Novel optimization techniques for robust beamforming and antenna selection

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

**NOVEL OPTIMIZATION TECHNIQUES FOR ROBUST
BEAMFORMING AND ANTENNA SELECTION**

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**NAI SIEW ENG
SCHOOL OF ELECTRICAL AND ELECTRONIC
ENGINEERING**

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NOVEL OPTIMIZATION TECHNIQUES FOR ROBUST BEAMFORMING AND ANTENNA SELECTION

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Abstract

The research in this thesis is emphasized on the investigation of optimization techniques for robust beamforming (against large steering direction errors and interferences) and antenna selection.

Recently, some adaptive beamformers based on worst-case performance optimization are shown to be more robust against steering vector errors than the traditional ones. Hence, this thesis studies the robust beamformers in the presence of large steering direction errors. The findings reveal that when there are large steering direction errors, the uncertainty set of the desired array steering vector is increased which can limit the output signal-to-interference-plus-noise ratio (SINR) performances of these methods. In this thesis, a technique is proposed to make use of a small uncertainty sphere (and a small flat ellipsoid) to search for the desired array steering vector iteratively. By preserving the beamformer's degrees-of-freedom (DOFs) and in turn, its interference-plus-noise suppression ability, and using the corrected desired array steering vector, the proposed technique achieves higher output SINR than the worst-case performance optimization based methods.

This thesis studies the adaptive beamformers based on worst-case performance optimization and that based on the covariance matrix taper (CMT) approach in scenarios with low snapshots and large steering direction errors. The findings show that these methods lack beampattern control that can lead to drastic output SINR performance degradation in the presence of unexpected or strong moving interferences (except the CMT beamformer which can be robust against strong moving

interferences). Thus, a beamforming optimization technique against (unexpected or strong moving) interferences via a general adaptive beamforming framework is proposed. Based on the use of a set of beampattern shaping constraints, the proposed beamformer achieves robustness against large steering direction errors and strong interferences that move during which the beamforming weights are applied. The proposed beamformer possesses adaptive interference rejection and direct sidelobe control capabilities where their relative proportions are automatically controlled by a weighing ratio in different scenarios without ambiguity.

Next, this thesis studies the direct application of the method of Davidson *et al.* with linear matrix inequality (LMI) constraints to select the minimum number of antennas required for beampattern designs. However, it is inefficient and the number of iterations before stopping is unknown. Therefore, a technique is proposed in this thesis to select the minimum number of antennas required systematically, while satisfying the LMI constraints imposed on the beampattern precisely. The proposed method is guaranteed to converge to the minimum number of antennas if this globally optimum solution lies in the search interval. This is easy to ensure at the start of the search.

Finally, the thesis studies the extension of the previously proposed technique to planar arrays. However, the findings show that the use of LMI constraints may be too computationally demanding given the number of antennas in planar arrays commonly used in practical applications. Therefore, this thesis proposes another antenna selection technique for planar arrays using conjugate symmetric beamforming weights. By minimizing a re-weighted objective function based on the magnitudes of the elements in the beamforming weight vector iteratively, the proposed technique selects certain antennas in an array to satisfy the prescribed beampattern specifications precisely. Interestingly, a sparse array with fewer antennas is obtained. Robust beampattern constraints are also derived.

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List of Abbreviations

BF	Beamformer
CMT	Covariance Matrix Taper
DOA	Direction-of-Arrival
DOF	Degree-of-Freedom
DRR	Dynamic Range Ratio
INR	Interference-to-Noise Ratio
IPM	Interior Point Method
IRMVB	Iterative Robust Minimum Variance Beamformer
KKT	Karush-Kuhn-Tucker
LHS	Left Hand Side
LMI	Linear Matrix Inequality
LMS	Least Mean Squares
LP	Linear Program
MSINR	Maximum Signal-to-Interference-Plus-Noise Ratio
MV	Minimum Variance
MVDR	Minimum Variance Distortionless Response
NP-hard	Non-Deterministic Polynomial-Time-hard
RAB-SL	Robust Adaptive Beamformer With Sidelobe Control
RAB-Motion	Robust Adaptive Beamformer Against Strong Interference Motion With Sidelobe Control

RB-SL	Robust Beamformer With Sidelobe Control
RHS	Right Hand Side
RLS	Recursive Least Squares
SDP	SemiDefinite Programming
SINR	Signal-to-Interference-Plus-Noise Ratio
SNR	Signal-to-Noise Ratio
SOCP	Second-Order-Cone Program
s.t.	subject to
ULA	Uniform Linear Array

Notation

In this thesis, the following conventions hold.

- Matrices and vectors are denoted by boldface capital and small letters, e.g., $\mathbf{X}$ and $\mathbf{x}$, respectively.
- Scalars are denoted by non-boldface italics.
- $\triangleq$ means “defined as”.
- $x \in X$ means “ x is in the set X ”.
- $A \cup B$ means “union of sets A and B ”.
- $[a, b]$, $[a, b)$, (a, b) , and $[a, b) \setminus (c, d)$ mean “ $\{x | a \leq x \leq b\}$ ”, “ $\{x | a \leq x < b\}$ ”, “ $\{x | a < x < b\}$ ”, and “ $\{x | a \leq x < b\}$ that excludes $\{x | c < x < d\}$ ”, respectively.
- $\forall$ means “for all”.
- $\exists$ means “exists”.
- $\mathbf{e}_1$ is the first column of a identity matrix.
- $\text{angle}(\cdot)$ obtains the angle of the argument.
- $\rightarrow 0$ means “tends to zero”.
- $\rightarrow \infty$ means “tends to infinity”.

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- $\log_2(\cdot)$ is logarithm to the base 2.
 - $\log_{10}(\cdot)$ is logarithm to the base 10.
 - $\sqrt{(\cdot)}$ is the square root operator.
 - $\cos(\cdot)$ and $\arccos(\cdot)$ are the cosine and inverse cosine operators, respectively.
 - $\sin(\cdot)$ and $\arcsin(\cdot)$ are the sine and inverse sine operators, respectively.
 - $\int_{\Theta}(\cdot)dx$ means the integral over the set Θ .
 - $\mathbb{E}\{\cdot\}$ is the statistical expectation operator.
 - $|x|$ is the absolute value of a complex scalar x .
 - $\lceil x \rceil$ is the ceiling operator of a real number x .
 - $\|\mathbf{x}\|_0$ is the l_0 -quasi norm.
 - $\|\mathbf{x}\|_1$ is the l_1 norm.
 - $\|\mathbf{x}\|$ is the l_2 (or Euclidean) norm.
 - $\|\mathbf{X}\|$ is the Frobenius norm of a complex matrix $\mathbf{X}$.
 - $\mathbf{I}$ is the identity matrix.
 - $\mathbb{C}^{M \times N}$ is the set of complex $M \times N$ matrices.
 - $\mathbb{R}^{M \times N}$ is the set of real $M \times N$ matrices.
 - $\mathbb{Z}_+$ is the set of positive integers.
 - j is the imaginary unit and has a value of $\sqrt{-1}$.
 - $e^{(\cdot)}$ is the exponential function.
 - $\Im\{\cdot\}$ is the imaginary operator.

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- $\Re\{\cdot\}$ is the real operator.
 - $\max\{\cdot\}$ obtains the maximum value of the argument(s).
 - $\min\{\cdot\}$ obtains the minimum value of the argument(s).
 - $\mathcal{O}(\cdot)$ is the order.
 - $\mathcal{P}\{\cdot\}$ obtains the principal eigenvector of a matrix.
 - $(\cdot)^*$ is the complex conjugate operator.
 - $(\cdot)^T$ is the transpose operator.
 - $(\cdot)^H$ is the Hermitian transpose operator.
 - $(\cdot)^{-1}$ is the inverse operator.
 - $[\mathbf{X}]_{ij}$ is the matrix element at the i th row and j th column.
 - $\mathbf{X} \succeq 0$ denotes that $\mathbf{X}$ is a positive semidefinite Hermitian matrix.
 - $\odot$ is the Schur-Hadamard element-by-element matrix product.
 - $\mathbf{0}$ denotes the vector of zeros of a conformable dimension.

Symbols

In this thesis, the following conventions hold.

x_i, y_i, z_i	x, y, and z positions of the i th antenna element, respectively (2.1)-(2.3)
ϕ	azimuth angle of the propagating wave with respect to the x-axis (2.5)
θ	polar angle of the propagating wave with respect to the z-axis (2.6)
$\mathbf{r}_k$	k th spatial position in the coordinate system
t, t_0	time index
$f(t, \mathbf{r}_k)$	time varying field of an electromagnetic plane wave at time t and position $\mathbf{r}_k$ (2.7)
$\boldsymbol{\alpha}$	slowness vector (2.8)
$\mathbf{u}(\theta, \phi)$	unit vector in the direction of (θ, ϕ) (2.8)
c	propagation speed of the wave
d, d_x, d_y	inter-element spacing between antenna elements, inter-element spacing in the x-direction between antenna elements, and inter-element spacing in the y-direction between antenna elements, respectively
N, M	number of antennas

$\tau_k(\theta)$	time difference of the wavefront at the k th antenna relative to the array origin at z_0 (2.9)
$A(t)$	complex modulating function (2.10)
f_c	carrier frequency of the propagating wave (2.10)
$\mathbf{k}$	wavelength vector (2.11)
$\mathbf{s}(\boldsymbol{\alpha}), \mathbf{s}(\theta), \mathbf{s}(\theta_i), \mathbf{s}(\Omega)$	array steering vectors (2.12), (2.13), (2.15), (5.2)
$\mathbf{s}(\theta_0), \mathbf{s}_0$	true desired array steering vectors (2.15), (2.18)
f_s, T_s	sampling frequency and sampling period of the propagating wave
$\mathbf{x}(n), y(n)$	n th sample of the received array snapshot vector and the beamformer output, respectively (2.14)
$\mathbf{w}$	complex beamforming weight vector (2.14)
J	number of interferences
θ_0, θ_i	DOAs of the desired signal and the i th interference, respectively
$z_0(n), z_i(n)$	waveforms of the desired signal and the i th interference, respectively
$\mathbf{n}(n)$	spatially white Gaussian noise
$\mathbf{R}$	theoretical array covariance matrix
σ_0^2	desired signal power
$\mathbf{R}_{\text{in}}$	covariance matrix of the interference-plus-noise components (2.17)
$\hat{\sigma}_0^2$	estimate of the desired signal power (2.22)
$\hat{\mathbf{R}}$	sample array covariance matrix (2.23)
N_s	number of array snapshots collected
SINR_{opt}	optimal SINR (2.24)
$\bar{\mathbf{s}}_0$	presumed desired array steering vector (2.26)

$\bar{\mathbf{s}}$	presumed array steering vector
$G(\theta)$	beampattern of a beamformer (2.25)
β	normalization constant (2.27)
$\hat{\mathbf{U}}_{\text{s}}, \hat{\mathbf{\Lambda}}_{\text{s}}$	matrix that contains the eigenvectors which span the estimated signal-plus-interference subspace and diagonal matrix that contains the corresponding estimated eigenvalues, respectively (2.28)
$\hat{\mathbf{U}}_{\text{n}}, \hat{\mathbf{\Lambda}}_{\text{n}}$	matrix that contains the eigenvectors which span the estimated noise subspace and diagonal matrix that contains the corresponding estimated eigenvalues, respectively (2.28)
$\hat{\mathbf{U}}_{\text{s}} \hat{\mathbf{U}}_{\text{s}}^H$	projection matrix onto the signal-plus-interference subspace (2.29)
γ	white noise gain (2.31c)
ζ	diagonal loading factor (2.32)
$\mathcal{A}(\epsilon)$	structured set of a continuum of steering vectors (2.34)
$\mathbf{s}$	array manifold
$\mathbf{e}$	complex error vector (2.34)
ϵ, ε	specified thresholds
c_1, c_2, c_3	constants
$\hat{\mathbf{s}}_0$	estimated desired array steering vector
$g, \check{g}$	Lagrange multipliers (2.37) and (3.29), respectively
$l(g), l(\check{g})$	Lagrange functions (3.2) and (3.27), respectively
Δ_1, Δ_2	Hermitian error matrices (2.41), (2.42) respectively
$\bar{\mathbf{R}}_{\text{s}}$	presumed desired signal covariance matrix (2.42)
$e(\theta)$	error function (2.45)

$\mathbf{w}_d$	desired beamforming weight vector (2.45)
G_d	desired beampattern
μ	specified gain (2.48)
Θ	beampattern region
E	total error (2.46)
$h(\theta)$	scalar weighting function for placing minimization emphasis at different beampattern regions
$\mathbf{Z}$	weighted covariance matrix (2.47)
k^2, χ^2	weighting scalars (2.48)
$\mathbf{r}_d$	cross-correlation vector (2.52)
$\hat{\mathbf{R}}_T$	tapered sample array covariance matrix (2.53)
$\mathbf{T}$	taper matrix (2.54)
ξ	taper parameter (2.54)
$\hat{\mathbf{R}}_{\text{mod}}$	modified sample array covariance matrix (2.56)
L	order of constraints (2.56)
$\mathbf{B}$	diagonal matrix of antenna element coordinates (2.56)
ϱ_l	tradeoff between the null depth and width (2.56)
$\varepsilon_1, \varepsilon_2$	uncertainty sphere size
$\varepsilon_1^f, \varepsilon_2^f$	uncertainty flat ellipsoid size
λ	wavelength of the propagating wave
$\bar{\theta}_0$	presumed desired signal direction
ψ	phase
$\Delta\theta$	DOA uncertainty range of the desired signal
$\hat{\mathbf{s}}_0^i (\hat{\mathbf{s}}_0^i), g^i$	estimate (normalized estimate) of the desired array steering vector and Lagrange multiplier at the i th it- eration

$\hat{\theta} \ (\hat{\theta}^i)$	generalized angle between the calculated desired array steering vector of the proposed IRMVB and the true desired array steering vector (at the i th iteration)
SINR_o	output SINR of the MV beamformer (3.6)
$\cos^2(\hat{\theta}; \mathbf{R}_{\text{in}}^{-1})$	cosine-squared of generalized angle $\hat{\theta}$ (3.7)
$\mathcal{H}(\mathbf{R}_{\text{in}}^{-1})$	space defined by $\bar{\mathbf{s}}_0^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0$
$\ \mathbf{x}\ _{\mathbf{R}}^2$	extended vector norm-squared
λ_k	eigenvalues (3.10)
K	total number of signals
$\mathbf{c}$	coefficient vector
$\mathbf{R}_i$	interference covariance matrix
σ_n^2	white noise variance
$\mathbf{B}$	$N \times L$ matrix with full column rank L (3.25)
$\mathbf{u}$	$L \times 1$ vector (3.25)
$\bar{\mathbf{s}}_0^i$	updated presumed desired array steering vector at the i th iteration
σ_i^2	interference power (3.31)
Θ_i	set of interference DOAs (3.31)
$f(\cdot), p(\cdot), q(\cdot)$	functions (3.33)
a, b	perturbation constants (3.33)
α, β	Lagrange multipliers (3.34)
$h^*(a, b)$	optimal value of the perturbed problem (3.35)
α^*, β^*	optimal Lagrange multipliers (3.36)
δ	difference between two consecutively obtained Lagrange multipliers (3.38)

$\mathbf{s}_t$	array steering vector that corresponds to an angle of $\bar{\theta}_0 + \Delta\theta$ or $\bar{\theta}_0 - \Delta\theta$ (3.38)
$\tilde{\mathbf{s}}$	true perturbed array steering vector
$\Delta\mathbf{s}$	array steering vector perturbation error
$L(\theta), l, L(\Omega)$	lower magnitude response limits
$U(\theta), u, U(\Omega)$	upper magnitude response limits
$\mathbf{r}_w$	array weight autocorrelation sequence (4.2)
R_w	Fourier transform of $\mathbf{r}_w$ (4.3)
$\mathbf{a}(\theta)$	extended array steering vector (4.3)
$\Theta_{\text{ML}}, \Theta_{\text{SL}}, \Theta_{\text{N}}$	mainlobe, sidelobe, and null regions respectively (4.7)
$\mathbf{r}_x, \hat{\mathbf{r}}_x$	true and estimated array snapshot autocorrelation sequences, respectively
ε_r	relative regularization factor
r_{dB}	response ripple
ρ	weighing parameter (4.7)
τ	sidelobe level (4.7)
τ_n	interference rejection level in Θ_{N} (4.7)
$\check{\theta}$	discretized angle
ν	small positive constant
ρ_r	weighing ratio (4.8)
Ω	electrical angle
P	number of discretized $\hat{\theta}$ in Θ
$\tilde{\mathbf{r}}_w$	modified array weight autocorrelation sequence (5.2)
$\tilde{\Omega}_{(\cdot)}$	set of Ω in an angular region defined by the subscript (5.3)
$\mathbf{p}$	complex vector (5.4)
Ω_l, Ω_u	lower and upper boundaries of a particular Ω region

$L(\cdot), \bigwedge(\cdot)$	linear operators
$\xi, \xi_1, \dots, \xi_5$	arbitrary real scalars
$\mathbf{X}, \mathbf{X}_1, \dots, \mathbf{X}_5$	positive semidefinite Hermitian matrices
$\mathbf{Z}, \mathbf{Z}_1, \dots, \mathbf{Z}_4$	positive semidefinite Hermitian matrices
$\Omega_{\text{ml}}, \Omega_{\text{sl}}, \Omega_{\text{nl}}$	lower boundaries of $\tilde{\Omega}_{\text{ML}}, \tilde{\Omega}_{\text{SL}},$ and $\tilde{\Omega}_{\text{N}},$ respectively (5.6)
$\Omega_{\text{mu}}, \Omega_{\text{su}}, \Omega_{\text{nu}}$	upper boundaries of $\tilde{\Omega}_{\text{ML}}, \tilde{\Omega}_{\text{SL}},$ and $\tilde{\Omega}_{\text{N}},$ respectively (5.6)
Ω_0	desired electrical angle (5.7)
m_1 and m_2	positive integers (5.8)
α	number between 0 and 1 (5.8)
M_c, M_l, M_u, M'_u	number of antennas
M_{opt}	optimal number of antennas
u_x, u_y	direction cosines
$G(u_x, u_y)$	beampattern of a planar array (6.1)
$\mathbf{s}(u_x, u_y)$	array manifold of a rectangular planar array
$U_{\text{ML}}, U_{\text{SL}}, U_{\text{N}}$	mainlobe, sidelobe, and null regions, respectively (6.6)
u_p	specified constant
$\varphi(\cdot)$	weight assigned to each element of the beamforming weight vector (6.9)
δ	threshold
$p(u_x, u_y), p(\theta)$	antenna element patterns
$\mathbf{C}$	coupling matrix
$\tilde{\mathbf{s}}(u_x, u_y)$	true array manifold of a planar array with array im- perfections or mutual coupling effects (6.10)

$\Upsilon(\eta)$	uncertainty set of the actual array manifold of a planar array with mutual coupling effects (6.10)
$\mathbf{e}(u_x, u_y)$	array manifold error of a planar array
η, η_m, η_s	constants
$\mathbf{v}_{-2}, \dots, \mathbf{v}_2$	array vectors (6.19)
r	radius of circular array (6.20)
$[T_{k,N}]_{i,j}$	Toeplitz matrix (A.1)
$\langle \mathbf{T}_{k,N}, \mathbf{X} \rangle$	sum of the elements on the k th lower off-diagonal of $\mathbf{X}$.
$\mathbf{d}(\Omega_l, \Omega_u)$	vector (A.2)

Chapter 1

Introduction

1.1 Background and Motivation

Antenna arrays have been used ubiquitously in many applications ranging from radar, sonar, satellite, and wireless communications, for the transmission and reception of wave energy [9–14]. In this thesis, receiving arrays are studied. Array signal processing involves the manipulation of signals induced on the antennas in an array in order to protect and enhance a desired signal in the presence of interferences and noise, or to detect the presence of signals and extract useful information from them. The useful information content may be in the form of the spatial locations of signal sources [15, 16] or the signal waveforms [1, 17]. One of the most important tasks in array signal processing is beamforming and the application areas for beamforming systems are continually expanding. Examples of new applications include hand-held ultrasound imaging systems, directional hearing aids, and smart antennas in wireless communications [18–22].

Beamforming systems can be broadly classified into two categories, namely fixed (data independent) or adaptive, depending on how the beamforming weight vector is designed. Fixed beamforming systems are usually designed so that the beampattern response approximates the desired one according to some pre-

scribed design specifications. Due to their implementation simplicity and robustness, fixed beamforming systems have been extensively used in many applications [11, 13, 23, 24]. On the contrary, adaptive beamforming systems are designed based on the received array data or its statistics and the design objective is usually to maximize the output SINR performance or to minimize the array output power subject to a distortionless response towards the desired signal. Compared to fixed beamforming systems, adaptive beamforming systems can achieve superior interference cancellation performance [19].

The theory of adaptive beamforming is well developed and a variety of advanced algorithms has been proposed [9–11, 25]. However, most of these techniques are based on assumptions of precise knowledge of the signal propagation model and antenna array characteristics. Additionally, these methods usually make use of several restrictive assumptions on the signal sources. As a result, the performances of conventional adaptive beamforming techniques can become severely degraded in practical situations where the exploited assumptions become wrong or imprecise. This degradation is primarily due to a high sensitivity of the conventional adaptive beamforming techniques to the mismatches between the presumed and actual characteristics of the propagation medium, signal sources, and the receiving array. Such mismatches can occur as a consequence of environmental fluctuations and non-stationarity, array manifold errors, array imperfections, unknown fading, scattering, and multipath propagation effects [18, 19, 26].

One of the most typical causes of performance degradation in adaptive beamforming systems is a mismatch between the nominal (presumed) and actual desired array steering vectors, resulting in steering vector errors. Adaptive beamforming techniques are known to be very sensitive to slight errors of this type because the adaptive beamformer then tends to misinterpret the desired signal in the array snapshots as an interference and attempts to suppress it by means of adaptive

nulling instead of maintaining a distortionless response towards it. This phenomenon is referred to as signal cancellation or signal self-nulling [18, 19].

The aforementioned steering vector errors can occur frequently in practice as a consequence of steering direction errors, imperfect array calibration, and distorted antenna shape. Other common causes of steering vector errors include array manifold mis-modelling due to source wavefront distortions resulting from environmental inhomogeneities, local scattering of the signal sources, etc. For example, in wireless communications, the source spatial signatures are estimated in the uplink mode at the base station antenna array using training sequences¹ transmitted by the mobile users. As the percentage of training symbols in the transmitted data stream is usually limited by the system bandwidth efficiency requirements, due to a high variability of the communication channels in time and a high user mobility rate, the accuracy of such spatial signature estimates can rapidly worsen after the termination of the training sequence [18, 19].

Another typical cause of performance degradation in adaptive beamforming systems is due to the non-stationary behaviour of the environment, receiving array, and signal sources. This may be caused by a rapidly changing propagation channel, antenna array motion or vibration, or interference motion which occur commonly in radar, sonar, and wireless communications [6, 26–28]. There are several undesirable effects associated with such non-stationarity. First, it naturally limits the availability of the array snapshots that can be collected and this finite snapshot support problem can lead to severe performance deterioration of many adaptive beamforming algorithms, especially when the array snapshots contain the desired signal [2, 18, 26, 29]. Low snapshot support can result in an inaccurate estimation of the array covariance matrix, giving rise to high sidelobes in adaptive beamforming systems. High sidelobes can lower the system’s tolerance to white

¹Note that the training sequences transmitted by the mobile users in wireless communications are different from the beamformer training data [2].

noise and they are also unacceptable in applications like radar due to the need to safeguard against unexpected interferences or intentional jammers [30–32]. In radar, sonar, and wireless communications where strong moving interferences are typically encountered, the beamforming weights may not be adapted fast enough to compensate for the interference motion [28, 33, 34]. As a result, the interferences may be located outside the beampattern nulls and thus, leak to the output of the adaptive beamformer through the sidelobes [19]. The same situation can occur if moving antenna arrays are used, e.g., towed arrays of hydrophones in sonar [34] or moving antenna platforms in airborne applications [18].

In the past few decades, many approaches have been proposed to improve the robustness of conventional adaptive beamforming systems in non-ideal conditions. In order to prevent the drastic performance degradation of adaptive beamformers due to steering direction errors, multiple point constraints are suggested to be imposed on the mainlobe in the vicinity of the presumed direction-of-arrival (DOA) of the desired signal [35]. Derivative constraints have also been proposed to flatten the mainlobe response in the region of the presumed desired signal's DOA. These constraints can also be applied to broaden beampattern nulls [11]. However, for every equality constraint imposed, the adaptive beamformer loses one DOF for interference cancellation. This loss of interference cancellation can have a significant adverse impact on small arrays [11]. On the other hand, inequality constraints have been used in adaptive beamforming [36, 37] and are referred to as soft constraints [36].

Several other approaches have been developed to partly overcome the problem of steering vector errors. One of the most popular approaches is the quadratically constrained beamformers of Abramovich [38], Carlson [39], and Cox *et al.* [40]. Though [38–40] are formulated independently and differently, they lead to diagonal loading of the sample array covariance matrix. However, it is not known how to

choose the optimal diagonal loading factor in [38–40] based on the uncertainty level or error in the desired array steering vector.

Another popular approach is the eigenspace based beamformers which exploit the orthogonality property between the signal-plus-interference and noise subspaces [41, 42]. Although the eigenspace based beamformers are highly effective in combating steering vector errors, they are inherently restricted in scenarios with low signal-to-noise-ratios (SNRs) and where the dimension of the signal-plus-interference subspace is large and/or unknown. These limitations render the eigenspace based beamformers unusable for many applications like wireless communications where the number of signal sources can be uncertain and large due to local scattering effects [2, 43, 44].

Recently, some promising adaptive beamforming systems have been developed which are robust against steering direction errors [45–47] and steering vector errors [1–3, 43, 48–57]. In [45, 46], the desired signal’s DOA is modelled as a discrete random variable with known a priori probability density function that describes its level of uncertainty. The resulting Bayesian beamformer of [45, 46] can be viewed as a mixture of directional beamformers combined according to the a posteriori distribution of the DOA given the array snapshots. The beamformer of Chen and Vaidyanathan [47] quadratically constrains the magnitude responses at two array steering vectors and then applies diagonal loading to force the magnitude response at a range of DOAs between these two array steering vectors to exceed unity. The diagonal loading factor used in [47] is determined by an iterative algorithm.

Based on worst-case performance optimization, the robust adaptive beamformers of Vorobyov *et al.* [43, 48], Li *et al.* [1, 49, 50], and Lorenz and Boyd [51, 52] explicitly model the uncertainty set of the desired array steering vector as a sphere or an ellipsoid. Although these beamformers are formulated using different approaches, they can be categorized under the diagonal loading method, except that

the optimal loading factor in these beamformers can be computed precisely based on the uncertainty set of the desired array steering vector. When the uncertainty set of the desired array steering vector is modelled as a sphere, the beamformers of [1, 43, 52] are proven to be equivalent [1]. Another interesting and less conservative design method involves encompassing the uncertainty set by a polyhedral cone [58]. As noted in [52], the complexity of the problem in [58] grows quickly with the number of vertices. The beamformer of Shahbazpanahi *et al.* [2] extends robust adaptive beamforming to signal models of a general rank and models errors in both the presumed signal covariance matrix and the sample array covariance matrix (to improve robustness against finite snapshot support), which surprisingly, yields closed-form solutions. In [3], Yu *et al.* model the error in the array snapshot autocorrelation sequence and in [55], robust magnitude response constraints on the beampattern are derived². In [56, 60], Yu *et al.* alternatively transform the adaptive beamforming problem with magnitude response constraints into a semidefinite programming (SDP) problem. Other application examples formulated with the SDP³ approach include [66–69].

Several popular adaptive beamforming approaches with sidelobe control are the penalty function method of [31] by Hughes and Whirter as well as the iterative method of [30] by Bell and Van Trees where quadratic constraints are used to minimize deviations between the adapted beampattern and the desired one. However, the methods of [30, 31] cannot control the mainlobe beamwidth and sidelobe level precisely. More recently, Liu *et al.* [32] have also considered adaptive beamforming with precise sidelobe control by means of multiple quadratic inequality constraints on the sidelobe region. Lu *et al.* [70] have extended this idea using a different objective function. However, [32, 70] do not incorporate robustness against steering vector errors in their problem formulations.

²In [3, 7, 8, 55, 59], the optimization variable is the array weight autocorrelation sequence but the beamformers of [7, 8, 59] are non-adaptive.

³For a comprehensive review on SDP, please refer to [61–65].

The ability to achieve broadened beampattern nulls in the interfering directions is very useful in practice to achieve robustness against moving interferences. This is particularly so for adaptive beamforming systems with limited computational resources as they may not be able to update the beamforming weights fast enough to cancel the moving interferences. To this aim, several methods have been proposed. For example, the non-adaptive beamformer of Er [71] obtains broad beampattern nulls by solving a least-squares null constrained optimization problem. Different from [71], the adaptive beamformers of Mailloux [72], Zatman [73], Riba *et al.* [33], and Guerci [5] achieve broadened beampattern nulls in the interfering directions adaptively based on the CMT approach. The CMT approach is also used to broaden the mainlobe in [33] where the desired signal direction is assumed to be known accurately. In other words, the method of [33] is not robust against steering direction errors. In [5], diagonal loading is also added to the sample array covariance matrix to help stabilize the sidelobes.

This thesis also investigates another important aspect of beamforming system design; that is antenna selection to optimize the number of antennas in the array. In most practical situations, the designer is given a set of prescribed specifications. Usually, one of the most important objectives is to minimize the number of antennas required, in order to reduce the cost of the beamforming system since the cost of an antenna element can be significant due to the antenna element itself and the associated electronics. The need to minimize the number of antennas used in the beamforming systems may be due to other constraints, for instance, the physical dimension of the array supporting structure, i.e., a radio tower, a mast on a ship, the fuselage of an aeroplane, and a towed array [11]. The weight of the beamforming array may also be strictly limited in certain applications like satellite communications. Furthermore, the lesser the number of antennas used, the lower the computational complexity of the resulting beamforming system. The chal-

lenge is to select as few antennas in an array as possible without violating the beampattern requirements.

For beampattern design, the requirements are typically to achieve a mainlobe with controllable beamwidth and response ripple (for robustness against large steering direction errors) as well as to achieve completely arbitrary sidelobe levels. The last requirement is advantageous when there are prior information about the approximate locations of strong moving interferences, as broadened deepened nulls can be incorporated into the beampattern design to eliminate the degradation they introduce to the beamforming system. However, classical beampattern design methods of Dolph [74], Tseng and Cheng [75], and Kim and Elliott [76] cannot design completely arbitrary sidelobe levels. The method of Woodward and Lawson [77] can achieve an arbitrary beampattern. However, [74–77] are all restricted to uniformly spaced arrays. Though applicable to non-uniformly spaced arrays, the methods of Zhou and Ingram [78] as well as Shi and Feng [79] cannot control the beampattern precisely according to the prescribed design specifications.

In addition, array imperfections and mutual coupling exist in a practical antenna array [80–82]. Therefore, the actual array manifold differs from the ideal presumed one. However, many beamforming methods in [11, 23] do not consider these effects. Consequently, their optimized beampatterns can violate the beampattern specifications in the presence of array imperfections and mutual coupling effects. Yan and Hovem of [83] have considered array manifold uncertainty possibly due to mutual coupling or other undesirable effects such as different element patterns and designed two beamformers with different optimization criteria, i.e., l_1 and l_2 regularization. From the results in [83], the two derived beamformers exhibit good robustness against array manifold uncertainties. However, the beamformers of [83] cannot achieve a mainlobe of an arbitrary beamwidth and response ripple.

1.2 Objectives

The objectives of the research in this thesis is to investigate optimization techniques for robust beamforming and antenna selection in arrays. In particular, optimization techniques against large steering direction errors and interferences are studied.

The adaptive beamformers of [1, 43, 52] are equivalent when the uncertainty set of the desired array steering vector is modelled as a sphere. Therefore, in Chapter 3, the beamformer of [1] is studied to see the effects of two factors; first, the SNR of the desired signal and second, the magnitude of steering direction errors, have on the optimal size of the uncertainty sphere which, in turn, affects the output SINR performance of the beamformer of [1].

In addition to large steering direction errors, low snapshot support and strong interferences that move during which the beamforming weights are applied, are frequently encountered in practical beamforming applications. However, the worst-case performance optimization based adaptive beamformers of [1–3, 43, 52] and [5] based on the CMT approach have not been studied in such scenarios. This thesis studies their beampattern behaviour at low snapshots and in the presence of large steering direction errors.

The last objective is to study beamforming optimization methods on antenna selection in linear and planar arrays so that the minimum number of antennas (or as few antennas as possible) are selected to achieve a mainlobe with controllable beamwidth and response ripple as well as completely arbitrary sidelobe levels according to the prescribed specifications. The method of Davidson *et al.* [7, 8] with LMI constraints is studied as the constraints can represent the semi-infinite magnitude response (upper and lower bound) constraints on the beampattern in a finite and convex manner.

1.3 Major Contributions of the Thesis

The major contributions of the thesis are as follows.

1. In Chapter 3, the findings show that if the desired signal is a dominant signal and that the steering direction error is small, the optimal output SINR performance of the beamformer of [1] is relatively insensitive to the size of the uncertainty sphere provided that it is larger than a certain threshold. If the desired signal is a weak signal, the size of the uncertainty sphere is critical for the optimal output SINR performance of the beamformer of [1]. Also, when large steering direction errors occur, the optimal size of the uncertainty sphere has to increase in order to adequately describe the error compared to that with small steering direction errors. However, a large uncertainty sphere rapidly reduces the output SINR performance of the beamformer of [1] as its interference-plus-noise suppression ability is traded off.
2. Therefore, in Chapter 3, to alleviate the problem of reduced output SINR performances of adaptive beamforming systems in the presence of large steering direction errors, a beamforming optimization method is proposed by using a small uncertainty sphere (and a small flat ellipsoid) to search for the desired array steering vector iteratively. This preserves the DOFs of the proposed beamformer and in turn, its interference-plus-noise suppression ability, and by using the corrected desired array steering vector, a better output SINR performance is achieved in the presence of large steering direction errors than the beamformer of [1]. Chapter 3 also derives theoretical results to support the effectiveness of the proposed beamformer.
3. In Chapter 4, the adaptive beamformers of [1–3, 43, 52] based on worst-case performance optimization and [5] based on the CMT approach are investigated in scenarios with large steering direction errors and low snapshots.

