Analysis, modelling and simulation of pop-up laminar structures

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

**ANALYSIS, MODELLING & SIMULATION OF
POP-UP LAMINAR STRUCTURES**

**MAK KIN WENG
SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING
2010**

**ANALYSIS, MODELLING & SIMULATION OF POP-
UP LAMINAR STRUCTURES**

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Abstract

This study aims to build a comprehensive and essential foundation in pop-up topology and geometry. It also proposes a computer-aided, systematic approach for pop-up designs. The research establishes a classification model for pop-up laminar structures, which organizes types of elemental pop-ups in a hierarchical framework. This in turn leads to the establishment of graph representations and mathematical relationships between the pop-up elements.

Through the analysis, pop-up layers were modelled by the Cartesian MV models and mathematical relationships between the mountain and valley creases. They defined the boundary regions for feasible pop-up designs. The cut edges and glue edges of planar vertex-edge G graphs and L graphs were used to establish topological conditions for governing the composition of the structures. Investigation into geometry of the tent and box formations led to the derivation of conditions for flat folding. While the graphs enable simple representation of pop-up structures without the complexity of geometric designs, the mathematical relationships permit the validation of the structures' composition and their ability to flat fold.

The graph structures and the mathematical relationships form the basis of a modular, object-oriented software system developed for this research. The workings of the system, and thus the theories on which it was built, are demonstrated through the examples shown in the various chapters in the thesis, and case studies presented in Chapter Seven.

Glossary

Some of the technical terms in the glossary are referenced to Figures A to D. These figures are found at the end of glossary on page xxv and xxvi.

90° pop-up structure

A pop-up structure that fully erects to its designated position when the two base pages are 90° apart and flattens when the base pages are 180° apart.

180° pop-up structure

A pop-up structure that fully erects to its designated position when the two base pages are at 180° apart.

Accordion pleats

Flat origami folds that comprise alternating mountain and valley creases with a minimum of three creases.

Angle fold

A tent-like pop-up structure with creases converging to a vertex on its body.

Angle of convergence

The angle between the gutter crease and a primary crease on the base page.

Angle of depression

The angle between two primary creases on a face of a tent pop-up layer.

Angle pleats

Pleats with creases that converge to a vertex on the paper.

Base pages

The pair of planar faces that forms the base of the structure, upon which a pop-up structure is built. See Figure B.

Base graph

A planar graph that represents a pop-up layer of the one-piece pop-up structure.

Boundary edge

A cut edge that bounds the base page. See Figure A.

Box fold

A multi-piece, box-like structure with the pop-up layers perpendicular to the base pages when fully erected.

Box formation

The essential configuration of one pair of primary pop-up layer and one pair of secondary pop-up layer to form a box fold.

Box pop-up layer

A pop-up layer of the box fold, comprising both primary and secondary pop-up faces. See Figure B.

Characteristic ratio R

A ratio that differentiate types of tent formations by their cross-sectional distances and heights.

Crease edge

A line produced by folding and incident to a crease vertex. See Figure A.

Crease initiation

The development of a new pop-up layer over a crease or a paper fold from a crease.

Crease vertex

A point of intersection of solely crease edges. See Figure A.

Cut edge

A line formed by a cut or slit on the paper and incident to a cut vertex. See Figure A.

Cut face

An area bounded by only cut edges and represents a slit or a cutout within the paper. It can be seen as an imaginary plane that may be planar or non-planar. See Figure A.

Cut vertex

A point of intersection of crease edges with cut edges or boundary edges. See Figure A.

Cut-and-crease pattern

The pattern of cut lines and crease lines on a one-piece pop-up structure when the base pages are opened to 180° .

Domain

A group of pop-up structures with common features, e.g. the 180° multi-piece tent pop-up structures.

Double-slit

A one-piece parallel or non-parallel fold constructed by two slits.

Flat foldability

The ability for a pop-up or paper fold to fold flat.

Flat vertex fold

A flat origami fold with creases intersecting at a vertex on the paper.

Floating layer

A seven-plane 180° pop-up structure made up of two 90° pop-up structures.

G-graph

A planar graph that illustrates the arrangement of vertices and edges on a one-piece pop-up structure.

Glue edge

A cut edge on the pop-up layer from which a glue tab may be extended. See Figure D.

Glue tab

One planar piece that extends from the edge of a pop-up layer to enable gluing to another layer or the base pages.

Gutter crease

The crease that separates a pair of base pages. See Figure C.

Inter-layer glue edge

An edge on a pop-up layer where a glue tab can be extended to join to the base pages or another pop-up layer. See Figure D.

Intra-layer glue edge

An edge on a pop-up layer where a glue tab can be extended to join its ends in a loop. See Figure D.

Inward fold

A secondary pop-up layer that folds beneath the primary pop-up layer that it is attached to when the pop-up structure is flat folded.

Layer component

A pop-up layer or a paper fold that constitutes to the layering of a pop-up structure.

L-graph

A planar graph that illustrates the arrangement of vertices and edges on a pop-up layer for a multi-piece pop-up structure.

Mountain crease

The assignment of a crease on a primary pop-up layer when its two adjacent faces are down-facing and make an angle of less than 180° between them.

Mountain crease initiation

The development of a new pop-up layer over a mountain crease or a paper fold from a valley crease.

Multi-piece pop-up structure

A pop-up structure constructed with more than one paper piece by cutting and gluing.

MV slope

The gradient defined by the increment of valley creases over that of mountain creases when a pop-up layer is added.

Non-parallel fold

A tent-like pop-up structure with converging creases that intersects outside its body.

Non-parallel pleats

Pleats with non-parallel creases but do not converge to a vertex on the paper.

One-piece pop-up structure

A pop-up structure constructed with one paper piece by slitting.

Outward fold

A secondary pop-up layer that folds away from the primary pop-up layer that it is attached to when the pop-up structure is flat folded.

Parallel fold

A tent-like pop-up structure with parallel creases.

Parallel pleats

Pleats with parallel creases.

Pop-up face

A movable plane bounded by creases and cut lines on the body of a pop-up. It can be a primary or secondary pop-up face. See Figure B.

Pop-up layer

It refers to connected pop-up planar faces on the pop-up structure. It can be a primary, secondary or box pop-up layer. See Figure B.

Primary crease

A crease on the primary pop-up layer that connects its primary pop-up face to another primary pop-up face or the base pages. See Figure C.

Primary pop-up face

A plane of the primary pop-up layer. See Figure B.

Primary pop-up layer

A pair of pop-up planar faces that is attached to the faces of the base pages or other layers. See Figure B.

Primary-secondary crease (*p-s* crease)

A crease that connects a primary pop-up layer to a secondary pop-up layer. See Figure C.

Radial pleats

Pleats with converging creases, namely non-parallel pleats and angle pleats.

Secondary crease

A crease that connects two secondary pop-up faces. See Figure C.

Secondary pop-up face

A plane of the secondary pop-up layer. See Figure B.

Secondary pop-up layer

A pair of pop-up planar faces that is jointed to the edges of a primary pop-up layer. See Figure B.

Single-slit

A one-piece pop-up structure constructed by one slit.

Solid face

An area bounded by crease edges and cut edges, and represents a planar pop-up face or a base page. See Figure A.

Tent formation

The essential configuration of one pair of primary pop-up layer to form an angle fold, a parallel fold or a non-parallel fold.

Valley crease

The assignment of a crease on a primary pop-up layer when its two adjacent planar faces are up-facing and make an angle of less than 180° between them.

Valley crease initiation

The development of a new pop-up layer over a valley crease or a paper fold from a valley crease.

Unbounded face

The area on the exterior of a pop-up structure surrounded by boundary edges.

Figure A. Vertices, edges and faces.

Figure B. Faces and layers

Figure C. Creases

Figure D. Glue edges

Publications

- 1 Tor, S. B., Mak, K.W. & Lee, Y.T. (2004) A Study on the Boundary Conditions of 90° Paper Pop-Up Structures, *Innovation in Manufacturing Systems and Technology (IMST)*, MIT DSpace. Website: <http://hdl.handle.net/1721.1/3921> (Last retrieved on 3 Oct 2010).
- 2 Mak, K.W., Tor, S. B. & Lee, Y.T. (2005) Analysis of 90° Pop-Up Structures for CAD Systems, *Proceedings of International Conference of Engineering Design (ICED)*, Melbourne, Australia.
- 3 Mak, K.W., Tor, S. B. & Lee, Y.T. (2010) Graph Analysis of Pop-Up Laminar Structures. (In preparation for submission to *Journal of Graph Theory*.)

Chapter One

Introduction

In computing, ‘pop-up’ often refers to a window, containing a message or menu, that appears superimposed over the window in use [1] when the button of the mouse is clicked or a special key is pressed. In geographic and seismologic studies, the term is associated with landslides and earthquakes. But when pop-ups refer to laminar structures, they are structures made up of thin plates and erected in 3D. These are used in engineering mechanisms and architectural designs. A more historical and popular reference lies in paper pop-ups. Montanaro [2] presented the history of these paper mechanical devices, which dates back to hundreds of years. Apart from being a traditional art form, pop-up structures are used in practical applications such as the collapsible design of French fry containers at fast food chains. In places like movie theatres and retail stores, layered structures of stationary pop-ups, such as those in Figure 1, seize the attention of passers-by.

(a)

(b)

Figure 1.1. Stationary pop-ups. (a) A door advertisement and (b) a movie stand.

However, pop-ups enthrall most when they are tailored to move, and many of these are designed for cards and books. Taylor and Bluemel [3] provided a comprehensive introductory guide to these pop-up books, detailing their histories, design techniques and resources for collectors. The Movable Book Society has also documented the bibliography of these publications [4]. Pop-up books can be very expensive due to the high cost of manufacturing, and some have been categorized as luxurious craft items. These books aside, guidebooks have been published to teach novice enthusiasts pop-up construction. Inside the books, pages illustrate step-by-step pictorial instructions with supplementary templates. However, hobbyists often make structures from specific designs without understanding crafting techniques behind those constructions. Hence in recent years, several books highlighting geometric rules, such as *The Elements of Pop-up* [5] and *Pop-up! A Manual of Paper Mechanisms* [6], were published.

1.1 Roles in pop-up design

The design of movable pop-ups not only requires talented hands in art but also engineering skills to accomplish precise folds in paper mechanisms. Figure 1.2 illustrates a movable pop-up engaged by folding mechanisms. However, pop-up construction has hitherto deployed traditional manual crafting methods. The efficiency in such designs largely depends on technical experiences of specialists, known as paper engineers, who design such paper crafts.

Figure 1.2. A moveable paper pop-up structure [7].

Hiner [8] and Baron [9] described the roles of paper engineers, which include simplifying and strengthening pop-ups using a dummy and shaping mechanism for ease of production. They also have to establish nesting areas required for production, prepare outline for the illustrators and construct die lines for making the blades that cut printed sheets into individual paper pieces. As these engineers form a very small band of skilled workers in the industry, such specialized skills inevitably come at a high cost.

Collaborations between artists, illustrators, and publishers are crucial. Very often, each in the team has to manually create samples for their respective tasks. Changes to pop-up mechanisms would result in redistribution of the samples. In addition, the team is likely to be based at different locations. So samples have to be dispatched via courier services. Hence, without a common platform that the design team can easily connect to, the design process can be slow and cumbersome.

As the manufacturing of pop-up books and cards largely entails manual work, a huge labour force is required at the production lines. While paper engineers,

illustrators and editors usually locate in US and European countries, the production factories are generally located in developing countries in Asia and South America such as China, Thailand and Ecuador, where wages are lower.

1.2 The production of pop-up books and cards

A typical pop-up book will usually have a first printing of about twenty thousand to thirty thousand, and a popular book can have a first printing of half a million [10]. About five hundred to one thousand and five hundred workers are required for assembling pop-up books in a typical hand assembly plant. The manufacturing cost of a pop-up book is about one quarter of the selling price. It decreases as the quantity of books increases.

The current production of a pop-up book or card can be divided into two stages: the design stage and the manufacturing stage. Figure 1.3 summarizes the production process. From an idea to the end product, Hiner [8] illustrated the entire process, which highlighted costing as a vital factor to consider during the production. Rubin [11] also shared an insightful account of the pop-up book assembling process, from which some illustrations are shown in Figure 1.4.

Due to the extensive manual process, trial and error becomes evident in the design stage, particularly in the sizing of the sketches, folding and aligning the paper pieces. A misfit of the pop-ups on the pages would result in repeated work. As such, the production of pop-up is labour intensive and time consuming. Therefore, it is desirable to minimize unnecessary work to save time and cost.

Figure 1.3. The production process of a pop-up book.

Figure 1.4. A production line for pop-up books [11].

With the advent of computer technology, many computer-aided design (CAD) tools have aided traditional craftworks and enhanced digital living. Examples

include architectural drawing and tailoring. In the research arena, Chua and Chow [12] have studied how computer and automation can take over lengthy and tedious work carried out by skilled engravers from scanning of 2D artwork to 3D surface and relief generation. Likewise for paper crafts, CAD tools for paper models and origami have been developed in recent years. These developments would be further discussed in Chapter Two. In pop-up designs, paper engineers employ graphics software like Macromedia FreeHand in place of manual drawing, tracing and colouring. But for the engineering work in creating pop-up mechanisms, a reduction of manual effort is still advantageous.

1.3 Objectives

This research aims to investigate intrinsic properties of pop-up structures. With the findings, the study seeks to build a comprehensive and essential foundation in pop-up topology and geometry, so as to achieve a better scientific understanding of the craft.

From this foundation, the study also aims to build a prototype to illustrate a systematic approach for computer-aided pop-up designs as an alternative to the manual-orientated method. This, in turn, will improve the efficiency of pop-up construction and enhance the learning of this craft using digital tools.

1.4 Scope

This study focuses on collapsible pop-up laminar structures that are found in most pop-up books and cards. The collapsible structures comprise planar paper pieces and are erected by the action of opening pages. As such, the investigation

of deformable curled paper sheets, 2D mechanisms such as the cam and the use of materials like threads to aid the erection of pop-up structures are excluded in this study.

In particular, this research establishes the classification of the structures through the exploration of their topological and geometrical characteristics. This is done by examining elemental components of the structures such as crease lines and faces of glue tabs. Through the classification, the study discusses the modelling of the structures and the eventual realization of an animation prototype to illustrate opening of pop-up structures and interaction between the planar pieces.

1.5 Organization of the thesis

In the subsequent chapters, the term ‘pop-up laminar structures’ is referred to as ‘pop-up structures’. Chapter Two reviews work associated with paper pop-up structures. These include collapsible designs, types of paper crafts and kinematics. The chapter further surveys related work in science-craft integration, which leads to the focus of this study. Chapter Three elaborates the classification of pop-up structures and describes the elementary elements representing the structures. Chapter Four discusses the crease properties in multi-layered pop-up structures. In particular, it discusses boundary regions for feasible pop-up construction. Chapter Five extends the investigation of crease and cut lines using Graph Theory. Chapter Six examines the geometric attributes of pop-up structures and conditions for their construction. Chapter

Seven looks into the software architecture and case studies for modelling pop-up structures. Chapter Eight concludes with the future work of this research.

Chapter Two

Literature review

Pop-up structures, when closely examined, provide ample ground for exploitation in three areas. They are movable books and toys, collapsible products and paper pop-up construction, as described below.

Pop-up structures are part of movable books and toys. Movable books include those with cut-outs, lift-up flaps as well as rotating wheels, cams and pull-tabs. Many books do not solely consist of pop-up structures but contain a combination of these designs. The Christmas Alphabet [7] and Whales, Mighty Giants of the Sea [13] in Figure 2.1, which have pop-up structures attached to lift-up flaps, better exemplify this case. Jackson [14] defines a pop-up as a self-erecting, three-dimensional structure, formed by the action of opening a crease. Although this study follows Jackson's definition, it is noted that some collapsible pop-up structures require manual assistance to raise them because they do not self-erect.

(a)

(b)

Figure 2.1. Lift up flaps with pop-up structures. (a) The Christmas Alphabet and (b) Whales, Mighty Giants of the Sea

Another attribute of pop-up structures is its collapsible design. Crease lines are a vital part in collapsible paper products. Collapsible designs appear in a wide range of products, which include foldable chairs, retractable antennas and deployable medical instruments. Section 2.1 discusses characteristics of such collapsible structures.

Undoubtedly, one characteristic of pop-up structures lies in its pop-up mechanisms. Pop-up construction has been used in many other types of paper crafts. Pop-up structures can be constructed as part of a foldable paper sculpture or exist as glued-on pieces on paper structures. Though not common, the use of pop-up structures can also be found in paper cutting and origami. Their applications in paper crafts are explored in details in Section 2.2. The Venn diagram in Figure 2.2 describes the association of pop-up structures with the three areas discussed above.

Figure 2.2. Association of pop-up structures with related areas.

It should be noted that examples given in the groups or sub-groups in Figure 2.2 do not represent all paper crafts of the same type. For example, it is not true to classify all paper sculptures as non-collapsible, as some can be made collapsible. On closer scrutiny, paper crafts like origami have been found to hold interesting properties that aid developments in mathematics and science. Section 2.4 expounds these recent developments.

2.1 Collapsible design

When examined closely, the McDonald's French fries container has an interesting feature that not many would instantly take note of. Figure 2.3 shows this common example of collapsible products, whose creases enable its body to flatten when not in use. In fact, numerous daily products have collapsible designs. These include table calendars, umbrellas and gusseted plastic bags. Yet, many users have taken them for granted, without realizing the ingenuity of their designs.

Figure 2.3. A French fries container. (a) Fully erected and (b) collapsed flat.

2.1.1 Characteristics of collapsible design

Collapsible bodies have three predominant features: a jointed structure, a storage intent and the capability to reverse in form. These features will also define the collapsibility of pop-up designs examined in this study.

(a) A jointed structure

Structures with collapsible designs have joints, both fixed and movable. They can be categorized into three main forms: foldable, telescopic and a combinational form of first two types. They also do not have parts that are removable. Some rotatable arms of collapsible bodies can be detached at one of their ends to enable folding but they cannot be removed from the bodies.

(b) A storage intent

Though collapsible bodies have joints, it would be erroneous to assume that all bodies with joints are collapsible. An exclusion from collapsible bodies is the hinged wooden door, which does not transform in shape or size. An adjustable office chair further exemplifies the point. Its sliding joints allow its height to be altered but they do not enable it to be compactly stored. Therefore, collapsible bodies not only have joints but they can also be stored in compact size. Hence without a storage intent, objects with foldable or telescopic parts are not collapsible bodies.

(c) Capability to reverse in form

Another feature that governs the collapsible design lies in the reversibility of their bodily forms. A change in form here refers to the alteration of shape and

volume. The condition of form reversibility is that after collapsing to its storage form, the body should be able to transform back to its initial form. For example, used aluminium cans or empty plastic bottles can be crushed prior to disposal to maximize storage space in trash bags. This action results in disorientated folds on the cans and bottles. However, the deformed objects cannot be reverted to their original forms. Therefore, such bodies are not classified as collapsible ones.

2.1.2 Classification of collapsible design

With a wide array of collapsible designs, some are practically simple while others are complex; it becomes difficult to classify them. One of the better approaches to organize them is to investigate their joints. Collapsible designs, by and large, can be covered by two joint types: definite joints and indefinite joints.

(a) Definite joints

In the field of kinematics, joints can be classified into two basic types: the revolute and the prismatic joints [15]. They produce purely rotational and linear motions along their joint axes respectively. Foldable bodies contain revolute joints while telescopic ones use prismatic joints or sliding mechanisms. Collapsible bodies with revolute joints include baby prams, cardboard boxes, foldable chairs, extension ladders, Chinese lanterns and wheelchairs. Those that make use of prismatic joints include extendable tubes and poles such as reception antennas of radio sets and extendable clothes-hanging poles. Figure

2.4 shows some common collapsible products with revolute joints and prismatic joints.

Figure 2.4. Collapsible designs. (a) Foldable chairs, (b) cardboard holders and (c) a retractable radio antenna.

Some objects have collapsible designs that incorporate both revolute and prismatic joints at different body parts. For example, an umbrella has extendable metal stems, yet its out-stretched arms have revolute joints. Another example is the legs of a camera tripod stand. See Figure 2.5. However, the incorporated use of revolute and prismatic joints is not to be mistaken as that of screw joints, which comprises a simultaneous rotation and linear motion. Screw joints are rarely found in collapsible product designs.

Figure 2.5. Integrated use of joints. (a) An umbrella's metal frame and (b) a camera tripod stand.

(b) Indefinite joints

There is a unique group of objects that can be considered collapsible bodies. Though foldable, they are made of non-rigid materials and do not have definite joints. Examples include clothes and inflatable objects such as the air-filled chair. A pair of trousers properly folded in sections and kept in the wardrobe would result in joints formed at the fold positions. But as fabric can be folded in numerous ways, they do not possess fixed joints. Two assumptions have to be made for these collapsible bodies. Firstly, folds are completely removed after a reversal to the incompact form, e.g. no crisp wrinkles on clothes. Secondly, the folds have sharp turning points. In other words, they are not curved. An attempt was made to group them under the classification of joint types in Figure 2.6.

Figure 2.6. Classification of collapsible bodies.

The above classification defines a clearer scope on collapsible design. The challenge would be to devise a measure for the extent of collapsibility. An approach to obtain this measure would be to formulate mathematical models that facilitate the three collapsibility conditions mentioned in the preceding discussion. Then design work can be efficiently assisted. Since interferences

can be oblivious to the eyes and are often undetected till the prototyping phase, such models can simulate motion of collapsible products for design verifications. Pop-up structures provide a good foundation for such collapsible studies.

2.1.3 Collapsibility in cards and books

One distinctive collapsible product available commercially is Santoro Graphics' Swing Card, as shown in Figure 2.7. The Swing Card is characteristically similar to the pop-up card as both can turn into three-dimensional structures and also collapse flat. It also has 'a multiplicity of moving parts that allow the objects in the center to swing back and forth', as quoted from Santoro's online site [16]. However, different approaches are used to erect both types of cards. While a pop-up card requires the opening of pages to erect its 3D structure, the swing card purely uses gravity to position its paper pieces in their 3D forms.

Figure 2.7. Examples of Swing Cards.

In addition, the pop-up structure is transformed into 3D by creases whereas the Swing Card uses slots and hooks. When the card is placed vertically, the hooks and slots enable the pieces to move and balance with one another in the air,

thereby forming a 3D structure. Santoro's trademarked products are however not to be mistaken for other swing cards commonly found in craft books and websites. The term 'swing card' is generally used to describe cards with rotatable parts like the Flip Flop card.

Among all the paper products with collapsible designs, the pop-up book deserves notable mention. It mainly aims to create visual interest along a storyline, bringing surprise 3D elements over ordinary 2D texts and graphics to the readers. A pop-up book can also be seen as a collection of pop-up cards attached together. Therefore, pop-up books and cards have the same technical considerations. The three fundamental considerations are firstly, the pop-up structures are stored as flat-folded planar pieces between pages when the book is closed. Secondly, the number of layers in a pop-up structure contributes to the thickness between two pages, and thus the thickness of the whole book. Thirdly, the size of the pages restricts the dimensions of a pop-up structure. The pop-up pieces cannot jut out of the pages when the book is closed. Else, they are unsightly and are prone to damages.

There are two basic parameters to describe collapsible pop-up structures found in cards and books, specifically the number of paper pieces required to make a pop-up and the angle between the base pages on which the pop-up can erect fully. Table 2.1 describes common pop-up structures introduced by Jackson [14] using these two parameters. For multi-piece pop-up structures, a pair of base pages is considered as one paper piece used. Pop-up structures are usually designed to erect fully at 90° and 180° though they can erect at other angles.

Most 90° pop-up structures are also one-piece structure, as shown in Table 2.1.

Chapter Three elaborates these parameters.

Table 2.1. Descriptions of collapsible pop-up structures.

Collapsible pop-up structures	No. of paper pieces used	Angle at which pop-up structures are fully erected
Single-Slit	1	90°
Double-Slit	1	90°
Wings	1	180°
Boxes	2	180°
Cylinders	2	180°
Floating Layers	5	180°
Horizontal V	2	180°
Pivots	2-3	180°
Scenery Flats	3-4	90°
Straps	4	180°

2.2 Paper crafts and their construction techniques

This section examines and compares common forms of paper crafts. The paper crafts discussed here are paper pop-up structures, paper cuttings, origami, paper sculptures and 3D paper models.

2.2.1 Paper pop-up structures

Pop-up structures that paper engineers complexly mold in varied forms can be described by their distinct structural compositions. These pop-up structures are most perceptible when they are broken up into simpler elements of creases and rotatable planes. These elements also differentiate the collapsible structures from the non-collapsible forms like stationary advertisement stands and many paper sculptures.

In addition, pop-up structures differ in crafting techniques. While collapsible pop-up structures are constructed by folding, scoring, tabbing and fitting through slots, non-collapsible ones are mounted by sticking thick foams and springs at their backs. Paper engineers also make pop-up structures by selecting techniques that best represent or highlight the motions of the objects or scenes. Pop-up crafting techniques are, in essence, methods of coordinated creasing, cutting and gluing.

Paper pop-up structures belong to the same category of paper crafts as paper cutting, origami and paper sculpture. Their physical attributes and crafting techniques have interesting similarities as well as dissimilarities. For example, though a pop-up may make a paper craft three dimensional, not all 3D crafts can be considered pop-up structures. Unless they have been constructed with pop-up techniques and demonstrate their attributes of collapsibility, origami and paper models in general are not considered pop-up structures. However, there are exceptional cases where pop-up structures techniques are applied to other types of paper crafts. For example, a paper model may contain parts developed from pop-up techniques. Each type of paper crafts is further discussed in the following sub-sections.

2.2.2 Paper cuttings

From Wayang Kulit in Indonesia to Papel Picado in Mexico, paper cutting has been a traditional folk art that has its origin in religious and ceremonial occasions, and is part of cultural heritages in countries like China. Hand-made by artisan families, delicate cutouts have been widely used as banners, masks

and ornaments. Other than paper, lamina wood pieces and plastic sheets provide alternate working media for art. Intricate Chinese wooden bookmarks have been carved out using similar paper cutting techniques. A practical use of paper cutting is in the making of stencils. Figure 2.8 shows examples of products created by paper cutting.

Figure 2.8. (a) Chinese paper cuts and (b) a wooden bookmark.

Cutting techniques in paper cutting can vary from chiseling, hole punching to techniques using traditional tools like craft knives and scissors. In paper cutting, folding is often done prior to cutting symmetrical shapes and patterns. On the other hand, creases on pop-up structures are created as joints. Hence, folds are used with different intentions. Sometimes, cutouts are affixed together to create 3D hanging ornaments and luminaries. Though not frequent, small, simple pop-up structures are made on cutouts to form interesting patterns and textures.

