in Fig. 4.30, a branched crack is located in a square plate with dimensions of $2W \times 2W = 10 \times 10$, and subjected to a unity uniform tension.

Fig. 4.30 A branched crack in a square plate under tension

The crack dimensions are $a = b = 1.0$ and the branched angle is $\theta = \pi/4$. Three cover sizes, i.e. $h = 0.56, 0.44, 0.34$ are used, respectively. Accordingly, the crack growth increments are all taken to be $da = 0.6h$ for the corresponding three cover sizes.

The calculated crack growth paths are shown in Fig. 4.31. It is seen that the growth paths are convergent as h decreases. As expected, the crack propagates in the cleavage mode (mode I), perpendicular to the maximal principal stress due to the applied load. Once again, this example shows that the present NMM is efficient and accurate when simulating the crack growth in complex crack problems.

Fig. 4.31 Influence of the cover size h on the crack path

4.4 Summary

This chapter extends the numerical manifold method (NMM) for complex crack problems. The singular physical covers are introduced to represent the physical covers which contain the crack tips inside. The asymptotic crack tip functions are incorporated into the local functions of singular physical covers. By such a modification, the crack tip singularity can be well represented and the crack tips can stop at arbitrary positions within a manifold element. The discrete equations are formulated. The Gauss quadrature is implemented to carry out the integrations. The domain form of interaction integral and the maximum circumferential stress criterion are adopted to evaluate the stress intensity factors and predict the crack growth, respectively. Eight examples conducted demonstrate the accuracy and robustness of the extended NMM for complex crack problems and their growth.

CHAPTER 5 COMPARISON BETWEEN THE NMM AND THE XFEM/GFEM IN CRACK MODELING

5.1 Introduction

Modeling cracks and their growth is highly important in both engineering application and academic research.

The FEM has been widely used in crack modeling. However, the conventional FEM requires the finite element mesh to be consistent with cracks, which complicates the meshing task and makes the remeshing inevitable for crack growth. In order to overcome such inconveniences, various modifications to the conventional FEM have been developed based on the partition of unity. Typical examples include the XFEM (Moes, et al., 1999; Daux et al., 2000), in which the finite element mesh is constructed regardless of cracks, and then discontinuous enrichment functions are incorporated to account for the presence of cracks, the GFEM (Duarte et al., 2007), which incorporate different enrichment functions from those used in the XFEM to deal with the cracks.

The NMM can also be viewed as a modification to the conventional FEM based on partition of unity. In Chapter 4, the NMM has been extended for complex crack problems and their growth. Several typical problems have been simulated to calibrate the developed program and to demonstrate the efficiency and robustness of the extended NMM for complex crack problems.

In this chapter, a detailed comparison between the NMM and the XFEM/GFEM in crack modeling is presented. How the two sets of covers in the NMM favor the complex crack modeling is fully discussed. Inspired by the advantageous modeling strategy of the NMM, a modification to the XFEM/GFEM is proposed. Modified XFEM/GFEM is termed as the parasitic finite element method (PFEM) since its basic idea is to add parasitic nodes to the original node if its nodal support is completely cut by the cracks into several isolated pieces. The efficiency of the PFEM is demonstrated through two typical examples.

The PFEM shares some similarities to Hansbo and Hansbo (2004), but they are different in essence, which will be clarified in Section 5.4.2.

5.2 Comparison between the NMM and the XFEM/GFEM

Modeling a single crack with the NMM is illustrated in Fig. 5.1a, where each mathematical cover with squared star is completely cut by the crack surface into two pieces and forms two physical covers, while each mathematical cover with circled star is partially cut by the crack surface and forms one singular physical cover. Each physical cover has an independent local function. Singular physical covers are enriched with the asymptotic crack tip functions given in Eq. (4.7). The displacement field can be described by Eq. (4.8) with $m_i = 2$.

Modeling the same single crack with the XFEM is illustrated in Fig. 5.1b, where each squared node whose support is completely cut by the crack surface into two isolated pieces is enriched with the generalized Heaviside function $H(\mathbf{x})$, while each circled node whose support is partially cut by the crack surface is enriched with the asymptotic crack tip functions given in Eq. (4.7). The corresponding displacement approximation is

$$\mathbf{u}^h(\mathbf{x}) = \sum_i \varphi_i(\mathbf{x}) \left(\mathbf{u}_i + \underbrace{H(\mathbf{x})\mathbf{b}_i}_{i \in n_\Gamma} + \underbrace{\sum_{\alpha=1}^4 \Phi_\alpha \mathbf{a}_i^\alpha}_{i \in n_\Lambda} \right) \quad (5.1)$$

where n_Γ is a set of nodes whose support is completely cut by the crack surface (i.e. squared nodes), n_Λ is a set of nodes whose support is partially cut by the crack surface (i.e. circled nodes).

Noticing the relation between the generalized Heaviside function $H(\mathbf{x})$ and the generalized Kronecker Delta functions δ_i^1 and δ_i^2

$$H(\mathbf{x}) = \delta_i^1 - \delta_i^2 \quad \text{if } \mathbf{x} \in \omega_i \quad (5.2)$$

Fig. 5.1 Modeling a single crack with: (a) the NMM; (b) the XFEM; (c) the GFEM

Viewing $\mathbf{u}_i$ as a union of two parts, i.e. $\mathbf{u}_i (i \notin n_T)$ and $\mathbf{u}_i (i \in n_T)$, and noticing the fact that

$$\delta_i^1 + \delta_i^2 \equiv 1 \quad \text{within } \omega_i \quad (5.3)$$

for the single crack problem, Eq. (5.1) becomes

$$\begin{aligned}
\mathbf{u}^h &= \sum_{i \in n_T} \varphi_i(\mathbf{x}) \left(\underbrace{\mathbf{u}_i}_{i \in \tilde{n}_T} + \underbrace{\mathbf{u}_i + \delta_i^1 \mathbf{b}_i - \delta_i^2 \mathbf{b}_i}_{i \in n_T} + \underbrace{\sum_{\alpha=1}^4 \Phi_\alpha \mathbf{a}_i^\alpha}_{i \in n_\Lambda} \right) \\
&= \sum_{i \in n_T} \varphi_i(\mathbf{x}) \left(\underbrace{\mathbf{u}_i}_{i \in \tilde{n}_T} + \underbrace{\delta_i^1 (\mathbf{u}_i + \mathbf{b}_i) + \delta_i^2 (\mathbf{u}_i - \mathbf{b}_i)}_{i \in n_T} + \underbrace{\sum_{\alpha=1}^4 \Phi_\alpha \mathbf{a}_i^\alpha}_{i \in n_\Lambda} \right)
\end{aligned} \tag{5.4}$$

If rearranging the unknowns as

$$\begin{cases} \mathbf{u}_i^1 \triangleq \mathbf{u}_i + \mathbf{b}_i \\ \mathbf{u}_i^2 \triangleq \mathbf{u}_i - \mathbf{b}_i \end{cases} \quad \text{for any } \mathbf{x}_i \ (i \in n_T) \tag{5.5}$$

Then, the XFEM approximation in Eq. (5.4) will turn into the NMM approximation in Eq. (4.8) for a single crack problem. In addition, we can easily find the independence of $\mathbf{u}_i^1$ and $\mathbf{u}_i^2$ from Eq. (5.5).

Modeling the same single crack with the GFEM is illustrated in Fig. 5.1c. All the nodal supports completely cut by the crack form Ω_{crack} , and the crack divides Ω_{crack} into two disjoint regions R_1 and R_2 . Two discontinuous functions $H_\alpha(\mathbf{x})$ ($\alpha=1,2$) are then defined corresponding to R_α ($\alpha=1,2$), where $H_\alpha(\mathbf{x})=1$ when $\mathbf{x}$ within R_α and $H_\alpha(\mathbf{x})=0$ elsewhere. The nodes whose support intersects ∂R_α (i.e. boundary of the region R_α) are enriched with $H_\alpha(\mathbf{x})$. The support of each squared node in Fig. 5.1c intersects both ∂R_1 and ∂R_2 , so the squared nodes should be enriched with both $H_1(\mathbf{x})$ and $H_2(\mathbf{x})$. However, noticing that the two enrichment functions $H_\alpha(\mathbf{x})$ ($\alpha=1,2$) are not linearly independent, but have the following relation

$$H_1(\mathbf{x}) + H_2(\mathbf{x}) = 1 \quad \text{for } \mathbf{x} \in \Omega_{\text{crack}} \tag{5.6}$$

Therefore, the squared nodes only need to be enriched by $H_1(\mathbf{x})$ or $H_2(\mathbf{x})$. Here, without losing generality, we choose $H_1(\mathbf{x})$. The GFEM approximation reads

$$\mathbf{u}^h = \sum_i \varphi_i(\mathbf{x}) \left(\mathbf{u}_i + \underbrace{H_1(\mathbf{x}) \mathbf{b}_{1i}}_{i \in n_T} \right) \quad (5.7)$$

where n_T is the set of the nodes whose support is completely cut by the crack surface (i.e. squared nodes).

Compare Eq. (5.1) and Eq. (5.7), we find that the GFEM is in essence same to the XFEM, but replaces $H(\mathbf{x})$ in the XFEM with H_1 to model the displacement discontinuity across the crack surface.

From their definitions, we have the following relation

$$\begin{cases} H_1(\mathbf{x}) = \delta_i^1 \\ H_2(\mathbf{x}) = \delta_i^2 \end{cases} \quad \text{if } i \in n_T \quad (5.8)$$

Using δ_i^1 and δ_i^2 and considering Eq. (5.8), Eq. (5.7) becomes

$$\mathbf{u}^h = \sum_{i \in n_T} N_i(\mathbf{x}) \left(\underbrace{\mathbf{u}_i}_{i \in \bar{n}_T} + \underbrace{\delta_i^1 (\mathbf{u}_i + \mathbf{b}_{1i}) + \delta_i^2 \mathbf{u}_i}_{i \in n_T} \right) \quad (5.9)$$

If rearranging the unknowns as

$$\begin{cases} \mathbf{u}_i^1 \triangleq \mathbf{u}_i + \mathbf{b}_{1i} & \text{if } \mathbf{x} \in R_1 \\ \mathbf{u}_i^2 \triangleq \mathbf{u}_i & \text{if } \mathbf{x} \in R_2 \end{cases} \quad (5.10)$$

then the GFEM approximation in Eq. (5.9) will identically turn into the NMM approximation in Eq. (4.8) for a single crack problem if enriching the singular physical covers containing the crack tips is not considered.

Modeling a branched crack with the NMM is illustrated in Fig. 5.2a, where each mathematical cover cut by the crack surfaces into two isolated pieces forms two physical covers, each mathematical cover cut by the crack surfaces into three isolated pieces forms three physical covers, and each mathematical cover partially cut by the crack surfaces forms one singular physical cover. Each physical cover has an independent local function. The singular physical covers are enriched by the asymptotic crack tip functions given in Eq. (4.7). The displacement field is still approximated by Eq. (4.8).

(a)

(b)

(c)

Fig. 5.2 Modeling a branched crack with: (a) the NMM; (b) the XFEM; (c) the GFEM

In the XFEM, the generalized Heaviside function $H(\mathbf{x})$ has only two values, i.e. +1 when $\mathbf{x}$ above the crack surface and -1 when $\mathbf{x}$ below the crack surface, thus can only describe the displacement jump across a single crack surface. In order to represent branched and intersecting cracks, additional discontinuous functions need to be introduced. Modeling the branched crack with the XFEM is illustrated in Fig. 5.2b. The branched crack is treated as the intersection of a main crack, crack I and a secondary crack, crack II. Three discontinuous functions $H_I(\mathbf{x})$, $H_{II}(\mathbf{x})$, and $J(\mathbf{x})$ are defined corresponding to crack I, crack II, and the junction of crack I and crack II, as shown in Fig. 5.2b. The enrichment of the nodes is as follows: nodes whose support is completely cut by crack I are enriched with $H_I(\mathbf{x})$, nodes whose support is completely cut by crack II are enriched with $H_{II}(\mathbf{x})$, nodes whose support contains the junction of crack I and crack II are enriched with $J(\mathbf{x})$, and nodes whose support is partially cut by the cracks are enriched with asymptotic crack tip functions in Eq. (4.7).

The displacement approximation is expressed as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i \in n_T} N_i(\mathbf{x}) \left(\underbrace{\mathbf{u}_i}_{i \in n_T} + \underbrace{H_I \mathbf{b}_{Ii}}_{i \in n_{\Gamma I}} + \underbrace{H_{II} \mathbf{b}_{IIi}}_{i \in n_{\Gamma II}} + \underbrace{J \mathbf{c}_i}_{i \in n_J} + \underbrace{\sum_{\alpha=1}^4 \Phi_\alpha \mathbf{a}_i^\alpha}_{i \in n_\Lambda} \right) \quad (5.11)$$

where $\mathbf{b}_{Ii}$ is the nodal enriched degrees of freedom associated with the $H_I(\mathbf{x})$, $\mathbf{b}_{IIi}$ is the nodal enriched degrees of freedom associated with the $H_{II}(\mathbf{x})$, $\mathbf{c}_i$ is the nodal enriched degrees of freedom associated with $J(\mathbf{x})$, $n_{\Gamma I}$ is the set of nodes whose support is completely cut by crack I, $n_{\Gamma II}$ is the set of nodes whose support is completely cut by crack II, n_J is the set of nodes whose support contains the junction. It should be noted that from their definitions we have $n_J \subset n_{\Gamma I}$ but $n_J \not\subset n_{\Gamma II}$, which reflects the difference between the main crack and the secondary crack(s).

On the other hand, following divisions are highlighted. For $i \in n_{\Gamma I}$ but $i \notin n_J$,

ω_i is divided into two disjoint sub-supports. For $i \in n_{\Gamma\text{II}}$, ω_i is divided into two disjoint sub-supports as well. And for $i \in n_J$, ω_i is divided into three disjoint sub-supports. So, from their definitions and Fig. 5.2b, we have

$$\begin{cases} H_I(\mathbf{x}) = \delta_i^1 - \delta_i^2 & \text{if } i \in n_{\Gamma\text{I}} \text{ \& } i \notin n_J \\ H_{\text{II}}(\mathbf{x}) = \delta_i^1 - \delta_i^2 & \text{if } i \in n_{\Gamma\text{II}} \end{cases} \quad (5.12a)$$

and

$$\begin{cases} H_I(\mathbf{x}) = \delta_i^1 - \delta_i^2 - \delta_i^3 \\ J(\mathbf{x}) = \delta_i^2 - \delta_i^3 \end{cases} \quad \text{if } i \in n_J \quad (5.12b)$$

Then, Eq. (5.11) can be rewritten as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i \in n_T} N_i(\mathbf{x}) \left(\underbrace{\mathbf{u}_i}_{i \in n_T} + \underbrace{\delta_i^1 \mathbf{b}_{li} - \delta_i^2 \mathbf{b}_{li}}_{i \in n_{\Gamma\text{I}} \text{ \& } i \notin n_J} + \underbrace{\delta_i^1 \mathbf{b}_{\text{II}i} - \delta_i^2 \mathbf{b}_{\text{II}i}}_{i \in n_{\Gamma\text{II}}} + \underbrace{\delta_i^1 \mathbf{b}_{li} - \delta_i^2 (\mathbf{b}_{li} - \mathbf{c}_i) - \delta_i^3 (\mathbf{b}_{li} + \mathbf{c}_i)}_{i \in n_J} + \underbrace{\sum_{\alpha=1}^4 \Phi_\alpha \mathbf{a}_{(i)}^\alpha}_{i \in N_\Lambda} \right) \quad (5.13)$$

Re-arranging the node sets, Eq. (5.13) is further written as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i \in n_T} N_i(\mathbf{x}) \left(\underbrace{\mathbf{u}_i}_{i \in \bar{n}_T} + \underbrace{\delta_i^1 (\mathbf{u}_i + \mathbf{b}_{li}) + \delta_i^2 (\mathbf{u}_i - \mathbf{b}_{li})}_{i \in n_{\Gamma\text{I}} \text{ \& } i \notin n_J} + \underbrace{\delta_i^1 (\mathbf{u}_i + \mathbf{b}_{\text{II}i}) + \delta_i^2 (\mathbf{u}_i - \mathbf{b}_{\text{II}i})}_{i \in n_{\Gamma\text{II}}} + \underbrace{\delta_i^1 (\mathbf{u}_i + \mathbf{b}_{li}) + \delta_i^2 (\mathbf{u}_i - \mathbf{b}_{li} + \mathbf{c}_i) + \delta_i^3 (\mathbf{u}_i - \mathbf{b}_{li} - \mathbf{c}_i)}_{i \in n_J} + \underbrace{\sum_{\alpha=1}^4 \Phi_\alpha \mathbf{a}_{(i)}^\alpha}_{i \in N_\Lambda} \right) \quad (5.14)$$

where $n_T = n_{\Gamma\text{I}} \cup n_{\Gamma\text{II}}$ and $\bar{n}_T$ is the complement of n_T in terms of n_T .

If rearranging the unknowns as

$$\begin{cases} \mathbf{u}_i^1 \triangleq \mathbf{u}_i + \mathbf{b}_{li} \\ \mathbf{u}_i^2 \triangleq \mathbf{u}_i - \mathbf{b}_{li} \end{cases} \quad \text{if } i \in n_{\Gamma\text{I}} \text{ \& } i \notin n_J \quad (5.15a)$$

$$\begin{cases} \mathbf{u}_i^1 \triangleq \mathbf{u}_i + \mathbf{b}_{\text{II}i} \\ \mathbf{u}_i^2 \triangleq \mathbf{u}_i - \mathbf{b}_{\text{II}i} \end{cases} \quad \text{if } i \in n_{\Gamma\text{II}} \quad (5.15b)$$

$$\begin{cases} \mathbf{u}_i^1 \triangleq \mathbf{u}_i + \mathbf{b}_{li} \\ \mathbf{u}_i^2 \triangleq \mathbf{u}_i - \mathbf{b}_{li} + \mathbf{c}_i \\ \mathbf{u}_i^3 \triangleq \mathbf{u}_i - \mathbf{b}_{li} - \mathbf{c}_i \end{cases} \quad \text{if } i \in n_J \quad (5.15c)$$

Then, the XFEM approximation in Eq. (5.14) will be identical to the NMM approximation in Eq. (4.8) for a branched crack problem.

Modeling the same branched crack with the GFEM is illustrated in Fig. 5.2c. Unlike that in the XFEM, it is no more necessary to artificially realize a branched crack as a main crack and several secondary cracks, and all the enrichment functions are defined through the polycrystalline aggregate topology. A domain Ω_{crack} as the neighborhood of the crack is partitioned into three non-overlapping regions by the branched crack, denoted as R_1 , R_2 , and R_3 . Three discontinuous enrichment functions H_α ($\alpha=1,2,3$) are therefore involved, in which H_α are defined so that $H_\alpha=1$ within R_α and $H_\alpha=0$ otherwise.

The enrichment functions are chosen as follows. For the nodes whose supports intersect ∂R_α , they are enriched with $H_\alpha(\mathbf{x})$. If one more enrichment functions can be chosen, one of them will be excluded due to their linear dependence. In implementation, we regulate that for nodes $\mathbf{x}_i$ ($i \in n_{\Gamma 1,2}$) whose supports are completely cut by both ∂R_1 and ∂R_2 , they are enriched with $H_1(\mathbf{x})$. For nodes $\mathbf{x}_i$ ($i \in n_{\Gamma 2,3}$) whose supports are completely cut by both ∂R_2 and ∂R_3 , they are enriched with $H_2(\mathbf{x})$. For nodes $\mathbf{x}_i$ ($i \in n_{\Gamma 3,1}$) whose supports are completely cut by both ∂R_3 and ∂R_1 , they are enriched with $H_3(\mathbf{x})$. And for nodes $\mathbf{x}_i$ ($i \in n_{\Gamma 1,2,3}$) whose supports are cut by ∂R_1 , ∂R_2 and ∂R_3 , they are simultaneously enriched with both $H_1(\mathbf{x})$ and $H_2(\mathbf{x})$. So, the displacement approximation is written as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i \in n_T} \varphi_i(\mathbf{x}) \left(\underbrace{\mathbf{u}_i}_{i \in n_T} + \underbrace{H_1 \mathbf{b}_{1i}}_{i \in n_{\Gamma 1,2}} + \underbrace{H_2 \mathbf{b}_{2i}}_{i \in n_{\Gamma 2,3}} + \underbrace{H_3 \mathbf{b}_{3i}}_{i \in n_{\Gamma 3,1}} + \underbrace{H_1 \mathbf{b}_{1i} + H_2 \mathbf{b}_{2i}}_{i \in n_{\Gamma 1,2,3}} \right) \quad (5.16)$$

where $\mathbf{b}_{1i}$ is the nodal enriched degrees of freedom associated with H_1 , $\mathbf{b}_{2i}$ associated with H_2 , $\mathbf{b}_{3i}$ associated with H_3 .

Note that, for node $\mathbf{x}_i$, $i \in n_{\Gamma_{1,2}} \cup n_{\Gamma_{2,3}} \cup n_{\Gamma_{3,1}}$, ω_i is divided into two disjoint sub-supports, and for node $\mathbf{x}_i$, $i \in n_{\Gamma_{1,2,3}}$, ω_i is divided into three disjoint sub-supports. Thus, from their definitions and Fig. 5.2c, we have

$$\begin{cases} H_1(\mathbf{x}) = \delta_i^1 \\ H_2(\mathbf{x}) = \delta_i^2 \end{cases} \quad \text{if } i \in n_{\Gamma} \quad (5.17)$$

where $n_{\Gamma} = n_{\Gamma_{1,2}} \cup n_{\Gamma_{2,3}} \cup n_{\Gamma_{3,1}} \cup n_{\Gamma_{1,2,3}}$.

Noticing Eq. (5.3) and the relations between the node sets, Eq. (5.16) is further rewritten as

$$\begin{aligned} \mathbf{u}^h(\mathbf{x}) &= \sum_{i \in n_T} \varphi_i(\mathbf{x}) \left(\underbrace{\mathbf{u}_i}_{i \in n_T} + \underbrace{\delta_i^1 \mathbf{b}_{1i}}_{i \in n_{\Gamma_{1,2}}} + \underbrace{\delta_i^1 \mathbf{b}_{2i}}_{i \in n_{\Gamma_{2,3}}} + \underbrace{\delta_i^1 \mathbf{b}_{3i}}_{i \in n_{\Gamma_{3,1}}} + \underbrace{\delta_i^1 \mathbf{b}_{1i} + \delta_i^2 \mathbf{b}_{2i}}_{i \in n_{\Gamma_{1,2,3}}} \right) \\ &= \sum_{i \in n_T} \varphi_i(\mathbf{x}) \left(\underbrace{\mathbf{u}_i}_{i \in \bar{n}_{\Gamma}} + \underbrace{\delta_i^1 (\mathbf{u}_i + \mathbf{b}_{1i}) + \delta_i^2 \mathbf{u}_i}_{i \in n_{\Gamma_{1,2}}} + \underbrace{\delta_i^1 (\mathbf{u}_i + \mathbf{b}_{2i}) + \delta_i^2 \mathbf{u}_i}_{i \in n_{\Gamma_{2,3}}} + \underbrace{\delta_i^1 (\mathbf{u}_i + \mathbf{b}_{3i}) + \delta_i^2 \mathbf{u}_i}_{i \in n_{\Gamma_{3,1}}} \right. \\ &\quad \left. + \underbrace{\delta_i^1 (\mathbf{u}_i + \mathbf{b}_{1i}) + \delta_i^2 (\mathbf{u}_i + \mathbf{b}_{2i}) + \delta_i^3 \mathbf{u}_i}_{i \in n_{\Gamma_{1,2,3}}} \right) \end{aligned} \quad (5.18)$$

where $\bar{n}_{\Gamma}$ is the complement of n_{Γ} in terms of n_T .

If rearranging the unknowns as

$$\begin{cases} \mathbf{u}_i^1 \triangleq \mathbf{u}_i + \mathbf{b}_{1i} \\ \mathbf{u}_i^2 \triangleq \mathbf{u}_i \end{cases} \quad \text{if } i \in n_{\Gamma_{1,2}} \quad (5.19a)$$

$$\begin{cases} \mathbf{u}_i^1 \triangleq \mathbf{u}_i + \mathbf{b}_{2i} \\ \mathbf{u}_i^2 \triangleq \mathbf{u}_i \end{cases} \quad \text{if } i \in n_{\Gamma_{2,3}} \quad (5.19b)$$

$$\begin{cases} \mathbf{u}_i^1 \triangleq \mathbf{u}_i + \mathbf{b}_{3i} \\ \mathbf{u}_i^2 \triangleq \mathbf{u}_i \end{cases} \quad \text{if } i \in n_{\Gamma_{3,1}} \quad (5.19c)$$

$$\begin{cases} \mathbf{u}_i^1 \triangleq \mathbf{u}_i + \mathbf{b}_{1i} \\ \mathbf{u}_i^2 \triangleq \mathbf{u}_i + \mathbf{b}_{2i} \\ \mathbf{u}_i^3 \triangleq \mathbf{u}_i \end{cases} \quad \text{if } i \in n_{\Gamma_{1,2,3}} \quad (5.19d)$$

Then, the GFEM approximation in Eq. (5.18) eventually turns into the NMM

approximation in Eq. (4.8) for this branched crack problem if enriching the nodes whose support contains the crack tips is not considered.

Both the NMM and the XFEM allow the cracks arbitrarily align within the finite element mesh. In addition, the NMM shape functions span the same space as those of the XFEM. However, the implementation of the NMM is much convenient than that of the XFEM, especially for complex crack problems. The XFEM incorporates discontinuous enrichment functions to account for cracks. Modeling a single crack is easy. However, it gets complicated when complex cracks are involved. For example, a branched crack needs to be treated as the intersection of a main crack and a secondary crack, three enrichment functions, i.e. $H_I(\mathbf{x})$, $H_{II}(\mathbf{x})$, and $J(\mathbf{x})$ are defined corresponding to crack I, crack II, and the junction of crack I and crack II, respectively. The nodes are enriched with the three discontinuous enrichment functions accordingly. Its implementation is complex. Modeling an intersecting crack with the XFEM is depicted in Fig. 5.3. The intersecting crack needs to be treated as the intersection of a main crack, crack I and two secondary cracks, crack II and crack III. Five discontinuous functions, $H_I(\mathbf{x})$, $H_{II}(\mathbf{x})$, $H_{III}(\mathbf{x})$, $J_{I,II}(\mathbf{x})$, $J_{I,III}(\mathbf{x})$ are defined corresponding to crack I, crack II, crack III, junction of crack I and crack II, and junction of crack I and crack III, respectively, as shown in Fig. 5.3. The enrichment of the nodes is as follows: nodes whose support is completely cut by crack I are enriched with $H_I(\mathbf{x})$, nodes whose support is completely cut by crack II are enriched with $H_{II}(\mathbf{x})$, nodes whose support is completely cut by crack III are enriched with $H_{III}(\mathbf{x})$, nodes whose support contains the junction of crack I and crack II are enriched with $J_{I,II}(\mathbf{x})$, nodes whose support contains the junction of crack I and crack III are enriched with $J_{I,III}(\mathbf{x})$, and nodes whose support is partially cut by the cracks are enriched with asymptotic crack tip functions given in Eq. (4.7). Its implementation becomes more complicated. It can be imagined that the situation may get even worse or too complicated to be handled when arbitrarily complex crack are encountered. However, the NMM models the cracks by simply splitting a mathematical cover into several physical covers and assigning each physical cover an

independent local function. By such a way, complex cracks can be modeled in an exactly same way as the modeling of a single crack. Its implementation is simple and straightforward.