The findings show that these beamformers of [1–3, 5, 43, 52] lack beam-pattern control. This increases the risk of beamforming system breakdown of [1–3, 5, 43, 52] if there are unexpected or strong moving interferences. The CMT beamformer [5] can be robust against moving interferences but not against steering direction errors. Therefore, in Chapter 4, an adaptive beamforming framework is proposed with beampattern shaping constraints to achieve robustness against large steering direction errors and strong interferences that move during which the beamforming weights are applied. Achieving adaptive interference rejection and sidelobe control are two conflicting goals as they consume DOFs of the beamforming systems (usually a fixed variable). Thus, a weighing ratio is proposed to automatically control their relative proportions in different scenarios without ambiguity. This is possible by making use of the estimated input array power. The proposed framework encompasses different optimization strategies and three beamformers have been derived from it where they have shown significant output SINR performance improvement over the beamformers of [1–3, 5, 43, 52] in scenarios with low snapshots, large steering direction errors, and strong moving interferences.

4. Two optimization techniques on antenna selection in arrays have been proposed in Chapters 5 and 6. In Chapter 5, the method of Davidson *et al.* [7, 8] is studied. The direct application of [7, 8] to compute the minimum number of antennas required for beampattern design is not systematic and the number of iterations before it stops is unknown. Therefore, a systematic technique is proposed in Chapter 5 to compute the minimum number of antennas required for beampattern design. In order to satisfy all the prescribed specifications of the beampattern precisely, the proposed method imposes LMI constraints as developed by Davidson *et al.* [7, 8] on the beam-

pattern, which remove the need to discretize the beampattern region. As the proposed formulation is quasi-convex, an iterative procedure is used to decompose it into a systematic sequence of convex feasibility problems, in order to find the minimum number of antennas required. The proposed method guarantees convergence if the globally optimum solution lies in the search interval. This is easy to ensure at the start of the search. Unlike the direct application of the method of [7, 8] to compute the minimum number of antennas required for beampattern design, the proposed method is systematic and the number of iterations before it stops is known.

5. It is found that the use of LMI constraints in the proposed antenna selection technique of Chapter 5 may be too computationally demanding for the number of antennas in planar arrays that are commonly employed in practice. Therefore, an alternative antenna selection technique is proposed for planar arrays in Chapter 6 (also applicable to linear arrays). The proposed technique imposes conjugate symmetric beamforming weights so that both the upper and nonconvex lower magnitude response constraints on the beampattern can be made convex. As such, the proposed method allows arbitrary beampatterns to be achieved precisely, unlike the beamforming methods of [74–76]. By minimizing a re-weighted objective function based on the magnitudes of the elements in the beamforming weight vector iteratively, the proposed technique selects certain antennas in an array to satisfy the prescribed specifications strictly. Interestingly, a sparse array with fewer number of antennas is obtained compared to the beamforming methods of [76, 79]. The proposed method can also be used to design non-uniformly spaced arrays with inter-element spacings larger than one half-wavelength, without the appearance of grating lobes in the resulting beampattern. Simulations are conducted on arrays with up to a few hundred antennas to show the

practicality of the proposed antenna selection method of Chapter 6.

6. In addition, the proposed antenna selection method of Chapter 6 is studied in the presence of array imperfections and mutual coupling effects. According to the findings of the numerical studies, the obtained beampatterns can violate the beampattern specifications in the presence of array imperfections and mutual coupling effects if they are not taken into consideration. This is because the actual array manifold can differ from the ideal presumed one. Therefore, robust beampattern constraints are further proposed in Chapter 6 which are derived based on the uncertainty set of the actual array manifold and worst-case performance optimization [43, 63]. Simulation results show that the proposed antenna selection method with the robust beampattern constraints is able to satisfy the specifications wholly in the presence of array imperfections and mutual coupling effects without incurring additional selected antennas.

1.4 Organization of the Thesis

This thesis is organized as follows. Chapter 2 gives an overview of the array system model that includes the coordinate system, the array geometries commonly used in beamforming applications, the spatial and temporal characteristics of the signal sources, and the array steering vector model. The concept of beamforming and the performance measures for beamforming systems which are used in this thesis are given. Typical causes in practice which result in mismatches between the exploited assumptions and the actual characteristics of the environment, antenna array, and signal sources are discussed. Several well-known beamformers that can achieve robustness against various types of mismatches or achieve beampattern control are described.

Chapters 3 to 6 contain the main contributions of this thesis. In Chapter 3, the beamformer of [1] is investigated and the findings are reported. Based on the findings of the study, when there is a large steering direction error, the uncertainty set of the desired array steering vector has to expand to accommodate the error but a large uncertainty set also rapidly degrades the output SINR performance of [1]. Thus, Chapter 3 proposes an iterative robust minimum variance beamformer (IRMVB) which uses either a small sphere or a small flat ellipsoid to search for the desired array steering vector. Theoretical analysis as well as several simulation results to compare the output SINR performances of the proposed IRMVB with the worst-case performance optimization based beamformers of [1–3, 43, 52] and [4] are presented to show the effectiveness of the proposed IRMVB.

In Chapter 4, the worst-case performance optimization based beamformers of [1–3, 43, 52] and [5] based on the CMT approach are studied in scenarios with large steering direction errors and low snapshots to see their beampattern behaviour. The findings reveal that these beamformers lack beampattern control. Therefore, Chapter 4 proposes an adaptive beamforming framework based on the use of a set of beampattern shaping constraints to achieve robustness against large steering direction errors, high sidelobes occurring at low snapshots, and strong interferences that move during which the beamforming weights are applied. A weighing ratio is also proposed in the framework to automatically control the relative proportions of adaptive interference rejection and sidelobe suppression unambiguously in different scenarios. The framework encompasses different optimization strategies and three beamformers have been derived with specific robustness goals. Several simulation results are presented to compare the output SINR performances of the proposed beamformers with the methods of [1–3, 5, 43, 52]. The tradeoffs of incorporating a broadened deepened null constraint to safeguard against strong moving interferences relative to the case without such a constraint as well as varying mainlobe

roll-off widths on the achievable minimal sidelobe level are also illustrated.

Chapter 5 studies the method of Davidson *et al.* [7, 8] with LMI constraints. The direct application of [7, 8] to compute the minimum number of antennas needed is not systematic and the number of iterations before stopping is unknown. Therefore, Chapter 5 proposes a more systematic antenna selection method guaranteed to converge to the minimum number of antennas provided that this solution lies in the search interval. This is easy to ensure at the start of the search. The number of iterations before the proposed method stops is known. Simulation examples are presented to compare the proposed method with the direct application of the method of Davidson *et al.* [7, 8].

In Chapter 5, it is found that the use of LMI constraints in the proposed antenna selection method can be too computationally demanding for the number of antennas in planar arrays that are typically considered for practical usage. Hence, Chapter 6 proposes an alternative antenna selection method (also applicable to linear arrays) by formulating the upper and lower bound magnitude response beampattern constraints using conjugate symmetric beamforming weights. By minimizing a re-weighted objective function based on the magnitudes of the elements in the beamforming weight vector iteratively, the proposed method obtains sparse arrays with fewer number of antennas required for beampattern design compared to the methods of [76, 79]. Next, the proposed antenna selection is studied in the presence of array imperfections and mutual coupling effects separately. The findings reveal that the obtained beampatterns violate the specifications. Robust beampattern constraints are derived based on the uncertainty set of the actual array manifold and worst-case performance optimization to compensate for the undesirable effects in practice. Several simulation results are given to compare the proposed antenna selection method with the methods of [76, 79] and to show the effectiveness of the proposed robust beampattern constraints.

Chapter 7 concludes this thesis and proposes some suggestions for possible future research.

Chapter 2

Array Signal Processing

2.1 Introduction

In this chapter, the array fundamentals and beamforming background are described as the underlying foundation for the studies in subsequent chapters. This chapter is organized as follows. Section 2.2 introduces the system model which includes the coordinate system, the array geometries commonly used in beamforming applications, the spatial and temporal characteristics of signal sources, and the array steering vector model. Section 2.3 describes the concept of beamforming and introduces some important performance measures for beamformers. Sections 2.4 and 2.5 discuss several well-known adaptive beamforming and beampattern control methods, respectively to counteract the detrimental effects brought about by the various mismatches between the exploited assumptions and the actual characteristics of the environment, antenna array, and signal sources.

2.2 System Model

The cartesian and spherical coordinate systems are frequently encountered in array systems to define the positions of the array antennas or the spatial locations in the

Figure 2.1: Spherical coordinate system with x , y , and z axes. θ and ϕ are the polar and azimuth angles defining the source direction, respectively. The distance of a point from the array origin is r .

radiation field of the propagating waves. Fig. 2.1 shows the coordinate system of interest where θ and ϕ denote the polar and azimuth angles defining the direction of the propagating wave, respectively [11]. The cartesian and spherical coordinates are related by the following equations

$$x = r \sin \theta \cos \phi, \quad (2.1)$$

$$y = r \sin \theta \sin \phi, \quad (2.2)$$

$$z = r \cos \theta, \quad (2.3)$$

and conversely,

$$r = \sqrt{x^2 + y^2 + z^2}, \quad (2.4)$$

$$\phi = \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right) = \arcsin\left(\frac{y}{\sqrt{x^2 + y^2}}\right), \quad (2.5)$$

$$\theta = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right). \quad (2.6)$$

At spatial position $\mathbf{r}_k = [x_k \ y_k \ z_k]^T$ where $(\cdot)^T$ is the transpose operator, the time varying field of an electromagnetic plane wave at time t can be expressed as

$$f(t, \mathbf{r}_k) = f(t - \boldsymbol{\alpha}^T \mathbf{r}_k) \quad (2.7)$$

where the slowness vector $\boldsymbol{\alpha}$ is given by

$$\begin{aligned} \boldsymbol{\alpha} &= -\frac{1}{c} [\sin \theta \cos \phi \ \sin \theta \sin \phi \ \cos \theta]^T \\ &= -\frac{1}{c} \mathbf{u}(\theta, \phi) \end{aligned} \quad (2.8)$$

and $\mathbf{u}(\theta, \phi)$ is the unit vector in the direction of (θ, ϕ) . The slowness vector points

Figure 2.2: A linear array with N antenna elements on the z axis. A plane wave impinges at θ angle from the array axis.

in the direction of the propagating wave with a magnitude of $1/c$ where c is the propagation speed of the wave [9]. In this thesis, the common assumption that the signal sources are located in the far-field of the antenna array is used [11]. Thus, the propagating wave incident on the array has a plane wavefront instead of a spherical one. The plane wavefront means that at any time t_0 , $f(t_0, \mathbf{r}_k)$ has the same value for all the points on the plane where $\boldsymbol{\alpha}^T \mathbf{r}_k$ is a constant. The planes where $\boldsymbol{\alpha}^T \mathbf{r}_k$ is a constant, are perpendicular to $\boldsymbol{\alpha}$ [9].

2.2.1 Array Geometry

The linear array, rectangular planar array, and circular planar array are some of the most common types of array geometries used in beamforming applications. Other types of array geometries exist, such as the hexagonal planar array and the cylindrical non-planar array which are described in greater details in [11].

In this thesis, the linear array and rectangular planar array are studied. As the linear array is a simple structure, it is most extensively studied in array signal processing literature. However, in certain applications, there may be two or more signals impinging from different spatial locations but at the same angle with respect to the linear array axis. When this happens, the linear array cannot distinguish

these signals as there exists a cone-shaped ambiguity along the array axis [9]. On the other hand, the rectangular planar and circular planar arrays can resolve such signals, as the region of ambiguity for planar arrays lies above and below the array plane. For this reason, the rectangular planar array is also studied in this thesis and it is introduced in Chapter 6 for the sake of brevity.

For a linear array, the antennas are assumed to be positioned on the z -axis of the coordinate system in Fig. 2.1. A linear array with N isotropic antennas is shown in Fig. 2.2. For a uniform linear array (ULA), the inter-element spacings are equal to d . The time difference of the wavefront at the k th antenna relative to the array origin at z_0 is

$$\tau_k(\theta) = -\frac{z_k \cos \theta}{c}. \quad (2.9)$$

2.2.2 Spatial and Temporal Characteristics of Signals

Under the narrowband signal model, the maximum time delay for a signal to traverse the entire array is much smaller than its correlation time (the reciprocal of its bandwidth). Thus, the variation of the signal envelope over the array aperture is negligible. At time t and spatial position $\mathbf{r}_k$, the narrowband signal is usually modelled as

$$f(t, \mathbf{r}_k) = A(t)e^{j(2\pi f_c t - \mathbf{k}^T \mathbf{r}_k)} \quad (2.10)$$

where $A(t)$ is a complex modulating function, $j = \sqrt{-1}$ is the imaginary unit, and f_c is the carrier frequency. The wavenumber vector $\mathbf{k}$ is given by

$$\mathbf{k} = 2\pi f_c \boldsymbol{\alpha} \quad (2.11)$$

and $\boldsymbol{\alpha}$ is given in (2.8).

Figure 2.3: A narrowband beamforming structure with N antenna elements. The received array data at the k th antenna at the n th snapshot is $x_k(n)$ which is then multiplied with w_k^* and summed up for $k = 1, \dots, N$ to form the beamformer output $y(n)$.

2.2.3 Array Steering Vector Model

According to the narrowband signal model in (2.10), the response of each antenna element to the impinging plane wave $A(t)e^{j2\pi f_c t}$ gives rise to the following vector

$$\mathbf{s}(\boldsymbol{\alpha}) = [e^{-j2\pi f_c \boldsymbol{\alpha}^T \mathbf{r}_0} \quad e^{-j2\pi f_c \boldsymbol{\alpha}^T \mathbf{r}_1} \quad \dots \quad e^{-j2\pi f_c \boldsymbol{\alpha}^T \mathbf{r}_{N-1}}]^T \quad (2.12)$$

which is the array steering vector. The steering vector for a linear array can be rewritten as

$$\mathbf{s}(\theta) = [e^{-j2\pi f_c \tau_0(\theta)} \quad e^{-j2\pi f_c \tau_1(\theta)} \quad \dots \quad e^{-j2\pi f_c \tau_{N-1}(\theta)}]^T \quad (2.13)$$

where $\tau_k(\theta)$ is given in (2.9).

2.3 Beamforming

The signal received by the k th antenna is sampled with a sampling frequency $f_s = 1/T_s$ where T_s is the sampling period. The n th sample of the received signal vector is $\mathbf{x}(n) = [x_1(n) \quad x_2(n) \quad \dots \quad x_N(n)]^T$. The signal received by the k th

antenna is weighed by w_k^* where $(\cdot)^*$ is the complex conjugate operator. The output of the beamformer at the n th sample is

$$y(n) = \mathbf{w}^H \mathbf{x}(n) \quad (2.14)$$

where $\mathbf{w}$ is the $N \times 1$ complex beamforming weight vector and $(\cdot)^H$ is the Hermitian transpose operator. The received array snapshot vector $\mathbf{x}(n)$ usually contains the desired signal, J interferences, and spatially white Gaussian noise, i.e.,

$$\mathbf{x}(n) = z_0(n)\mathbf{s}(\theta_0) + \sum_{i=1}^J z_i(n)\mathbf{s}(\theta_i) + \mathbf{n}(n) \quad (2.15)$$

where θ_0 and θ_i refer to the DOAs of the desired signal and the i th interference, respectively. The terms $z_0(n)$ and $z_i(n)$ represent the waveforms of the desired signal and the i th interference, respectively. The spatially white Gaussian noise is denoted by $\mathbf{n}(n)$. The desired signal and J interferences are assumed to be zero mean, stationary, and mutually independent [9]. The array steering vector $\mathbf{s}(\theta_i)$ corresponds to that of the i th signal source impinging from an angle of θ_i where $i = 0, \dots, J$. The total number of signal sources is $J + 1$. The array output power is $\mathbb{E}\{|y(n)|^2\} = \mathbb{E}\{|\mathbf{w}^H \mathbf{x}(n)|^2\} = \mathbf{w}^H \mathbf{R} \mathbf{w}$ where $\mathbb{E}\{\cdot\}$ is the statistical expectation operator, $|\cdot|$ is the absolute operator, and $\mathbf{R} = \mathbb{E}\{\mathbf{x}(n)\mathbf{x}^H(n)\}$ is the covariance matrix of the array snapshots or the array covariance matrix.

One of the most important performance measures of beamforming systems is the output SINR performance which is defined as the ratio between the output power of the desired signal and the output power of the interference-plus-noise components. It is expressed as

$$\begin{aligned} \text{SINR} &= \frac{\mathbb{E}\{|z_0(n)\mathbf{w}^H \mathbf{s}(\theta_0)|^2\}}{\mathbb{E}\{|\sum_{i=1}^J z_i(n)\mathbf{w}^H \mathbf{s}(\theta_i) + \mathbf{w}^H \mathbf{n}(n)|^2\}}, \\ &= \frac{\sigma_0^2 |\mathbf{w}^H \mathbf{s}(\theta_0)|^2}{\mathbf{w}^H \mathbf{R}_{\text{in}} \mathbf{w}} \end{aligned} \quad (2.16)$$

where $\sigma_0^2 = \mathbb{E}\{|z_0(n)|^2\}$ is the desired signal power and

$$\mathbf{R}_{\text{in}} = \mathbb{E}\left\{\left[\sum_{i=1}^J z_i(n)\mathbf{s}(\theta_i) + \mathbf{n}(n)\right]\left[\sum_{i=1}^J z_i(n)\mathbf{s}(\theta_i) + \mathbf{n}(n)\right]^H\right\} \quad (2.17)$$

is the covariance matrix of the interference-plus-noise components.

There are several design criteria for the optimal beamformer weight vector such as the Minimum Mean Square Error, Minimum Variance Distortionless Response (MVDR), and Maximum SINR (MSINR) [11, 12, 84]. The MSINR criterion is probably the most widely used. By maximizing the output SINR (2.16) subject to (s.t.) a distortionless response to the desired signal, the problem can be formulated as

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \mathbf{R}_{\text{in}} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{s}_0 = 1 \end{aligned} \quad (2.18)$$

where, without loss of generality, $\mathbf{s}_0$ is used to represent $\mathbf{s}(\theta_0)$, assuming there are no errors in the steering vector such as gain or phase perturbations. The solution to (2.18) can be obtained by applying the Lagrange multiplier methodology as

$$\mathbf{w} = \frac{\mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0}{\mathbf{s}_0^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0}. \quad (2.19)$$

The solution in (2.19) is also referred to as the MVDR beamformer [26, 43].

An equivalent formulation to (2.18) (when $\mathbf{R}_{\text{in}}$ is not available) with a different objective function is the minimization of the array output power subject to the same constraint of a distortionless response towards the desired signal [33], i.e.,

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \mathbf{R} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^H \mathbf{s}_0 = 1 \end{aligned} \quad (2.20)$$

and the solution to the problem (2.20) is

$$\mathbf{w} = \frac{\mathbf{R}^{-1}\mathbf{s}_0}{\mathbf{s}_0^H \mathbf{R}^{-1} \mathbf{s}_0}. \quad (2.21)$$

The solution in (2.21) is widely known as the Capon beamformer [85] or the minimum variance (MV) beamformer [52]. For the sake of consistency, (2.21) is referred to as the MV beamformer in the sequel. The array output power obtained with the MV beamformer (2.21) is the estimate of the desired signal power σ_0^2 [1] which is given by

$$\hat{\sigma}_0^2 = \frac{1}{\mathbf{s}_0^H \mathbf{R}^{-1} \mathbf{s}_0}. \quad (2.22)$$

However, in practical applications, neither an accurate knowledge of $\mathbf{R}_{\text{in}}$ nor $\mathbf{R}$ is available. Thus, the sample array covariance matrix

$$\hat{\mathbf{R}} = \frac{1}{N_s} \sum_{n=1}^{N_s} \mathbf{x}(n) \mathbf{x}^H(n) \quad (2.23)$$

is used to estimate $\mathbf{R}$ and N_s is the number of array snapshots collected for the estimation. It is well-known that if the desired signal is absent from the collected array snapshots (or training snapshots) $\mathbf{x}(n)$ used for the calculation of $\hat{\mathbf{R}}$, the MV beamformer converges rapidly to the optimal SINR given by

$$\text{SINR}_{\text{opt}} = \sigma_0^2 \mathbf{s}_0^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0. \quad (2.24)$$

In fact, Reed *et al.* [29] have shown that if the number of collected array snapshots $N_s \geq 2N$, then the average SINR achieved is only less than 3dB from SINR_{opt} in (2.24) under the assumption that the desired signal is not present in $\mathbf{x}(n)$.

The traditional approach to the design of adaptive beamforming systems assumes that the desired signal is absent from the training snapshots. In such cases,

the adaptive beamformer is known to be robust against steering vector errors and finite snapshot support, and a variety of rapidly converging algorithms has been developed for this particular case [10, 29]. Although the assumption of the absence of the desired signal from the training snapshots may be relevant in some specific cases, e.g., radar and active sonar, there are many other applications whereby the training snapshots always contain the desired signal, e.g., mobile communications, passive sonar, microphone array speech processing, and medical imaging.

If the desired signal is present in the collected array snapshots used for the computation of $\hat{\mathbf{R}}$ in (2.23), then the convergence rate of the MV beamformer is much slower and generally requires $N_s \gg N$ [26, 41, 86]. In this thesis, the desired signal is assumed to be always present in the collected array snapshots since this is usually the case in many practical applications.

Apart from using the output SINR in (2.16) as the performance measure for beamforming systems, another commonly used performance measure is the beam-pattern which is defined as the beamformer's response to a plane wave as a function of its DOA θ . The beampattern of a beamformer is expressed as

$$G(\theta) = \mathbf{w}^H \mathbf{s}(\theta) \quad (2.25)$$

and its magnitude response in dB is $20 \log_{10} |G(\theta)|$.

2.4 Adaptive Beamforming Systems

The MV beamformer is derived based on some restrictive assumptions about the environment, antenna array, and signal sources. As discussed in Chapter 1, one of the most typical causes of performance degradation of adaptive beamforming systems is due to a mismatch between the presumed and actual array response to the desired signal, resulting in steering vector errors. Such errors can result

from steering direction errors, imperfect array calibration, unknown wavefront distortions, signal fading, local scattering, etc., and these mismatches are usually not precisely known to the beamforming system. As a result, the ideal presumed desired array steering vector $\bar{\mathbf{s}}_0$ is used instead of the actual one $\mathbf{s}_0$.

Due to the rapid variations of the propagation channel, signal source motion, and antenna array movement or vibration, the number of array snapshots that can be collected may be limited. Thus, the array covariance matrix $\mathbf{R}$ is replaced by a poor estimate $\hat{\mathbf{R}}$ by (2.23) and the optimization problem solved in practice is

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^H \bar{\mathbf{s}}_0 = 1. \end{aligned} \quad (2.26)$$

The solution to (2.26) is given by

$$\mathbf{w} = \beta \hat{\mathbf{R}}^{-1} \bar{\mathbf{s}}_0 = \frac{\hat{\mathbf{R}}^{-1} \bar{\mathbf{s}}_0}{\bar{\mathbf{s}}_0^H \hat{\mathbf{R}}^{-1} \bar{\mathbf{s}}_0}. \quad (2.27)$$

The term $\beta = 1/\bar{\mathbf{s}}_0^H \hat{\mathbf{R}}^{-1} \bar{\mathbf{s}}_0$ is a normalization constant which does not affect the output SINR and is omitted in the sequel for the sake of brevity.

The MV beamformer of (2.27) is extremely sensitive to even slight mismatches between the presumed and actual array response to the desired signal ($\bar{\mathbf{s}}_0 \neq \mathbf{s}_0$) because the beamformer interprets the desired signal component in the array snapshots as an additional interfering source and therefore, suppresses it by adaptively nulling instead of maintaining a distortionless response towards it. The phenomenon is known as signal cancellation or signal self-nulling. Additionally, limited snapshot support results in an inaccurate estimate $\hat{\mathbf{R}}$ which can lead to high sidelobes in the MV beamformer. High sidelobes reduce the system tolerance to white noise and can also cause substantial performance degradation of the adaptive beamforming systems in the presence of unexpected interferences or in-

tentional jammers. This is because after the appearance of the new interferences, the MV beamformer requires some transition time to adaptively suppress them and it is during this time interval that its performance can break down [32].

In conclusion, there is a strong demand for optimization techniques for robust beamforming systems to protect the desired signal in the presence of the aforementioned problems and other undesirable effects. For a comprehensive review of some other causes of performance degradation of adaptive beamforming systems and the methods proposed in the literature to counteract or reduce the performance deterioration, please refer to [11, 12, 14, 18, 19, 26]. In the next section, some of the well-known beamformers in the literature are discussed.

2.4.1 Eigenspace Based Beamformer

One of the most effective adaptive beamforming approaches against steering vector errors is the eigenspace based beamformer [41, 42]. Instead of using the presumed desired array steering vector $\bar{\mathbf{s}}_0$ in the MV beamformer (2.27), a more accurate estimate of the desired array steering vector is used. This is obtained by projecting $\bar{\mathbf{s}}_0$ onto the signal-plus-interference subspace obtained from the eigen-decomposition of the sample array covariance matrix $\hat{\mathbf{R}}$. The sample array covariance matrix can be decomposed into

$$\hat{\mathbf{R}} = \hat{\mathbf{U}}_s \hat{\mathbf{\Lambda}}_s \hat{\mathbf{U}}_s^H + \hat{\mathbf{U}}_n \hat{\mathbf{\Lambda}}_n \hat{\mathbf{U}}_n^H \quad (2.28)$$

where the $N \times (J+1)$ matrix $\hat{\mathbf{U}}_s$ contains the eigenvectors which span the estimated signal-plus-interference subspace and the $(J+1) \times (J+1)$ diagonal matrix $\hat{\mathbf{\Lambda}}_s$ contains the corresponding estimated eigenvalues. Note that the total number of signal sources is $J+1$ assumed to be known. Similarly, the $N \times (N-J-1)$ matrix $\hat{\mathbf{U}}_n$ contains the eigenvectors which span the estimated noise subspace and the $(N-J-1) \times (N-J-1)$ diagonal matrix $\hat{\mathbf{\Lambda}}_n$ contains the corresponding

estimated eigenvalues. The eigenspace based beamformer solves

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^H (\hat{\mathbf{U}}_s \hat{\mathbf{U}}_s^H \bar{\mathbf{s}}_0) = 1 \end{aligned} \quad (2.29)$$

where $\hat{\mathbf{U}}_s \hat{\mathbf{U}}_s^H$ is the projection matrix onto the signal-plus-interference subspace. The solution to the problem in (2.29) is given by

$$\mathbf{w} = \hat{\mathbf{U}}_s \hat{\mathbf{\Lambda}}_s^{-1} \hat{\mathbf{U}}_s^H \bar{\mathbf{s}}_0. \quad (2.30)$$

Although the eigenspace based beamformer [41, 42] is effective against steering vector errors, it is not efficient at low SNRs as there is a high probability of subspace swapping [2, 48]. It is also not efficient in scenarios where the number of signal sources $J + 1$ is large or unknown. These drawbacks make the eigenspace based beamformer difficult to employ in some applications, e.g., wireless communications where the total number of signal sources can be large and uncertain due to local scattering effects [2, 43, 44].

2.4.2 Diagonally Loaded Beamformer

One of the most popular adaptive beamforming approaches against steering vector errors is the diagonally loaded beamformers [38–40] (or quadratically constrained beamformers) which impose an inequality constraint on the beamforming weight vector (2.31c) to control the white noise gain as

$$\min_{\mathbf{w}} \quad \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad (2.31a)$$

$$\text{s.t.} \quad \mathbf{w}^H \bar{\mathbf{s}}_0 = 1, \quad (2.31b)$$

$$\|\mathbf{w}\|^2 \leq \gamma \quad (2.31c)$$

where $\|\cdot\|$ is a vector norm when the argument is a complex vector and γ is a specified threshold. The solution to the problem in (2.31) is

$$\mathbf{w} = (\hat{\mathbf{R}} + \zeta \mathbf{I})^{-1} \bar{\mathbf{s}}_0 \quad (2.32)$$

where $\mathbf{I}$ is an identity matrix and the diagonal loading factor ζ is adjusted iteratively so that the weight vector norm constraint in (2.31c) is satisfied [40]. This is because the optimal loading factor cannot be directly expressed as a function of the constraint (2.31c) and has to be solved numerically [11, 19].

The diagonally loaded beamformer (2.32) resembles the MV beamformer in (2.27) except that a scaled identity matrix $\zeta \mathbf{I}$ is added to $\hat{\mathbf{R}}$. This has the interpretation of injecting white noise into $\hat{\mathbf{R}}$. The effects of diagonal loading are as follows. First, it ensures that the matrix $\hat{\mathbf{R}} + \zeta \mathbf{I}$ can be inverted regardless of whether $\hat{\mathbf{R}}$ is singular or not. It also helps stabilize and lower the sidelobes to some extent. Hence, diagonal loading improves the robustness of the MV beamformer against high sidelobes at low snapshots. Second, diagonal loading improves the robustness of the MV beamformer against steering vector errors because it puts more effort in suppressing the white noise rather than the interferences [39, 47].

On the other hand, the main disadvantage with the diagonal loading approach is that the diagonal loading factor ζ is not directly related to the amount of uncertainty or error in the desired array steering vector or the weight vector norm constraint (2.31c). If the diagonal loading factor ζ is chosen too large, the MV beamformer can fail to cancel strong interferences as it concentrates its effort in suppressing the white noise. A convenient and common practice in many beamforming applications is to add the diagonal loading factor ζ to $\hat{\mathbf{R}}$ in (2.32) without undergoing any iterative procedure [2, 11, 43, 45]. The standard choice of ζ is 10 to 12dB above the antenna noise level [2, 43, 45].

2.4.3 Beamformer Based on Second-Order-Cone Program (SOCP) Approach

Recently, theoretically rigorous advancements in adaptive beamforming robust against steering vector errors [1–3, 18, 19, 43, 50, 52–55] have been made¹. Based on worst-case performance optimization and the uncertainty set of the desired array steering vector [1, 43, 50, 52] or the desired signal covariance matrix [2, 53, 54], respectively, these beamformers belong to the class of diagonal loading approach. However, unlike [38–40], the amount of diagonal loading can be calculated precisely based on the uncertainty sets. As mentioned in Chapter 1, though formulated differently, the beamformers of [1, 43, 52] turned out to be equivalent when the uncertainty set is modelled as a sphere as proven in [1]. In the following, the differences amongst the formulations of [1, 43, 52] are discussed.

The beamformer of Vorobyov *et al.* [43, 48] minimizes the array output power while maintaining distortionless response for a continuum of all steering vectors in a structured set $\mathcal{A}(\epsilon)$ by inequality constraints, i.e.,

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \\ \text{s.t.} \quad & \min_{\mathbf{s} \in \mathcal{A}(\epsilon)} |\mathbf{w}^H \mathbf{s}| \geq 1. \end{aligned} \quad (2.33)$$

where

$$\mathcal{A}(\epsilon) \triangleq \{\mathbf{s} | \mathbf{s} = \bar{\mathbf{s}}_0 + \mathbf{e}, \|\mathbf{e}\| \leq \epsilon\}, \quad (2.34)$$

$\triangleq$ is “defined as” and $\mathbf{e}$ is a complex error vector. The constraint in (2.33) guarantees that distortionless response can be maintained in the worst-case scenario, i.e., for the particular vector $\mathbf{s}$ that corresponds to the smallest value of $|\mathbf{w}^H \mathbf{s}|$. In this way, the beamformer in (2.33) can be robust against steering vector errors provided

¹The discussion on the beamformer of [3] is deferred to Section 4.2.2.

that the desired array steering vector $\mathbf{s}_0$ belongs to $\mathcal{A}(\epsilon)$. However, (2.33) is non-convex [43] and the general non-convex quadratically constrained quadratic programming problem, e.g., (2.33) is Non-deterministic Polynomial-time-hard (NP-hard²) to solve and thus, intractable [43].

Through a series of transformations detailed in [43], the authors derived an equivalent convex optimization problem³ to (2.33) based on the second-order-cone program (SOCP) approach [63, 65, 87], i.e.,

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^H \bar{\mathbf{s}}_0 \geq \epsilon \|\mathbf{w}\| + 1, \\ & \Im\{\mathbf{w}^H \bar{\mathbf{s}}_0\} = 0 \end{aligned} \tag{2.35}$$

where $\Im\{\cdot\}$ is an operator that takes the imaginary part of its argument. The problem (2.35) is a (convex) SOCP which can be solved globally and efficiently in polynomial-time via the well established interior point method (IPM) using open source solvers, e.g., [88, 89].

The solution to (2.35) takes the form of $c_1(\hat{\mathbf{R}} + c_2\mathbf{I})^{-1}\bar{\mathbf{s}}_0$ for some appropriate constants c_1 and c_2 [43]. Hence, the beamformer of [43] can be viewed as a diagonal loading approach. Further extensions of the beamformer of [43] include [4, 90, 91]. Different from [43] which uses an isotropic uncertainty set for the desired array steering vector, Lorenz and Boyd have extended the robust adaptive beamforming problem to the case where the uncertainty set is anisotropic [51, 52].

²NP-hard problems are “extremely difficult problems with no known polynomial-time solutions” [43].

³Convex optimization problems involve the minimization of a convex objective function subject to convex constraints. Such problems have the property that a local optimum is the global optimum [63].

