Autonomous vehicle navigation in neighbourhood environments : a map assisted and a feature-based approach

Lee, Kwang Wee

2007

Lee, K. W. (2007). Autonomous vehicle navigation in neighbourhood environments : a map assisted and a feature-based approach. Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/4575>

<https://doi.org/10.32657/10356/4575>

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**Autonomous Vehicle Navigation
in Neighbourhood Environments:
A Map Assisted and a Feature-based Approach**

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A thesis submitted to the Nanyang Technological University
in fulfillment of the requirement for the degree of
Doctor of Philosophy

2007

Acknowledgements

First of all, I would like to thank my two supervisors. To Dr. W. Sardha Wijesoma thanks for his guidance and support throughout the years. Being patient and encouraging he has taught me how to face the challenges of research. To Dr Javier Ibañez Guzmán thanks for his availability and support during this research. Through our several discussions, they have decidedly influenced the way I think about the problems discussed here. I also wish to thank both of them for their useful comments and positive criticisms on the development of this thesis.

Special thanks goes to Mr Tan Chai Soon and other technical staff of M&E workshop and Intelligent Robotics Lab for their continuous support. I also wish to thank the students and my friends of Nanyang Technological University (NTU) for assisting me on various occasions. I am grateful to NTU for providing me the financial support and providing the best environment to carry out the research work. I would also like to thank the examiners for their positive criticisms on the draft of the thesis.

Finally, I would like to thank my parents and wife, Sok Eng. I could not have done this if it were not for their support, understanding and love. I would like to dedicate this work to my parents, wife and twin baby sons, Yu Xiang and Yu Sen.

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Summary

For several decades, autonomous vehicle guidance has been a topic of intense research. Several national and international research programs have been initiated all around the world. The main objectives for transport industries are to improve road safety, and optimise the usage of transportation networks and energy consumption. In the case of military's interest, the aim is to lower the number of army casualties in conflict zones by deploying ground vehicles with autonomous capabilities.

The research community has theoretically and experimentally addressed the problem of vehicle guidance with different levels of success and has revealed that one of the fundamental problems for vehicle guidance is localisation. The importance of vehicle localisation has been recognised for some time and in the past decades, much research effort has been directed to achieve localisation for vehicle navigation. These efforts have resulted in significant gains in terms of localisation performance. However, no systematic studies or theoretical analysis of these approaches have been carried out to fully appreciate and understand the properties of the resulting estimation errors and the limitations of such techniques. In most of these approaches, the issues of observability and its implications on estimation errors have been neglected or taken for granted. In cases where such analysis is carried out, inappropriate or incorrect methods of analysis involving linearised models and tools have been applied.

In this thesis, the objectives are to establish and apply an appropriate non-linear framework of analysis of observability for vehicle localisation. More specifically, the methodology is applied to analyse the observability of various localisation approaches of

ground vehicles, which includes Feature-based Localisation, Path Constrained Localisation, and Concurrent Mapping and Localisation (CML). Generally the vehicle localisation problem is highly non-linear. Thus, unlike in linear systems, the effects of inputs on the observability of coupled non-linear systems cannot be ignored. In order to correctly analyse observability, one needs to apply tools that can accommodate the effects of inputs, the significant coupling and non-linear effects of the system.

By applying the appropriate analysis, the limitation of localisation systems using spatial information in terms of feature points and path constraints are theoretically addressed. The theoretical work also extends to prove the necessity of absolute information in the CML problem and determines the minimum amount of absolute measurements required for achieving an observable localisation system. To devise an observable system plus the feasibility to implement it in a large-scale environment, the Path Constrained localisation approach is integrated within a CML framework. A general framework based on standard CML using *a priori* information describing the vehicle's traversable path is thus proposed. The proposed approach's observability is obtainable. Based on the simulation results, the error when the vehicle is travelling along a path interval is shown to be bounded, contrary to the results on Path Constrained localisation approaches alone where the error is cumulative.

In this thesis, the spatial information is in the form of an approximate skeletal digital road map. The central notion is how the knowledge of a vehicle's presence on a known road segment is used to constrain the vehicle pose estimates, and the refinement of the pose through simultaneously mapping and tracking unknown features. Simply knowing the road segment on which the vehicle is plying helps to assist in constraining

the lateral and orientation errors, and the longitudinal errors when the vehicle turns into a subsequent road segment. Along straight roads however, it is established that path constraints are insufficient to prevent the vehicle from accumulating longitudinal positional errors. The CML framework proposed in this thesis bounds this accumulation of longitudinal error.