Fig. 5.3 Modeling an intersecting crack with the XFEM

Both the NMM and the GFEM allows the finite element mesh be constructed totally independent of the problem domain, thus a regular finite element mesh is always applicable in both the NMM and the GFEM. In addition, the NMM shape functions also span the same space as those of the GFEM if the enrichments to the singular physical covers are not considered. The GFEM adopts different enrichment functions from those used in the XFEM to represent the cracks. Compared with the XFEM, the enrichment functions in the GFEM can be easily constructed. Complex cracks with arbitrary number of branched can be easily handled. However, enriching to the nodes in the GFEM needs to be properly designed since the enrichment functions $H_a(\mathbf{x})$ are not linearly independent. In addition, its enrichment functions are derived from the aggregate topology, thus can only allow the crack tips stop at the sides of elements. In contrast, the NMM incorporates asymptotic crack tip functions given in Eq. (4.7) into the singular physical covers and does not suffer from such limitations.

In the following section, the XFEM and the GFEM will be re-examined from the

perspective of nodal support. The key differences between the NMM and the XFEM/GFEM will be discussed.

5.3 Key differences between the NMM and the XFEM/GFEM in crack modeling

In the conventional FEM, the shape function $\varphi_i(\mathbf{x})$ of node $\mathbf{x}_i$ has a compact support defined by $\omega_i = \{\mathbf{x} : \varphi_i(\mathbf{x}) \neq 0\}$. The support ω_i can be interpreted as the region where node $\mathbf{x}_i$ is effective in interpolating the unknown field. The support ω_i is formed by collecting all the finite elements containing $\mathbf{x}_i$ as a node.

Careful observation on the examples in Figs. 5.1b, 5.1c, 5.2b, 5.2c, 5.3 reveals that the XFEM and the GFEM defines the support ω_i of node $\mathbf{x}_i$ as the union of all the finite elements sharing node $\mathbf{x}_i$. In the following part, we will take an examination on the nodal supports cut by a single crack, a branched crack, and an intersecting crack, respectively, and figure out the aims of the introduced discontinuous enrichment functions.

Assume the support ω_i of node $\mathbf{x}_i$ is completely cut by a single crack into two pieces, denoted as ω_i^1 (above the crack surface) and ω_i^2 (below the crack surface), as shown in Fig. 5.4a. In the XFEM, node $\mathbf{x}_i$ will be enriched with the generalized Heaviside function $H(\mathbf{x})$ ($H(\mathbf{x})=1$ when $\mathbf{x} \in \omega_i^1$, and $H(\mathbf{x})=-1$ when $\mathbf{x} \in \omega_i^2$). Each node $\mathbf{x}_i$ initially has a unique unknown $\mathbf{u}_i$. After enrichment, node $\mathbf{x}_i$ will have an enriched unknown $\mathbf{u}_i + H(\mathbf{x})\mathbf{b}_i$, which takes two independent values $\mathbf{u}_i + \mathbf{b}_i$ and $\mathbf{u}_i - \mathbf{b}_i$, effective in ω_i^1 and ω_i^2 , respectively. In the GFEM, $\omega_i^1 \subset R_1$, $\omega_i^2 \subset R_2$. Since the ω_i intersects both ∂R_1 and ∂R_2 , node $\mathbf{x}_i$ will be enriched with $H_1(\mathbf{x})$ or $H_2(\mathbf{x})$. Here, without losing generality, we choose $H_1(\mathbf{x})$, where $H_1(\mathbf{x})=1$ when $\mathbf{x} \in \omega_i^1$, and $H_1(\mathbf{x})=0$ when $\mathbf{x} \in \omega_i^2$. After enrichment, node $\mathbf{x}_i$ will have an enriched unknown $\mathbf{u}_i + H_1(\mathbf{x})\mathbf{b}_{1i}$, which takes two independent values $\mathbf{u}_i + \mathbf{b}_{1i}$ and

$\mathbf{u}_i$, effective in ω_i^1 and ω_i^2 , respectively.

Assume the support ω_j of node $\mathbf{x}_j$ is completely cut by a branched crack into three pieces, denoted as ω_j^1 , ω_j^2 , and ω_j^3 , as shown in Fig. 5.4b. In the XFEM, node $\mathbf{x}_j$ will be enriched with the generalized Heaviside function $H_1(\mathbf{x})$ and a junction function $J(\mathbf{x})$. Each node $\mathbf{x}_i$ initially has a unique unknown $\mathbf{u}_i$. After enrichment, node $\mathbf{x}_j$ will have an enriched unknown $\mathbf{u}_j + H_1(\mathbf{x})\mathbf{b}_j + J(\mathbf{x})\mathbf{c}_j$, which takes three independent values $\mathbf{u}_j + \mathbf{b}_j$ within ω_j^1 , $\mathbf{u}_j - \mathbf{b}_j + \mathbf{c}_j$ within ω_j^2 , $\mathbf{u}_j - \mathbf{b}_j - \mathbf{c}_j$ within ω_j^3 . In the GFEM, $\omega_j^1 \subset R_1$, $\omega_j^2 \subset R_2$, $\omega_j^3 \subset R_3$. Since the ω_j intersects ∂R_1 , ∂R_2 , and ∂R_3 , node $\mathbf{x}_j$ will be enriched with $H_1(\mathbf{x})$ and $H_2(\mathbf{x})$ or $H_2(\mathbf{x})$ and $H_3(\mathbf{x})$, or $H_3(\mathbf{x})$ and $H_1(\mathbf{x})$. Without losing generality, we choose $H_1(\mathbf{x})$ and $H_2(\mathbf{x})$ here. After enrichment, the node $\mathbf{x}_j$ will have an enriched unknown $\mathbf{u}_j + H_1(\mathbf{x})\mathbf{b}_{1j} + H_2(\mathbf{x})\mathbf{b}_{2j}$, which takes three different values, $\mathbf{u}_j + \mathbf{b}_{1j}$ within ω_j^1 , $\mathbf{u}_j + \mathbf{b}_{2j}$ within ω_j^2 , and $\mathbf{u}_j$ within ω_j^3 .

Assume the support ω_k of node $\mathbf{x}_k$ is completely cut by an intersecting crack into four pieces, denoted as ω_k^1 , ω_k^2 , ω_k^3 and ω_k^4 , as shown in Fig. 5.4c. In the XFEM, node $\mathbf{x}_k$ will be enriched with the generalized Heaviside function $H_1(\mathbf{x})$ and two junction functions $J_{\text{I,II}}(\mathbf{x})$, $J_{\text{I,III}}(\mathbf{x})$, also defined in Fig. 5.4c. Each node $\mathbf{x}_i$ initially has a unique unknown $\mathbf{u}_i$. After enrichment, node $\mathbf{x}_k$ will have an enriched unknown $\mathbf{u}_k + H_1(\mathbf{x})\mathbf{b}_k + J_{\text{I,II}}(\mathbf{x})\mathbf{c}_k + J_{\text{I,III}}(\mathbf{x})\mathbf{d}_k$, which takes four different values corresponding to four isolated pieces, i.e. $\mathbf{u}_k + \mathbf{b}_k + \mathbf{c}_k$ within ω_k^1 , $\mathbf{u}_k + \mathbf{b}_k - \mathbf{c}_k$ within ω_k^2 , $\mathbf{u}_k - \mathbf{b}_k + \mathbf{d}_k$ within ω_k^3 , $\mathbf{u}_k - \mathbf{b}_k - \mathbf{d}_k$ within ω_k^4 . In the GFEM, $\omega_k^1 \subset R_1$, $\omega_k^2 \subset R_2$, $\omega_k^3 \subset R_3$, $\omega_k^4 \subset R_4$. Since its support intersects ∂R_1 , ∂R_2 , ∂R_3 and ∂R_4 , node $\mathbf{x}_k$ will be enriched with $H_1(\mathbf{x})$, $H_2(\mathbf{x})$ and $H_3(\mathbf{x})$ or $H_2(\mathbf{x})$, $H_3(\mathbf{x})$ and $H_4(\mathbf{x})$, or $H_3(\mathbf{x})$, $H_4(\mathbf{x})$ and $H_1(\mathbf{x})$. Without losing generality, we choose $H_1(\mathbf{x})$, $H_2(\mathbf{x})$ and $H_3(\mathbf{x})$ here. After enrichment, the node $\mathbf{x}_k$ will have

an enriched unknown $\mathbf{u}_k + H_1(\mathbf{x})\mathbf{b}_{1k} + H_2(\mathbf{x})\mathbf{b}_{2k} + H_3(\mathbf{x})\mathbf{b}_{3k}$, which takes four different values corresponding to four isolated pieces, i.e. $\mathbf{u}_k + \mathbf{b}_{1k}$ within ω_k^1 , $\mathbf{u}_k + \mathbf{b}_{2k}$ within ω_k^2 , $\mathbf{u}_k + \mathbf{b}_{3k}$ within ω_k^3 , and $\mathbf{u}_k$ within ω_k^4 .

Fig. 5.4 Illustration of nodal supports cut by: (a) a single crack; (b) a branched crack; (c) an intersecting crack

The above detailed examination shows that the XFEM defines the support ω_i of node $\mathbf{x}_i$ as the union of all the finite elements sharing node $\mathbf{x}_i$. Each node $\mathbf{x}_i$ initially has a unique unknown $\mathbf{u}_i$. If the support ω_i of node $\mathbf{x}_i$ is completely cut by the crack surfaces into several pieces, then node $\mathbf{x}_i$ will be enriched with $H(\mathbf{x})$

and/or $J(\mathbf{x})$. The enriched node has several independent unknowns, each of which is effective in one piece of its nodal support.

The GFEM also defines the support ω_i of node $\mathbf{x}_i$ as the union of all the finite elements sharing node $\mathbf{x}_i$. Each node $\mathbf{x}_i$ initially has a unique unknown $\mathbf{u}_i$. If the support ω_i of node $\mathbf{x}_i$ is completely cut by the crack surfaces into several pieces, then node $\mathbf{x}_i$ will be enriched with $H_i(\mathbf{x})$. The enriched node has several independent unknowns, each of which is effective in one piece of its nodal support.

In contrast to the XFEM and the GFEM, the NMM cuts each mathematical cover which is intersected by crack surfaces into several pieces into several physical cover and directly assign each physical cover an independent local function.

Both the XFEM and the GFEM introduce several independent unknowns to the node whose support is completely cut by the crack surfaces through using discontinuous enrichment functions. Defining the enrichment functions and enriching the nodes are not convenient. The NMM directly introduces independent local functions to each physical cover, which makes the arbitrarily complex cracks be modeled in a straightforward manner. Inspired by the advantageous modeling strategy of the NMM, a modification to the XFEM/GFEM is proposed. The basic idea is if the support of a node is completely cut by the crack surfaces into m parts, m independent unknowns will be directly assigned to this node, such that each unknown is effective in one isolated part. The new technique is termed as the parasitic finite element method (PFEM). Details of the PFEM will be presented in the next section.

5.4 The parasitic finite element method (PFEM)

5.4.1 The parasitic finite element method (PFEM)

Inspired by the advantageous modeling strategy of the NMM, a modification to the XFEM/GFEM is proposed. The basic idea is if the support ω_i of node $\mathbf{x}_i$ is completely cut by crack surfaces into m_i isolated sub-supports, i.e. $\omega_i = \bigcup_{j=1 \sim m_i} \omega_i^j$,

$m_i - 1$ additional nodes are added to the same position of node $\mathbf{x}_i$, such that each of

the m_i nodes is effective in one isolated sub-support. Each node has an independent unknown. The additional nodes are parasitic to the original node, thus termed as parasitic nodes, while the new technique is termed as the parasitic finite element method (PFEM).

For convenience of interpretation, the original node $\mathbf{x}_i$ is re-denoted by $\mathbf{x}_i^1$ and the other $m_i - 1$ parasitic nodes are denoted by $\mathbf{x}_i^j$ ($j = 2 \sim m_i$), such that $\mathbf{x}_i^j$ ($j = 1 \sim m_i$) is effective in sub-support ω_i^j ($j = 1 \sim m_i$).

Using multiple nodes at the same position is not invented by the author, but has long been adopted in the conventional FEM.

Shown in Fig. 5.5 is a conventional FEM model for a continuous problem with nine nodes and four elements.

Fig. 5.5 A conventional FEM model for a continuous problem

Then, we turn to consider a discontinuous problem due to the advent of an edge crack. Within the conventional FEM framework, the finite element mesh must be consistent with the crack. A discontinuous model satisfying this requirement is established in Fig. 5.6a.

Compared with the continuous model in Fig. 5.5, one more node $\mathbf{x}_{10}$ is added to the same position as node $\mathbf{x}_4$. Node $\mathbf{x}_4$ is effective in element I, while node $\mathbf{x}_{10}$ is effective in element III. By using double nodes at the same position, the displacement discontinuity across the edge crack is eventually modeled in the conventional FEM.

Taking an examination on the nodal supports, we see that the introduced edge

crack completely divides the original ω_4 into two isolated parts, as shown in Fig. 5.6b. Based on the PFEM, one parasitic node $\mathbf{x}_4^2$ is added to node $\mathbf{x}_4$, and meanwhile node $\mathbf{x}_4$ is re-denoted by $\mathbf{x}_4^1$. It is not difficult to figure it out if we denote $\mathbf{x}_4 \triangleq \mathbf{x}_4^1$ and $\mathbf{x}_{10} \triangleq \mathbf{x}_4^2$, the conventional FEM model in Fig. 5.5a turns into a PFEM model.

Fig. 5.6 A conventional FEM model for a discontinuous problem with an edge crack:
(a) nodes and elements; (b) supports intersected by cracks

If a branched crack is introduced into the continuous model, within the framework of the conventional FEM, a discontinuous model is constructed in Fig. 5.7a.

Compared with the continuous model in Fig. 5.5, one additional node $\mathbf{x}_{10}$ is added to the same position as node $\mathbf{x}_2$ such that node $\mathbf{x}_2$ is effective in element I and node $\mathbf{x}_{10}$ is effective in element II. One additional node $\mathbf{x}_{11}$ is added to the same position of node $\mathbf{x}_4$, such that node $\mathbf{x}_4$ is effective in element I and node $\mathbf{x}_{11}$

is effective in element III. Two additional nodes $\mathbf{x}_{12}$ and $\mathbf{x}_{13}$ are added to the same position of node $\mathbf{x}_5$, such that node $\mathbf{x}_5$ is effective in element I, node $\mathbf{x}_{12}$ is effective in elements II and IV, node $\mathbf{x}_{13}$ is effective in element III. One additional node $\mathbf{x}_{14}$ is added to the same position of node $\mathbf{x}_8$, such that node $\mathbf{x}_8$ is effective in element III, node $\mathbf{x}_{14}$ is effective in element IV. Using double nodes at the positions of original nodes 2, 4, 8, and triple nodes at the position of the original node 5, the discontinuity jump across the branched crack is modeled in the conventional FEM.

Fig. 5.7 A conventional FEM model for problem with a branched crack: (a) nodes and elements; (b) crack intersected supports

Taking an inspection on the nodal supports, we see that the branched crack cuts each of the original supports ω_i ($i = 2, 4, 8$) into two isolated parts and the original support ω_5 into three isolated parts, shown in Fig. 5.7b.

Based on the PFEM, parasitic node $\mathbf{x}_2^2$ is added to the node $\mathbf{x}_2$ (re-denoted by

$\mathbf{x}_2^1$); parasitic node $\mathbf{x}_4^2$ to node $\mathbf{x}_4$ (re-denoted by $\mathbf{x}_4^1$); parasitic node $\mathbf{x}_8^2$ to node $\mathbf{x}_8$ (re-denoted by $\mathbf{x}_8^1$); $\mathbf{x}_5^2$ and $\mathbf{x}_5^3$ to node $\mathbf{x}_5$ (re-denoted by $\mathbf{x}_5^1$). By introducing such parasitic nodes, the displacement jump across the branched crack is modeled. If denoting $\mathbf{x}_2 \triangleq \mathbf{x}_2^1$, $\mathbf{x}_{10} \triangleq \mathbf{x}_2^2$, $\mathbf{x}_4 \triangleq \mathbf{x}_4^1$, $\mathbf{x}_{11} \triangleq \mathbf{x}_4^2$, $\mathbf{x}_5 \triangleq \mathbf{x}_5^1$, $\mathbf{x}_{12} \triangleq \mathbf{x}_5^2$, $\mathbf{x}_{13} \triangleq \mathbf{x}_5^3$, $\mathbf{x}_8 \triangleq \mathbf{x}_8^1$, and $\mathbf{x}_{14} \triangleq \mathbf{x}_8^2$, the conventional FEM model in Fig. 5.7a turns into a PFEM model.

Further, if an intersecting crack is introduced into the continuous model. Within the framework of the conventional FEM, a discontinuous model is constructed in Fig. 5.8a.

Fig. 5.8 A conventional FEM model for problem with an intersecting crack: (a) nodes and elements; (b) crack intersected supports

Compared with the continuous model in Fig. 5.4, one additional node $\mathbf{x}_{10}$ is

added to the same position as node $\mathbf{x}_2$. One additional node $\mathbf{x}_{11}$ is added to node $\mathbf{x}_4$. Three additional nodes $\mathbf{x}_{12}$, $\mathbf{x}_{13}$, and $\mathbf{x}_{14}$ are added to node $\mathbf{x}_5$. One additional node $\mathbf{x}_{15}$ is added to node $\mathbf{x}_6$. One additional node $\mathbf{x}_{16}$ is added to node $\mathbf{x}_8$. Using double nodes at the positions of original nodes 2, 4, 6, 8, and four nodes at the position of original node 5, the displacement jump across the intersecting crack is modeled in the conventional FEM.

Taking an inspection on the nodal supports, we see that the branched crack cuts each of the original supports ω_i ($i = 2, 4, 6, 8$) into two isolated parts and the original support ω_5 into four isolated parts, shown in Fig. 5.8b.

Based on the PFEM, parasitic node $\mathbf{x}_2^2$ is added to the node $\mathbf{x}_2$ (re-denoted by $\mathbf{x}_2^1$); parasitic node $\mathbf{x}_4^2$ to node $\mathbf{x}_4$ (re-denoted by $\mathbf{x}_4^1$); parasitic node $\mathbf{x}_6^2$ to node $\mathbf{x}_6$ (re-denoted by $\mathbf{x}_6^1$); parasitic node $\mathbf{x}_8^2$ to node $\mathbf{x}_8$ (re-denoted by $\mathbf{x}_8^1$); parasitic nodes $\mathbf{x}_5^2$, $\mathbf{x}_5^3$ and $\mathbf{x}_5^4$ to node $\mathbf{x}_5$ (re-denoted by $\mathbf{x}_5^1$). If denoting $\mathbf{x}_2 \triangleq \mathbf{x}_2^1$, $\mathbf{x}_{10} \triangleq \mathbf{x}_2^2$, $\mathbf{x}_4 \triangleq \mathbf{x}_4^1$, $\mathbf{x}_{11} \triangleq \mathbf{x}_4^2$, $\mathbf{x}_5 \triangleq \mathbf{x}_5^1$, $\mathbf{x}_{12} \triangleq \mathbf{x}_5^2$, $\mathbf{x}_{13} \triangleq \mathbf{x}_5^3$, $\mathbf{x}_{14} \triangleq \mathbf{x}_5^4$, $\mathbf{x}_6 \triangleq \mathbf{x}_6^1$, $\mathbf{x}_{15} \triangleq \mathbf{x}_6^2$, $\mathbf{x}_8 \triangleq \mathbf{x}_8^1$, and $\mathbf{x}_{16} \triangleq \mathbf{x}_8^2$, the conventional FEM model turns into a PFEM model.

Through the above three discontinuous examples, it is clearly seen that the conventional FEM uses multiple nodes at the same position to describe the displacement jump across crack surfaces. The PFEM is identical to the conventional FEM for the cases where the finite element mesh coincides with the cracks. However, the PFEM is not limited to such cases.

Next, we examine two more cases in which the cracks arbitrarily align within the finite element mesh. To this end, we introduce a generalized Kronecker Delta function

$$\delta_i^j$$

$$\delta_i^j(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \omega_i^j \\ 0 & \text{if } \mathbf{x} \notin \omega_i^j \end{cases} \quad (5.20)$$

With this function, node $\mathbf{x}_i^j$, which is effective in ω_i^j , possesses the shape

function $N_i(\mathbf{x})\delta_i^j$ and the unknown $\mathbf{u}_i^j$.

See a general case shown in Fig. 5.9a, where the edge crack does not align with the finite element mesh.

Due to the fact that the ω_1 is completely split into two sub-supports ω_1^1 and ω_1^2 by the crack surface, one parasitic node $\mathbf{x}_1^2$ is added to the same position of node $\mathbf{x}_1$ (re-denoted as $\mathbf{x}_1^1$). Similarly, the ω_4 is completely split into two sub-supports ω_4^1 and ω_4^2 , thus one parasitic node $\mathbf{x}_4^2$ is added to the same position of node $\mathbf{x}_4$ (re-denoted as $\mathbf{x}_4^1$).

(a)

(b)

Fig. 5.9 A PFEM model for a problem with an edge crack not aligned with the finite element mesh: (a) nodes and elements; (b) supports intersected by the crack

The displacement jump across the crack surface in element I is obtained as

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = \sum_{i \in \{1,4\}} N_i(\mathbf{x}) (\delta_i^1 \mathbf{u}_i^1 - \delta_i^2 \mathbf{u}_i^2) \quad (5.21)$$

where $\llbracket \cdot \rrbracket$ represents the jump of a function, and $\mathbf{u}_i^j$ ($i=1,4; j=1,2$) are the unknowns attached to node $\mathbf{x}_i^j$ ($i=1,4; j=1,2$). Considering the property of δ_i^j (i.e. $\delta_i^1=1$ and $\delta_i^2=0$ for the upper crack surface, $\delta_i^1=0$ and $\delta_i^2=1$ for the lower crack surface), Eq. (5.21) becomes

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = \sum_{i \in \{1,4\}} N_i (\mathbf{u}_i^1 - \mathbf{u}_i^2) \quad (5.22)$$

Due to the fact that $\mathbf{u}_i^1$ and $\mathbf{u}_i^2$ are independent unknowns associated with different nodes $\mathbf{x}_i^1$ and $\mathbf{x}_i^2$, the displacement discontinuity across the crack surface in element I is successfully manifested.

See a more general case shown in Fig. 5.10, where the finite element mesh does not align with the finite element mesh and the crack tip does not stop at the sides of finite elements.

Fig. 5.10 A PFEM model for a problem with an edge crack whose crack tip stopping

inside a finite element: (a) nodes and elements; (b) supports intersected by the crack

Due to the fact that ω_1 is completely split into two sub-supports, one parasitic node $\mathbf{x}_1^2$ is added to the same position as node $\mathbf{x}_1$ (re-denoted as $\mathbf{x}_1^1$). The ω_4 is completely split into two sub-supports ω_4^1 and ω_4^2 , thus one parasitic node $\mathbf{x}_4^2$ is added to the same position of node $\mathbf{x}_4$ (re-denoted as $\mathbf{x}_4^1$).

The edge crack introduces displacement jump in finite elements I and II. The situation for element I is exactly same as in Fig. 5.9a, thus the displacement jump across the crack surface in this element can be similarly described. Then, we turn to take an examination on the displacement field in element II. There are four nodes, i.e. $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_5, \mathbf{x}_6$ associated with element II. No parasitic nodes are added to these four nodes since their supports ω_i ($i = 2, 3, 5, 6$) are only partially cut by the crack surface. The displacement field in element II must be continuous. The displacement jump across the crack at the left side of point p is well described by adding parasitic nodes while the displacement jump across the crack beyond point p is still pending.

To consistently model the displacement jump across the entire crack surface and to accurately describe the crack tip singularity as well, a node whose support contains a crack tip is enriched with the asymptotic crack tip functions Φ_α ($\alpha = 1 \sim 4$) in Eq. (4.7).

So far, the displacement approximation in a general sense for the PFEM is summarized as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i \in n_T} N_i(\mathbf{x}) \left(\underbrace{\mathbf{u}_i}_{i \in \bar{n}_T} + \underbrace{\sum_{j=1}^{m_i} \delta_i^j \mathbf{u}_i^j}_{i \in n_T} + \underbrace{\sum_{\alpha=1}^4 \Phi_\alpha \mathbf{a}_i^\alpha}_{i \in n_\Lambda} \right) \quad (5.23)$$

where n_T is the set of all the nodes in the finite element mesh, n_T and n_Λ are the two disjoint subsets of n_T , whose supports are completely cut by crack surfaces for the former and contain the crack tips inside for the latter, respectively, $\bar{n}_T$ denotes the complementary set of n_T in terms of n_T , $\mathbf{a}_i^\alpha$ are the enriched unknowns

associated with the enrichment bases for node $\mathbf{x}_i$ belonging to n_Λ , and Φ_α denotes the matrix of enrichment bases.

With Eq. (5.23), a multiply branched crack can be modeled by the PFEM in the manner shown in Fig. 5.11. Squares denote the positions where parasitic nodes are added, beside which the numbers indicates how many parasitic nodes are added in this position. Circles denote the nodes containing crack tips and to be enriched with Eq. (4.7). In addition, from Eq. (5.23) and the definition of ω_i^j , $\mathbf{u}_i^j$ is just the displacement of node $\mathbf{x}_i^j$, and therefore has a clear physical meaning.

Fig. 5.11 A PFEM model for a problem with a multiple branched crack

5.4.2 PFEM vs. Hansbo and Hansbo (2004)

In this section, the PFEM will be dissected from Hansbo and Hansbo (2004) in modeling cracks.

Hansbo and Hansbo (2004) construct their approximation at element level. Suppose a triangular element K is intersected by a crack into two parts, K_1 and K_2 , illustrated in Fig. 5.12.

Hansbo and Hansbo (2004) define the displacement field $\mathbf{u}(\mathbf{x})$ in element K as

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \mathbf{u}_1(\mathbf{x}) & \text{in } K_1 \\ \mathbf{u}_2(\mathbf{x}) & \text{in } K_2 \end{cases} \quad (5.24)$$

$\mathbf{u}_1(\mathbf{x})$ and $\mathbf{u}_2(\mathbf{x})$ are two independent functions. One needs three degrees of freedom (DOFs) to determine the linear function $\mathbf{u}_1(\mathbf{x})$ and another three DOFs to determine $\mathbf{u}_2(\mathbf{x})$. It is convenient to choose the nodal unknowns of element K to represent $\mathbf{u}_1(\mathbf{x})$ and $\mathbf{u}_2(\mathbf{x})$. Denote the three nodes of element K as $\mathbf{x}_{(i)}, i=1 \sim 3$, where i is the local index, shown in Fig. 5.12, () transfers the local index i into a global index. Denote the nodal unknowns associated with $\mathbf{u}_1(\mathbf{x})$ as $\mathbf{u}_{(i)}^1, i=1 \sim 3$, and the nodal unknowns associated with $\mathbf{u}_2(\mathbf{x})$ as $\mathbf{u}_{(i)}^2, i=1 \sim 3$.