2.4.4 Beamformer Based on Covariance Fitting Approach

Based on the covariance fitting approach, the beamformer of Li *et al.* [1, 49] solves the MV beamforming problem coupled with the uncertainty set of the desired array steering vector $\mathbf{s}_0$, i.e.,

$$\min_{\mathbf{s}} \quad \mathbf{s}^H \hat{\mathbf{R}}^{-1} \mathbf{s} \quad (2.36a)$$

$$\text{s.t.} \quad \|\mathbf{s} - \bar{\mathbf{s}}_0\|^2 \leq \varepsilon. \quad (2.36b)$$

It is assumed that $\mathbf{s}_0$ belongs to the sphere in (2.36b) centred at the presumed desired array steering vector $\bar{\mathbf{s}}_0$ where ε is a user parameter that specifies the Euclidean distance between a steering vector $\mathbf{s}$ and $\bar{\mathbf{s}}_0$. Assume strong duality is achieved in (2.36), let $g \geq 0$ be the Lagrange multiplier that corresponds to the inequality constraint (2.36b). Due to the complementary slackness condition in the Karush-Kuhn-Tucker (KKT) conditions [63], either $g = 0$ and $\|\mathbf{s} - \bar{\mathbf{s}}_0\|^2 < \varepsilon$ will occur or $g > 0$ and $\|\mathbf{s} - \bar{\mathbf{s}}_0\|^2 = \varepsilon$ will occur. In the first case, the constraint (2.36b) is inactive and the optimal solution is the eigenvector of $\hat{\mathbf{R}}$ corresponding to the maximum eigenvalue provided that the presumed desired array steering vector $\bar{\mathbf{s}}_0$ is near to the true one, the interferences are well-separated from the mainlobe region, and that the desired signal is dominant. Otherwise, the constraint (2.36b) is active, by applying the Lagrange multiplier methodology to (2.36), the estimated desired array steering vector $\hat{\mathbf{s}}_0$ is

$$\begin{aligned} \hat{\mathbf{s}}_0 &= (g^{-1} \hat{\mathbf{R}}^{-1} + \mathbf{I})^{-1} \bar{\mathbf{s}}_0, \\ &= \bar{\mathbf{s}}_0 - (\mathbf{I} + g \hat{\mathbf{R}})^{-1} \bar{\mathbf{s}}_0 \end{aligned} \quad (2.37)$$

where the matrix inversion lemma [92] is used to obtain the second equality [1]. The Lagrange multiplier $g \geq 0$ is found by solving the following constraint equation

$$l(g) \triangleq \|(\mathbf{I} + g\hat{\mathbf{R}})^{-1}\bar{\mathbf{s}}_0\|^2 = \varepsilon, \quad (2.38)$$

which is obtained by replacing $\mathbf{s}$ with the estimated desired array steering vector $\hat{\mathbf{s}}_0$ (2.37) in the constraint $\|\mathbf{s} - \bar{\mathbf{s}}_0\|^2 = \varepsilon$, after which $\hat{\mathbf{s}}_0$ can be found by (2.37) with the obtained g . Stoica *et al.* of [49] recommends to remove the scaling ambiguity by using $\hat{\mathbf{s}}'_0 = \sqrt{N}\hat{\mathbf{s}}_0/\|\hat{\mathbf{s}}_0\|$. The beamformer of [1] is obtained by using the estimated desired array steering vector $\hat{\mathbf{s}}'_0$ in the MV beamformer (2.27) as

$$\mathbf{w} = \hat{\mathbf{R}}^{-1}\hat{\mathbf{s}}'_0 \quad (2.39)$$

$$= c_3(\hat{\mathbf{R}} + \frac{1}{g}\mathbf{I})^{-1}\bar{\mathbf{s}}_0 \quad (2.40)$$

for some constant c_3 . Note that the solution of [1] in (2.40) also takes the diagonal loading form and it is computationally more efficient [1] than the SOCP approach of [43, 52]. The beamformer of Li *et al.* [1] also models the uncertainty set of the desired array steering vector as a flat ellipsoid which is introduced in Section 3.3.2 for the sake of brevity.

2.4.5 Beamformer for General Rank Signal Models

The beamformer of Shahbazpanahi *et al.* [2, 93] considers robust adaptive beamforming against errors in signal models of a general rank. The actual array covariance matrix $\mathbf{R}$ and the desired signal covariance matrix $\mathbf{R}_s$ are modelled as

$$\mathbf{R} = \hat{\mathbf{R}} + \Delta_1, \quad \|\Delta_1\| \leq \zeta \quad (2.41)$$

$$\mathbf{R}_s = \bar{\mathbf{R}}_s + \Delta_2, \quad \|\Delta_2\| \leq \epsilon \quad (2.42)$$

respectively where $\bar{\mathbf{R}}_s$ is the presumed desired signal covariance matrix. Δ_1 and Δ_2 are unknown Hermitian matrices that describe the error between $\mathbf{R}$ and $\hat{\mathbf{R}}$, and the error between $\mathbf{R}_s$ and $\bar{\mathbf{R}}_s$, respectively. The parameters ζ and ϵ are not the same as those in the previously introduced approaches and $\|\cdot\|$ is the Frobenius norm of a matrix when the argument is a complex matrix.

Based on worst-case performance optimization, the formulation in [2] is

$$\begin{aligned} \min_{\mathbf{w}} \max_{\|\Delta_1\| \leq \zeta} \quad & \mathbf{w}^H (\hat{\mathbf{R}} + \Delta_1) \mathbf{w} \\ \text{s.t.} \quad & \mathbf{w}^H (\bar{\mathbf{R}}_s + \Delta_2) \mathbf{w} \geq 1, \quad \forall \|\Delta_2\| \leq \epsilon. \end{aligned} \quad (2.43)$$

The constraint in (2.43) guarantees that distortionless response can be maintained for the worst-case mismatch, i.e., the mismatch which corresponds to the smallest value of $\mathbf{w}^H (\bar{\mathbf{R}}_s + \Delta_2) \mathbf{w}$. The closed-form solution to (2.43) is

$$\mathbf{w} = \mathcal{P}\{(\hat{\mathbf{R}} + \zeta \mathbf{I})^{-1}(\bar{\mathbf{R}}_s - \epsilon \mathbf{I})\} \quad (2.44)$$

where $\mathcal{P}\{\cdot\}$ is an operator that obtains the principal eigenvector of a matrix, i.e., the eigenvector that corresponds to the maximum eigenvalue of the matrix. Note that this beamformer of Shahbazpanahi *et al.* [2] is the only closed-form method among the worst-case performance optimization based beamformers of [1, 2, 43, 52]. Extensions of the beamformer of [2] can be found in [53, 54].

2.5 Beamformers With Beampattern Control

In Section 2.4, most of the beamforming methods concentrate on stabilizing the mainlobe [38–40] or maintaining a unity gain response to the worst-case error in the desired array steering vector [1, 43, 50, 52] or the desired signal covariance matrix [2, 53, 54], in order to avoid signal cancellation in the presence of steering vector

errors. In these methods, beampattern control is neglected. This can give rise to severe consequences in practical situations where the number of array snapshots available for collection is limited. This can lead to an inaccurate estimation of the array covariance matrix and high sidelobes can appear, resulting in an increase in the white noise gain. Also, in scenarios with intentional jamming or strong moving interferences, adaptive beamforming systems without beampattern control may suffer drastic performance degradation.

2.5.1 Beamformer Based on Penalty Function Approach

In order to prevent high sidelobes from developing in adaptive beamforming systems due to finite snapshot support, Hughes and McWhirter [31] define an error function

$$e(\theta) = |\mathbf{s}^H(\theta)\mathbf{w} - \mathbf{s}^H(\theta)\mathbf{w}_d|^2 \quad (2.45)$$

which is the difference between the actual beampattern and the desired one $G_d(\theta) = \mathbf{s}^H(\theta)\mathbf{w}_d$ where $\mathbf{w}_d$ is the beamforming weight vector that achieves the desired beampattern. Then, the total error over the beampattern region Θ is

$$\begin{aligned} E &= (\mathbf{w} - \mathbf{w}_d)^H \left[\int_{\Theta} h(\theta) \mathbf{s}(\theta) \mathbf{s}^H(\theta) d\theta \right] (\mathbf{w} - \mathbf{w}_d) \\ &= (\mathbf{w} - \mathbf{w}_d)^H \mathbf{Z} (\mathbf{w} - \mathbf{w}_d) \end{aligned} \quad (2.46)$$

where $h(\theta)$ is a scalar weighting function for placing minimization emphasis at different beampattern regions and

$$\mathbf{Z} = \int_{\Theta} h(\theta) \mathbf{s}(\theta) \mathbf{s}^H(\theta) d\theta \quad (2.47)$$

is a weighted covariance matrix. The authors of [31] impose a gain constraint towards the desired signal ($\mathbf{w}^H \bar{\mathbf{s}}_0 = \mu$) and proposed to minimize the following objective function

$$\min_{\mathbf{w}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} + k^2 (\mathbf{w} - \mathbf{w}_d)^H \mathbf{Z} (\mathbf{w} - \mathbf{w}_d) + \chi^2 (\mathbf{w}^H \bar{\mathbf{s}}_0 - \mu) (\mathbf{w}^H \bar{\mathbf{s}}_0 - \mu)^* \quad (2.48)$$

where k^2 and χ^2 are independent scalars to weigh the importance of the total error E and the constraint that $\mathbf{w}^H \bar{\mathbf{s}}_0 - \mu = 0$, respectively. The solution to the unconstrained problem (2.48) is given by

$$\mathbf{w} = (\hat{\mathbf{R}} + k^2 \mathbf{Z} + \chi^2 \bar{\mathbf{s}}_0 \bar{\mathbf{s}}_0^H)^{-1} (k^2 \mathbf{Z} \mathbf{w}_d + \chi^2 \mu \bar{\mathbf{s}}_0). \quad (2.49)$$

The beamformer of Hughes and McWhirter [31] is general and flexible; it simplifies to the MV beamformer by choosing $\chi \rightarrow \infty$ and $k \rightarrow 0$ (the matrix inversion lemma is required to expand (2.49) to see this relationship) while it simplifies to a data-independent beamformer $\mathbf{w}_d$ by choosing $k \rightarrow \infty$ in (2.49). By a judicious choice of these parameters, a beamformer that jointly achieves interference cancellation, sidelobe control, and a specified gain towards the desired signal, can be obtained simultaneously. Examples are provided in [31] where the obtained beampatterns form a narrow mainlobe. Though such a beampattern is desirable for many applications, there are also many situations which require beampatterns to have a broad mainlobe with a specified beamwidth and response ripple.

2.5.2 Beamformer Based on Iterative Weighted Least Squares Approach

Zhou and Ingram have developed a data-independent method [78, 94] that generalizes the work of Olen and Compton [95] by extending the beampattern control ability to the mainlobe region. The method of [78, 94] is based on iteratively min-

imizing the weighted least squares errors between the actual beampattern and the desired one $G_d(\theta)$ as

$$\begin{aligned} \min_{\mathbf{w}} \quad & \int_{\Theta} h(\theta) |\mathbf{s}^H(\theta) \mathbf{w} - G_d(\theta)|^2 d\theta \\ \text{s.t.} \quad & \mathbf{C} \mathbf{w} = \mathbf{h} \end{aligned} \quad (2.50)$$

where $\mathbf{C}$ and $\mathbf{h}$ refer to the constraint matrix and constraint vector, respectively. The solution to (2.50) is

$$\mathbf{w} = \mathbf{Z}^{-1} \mathbf{r}_d + \mathbf{Z}^{-1} \mathbf{C}^H (\mathbf{C} \mathbf{Z}^{-1} \mathbf{C}^H)^{-1} (\mathbf{h} - \mathbf{C} \mathbf{Z}^{-1} \mathbf{r}_d) \quad (2.51)$$

where the matrix $\mathbf{Z}$ is given by (2.47) and the cross-correlation vector $\mathbf{r}_d$ is

$$\mathbf{r}_d = \int_{\Theta} h(\theta) G_d(\theta) \mathbf{s}(\theta) d\theta. \quad (2.52)$$

The weighting function $h(\theta)$ is updated iteratively for both the mainlobe and sidelobe regions separately according to the equations in [78, 94], and the iterative algorithm stops when a satisfactory beampattern is achieved. An similar approach to [78, 94, 95] but with a different formulation is developed by Bell and Van Trees [30] which is also iterative.

Some comments on the methods of [31] based on the penalty function approach and [30, 78, 94, 95] based on the iterative weighted least squares approach, are in order. Although they are applicable to arbitrary arrays and can achieve an arbitrary desired beampattern, they cannot ensure that the mainlobe beamwidth and sidelobe levels can be precisely controlled according to the design specifications.

Next, two adaptive beamformers with null control capability are described to tackle the problem of moving interferences.

2.5.3 Beamformer Based on Covariance Matrix Taper (CMT)

Approach

The CMT based beamformers of [5, 72, 73] can be robust against moving interferences by artificially broadening the beampattern nulls in the interfering directions through a tapered sample array covariance matrix $\hat{\mathbf{R}}_T$ as

$$\hat{\mathbf{R}}_T = \mathbf{T} \odot \hat{\mathbf{R}}, \quad (2.53)$$

$$[\mathbf{T}]_{mn} = \frac{\sin(\xi(m-n))}{\xi(m-n)} \quad (2.54)$$

where $\mathbf{T}$ is a taper matrix, $\odot$ defines the Schur-Hadamard element-by-element matrix product, and ξ is a taper parameter that determines the null width. The CMT beamformer is obtained as

$$\mathbf{w} = \hat{\mathbf{R}}_T^{-1} \hat{\mathbf{s}}_0. \quad (2.55)$$

2.5.4 Beamformer With Data Dependent Derivative Constraints

Another different approach with the same objective of broadening the beampattern nulls in the interfering directions is by data dependent derivative constraints. Gershman *et al.* [27] have proposed to replace the sample array covariance matrix $\hat{\mathbf{R}}$ by a modified one, i.e.,

$$\hat{\mathbf{R}}_{\text{mod}} = \hat{\mathbf{R}} + \sum_{l=1}^L \varrho_l \mathbf{B}^l \hat{\mathbf{R}} \mathbf{B}^l \quad (2.56)$$

where $\mathbf{B}$ is a known diagonal matrix of antenna element coordinates, ϱ_l determines the tradeoff between the null depth and width, and L is the order of the constraints. An interesting relationship has been found in [96] that the matrix in (2.56) can be

represented as the Schur-Hadamard matrix product (2.53) with a particular taper matrix $\mathbf{T}$. Therefore, the data dependent derivative constraint method [27] can be viewed as a particular form of CMTs [26].

In the next chapter, the problem of large steering direction errors is looked into and an iterative solution is proposed.

Chapter 3

Beamforming Optimization Against Large Steering Direction Errors by an Iterative Approach

3.1 Objectives

Based on worst-case performance optimization, the robust adaptive beamformers of [1, 43, 52] are promising techniques against steering vector errors. The objective of the research in this chapter is to investigate the behaviour of these beamformers in the presence of large steering direction errors as compared to that with small steering direction errors. Since these beamformers of [1, 43, 52] are equivalent when the uncertainty set of the desired array steering vector is modelled as a sphere [1], the beamformer of Li *et al.* [1] is chosen as the representative in our study. It is studied in scenarios where the desired signal is either a dominant signal or a weak signal in the presence of small and large steering direction errors, respectively.

The findings of the study reveal that in the presence of large steering direction error, a larger uncertainty sphere is required to accommodate the increased error than if the steering direction error is small. Robustness of the beamformer of [1]

against large steering direction errors is obtained but at the expense of a reduced output SINR performance due to the degradation of its interference-plus-noise suppression ability.

The findings also show that there is room for improvement in terms of the output SINR performance of adaptive beamforming systems in the presence of large steering direction errors. Hence, the research in this chapter is directed to improve this aspect by proposing an IRMVB which uses a small uncertainty sphere (and a small flat ellipsoid) to search for the desired array steering vector iteratively. In this way, the interference-plus-noise suppression ability of the proposed beamformer can be preserved by preserving its DOFs and by using the corrected desired array steering vector, the proposed IRMVB achieves higher output SINR than the beamformers of [1–3, 43, 52]. Different from [97] which uses only one stopping criterion, the proposed IRMVB of this chapter applies two stopping criteria. Due to one of the stopping criteria, the calculated steering vector by the proposed IRMVB is not allowed to converge to the steering vectors of the interferences. This problem is not dealt with in [97]. Unlike [97], this chapter proposes another IRMVB which uses a small flat ellipsoid to search for the desired array steering vector iteratively. Theoretical analysis and simulation results show the effectiveness of the proposed method.

3.2 Numerical Study of the Beamformer of Li *et al.*

The beamformer of Li *et al.* [1] achieves robustness against steering vector errors by maximizing the array output power of the MV beamformer $P = \mathbf{s}^H \hat{\mathbf{R}}^{-1} \mathbf{s}$ or

equivalently, minimizing its denominator and solving

$$\min_{\mathbf{s}} \quad \mathbf{s}^H \hat{\mathbf{R}}^{-1} \mathbf{s} \quad (3.1a)$$

$$\text{s.t.} \quad \|\mathbf{s} - \bar{\mathbf{s}}_0\|^2 \leq \varepsilon_1 \quad (3.1b)$$

once where ε_1 is a user parameter.

Assume strong duality is achieved in (3.1), let $g \geq 0$ be the Lagrange multiplier that corresponds to the inequality constraint (3.1b). Due to the complementary slackness condition in the KKT conditions, either $g = 0$ and $\|\mathbf{s} - \bar{\mathbf{s}}_0\|^2 < \varepsilon_1$ will occur or $g > 0$ and $\|\mathbf{s} - \bar{\mathbf{s}}_0\|^2 = \varepsilon_1$ will occur. In the first case, the constraint (3.1b) is inactive and the optimal solution is the eigenvector of $\hat{\mathbf{R}}$ corresponding to its maximum eigenvalue provided that the presumed desired array steering vector is near to the true one, the interferences are well-separated from the mainlobe region, and that the desired signal is dominant. Otherwise, the constraint (3.1b) is active and the solution will occur at the boundary of the constraint set under the assumption that $\|\bar{\mathbf{s}}_0\|^2 > \varepsilon_1$. By applying the Lagrange multiplier methodology to (3.1),

$$l(g) = \mathbf{s}^H \hat{\mathbf{R}}^{-1} \mathbf{s} + g(\|\mathbf{s} - \bar{\mathbf{s}}_0\|^2 - \varepsilon_1) \quad (3.2)$$

is obtained. Setting the differentiation of (3.2) with respect to $\mathbf{s}$ to zero gives the calculated desired array steering vector as

$$\hat{\mathbf{s}}_0 = (g^{-1} \hat{\mathbf{R}}^{-1} + \mathbf{I})^{-1} \bar{\mathbf{s}}_0 \quad (3.3)$$

$$= \bar{\mathbf{s}}_0 - (\mathbf{I} + g \hat{\mathbf{R}})^{-1} \bar{\mathbf{s}}_0 \quad (3.4)$$

where g is obtained by replacing $\mathbf{s}$ with $\hat{\mathbf{s}}_0$ in $\|\mathbf{s} - \bar{\mathbf{s}}_0\|^2 = \varepsilon_1$ and solving

$$\|\hat{\mathbf{s}}_0 - \bar{\mathbf{s}}_0\|^2 \triangleq \|(\mathbf{I} + g\hat{\mathbf{R}})^{-1}\bar{\mathbf{s}}_0\|^2 = \varepsilon_1, \quad (3.5)$$

after which $\hat{\mathbf{s}}_0$ is found by (3.4) with the obtained g from (3.5).

Next, the output SINR performance of the beamformer of Li *et al.* [1] is studied in two scenarios with small (3°) and large (6°) steering direction errors, respectively. A ULA of 10 isotropic elements with a 0.5λ spacing is used where λ is the wavelength of the impinging signals. In both scenarios, two interferences arrive with DOAs and interference-to-noise ratios (INRs) of $[110^\circ, 20\text{dB}]$ and $[120^\circ, 20\text{dB}]$ in spatially white Gaussian noise of unit variance, respectively. In each scenario, there are two cases; in Case 1, the desired signal is a dominant signal with $\text{SNR} = 30\text{dB}$ and in Case 2, the desired signal is a weak signal with $\text{SNR} = 10\text{dB}$, respectively. In the first scenario, the desired signal arrives from $\theta_0 = 93^\circ$ but it is presumed to be from $\bar{\theta}_0 = 90^\circ$. Hence, there is a small steering direction error of 3° . The effect of the uncertainty level in the desired array steering vector on the output SINR performance of the beamformer of [1] is illustrated in Fig. 3.1.

Next, in the second scenario, all the parameters in the first scenario remain the same except that the desired signal now arrives from $\theta_0 = 96^\circ$, resulting in a large steering direction error of 6° . The effect of the uncertainty level in the desired array steering vector on the output SINR performance of the beamformer of [1] (in the second scenario) is shown in Fig. 3.2.

From Figs. 3.1 and 3.2, the output SINR performances of the beamformer of [1] are poor at low uncertainty values (ε) because the uncertainty sphere does not enclose the desired array steering vector. In Fig. 3.1 with small steering direction errors, if the desired signal is dominant (Case 1), the output SINR remains relatively constant after a certain ε is exceeded. This is because the optimization problem in the beamformer of [1] becomes a principal component analysis

Figure 3.1: Optimal SINRs and output SINRs of the beamformer of Li *et al.* [1] in Case 1 with dominant desired signal of SNR = 30dB and Case 2 with weak desired signal of SNR = 10dB in the presence of steering direction error of 3° . There are two interferences with DOAs and INRs of $[110^\circ, 20\text{dB}]$ and $[120^\circ, 20\text{dB}]$, respectively.

Figure 3.2: Optimal SINRs and output SINRs of the beamformer of Li *et al.* [1] in Case 1 with dominant desired signal of SNR = 30dB and Case 2 with weak desired signal of SNR = 10dB in the presence of steering direction error of 6° . There are two interferences with DOAs and INRs of $[110^\circ, 20\text{dB}]$ and $[120^\circ, 20\text{dB}]$, respectively.

one and the optimal calculated desired array steering vector $\hat{\mathbf{s}}_0$ is the eigenvector corresponding to the largest eigenvalue of $\hat{\mathbf{R}}$ [98]. Thus, the optimized solution is the desired array steering vector. In Fig. 3.2 with large steering direction errors, if the desired signal is dominant (Case 1), the output SINR is higher at larger ε values as they describe the uncertainty set of the desired array steering vector more adequately.

However, in Case 2, if the desired signal is weak, the output SINR performance of the beamformer of [1] is very much different from that when the desired signal is dominant. When the uncertainty sphere is larger than the optimal ε , the optimized solution may not be near to the desired array steering vector and some interference components may be included in the array output signal, thereby reducing the output SINR performance of the beamformer of [1]. Figs. 3.1 and 3.2 also reveal that when there are large steering direction errors, a larger uncertainty sphere should be chosen for the optimal output SINR performance of the beamformer of [1] in order to adequately describe the increased error. Robustness of [1] against large steering direction errors can be achieved by sacrificing some of its interference-plus-noise suppression ability which leads to a poorer output SINR performance of [1]. This is evident from the different optimal output SINRs of the beamformer of [1] in the two scenarios for the respective two cases.

In essence, the selection of ε is critical when the desired signal is weak and it should be as small as possible in order to preserve the interference-plus-noise suppression ability of the beamformer of [1] while being adequately large to describe the amount of uncertainty in the desired array steering vector.

Figure 3.3: Concept of the proposed IRMVB using a small red sphere of size ϵ_2 to search for the desired array steering vector $\mathbf{s}_0$ iteratively, in the presence of steering direction error. The desired array steering vector $\mathbf{s}_0$ corresponds to an angle of θ_0 while the presumed one is $\bar{\mathbf{s}}_0$ which corresponds to an angle of $\bar{\theta}_0$. On the other hand, the beamformer of Li *et al.* [1] uses a much bigger green sphere of size ϵ_1 to find $\mathbf{s}_0$ in one step.

3.3 Proposed Iterative Robust Minimum Variance Beamformer (IRMVB)

In practice, signal source movement, antenna array motion, etc., can result in a large error or uncertainty in the desired array steering vector [14, 26]. In order to enhance the output SINRs of adaptive beamformers in the presence of large steering direction errors, an IRMVB is proposed which uses a smaller uncertainty sphere (and a smaller flat ellipsoid) than that used by [1] to search for the desired array steering vector iteratively.

3.3.1 Spherical Uncertainty Set

The concept of the proposed IRMVB (with spherical uncertainty set) is shown in Fig. 3.3. When there is a steering direction error, the desired array steering vector $\mathbf{s}_0$ (corresponding to the desired signal direction θ_0) and the presumed one

$\bar{\mathbf{s}}_0$ (corresponding to the presumed desired signal direction $\bar{\theta}_0$) do not coincide. If this error is large, the uncertainty sphere (green sphere) of size ε_1 used in the beamformer of [1] has to be large. This consumes the DOFs of the beamformer of [1] and weakens its interference-plus-noise suppression ability, resulting in more interference and noise components being included in the beamformer output $y(k)$ which, in turn, reduces the output SINR of [1].

In order to overcome this problem, the proposed IRMVB uses a small uncertainty sphere (red sphere) of size ε_2 (where $\varepsilon_2 \ll \varepsilon_1$) to adjust the steering vector from $\bar{\mathbf{s}}_0$ to approach $\mathbf{s}_0$. This is done by imposing the constraint (3.1b) (with ε_2 in place of ε_1) centred at the presumed desired array steering vector $\bar{\mathbf{s}}_0$ at the first iteration, i.e., $\|\mathbf{s} - \bar{\mathbf{s}}_0\|^2 = \varepsilon_2$ and solving for the corrected desired array steering vector. After each iteration, the calculated steering vector by the proposed IRMVB is scaled so that it has a norm of $\sqrt{N}$ to prevent scaling ambiguity [1]. Again, the spherical constraint is imposed but centred at the calculated steering vector of the previous iteration of the proposed IRMVB to solve for the next steering vector. This process is repeated until the desired array steering vector is reached. This can be achieved by using the proposed stopping criteria in (3.38) introduced later in Section 3.3.3. The proposed IRMVB weight can then be obtained by using the converged steering vector to replace $\bar{\mathbf{s}}_0$ in (2.27).

It should be noted that Li *et al.* [50] suggest that ε_1 can be chosen as $\varepsilon_1 = \min_{\psi} \|\mathbf{s}_0 e^{j\psi} - \bar{\mathbf{s}}_0\|^2$ where ψ is a phase. However, in practice, $\mathbf{s}_0$ is unknown and is replaced by $\mathbf{s}(\bar{\theta}_0 \pm \Delta\theta)$ for the estimation of ε_1 where $\Delta\theta$ is the DOA uncertainty range of the desired signal. Thus, such a ε_1 choice is likely to be sub-optimal; the over-estimation or under-estimation of the optimal ε_1 can result in a degraded output SINR of the beamformer of Li *et al.* [1]. In contrast, the proposed method makes use of this coarse ε_1 estimate and choose a much smaller ε_2 to search for the desired array steering vector. As presented later in Section 3.4.3, the simulation

results show that the output SINR of the proposed IRMVB is insensitive to a wide range of ε_2 . The ε_2 used are smaller than the coarse ε_1 estimation for all the respective simulation scenarios.

The proposed IRMVB algorithm (with spherical uncertainty set) is summarized. Let the steering vector determined at the i th iteration of the proposed algorithm and the corresponding Lagrange multiplier be $\hat{\mathbf{s}}_0^i$ and g^i , respectively.

1. At $i = 0$, initialize $\hat{\mathbf{s}}_0^0 = \bar{\mathbf{s}}_0$.
2. When $i \geq 1$, solve equation (3.5) with $\hat{\mathbf{s}}_0^{i-1}$ and ε_2 (instead of $\bar{\mathbf{s}}_0$ and ε_1 , respectively) to obtain g^i . Find $\hat{\mathbf{s}}_0^i$ by (3.4) with the obtained g^i and $\hat{\mathbf{s}}_0^{i-1}$ (instead of g and $\bar{\mathbf{s}}_0$, respectively). Obtain $\hat{\mathbf{s}}_0^i = \sqrt{N}\hat{\mathbf{s}}_0^i/\|\hat{\mathbf{s}}_0^i\|$ so that $\|\hat{\mathbf{s}}_0^i\| = \sqrt{N}$. The value $\varepsilon_2 = 0.1$ is used; this will be discussed later in Section 3.4 (Simulation Results and Discussion).
3. Check if the stopping criteria are reached. The stopping criteria in (3.38) are discussed in Section 3.3.3 (Design of the Stopping Criteria). If they are satisfied, go to step (4). If they are not satisfied, assign $\hat{\mathbf{s}}_0^i$ to $\hat{\mathbf{s}}_0^i$ and repeat step (2).
4. Use the converged $\hat{\mathbf{s}}_0^{i-1}$ to replace $\bar{\mathbf{s}}_0$ in the MV beamformer (2.27) to obtain the proposed IRMVB weight.

The proposed IRMVB works by searching for a steering vector at each iteration to approach the desired array steering vector. The theoretical result of this chapter (Theorem 1) shows that the proposed IRMVB (with spherical uncertainty set) can increase the output SINR with each iteration. This is achieved because the generalized angle $\hat{\theta}$ between the calculated steering vector of the proposed IRMVB and the desired array steering vector is reduced with each iteration. In order to see this, Lemma 1 is required [99, 100].

Lemma 1. *In the presence of steering vector errors ($\bar{\mathbf{s}}_0$ is used instead of $\mathbf{s}_0$) and assuming that the theoretical array covariance matrix $\mathbf{R}$ is available, the output SINR of the MV beamformer, i.e., $\mathbf{w} = (\bar{\mathbf{s}}_0^H \mathbf{R}^{-1} \bar{\mathbf{s}}_0)^{-1} \mathbf{R}^{-1} \bar{\mathbf{s}}_0$ is*

$$\text{SINR}_o = \frac{\text{SINR}_{\text{opt}} \cos^2(\hat{\theta}; \mathbf{R}_{\text{in}}^{-1})}{1 + \sin^2(\hat{\theta}; \mathbf{R}_{\text{in}}^{-1})[2\text{SINR}_{\text{opt}} + \text{SINR}_{\text{opt}}^2]} \quad (3.6)$$

where SINR_{opt} is the optimal SINR given in (2.24). Here, $\hat{\theta}$ is the generalized angle between the presumed desired array steering vector $\bar{\mathbf{s}}_0$ and the true one $\mathbf{s}_0$; the cosine-squared of which is given by

$$\cos^2(\hat{\theta}; \mathbf{R}_{\text{in}}^{-1}) = \frac{|\bar{\mathbf{s}}_0^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0|^2}{\|\bar{\mathbf{s}}_0\|_{\mathbf{R}}^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2} \quad (3.7)$$

in the space $\mathcal{H}(\mathbf{R}_{\text{in}}^{-1})$ defined by the inner product between $\bar{\mathbf{s}}_0$ and $\mathbf{s}_0$, i.e., $\bar{\mathbf{s}}_0^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0$. It is appropriate to mention that $0 \leq \cos^2(\hat{\theta}; \mathbf{R}_{\text{in}}^{-1}) \leq 1$ due to Schwarz inequality. $\|\mathbf{x}\|_{\mathbf{R}}^2 = \mathbf{x}^H \mathbf{R}_{\text{in}}^{-1} \mathbf{x}$ is the extended vector norm-squared, i.e., the length of $\mathbf{x}$ in $\mathcal{H}(\mathbf{R}_{\text{in}}^{-1})$ is $(\mathbf{x}^H \mathbf{R}_{\text{in}}^{-1} \mathbf{x})^{1/2}$ where $\mathbf{x} \in \mathbb{C}^N$. When $\cos^2(\hat{\theta}; \mathbf{R}_{\text{in}}^{-1}) = 0$, it means that the vectors $\bar{\mathbf{s}}_0$ and $\mathbf{s}_0$ are orthogonal in $\mathcal{H}(\mathbf{R}_{\text{in}}^{-1})$; when $\cos^2(\hat{\theta}; \mathbf{R}_{\text{in}}^{-1}) = 1$, it means that the vectors $\bar{\mathbf{s}}_0$ and $\mathbf{s}_0$ are aligned perfectly in that one is a scalar multiple of the other.

One important observation from Lemma 1 is that (3.6) is a monotonically increasing function of $\cos^2(\hat{\theta}; \mathbf{R}_{\text{in}}^{-1})$.

Theorem 1. *Let the steering vector found by the proposed IRMVB at the i th iteration be $\hat{\mathbf{s}}_0^i$ and its scaled version be $\hat{\mathbf{s}}_0^i$ (to prevent scaling ambiguity). Let the generalized angle between $\hat{\mathbf{s}}_0^{i-1}$ and $\mathbf{s}_0$ be $\hat{\theta}^{i-1}$ and that between $\hat{\mathbf{s}}_0^i$ and $\mathbf{s}_0$ be $\hat{\theta}^i$, respectively. Assuming that the interferences are not located near the protected mainlobe region, this chapter shows that*

$$\cos^2(\hat{\theta}^i; \mathbf{R}_{\text{in}}^{-1}) = \frac{|\hat{\mathbf{s}}_0^{iH} \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0|^2}{\|\hat{\mathbf{s}}_0^i\|_{\mathbf{R}}^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2} \geq \frac{|\hat{\mathbf{s}}_0^{(i-1)H} \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0|^2}{\|\hat{\mathbf{s}}_0^{i-1}\|_{\mathbf{R}}^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2} = \cos^2(\hat{\theta}^{i-1}; \mathbf{R}_{\text{in}}^{-1}) \quad (3.8)$$

which means that the generalized angle between the calculated steering vector of the proposed IRMVB and the true one is reduced with each iteration, thereby increasing the output SINR of the IRMVB.