2.2.3 Origami

Origami can extend beyond the scope of art. The napkin folds found on dining tables in restaurants is an example of many practical uses that origami provides. The purpose of origami is to attain designated shapes. However, from the perspective of an origami purist, it means more than that: An origami figure

from an uncut sheet was better than one from a cut sheet; a single sheet was more desirable than two; and a fold from a square was preferred over any other starting shape [18].

(a) Comparison between origami and pop-up structures

Just as origami uses one or more sheets, pop-up structures can be made from a single piece or from multiple pieces. Origami deploys complex folds like Sink Fold and Inside Out, which are not used in pop-up construction. While origami does not necessarily involve cutting, both pop-up structures and origami have similar characteristics like their ability to flat fold and the differentiation of mountain and valley creases. These are further discussed in Section 2.4.3. Like pop-up structures, origami can produce 3D folds such as the ‘cube of balloon’ described by Kawasaki [17]. In some books, readers are taught creative folds to construct pop-up cards. For example, Shafer [18] introduced pop-up cards that are made by origami using a single piece of paper. In some paper artwork, origami-folded figures are added onto pop-up structures.

(b) Pleats

Pleats, a common type of origami characterized by a chain of repeated folds, can be found on pleated clothes. Practical uses are found in collapsible objects such as Chinese paper fans, water bottles and lanterns. Accordion pleats can be incorporated into pop-up designs though their use is rare. These pleats comprise a train of zigzagged folds, and can be dichotomized into parallel pleats and radial pleats. In this study, radial pleats that converge to a real vertex are known

as angle pleats. Radial pleats that do not converge to a real vertex are termed non-parallel pleats. Section 4.6 discusses more about these pleats.

(c) Modular origami

Modular origami entails the folding of similar paper pieces known as units. These units are then assembled together, usually without using glue, to form polyhedral or other three-dimensional geometric structures like spiky balls, as illustrated in Figure 2.9a. In his book on modular origami, Mitchell [19] introduced a comprehensive collection of such structures such as The Rhombic Tetrahedron and the Great Dodecahedron. The Chinese modular origami, also known as block folding origami by the Japanese, falls within this class of origami. Though the repeating units are joined together, the products often do not have regular shapes. They are also not flat foldable. For example, models of animals like the swan and the peacock, as shown in Figure 2.9b, are common structures made with modular origami.

(a)

(b)

Figure 2.9. (a) A spiky icosahedron [19] and (b) a model of a peacock [20].

2.2.4 Paper sculptures

Jackson [22] described a paper sculpture as many cut-out pieces of coloured paper, which are then rolled, creased, bent or otherwise mangled, and then glued together to form a picture. See Figure 2.10. Paper sculptures have very similar crafting techniques as that of pop-up structures: both paper sculptures and pop-up structures frequently employ techniques such as scoring and tabbing. In fact, many paper sculptures contain non-collapsible pop-up structures, and these pop-up structures are used to create textures and patterns. In addition, paper sculptures and pop-up structures use curve creasing while origami can only be constructed with straight creases. Very often, contrasting backing colours or the use of light are used to enhance both pop-up cards and paper sculptures.

Figure 2.10. Forms of paper sculptures [22, 23].

2.2.5 3D paper models

Unlike paper sculptures which project perspective view using planar layered paper pieces, 3D paper models are proportionally scaled-down, identical replicas of real objects. To make a paper model, usually one cut piece of paper is used to form most of the model's body. Creasing, tabbing and gluing are

common techniques applied in paper models. Touch-3D [24], a modelling program, enables users to design 3D objects that can be unfolded into 2D planar pieces. The Canon website [25] also offers a range of paper model designs that Internet users can print and make on their own.

Figure 2.11. A paper model of a glued airplane.

Figure 2.11 illustrates a model constructed using templates from the website. Glued models are generally not collapsible. But those that are collapsible typically have symmetrical shapes and planar faces. Such models can be found in pop-up books. Table 2.2 reviews the characteristics and construction techniques commonly used for the above mentioned paper crafts.

Table 2.2. Commonly used crafting techniques in paper crafts.

Types of Paper Crafts		Collapsible Pop-up structures	Paper Cutting	Origami	Paper Sculpture	Paper Model
Commonly Used Crafting Technique	Creasing	Yes	Yes	Yes	Yes	Yes
	Indenting	Yes	No	No	Yes	Yes
	Scoring	Yes	No	No	Yes	No
	Cutting	Yes	Yes	Yes	Yes	Yes
	Chiseling	No	Yes	No	No	No
	Pleating	Yes	Yes	Yes	Yes	No
	Crimping	No	No	No	Yes	Yes
	Tabbing	Yes	No	No	Yes	Yes
	Gluing	Yes	Yes	No	Yes	Yes
	Curling	Yes	No	Yes	Yes	Yes
	Bending	Yes	No	Yes	Yes	Yes

2.2.6 Uses of light and sound

Paper crafts are often accompanied by techniques in light and sound. Paper and light produce stunning effects when they interact. Paper structures are often highlighted when light shines on them. For the purpose of decoration, many paper crafts like paper cut ornaments and pop-up structures illuminate and cast fascinating shadow imageries on the surroundings. Light also enhances the three-dimensional effect of structures like those in pop-up cards and paper sculptures.

A more interesting application of papers in paper crafts is to use them as sound-producing elements. Sound provides an entertaining way to emphasize the moving of pop-up mechanisms. To do that, paper engineers make use of interfering paper pieces, to produce clicking or clanking sounds as pages are turned. These pieces normally have teeth-liked edges that slice angularly onto one another. The Elements of Pop-up [5] named such pop-up structures as Noise Makers. Beside pop-up structures, paper crafts like origami also enable sound production. An example would be origami toys that snap when two paper pieces hit against each other. Table 2.3 summaries the uses of light and sound in paper crafts.

Table 2.3. Uses of light, movement and sound in paper crafts.

Types of Paper Crafts	Collapsible Pop-up structures	Paper Cutting	Origami	Paper Sculpture	Paper Model
Use of light	Yes	Yes	Yes	Yes	No
Use of sound	Yes	No	Yes	No	No

Figure 2.12 shows the association of pop-up structures with origami, paper cutting, paper sculptures and paper models. In conclusion, pop-up structures cannot be regarded as stand-alone crafts. Pop-up structures are used in the crafting paper structures and origami can be folded into pop-up structures. At the same time, the feasibility of other combinations, like that between origami and paper sculptures, cannot be ruled out, because they apply common techniques.

Figure 2.12. Association of pop-up structures with other paper crafts.

2.3 Kinematics analysis on pop-up structures

In Section 2.1, joints have been used to describe creases of the collapsible pop-up structures. On the ground of mechanics, the linkages in pop-up structures refer to the planar paper pieces. In this area, Winder et. al. [26] has established planar and spherical kinematics in common pop-up mechanisms. Linkages in pop-up structures, however, do not have to be constructed by flat, rotatable pieces, as shown in Figure 2.13a. Instead, some pop-up structures have curved

planes, as shown in Figure 2.13b. The cylinder structure mentioned in [5], for example, causes a flat piece of paper to bend and rise as the pages are opened.

Figure 2.13. Linkage on pop-up structures. (a) A tent structure with rotatable planes and (b) a cylindrical structure with a curved plane.

2.3.1 Bar linkages

Most pop-up structures for both one-piece and multi-piece types originate from four-bar linkage mechanisms. Figure 2.14 uses a basic double-slit structure as an example to illustrate the four-bar linkage of a one-piece pop-up structure. Assuming that one of the base pages does not move, the adjacent base page that opens up would form the lever or rocker of a four-bar linkage while one of the two protruding pop-up pieces would act as a coupler [27]. Of those with more than four linkages, one example is the box fold structure - a formation of an eight-bar linkage. Multi-layer pop-up structures like Steps and Trellises, as shown in Figure 2.15, are multi-loop linkages. The base pages actuate the layers connected to them. In turn, these layers actuate the other layers on the mechanisms. Most pop-up structures such as the double-slit and Trellises are planar mechanisms that allow rotary motion about their creases and have one degree of freedom. This means that one input to any one link on a mechanism will yield definite motions in all the links.

Figure 2.14. (a) A double-slit structure and (b) the view of the four-bar linkage from its side.

Figure 2.15. (a) Steps and (b) trellises.

In particular, the parallel box fold produces motion that is similar to the Sarrut mechanism discussed by Bennett [28]. It has six connected planes and creases that act like parallel hinges like the Sarrut mechanism. Among these planes, a pair has movement that is rectilinear to each other upon opening or closing the base pages, producing one degree of freedom. The generalized form of the Sarrut mechanism, comprising all-concurrent hinges, can also be used to describe the mechanism of non-parallel box folds.

2.3.2 Planar and spatial linkages

When all the creases parallel to one another, a pop-up structure produces planar movement. Hence, a double-slit structure forms a planar four-bar linkage. Conversely, if any one of the creases is not parallel to the others, a spatial linkage mechanism is formed. For example, the Horizontal V generates a spatial four-bar linkage because its creases converge to a point. Figure 2.16 gives examples of pop-up structures with planar and spatial movements. It is also possible to construct a multi-layer pop-up structure with a combination of planar and spatial movements, such as one comprising scenery flats and a box fold. Table 2.4 tabulates the linkage characteristics of various types of pop-up structures.

Figure 2.16. Pop-up structures with (a) planar movement and (b) spatial movement

Table 2.4. Linkage characteristics of pop-up structures.

Collapsible Pop-up structures	Type of Linkage Mechanism	No of bar linkages	No of joints
Single-slit	Spatial	4	4
Double-slit	Planar / Spatial	4	4
Horizontal V	Spatial	4	4
Scenery Flat	Planar	4	4

2.4 The integration of craft and science

Although for hundreds of years, paper crafts have been mainly perceived as art forms, they are now deeply surveyed through the eyes of science. The integration of craft and science results in a fresh two-way development, as depicted in Figure 2.17.

Figure 2.17. The integration of craft and science.

2.4.1 Craft development for science

Paper crafts have alternatively been used in mathematics education and physics applications. Such developments have rapidly grown over the past decade, particularly in origami. For example, Murakami [29] deployed paper folding operations to describe the trisection of an arbitrary angle. Alperin [30] used origami constructions to explain the relation between field theory and geometry. Similarly, Huzita [31] has developed a set of origami axioms that describe geometry constructions usually done with compass and rulers.

In applications, origami's crease patterns have been explored for foldable designs in telescopes and safety airbags in vehicles [32]. A vital aspect in the design of these applications would be the ability to store in compactness. Buri and Weinand [33] examined folded plate architecture with origami patterns.

Nojima [34] also modelled the folding of thin flat sheets and cylindrical shells after origami. Possible applications of such models are in the construction of deployable space structures and the manufacture of daily products like bottles for soft drinks. Hence with its potential benefits, there is a necessitated call to look into paper creasing.

In the craft of pop-up structures, it has been used as an educational aid to provide visual effects in many children books. However, the craft's development for scientific uses has been less common. One case in point is Diego Uribe's [35] attempt to stimulate children's interests in fractals through pop-up card making. On the same platform, Etheredge [36] investigated the use of pop-up books to educate students in engineering designs.

2.4.2 Scientific development for crafts

This area of development is significantly represented by a growing number of software applications developed to design crafting work. Blauvelt, Wrensch and Eisenberg [37] envisioned such development in computer-aided design (CAD) as a strategy for blending crafts and computers. Eisenberg and his research group [38, 39] described their collective effort in developing intelligent toys, specifically computation and mathematical crafts that are suitable for children education. In particular, Nishioka and Eisenberg [40] have developed HyperGami, an educational Java based program for creating paper polyhedral models. Programs like Touch 3D [24] and Pepakura Designer [41] are equipped with the ability to create 3D models as well as unfolding functions for printing and subsequent jointing purposes.

Based on paper folding methods, Glassner [42, 43] developed a new approach to construct polyhedral models of platonic solids like the cube and the octahedron without cutting and gluing. Bangay [44] discussed the use of heuristics and algorithms to solve paper folding problems for complex polyhedral models using a sheet of paper. Likewise in the field of origami, programs like Cosmo Player [45] and White Dune [46] have been built to generate Virtual Reality Modelling Language (VRML) models of modular origami as visual aids for origami assembly. Miyasaki [47] also constructed a virtual manipulation system that simulated the folding of a paper face on a graphic screen by selecting its corner vertex with the mouse. Similarly, Mitani introduced computer-aided design for origami architecture models using Voxel data structures [48] and polygonal representation [49]. In addition, Robert Lang [50] developed a unique program, TreeMaker [50, 51], which can compute complex crease patterns for origami bases, as shown in Figure 2.18.

Figure 2.18. Origami developed with TreeMaker [50] (a) A planar graph, (b) the crease pattern and (c) the final origami fold.

Apart from the dedicated attention given to virtual paper folding, there has been in-depth exploration in recognizing and converting 2D origami drawings into interactive digital forms. For example, Kato [52] devised algorithms to

recognize dashed lines and arrows in origami books, and Shimauchi [53] created a way to recognize crease patterns from 2D sketches. Company [54] also developed an engine that transforms 2D sketches to 3D polyhedral and wire-frame objects.

The construction of paper pop-up structures has also not been left out from the realm of computational science. Uehara and Teramoto [55] studied the complexity of a pop-up book by examining the opening and closing of the book. Mitani created the 3D Card Maker [56], subsequently renamed as Pop-up Card Designer, that generates one-piece, 90° pop-up structures. Zhang [57] followed on with the development of an origamic architecture for one-piece pop-up structures. Chapter Seven further assesses programs for pop-up design.

As a common technique used in many paper crafts, creasing serves an important role in origami, paper cutting and pop-up structures. Creases of paper crafts have been technically explored in the last two decades. The following sections highlight the developments in flat origami, paper cutting techniques and modelling of pop-up structures.

2.4.3 Flat origami

The ability to flat fold has always been of interest in paper folding. The analysis on origami's crease patterns has yielded interesting results. For example, Bern [58] examined given crease patterns of flat origami. Geometrical properties of creases such as the angles between intersecting creases and the mountain-valley

crease attributes have been investigated for flat origami. Two theorems highlighting these properties are presented below.

a) Maekawa-Justin theorem [59]

If M is the number of mountain creases and V is the number of valley creases met to form a vertex in a flat folded origami, then

$$|M - V| = 2 \quad (2.1)$$

Maekawa stated that the mountains and valley folds of a flat single vertex fold pattern differs by two in order to achieve flat foldability. This theorem has also been asserted by mathematician Jacques Justin [60]. Conversely, it is not true to assume that the mountain-valley crease relation determines a flat vertex fold. Figure 2.19 gives a basic illustration of a flat vertex fold. Dash-dot lines and dash lines represent mountain and valley creases respectively.

Figure 2.19. (a) The crease pattern of a flat vertex fold and (b) its folded state.

In his paper, Thomas Hull [61] discussed the case of a non-intersecting single crease, as shown in Figure 2.20, which leads to $|M - V| = 1$. For Maekawa-Justin theorem to stand, Hull considered such a crease to be made up of two lines joined at a vertex. Hence, the crease either exists as two mountain creases or two valley creases, and this aligns with the theorem. However, another view

is not to consider cases with non-intersecting creases as flat vertex folds. A constraint can be set up on par with the theorem. This is further elaborated in Section 4.6.2.

Figure 2.20. (a) The crease pattern of a non-intersecting crease and (b) its folded state.

b) Kawasaki-Justin theorem [59]

Let $2n$ crease lines meet at a vertex and α be the angle between 2 crease lines in an origami. If $\alpha_1, \alpha_2, \dots, \alpha_{2n}$ are consecutive angles, then the folding of the crease lines would result in a flat vertex fold if and only if

$$\alpha_1 - \alpha_2 + \alpha_3 - \dots - \alpha_{2n} = 0 \quad (2.2)$$

Figure 2.21 illustrates the angles from a folded piece of paper. Hull [61] stated that the sum of the alternate angles about the vertex is π . Kawasaki has also referenced this relation to Husimi's theorem [19].

Figure 2.21. (a) The angles on a crease pattern of a flat vertex fold and (b) its folded state.

However, Hull reviewed that the above two theorems do not describe more than one vertex fold in general. In flat multiple-vertex folds, the theorems hold for each local vertex but assigning mountain-valley attributes to all the creases on the crease pattern may lead to ambiguous results.

c) Counting of mountain-valley assignments [59, 62]

Hull developed recursive functions C that enable the counting of possible mountain-valley crease combinations for a flat vertex fold. The theorem states that if $\alpha_i = \alpha_{i+1} = \alpha_{i+2} = \dots = \alpha_{i+k}$ and $\alpha_{i-1} > \alpha_i$ and $\alpha_{i+k+1} > \alpha_{i+k}$ for some i and k ,

and if k is odd, then

$$C(\alpha_1, \dots, \alpha_{2n}) = \binom{k+2}{\frac{k+1}{2}} C(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+k+1}, \dots, \alpha_{2n}). \quad (2.3)$$

else if k is even, then

$$C(\alpha_1, \dots, \alpha_{2n}) = \binom{k+2}{\frac{k+2}{2}} C(\alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} - \alpha_i + \alpha_{i+k+1}, \alpha_{i+k+2}, \dots, \alpha_{2n}) \quad (2.4)$$

The functions operate by observing the conditions given in the theorem, i.e. locating the smallest angle in the angle sequence $\{\alpha_1, \dots, \alpha_{2n}\}$ from a flat vertex fold. The angles combine according to the functions, and the length of the sequence is reduced till all the angles in it have equal values. The sum of the angles in the sequence is always 360° . The basis of the recursive functions lies in the Maekawa-Justin and Kawasaki-Justin theorems. In another paper, Hull [62] provided the following example of six crease lines with angles 100° , 70° , 50° , 40° , 30° and 70° .

$$\begin{aligned}
C(100, 70, 50, 40, 30, 70) &= \binom{2}{1} C(100, 70, 50, 40 - 30 + 70) \\
&= \binom{2}{1} C(100, 70, 50, 80) \\
&= \binom{2}{1} \binom{2}{1} C(100, 70 - 50 + 80) \\
&= \binom{2}{1} \binom{2}{1} C(100, 100) \\
&= \binom{2}{1} \binom{2}{1} 2 = 8
\end{aligned}$$

Thus, for the given angle combination, there are eight ways to assign the mountain-valley attributes to the crease lines. Based on previous ideas of vertex folds, Gibson and Huband [63] also discovered other properties of fold-cut angles. Likewise, different combinations of mountain-valley assignments for pop-up structures' creases exist. This is further discussed in Chapter Four.

2.4.4 Paper cut polygons with one straight cut

Demaine [64] developed a method to achieve any desired cut polygons by one straight cut after a piece of paper is folded flat. The polygon is described as a planar cut graph on the paper. Figure 2.22 demonstrates the method.

Figure 2.22. Folding a paper to make a star from one straight cut [63].

Before executing the single straight cut, a skeleton has to be drawn on the cut graph, as shown in Figure 2.23. The skeleton comprises bisectors passing through each pair of adjacent edges on the cut graph. The adjacent bisectors then meet within the cut graph to form the eventual skeleton. By folding the crease pattern of the skeleton, the edges of the cut graph will align as a single line in the flat folded stage. This single line produces the desired shape when cut.

Figure 2.23. (a) A cut graph of a cross (thick lines) and (b) its skeletons (thin lines).

However in many cases, the skeleton does not allow flat folding because three bisectors meet at the vertices. To achieve flat folding, the number of lines that meet at the vertices has to be even. This can be derived from Maekawa-Justin Theorem, which Hull also provides a proof [62]. If N is the number of creases,

$$\begin{aligned}
 N &= M + V \\
 &= (M - V) + 2V \\
 &= 2 + 2V
 \end{aligned}
 \tag{2.5}$$

where M and V are the number of mountain and valley creases.

Hence, cut edge perpendiculars are added so that the number of creases at each vertex is even and flat folding is permissible. Hence paper cut, similar to origami, can be described with geometrical studies using crease patterns and

application of theorems for flat folds. As shown in Figure 2.24, the perpendiculars cut the edges of the cut graph at 90° .

Figure 2.24. Perpendiculars on the cut graph (dashed lines).

The real perpendicular, as Demaine defined, is incident to a skeleton vertex and folding is done along this line. With the addition of real perpendiculars, the number of crease line becomes even.

2.4.5 Pop-up modelling

In the modelling of pop-up structures, Lee, Tor and Soo and Glassner studied the motion of the paper craft for the purpose of creating a computer-aided design system. In the work of Lee, Tor and Soo [65], pop-up mechanisms have been categorized according to their behaviour upon opening the pages, namely pop-up pieces that remain planar and those that are allowed to bend. The focus on their analysis lies on the former behaviour. For pop-up structures to be non-deformable, the group introduced the following three conditions.

- Every crease line has to be straight.
- At least four creases and four planar pieces are needed to construct the pop-up.

- Crease lines have to be all parallel or all concurrent for pieces to be always planar foldable.

While the group did not elaborate on the second condition, this crease-planar face relation is further discussed in the Chapter Four. The group also provided a proof for the third condition, which refers to the flat foldability of pop-up structures. The similarity in flat foldability of origami and pop-up structures can be drawn from the case of the flat vertex fold and the pop-up's instance of all concurrent crease lines. Both have even numbers of crease lines incident at a vertex, which contribute to flat foldability.

Figure 2.25. Locating angles on a pop-up structure with parallel creases [64].

In the mathematical derivation for the modelling for pop-up folding and unfolding motions, Lee, Tor and Soo separated the investigation into two general cases: parallel crease lines and crease lines that converge to a point. Figure 2.25 illustrates one of the cases. While they sought angles between base pages and their adjacent pop-up pieces to establish the positions of the structures, Glassner [66, 67] located the loci of a pop-up's motion using three

overlapping spheres. In his work, Glassner derived pop-up geometry using the single-slit and the V fold, also known as the Horizontal V. Both have creases that converge to a point. His technique lies in finding the intersecting point of the three spheres, which locates a moving vertex on the pop-up. Figure 2.26 illustrates the case for the single-slit.

Figure 2.26. Locating a moving vertex using intersecting spheres [65].

Lee, Tor and Soo further discussed the animation for a pop-up structure with multiple levels. Likewise, Glassner described the construction of pop-up layers using his pop-up design assistant software as ‘generation’. In the area of educational technology, Hendrix [68, 69] developed Popup Workshop to encourage elementary students to experiment the craft with complex, aesthetical designs.

In another dimension of modelling, the MagicBook [70] has catered to the study and development for augmented reality (AR), from which readers view animated 3D virtual models from the pages using an AR display as they browse the book. Billingham et al. referred the AR views in MagicBook as an enhanced version of the traditional 3D pop-up book. Dünser and Hornecker [71] also investigated augmented books as an educational tool for young

children. The interactive nature of these digitally enhanced books has also paved the way for Mixed Reality (MR) systems such as the Little Red project [72]. Despite their capability to generate and animate 3D virtual objects, augmented books do not fully model pop-up books. Their characterization of pop-up structures falls short of the deployment of pop-up mechanisms.

2.4.6 Mathematical properties of pop-up structures

In recent years, an increasing number of books such as *The Elements of Pop-up* [5] and *Paper Engineering for Pop-up Books and Cards* [73] have been published to educate craft enthusiasts on simple mathematical relationships applied to pop-up construction. Figure 2.27 illustrates two simple geometric properties of pop-up structures found in such books. Very often, these mathematical relationships are limited to the construction of basic pop-up structures. A more thorough study of the mathematics intrinsic in pop-up structures and how they could aid the craft enthusiasts in pop-up design would be useful in developing software for pop-up constructions.

Figure 2.27. Simple geometric properties in pop-up structures. (a) $\angle\alpha = \angle\beta$ on an angle fold and (b) $a = b$ on a parallel fold.

2.5 Research focus

Thus far, much of the research in pop-up design has primarily delved in one-piece pop-up structures. Multi-layer pop-up structures have also not been deeply surveyed. The categorization of pop-up structures based on their topological and geometrical properties remains to be explored. The subsequent chapters, examines the above-mentioned areas in detail.

Figure 2.28. Research in collapsible pop-up structures.

Figure 2.28 shows a diagram illustrating regions of research areas for collapsible pop-up structures. It also illustrates how they associate with other paper crafts and how the software architecture can fit into this research. The areas worth investigating are the classification of pop-up structures, layering, pop-up topology and geometry and 3D modelling. The next chapter introduces elements of pop-up structures and classifies the types of structures with a proposed model.

Chapter Three

Definition, classification and representation scheme of pop-up structures

When art meets engineering mechanisms, the resultant pop-up structures can be very elaborate and complex; they take many forms. In the literatures, varying terms have been used to denote the same pop-up structures. Despite the variations in design, all structures contain a fundamental set of topological and geometrical features. These features are vital for the classification for pop-up structures. In turn, the classification can facilitate a systematic approach to design the structures using software. The following sections clarify and explain basic definitions and features of pop-up structures.

3.1 Basic definitions

A pop-up structure comprises pop-up layers that are linked by creases. It is also made up of topological elements, comprising vertices, edges and faces. This section describes these elements of a pop-up structure. The glossary in this report also defines these terms.

3.1.1 Pop-up layers

Prior to the construction of pop-up layers, the base of the structure, upon which the layers are built, has to be created. The base consists of a pair of planar sheets, known as *base pages*, and is divided by the gutter crease. Pop-up layers form the collapsible portion of a pop-up structure. The number of layers affects the thickness of the structure when it is folded. A pop-up layer is made up of planar pop-up faces. Figure 3.1 shows two types of pop-up layers.

Figure 3.1. Basic pop-up structures.

To permit movement of the structure, the two conditions hereunder have to be satisfied. Firstly, a pop-up layer must form a closed linkage with the base pages, another pop-up layer or a combination of both. Secondly, the two edges of a pop-up layer joining to the base pages or another pop-up layer should not be connected to the same planar face. The first layer, for example, has to connect the left base page and the right base page, alternatively described as a bridge over the gutter crease. Figure 3.2 illustrates these conditions. As a general concept, a pop-up layer has to be built over a crease to enable motion.

Figure 3.2. Connection of pop-up layers.

The numbering of pop-up layers in this report is arbitrary and not an indication of order. In Figure 3.3, for example, a layer developed over a crease on the first layer can be the third layer. More than one layer can also be created over the same crease, as shown by the first and fourth layers in the figure.

Figure 3.3. Arbitrary numbering of pop-up layers.

While layers of one-piece pop-up structures are constructed from the same piece of paper, layers of multi-piece pop-up structures are separate paper pieces affixed on base pages and other pop-up layers. In her work, Hendrix [68, 69] referred to layers on multi-piece pop-up structures as applied elements.