Fig. 5.12 Illustration of a triangular element intersected by a crack

Denote the standard shape functions for nodes $\mathbf{x}_{(i)}, i=1 \sim 3$ as $N_{(i)}(\mathbf{x}), i=1 \sim 3$, then the shape functions associated with $\mathbf{u}_{(i)}^1$ and $\mathbf{u}_{(i)}^2$ are defined as

$$N_i^1(\mathbf{x}) = \begin{cases} N_i(\mathbf{x}) & \text{in } K_1 \\ 0 & \text{in } K_2 \end{cases} \text{ for } \mathbf{u}_i^1, i=1 \sim 3 \quad (5.25a)$$

and

$$N_i^2(\mathbf{x}) = \begin{cases} 0 & \text{in } K_1 \\ N_i(\mathbf{x}) & \text{in } K_2 \end{cases}, \text{ for } \mathbf{u}_i^2, i=1 \sim 3 \quad (5.25b)$$

Then, the discontinuous displacement field $\mathbf{u}(\mathbf{x})$ in element K is obtained as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i=1}^3 \left(\underbrace{N_i^1(\mathbf{x}) \mathbf{u}_i^1}_{\mathbf{x} \in K_1} + \underbrace{N_i^2(\mathbf{x}) \mathbf{u}_i^2}_{\mathbf{x} \in K_2} \right) \quad (5.26)$$

Each cracked element has six DOFs.

The method proposed by Hansbo and Hansbo (2004) is convenient in modeling arbitrary strong and weak discontinuities. However, it also suffers from some limitations. For example, the method by Hansbo and Hansbo (2004) cannot handle partially cracked elements. See the example in Fig. 5.13. The displacement discontinuities across the crack surface in elements 1 and 8 cannot be described, thus part of the crack (represented by dashed line) is trimmed, and the crack tips are restricted to the sides of elements 2 and 7.

(a)

(b)

Fig. 5.13 Modeling an internal crack with the method proposed by Hansbo and Hansbo (2004): (a) the model; (b) zoom in element 7

Another problem will arise when Eq. (5.26) is applied to elements 2 and 7. Take the modeling of element 7 for an example. The crack tip lies in the line A-B. The displacement field along the line A-B should be continuous. However, Eq. (5.26) results in two completely independent displacement fields in K_1 and K_2 , thus the displacement field is discontinuous across crack tip D along the line A-B. In order to satisfy the displacement compatibility condition at crack tip D, additional techniques

must be adopted.

In contrast, the PFEM does not suffer from the above two limitations. Still take the single crack problem in Fig. 5.13 for an example. We first investigate the displacement field in element 3. Denote its three nodes as $\mathbf{x}_{(i)}, i=1 \sim 3$. The nodal support of $\mathbf{x}_{(1)}$ is completely cut by the crack into two parts, thus one parasitic node $\mathbf{x}_{(1)}^2$ is added to $\mathbf{x}_{(1)}$, which is re-denoted as $\mathbf{x}_{(1)}^1$. Similarly, one parasitic node $\mathbf{x}_{(2)}^2$ is added to $\mathbf{x}_{(2)}$, which is re-denoted as $\mathbf{x}_{(2)}^1$. One parasitic node $\mathbf{x}_{(3)}^2$ is added to $\mathbf{x}_{(3)}$, which is re-denoted as $\mathbf{x}_{(3)}^1$. The displacement field in element 3 is obtained as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i=1}^3 \left(\underbrace{N_{(i)}(\mathbf{x})\delta_{(i)}^1 \mathbf{u}_{(i)}^1}_{\mathbf{x} \in K_1} + \underbrace{N_{(i)}(\mathbf{x})\delta_{(i)}^2 \mathbf{u}_{(i)}^2}_{\mathbf{x} \in K_2} \right) \quad (5.27)$$

Considering the definitions of $\delta_{(i)}^1$ and $\delta_{(i)}^2$, the PFEM approximation in Eq. (6.33) is exactly same with Hansbo and Hansbo (2004)'s approximation in Eq. (5.26) in modeling element 3. Both the PFEM and Hansbo and Hansbo (2004) adopt six DOFs to describe the discontinuous displacement field in element 3. Similar conclusions apply to elements 4 and 6.

However, the differences of these two methods will manifest in modeling elements 2 and 7. Take element 7 for an example. For comparison purpose, here we also consider the situation in which crack tip stop at the side of element 7. One parasitic node $\mathbf{x}_{(3)}^2$ is added to $\mathbf{x}_{(3)}$, re-denoted as $\mathbf{x}_{(3)}^1$ and no parasitic nodes are added to $\mathbf{x}_{(1)}$ and $\mathbf{x}_{(2)}$. Then, the displacement field in element 7 is

$$\mathbf{u}^h(\mathbf{x}) = \underbrace{N_{(1)}(\mathbf{x})\mathbf{u}_{(1)} + N_{(2)}(\mathbf{x})\mathbf{u}_{(2)}}_{\mathbf{x} \in K} + \underbrace{N_{(3)}(\mathbf{x})\delta_{(3)}^1 \mathbf{u}_{(3)}^1}_{\mathbf{x} \in K_1} + \underbrace{N_{(3)}(\mathbf{x})\delta_{(3)}^2 \mathbf{u}_{(3)}^2}_{\mathbf{x} \in K_2} \quad (5.28)$$

Since $\mathbf{u}_{(3)}^1$ and $\mathbf{u}_{(3)}^2$ are two independent unknowns, the displacement jump across the crack surface in element 7 is obviously obtained. In addition, since $N_{(3)}(\mathbf{x})$ vanishes along the line A-B, the displacement field along line A-B is naturally continuous. From Eq. (5.28), we know that the PFEM uses 4 DOFs to describe the

discontinuous displacement field in element 7, in contrast to the 6 DOFs used by Hansbo and Hansbo (2004). Same conclusion applies to element 2.

The PFEM allows the crack tips stop within the elements. Denote the three nodes of a partially cracked element as $\mathbf{x}_{(i)}, i=1 \sim 3$. No parasitic nodes are added to $\mathbf{x}_{(i)}, i=1 \sim 3$ since their nodal supports are only partially intersected. As discussed in Section 5.4.1, in order to describe the displacement discontinuity in the partially cracked element, $\mathbf{x}_{(i)}, i=1 \sim 3$ are enriched with the crack tip functions in Eq. (4.7). The displacement field in partially cracked element is

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i=1}^3 N_i(\mathbf{x}) \left(\mathbf{u}_i + \sum_{\alpha=1}^4 \Phi_{\alpha} \mathbf{a}_i^{\alpha} \right) \quad (5.29)$$

Then, the displacement discontinuity in partially cracked elements is modeled. From Eq. (5.29), we know that the PFEM uses 15 DOFs to describe a partially cracked element.

To sum up, Hansbo and Hansbo (2004) adopts six DOFs for all the cracked triangular elements. Partially cracked elements cannot be handled. The displacement compatibility condition at crack tips needs to be enforced by additional techniques. The PFEM constructs its approximation through the analysis of nodal support of each node. The DOFs used in different kinds of cracked elements may be different. The PFEM is equivalent to Hansbo and Hansbo (2004) in some fully cracked elements, but differs from Hansbo and Hansbo (2004) for elements with crack tips inside or at its sides. In addition, the PFEM does not suffer from the limitations encountered by Hansbo and Hansbo (2004).

5.4.3 Illustration examples

In this section, two numerical examples are modeled with the PFEM. A symmetrically branched crack problem is first studied. The numerical results for stress intensity factors (SIFs) are compared with the reference solutions. Then, a more complex case with a multiple branched crack is investigated. The convergence of the stress intensity factors at the crack tips with the refinement of the mesh is studied. In

these two examples, the domain form of interaction integral is adopted to evaluate the SIFs at crack tips.

A symmetrically branched crack in a finite plate

As the first example, we investigate a finite plate with a symmetrically branched crack under uniform tension in plane strain state as shown in Fig. 5.14.

Fig. 5.14 A finite plate with a symmetrically branched crack under uniaxial tension

The plate dimensions are $W = 5$ and $H = 4$, and material constants are Young's modulus $E = 1000$ and the Poisson's ratio $\nu = 0.3$. The uniform tension is assumed to be $\sigma = 1.0$. The normalized stress intensity factors for tips A and B are defined as

$$F_I^A = K_I^A / \sigma \sqrt{\pi c}, \quad F_I^B = K_I^B / \sigma \sqrt{\pi c}, \quad F_{II}^B = K_{II}^B / \sigma \sqrt{\pi c} \quad (5.30)$$

with $c = (a + b \cos \theta) / 2$ (see Fig. 5.14).

During our simulation, the crack parameters are chosen as $a = b = 1.0$, $\theta = \pi / 4$. SIFs for tip A and B are calculated with three different meshes. To save pages, only one mesh is shown in Fig. 5.15.

Fig. 5.15 A representative mesh for the branched crack problem (832 original nodes, 18 parasitic nodes and 775 elements)

Parameters for the three meshes are listed in Table 5.1. The normalized results for the SIFs obtained from the PFEM are summarized in Table 5.2. The comparison with the XFEM result (Daux et al., 2000) shows that the results from the PFEM are satisfactory.

Table 5.1 The parameters of the three meshes for the problem in Fig. 5.14

Mesher	Original node number	Enriched node number	Added parasitic node number	Total parasitic node number*
Mesh-1	832	12	18	850
Mesh-2	2184	12	38	2222
Mesh-3	8736	12	72	8808

*including both the original nodes and the added parasitic nodes.

Table 5.2 Normalized SIFs for the symmetrically branched crack problem in Fig. 5.14

SIFs	PFEM			XFEM (Daux et al., 2000)
	Mesh-1(832)	Mesh-2(2184)	Mesh-3(8436)	
F_{IA}	1.074	1.084	1.087	1.091
F_{IB}	0.499	0.506	0.509	0.511
F_{IIB}	0.528	0.534	0.537	0.542

A multiple branched crack in a finite plate under bi-axial tension

This example intends to show the capability of the PFEM in solving complex crack problems. As shown in Fig. 5.16, a square plate with a multi-branched crack under biaxial tension is considered.

Fig. 5.16 A square plate with multiple a branched crack subjected to biaxial tension

In our simulation, the dimensions are $W = H = 5.0$, $a = 1.5$, $b = 1.0$, $c = 3.5$, $d = 1.25$, $e = 2.0$, $\alpha = 45^\circ$, and $\beta = 30^\circ$. The external load is biaxial uniform tension with $\sigma = 1.0$. The material constants are same as those in the previous example.

To the author's knowledge, the reference solution to this problem is not available. So, the current work focuses on the convergence test for such a problem. For simplicity, only two of the crack tips labeled by A and B in Fig. 5.16 are investigated.

The normalized SIFs for these two crack tips are expressed as

$$F_I^M = K_I^M / \sigma\sqrt{\pi a}, \quad F_{II}^M = K_{II}^M / \sigma\sqrt{\pi a} \quad (M = A, B) \quad (5.31)$$

The SIFs are calculated for different meshes. A representative mesh (784 nodes, 122 parasitic nodes, and 729 elements) is shown in Fig. 5.17.

Fig. 5.17 A representative mesh for the multiple branched crack problem (784 nodes, 122 parasitic nodes and 729 elements)

The SIFs results for different value of average element size h are listed in Table 5.3. It is seen that, even for such a complex problem, the PFEM shows a consistent convergence as the mesh is gradually refined.

Table 5.3 Normalized SIFs for a multiply branched crack problem

h	0.50	0.46	0.35	0.28	0.24	0.20	0.17
K_I^A	4.978	4.997	5.033	5.041	5.051	5.062	5.067
K_{II}^A	1.783	1.792	1.800	1.804	1.807	1.810	1.811
K_I^B	2.463	2.471	2.468	2.474	2.470	2.471	2.470
K_{II}^B	1.240	1.233	1.226	1.218	1.219	1.217	1.216

5.5 Summary

This chapter conducts a detailed comparison between the NMM and the XFEM/GFEM.

Both the NMM and the XFEM allow the cracks arbitrarily align within the finite element mesh. The NMM shape functions span the same space as the XFEM shape functions. However, the implementation of the NMM is much more convenient than that of the XFEM. The XFEM incorporates discontinuous enrichment functions to account for the cracks. When complex cracks with multiple branches are encountered, enriching the nodes becomes tedious, or even impossible to handle. The NMM partitions each mathematical cover intersected by the crack into several physical covers and assigns each physical cover an independent local function. By such a way, complex cracks are treated as easy as a single crack.

Both the NMM and the GFEM allow the finite element mesh be constructed totally independent of the problem domain, thus a regular finite element mesh is always applicable in both methods. The GFEM adopts different discontinuous enrichment functions from those used in the XFEM to describe the cracks. Its implementation is simpler than the XFEM. Cracks with multiple branches can be dealt with easily. However, enrichments to the nodes should be carefully designed since the enrichment functions $H_\alpha(\mathbf{x})$ are not linearly independent. In addition, its enrichment functions are derived based on aggregate topology, thus can only allow the crack tips stopping at the sides of the elements. In contrast, the NMM incorporates the asymptotic crack tip functions into the singular physical covers and thus does not suffer from the above limitations.

Inspired by the advantageous modeling strategy of the NMM, a modification to the XFEM/GFEM is proposed. Its basic idea is to add parasitic nodes to the original node if its support is completely cut by the cracks into several pieces. The new technique is termed as the parasitic finite element method (PFEM). Through two typical examples, the PFEM is proved valid and efficient. The PFEM can reduce to the XFEM/GFEM after rearranging its unknowns. However, the PFEM is over the XFEM/GFEM in the following three aspects.

First, the PFEM is conceptually suitable for arbitrary crack problems, unlike the XFEM restrained by its enrichment functions. With the PFEM, complex cracks with arbitrary number of branches can be easily handled, and therefore, the PFEM is superior to the XFEM in suitability.

Second, the shape functions together with the attached unknowns in the PFEM avoid the linear dependency often encountered in the enrichment functions in the GFEM.

Third, the unknowns attached to the parasitic nodes in the PFEM have a clear interpretation in physics. However, this is not the case for any other modifications.

The PFEM shares some similarities with Hansbo and Hansbo (2004). However, they are different in essence. Hansbo and Hansbo (2004) constructs its approximation at element level. It adopts six degrees of freedom for all the cracked triangular elements. It cannot handle partially cracked elements. The displacement compatibility condition at crack tips needs to be enforced by additional techniques. The PFEM constructs its approximation through the analysis of nodal support of each node. The DOFs used in different kinds of elements may be different. The PFEM is equivalent to Hansbo and Hansbo (2004) in some fully cracked elements. However, the PFEM does not suffer from the limitations encountered by Hansbo and Hansbo (2004).

CHAPTER 6 NMM COUPLED WITH A FRACTURING ALGORITHM BASED ON MOHR-COULOMB CRITERION WITH A TENSILE CUTOFF FOR ROCK FAILURE ANALYSIS

6.1 Introduction

The practical rock failure is a complex progressive evolution involving initially elastic deformation, crack initiation, propagation of both the newly-initiated cracks and the pre-existing cracks, crack coalescence, formation of a complete failure surface, large-scale displacement and even movement of a discrete block system. However, the current continuum-based methods and discontinuum-based methods can only simulate certain stages of the whole process. Thus, a new approach which is able to account for the whole process is essential.

In the following part, taking the modeling of the rock slope failure for an example, the existing numerical methods are reviewed and their limitations are discussed.

Limit equilibrium techniques have been widely used in rock slope failure analysis.

Certain rock slope failures are kinematically controlled and can be treated as a limit equilibrium problem. One example is the translational failure where basal shear, lateral and rear release all take place on persistent joints, particularly at the residual angle of friction. However, such conditions are rare in practice. In most cases, major rock slope instabilities undergo some degree of progressive shear plane development, deformation and extensive internal disruption of the slope, which bears little resemblance to the rigid block assumption adopted in limit equilibrium methods. The initiation and eventual failure are complex, thus cannot be well described through a simple static analysis.

In contrast, numerical methods are able to provide approximate solutions to

problems which cannot be solved using limit equilibrium techniques alone. They are able to treat rock slope stability problems involving complexity relating to geometry, material anisotropy, non-linear behavior, the presence of in-situ stresses and coupled processes (e.g. pore pressures, seismic loading, etc). Numerical techniques used for rock slope analyses are generally divided into continuum approaches, discontinuum approaches, as well as hybrid approaches.

Continuum approaches used in the slope stability analysis include the finite element method (FEM) and finite difference method (FDM). Continuum methods provide approximate solutions to problems involving intact rock deformation during rock slope failure. Early numerical analysis of rock slopes were predominately undertaken using continuum-based finite element methods. For example, Kalkani and Piteau (1976) analyzed the toppling of rock slopes at Hells Gate in British Columbia, Canada. Krahn and Morgenstern (1976) undertook preliminary modeling of the Frank Slide in Alberta, Canada. Radbruch-Hall et al. (1976) simulated the stress distributions in rock slopes with the aim to investigate high mountain deformation mechanisms and ‘sackung’ formation. Stacey et al. (2003) presented an innovative analysis of extensile strain distributions associated with deep open pit mines. The use of finite difference methods has been predominately involved the use of FLAC 2-D and 3-D codes (Itasca, 2004). For example, Board et al. (1996) demonstrated the potential for using continuum-based FDM codes in conjunction with discontinuum methods in the analysis of high rock slopes at the Chuquicamata open pit mine in Chile. Coggan et al. (2000) showed the use of both 2D and 3D FDM analyses in the back analysis of highly kaolinised china clay slopes. Guadagno et al. (2003) adopted a FDM approach to investigate the influence of cut-and-fill works on slopes in Campania, southern Italy. Kinakin and Stead (2005) investigated ‘sackung’ formation using an elasto-plastic FDM model of varied rock slope ridge geometries associated with ‘sackung’ features in British Columbia, Canada. Both continuum-based FEM and FDM models are routinely used in engineering landslide investigations. These methods are most appropriate to analyze the slopes where failure is controlled by intact material deformation or through a restricted number of discrete discontinuities, e.g. bedding planes and/or faults.

However, when the stability of the rock slope is predominately controlled by movement of joint-bounded blocks and/or intact rock deformation, the use of discontinuum methods should be considered. Discontinuum methods treat the problem domain as an assemblage of distinct, interacting bodies or blocks which are subjected to external loads and are expected to undergo significant motion with time. Joints are viewed as interfaces between the blocks and are treated as a boundary condition rather than a special element in the model. Discontinuum methods are extremely suitable for rock slope instability problems since they are capable of simulating large displacements due to slip and/or opening along discontinuities, and they are capable of modeling the internal deformation of joint-bounded intact rock blocks. Two methods mostly in use are distinct element method (Hart, 1993) and the discontinuous deformation analysis (DDA) (Shi, 1988). The DDA has been used to analyze the Vaiont rock slide by Sitar and Maclaughlin (1997), and a major rockfall in Japan by Chen and Ohnishi (1999). The predominant discontinuum method in use, however, is the distinct element code UDEC (Itasca, 2004). UDEC has been used to investigate a wide range of rock slope failures from simple planar slope failure mechanism (Costa et al., 1999) to complex deep-seated toppling instability (Board et al., 1996; Benko and Stead, 1999; Hutchison et al., 2000; Nichol et al., 2002) and buckling (Stead and Eberhardt, 1997). Their results demonstrated the need to consider both intact material deformation and joint-controlled displacements in the analysis of complex rock slope instabilities.

Although both continuum and discontinuum methods are useful tools to analyze rock slope stability problems, practical rock slope failures involve an additional complexity relating to the brittle fracturing of intact rock. Early work to address the influence of fracturing in rock engineering include Williams et al. (1985) and Mustoe (1989) who incorporated fracturing process into a discrete element code. The fracturing was fulfilled through a Mohr-Coulomb criterion with a tension cut-off. With this, an element could fracture through its centroid or along an element edge once the criterion is satisfied. Cundall and Strack (1979) proposed a variation of the discrete element code to simulate particulate material behavior, which led to the development of the Particle Flow Code (PFC) (Itasca, 2004), in which particles are

bonded together to represent joint-bounded blocks, and fracturing of intact rock blocks are fulfilled through the stress-induced breakage of bonds between blocks. The PFC was adopted to analyze heavily jointed rock slopes by Wang et al. (2003). A major shortcoming of the above-mentioned methods is that they only imitate intact rock fracture. Another example is the coupled finite-discrete element methods. One finite-discrete code is the ELFEN (Rockfield, 2004). Starting from a continuum representation of the joint-bounded blocks by finite elements, fracturing in ELFEN is controlled according to a fracturing criterion specified through a constitutive model (e.g. Rankine tension, rotating crack, Mohr-Coulomb, etc). At some point in the analysis, a failure band may be formed within a single element or between elements. The load bearing capacity across such localized bands decreases to zero as damage increases until eventually the energy needed to form a discrete fracture is released. The mesh topology is updated, initially leading to fracture propagation within a continuum and eventually resulting in the formation of discrete blocks. Subsequent motion of these discrete blocks and further fracturing of both the remaining continuum and previously created discrete blocks is then simulated. This evolution process is continued until either the system comes to equilibrium or up to the time of interest. By applying such techniques, it is possible to model both the behavior of a continuum and a discontinuum, and the transformation of the rock mass from a continuum to a discontinuum. The ELFEN has been successfully applied to the simulation of intense fracturing associated with surface mine blasting, mineral grinding, retaining wall failure and underground rock caving (Munjiza et al., 1995), asperity degradation and damage in the direct shear test (Karami and Stead, 2008), discrete fracture in quasi-brittle materials (Klerck et al., 2004), realistic slope failure and various rock slope failure mechanisms (Stead et al., 2004; Stead et al., 2006).

Another example of the hybrid continuum-discontinuum methods is the newly developed numerical manifold method (NMM). The NMM presents a unique way for discontinuity modeling. It treats continuous block, fractured block and discrete block system in a unified framework. The NMM seems promising for practical rock failure analysis. The current NMM can simulate the continuum, the fractured body, and the discontinuum well. However, before we apply it to practical rock failure analysis, a

fracturing algorithm which fulfills the transition from continua to discontinua must be implemented.

In Chapter 4, a fracturing algorithm based on the fracture mechanics has been implemented into the NMM for crack problems. The extended NMM is proved efficient and robust for complex cracks and their growth. A comparison indicates that the extended NMM is obviously superior to the XFEM/GFEM for complex crack problems. However, the fracturing algorithm based on fracture mechanics is not appropriate for practical rock failure analysis. The theory of fracture mechanics can explain the stress concentration near crack tips and predict the possible direction of crack propagation, but are unable to deal with the initiation of cracks. The stress intensity factors are evaluated by the domain form of the interaction integral, which assumes a traction-free crack surface. In rock failure analysis, compressive loading condition is usually encountered. The joint surfaces usually involve frictional contact. Although some efforts have been done to account for the traction along the crack surface in the literature, it is still not an easy task. In addition, the domain form of the interaction integral needs a domain around the crack tip to evaluate the stress intensity factors, the domain must not be intersected by other cracks, thus it is not applicable to the case where the crack tip is close to another crack surface or crack tip. In a rock failure analysis, crack coalesce is essential to form a complete failure surface to allow the final kinematic release.

In the literature, many complex numerical models invoking the concepts of elasto-plasticity, damage mechanics and statistical techniques have been presented. However, most of them are not widely used due to the difficulty in determining material parameters and the high level of technical expertise required by the user (Klerck et al., 2004).

The aim of this chapter is to present a relatively simple numerical model to represent the fracturing of intact rock in compression.

To this end, the widely used Mohr-Coulomb failure criterion with a tensile cutoff is adapted. The Mohr-coulomb criterion is able to recover the salient features of the quasi-brittle response within engineering accuracy, including dilation and has the advantage of being based on parameters that are easily determined experimentally

(Klerck et al., 2004).

The remainder of this chapter is organized as follows. Section 6.2 briefly discusses the applicability of the conventional NMM for rock failure analysis. A fracturing algorithm based on the Mohr-Coulomb criterion with a tensile cutoff is implemented into the NMM in Section 6.3. Section 6.4 summarizes the main work of this chapter.

6.2 Applicability of the conventional NMM for rock failure analysis

A practical rock failure analysis involves the modeling of the continuum, the pre-existing discontinuities, the fracturing process which fulfills the transition from a continuum to a discontinuum, and the interactions among multiple discrete blocks.

The studies in Section 3.3 have demonstrated that the conventional NMM is capable of describing both the continuum and the discontinuum well.

The NMM is also convenient in dealing with arbitrarily complex cracks by splitting covers. With reference to a single crack problem in Fig. 6.1, the discontinuity modeling in the NMM is briefly recalled.

The single crack introduces displacement jumps in finite elements 1 to 8 in Fig. 6.1a. Among them, elements 2 to 7 are completely cut by the crack surface, while elements 1 and 8 are partially cut by the crack surface. In the following part, we will show how the displacement jumps in these elements are modeled within the NMM framework.

(a)

Fig. 6.1 A single crack problem: (a) NMM model; (b) local index

We first investigate the fully cracked elements. Take element 2 for an example. There are three mathematical covers associated with element 2, denoted as $M_{(1)}$, $M_{(2)}$ and $M_{(3)}$, where i ($i=1\sim 3$) are the local indices of the three mathematical covers given in Fig. 6.1b, and the subscript $()$ transforms the local index i into a global index. The $M_{(1)}$ forms one physical cover $P_{(1)}^1$. The $M_{(2)}$ forms two physical covers $P_{(2)}^1$ (upper one) and $P_{(2)}^2$ (lower one). The $M_{(3)}$ forms one physical cover $P_{(3)}^1$. The displacement jump across the crack surface in element 2 is obtained as

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = \varphi_{(2)}(\mathbf{x}) (\delta_{(2)}^1 \mathbf{u}_{(2)}^1 - \delta_{(2)}^2 \mathbf{u}_{(2)}^2) \quad (6.1)$$

Considering $\delta_{(2)}^1 = 1$, $\delta_{(2)}^2 = 0$ at the upper crack surface, $\delta_{(2)}^1 = 0$, $\delta_{(2)}^2 = 1$ at the lower crack surface, Eq. (6.1) is further re-written as

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = \varphi_{(2)}(\mathbf{x}) (\mathbf{u}_{(2)}^1 - \mathbf{u}_{(2)}^2) \quad (6.2)$$

Since $\mathbf{u}_{(2)}^1$ and $\mathbf{u}_{(2)}^2$ are two independent local functions with different unknowns, the displacement jump across the crack surface in element 2 is modeled.

Following a similar way, the displacement jump in elements 3 to 7 can be modeled.

Then, we turn to take an examination on the partially cracked elements. Take element 1 for an example. There are also three mathematical covers associated with element 1, still denoted as $M_{(1)}$, $M_{(2)}$ and $M_{(3)}$. All the three mathematical covers are partially cut by the crack surface, thus each forms only one singular physical cover.