Proof: Assume that the theoretical array covariance matrix $\mathbf{R}$ is used in place of $\hat{\mathbf{R}}$. Initialize $\hat{\mathbf{s}}_0^0 = \bar{\mathbf{s}}_0$ and at the 1st iteration ($i = 1$), $\hat{\mathbf{s}}_0^1 = \mathbf{s}_1$. As noted in [1], normalization is done to remove the scaling ambiguity, thus $\mathbf{s}'_1 = \sqrt{N}\mathbf{s}_1/\|\mathbf{s}_1\|$ so that $\|\mathbf{s}'_1\| = \sqrt{N}$. Let the generalized angle between $\bar{\mathbf{s}}_0$ and $\mathbf{s}_0$ be $\hat{\theta}^0$ and that between $\mathbf{s}_1$ and $\mathbf{s}_0$ be $\hat{\theta}^1$. The following sets out to prove that

$$\cos^2(\hat{\theta}^1; \mathbf{R}_{\text{in}}^{-1}) = \frac{|\mathbf{s}'_1{}^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0|^2}{\|\mathbf{s}'_1\|_{\mathbf{R}}^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2} \geq \frac{|\bar{\mathbf{s}}_0{}^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0|^2}{\|\bar{\mathbf{s}}_0\|_{\mathbf{R}}^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2} = \cos^2(\hat{\theta}^0; \mathbf{R}_{\text{in}}^{-1}) \quad (3.9)$$

where the output SINR in (3.6) is a monotonically increasing function of $\cos^2(\hat{\theta}; \mathbf{R}_{\text{in}}^{-1})$, meaning that the proposed IRMVB increases the output SINR after one iteration.

Eigen-decomposition is applied on $\mathbf{R}$ in (3.3), so

$$\mathbf{s}_1 = \mathbf{U} \begin{bmatrix} \frac{g\lambda_1}{1+g\lambda_1} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \frac{g\lambda_N}{1+g\lambda_N} \end{bmatrix} \mathbf{U}^H \bar{\mathbf{s}}_0 \quad (3.10)$$

where the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{R}$ and λ_k where $k = 1, \dots, N$ are the corresponding eigenvalues.

In the presence of K strong signals and spatially white Gaussian noise, $\lambda_1 \geq \lambda_2 \geq \dots \lambda_K \gg \lambda_{K+1} = \dots = \lambda_N$, $\frac{g\lambda_k}{1+g\lambda_k} \approx 1$ for $k = 1, 2, \dots, K$ and $\frac{g\lambda_k}{1+g\lambda_k} \approx 0$ for $k = K+1, K+2, \dots, N$. Thus, (3.10) can be approximated as

$$\mathbf{s}_1 \approx \mathbf{U}_s \mathbf{U}_s^H \bar{\mathbf{s}}_0 \quad (3.11)$$

where the columns of $\mathbf{U}_s$ are the eigenvectors corresponding to the largest K eigenvalues of $\mathbf{R}$. The desired array steering vector $\mathbf{s}_0$ is spanned by $\mathbf{U}_s$, i.e.,

$\mathbf{s}_0 = \mathbf{U}_s \mathbf{c}$ where $\mathbf{c}$ is a coefficient vector. Since the space of the interferences is a subspace of $\mathbf{U}_s$, the interference-plus-noise covariance matrix $\mathbf{R}_{\text{in}}$ can be expressed as [100]

$$\mathbf{R}_{\text{in}} = \mathbf{R}_i + \sigma_n^2 \mathbf{I} = [\mathbf{U}_s \quad \mathbf{U}_n] \begin{bmatrix} \mathbf{A}_i & \mathbf{0} \\ \mathbf{0} & \sigma_n^2 \mathbf{I} \end{bmatrix} [\mathbf{U}_s \quad \mathbf{U}_n]^H \quad (3.12)$$

where $\mathbf{R}_i$ is the interference covariance matrix, the $K \times K$ matrix $\mathbf{A}_i$ may not necessarily be diagonal [100], σ_n^2 is the variance of the white noise, and the columns of $\mathbf{U}_n$ are the eigenvectors corresponding to the smallest $N - K$ eigenvalues of $\mathbf{R}$. Working on the left hand side (LHS) of the inequality of (3.9),

$$\mathbf{s}_1'^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0 = \mathbf{s}_1'^H [\mathbf{U}_s \quad \mathbf{U}_n] \begin{bmatrix} \mathbf{A}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \sigma_n^{-2} \mathbf{I} \end{bmatrix} [\mathbf{U}_s \quad \mathbf{U}_n]^H \mathbf{U}_s \mathbf{c} \quad (3.13)$$

$$= \frac{\sqrt{N} \bar{\mathbf{s}}_0^H \mathbf{U}_s}{\|\mathbf{U}_s \mathbf{U}_s^H \bar{\mathbf{s}}_0\|} [\mathbf{I} \quad \mathbf{0}] \begin{bmatrix} \mathbf{A}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \sigma_n^{-2} \mathbf{I} \end{bmatrix} [\mathbf{I} \quad \mathbf{0}]^H \mathbf{c} \quad (3.14)$$

$$= \frac{\sqrt{N} \bar{\mathbf{s}}_0^H \mathbf{U}_s \mathbf{A}_i^{-1} \mathbf{c}}{\|\mathbf{U}_s \mathbf{U}_s^H \bar{\mathbf{s}}_0\|} \quad (3.15)$$

and

$$\|\mathbf{s}_1'\|_{\mathbf{R}}^2 = \mathbf{s}_1'^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_1' \quad (3.16)$$

$$= \frac{N \bar{\mathbf{s}}_0^H \mathbf{U}_s \mathbf{A}_i^{-1} \mathbf{U}_s^H \bar{\mathbf{s}}_0}{\|\mathbf{U}_s \mathbf{U}_s^H \bar{\mathbf{s}}_0\|^2}. \quad (3.17)$$

Finally, the LHS of the inequality of (3.9) is

$$\frac{|\mathbf{s}_1'^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0|^2}{\|\mathbf{s}_1'\|_{\mathbf{R}}^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2} = \frac{|\bar{\mathbf{s}}_0^H \mathbf{U}_s \mathbf{A}_i^{-1} \mathbf{c}|^2}{(\bar{\mathbf{s}}_0^H \mathbf{U}_s \mathbf{A}_i^{-1} \mathbf{U}_s^H \bar{\mathbf{s}}_0) \|\mathbf{s}_0\|_{\mathbf{R}}^2}. \quad (3.18)$$

Working on the right hand side (RHS) of the inequality of (3.9),

$$\bar{\mathbf{s}}_0^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0 = \bar{\mathbf{s}}_0^H \begin{bmatrix} \mathbf{U}_s & \mathbf{U}_n \end{bmatrix} \begin{bmatrix} \mathbf{A}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \sigma_n^{-2} \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{U}_s & \mathbf{U}_n \end{bmatrix}^H \mathbf{U}_s \mathbf{c} \quad (3.19)$$

$$= \bar{\mathbf{s}}_0^H \mathbf{U}_s \mathbf{A}_i^{-1} \mathbf{c} \quad (3.20)$$

and

$$\|\bar{\mathbf{s}}_0\|_{\mathbf{R}}^2 = \bar{\mathbf{s}}_0^H \mathbf{R}_{\text{in}}^{-1} \bar{\mathbf{s}}_0 \quad (3.21)$$

$$= \bar{\mathbf{s}}_0^H \mathbf{U}_s \mathbf{A}_i^{-1} \mathbf{U}_s^H \bar{\mathbf{s}}_0 + \sigma_n^{-2} \bar{\mathbf{s}}_0^H \mathbf{U}_n \mathbf{U}_n^H \bar{\mathbf{s}}_0. \quad (3.22)$$

Finally, the RHS of the inequality of (3.9) is

$$\frac{|\bar{\mathbf{s}}_0^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0|^2}{\|\bar{\mathbf{s}}_0\|_{\mathbf{R}}^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2} = \frac{|\bar{\mathbf{s}}_0^H \mathbf{U}_s \mathbf{A}_i^{-1} \mathbf{c}|^2}{(\bar{\mathbf{s}}_0^H \mathbf{U}_s \mathbf{A}_i^{-1} \mathbf{U}_s^H \bar{\mathbf{s}}_0) \|\mathbf{s}_0\|_{\mathbf{R}}^2 + \sigma_n^{-2} \|\mathbf{U}_n^H \bar{\mathbf{s}}_0\|^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2}. \quad (3.23)$$

Since $\sigma_n^{-2} \|\mathbf{U}_n^H \bar{\mathbf{s}}_0\|^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2 \geq 0$,

$$\cos^2(\hat{\theta}^1; \mathbf{R}_{\text{in}}^{-1}) = \frac{|\mathbf{s}_1'^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0|^2}{\|\mathbf{s}_1'\|_{\mathbf{R}}^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2} \geq \frac{|\bar{\mathbf{s}}_0^H \mathbf{R}_{\text{in}}^{-1} \mathbf{s}_0|^2}{\|\bar{\mathbf{s}}_0\|_{\mathbf{R}}^2 \|\mathbf{s}_0\|_{\mathbf{R}}^2} = \cos^2(\hat{\theta}^0; \mathbf{R}_{\text{in}}^{-1}), \quad (3.24)$$

this shows that the proposed IRMVB does improve the output SINR after one iteration as the generalized angle between the calculated steering vector and the desired array steering vector is reduced. For subsequent iterations ($i > 1$), similar reasoning applies for $\hat{\mathbf{s}}_0^i$ and $\hat{\mathbf{s}}_0^{i-1}$. Therefore, Theorem 1, i.e., $\cos^2(\hat{\theta}^i; \mathbf{R}_{\text{in}}^{-1}) \geq \cos^2(\hat{\theta}^{i-1}; \mathbf{R}_{\text{in}}^{-1})$ is arrived, indicating that the IRMVB reduces the generalized angle between its calculated steering vector and the desired array steering vector with each iteration, thereby increasing its output SINR. $\square$

Next, the proposed IRMVB which uses a small flat ellipsoid to search for the desired array steering vector iteratively is presented before the design of the stopping criteria is discussed.

3.3.2 Flat Ellipsoidal Uncertainty Set

As considered in [1, 52], if there is prior information, the uncertainty set of the desired array steering vector can be made tighter by modelling it as a flat ellipsoid, i.e., $\mathbf{s} = \mathbf{B}\mathbf{u} + \bar{\mathbf{s}}_0$ where $\mathbf{B}$ is a $N \times L$ matrix with full column rank ($L < N$) and $\mathbf{u}$ is a $L \times 1$ vector. The beamformer of Li *et al.* [1] solves

$$\min_{\mathbf{u}} (\mathbf{B}\mathbf{u} + \bar{\mathbf{s}}_0)^H \hat{\mathbf{R}}^{-1} (\mathbf{B}\mathbf{u} + \bar{\mathbf{s}}_0) \quad (3.25a)$$

$$\text{s.t.} \quad \|\mathbf{u}\| \leq (\varepsilon_1^f)^{1/2} \quad (3.25b)$$

once with $(\varepsilon_1^f)^{1/2} = 1$ where the superscript “f” denotes the case for flat ellipsoidal constraint. In contrast, the proposed IRMVB solves (3.25) iteratively using $(\varepsilon_2^f)^{1/2} \ll 1$ in place of $(\varepsilon_1^f)^{1/2}$. Let

$$\check{\mathbf{R}} = \mathbf{B}^H \hat{\mathbf{R}}^{-1} \mathbf{B}, \quad \check{\mathbf{s}}_0 = \mathbf{B}^H \hat{\mathbf{R}}^{-1} \bar{\mathbf{s}}_0, \quad (3.26)$$

and applying the Lagrange multiplier methodology to (3.25),

$$\check{l}(\check{g}) = \mathbf{u}^H \check{\mathbf{R}} \mathbf{u} + \check{\mathbf{s}}_0^H \mathbf{u} + \mathbf{u}^H \check{\mathbf{s}}_0 + \check{g}(\mathbf{u}^H \mathbf{u} - \varepsilon_1^f) \quad (3.27)$$

where $\check{g} \geq 0$ is the Lagrange multiplier. Setting the differentiation of (3.27) with respect to $\mathbf{u}$ to zero gives

$$\check{\mathbf{u}} = -(\check{\mathbf{R}} + \check{g}\mathbf{I})^{-1} \check{\mathbf{s}}_0. \quad (3.28)$$

If $\|\check{\mathbf{R}}^{-1} \check{\mathbf{s}}_0\|^2 > \varepsilon_1^f$, $\check{g} > 0$ is the root of the constraint equation

$$\|\check{\mathbf{u}}\|^2 \triangleq \|(\check{\mathbf{R}} + \check{g}\mathbf{I})^{-1} \check{\mathbf{s}}_0\|^2 = \varepsilon_1^f. \quad (3.29)$$

Otherwise, $\check{g} = 0$. With the obtained $\check{g}$ in $\check{\mathbf{u}}$ of (3.28), the calculated desired array steering vector is

$$\check{\mathbf{s}}_0 = \mathbf{B}\check{\mathbf{u}} + \bar{\mathbf{s}}_0. \quad (3.30)$$

The proposed IRMVB algorithm (with flat ellipsoidal uncertainty set) is summarized. Let the steering vector determined at the i th iteration of the proposed algorithm and the corresponding Lagrange multiplier be $\check{\mathbf{s}}_0^i$ and $\check{g}^i$, respectively. Given $\mathbf{B}$, let $\check{\mathbf{s}}_0^i = \mathbf{B}^H \hat{\mathbf{R}}^{-1} \bar{\mathbf{s}}_0^i$ where $\bar{\mathbf{s}}_0^i$ is the updated presumed desired array steering vector at the i th iteration of the proposed algorithm.

1. At $i = 0$, initialize $\bar{\mathbf{s}}_0^0 = \bar{\mathbf{s}}_0$ and $\check{\mathbf{s}}_0^0 = \mathbf{B}^H \hat{\mathbf{R}}^{-1} \bar{\mathbf{s}}_0^0$.
2. When $i \geq 1$, if $\|\hat{\mathbf{R}}^{-1} \check{\mathbf{s}}_0^{i-1}\|^2 \leq \varepsilon_2^f$, set $\check{g}^i = 0$. If $\|\hat{\mathbf{R}}^{-1} \check{\mathbf{s}}_0^{i-1}\|^2 > \varepsilon_2^f$, solve equation (3.29) with $\check{\mathbf{s}}_0^{i-1}$ and ε_2^f (instead of $\check{\mathbf{s}}_0$ and ε_1^f , respectively) to obtain $\check{g}^i$. Calculate $\check{\mathbf{u}}^i$ by (3.28) using $\check{\mathbf{s}}_0^{i-1}$ and $\check{g}^i$ (instead of $\check{\mathbf{s}}_0$ and $\check{g}$, respectively). Next, obtain $\check{\mathbf{s}}_0^i$ by (3.30) with $\check{\mathbf{u}}^i$ and $\bar{\mathbf{s}}_0^{i-1}$ (instead of $\check{\mathbf{u}}$ and $\bar{\mathbf{s}}_0$, respectively). Obtain $\check{\mathbf{s}}_0^i = \sqrt{N} \check{\mathbf{s}}_0^i / \|\check{\mathbf{s}}_0^i\|$ so that $\|\check{\mathbf{s}}_0^i\| = \sqrt{N}$. The value of $\varepsilon_2^f = 0.1$ is set (same as ε_2).
3. Check if the stopping criteria in (3.38) are reached. If they are satisfied, go to step (4). If they are not satisfied, assign $\check{\mathbf{s}}_0^i$ to $\bar{\mathbf{s}}_0^i$ to update the presumed desired array steering vector, calculate $\check{\mathbf{s}}_0^i = \mathbf{B}^H \hat{\mathbf{R}}^{-1} \bar{\mathbf{s}}_0^i$ and repeat step (2).
4. Use the converged $\check{\mathbf{s}}_0^{i-1}$ to replace $\bar{\mathbf{s}}_0$ in the MV beamformer (2.27) to obtain the proposed IRMVB weight.

3.3.3 Design of the Stopping Criteria

If the iterative process is allowed to continue and if the interference power σ_i^2 is higher than the desired signal power σ_0^2 , i.e.,

$$\sigma_i^2 = \frac{1}{\mathbf{s}^H(\theta_i) \hat{\mathbf{R}}^{-1} \mathbf{s}(\theta_i)} > \frac{1}{\mathbf{s}^H(\theta_0) \hat{\mathbf{R}}^{-1} \mathbf{s}(\theta_0)} = \sigma_0^2 \quad (3.31)$$

where $\theta_i \in \Theta_i$ (the set of the interferences' DOAs), then the output SINR will eventually decrease as the calculated steering vector converges to the steering vectors of the interferences. This is because the proposed IRMVB seeks to find a solution to minimize its objective function and

$$\mathbf{s}^H(\theta_i) \hat{\mathbf{R}}^{-1} \mathbf{s}(\theta_i) < \mathbf{s}^H(\theta_0) \hat{\mathbf{R}}^{-1} \mathbf{s}(\theta_0) \quad (3.32)$$

is achieved when the calculated steering vector by the proposed IRMVB converges to the steering vectors of the interferences. Thus, stopping criteria are required to interrupt the iterative algorithm once the desired array steering vector is reached.

In order to help shed light on the design of the stopping criteria in the proposed IRMVB, the sensitivity analysis in optimization problems is used [63]. Consider the standard optimization problem subject to perturbations a, b :

$$\min_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad p(\mathbf{x}) \leq a, \quad q(\mathbf{x}) = b \quad (3.33)$$

where $f(\cdot)$, $p(\cdot)$, and $q(\cdot)$ are functions and the variable $\mathbf{x} \in \mathbb{C}^N$. This corresponds to the standard optimization problem when $a = 0$ and $b = 0$, i.e., no perturbation. The Lagrangian associated with the standard optimization problem is

$$L(\alpha, \beta) = f(\mathbf{x}) + \alpha p(\mathbf{x}) + \beta q(\mathbf{x}) \quad (3.34)$$

where α (≥ 0) and β are the dual variables or Lagrange multipliers. If there is strong duality and dual optimum is achieved, let (α^*, β^*) be optimal for the dual of the standard optimization problem, then, for all a and b ,

$$h^*(a, b) \geq h^*(0, 0) - \alpha^* a - \beta^* b \quad (3.35)$$

where $h^*(a, b)$ is the optimal value of the perturbed problem. Assuming $h^*(a, b)$ is differentiable at $a = 0$, $b = 0$, the optimal Lagrange multipliers α^* and β^* are related to the gradient of h^* at $a = 0$, $b = 0$ as

$$\alpha^* = -\frac{\partial h^*(0, 0)}{\partial a}, \quad \beta^* = -\frac{\partial h^*(0, 0)}{\partial b}. \quad (3.36)$$

The optimal α^* and β^* are the local sensitivities of the optimal value $h^*(0, 0)$ with respect to the constraint perturbations. In other words, they are measures of how active the constraints are at the optimal $\mathbf{x}^*$. Suppose $p(\mathbf{x}^*) = 0$, then the inequality constraint is active and α^* indicates how active this constraint is. If α^* is large, the impact on the optimal value is large even if the constraint is tightened or relaxed slightly. If α^* is small, the constraint can be tightened or relaxed slightly without much impact on the optimal value.

The previous analysis can be used to design the stopping criteria of the IRMVB. There is one inequality constraint in (3.1) and (3.25). Without loss of generality, (3.1) (with spherical uncertainty set) is used for subsequent discussion. At i th iteration of the proposed IRMVB algorithm, if the Lagrange multiplier g^i is large, it suggests that the objective function value may be decreased further since a slight adjustment in the constraint will have a large impact on the optimal value. In contrast, if g^i is small, it suggests that the optimal value is reached as a slight adjustment in the constraint will not impact the optimal value much.

Further insights can be gained from considering the $\mathbf{s}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{s}(\theta)$ spectrum

(reciprocal of the power spectrum). At high desired signal's SNRs, a clear minimum/trough is formed at the desired signal direction. The proposed IRMVB starts at the presumed desired array steering vector. By minimizing the objective function, the steering vectors calculated by the proposed IRMVB approach the desired array steering vector $\mathbf{s}_0$ iteratively and the Lagrange multiplier g is decreased at each iteration. When the desired array steering vector is reached, this minimum of $\mathbf{s}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{s}(\theta)$ is reached and the corresponding Lagrange multiplier g is very small. As the proposed IRMVB uses a small sphere, the current steering vector obtained by the proposed IRMVB is very near to that of the next iteration. Again, the next Lagrange multiplier g will also be very small. Therefore, the stopping of the proposed algorithm is triggered once $|g^i - g^{i-1}| \leq \delta$ is satisfied where δ is a threshold to determine that the difference between two consecutively obtained g values is small enough such that the two corresponding calculated $\hat{\mathbf{s}}_0^i$'s are very near to $\mathbf{s}_0$.

On the contrary, at low desired signal's SNRs, the previous stopping criterion does not work well as the $\mathbf{s}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{s}(\theta)$ spectrum does not show a clear minimum/trough at the desired signal direction and there are other minimums/troughs corresponding to the interferences with higher powers. Hence, the Lagrange multiplier can be large when the proposed IRMVB reaches the desired array steering vector because there seems no trough/minimum corresponding to the desired signal and that the optimal value can be further reduced with more iterations. If the iterative algorithm continues, the calculated steering vector will converge to the steering vectors of the interferences instead and the output SINR will be very poor. Hence, an additional stopping criterion is needed. The algorithm can be stopped when the inner product between $\hat{\mathbf{s}}_0^i$ calculated by the proposed IRMVB at the i th iteration and $\bar{\mathbf{s}}_0$ is equal or less than the inner product between $\mathbf{s}_t$ and $\bar{\mathbf{s}}_0$ (presumed desired array steering vector). $\mathbf{s}_t$ is a steering vector that corresponds to an

angle of $\bar{\theta}_0 + \Delta\theta$ or $\bar{\theta}_0 - \Delta\theta$, whichever results in a smaller inner product with $\bar{\mathbf{s}}_0$. $\Delta\theta$ is related to the DOA uncertainty range of the desired signal and it is usually given as a system design requirement for a specific application [64]. For example, in wireless communications, a coarse knowledge of the beamforming scenario can be available from field trials or measurement campaigns [2] and information such as the average mobility rate of the mobile users can be exploited to obtain $\Delta\theta$. It is assumed that no interference will arrive in the DOA uncertainty region of the desired signal $[\bar{\theta}_0 - \Delta\theta, \bar{\theta}_0 + \Delta\theta]$ where the desired signal is expected to arrive from. Note that the DOA uncertainty range of the desired signal is also used to implement the beamformers of [3, 4]. The second stopping criterion prevents the angle θ^i between $\hat{\mathbf{s}}_0^i$ calculated by the proposed IRMVB and $\bar{\mathbf{s}}_0$ from exceeding $\Delta\theta$ where $0^\circ \leq \Delta\theta \ll 90^\circ$, i.e.,

$$\cos \theta^i = \frac{|\hat{\mathbf{s}}_0^{iH} \bar{\mathbf{s}}_0|}{|\hat{\mathbf{s}}_0^i| |\bar{\mathbf{s}}_0|} > \frac{|\mathbf{s}_t^H \bar{\mathbf{s}}_0|}{|\mathbf{s}_t| |\bar{\mathbf{s}}_0|} = \min\{\cos(\bar{\theta}_0 \pm \Delta\theta)\}. \quad (3.37)$$

Note that the θ^i used in the second stopping criterion differs from the generalized angle $\hat{\theta}^i$ of (3.7). The former (θ^i) is the angle between the calculated steering vector of the proposed IRMVB $\hat{\mathbf{s}}_0^i$ and the *presumed* desired array steering vector while the latter ($\hat{\theta}^i$ of (3.7)) is the angle between $\hat{\mathbf{s}}_0^i$ and the *true* desired array steering vector.

The second stopping criterion can be used when the desired array steering vector suffers from steering direction error or array calibration errors or both. For brevity sake, Fig. 3.4 illustrates the concept of the second stopping criterion for the case of array calibration errors only where the desired array steering vector lies in the green sphere of size ε_1 centred at the presumed one $\bar{\mathbf{s}}_0$. Although $\Delta\theta$ is not directly related to the non-angular distortions of the desired array steering vector (which is difficult to estimate in practice), the second stopping criterion ensures that at low SNRs of the desired signal, the calculated steering vector by

Figure 3.4: Concept of the second stopping criterion in the proposed IRMVB in the presence of array calibration errors only. Due to calibration errors, the desired array steering vector is located in the green sphere of size ε_1 centred at the presumed one $\bar{\mathbf{s}}_0$. The proposed IRMVB uses a small red sphere of size ε_2 to search for $\mathbf{s}_0$ iteratively. At low SNRs of the desired signal, the second stopping criterion prevents the steering vectors calculated by the proposed IRMVB from going out of the dotted cone which describes the DOA uncertainty region of the desired signal $[\theta_0 - \Delta\theta, \theta_0 + \Delta\theta]$.

the proposed IRMVB does not go out of the dotted cone in Fig. 3.4 which describes the DOA uncertainty region of the desired signal $[\theta_0 - \Delta\theta, \theta_0 + \Delta\theta]$. In this way, the final converged steering vector by the proposed IRMVB can be ensured to be still near to the desired array steering vector. In the same case at high desired signal's SNRs, the first stopping criterion in (3.38) is in effect.

In summary, the proposed IRMVB stops upon reaching

$$(g^{i-1} < 1 \text{ and } |g^i - g^{i-1}| \leq \delta) \text{ or } \left(\frac{|\hat{\mathbf{s}}_0^{iH} \bar{\mathbf{s}}_0|}{|\hat{\mathbf{s}}_0^i| |\bar{\mathbf{s}}_0|} \leq \frac{|\mathbf{s}_t^H \bar{\mathbf{s}}_0|}{|\mathbf{s}_t| |\bar{\mathbf{s}}_0|} \right). \quad (3.38)$$

The first stopping criterion is triggered only when the Lagrange multiplier g starts to become small, i.e., $g^{i-1} < 1$. At high SNRs of the desired signal, the first stopping criterion is expected to be triggered before the second one. Note that the knowledge of the desired signal's SNR is not required to implement the proposed IRMVB and it can be easily implemented by solving (3.5) or (3.29) with a Newton's

method (similar to [1]) while [4, 43, 53, 91, 101] require specialized IPM solvers, e.g., [88]. From the simulation results, the IRMVB always converges quickly with few iterations.

3.4 Simulation Results and Discussion

The proposed IRMVB is tested on a ULA of 10 isotropic elements with a 0.5λ spacing. The noise is spatially white Gaussian with unit variance. The desired signal is always present in the array snapshots with DOA and SNR of $[96^\circ, 0\text{dB}]$, respectively but it is presumed to be at $\bar{\theta}_0 = 90^\circ$. There is a steering direction error of 6° and the DOA uncertainty region of the desired signal is assumed to be $[83^\circ, 97^\circ]$ where $\Delta\theta = 7^\circ$. There are two interferences with DOAs and INRs of $[110^\circ, 30\text{dB}]$ and $[120^\circ, 30\text{dB}]$, respectively. The abbreviation “BF” in the plots stands for “Beamformer” and 100 Monte Carlo trials are used to obtain each output SINR point. The proposed IRMVB uses the spherical constraint unless stated otherwise, with $\delta = 0.01$ and $\varepsilon_2 = 0.1$ (this choice is discussed later).

3.4.1 Output SINR of IRMVB Versus Number of Snapshots

In the first example, the output SINR of the proposed IRMVB versus the number of snapshots is shown in Fig. 3.5. The beamformer of Li *et al.* [1] is tested with optimal $\varepsilon_1 = 8.5$. The beamformer of Shahbazpanahi *et al.* [2] is tested with diagonal loading¹ = 16 added to $\hat{\mathbf{R}}$ and optimal $\mathbf{R}_s$ loading = -7 . The beamformer of Yu *et al.* [3] is tested with optimal relative regularization factor = 0.1. The beamformer of Hassanien *et al.* [4] is tested with the optimal number of eigenvectors of dominant eigenvalues = 4 and diagonal loading = 10 is used to

¹The rule of thumb for choosing the diagonal loading added to $\hat{\mathbf{R}}$ is 10dB to 12 dB above the noise level [2].

Figure 3.5: Optimal SINR and output SINRs of the proposed IRMVB, the beamformer of Li *et al.* [1], the beamformer of Shahbazpanahi *et al.* [2], the beamformer of Yu *et al.* [3], the beamformer of Hassanien *et al.* [4], and the MV beamformer. There is a steering direction error of 6° .

control the sidelobes. The same DOA uncertainty region is used in [3, 4]. The MV beamformer is tested with $\bar{\mathbf{s}}_0$ in (2.27). The other parameters remain the same.

Overall, the proposed IRMVB achieves the best output SINR in Fig. 3.5. When the number of snapshots is 500 or more, the output SINR is about 0.3dB from the optimal SINR. The performance of the beamformer of [4] is very near to that of the proposed IRMVB and their *normalized* beampatterns in Fig. 3.6 at 100 snapshots show that they are able to find the desired array steering vector by pointing their mainlobes to the desired signal direction at 96° instead of the presumed one (shown as a dashed vertical line). On the other hand, the beamformers of [1, 2] point their mainlobes towards the presumed desired signal direction. The beamformer of [3] forms a broad mainlobe with a controlled response ripple according to the DOA uncertainty range and inevitably includes more noise in the beamformer output.

Figure 3.6: Normalized beampatterns of the proposed IRMVB, the beamformer of Li *et al.* [1], the beamformer of Shahbazpanahi *et al.* [2], the beamformer of Yu *et al.* [3], and the beamformer of Hassanien *et al.* [4] at 100 snapshots in the presence of steering direction error of 6° and SNR= 0dB for one realization. There are two interferences with DOAs and INRs of $[110^\circ, 30\text{dB}]$ and $[120^\circ, 30\text{dB}]$, respectively. Solid vertical lines indicate the impinging signals' DOAs and dashed vertical line indicates the presumed desired signal's DOA.

Figure 3.7: Optimal SINR and output SINRs of the proposed IRMVB, the beamformer of Li *et al.* [1], the beamformer of Shahbazpanahi *et al.* [2], the beamformer of Yu *et al.* [3], the beamformer of Hassanien *et al.* [4], and the MV beamformer. There is a steering direction error of 6° .

3.4.2 Output SINR of IRMVB Versus SNR

In the second example, the output SINR of the proposed IRMVB versus the desired signal's SNR is shown in Fig. 3.7. The number of snapshots is 100. The other parameters remain the same. The performances of the proposed IRMVB and the beamformer of [4] are very similar at high SNRs but the proposed IRMVB outperforms [4] at low SNRs. This is because the orthogonal matrix projection operation in [4] increases its noise power at low SNRs. The proposed IRMVB also outperforms the beamformer of [1] due to its improved interference-plus-noise suppression ability derived from the use of a small uncertainty sphere iteratively.

Figure 3.8: Optimal SINR and output SINRs of the proposed IRMVBs with $\epsilon_2 = 0.01, 0.1, 0.5$, and 1 at $\text{SNR} = -10\text{dB}$. There is a steering direction error of 6° . No stopping criteria are imposed in these IRMVBs. For each ϵ_2 , a marker is used to indicate the iteration index at which the proposed stopping criteria in (3.38) would have stopped at and the corresponding output SINR if they were implemented.

3.4.3 Output SINR of IRMVBs Versus ϵ_2

In the third example, the effect of ϵ_2 on the output SINR of the proposed IRMVB at different desired signal's SNRs (-10dB and 6dB) is shown. The choices of ϵ_2 are $0.01, 0.1, 0.5$, and 1 . The other parameters remain the same as the second example. Note that the stopping criteria in (3.38) are *not* imposed in the IRMVBs as the purpose is to study their behaviour as the iterative processes continue. The iteration index at which the stopping criteria in (3.38) would have stopped the IRMVB algorithm (if they were implemented) are indicated at the corresponding output SINRs using different markers in the plots.

From Figs. 3.8 and 3.9, the optimal output SINR of the proposed IRMVB is insensitive to ϵ_2 provided that $\epsilon_2 \ll \epsilon_1$. This is an attractive property. A small ϵ_2

Figure 3.9: Optimal SINR and output SINRs of the proposed IRMVBs with $\epsilon_2 = 0.01, 0.1, 0.5$, and 1 at $\text{SNR} = 6\text{dB}$. There is a steering direction error of 6° . No stopping criteria are imposed in these IRMVBs. For each ϵ_2 , a marker is used to indicate the iteration index at which the proposed stopping criteria in (3.38) would have stopped at and the corresponding output SINR if they were implemented.

reduces the sensitivity of the proposed IRMVB to output SINR degradation if the IRMVB algorithm is not stopped precisely at the optimal output SINR but it also increases the number of iterations to reach the optimal output SINR. In contrast, a large ε_2 allows the IRMVB to converge quickly to the optimal output SINR but it is important to stop the IRMVB algorithm precisely as the output SINR can rapidly decrease with further iterations.

At low desired signal's SNR, i.e., -10dB in Fig. 3.8, the second stopping criterion in (3.38) is in effect. At high desired signal's SNR, i.e., 6dB in Fig. 3.9, the first stopping criterion in (3.38) is in effect. From Figs. 3.8 and 3.9, the proposed stopping criteria in (3.38) are effective in stopping the IRMVBs at nearly the optimal output SINRs at various SNRs and ε_2 values. Finally, $\varepsilon_2 = 0.1$ is recommended in the proposed IRMVB (plotted in red with a circle marker in Figs. 3.8 and 3.9). Compared to other ε_2 values, $\varepsilon_2 = 0.1$ offers both robustness against output SINR degradation and fast convergence where the proposed IRMVB typically stops between 10 – 20 iterations for a large steering direction error of 6° .

3.4.4 Output SINR of IRMVB versus Choice of $\Delta\theta$

In the fourth example, Fig. 3.10 shows the effect of $\Delta\theta$ in the second stopping criterion of (3.38) which affects the iteration index at which the proposed algorithm is stopped, at *low* SNR = -10dB . Fig. 3.10 is actually the red curve with circle marker of Fig. 3.8. As in [3, 4], the DOA uncertainty range $[\bar{\theta}_0 - \Delta\theta, \bar{\theta}_0 + \Delta\theta]$ is the region where the desired signal is expected to arrive from. Fig. 3.10 shows an interesting observation where the output SINR of the proposed IRMVB increases as $\Delta\theta$ increases to 8° even though the steering direction error is 6° . This is because after $\Delta\theta > 6^\circ$, the generalized angles $\hat{\theta}$ between the subsequent steering vectors calculated by the proposed IRMVB and the desired array steering vector continue to decrease which, in turn, increases the output SINR of the proposed IRMVB.

Figure 3.10: Optimal SINR and output SINR of the proposed IRMVB with $\Delta\theta = 6^\circ, 7^\circ, 8^\circ$, and 9° at $\varepsilon_2 = 0.1$ and $\text{SNR} = -10\text{dB}$. There is a steering direction error of 6° . No stopping criteria are imposed in the IRMVB. For each $\Delta\theta$, a marker is used to indicate the iteration index at which the proposed stopping criteria in (3.38) would have stopped at and the corresponding output SINR if they were implemented.