3.1.2 Primary and secondary elements

A *primary pop-up layer* is a layer attached to the faces of the base pages or other layers. It is made up of *primary pop-up faces*. A *secondary pop-up layer* is a layer jointed to the edges of a primary pop-up layer. It is made up of *secondary pop-up faces*. The *box pop-up layer* comprises both primary and secondary pop-up faces. Figure 3.4 illustrates the layers and faces. The terms *primary creases* and *secondary creases* define creases on primary and secondary pop-up layers respectively. Creases connecting primary and

secondary pop-up layers are known as *primary-to-secondary creases*, which we denote in short as *p-s creases*. Figure 3.5 illustrates these creases.

Figure 3.4. Pop-up layers and faces on pop-up structures.

Figure 3.5. Creases on pop-up structures.

There are two types of secondary pop-up layers: the *outward fold* and the *inward fold*. The former type folds away from the primary pop-up layer while the latter type folds beneath the primary pop-up layer when the structure is flat folded. See Figure 3.6.

Figure 3.6. Secondary pop-up layers. (a) The outward fold and (b) the inward fold.

Note that the reverse, where a primary pop-up layer is constructed over a crease of the secondary pop-up layer, is also viable. In addition, primary and secondary pop-up layers can be added to one-piece pop-up structures. But base pages cannot open at 180° if secondary pop-up layers are added to these pop-up structures.

3.13 Crease assignments

Gutter creases, primary creases, secondary creases and *p-s* creases all have directional assignments, which determine each crease as either a *mountain crease* or a *valley crease*. As defined for primary pop-up layers in [74], if two adjacent pop-up faces are up-facing and make an angle less than 180° between them, the crease connecting the pair is a valley crease. On the other hand, if two adjacent pop-up faces are down-facing and make an angle less than 180° between them, the crease is a mountain crease. See Figure 3.7. For creases on secondary pop-up layers, the directions to identify crease assignments are different. Chapter Four further differentiates the directional parameters of the

creases. We take the gutter crease to be a valley crease for all the cases in this study, unless otherwise stated.

Figure 3.7. (a) A mountain crease, (b) a valley crease and (c) a two-layer structure.

A pop-up layer can be developed over a mountain crease or a valley crease. This development is termed as an *initiation*. The creation of a new layer over a mountain crease is known as a *mountain crease initiation*. Similarly, that over valley creases is known as a *valley crease initiation*. Since the gutter crease is taken to be a valley crease, all first primary pop-up layers are formed by valley crease initiations, as shown in Figure 3.7c. In the same figure, the second layer results from a mountain crease initiation. Crease initiations are only applicable to primary pop-up layers, not secondary pop-up layers. The concept of crease initiations is also applied to origami and this is further discussed in Chapter Four.

3.1.4 Vertices, edges and faces

More fundamentally, each pop-up structure can be represented by vertices, edges and faces, as illustrated in Figure 3.8. These topological elements are applied in the investigation using Graph Theory in Chapter Five. Their definitions are given as follows.

Figure 3.8. Edges, vertices and faces on a one-piece 90° pop-up structure.

(a) Vertices

A vertex forms at the intersection of two or more edges. Two distinct vertex types exist in the topology of pop-up structures: the *crease vertex* and the *cut vertex*. A crease vertex is a point of intersection of solely crease edges. It is located in the interior of the paper sheet, as shown on Figure 3.8. A cut vertex is a point of intersection of crease edges with cut edges or boundary edges. It lies in the interior or on the boundary of the paper sheet.

(b) Edges

An edge refers to the side of a pop-up face or a base page. Two types of edges exist: the *crease edge* and the *cut edge*. A crease edge is a line produced by folding and is incident to a crease vertex. A cut edge is a line formed by a cut or slit on the paper and is incident to a cut vertex. A cut edge is termed a *glue edge* if it is tabbed for gluing. There are two types of glue edges: the *intra-layer glue edge* and the *inter-layer glue edge*. Both are discussed in Section 5.6.2. *Boundary edges* are also unique cut edges that bound the base pages, and are introduced in Section 3.15. In this thesis, crease edges, cut edges, glue edges

and boundary edges are differentiated by solid lines, dash lines, dotted lines and bold solid lines respectively in diagrams.

(c) Faces

A face refers to the area bounded by crease edges and cut edges. There are three types of faces. A *solid face* is an area bounded by crease edges and cut edges, and represents a planar pop-up face or a base page. A *cut face* is an area bounded by only cut edges, and represents a slit or a cutout within the paper. It can be seen as an imaginary plane that is planar or non-planar. The *unbounded face* refers to the exterior of a pop-up structure and is surrounded by boundary edges. The unbounded face is applied in conjunction with Hamiltonian graphs in Chapter Five.

3.1.5 Boundaries of base pages

A pair of base pages for a pop-up structure is usually a piece of rectangular paper folded along a gutter crease, as shown in Figure 3.9a. But the base pages can also be other shapes, consisting of a number of edges. See Figure 3.9b. This raises an issue of how edges and vertices can be examined if there can be numerous variations to the shape of the base pages.

Figure 3.9. Graphs of (a) a rectangular paper, (b) a multiple-edged paper and (c) a paper with two boundary edges.

As the main interest of this study lies in the crease edges and cut edges that lead to the development of the structures, the numbers of outer edges and vertices on the base pages do not affect the topological attributes of the structures. Therefore, the shape of the paper sheet carries little importance. As such, boundary edges, represented by bold solid lines, are drawn to define the faces of the base pages. They are illustrated in Figure 3.9c.

3.16 Part classification of a pop-up structure

The components of a pop-up structure examined thus far can be summarized in a part classification in Figure 3.10. This part classification is separated into two tiers. The primary tier encompasses topological primitives of vertices, edges and faces. The secondary tier comprises elemental parts of the structure that are derived from the primitives.

Figure 3.10. Part classification of a pop-up structure.

3.2 Categorization of pop-up structures

The classification enables consistency in dealing with the attributes of the structures. It also arranges the structures in domains, builds a hierarchy for an array of modular structures, and facilitates multiple layering. Six features of topological and geometrical characteristics are analyzed for the classification of pop-up structures. The topological features comprise the number of paper pieces needed for a structure, the type of pop-up faces required for construction and the basis of linkages in the structures. The geometrical attributes are the intended angle to which the base pages open for a full erection of the structure, the symmetry about the gutter crease and the convergence of creases.

3.2.1 Topological compositions

The investigation into the topological facet of pop-up structures concerns the configurations of elements put together to form the structures. Elements like

pop-up faces and linkages can vary and be integrated to form different structures. Some of these have been introduced earlier and are elaborated below.

(a) Number of paper pieces applied

A pop-up structure can be created by applying one single piece or multiple pieces of papers. The number of pieces is often used in labeling and describing pop-up structures. More notably, it distinguishes and broadly categorizes the structures into two types, as shown in Figure 3.11. A one-piece pop-up structure is crafted by creasing and cutting one piece of paper, and can fully erect at 90° . On the other hand, a multi-piece pop-up structure has two or more paper pieces and requires tabbing and gluing.

Figure 3.11. (a) A one-piece pop-up structure and (b) a multi-piece pop-up structure.

Multi-piece pop-up structures can erect at both 90° and 180° . One-piece pop-up structures are thus unique cases and can be taken as subsets of their multi-piece counterparts. For example, the one-piece parallel double-slit in Figure 3.11a is a subtype of the multi-piece 90° parallel fold in Figure 3.11b. Likewise, the one-

piece single-slit is a subtype of the multi-piece 90° angle fold. The definitions of these pop-up structures are found in the glossary.

(b) Essential pop-up faces

The earlier section defines pop-up layers as a composition of primary and secondary pop-up faces. They are useful in differentiating the types of structure formation. For the tent formation, only primary pop-up faces are required for its construction, as shown in Figure 3.12a. For the box formation, the use of both primary and secondary pop-up faces is necessary. Without secondary pop-up faces, the box formation is incomplete and its primary pop-up faces would not erect. See Figure 3.12b. If the secondary and *p-s* creases of the box formation converge to a point, the pop-up structure becomes a tent formation. See Figure 3.13. As such, the tent formation can be regarded as a subset of the box formation.

Figure 3.12. (a) Primary pop-up faces in the tent formation. (b) Both primary and secondary pop-up faces in the box formation.

Figure 3.13. Convergence of creases on secondary pop-up faces.

(c) Basis of linkages

Chapter Two introduces pop-up structures as bar linkages. To more appropriately describe the structures, they are termed *plane linkages* in the context of this study. The basis of linkages refers to the minimum number of linkages sufficient to erect a pop-up structure. A pair of base pages forms a two-plane linkage while a tent formation has the basis of a four-plane closed loop linkage. Though a 180° box fold usually has eight planes, a box formation just requires the basis of a six-plane closed loop linkage to erect. See Figure 3.14.

(a)

(b)

Figure 3.14. Plane linkages. (a) A four-plane linkage and (b) a six-plane linkage.

A three-plane linkage forms a truss and does not permit flat folding and movement. Other linkages, such as a five-plane linkage, are possible

frameworks for a pop-up structure but they do not generate fixed loci for the structure's movement due to a higher degree of mobility. Essentially, elemental pop-up structures can be distinguished by a basis of four-plane linkages and a basis of six-plane linkages. As the basis of linkages is pertinent to the attributes of essential pop-up faces, they have a mutual dependency. This is elaborated in Section 3.3.

3.2.2 Geometrical attributes

The geometrical attributes are concerned with the measurable parameters of the pop-up elements, which determine the positioning and alignment of pop-up layers. These include the angle of opening base pages, the symmetry of primary creases and convergence of primary creases.

(a) Angles between base pages

Like the number of paper pieces in a pop-up structure, the angle between the base pages is widely used to describe pop-up structures, such as 90° pop-up structures and 180° pop-up structures. The notations, 90° and 180° , define the angle between the base pages when the structure is fully erect, as shown in Figure 3.15.

Figure 3.15. (a) A 90° pop-up structure. (b) and (c) are 180° pop-up structures.

Geometrically, it is feasible to have pop-up structures fully erected at other angles but that is not the norm for design specifications of pop-up cards and books. Nevertheless, some books such as *The Farmhouse Carousel* [75] have base pages that open up to 270° and 360° . They incorporate multiple layers of 90° and 180° pop-up designs.

On a 180° pop-up structure of the tent formation such as that in Figure 3.15b, the angle between its two pop-up faces can vary between 0° and 180° . If the primary pop-up faces and the base pages are coplanar, a 90° pop-up structure such as that in Figure 3.15a would be formed. Thus, the 90° pop-up structures can be categorized as a subtype of the 180° pop-up structures of the tent formation.

Unlike 90° pop-up structures, most 180° pop-up structures do not contain multiple identical layers because the layers would not erect fully when the base pages are 180° apart. One exceptional case is the floating layer, which comprises two 90° pop-up structures that share a common plane. Figure 3.16a illustrates a pop-up structure with two floating layers. But 90° pop-ups can be built at the side of a 180° pop-up structure, as shown in Figure 3.16b.

Figure 3.16. (a) A floating layer on top of another and (b) a 90° pop-up layer adjoined to a box layer.

(b) Symmetry of primary creases

Another geometric property to observe is the symmetry of primary creases on a primary pop-up layer. For the first layer, the line of symmetry is the gutter crease. For a successive pop-up layer, the line of symmetry is the primary crease that it bridges over. Figure 3.17 gives examples of a pair of primary creases' symmetry about the gutter crease. There are two types of symmetry: the position symmetry and the shape symmetry. The position symmetry concerns the distances and angles of two primary creases from the line of symmetry. It affects how the pop-up layer moves. On the other hand, the shape symmetry concerns the lengths of two primary creases but it does not affect how the pop-up layer moves. A pop-up structure is asymmetrical if it does not have position symmetry or shape symmetry. Figures 3.17b illustrates a case where both types of symmetry are not satisfied.

(a) Symmetric (b) Asymmetric
Figure 3.17. Symmetry of primary creases about the gutter crease.

The tent formation can be symmetrical and asymmetrical. But for the box formation, secondary pop-up faces will distort upon folding if the primary creases are not equidistant from the gutter crease, as shown in Figure 3.18. Hence, position symmetry is necessary for the box formation.

Figure 3.18. A box fold without position symmetry.

But if its primary creases are equidistant from the gutter crease but are shape asymmetric with different lengths, the box formation can flat fold without deformation. This is achievable due to a geometric condition for flat folding that states the sum of the widths of the primary and secondary pop-up faces of the box pop-up layer on the left base page has to be equal to that on the right base page [5, 76]. Geometric conditions for flat folding are further discussed in Section 6.2. Hence, the symmetry of the primary creases distinguishes the tent formation and the box formation.

(c) Convergence of primary creases

A pop-up structure can have parallel primary creases in which case, it is known as a parallel fold. Figure 3.19a illustrates the structure. Its creases converge at infinity. Conversely, the primary creases of a non-parallel fold or an angle fold

converges at a point, as shown in Figure 3.19b and Figure 3.19c respectively. Lee, Tor and Soo [65] also defined non-deformable pop-up structures using all parallel and all concurrent creases. More specifically for concurrent creases, the convergence point of a non-parallel fold lies out of the pop-up structure whereas the convergence point of an angle fold lies on the physical creases. Therefore, the position of convergence point distinguishes the parallel fold, the non-parallel fold and the angle fold. The non-parallel fold can be taken to be the generalized form for both the angle fold and the parallel fold, which are constraint cases. As the use of varying convergence of primary creases is prevalent in all types of pop-up structures, including the one-piece pop-ups and box folds, it is an important feature in classifying the structures. Furthermore, the convergence of primary creases is necessary for flat folding.

Figure 3.19. (a) Parallel primary creases converge at infinity. Primary creases converge to a point (b) outside the structure and (c) on the structure.

3.3 A classification model for pop-up structures

The six topological and geometrical features can be deployed in grouping pop-up structures. More significantly, the features enable the categorization of pop-up structures using a model, as given in Appendix A - The Classification Model for Pop-up Structures. The features distinguish eleven pop-up structures such that each exhibits a unique set of attributes, as illustrated in Figure 3.20. For

example, a parallel double-slit can be described as a one-piece 90° pop-up structure. Structurally, it is a four-plane linkage made up of primary pop-up faces. It also has parallel creases with no symmetry restriction. In addition, the eleven pop-up structures in the proposed model are elemental forms that enable the construction of more complex layered designs.

Figure 3.20. Feature categorization of pop-up structures

Some of the features are also found to exhibit interdependency. In particular, the essential pop-up faces, the basis of linkages and the symmetry about primary creases are interdependent. If the basis of linkages in a structure is six, the design requires the construction of both primary and secondary pop-up faces, and vice versa. The primary creases also have to be position symmetric for the basis of a six-plane linkage. This interdependency is also shown in Figure 3.20.

3.3.1 Domains of pop-up structures

In the model, the pop-up structures are arranged in a tabular form. As shown in Figure 3.21, the columns separate the structures into domains, namely the one-piece 90° pop-up structures, the multi-piece 90° pop-up structures, the multi-piece 180° tent pop-up structures and the multi-piece 180° box pop-up structures. The rows partition the structures according to the convergence of primary creases. This feature divides the pop-up structures in each domain into angle folds, non-parallel folds and parallel folds. The domains in Figure 3.21 are arranged in such a way to facilitate a hierarchical system, which is discussed in the following sub-sections.

Angle folds		180° angle fold	90° angle fold	Single-slit angle fold
Non-parallel folds	180° non-parallel box fold	180° non-parallel fold	90° non-parallel fold	Single/double-slit Non-parallel fold
Parallel folds	180° parallel box fold	180° parallel fold	90° parallel fold	Single/double-slit parallel fold
	Multi-piece 180° box pop-up structures	Multi-piece 180° tent pop-up structures	Multi-piece 90° pop-up structures	One-piece 90° pop-up structures

Figure 3.21. Subdivisions of pop-up structures

In the table, the cell under the multi-piece 180° box pop-up structures and the angle folds is vacant. The triangular box fold illustrated in Figure 3.22, comprising two primary pop-up faces and two secondary pop-up faces, seems to fit the cell. Its primary creases on the base page converge to a crease vertex, thereby describing the feature of the angle folds. But its secondary pop-up faces are dispensable because its primary pop-up faces do not depend on them to

erect. Therefore, the structure does not comply with the attribute of essential pop-up faces for box formation. Furthermore, the triangular box fold is not an elemental structure but a varied design of the 180° angle fold with a secondary layer. Hence, it does not belong to the classification in the table.

Figure 3.22. A triangular box fold.

3.3.2 Supertype-subtype associations

The supertype refers to a general type and the subtype is a specialization of that type [77]. A supertype and its corresponding subtype are linked by an association. A type in this study refers to a domain specified in the Section 3.3.1. e.g. multi-piece 90° pop-up structures. Each type comprises many instances. For example, an instance of the multi-piece 90° pop-up structures is a three-layer 90° pop-up structure with specific angles and heights. These associations enable clear delineations of parent-child relationships between types of pop-up structures. They also enable the domains to form a hierarchy. As three of the six features are interdependent, associations are established with only four of them, and these are summarized in Table 3.1.

Table 3.1. The subtype-supertype associations

Feature	Supertype	Subtype
Number of paper pieces applied	Multi-piece	One-piece
Essential pop-up faces	Box formation (Primary and secondary pop-up faces)	Tent formation (Primary pop-up faces)
Angle between base pages	180° pop-up structures	90° pop-up structures
Convergence of primary creases	Non-parallel fold	Angle fold & parallel fold

The number of paper pieces applied, the essential pop-up faces and the angle between base pages form associations between domains of pop-up structures. The convergence of primary creases forms localized relationships between structures in each domain.

3.3.3 A hierarchical representation for pop-up structures

To configure the hierarchy for the classification, the suggested approach is to arrange the domains of pop-up structures using the supertype-subtype associations. The arrangement, in turn, builds a taxonomy schema, as shown in Figure 3.23. In the schema, the domains represent taxa, and arrows illustrate the subtype-supertype associations. As with the convention of bottom-up hierarchies, the unidirectional arrows emanate from the child to the parent to indicate the child's inheritance of properties from the parent. The schema proposes the multi-piece 180° box pop-up structures as the general form of all types of elemental pop-up structures while the one-piece 90° pop-up structures are the most constrained forms in the family. The semantics of the association also requires the domains to be transitive [77]. Therefore, the one-piece 90° pop-up structures are a subtype of the multi-piece 180° box pop-up structures.

Figure 3.23. A taxonomy schema for domains of pop-up structures

The schema can be alternatively elaborated in a cascaded dendrogram, or a bottom-up category tree, in Figure 3.24. In the dendrogram, the features of pop-up structures are separated as mutually exclusive pairs at each hierarchical division. However, without the proposed taxonomy schema above, the dendrogram itself is a poor representation of the associations for the domains. This is because the domains can be rearranged in various hierarchical divisions, including that in Figures 3.25.

Figure 3.24. A dendrogram for the classification of domains

Figure 3.25. A dendrogram not adhering to the taxonomy schema

The dendrogram hitherto does not illustrate the inter-domain associations by convergence of primary creases. To make the representation inclusive of these associations, the taxonomy schema in Figure 3.23 is better represented by substituting the domains with their corresponding sets of elemental pop-up structures. This results in a latticed hierarchy, as shown in the category tree in Figure 3.26.

Figure 3.26. A latticed hierarchy for pop-up structures

In the latticed hierarchy, the 180° non-parallel box fold sits at the root node, and the single/double-slit parallel fold and the single-slit angle fold lie at its lowest level. The arrows specifically define the parent-child relationship between each pair of pop-up structures. For example, the single-slit angle fold is the child of the single-slit non-parallel fold, and inherits attributes of non-parallel creases and cut faces for one-piece structures. At the same time, it is also a child of the 90° angle fold, and inherits creases that are incident to a crease vertex on the structure. This latticed hierarchy is adopted in the classification model, as shown in Figure 3.27.

Figure 3.27. The latticed hierarchy in the classification model

3.4 Summary

There are other feasible methods to categorize pop-up designs such as classifications by detailed crafting methods and kinematic properties. Besides the supertype-subtype association, other types of relationships such as the part-whole relation [78] can also be deployed for classifying pop-up structures. While these have been taken into account, the classification model in this study specifically divides pop-up structures into categories of primitive building units,

which are essential for rudimentary and advanced designs. The bottom-up hierarchical approach is applied for three reasons. Firstly, it provides a conceptual framework for the development of a software architecture that supports object-oriented programming. Secondly, it explains with clarity the associative paths between the categories of pop-up structures. Thirdly, it identifies common attributes in pop-up structures and facilitates class modelling. The next chapter extends the discussion to the layering of these elemental structures.

Chapter Four

Pop-up layers and their model representations

As the lowest sub-family in the classification, the one-piece pop-up structure exhibits the simplest form of all pop-up designs. Made solely from a sheet of paper, its construction techniques are limited to creasing and cutting, often straight-lined slitting. The structures are in tent formation and usually only comprise primary pop-up layers. Since the layers are not detachable from the base pages, their orientations and geometry are constrained. Hence, one-piece pop-up structures are useful for investigating topology of primary pop-up layers. The box formation is excluded from this study as its structure does not permit successive layering. In this chapter, Sections 4.1 to 4.4 examine crease characteristics of primary pop-up layers on one-piece structures. Section 4.5 extends the study to secondary pop-up layers. Section 4.6 further considers paper folds, or origami, as elements of pop-ups. Section 4.7 examines the composite layering of pop-up layers and paper folds. Section 4.8 discusses models of pop-up structures with composite layers.

4.1 Creasing on primary pop-up layers

To understand creasing on one-piece pop-up structures, let us fold a piece of paper into half to form the gutter crease and two base pages. Then make a pair of identical parallel cuts perpendicularly across the gutter crease. Crease two parallel lines that connect the end points of the cut lines, and close the base pages. The above describes the construction of a double-slit pop-up structure. An example is given in Figure 4.1. In the figure, creases and cut lines are

represented by solid lines and dashed lines within the boundary of the paper respectively.

Figure 4.1. A symmetrical double-slit. (a) Its cut-and-crease pattern and (b) its side view at a 90° erection.

The double-slit reveals four parallel creases and four planar pieces, of which two are the base pages. The creases may not be obvious when viewed from the flat cut-and-crease patterns, as shown in Figure 4.1a. But when the base pages are set to 90° on the side view, as shown in Figure 4.1b, the creases are easily identifiable. The cut-and-crease pattern in Figure 4.1a can be taken as a special case, in which two creases are collinear because of the symmetry about the gutter crease. However, if an asymmetrical double-slit is constructed, the creases would be clearly visible on the cut-and-crease pattern, as illustrated in Figure 4.2.

Figure 4.2. An asymmetrical double-slit. (a) Its cut-and-crease pattern and (b) its side view at a 90° erection.

4.2 Crease-layer relation

Both Figures 4.1 and 4.2 exemplify pop-up structures with the first primary pop-up layer. When the next layer is built on the structure in Figure 4.2, three new creases are created, thereby giving rise to a total of seven creases, as shown in Figure 4.3.

Figure 4.3. Two layers of double-slit. (a) Its cut-and-crease pattern and (b) its side view at a 90° erection.

As three creases are added on each subsequent layer, the total number of creases abides by the sequence $\{1, 4, 7, 10, 13, \dots, 3n+1\}$ [79] according to the number of pop-up layers created, n . Therefore, if N is the total number of creases on a pop-up structure with n th layers, then

$$N = 3n + 1, \quad n \in \mathbb{Z}^+ \quad (4.1)$$

The existence of a gutter crease is required before any pop-up can be produced. Thus, the constant ‘1’ in the equation represents the gutter crease, i.e. $N = 1$ when $n = 0$.

4.3 Crease-face relation

One-piece pop-up structures are tent formations, and thus they have a basis of four-plane linkages, as discussed in Section 3.2.1a. Since each pop-up layer of a one-piece pop-up structure comprises two planes, the number of plane linkages

on the structure is even regardless of its number of layers. This condition is necessary for flat folding of one-piece pop-up structures and can be mathematically expressed in a relationship between the number of planar faces, F , and the number of pop-up layers, n . That is

$$F = 2n + 2, n \in \mathbb{Z}^+. \quad (4.2)$$

As the structure consists of only two base pages when there is no layer, $F = 2$ when $n = 0$. Combining (4.1) and (4.2) yields another relationship between the number of faces, F , and the number of creases, N .

$$F = \frac{2}{3}(N + 2), F \in \mathbb{Z}^+, \text{ for some } N \geq 1 \quad (4.3)$$

(4.3) is used to derive the boundary constraints in the next section. This expression can also be derived from Gruebler's Equation [80] with one degree of freedom and one grounded link.

4.4 Crease boundary conditions for primary pop-up layers

A one-piece pop-up structure can take many multi-layer designs. There are, however, instances where a structure cannot erect or flat fold. A question thereupon arises from such cases: What are the arithmetical constraints that determine the flat foldability of a pop-up structure?

In the early 1980s, Maekawa demonstrated that the difference between the numbers of mountain and valley creases is always two in order to achieve flat foldability in a flat vertex fold. This has since been known as the Maekawa's Theorem [32, 59]. Justin [60] also contributed similar findings. Likewise, crease assignments on a pop-up structure affect its ability to collapse flat. Other causes to a pop-up structure's inability to fold flat include the material used for

constructions and interferences with other paper pieces.

4.4.1 Successive layering

As the gutter crease is taken as a valley crease, the first pop-up layer is always developed over a valley crease, i.e. a valley crease initiation. But subsequent layers can be constructed over either mountain or valley creases. Thus, we look into two boundary cases where successive pop-up layers are developed solely from mountain crease initiations and solely from valley crease initiations.

Let us denote the number of mountain and valley creases with M_P and V_P respectively, where the subscripted ‘ P ’ refers to the primary pop-up layers. For the same type of crease initiations for subsequent layers, the increment of the number of creases is constant, as described earlier on the crease-layer relation. Hence F , M and V forms a linear expression

$$F = aM_P + bV_P, \quad (4.4)$$

where a and b are arbitrary integer constants. Derivations using sequential number patterns generated from pop-up layers will result in two important expressions, (4.5) and (4.6), which describe the two boundary cases. Their derivations are in Appendix B.

For successive mountain crease initiations, the boundary relationship is

$$F = \frac{2}{5}(3V_P + M_P). \quad (4.5)$$

For successive valley crease initiations, the boundary relationship becomes

$$F = 2V_P - 2M_P. \quad (4.6)$$

Both relationships still hold when the paper structure is flipped over. But valley creases becomes mountain creases and vice-versa. In such cases, the variables M and V are interchanged.