If taking the usually used polynomials as local approximation space, the displacement field in element 1 will be continuous. In other words, the crack in partially cracked element cannot be described. To solve this problem, we have proposed to enrich each singular physical cover with the asymptotic crack tip functions in Eq. (4.7) in Chapter 4. This can be done because of the partition of unity functions used in the NMM. The first term in Eq. (4.7) is discontinuous across the crack surface (i.e. $\theta = \pm\pi$), thus the displacement jump in partially cracked elements can be modeled. Detailed discussion can be referred to Section 4.2.3.

However, to make it simple, the enrichments to the singular physical covers will not be considered in this chapter. The crack tips are restricted to the sides of the manifold elements. Such a treatment is acceptable within an engineering accuracy.

To sum up, the NMM can simulate the continuum, the pre-existing discontinuities, the interactions among multiple discrete blocks well. Once a fracturing algorithm is implemented, the NMM seems promising for practical rock failure analysis.

6.3 A fracturing algorithm based on Mohr-Coulomb criterion and a tensile cutoff

6.3.1 Mohr-Coulomb criterion

The widely used Mohr-Coulomb criterion with a tensile cutoff is chosen as the failure criterion of intact rock material. The Mohr-Coulomb criterion is briefly introduced in this section. The equations are after Bray and Brown (2004).

According to the Mohr-Coulomb failure criterion, the shear strength on a plane such as ab in Fig. 6.2 is

$$s = c + \sigma_n \tan \phi \quad (6.3)$$

where c is the cohesion, ϕ is the angle of internal friction, σ_n is the normal stress.

Applying equilibrium condition to the upper part of Fig. 6.2 gives the normal stress as

$$\sigma_n = \frac{1}{2}(\sigma_1 + \sigma_3) + \frac{1}{2}(\sigma_1 - \sigma_3)\cos 2\beta \quad (6.4a)$$

and the shear stress as

$$\tau = \frac{1}{2}(\sigma_1 - \sigma_3) \sin 2\beta \quad (6.4b)$$

Fig. 6.2 Normal and shear stresses on plane ab

Substituting σ_n , letting $s = \tau$ in Eq. (6.3) and rearranging terms yield the limit stress condition on any plane defined by β as

$$\sigma_1 = \frac{2c + \sigma_3 (\sin 2\beta + \tan \phi (1 - \cos 2\beta))}{\sin 2\beta - \tan \phi (1 + \cos 2\beta)} \quad (6.5)$$

There will be a critical plane on which the available shear strength will be first reached as σ_1 is increased. The Mohr circle construction in Fig. 6.3a gives the orientation of this critical plane as

$$\beta = \frac{\pi}{4} + \frac{\phi}{2} \quad (6.6)$$

which may also be obtained by solving $d(s - \tau)/d\beta = 0$.

For the critical plane, we have $\sin 2\beta = \cos \phi$, $\cos 2\beta = -\sin \phi$, then Eq. (6.5) reduces to

$$\sigma_1 = \frac{2c \cos \phi + \sigma_3 (1 + \sin \phi)}{1 - \sin \phi} \quad (6.7)$$

This linear relation between σ_3 and the peak value of σ_1 is shown in Fig. 6.3b. Note that the slope of this envelope is related to ϕ by the following equation

$$\tan \psi = \frac{1 + \sin \phi}{1 - \sin \phi} \quad (6.8)$$

and that the uniaxial compressive strength is related to c and ϕ by

$$\sigma_c = \frac{2c \cos \phi}{1 - \sin \phi} \quad (6.9)$$

(a)

(b)

Fig. 6.3 Mohr-Coulomb strength envelopes in: (a) $\sigma_n - \tau$ plane; (b) $\sigma_1 - \sigma_3$ plane

If the Mohr-Coulomb envelop shown in Fig. 6.3b is extrapolated to $\sigma_1 = 0$, it will intersect the σ_3 axis at an apparent value of uniaxial tensile strength of the material given by

$$\sigma_T = \frac{2c \cos \phi}{1 + \sin \phi} \quad (6.10)$$

The experimentally measured uniaxial tensile strength is generally lower than those predicted by Eq. (6.10). For this reason, a tensile cutoff is usually applied at a selected value of uniaxial tensile stress, T_0 , as shown in Fig. 6.4. The tensile failure criterion reads

$$-\sigma_3 = T_0 \quad (6.11)$$

(a)

(b)

Fig. 6.4 Mohr-Coulomb strength envelopes with a tensile cutoff in: (a) $\sigma_n - \tau$ plane;

(b) $\sigma_1 - \sigma_3$ plane

Mohr-Coulomb criterion is chosen as the failure criterion for intact rock in the current study, because: 1) it is widely applied; it is able to recover the salient features of the quasi-brittle response within engineering accuracy (Klerck et al., 2004); 2) its parameters can be conveniently determined by conventional uniaxial compression (with axial and lateral strain measurement) and indirect tensile testing.

6.3.2 Crack initiation and propagation

In this section, we propose a way to determine the location and the orientation of a crack surface within the framework of the NMM. In some other FEM-based methods, fracturing is usually established by dividing finite elements. However, in the present study, instead of dividing elements, the cover-division strategy firstly proposed by Terada et al. (2007) is adopted.

Here we regulate that the covers are generated from a triangular finite element mesh, the partition of unity functions are chosen as the triangular finite element shape functions, the local functions for each physical covers are chosen as constant, thus the stress field computed from the gradients of the displacement fields are constant over each manifold element and discontinuous across the crack surfaces.

Then, we rely on physical covers rather than the elements to determine the location and the orientation of a crack surface. In order to judge the failure initiation with reference to tensile failure criterion (6.11) or shear failure criterion (6.7), we need to obtain a reliable stress distribution over the problem domain and then relate it to each physical cover. In this respect, we take the weighted average of the stresses of all the associated manifold elements as the stress of a physical cover, expressed as

$$\sigma = \frac{\sum_{k=1}^m A_k \sigma_k}{\sum_{k=1}^m A_k} \quad (6.12)$$

where m is the number of manifold elements associated with this physical cover, A_k and σ_k are the area and the constant stress of manifold element E_k . This type of approximation is termed as Winslow's smoothing scheme (taking areas being weights)

in the context of adaptive FEM and may be regarded as a domain-integral approach for evaluating non-local stress/strain (Terada et al., 2007). Due to the non-local nature of stress, it is expected that the mesh dependency can be avoided to some extent in the judgment of failures with the failure criterion (6.7) and (6.11).

During the fracturing process, two types of failure patterns are involved, i.e. crack initiation and crack propagation.

Crack initiation

We first evaluate the smoothed stress on all the physical covers by using Eq. (6.12), compute the principal stresses together with the corresponding principal directions, and locate those physical covers violating the failure criterion (6.7) or (6.11). Here, suppose we find one physical cover as shown in Fig. 6.5. With only the failure direction, the location of a crack surface is still indeterminate. Thus, we further assume that the crack surface passes through the center of the physical cover since the stress value we obtain from Eq. (6.12) is regarded as representative of that at the center of the physical cover. If the failure criterion (6.11) is met, mode I fracturing takes place with the crack direction perpendicular to σ_3 , as shown in Fig. 6.5a. If the principal stresses of the physical cover satisfies condition expressed in Eq. (6.7), mode II crack initiates with the direction inclined at $\pm(\pi/4 + \phi/2)$ with respect to σ_3 , as shown in Fig. 6.5b. One of the two directions can be specified by the user (Lin et al, 1996).

Fig. 6.5 Illustration of crack initiation based on Mohr-Coulomb criterion: (a) mode I; (b) mode II (O represents center of the physical cover; red line represents new crack)

In this study, we choose the cover-division strategy rather than the element-division strategy. It is partly due to the fact that the mesh dependency can be avoided to some extent because of the non-local nature of the stress. There is another reason for us to do such a choice, which will be explained with an example shown in Fig. 6.6.

(a)

(b)

Fig. 6.6 Comparison between element-division strategy and cover-division strategy in crack initiation modeling: (a) element-division; (b) cover-division

In Fig. 6.6a, element-division strategy is adopted. The newly initiated crack divides a manifold element into two parts. Then, we take an examination on the stress field in this cracked element. There are totally three mathematical covers associated with this element, denoted as $M_{(i)}, i = 1 \sim 3$. All of these three mathematical covers are only partially cut by the crack surface, thus each of which forms one physical cover, denoted as $P_{(1)}^I$, $P_{(2)}^I$, and $P_{(3)}^I$, respectively. Since we adopt the polynomials

as the local approximation space for each physical cover, the displacement jump across the newly initiated crack surface cannot be reflected. In order to alleviate this problem, we can do the remeshing or enriching the singular physical covers. Both of them will complicate the simulation. However, the cover-based division strategy as shown in Fig. 6.6b does not suffer from such a problem.

Crack propagation

Assume one physical cover contains the forefront of the crack surface that might be evolving, as shown in Fig. 6.7.

Suppose that we have evaluated smoothed stresses on physical covers by Eq. (6.12) with an approximate number of associated manifold elements m . If the stress state of this physical cover violates the tensile failure criterion (6.11) or the shear failure criterion (6.7), then we divide this physical cover from the crack tip in the failure direction, as indicated in Fig. 6.7.

Fig. 6.7 Illustration of crack propagation based on the Mohr-Coulomb criterion: (a) mode I; (b) mode II (red line represents new crack surface)

6.3.3 Treatment of manifold elements

If the fracturing criterion is met, some manifold elements will be split into two manifold elements. The manifold elements generated by this way may have irregular shape, or even with large aspect ratio. Such manifold elements were termed as

‘generalized elements’ by Terada and Kurumatani (2004). By a series of numerical experiments, Terada and Kurumatani (2004) concluded that the performance of the generalized elements in the NMM was equivalent to or slightly superior to that of the standard finite elements. In addition, integration of the weak form over such irregular manifold elements can be directly evaluated by the simplex integration method. Because of the above two reasons, once the failure criterion is met, we just need to simply split the corresponding physical covers and thus simply split the manifold elements.

Treatment of elements in the NMM is much convenient than that in other methods.

The conventional FEM requires the finite element mesh coincides with the cracks. When crack growth needs to be modeled, remeshing is inevitable, which makes the simulation tedious.

In the XFEM, if unconditioned elements are generated, remeshing must be done to achieve an acceptable element topology. In addition, the crack growth results in elements with arbitrary shape. Such elements need to be partitioned into a set of triangles to do the integration, as illustrated in Fig. 6.8.

Fig. 6.8 Illustration of element partitioning in the XFEM for integration purpose

In the finite-discrete element code ELFEN, once the nodal failure factor is greater than one, two methods are proposed to insert the discrete crack. The dashed line in Fig. 6.9a gives the actual discrete crack orientation (i.e. weighted average fracture direction). Intra-element fracturing (Fig. 6.9b) inserts the discrete crack in the actual crack direction, which may results in the creation of unfavorably dimensioned silver elements, therefore requires local adaptive mesh refinement to achieve an acceptable

element topology. In contrast, the inter-element fracturing (Fig. 6.9c) snaps the discrete crack to the most favorably orientated existing element side, which does not require remeshing any more. However, it requires very fine meshes to capture the fracture orientation accurately.

Fig. 6.9 Dividing elements in ELFEN: (a) actual crack direction; (b) intra-element fracturing; (c) inter-element fracturing

6.3.4 Treatment of physical covers

When new crack surface is introduced, some of the physical covers may be intersected. We use path connectivity to determine whether the intersected cover is connected or not. The cover is path-connected if a continuous path can be found between any two points, which is weaker than the requirement of simply-connected. Fig. 6.10 shows two covers that each has a circular hole and a notch.

Fig. 6.10 Illustration of path-connectivity criterion: (a) path-connected; (b) path-disconnected

Both covers are not simply-connected. Based on the path-connectivity criterion, the cover on the left is path-connected while the cover on the right is not. The path-connected cover remains one cover. The path-disconnected cover may be partitioned into several covers such that each of which is path-connected.

6.3.5 Updating physical covers for each manifold element

Once a physical cover is split into two physical covers, the associated physical covers for the related manifold elements must be updated. Assume the physical cover P_i be split into two physical covers P_i^1 and P_i^2 . Updating the physical covers for the manifold elements can be fulfilled by two steps. Firstly, find all the manifold elements associated with physical cover P_i . Secondly, find an arbitrary point in each manifold element. If this point is within the physical cover P_i^1 , which indicates this manifold element is covered by physical cover P_i^1 , then we use P_i^1 to replace the original P_i ; If not, this manifold element must be covered by physical cover P_i^2 , then we use P_i^2 to replace the original P_i .

To determine whether a point is within a given physical cover or not, the point method (Shi, 1988) illustrated in Fig. 6.11 can be used.

Fig. 6.11 Illustration of the point method: (a) the number of intersection points between the ray and the loop is 1 (odd), the point is within the physical cover; (b) the number of intersection points is 4 (odd), the point is outside the physical cover

The ordered vertices of the physical cover form a loop. From the point we draw a ray in any direction. If the number of the intersection points of the ray and the loop is even, the point is outside the physical cover. If the number of the intersection points of the ray and the loop is odd, the point is inside the physical cover.

A simple example shown in Fig. 6.12 is adopted to illustrate the treatments of manifold elements and physical covers during the crack initiation and propagation.

Fig. 6.10a shows the initial NMM model with 4 manifold elements and 6 physical covers.

Fig. 6.12 An example to illustrate the treatment of elements and covers during the crack initiation and propagation

In Fig. 6.12b, a new crack initiates and divides the original physical cover 4 into two physical covers, denoted as 4^1 (upper one) and 4^2 (lower one). The physical covers 2 and 5 are only partially intersected, thus each remains one single physical cover. The initiated crack also cuts the original manifold element II into two elements, denoted as II^1 (upper one) and II^2 (lower one). These two new manifold elements are first assigned the physical covers 2, 4, 5 of the original manifold element II. The manifold elements II^1 and II^2 are associated with the original physical cover 4. Then, based on the point method illustrated in Fig. 6.11, the physical cover 4^1 is used to replace 4 in the element II^1 and the physical cover 4^2 is used to replace 4 in the

element II^2 .

In Fig. 6.12c, the crack propagates and cuts the original physical cover 2 into two physical covers, denoted as 2^1 (upper one) and 2^2 (lower one). The physical covers 3 and 5 are only partially intersected, thus each remains a single physical cover. The propagated crack also cuts the manifold element III into two elements III^1 (upper one) and III^2 (lower one). These two new manifold elements are initially assigned the physical covers 2, 3, 5 of the original manifold element III. Then, we will find that the manifold elements I, II^1 , II^2 , III^1 , III^2 are associated with the original physical cover 2. Then, based on the point method, we use the physical cover 2^1 to replace 2 in the manifold elements I, II^1 , III^1 and the physical cover 2^2 to replace 2 in the manifold elements II^2 , III^2 .

In Fig. 6.12d, the crack further propagates and divides the physical cover 3 into two physical covers 3^1 (upper one) and 3^2 (lower one), the physical cover 5 into two physical covers 5^1 (upper one) and 5^2 (lower one), the physical cover 6 into two physical covers 6^1 (upper one) and 6^2 (lower one). The propagated crack also cuts the manifold element IV into two elements, denoted as IV^1 (upper one) and IV^2 (lower one). These two new elements are first assigned the physical covers 3, 5, 6 of the original manifold element IV. Then, we can find that the manifold elements I, III^1 , III^2 , IV^1 , IV^2 are associated with the original physical cover 3. Based on the point method, the physical cover 3^1 is assigned to I, III^1 , IV^1 , and the physical cover 3^2 is assigned to III^2 , IV^2 . The manifold elements II^1 , II^2 , III^1 , III^2 , IV^1 , IV^2 are associated with the original physical cover 5. Based on the point method, the physical cover 5^1 is used to replace 5 in the manifold elements II^1 , III^1 , IV^1 , and the physical cover 5^2 is used to replace 5 in the manifold elements II^2 , III^2 , IV^2 . The manifold elements IV^1 and IV^2 are associated with the original physical cover 6. Based on the point method, the physical cover 6^1 is used to replace 6 in the manifold element IV^1 and the physical cover 6^2 is used to replace 6 in the manifold element IV^2 .

In the following part, we will examine whether the displacement jump across the crack surface and the displacement compatibility condition along the common side of adjacent elements are satisfied. Here take the configuration in Fig. 6.12d for an example. Firstly, we examine the displacement field in element II. The displacement

jump across the crack surface in element II can be obtained as

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = (\varphi_2 \delta_2^1 \mathbf{u}_2^1 + \varphi_4 \delta_4^1 \mathbf{u}_4^1 + \varphi_5 \delta_5^1 \mathbf{u}_5^1) - (\varphi_2 \delta_2^2 \mathbf{u}_2^2 + \varphi_4 \delta_4^2 \mathbf{u}_4^2 + \varphi_5 \delta_5^2 \mathbf{u}_5^2) \quad (6.13)$$

Considering $\delta_2^1 = 1$, $\delta_2^2 = 0$, $\delta_4^1 = 1$, $\delta_4^2 = 0$, $\delta_5^1 = 1$, $\delta_5^2 = 0$ at the upper crack surface, and $\delta_2^1 = 0$, $\delta_2^2 = 1$, $\delta_4^1 = 0$, $\delta_4^2 = 1$, $\delta_5^1 = 0$, $\delta_5^2 = 1$ at the lower crack surface, Eq. (6.13) can be further written as

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = \sum_{i=2,4,5} \varphi_i (\mathbf{u}_i^1 - \mathbf{u}_i^2) \quad (6.14)$$

With the knowledge that $\mathbf{u}_i^1$ and $\mathbf{u}_i^2$, $i = 2, 4, 5$ are independent local functions with independent unknowns, the displacement jump in element II is obviously manifested.

Similarly, the displacement jump across the crack surface in element III can be obtained as

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = \sum_{i=2,3,5} \varphi_i (\mathbf{u}_i^1 - \mathbf{u}_i^2) \quad (6.15)$$

Since $\mathbf{u}_i^1$ and $\mathbf{u}_i^2$, $i = 2, 3, 5$ are independent local functions with independent unknowns, the displacement jump in element III is also obtained.

The displacement jump across the crack surface in element IV can be proved in an exactly same way.

Then, we turn to investigate the displacement field across the common side of adjacent elements. Take element II¹ and III¹ for an example. The displacement jump across their common side is

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = (\varphi_2 \delta_2^1 \mathbf{u}_2^1 + \varphi_4 \delta_4^1 \mathbf{u}_4^1 + \varphi_5 \delta_5^1 \mathbf{u}_5^1) - (\varphi_2 \delta_2^1 \mathbf{u}_2^1 + \varphi_3 \delta_3^1 \mathbf{u}_3^1 + \varphi_5 \delta_5^1 \mathbf{u}_5^1) \quad (6.16)$$

We know that φ_3 and φ_4 vanish at the common side of the element II¹ and III¹, thus Eq. (6.16) is further written as

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = (\varphi_2 \delta_2^1 \mathbf{u}_2^1 + \varphi_5 \delta_5^1 \mathbf{u}_5^1) - (\varphi_2 \delta_2^1 \mathbf{u}_2^1 + \varphi_5 \delta_5^1 \mathbf{u}_5^1) = 0 \quad (6.17)$$

The displacement compatibility along the common side of the elements II¹ and

III^1 is obviously satisfied.

Take elements III^1 and IV^1 for another example. The displacement jump cross their common side is

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = (\varphi_2 \delta_2^1 \mathbf{u}_2^1 + \varphi_3 \delta_3^1 \mathbf{u}_3^1 + \varphi_5 \delta_5^1 \mathbf{u}_5^1) - (\varphi_3 \delta_3^1 \mathbf{u}_3^1 + \varphi_5 \delta_5^1 \mathbf{u}_5^1 + \varphi_6 \delta_6^1 \mathbf{u}_6^1) \quad (6.18)$$

With the knowledge that φ_2 and φ_6 vanish along the common side of the elements III^1 and IV^1 , Eq. (6.18) is further written as

$$\llbracket \mathbf{u}^h(\mathbf{x}) \rrbracket = (\varphi_3 \delta_3^1 \mathbf{u}_3^1 + \varphi_5 \delta_5^1 \mathbf{u}_5^1) - (\varphi_3 \delta_3^1 \mathbf{u}_3^1 + \varphi_5 \delta_5^1 \mathbf{u}_5^1) = 0 \quad (6.19)$$

The displacement compatibility condition between the elements III^1 and IV^1 is obviously satisfied.

The displacement compatibility condition between other adjacent elements can be verified in an exactly same way.

To sum up, the algorithms we proposed to treat the elements and covers during the crack initiation and propagation can guarantee the displacement jump across the crack surface and the displacement compatibility condition along the common side of two adjacent elements.

6.3.6 Treatment of loops

The contact response of the newly created crack surfaces is of vital importance to the stability of the solutions after crack initiation and/or propagation. The NMM possesses robust contact detection and modeling algorithm. Its effectiveness and efficiency has been demonstrated in four examples presented in Sections 3.3.3, 3.3.4, 3.3.5 and 3.3.6.

In the NMM, each boundary with ordered vertex is termed as a loop. Its advance direction is defined such that material is always at its left hand side. Fig. 6.13 shows a block with an internal hole. In this case, two loops are defined.

In the NMM, contact detection is fulfilled through two steps. Firstly, a global search is carried out to find all the loop pairs which may come into contact within this time step by representing the loops by bounding boxes. If the distance between the

bounding boxes of any two loops is less than 2ρ , where ρ is the maximum displacement allowed in each time step, they might come into contact. Then, a local search is conducted to figure out all the vertex-edge contact pairs of each possible loop pair.

Fig. 6.13 Illustration of loops in the NMM

Details on the contact detection in the NMM can be referred to Shi (1991; 1992; 1995; 1996a; 1997).

In the NMM, the contact detection is based on loops. Therefore, once crack initiation and/or propagation occurs, we only need to update the loops and then can use the existing contact algorithm to handle the contact response between the two sides of a crack surface or between two blocks.

(c)

(d)

Fig. 6.14 Updating loops during the crack initiation and propagation: (a) Type I, a new loop is generated; (b) Type II, a crack initiates from an existing loop, more ordered vertices added to the loop; (c) Type III, a crack cuts an exiting loop into two loops; (d) Type IV, a crack combines two existing loops into a single loop

The modifications to the loops can be categorized into the following four types:
 (1) Type I: a new crack initiates and does not connect to any other existing loops, thus

a new loop is added (Fig. 6.14a); (2) Type II: a new crack initiates from an existing loop, then more ordered vertices are added to this loop (Fig. 6.14b); (3) Type III: a crack cuts an existing loop into two loops (Fig. 6.14c); (4) Type IV: a crack combines two existing loops into a single loop (Fig. 6.14d).

6.3.7 Evaluation of block size with the simplex integration method

Once the geo-structure finally becomes a discrete block system, we may need to know the size of each block. The block size can be expressed as

$$\iint_A dx dy \quad (6.20)$$

where A represents the domain occupied by the block.

In Section 3.2.3, the simplex integration method is applied to evaluate the integration of the weak form over each manifold element. Here, the simplex integration method is adopted to evaluate the block size.

Consider an arbitrary block with ordered vertices $V_1, V_2, \dots, V_6, V_7 = V_1$. Then, a set of oriented triangles are obtained as $\Delta_{V_0V_1V_2}, \Delta_{V_0V_2V_3}, \dots, \Delta_{V_0V_6V_7}$ (illustrated in Fig. 6.15), where V_0 is an arbitrary point. The orientation of each triangle is marked by an oriented circle. ‘+’ means the simplex area of the oriented triangle (or simplex) is positive. ‘-’ means the simplex area of the oriented triangle is negative. ‘+’ and ‘-’ of an oriented triangle (or simplex) is determined by the sign of the Jacobi determinant J , where J will be defined later in Eq. (6.25). In ordinary integration, integration domain has no orientation. Area is always positive. Area integration has no algebraic addition. In simplex integration, orientation to each triangle is introduced. Area may be positive or negative. Algebraic addition becomes applicable. In Fig. 6.15, simplex area of triangles $\Delta_{V_0V_1V_2}, \Delta_{V_0V_4V_5}, \Delta_{V_0V_5V_6}$, and $\Delta_{V_0V_6V_7}$ are positive, while simplex area of triangles $\Delta_{V_0V_2V_3}$ and $\Delta_{V_0V_3V_4}$ are negative. It can be easily seen that sum of the simplex areas of all the oriented triangles yields the area of the block.

Eq. (6.20) is further written as

$$\iint_A dx dy = \sum_{k=1}^n \int_{V_0 V_k V_{k+1}} 1 D(x, y) \quad (6.21)$$

where $D(x, y)$ represent the two-dimensional simplex integration, defined as the normal integration $dx dy$ multiplied by the sign of the Jacobi determinant J , where the expression of J will be provided later in Eq. (6.25), n is the number of edges of the arbitrary polygonal block.

Integration over each simplex can be evaluated analytically. Its derivation follows two steps. First, a basic integration in terms of area coordinates, u_0 , u_1 and u_2 over a coordinate simplex with vertices $U_0 (0,0)$, $U_1 (1,0)$, and $U_2 (0,1)$, is obtained as

$$S_1 = \iint_{U_0 U_1 U_2} u_0^{i_0} u_1^{i_1} u_2^{i_2} du_1 du_2 = \frac{i_0! i_1! i_2!}{(i_0 + i_1 + i_2 + 2)!} \quad (6.22)$$

Fig. 6.15 Triangulation of an arbitrary block to evaluate its size with the simplex integration

Second, a general integration in normal coordinate over a simplex with vertices of $V_0 (x_0, y_0)$, $V_k (x_k, y_k)$, and $V_{k+1} (x_{k+1}, y_{k+1})$ is derived by the following coordinate

transformation

$$\begin{aligned} 1 &= u_0 + u_k + u_{k+1} \\ x &= x_0 u_0 + x_k u_k + x_{k+1} u_{k+1} \\ y &= y_0 u_0 + y_k u_k + y_{k+1} u_{k+1} \end{aligned} \quad (6.23)$$

such that

$$\int_{V_0 V_k V_{k+1}} x^{m_1} y^{m_2} D(x, y) = \text{sign}(J) \iint_{V_0 V_k V_{k+1}} x^{m_1} y^{m_2} dx dy = J \iint_{U_0 U_1 U_2} x^{m_1} y^{m_2} du_1 du_2 \quad (6.24)$$

where J is the Jacobi determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u_k} & \frac{\partial x}{\partial u_{k+1}} \\ \frac{\partial y}{\partial u_k} & \frac{\partial y}{\partial u_{k+1}} \end{vmatrix} = \begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_k & y_k \\ 1 & x_{k+1} & y_{k+1} \end{vmatrix} \quad (6.25)$$

Then, Eq. (6.21) can be further written as

$$\begin{aligned} \iint_A dx dy &= \sum_{k=1}^n \int_{V_0 V_k V_{k+1}} 1 D(x, y) = \sum_{k=1}^n J \iint_{U_0 U_1 U_2} du_1 du_2 \\ &= \sum_{k=1}^n J \frac{0!0!0!}{(0+0+0+2)!} = \sum_{k=1}^n \frac{1}{2} J = \sum_{k=1}^n \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_k & y_k \\ 1 & x_{k+1} & y_{k+1} \end{vmatrix} \\ &= \sum_{k=1}^n \frac{1}{2} \begin{vmatrix} x_k & y_k \\ x_{k+1} & y_{k+1} \end{vmatrix} \end{aligned} \quad (6.26)$$

Once the vertices of a block are known, the size of the block can be easily evaluated no matter whatever shape it is. For the block formed by more than one loop, simplex integration is carried out in each loop and then the summation will be the size of the block. Take the block formed by loop 1 and loop 2 in Fig. 6.11 for an example. The size of the block is the summation of simplex integrations of constant function 1 over loop 1 and loop 2.