Figure 3.11: Optimal SINR and output SINRs of the proposed IRMVB, the beamformer of Li *et al.* [1], the beamformer of Shahbazpanahi *et al.* [2], the beamformer of Yu *et al.* [3], the beamformer of Hassanien *et al.* [4], and the MV beamformer. There are array calibration errors.

Refer to (3.6) and (3.7). Though the steering direction error of 6° is unknown, Fig. 3.10 shows that the proposed IRMVB is rather robust to the choice of $\Delta\theta$, i.e., the difference in the output SINRs with $\Delta\theta = 6^\circ$ and $\Delta\theta = 8^\circ$ is less than 1dB.

3.4.5 Output SINR of IRMVB With Array Calibration Errors

In the fifth example, only array calibration errors (sensor amplitude, phase, and position errors) are considered by perturbing each element of the steering vector of each impinging signal with a zero-mean circularly symmetric complex Gaussian random variable, i.e., $\tilde{\mathbf{s}} = \bar{\mathbf{s}} + \Delta\mathbf{s}$ with $\|\Delta\mathbf{s}\|^2 = 0.2\|\bar{\mathbf{s}}\|^2$ where $\tilde{\mathbf{s}}$ and $\bar{\mathbf{s}}$ are the true and presumed array steering vectors, respectively. The perturbing Gaussian

random variables are independent of each other. The beamformer of Li *et al.* [1] uses optimal $\varepsilon_1 = 5.5$. The beamformer of Shahbazpanahi *et al.* [2] uses optimal $\mathbf{R}_s$ loading = -5.5 . There are no steering direction error but $\Delta\theta = 3^\circ$ is still used. The DOA uncertainty region $[87^\circ, 93^\circ]$ is used in the proposed IRMVB, the beamformer of Yu *et al.* [3], and the beamformer of Hassanien *et al.* [4]. The beamformer of [3] uses optimal relative regularization factor = 3.2. From Fig. 3.11, the proposed IRMVB achieves the best output SINR among the tested beamformers in the presence of array calibration errors and it is found to be insensitive to $3^\circ \leq \Delta\theta \leq 5^\circ$ where the output SINR difference is only $< 0.5\text{dB}$.

3.4.6 Output SINRs of IRMVBs With Flat Ellipsoidal and Spherical Constraints

In the sixth example (based on an example in [1]), the output SINRs of the proposed IRMVB using flat ellipsoidal and spherical constraints, respectively are compared in the presence of steering direction errors. There are eight interferences with DOAs of $[15^\circ, 30^\circ, 45^\circ, 60^\circ, 80^\circ, 115^\circ, 125^\circ, 140^\circ]$, all at INR= 50dB. The desired signal is at 100° but it is presumed to be at 102° . $\Delta\theta = 3^\circ$ is assumed. Finite snapshot effect is not considered here. The beamformer of Li *et al.* [1] uses optimal $\varepsilon_1 = 6.5$ (for spherical constraint). For the proposed IRMVB and the beamformer of Li *et al.* [1], both using flat ellipsoidal constraints, let $\mathbf{s}(\bar{\theta}_0) - \mathbf{s}(\bar{\theta}_0 - \Delta\theta)$ and $\mathbf{s}(\bar{\theta}_0) - \mathbf{s}(\bar{\theta}_0 + \Delta\theta)$ be the first and second columns of $\mathbf{B}$ matrix, respectively.

In Fig. 3.12, the proposed IRMVB with spherical constraint outperforms the beamformer of Li *et al.* [1] with spherical constraint. At high SNRs ($\geq 30\text{dB}$), the proposed IRMVB with flat ellipsoidal constraint outperforms the beamformer of Li *et al.* [1] with flat ellipsoidal constraint which, in turn, outperforms the proposed IRMVB with spherical constraint. As the proposed IRMVB with flat ellipsoidal constraint has the best output SINR at SNR= 35dB, the beampatterns of the

Figure 3.12: Optimal SINR and output SINRs of the proposed IRMVBs using flat ellipsoidal and spherical constraints, respectively, the beamformers of Li *et al.* [1] using flat ellipsoidal and spherical constraints, respectively, and the MV beamformer. There is a steering direction error of 2° .

Figure 3.13: Beam patterns of the proposed IRMBVs using flat ellipsoidal and spherical constraints, respectively, and the beamformers of Li *et al.* [1] using flat ellipsoidal and spherical constraints, respectively at $\text{SNR} = 35\text{dB}$. Solid vertical lines indicate the impinging signals' DOAs. Dotted horizontal line indicates the 0dB gain level.

beamformers at SNR= 35dB are examined in Fig. 3.13. The beampatterns of the proposed IRMVB and the beamformer of Li *et al.* [1], both using flat ellipsoidal constraints, are similar. However, the output SINR of the proposed IRMVB with flat ellipsoidal constraint is better than [1] due to its superior interference rejection ability as a result of using a small flat ellipsoid to search for the desired array steering vector. This is evident from the deeper nulls formed at the interferences' DOAs by the proposed IRMVB with flat ellipsoidal constraint. This example suggests that in some cases, if there exists prior information about the beamforming scenario at hand, using the proposed IRMVB with the small flat ellipsoid to search for the desired array steering vector can be more beneficial in achieving a higher output SINR.

3.4.7 Power Spectrum of IRMVB With Array Calibration Errors

In the final example based on an “imaging” example in [1], the DOA estimation performance of the proposed IRMVB is illustrated via the power spectrum. There are five signals with DOAs of $[55^\circ, 75^\circ, 90^\circ, 100^\circ, 130^\circ]$ and SNRs of $[30, 15, 40, 35, 20]$ dB, respectively. There are array calibration errors (as in the fifth example) with $\|\Delta \mathbf{s}\|^2 = 0.1$ and the number of snapshots is 1000. The beamformer of Li *et al.* [1] uses $\varepsilon_1 = 0.1$ and the proposed IRMVB uses $\varepsilon_2 = 0.01$ and $\Delta\theta = 0.5^\circ$. The delay-and-sum beamformer [14] is also tested. Fig. 3.14 shows that the proposed IRMVB also forms high but narrower peaks in the power spectrum compared to those of [1]. Although the spectrum peak of the proposed IRMVB at 130° is lower than that of [1], IRMVB give accurate DOA estimates. For [1], there is a false spectrum peak at 99° which should be at 100° . Comparatively, the resolution of the MV beamformer deteriorates in the presence of array calibration errors. The delay-and-sum beamformer gives the worst (poorest) resolution and also, results

Figure 3.14: Power spectrums of the proposed IRMVB, the beamformer of Li *et al.* [1], the MV beamformer, and the delay-and-sum beamformer. There are array calibration errors. Dotted vertical lines indicate the impinging signals' DOAs.

in false peaks.

3.4.8 Output SINR of IRMVB at Very Low Snapshots

In the final example, the same simulation parameters in the first example are used except that the number of array snapshots is now varied from 4 to 20 to see the output SINR performance of the proposed IRMVB. Fig. 3.15 shows that the proposed IRMVB still has room for further output SINR improvement at very low snapshots due to the occurrence of high sidelobes.

3.5 Summary

An IRMVB which uses a small uncertainty sphere (and a small flat ellipsoid) to search for the desired array steering vector iteratively has been proposed. By

Figure 3.15: Optimal SINR and output SINRs of the proposed IRMVB, the beamformer of Li *et al.* [1], the beamformer of Shahbazpanahi *et al.* [2], the beamformer of Yu *et al.* [3], the beamformer of Hassanien *et al.* [4], and the MV beamformer at very low snapshots in the presence of steering direction error of 6° .

preserving its DOFs and in turn, its interference-plus-noise suppression ability, and by using the corrected desired array steering vector, the proposed IRMVB achieves higher output SINR than the worst-case performance optimization based beamformers. Theoretical analysis and simulation results have been presented to support the effectiveness of the proposed beamformer.

Chapter 4

Beamforming Optimization Against Interferences by a General Adaptive Beamforming Framework

4.1 Objectives

One of the objectives of the research in this chapter is to study the robust adaptive beamformers of [1–3, 5, 43, 52] in the presence of large steering direction errors and at low snapshots¹. These problems are frequently encountered in practical beamforming applications. The findings of the study in this chapter reveal that the sidelobes of the beamformers of [1–3, 5, 43, 52] are not controlled and can be high at low snapshots. This can cause system performance breakdown of [1–3, 5, 43, 52] when there are unexpected interferences or strong interferences that move during which the beamforming weights are applied and are located outside

¹The number of array snapshots is considered low when it is about the same as the number of antenna elements [50].

the beampattern nulls, leaking to the array output via the sidelobes. The CMT beamformer of [5] can be robust against moving interferences but not against steering direction errors.

As mentioned earlier in the Introduction of the thesis, achieving adaptive interference rejection and beampattern control are two conflicting goals for beamforming systems as they consume DOFs (usually fixed in most cases). Therefore, another objective of the research in this chapter is to achieve these two capabilities and find a way to automatically balance or adjust them so that the resulting beamformer is easily applicable to different scenarios with a different number of signals and/or different SNRs.

This chapter proposes a framework based on the use of a set of beampattern shaping constraints. A weighing ratio is also proposed to automatically control the relative proportions of adaptive interference rejection and sidelobe suppression in different scenarios without ambiguity. The proposed framework encompasses different optimization strategies leading to various beamformers with specific robustness goals. Three examples of beamformers are derived to handle typical application needs: (a) Reception of desired signal with sidelobe suppression, (b) Reception of desired signal with joint interference and sidelobe suppression, and (c) Reception of desired signal with joint interference and sidelobe suppression combined with robustness against strong moving interferences. Similar to the beamformers of [1, 2, 43, 52, 85], the three proposed beamformers require prior knowledge of the desired signal's approximate DOA but the proposed beamformers are more robust against large steering direction errors. The third proposed beamformer requires additional information about the DOAs of strong moving interferences only and these estimated DOAs do not need to be highly accurate. The proposed framework is a SOCP which can be solved globally and efficiently in polynomial-time by the IPM using open-source solvers, e.g., [88, 89].

4.2 Background

In this section, the background on beampattern design and a recently developed adaptive beamformer of Yu *et al.* [3, 102] is described as the underlying foundation for the subsequent discussion in this chapter.

4.2.1 Beampattern Design

Consider a ULA of N isotropic elements with d inter-element spacing. A plane wave of λ wavelength impinges on it at θ degrees from the array axis. The elements sample the wave and the resulting signals are linearly combined with the complex array weights w_i for $i = 0, \dots, N - 1$ to generate the array output. The array beampattern can be expressed as $G(\theta) = \sum_{k=0}^{N-1} w_k e^{j \frac{2\pi}{\lambda} k d \cos \theta}$. A typical beampattern design constraint can be posed as

$$L(\theta) \leq |G(\theta)| \leq U(\theta), \quad \theta \in [0^\circ, 180^\circ] \quad (4.1)$$

where $L(\theta)$ and $U(\theta)$ denote the lower and upper magnitude response limits, respectively. However, the lower bound of this constraint $L(\theta) \leq |G(\theta)|$ is non-linear, non-convex, and thus, NP-hard to solve [3, 7, 8, 59, 103]. A change of variables is required to transform (4.1) into a convex constraint.

The autocorrelation sequence of the complex weights is

$$r_w(k) = \sum_{i=-(N-1)}^{N-1} w_i w_{i+k}^*, \quad (4.2)$$

where $k = -(N - 1), \dots, 0, \dots, N - 1$ and $w_i = 0$ for $i < 0$ and $i > N - 1$ [59].

The Fourier Transform of $r_w(k)$ is

$$R_w(\theta) = \sum_{k=-(N-1)}^{N-1} r_w(k) e^{j k \frac{2\pi}{\lambda} d \cos \theta}$$

$$= \mathbf{a}^T(\theta) \mathbf{r}_w = |G(\theta)|^2 \quad (4.3)$$

where $\mathbf{a}(\theta) = [e^{-j(N-1)\frac{2\pi}{\lambda}d\cos\theta} \ \dots \ 1 \ \dots \ e^{j(N-1)\frac{2\pi}{\lambda}d\cos\theta}]^T$ is the $(2N-1) \times 1$ extended array steering vector which is different from the $N \times 1$ steering vector $\mathbf{s}(\theta)$ defined in (2.13) and the array weight autocorrelation sequence is $\mathbf{r}_w = [r_w(-(N-1)) \ \dots \ r_w(0) \ \dots \ r_w(N-1)]^T$ [3, 59]. The non-convex beam-pattern design constraint (4.1) can now be transformed into

$$L^2(\theta) \leq \mathbf{a}^T(\theta) \mathbf{r}_w \leq U^2(\theta), \quad \theta \in [0^\circ, 180^\circ] \quad (4.4)$$

which is convex and thus, a global solution can be obtained. For each θ , the constraint (4.4) is a pair of linear inequalities in $\mathbf{r}_w$ and hence, affine over $\mathbf{r}_w$.

4.2.2 Adaptive Beamformer of Yu *et al.*

One way to tackle the signal cancellation problem in the MV beamformer [85] due to steering direction errors is to impose (4.4) to broaden the mainlobe region Θ_{ML} and control its response [3] as

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^H \mathbf{R} \mathbf{w} \\ \text{s.t.} \quad & l \leq \mathbf{a}^T(\theta) \mathbf{r}_w \leq u, \quad \theta \in \Theta_{\text{ML}} \end{aligned} \quad (4.5)$$

where the array output power $\mathbf{w}^H \mathbf{R} \mathbf{w}$ is minimized. $l = L^2(\theta)$ and $u = U^2(\theta)$ are constants to define the response ripple of Θ_{ML} in a simple manner. Due to the non-linear relationship between the array weight autocorrelation sequence $\mathbf{r}_w$ and beamforming weight vector $\mathbf{w}$, it may be difficult to solve (4.5).

Hence, in the adaptive beamformer of Yu *et al.* [3], the array output power is derived as $\mathbf{r}_x^T \mathbf{r}_w$ which is linear in $\mathbf{r}_w$ where the array snapshot autocorrelation sequence is $\mathbf{r}_x = [r_x(-(N-1)) \ \dots \ r_x(0) \ \dots \ r_x(N-1)]^T$. However, the

theoretical $\mathbf{r}_x$ cannot be obtained in practice due to array imperfections and the requirement of an infinite number of array snapshots $\mathbf{x}(n)$. As such, the authors of [3] applied the worst-case performance optimization approach [43, 63] and obtained

$$\begin{aligned} \min_{\mathbf{r}_w} \quad & \hat{\mathbf{r}}_x^T \mathbf{r}_w + \varepsilon \|\mathbf{r}_w\| \\ \text{s.t.} \quad & l \leq \mathbf{a}^T(\theta) \mathbf{r}_w \leq u, \quad \theta \in \Theta_{\text{ML}} \\ & \mathbf{a}^T(\theta) \mathbf{r}_w \geq 0, \quad \theta \in \Theta \\ & r_w(-k) = r_w^*(k), \quad k = 1, \dots, N-1. \end{aligned} \tag{4.6}$$

The theoretical $\mathbf{r}_x$ belongs to the set $\mathbf{r}_x = \hat{\mathbf{r}}_x + \mathbf{e}$, $\|\mathbf{e}\| \leq \varepsilon$, and $\varepsilon > 0$ where $\hat{\mathbf{r}}_x$ is the estimate of $\mathbf{r}_x$. Without the second-order term $\|\mathbf{r}_w\|$ in the objective function, (4.6) is a (convex) linear program (LP) with linear objective function and constraints in $\mathbf{r}_w$. With $\|\mathbf{r}_w\|$, (4.6) is a SOCP problem. The constraint $\mathbf{a}^T(\theta) \mathbf{r}_w \geq 0$ in (4.6) is sufficient to ensure that the complex weights w_i can be extracted from $\mathbf{r}_w$ by spectral factorization [7, 8, 59]. Here, minimum phase spectral factorization is used to obtain unique weights.

4.3 Numerical Study of Beamformers at Low Snapshots and Large Steering Direction Errors

In this section, the beampattern behaviour of the beamformers of [1, 2, 5, 85] and [3] introduced in Section 4.2.2 are investigated in the presence of large steering direction errors and at low snapshots. The simulation details of this example are the same as those in the third example of Section 4.6 (Simulation Results and Discussion). A ULA of 16 isotropic antennas with $d = 0.5\lambda$ spacing is used. The noise is spatially white Gaussian with unit variance. The desired signal is always present in the array snapshots with its DOA and SNR fixed at $[96^\circ, 0\text{dB}]$,

Figure 4.1: Beampatterns (unnormalized) of the beamformers of Yu *et al.* [3] with $\varepsilon_r = 0.3$ and 1.3, the beamformer of Shahbazpanahi *et al.* [2], the beamformer of Li *et al.* [1], and the MV beamformer at 16 snapshots in the presence of steering direction error of 6° and SNR= 0dB for one realization. There are two interferences with DOAs and INRs of $[30^\circ, 20\text{dB}]$ and $[140^\circ, 25\text{dB}]$, respectively. The DOAs of the impinging signals are denoted with black vertical lines.

respectively. However, it is presumed to be at 90° so there is a large steering direction error of 6° . The DOAs and INRs of two interferences are $[30^\circ, 20\text{dB}]$ and $[140^\circ, 25\text{dB}]$, respectively. The number of array snapshots collected is only 16. The tested beamformers are:

1. Beamformers of Yu *et al.* [3] with $\varepsilon_r = 0.3$ and 1.3;
2. Beamformer of Shahbazpanahi *et al.* [2] with $\hat{\mathbf{R}}$ loading = 25 and $\bar{\mathbf{R}}_s$ loading = -15;
3. Beamformer of Li *et al.* [1] with uncertainty sphere size = 15.89;
4. MV Beamformer.

The beamformer of Yu *et al.* [3] is tested with two relative regularization factors ε_r ($\varepsilon_r = \varepsilon/\hat{r}_x(0)$ and its purpose is explained in Section 4.6.3); $\varepsilon_r = 0.3$ is used to compare with the proposed beamformers introduced later and $\varepsilon_r = 1.3$ is the optimal value. In the beamformer of [3], $\Theta_{\text{ML}} = \pm 7^\circ + 90^\circ$ and response ripple $r_{dB} = \pm 1.3\text{dB}$ are set. The optimal robustness parameters of [1, 2] are used. The beampatterns of the tested beamformers for one realization are shown in Fig. 4.1. Apart from the MV beamformer which nulls the desired signal, all the other beamformers maintain approximately a unity gain (0dB) to the desired signal but they have very high sidelobes. For example, their highest sidelobes are only about -10dB to -15dB from their peak magnitude responses at 90° .

In the next example, the DOAs and INRs of two interferences are $[30^\circ, 20\text{dB}]$ and $[140^\circ, 40\text{dB}]$, respectively. The simulation details of this example are the same as those in the fifth example of Section 4.6 (Simulation Results and Discussion) where the 40dB interference starts to move out of the beampattern null during which the beamforming weights are applied. The tested beamformers are:

1. Beamformers of Yu *et al.* [3] with $\varepsilon_r = 0.3$ and 13.2;

Figure 4.2: Normalized beampatterns of the beamformers of Yu *et al.* [3] with $\varepsilon_r = 0.3$ and 13.2 and the CMT beamformer [5] at 16 snapshots in the presence of steering direction error of 6° and SNR=0dB for one realization. There are two interferences with DOAs and INRs of $[30^\circ, 20\text{dB}]$ and $[140^\circ, 40\text{dB}]$, respectively. The DOAs of the impinging signals are denoted with black vertical lines.

2. CMT Beamformer [5] with taper parameter = 0.02.

Their normalized beampatterns for one realization are shown in Fig. 4.2 before the 40dB interference starts to move. From Fig. 4.2, the CMT beamformer [5] has broadened the beampattern nulls compared to the narrow beampattern nulls of the beamformers of Yu *et al.* [3]. As such, the CMT beamformer [5] is more robust against moving interferences. However, it is not robust against steering direction errors and high sidelobes also appear at low snapshots.

From Figs. 4.1 and 4.2, the beamformer of [3] has flexible mainlobe control while achieving interference rejection. With the use of an optimal ε in the beamformer of [3], robustness against errors in $\hat{\mathbf{r}}_x$ incurred at low snapshots can be enhanced and the sidelobes are more stabilized compared to that with a small ε . However, the beamformer of [3] is still non-robust against moving interferences

when its beamforming weights are not updated fast enough to suppress the interferences which move out of the sharp beampattern nulls during which the weights are applied.

4.4 Proposed Adaptive Beamforming Framework With Beampattern Shaping Constraints

Based on the findings of the numerical studies in Section 4.3, a beamformer design is proposed which can maintain low sidelobes at low snapshots while nulling out interferences automatically. In addition, the designed beamformer can be robust against large steering direction errors and strong interferences that move out of the beampattern nulls during which the beamforming weights are held fixed. The latter is useful in scenarios where the beamforming weights are designed using array snapshots taken over some time interval and then used to receive signals over a generally longer time interval before being updated despite changes in the scenario (“frozen weights” scenario) [33]. A straightforward way is also proposed to automatically control the relative proportions of adaptive interference rejection and sidelobe control in different scenarios without ambiguity.

4.4.1 Problem Formulation

The beampattern shaping constraints (4.7b)-(4.7d) segregate the entire beampattern explicitly into the mainlobe (Θ_{ML}), sidelobe (Θ_{SL}), and null (Θ_{N}) regions, respectively for control:

$$\min_{\mathbf{r}_w, \tau} (\hat{\mathbf{r}}_x^T \mathbf{r}_w + \varepsilon \|\mathbf{r}_w\|) + \rho \tau \quad (4.7a)$$

$$\text{s.t.} \quad l \leq \mathbf{a}^T(\theta) \mathbf{r}_w \leq u, \quad \theta \in \Theta_{\text{ML}} \quad (4.7b)$$

$$\mathbf{a}^T(\theta) \mathbf{r}_w \leq \tau, \quad \theta \in \Theta_{\text{SL}} \quad (4.7c)$$

$$\mathbf{a}^T(\theta)\mathbf{r}_w \leq \tau_n, \quad \theta \in \Theta_N \quad (4.7d)$$

$$\mathbf{a}^T(\theta)\mathbf{r}_w \geq 0, \quad \theta \in \Theta \quad (4.7e)$$

$$r_w(-k) = r_w^*(k), \quad k = 1, \dots, N-1. \quad (4.7f)$$

The scalar ρ is a weighing parameter that weighs the minimization importance between the compensated array output power ($\hat{\mathbf{r}}_x^T \mathbf{r}_w + \varepsilon \|\mathbf{r}_w\|$) and the sidelobe level (τ). The term τ in (4.7a) is an optimization variable whose value depends on ρ . This framework directly minimizes τ to obtain low sidelobes so there is no need to specify τ . τ_n in (4.7d) refers to the interference rejection level in Θ_N and is user-specified. The proposed framework achieves uniformly low sidelobes while preserving the beamformer's interference rejection ability by using a small ε value.

As is in (4.6), the angle θ in the proposed framework (4.7) is discretized as $\check{\theta}$. After the optimization, if $\mathbf{a}^T(\theta)\mathbf{r}_w \geq 0$ is not satisfied over the entire Θ region (for example, between two $\check{\theta}$ samples), then spectral factorization is not possible; there are no beamforming weights w_i that have $\mathbf{r}_w$ as its autocorrelation. When this happens, a simple heuristic solution is to add a “safety margin” [59], i.e., $\mathbf{a}^T(\check{\theta})\mathbf{r}_w \geq \nu$ where ν is small and positive, and solve the discretized version of (4.7) by increasing ν until spectral factorization is successful [59].

4.4.2 Weighing Ratio ρ_r

The higher the ρ value in the objective function (4.7a), the more emphasis the beamformer places in minimizing the sidelobes and the lower the sidelobe level is. If the scenario changes with a different number of signals and/or signal powers, the ($\hat{\mathbf{r}}_x^T \mathbf{r}_w + \varepsilon \|\mathbf{r}_w\|$) term in (4.7a) is affected, but not the τ term in (4.7a). As a result, to maintain a specific relative minimization importance between ($\hat{\mathbf{r}}_x^T \mathbf{r}_w + \varepsilon \|\mathbf{r}_w\|$) and τ in different scenarios, ρ has to be re-calculated for each of them. This is ambiguous and cumbersome. Therefore, ρ is proposed to vary with the input signal

power as

$$\rho = \rho_r \hat{r}_x(0) \quad (4.8)$$

where ρ_r is a weighing ratio and $\hat{r}_x(0)$ is the estimated input signal power. Whenever the scenario changes with a different number of signals and/or signal powers, it is reflected in $\hat{r}_x(0)$ and at the same time, $\rho_r \hat{r}_x(0)\tau$ in (4.7a) gets affected by $\hat{r}_x(0)$ proportionately. In this way, ρ enables fixed relative interference rejection and sidelobe control capabilities to be maintained in different scenarios by a fixed ρ_r value without ambiguity.

The weighing ratio ρ_r varies in the range of $[0, \infty)$. As $\rho_r \rightarrow \infty$, the objective function (4.7a) becomes

$$\min_{\mathbf{r}_w, \tau} \tau. \quad (4.9)$$

In this case, the resulting non-adaptive beamformer (4.9) achieves the lowest sidelobes which help safeguard against unexpected interferences or intentional jammers. Conversely, as $\rho_r \rightarrow 0$, (4.7a) becomes

$$\min_{\mathbf{r}_w} \hat{\mathbf{r}}_x^T \mathbf{r}_w + \varepsilon \|\mathbf{r}_w\|. \quad (4.10)$$

This beamformer (4.10) is fully adaptive which rejects interferences automatically (assuming a small ε). As such, ρ_r can be used to tune the responsiveness of the proposed beamformer to achieve different optimization strategies.

A more robust beamformer is expected when more constraints are imposed but they consume the beamformer's DOFs and tradeoff the output SINR. However, (4.7b)-(4.7e) in the proposed framework are inequality constraints, not all are active during optimization and hence, a high output SINR is achievable.

4.5 Special Cases of the Proposed Framework

In this section, three beamformers are derived from the framework (4.7) for three typical beamforming scenarios, which are simulated in Section 4.6 to show the importance of sidelobe and null control.

4.5.1 Robust Beamformer With Sidelobe Control (RB-SL)

It is important in radar, for example, to safeguard against unexpected interferences or intentional jammers [30–32]. In (4.7), $\rho_r \rightarrow \infty$ is set (See (4.9)), constraint (4.7d) is removed so as to obtain the Robust Beamformer with SideLobe control (RB-SL). The non-adaptive RB-SL achieves the minimal sidelobe level for fixed mainlobe constraints [59] while the methods in [30, 104] cannot control the mainlobe beamwidth and sidelobe level precisely according to the prescribed specifications [66].

4.5.2 Robust Adaptive Beamformer With Sidelobe Control (RAB-SL)

In order to incorporate joint interference rejection and sidelobe control in the beamformer, $0 < \rho_r < \infty$ is set in (4.7). Again, constraint (4.7d) is not used and the Robust Adaptive Beamformer with SideLobe control (RAB-SL) is obtained. The beamformer of [3] is a special case of RAB-SL; [3] does not consider sidelobe control. Hence, the beamformer of [3] may possibly achieve a better interference rejection but it can easily breakdown when there are unexpected interferences whereas the RAB-SL brings about a more balanced performance with joint interference rejection and sidelobe control.

Figure 4.3: RAB-Motion block diagram. Vector input/output are indicated by bold arrowed lines.

4.5.3 Robust Adaptive Beamformer Against Strong Interference Motion With Sidelobe Control (RAB-Motion)

One way to tackle strong moving interferences may be by fast and continuous weight adaptation [71] but this can be prohibitive in some applications. Thus, applications are considered where frozen beamforming weights are used, i.e., weights that are designed using array snapshots collected over a time interval and then used over the next time interval despite changes in the scenario [33]. The low sidelobes of RB-SL and RAB-SL can attenuate moving interferences but they may not be effective against the stronger ones. Hence, (4.7d) is used to additionally shape the beampattern in the vicinities of strong moving interferences to have broadened deepened nulls. This design (4.7) is called the Robust Adaptive Beamformer against strong interference Motion with sidelobe control (RAB-Motion).

Fig. 4.3 shows the RAB-Motion block diagram and the beamforming weight vector $\mathbf{w}$ can be arbitrarily initialized to all ones. The impinging signals are received as array snapshots $\mathbf{x}(n)$ where $n = 1, \dots, N_s$, after which an estimated array snapshot autocorrelation sequence $\hat{\mathbf{r}}_x$ is formed. At the same time, $\mathbf{x}(n)$

is input to the ‘Estimate DOAs of strong moving interferences’ module where the MVDR-based techniques in [28] may be used to estimate the strong moving interferences’ DOAs. Note that these DOA estimates do not need to be very accurate as broad nulls will be formed on them subsequently. Next, the DOA estimates are input to the ‘parameter settings/design specifications’ module to adjust Θ_N . For example, a Θ_N can be centred at the estimated DOA of a strong moving interference. Similarly, in the ‘design specifications’ module, Θ_{ML} can be centred at the presumed desired signal’s DOA. The widths of Θ_{ML} and Θ_N are set based on system design specifications/error tolerances pertaining to a particular application [64]. Θ_{SL} is the region outside Θ_{ML} and Θ_N ². The angle θ in (4.7) is discretized as $\check{\theta}$ by uniformly discretizing the electrical angle $\Omega = 2\pi d \cos \theta / \lambda$ which does not affect the convexity of the constraints in (4.7) [7, 8, 59].

In the ‘design specifications’ module, l and u are the mainlobe’s lower and upper magnitude response limits, respectively and should not deviate too far from 0dB for applications like signal power estimation [1, 19, 66]. τ_n can be chosen based on the highest interference power that can be expected for the application. The choices of ε_r and ρ_r are discussed in Section 4.6 (Simulation Results and Discussion). With the constrained parameters and $\hat{\mathbf{r}}_x$, the RAB-Motion (4.7) is solved using the IPM³ to obtain the array weight autocorrelation sequence $\mathbf{r}_w$ after which spectral factorization is applied to derive the beamforming weight vector $\mathbf{w}$. The weight vector is then applied on $\mathbf{x}(n)$ for $n = N_s + 1, \dots, 2N_s$, to obtain the corresponding array output $y(n)$ for $n = N_s + 1, \dots, 2N_s$. When the next block of array snapshots is received, the above process is repeated to adapt $\mathbf{w}$.

For the beamformers of [1, 2, 43, 52, 85] and the proposed beamformers (RB-

²It is required that Θ_{ML} and Θ_N do not overlap.

³The IPM is used instead of the traditional gradient-based optimization techniques because the latter can suffer from slow convergence and are sensitive to algorithm initialization and stepsize. In comparison, the IPM is not hampered by these issues and solves convex problems globally and efficiently in polynomial-time [105]. The interested reader can refer to [63, Ch. 11] for a discussion on IPM.

SL, RAB-SL, and RAB-Motion), all of them require prior knowledge of the desired signal's DOA (which may deviate from its true value). The other signals present are considered as interferences. Similar to the beamformers of [1, 2, 43, 52, 85], the proposed adaptive beamformers (RAB-SL and RAB-Motion) can automatically reject interferences without the knowledge of their DOAs. The RAB-Motion requires additional information about the DOAs of strong moving interferences only, in order to place broadened deepened nulls on them. Hence, an additional DOA estimation module for this purpose is required in RAB-Motion as shown in Fig. 4.3, compared to the beamformers of [1, 2, 43, 52, 85]. However, the advantage of the RAB-Motion is that it can provide a much higher output SINR performance than the beamformers of [1, 2, 43, 52, 85] in a “frozen-weights” scenario with strong moving interferences.

4.5.4 Complexity of the Proposed Beamformers

Similar to the beamformer of [43], the weights of our proposed beamformers cannot be easily updated with each snapshot and has to be recomputed numerically [3] with every new block of array snapshots. This is an inherent limitation of the proposed framework. However, from the simulation results presented later in Fig. 4.5, the proposed RAB-SL achieves a high output SINR even at very low snapshots, i.e., 4 snapshots. This shows that the proposed beamformers are able to respond quickly to changes in the scenarios requiring only a few snapshots. The overall worst-case⁴ complexity of the proposed beamformers to process a block of array snapshots is $\mathcal{O}(N^{3.5} + PN^{2.5})$ [87] where P is the number of discretized $\check{\theta}$ in Θ but the actual complexity is much lower since most of the P inequality constraints on Θ are inactive. Furthermore, the proposed formulation (4.7) can be more efficiently implemented by exploiting its problem structure [63]. In comparison,

⁴The worst-case complexity assumes all P inequality constraints on Θ to be active.

the complexity of the MV beamformer to process a block of array snapshots is $\mathcal{O}(N^3)$ and if updating of weights per snapshot is allowed by the Recursive Least Squares (RLS) and Least Mean Squares (LMS) algorithms, then the complexities are $\mathcal{O}(N^2)$ and $\mathcal{O}(N)$ per updating step, respectively [12] but they are still non-robust against steering direction errors, for example. In order to operate in a practical scenario with moving interferences, the RLS and LMS algorithms have to continuously update the beamforming weights, which incur computations at each updating step. On the other hand, the RAB-Motion can allow the designed weight vector $\mathbf{w}$ to be used for a longer time interval without updating despite the presence of strong moving interferences.

4.6 Simulation Results and Discussion

The proposed beamformers are tested using a ULA of 16 isotropic antennas with a 0.5λ spacing. The noise is spatially white Gaussian with unit variance. The desired signal is always present in the array snapshots with its DOA and SNR fixed at $[96^\circ, 0\text{dB}]$, respectively. However, it is presumed to be at 90° resulting in a 6° steering direction error. $\Theta_{\text{ML}} = \pm 7^\circ + 90^\circ$ and $r_{dB} = \pm 1.3\text{dB}$ are set. Since ε is the error norm in $\hat{\mathbf{r}}_x$, it depends on the input signal power and the relative regularization factor $\varepsilon_r = \varepsilon/\hat{r}_x(0)$ is used to avoid ambiguity [3]. Θ_{SL} is defined as $[0^\circ, 75^\circ] \cup [105^\circ, 180^\circ]$ for the proposed beamformers where a constant mainlobe roll-off width of 8° is used. Θ_{SL} is unconstrained in the beamformer of [3].