4.4.2 Crease constraints

The design of one-piece pop-up structures is bounded by three crease constraints. Two of them are derived from (4.5) and (4.6), as introduced in Section 4.4.1. Since the total number of creases, N , is the sum of mountain and valley creases, (4.3) can be expanded to

$$F = \frac{2}{3}(M_p + V_p + 2). \quad (4.7)$$

Equating (4.5) and (4.7), a new expression on mountain and valley creases is formed.

$$M_p = 2V_p - 5, \text{ for } V_p \geq 3 \quad (4.8)$$

(4.8) is the crease constraint between mountain creases and valley creases for the boundary case of successive mountain crease initiations. Similarly, another crease constraint is obtained by looking into the boundary case of successive valley crease initiations. Equating (4.6) and (4.7) gives

$$V_p = 2M_p + 1, \text{ for } M_p \geq 1 \quad (4.9)$$

As the first primary pop-up layer has a minimum of one mountain crease and three valley creases, the limits in (4.8) and (4.9) are always satisfied. By the same deduction, the total number of creases on the first layer has to be four. Therefore,

$$M_p + V_p \geq 4 \quad (4.10)$$

(4.10) forms the third crease constraint for primary pop-up layers. As the

minimum number of faces on a one-piece pop-up structure is four, (4.10) can also be derived from (4.7).

$$\frac{2}{3}(M_P + V_P + 2) \geq 4$$

$$M_P + V_P \geq 4$$

4.4.3 Model of mountain and valley creases

The three crease constraints form a semi-infinite region R_P on a mountain-valley crease model, as illustrated in Figure 4.4. In short, the model is termed the MV model. In the figure, the coordinates $(0, 1)$ represents the gutter crease and $(1, 3)$ represents the first primary pop-up layer. However, only specific integer coordinates in the boundary region represent the number of mountain and valley creases for the primary pop-up layers. To locate these coordinates, two types of lines, the layer lines and the crease initiation lines, are examined. These lines intersect the integer coordinates on the MV model.

Figure. 4.4. The boundary region for primary pop-up layers.

(a) Layer lines

Layer lines represent the pop-up layers. The lines are given by the linear relationship

$$M_p + V_p = N \quad (4.11)$$

and N is determined from (4.1). The third crease constraint $M_p + V_p = 4$ is the layer line for the first layer. Possible integer coordinates are located on the family of parallel layer lines, as shown in Figure 4.5a.

Figure 4.5. (a) Layer lines $M_p + V_p = N$ and (b) crease initiation lines.

(b) Crease initiation lines

Crease initiation lines represent how a primary pop-up layer develops. The crease initiation lines are parallel and are inclusive of the crease constraint lines belonging to the two boundary cases. As there are two types of crease initiations, the lines for mountain crease initiations and valley crease initiations differ in gradients on the MV model. The gradients are termed MV slopes, and

can be expressed as $\frac{V_{p,n+1} - V_{p,n}}{M_{p,n+1} - M_{p,n}}$, where the subscripted notations denote two

different but consecutive primary pop-up layers. For a mountain crease initiation, there is an increment of two mountain creases and one valley crease. Hence, its MV slope is $\frac{1}{2}$. The general solution for the equations of the mountain crease initiation is given by

$$V_p = \frac{1}{2}M_p + \frac{3}{2}\alpha + 1 \quad (4.12)$$

for some positive constant α . Therefore, the family of solution lines, represented by dotted lines in Figure 4.5b, is in the form $V_p = \frac{1}{2}M_p$ shifted vertically for different values of α . For a valley crease initiation, there is an increment of one mountain crease and two valley creases. The MV slope is thus 2. The general solution for the equations of valley crease initiation is given by

$$V_p = 2M_p - 3\beta + 4 \quad (4.13)$$

for some positive constant β . Hence, its family of solution lines, represented by dashed lines in Figure 4.5b, is in the form $V_p = 2M_p$ shifted vertically for different values of β . The same figure also shows the intersections of crease initiation lines and layer lines, i.e. positions of the MV coordinates. The model thus enables the deduction of the number of mountain and valley creases required for the layers.

Properties of creases in primary pop-up layers enable model representations of multi-layer pop-up structures. Figures 4.6 and 4.7 show examples of three-layer one-piece pop-up structures and their corresponding MV models. However, the study of pop-up layers is not complete without investigation into secondary pop-up layers and paper folds.

Figure 4.6. A pop-up structure constructed by successive valley crease initiations.

Figure 4.7. A pop-up structure constructed by one valley and two mountain crease initiations.

4.5 Creasing on secondary pop-up layers

In this section, the investigation is extended to secondary pop-up layers. As these layers are attached to primary pop-up layers, the construction thus gives rise to a multi-piece pop-up structure. As with most pop-up designs, typically not more than one secondary pop-up layer is attached to a primary pop-up layer for effective folding. A secondary pop-up layer is also possibly redundant if it is coplanar with another. Hence, layering of secondary pop-up layers on common edges is seldom applied. Even so, multiple secondary pop-up layers can still be

attached to a structure with more than one primary pop-up layer or to other secondary pop-up layers. The examination of secondary pop-up layers is therefore advantageous, particularly on how their crease characteristics relate with the entire crease embodiment on a pop-up structure.

4.5.1 Directions of crease assignments

As a pop-up structure with secondary pop-up layers produces large spatial motions, crease assignments for p -s and secondary creases are difficult to identify, and would lead to ambiguity if those defined for primary pop-up layers are applied. The inward fold and the outward fold, as introduced in Section 3.1.2, would also not be clearly distinguishable. To remedy this issue, one approach is to establish a new set of directional parameters for investigating characteristics of p -s and secondary creases.

Figure 4.8. Directional parameters for creases.

Let us therefore denote the number of mountain creases and the number of valley creases on secondary pop-up layers as M_S and V_S respectively, where the notation 'S' indicates secondary pop-up layers. The direction defining mountain creases and valley creases on secondary pop-up layers is parallel to the gutter

crease, as illustrated in Figure 4.8. The secondary crease is taken to be a mountain crease if it points away from the primary pop-up layer. It is a valley crease if it points towards the primary pop-up layer. If the primary pop-up layer, such as the double-slit, has two open ends that a secondary pop-up layer can attach to, these crease assignments are still applicable to either end. In accordance with this specification, all p - s creases are mountain creases.

4.5.2 Model of mountain and valley creases

Secondary pop-up layers are not developed over creases. They are thus different from primary pop-up layers, which are created by means of successive crease initiations. Nevertheless, the arithmetical principles of the MV model apply. In the layering of secondary layers, three new creases are formed whenever a secondary pop-up layer is added to the structure. Hence, the total number of creases, N , can be expressed in term of the number of layers of secondary pop-up layers, n , as

$$N = 3n, n \in \mathbb{Z}^+ \quad (4.14)$$

Figure 4.9. Mountain and valley creases on (a) an outward fold and (b) an inward fold.

Figure 4.9 illustrates the types of creases on an outward fold and an inward fold. In the case of an outward fold, there is an increment of three mountain creases. For successive layers of outward fold, its equation is

$$V_s = 0, M_s = 3n, n \in \mathbb{Z}^+ \quad (4.15)$$

where n is the number of secondary pop-up layers. Its MV slope is 0. In the case of an inward fold, there is an increment of two mountain creases and one valley crease. Considering successive layers of the inward fold, the equation for its crease constraint is

$$V_s = \frac{1}{2}M_s \quad (4.16)$$

Thus its MV slope is $\frac{1}{2}$. These two equations serve as constraints of a boundary region R_s for the p -s and secondary creases. Figure 4.9 illustrates the two crease constraints and the boundary region in an MV model. The origin represents the case where there is no secondary pop-up layer. Likewise, only specific integer coordinates in the boundary region represent the number of mountain and valley creases feasible for flat foldability of secondary pop-up layers. To locate these coordinates, the layer lines and the development lines for inward and outward folds are determined.

Figure 4.10. The boundary region for secondary pop-up layers.

The expression for layer lines is similar to that of the primary pop-up layers, and it is given by

$$M_s + V_s = N \quad (4.17)$$

where N is determined by (4.14). Within the region, the development of inward folds and outward folds are differentiated by two types of lines: the inward fold lines and the outward fold lines. The general solution for the equations of the inward fold lines is given by

$$V_s = \frac{1}{2}M_s - \frac{3}{2}\lambda + \frac{3}{2} \quad (4.18)$$

for some positive λ . Likewise, the general solution for the equations of the outward fold lines is given by

$$V_s = \gamma - 1 \quad (4.19)$$

for some positive γ . Figure 4.11 illustrates the lines in the boundary region R_S . In the figure, the inward fold lines and the outward fold lines are represented by dashed lines and dotted lines respectively. MV coordinates for secondary pop-up layers are located at the intersections of the lines. The combined model representation of primary and secondary pop-up layers is further discussed in Section 4.9.

Figure 4.11. (a) Layer lines $M_s + V_s = N$. (b) Inward fold lines and outward fold lines.

4.6 Pop-up effects by paper folds

Even without the presence of pop-up structures, paper folds alone are capable of generating pop-up effects. They can be achieved without processes like cutting or gluing multiple paper pieces together. This section examines pop-up effects by paper folding on a single sheet of paper. Figure 4.12 illustrates effects by purely paper folding.

Figure 4.12. Pop-up effects by paper folding.

Paper folds are examined for three other reasons. Firstly, creasing is a technique essential in both origami and pop-up making. In flat origami, the elementary paper folds consist of flat vertex folds and accordion pleats, as shown in Figure

4.13. As introduced in Section 2.2.3b, accordion pleats comprise angle pleats, non-parallel pleats and parallel pleats.

Figure 4.13. (a) A flat vertex fold and (b) accordion pleats.

Secondly, creases of flat origami and pop-up structures have similar properties. For example, both the single-slit pop-up structure and the flat vertex fold have converging creases that meet at a vertex. Parallel creases are essential to both the double-slit pop-up structure and parallel pleats. Thirdly and of utmost importance, paper folds are added to pop-up designs to increase pop-up effects. Therefore, an examination on flat origami would be helpful in construing crease characteristics of pop-up structures. Figure 4.14 categorizes these basic types of flat origami.

Figure 4.14. Types of flat origami.

4.6.1 Crease assignments, folded states and creasing techniques

This sub-section defines the crease assignments and folded states of paper folds. It also introduces and distinguishes two creasing techniques applicable to flat vertex folds and pleats.

(a) Crease assignments

The mountain-valley crease assignments for paper folds are determined by examining the crease pattern when the paper is completely unfolded. In this study, the crease assignments are examined from the top face of the paper. For all cases described hereafter, the first crease line is taken to be a valley crease.

(b) Folded states

Like pop-up structures, paper folds can be constructed with multiple layers. A folded state refers to the state after a step of flat folding is performed, as graphically explained in Figure 4.15. It does not refer to the number of planar faces in an origami structure.

Figure 4.15. Two folded states: from one state to the next.

(c) Creasing techniques

In general, there are two creasing techniques to create a folded state. The first technique is to fold across a face to create new creases, as illustrated Figure

4.15. The second technique is by initiating folds on the two sides of an existing crease. In this technique by crease initiation, the crease assignment of the existing crease changes, as shown in Figure 4.16. In turn, two creases, one on each side of the existing crease, are created. The two new creases have the same assignment but are different from that of the existing crease. In origami jargon, this technique is known as reverse folding. To generate pop-up effects, the latter technique has to be applied. This is because the former restricts rotational movement of the paper pieces, but the latter does not.

Figure 4.16. Folding by initiating from an existing crease.

4.6.2 Flat vertex folds

Maekawa [59], Jacques [60] and Hull [61] have established the relationship between mountain and valley creases for localized vertices, as discussed in Section 2.4.3. The following section discusses the layering of multiple flat vertex folds.

(a) The first folded state

A gutter crease has to be constructed prior to the creation of a flat vertex fold. The flat vertex fold, i.e. the subsequent folded state, is created by reverse folding a portion of the gutter crease. This results in three new creases. Figure

4.17 illustrates the process. When the paper is unfolded, a total of one mountain crease and three valley creases can be seen. This aligns with Maekawa's theorem [59], and the equation $V - M = 2$ describes the crease property of the flat vertex fold.

Figure 4.17. Crease patterns of (a) the gutter crease and that of (b) the first folded state of the flat vertex fold.

Hull's case of a single crease line mentioned in Section 2.4.3a suggests that the mathematic relationship can be generalized to include the gutter crease. This is represented by $(0, 2)$ in Figure 4.18a. However, for the analysis here, the mathematical relationship is constrained such that

$$V - M = 2, \text{ for some } M > 0 \quad (4.20)$$

since a flat vertex fold is only achieved when four creases meet upon folding, i.e. $M + V = 4$. As such, the model representation of (4.20), as illustrated in Figure 4.18b, is different from that in Figure 4.18a. The coordinates $(0, 1)$ represent the gutter crease while $(1, 3)$ give the first folded state of a flat vertex fold.

Figure 4.18. MV models for the flat vertex fold. (a) The generalized case and (b) the constraint case for the first folded state.

(b) Successive folded states

In Maekawa's Theorem, the subsequent folded states increase about the localized vertex, i.e. new creases are incident to the same vertex for all folded states, as shown in Figure 4.19a. But this produces angle pleats, not new flat vertex folds. In the context of this study, a new vertex is formed with each successive folded state. Thus, as Figure 4.19b illustrates, each successive flat vertex fold produces new three creases.

Figure 4.19. Crease patterns of (a) new angle pleats on a flat vertex fold and (b) a new flat vertex fold.

If each successive flat vertex fold is initiated from a valley crease, an increment of one mountain crease and two valley creases is noted. The MV slope for successive valley crease initiations is thus 2. On the other hand, if the successive fold is initiated from a mountain crease, the MV slope is $\frac{1}{2}$. With the inference to its MV slopes, a semi-infinite boundary region is defined for the flat vertex fold and its successive layers. The region, as shown in Figure 4.20, is identical to that of the primary pop-up layers. Likewise, the MV model here represents feasible construction of mountain and valley creases for multiple flat vertex folds.

Figure 4.20. The boundary region for the flat vertex fold.

When related with Maekawa's Theorem, the line of the equation $V - M = 2$ is an angle bisector of the region. The line can be seen as an adherence to a region of flat foldability. But the theorem is more pertinent to pleats, as analyzed in the next subsection.

4.6.3 Accordion pleats

In this study, an accordion pleating pattern is defined as an alternation of mountain and valley creases with at least three creases since pleats comprise

repeated creases with the same assignment. In pop-up structures, pleating cannot be carried out from any edges of pop-up layers since they are connected to the base pages or other pop-up layers. Only the creasing technique of crease initiations can be applied. Figure 4.21 demonstrates the construction of pleats by initiating from a crease.

Figure 4.21. Crease patterns of pleats. (a) An existing crease and (b) the first folded state of pleats.

Figure 4.21a shows a gutter crease, which is a valley crease. Assuming both ends of the paper are fixed, two new creases, one on each side of the existing crease, are formed upon a valley crease initiation. The initial valley crease is reversed and becomes a mountain crease, and the two new creases take valley crease assignments, as shown in Figure 4.21b. This becomes the first folded state of the pleats. The next folded state is formed when another crease initiation is performed. Though the illustrations take parallel pleats as an example, the outcome is the same for radial pleats, which are discussed in Sections 2.2.2 and 4.6. Regardless of the types of crease initiations, there is always an increment of one mountain crease and one valley crease. This implies that the MV slope for pleats is 1. Comparing with Maekawa's flat vertex folds

which apply radial pleats after the first vertex fold, the relationship $V - M = 2$ also reveals an MV slope of 1.

Figure 4.22. The MV model for pleats.

The crease properties for pleats can be more clearly represented on the MV model, as shown in Figure 4.22. In the figure, the coordinates $(0, 1)$ refer to the gutter crease. Pleats created by the creasing technique of initiating from existing creases can thus be represented by

$$V = M + 1, \text{ for } M > 0. \quad (4.21)$$

4.7 Composite layering of pop-up structures

This chapter has thus far individually discussed successive layering of primary pop-up layers, secondary pop-up layers, flat vertex folds and pleats. Collectively, these four layer components give rise to myriad of ways to layer a pop-up structure, or concisely described as composite layering. For example, a secondary pop-up layer can precede a primary pop-up layer. Maekawa's localized flat vertex folds are derived from the composite layering of a flat vertex fold and radial pleats. More intricately, a primary pop-up layer can be

decorated with flat vertex folds and pleats. Figure 4.23 shows two pop-up structures with composite layers.

Figure 4.23. (a) A single-slit with a vertex fold and (b) a double slit with parallel pleats.

Three suppositions have to be adhered to for composite layering. Firstly, the first pop-up layer is always a primary pop-up layer. Secondly, paper folds are only constructed on pop-up layers and not on base pages. Thirdly, consecutive layer components do not have to be of different types. For example, a pop-up structure can comprise successive primary pop-up layers, followed by a flat vertex fold. Figure 4.24 illustrates the conceptual representation for composite layering. Arrows point from preceding layer components to feasible subsequent layer components.

Figure 4.24. Conceptual representation of composite layering.

While the four layer components can generally be built on one another, there are cases where one is not compatible with the other. Specifically, the construction of the secondary pop-up layer and pleats are dependent on structural compositions of preceding layer components. Two compatibility conditions are listed as follows.

1. The parallel fold cannot be a secondary pop-up layer if it is to attach to a primary pop-up layer of the tent formation. This is because the p - s creases have to converge to a vertex on the structure, and thus the parallel edges of the parallel fold cannot be attached to the primary pop-up layer.
2. Parallel pleats and non-parallel pleats can only be constructed from a crease that does not converge with other creases to a real vertex. Therefore, the flat vertex fold, angle pleats and the angle fold are incompatible preceding components of parallel and non-parallel pleats.

4.7.1 The iterative layering model

Taking account of the compatibility conditions, Figure 4.25 summarizes the

composite layering of the four components in an iterative layering model. This model is expanded from the conceptual representation in Figure 4.24. The model consists of three intra-cycles and one inter-cycle. Of the three intra-cycles, one demonstrates iterative layering between the pop-up layers and the other two between the paper folds. An example of layering in an intra-cycle is in the order {parallel fold, angle fold, angle fold, non-parallel fold}. The inter-cycle illustrates iterative layering between pop-up layers and the paper folds. An example of layering in an inter-cycle is in the order {non-parallel fold, parallel pleats, angle fold, non-parallel pleats, flat vertex fold}. Within the paper folds, the unidirectional arrow reflects the compatibility condition for parallel and non-parallel pleats. This model is useful in structuring and managing complex pop-up structures with composite layers.

*Due to compatibility condition for pleats, the subsequent layer component built on angle folds cannot be parallel pleats.

Figure 4.25. The iterative layering model.

4.8 Models for composite pop-up layering

The boundary regions of both primary and secondary pop-up layers are mutually exclusive since they deploy different sets of crease assignments. Paper folds added to a pop-up structure are also not governed by mathematical relationships of both types of pop-up layers though it can be deduced that *MV* coordinates of a primary pop-up layer with flat vertex folds and pleats would lie within its boundary region. This is because the *MV* slopes of supplementary paper folds lie between the *MV* slopes of primary pop-up layers of $\frac{1}{2}$ and 2, as shown in the summary of *MV* slopes in Table 4.1. The *MV* models of all these four layer components can be integrated to model a pop-up structure with composite layers.

Table 4.1. Summary of *MV* Slopes.

Type of layer components	Types of development	<i>MV</i> slope
Primary pop-up layers	Mountain crease initiation	$\frac{1}{2}$
	Valley crease initiation	2
Secondary pop-up layers	Outward fold attachment	0
	Inward fold attachment	$\frac{1}{2}$
Flat vertex folds	Mountain crease initiation	$\frac{1}{2}$
	Valley crease initiation	2
Pleats	Mountain and valley crease initiation	1

4.8.1 *MV* models

Let us consider the following case of a pop-up structure with seven layer components, as shown in Figure 4.26. The layering of the components is consistent with the iterative layering model.

Figure 4.26. A pop-up structure with composite layers.

In the figure, the components are numbered according to the order of their construction, and labeled with abbreviations of its component type in brackets. For example, '3 (n_1)' refers to the third layer constructed on the pop-up structure. Its component type, n_1 , is the first non-parallel fold on the structure. The abbreviations of all types of layer components are listed in Table 4.2. They are also used for illustration in *MV* models in the following sections. The characteristics of the layer components of the pop-up structure in Figure 4.26 are described in Table 4.3.

Table 4.2. Abbreviations of layer components.

Layer component	Type of layer component	Abbreviation
Primary pop-up layer	Angle fold	a
	Non-parallel fold	n
	Parallel fold	p
Secondary pop-up layer	Inward fold	i
	Outward fold	o
Paper fold	Flat vertex fold	f
	Angle pleats	a
	Non-parallel pleats	n
	Parallel pleats	p

Table 4.3. Description of a composite layered pop-up structure.

Layer No.	Joint to	Layer Component	Type of layer component	Type of development
1	-	Primary pop-up layer	Parallel fold (p_1)	Valley crease initiation
2	1	Primary pop-up layer	Parallel fold (p_2)	Mountain crease initiation
3	1	Primary pop-up layer	Non-parallel fold (n_1)	Valley crease initiation
4	1	Secondary pop-up layer	Outward fold (o)	Outward fold attachment
5	4	Pleats	Angle pleats (a)	Mountain crease initiation
6	3	Flat vertex fold	Flat vertex fold (f)	Mountain crease initiation
7	5	Primary pop-up layer	Non-parallel fold (n_2)	Valley crease initiation

To model the pop-up structure, MV models of the primary and secondary pop-up layers are superimposed on a global MV coordinate system (M_G, V_G). Figure 4.27 illustrates the MV model of the given case. Coordinates of the layer components are represented by nodes and marked with abbreviation. Note that the primary pop-up layers and pleats use the same set of abbreviations but are differentiated by their MV slopes. The MV slope of 1 is unique to pleats.

MV models of secondary pop-up layers have their origins anchored at the nodes of the primary pop-up layers that they are built on. The overlapping of boundary regions is possible but in the given case, the regions do not overlap. *MV* models of the flat vertex fold and pleats can be added to either region of the pop-up layers according to where they are constructed upon. In the given case, the flat vertex fold was created on a primary pop-up layer and the angle pleats were added to the secondary pop-up layer.

Figure 4.27. The *MV* model of the pop-up structure in Figure 4.26.

Inverted pop-up structures, including those with supplementary paper folds, can be represented on *MV* models as a reflection along the line $V = M$. The inversion can also be applied using a transformation matrix, $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, such that

$$\begin{pmatrix} M \\ V \end{pmatrix}_{Inverted} = T \begin{pmatrix} M \\ V \end{pmatrix}. \quad (4.22)$$

4.8.2 Applications of MV models

The application of the MV models is two-fold. Firstly, it permits the user to cap the number of pop-up layers for a design, as determined by design requirements. The layer line, together with the crease constraints, encloses the boundary region where conceivable MV coordinate pairs and crease initiations can be identified.

Figure 4.28. Possible routes to develop pop-up structures.

In the example where three primary pop-up layers are required for a design, the boundary region is bounded by the layer line $M_p + V_p = 10$, as shown in Figure 4.28. There are two possible routes of crease initiations to reach the MV coordinates (4, 6) in Figure 4.28a. However, the eventual pop-up designs are different for the two routes of development although the number of creases is the same. In Figure 4.28b, the MV coordinates (5, 5) lie on the crease constraint $M_p = 2V_p - 5$. There is only one possibility in the combination of crease initiations for a pop-up structure with this combination of creases.

Secondly, the *MV* model can be used as a form of heuristic to illustrate the construction process for a pop-up structure, highlighting the number of layers and the type of crease initiations applied. The model can be enhanced by attaching sets of parameters and symbols at the coordinate pairs to denote the type of elemental pop-up structures used, as shown in Figure 4.27. Lastly, it can be incorporated as a verification tool to check if structures with a designated number of mountain and valley creases are flat foldable.

However, the *MV* model contains several drawbacks. Firstly, it cannot identify the particular crease that a component is built upon if there is more than one crease of the same crease assignment on the preceding layer. For example in Figure 4.26, the non-parallel fold (n_1) is constructible upon either of the two valley creases of the parallel fold (p_1). To improve the *MV* models in such cases, creases can be tagged and labeled on the models. Secondly, the model can be constructed in three dimensions instead of the present 2D Cartesian coordinate system. In the higher dimension, the sets of *MV* coordinates for the primary and the secondary pop-up layers would be clearly illustrated in separate Cartesian planes. The two planes can joint at common nodes like p_1 in the given case. This allows a clearer presentation of the model when the complexity of composite layers increases. Another problem arises when a pop-up structure has a couple of identical layer components connected with the same preceding component. The crease initiation or development lines would coincide on the *MV* model, and the representation would not be useful. One solution is to assign an integer value to the lines to indicate the number of identical components.

4.8.3 MV Trees

The use of *MV* models motivates another approach to model pop-up structures with composite layers. That is to represent it as a weighted tree, specifically a tree with weighted edges. In Graph Theory, a tree is defined as a connected graph that contains no cycle [81]. On the tree, the edges are directed and assigned with values of *MV* slopes. Like *MV* models, nodes represent layer components, and are labeled with abbreviations, as shown in Table 4.3. The tree is laid out as an acyclic skeleton that preserves the layering sequence. Figure 4.29a illustrates the weighted tree T of the case in Figure 4.26 where the vertex set is $V(T) = \{p_1, n_1, v, n_2, p_2, o, a\}$.

Figure 4.29. Tree representations of a pop-up structure with composite layers.

By further breaking down T into a simpler form, its nodes can be augmented to form another tree T_{MV} , whose nodes represent crease assignments, as shown in Figure 4.29b. Nodes in black and white symbolize mountain creases and valley creases respectively. Note that the black nodes in T (Figure 4.29a) do not represent valley creases. In T_{MV} , the *MV* slopes are now depicted by the

increase in the number of mountain and valley creases from one node to adjacent nodes. The groups of nodes in dashed rectangles correspond to the nodes of T . As such, each layer component can be represented by a set of vertices, each marked with m or v . For example, $p_I = \{m_I, v_I, v_2, v_3\}$, as shown in Figure 4.29b. T_{MV} can also be divided into two subgraphs, known as spanning trees [81], T_p and T_s , such that the edge set $E(T_{MV}) = E(T_p) \cup E(T_s)$. T_p and T_s represent the primary and secondary pop-up layers with supplementary paper folds respectively. Figure 4.30 illustrates the two spanning trees.

Figure 4.30. Tree representations of primary and secondary pop-up layers.

Since pop-up layers can be modelled as trees, they also adopt properties of trees. As such, T_{MV} exhibits the properties in Table 4.4.

Table 4.4. Properties of T_{MV} .

Properties	Implications
It is a connected graph, i.e. every two vertices are joined by a path.	If an isolated-vertex exists, it represents an alien crease, which may not be part of the pop-up structure.
It does not have cycles, i.e. no vertex is repeated.	If cycles emerge, the pop-up structure has interfering layer components and may not be erectable or collapsible.
Every two vertices are joined by a unique path, i.e. there are no parallel edges.	If parallel edges exist, layer components may be coplanar and are thus redundant.
The number of vertices is one more than the number of edges.	The relationship can be used to check if a graph is a tree, and thus validates the composition of a pop-up structure.