6.4 Summary

In this chapter, the applicability of the conventional NMM for practical rock failure analysis is firstly investigated. Its limitations are pointed out. Then, a fracturing algorithm is proposed. Computer implementation details including the crack initiation and propagation criterion, treatment of manifold elements, treatment of physical covers, updating physical covers for each manifold element, updating loops for contact detection and modeling during the fracturing process, and evaluation of the rock debris size with the simplex integration method are presented. The Mohr-Coulomb criterion with a tensile cutoff is adopted to predict the crack initiation and propagation. The Mohr-coulomb criterion is able to recover the salient features of the quasi-brittle response within engineering accuracy, including dilation and has the advantage of being based on parameters that can be conveniently determined experimentally. The cover-division strategy rather than the usually adopted element-division strategy is adopted, which makes the mesh dependency be avoided to some extent because of the non-local nature of the stress. In addition, cover-division strategy does not suffer from the limitations of the element-division strategy for crack initiation. The way we propose to treat the manifold elements and the physical covers guarantees the displacement jump across the crack surface and displacement compatibility condition along the common side of two adjacent elements. An algorithm is proposed to update the loops during the crack initiation and propagation, such that the original robust contact detection and modeling techniques can be directly adapted to handle the contact response between the two sides of a crack surface or between two blocks.

CHAPTER 7 CALIBRATION OF THE NMM ENABLING FRACTURING IN CRACK TRAJECTORY PREDICTION

7.1 Introduction

In Chapter 6, a fracturing algorithm based on the Mohr-Coulomb criterion with a tensile cutoff is proposed and implemented into the NMM.

This chapter firstly presents a calibration of the NMM enabling fracturing through several typical crack initiation and propagation problems. Then, through a simple crack propagation example, a comparison between the NMM enabling fracturing and the sub-block DDA method is carried out. The problems which may arise in the sub-block DDA method for fracturing modeling are pointed out. The reasons for us to choose the NMM rather than the sub-block DDA for the practical rock failure analysis are discussed. Lastly, several examples for progressive failures in rock slopes with non-persistent joints are simulated to demonstrate the potentials of our developed program for such problems.

7.2 Calibration of the NMM enabling fracturing

In this section, several examples are presented to calibrate our developed NMM program in crack initiation and propagation modeling.

7.2.1 An edge crack in a finite plate under uniaxial tension

As the first example, a problem with an edge crack in a finite plate under uniaxial tension shown in Fig. 7.1 is investigated. It is a typical mode I crack problem. The crack is expected to propagate straight ahead.

In calculation, the plate size is taken to be $W = H = 1.0$ m. The crack length is taken to be $a = 0.2$ m. The uniform uniaxial tension $\sigma = 6.0$ MPa. The material parameters are 80 GPa for Young's modulus, 0.3 for Poisson's ratio, and 5 MPa for

tensile strength.

Fig. 7.1 An edge fractured plate under uniaxial tension

(a)

(b)

Fig. 7.2 Crack growth process in an edge fractured plate under uniaxial tension

A set of uniform hexagonal mathematical covers are used. Fig. 7.2a shows the initial configuration of the NMM model, while Figs. 7.2b-d give the crack growth process. The crack growth trajectory agrees well with the theoretical prediction.

It should be noted that, when the crack meets the boundary of the plate, the material volumes at the two sides of the crack can completely detach from each other, shown in Fig. 7.2d. The transition from a fractured body to a two-block system is fulfilled.

7.2.2 An inclined crack in a plate under uniaxial tension

A plate with an initial inclined edge crack is submitted to the pure tensile test as shown in Fig. 7.3.

The plate size is taken to be $W = H = 1.0$ m. The crack length is taken to be $a = 0.2$ m in length with an inclination angle θ as 45° . The plate is subjected to a uniform uniaxial tension. The uniform uniaxial tension $\sigma = 6.0$ MPa. The material parameters are 80 GPa for Young's modulus, 0.3 for Poisson's ratio, and 5 MPa for tensile strength.

Due to the mode I loading condition, the crack is theoretically expected to

propagate perpendicular to the maximum principal stress which is in the vertical direction.

Fig. 7.3 An inclined edge crack in a finite plate under uniform uniaxial tension

A uniform hexagonal mathematical cover system generated from a regularly-patterned triangular finite element mesh is used.

Fig. 7.4 Final crack propagation trajectory in the plate with an inclined edge crack

The final crack growth pattern is shown in Fig. 7.4, where the crack propagates towards the preferential growth direction due to mode I loading condition, perpendicular to the maximum principal stress which is in the vertical direction due to the applied pure tension loading.

The NMM calculated propagation trajectory agrees with the theoretical prediction and the previous numerical results by the FEM with auto-remeshing (Bouchard et al., 2003).

7.2.3 Crack in an edge fractured plate with a circular hole under tension.

The NMM gives good results on crack growth trajectories of the last two simple examples. In this example, a little more complex problem of an edge fractured plate with an off-center circular hole is under consideration.

The dimension for the plate is 2 m $\times$ 4m. The crack is located at the left edge of the mid-plane of the plate with the size of 0.2 m. The circular hole has a 0.5 m offset from the mid-plane of the plate and a radius of 0.8 m. The plate is subjected to uniform uniaxial tension at top and bottom edges. The uniaxial tension is $\sigma = 5.5$ MPa. The material parameters are 80 GPa for Young's modulus, 0.3 for Poisson's ratio, and 5 MPa for tensile strength.

The purpose of this example is to investigate the influence of the hole on the crack growth path.

A similar problem have been studied by Rashid (1998) with the proposed arbitrary local mesh replacement method and Bouchard et al. (2000; 2003) with the FEM combining advanced remeshing technique. They found that the crack is moving towards the hole since it creates a stress drop. Once the crack tip moves beyond the hole, the crack reorients horizontally under the mode I loading.

Fig. 7.5a shows the initial configuration of the NMM model for the problem of a plate with an off-center hole. Figs. 7.5b,c give the crack growth process. Similar results have been obtained with the propagating crack attracted by the hole.

Instead of the local mesh refinement in Rashid (1998) and the auto-remeshing in Bouchard et al. (2000; 2003), the NMM uses uniform mathematical covers, thus is more convenient and saves a lot of computational efforts while without losing

accuracy of the simulation results.

Fig. 7.5 Crack growing process in the edge fractured plate containing an offset circular hole under tension

7.2.4 Crack growth from a fillet

This example shows the growth of a crack from a fillet in a structural member. The configuration to be studied is taken from the experimental work carried out by Sumi et al. (1995).

Shown in Fig. 7.6a is experimental setup with the actual domain modeled by the NMM outlined with a dashed line box. The purpose of this example is to investigate the effect of the bending stiffness of the bottom I-beam on the crack growth path. Two extreme cases, i.e. a rigid constraint (to represent very thick bottom beam) and flexible constraint (to represent very thin beam) are considered.

The material is assumed linear elastic with Young's modulus and Poisson's ratio of $E = 200\text{GPa}$, and $\nu = 0.3$, respectively. The applied load is $P = 20\text{ kN}$. Fillet radius is $\rho = 20\text{ mm}$. The initial crack length $a_0 = 5\text{ mm}$.

Fig. 7.6b shows the NMM model for the fillet problem. A regularly-patterned

triangular finite element mesh is adopted to generate the mathematical covers. It is the advantage of the NMM method that a regular finite element mesh can always be used. The meshing task in the NMM is very convenient.

The effects of the stiffness of the bottom I-beam are incorporated into the model through boundary conditions. For the rigid bottom I-beam, the entire bottom edge is fixed while a flexible bottom beam is idealized by fixed the displacement at only two endpoints of the bottom edge.

(a)

(b)

Fig. 7.6 A fillet problem: (a) experimental setup; (b) NMM model

The NMM results in Fig. 7.7 show the influence of the bottom I-beam rigidity on the crack growth path, which are consistent with both experimental results (Sumi et al., 1995) and previous numerical results using the EFG method (Fleming et al., 1997) and the remeshing method (Bouchard et al., 2003).

Fig. 7.7 Crack growth process in a fillet: (a) rigid I-beam; (b) flexible I-beam

7.2.5 Three-point beam bending

The three-point bending flexural test is initially designed to provide values for flexural stress σ_f , flexural strain ε_f , the modulus of elasticity in bending E_f . Assume L as the support span (mm), b as the width of the test beam, d as the depth of the tested beam, P as the concentrated load, D as the maximum deflection of the center of the beam, m as the gradient of the initial straight-line portion of the load

deflection curve. Then, the flexural stress is obtained as

$$\sigma_f = \frac{3PL}{2bd^2} \quad (7.1)$$

The flexural strain is obtained as

$$\varepsilon_f = \frac{6Dd}{L^2} \quad (7.2)$$

The flexural modulus is calculated as

$$E_f = \frac{L^3 m}{4bd^3} \quad (7.3)$$

Here, the three-point beam bending is numerically simulated with the purpose to investigate the crack initiation and propagation trajectory during the loading process.

The initial configuration of the NMM model is depicted in Fig. 7.8a. A simply supported beam is subjected to a concentrated loading at the center point. The left end point of the beam is fixed and the right end point of the beam is located on a rigid block to represent the simply-supported boundary condition. The beam deforms by driving the third central point downwards.

The material properties are: Young's modulus, $E = 10\text{G Pa}$, Poisson's ratio $\nu = 0.2$.

The loading span of the tested beam is 1.0 m, the depth of the tested beam is 0.2 m. The concentrated loading is 520 kN. The tensile strength of the beam material is 10 MPa.

Fig. 7.8a shows the initial configuration of the tested beam, while Figs. 7.8b-d give the crack initiation and propagation process.

Due to symmetry, the centerline of the process zone is straight and the failure mechanism corresponds to pure mode-I fracture. The maximum stress is located at the center.

From Fig. 7.8, it can be seen that the numerically obtained crack initiation and propagation is consistent with the theoretical prediction.

(a) Step 0

(b) Step 13

(c) Step 22

(d) Step 33

Fig. 7.8 Crack initiation and propagation in a three-point loaded beam

7.2.6 Brazilian test

Brazilian test (also known as splitting tensile strength test or diametral compression test) is used to indirectly determine the tensile strength of rock-like brittle materials. A circular cylindrical specimen is compressed along its diameter, failure occurs by an extension fracture in the loaded diametral plane at some value of the applied load, as illustrated in Fig. 7.9. With this load and the dimension of the specimen, the uniaxial tensile strength can be estimated.

In this study, Brazilian test is numerically simulated with the NMM enabling

fracturing.

The diameter of the specimen is 0.2 m. The applied loading is 3600 kN.

The constitutive properties of the materials are: Young's Modulus $E = 20$ GPa, Poisson's ratio $\nu = 0.2$. The tensile strength of the material is set to be 2MPa.

Fig. 7.9 Illustration of the Brazilian test

Fig. 7.10a shows the initial configuration of the NMM model for the Brazilian test. Figs. 7.10b-e show the crack initiation and propagation process. At time step 15, the crack initiates at the center along its diameter. Then, the crack further propagates until it reaches the most upper and lower part of the specimen and finally stops because of the compressive stress there.

The crack initiation and propagation trajectory is consistent with the theoretical prediction.

(a) Step 0

(b) Step 15

(c) Step 21

(d) Step 51

(e) Step 81

Fig. 7.10 Crack initiation and propagation in a diametrically loaded disc obtained from the NMM enabling fracturing

7.3 Comparison between the NMM and the sub-block DDA method in fracturing modeling

The original DDA simulates each discrete block with only six degrees of freedom (DOFs), (i.e. normal and shear strains of the block, rigid body translation of a specified point within the block, and the rotation angle of the block with the rotation center at the specified point within the block) in a two-dimensional setting, which results in a linear displacement field and a constant stress field within each block. Such an approximation may not be suitable for problems in which stress or displacement variations within each block are important and/or fracturing within each block is desired. In order to resolve the stress variations within each block and to account for the possible fracturing within each block, various modifications to the DDA have been proposed.

For example, each block can be discretized into a finite element mesh or a finite difference mesh. Each mesh line is considered as a potential crack surface. Once a given criterion is met, a potential crack surface will convert to a real crack surface.

Similarly, each block can be discretized into a finite number of sub-blocks. The sub-block boundaries are treated as artificial joints. Sub-blocks are connected together to represent a single block through contacts along the artificial joints. The

continuity condition within each block is preserved through assigning properly designed parameters for each artificially joint. Again, once a given criterion (say the Mohr-Coulomb criterion with a tensile cutoff) is satisfied, an artificial joint will convert to a real joint.

The NMM can also be viewed as a modification to the DDA by discretizing each block into a finite number of covers. The stress variation within each block can be well resolved. Coupled with the fracturing algorithm proposed in Chapter 6, fracturing process within each block can be simulated.

The first three modifications to the original DDA (i.e. discretizing each block by a finite element mesh or a finite difference mesh or a set of sub-blocks) share some similarities. For example, in all the three methods, the crack opening and sliding can only occur along the pre-defined mesh line or the sub-block boundaries. All the three methods eliminate completely the need of remeshing. However, all the three methods suffer from the same problem, i.e. the crack propagation trajectory is mesh-dependent. Due to these similarities, this study only takes an examination on the DDA method with each block discretized into a set of sub-blocks (or the sub-block DDA method in short). The sub-block DDA method is compared with the NMM enabling fracturing in fracturing modeling. The reasons for us to choose the NMM enabling fracturing rather than the sub-block DDA for the practical rock failure analysis are discussed.

For easy comparison, a simple problem with an edge crack in a finite plate under uniaxial tension shown in Fig. 7.1 is investigated.

It is a typical mode I crack problem. The crack is expected to propagate straight ahead.

In the calculation, the plate size is taken to be $W = H = 1.0$ m, crack length $a = 0.2$ m. The uniform uniaxial tension $\sigma = 6.0$ MPa.

The material parameters are 80 GPa for Young's modulus, 0.3 for Poisson's ratio.

7.3.1 DDA with sub-block algorithm for fracturing modeling

In this sub-section, modeling the edge crack problem in Fig. 7.1 with the sub-block DDA is presented.

In the simulations, it is found that a parameter called the maximum displacement ratio (equivalent to the time interval) greatly influences the final crack propagation trajectory. Thus this parameter must be carefully chosen. In each simulation, the maximum displacement ratio is gradually reduced until a consistent crack propagation trajectory can be obtained.

Fig. 7.11a shows one DDA model for the edge crack problem.

Fig. 7.11 DDA predicted crack growth process in an edge fractured plate under uniaxial tension, sub-block configuration 1 (sub-block size = 0.1m)

The frictional angle, the cohesion and the tensile strength of the artificial joints are set to 35° , 4 MPa, and 0.6 MPa, respectively. The maximum displacement ratio is set to 0.0001. The penalty value of contact spring is set to 400 GPa.

Figs. 7.11b-d give the crack propagation process. Due to mode I loading, the crack is expected to propagate straight ahead. In the DDA simulation, crack opening can only occur along the pre-defined artificial joints. When all the parameters are chosen properly, the crack propagates in a path which is closest to the theoretical solution among all the possible crack growth paths. However, when the sub-block size is relatively large, the DDA predicted crack propagation trajectory is far away from the theoretical prediction.

The crack propagation trajectory is sensitive to the sub-block configuration. Even with the same sub-block size, different sub-block configurations may lead to different crack propagation trajectories.

In order to investigate the sub-block configuration effect, except the sub-block configuration (numbered as configuration 1) used in Fig. 7.11, three more sub-block configurations are designed and shown in Figs. 7.12b-d.

The maximum displacement ratio for the sub-block configurations 2, 3, 4 is 0.00003. The final crack propagation trajectories of the four cases are compared in Fig. 7.12.

Fig. 7.12 Comparison of the crack propagation trajectories of different DDA models with different sub-block configurations: (a) configuration 1; (b) configuration 2; (c) configuration 3; (d) configuration 4 (sub-block size = 0.1m)

In all the four cases, the cracks propagate along pre-defined artificial joints. Their crack trajectories differ from each other due to the different sub-block configurations. Since the sub-block size is large, the DDA predicted crack trajectories in the four cases deviate from the theoretical prediction.

In order to improve the simulation results, one possible way is to decrease the sub-block size. In order to check the sub-block size effect, the DDA model with sub-block configuration 4 but a different sub-block size 0.06 m is simulated. The final crack propagation trajectories for the two cases are compared in Fig. 7.13. The comparison indicates that smaller sub-block size will lead to a better result. Theoretically speaking, when the sub-block size is infinitely small, the DDA predicted crack propagation trajectory will be consistent with the theoretical prediction. However, refinement of the sub-blocks implies the increase of the computational cost.

In order to save the computational time, one possible way is to locally refine the sub-blocks. Still take the configuration 4 for an example. The size of the sub-blocks in the region near the crack and its potential propagation path is reduced from 0.1 m to 0.05 m. The comparison between the crack propagation trajectories before and after

local refinement is shown in Fig. 7.14.

Fig. 7.13 Comparison of the crack propagation trajectories of different DDA models with different sub-block sizes, sub-block configuration 4: (a) sub-block size = 0.1m; (b) sub-block size = 0.06m

Fig. 7.14 Comparison of the crack propagation trajectories of different DDA models before and after local refinements, sub-block configuration 4: (a) before local refinement; (b) after local refinement

The comparison in Fig. 7.14 shows that the local refinement can improve the DDA predicted crack trajectory and also save the computational time compared with the refinement of the whole model.

(a)

(b)

Fig. 7.15 DDA model with a biased sub-block configuration and the final crack propagation trajectory

Actually, if the potential crack propagation path is prior known, then a sub-block configuration that is biased in the sense that the sub-block sides are pre-aligned with

the anticipated propagation path can be adopted. A DDA model with such a biased sub-block configuration is shown in Fig. 7.15a. Its final crack propagation path is given in Fig. 7.15b. The crack propagation trajectory obtained from the DDA model with a sub-block configuration that is biased in the sense that the sub-block sides are pre-aligned with the expected propagation path is exactly same as the theoretical prediction.

7.3.2 NMM for fracturing modeling

In this sub-section, modeling the edge crack problem in Fig. 7.1 with the NMM enabling fracturing is presented.

The edge crack problem shown in Fig. 7.1 is a typical mode I crack problem. Theoretically, the edge crack should propagate straight ahead under the mode I loading condition.

In the NMM, a finite element mesh is usually adopted to generate the mathematical covers. Different from that used in the conventional FEM, the finite element mesh in the NMM does not need to conform to neither the external boundaries nor the internal cracks. A regular finite element mesh is always applicable in the NMM, which makes the meshing in the NMM very convenient.

In all the simulations in this sub-section, a regularly-patterned triangular finite element mesh is adopted to generate the mathematical covers. The mathematical cover system is rotated and/or vertically moved to generate different NMM models. The numerically obtained crack trajectories are compared with the theoretical predication. The influence of rotation and translation of the mathematical cover system on the final crack trajectories are discussed.

Fig. 7.16a shows one NMM model (with mesh 1) for the edge crack problem in Fig. 7.1. In this case, the maximum displacement ratio is set to 0.0001. Figs. 7.16b-d give the crack propagation process predicted by the NMM enabling fracturing. The crack propagates straight ahead under mode I loading condition. The NMM predicted crack trajectory is consistent with the theoretical prediction.

Fig. 7.16 NMM predicted crack growth process in an edge fractured plate under uniaxial tension, mesh 1

Then, we move the mathematical cover system in the vertical direction until it coincides with the crack. This mesh configuration is numbered as mesh 2. The NMM model generated from mesh 2 is given in Fig. 7.17a, while the numerically predicted crack propagation trajectory is shown in Fig. 7.17b. Again, the crack propagates straight ahead, consistent with the theoretical prediction.

In this case, we design the mesh in such a way that the finite element mesh coincides with the crack surface. However, it is actually different from the DDA model in Fig. 7.15. In the sub-block DDA method, the crack opening only occurs along the pre-defined artificial joints. Different sub-block topologies result in different crack propagation trajectories. If the sub-block system is designed in such a way that the sub-block sides coincide with the anticipated crack propagation path, as in Fig. 7.15, the crack will propagate along this pre-defined path. In the NMM, crack opening is modeled by splitting covers. In Fig. 7.17b, the crack propagates straight ahead along the mesh, which is not due to our pre-defined mesh topology but because of the stress status of the physical covers.

Fig. 7.17 NMM predicted crack propagation trajectory in an edge fractured plate under uniaxial tension, mesh 2: (a) initial configuration; (b) final crack propagation trajectory

Then, the mathematical cover system is rotated by 90° , which is numbered as mesh 3. The NMM models generated from mesh 3 is shown in Fig. 7.18a, while the corresponding crack propagation trajectory is given in Fig. 7.18b. Again, the crack propagates in the horizontal direction, which agrees well with the theoretical prediction.

Fig. 7.18 NMM predicted crack propagation trajectory in an edge fractured plate under uniaxial tension, mesh 3: (a) initial configuration; (b) final configuration

Then, mesh 3 is further moved vertically. This mesh is numbered as mesh 4. The NMM model generated from mesh 4 is shown in Fig. 7.19a. The final crack propagation trajectory is given in Fig. 7.19b, which is also consistent with the theoretical solution.

Fig. 7.19 NMM predicted crack propagation trajectory in an edge fractured plate under uniaxial tension, mesh 4: (a) initial configuration; (b) final configuration

7.3.3 Comparison between the NMM and the sub-block DDA

Modeling of a simple edge crack problem with the sub-block DDA and the NMM have been presented in the last two sub-sections, respectively. In this sub-section, a comparison between the sub-block DDA and the NMM in fracturing modeling is carried out. The applicability of these two methods for practical rock failure analysis is discussed.

In the sub-block DDA, each block is discretizing into a finite number of small sub-blocks. Sub-block boundaries are treated as artificial joints. All the sub-blocks are connected together to represent the whole block through contacts at the artificial joints. The continuity condition within each block is enforced by assigning each artificial joint proper parameters. By such a way, the displacement field and stress field within each block can be well resolved. In addition, fracturing in each block can be fulfilled by converting the artificial joints into real crack surfaces once a given criterion is satisfied. Both crack opening and crack sliding can be modeled. Both crack initiation and propagation are possible. However, crack opening and sliding can only occur along the pre-defined artificial joints. Such an assumption completely avoids the need for remeshing. However, it suffers from some limitations. The numerically obtained crack propagation trajectory is obviously dependent on the sub-block topology. Different sub-block configurations will lead to different crack propagation trajectories. When the sub-block system is coarse, the numerically obtained crack trajectory may be far away from the theoretical prediction. Only when a high-density sub-block system is employed, a satisfactory result can be obtained. However, refinement of the sub-blocks implies the increase of the computational cost. In order to save the computational cost, a local refinement can be adopted. However, it is only applicable to the cases where the possible crack trajectory is known. In practical rock failure analysis, due to the complexities of both the problem geometries and loading conditions, the possible trajectories are hard to known beforehand. Under such circumstances, auto-refinement near the crack tips is desired, which obviously complicates the simulation. Another way to get a satisfactory result is to use a sub-block system which is biased in the sense that the sub-block sides are pre-aligned

with the anticipated propagation path. However, it is also only applicable to the cases where the exact crack trajectory is known. It is obviously impossible for the practical rock failure process.

The NMM can also be viewed as a modification to the original DDA. Instead of using sub-blocks, the NMM uses a set of covers to resolve the displacement field and stress field in each block. The fracturing algorithm proposed in Chapter 6 can be adopted to predict the crack initiation and propagation and finally the transition from continua to discontinua. As indicated in Section 7.3.2, the NMM obtained crack propagation trajectory is not sensitive to the mesh topology. Rotation and the translation of the mathematical cover system have little influence on numerical results. Thus, the NMM enabling fracturing is more suitable for the practical rock failure analysis than the sub-block DDA method.

7.4 Modeling progressive failure in rock slopes involving non-persistent joints

Discontinuity (or referred to as joint) persistence is one of the most important parameters affecting the rock failures (Einstein et al, 1983). However, in the traditional rock engineering analyses, persistent discontinuities are often assumed. It is part due to post-failure investigations that fully persistent discontinuities are consistent with the failure surface, but mainly due to the constraints of the analysis techniques employed. For example, many analysis techniques, such as limit equilibrium wedge or planar analysis, distinct element code UDEC (Itasca, 2004), discontinuous deformation analysis (DDA) (Shi, 1988), etc, require the input of fully persistent discontinuities.

Such an assumption is only valid in cases where major persistent faults and/or bedding planes are present. However, in massive natural rock slopes and deep, engineered slopes (e.g. open pit mines), it is highly unlikely that such a network of fully persistent discontinuities exist.

Many researchers (Terzaghi, 1962; Robertson, 1970; Einstein et al, 1983) suggest that the persistence of key discontinuity sets is in reality more limited and that a

complex interaction between existing natural discontinuities and brittle fracture propagation through intact rock bridges is required to bring the slope to failure.

In contrast to the sudden failure happened in rock mass with persistent discontinuities, the failure in rock mass with non-persistent discontinuities is termed as progressive failure. The key to progressive failure is the propagation of fractures through intact rock bridges between existing discontinuities.

The NMM enabling fracturing is extremely suitable for progressive failure analysis because: (1) regular mathematical cover system is always applicable, which makes the meshing very convenient and remeshing be totally avoided for fracturing; (2) its two sets of covers make it convenient to describe complex cracks; (3) the fracturing algorithm proposed in Chapter 6 and verified in this chapter makes the transition from continua to discontinua possible; (4) its mature contact logic makes the interactions among the finally generated discrete block system be modeled in a realistic manner.

In this section, the NMM enabling fracturing is applied to investigate the progressive failure in rock slopes with non-persistent joints.

The intact rock material is assumed elastic with the material properties are given as Young's Modulus $E = 20$ GPa, Poisson's ratio $\nu = 0.25$, tensile strength $T = 1$ MPa.

The pre-existing joints are assumed as contact surfaces with zero-thickness. The friction angle $\phi = 10^\circ$, tensile strength and cohesion of the joints are assumed as zero.

In the simulation, only gravitational loading is applied to the footwall slopes. The gravitational loading is increased by 10 times to bring the slope to failure. It is equivalent to the strength reduction method usually adopted by various researchers. However, the current approach can greatly save computational time.

The fracturing algorithm based on the Mohr-Coulomb criterion with a tensile cutoff presented in Chapter 6 and verified in this chapter is adopted to predict the fracturing in intact rock bridge. Both tensile failure and shear failure can be captured.