4.6.1 Significance of the Weighing Ratio ρ_r

In the first example, the RAB-SL is used to show the effectiveness of the proposed weighing ratio ρ_r to control the relative amounts of interference rejection and side-lobe suppression in three scenarios without ambiguity. There are two interferences

Table 4.1: Sidelobe levels of two RAB-SLs in three scenarios.

Scenario	1	2	3
First RAB-SL (fixed $\rho_r = 1$)	-31.95dB	-32.09dB	-33.05dB
Second RAB-SL (fixed $\rho = 103$)	-31.95dB	-31.74dB	-15.57dB

and one desired signal in each scenario. Two RAB-SLs are tested using the true array snapshot autocorrelation sequence $\mathbf{r}_x$. The first RAB-SL uses $\rho = \rho_r r_x(0)$ with a fixed ρ_r while the second RAB-SL uses a fixed ρ in all the scenarios. In the first scenario, the DOAs and INRs of two interferences are $[30^\circ, 0\text{dB}]$ and $[140^\circ, 20\text{dB}]$, respectively. Thus, the resulting input signal power $r_x(0)$ is 103. In the first RAB-SL, $\rho_r = 1$ is used to obtain equal relative amounts of interference rejection and sidelobe control for all the scenarios. In the second RAB-SL, $\rho = 1 \times r_x(0) = 103$ is used but this value remains fixed for the next two scenarios. In the second scenario, the DOAs and INRs of the interferences are $[30^\circ, 20\text{dB}]$ and $[140^\circ, 20\text{dB}]$, respectively. In the third scenario, the DOAs and INRs of the interferences are $[30^\circ, 50\text{dB}]$ and $[140^\circ, 50\text{dB}]$, respectively. The sidelobe levels for the two RAB-SLs in these scenarios are tabulated in Table 6.2. From Table 6.2, the sidelobe levels of the first RAB-SL are well-managed in all the scenarios, even in the third scenario where the input signal power has increased by a large amount. In the second RAB-SL, the importance of sidelobe control is diminished by the use of a fixed ρ , resulting in a much higher sidelobe level of -15.57dB in the third scenario. This large variability in sidelobe levels at different situations is clearly undesirable.

4.6.2 Output SINR of RB-SL Versus Number of Unexpected Interferences

In the second example, the robustness of the non-adaptive RB-SL against unexpected interferences is demonstrated. 16 snapshots (equal to the number of antennas) are collected when there are no interferences. The interferences appear

Figure 4.4: Optimal SINR and output SINRs of the proposed RB-SL and the delay-and-sum beamformer versus the number of unexpected interferences that appear in Θ_{SL} during which the beamforming weights are applied, at 16 snapshots and in the presence of steering direction error of 6° .

unexpectedly in Θ_{SL} during which the weights of the tested beamformers are applied and they increase from 1 to 15 with equal $INR = 20\text{dB}$ and assume a uniform distribution in Θ_{SL} . The tested beamformers are:

1. Proposed RB-SL;
2. Delay-and-sum Beamformer.

The conventional beamformer (or delay-and-sum beamformer in [14]) is tested which uses the presumed desired array steering vector $\bar{\mathbf{s}}$ in its weight vector. 100 Monte Carlo simulations are used to obtain each output SINR point. From Fig. 4.4, the proposed RB-SL outperforms the delay-and-sum beamformer significantly.

4.6.3 Output SINR of RAB-SL Versus Number of Snapshots

In the third example, the joint adaptive interference rejection and sidelobe control abilities of the proposed RAB-SL versus the number of array snapshots is given in Fig. 4.5. The DOAs and INRs of the interferences are $[30^\circ, 20\text{dB}]$ and $[140^\circ, 25\text{dB}]$, respectively. The tested beamformers are:

1. Proposed RAB-SL with $\varepsilon_r = 0.3$ and $\rho_r = 2$;
2. Beamformers of Yu *et al.* [3] with $\varepsilon_r = 0.3$ and 1.3;
3. Beamformer of Shahbazpanahi *et al.* [2] with $\hat{\mathbf{R}}$ loading = 25 and $\bar{\mathbf{R}}_s$ loading = -15;
4. Beamformer of Li *et al.* [1] with uncertainty sphere size = 15.89;
5. MV Beamformer.

Two different ε_r values are used for the beamformer of [3] to see its effect on the output SINR performance: the first $\varepsilon_r = 0.3$ is fixed to compare with the proposed beamformers; the second ε_r is the optimal value selected from the average of 100 Monte Carlo simulations at 16 snapshots. The optimal $\rho_r = 2$ for RAB-SL in this scenario is used. Note that small (arbitrary) $\varepsilon_r = 0.3$ is used in RAB-SL to provide for some robustness against errors in $\hat{\mathbf{r}}_x$. A diagonal loading of 25 is added to $\hat{\mathbf{R}}$ and the optimal $\bar{\mathbf{R}}_s$ loading is chosen for each simulation [2]. The optimal uncertainty sphere size for the beamformer of [1] is used for each simulation. The MV beamformer uses $\bar{\mathbf{s}}$ in its weight vector (2.27). From Fig. 4.5, the proposed RAB-SL achieves the best convergence rate and output SINR performance among the tested beamformers.

The normalized beampatterns of the tested beamformers at 16 snapshots are shown in Fig. 4.6 where the proposed RAB-SL has the lowest sidelobe level at low

Figure 4.5: Optimal SINR and output SINRs of the proposed RAB-SL, the beamformers of Yu *et al.* [3] with $\epsilon_r = 0.3$ and 1.3, the beamformer of Shahbazpanahi *et al.* [2], the beamformer of Li *et al.* [1], and the MV beamformer versus the number of snapshots in the presence of steering direction error of 6° and $\text{SNR} = 0\text{dB}$.

Figure 4.6: Normalized beampatterns of the proposed RAB-SL, the beamformers of Yu *et al.* [3] with $\varepsilon_r = 0.3$ and 1.3 , the beamformer of Shahbazpanahi *et al.* [2], the beamformer of Li *et al.* [1], and the MV beamformer at 16 snapshots in the presence of steering direction error of 6° and $\text{SNR} = 0\text{dB}$ for one realization. There are two interferences with DOAs and INRs of $[30^\circ, 20\text{dB}]$ and $[140^\circ, 25\text{dB}]$, respectively. The impinging signals are indicated with black vertical lines.

snapshots while maintaining a broad mainlobe that is nearly the same as that of the beamformer of [3].

4.6.4 Output SINR of RAB-SL Versus SNR

In the fourth example, the same beamformers are tested at 16 snapshots by varying the desired signal's SNR. Fig. 4.7 shows that the proposed RAB-SL has the best output SINR performance.

Figure 4.7: Optimal SINR and output SINRs of the proposed RAB-SL, the beamformers of Yu *et al.* [3] with $\epsilon_r = 0.3$ and 1.3, the beamformer of Shahbazpanahi *et al.* [2], the beamformer of Li *et al.* [1], and the MV beamformer versus SNR in the presence of steering direction error of 6° at 16 snapshots.

4.6.5 Output SINR of RAB-Motion With Strong Moving Interference

In the fifth example, the performance of the proposed RAB-Motion (4.7) in a “frozen-weights” scenario is illustrated whereby a 20dB stationary interference is present at 30° and a 40dB interference is present at 140° initially. 16 snapshots are collected when the 40dB interference is at 140° . During which the beamforming weights are applied, the 40dB interference moves with its DOA changes as $\theta = 140^\circ + 4^\circ \sin(t/60)$. The tested beamformers are:

1. Proposed RAB-Motion with $\varepsilon_r = 0.3$, $\rho_r = 1.6$, $\Theta_N = \pm 3.5^\circ + 140^\circ$, and $\tau_n = -50\text{dB}$;
2. Beamformers of Yu *et al.* [3] with $\varepsilon_r = 0.3$ and 13.2;
3. Beamformer of Shahbazpanahi *et al.* [2] with $\hat{\mathbf{R}}$ loading = 25 and $\bar{\mathbf{R}}_s$ loading = -15;
4. CMT Beamformer [5] with taper parameter = 0.02;
5. Beamformer of Li *et al.* [1] combined with CMT using uncertainty sphere size = 15.89 and taper parameter = 0.02;
6. Beamformer of Vorobyov *et al.* [6] (combined with CMT) using uncertainty level = 3.92 and taper parameter = 0.02.

For the RAB-Motion, the interference rejection level is set at $\tau_n = -50\text{dB}$ arbitrarily in $\Theta_N = \pm 3.5^\circ + 140^\circ$. The optimal $\rho_r = 1.6$ for the initial stationary scenario is used. For the CMT beamformer [5], the taper parameter is set to 0.02 in $\mathbf{T}$ of (2.54) to obtain approximately a beampattern null width of $\pm 3.5^\circ + 140^\circ$, assuming an exact knowledge of $\mathbf{R}$ and the absence of steering direction error. It is interesting to compare the RAB-Motion with the beamformer of Li *et al.* [1]

combined with CMT, as the beamformer of [1] is robust against steering vector errors whereas the CMT beamformer is robust against moving interferences. The weight vector is obtained by replacing $\bar{\mathbf{s}}_0$ in (2.55) by $\hat{\mathbf{s}}$ which is the desired array steering vector calculated by the beamformer of [1]. The same taper parameter of 0.02 is used. The beamformer of Vorobyov *et al.* [6] is also tested which recommended combination with CMT in [6] for robustness against moving interferences. The sliding window technique is not used and the data matrix loading [6] is set to zero in order to be consistent with the “frozen-weights” setting of this example. The same taper parameter of 0.02 is used and the optimal uncertainty level of 3.92 in the beamformer of [6] (combined with CMT) is used.

The normalized beampatterns of the proposed RAB-Motion, the beamformers of Yu *et al.* [3], and the CMT beamformer of [5] are shown in Fig. 4.8 before the 40dB interference starts to move. The proposed RAB-Motion is able to control the beampattern null width and depth precisely to safeguard against the moving 40dB interference. From Fig. 4.9, the proposed RAB-Motion achieves the best output SINR performance.

Lastly, the proposed beamformers (RAB-SL and RAB-Motion) are not sensitive to the exact choice of ρ_r . Even if the optimal ρ_r is not used, a wide range of $0.2 \leq \rho_r \leq 8$ in RAB-SL and RAB-Motion also gives better output SINR performances than the other tested beamformers in the scenarios of this chapter.

4.6.6 Output SINR of RAB-SL With Mainlobe Interference

In the sixth example, the performance of the proposed RAB-SL is studied again in the presence of a mainlobe interference located at 84° . The mainlobe interference's INR is the same as the desired signal's SNR. The other parameters remain the same as those in the third example. The tested beamformers are:

Figure 4.8: Normalized beampatterns of the proposed RAB-Motion, the beamformers of Yu *et al.* [3] with $\epsilon_r = 0.3$ and 13.2, and the CMT beamformer [5] in a “frozen weights” scenario with a steering direction error of 6° and SNR= 0dB for one realization. The 40dB interference is at 140° . The DOAs of the impinging signals are denoted with black vertical lines. The vertical dotted lines denote the Θ_N region in the RAB-Motion.

Figure 4.9: Optimal SINR and output SINRs of the proposed RAB-Motion, the beamformers of Yu *et al.* [3] with $\epsilon_r = 0.3$ and 13.2, the beamformer of Shahbazpanahi *et al.* [2], the CMT beamformer [5], the beamformer of Li *et al.* [1] and CMT, and the beamformer of Vorobyov *et al.* [6] (combined with CMT) in a “frozen weights” scenario where a 40dB interference moves out of the beam pattern null during which the beamforming weights are applied.

Figure 4.10: Optimal SINR and output SINRs of the proposed RAB-SL, beamformers of Yu *et al.* [3] with $\epsilon_r = 0.3$ and 1.7, the beamformer of Shahbazpanahi *et al.* [2], the beamformer of Li *et al.* [1], and the MV beamformer versus the INR of the mainlobe interference at 84° . Its INR is the same as the SNR of the desired signal.

1. Proposed RAB-SL with $\epsilon_r = 0.3$ and $\rho_r = 2.2$;
2. Beamformers of Yu *et al.* [3] with $\epsilon_r = 0.3$ and 1.7;
3. Beamformer of Shahbazpanahi *et al.* [2] with $\hat{\mathbf{R}}$ loading = 25 and $\bar{\mathbf{R}}_s$ loading = -15;
4. Beamformer of Li *et al.* [1] with uncertainty sphere size = 15.83;
5. MV Beamformer.

From Fig. 4.10, the proposed RAB-SL achieves the highest output SINR performance when the mainlobe interference's INR ≤ -5 dB. However, the proposed RAB-SL fails to perform when the mainlobe interference gets stronger (≥ 0 dB) as the mainlobe constraints (4.7b) prevent the RAB-SL from forming a null to cancel

Figure 4.11: Tradeoff of the sidelobe level at different mainlobe roll-off widths, with null constraints at different interference rejection levels τ_n and without the null constraint (RB-SL), respectively.

it. Indeed, when an adaptive beamformer with a large robust mainlobe region is used for an application, it is usually assumed that there exists prior information that there are no interferences arriving in the protected mainlobe region.

4.6.7 Beampattern Shaping Design Tradeoffs

In the proposed framework (4.7), many inequality constraints have resulted due to the discretization process. According to the complementary slackness concept in optimization problems [63], if an inequality constraint is active, it is transformed into an equality one. Many factors can affect the number of active constraints [8].

In the final example, the effects of two design parameters, i.e., mainlobe roll-off width and interference rejection level τ_n on the achievable minimal sidelobe level are investigated. The formulation (4.7) is used but $\rho_r \rightarrow \infty$ is set so that the beamformer minimizes the sidelobe level only. The mainlobe roll-off width

varies from 7° to 9° and τ_n which varies from -45 to -65dB is imposed on $\Theta_N = \pm 5^\circ + 140^\circ$. It is desirable to have as narrow a mainlobe roll-off width as possible in order to quickly attenuate interferences which may arrive at directions very near to the protected mainlobe region. The RB-SL (with no null constraint) is also compared.

From Fig. 4.11, as the mainlobe roll-off width decreases (or as τ_n becomes more stringent), the sidelobe level increases due to a reduction in the number of active constraints for sidelobe suppression. The RB-SL (plotted in black in Fig. 4.11) shows that the lowest sidelobe level is achieved at the various mainlobe roll-off widths since it does not employ any null constraint. This means that though the use of null constraint (4.7d) provides robustness against strong moving interferences, they involve a compromise, in this case, the sidelobe level. In order to achieve the best output SINR performance, the user should select the specific beamformer from the proposed framework, tailored to the application needs.

4.7 Summary

Based on the use of a set of beampattern shaping constraints, the proposed framework achieves robustness against large steering direction errors and strong interferences that move out of the beampattern nulls during which the beamforming weights are applied. It also achieves adaptive interference rejection and direct sidelobe control where both abilities are controlled automatically by a weighing ratio in different scenarios without ambiguity. Three beamformers have been derived from the proposed framework and they have consistently shown significant output SINR performance improvement over the other tested beamformers.

Chapter 5

Beamforming Optimization on Antenna Selection With the Use of Linear Matrix Inequality (LMI) Constraints

5.1 Objectives

The objective of the research in this chapter is to study antenna selection optimization techniques for beamforming systems so as to determine the minimum number of antennas required for beampattern designs to satisfy the prescribed specifications accurately. Many beampattern design methods of [59, 70, 101] make use of discretization techniques so as to impose multiple constraints on the beampattern which inevitably, incur some heuristic approximation. Therefore, the method of Davidson *et al.* in [7, 8] with LMI constraints is studied as the constraints are able to represent the semi-infinite magnitude response (upper and lower bound) constraints on the beampattern in a finite and convex manner. This eliminates

the need to discretize the beampattern region like in Chapter 4 and hence, avoids the heuristic approximation involved¹ [7, 8]. However, the direct application of the method [7, 8] to compute the minimum number of antennas required is not systematic and the number of iterations before stopping is unknown.

In this chapter, a formulation is proposed to find the minimum number of antennas required to achieve the prescribed beampattern specifications and the corresponding beamforming weights systematically². The synthesized beampattern by the proposed antenna selection method conforms to the prescribed specifications of an arbitrary beampattern strictly, unlike [77]. An alternative LMI representation of the beampattern constraints is recently reported in [68].

The proposed formulation is quasi-convex which means that it can have stationary points that are suboptimum solutions, thus an iterative procedure is used to decompose the formulation into a systematic sequence of convex feasibility problems. At each step, a feasibility problem is solved efficiently via the IPM using open-source solvers, e.g., [88, 89], which determines if a set of beamforming weights exists, for a particular number of antennas. As such, the proposed antenna selection method is guaranteed to find the minimum number of antennas to achieve the prescribed beampattern specifications so long as this solution is in the search interval. This is easy to ensure at the start of the search.

5.2 Proposed Antenna Selection Method

The data model used in this chapter is the same as that in Section 4.2.1 where a ULA of $N + 1$ isotropic antenna elements is used and the beampattern design constraint in (4.4) is applied. Here, the aim is to find the minimum number of

¹The LMI constraints are not used to control the beampattern in Chapter 4 as a large number of auxiliary variables is involved and the computational complexity may not be suitable for adaptive beamforming applications that require beamforming weights to be computed quickly in response to the changes in the scenario.

²A related problem arises in minimum order filter design [69].

antenna elements $N + 1$ and the corresponding weights w_i for beampattern design. For notational convenience, let $M = N + 1$. The semi-infinite constraint (4.4) is used to specify beampattern requirements explicitly on Θ_{ML} , Θ_{SL} , and Θ_{N} , respectively via (5.1b)-(5.1c). The proposed problem is formulated as

$$\min_{M, \mathbf{r}_w} M \quad (5.1a)$$

$$\text{s.t. } L^2(\theta) \leq \mathbf{a}^T(\theta) \mathbf{r}_w \leq U^2(\theta), \quad \theta \in \Theta_{\text{ML}} \quad (5.1b)$$

$$\mathbf{a}^T(\theta) \mathbf{r}_w \leq U^2(\theta), \quad \theta \in \Theta_{\text{SL}} \cup \Theta_{\text{N}} \quad (5.1c)$$

$$\mathbf{a}^T(\theta) \mathbf{r}_w \geq 0, \quad \theta \in \Theta \quad (5.1d)$$

$$r_w(-k) = r_w^*(k), \quad k = 0, \dots, M - 1 \quad (5.1e)$$

$$M \in \mathbb{Z}_+, \quad (5.1f)$$

where “ $A \cup B$ ” means “union of sets A and B” and $\mathbb{Z}_+$ denotes the set of positive integers. The optimization variable in (5.1) is the array weight autocorrelation sequence $\mathbf{r}_w$. The constraint (5.1c) requires that the response in Θ_{SL} (or Θ_{N}) is at most $U^2(\theta)$. The constraint (5.1d) is sufficient to ensure that the complex weights w_i can be extracted (though not uniquely) from the obtained $\mathbf{r}_w$ by spectral factorization [59]. Here, minimum phase spectral factor is used³.

5.3 Reformulated Proposed Antenna Selection Method With LMI Constraints

It is a common practice to approximate the semi-infinite constraints (5.1b)-(5.1d) by discretizing θ , like in [59, 101]. This does not affect the convexity of the resulting constraints. Though the discretization grid can be made very fine, unfortunately, the discretized version of the problem (5.1) can have numerical difficulties [7, 8]

³Many choices exist for the spectral factor and the minimum phase spectral factor may not always be the most appropriate choice.

and still, the resulting constraints are approximations of (5.1b)-(5.1d) at best. In order to avoid these drawbacks, (5.1b)-(5.1d) are expressed as LMI constraints by rewriting the beampattern expression in (4.4) as

$$\mathbf{a}^T(\theta)\mathbf{r}_w = \Re\{\mathbf{s}^H(\Omega)\tilde{\mathbf{r}}_w\} \quad (5.2)$$

where $\mathbf{s}(\Omega) = [1 \ e^{j\Omega} \ \dots \ e^{jN\Omega}]^T$, $\tilde{\mathbf{r}}_w = [r_w(0) \ 2r_w(1) \ \dots \ 2r_w(N)]^T$ [106], and $\Re\{\cdot\}$ is a real operator. With (5.2), the constraints (5.1b)-(5.1d) can be written as (5.3a)-(5.3c) according to the specific beampattern region:

$$\Re\{\mathbf{s}^H(\Omega)(-\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1)\} \geq 0, \Omega \in \tilde{\Omega}_{\text{ML}} \cup \tilde{\Omega}_{\text{SL}} \cup \tilde{\Omega}_{\text{N}} \quad (5.3a)$$

$$\Re\{\mathbf{s}^H(\Omega)(\tilde{\mathbf{r}}_w - L^2(\Omega)\mathbf{e}_1)\} \geq 0, \Omega \in \tilde{\Omega}_{\text{ML}} \quad (5.3b)$$

$$\Re\{\mathbf{s}^H(\Omega)\tilde{\mathbf{r}}_w\} \geq 0, \forall \tilde{\Omega} \quad (5.3c)$$

where $\tilde{\Omega}_{(\cdot)}$ denotes the set of Ω in an angular region defined by the subscript and the unit vector $\mathbf{e}_1$ is the first column of a $M \times M$ identity matrix. Note the change in the arguments of $L^2(\cdot)$ and $U^2(\cdot)$. Next, Theorem 2 [7, 8] is applied to transform the constraints in (5.3) into their equivalent LMI forms⁴.

Theorem 2. *With a $\mathbf{p} \in \mathbb{R} \times \mathbb{C}^M$ where $\mathbb{R}$ and $\mathbb{C}^M$ denote the sets of real numbers and complex $M \times 1$ vectors, respectively and that $0 \leq \Omega_l < \Omega_u < 2\pi$, these sets*

$$\mathcal{K}(\Omega_l, \Omega_u) = \{\mathbf{p} | \Re\{\mathbf{s}^H(\Omega)\mathbf{p}\} \geq 0, \Omega \in [\Omega_l, \Omega_u]\} \quad (5.4a)$$

$$\bar{\mathcal{K}}(\Omega_l, \Omega_u) = \{\mathbf{p} | \Re\{\mathbf{s}^H(\Omega)\mathbf{p}\} \geq 0, \Omega \in [0, 2\pi) \setminus (\Omega_l, \Omega_u)\} \quad (5.4b)$$

$$\mathcal{K}(0, 2\pi) = \{\mathbf{p} | \Re\{\mathbf{s}^H(\Omega)\mathbf{p}\} \geq 0, \Omega \in [0, 2\pi)\} \quad (5.4c)$$

describe trigonometric polynomials that are non-negative over a segment of the unit circle $[\Omega_l, \Omega_u]$, the complement of that segment $[0, 2\pi) \setminus (\Omega_l, \Omega_u)$, and on the unit

⁴For related development on trigonometric polynomials, please refer to [68, 107, 108].

circle $[0, 2\pi)$, respectively. By a generalized Positive Real Lemma [7, 8], (5.4a)-(5.4c) can be converted into equivalent LMI forms as

$$\mathcal{K}(\Omega_l, \Omega_u) = \{\mathbf{p} | \mathbf{p} + j\xi \mathbf{e}_1 = \bar{L}(\mathbf{X}) + \bar{\Lambda}(\mathbf{Z}; \Omega_l, \Omega_u), \exists \mathbf{X}, \mathbf{Z} \succeq 0\}, \quad (5.5a)$$

$$\bar{\mathcal{K}}(\Omega_l, \Omega_u) = \{\mathbf{p} | \mathbf{p} + j\xi \mathbf{e}_1 = \bar{L}(\mathbf{X}) - \bar{\Lambda}(\mathbf{Z}; \Omega_l, \Omega_u), \exists \mathbf{X}, \mathbf{Z} \succeq 0\}, \quad (5.5b)$$

$$\mathcal{K}(0, 2\pi) = \{\mathbf{p} | \mathbf{p} = \bar{L}(\mathbf{X}), \exists \mathbf{X} \succeq 0\}, \quad (5.5c)$$

respectively where ξ is an arbitrary real scalar and $\mathbf{X}, \mathbf{Z} \succeq 0$ denotes positive semidefinite Hermitian matrices. $\bar{L}(\cdot)$ and $\bar{\Lambda}(\cdot)$ are linear operators where their definitions have been deferred to Appendix A.

Applying Theorem 2 to the beamforming problem of interest, the proposed formulation (5.1) is transformed into

$$\min_{M, \tilde{\mathbf{r}}_w} M \quad (5.6a)$$

$$\text{s.t.} \quad -\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_1\mathbf{e}_1 = \bar{L}(\mathbf{X}_1) - \bar{\Lambda}(\mathbf{Z}_1; \Omega_{ml}, \Omega_{mu}), \Omega \in \tilde{\Omega}_{ML} \quad (5.6b)$$

$$\tilde{\mathbf{r}}_w - L^2(\Omega)\mathbf{e}_1 + j\xi_2\mathbf{e}_1 = \bar{L}(\mathbf{X}_2) - \bar{\Lambda}(\mathbf{Z}_2; \Omega_{ml}, \Omega_{mu}), \Omega \in \tilde{\Omega}_{ML} \quad (5.6c)$$

$$-\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_3\mathbf{e}_1 = \bar{L}(\mathbf{X}_3) + \bar{\Lambda}(\mathbf{Z}_3; \Omega_{sl}, \Omega_{su}), \Omega \in \tilde{\Omega}_{SL} \quad (5.6d)$$

$$-\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_4\mathbf{e}_1 = \bar{L}(\mathbf{X}_4) + \bar{\Lambda}(\mathbf{Z}_4; \Omega_{nl}, \Omega_{nu}), \Omega \in \tilde{\Omega}_N \quad (5.6e)$$

$$\tilde{\mathbf{r}}_w = \bar{L}(\mathbf{X}_5), \forall \tilde{\Omega} \quad (5.6f)$$

$$M \in \mathbb{Z}_+, \forall \mathbf{X} \succeq 0, \forall \mathbf{Z} \succeq 0 \quad (5.6g)$$

where Ω_{ml} and Ω_{mu} define the lower and upper boundaries of $\tilde{\Omega}_{ML}$, respectively. This notational convention applies to $\tilde{\Omega}_{SL}$ and $\tilde{\Omega}_N$ as well. The mainlobe constraints in (5.6) obtain a broad mainlobe and assume it is centred at 90° hence (5.5b) is used. Since $\tilde{\Omega}_{SL}$ and $\tilde{\Omega}_N$ fall outside this region, then (5.5a) is used. The LMI form of (5.1d) is given by (5.6f). The sets of $\mathbf{X}$, $\mathbf{Z}$, and ξ with different subscripts are used to differentiate the constraints in (5.6). The optimization variable

here is $\tilde{\mathbf{r}}_w$ and not $\mathbf{r}_w$ as in (5.1), so the constraint (5.1e) can be omitted from (5.6). After (5.6) is solved, the solution $\tilde{\mathbf{r}}_w$ is rearranged to obtain $\mathbf{r}_w$ according to (5.1e) and (5.2), after which spectral factorization is applied on $\mathbf{r}_w$ to derive the beamforming weights w_i .

If beampattern requirements specify a mainlobe with unity gain in a desired direction θ_0 (or equivalently Ω_0), then the mainlobe constraints in (5.6) are not applicable. They should be replaced by $\Re\{\mathbf{s}^H(\Omega_0)\tilde{\mathbf{r}}_w\} = 1$ as

$$\min_{M, \tilde{\mathbf{r}}_w} M \quad (5.7a)$$

$$\text{s.t.} \quad \Re\{\mathbf{s}^H(\Omega_0)\tilde{\mathbf{r}}_w\} = 1, \quad (5.7b)$$

$$-\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_1\mathbf{e}_1 = \bar{L}(\mathbf{X}_1) + \bar{\Lambda}(\mathbf{Z}_1; \Omega_{sl}, \Omega_{su}), \Omega \in \tilde{\Omega}_{\text{SL}} \quad (5.7c)$$

$$-\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_2\mathbf{e}_1 = \bar{L}(\mathbf{X}_2) + \bar{\Lambda}(\mathbf{Z}_2; \Omega_{nl}, \Omega_{nu}), \Omega \in \tilde{\Omega}_{\text{N}} \quad (5.7d)$$

$$\tilde{\mathbf{r}}_w = \bar{L}(\mathbf{X}_3), \forall \tilde{\Omega} \quad (5.7e)$$

$$M \in \mathbb{Z}_+, \forall \mathbf{X} \succeq 0, \forall \mathbf{Z} \succeq 0. \quad (5.7f)$$

Different from (5.1), the proposed formulations (5.6) and (5.7) impose finite LMI constraints on the beampattern. However, (5.1), (5.6) and (5.7) are all quasi-convex optimization problems consisting of convex constraints and a quasi-convex objective function. The discontinuous objective function $f(M) = M$ is quasi-convex since the modified Jensen's inequality is proven to hold:

$$f(\alpha m_1 + (1 - \alpha)m_2) \leq \max\{f(m_1), f(m_2)\}, \quad (5.8)$$

$$\alpha m_1 + (1 - \alpha)m_2 \leq \max\{m_1, m_2\}, \quad (5.9)$$

$$\alpha(m_1 - m_2) + m_2 \leq m_2, \quad (5.10)$$

where $0 \leq \alpha \leq 1$, $\max\{\cdot\}$ is an operator that gives the maximum value of its arguments, m_1 and m_2 are elements in the domain of the objective function. The

last two inequalities hold with $m_1 < m_2$.

5.4 Implementation of the Proposed Antenna Selection Method

The sub-level sets of quasi-convex optimization problems are convex [63], thus the globally optimum solutions of the proposed formulations (5.1), (5.6) and (5.7) can be found by decomposing them into a sequence of convex feasibility problems. (5.7) is used as an example. Suppose M_{opt} is the globally optimum solution of (5.7). Consider the decomposition of (5.7) into (5.11) with the same constraints but at M_c number of antennas.

$$\begin{aligned}
 &\text{Find} && \tilde{\mathbf{r}}_w && (5.11) \\
 &\text{s.t.} && \Re\{\mathbf{s}^H(\Omega_0)\tilde{\mathbf{r}}_w\} = 1, \\
 &&& -\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_1\mathbf{e}_1 = \bar{L}(\mathbf{X}_1) + \bar{\Lambda}(\mathbf{Z}_1; \Omega_{sl}, \Omega_{su}), \Omega \in \tilde{\Omega}_{SL} \\
 &&& -\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_2\mathbf{e}_1 = \bar{L}(\mathbf{X}_2) + \bar{\Lambda}(\mathbf{Z}_2; \Omega_{nl}, \Omega_{nu}), \Omega \in \tilde{\Omega}_N \\
 &&& \tilde{\mathbf{r}}_w = \bar{L}(\mathbf{X}_3), \forall \tilde{\Omega} \\
 &&& \forall \mathbf{X} \succeq 0, \forall \mathbf{Z} \succeq 0.
 \end{aligned}$$

The problem (5.11) is convex which can be solved via an IPM solver to determine if the constraints are feasible at M_c number of antennas. If so, there exists a non-empty feasible set and it finds a solution point $\tilde{\mathbf{r}}_w$ in the set, implying that $M_{opt} \leq M_c$. Feasible beamforming weights w_i can be found after spectral factorization is performed on $\mathbf{r}_w$ (by transforming $\tilde{\mathbf{r}}_w$). Otherwise, the feasible set is empty and the solver issues a certificate of infeasibility implying that $M_{opt} > M_c$.

As such, the proposed formulations can be solved by an iterative procedure known as the bisection search. It starts with a search interval $[M_l, M_u]$ presumed

to contain M_{opt} where $M_l < M_u$. The feasibility problem (5.11) is solved at $M_c = \frac{(M_l + M_u)}{2}$ to determine if M_{opt} resides in the lower or upper half of the interval, after which the search interval is updated accordingly. This produces a new interval containing M_{opt} at half of the previous interval width. The above steps are repeated until the stopping criterion is reached where the search interval converges to a globally optimum M_{opt} value, i.e., $M_u - M_l < 1$.

5.5 Convergence of the Proposed Antenna Selection Method

In the previous section, it is presumed that M_{opt} exists in the search interval $[M_l, M_u]$. This insinuates that the problem (5.11) has to be feasible at M_u antennas. Otherwise, M_{opt} is not located in $[M_l, M_u]$ and (5.11) will be infeasible at all the tested values in $[M_l, M_u]$.

In order to prevent the proposed antenna selection method from searching through $[M_l, M_u]$ which does not contain M_{opt} , a simple suggestion is to check the feasibility of (5.11) at M_u number of antennas at the start of the search. If (5.11) is infeasible at M_u antennas, this infeasible M_u value is assigned to M_l and M_u is re-assigned with M'_u ($M_u < M'_u$) so that the new search interval $[M_l, M_u]$ does not widen unnecessarily. Given the nature of the bisection search, an appropriate choice of M'_u can be twice the value of M_u . This is done until the feasibility check is passed at M_u number of antennas. In so doing, the proposed method ensures that it is searching through an interval which contains M_{opt} and thereby guarantees convergence. The implementation procedure of the proposed antenna selection method is summarized here.

1. Choose a search interval $[M_l, M_u]$. M_l can be set to 1.
2. Check the feasibility of equation (5.11) at M_u number of antennas. If it

is feasible, proceed to step (3). Otherwise, update M_l with the value of M_u . Update M_u with M'_u whose value is greater than M_u . Repeat step (2). This ensures that the proposed method is searching through $[M_l, M_u]$ where $M_l < M_{opt} \leq M_u$ and that the search interval does not widen unnecessarily.

3. Let $M_c = \lceil \frac{M_l + M_u}{2} \rceil$. Since the number of antennas has to be an integer, a $\lceil \cdot \rceil$ operator is used to round up $(\frac{M_l + M_u}{2})$ which can be a non-integer.
4. Check the feasibility of equation (5.11) at M_c number of antennas.
5. If equation (5.11) is reported feasible, update M_u with the value of M_c . Otherwise, update M_l with the value of M_c .
6. Repeat steps (3)-(5) until the stopping criterion $(M_u - M_l < 1)$ is reached.