The approach of MV trees eliminates the concerns with separate MV coordinate sets for primary and secondary pop-up layers and the representation of identical components on the Cartesian coordinate system. The representations of pleats and outward folds in T_{MV} are distinct as the tree patterns of their crease assignments are unique. These are illustrated in the angle pleats (a) and the outward fold (o) in Figure 4.29b. However, other layer components cannot be easily distinguished by T_{MV} models unless their nodes are grouped and labeled in vertex sets. Both the MV model and the MV tree have their pros and cons but both are useful models for multi-layer pop-up structures.

4.9 Summary

A pop-up structure can comprise primary pop-up layers, secondary pop-up layers, flat vertex folds and pleats. Each of these layer components are

examined using crease properties and placed together for composite layering, which is governed by the iterative layering model. This chapter describes two models that can represent composite layering, the *MV* model and the *MV* tree. The former uses the Cartesian coordinates system while the latter deploys weighted edges. Both models assess mountain-valley crease assignments, the type of layer components used and their order of construction. Though the analysis in this chapter is done with one-piece pop-up structures, both models are applicable to pop-up structures of the tent formation, both one-piece and multi-piece. As the iterative layer model can produce a diversity of pop-up designs, individualized models will result. Figure 4.31 illustrates the summary for the analysis done in this chapter.

Figure 4.31. Summary of analysis on composite layering of pop-up structures.

One significant finding in this chapter is the identification of boundary regions where construction of pop-up structures is feasible. A pop-up structure that is flat foldable satisfies the boundary conditions. But a graph constructed within the boundary region does not necessary lead to the creation of a flat-foldable pop-up structure. In addition, paper folds and pop-up structures carry similar crease properties, as shown in Table 4.5. In particular, the line $V - M = 2$ which describes the flat foldability in flat vertex fold lies within the boundary

region for primary pop-up layers. The weighted MV tree stemmed from the concept of Graph Theory. The next chapter further expounds compositions of pop-up structures using the theory.

Table 4.5. Comparison between paper folds and one-piece pop-up structures.

Properties of creases	Paper folds	Pop-up structures
Converging creases intersect on paper	Flat vertex fold	Single-slit, angle fold
Converging creases do not intersect on paper	Non-parallel pleats	Single/ double-slit, non-parallel fold
Parallel creases	Parallel pleats	Single/ double-slit, parallel fold
Ability to flat fold	The line $V - M = 2$	Boundary region which covers the line $V - M = 2$

Chapter Five

Planar graphs of pop-up structures

A graph is planar if there exists a drawing of it on the Euclidean plane such that no two edges intersect other than at a vertex. In the context of pop-up design, the elemental composition of pop-up faces, and thus their crease edges and cut edges, do not cross one another. If edges cross, topological representations of the pop-up structures can aid in detecting interference or distortion of jointed paper pieces, particularly when used in modelling. Thus, this propels the reason in exploring planar graph models of pop-up structures.

At the same time, the focus of the investigation now shifts from the secondary tier to the primary tier of the part classification, as discussed in Section 3.16. While the previous chapter primarily examines components in the secondary tier such as pop-up layers and types of creases, this chapter focuses on primitives – vertices, edges and faces of pop-up structures. The following sections discuss how planar graphs can be used to describe the topology of pop-up structures.

Sections 5.1 to 5.6 look into the graph forms of one-piece pop-up structures. In particular, Section 5.2 examines modular sub-graphs that build up the graphical representation of the pop-up structures. Section 5.3 examines the validation of the structures' topology with known properties of planar graphs. Section 5.4 further establishes a set of topological rules for crease and cut edges. Section 5.5 highlights geometric constraints using a degree-four vertex case. Section 5.6 examines the connectedness of faces using dual graphs. Section 5.7 extends the

study to multi-piece pop-up structures and analyzes in particular the gluing technique applied to these structures.

5.1 Planar graphs of one-piece pop-up structures

If the faces and edges of an unfolded one-piece pop-up structure are to be mapped as a graph, a planar graph [82], as shown in Figure 5.1, will be formed. The planar graph is also a plane graph, since the structure is made from a planar piece of paper. As the unfolded structure does not produce edges with crossings and overlapping faces, the planarity of its graph corresponds to a structure's ability to unfold flat.

Figure 5.1. (a) A one-piece pop-up structure and (b) a graph that illustrates its edges.

Though non-overlapping faces are necessary for flat folding of the one-piece pop-up structure, its ability to unfold flat does not mean that it is flat foldable. As the geometry of the structure affects flat folding, the planarity of a graph can be deemed as an essential but not sufficient condition for flat foldability. The subsequent sections analyze the properties of planar graphs for one-piece pop-up structure; a planar graph is denoted by G .

5.1.1 Graph forms of pop-up structures

The planar graph of a structure can represent both the erected and the collapsed states as the topology of the structure does not change. Figure 5.2 illustrates graph forms that are produced by unfolded one-piece pop-up structures. Figure 5.2a shows the graph of the single-slit and Figure 5.2b that of the double-slit when the base pages are 180° apart. On G , a single-slit is characterized by a cut face, a crease vertex and a cut vertex. A double-slit has two cut faces and two cut vertices. The cut faces created by slits are bounded by cut edges, which are represented by dashed lines in the figures. As discussed in Chapter Three, boundary edges are cut edges that represent the boundary of the base pages. They are represented by thick solid lines. If paper folds are added on the structures, they will result in additional vertices and edges on G .

Figure 5.2. Unfolded one-piece pop-up structures and their corresponding graphs.

(a) Single-slit and (b) double-slit

(a) Graph of the single-slit

To better represent erected structures, isomorphic forms of the planar graphs are examined. A pair of graphs is isomorphic if there are one-to-one

correspondences between vertices and edges of both graphs. For example, Figure 5.3a shows a graph of the erected single-slit, which is isomorphic to that in Figure 5.2a. In addition, every convex polyhedron can be represented in the plane or surface of a sphere by a planar graph, which is known as the polyhedral graph. In Figure 5.3, the graph of the single-slit and the polyhedral graph of the semi-octahedron or pyramid both contain a common sub-graph. Hence, the graph in Figure 5.3a provides an appropriate representation of the single-slit, which erects a pyramid-like pop-up layer.

Figure 5.3. (a) The single-slit graph, (b) a graph of the pyramid and (c) a sub-graph of the single-slit.

However, a single-slit does not always have a pyramid-like form, as illustrated in Figure 5.4b. Suppose this graph becomes contractible. That is, two boundary edges are contracted and the three vertices incident to them merge. Then the resulting graph, as shown in Figure 5.4c, becomes another variation of the single-slit. The single-slit in Figure 5.4a and 5.4c are angle folds whereas that in Figure 5.4b is a non-parallel fold. The graph for a single-slit parallel fold is also the same as that in Figure 5.4b. The faces of the pop-up structures are numbered to correspond to the faces on the graphs.

Figure 5.4. Variations of the single-slit (top) and their corresponding graphs (bottom).

(b) Graph of the double-slit

Figure 5.5a shows a graph of the double-slit that is isomorphic to that in Figure 5.2b. Similarly, its sub-graph (Figure 5.5c) is also a sub-graph of the cube graph (Figure 5.5b). Thus, the graph in Figure 5.5a can represent the double-slit in its erected form, which has a cubical form.

Figure 5.5. (a) The double-slit graph, (b) a cube graph and (c) a sub-graph of the double-slit.

5.1.2 Base graphs

The graphs in Figure 5.6 represent pop-up layers of one-piece pop-up structures and are sub-graphs of the single-slit and double-slit. As pop-up layers are the building blocks of pop-up structures, let us term the graph that represents a pop-up layer of a one-piece pop-up structure as a *base graph*.

Figure 5.6. Base graphs of (a) the single-slit and (b) the double-slit.

Additional pop-up layers can be represented by attaching base graphs onto crease edges of an existing planar graph. Figure 5.7 shows a two-layer pop-up structure comprising double-slits. The numbers in the figure show a correspondence between the planar faces of the structure to the solid faces on the graph. Its cut faces are not numbered. Figure 5.8 illustrates the development of a planar graph that models a three-layer pop-up structure with two double-slits and one single-slit.

Figure 5.7. A two-layer one-piece pop-up structure and its corresponding graph.

Figure 5.8. The development of a one-piece pop-up structure with (a) the first layer, (b) the second layer and (c) the third layer.

To validate if multi-layer G graphs are planar, we apply this rule: a graph is planar if its sub-graphs are planar [82]. Since base graphs are sub-graphs of polyhedral graphs and polyhedral graphs are known to be planar [82], base graphs are therefore planar. Base graphs are attached to crease edges of a planar graph to form multi-layer one-piece pop-up structures, as Figure 5.8 illustrates. Therefore, multi-layer G graphs are also planar.

As shown in Figure 5.8, the attachment of base graphs is achieved by firstly dividing a crease edge on the existing planar graph into two, and secondly joining their ends with edges of the base graphs. But as only specific edges of these graphs can be conjoined, it is necessary to validate the topological compositions of G graphs. This is discussed in next section.

5.2 Validation of topological compositions in G graphs

G is a plane graph. One characteristic of plane graphs is the Hamiltonian cycle, which is a closed trail that passes through all vertices on G [82]. Figure 5.9 shows possible trails for Hamiltonian cycles in arrows.

Figure 5.9. Hamiltonian cycles of (a) a single-slit, (b) a double-slit and (c) a three-layer pop-up structure.

To verify that G is Hamiltonian, Grinberg's Theorem [83, 84] is applied. The theorem states that if a loopless plane graph has a Hamiltonian cycle C , then

$$\sum_{i=2}^n (i-2)(\phi'_i - \phi''_i) = 0 \quad (5.1)$$

where ϕ'_i and ϕ''_i are the number of faces of degree i contained in the interior of C and exterior of C respectively. The degree of a face is the number of edges that bounds the face. If the theorem is not satisfied, G is not Hamiltonian and may contain vertices and edges that result in flawed compositions of pop-up structures. For example, Figure 5.10a illustrates a non-Hamiltonian graph that cannot model a pop-up structure, and Figure 5.10b illustrates one with two redundant crease edges.

Figure 5.10. (a) and (b) are non-Hamiltonian graphs. (c) The numbers indicate the degree of the faces on the graph.

To illustrate the application of the theorem, the Hamiltonian cycle, C , on the graph in Figure 5.10c is analyzed. The interior of C has two faces, degrees 4 and 5. On its exterior, including the unbounded face, there are four faces: one of degree-two, two of degree-three and one of degree-five. In applying the formula, the example is found to be Hamiltonian, as shown below.

$$\begin{aligned}
 & \sum_{i=2}^n (i-2)(\phi'_i - \phi''_i) \\
 &= (2-2)(0-1) + (3-2)(0-2) + (4-2)(1-0) + (5-2)(1-1) \\
 &= 0
 \end{aligned}$$

As G is planar, its Euler's characteristic $v - e + f$ is 2, where v is the number of vertices, e is the number of edges and f is the number of faces. In addition, having properties of polyhedral graphs [85], G also follows $\sum_{n \geq 3} nv_n = 2e$, where v_n is the number of vertices of degree n . For example, the graph in Figure 5.11 has five vertices of degree-three and one vertex of degree-four. The graph satisfies the formula since the number of edges, e , is 10, as shown in the calculation.

$$\begin{aligned}
 LHS &= \sum_{n \geq 3} nv_n = 3(5) + 4(1) = 20 = 2(10) = 2e \\
 &= RHS
 \end{aligned}$$

Figure 5.11. Degrees of vertices on a graph.

The theorems above offer feasible methods to validate the graph structures. However, such validations are not adequate because they do not inspect the type of edges, in particular the localized configurations of crease edges and cut edges. Hence, topological relationships between these edges have to be examined.

5.3 Topological conditions for crease and cut edges

This section establishes conditions that ensure local edge topologies are valid for pop-up structures. These conditions govern the developments of successive pop-up layers and supplementary paper folds. They also describe how a fold and a cut are applied together to develop one-piece pop-up structures. The cut edges described in the conditions refer to slits and boundary edges of the paper. Adjacent edges refer to edges incident to the same vertex.

Condition 5.1

Cut edges within boundary edges exist if and only if G contains a n -cycle of cut edges such that $n \geq 4$, $n = 2k$, $k \in \mathbb{Z}^+$.

A n -cycle of cut edges refer to n numbers of cut edges that are connected in a loop to form a cut face on the pop-up structure. Each end of an edge is only connected to one other edge. It has been shown that the number of solid faces is always even [74], and at least four plane-linkages are required to form a pop-up structure of the tent formation, as discussed in Section 3.2.1c. When a paper is slit, a cut edge is created at one side of each face, as shown in Figure 5.12. Hence, the number of cut edges must also be even and at least four, i.e. $n \geq 4$, $n = 2k$.

Figure 5.12. Vertices and edges formed as a result of slitting.

Condition 5.2

At each cut vertex of G , the number of adjacent cut edges is always two.

A slit across a crease edge creates a pair of cut edges and a cut vertex, which is the intersection of the edges, as illustrated in Figure 5.12. Additional cut edges incident at the cut vertex do not produce pop-up layers but are redundant cuts on the paper.

As a corollary to Conditions 5.1 and 5.2, *two cycles on G cannot share a common cut edge*, for this will create three adjacent cut edges. Adjacent cut faces are also unattainable.

Condition 5.3

At each cut vertex of G , there is at least one crease edge adjacent to the pair of cut edges.

In order for a pop-up layer to fold at the cut vertex, at least one crease edge is incident to the vertex. The number of crease edges on the cut vertex can be increased by constructing paper folds at the slit.

Condition 5.4

At each crease vertex of G , the number of adjacent crease edges is even and at least four.

Adjacent crease edges incident to crease vertices lead to flat vertex folds. Thus, the condition can be retraced to Maekawa's theorem and Section 4.6.2.

As a corollary to Conditions 5.3 and 5.4, *the degree of each vertex on G is at least three, i.e. $d_G(v) \geq 3$* . Two cut edges and at least one crease edge are incident to a cut vertex. At least four edges are incident to a crease vertex is. Hence, the degree of each vertex in G is at least three.

This corollary is analogous to a definition of the polyhedral graph, which states that *the degree of every vertex is at least three and the degree of every face is at least three* [85]. The portion relating to the polyhedral face is, however, not

applicable to G as the pop-up structure is not a convex polyhedron though it yields a three-dimensional form. Each face on G has a minimum degree of two.

Condition 5.5

If crease edges are not adjacent, G is 3-regular.

A graph is 3-regular if all its vertices are of degree-three. Of the three variations of the single-slit graph shown in Figure 5.13, the graphs in Figures 5.13a and 5.13b have adjacent crease edges. The former gives a vertex of degree-four and the latter a vertex of degree-five. Therefore, they are not 3-regular. But the case illustrated in Figure 5.13c is a 3-regular graph, i.e. all its vertices are of degree-three. It is also known as a cubic graph.

Figure 5.13. (a) and (b) are non-regular graphs of the single-slit. (c) The cubic graph of the single-slit.

Likewise, the graph of a double-slit is also 3-regular as its crease edges are not adjacent. The condition is also applicable to G graphs of multi-layer one-piece pop-up structures. For example, the graph of the degree-four vertex case in Figure 5.14 is not regular. In summary, the degree of vertices resulting from adjacent creases differentiates variations of the graphs, particularly for single-slits, as shown in Table 5.1.

Table 5.1. Type of G graphs

Type of G graphs (without added paper folds)	Adjacent crease edges	Highest degree for vertices
A single-slit, angle fold	Yes	Degree-four
A single-slit, angle fold touching boundary edges	Yes	Degree-five
A single-slit, parallel and non-parallel fold	No	3-regular
A double-slit, parallel and non-parallel fold	No	3-regular

5.4 A geometric condition for a degree-four cut vertex case

The conditions discussed up to this point are purely topological. This section highlights a geometric condition arising from a case of two pop-up layers sharing two cut vertices. This means that the two pop-up layers share a common slit. In this case, each cut vertex is of degree-four. As illustrated in Figure 5.14, two crease edges and two cut edges are incident to each cut vertex. The condition for the case concerns the redundancy of slits and interference between pop-up layers, as described in Condition 5.6.

Figure 5.14. A two-layer pop-up structure with degree-four vertices.

Condition 5.6

At each degree-four cut vertex, the pair of crease edges has to form an angle of more than 180° on a base page or a pop-up face.

The angle between the two crease edges is denoted by θ in Figure 5.14. The assumption for this condition requires that no crafting is performed on the slit after it is created. The condition is elaborated in two parts as follows.

a) If the pair of crease edges at the degree-four cut vertex forms an angle of 180° , the common slit between the two pop-up layers is redundant.

A slit is redundant if an identical pop-up structure can be produced without making that slit. Pop-up structures with redundant slits are shown in Figure 5.15. When the two crease edges make an angle of 180° , they become collinear. Collinear crease edges incident to degree-four cut vertices are highlighted by bold lines in the figure.

Figure 5.15. Redundant slits on pop-up structures.

If pop-up layers are created from the gutter crease and are joined to the boundary edges, the gutter crease becomes non-distinct. As shown in Figure 5.15c, the structure can be constructed solely by paper folding. Figure 5.15 has so far illustrated pop-up structures, each with two pairs of collinear crease edges incident to degree-four vertices. Figure 5.16a shows a pop-up structure with only one pair of collinear crease edges. Without the assumption of the condition, a slit can be further folded and cut, as shown in Figure 5.16b. In this case, the slit is not redundant.

Figure 5.16. (a) A structure with one pair of collinear crease edges and (b) a structure with a non-redundant slit.

b) If the pair of crease edges at the degree-four cut vertex forms an angle of less than 180° , the two pop-up layers will interfere with each other when they move.

As mentioned earlier in the chapter, correct topology alone is not sufficient for pop-up layers to move without interference. Figure 5.17a illustrates the interference of two pop-up layers whose crease edges make an angle of less than 180° . Without the assumption for the condition, the interference may be removed by adding cuts and creases to the pop-up layers like those in Figure

5.16b. The pop-up spinner [86], as shown in Figure 5.17b, is an example of a structure constructed with degree-four vertices and satisfies Condition 5.6. Its pop-up layers do not interfere with one another despite its complex crease pattern.

Figure 5.17. (a) A pop-up structure with interfering pop-up layers and (b) a pop-up spinner.

5.5 Duals of G graphs

The dual G^* of the planar graph G has vertices that correspond to the faces of G and faces that correspond to vertices of G [87]. Figure 5.18 provides two examples of such graphs. In each example, the face on G corresponding to a vertex on G^* is indicated by the same number on both G and G^* . The dual graphs are useful for studying the connectedness of pop-up faces and identifying types of faces on G . By applying the method of vertex colouring on G^* , solid faces on G can be easily distinguished from its cut faces. The following two conditions deploy the use of dual graphs.

Figure 5.18. Corresponding dual graphs of (a) a single-slit and (b) a double-slit.

Condition 5.7

If a vertex set $V_{Cut}(G^)$ corresponds to the cut faces, including the unbounded face, on G , then the vertices in $V_{Cut}(G^*)$ cannot be joined by one edge.*

Cut faces in G cannot be adjacent in order to satisfy the corollary of Condition 5.2. This can be represented by 3-colouring of G^* such that vertices in the set $V_{Cut}(G^*)$ are assigned the same colour. Replacing colours with numbers, Figure 5.19 illustrates two cases where $V_{Cut}(G^*)$ is represented by the number '1'. Vertices corresponding to unbounded faces are abbreviated as 'u'. Vertices corresponding to solid faces are indicated by '2' and '3'. Therefore, if a vertex of $V_{Cut}(G^*)$ is identified, the rest of the set can be determined. As the vertices of G^* can be represented by a minimum of three colours, the duals are characterized by a chromatic number $\chi(G^*)$ of 3.

Figure 5.19. Vertex colouring of G^* . (a) A single-slit and (b) the case of degree-four vertex.

Condition 5.8

A vertex v^* on G^* corresponds to cut faces on G if

$$d_{G^*}(v^*) = \begin{cases} 2k, k \in \mathbb{Z}^+ & \text{for the unbound face (outside the boundary edges).} \\ 2k+2, k \in \mathbb{Z}^+ & \text{for all cut faces within the boundary edges.} \end{cases}$$

The degree of the vertex $d_{G^*}(v^*)$ corresponds to the number of edges that bind a cut face in G . Therefore, the degree is even because cut faces are bounded by an even cycle of cut edges, as discussed in Condition 5.1. To enable flat folding, the minimum number of cut edges within the boundary edge is four while that of the boundary edge is two. The examples in Figure 5.19 clearly differentiate the unbound face and cut faces within the boundary edges of G . However, Figure 5.20 shows two distinct duals whose vertex colours are interchangeable and still represent the same face configurations. In the figure, the numbers represent the faces, as explained in Condition 5.7.

Figure 5.20. Duals with interchangeable vertex colours. (a) A 4-regular dual
and
(b) a dual with vertices of degree-six.

5.6 Planar graphs of multi-piece pop-up structures

Unlike one-piece pop-up structures which use the cutting or slitting technique, multi-piece pop-up structures are formed by attaching glue tabs of pop-up faces onto other pieces. The glue tab is an extended portion on a pop-up face and does not contribute to the structural design of a pop-up structure. A multi-piece pop-up structure can be divided into two parts: the base pages and the pop-up layers. The graph of the base pages is the same as that of the one-piece structure. Figure 5.21 illustrates the graphs of common pop-up layers of multi-piece pop-up structures. Solid and dashed lines represent crease edges and cut edges respectively.

Figure 5.21. Graphs of common pop-up layers of multi-piece pop-up structures.

In this study, we assume that a pop-up layer of the multi-piece pop-up structure is not formed by gluing individual pop-up faces together but by cutting and

creasing a paper sheet to form connected pop-up faces. The latter approach aligns with the practices of the manufacturing industry whereas the former approach is a tedious and labour intensive process. Hence, the analysis here does not look into the graph of a pop-up face. Note that there are, however, cases where individual pop-up faces are attached to a pop-up structure to aid the movement of its mechanisms.

5.6.1 Graphs of pop-up layers

Let us denote the graph of pop-up layers as L . L graphs are planar since edges do not cross one another. But they are different from G graphs as only layers are modelled in L graphs while base pages and all layers of one-piece pop-up structures are represented in G graphs. Figure 5.21 also shows that L graphs contain degree-two vertices while the G graphs have a minimum of three degrees. As such, properties of L and G graphs differ.

Table 5.2. Application of G graphs' conditions to L graphs.

Conditions	Application to L graphs
5.1	L contains a cycle of cut edges which satisfies this condition.
5.2	L contains adjacent cut edges which satisfies this condition.
5.3	L contains crease edges adjacent to cut edges which satisfies this condition. Solely adjacent cut edges, however, exist.
5.4	Without additional paper folds, L does not contain crease vertices. With additional paper folds, L satisfies this condition.
5.5	L does not satisfy this condition.
5.6	The geometric condition is applicable to multi-piece pop-up structures of the tent formation.
5.7 and 5.8	L comprises only solid faces. But the conditions are satisfied if multi-piece pop-up structures of the tent formation are represented as G graphs.

L graphs are Hamiltonian. As they are planar, they follow Euler's characteristic $v - e + f = 2$. The relationship between vertices and edges is $\sum_{n \geq 2} nv_n = 2e$ since the vertices have a minimum of two degrees whereas vertices of polyhedral graphs and G graphs have a minimum of three degrees. Table 5.2 compares L graphs with conditions established for G graphs in earlier sections.

5.6.2 Glue edges on L graphs

The glue tabs are not modelled as part of L graphs since they do not affect the structural designs of the pop-up structures. However, it is beneficial to investigate cut edges on L that give rise to glue tabs and conditions that govern their locations. The *glue edge* is defined as a cut edge on the pop-up layer from which a tab may be extended. It is represented by dotted lines, as shown in Figure 5.22.

Figure 5.22. Glue edges on L graphs.

Glue edges on a primary pop-up layer are attached to the face of a base page or a pop-up layer. On the other hand, glue edges on a secondary pop-up layer are aligned to the edges of primary pop-up layers. After a glue edge is attached to a

face or an edge, it becomes a crease edge. This transition is illustrated in Figure 5.23. Figure 5.24 shows some possible L graphs superimposed on the graph of the base pages. In the figure, the glue edges have become crease edges. Note that geometric considerations are omitted from these models.

Figure 5.23. The transition of a cut edge to a crease edge.

Figure 5.24. L graphs on graphs of base pages.

There are two types of glue edges: the *inter-layer glue edge* and the *intra-layer glue edge*. An inter-layer glue edge joins a pop-up layer to the base pages or another pop-up layer. An intra-layer glue edge connects a box pop-up layer in a loop, and is only used in box folds.

5.6.3 Topological conditions for glue edges

Glue edges on L comply with conditions described below. In particular, Conditions 5.9 and 5.10 govern intra-layer glue edges while Conditions 5.11 to 5.13 govern inter-layer glue edges.

Condition 5.9

An intra-layer glue edge exists if and only if L has n connected faces such that $n \geq 4$, $n = 2k$, $k \in \mathbb{Z}^+$.

As mentioned earlier, intra-layer glue edges are found only in box folds. A box pop-up layer usually contains six connected faces. But a minimum of four faces, comprising two primary and two secondary pop-up faces, is required to support erection, as discussed in Section 3.2.1c. Hence, $n \geq 4$. It is also established that the sum of the widths of the primary and secondary pop-up faces of the box pop-up layer on the left base page has to be equal to that on the right side [5, 76] for flat folding. So an increase of one pop-up face on one side of the layer has to be accompanied by another pop-up face on the other side to achieve equal widths. L , therefore, has an even number of faces i.e. $n = 2k$.

Condition 5.10

An intra-layer glue edge exists if it is not adjacent to any crease edge on L .

The layer is not flat foldable if the crease edge and the intra-layer glue edge are adjacent. This is because creases have to be all parallel or all concurrent for the faces to be always planar foldable [65]. A glue edge will become a crease edge when it is attached to the other end of the layer, as explained in Figure 5.23. Hence, if the glue edge is adjacent to a crease edge, the eventual creases on the layer are not all parallel or all concurrent. Figure 5.25 illustrates an example.

Figure 5.25. Cut edges adjacent to crease edges.

A corollary of Condition 5.10 states that *an intra-layer glue edge exists if it is incident to two vertices, each of degree two on L* . This condition does not allow glue edges to be adjacent to crease edges because crease edges are incident to vertices with degree-three or more, as discussed in Conditions 5.2 and 5.3. Figure 5.26a illustrates an example of L graphs that does not contain any intra-layer edge as there is no cut edge incident to two degree-two vertices. The numbers in the figure refer to the degrees of the vertices. Figure 5.26b shows an L graph with an intra-layer glue edge, which satisfies Conditions 5.9 and 5.10.

Figure 5.26. An L graph (a) without intra-layer glue edges and (b) with an intra-layer glue edge.