7.4.1 Step-path failure

Step-path failure has been undertaken in the past using a combination of limit

equilibrium and fracture mechanics theory (Stead et al., 2006). The use of the NMM combined with a fracturing algorithm seems particularly promising for the simulation of step-path geometries.

Fig. 7.20 shows a rock slope with three parallel joints. A translational failure of this slope requires fracturing in intact rock bridges between two discrete joints to allow kinematic release.

Fig. 7.20 A slope with one joint set comprising of three non-persistent joints

Here, a regularly-patterned triangular finite element mesh is adopted to generate the NMM models for the problem in Fig. 7.20, as shown in Fig. 7.21. In the simulation, the mesh size must be small enough to capture the displacement/stress field well, but cannot too small otherwise the simulation cannot be finished in a reasonable time.

Fig. 7.21 NMM model for the problem in Fig. 7.20

Fig. 7.22 depicts the slope failure process obtained from the NMM combined with fracturing algorithm.

From Fig. 7.22, it is clearly seen that excess shearing stresses along a discontinuity result in the propagation of orthogonal tensile fractures until eventually cross-over fractures provide a continuous, but stepped failure surface and overall failure occurs.

It is evident that joint persistence, spacing, frictional angle and cohesion of joint, and tensile strength of intact rock all influence the step-path failure generation. However, a thorough parametric study with regarding to all these parameters is beyond the scope of this study.

(a) Step 0

(b) Step 40

(c) Step 50

(d) Step 200

(e) Step 400

Fig. 7.22 Step-path failure modeled by the NMM enabling fracturing

Fig. 7.23a shows another rock slope. Except the predominant joint set in Fig. 7.20, another joint set with various non-persistent joints are introduced.

Again, a regularly-patterned mathematical cover system is adopted. Since the NMM model is similar to the one shown in Fig. 7.21, it is omitted from here to save pages.

Figs. 7.23b-e give the failure process of the slope. The propagation of orthogonal tensile fractures eventually form a continuous but stepped failure surface along the predominant joint set.

(a) Step 0

(b) Step 100

(c) Step 200

(d) Step 400

(e) Step 600

Fig. 7.23 A more complex step-path failure modeled by the NMM enabling fracturing

But different from Fig. 7.22, the propagations of the non-persistent joints in the

other joint set divide the whole rock block into several fragments. The sizes of all the rock fragments are evaluated by the virtue of simplex integration method and listed in Fig. 7.24.

Fig. 7.24 Rock fragments generated from the step failed slope and their sizes

7.4.2 Bilinear failure

Bilinear failures form the most common mechanism in slope failures and have also been described frequently (Stead et al, 1997). The bilinear failure surface usually comprises an upper failure surface either along bedding or along a major structural plane of weakness. The lower failure plane or cross-over surface may be low angle delighting thrust plane, low angle plane dipping away from the excavation, or intact

rock shear at the slope toe.

Fig. 7.25a shows a slope with a persistent bilinear failure surface and another joint set with various non-persistent joints. Again, a regularly-patterned mathematical cover system is adopted to generate the NMM model. Fig. 7.25b-d shows the slope failure process. The massive slope moves downward along the pre-defined upper failure surface and the lower failure surface. At the same time, the non-persistent joints propagate and finally coalesce with other joints or the bi-linear failure surface and divide the slope into several fragments.

(a) Step 0

(b) Step 100

(c) Step 300

(d) Step 600

(e) Step 1000

Fig. 7.25 Bi-linear slope failure modeled by the NMM enabling fracturing

The final generated discrete blocks and their sizes are listed in Fig. 7.26.

Fig. 7.26 Rock fragments generated from the bilinear failed slope and their sizes

7.5 Summary

In this chapter, the NMM enabling fracturing is firstly calibrated through several typical crack initiation and propagation examples. The effectiveness and efficiency of the NMM enabling fracturing are demonstrated. Then, the NMM enabling fracturing and the sub-block DDA are compared in fracturing modeling. Both methods can be viewed as modifications to the original DDA method. Both methods can resolve the displacement field and stress field within each block and can allow fracturing in each block. However, crack propagation trajectory predicted by the sub-block DDA is highly mesh dependent. In contrast, the NMM does not suffer from such a limitation, thus is more suitable for the practical rock failure analysis. Lastly, the NMM enabling fracturing is applied to simulate the progressive failure in rock with non-persistent joints. Numerical results indicate that the NMM enabling fracturing is able to capture the fracturing in intact rock bridge and finally allow the kinematic release. The NMM enabling fracturing is promising for such problems.

CHAPTER 8 NMM ENABLING FRACTURING FOR FOOTWALL SLOPE FAILURE IN SURFACE COAL MINES

8.1 Introduction

Footwall slope instability frequently occurs in surface coal mines situated in areas of steeply dipping strata (Stead and Eberhardt, 1997). An example is given in Fig. 8.1, which is a footwall slope failure occurred at the Line Creek Mine in British Columbia, Canada. The failed sandstone slab is 7.5-9.0 m in thickness and up to 300 m in height. The dip of the bedding is about 50° . The failure is progressive. The main slab failure lasts for 3 days, followed by minor rockfalls for 2 days.

Fig. 8.1 Footwall failure at a Western Canadian coal mine, after Stead and Eberhardt (1997)

Footwall slopes are often excavated parallel to the dip, which results in extensive slope lengths along both strike and down-dip. The mining continues down-dip, which increases the length of the exposed footwall slope and leads to potential instability problems in the thinly bedded strata. The importance of footwall slope instability has long been recognized (Stead and Eberhardt, 1997).

Limit equilibrium techniques have been successfully undertaken for routine

surface coal mine design. However, such methods are limited especially in the aspect of the degree of complexity that can be incorporated into the analysis in both problem geometry and material behavior. They only allow a static analysis of footwall slab movement along the pre-defined discontinuities but fail to represent the intact rock deformation which is also essential to the final slope failure. For complex problems involving both sliding along pre-existing joints and intact rock deformation and/or fracturing, the use of numerical techniques may provide an additional tool.

The finite element method (FEM) has been widely applied to open pit coal mining with the assumption of a continuum and constitutive criteria. The spoil pile stability and the influence of subsidence and old workings on the footwall slope deformation have been investigated with the FEM (Stead and Eberhardt, 1997).

The finite difference program FLAC (Itasca, 1993a) has also been applied to analyze the potential failure mechanisms in footwall slopes by Eberhardt and Stead (1993). The footwall slab was assumed continuous. The excavation of the slope was simulated by removing elements representing overburden in specified lifts such that the length of the exposed footwall slab was gradually increased. Parametric studies regarding to bed thickness and dip were carried out. Both elastic constitutive criterion and elasto-plastic Mohr-Coulomb criterion were assumed. Deformation of the footwall slab was obtained. Failure mechanism was analyzed.

Although the finite difference program FLAC also has the ability to model the discrete discontinuities using 'interfaces' along which normal and shear stiffness of joints are assumed, modeling jointed footwall slope is more appropriately undertaken using discontinuum techniques such as the distinct element code UDEC. Comparatively, the UDEC is more applicable in analysis of surface coal mine slopes since it directly represents the jointed footwall slopes using discrete blocks. The blocks can be rigid or deformable by discretized into a finite difference mesh. Various constitutive models can be adopted to analyze movements along discontinuities and deformations of blocks. Displacements along joints are represented by spring-slider systems. Using the UDEC (Itasca, 1993b), Stead and Eberhardt (1997) has reproduced all the major failure mechanisms observed in surface coal mine footwall slopes. However, UDEC also suffers from some limitations. As can be seen from the two

examples in Figs. 8.2 and 8.3, the UDEC can only describe the joint-bounded blocks. The joints are always assumed persistent, while the relative sliding occurs along the pre-defined discontinuities. Fracturing in the intact rock is not allowed.

Fig. 8.2 Distinct element analysis of a UK bilinear footwall slope instability, after Stead and Eberhardt (1997)

Fig. 8.3 Distinct element analysis of a ploughing instability, curved slope, after Stead and Eberhardt (1997)

In this chapter, the NMM enabling fracturing proposed in Chapter 6 and verified in Chapter 7 is applied to investigate the potential failure mechanisms of footwall slopes with non-persistent joints.

The remainder of the chapter is organized as follows. The major failure mechanisms of footwall slopes are briefly reviewed in Section 8.2. Modeling the footwall slope failures with the NMM enabling fracturing is presented in Section 8.3. Parametric studies regarding to the orientation of the bedding plane, orientation of the joint set, slope profile, joint persistence, etc, are carried out. Section 8.4 summarizes the main work of this chapter.

8.2 Failure mechanisms of footwall slopes in surface coal mines

The predominant footwall failure mechanisms have been summarized by Hawley et al. (1986), Cavers et al. (1986), Walton and Atkinson (1978), Walton and Coates (1980), and Serra de Renobales (1987). The major footwall failure mechanisms are briefly reviewed in the following part.

Bilinear slab failure (Fig. 8.4). The bilinear failure surface usually comprises an upper failure surface either along bedding or along a major structural weak plane. The bilinear slab failure may be further subdivided into the following types according to their low failure surface: (1) Type I (Fig. 8.4a), failure along low angle basal surface dipping into cut; active-passive mechanism; (2) Type II (Fig. 8.4b), failure along low angle planes dipping away from the excavation; quasi-rotational mechanism; (3) Type III (Fig. 8.4c), failure involves toe-release through shear of intact rock/joints or weak materials; (4) Type IV (Fig. 8.4d), complex failures involving translation, buckling and/or ploughing.

Fig. 8.4 Mechanisms of bilinear slab failure in footwall slopes: (a) Type I; (b) Type II; (c) Type III; (d) Type IV, after Stead and Eberhardt (1997)

Ploughing (Fig. 8.5). Failure occurs due to sliding along the bedding plane with toe release by a ploughing action along steeply inward-dipping joints sub-parallel to the slope. The passive toe block is lifted and rotated out of the footwall slope due to the movement of the active or driving footwall slab.

Fig. 8.5 Ploughing failure in footwall slopes, after Stead and Eberhardt (1997)

Buckling. Illustration of the buckling failure is illustrated in Fig. 8.6.

Fig. 8.6 Buckling failure in footwall slopes: (a) planar footwall; (b) curved footwall, after Stead and Eberhardt (1997)

Fig. 8.7 Step-path failure in footwall slopes, after Stead and Eberhardt (1997)

Step-path (Fig. 8.7). This type of failure mechanism may involve sliding on varying joint set orientations, separation along sub-vertical joints and fracturing in intact rock bridges.

Planar failure (Fig. 8.8). Planar failures rarely occur in practice.

Fig. 8.8 Planar failure in footwall slopes, after Stead and Eberhardt (1997)

Detailed description of the major failure mechanisms and the key factors influencing footwall slope instability in surface coal mining can be found in Stead and Eberhardt (1997).

This study will focus on the application of the NMM enabling fracturing to reproduce the major failure mechanisms of footwall slopes observed in surface coal mines.

8.3 Modeling potential failures of footwall slopes with the NMM

8.3.1 Construction of NMM models of footwall slopes

The intact rock material is assumed elastic with the material properties are given as Young's Modulus $E = 20$ GPa, Poisson's ratio $\nu = 0.25$, tensile strength $T = 1$ MPa.

The pre-existing joints are assumed as contact surfaces with zero-thickness. Their frictional angles are set to 10° . The tensile strength and cohesion of the joints are assumed as zero.

In the simulation, only gravitational loading is applied to the footwall slopes. The gravitational loading is increased by 10 times to bring the footwall slope to failure. It

is equivalent to the strength reduction method usually adopted by various researchers. However, the current approach can greatly save computational time.

A regularly-patterned mathematical cover system is adopted in the NMM models of footwall slopes. In the simulation, the cover size must be small enough to capture the displacement/stress field well, but cannot be too small otherwise the simulation cannot be finished in a reasonable time. To save pages, only the NMM model of one case is depicted in Fig. 8.9 for illustration purpose.

Fig. 8.9 Illustration of the mathematical cover system in the NMM models of footwall slopes

The fracturing algorithm has been fully described in Chapter 6. The crack propagation is based Mohr-Coulomb criterion with a tensile cutoff. Both tensile failure and shear failure can be captured.

The footwall slope failure may be sensitive to various parameters such as joint shear stiffness, joint friction angle, joint cohesion, deformation properties of intact rock material, in situ stresses, groundwater pressures, etc. An exhaustive parametric study of the factors influencing the footwall failure is beyond the scope of the present

study. Although considerable effort such as the installation of drainage measures is often exerted to ensure that instability is not promoted by high pore water pressures (Stead and Eberhardt, 1997), in the analyses presented here the footwall slopes are assumed dry.

The parameters under investigation only include the geometry-related parameters, such as the orientation of the bedding plane, orientation of the predominant joint set, slope profile, joint persistence, etc. Their influences on the footwall failure mechanisms are investigated in the following few sub-sections.

In this study, the non-persistent joints rather than the persistent joints usually adopted in the UDEC and the DDA are assumed.

8.3.2 Influence of the orientation of bedding plane

Surface coal mining, particularly in the areas of mountainous topography, often involves the formation of extensive footwall slopes parallel to the strata dip. The stability of the thinly bedded strata may be highly influenced by the bedding dip (or the orientation of the bedding plane). Thus, the first parameter under investigation is the orientation of the bedding plane. Its influence on the failure mechanism of footwall slopes in surface coal mines is analyzed.

In order to do so, two groups of cases are designed. The first group of cases has the angle from the x-axis to the bedding plane as 81° , less than 90° . The other group of cases has the angle from the x-axis to the bedding plane as 132° , larger than 90° .

For easy interpretation, the angle from the x-axis to the bedding plane anticlockwise is denoted as θ_1 .

Through a series of numerical experiments, we find that the orientation of bedding plane has a significant influence on the failure modes of the footwall slopes. When θ_1 is less than 90° , toppling is observed.

Configuration of one case in the first group (numbered as case 01) is illustrated in Fig. 8.10.

Fig. 8.10 Configuration of case 01, $\theta_1 = 81^\circ$

The failure process of case 01 is depicted in Fig. 8.11. Topping failure mechanism is clearly observed. Firstly, cracks propagate from the tips of pre-existing non-persistent joints. Then, cracks coalesce with the bedding plane and forms several discrete blocks (Fig. 8.11b). The top block topples down. Large movements among discrete blocks are observed (Fig. 8.11c). Some cracks further propagate in each discrete block and divide it into several sub-blocks (Fig. 8.11d, e). The movement of each discrete block continues until its velocity becomes zero and finally the discrete block system settle down.

(a) Step 0

(b) Step 501

(c) Step 612

(d) Step 692

(e) Step 854

Fig. 8.11 Toppling failure process of case 01, $\theta_1 = 81^\circ$

Change of the predominant joint set, toppling may occur at different positions. Another case (numbered as case 02) with $\theta_1 = 81^\circ$ but different joint set from case 01 is simulated. Fig. 8.12a shows the initial configuration of case 02, while Figs. 8.12b-d give the crack propagation and the final toppling process. The toppling occurs at the toe of the slope in this case.

(a) Step 0

(b) Step 170

(c) Step 403

Fig. 8.12 Toppling failure process of case 02, $\theta_1 = 81^\circ$

8.3.3 Influence of the orientation of the predominant joint set

During the case studies in the last section, we find that when θ_1 is less than 90° , toppling failure mechanism can be observed. However, when θ_1 is larger than 90° , the failure mechanisms of the footwall slopes are not fixed, but change with the variation of the orientation of the predominant joint set. Thus, in this section, the influence of the orientation of the predominant joint set on the failure mechanisms of the footwall slopes is investigated.

The orientation of bedding plane is fixed as a constant $\theta_1 = 132^\circ$. Orientation of the predominant joint set varies with regard to the bedding plane. In the current study, five different orientations are considered with the angle from the bedding plane to the joint set anticlockwise as 23° , 36° , 86° (sub-vertical), 130° , 144° , respectively, as illustrated in Fig. 8.13.

Fig. 8.13 Variation of the joint set orientation with respect to the bedding plane, θ_2 is the angle from the bedding plane to the joint set anticlockwise

For easy interpretation, the angle from the bedding plane to the joint set anticlockwise is denoted as θ_2 .

With the five orientations of the joint sets as shown in Fig. 8.13, five cases are designed and numbered as cases 03 to 08.

The configuration of case 03 with $\theta_1 = 132^\circ$ and $\theta_2 = 23^\circ$ is depicted in Fig.

8.14.

Fig. 8.14 Configuration of case 03, $\theta_1 = 132^\circ$, $\theta_2 = 23^\circ$

The failure process of case 03 is given in Fig. 8.15. Fig. 8.15a shows the initial configuration of case 03. Figs. 8.15b-e give the failure process. Firstly, cracks propagate from the tips of pre-existing non-persistent joints (Fig. 8.15b). Then, propagated cracks coalesce with the bedding plane and form several discrete blocks (Fig. 8.15b). The bedding plane works as the upper failure surface, while the propagated joint plane acts as the lower failure surface. A bilinear failure mechanism is obtained (Figs. 8.15c, d, e).

(a) Step 0

(b) Step 29

(c) Step 101

(d) Step 201

(e) Step 301

Fig. 8.15 Bilinear failure mechanism for case 03, $\theta_1 = 132^\circ$, $\theta_2 = 23^\circ$

Case 04 has the same footwall slope profile $\theta_1 = 132^\circ$, but a different joint set orientation $\theta_2 = 36^\circ$.

The failure process of case 04 is depicted in Fig. 8.16. Fig. 8.16a shows the initial configuration of the footwall slope. Figs. 8.16b-e give the failure evolution. Firstly, cracks propagate from the tips of pre-existing non-persistent joints (Fig. 8.16b). Then, propagated cracks coalesce with the bedding plane and form several discrete blocks (Fig. 8.16c). Taking the bedding plane as the upper failure surface and the propagated joint plane as the lower failure surface, bilinear failure mechanism is observed in Figs. 8.16c-e.

(a) Step 0

(b) Step 20

(c) Step 100

(d) Step 200

(e) Step 300

Fig. 8.16 Bilinear failure process of case 04, $\theta_1 = 132^\circ$, $\theta_2 = 36^\circ$

Keep the slope profile same as the cases 03 and 04, we further change the orientation of the joint set to $\theta_2 = 86^\circ$ (i.e. the predominant joint set is sub-vertical to the bedding plane). This case is numbered as case 05.

The failure evolution process of case 05 is given in Fig. 8.17. Fig. 8.17a shows the initial configuration of case 05, while Fig. 8.17b gives the final configuration of the footwall slope after a sufficient time. No obvious failure is observed.

(a) Step 0

(b) Step 3000

Fig. 8.17 Failure process of case 05, $\theta_1 = 132^\circ$, $\theta_2 = 36^\circ$

Keeping the slope profile same as the cases 03, 04 and 05, the orientation of the joint set is further changed to $\theta_2 = 130^\circ$. This case is numbered as case 06.

Fig. 8.18a shows the initial configuration of case 06, while the Figs. 8.18b-e give the failure process of the corresponding footwall slope. The joint marked by the circle is pre-defined as persistent. With the upper footwall slab above this persistent joint slides downward along the bedding plane, the lower (or passive) block is gradually lifted and rotated out of the footwall slope, as shown in Figs. 8.18b-e. A typical ploughing failure mechanism is reproduced.

(a) Step 0

(b) Step 80

(c) Step 150

(d) Step 247

(e) Step 342

Fig. 8.18 Ploughing failure process of case 06, $\theta_1 = 132^\circ$, $\theta_2 = 130^\circ$

Keeping the footwall slope profile same as in the cases 03, 04, 05, 06, the orientation of the predominant joint set is further increased to $\theta_2 = 144^\circ$. This case is numbered as case 07.

(a) Step 0

(b) Step 53

(c) Step 100

(d) Step 116

Fig. 8.19 Ploughing failure mechanism of case 07, $\theta_1 = 132^\circ$, $\theta_2 = 144^\circ$

Fig. 8.19a shows the initial configuration of case 07, while the Fig. 8.19b-d gives the failure process. Again, with the upper footwall slab sliding downward along the bedding plane, the lower footwall slab is gradually lifted and rotated out of the footwall. Typical ploughing failure mechanism is obtained.

In this section, five different cases with various different orientations of predominant joint sets with θ_2 as 23° , 36° , 86° , 130° , 144° , respectively, are investigated. The failure mechanisms observed in the five cases are compared in Fig. 8.20.

It can be clearly seen that the orientation of predominant joint set has a significant influence on the failure patterns. When θ_2 is less than 90° (i.e. case 03 and case 04, shown in Figs. 8.20a, b), a bilinear failure mechanism is observed. When θ_2 is larger than 90° (i.e. case 05 and case 06, shown in Figs. 8.20d, e), a ploughing failure mechanism is observed. When θ_2 is around 90° (i.e. the predominant joint set is sub-vertical to the bedding plane; case 05, shown in Fig. 8.20c), no obvious failure is observed. Actually, with the joint set sub-vertical to the bedding plane, a buckling failure mechanism is expected. Thus, in the next sub-section, we will focus on the cases where the joint set is sub-vertical to the bedding plane and try to figure out the key factors which influence the buckling failure mechanism.

Fig. 8.20 Comparison of the failure modes of cases 03 to 07 with same θ_1 as 132° , but different θ_2 as 23° , 36° , 86° , 130° , 144° , respectively

8.3.4 Factors influencing the buckling failure mechanism

With the joint set sub-vertical to the bedding plane, buckling is expected to occur. However, in the studies in the last sub-section, no obvious failure is observed. Thus, in this section, we change various parameters to check their influences on the buckling failure mechanism.

Firstly, we decrease the orientation of the bedding plane θ_1 from 132° to 112° (i.e. increases the bedding dip from 48° to 68°) and decreases the bed thickness from 0.74 m to 0.46 m. This case is numbered as case 08. Fig. 8.21a gives the initial configuration of case 08, while Figs. 8.21b-d give the failure process.

(a) Step 0

(b) Step 21

(c) Step 145

(d) Step 1135

Fig. 8.21 Failure process of case 08, $\theta_1 = 112^\circ$, $\theta_2 \doteq 90^\circ$

We can observe crack propagation and coalescence with the bedding plane and the formation of discrete blocks, but not buckling failure mechanism.

Based on case 05 and case 08, we conclude that buckling is unlikely occur in footwall slopes with a planar bedding plane. This result is consistent with the earlier researches' finding that that buckling of elastic footwall slabs does not appear to be a likely failure mechanism unless very thin bed thicknesses are assumed (Stead and Eberhardt, 1997).

Then, we change the footwall slope profile by introducing a bedding roll to the footwall slope and investigate its influence on the buckling failure mechanism. The case is numbered as case 09. Configuration of case 09 is illustrated in Fig. 8.21, with the bedding roll marked by a circle.

Fig. 8.22 Configuration of case 09, $\theta_1 = 102^\circ$ below the bedding roll, $\theta_1 = 117^\circ$ above the bedding roll, joint set sub-vertical to the bedding plane

The θ_1 of the bedding plane below the bedding roll is 102° , while the θ_1 of the bedding plane above the bedding roll is 117° . The joint set is assumed sub-vertical to the bedding plane.

The failure process of case 09 is depicted in Fig. 8.23. With the introduction of the bedding roll, buckling failure mechanism is observed.

(a) Step 0

(b) Step 70

(c) Step 127

(d) Step 161

Fig. 8.23 Buckling failure process of case 09, a slope with a bedding roll, $\theta_1 = 102^\circ$ below the bedding roll, $\theta_1 = 117^\circ$ above the bedding roll, joint set sub-vertical to the bedding plane

In case 10, we slightly change the joint near the bedding roll in case 09 (marked by dashed line in Fig. 8.24) to the new position marked by the solid line in Fig. 8.24.

Fig. 8.24 Configuration of case 10, A slope with a bedding roll, $\theta_1 = 102^\circ$ below the bedding roll, $\theta_1 = 117^\circ$ above the bedding roll, joint set sub-vertical to the bedding plane, the joint near the bedding roll is slightly changed compared with case 09

The failure process of case 10 is depicted in Fig. 8.25. Fig. 8.25a shows the initial configuration of case 10, while Fig. 8.25b-g give the failure process. It is interesting to find that the failure mode is different from the one of case 09 shown in Fig. 8.23. In this case, with the upper footwall slab moving downward along the bedding plane, one block is squeezed out (Fig. 8.25b). After that, the upper footwall slab further moves downward along the bedding plane under the gravitational loading (Fig. 8.25c). The upper footwall slab meets the lower footwall slab in Fig. 8.25d. Then, the blocks near the bedding roll move out of footwall slope and collapse of the footwall slope is observed.

(a) Step 0

(b) Step 113

(c) Step 201

(d) Step 230

(e) Step 247

(f) Step 260

(g) Step 300

Fig. 8.25 Failure process of case 10, with a bedding roll, $\theta_1 = 102^\circ$ below the bedding roll, $\theta_1 = 117^\circ$ above the bedding roll, joint set sub-vertical to the bedding plane, the joint near the bedding roll is slightly changed compared with case 09

Then, we change one persistent joint in case 09 (marked by dash line in Fig. 8.26) to a non-persistent joint marked by the solid line in Fig. 8.26. This case is numbered as case 11.

Fig. 8.26 Configuration of case 11, with a bedding roll, $\theta_1 = 102^\circ$ below the bedding roll, $\theta_1 = 117^\circ$ above the bedding roll, joint set sub-vertical to the bedding plane, a persistent joint in case 09 is changed to a non-persistent joint

The failure process of case 11 is given in Fig. 8.27. Fig. 8.27a shows the initial configuration, while Figs. 8.27b-f give the failure evolution. It can be seen that the joint persistence has a great influence on the failure mode. The failure mechanism changes a lot from case 09. In this case, with the upper footwall slab moves downwards along the bedding plane, the lower footwall slab is lifted and rotated out of the footwall slope (Figs. 4.27b,c). The failure mechanism is more like a ploughing rather than buckling. After that, the whole footwall slope collapse under the gravitational loading (Figs. 4.27d-f).

(a) Step 0

(b) Step 59

(c) Step 123

(d) Step 201

(e) Step 255

(f) Step 300

Fig. 8.27 Failure process of case 11, a slope with a bedding roll, $\theta_1 = 102^\circ$ below the bedding roll, $\theta_1 = 117^\circ$ above the bedding roll, joint set sub-vertical to the bedding plane, a persistent joint in case 09 is changed to a non-persistent joint

The configuration of our last case. i.e. case 12, is illustrated in Fig. 8.28. In this case, we further change another persistent joint in case 11 (marked by dashed line in Fig. 8.28) into a non-persistent joint (marked by solid line in Fig. 8.28).

Fig. 8.28 Configuration of case 12, a slope with a bedding roll, $\theta_1 = 102^\circ$ below the bedding roll, $\theta_1 = 117^\circ$ above the bedding roll, joint set sub-vertical to the bedding plane, a persistent joint in case 11 is changed to a non-persistent joint

The failure process of case 12 is given in Fig. 8.29. Fig. 8.29a shows the initial configuration of case 12, while Figs. 8.29b-f give the failure evolution. It can be seen that the failure modes changes again from the last three cases. Two cracks propagate, then coalesce with the bedding plane and finally divide the footwall slope into two discrete slabs (Fig. 8.29b). With the upper footwall slab moves downward along the bedding plane under gravitational loading, the lower footwall slab is lifted and rotated out of the bedding plane (Fig. 8.29c). The upper footwall slab continues moving downwards under the gravitational loading. Some cracks in the upper footwall slab propagate and further divide it into several discrete slabs (Figs. 8.29d-f).