Suppose the proposed method begins with $[M_l, M_u]$ known in advance to contain M_{opt} , then the number of iterations before it stops is $\lceil \log_2(M_u - M_l) \rceil$ [63].

5.6 Simulation Results and Discussion

The proposed formulations (5.6) and (5.7) are applied to find the minimum number of antennas needed for two different beampattern designs (specifications are shown in dashed lines) and their corresponding beamforming weights. A ULA of M isotropic antennas with $d = 0.5\lambda$ spacing is used. The search interval is set to $[M_l, M_u] = [1, 32]$ where both designs are tested feasible at 32 antennas. The beampatterns for both designs using $M_u = 32$ antennas by the method of [7, 8] are also shown.

Figure 5.1: Beampatterns obtained by the method of Davidson *et al.* [7, 8] with 32 antennas and the proposed antenna selection method (5.6) with 14 antennas. The beampattern has to be lower than the outer dashed lines and higher than the inner dashed lines.

5.6.1 Beampattern Design With Broad Mainlobe and Controlled Sidelobe Level

First, a beampattern with a broad mainlobe and suppressed sidelobe level is desired. The sidelobe region is $[0^\circ, 69.4^\circ] \cup [110.6^\circ, 180^\circ]$ to be suppressed by -25dB . The mainlobe width is set to 22° and its response ripple is to be within 0.48dB . The proposed formulation (5.6) (without null constraint) is used to achieve the beampattern (plotted in red) in Fig. 5.1 and the minimum number of antennas required is $M_{opt} = 14$. The corresponding complex beamforming weights (magnitude and phase) are plotted in Fig. 5.2.

Figure 5.2: Complex beamforming weights (magnitude and phase) obtained by the proposed antenna selection method (5.6) to achieve the beampattern (plotted in red) in Fig. 5.1.

5.6.2 Beampattern Design With Narrow Mainlobe, Controlled Sidelobe Level, and Broad Nulls

Next, a beampattern with narrow mainlobe, controlled sidelobe level, and broad nulls is desired, so the proposed formulation (5.7) is used. The sidelobe region is $[0^\circ, 79.55^\circ] \cup [100.45^\circ, 180^\circ]$ to be suppressed by -40dB . The null regions are $[50^\circ, 60^\circ] \cup [120^\circ, 130^\circ]$ with an attenuation level of -55dB . The resulting beampattern is shown in Fig. 5.3 and $M_{opt} = 20$. The corresponding complex beamforming weights (magnitude and phase) are plotted in Fig. 5.4.

Given the same prior information that the optimum solution M_{opt} lies in $[M_l, M_u] = [1, 32]$, the direct application of the method of [7, 8] to find the minimum number of antennas needed for beampattern designs would involve either increasing the number of antennas one by one from $M_l = 1$ or reducing the number

Figure 5.3: Beam patterns obtained by the method of Davidson *et al.* [7, 8] with 32 antennas and the proposed antenna selection method (5.7) with 20 antennas. The beampattern has to be lower than the dashed lines.

of antennas one by one from $M_u = 32$. However, this results in a large number of iterations (20 and 14 iterations for the first and second designs, respectively for the latter case). Both ways are not efficient and the number of iterations required before stopping is unknown. In contrast, the advantages of the proposed antenna selection method are that the minimum number of antennas for beampattern designs is computed in a systematic way via the bisection search and the number of iterations before it stops, is known. For the proposed method, only 5 iterations are required in both cases.

Note that in the case of applying the method of [7, 8] using more antennas than required, the extra DOFs are well-utilized as the generated beampatterns in Figs. 5.1 and 5.3 have exceeded the beampattern specifications. Such a performance is due to the properties of the IPM.

Another remark is that applying the LMI constraints as in the proposed an-

Figure 5.4: Complex beamforming weights (magnitude and phase) obtained by the proposed antenna selection method (5.7) to achieve the beampattern (plotted in red) in Fig. 5.3.

Antenna selection method requires a large number of auxiliary variables and as noted in [68], the computational complexity does not grow gracefully with the problem size. This makes the proposed antenna selection method difficult to implement for large arrays. From the simulation experience, the beampattern design for an array with more than 50 antenna elements cannot be performed on a personal computer.

Finally, for non-uniformly spaced arrays, the LMI forms in (5.5) cannot be applied. However, if the non-uniformly spaced array is symmetric, then by imposing conjugate symmetric beamforming weights, another LMI characterization for trigonometric polynomials [68] can be used on the non-uniformly spaced symmetric array. However, the search for the minimum number of array antennas has to be done exhaustively. Alternatively, after the first optimization step with the original non-uniformly spaced symmetric array to obtain the beamforming weights, the pair of symmetric antennas with the smallest weight magnitude can be removed.

At the next step, the optimization process is repeated on the new array in order to obtain another set of beamforming weights. This process is repeated until the LMI constraints on the beampattern cannot be maintained. Note that the suggested approach is not guaranteed to converge to the minimum number of antennas.

5.7 Summary

An antenna selection method for beamforming systems has been proposed to find the minimum number of antennas needed for beampattern designs and the corresponding beamforming weights, systematically. LMI constraints are imposed on the beampattern so that the prescribed specifications are satisfied precisely. The proposed quasi-convex formulation is decomposed into a systematic sequence of convex feasibility problems and it is guaranteed to find the minimum number of antennas required so long as this solution lies in the search interval, which can be easily ensured at the start of the search. Simulation results are given to show the effectiveness of the proposed method.

Chapter 6

Beamforming Optimization on Antenna Selection for Planar Arrays With Conjugate Symmetric Beamforming Weights

6.1 Objectives

One of the objectives of the research in this chapter is to study the proposed antenna selection method of Chapter 5 for use on planar arrays. This would require the reformulation of the beampattern expression for planar arrays into the LMI form of [7, 8]. Based on the simulation experience in Chapter 5, the implementation of the LMI constraints in arrays with more than 50 antennas is not possible on a personal computer due to the large number of auxiliary variables involved [68]. Most importantly, the proposed method of Chapter 5 is restricted to uniformly spaced arrays.

Therefore, this chapter proposes an antenna selection method for planar ar-

rays with the use of conjugate symmetric beamforming weights in order that the non-convex lower bound magnitude response constraint on the beampattern can be affine (thus convex). The resulting upper bound beampattern constraint is also affine. This enables the design of a mainlobe with controllable beamwidth and response ripple (to achieve robustness against large steering direction errors). In addition, arbitrary sidelobe levels can also be designed (to achieve robustness against unexpected or strong moving interferences). The proposed method minimizes a re-weighted objective function based on the magnitudes of the elements in the beamforming weight vector iteratively to suppress its small elements to zero. A zero element in the beamforming weight vector implies that the corresponding antenna is redundant. This results in a more efficient beampattern requiring fewer antennas that satisfies the same specifications as that achieved by a non-sparse array. The proposed method can design non-uniformly spaced arrays (with inter-element spacings larger than one half-wavelength) for an arbitrary beampattern precisely without suffering from grating lobes.

Another objective of the research in this chapter is to study the robustness of the proposed antenna selection method in the presence of array imperfections and mutual coupling effects. The findings show that the beampatterns produced by the proposed method of this chapter can violate the beampattern specifications because the actual array manifold differs from the ideal presumed one due to array imperfections or mutual coupling effects. Therefore, this chapter further proposes robust beampattern constraints for the antenna selection method based on the uncertainty set of the actual array manifold and worst-case performance optimization [43, 63]. Simulation results show that the resulting beampatterns are robust against array imperfections and mutual coupling effects while achieving specifications wholly without increasing the number of selected antennas. The proposed antenna selection method with the robust beampattern constraints is a

SOCP that can be solved by the IPM efficiently in polynomial-time with open-source solvers, e.g., [88, 89]. The proposed methods can also be used on linear arrays.

6.2 Data Model

A rectangular planar array is considered with N and M isotropic antennas in the x and y directions, respectively as shown in Fig. 6.1. The inter-element spacings are d_x and d_y in the x and y directions, respectively. A linear array can be easily derived. A plane wave of λ wavelength impinges on the array in a (θ, ϕ) direction where θ and ϕ denote the polar and azimuth angles, respectively. With direction cosines $u_x = \sin \theta \cos \phi$ and $u_y = \sin \theta \sin \phi$ [11], respectively, the array beampattern is defined as

$$G(u_x, u_y) = \sum_{n=1}^N \sum_{m=1}^M w_{n,m} e^{j \frac{2\pi}{\lambda} n d_x u_x + j \frac{2\pi}{\lambda} m d_y u_y} \quad (6.1)$$

with complex beamforming weights $w_{n,m}$ for $n = 1, \dots, N$ and $m = 1, \dots, M$. A typical mainlobe design constraint is

$$l \leq |G(u_x, u_y)| \leq u, \quad u_x, u_y \in [-1, 1]. \quad (6.2)$$

Such a constraint offers flexibility in controlling the mainlobe beamwidth and response ripple. Although (6.2) is intuitive, it is not widely employed in beampattern design applications as the lower bound constraint $l \leq |G(u_x, u_y)|$ is non-linear, non-convex, and thus, NP-hard to solve.

Figure 6.1: A uniformly spaced rectangular planar array with N and M antennas (indicated by the black circles) in the x and y directions, respectively. The inter-element spacings are d_x and d_y in the x and y directions, respectively. The source impinges in a (θ, ϕ) direction where θ and ϕ are the polar and azimuth angles, respectively.

6.3 Proposed Beampattern Constraints

The aim here is to develop convex lower and upper magnitude response constraints on the beampattern¹. One possible approach is to use the array weight autocorrelation sequence $\mathbf{r}_w$ as is in Chapters 4 and 5 to transform the constraint in (6.2) into an affine one. However, this approach requires spectral factorization in order to obtain the beamforming weights and hence, it does not allow operations to be performed on the beamforming weights directly. Moreover, the use of $\mathbf{r}_w$ is restricted to uniformly spaced arrays.

Considering that the array manifolds of non-uniformly spaced but symmetric linear or planar arrays have conjugate symmetric structure (by placing the centre of the array at the origin of the coordinate system), the beampattern function in (6.2) can be transformed into a real function by using conjugate symmetric beamforming weights; $w_{i,j} = w_{N-i+1,M-j+1}^*$, i.e., $w_{1,1} = w_{N,M}^*$, $w_{1,2} = w_{N,M-1}^*$, and so on. Any lower or upper bound constraint on a real function is affine. Therefore, the beampattern in (6.1) can be written as

$$\begin{aligned} G(u_x, u_y) = & e^{j\frac{2\pi}{\lambda}(\frac{N+1}{2})d_x u_x + j\frac{2\pi}{\lambda}(\frac{M+1}{2})d_y u_y} \left[w_{1,1} e^{-j\frac{2\pi}{\lambda}(\frac{N-1}{2})d_x u_x - j\frac{2\pi}{\lambda}(\frac{M-1}{2})d_y u_y} \right. \\ & + w_{1,2} e^{-j\frac{2\pi}{\lambda}(\frac{N-1}{2})d_x u_x - j\frac{2\pi}{\lambda}(\frac{M-1}{2}-1)d_y u_y} + \cdots + w_{(\frac{N-1}{2}+1),(\frac{M-1}{2}+1)} + \cdots \\ & \left. + w_{1,2}^* e^{j\frac{2\pi}{\lambda}(\frac{N-1}{2})d_x u_x + j\frac{2\pi}{\lambda}(\frac{M-1}{2}-1)d_y u_y} + w_{1,1}^* e^{j\frac{2\pi}{\lambda}(\frac{N-1}{2})d_x u_x + j\frac{2\pi}{\lambda}(\frac{M-1}{2})d_y u_y} \right]. \end{aligned} \quad (6.3)$$

The magnitude response $|G(u_x, u_y)|$ is then expressed as

$$\begin{aligned} |G(u_x, u_y)| = & 2\Re\left\{ w_{1,1} e^{-j\frac{2\pi}{\lambda}(\frac{N-1}{2})d_x u_x - j\frac{2\pi}{\lambda}(\frac{M-1}{2})d_y u_y} \right. \\ & \left. + w_{1,2} e^{-j\frac{2\pi}{\lambda}(\frac{N-1}{2})d_x u_x - j\frac{2\pi}{\lambda}(\frac{M-1}{2}-1)d_y u_y} + \cdots + w_{(\frac{N-1}{2}+1),(\frac{M-1}{2}+1)} \right\}. \end{aligned} \quad (6.4)$$

¹It is worth noting that if only the upper magnitude response constraint on the beampattern is required, then quadratic inequality constraints can be used as is in [32, Eq. 5].

Odd N and M are assumed. The extension to even N and M follows similarly. With conjugate symmetric beamforming weights, the imaginary parts of the terms in the square brackets of (6.3) cancel out. Thus, $|G(u_x, u_y)|$ is a real function in (6.4) which can be rewritten as $|G(u_x, u_y)| = \mathbf{s}^T(u_x, u_y)\mathbf{w}$. The non-convex constraint (6.2) is now equivalent to

$$l \leq \mathbf{s}^T(u_x, u_y)\mathbf{w} \leq u, \quad u_x, u_y \in [-1, 1] \quad (6.5)$$

where

$$\begin{aligned} \mathbf{s}(u_x, u_y) = & \begin{bmatrix} e^{-j\frac{2\pi}{\lambda}(\frac{N-1}{2}d_x u_x - j\frac{2\pi}{\lambda}(\frac{M-1}{2})d_y u_y} & e^{-j\frac{2\pi}{\lambda}(\frac{N-1}{2}d_x u_x - j\frac{2\pi}{\lambda}(\frac{M-1}{2}-1)d_y u_y} & \dots & 1 & \dots \\ e^{j\frac{2\pi}{\lambda}(\frac{N-1}{2}d_x u_x + j\frac{2\pi}{\lambda}(\frac{M-1}{2}-1)d_y u_y} & e^{j\frac{2\pi}{\lambda}(\frac{N-1}{2}d_x u_x + j\frac{2\pi}{\lambda}(\frac{M-1}{2})d_y u_y} & \dots & 1 & \dots \end{bmatrix}^T \end{aligned}$$

is the array manifold for a rectangular planar array. For every (u_x, u_y) , the constraint (6.5) is a pair of linear inequalities in $\mathbf{w}$. Thus, (6.5) is affine over $\mathbf{w}$.

The constraint (6.5) is used to provide direct control over the mainlobe region (U_{ML}) in (6.6b), and modified for the sidelobe (U_{SL}) and null (U_{N}) regions, respectively in (6.6c)-(6.6d) as

$$\min_{\mathbf{w}} \quad \tau \quad (6.6a)$$

$$\text{s.t.} \quad l \leq \mathbf{s}^T(u_x, u_y)\mathbf{w} \leq u, \quad \{u_x, u_y\} \in U_{\text{ML}} \quad (6.6b)$$

$$\mathbf{s}^T(u_x, u_y)\mathbf{w} \leq \tau, \quad \{u_x, u_y\} \in U_{\text{SL}} \quad (6.6c)$$

$$\mathbf{s}^T(u_x, u_y)\mathbf{w} \leq \tau_n, \quad \{u_x, u_y\} \in U_{\text{N}}. \quad (6.6d)$$

Though the objective of this chapter is to develop an antenna selection method, for completeness, an important beampattern design aim (sidelobe level minimization) is briefly shown in (6.6). The problem (6.6) minimizes τ which defines the magnitude response of U_{SL} , although other beampattern parameters may be minimized. τ_n is a user-specified parameter which defines the magnitude response in

U_N . The problem (6.6) is a (convex) LP with affine objective function and constraints. Given that (6.6) is a LP, it can be solved globally by the IPM². The solution τ thus obtained is the best that can be achieved, even by other methods with the same constraints.

The designed constraints (6.6b) and (6.6c)-(6.6d) can generate completely arbitrarily shaped mainlobe and arbitrary sidelobe levels with specified magnitude responses (precisely), respectively. This capability is lacking in many methods of [74–76]. For example, to obtain a diamond-shaped or circular-shaped mainlobe, the mainlobe region in (6.6b) is defined as $\{(u_x, u_y) \mid |u_x| + |u_y| \leq u_p\} \in U_{ML}$ or $\{(u_x, u_y) \mid u_x^2 + u_y^2 \leq u_p^2\} \in U_{ML}$, respectively where u_p is a specified constant.

6.4 Proposed Antenna Selection Method

This section aims to find an array with as few antennas as possible to achieve the beampattern constraints (6.6b)-(6.6d).

6.4.1 Problem Development

The problem of interest is related to finding the sparsest $\mathbf{w}$:

$$\min_{\mathbf{w}} \quad \|\mathbf{w}\|_0 \quad (6.7a)$$

$$\text{s.t.} \quad l \leq \mathbf{s}^T(u_x, u_y)\mathbf{w} \leq u, \quad \{u_x, u_y\} \in U_{ML} \quad (6.7b)$$

$$\mathbf{s}^T(u_x, u_y)\mathbf{w} \leq \tau, \quad \{u_x, u_y\} \in U_{SL} \quad (6.7c)$$

$$\mathbf{s}^T(u_x, u_y)\mathbf{w} \leq \tau_n, \quad \{u_x, u_y\} \in U_N \quad (6.7d)$$

where the l_0 -quasi norm $\|\cdot\|_0$ is the count of the number of non-zero elements of its argument³. For (6.7)-(6.9), τ is a user-specified term. Though (6.7b)-(6.7d)

²The IPM has the advantage of polynomial complexity over the simplex algorithm in solving LPs [109].

³ l_0 -quasi norm is not a valid norm as it violates the triangle inequality.

are convex constraints, (6.7) is a NP-hard combinatorial optimization problem due to the non-convex objective function. In order to find the globally optimal solution, an exhaustive combinatorial search through $2^{NM/2}$ sparsity patterns of $\mathbf{w}$ is required because each w_k can either take on values of zero or non-zero. For each of these combinations, a convex problem results but the exhaustive search is easily intractable even for a modest-sized array.

In order to circumvent the intractable problem (6.7), a good approach is to approximate or relax it in a convex manner by solving

$$\min_{\mathbf{w}} \quad \|\mathbf{w}\|_1 \quad (6.8a)$$

$$\text{s.t.} \quad l \leq \mathbf{s}^T(u_x, u_y)\mathbf{w} \leq u, \quad \{u_x, u_y\} \in U_{\text{ML}} \quad (6.8b)$$

$$\mathbf{s}^T(u_x, u_y)\mathbf{w} \leq \tau, \quad \{u_x, u_y\} \in U_{\text{SL}} \quad (6.8c)$$

$$\mathbf{s}^T(u_x, u_y)\mathbf{w} \leq \tau_n, \quad \{u_x, u_y\} \in U_{\text{N}} \quad (6.8d)$$

where $\|\mathbf{x}\|_1 = \sum_{k=1}^{NM} |x_k|$ is known as the l_1 -norm. Indeed, the l_1 -norm is the closest convex function to the l_0 -quasi norm and the l_1 -norm is known to produce sparse solutions for many applications⁴. The problem (6.8) is a SOCP.

In order to further increase the sparsity of $\mathbf{w}$, the algorithm in [111, 112] is modified for our beampattern design problem⁵. In the proposed antenna selection method (6.9), a re-weighted l_1 -norm of the vector $\mathbf{w}$ is minimized at each step as

$$\min_{\mathbf{w}^i} \quad \sum_{k=1}^{NM} \varphi(w_k^{i-1})|w_k^i| \quad (6.9a)$$

$$\text{s.t.} \quad l \leq \mathbf{s}^T(u_x, u_y)\mathbf{w}^i \leq u, \quad \{u_x, u_y\} \in U_{\text{ML}} \quad (6.9b)$$

⁴Theoretical analysis on the sparsity-promoting properties of the l_1 -norm can be found in [110] and the references therein.

⁵The algorithm is used for sparse signal recovery [111] and for portfolio optimization with real-valued variables [112]. In this chapter, the algorithm is modified for a new problem for beampattern design applications where the optimized beamforming weight vector is complex-valued.

$$\mathbf{s}^T(u_x, u_y) \mathbf{w}^i \leq \tau, \quad \{u_x, u_y\} \in U_{\text{SL}} \quad (6.9c)$$

$$\mathbf{s}^T(u_x, u_y) \mathbf{w}^i \leq \tau_n, \quad \{u_x, u_y\} \in U_{\text{N}} \quad (6.9d)$$

where the weightings $\varphi(w_k^{i-1}) = 1/(|w_k^{i-1}| + \delta)$ are assigned to each of the elements of $\mathbf{w}^i$ ($\mathbf{w}$ at i th iteration). The threshold $\delta > 0$ provides numerical stability and helps determine if a particular w_k should be considered as zero. δ is small and fixed during the iterations. In the first iteration ($i = 1$), the weightings $\varphi(w_k^0)$ can be initialized to all ones and the resulting objective function (6.9a) becomes $\sum_{k=1}^{NM} |w_k| = \|\mathbf{w}\|_1$. Thus, the equivalent problem to solve in the first iteration is (6.8) which is a good initializer for (6.9).

With the obtained w_k^i , those of small magnitudes are given larger weightings $1/(|w_k^i| + \delta)$ in the next iteration and vice versa. At each iteration, a positive weighted sum of the magnitudes of the elements in $\mathbf{w}$ is minimized to produce a new $\mathbf{w}$ hence the proposed method (6.9) is still a SOCP. The iterative procedure repeats until the maximum number of iterations is reached. This value can be readily obtained by simple trial-and-error during the design process. By extensive simulations, it is found suitable to set it as 20. The result is that the small entries in $\mathbf{w}$ are suppressed to zero, as far as the constraints allow and thus, yielding a sparse $\mathbf{w}$.

The proposed method (6.9) can help reduce the Dynamic Range Ratio (DRR) of $\mathbf{w}$ which eases the design and implementation of the feed networks. A direct way to lower the DRR may be to eliminate antennas with small weights but the resulting beampattern may violate the specifications. Instead, $\mathbf{w}$ is continually adjusted at each iteration of the proposed method (6.9) according to the magnitudes of its elements (to increase its sparsity) while satisfying the specifications precisely.

6.4.2 Algorithm Summary and Remarks

The proposed antenna selection method (6.9) is summarized.

1. In the first iteration ($i = 1$), initialize $\varphi(w_k^{i-1}) = \varphi(w_k^0) = 1$ for all k .
Therefore, the equivalent problem to solve is equation (6.8).
2. For subsequent iterations ($i > 1$), with a fixed $\delta = 10^{-5}$ (the choice of δ is discussed in Section 6.7 (Simulation Results and Discussion)) and the obtained w_k^{i-1} , set $\varphi(w_k^{i-1}) = 1/(|w_k^{i-1}| + \delta)$ and solve equation (6.9).
3. Repeat step (2) until the maximum number of iterations of 20 is reached.

Some remarks on the proposed methods (6.6) and (6.9) are:

1. The antenna selection method (6.9) can be used to switch off antennas in existing arrays for different beampattern designs. Next, compared to uniformly spaced arrays, an array with the same number but arbitrarily spaced antennas has more DOFs, suggesting that the design of non-uniformly spaced arrays can lessen the number of antennas required [113]. (6.9) can be used to design such arrays for prescribed beampattern specifications before the actual array is mounted.
2. While the antenna selection method (6.9) is not guaranteed to reach the globally optimal solution⁶ for (6.7), it converges very quickly with a few iterations as observed from the simulation examples in Section 6.7 (Simulation Results and Discussion).
3. Unlike [74], the proposed methods (6.6) and (6.9) still apply directly when all the antennas have the same directive pattern $p(u_x, u_y)$ by modifying the proposed constraint as $l \leq p(u_x, u_y)[\mathbf{s}^T(u_x, u_y)\mathbf{w}] \leq u$. The polarization of

⁶A combinatorial search through $2^{NM/2}$ sparsity patterns of $\mathbf{w}$ to find the globally optimal solution is intractable for the size of arrays that are considered in this chapter.

the array antennas is considered to match that of the impinging wave so as to extract the maximum power from it [80]. It should be noted that (6.6) and (6.9) are not restricted to arrays with the same number of antennas in both x and y directions ($N \neq M$), unlike [75]. The inter-element spacings can also be different ($d_x \neq d_y$).

6.5 Numerical Study of the Proposed Antenna Selection Method

In this section, the robustness of the proposed antenna selection method (6.9) is studied in the presence of array imperfections and mutual coupling effects separately. Specifications are shown in dotted lines.

In this example, the same (third) example of Section 6.7 (Simulation Results and Discussion) is used. A non-uniformly spaced 41-antenna array from [67] is used to consider array imperfection effects where the antenna positions are given in Table 6.1. The specifications are that the mainlobe beamwidth is 40° and $r_{\text{dB}} = 1\text{dB}$. The sidelobe region is $[0^\circ, 63^\circ] \cup [117^\circ, 180^\circ]$ and $\tau = -30\text{dB}$. All the antennas are assumed to have the same element pattern $p(\theta) = \sin \theta$. The antenna selection method chooses 19 antennas (obtained at the 2nd iteration) to achieve the specifications. However, array imperfections exist and the element pattern of the 20th antenna (with weight magnitude $> \delta = 10^{-5}$) is $\sin^{1.2} \theta$. Thus, the resulting beampattern in Fig. 6.2 violates the specifications.

Next, the same fourth example of Section 6.7 (Simulation Results and Discussion) is used. A $\lambda/2$ spaced ULA of thin half-wavelength dipoles is used to consider mutual coupling effects. The dipoles are located along the z -axis and aligned with the x -axis. Accordingly, the coupling matrix $\mathbf{C}$ is calculated by the induced electromotive force approach [80, Ch. 8]. The beampattern specifications are the same

Figure 6.2: Beampattern achieved by the proposed antenna selection method (6.9) using 19 antennas in the presence of array imperfections. The element pattern of the 20th antenna is $\sin^{1.2} \theta$ which is different from the other antennas. Top: A close-up view of the mainlobe. Bottom: A close-up view of the sidelobe. The beampattern has to be lower than the outer dotted lines and higher than the inner dotted lines.

Table 6.1: Antenna positions of a non-uniformly spaced 41-antenna linear array

Antenna element	Position (λ)	Antenna element	Position (λ)
1, 41	± 10.0000	11, 31	± 5.0000
2, 40	± 9.6065	12, 30	± 4.6065
3, 39	± 8.8098	13, 29	± 3.8098
4, 38	± 8.2995	14, 28	± 3.2995
5, 37	± 7.8973	15, 27	± 2.8973
6, 36	± 7.3497	16, 26	± 2.3497
7, 35	± 6.8494	17, 25	± 1.8494
8, 34	± 6.5302	18, 24	± 1.5302
9, 33	± 5.6299	19, 23	± 0.6299
10, 32	± 5.3749	20, 22	± 0.3749
-	-	21	0

as the previous example. First, mutual coupling is ignored. The proposed beam-pattern design method (6.6) is tested and at least 23 antennas are needed in a ULA to achieve the specifications. The resulting beampattern in the presence of mutual coupling is plotted in blue in Fig. 6.3. Note that the mainlobe specifications are violated slightly. Again, mutual coupling is ignored and the proposed antenna selection method (6.9) chooses only 18 antennas (obtained at the 2nd iteration) out of a 24-antenna ULA, resulting in a sparse linear array to achieve the same specifications. The resulting beampattern in the presence of mutual coupling is plotted in green in Fig. 6.3 which violates the sidelobe specifications.

From the findings of the numerical studies, there is a need to develop an effective solution to ensure the robustness of the proposed antenna selection method against the undesirable effects that exist in the array in practice.

Figure 6.3: Beam patterns achieved by the proposed beam pattern design method (6.6) with 23-antenna ULA and the proposed antenna selection method (6.9) with 18 selected antennas, respectively in the presence of mutual coupling effects. Top: A close-up view of the mainlobe. Bottom: A close-up view of the sidelobe. The beam pattern has to be lower than the outer dotted lines and higher than the inner dotted lines.

6.6 Robustness Issues

Due to mutual coupling or array imperfections, the actual array manifold $\tilde{\mathbf{s}}(u_x, u_y)$ can be unknown and it can differ from the ideal presumed one $\mathbf{s}(u_x, u_y)$. Thus, the synthesized beampattern by (6.6) or (6.9) may violate the specifications. In order to improve the robustness of the proposed methods (6.6) and (6.9) against various undesirable effects, the worst-case performance optimization technique [43, 63] is applied. In order to illustrate, mutual coupling effects are considered by means of a coupling matrix $\mathbf{C}$. The actual array manifold is $\tilde{\mathbf{s}}(u_x, u_y) = \mathbf{C}\mathbf{s}(u_x, u_y)$ which belongs to an uncertainty set modelled as

$$\Upsilon(\eta) = \{\tilde{\mathbf{s}}(u_x, u_y) | \tilde{\mathbf{s}}(u_x, u_y) = \mathbf{s}(u_x, u_y) + \mathbf{e}(u_x, u_y)\} \quad (6.10)$$

where the complex error $\mathbf{e}(u_x, u_y)$ is norm-bounded by a known constant, i.e., $\|\mathbf{e}(u_x, u_y)\| \leq \eta(u_x, u_y) = \eta$ (for simplicity). The proposed mainlobe constraint (6.5) is rewritten as

$$l \leq \min |\tilde{\mathbf{s}}^T(u_x, u_y)\mathbf{w}| \leq \max |\tilde{\mathbf{s}}^T(u_x, u_y)\mathbf{w}| \leq u. \quad (6.11)$$

Applying the triangle and Cauchy-Schwarz inequalities yields

$$|\tilde{\mathbf{s}}^T(u_x, u_y)\mathbf{w}| = |\mathbf{s}^T(u_x, u_y)\mathbf{w} + \mathbf{e}^T(u_x, u_y)\mathbf{w}| \quad (6.12)$$

$$\geq |\mathbf{s}^T(u_x, u_y)\mathbf{w}| - |\mathbf{e}^T(u_x, u_y)\mathbf{w}| \quad (6.13)$$

$$\geq \mathbf{s}^T(u_x, u_y)\mathbf{w} - \eta\|\mathbf{w}\| \quad (6.14)$$

where $\mathbf{s}^T(u_x, u_y)\mathbf{w} > \eta\|\mathbf{w}\|$, equality holds with $\mathbf{e}(u_x, u_y) = -\eta(\mathbf{w}/\|\mathbf{w}\|)e^{j\psi}$ and $\psi = \angle\{\mathbf{s}^T(u_x, u_y)\mathbf{w}\} = 0$ as $\mathbf{s}^T(u_x, u_y)\mathbf{w}$ is real. Thus, $\min |\tilde{\mathbf{s}}^T(u_x, u_y)\mathbf{w}| =$

$\mathbf{s}^T(u_x, u_y)\mathbf{w} - \eta\|\mathbf{w}\|$. Also, similarly,

$$|\tilde{\mathbf{s}}^T(u_x, u_y)\mathbf{w}| \leq \mathbf{s}^T(u_x, u_y)\mathbf{w} + |\mathbf{e}^T(u_x, u_y)\mathbf{w}| \quad (6.15)$$

$$\leq \mathbf{s}^T(u_x, u_y)\mathbf{w} + \eta\|\mathbf{w}\| \quad (6.16)$$

where equality holds with $\mathbf{e}(u_x, u_y) = \eta(\mathbf{w}/\|\mathbf{w}\|)$. Since $\max |\tilde{\mathbf{s}}^T(u_x, u_y)\mathbf{w}| = \mathbf{s}^T(u_x, u_y)\mathbf{w} + \eta\|\mathbf{w}\|$, (6.11) becomes

$$\mathbf{s}^T(u_x, u_y)\mathbf{w} - \eta\|\mathbf{w}\| \geq l, \quad \{u_x, u_y\} \in U_{\text{ML}} \quad (6.17a)$$

$$\mathbf{s}^T(u_x, u_y)\mathbf{w} + \eta\|\mathbf{w}\| \leq u, \quad \{u_x, u_y\} \in U_{\text{ML}}. \quad (6.17b)$$

(6.17b) can be extended to U_{SL} and U_{N} . When these robust second-order-cone constraints in (6.17) are used in (6.6) and (6.9), the resulting (6.6) and (6.9) become SOCPs.

6.7 Simulation Results and Discussion

This section tests the proposed beampattern design method (6.6) and antenna selection method (6.9). For (6.9), the maximum number of iterations is 20 and $\delta = 10^{-5}$ (this choice is discussed later). The antennas used are isotropic unless stated otherwise.

6.7.1 Linear Arrays

In the first example, the antenna selection method (6.9) is used on a non-uniformly spaced 41-antenna array from [67, 79]. Antenna positions are given in [67] and Table 6.1. A two-step least-squares approach is developed in [79] and the same beampattern specifications are used here. The mainlobe beamwidth and r_{dB} are 40° and 0.4455dB, respectively. The sidelobe region is $[0^\circ, 65^\circ] \cup [115^\circ, 180^\circ]$ and

Figure 6.4: Beampattern achieved by the proposed antenna selection method (6.9) using 31 selected antennas in a non-uniformly spaced linear array. A close-up view of the mainlobe is shown in the inset. The beampattern has to be lower than the outer dotted lines and higher than the inner dotted lines.

$\tau = -30\text{dB}$. Specifications are shown in dotted lines. In the first iteration, a l_1 -norm version of (6.9) is solved and the solution is 39 antennas. The final solution found by (6.9) is 31 antennas (obtained at the 4th iteration) and the resulting beampattern is shown in Fig. 6.4. The resulting sparse array with 31 selected antennas is shown in Fig. 6.5. Compared to [79], 10 antennas are saved.

In the second example, the antenna selection method (6.9) is used on the same non-uniformly spaced array as the first example but with directive antennas of element pattern $p(\theta) = \sin \theta$, e.g., short dipoles located on and aligned with the z -axis. The same specifications are used except that $r_{\text{dB}} = 1\text{dB}$ and the sidelobe region is $[0^\circ, 43^\circ] \cup [97^\circ, 180^\circ]$. Case 1 ignores $p(\theta)$ and 27 antennas are selected by (6.9) (obtained at the 3rd iteration). Case 2 considers $p(\theta)$ in the optimization and only 25 antennas are selected by (6.9) (obtained at the 6th iteration). Fig.

Figure 6.5: 31 antennas selected by the proposed antenna selection method (6.9) in a 41-antenna non-uniformly spaced linear array. Selected antennas are indicated with red circles. Unselected antennas are indicated with black crosses. This optimized array achieves the beampattern in Fig. 6.4.