Condition 5.11

Inter-layer glue edges exist in pairs on L .

Two and only two inter-layer glue edges have to be affixed on different faces or edges of a pop-up layer to engage the folding mechanism. Figure 5.27

illustrates examples of locations of inter-layer glue edges on L , highlighted by dotted lines.

Figure 5.27. L graphs with inter-layer glue edges.

Condition 5.12

Inter-layer glue edges exist if they are separated by two cut edges, which are adjacent to a common crease edge on L .

The inter-layer glue edges have to be separated by an even number of connected cut edges to enable flat folding of the pop-up layer. For the box pop-up layer, the two cut edges represent the edges of its secondary pop-up faces. Examples in Figure 5.27 also satisfy this condition. The condition also prevents glue edges from locating on opposite sides of L , as shown in Figure 5.28. In the figure, the inter-layer glue edges are separated by two cut edges that lack a common adjacent crease edge. Hence, it does not satisfy the condition.

Figure 5.28. An L graph that does not satisfy Condition 5.12.

Condition 5.13

An inter-layer glue edge exists if it is incident to at least one vertex of degree-three on L .

This condition complements Condition 5.12. It ensures that the inter-layer glue edge does not take the location of the intra-layer glue edge, which is incident with vertices of degree-two and explained in Condition 5.10. The examples in Figure 5.29 satisfy this condition and Condition 5.12.

Figure 5.29. L graphs that satisfy both Conditions 5.12 and 5.13.

The combination of intra-layer and inter-layer glue edges determines the types of structure the layers would take. For example, in Figure 5.30, the pair of identical graphs satisfies conditions for the inter-layer glue edges. However, the existence of an intra-layer glue edge in Figure 5.30b would differentiate the

pop-up structures that eventually develop. The graph in Figure 5.30a leads to a parallel or non-parallel box fold but that in Figure 5.30b results in an angle fold with a secondary pop-up layer. The topological conditions, thus, identify and assign appropriate glue edges on pop-up layers to ensure accurate constructions of multi-piece pop-up structures.

Figure 5.30. L graphs of (a) a box layer and (b) an angle fold with a secondary pop-up layer.

5.7 Summary

By and large, the study has established topological conditions for the elemental composition of pop-up structures, as summarized in Figure 5.31. The composition of one-piece pop-up structures is different from that of multi-piece pop-up structures, essentially made up of individual pop-up layers glued together. Common conditions underlying the pop-up structures analyzed in this research are shown in Table 5.2. However, there is not a governing rule for all types of possible pop-up structures as each type of pop-up structures are topologically and geometrically unique, as illustrated in the classification model for pop-up structures (Appendix A).

Figure 5.31. Summary of analysis of vertex-edge-face compositions.

The conditions can be translated and applied as evaluation tools for software modelling of pop-up structures. For one-piece pop-up structures, while conditions for crease and cut edges enable detection of defective structural composition on, duals of G graphs identify their solid and cut face configurations. For pop-up layers of multi-piece pop-up structures, conditions on L graphs locate glue edges suitable for tabbing. However, the use of glue tabs would not be effective if parts of a multi-piece pop-up structure cannot fit well together. Hence, the next chapter will focus on geometries that enable the joining of layers in multi-piece pop-up structures.

Chapter Six

Geometrical properties of pop-up structures

Geometry and topology form a complementary whole of a pop-up design. The previous chapter is concerned with the topological representation of pop-up structures. In this chapter, geometry on the positioning and angular displacements of pop-up layers is examined. Geometric attributes are crucial in the construction of multi-piece pop-up structures to ensure flat folding between pop-up layers whereas the freedom of geometric design is limited in one-piece pop-up structures. However, the findings are still applicable to one-piece pop-up structures with added secondary pop-up layers. Section 6.1 examines a ratio on pop-up structures to distinguish pop-up structures of the tent formation. Section 6.2 looks into geometric conditions for flat folding. Section 6.3 further analyzes other geometric relationships, particularly those relating secondary pop-up layers to primary pop-up layers. Section 6.4 provides an analysis of the geometric relationships with an example.

6.1 Characteristic ratio of pop-up structures

Let us consider pop-up structures made of the tent formation. Such a structure consists of a basis of two base pages and one pop-up layer, whose cross-section is shown in Figure 6.1.

Figure 6.1. Cross sectional view of a pop-up structure.

Let b_l be the distance between the gutter crease and the crease on the left base page and similarly, b_r for that on the right base page. b_l and b_r are perpendicular to the gutter crease. h_l and h_r are the cross-sectional heights of the left plane and right plane of the pop-up layer respectively. Let ϕ be the angle between the two planes of the pop-up layer and θ be the angle between the base pages.

6.1.1 Multi-piece 180° pop-up structures

For a structure to be fully erected at 180°,

$$b_l < h_r, b_r < h_l, \phi < \theta.$$

A proof for $b_l < h_r$ when $b_r < h_l$ is shown in Appendix C. Thus,

$$b_l + b_r < h_l + h_r \quad (6.1)$$

Squaring both sides,

$$b_l^2 + b_r^2 < (h_l + h_r)^2 - 2b_l b_r \quad (6.2)$$

If x is the distance between the two creases shown in Figure 6.1, applying Cosine rule on the base pages gives

$$x^2 = b_l^2 + b_r^2 - 2b_l b_r \cos \theta \quad (6.3)$$

Similarly, with the pop-up layer,

$$x^2 = h_l^2 + h_r^2 - 2h_l h_r \cos \phi \quad (6.4)$$

Combining (6.3) and (6.4),

$$b_l^2 + b_r^2 = 2b_l b_r \cos \theta + h_l^2 + h_r^2 - 2h_l h_r \cos \phi \quad (6.5)$$

Substituting (6.5) into (6.2),

$$2b_l b_r \cos \theta + h_l^2 + h_r^2 - 2h_l h_r \cos \phi < (h_l + h_r)^2 - 2b_l b_r$$

$$\cos \phi > \frac{b_l b_r}{h_l h_r} (\cos \theta + 1) - 1 \quad (6.6)$$

Appendix B provides the detailed derivation of Equation 6.6. Let

$$R = \frac{b_l b_r}{h_l h_r} \quad (6.7)$$

and have it defined as the *characteristic ratio* of pop-up structures. This ratio would be useful in verifying the feasibility of various dimensions of pop-up structures. An example of the dimensioning of multi-piece parallel pop-up structures is elaborated in Section 6.4.

In addition, since $b_l < h_r$ and $b_r < h_l$,

$$\frac{b_l b_r}{h_l h_r} < 1$$

$$R < 1$$

R also has to be a positive value since each of its variables is positive. Hence, if $0 < R < 1$, a pop-up structure can be erected till the base pages are opened at 180° , and the angle ϕ can be related to the angle θ such that $\cos \phi > R(\cos \theta + 1) - 1$.

6.1.2 Multi-piece 90° pop-up structures

On the other hand, if the structure is to be fully erected at 90° ,

$$b_l = h_r, \quad b_r = h_l, \quad \phi = \theta.$$

Therefore,

$$\frac{b_l b_r}{h_l h_r} = \frac{(\cos \phi + 1)}{(\cos \theta + 1)} = 1 \quad (6.8)$$

Hence, if $R=1$, a pop-up structure can be fully erected at 90° and it flattens at 180° .

6.1.3 Unique cases

In unique cases where the pop-up layer becomes coplanar and the structure cannot erect or flatten at 180° , as shown in Figure 6.2,

$$b_l > h_r, b_r > h_l, \phi > \theta,$$

$$R > 1.$$

The movement of the pop-up structure would cease at an angle of θ on opening the base page. Figure 6.2b and 6.2c may be considered unique cases of 90° pop-up structures since their pop-up layers erect when the base pages are 90° apart albeit not following the conditions stated in Section 6.1.2.

Figure 6.2. Unique cases.

The maximum angle θ can be found with

$$\cos \theta = \frac{b_l^2 + b_r^2 - (h_l + h_r)^2}{2b_l b_r} \quad (6.9)$$

Thus, if $R > 1$, a pop-up structure can only be erected up to an angle θ when the pop-up layer becomes coplanar, i.e. $\phi = 180^\circ$. Table 6.1 summarizes the inference of the characteristic ratio.

Table 6.1 Summary of the characteristic ratio.

Characteristic ratio R	Opening of base pages at 180°	Possible types of pop-up structures formed
$R < 0$	Not possible	None
$0 < R < 1$	The structure erects when $\theta = 180^\circ$.	Multi-piece 180° pop-up structures. Not applicable to one-piece structures.
$R = 1$	The structure flattens when $\theta = 180^\circ$.	Multi-piece 90° pop-up structures. Applicable to one-piece structures.
$R > 1$	The base pages do not open to 180° .	Unique cases of multi-piece 90° pop-up structures. Not applicable to one-piece structures.

6.2 Conditions for flat folding of multi-piece pop-up structures

Sections 6.2.1 and 6.2.2 examine fundamental geometric conditions required for the flat folding of the tent formation and the box formation respectively. The tent formation comprises the angle fold, the non-parallel fold and the parallel fold while the box formation comprises the non-parallel box fold and parallel box fold. Though explained with multi-piece 180° pop-up structures, these properties are also applicable to multi-piece 90° and one-piece pop-up structures. For a pop-up structure to be planar foldable, the creases have to be all parallel or all concurrent [65]. Hence, the creases described in the conditions for flat folding are all parallel or all concurrent. But there are more than one set of all parallel or all concurrent creases on pop-up structures with secondary pop-up faces. Figure 6.3 illustrates this.

Figure 6.3. Sets of all parallel or all concurrent creases.

6.2.1 Flat folding for the tent formation

Condition 6.1

The sums of angles about an intersecting point of four creases on opposite sides of the structure are equal, i.e.

$$p_{1l} + s_{1l} = p_{1r} + s_{1r} \quad (6.10)$$

where p_{1l} and p_{1r} are angles on the primary pop-up layer, and s_{1l} and s_{1r} are angles on the secondary pop-up layer, as shown in Figure 6.4.

Figure 6.4. Locations of s_{1l} , s_{1r} , p_{1l} and p_{1r} .

p_{1l} and s_{1l} are adjacent to the left p - s crease while p_{1r} and s_{1r} are adjacent to the right p - s crease. The secondary crease separates s_{1l} and s_{1r} on the secondary pop-up layer. This condition ensures the flat folding of a jointed pair of primary and secondary layers.

However, for p - s creases to fold and pop-up layers to remain planar throughout the structure's movement, Equation 6.10 is constrained such that $p_{1l} + s_{1l} < 180^\circ$ and $p_{1r} + s_{1r} < 180^\circ$. This implies that the sum of angles about the vertex incident to the creases, as shown in Figure 6.4, is less than 360° . Mathematically, $p_{1l} + s_{1l} + p_{1r} + s_{1r} < 360^\circ$. If $p_{1l} + s_{1l} = p_{1r} + s_{1r} = 180^\circ$, the secondary layer would become a protrusion from the primary pop-up layer. The p - s creases would also not fold when the primary layer rotates and are thus redundant. If $p_{1l} + s_{1l} = p_{1r} + s_{1r} > 180^\circ$, the secondary pop-up layer would be deformed by the movement of the primary pop-up layer.

Figure 6.5. An inward fold of the secondary pop-up layer.

s_{1l} and s_{1r} would interchange when the secondary crease is reverse-folded, i.e. the secondary pop-up layer changes from an outward fold to an inward fold, as shown in Figure 6.5. The secondary crease would also change its position.

However, the position of the crease remains unchanged if it is an angle bisector between the two p - s creases. Section 6.3.1 would further discuss s_{1l} and s_{1r} and express them mathematically.

Condition 6.2

The sum of the cross-sectional distances, perpendicular to the gutter crease, between the gutter crease and the crease on the primary pop-up layer should be equal on both sides to enable flat folding. That is,

$$b_l + h_l = b_r + h_r \quad (6.11)$$

The length b and the cross sectional height h have been introduced in the Characteristic Ratio in Section 6.1. These dimensions are crucial in keeping the flat folded structure within its base pages, and would be further examined in Section 6.3.

Condition 6.3

If the primary creases are all concurrent, the sum of the angles of depression and convergence on one side of the structure is equal to that of the adjacent side, such that

$$c_l + d_l = c_r + d_r \quad (6.12)$$

where c_l are c_r the angles of convergence on the left side and right side of the base pages or the preceding primary pop-up layer respectively, and d_l and d_r are the angles of depression on the left side and right side of the primary pop-up layer respectively.

Figure 6.6. Angles of convergence and depression on an angle fold with base pages opened at 180° . The gutter crease is shown as a dotted line.

As shown in Figure 6.6, the angles of convergence are located on base pages, each between the primary crease and the gutter crease. The angles of depression are located on primary pop-up layer, each between two primary creases of the same plane. Similar to Condition 6.1, this rule enables the structure to flat fold when the base pages are closed. Multi-piece 90° pop-up structures without secondary pop-up layers also collapse flat when the base pages are 180° apart. They are further constrained such that *the sum of angles of convergence is equal to the sum of angles of depression*,

$$c_l + c_r = d_l + d_r. \quad (6.13)$$

When combined with Equation 6.12, $c_l = d_r$ and $c_r = d_l$. This pair of equations is often referred to in pop-up crafting books such as [5] as the necessary conditions for creating angle folds and non-parallel folds.

6.2.2 Flat folding for the box formation

All concurrent creases on the box folds do not meet at real vertices though they are converging. Besides the requirement of all parallel or all concurrent creases, primary creases on the box folds have to be position symmetric about the gutter crease or a crease on the preceding layer, which it is built upon, as described in Section 3.2.2b. In addition, the two following conditions have to be considered for flat folding.

Condition 6.4

If creases on the secondary pop-up faces are all parallel, the sum of the widths of its primary and secondary pop-up faces on the left base page is equal to that on the right base page, i.e.

$$w_{p1} + w_{s1} + w_{s3} = w_{p2} + w_{s2} + w_{s4} \quad (6.14)$$

where w_p and w_s refer to the widths of the primary and secondary pop-up faces respectively. See Figure 6.7.

Figure 6.7. (a) An enclosed box fold and (b) an open box fold.

The condition for the enclosed box fold in Figure 6.7a is also examined in [5, 76]. For the case in Figure 6.7b, the primary pop-up faces have different widths, and their two edges indicated in the illustration are assumed to be parallel to the p - s and secondary creases. Also shown in the figure, w_{p2} is the width on the right primary pop-up face referenced from the edge of the shorter w_{p1} . Without two secondary pop-up faces, Equation 6.14 is reduced to

$$w_{p1} + w_{s1} = w_{p2} + w_{s2} \quad (6.15)$$

The widths of adjacent secondary pop-up faces will interchange if the outward fold becomes an inward fold, and vice versa. Hence, w_{s1} and w_{s2} will interchange, and so do w_{s3} and w_{s4} .

Condition 6.5

If creases on the secondary pop-up faces are all concurrent, the sum of angles between each crease on the pop-up faces on the left base page is equal to that on the right base page.

$$\alpha_l + \beta_l + \gamma_l = \alpha_r + \beta_r + \gamma_r \quad (6.16)$$

where α , β and γ are angles between creases on each pop-up face, as shown in Figure 6.8. l and r denote the left and right base pages respectively. The two secondary creases make an angle δ when the box pop-up layer is flattened such that $\delta = \alpha + \beta + \gamma$, as shown in Figure 6.8b. In the case where the box pop-up layer has only one pair of secondary pop-up faces, Equation (6.16) is reduced to

$$\alpha_l + \beta_l = \alpha_r + \beta_r \quad (6.17)$$

Similar to Condition 6.1, the angles on adjacent secondary pop-up faces will interchange if an outward fold becomes an inward fold, and vice versa. Hence, the angle pair, α_l and α_r , will interchange, and similarly for γ_l and γ_r .

Figure 6.8. (a) A box fold with all concurrent creases and (b) its pop-up faces when flattened.

In all, three conditions are required to govern flat folding of a box fold. For pop-up structures of the tent formation, two are required. If secondary pop-up layers are added, they are governed by an additional condition. They are listed as follows.

(a) Flat folding conditions for the tent formation

1. The pop-up structure possesses all parallel or all concurrent creases.
2. The pop-up structure satisfies Condition 6.2 for parallel folds or Condition 6.3 for non-parallel and angle folds
3. The pop-up structure satisfies Condition 6.1 if secondary pop-up layers exist.

(b) Flat folding conditions for the box formation

1. The pop-up structure possesses all parallel or all concurrent creases.
2. The pop-up structure is position symmetric about the crease that it is built upon.
3. The pop-up structure satisfies Condition 6.4 for parallel box folds or Condition 6.5 for non-parallel box folds.

The conditions of all parallel and all concurrent creases are linked to the conditions of flat folding discussed in this section, as shown in Figure 6.9. It is a generic condition that applies to pop-up structures, whether the tent or the box formation. On the other hand, Conditions 6.1 to 6.5 are applied to specific pop-up layers and creases.

Figure 6.9. Summary for flat folding conditions.

6.3 Other geometric parameters on pop-up layers

Angles, heights and lengths of pop-up layers are vital for the construction of a multi-piece pop-up structure. This section discusses the mathematical relationships between these parameters, in particular those that relate dimensions of secondary pop-up layers to those of primary pop-up layers. These mathematical expressions enable calculations of other dimensions and verifications on a computer-aided platform.

6.3.1 Parameters on the secondary pop-up layer

(a) Angles on the secondary pop-up layer

If the two p - s creases do not meet in line when the pop-up layer is flattened, then the secondary crease makes different angles s_{1l} and s_{1r} with each of the two p - s creases. In other words, the secondary crease is not an angle bisector. When the pop-up layers are flat folded, the angular difference between s_{1l} and s_{1r} is also the angle bounded by the two p - s creases. Equation 6.10 can be rearranged to represent this relationship.

$$p_{1l} - p_{1r} = s_{1r} - s_{1l} \quad (6.18)$$

With the above equation, s_{1l} and s_{1r} can be determined if l_1 , l_2 , l_3 , p_{1l} and p_{1r} , are known. l_1 and l_2 are the lengths of the two p - s creases while l_3 is the distance between the end points of the p - s creases on the base pages when they are opened at 180° . See Figure 6.10. The edges of the secondary pop-up layer are assumed to touch the line l_3 when the structure fully erects. Note that for a multi-piece 180° structure to rise above the base pages when they are opened to 180° , $l_3 < l_1 + l_2$.

Figure 6.10. Lengths of p - s creases and the distance between their end points.

By Cosine rule,

$$s_{1l} = \frac{1}{2} \left[p_{1r} - p_{1l} + \cos^{-1} \left(\frac{l_1^2 + l_2^2 - l_3^2}{2l_1l_2} \right) \right] \quad (6.19)$$

and

$$s_{1r} = \frac{1}{2} \left[p_{1l} - p_{1r} + \cos^{-1} \left(\frac{l_1^2 + l_2^2 - l_3^2}{2l_1l_2} \right) \right] \quad (6.20)$$

If the secondary crease is an angle bisector, $p_{1l} = p_{1r}$ and

$$s_{1l} = s_{1r} = \frac{1}{2} \left[\cos^{-1} \left(\frac{l_1^2 + l_2^2 - l_3^2}{2l_1l_2} \right) \right] \quad (6.21)$$

To enable folding of p - s creases on a multi-piece 180° pop-up structure and ensure no deformation on its secondary layer, $s_{1l} + s_{1r} \leq 180^\circ$. For a multi-piece 90° pop-up structure with a secondary pop-up layer to fully erect at 90° , $s_{1l} + s_{1r} \geq 90^\circ$.

Figure 6.11 illustrates the other angles on the secondary pop-up layer, s_{2l} , s_{2r} , s_{3l} and s_{3r} . These angles vary with s_{1l} and s_{1r} and can be calculated using the sum of interior angles in a triangle. The derivation of their equations, including Equations 6.19 and 6.20, are found in Appendix B. Section 6.4 discusses the application of these formulas.

Figure 6.11. Angles on the secondary pop-up layer.

(b) Lengths on the secondary pop-up layer

l_3 can be further used to determine dimensions on the secondary pop-up layer.

Let l_4 be the length of the secondary crease, which divides l_3 into l_5 and l_6 such that $l_5 + l_6 = l_3$. See Figure 6.12. Similarly, l_4 , l_5 and l_6 can be expressed as Equations 6.22, 6.23 and 6.24 respectively. Their derivations are found in Appendix B.

$$l_4 = \frac{l_1 \sin s_3}{\sin(s_1 + s_3)} \quad (6.22)$$

$$l_5 = \frac{l_1 \sin s_1}{\sin(s_1 + s_3)} \quad (6.23)$$

$$l_6 = l_3 - \frac{l_1 \sin s_1}{\sin(s_1 + s_3)} \quad (6.24)$$

Figure 6.12. Other lengths on the secondary pop-up layer.

6.3.2 Parameters on the primary pop-up layer

The height perpendicular to the primary crease on the base page on the left side of the primary pop-up layer is denoted by h_l and on the right h_r . See Figure 6.13. These heights are important to the measurements of the pop-up structure. h should be smaller than the distance between the edge of the base page and its primary crease, d_b , so that the structure would not protrude from the base pages when flat folded.

Figure 6.13. Heights on the primary pop-up layer.

On each side of a primary pop-up layer, the height h and the length of the p -s crease l are related such that

$$h = l \sin p \text{ if } p \leq 90^\circ \quad (6.25)$$

or

$$h = l \cos\left(p - \frac{\pi}{2}\right) \text{ if } p \geq 90^\circ \quad (6.26)$$

where p refers to p_{ll} or p_{lr} , depending on the side of the primary pop-up layer.

However, angle folds and non-parallel folds are structurally dissimilar from parallel folds. The following two sub-sections elaborate the geometries of primary pop-up layers of these folds.

(a) Length on the parallel fold

The following formula concerns only the 180° parallel fold. It has primary creases that are parallel to the gutter crease. By Pythagoras Theorem,

$$l_3 = \sqrt{(b_l + b_r)^2 + \left(\sqrt{l_2^2 - h_r^2} - \sqrt{l_1^2 - h_l^2}\right)^2} \quad (6.27)$$

Figure 6.14 illustrates the parameters used in the formulation of the equation.

Thus, if given the perpendicular heights h_l and h_r , lengths l_1 and l_2 , the distance on one base page b_l , the length l_3 could be deduced. The derivation of Equation 6.27 is found in Appendix B.

Figure 6.14. Parameters on the 180° parallel fold.

(b) Lengths on angle and non-parallel folds

These pop-up structures have crease lines that converge to a point. The location of the point of convergence and the height of the structures vary with angles of depression on the primary creases located on the base pages. Figure 6.15a shows the parameters of the non-parallel fold and Figure 6.15b its plan view. If b_l and b_r are the horizontal distances perpendicular to the gutter crease and $|\Delta x| = |x_r - x_l|$ is the absolute difference in vertical distance between the two endpoints of l_3 ,

$$l_3 = \sqrt{(b_l + b_r)^2 + \left(\frac{|\Delta x|}{\cos c_i}\right)^2} + 2|\Delta x|(b_l + b_r)\tan c_i \quad (6.28)$$

where c_i is the angle of convergence adjacent to the longer primary crease on the base pages. The derivation of Equation 6.28 is found in Appendix B. If the angle of convergence is zero, i.e. $c_i = 0$,

$$l_3 = \sqrt{(b_l + b_r)^2 + (|\Delta x|)^2} \quad (6.29)$$

This is the basis of the expression for parallel folds in Equation 6.27. Hence, Equation 6.29 is a generalized form for the length l_3 .

Figure 6.15. (a) Parameters on a non-parallel fold and (b) its plan view.

(c) Heights on angle and non-parallel folds

Unlike the parallel fold, the height of primary pop-up layer on the angle fold or the non-parallel fold is not constant. Figure 6.16 illustrates a side of the primary pop-up layer in Figure 6.15. h_0 is the largest h on the primary pop-up layer, assuming the peak at which h_0 is located is on the edge of the pop-up layer. If d is the angle of depression, h_0 can be calculated by

$$\begin{aligned} h_0 &= l \sin(180^\circ - p - d) \\ &= l \sin(p + d) \end{aligned} \quad (6.30)$$

where l and p refer to l_1 and p_{1l} or l_2 and p_{1r} , depending on the side of the pop-up layer examined.

Figure 6.16. h_0 and h_{ang} on the side of the primary pop-up layer.

The height h changes with the distance $\frac{x}{\cos c}$ along the primary crease on the base page such that

$$h = h_0 - \frac{x \tan d}{\cos c} \quad (6.31)$$

where c is the angle of convergence of the same piece.

To gauge if the structure on angle or non-parallel fold protrudes from the base pages when they are closed, h_{ang} is measured. See Figure 6.16. Unlike the parallel fold, h_{ang} is not perpendicular to the primary creases due to the angular positioning of the primary creases. It is mathematically expressed as

$$\begin{aligned} h_{ang} &= \frac{h_o}{\cos c} \\ &= \frac{l \sin(p + d)}{\cos c} \end{aligned} \quad (6.32)$$

6.4 Feasibility check and application of formulas

As mentioned in the previous section, suppose the heights h_1 and h_2 , the angles p_1 and p_2 , and the distance b_l are known, these parameters can be used to define the orientation and positioning of the pop-up structure. Eventually, other dimensions on the structure can be deduced with the governing equations.

R , the characteristic ratio defined in Section 6.1, acts as an indicator to determine acceptable dimensions for pop-up structures. Hence, for dimensions to be valid, $0 < R \leq 1$. This excludes unique cases in Section 6.1.3. In addition, let w represents the minimum width for either of the base pages such that the base pages conceal the structure when they are closed. As shown in Figure 6.17, the minimum width for each base page would be the same due to Condition 6.2 since $b_l + h_l = b_r + h_r$. Hence, on each side of the primary pop-up layer,

$$w = b + h \quad (6.33)$$

Figure 6.17. The minimum width for base pages.

6.4.1 A case study with the parallel fold

Using the parallel fold as an example, dimensions for a secondary pop-up layer were calculated using dimensions from its primary pop-up layer. Given a fixed

input value for heights $h_l = 45\text{mm}$, $h_r = 66\text{mm}$ and a range for b_l between 10mm and 100mm, the feasible range of b_l to build a pop-up structure was first deduced from the characteristic ratio. The other inputs were angles $p_{lr} = 120^\circ$ and $p_{lr} = 100^\circ$, as shown in Figure 6.18. Table 6.2 illustrates the computation extracted from a MS Excel worksheet.

Figure 6.18. Dimensions of the primary pop-up layer on the case study.

Table 6.2. Parameter inputs and feasibility check on input dimensions.