(a) Step 0

(f) Step 296

Fig. 8.29 Buckling failure process of case 12, a slope with a bedding roll, $\theta_1 = 102^\circ$ below the bedding roll, $\theta_1 = 117^\circ$ above the bedding roll, joint set sub-vertical to the bedding plane, a persistent joint in case 11 is changed to a non-persistent joint

The failure mode of the footwall slope in case 12 is also more like ploughing rather than buckling, same as in case 11. However, the ploughing occurs at a different position compared with case 11.

Fig. 8.30 Comparison of failure modes of: (a) case 05, planar bedding; (b) case 08, planar bedding but dipper bedding and thinner bed thickness; (c) case 09, with a bedding roll; (d) case 10, with a bedding roll, the joint near bedding roll slightly changed; (e) case 11, with a bedding roll, the joint near bedding roll changed from persistent to non-persistent; (f) case 12, with a bedding roll, another joint changed from persistent to non-persistent

In this section, the situation where the joint set is sub-vertical to the bedding plane is investigated by changing a few parameters. The failure mechanisms of different cases with different parameters are compared in Fig. 8.30.

The buckling failure mechanism does not likely occur in footwall slopes with a planar bedding plane. Cracks may propagate and coalesce with the bedding plane and divide the footwall slope into several discrete slabs. After that, the footwall slope keeps stable. Increase of the bedding dip and decrease of the bed thickness do not tend to turn the footwall slope to failure. When a bedding roll is introduced into the bedding plane, buckling failure mechanism is easily observed. Joint orientation has a great influence on the failure mode. Slight change of the orientation of the joint around the bedding roll may lead to total different failure mode. Joint persistence also greatly affects the failure mode of footwall slopes. Change the persistence of one or more joints, the failure modes will be totally different.

8.4 Summary

In this chapter, the NMM enabling fracturing is applied to analyze the potential failure mechanisms of footwall slopes in surface coal mines. Non-persistent joints rather than the persistent joints usually adopted in the UDEC and the DDA are considered. Numerical studies indicate that the NMM enabling fracturing can reproduce the major failure mechanisms observed in footwall slopes. Parametric studies regarding to orientation of bedding plane, orientation of predominant joint set, bedding profile (with or without bedding roll), joint persistence are carried out. Denote the angle from the x-axis to the bedding plane anticlockwise as θ_1 , and the angle from the bedding plane to the joint set anticlockwise as θ_2 . Numerical experiments show that the orientation of bedding plane has a significant influence on the footwall failure mechanism. When $\theta_1 < 90^\circ$, toppling failure mechanism is observed. When $\theta_1 > 90^\circ$, orientation of the predominant joint set dominates the footwall slope failure mechanisms. With $\theta_1 > 90^\circ$ and $\theta_2 < 90^\circ$, bilinear failure mechanism is obtained. The bedding plane works as the upper failure surface while the propagated joint plane acts as the lower failure surface. With $\theta_1 > 90^\circ$ and $\theta_2 > 90^\circ$, ploughing is observed. With the upper footwall slab moves downwards along the bedding plane under gravitational loading, the lower footwall slab is lifted and rotated out of the footwall slope. When $\theta_1 > 90^\circ$ and $\theta_2 \doteq 90^\circ$ (i.e. joint set is sub-vertical to the bedding plane), buckling failure mechanism is expected to occur. However, the numerical studies show that the buckling does not likely occur in footwall slopes with a planar bedding plane. Cracks may propagate and coalesce with the bedding plane and divide the footwall slope into several discrete slabs. After that, the footwall slope keeps stable. Increase of the bedding dip and decrease of the bed thickness do not tend to turn the footwall slope to failure. When a bedding roll is introduced into the bedding plane, buckling failure mechanism is easily observed. Both joint orientation and joint persistence affect the failure modes observed.

CHAPTER 9 CONCLUSIONS AND RECOMMENDATIONS

9.1 Conclusions

This thesis focuses on the extension of the numerical manifold method (NMM) for engineering failure analysis. Based on the work in this thesis, the following conclusions can be drawn:

1. A detailed comparison between the NMM and various other numerical methods indicates that the NMM has the following distinct features: (1) its mathematical covers can be constructed totally independent of the problem domain; a regular mathematical cover system is always applicable, which makes its meshing task very convenient and the remeshing be totally avoided for discontinuity evolution; (2) its two sets of covers make the displacement jump across complex cracks with arbitrary number of branches modeled in a straightforward manner; (3) its simplex integration scheme makes the integration over irregular manifold elements be evaluated directly without the need of element partitioning which is usually adopted in the XFEM; (4) its partition of unity framework makes the incorporation of any prior known knowledge about the local behavior of the solution of boundary value problems possible; (5) its partition of unity framework also makes the high-order approximations be realized easily without introducing any edge and middle nodes which is usually adopted in the p -FEMs; (6) its contact detection and modeling algorithm makes the interactions among multiple discrete blocks be modeled in a realistic manner. The above outstanding features make the NMM extremely suitable for discontinuity modeling.
2. Because of its convenient meshing, its unique way for discontinuity modeling, and its simplex integration scheme, the NMM seems extremely suitable for crack problems. However, the conventional NMM adopts polynomials as local approximation spaces for each physical cover. The

singularities around the crack tips cannot be well resolved. The crack tips are only allowed to stop at the sides of manifold elements. In order to overcome such inconveniences, modifications to the conventional NMM have been done. A new concept “singular physical cover” is introduced to represent the physical cover which is partially cut by the crack surface and contains the crack tip singularities. Asymptotic crack tip functions extracted from the analytical solution are incorporated into the local approximation spaces of singular physical covers based on partition of unity. The domain form of the interaction integral and the maximum circumferential stress criterion are employed to evaluate the stress intensity factors at the crack tips and to predict the crack propagation direction, respectively. By such extensions, the stress singularities around the crack tips can be well captured, the crack tips are allowed to stop at arbitrary positions within a manifold element. Several typical crack problems are simulated. Numerical results indicate that the extended NMM is efficient and robust for complex crack problems.

3. The NMM is compared with the extended finite element method (XFEM) and the generalized finite element method (GFEM) in crack modeling. Both the NMM and the XFEM allow the mesh be constructed independent of the cracks. In addition, the NMM shape functions span the same space as the XFEM shape functions. However, the NMM is more convenient than the XFEM for complex crack problems. The XFEM incorporates discontinuous enrichment functions to account for the cracks. Modeling a single crack can be easily done by employing the generalized Heaviside function $H(\mathbf{x})$. However, when complex cracks with multiple branches are encountered, defining the discontinuous enrichment functions and enriching the nodes become complicated, or even impossible to handle. The NMM describes the displacement jump across the crack surfaces by splitting covers. Complex cracks are modeled in an exactly same way as the modeling of a single crack. The GFEM also constructs the finite element mesh totally independent of problem domain, thus a regular finite element mesh is always applicable. The GFEM also adopts discontinuous enrichment functions to represent the

cracks. Its enrichment functions are different from those used in the XFEM. Its implementation is easier than that of the XFEM. Complex cracks with arbitrary number of branches can be easily handled. However, its discontinuous enrichment functions must be carefully chosen because they are not linearly independent. In addition, its enrichment functions are derived based on aggregate topology, thus can only allow the crack tips stopping at the sides of the elements. In contrast, the extended NMM incorporates the asymptotic crack tip functions into the singular physical covers, thus does not suffer from such a limitation.

4. Inspired by the advantageous modeling strategy of the NMM, a modification to the XFEM/FEM is proposed. The modified XFEM/GFEM does not need enrichment functions to account for the cracks. Its basic idea is to add additional nodes to the node whose support is completely cut by the cracks into several pieces. The additional nodes are termed as parasitic nodes since they are parasitic to the original node. This new approach is termed as the parasitic finite element method (PFEM). Through two typical examples, the PFEM is proved valid and efficient. The PFEM can be viewed as a special case of the NMM.
5. The NMM combines the continuum-based finite element method (FEM) and the block-orientated discontinuous deformation analysis (DDA) in a unified framework. As a hybrid continuum-discontinuum method, the NMM seems promising for practical rock failure analysis, which involves the crack initiation, propagation of both newly initiated cracks and pre-existing cracks, crack coalescence, formation of complete failure surface, and lastly the large movement of discrete rock block system. To apply the NMM for such problems, a fracturing algorithm is essential to fulfill the transition from a continuum to a discontinuum. Although a fracturing based on fracture mechanics has already developed, it cannot meet the complexities encountered in the rock failure process. For example, the fracture mechanics can predict the crack propagation well; however, it cannot predict the crack initiation. The loading to the rock mass is always compressive, which

violates the assumption of traction-free crack surface. In addition, cracks may propagate and intersect other cracks, which makes the domain form of interaction integral not applicable any more. To this end, a fracturing algorithm based on the Mohr-Coulomb criterion and a tensile cutoff is implemented. The Mohr-coulomb criterion is able to recover the salient features of the quasi-brittle response within engineering accuracy, and has the advantage of being based on parameters that are easily determined experimentally. With the Mohr-coulomb criterion, both tensile cracks and shear cracks can be modeled. A cover-division strategy is adopted to fulfill the crack initiation and propagation. Compared with the element-division strategy which is usually adopted in various FEM models, the cover-division strategy can avoid the mesh dependency to some extent because of the non-local nature of the stress. The computer implementation details including treatment of elements, treatment of covers, updating physical covers for each manifold element, updating loops for contact detection and modeling during the fracturing process, evaluation of rock debris size with the simplex integration method are presented. The developed program is calibrated through several typical crack initiation and propagation examples.

6. Joint persistence has a significant influence on the rock failure. However, in the traditional rock failure analysis, persistent joints are often assumed, which is mainly due to the constraints that many analysis techniques, such as the limit equilibrium wedge or planar analysis, the distinct element code UDEC, the discontinuous deformation analysis (DDA) method, etc, require the input of fully persistent discontinuities. In this study, the NMM enabling fracturing is applied to simulate the progressive failure of rock slopes with non-persistent joints. Numerical examples indicate that it is able to capture the fracturing in intact rock bridge and finally allow the kinematic release.
7. The developed program is also applied to investigate the potential failure mechanisms of footwall slopes in surface coal mining. Numerical results indicate that the developed program can reproduce the major failure mechanisms observed in footwall slopes. Parametric studies regarding to

orientation of bedding plane, orientation of predominant joint set, bedding profile (with or without bedding roll), joint persistence are carried out. Numerical experiments show that the orientation of bedding plane has a significant influence on the footwall failure mechanism. When $\theta_1 < 90^\circ$, where θ_1 represents the angle from the x-axis to the bedding plane anticlockwise, toppling failure mechanism is observed. When $\theta_1 > 90^\circ$, orientation of the joint set dominates the footwall slope failure mechanisms. With $\theta_1 > 90^\circ$ and $\theta_2 < 90^\circ$, where θ_2 represents the angle from the bedding plane, bilinear failure mechanism is obtained. The bedding plane works as the upper failure surface while the propagated joint plane acts as the lower failure surface. With $\theta_1 > 90^\circ$ and $\theta_2 > 90^\circ$, ploughing is observed. With the upper footwall slab moves downwards along the bedding plane under gravitational loading, the lower footwall slab is lifted and rotated out of the footwall slope. When $\theta_1 > 90^\circ$ and $\theta_2 \doteq 90^\circ$ (i.e. joint set is sub-vertical to the bedding plane), buckling failure mechanism is expected to occur. However, the numerical studies show that the buckling does not likely occur in footwall slopes with a planar bedding plane. Cracks may propagate and coalesce with the bedding plane and divide the footwall slope into several discrete slabs. After that, the footwall slope keeps stable. Increase of the bedding dip and decrease of the bed thickness do not tend to turn the footwall slope to failure. When a bedding roll is introduced into the bedding plane, buckling failure mechanism is easily observed. Both joint orientation and joint persistence affect the failure modes observed.

9.2 Recommendations

The studies in Chapters 4 and 5 have demonstrated the efficiency and robustness of the NMM to account for the complex cracks and their growth. The following work can be done to further improve its capability:

- to revise the current domain form of interaction integral to make its applicable to cracks involving frictional contact.

The preliminary studies in Chapters 6 to 8 have demonstrated the great potential of the NMM enabling fracturing for practical rock failure analysis. However, to realistically characterize the practical rock failure process, the following work must be done:

- To provide sufficient data to allow for the use of numerical methods. It is essential to record geometric data on joint sets (e.g., orientation, persistence and spacing) and shear strength (e.g., roughness, infill, water, etc.), and combine field and laboratory data to assess scaled rock mass properties and allow characterization of the rock mass, for example, according to the Geological Strength Index, GSI;
- To incorporate probabilistic algorithms in which variations in joint geometrical characteristics (e.g., dip, dip direction, etc.) can be assessed for their influences on the stability of rock mass structures;
- To replace the currently used elastic model with more advanced elasto-plastic constitutive models to describe the intact rock behavior;
- To implement a fracture energy release rate-based strain softening criterion to simulate internal strength degradation and damage contributing to massive rock mass failure;
- fracture energy approach to account for the fracturing, damage and associated softening within a rock mass prior to and during failure;
- To couple hydro-mechanical modeling;
- To incorporate support interaction.

In this study, two fracturing algorithms are proposed. One is based on the fracture mechanics. The other one is based on the Mohr-Coulomb criterion with a tensile cutoff. In the future study, these two methods can be coupled together to inherit both methods' advantages while avoiding both methods' shortcomings. The Mohr-Coulomb criterion is adopted to predict the crack initiation, while the fracture mechanics is adopted to deal with the crack propagation.

REFERENCES

Amadei B, Editor (1999), *Proceedings of the Third International Conferences on Analysis of Discontinuous Deformation (ICADD-3)*, Vail, Colorado, USA.

An XM, Li LX, Ma GW, Zhang HH (2010), “Prediction of rank deficiency in partition of unity-based methods”, *Computer Methods in Applied Mechanics and Engineering* (under review).

An XM, Ma GW (2010), “Numerical manifold method based on modified finite element partitions of unity”, *Engineering Analysis with Boundary Elements* (under review).

Babuska I, Caloz G, Osborn J (1994), “Special finite element methods in a class of second order elliptic problems with rough coefficients”, *SIAM Journal on Numerical Analysis*, 31(4), 945-981.

Babuska I, Melenk JM (1997), “The partition of unity method”, *International Journal for Numerical Methods in Engineering*, 40, 727-758.

Babuska I, Strouboulis T, Copps K (2000), “The design and analysis of the generalized finite element method”, *Computer Methods in Applied Mechanics and Engineering*, 181(1), 43-69.

Barsoum R (1977), “Triangular quarter-point elements as elastic and perfectly-plastic crack tip elements”, *International Journal for Numerical Methods in Engineering*, 11, 85-98.

Bates B (2004), “Development and DDA modeling of benchmark tests”, B.S.c. Thesis, Department of Civil and Engineering, University of Cape Town.

Belytschko T, Black T (1999), “Elastic crack growth in finite elements with minimal remeshing”, *International Journal for Numerical Methods in Engineering*, 45(5),

REFERENCES

601-620.

Belytschko T, Krongauz Y, Organ D, Fleming M, Krysl P (1996), “Meshless methods: an overview and recent developments”, *Computer Methods in Applied Mechanics and Engineering*, 139, 3-47.

Belytschko T, Lu YY, Gu L (1994), “Element-free Galerkin method”, *International Journal for Numerical Methods in Engineering*, 37, 229-256.

Belytschko T, Moes N, Usui S, Parimi C (2001), “Arbitrary discontinuities in finite elements”, *International Journal for Numerical Methods in Engineering*, 50(4), 993-1013.

Benko B, Stead D (1999), “Analysis of two landslide case studies using numerical modeling”, *Proceedings of 13th Annual Vancouver Geotechnical Society Symposium*, Vancouver, pp.19-29.

Bicanic N Editor (2001), *Proceedings of the Forth International Conferences on Analysis of Discontinuous Deformation (ICADD-4)*, University of Glasgow Scotland, UK.

Bittencourt TN, Wawrzynek PA, Ingraffea AR (1996), “Quasi-automatic simulation of crack propagation for 2D LEFM problems”, *Engineering Fracture Mechanics*, 55, 321-334.

Board M, Chacon E, Varona P, Lorig L (1996), “Comparative analysis of toppling behaviour at Chuquicamata open-pit mine, Chile”, *Transactions of the Institution of Mining and Metallurgy Section A-Mining Industry*, 105, A11-A21.

Bouchard P, Bay F, Chastel Y (2003), “Numerical modeling of crack propagation: automatic remeshing and comparison of different criteria”, *Computer Methods in Applied Mechanics and Engineering*, 192, 3887-3908.

Bouchard PO, Bay F, Chastel Y, Tovenia I (2000), “Crack propagation modelling using an advanced remeshing technique”, *Computer Methods in Applied Mechanics*

and Engineering, 89,723-742.

Bray BHG, Brown ET (2004), “Rock mechanics for underground mining”, Springer.

Cavers DS, Baldwin GJ, Hannah T, Singhal RK (1986), “Design methods for open pit mine footwalls”, *Proc Int Symp on Geotechnical Stability in Surface Mining*, Calgary, pp.79-86.

Chen G, Ohnishi Y (1999), “Slope stability analysis using discontinuous deformation analysis”, *Proceedings of 37th US Symposium on Rock Mechanics (USRMS)*, June 7-9, Balkema, Rotterdam, pp.535-541.

Chen GQ, Ohnishi Y, Ito T (1998), “Development of high-order manifold method”, *International Journal for Numerical Methods in Engineering*, 43, 685-712.

Chen JS, Pan C, Roque MOL, Wang HP (1998), “A Lagrangian reproducing kernel particle method for metal forming analysis”, *Computational Mechanics*, 22, 289-307.

Chen JS, Pan C, Wu CT, Liu WK (1996), “Reproducing kernel particle methods for large deformation analysis of non-linear structures”, *Computer Methods in Applied Mechanics and Engineering*, 139, 195-227.

Chen Y, Hasebe N (1995), “New integration scheme for the branch crack problem”, *Engineering Fracture Mechanics*, 52, 791-801.

Cheng YM, Zhang YH (2008), “Formulation of a three-dimensional numerical manifold method with tetrahedron and hexahedron elements”, *Rock Mechanics and Rock Engineering*, 41(4), 601-628.

Cheng YM, Zhang YH, Chen WS (2002), “Wilson non-conforming element in numerical manifold method”, *Communications in Numerical Methods in Engineering*, 18, 877-884.

Cheung Y, Wang Y, Woo C (1984), “A general method for multiple crack problems in a finite plate”, *Computational Mechanics*, 20, 583-589.

REFERENCES

- Chiou YJ, Lee YM, Tsay RJ (2002), “Mixed mode fracture propagation by manifold method”, *International Journal of Fracture*, 114, 327-347.
- Cleary PW (2004), “Large scale industrial DEM modelling”, *Engineering Computations: Int J for Computer-Aided Engineering*, 21(2-3), 169-204.
- Coggan JS, Stead D, Howe JH (2000), “Characterisation of a structurally controlled flowslide in a kaolinised granitic slope”, *Landslides in research, theory and practice, 8th International Symposium on Landslides*, Cardiff, Thomas Telford, pp.299-304.
- Costa M, Coggan JS, Eyre JM (1999), “Numerical modeling of slope behavior at Delabole slate quarry”, *International Journal of Surface Mining, Reclamation and Environment*, 13, 11-18.
- Cundall PA (1971), “A computer model for simulating progressive, large scale movements in blocky rock systems”, *Proceedings, International Symposium on Rock Fracture*, Nancy, France, II-8.
- Cundall PA, Strack ODL (1979), “A discrete numerical model for granular assemblies”, *Geotechnique*, 29(1), 47-65.
- Daux C, Moes N, Dolbow J, Sukumar N, Belytschko T (2000), “Arbitrary branched and intersecting cracks with the extended finite element method”, *International Journal for Numerical Methods in Engineering*, 48, 1741–1760.
- De Borst R, Remmers JJC, Needleman A, Abellan MA (2004), “Discrete vs smeared crack models for concrete fracture: bridging the gap”, *International Journal for Numerical Methods in Engineering*, 28, 583-607.
- De S, Bathe KJ (2000), “The method of finite spheres”, *Computational Mechanics*, 25, 329-345.
- Desai CS, Zamman MM, Lightner JG, Siriwardane HJ (1984), “Thinlayer element for interfaces and joints”, *International Journal for Numerical and Analytical Methods in Geomechanics*, 8, 19-43.

Dolbow J, Moes N, Belytschko T (2000a), “Discontinuous enrichment in finite elements with a partition of unity method”, *Finite Elements in Analysis and Design*, 36, 235-260.

Dolbow J, Moes N, Belytschko T (2000b), “Modeling fracture in Mindlin-Reissner plates with the extended finite element method”, *International Journal of Solids and Structures*, 57(48-50), 7161-7183.

Dolbow J, Moes N, Belytschko T (2001), “An extended finite element method for modeling crack growth with frictional contact”, *Computer Methods in Applied Mechanics and Engineering*, 190(51-52), 6825-6846.

Duarte A (1996), “The hp cloud method”, Ph.D dissertation, University of Texas at Austin, Austin, TX.

Duarte CA, Babuska I, Oden JT (2000), “Generalized finite element methods for three-dimensional structural mechanics problems”, *Computers and Structures*, 77, 215-232.

Duarte CA, Hamzeh ON, Liszka TJ, Tworzydło WW (2001), “A generalized finite element method for the simulation of three-dimensional dynamic crack propagation”, *Computer Methods in Applied Mechanics and Engineering*, 190, 2227-2262.

Duarte CA, Reno LG, Simone A (2007), “A high-order generalized FEM for through-the-thickness branched cracks”, *International Journal for Numerical Methods in Engineering*, 72, 325-351.

Duarte CAM, Oden JT (1996a), “An hp adaptive method using clouds”, *Computer Methods in Applied Mechanics and Engineering*, 139, 237-262.

Duarte CAM, Oden JT (1996b), “Hp clouds-an hp meshless method”, *Numerical Methods for Partial Differential Equations*, 12, 673-705.

Eberhardt E, Stead D (1993), “Numerical modeling of buckling and ploughing slope instability in surface coal mines”, *Proceedings of the First Canadian Symposium on*

REFERENCES

Numerical Modeling Applications in Mining and Geomechanics, Montreal, pp. 176-185.

Einstein HH, Veneziano D, Baecher GB, O Reilly KJ (1983), “The effect of discontinuity persistence on rock slope stability”, *International Journal of Rock Mechanics and Mining Sciences and Geomechanics Abstracts*, 20(5), 227-236.

Erdogan F, Sih G (1963), “On the crack extension in plates under plane loading and transverse shear”, *Journal of Basic Engineering*, 85, 519-527.

Ewalds H, Wanhill R (1989), “Fracture mechanics”, New York: Edward Arnold.

Fang XS, Tu GF, Wang Y, Shi GH (2005), “Data compression using manifold method”, *Proceedings of the Seventh International Conferences on Analysis of Discontinuous* (ICADD-7), Honolulu, Hawaii, pp.191-198.

Fleming M, Chu BM, Belytschko T (1997), “Enriched element-free Galerkin methods for singular fields”, *International Journal for Numerical Methods in Engineering*, 40, 1483-1504.

Fries TP, Belytschko T (2006), “The intrinsic XFEM: a method for arbitrary discontinuities without additional unknowns”, *International Journal for Numerical Methods in Engineering*, 68, 1358-1385.

Gao HF, Cheng YM (2010), “A complex variable meshless manifold method for fracture problems”, *International Journal of Computational Methods*, 7(1), 55-81.

Gasser TC, Holzapfel GA (2005), “Modeling 3D crack propagation in unreinforced concrete using PUFEM”, *Computer Methods in Applied Mechanics and Engineering*, 194, 2859–2896.

Ghaboussi J, Wilson EL, Isenberg J (1973), “Finite element for rock joints and interfaces”, *Journal of the Soil Mechanics and Foundations, ASCE*, 99(10), 849-862.

Goodman RE, Taylor RL, Brekke TL (1968), “A model for the mechanics of jointed rock”, *Journal of the Soil Mechanics and Foundations Division, ASCE*, 94, 637-659.

Guadagno FM, Martino S, Scarascia Mugnozza G (2003), "Influence of man-made cuts on the stability of pyroclastic covers (Campania, southern Italy): a numerical modeling approach", *Environmental Geology*, 43, 371-384.

Hansbo A, Hansbo P (2004), "A finite element method for the simulation of strong and weak discontinuities in solid mechanics", *Computer Methods in Applied Mechanics and Engineering*, 193, 3523-3540.

Hart RD (1993), "An introduction to distinct element modeling for rock engineering", *Comprehensive Rock Engineering: Principles, Practice and Projects*, vol. 2. Pergamon Press, Oxford, pp.245-261.

Hatzor YH, Editor (2002), *Proceedings of the Fifth International Conferences on Analysis of Discontinuous Deformation (ICADD-5)*, Ben-Gurion University of the Negev, Beer Sheva, Israel.

Hawley PM, Martin DC, Acott CP (1986), "Failure mechanics and design considerations for footwall slopes", *CIM Bull*, 79, 47-53.

He L, Ma GW (2010), "Development of 3-D numerical manifold method", *International Journal of Computational Methods*, 7(1), 107-129.

Hellen TK (1975), "On the method of virtual crack extensions", *International Journal for Numerical Methods in Engineering*, 9, 187-207.

Huang R, Sukumar N, Prevost JH (2003), "Modeling quasi-static crack growth with the extended finite element method, Part II: numerical applications", *International Journal of Solids and Structures*, 40, 7539-7552.

Hutchison B, Dugan K, Coulthard MA (2000), "Analysis of flexural toppling at Australian bulk minerals Savage River mine", *Proceedings of GeoEng 2000, International Conference on Geotechnical & Geological Engineering*, Melbourne.

Itasca (1993a), FLAC version 3.2, Users manual, Vols. 1-3.

Itasca (1993b), UDEC version 1.18, Users manual, Vols. 1-2.

REFERENCES

Itasca (2004), “Itasca Software Products—FLAC, FLAC3D, UDEC, 3DEC, PFC, PFC3D”, Itasca Consulting Group Inc., Minneapolis.

Jiang QH, Zhou CB, Liu DQ (2009), “A three-dimensional numerical manifold method based on tetrahedral meshes”, *Computers and Structures*, 87(13-14), 880-889.

Jing L (2003), “A review of techniques, advances and outstanding issues in numerical modeling for rock mechanics and rock engineering”, *International Journal of Rock Mechanics & Mining Sciences*, 40, 283-353.

Ju Y, Fang XS, Bian HY, Editors (2007), *Proceedings of the Eighth International Conferences on Analysis of Discontinuous Deformation (ICADD-8)*, Beijing, China.

Kalkani EC, Piteau DR (1976), “Finite element analysis of toppling failure at Hell’s Gate Bluffs”, *British Columbia, Bulletin of the Association of Engineering Geologists*, XIII (4), pp.315-327.