Figure 6.6: Beam patterns achieved by the proposed antenna selection method (6.9) in Case 1 and Case 2, without and with antenna element pattern $p(\theta) = \sin \theta$ consideration, respectively. 27 and 25 antennas are selected in a non-uniformly spaced linear array for Case 1 and Case 2, respectively. A close-up view of the mainlobe is shown in the inset. The beam pattern has to be lower than the outer dotted lines and higher than the inner dotted lines.

6.6 shows that the beam pattern in Case 1 (plotted in blue) violates the mainlobe specifications whereas the beam pattern in Case 2 (plotted in red) satisfies all the specifications. The resulting sparse arrays with 27 and 25 selected antennas for Case 1 and Case 2, respectively are shown in Fig. 6.7.

In the third example, the same 41-antenna array and specifications as the second example are used except that the sidelobe region is $[0^\circ, 63^\circ] \cup [117^\circ, 180^\circ]$. All the antennas are assumed to have the same element pattern $p(\theta) = \sin \theta$. In Case 3, the antenna selection method chooses 19 antennas (obtained at the 2nd iteration) to achieve the specifications. However, there are array imperfections and the element pattern of the 20th antenna (with weight magnitude $> \delta = 10^{-5}$) is $\sin^{1.2} \theta$. Thus, the resulting beam pattern (plotted in blue) for Case 3 in Fig.

Figure 6.7: Top: 27 and bottom: 25 antennas selected by the proposed antenna selection method (6.9) in a 41-antenna non-uniformly spaced linear array in Case 1 and Case 2, without and with antenna element pattern $p(\theta) = \sin(\theta)$ consideration, respectively. Selected antennas are indicated with red circles. Unselected antennas are indicated with black crosses. These optimized arrays in Case 1 and Case 2 achieve the beampatterns (plotted in blue and red) in Fig. 6.6, respectively.

Figure 6.8: Beampatterns achieved by the proposed antenna selection method (6.9) using 19 antennas, without and with robust constraints in Case 3 and Case 4, respectively in the presence of array imperfections. The element pattern of the 20th antenna is $\sin^{1.2} \theta$ which is different from the other antennas. A close-up view of the mainlobe is shown in the inset. The beampattern has to be lower than the outer dotted lines and higher than the inner dotted lines.

6.8 violates the specifications. In Case 4, the proposed robust constraints for the mainlobe and sidelobe regions are used with the antenna selection method where η 's for mainlobe and sidelobe regions are $\eta_m = 0.001$ and $\eta_s = 0.007$, respectively. The resulting beampattern (plotted in red) in Fig. 6.8 satisfies all the specifications using the same 19 antennas (obtained at the 3rd iteration). The antenna selection method (6.9) with the robust constraints is found to be insensitive to η_m and η_s in $0.001 \leq \eta_m \leq 0.023$ and $0.0065 \leq \eta_s \leq 0.007$, respectively. The resulting sparse arrays with 19 selected antennas for Case 3 and Case 4, respectively are shown in Fig. 6.9.

In the fourth example, a $\lambda/2$ spaced ULA of thin half-wavelength dipoles is used to consider mutual coupling effects. They are located along the z-axis and

Figure 6.9: 19 antennas selected by the proposed antenna selection method (6.9) in a 41-antenna non-uniformly spaced linear array, without and with robust constraints in Case 3 and Case 4, respectively in the presence of array imperfections. Selected antennas are indicated with red circles. Unselected antennas are indicated with black crosses.

aligned with the x-axis. Accordingly, the coupling matrix $\mathbf{C}$ is calculated by the induced electromotive force approach [80, Ch. 8]. The same specifications in the third example are used. First, mutual coupling is ignored. The beam pattern design method (6.6) is tested and at least 23 antennas are needed in a ULA to achieve the specifications. The resulting beam pattern in the presence of mutual coupling is plotted in blue in Fig. 6.10 which violates the mainlobe specifications. Again, mutual coupling is ignored and the antenna selection method (6.9) selects only 18 antennas (obtained at the 2nd iteration) out of a 24-antenna ULA to achieve the same specifications. The resulting beam pattern in the presence of mutual coupling is plotted in green in Fig. 6.10 which violates the specifications. In order to compensate for mutual coupling effects, the proposed robust constraints for the mainlobe and sidelobe regions are used with the antenna selection method (6.9) where $\eta_m = 0.001$ and $\eta_s = 0.003$. The resulting beam pattern (plotted in red) in Fig. 6.10 satisfies all the specifications in the presence of mutual coupling effects with the same 18 antennas (obtained at the 2nd iteration). The antenna selection method (6.9) with the robust constraints is found to be insensitive to η_m and η_s in $0.0001 \leq \eta_m \leq 0.001$ and $0.003 \leq \eta_s \leq 0.006$, respectively. The resulting optimized arrays with 23, 18, and 18 selected antennas are shown in Fig. 6.11.

The inclusion of the antenna pattern $p(\theta)$ and compensation for the array imperfections and mutual coupling effects are shown for linear arrays but they can be extended in the same way to planar arrays.

6.7.2 Planar Arrays

Next, $\lambda/2$ spaced rectangular planar arrays are considered and beam patterns are plotted in $u_x - u_y$ axes [11, Ch. 4]. In the fifth example, a beam pattern with a diamond-shaped mainlobe, controlled sidelobe level, and circular-shaped null region is designed by the beam pattern design method (6.6). A 14×14 array is used.

Figure 6.10: Top: Beam patterns achieved by the proposed beam pattern design method (6.6) with a 23-antenna ULA, proposed antenna selection method (6.9) without and with robust constraints using 18 selected antennas, respectively in the presence of mutual coupling effects. Middle: A close-up view of the mainlobe. Bottom: A close-up view of the sidelobe. The beam pattern has to be lower than the outer dotted lines and higher than the inner dotted lines.

Figure 6.11: 23-antenna ULA for the proposed beampattern design method (6.6) and 18 antennas selected by the proposed antenna selection method (6.9) in a 24-antenna ULA, without and with robust constraints, respectively in the presence of mutual coupling effects. Selected antennas are indicated with red circles. Unselected antennas are indicated with black crosses.

Figure 6.12: Beampattern with diamond-shaped mainlobe, controlled sidelobe level, and circular-shaped null achieved by the proposed beampattern design method (6.6) with a 14×14 array.

The mainlobe is $|u_x - 0.2| + |u_y - 0.2| \leq 0.2$ with $r_{\text{dB}} = 1\text{dB}$. The sidelobe region is $|u_x - 0.2| + |u_y - 0.2| \geq 0.4$. The null region is $(u_x + 0.5)^2 + (u_y + 0.5)^2 \leq 0.1^2$ with $\tau_n = -50\text{dB}$. The resulting beampattern in Fig. 6.12 meets all the requirements (including arbitrary sidelobe levels) and the minimal sidelobe level achieved is $\tau = -24.30\text{dB}$.

In the sixth example, the beampattern design method (6.6) is used to design a circular-shaped mainlobe with controlled sidelobe level. A 11×11 array is used. The total number of antennas is 121. The mainlobe region is $u_x^2 + u_y^2 \leq 0.2^2$ with $r_{\text{dB}} = 1\text{dB}$. The sidelobe region is $u_x^2 + u_y^2 \geq 0.4^2$. The minimal sidelobe level achieved is $\tau = -25.85\text{dB}$. For brevity, the resulting beampatterns in the sixth and seventh examples are not shown. A final representative beampattern (with same specifications) is shown in the eighth example by Fig. 6.15. All the 121 antennas here are used (with magnitudes $> \delta = 10^{-5}$). Element magnitudes in

Figure 6.13: 85 antennas selected by the proposed antenna selection method (6.9) in a 11×11 array with original inter-element spacing of $\lambda/2$. Selected antennas are indicated with red circles. Unselected antennas are indicated with black crosses.

$\mathbf{w} \leq 10^{-5}$ are considered zero, for a fair comparison with the antenna selection method (6.9).

In the seventh example, the antenna selection method (6.9) is used on the same 11×11 array to produce a sparse one which satisfies the same specifications in the sixth example. In the first iteration, a l_1 -norm version of (6.9) is solved and the solution is 105 antennas. The final solution found by (6.9) is 85 antennas (obtained at the 2nd iteration). The $\lambda/2$ spaced 11×11 array with the selected antennas is shown in Fig. 6.13, accounting for 70% of those in the original array. The selected antennas are indicated with red circles while the unselected ones are indicated with black crosses.

In the eighth example, the antenna selection method (6.9) is tested on a $\lambda/4$ spaced array and the same specifications as in the sixth and seventh examples are used. In order to maintain the same aperture as the 11×11 array for subsequent

Figure 6.14: 76 antennas selected by the proposed antenna selection method (6.9) in a 21×21 array with original inter-element spacing of $\lambda/4$. Selected antennas are indicated with red circles. Unselected antennas are indicated with black crosses. This optimized array achieves the beampattern in Fig. 6.15.

comparison, a 21×21 array is used. The total number of antennas is 441. The purpose of this example is to study if the proposed method will produce a similar array as the seventh example with the same number of selected antennas, or an array with fewer selected antennas and larger inter-element separations or fail. In the first iteration, a l_1 -norm version of (6.9) is solved and the solution is 128 antennas. The final solution found by (6.9) is 76 antennas (obtained at the 4th iteration). The optimized array is shown in Fig. 6.14 with many inter-element spacings much greater than $\lambda/2$. Thus, sidelobe control is especially critical as grating lobes may appear. The resulting beampattern in Fig. 6.15 still has its sidelobe level strictly maintained at -25.85dB . Only 63% of the antennas in the original 11×11 array are selected by (6.9) in this example.

The antenna selection method (6.9) can reduce the DRR of $\mathbf{w}$, i.e., $\text{DRR} =$

Figure 6.15: Beampattern with circular-shaped mainlobe and controlled sidelobe level achieved by the proposed antenna selection method (6.9) using 76 antennas in the array of Fig. 6.14. Top: View of beampattern in u_x direction. Bottom: View of beampattern in u_y direction.

Table 6.2: DRRs of beamforming weight vectors in various examples

Simulation example	Number of selected antennas	DRR
Example 6: Eq. (6.6) on $\lambda/2$ spaced 11×11 array	121	177.36
Example 7: Eq. (6.9) on $\lambda/2$ spaced 11×11 array	85	33.27
Example 8: Eq. (6.9) on $\lambda/4$ spaced 21×21 array	76	34.22

$\max |w_i| / \min |w_i|$ by omitting elements with magnitudes $\leq \delta$. The DRRs of the beamforming weight vectors using (6.6) in the sixth example, as well as (6.9) in the seventh and eighth examples, are tabulated in Table 6.2 (same specifications are imposed in these examples). The antenna selection method (6.9) has lowered the DRRs of the beamforming weight vectors in the seventh and eighth examples compared to that obtained by the beampattern design method (6.6) in the sixth example.

The choice of δ in the antenna selection method (6.9) is discussed. A wide range of δ values (10^{-4} to 10^{-7}) is tested, using the seventh ($\lambda/2$ spaced 11×11 array) and eighth ($\lambda/4$ spaced 21×21 array) examples. From Fig. 6.16, there is almost no difference in the final solution or the number of iterations needed to reach the final solution. The convergence of (6.9) is slower with smaller δ values. In the 21×21 array example, (6.9) does not converge to the same final solution with $\delta = 10^{-4}$ as with the other δ 's since the elements in $\mathbf{w}$ with magnitudes $\leq 10^{-4}$ are considered unused, so more antennas are needed to meet the same specifications. In essence, (6.9) is not very sensitive to the choice of δ and $\delta = 10^{-5}$ is found suitable for all the examples and they converge in a few iterations.

Finally, the antenna selection method (6.9) is used to design a non-uniformly spaced array for a circular-shaped mainlobe and controlled sidelobe level with specifications from [76]. The mainlobe region is $u_x^2 + u_y^2 \leq 0.29^2$ with $r_{\text{dB}} = 1\text{dB}$ and the sidelobe region is $u_x^2 + u_y^2 \geq 0.44^2$ with $\tau = -20\text{dB}$. The method of [76] uses all the 100 antennas in a uniform $\lambda/2$ spaced 10×10 array. The antenna

Figure 6.16: Convergence of the proposed antenna selection method (6.9) at different δ values, for the seventh and eighth examples. Top: Seventh example operates on the $\lambda/2$ spaced 11×11 array. Bottom: Eighth example operates on the $\lambda/4$ spaced 21×21 array.

Figure 6.17: 93 antennas selected by the proposed antenna selection method (6.9) in a 15×15 array with original inter-element spacing of $\lambda/2$. Selected antennas are indicated with red circles. Unselected antennas are indicated with black crosses.

selection method (6.9) is tested on the same type of array and the result with the least number of selected antennas is reported. Only 93 antennas are selected by (6.9) (obtained at the 5th iteration) in a 15×15 array to achieve the same specifications (beam pattern not shown for brevity). Compared to [76], (6.9) saves 7 antennas. The optimized array is shown in Fig. 6.17.

Some final comments are in order. While the proposed methods (6.6) and (6.9) are tested on rectangular planar arrays, they also apply to non-rectangular planar arrays such as hexagonal or circular arrays. In order to illustrate, the example of a 19-antenna hexagonal array on the x - y plane as shown in Fig. 6.18 is used where $d_x = \frac{\lambda}{2}$ and $d_y = \frac{\sqrt{3}d_x}{2}$. The array manifold of the 19-antenna hexagonal array is

$$\mathbf{s}(u_x, u_y) = [\mathbf{v}_2^T \quad \mathbf{v}_1^T \quad \mathbf{v}_0^T \quad \mathbf{v}_{-1}^T \quad \mathbf{v}_{-2}^T]^T \quad (6.18)$$

Figure 6.18: A 19-antenna hexagonal array on the x - y plane. The index m where $m = -2, -1, \dots, 2$ indicates the set of antennas (indicated by black circles) located at the same distance from the x -axis. The distance between two antennas is d_x and that between two neighbouring rows of antennas is d_y .

where the vectors $\mathbf{v}_m$ with $m = -2, -1, \dots, 2$ are

$$\mathbf{v}_2 = e^{j\pi\sqrt{3}u_y} [e^{-j\pi u_x} \quad 1 \quad e^{j\pi u_x}]^T, \quad (6.19a)$$

$$\mathbf{v}_1 = e^{j\pi\frac{\sqrt{3}}{2}u_y} [e^{-j\pi\frac{3}{2}u_x} \quad e^{-j\pi\frac{1}{2}u_x} \quad e^{j\pi\frac{1}{2}u_x} \quad e^{j\pi\frac{3}{2}u_x}]^T, \quad (6.19b)$$

$$\mathbf{v}_0 = [e^{-j\pi 2u_x} \quad e^{-j\pi u_x} \quad 1 \quad e^{j\pi u_x} \quad e^{j\pi 2u_x}]^T, \quad (6.19c)$$

$$\mathbf{v}_{-1} = e^{-j\pi\frac{\sqrt{3}}{2}u_y} [e^{-j\pi\frac{3}{2}u_x} \quad e^{-j\pi\frac{1}{2}u_x} \quad e^{j\pi\frac{1}{2}u_x} \quad e^{j\pi\frac{3}{2}u_x}]^T, \quad (6.19d)$$

$$\mathbf{v}_{-2} = e^{-j\pi\sqrt{3}u_y} [e^{-j\pi u_x} \quad 1 \quad e^{j\pi u_x}]^T, \quad (6.19e)$$

respectively [11]. From (6.18) and (6.19), the array manifold of a hexagonal array is conjugate symmetric. By using conjugate symmetric beamforming weights, the same constraint in (6.5) can also be imposed on a hexagonal array to control the lower and upper magnitude response bounds on the beampattern. Therefore, the proposed methods of this chapter applies directly on a hexagonal array.

Next, a 6-antenna circular array on the x - y plane as shown in Fig. 6.19 is used as another example. The array manifold of the 6-antenna circular array is

Figure 6.19: A 6-antenna circular array on the x - y plane where the antennas (indicated by black circles labelled from 0 to 5) are located uniformly on a circle of radius r .

$$\mathbf{s}(\theta, \phi) = \begin{bmatrix} e^{-j\frac{2\pi r}{\lambda} \sin \theta \cos \phi} & e^{-j\frac{2\pi r}{\lambda} \sin \theta \cos(\phi - \frac{\pi}{3})} & e^{-j\frac{2\pi r}{\lambda} \sin \theta \cos(\phi - \frac{2\pi}{3})} \\ e^{-j\frac{2\pi r}{\lambda} \sin \theta \cos(\phi - \pi)} & e^{-j\frac{2\pi r}{\lambda} \sin \theta \cos(\phi - \frac{4\pi}{3})} & e^{-j\frac{2\pi r}{\lambda} \sin \theta \cos(\phi - \frac{5\pi}{3})} \end{bmatrix}^T \quad (6.20)$$

where r is the radius of the circular array. By expanding the trigonometric terms and using $u_x = \sin \theta \cos \phi$ and $u_y = \sin \theta \sin \phi$, (6.20) becomes

$$\mathbf{s}(u_x, u_y) = \begin{bmatrix} e^{-j\frac{2\pi r}{\lambda} u_x} & e^{-j\frac{2\pi r}{\lambda} \frac{1}{2} u_x - j\frac{2\pi r}{\lambda} \frac{\sqrt{3}}{2} u_y} & e^{j\frac{2\pi r}{\lambda} \frac{1}{2} u_x - j\frac{2\pi r}{\lambda} \frac{\sqrt{3}}{2} u_y} \\ e^{j\frac{2\pi r}{\lambda} u_x} & e^{j\frac{2\pi r}{\lambda} \frac{1}{2} u_x + j\frac{2\pi r}{\lambda} \frac{\sqrt{3}}{2} u_y} & e^{-j\frac{2\pi r}{\lambda} \frac{1}{2} u_x + j\frac{2\pi r}{\lambda} \frac{\sqrt{3}}{2} u_y} \end{bmatrix}^T. \quad (6.21)$$

From (6.21), the first and fourth terms, second and fifth terms, and third and last terms are conjugate symmetric, respectively. Instead of imposing $w_i = w_{N-i+1}^*$ similar to that in Section 6.3, the beamforming weights of the 6-antenna circular array is constrained to be

$$\mathbf{w} = [w_0 \quad w_1 \quad w_2 \quad w_0^* \quad w_1^* \quad w_2^*]^T \quad (6.22)$$

so that the corresponding beampattern expression $|G(u_x, u_y)| = \mathbf{s}^T(u_x, u_y)\mathbf{w}$ is real. As a result, any lower and upper magnitude response bound constraints on the beampattern is affine and the proposed methods of this chapter applies directly to a circular array.

6.8 Summary

A convex optimization based beampattern design method with antenna selection has been proposed for planar arrays (also applicable to linear arrays). Thus, a sparse array with fewer antennas can be obtained that satisfies precisely the same beampattern specifications achieved by a non-sparse array. This method can achieve a mainlobe of an arbitrary beamwidth and response ripple as well as completely arbitrary sidelobe levels. It can design non-uniformly spaced arrays with inter-element spacings greater than one half-wavelength, without grating lobes appearing in the resulting beampattern. Robust beampattern constraints are derived to compensate for the undesirable effects in practice. Simulations are conducted on arrays with up to a few hundred antennas to show the practicality of the proposed method.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

The research studies reported in this thesis have focused on the investigation of optimization techniques for robust beamforming against large steering direction errors and (unexpected or strong moving) interferences because these problems are unavoidable and frequently encountered in practical beamforming applications. Furthermore, this thesis studies optimization techniques on antenna selection for the efficient use of the antennas in arrays.

In order to better understand the recently developed adaptive beamformers of [1, 43, 52] based on worst-case performance optimization, in Chapter 3, numerical studies have been conducted on the beamformer of [1] (chosen as the representative of [1, 43, 52]) to see its output SINR performance in various scenarios with different magnitudes of steering direction errors and different SNRs. The findings show that the optimal output SINR performance of [1] is relatively insensitive to the size of the uncertainty sphere if the desired signal is a dominant signal and the steering direction error is small, provided that the size of the uncertainty sphere is larger than a certain threshold. On the contrary, the size of the uncertainty sphere is critical for the optimal output SINR performance of [1] when the desired signal

is a weak signal. The findings also reveal that in the presence of large steering direction errors, the optimal size of the uncertainty sphere has to expand in order to accommodate the increased error compared to that with small steering direction errors.

Based on the findings of Chapter 3, a technique is proposed to improve the output SINR performances of adaptive beamforming systems in the presence of large steering direction errors. Instead of having to use a large uncertainty sphere in the beamformer of [1], the proposed technique makes use of a small uncertainty sphere (and a small flat ellipsoid) to search for the desired array steering vector iteratively. Consequently, by preserving its DOFs and in turn, its interference-plus-noise suppression ability, and by using the corrected desired array steering vector, the proposed IRMVB achieves a much higher output SINR performance than the beamformer of [1]. The effectiveness of the proposed beamformer is supported by the theoretical analysis and simulation results of Chapter 3.

In Chapter 4, numerical studies have been conducted on the beamformers of [1–3, 43, 52] based on worst-case performance optimization and [5] based on the CMT approach to study their beampattern behaviour at low snapshots and in the presence of large steering direction errors. The findings of Chapter 4 show that the sidelobes of these beamformers are uncontrolled and can be very high at low snapshots. High sidelobes increase the risk of beamforming system breakdown when there are unexpected interferences or strong interferences that move out of the beampattern nulls during which the beamforming weights are applied. The CMT beamformer of [5] can be robust against moving interferences but it is not robust against steering direction errors. Therefore, a general adaptive beamforming framework is proposed. Based on the use of a set of beampattern shaping constraints, the proposed beamformer is robust against large steering direction errors and strong interferences that move during which the beamforming weights are ap-

plied. The proposed beamformer also achieves joint adaptive interference rejection and direct sidelobe suppression where a weighing ratio is proposed to automatically control their relative proportions in different scenarios without ambiguity. Compared to the beamformers of [1–3, 5, 43, 52], the proposed beamformer achieves a higher output SINR performance in the presence of large steering direction errors and strong moving interferences, and at low snapshots.

One important aspect of beamforming system design is the efficient use of the antennas in the array as the use of the least number of antennas leads to savings in terms of the cost, weight, and computational complexity of the beamforming system. Therefore, this leads to the investigation of optimization techniques on antenna selection in arrays. The challenge is how to select the minimum number of antennas or as few antennas as possible for beampattern designs without violating the prescribed beampattern specifications. The method of Davidson *et al.* [7, 8] with LMI constraints is studied in Chapter 5 as the LMI constraints can represent the semi-infinite magnitude response (lower and upper bound) constraints on the beampattern in a finite and convex manner. However, the direct application of the method of [7, 8] to find the minimum number of antennas required is not systematic and moreover, the number of iterations before stopping is unknown.

Therefore, in Chapter 5, a technique has been proposed to find the minimum number of antennas needed and the corresponding beamforming weights systematically. The proposed method employs LMI constraints on the beampattern so that the prescribed specifications can be satisfied precisely. The bisection search is used to decompose the proposed quasi-convex formulation into a systematic sequence of convex feasibility problems. The proposed method is guaranteed to find the minimum number of antennas required so long as this solution lies in the search interval, which can be easily ensured at the start of the search. Unlike the direct application of the method of [7, 8] to find the minimum number of antennas

required, the number of iterations before the proposed technique stops is known.

Following the antenna selection method proposed in Chapter 5, the next step is to extend this method for use on planar arrays. However, based on the simulation experience in Chapter 5, the implementation of the LMI constraints may be too computationally intensive on a personal computer given the number of antennas in planar arrays that are typically employed in practice. Furthermore, the proposed method of Chapter 5 is restricted to uniformly spaced arrays. As such, an alternative antenna selection technique is proposed for planar arrays (also applicable to linear arrays). In Chapter 6, the proposed antenna selection technique uses conjugate symmetric beamforming weights on a non-uniformly spaced symmetric array so that the original non-convex lower bound magnitude response constraint on the beampattern is affine, thus convex. The resulting upper bound magnitude response constraint on the beampattern is also affine. This then facilitates the proposed technique to achieve any arbitrary beampattern precisely according to the specifications. By minimizing a re-weighted objective function based on the magnitudes of the elements in the beamforming weight vector iteratively, the proposed antenna selection method selects certain antennas in an array, yielding a sparse array with fewer number of antennas for beampattern design as compared to [76, 79]. The proposed antenna selection technique can be used to design non-uniformly spaced symmetric arrays with inter-element spacings larger than one half-wavelength without the appearance of grating lobes in the resulting beampattern. Simulations have been conducted on arrays with up to a few hundred antennas to show the practicality of the proposed antenna selection method.

Array imperfections and mutual coupling effects do exist in a practical array [80–82] but many beamforming methods in [11, 23] do not consider them during beampattern design. In Chapter 6, the proposed antenna selection method is studied to see the stability of the resulting beampatterns in the presence of array

imperfections and mutual coupling effects. The findings of the numerical studies show that the beampatterns obtained by the proposed antenna selection method violate the specifications in the presence of array imperfections and mutual coupling effects because the actual array manifold differs from the ideal presumed one and which can be unknown. Therefore, in Chapter 6, based on the uncertainty set of the actual array manifold and worst-case performance optimization approach [43, 63], robust beampattern constraints are proposed for the antenna selection method. The resulting beampatterns optimized by the proposed antenna selection method with the robust beampattern constraints satisfy all the specifications precisely in the presence of array imperfections and mutual coupling effects without incurring additional selected antennas.

Although this thesis focuses on receiving antenna arrays, some of the proposed techniques of Chapters 5 and 6 are applicable to active arrays, i.e., radar arrays with transmitters and receivers. When active arrays are used in the transmitting mode, the beampatterns are required to have low sidelobes in certain regions to achieve a low probability of intercept or to reduce unwanted radiation in certain directions due to human safety reasons while beamforming towards a prescribed direction [114]. It may also be desirable to steer the nulls of the transmit beampattern to avoid illuminating directions that can give rise to multipath clutter. This can be achieved by a two-step approach; first, the array transmits a probing signal to determine the directions where the nulls should be steered to and second, the transmit beampattern is designed with nulls at the appropriate directions [115]. Hence, the proposed antenna selection techniques of Chapters 5 and 6 can be used to minimize the number of antennas required in order to achieve a beampattern with a mainlobe of some specific beamwidth and response ripple, prescribed sidelobe levels, as well as nulls. It is noted that unlike receive beampattern design, additional constraints that the transmit power of the antennas or the total trans-

mit power should be below than a certain threshold, may have to be imposed for transmit beampattern design applications [114]. Nevertheless, the proposed antenna selection methods of Chapters 5 and 6 can be applied by modifying them with the additional constraints.

7.2 Future Work

Based on the studies conducted, obtained findings, and the techniques proposed in this thesis, several suggestions for future research are outlined as follows.

The proposed IRMVB of Chapter 3 and the proposed adaptive beamforming framework of Chapter 4 are designed for narrowband beamforming to tackle problems such as large steering direction errors, occurrence of high sidelobes due to limited snapshot support, and strong moving interferences. These problems are also unavoidable in broadband beamforming applications, e.g., speech processing, teleconferencing, and acoustic surveillance [19]. Therefore, further study may be conducted to extend the proposed IRMVB of Chapter 3 to investigate the problem of large steering direction error of the desired broadband signal in the presence of other broadband interferences. Further study can be carried out to extend the proposed framework of Chapter 4 for broadband beamforming to obtain a broad mainlobe, controlled sidelobe level, and broad nulls in order to achieve robustness against large steering direction errors of the desired broadband signal, sudden or strong moving broadband interferences, respectively.

The antenna selection methods proposed in Chapters 5 and 6 are designed for linear and planar arrays, respectively. Further work may be done to extend the findings and methods developed in Chapters 5 and 6 to the investigation of the antenna selection problem in antenna arrays with different geometries such as concentric circular arrays and also conformal antenna arrays. A conformal antenna array is one that conforms to some prescribed shapes such as a cylinder, a sphere,

or some other irregular shapes which can form some part of an airplane, a high-speed train, or a vehicle [116]. Antenna selection may decrease the number of antennas required for beam pattern designs in conformal arrays which may help to reduce the weight and drag force, and in turn, decrease the fuel consumption. Another possible research area may be to study the antenna selection problem for conformal arrays on a smooth surface by approximating the surface with several planar antenna arrays [116].

The proposed antenna selection method of Chapter 6 makes use of conjugate symmetric beamforming weights in order to achieve convex upper and lower bound magnitude response constraints for non-uniformly spaced symmetric arrays. However, in practical applications, any of the antenna elements in the array can fail, i.e., the weight on the failed antenna is zero. When this happens, the resulting array is no longer symmetric and it is likely that the beam pattern obtained by the proposed method will violate the specifications. Further work may include investigating the effect of antenna failure on the beam patterns produced by the antenna selection method of Chapter 6. Another important area of study is to remove the restrictions of using conjugate symmetric beamforming weights and symmetric arrays and explore antenna selection methods for arrays with randomly deployed antennas that can achieve beam patterns with upper and lower bound magnitude response constraints.

Chapter 8

Author's Publications

Journal papers

1. S. E. Nai, W. Ser, Z. L. Yu, and S. Rahardja, "A robust adaptive beamforming framework with beampattern shaping constraints," *IEEE Trans. Antennas Propag.*, vol. 57, no. 7, pp. 2198-2203, Jul. 2009.
2. S. E. Nai, W. Ser, Z. L. Yu, and S. Rahardja, "Beampattern synthesis with linear matrix inequalities using minimum array sensors," *Progress Electromagn. Research M*, vol. 9, pp. 165-176, 2009.
3. S. E. Nai, W. Ser, Z. L. Yu, and H. Chen, "Beampattern synthesis for linear and planar arrays with antenna selection by convex optimization," *IEEE Trans. Antennas Propag.*, vol. 58, no. 12, pp. 3923-3930, Dec. 2010.
4. S. E. Nai, W. Ser, Z. L. Yu, and H. Chen, "Iterative robust minimum variance beamforming," accepted by *IEEE Trans. Signal Process.*, 2010.

Conference papers

1. S. E. Nai, W. Ser, Z. L. Yu, and S. Rahardja, "Iterative robust Capon beamformer," in *Proc. IEEE 14th Workshop Statistical Signal Process. (SSP'07)*, Madison, WI, Aug. 2007, pp. 542-545.

2. S. E. Nai, W. Ser, and Z. L. Yu, "Robust adaptive beamformers: a beam-pattern shaping approach," in *Proc. IEEE 5th Workshop Sensor Array Multichannel Signal Process. (SAM'08)*, Darmstadt, Germany, Jul. 2008, pp. 298-302.

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Appendix A

Definitions of Operators in Linear Matrix Inequality (LMI) Formulation of Chapter 5

In order to obtain LMI formulations of the sets in (5.4), the authors of [7, 8] defined a Toeplitz matrix as

$$[\mathbf{T}_{k,N}]_{i,j} = \begin{cases} 1, & \text{if } i = j + k, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i, j \in [0, 1, \dots, N]. \quad (\text{A.1})$$

The notation $\langle \mathbf{T}_{k,N}, \mathbf{X} \rangle = \sum_{l=0}^{N-k} X_{l+k,l}$ means the sum of the elements on the k th lower off-diagonal of $\mathbf{X}$. The adjoint operator $\bar{L}(\cdot) : \mathbb{C}^{(N+1) \times (N+1)} \rightarrow \mathbb{C}^{N+1}$ refers to $y = \bar{L}(\mathbf{X})$ with $y_0 = \langle \mathbf{T}_{0,N}, \mathbf{X} \rangle$ and $y_k = 2\langle \mathbf{T}_{k,N}, \mathbf{X} \rangle$ for $k = 1, \dots, N$. With

$0 \leq \Omega_l < \Omega_u < 2\pi$, the vector $\mathbf{d}(\Omega_l, \Omega_u)$ is defined as

$$\mathbf{d}(\Omega_l, \Omega_u) = \begin{cases} \begin{bmatrix} \cos \Omega_l + \cos \Omega_u - \cos(\Delta\Omega) - 1 \\ (1 - e^{j\Omega_l})(e^{j\Omega_u} - 1) \end{bmatrix}, & \text{if } \Omega_l > 0 \\ \begin{bmatrix} -\sin \Omega_u \\ j(1 - e^{j\Omega_u}) \end{bmatrix}, & \text{if } \Omega_l = 0 \end{cases} \quad (\text{A.2})$$

where $\Delta\Omega = \Omega_u - \Omega_l$. The adjoint operator $\bar{\Lambda}(\cdot) : \mathbb{C}^{(N+1) \times (N+1)} \rightarrow \mathbb{C}^{N+1}$ is given by $y = \bar{\Lambda}(\mathbf{X})$:

$$y_0 = d_0(\Omega_l, \Omega_u) \langle \mathbf{T}_{0,N-1}, \mathbf{X} \rangle + d_1^*(\Omega_l, \Omega_u) \langle \mathbf{T}_{1,N-1}, \mathbf{X} \rangle, \quad (\text{A.3})$$

$$y_k = 2d_0(\Omega_l, \Omega_u) \langle \mathbf{T}_{k,N-1}, \mathbf{X} \rangle + d_1(\Omega_l, \Omega_u) \langle \mathbf{T}_{k-1,N-1}, \mathbf{X} \rangle \\ + d_1^*(\Omega_l, \Omega_u) \langle \mathbf{T}_{k+1,N-1}, \mathbf{X} \rangle, \quad (\text{A.4})$$

$$y_{N-1} = 2d_0(\Omega_l, \Omega_u) \langle \mathbf{T}_{N-1,N-1}, \mathbf{X} \rangle + d_1(\Omega_l, \Omega_u) \langle \mathbf{T}_{N-2,N-1}, \mathbf{X} \rangle, \quad (\text{A.5})$$

$$y_N = d_1(\Omega_l, \Omega_u) \langle \mathbf{T}_{N-1,N-1}, \mathbf{X} \rangle, \quad (\text{A.6})$$

where $k = 1, \dots, N-2$, and d_0 and d_1 are the first and second elements of $\mathbf{d}(\Omega_l, \Omega_u)$, respectively.