No.	b_r (mm)	w (mm)	l_1 (mm)	l_2 (mm)	l_3 (mm)	l_4 (mm)	l_5 (mm)	l_6 (mm)
1	-11.0	55.0	52.0	67.0	14.4	#NUM!	#NUM!	#NUM!
2	-1.0	65.0	52.0	67.0	23.8	51.5	-1.0	24.9
3	9.0	75.0	52.0	67.0	41.6	52.8	08.4	33.2
4	19.0	85.0	52.0	67.0	60.7	49.6	17.7	43.0
5	29.0	95.0	52.0	67.0	80.3	43.1	27.5	52.8
6	39.0	105.0	52.0	67.0	100.0	32.0	38.1	61.9
7	49.0	115.0	52.0	67.0	119.9	#NUM!	#NUM!	#NUM!
8	59.0	125.0	52.0	67.0	139.7	#NUM!	#NUM!	#NUM!
9	69.0	135.0	52.0	67.0	159.6	#NUM!	#NUM!	#NUM!
10	79.0	145.0	52.0	67.0	179.6	#NUM!	#NUM!	#NUM!

#NUM! indicates invalid numeric values in the formula.

Values highlighted in bold are dimensions within the feasible range of R . In this case, the range $30\text{mm} \leq b_l \leq 60\text{mm}$ was secured for a viable pop-up

construction. With the inputs in Table 6.2, the minimum width of the base pages w , the length b_r and the lengths on the secondary pop-up layer, l_1 , l_2 , l_3 , l_4 , l_5 , and l_6 , were calculated using relationships established in Section 6.3. Lengths calculated with input values out of the feasible range gave negative values or error responses, as shown in Table 6.3. Similarly, angles on the secondary pop-up layer, s_{1l} , s_{1r} , s_{2l} , s_{2r} , s_{3l} , and s_{3r} , were generated. Angles using input values out of the feasible range also generated negative values or error responses, as shown in Table 6.4. To verify the accuracy of the calculations, the six angles in Table 6.4 should sum up to 360° by the principle of the sum of interior angles in a quadrilateral.

Table 6.3. Dimensions of lengths generated from input values.

No.	$b_r(mm)$	$w(mm)$	$l_1(mm)$	$l_2(mm)$	$l_3(mm)$	$l_4(mm)$	$l_5(mm)$	$l_6(mm)$
1	-11.0	55.0	52.0	67.0	14.4	#NUM!	#NUM!	#NUM!
2	-1.0	65.0	52.0	67.0	23.8	51.5	-1.0	24.9
3	9.0	75.0	52.0	67.0	41.6	52.8	08.4	33.2
4	19.0	85.0	52.0	67.0	60.7	49.6	17.7	43.0
5	29.0	95.0	52.0	67.0	80.3	43.1	27.5	52.8
6	39.0	105.0	52.0	67.0	100.0	32.0	38.1	61.9
7	49.0	115.0	52.0	67.0	119.9	#NUM!	#NUM!	#NUM!
8	59.0	125.0	52.0	67.0	139.7	#NUM!	#NUM!	#NUM!
9	69.0	135.0	52.0	67.0	159.6	#NUM!	#NUM!	#NUM!
10	79.0	145.0	52.0	67.0	179.6	#NUM!	#NUM!	#NUM!

#NUM! indicates invalid numeric values in the formula.

Table 6.4. Dimensions of angles generated from input values.

<i>No.</i>	<i>s_{1l} (deg)</i>	<i>s_{1r} (deg)</i>	<i>s_{2l} (deg)</i>	<i>s_{2r} (deg)</i>	<i>s_{3l} (deg)</i>	<i>s_{3r} (deg)</i>
1	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!
2	-1.01	18.99	119.67	42.35	61.34	118.66
3	9.16	29.16	90.86	50.83	79.98	100.02
4	19.89	39.89	72.52	47.69	87.59	92.41
5	31.93	51.93	56.09	40.05	91.98	88.02
6	46.92	66.92	37.79	28.37	95.29	84.71
7	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!
8	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!
9	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!
10	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!

#NUM! indicates invalid numeric values in the formula.

Table 6.5 lists the output parameters in the case study and the equations used to calculate them. The table also identifies the lower and upper limits of the parameters. The limits correspond to $0 < R < 1$, which is the range of R for a multi-piece parallel fold, given the input values of b_l , h_l and h_r . The output lengths l_1 and l_2 do not have lower and upper limits because their equations are solely dependent on the lengths h_l and h_r , and angles s_{1l} and s_{1r} , which are fixed input values. Though theoretically achievable, the lower limits of some parameters, such as b_r and l_5 , are too small for the construction of pop-up structures.

Table 6.5. Lower and upper limits of the parameters.

Parameter	Equation	Lower limit	Upper limit
R	6.7	0.01	0.99
b_r	6.11	0.70 mm	44.80 mm
W	6.33	66.70 mm	110.80 mm
l_1	6.26	NA	NA
l_2	6.26	NA	NA
l_3	6.27	26.45 mm	111.49 mm
l_4	6.22	52.21 mm	20.75 mm
l_5	6.23	0.61 mm	45.08 mm
l_6	6.24	25.84 mm	66.41 mm
s_{1l}	6.19	0.62 deg	59.39 deg
s_{1r}	6.20	20.62 deg	79.39 deg
s_{2l}	Appendix B part c	113.40 deg	23.33 deg
s_{2r}	Appendix B part c	45.36 deg	17.88 deg
s_{3l}	Appendix B part c	65.98 deg	97.28 deg
s_{3r}	Appendix B part c	114.02 deg	82.72 deg

The geometric relationships applied in this example are preliminary to analysis of more complex pop-up structures. Vital design requirements such as the minimum width for the base pages can be deduced. While tailoring the design of pop-up structures, dimensional changes can be predicted and adjusted without trials and errors. From the example, it can be observed that as b_l increased within its feasible range, lengths l_3 , l_5 and l_6 increased while l_4 decreased. Angles s_{1l} , s_{1r} and s_{3l} also increased while s_{2l} , s_{2r} and s_{3r} decreased. Likewise, other input parameters can be altered to determine selected output parameters.

6.5 Summary

This chapter has presented fundamental geometric relationships and considerations that are vital for the construction of pop-up structures. The characteristic ratio R can help to determine feasible dimensions for pop-up

structures of the tent formation. Conditions have been established for flat folding of pop-up structures. Relationships between parameters on pop-up layers are identified. They enable calculations and verifications of other dimensions on the same layer or adjacent layers. Figure 6.19 summarizes the analysis done in this chapter.

Figure 6.19. Summary of geometric properties examined in the chapter.

The analysis in this chapter does not apply to all types of pop-up structures as they differ geometrically. Hence, the characteristic ratio does not apply to the box folds, and the conditions for flat folding are different for the tent and box formations. Though this chapter illustrates pop-up structures with one primary and one secondary pop-up layers, the properties are pertinent and applicable to those with multiple layers. The next chapter introduces a software architecture that permits the 3D modelling of pop-up structures.

Chapter Seven

3D modelling and simulation of pop-up structures

Pop-up structures were assessed through modelling and simulation on the computer. This chapter discusses in particular the software organization and architecture for a CAD platform. It also highlights the implementation of the simulation engine and results with case studies.

7.1 Existing tools for modelling of pop-up structures

There has been investigation to incorporate pop-up design in existing CAD programs such as AutoCAD and Pro/Engineers [76, 88]. For these investigations, customized programs built upon existing library functions and interfaces of the software were created. The structures of the customized programs were made up of sub-routines, which called for the drawings of a small number of pop-up structures and utilities like scaling and positioning to manipulate them. The CAD programs readily provide an architecture framework and a library base that developers can tap upon. However, there are no comprehensive studies on the characteristics of pop-up structures, which would provide protocols to manage the functionality of a pop-up design application. Software development kits of existing software do not offer sufficient functions for the development of pop-up design programs.

As mentioned in Chapter Two, Glassner, Hendrix and Mitani have developed programs for pop-up design. Mitani's Pop-up Card Designer is a simple and user-friendly application, capable of developing structures with multiple double-slits. However, the program does not enable construction of other pop-

up forms like the multi-piece 180° structures. Both Glassner and Hendrix have substantially examined basic pop-up geometry and offered 3D modelling using different approaches. Hendrix's Popup Workshop is coded with object oriented Java classes. Glassner's pop-up design assistant enfolds geometry that was explicitly calculated, after which models were rendered with Autodesk 3ds Max [89]. Both programs enable an easier process in pop-up design. But they fall short of a broader classification of pop-up types.

7.2 Components for modelling of pop-up structures

For a computer-aided tool to effectively model pop-up structures, its core should essentially consist of three distinct components: an object-oriented environment, the simulation of pop-up mechanisms and the use of intrinsic mathematical properties. While the object-oriented environment provides the infrastructure required to model and simulate the pop-up structures, the mathematical properties govern the characteristics of these models. The following sub-sections further discuss the three components. Another component to consider is the convertor of 3D models to 2D drawings to facilitate post-modelling activities. This component is commonly found in CAD tools, and an example is illustrated in the Section 7.2.2.

7.2.1 An object-oriented environment

Garetti et al emphasized that the object-oriented paradigm provides an excellent approach to manage and express complex systems, enable realistic views, flexibility, extensibility and reusability [90]. Hence, it suits the development for 3D modelling of pop-up structures. A key feature in the desired CAD

environment is to enable a hierarchically structured logical representation that permits modular decomposition [91]. As analyzed in previous chapters, pop-up structures can be hierarchically categorized and broken up into layers - the building blocks of the structures. In particular, the eleven types of elemental pop-up structures in the classification model are found to exhibit distinct topological and geometrical attributes. So each type can be set as a modular entity in the program. Furthermore, attributes of these pop-up structures can be described by classes. The properties of the classes comprise structural attributes, constraints and operations [92]. Figure 7.1 gives examples of classes in modelling pop-up structures.

Figure 7.1. Correlation between class modelling and pop-up domain.

7.2.2 Simulation of pop-up mechanisms

For the purpose of archiving his work, Wolff [93] videotaped each page of his pop-up books as it opened and closed before they were shipped to the publishers. He also commented that the video could only present the action of the book opening and closing from a camera direction. His effort reveals the need for capturing pop-up mechanism in motion and the inadequacy of present practices. Hence, simulating the pop-up mechanism is crucial not just in the

design stage but also in post-design operations. Thus, the computer-aided tool for pop-up design should illustrate the dynamic behaviour of pop-up structures, which Lee et al. [65] and Glassner [66, 67] have assessed.

3D modelling, animation and rendering packages such as Blender [94] and Autodesk 3ds Max [89] are powerful in creating 3D objects whose geometry can be altered and parts animated. Though they can model folding of planes with basic plane geometries and rotational axes, these modelling tools are not customized to the technical needs of paper engineers. Investigation on Autodesk 3ds Max also showed that the folding of planes do not always follow the desired loci when animating models of pop-up structures. For example, in the case of a tent pop-up structure, when the geometric properties of pop-up structures are not defined, the left base page may intersect the right pop-up face during its animation, as illustrated in Figure 7.2. Additional functions that enable mountain-valley assignments for each crease as well as geometric relationships described in Chapter Six are required to control the rotation of the planes.

Figure 7.2. Animation of a pop-up structure without geometric constraints.

The closest that Blender has experimented with paper crafts is when paper models were created using Unfolder [95], a Python script that converts meshes into a flat net without deforming any faces. Therefore, just as Kergosien and his team [96] recognized the need of paper behaviour to simulate bending of paper-like sheets, behaviours of pop-up structures are needed for simulating the models. Figure 7.3 positions pop-up modelling tools in an overview of the 3D modelling tools discussed in the preceding sections.

Figure 7.3. An overview of the 3D modelling tools.

7.2.3 The use of topological and geometrical properties

The properties refer to the mathematical relationships and conditions derived from analysis in the previous chapters. They are not generic to the entire family of pop-up structures. For example, the crease boundary conditions can be applied to both one-piece and multi-piece tent pop-up structures. But the topological conditions for glue edges are only applicable to multi-piece pop-up structures. The properties in this research characterize compositions and behaviours of pop-up models but they are not sufficient for supporting a

computer-aided tool. Table 7.1 shows some of the mathematical properties with suggested applications for the computer-aided tool.

Table 7.1. Examples of mathematical properties and their suggested applications.

Properties	Types of pop-up structures	Applications
Crease boundary conditions	One-piece and multi-piece Tent formation	Check mountain-valley assignments for creases to enable flat-folding.
Conditions for vertices edges (G graphs)	One-piece Tent formation	Verify the feasibility of adding and removing creases on pop-up faces.
Conditions for glue edges (L graphs)	Multi-piece Tent and box formations	Locate edges on pop-up faces where glue tabs can be constructed.
Geometric relationships	One-piece and multi-piece Tent and box formations	Compute the orientation and dimension of creases and cut edges.

Figure 7.4 illustrates an example of how properties can be applied to verify if the composition of a model is feasible. Two sets of properties are used in the flow diagram; each described in a level of validation. The first level checks the edges of a model with topological conditions of crease and cut edges. The second level analyzes the assignments of crease types on the model. Any combination of mountain and valley creases that falls out of the boundary region would render the design inapplicable.

Figure 7.4. Application of properties on elements of 3D models [97].

7.3 Software architecture and organization

This section examines the software architecture for the pop-up modelling tool. Applying definitions by Gorton [97], the software architecture discussed in this section aims to define its structure, specify component communication and provide an abstraction to the architectural description. The architecture of the tool loosely adapts two architectural views based on work by Hofmeister et al. [99] and Bass et al. [100]. These two views are the conceptual view and the module view.

7.3.1 Conceptual architecture

The conceptual architecture is a UML-based view concerned with the structure of major design components and the connections between them. Using the model-view-controller (MVC) framework [101, 102], the proposed conceptual architecture is a vertical incorporation of three layers, namely the library, the controller and the graphical user interface (GUI), as illustrated in Figure 7.5.

Figure 7.5. Conceptual architecture of a modelling tool for pop-up structures.

The layers comprise sets of highly cohesive components connected by arrows, which depict dependencies between the components. For example, the operations of the simulator are significantly dependent on the OpenGL specifications and computation from the calculator.

(a) The library

The library houses libraries and databases, and supports the functioning of the controllers. Pop-up classes call upon pop-up topology and geometry databases to govern compositions and behaviours of pop-up structure. The object-oriented environment is supported by standard specifications of Open Graphics Library (OpenGL) [103, 104].

(b) The controller

The controller is the kernel of programming. The C++ core-code in this layer communicates and manages data between the library and GUI through three

processing components: the calculator, the simulator and the convertor. The calculator controls and verifies the relationships of the vertices, edges and planes, and tracks the build-up of pop-up layers. Upon verifying the structure composition, the data is sent to the simulator where images are rendered and constructed into models for animation. The other component in this layer is the convertor, which transforms the computed data into the 2D drawings, the logical view and the textual view.

(c) The user interface

The user interface displays four viewing components, specifically the logical view, the textual view, the 2D unfolded view and the 3D view. These views are not to be confused with the architecture views discussed earlier. The logical view provides the hierarchical view of how the pop-up model is constructed whereas the textual view displays the processed information as structured text. The 2D unfolded view plots unfolded polygonal meshes of the model. The 3D view enables the model to rotate under the camera projection and its pop-up faces to animate the pop-up mechanisms.

7.3.2 The simplified conceptual architecture

Figure 7.6 shows a simplified conceptual architecture that is used in the case studies of the 3D modelling application. The architecture is abridged to focus on the simulation of pop-up structures and avoid analysis paralysis [105], i.e. being overly concerned with the software organization and neglecting its functional development. So the architectural view is made minimal yet sufficient to fulfill its aims stated at the beginning of the chapter. As such, the

logical and textual views in the GUI and the converter are removed from the architecture.

Figure 7.6. Simplified conceptual architecture.

7.3.3 Module architecture

The purpose of the module architecture for the modelling tool is to enable modularization in the software organization. By partitioning the entire architecture into separate components, the system can be upgraded, maintained and extended easily as it increases in size. Figure 7.7 illustrates the modular decomposition in the module view. The module architecture can be broken down into twelve modules, concurring with the elemental pop-up structures in the classification model in Appendix A. However, there is one difference between the classification model and the module decomposition in Figure 7.7. The base pages in the classification model are conceived as an essential part of pop-up structures. But in the module view, the base pages are distinguished as a separate entity generic to all pop-up designs, and the base page module creates this entity.

Packages	Pop-up modules		
	Base page module		
The one-piece slit package	Single-slit angle fold module	Single/double-slit non-parallel fold module	Single/double-slit parallel fold module
The multi-piece 90° fold package	90° angle fold module	90° non-parallel fold module	90° parallel fold module
The multi-piece 180° tent fold package	180° angle fold module	180° non-parallel fold module	180° parallel fold module
The multi-piece 180° box fold package		Non-parallel box fold module	Parallel box fold module

Figure 7.7. Modular decomposition in the module view.

In addition, the modules are grouped into packages, as depicted by the horizontal bands in Figure 7.7. In this context, the package denotes a group of elements such as classes, associations, generalizations with a common theme, as defined by Blaha and Rumbaugh [106]. The vertical layering (from the lowest layer to the top layer) of the packages does not represent any semantics though there is geometric association in the classification model. The modules in each package carry out distinct operations and are described as follows.

(a) The base page module

As shown in Figure 7.7, the base page module does not fall under any package. It generates a pair of planes and precedes the activation of all other modules. Since the modules can be run in any sequence, it is possible not to follow this order. But as the construction of paper pop-up structures always commences

with the creation of two base pages, it is logical to initiate modelling with the base page module. Figure 7.8 illustrates the task of constructing base pages when broken down and viewed from two object diagrams. Figures 7.8a and 7.8b illustrate the plant and product domains respectively. In the diagrams, objects represent real world elements, which are in brackets. Arrows represent unidirectional associations. For example, the base page department works under the production plant, and ‘work for’ is the association name.

Figure 7.8. Object diagrams. (a) The plant domain and (b) the product domain.

The two domains are linked by two associations, one between the pop-up module and the Shape View, and another between the base page module and the base page model, as shown in Figure 7.9. In the diagram, the base page module performs two operations. Firstly, it receives instructions from the pop-up module to make the base pages. Secondly, it initiates the construction of the base pages. At the same time, the pop-up module acquires properties of the planes from the Shape View. This object diagram can be seen as the basic model to describe operations of the modules. In the following sub-sections, the diagram is used to explain the operations of the packages.

Figure 7.9. Object diagram of base pages' construction.

(b) The one-piece slit package

The one-piece slit package contains the single-slit angle fold module, the single-slit/double-slit non-parallel slits module and the single-slit/double-slit parallel module. In order to create slits in pop-up structures, the worker needs to work with a cutter. Likewise, the slit modules share the cutter that creates slits on the models. The object diagram for modules in this package is illustrated in Figure 7.10 with the single slit module as an example. The task of applying the cutter is shown by an association between the single-slit angle fold module and the cutter in the diagram. Note also that the object 'Shape View' in the diagram reflects the design of single slit, which is triangular in shape and different from that in Figure 7.9.

Figure 7.10. Object diagram of the single slit's construction.

(c) The multi-piece 90° and 180° tent fold packages

Multi-piece pop-up structures do not require slitting, and layers are added to construct the pop-up structures. Hence, the object diagrams of the modules in these packages do not use the cutter. They are thus identical to that of the base pages. For example, Figure 7.11 shows the object diagram of the 90° angle fold's construction.

Figure 7.11. Object diagram of the 90° angle fold's construction.

(d) The box fold package

The box fold package comprises the non-parallel box fold module and the parallel box fold module. A box fold structure has both primary and secondary

pop-up faces, as introduced in Section 3.1.2, whereas the modules mentioned in preceding sub-sections construct only primary pop-up faces. Hence, there is an additional task to build secondary pop-up faces, and this is illustrated in the object diagram of the parallel box fold module. See Figure 7.12.

Figure 7.12. Object diagram of a parallel box fold's construction.

The object diagrams presented so far illustrate instances of single-layer construction of pop-up structures. Values held by each object were not indicated in the diagrams since the main aim here is to build conceptual models with connected components that describe the software architecture and organization. Other model techniques like Object Role Model (ORM) [105, 107] are also known for developing conceptual models. Among all, object diagrams are less abstract yet providing sufficient details to model real instances, and effectively bridges architecture views to implementations. They also aid the explanation of relationships between classes applied in the implementation. The following section further discusses the classes.

7.4 Implementation and case studies

This section further explores a 3D modelling application implemented using model-driven architecture views discussed in Section 7.3. Using Visual C++ with Microsoft Foundation Classes (MFC), the software is designed using the three-layer conceptual architecture, as shown in Figure 7.13. In this architecture, the MFC Dialog is the controller of the program. It manages the view and enables interaction with the classes, which forms the library. The GUI is supported by OpenGL and PLIB. PLIB is a set of component libraries that includes the Standard Geometry (SG) Library and the Simple Scene Graph (SSG) Library [108]. It contains a 3D engine that supports object-oriented interface and complements OpenGL.

Figure 7.13. Architecture framework for the application.

7.4.1 Modularization for 3D modelling application

The software architecture created for the 3D modelling application is made up of seven modules. These modules perform operations described in the object diagrams in Section 7.3.3, and are linked to the pop-up module. Figure 7.14 shows an overview of these modules. Two tools deployed in the modelling, namely the cutter and the decorators, are highlighted in the figure. They are discussed in Section 7.4.2. The models of elemental pop-up structures created by these modules are shown in Appendix D.

Figure 7.14. Modules used in the application.

As discussed in the module view in Section 7.3.3c, operations of multi-piece 90° fold modules are the same as those of the multi-piece 180° fold modules. Their models are also very similar to those of the one-piece 90° folds. Hence, multi-piece 90° folds are duplicative and are thus omitted from the case studies. Modules of the single-slit/double-slit non-parallel folds and the non-parallel box fold are also not created for the case studies as their structures and operations are similar to the slit parallel module and the parallel box module respectively.

Figure 7.15 shows the interface of the application. The base page module is activated upon running the application. Hence, the model of the base pages is always on the 3D view. The model can be rotated in three-dimensional space about an endpoint on its gutter crease, which is pivoted at the centre of the

screen. The base pages are also set at 180° apart in their initial positions, and one of its planes will rotate about the gutter crease to simulate the folding of the base pages.

Figure 7.15. The interface of the application.

To create models of pop-up structures, the modules are activated by clicking buttons on the interface. Each button is linked to a module. There are also buttons to play and cease the simulation of the models as well as to reset to the initial view. Figure 7.16 explains the use of each button.

Figure 7.16. Functions of buttons on the interface.

In the algorithm, all the modules are made up of classes, and these are illustrated in the class diagram in Appendix E. The attributes and operations of

the classes are also described in the appendix. The classes of the modules inherit properties of the base class called the pop-up module. The pop-up module calls for the common operations of creating and positioning a left plane and a right plane for a model. Figure 7.17 shows the class diagram of the base page module. The structure of the diagram is similar to that of Figure 7.9 but the plane class and the ShapeView class are not shown. The base pages module class inherits properties from the pop-up module, and is linked to the base page class by a unidirectional association. As only a pair of base pages is required for the construction of a pop-up structure, the base page module only creates an instance of the base page. Hence, the multiplicity value at its unidirectional association is one. Figure 7.18 illustrates the operations of the base pages module in a sequence diagram.

Figure 7.17. Class diagram of the base pages module.

Figure 7.18. Sequence diagram of creating the base pages.

Apart from the parallel box module, other modules can create more than one identical model to simulate layering, as shown in Figure 7.19. Figure 7.19a shows the model of a two-layer pop-up structure comprising multi-piece 180° angle folds and Figure 7.19b that of a three-layer pop-up structure comprising multi-piece 180° parallel folds.

Figure 7.19. Layering of pop-up structures (a) Angle folds in a two-layer model and (b) parallel folds in a three-layer model.

Using the multi-piece 180° angle fold module as an example, its class diagram is shown in Figure 7.20. As illustrated in the multiplicity value on its association, the module can create zero or many angle folds. Similarly, the angle fold module class inherits properties from the pop-up module class. The angle fold module class inherits properties from the pop-up module class. The angle fold class also calculates the angles that permit folding of the planes. Figure 7.21 illustrates the operations of the angle fold module in a sequence diagram, which are similar to those of the base pages module.

Figure 7.20. Class diagram of the angle fold module.

Figure 7.21. Sequence diagram of creating an angle fold.

7.4.2 The cutter and the decorators

The cutter and the decorators are two tools deployed in the application to construct models. The former is used in one-piece pop-up structures while the latter is applied to multi-piece pop-up structures, as shown in Figure 7.14.

(a) The cutter

The cutter creates planes with cutout portions on the models of the slit package. Whenever a pop-up layer is added, planes that they are built upon change in shapes. Figure 7.22 illustrates the model of a three-layer one pop-up structure whose planes were reshaped as pop-up layers were added. The changes in shape of its base page are shown in Figure 7.23.

Figure 7.22. The model of a three-layer one-piece structure.

Figure 7.23. Reshaping of a base page.

In the algorithm, the modules of the slit package are linked to the cutter class, which contains the SSG triangulation functions [109]. Hence, these modules perform additional operations to create the cutouts on the planes as compared with other modules. Figure 7.24 shows the class diagram of the single-slit angle fold module and Figure 7.25 its sequence diagram.

Figure 7.24. Class diagram of the single-slit angle module.

Figure 7.25. Sequence diagram of the single-slit angle fold module.

(b) The decorators

While they are an essential part of the box fold structure, secondary pop-up faces can be added to pop-up structures of other types. Figure 7.26 illustrates the model of the multi-piece 180° angle fold with a secondary pop-up layer, comprising two pop-up faces. These faces are added by activating a toggle button on the interface. Figure 7.27 shows the model of a parallel box fold with the essential secondary pop-up faces.

Figure 7.26. An angle fold with secondary pop-up faces. (a) Fully erected and (b) simulating an outward fold.

Figure 7.27. A parallel box fold with secondary pop-up faces. (a) Fully erected and (b) simulating both the outward and inward folds.

Regardless if the secondary pop-up faces are essential, both box fold modules and modules of the multi-piece 180° tent package enables the creation of secondary pop-up faces by using the flap decorator and the shape decorator. For the parallel box fold, this is done by modifying the structure of the object diagram in Figure 7.12 to include the decorators.

Figure 7.28 shows the class diagram of the parallel box fold with decorators and Figure 7.29 that of the multi-piece 180° angle fold. Note that the module classes

are not shown in the two diagrams. The flap decorator class is a child class of the shape decorator class, which in turn is a child class of the ShapeView class. ShapeView is an abstract class that contains properties of the shapes used for pop-up faces.

Figure 7.28. Class diagram of the parallel box fold module.

While the shape decorator contains common properties required for constructing secondary pop-up faces, the flap decorator is polymorphic and creates specific secondary pop-up faces for different pop-up structures. For example, the function *InputTriangleVertex()* in the flap decorator class is applied to create triangular secondary pop-up faces for multi-piece 180° angle folds, parallel folds and non-parallel folds whereas *InputRectVertex()* creates rectangular secondary pop-up faces for the box folds.