Karami A, Stead D (2008), “Asperity degradation and damage in the direct shear test: a hybrid FEM/DEM approach”, *Rock Mechanics and Rock Engineering*, 41(2), 229-266.

Karihaloo BL (1973), “Fracture mechanics and structural concrete”, England: Longman Scientific & Technical.

Katona MG (1983), “A simple contact–friction interface element with applications to buried culverts”, *International Journal for Numerical and Analytical Methods in Geomechanics*, 7, 371-384.

Kinakin D, Stead D (2005), “Analysis of the distributions of stress in natural ridge forms: implications for the deformation mechanisms of rock slopes and the formation of sacking”, *Geomorphology*, 65, 85-100.

Kirsch G (1898), “Theory of elasticity and application in strength of materials”, *Zeitschrift des VDL*, 42(29), 797-807.

Klerck PA, Sellers EJ, Owen DRJ (2004), “Discrete fracture in quasi-brittle materials

under compressive and tensile stress states”, *Computer Methods in Applied Mechanics and Engineering*, 193, 3035-3056.

Krahn J, Morgenstern NR (1976), “The mechanics of the Frank Slide”, *Rock Engineering for Foundations and Slopes, Proceedings of ASCE Geotechnical Specialty Conference*, Boulder. ASCE, New York, pp.309-332.

Kurumatani M, Terada K (2005), “Finite cover method with mortar elements for elastoplasticity problems”, *Computational Mechanics*, 36, 45-61.

Kurumatani M, Terada K (2009), “Finite cover method with multi-cover layers for the analysis of evolving discontinuities in heterogeneous media”, *International Journal for Numerical Methods in Engineering*, 79(1), 1-24.

Kwon YW, Akin JE (1989), “Development of a derivative singular element for application to crack propagation problems”, *Computers and Structures*, 31(3), 467-471.

Li C, Wang CY, Sheng J, Editors (1995), *Proceedings of the First International Conferences on Analysis of Discontinuous Deformation (ICADD-1)*, National Central University, Chungli, Taiwan.

Li S, Cheng Y, Wu YF (2005), “Numerical manifold method based on the method of weighted residuals”, *Computational Mechanics*, 35, 470-480.

Li S, Liu WK (2000), “Numerical simulations of strain localization in inelastic solids using mesh-free methods”, *International Journal for Numerical Methods in Engineering*, 48, 1285-1309.

Li SC, Li SC, Cheng YM (2005), “Enriched meshless manifold method for two-dimensional crack modeling”, *Theoretical and Applied Fracture Mechanics*, 44, 234-248.

Lin CT, Amadei B, Jung J, Dwyer J (1996), “Extensions of discontinuous deformation analysis for jointed rock masses”, *International Journal of Rock Mechanics and*

REFERENCES

Mining Science & Geomechanics Abstracts, 33(7), 671-694.

Lin JS (2003), “A mesh-based partition of unity method for discontinuity modeling”, *Computer Methods in Applied Mechanics and Engineering*, 192, 1515-1532.

Lin JS, Koo CY, Chern JC (1999), “Manifold method analysis of rock masses containing joints of two different scales”, *Proceedings of the Third International Conferences on Analysis of Discontinuous Deformation (ICADD-3)*, Vail, Colorado, USA, pp.231-242.

Lin SZ, Qi YF, Su HD (2005), “Formulation of high-order numerical manifold method and fast simplex integration based on special matrix operations”, *Proceedings of the Seventh International Conferences on Analysis of Discontinuous Deformation (ICADD-7)*, Honolulu, Hawaii, pp.183-190.

Liszka TJ, Duarte CAM, Tworzydło WW (1996), “Hp-meshless cloud method”, *Computer Methods in Applied Mechanics and Engineering*, 139, 263-288.

Liu HY, Kou SQ, Lindqvist PA (2002), “Numerical simulation of the fracture process in cutting heterogeneity brittle material”, *International Journal for Numerical and Analytical Methods in Geomechanics*, 26, 1253-1278.

Liu KW, Chen Y, Uras RA, Chang CT (1996), “Generalized multiple scale reproducing kernel particle methods”, *Computer Methods in Applied Mechanics and Engineering*, 139, 91-157.

Liu KW, Jun S, Zhnag YF (1995), “Reproducing kernel particle methods”, *International Journal for Numerical Methods in Engineering*, 20, 1081-1106.

Liu WK, Li S, Belytschko T (1997), “Moving least-square reproducing kernel methods, Part I: methodology and convergence”, *Computer Methods in Applied Mechanics and Engineering*, 143, 113-154.

Lu M, Editor (2003), *Proceedings of the Sixth International Conferences on Analysis of Discontinuous Deformation (ICADD-6)*, Norwegian University of Science and

Technology, Trondheim, Norway.

Ma GW, An XM, Wang MY (2007b), “Numerical studies on dynamic friction mechanism of blocky rock system using manifold method”, *Proceedings of the Eighth International Conferences on Analysis of Discontinuous Deformation (ICADD-8)*, Beijing, China, pp.221-226.

Ma GW, An XM, Zhou YX (2007a), “Modeling shear resistance of heterogeneous rock joint by using manifold method”, *Proceedings of the Eighth International Conferences on Analysis of Discontinuous Deformation (ICADD-8)*, Beijing, China, pp.33-38.

Ma GW, Zhou YX, Editors (2009), *Proceedings of the 9th International Conferences on Analysis of Discontinuous Deformation (ICADD-9)*, 25-27 Nov, Singapore.

Maclaughlin MM, Sitar N, Editors (2005), *Proceedings of the Seventh International Conferences on Analysis of Discontinuous Deformation (ICADD-7)*, Honolulu, Hawaii.

Melenk JM (1992), “Finite element methods with harmonic shape functions for solving Laplace’s equation”, M.S. thesis, University of Maryland, College Park, MD.

Melenk JM (1995), “On generalized finite element methods”, Ph.D. dissertation, University of Maryland, College Park, MD.

Melenk JM, Babuska I (1996), “The partition of unity finite element method: basic theory and applications”, *Computer Methods in Applied Mechanics and Engineering*, 139, 289-314.

Melenk JM, Babuska I (1997), “Approximation with harmonic and generalized harmonic polynomials in the partition of unity method”, *Computer Assisted Mechanics and Engineering Sciences*, 4, 607-632.

Mergheim J, Kuhl E, Steinmann P (2005), “A finite element method for the computational modelling of cohesive cracks”, *International Journal for Numerical*

REFERENCES

Methods in Engineering, 63, 276–289.

Miki S, Sasaki T, Koyama T, Nishiyama S, Ohnishi Y (2010), “Development of coupled discontinuous deformation analysis and numerical manifold method (NMM-DDA)”, *International Journal of Computational Methods*, 7(1), 131-150.

Moes N, Belytschko T (2002), “Extended finite element method for cohesive crack growth”, *Engineering Fracture Mechanics*, 69(7), 813-833.

Moes N, Dolbow J, Belytschko T (1999), “A finite element method for crack growth without remeshing”, *International Journal for Numerical Methods in Engineering*, 46, 131-150.

Monaghan JJ (1988), “An introduction to SPH”, *Computer Physics Communications*, 48, 89-96.

Munjiza A, Owen DRJ, Bicanic N (1995), “A combined finite-discrete element method in transient dynamics of fracturing solids”, *Engineering Computations*, 12, 145-174.

Muravin B, Turkel E (2006), “Multiple crack weight for solution of multiple intersecting cracks by meshless numerical methods”, *International Journal for Numerical Methods in Engineering*, 67, 1146-1159.

Mustoe GGW (1989), “Special elements in discrete element analysis”, *Proceedings of 1st U.S. Conference on Discrete Element Methods*, Golden, Colorado.

Nayroles B, Touzot G, Villon P (1992), “Generalizing the finite element method: diffuse approximation and diffuse elements”, *Computational Mechanics*, 10, 307-318.

Nichol SL, Hungr O, Evans SG (2002), “Large scale brittle and ductile toppling of rock slopes”, *CAN Geotechnical*, 39 (4), 773-788.

Nuismer RJ (1975), “An energy release rate criterion for mixed mode fracture”, *International Journal of Fracture*, 11, 245-250.

Oden JT, Duarte CA, Zienkiewicz OC (1998), “A new cloud-based hp finite element method”, *Computer Methods in Applied Mechanics and Engineering*, 153, 117-126.

Oden JT, Duarte CAM (1997), “Solution of singular problems using h-p clouds”, *The Mathematics of Finite Elements and Applications*, John Wiley & Sons, NY, pp.35-54.

Ohnishi Y, Editor (1997), *Proceedings of the Second International Conferences on Analysis of Discontinuous Deformation (ICADD-2)*, Kyoto University, Kyoto, Japan.

Ohnishi Y, Tanaka M, Koyama T, Mutoh K (1999), “Manifold method in saturated-unsaturated unsteady groundwater flow analysis”, *Proceedings of the Third International Conferences on Analysis of Discontinuous Deformation (ICADD-3)*, Vail, Colorado, USA, pp.221-230.

Okazawa S, Terasawa H, Kurumatani M, Terada K, Kashiwama K (2010), “Eulerian finite cover method for solid dynamics”, *International Journal of Computational Methods*, 7(1), 33-54.

Onate E, Idelsohn S, Zienkiewicz OC, Taylor RL, Sacco C (1996), “Stabilized finite point method for analysis of fluid mechanics problems”, *Computer Methods in Applied Mechanics and Engineering*, 139, 315-346.

Paris PC, Sih GC (1965), “Stress analysis of crack”, *ASTM STP*, 381, 1-30.

Patton FD (1966). “Multiple modes of shear failure in rock”. Proc. 1st ISRM Congr. Lisbon, pp.509-513.

Radbruch-Hall DH, Varnes DJ, Savage WZ (1976), “Gravitational spreading of steep-sided ridges (“sacking”) in western United States”, *International Association of Engineering Geology Bulletin*, 14, pp.23-35.

Randles PW, Libersky LD (1996), “Smoothed particle hydrodynamics: some recent improvements and applications”, *Computer Methods in Applied Mechanics and Engineering*, 139, 375-408.

REFERENCES

- Rashid MM (1998), “The arbitrary local mesh replacement method: an alternative to remeshing for crack propagation analysis”, *Computer Methods in Applied Mechanics and Engineering*, 154, 133-150.
- Rice JR (1968), “A path independent integral and the approximate analysis of strain concentration by notches and cracks”, *Journal of Applied Mechanics*, 35, 379-386.
- Robertson AM (1970), “The interpretation of geological factors for use in slope theory”, In: *Planning Open Pit Mines, Proceedings*, Johannesburg, pp.55-71.
- Rockfield (2004), “ELFEN 2D/3D Numerical Modelling Package”, Rockfield Software Ltd., Swansea.
- Salami MR, Banks D, Editors (1996), *Proceedings of the First International Forum on Discontinuous Deformation Analysis (DDA) and Simulations of Discontinuous Media*, University of California, Berkeley, California, USA.
- Scheele F, Bates B (2005), “DDA benchmark testing at UCT-a summary”, *Proceedings of the Seventh International Conferences on Analysis of Discontinuous Deformation (ICADD-7)*, Honolulu, Hawaii, pp.59-69.
- Serra de Renobales T (1987), “Strata buckling in footwall slopes in coal mining”, *Proc Congress of International Society for Rock Mechanics*, Montreal, pp.527-531.
- Shewchuk JR (1997), “Adaptive precision floating-point arithmetic and fast robust geometric predicates”, *Discrete and Computational Geometry*, 18, 305-363.
- Shi GH (1988), “Discontinuous Deformation Analysis – A new numerical model for the static and dynamics of block systems”, PhD Dissertation, Department of Civil Engineering, U.C. Berkeley.
- Shi GH (1991), “Manifold method of material analysis”, *Trans. 9th Army Conf. on Applied Mathematics and Computing*, Minneapolis, Minnesota, pp.57-76.
- Shi GH (1992), “Modeling rock joints and blocks by manifold method”, *Rock Mechanics, Proc. 33rd U. S. Symposium*, Santa Fe, New Mexico, pp.639-648.

Shi GH (1995), "Numerical manifold method", *Proceedings of the First International Conferences on Analysis of Discontinuous Deformation (ICADD-1)*, Chungli, Taiwan, pp.187-222.

Shi GH (1996a), "Manifold method", *Proceedings of the First International Forum on Discontinuous Deformation Analysis (DDA) and Simulations of Discontinuous Media*, TSI Press, Albuquerque, New Mexico, USA, pp.52-204.

Shi GH (1996b), "Simplex integration for manifold method, FEM, DDA and analytical analysis", *Proceedings of the First International Forum on Discontinuous Deformation Analysis (DDA) and Simulations of Discontinuous Media*, TSI Press, Albuquerque, New Mexico, USA, pp.205-262.

Shi GH (1997), "Numerical manifold method", *Proceedings of the Second International Conferences on Analysis of Discontinuous Deformation (ICADD-2)*, Kyoto, pp.1-35.

Shih C, Asaro R (1988), "Elastic-plastic analysis of cracks on bimaterial interfaces: Part i-small scale yielding", *Journal of Applied Mechanics*, 55, 299-316.

Shyu K, Salami MR (1995), "Manifold with four-node isoparametric finite element method", *1st Working Forum on the Manifold Method of Material Analysis*. California, USA., pp.165-182.

Sih GC (1974), "Strain-energy-density factor applied to mixed mode crack problems", *International Journal of Fracture*, 10, 305-321.

Simone A, Duarte CA, Van der Giessen E (2006), "A generalized finite element method for polycrystals with discontinuous grain boundaries", *International Journal for Numerical Methods in Engineering*, 67, 1122-1145.

Sitar N, Maclaughlin MM (1997), "Kinematics and discontinuous deformation analysis of landslide movement", *II Panamerican Symposium on Landslides*, Rio de Janeiro, Nov. 10-14th.

REFERENCES

- Stacey TR, Xianbin Y, Armstrong R, Keyter GJ (2003), "New slope stability considerations for deep open pit mines", *Journal of the South African Institute of Mining and Metallurgy*, 103, 373-389.
- Stead D, Coggan JS, Eberhardt E (2004), "Realistic simulation of rock slope failure mechanisms: the need to incorporate principles of fracture mechanics", *International Journal of Rock Mechanics and Mining Science -SINOROCK2004*, 41, 466.
- Stead D, Eberhardt E (1997), "Developments in the analysis of footwall slopes in surface coal mining", *Engineering Geology*, 46(1), 41-61.
- Stead D, Eberhardt E, Coggan JS (2006), "Developments in the characterization of complex rock slope deformation and failure using numerical modeling techniques", *Engineering Geology*, 83, 217-235.
- Strouboulis T, Babuska I, Copps K (2000a), "The design and analysis of the generalized finite element method", *Computer Methods in Applied Mechanics and Engineering*, 181(1-3), 43-69.
- Strouboulis T, Copps K, Babuska I (2000b), "The generalized finite element method: an example of its implementation and illustration of its performance", *International Journal for Numerical Methods in Engineering*, 47(8), 1401-1417.
- Strouboulis T, Copps K, Babuska I (2001), "The generalized finite element method", *Computer Methods in Applied Mechanics and Engineering*, 190(32-33), 4081-4193.
- Strouboulis T, Zhang L, Babuska I (2003), "Generalized finite element method using mesh-based handbooks: application to problems in domains with many voids", *Computer Methods in Applied Mechanics and Engineering*, 192(28-30), 3109-3161.
- Su HD, Huang YY (2007), "Application of numerical manifold method in fluid-solid interaction harmonic analysis", *Proceedings of the Eighth International Conferences on Analysis of Discontinuous Deformation (ICADD-8)*, Beijing, China, pp.227-234.
- Su HD, Xie XL (2005), "Preliminary research on solving large displacement problems

by numerical manifold method with fixed mathematical meshes”, *Proceedings of the Seventh International Conferences on Analysis of Discontinuous Deformation* (ICADD-7), Honolulu, Hawaii, pp.175-182.

Su HD, Xie XL, Liang QY (2003), “Automatic programming for high-order numerical manifold method”, *Proceedings of the Sixth International Conferences on Analysis of Discontinuous Deformation* (ICADD-6), Trondheim, Norway, pp. 153-160.

Sukumar N, Chopp DL, Moes N, Belytschko T (2001), “Modeling holes and inclusions by level sets in the extended finite-element method”, *Computer Methods in Applied Mechanics and Engineering*, 190(46-47), 6183-6200.

Sukumar N, Chopp DL, Moran B (2003), “Extended finite element method and fast marching method for three dimensional fatigue crack propagation”, *Engineering Fracture Mechanics*, 70(1), 29-48.

Sukumar N, Moes N, Moran B, Belytschko T (2000), “Extended finite element method for three-dimensional crack modeling”, *International Journal for Numerical Methods in Engineering*, 48(11), 1549-1570.

Sukumar N, Moran B, Belytschko T (1998), “The natural element method in solid mechanics”, *International Journal for Numerical Methods in Engineering*, 43, 839-887.

Sukumar N, Prevost JH (2003), “Modeling quasi-static crack growth with the extended finite element method, Part I: computer implementation”, *International Journal of Solids and Structures*, 40, 7513-7537.

Sulsky D, Schreyer HL (1996), “Axisymmetric form of the material point method with applications to upsetting and Taylor impact problems”, *Computer Methods in Applied Mechanics and Engineering*, 139, 409-429.

Sumi Y, Yang C, Wang Z (1995), “Morphological aspects of fatigue crack propagation. Part II-effects of stress biaxiality and welding residual stresses”, Technical report,

REFERENCES

Department of Naval Architecture and Ocean Engineering, Yokohama National University, Japan.

Tang CA (1997), “Numerical simulation of progressive rock failure and associated seismicity”, *International Journal of Rock Mechanics & Mining Science*, 34(2), 149-261.

Taylor RL, Zienkiewicz OC, Onate E (1998), “A hierarchical finite element method based on the partition of unity”, *Computer Methods in Applied Mechanics and Engineering*, 152, 73-84.

Terada K, Asai M, Yamagishi M (2003), “Finite cover method for linear and non-linear analyses of heterogeneous solids”, *International Journal for Numerical Methods in Engineering*, 58, 1321-1346.

Terada K, Ishii T, Kyoya T, Kishino Y (2007a), “Finite cover method for progressive failure with cohesive zone fracture in heterogeneous solids and structures”, *Computational Mechanics*, 39, 191-210.

Terada K, Kurumatani M (2004), “Performance assessment of generalized elements in the finite cover method”, *Finite Element in Analysis and Design*, 41, 111-132.

Terada K, Kurumatani M (2005), “An integrated procedure for three-dimensional structural analysis with the finite cover method”, *International Journal for Numerical Methods in Engineering*, 63, 2102-2123.

Terada K, Maruyama A, Kurumatani M (2007b), “Eulerian finite cover method for quasi-static equilibrium problems of hyperelastic bodies”, *Communications in Numerical Methods in Engineering*, 23, 1081-1094.

Terzaghi K (1962), “Stability of steep slopes on hard unweathered rock”, *Geotechnique*, 12, 251-270.

Timoshenko SP, Goodier JN (1970), “Theory of elasticity”, third edition, McGraw-Hill, New York.

Tsay RJ, Chiou YJ, Chuang WL (1999), “Crack growth prediction by manifold method”, *Journal of Engineering Mechanics*, 125, 884-890.

Ventura G, Xu JX, Belytschko T (2002), “A vector level set method and new discontinuity approximations for crack growth by EFG”, *International Journal for Numerical Methods in Engineering*, 54, 923-944.

Walton G, Atkinson T (1978), “Some geotechnical considerations in the planning of surface coal mines”, *Transactions of the Institution of Mining and Metallurgy*, 87, 147-171.

Walton G, Coates H (1980), “Some footwall failure modes in South Wales opencast workings”, Proc 2nd International Conference on Ground Movements and Structures, Cardiff, pp.435-448.

Wang C, Tannant DD, Lilly PA (2003), “Numerical analysis of the stability of heavily jointed rock slopes using PFC2D”, *International Journal of Rock Mechanics and Mining Sciences*, 40(3), 415-424.

Weibull W (1951), “A statistical distribution function of wide applicability”, *Journal of Applied Physics*, 28, 293-297.

Wells GN, Sluys LJ (2001), “A new method for modelling cohesive cracks using finite elements”, *International Journal for Numerical Methods in Engineering*, 50, 2667–2682.

Williams JR, Hocking G, Mustoe GGW (1985), “The theoretical basis of the discrete element method”, *Proceedings of International Conference on Numerical Methods in Engineering: Theory and Applications*, Swansea, pp.897–906.

Xie M, Gerstle WH, Rahulkumar P (1995), “Energy-based automatic mixed-mode crack-propagation modeling”, *Journal of Engineering Mechanics ASCE*, 121, 914-923.

Yau J, Wang S, Corten H (1980), “A mixed-mode crack analysis of isotropic solids

REFERENCES

using conservation laws of elasticity”, *Journal of Applied Mechanics*, 47, 335-341.

Zhang GX, Sugiura Y, Hasegawa H (1999b), “Crack propagation by manifold and boundary element method”, *Proceedings of the Third International Conferences on Analysis of Discontinuous Deformation* (ICADD-3), Vail, Colorado, USA, pp. 273-282.

Zhang GX, Sugiura Y, Saito K (1999a), “Application of manifold method to jointed dam foundation”, *Proceedings of the Third International Conferences on Analysis of Discontinuous Deformation* (ICADD-3), Vail, Colorado, USA, pp.211-220.

Zhang GX, Zhao Y, Ha J (2007), “Structure failure simulation by numerical manifold method”, *Proceedings of the Eighth International Conferences on Analysis of Discontinuous Deformation* (ICADD-8), Beijing, China, pp.235-242.

Zhang GX, Zhao Y, Peng XC (2010), “Toppling failure simulation of rock slope by numerical manifold method”, *International Journal of Computational Methods*, 7(1), 167-189.

Zhang GX, Zhu BF, Lu ZC (2003), “Cracking simulation of the Wuqiangxi ship lock by manifold method”, *Proceedings of the Sixth International Conferences on Analysis of Discontinuous Deformation* (ICADD-6), Trondheim, Norway, pp.133-140.

Zhang HW, Zhou L (2006), “Numerical manifold method for dynamic nonlinear analysis of saturated porous media”, *International Journal for Numerical and Analytical Methods in Geomechanics*, 30, 927-951.

Zhao GF, Zhao J, Ma GW, Zhang HH (2010), “A numerical manifold method for plane micropolar elasticity”, *International Journal of Computational Methods*, 7(1), 151-166.

Zhou W, Kuo X, Yang R (1999), “Crack propagation using manifold method coupled with element free method”, *Proceedings of the Third International Conferences on Analysis of Discontinuous Deformation* (ICADD-3), Vail, Colorado, USA, pp.

283-289.

Zienkiewicz OC, Best B, Dullage C, Stagg K (1970), "Analysis of nonlinear problems in rock mechanics with particular reference to jointed rock systems", *Proceedings of the Second International Congress on Rock Mechanics*, Belgrade, pp.8-14.

PUBLICATIONS

Journal Papers:

1. Ma GW, An XM (2008), “Numerical simulation of blasting induced rock fractures”, *International Journal of Rock Mechanics and Mining Sciences*, 45, 966-975.
2. Ma GW, An XM, Wang MY (2009), Analytical study on dynamic friction mechanism of blocky rock system, *International Journal of Rock Mechanics and Mining Science*, 46(5), 946-951.
3. Ma GW, An XM, Zhang HH, Li LX (2009), “Modeling complex crack problems using the numerical manifold method”, *International Journal of Fractures*, 156, 21-35.
4. Zhang HH, Li LX, An XM, Ma GW (2010), “Numerical analysis of 2-D crack propagation problems using the numerical manifold method”, *Engineering Analysis with Boundary Elements*, 34, 41-50.
5. Ma GW, An XM, He L (2010), “The numerical manifold method: a review”, *International Journal of Computational Methods*, 7(1), 1-32.
6. An XM, Fu GY, Ma GW (2011), “A comparison between the NMM and the XFEM in discontinuity modeling”, *International Journal of Computational Methods* (accepted).
7. Ning YJ, Yang J, An XM, Ma GW (2011), “Modelling rock fracturing and blast-induced rock mass failure via advanced discretisation within the discontinuous deformation analysis”, *Computers and Geotechnics* (accepted).
8. An XM, Li LX, Ma GW, Zhang HH (2011), “Prediction of rank deficiency in partition of unity-based methods”, *Computer Methods in Applied Mechanics and Engineering* (accepted).

Conference Papers:

1. **An XM**, Ma GW (2006), “Numerical Study of Fracture Control Technique for Smooth Blasting”, ISRM International Symposium 2006, *4th Asian Rock Mechanics Symposium*, Rock Mechanics in Underground Construction, edited by C.F. Leung, Y.X. Zhou, World Scientific, 8-10 Nov. 2006, Singapore.
2. Ma GW, **An XM**, Zhou YX (2007), “Numerical Investigation of Shear Mechanism of Rock Joint Using Manifold Method”, *The 8th International Conference on Analysis of Discontinuous Deformation (ICADD-8)*, August 14-19, 2007, Beijing, China.
3. Ma GW, **An XM**, Wang MY (2007), “Numerical Studies on Dynamic Friction Mechanism of Blocky Rock System Using Manifold Method”, *The 8th International Conference on Analysis of Discontinuous Deformation (ICADD-8)*, August 14-19, 2007, Beijing, China.
4. **An XM**, Li JC, Ma GW (2008), “Vibrating Screen Modeling with the Numerical Manifold Method”, *the 1st Southern Hemisphere International Rock Mechanics Symposium (SHIRMS)*, Vol. 2, 205-214, September 16-19, 2008, Perth, Australia.
5. Li JC, **An XM**, Ma GW (2008), “Wave propagation and attenuation through filled rock joints”, *the 1st Southern Hemisphere International Rock Mechanics Symposium (SHIRMS)* Vol. 2, 215-228, September 16-19, 2008, Perth, Australia.
6. Ma GW, **An XM**, He L, Shi GH (2008), “Modeling arbitrary fracture propagation with manifold method”, *The 42nd U.S. Rock Mechanics Symposium (USRMS)*, June 29 - July 2, 2008, San Francisco, CA, USA.
7. **An XM**, Ma GW (2009), “Comparison between XFEM and NMM in modeling arbitrary discontinuities”, *The 9th International Conference on Analysis of Discontinuous Deformation (ICADD-9)*, 25-27 Nov, 2009, Singapore.
8. Ma GW, He L, **An XM** (2009), “Development of the numerical manifold method and its application in rock engineering”, *The 9th International Conference on*

PUBLICATIONS

- Analysis of Discontinuous Deformation (ICADD-9)*, 25-27 Nov, 2009, Singapore.
9. Ning YJ, **An XM**, Ma GW (2010), “Simulation of rock slope failures with the numerical manifold method”, 44th US Rock Mechanics Symposium, Salt Lake City, 27-30 June 2010.
 10. **An XM**, Ning YJ, Ma GW (2010), “Numerical modeling of footwall slopes in surface coal mining with the NMM”, 6th Asian Rock Mechanics Symposium, Oct 2010, New Delhi, India (accepted).