Figure 7.29. Class diagram of the angle fold module with decorators.

7.4.3 Composite layering of pop-up structures

The modular architecture enables the application to model and simulate composite layering of pop-up structures. In composite layering, multi-layer pop-up structures can be built with non-identical elemental structures and a combination of primary and secondary pop-up layers, as discussed in Section 4.8. Figure 7.30 shows the model of a one-piece pop-up structure that comprises one single-slit and two double-slits. In the application, secondary pop-up faces are not created on models of one-piece pop-up structures because the angle between base pages is constrained at 180° , and it is not feasible for

one-piece pop-up structures with secondary pop-up faces to collapse flat at that angle, as mentioned in Section 3.1.2.

(a) Collapsed flat at 180°

(b) Erected at 90°

Figure 7.30. A model of the one-piece pop-up structure.

Figure 7.31 illustrates a model of a multi-piece pop-up structure formed by a parallel fold, a non-parallel fold and an angle fold with a secondary pop-up layer. A model of the parallel box fold with another two layers added to its side is shown in Figure 7.32. The former model exemplifies composite layering with a secondary pop-up layer while the latter illustrates composite layering with a box layer.

(a) Erected at 180°

(b) On closing the base pages

Figure 7.31. A model of a multi-piece pop-up structure.

Figure 7.32. A model of a box fold with added layers.

Though the models illustrated here have four layers and less, the number of layers constructible can exceed four with the application. While it has shown that modularization enables the modelling of a range of pop-up structures, the application can be made more robust by allowing primary pop-up layers to be built on secondary pop-up layers. Its capability can be further expanded by permitting construction on the other base page and on pop-up layers above the base pages.

7.4.4 Modelling of real-life objects

This section examines the modelling of a pop-up card and a French fries container. These case studies were modelled using the module architecture. The relations, conditions and constraints described in Chapter Four to Six are applied to verify the topology and geometry of the models.

(a) The pop-up card

The pop-up card ‘Icicle’ [7] is a multi-piece 180° pop-up structure, comprising two layers of angle folds and the base pages, as shown in Figure 7.33. The

second layer of angle fold was built on the first layer. The pop-up structure was modelled with elemental forms without details like the irregular shape of the icicles and rendering of paper textures. The model can be rotated in 3D space and simulates the erection of the pop-up structure by moving the base pages. Figure 7.34 shows the various stages of opening the base pages. If the second layer is attached to the edges of the first layer, the model can be constructed with an angle fold with secondary pop-up faces instead.

Figure 7.33. The pop-up card 'Icicle' [7].

Figure 7.34. Simulation of the pop-up card.

(b) The French fries container

The McDonald's French fries container has the structure of a box fold. It can be modelled using planar faces by solely considering its topology. Hence, the model is geometrically different from the real product whose constrained geometry results in curved faces. These faces are not examined in the thesis.

There is more than one approach to model the foldable container as a pop-up structure. The first approach designates the base of the container as the base pages. Then the front and back walls of the container form the primary pop-up faces of a box layer while its two sides become secondary pop-up faces. See Figure 7.35a. The second approach designates the front and back walls of the model as the base pages, which extend to meet at an imaginary gutter crease. The base and sides of the model can be modelled with a parallel fold and non-parallel folds respectively. See Figure 7.35b. Other approaches include setting one of the two sides as the base pages.

Figure 7.35. Modelling of the container. (a) The first approach and (b) the second approach.

For this case study, the second approach was applied due to its simpler parent-child relationships between the layers. As the parallel fold and the non-parallel folds were connected to the base pages, the simulation of these folds depended solely on the motion of the base pages. The first approach, however, requires more complex calculation to obtain folding angles on the secondary pop-up faces, which are not directly connected to the base pages. These angles change with the motion of the primary pop-up faces, which in turn are dependent on the angle of opening the base pages. Figure 7.36 shows the model at various stages of simulation. As curled planes are not examined in this research, the curved walls of the container were modelled as flat planes.

Figure 7.36. Simulation of the French fries container.

Part of the base pages were set as invisible planes and intersected at the imaginary gutter crease as mentioned earlier. The points of convergence of the two non-parallel folds also lay on this crease, as shown in Figure 7.37. Hence, these points, which are essential for the construction of the non-parallel folds, are easily located. This is another reason for the preference of the second approach over the first.

Figure 7.37. Imaginary gutter crease and points of convergence.

(c) The pop-up structure with composite layers

A model based on the pop-up structure in Figure 4.26 was created to illustrate complex composite layering. Figure 7.38 illustrates the model when it is fully erected at 90° and when it is collapsing. The model has seven layers, including parallel folds and non-parallel folds as the primary pop-up layers. A flat vertex fold was created on the non-parallel fold and radial pleats on the secondary pop-up layer. Refer to Table 4.3 for a detailed description of the layer components. The model illustrates cut and glued elements as well as supplementary paper

folds. Its structure is thus different from the models of the pop-up card and the French fries container. For the ease of illustration of all the layers, the secondary pop-up layer in this model was added to the opposite side of the pop-up structure in Figure 4.26.

Figure 7.38. Simulation of the pop-up structure with composite layers.

7.5 Summary

Based on the classification model, a software application was built to study the 3D modelling of pop-up structures. The two tools created in the application, namely the cutter and the decorators, are analogous to the crafting techniques of cutting and gluing respectively. The object-oriented module architecture allows

the ease of modification and expansion in the number of layers on the models. It is thus useful for designing a wide range of multi-layer pop-up structures. In addition, pop-up structures such as the French fries containers can be modelled by a number of ways when they are broken down and represented by elemental types of pop-up structures. The complexity in simulation also depends on the structural relationships between these elements. The next chapter concludes the findings and discusses the limitations and future work of this research.

Chapter Eight

Conclusion, limitations and future work

This thesis establishes a classification model for pop-up structures, which organizes types of elemental pop-ups in a hierarchical framework. This in turn leads to the graph representations and mathematical models. These then pave the way for the creation of a software prototype that allows modular construction of pop-up structures. This research can be summarized into four main areas, as shown in Figure 8.1.

Figure 8.1. The modelling of pop-up structures.

Section 8.1 further summaries the analysis of pop-up structures. Section 8.2 discusses the contribution of the research while Section 8.3 identifies its limitations. The chapter ends with future work in Section 8.4.

8.1 Summary of analysis

The classification of pop-up structures identifies topological and geometrical attributes as well as elemental parts that describe their compositions. The Cartesian MV models and the vertex-edge graphs of MV trees, G graphs and L graphs enable simple representation of pop-up structures without the complexity of geometric designs. They are particularly useful in representing

the composition of multi-layer pop-up structures. The mathematical relationships and conditions provide a systematic method to examine and validate the makeup of the structures and their ability to flat fold. The main findings in graph representations and mathematical modelling are given in Figure 8.2.

Figure 8.2. Graph representations and mathematical modelling

The module architecture proposed for the 3D modelling of pop-up structures has led to the development of a software application and modelling of case studies. These case studies show that the modular design approach can effectively model a variety of pop-up designs. However, the software system also has limitations, which warrants further examination. Sections 8.3.3 and 8.4.4 respectively discuss the limitations and its future work.

8.2 Contributions

The main contributions of this research are in two areas: the design approach and the computer-assisted learning of the craft.

8.2.1 The design approach

As described in Chapter One, the design of pop-ups has been largely manual and characterized by trials and errors. By building up in-depth understanding of inherent properties of pop-up structures, a more systematic design approach using a modular architectural system to construct pop-up structures can be developed, as discussed in Chapter Seven. The system classifies different types of pop-up structures according to topological and geometric properties, enables parent-child relationships for composite layering, and deploys specific object-oriented tools like the cutter and the decorator to build models of pop-up structures. The relations, conditions and constraints discussed in Chapters Four to Six are applied to validate the topology and geometry of the models. This design approach reduces or eliminates trials and errors, thereby increasing time efficiency and allowing more efficient communication between designers. Eventually, the lead-time to manufacture would be shorter, and the end products would be produced at a lower cost.

8.2.2 The computer-assisted learning of the craft

Computational science has blended with traditional crafting methods to complement the learning of the craft. Blauvelt, Wrensch and Eisenberg [37] proposed the use of geometry-specific languages, alternatively described as computational languages created around particular modes of physical

movement, to explore behaviours of craft materials such as strings, hinges and rotatable tacks. Likewise, pop-up structures exhibit specific ‘pop-up’ behaviours determined by its structural composition and geometry, which are examined in Chapters Three to Six. Hence, the characteristics of pop-up structures discussed in this research can also be expressed in a geometry-specific language. They can be used in computer-aided tools like those illustrated in the case studies of Chapter Seven. A computer-aided tool replaces the need for specialized skills in pop-up making and reduces learning time. Both novices and the experienced designers would be able to exploit its usefulness for educational purposes and manufacturing.

8.3 Limitations

Of the pop-up structures inspected, more attention can be given to the box folds. The range of elemental pop-up structures is also not limited to those identified in the classification model. There is also a lack of information input from the paper engineers. The translation of mathematical findings into relevant applications can also be challenging. The following sub-sections elaborate.

8.3.1 The box formation

While the research has extensively examined the tent formation, the study of the box formation can be further explored. The box formation is primarily surveyed in the classification model, L graphs, the identification of glue edges and conditions for flat folding. Though multi-layer box folds can be constructed, subsequent layers cannot erect properly, as illustrated in Figure 8.3a. So the box formation does not fit into the study of layers in Chapters Four and Five.

However, the box formation can be described by MV models. It can also be built as the first layer, followed by multiple layers of the tent formation, as shown in Figure 7.32 and Figure 8.3b.

Figure 8.3. (a) Successive box layers and (b) a box layer with layers of the tent formation.

Similarly, the characteristic ratio and subsequent investigation on geometry of the primary and secondary pop-up layers exclude the box formation. Its conditions for flat folding are covered in Sections 3.2.2 and 6.2.2. Table 8.1 shows that the box formation can be further distinguished by the convergence of p - s creases and related to 90° and 180° tent pop-up structures with secondary layers.

Table 8.1. Convergence of creases.

		Convergence of p - s creases		
		At infinity	At a point exterior of structure	At a point on the physical creases
Convergence of primary creases	At a point on the physical creases	$90^\circ/180^\circ$ angle box fold with parallel p - s creases	$90^\circ/180^\circ$ angle box fold with non-parallel p - s creases	$90^\circ/180^\circ$ angle fold with secondary layer
	At a point exterior of structure	$90^\circ/180^\circ$ non-parallel box fold with parallel p - s creases	$90^\circ/180^\circ$ non-parallel box fold with non-parallel p - s creases	$90^\circ/180^\circ$ non-parallel fold with secondary layer
	At infinity	$90^\circ/180^\circ$ parallel box fold with parallel p - s creases	$90^\circ/180^\circ$ parallel box fold with non-parallel p - s creases	$90^\circ/180^\circ$ parallel fold with secondary layer

The box formation can also be seen as a body resulting from the truncation of the tent formation. Figure 8.4 illustrates two examples. In the figure, the points of convergence of p - s creases are joined by an imaginary line. This line is parallel with the primary creases on a parallel box fold, as shown in Figure 8.4a. But it is concurrent with the primary creases on a non-parallel box fold, as shown in Figure 8.4b. Hence, a more detailed examination of the box formation can bring about a thorough study on its geometry.

Figure 8.4. The imaginary line is (a) parallel to primary creases and (b) concurrent with primary creases.

8.3.2 Elemental structures for the classification model

Another concern is whether the classification model for pop-up structures has adequately identified all elemental pop-up structures. One case to highlight is the 90° box fold and how it fits into the classification. The 90° box fold can be seen as a variation of the 90° pop-up structures with secondary pop-up layers, as shown in Figure 8.5. Like other 90° pop-up structures with secondary pop-up layers, the base pages of the box fold cannot open up to 180° due to the restricted movement of its secondary layers. Hence, it can be classified under multi-piece 90° pop-up structures.

Figure 8.5. A 90° box fold.

However, the 90° box pop-up structures, comprising the 90° parallel box fold and 90° non-parallel box fold, can also be a domain, as discussed in Section 3.3.1. It can be an additional link between the two domains, the multi-piece 180° box and the multi-piece 90° pop-up structures, as illustrated in Figure 8.6. Thus, the classification of pop-up structures can be further reviewed to include the 90° box folds and other possible elemental pop-up structures.

Figure 8.6. An additional domain.

8.3.3 The software development

While the modular architecture enables the modelling of different types of pop-up structures, the software application in Chapter Seven does not contain the models of all the elemental pop-structures described in the classification model. It is also limited in its geometric capabilities and user interactions. For example,

the layering of the models is confined to one of the base pages, and the positions and shapes of each layer are not manipulatable. Constraints have to be added to prevent construction of pop-up structures on the exterior of the base pages. Besides these limitations, there has to be functional matches with the mental models of the users and mathematical properties of pop-up structures in the software development.

(a) Mental model of users

The modelling of pop-up structures can be substantiated with a study on the tasks of paper engineers. This is vital for matching design requirements of the users to the functions of the software application. In this research, insights of the work of paper engineers were only gained through email exchanges with several paper engineers, books and the Internet. The procedures taken through the design stages from concepts to details, the design specifications required in the industry, and how crafting is executed with tools such as circle templates and x-acto blades are not adequately examined. Thus, the software application presented in Chapter Seven may not reflect the intention of the users. However, this study should not be constrained by current practices and manual crafting techniques.

(b) Topological and geometrical properties

In addition, topological and geometrical properties need to match functions required for the operations of the tool. For example, both the Hamiltonian cycle and topological conditions for edges discussed in Sections 5.2 and 5.3 respectively can check for flaws in G graphs. But the latter is a better choice for

verifying the pop-up construction because it differentiates cut edges from crease edges while the former does not. Figure 8.7 depicts the functional match between the computer-aided tool and three essential components: the paper engineers, their working environment and the pop-up structures.

Figure 8.7. Functional match for the computer-aided tool.

8.4 Future Work

There are four primary areas for future work. They are an elaborated study on box folds, the inclusion of non-planar pop-ups, the study of non-laminar pop-up structures and the software development for end users.

8.4.1 The study of box folds

The box folds can be further assessed as follows.

- Examine variations of the box folds and their features, in particular their relations to the primary and *p-s* creases.
- Study geometric conditions and constraints of box folds apart from those for flat folding.

8.4.2 The inclusion of non-planar pop-up structures

The classification of pop-up structures can be expanded to include the following.

- Cover structures that the current model does not contain, including structures with deformable pop-up faces and curved edges.
- Investigate other methods like the use of threads, slots and gravity that aid the development of pop-up structures.

8.4.3 The study of non-laminar pop-up structures

This research can be further extended to examine non-laminar pop-up structures, i.e. structures that are not constructed with thin sheets, such as deployable furniture.

- Investigate how properties of laminar pop-up structures can be applied to the design of deployable products.
- Examine the modelling of non-laminar pop-up structures for computer-aided design.

8.4.4 The software development for end users

To develop an end-user interactive system for pop-up design and evaluation, the software development can include the following.

- Identify design requirements, specifications and variables that are important to pop-up designs.
- Expand the module software architecture in Chapter Seven to include the modelling of other elemental pop-up structures.

- Enable construction of pop-up layers that are not attached to the base pages.

Figure 8.8 shows an example experimented with the software Python [110].

- Enable geometric inputs and alterations for the 3D models.
- Extract information from surveys and task analysis of paper engineers that are useful for software development.

Figure 8.8. A simulation of a three-layer pop-up structure generated by Python.

8.4.5 The manufacturing process

Besides the design, the mass production of pop-up books and cards is another tedious and laborious process. For example, five hundred workers are required to join two hundred pieces in each copy of the pop-up book, *Knick-Knack Paddywhack!* [111]. As Baron [9] recorded, the completion of its first printing order required more than twenty-three million individual assembly steps. The manufacturing process for pop-up books is undoubtedly an area that technology can help.

One established paper engineer, David Carter, once indicated in an email exchange with the author his vision to mechanize the process in an assembly line with robotic units. To this end, one crucial issue to tackle is on how a mechanized method can be developed to assemble complex pop-up structures

that call for precision assembly. Studies on the design for manufacture and assembly are therefore appropriate. In the long run, an integrated CAD/CAM system can be realized to connect the design stage to the manufacturing stage where the die stamping and the assembling of pop-up parts are, too, shifted to the digital platform.

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Appendix A

The Classification Model for Pop-up Structures

Appendix A

Derivation of equations

a) Derivation of Equation 4.5

Table A. Crease initiations for three layers.

Layer n	No. of planar faces F	No. of creases N	No. of mountain creases M	No. of valley creases V	Type of crease initiations
1	4	4	1	3	Valley
2	6	7	3	4	Mountain
3	8	10	5	5	Mountain

Using values from the second and third layers as shown in Table A, the two following equations are formed when they are substituted in (4.4).

$$6 = 3a + 4b$$

$$8 = 5a + 5b$$

Solving the equations gives $a = \frac{2}{5}$ and $b = \frac{6}{5}$.

These two values are substituted in (4.4) and

$$F = \frac{2}{5}(3V + M) . \quad (4.5)$$

b) Derivation of Equation 6.6

Substituting (6.5) into (6.2),

$$2b_l b_r \cos \theta + h_l^2 + h_r^2 - 2h_l h_r \cos \phi < (h_l + h_r)^2 - 2b_l b_r$$

$$2b_l b_r \cos \theta + h_l^2 + h_r^2 - 2h_l h_r \cos \phi < h_l^2 + h_r^2 + 2h_l h_r - 2b_l b_r$$

$$b_l b_r \cos \theta + b_l b_r < h_l h_r \cos \phi + h_l h_r$$

$$(\cos \phi + 1) > \frac{b_l b_r}{h_l h_r} (\cos \theta + 1)$$

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$$\cos \phi > \frac{b_l b_r}{h_l h_r} (\cos \theta + 1) - 1 \quad (6.6)$$

c) Derivation of Equations 6.19 and 6.20

From Condition 6.1

$$p_{1l} - p_{1r} = s_{1r} - s_{1l} \quad (6.10)$$

By the Cosine rule,

$$\cos(s_{1l} + s_{1r}) = \frac{l_1^2 + l_2^2 - l_3^2}{2l_1 l_2}$$

$$s_{1r} = \cos^{-1}\left(\frac{l_1^2 + l_2^2 - l_3^2}{2l_1 l_2}\right) - s_{1l}$$

Substituting the above equation into (6.10)

$$\begin{aligned} p_{1l} - p_{1r} &= \cos^{-1}\left(\frac{l_1^2 + l_2^2 - l_3^2}{2l_1 l_2}\right) - s_{1l} - s_{1l} \\ s_{1l} &= \frac{1}{2} \left[p_{1r} - p_{1l} + \cos^{-1}\left(\frac{l_1^2 + l_2^2 - l_3^2}{2l_1 l_2}\right) \right] \end{aligned} \quad (6.19)$$

Similarly, substituting (6.19) into (6.10) gives

$$s_{1r} = \frac{1}{2} \left[p_{1l} - p_{1r} + \cos^{-1}\left(\frac{l_1^2 + l_2^2 - l_3^2}{2l_1 l_2}\right) \right] \quad (6.20)$$

By Cosine rule,

$$s_{2l} = \cos^{-1}\left(\frac{l_1^2 + l_3^2 - l_2^2}{2l_1 l_3}\right)$$

and

$$s_{2r} = \cos^{-1}\left(\frac{l_2^2 + l_3^2 - l_1^2}{2l_2 l_3}\right)$$

Appendix B

s_{3l} and s_{3r} can be expressed in terms of s_{1l} and s_{2l} such that

$$s_{3l} = 180^\circ - s_{1l} - s_{2l} \text{ and } s_{3r} = s_{1l} + s_{2l}.$$

Using the sum of interior angles in a triangle, $s_{1l} + s_{1r} + s_{2l} + s_{2r} = \pi$.

d) Derivation of Equations 6.22, 6.23 and 6.24

By the Sine rule,

$$\frac{l_4}{\sin s_{2l}} = \frac{l_1}{\sin s_{3l}}$$

To determine l_4 , since $s_{3l} = 180^\circ - s_{1l} - s_{2l}$,

$$\begin{aligned} l_4 &= \frac{l_1 \sin s_{2l}}{\sin s_{3l}} \\ &= \frac{l_1 \sin s_{2l}}{\sin(s_{1l} + s_{2l})} \end{aligned} \quad (6.22)$$

Similarly, by the Sine rule

$$\frac{l_5}{\sin s_{1l}} = \frac{l_1}{\sin s_{3l}}$$

To determine l_5

$$\begin{aligned} l_5 &= \frac{l_1 \sin s_{1l}}{\sin s_{3l}} \\ &= \frac{l_1 \sin s_{1l}}{\sin(s_{1l} + s_{2l})} \end{aligned} \quad (6.23)$$

Since $l_5 + l_6 = l_3$,

$$\begin{aligned} l_6 &= l_3 - l_5 \\ &= l_3 - \frac{l_1 \sin s_{1l}}{\sin s_{3l}} \\ &= l_3 - \frac{l_1 \sin s_{1l}}{\sin(s_{1l} + s_{2l})} \end{aligned} \quad (6.24)$$

Appendix B

e) Derivation of Equation 6.27

As referred to Figure 6.14, by Pythagoras Theorem,

$$l_3^2 = (b_l + b_r)^2 + (x_r - x_l)^2$$

where x_l and x_r are relative positions from the perpendicular heights.

Likewise, by Pythagoras Theorem,

$$x_l^2 = l_1^2 - h_l^2$$

and

$$x_r^2 = l_2^2 - h_r^2$$

Substituting the two equations into (6.26),

$$\begin{aligned} l_3^2 &= (b_l + b_r)^2 + \left(\sqrt{l_2^2 - h_r^2} - \sqrt{l_1^2 - h_l^2} \right)^2 \\ l_3 &= \sqrt{(b_l + b_r)^2 + \left(\sqrt{l_2^2 - h_r^2} - \sqrt{l_1^2 - h_l^2} \right)^2} \end{aligned} \quad (6.27)$$

f) Derivation of Equation 6.28

As referred to Figure 6.15b, by the Cosine rule,

$$\begin{aligned} l_3^2 &= (b_l + b_r)^2 + \left(\frac{|\Delta x|}{\cos c_i} \right)^2 - 2 \left(\frac{|\Delta x|}{\cos c_i} \right) (b_l + b_r) \cos(c_i + 90^\circ) \\ &= (b_l + b_r)^2 + \left(\frac{|\Delta x|}{\cos c_i} \right)^2 - 2 \left(\frac{|\Delta x|}{\cos c_i} \right) (b_l + b_r) (-\sin c_i) \\ &= (b_l + b_r)^2 + \left(\frac{|\Delta x|}{\cos c_i} \right)^2 + 2|\Delta x|(b_l + b_r)\tan c_i \\ l_3 &= \sqrt{(b_l + b_r)^2 + \left(\frac{|\Delta x|}{\cos c_i} \right)^2 + 2|\Delta x|(b_l + b_r)\tan c_i} \end{aligned} \quad (6.28)$$

where c_i is the angle of convergence adjacent to the longer primary crease on the base pages.

Appendix C

A proof for $b_l < h_r$ when $b_r < h_l$

In order for the pop-up structure of the tent formation to collapse flat when the base pages are closed, we apply Equation 6.11 under Condition 6.2. That is,

$$b_l + h_l = b_r + h_r$$

(6.11)

$$h_l = b_r + h_r - b_l$$

Substituting the above equation into $b_r < h_l$,

$$b_r < b_r + h_r - b_l$$

$$b_l < h_r$$

Thus $b_l < h_r$ when $b_r < h_l$.

The same approach can be used to show that $b_l = h_r$ and $b_r = h_l$ for 90° pop-up structures in Section 6.12, and $b_l > h_r$ and $b_r > h_l$ for structures that cannot fully erect in Section 6.13.

Models of elemental pop-up structures created by modules

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(a) Base pages, (b) a parallel box fold, (c) a single-slit angle fold, (d) a double-slit parallel fold, (e) a 180° angle fold, (f) a 180° parallel fold, and (g) a 180° non-parallel fold.

Appendix E

Class diagram of the 3D modelling application

Appendix E

Appendix E

Attributes

DistL: The distance on a left base page between the gutter crease and a left primary pop-up face that is attached to it.

DistR: The distance on a right base page between the gutter crease and a right primary pop-up face that is attached to it.

FlapAngle: The angle between a secondary pop-up face and an adjacent primary pop-up face.

FoldHeight: The height of a secondary pop-up face.

FoldL: The width of a left primary pop-up face.

FoldR: The width of a right primary pop-up face.

FoldWidth: The width of a secondary pop-up face.

HAngleL: The angle between a left primary pop-up face and the face that it is built upon.

HAngleR: The angle between a right primary pop-up face and the right base page.

LeftAngleShp: Properties of colours, positions and pitches of the left shape.

NoOfFold: The number of secondary pop-up layers.

prevHAngleR: The angle between a preceding right primary pop-up layer and the right base page.

RightAngleShp: Properties of colours, positions and pitches of the right shape.

ShapeNum: The number of pop-up faces on a model with secondary pop-up layers.

ShapeVecSize: The number of pop-up faces on a model.

Appendix E

Appendix E

Operations

AlterFlap() transforms the shape of a secondary pop-up face to fit into the 3D view.

AlterLeftShape() transforms the shape of a left primary pop-up face to fit into the 3D view.

AlterRightShape() transforms the shape of the right primary pop-up face to fit into the 3D view.

Basefold() creates the base pages.

CopyVertexArray() logs the vertices of a shape.

Create() internally calls for *Make_Shape()*.

CreateFlap() internally calls for *Make_Flap()*.

CreateLeftPc() creates a left primary pop-up face.

CreateRightPc() creates a right primary pop-up face.

FoldAngleL() obtains the angle between a left primary pop-up face and the preceding left primary pop-up face that it is built upon.

FoldAngleR() obtains the angle between a right primary pop-up face and the preceding right primary pop-up face that it is built upon.

getSlitPlane() creates the shape of a primary pop-up face with a cut face.

InputLeftCutout() creates the vertices of the cut face on the left primary pop-up face.

InputRectVertex() create the vertices of a rectangular secondary pop-up face.

InputRightCutout() creates the vertices of the cut face on the right primary pop-up face.

InputTriangleVertex() creates the vertices of a triangular secondary pop-up face.

InputVertex() creates the vertices of the base pages.

Appendix E

Appendix E

Make_Flap() creates a shape for a secondary pop-up face.

Make_Shape() creates a shape for the primary pop-up face.

PositionFlaps() determines the displacements and angles of a secondary pop-up face.

SetPosition() determines the displacements and angles of the primary pop-up face.

ssgtriangulate() triangulates a primary pop-up face.

triangulateConcave() triangulates a primary pop-up face with concave edges.

Appendix E