An algorithm incorporating *a priori* path constraints derived from the road map is developed within the CML framework by extending the conventional FastSLAM algorithm. The proposed Path Constrained FastSLAM clearly outperforms FastSLAM in accuracy on relatively large road sections involving loops with sparsely distributed features. The implementation of Path Constrained FastSLAM using a Rao-Blackwellised particle filter with adaptive particle sampling sizes is shown to be efficient and executable in real-time without any compromise in accuracy. These are demonstrated using experimental results from a mobile robot conducted in a campus road environment. The field experiments justify the viability of the proposed approach on using the CML mechanism with the aid of approximate skeletal road maps. A comparative approach such as FastSLAM is also implemented to benchmark the improvements made by the proposed novel approach. The results show that the Path Constrained CML approach is more viable and effective to solve the large-scale localisation problem in neighbourhood environments.

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Chapter 1

Introduction

1.1 Motivation

For several years now, the research community has been attempting to address the problem of vehicle guidance with varying levels of success. Both theoretical efforts and experimental results have shown that navigation presents a fundamental problem. Enabling a vehicle to move autonomously in real environments poses major challenges of real-time perception, control and decision-making. To successfully navigate an autonomous vehicle and have it reach a desired destination, it is necessary to determine its position, its course and the distance travelled while avoiding collisions, optimising energy use and meeting task constraints.

Presently, there is much interest in autonomous vehicle guidance and driving assistance systems for use in mass-market vehicles. The interest is fuelled by the successful deployment of Unmanned Aerial Vehicles and Search robots by the military in conflict zones. The military's interest in such systems is also increasing as the tolerance for army casualties decreases in advanced countries around the world. In the United States of America, by congressional mandate, one third of all the army ground vehicles must have robotic capabilities by the year 2015. This requirement has resulted in considerable funding for the effort and has even triggered the *Darpa Grand Challenge* –

a race called to build the first fully unmanned ground vehicle capable of competing on an off-road course in the desert [1].

Adding to the demand, are car manufacturers (largely European) who are eagerly exploring safety systems that go beyond the current passive security features such as airbags, chassis design and ergonomics. The manufacturers are working towards the development of active, perception based safety systems that they call advanced driving assistance systems (ADAS). A notable component of the effort is the PreVent Integrated Project which is a research programme sponsored by the European Union [2]. The programme's major aim is to reduce road accidents by 50 per cent by the year 2010. To reach this goal, it plans to develop, demonstrate, test and evaluate preventive safety applications using a combination of advanced sensors, communication and positioning technologies. These technologies are similar to those required for unmanned ground vehicles meant for military and commercial use and will be integrated into on-board systems for driver assistance.

Navigation for vehicle guidance is a large domain and the subfield of perception is the subject of much research. While perception remains a major challenge, perception alone is inadequate without context as it is context that defines the relationship between the vehicle and its environment within time and space. Specifically, context refers to the vehicle's position, altitude, kinematics and dynamic information and these measures have to be input in real-time with respect to either an identified environmental object or in relation to the vehicle's final destination. Without such information, a vehicle's onboard system would lack situational awareness and vehicle guidance functions will be difficult to implement.

Putting it simply, in order to guide a vehicle from Point A to Point B in an autonomous manner, it is necessary for the navigation system to “know” a few things, essentially, “Where am I?”, “Where am I going?” and “How should I get there?” [3]. First, through its system of perception, it should be able to tell which areas in front of the vehicle are not currently being obstructed. It should also be able to determine the state of the vehicle (e.g. position, direction, velocity) with regard to the environment and have an “awareness” of the arrival point (localisation). The information from these functions is then used for navigation, i.e., to plan the vehicle’s path towards the desired position. After the information is processed, heading and speed related commands are generated for the vehicle motion controller. However, in order for the autonomous vehicle to respond accordingly, it has to be controlled by the mobility function. The block diagram in Figure 1.1 shows a functional perspective of an autonomous vehicle and the inter-relationships between its different components.

Figure 1.1 Autonomous navigation systems (Functional diagram)

It is clear from the diagram (and indeed from a logical perspective) that localisation is crucial for vehicle guidance because it determines the context of the vehicle with respect to its environment and desired position. A failure to localise would not only keep it from reaching a target position, but can also cause it to move hazardingly.

Today, it is possible to determine the position of a vehicle with very good accuracy in real time either through the use of global navigation satellite systems (GNSS) and augmentation systems such as GPS + Egnos or RTK GPS systems. Other solutions include the use of proprioceptive sensors such as Inertial Measurement Units and Odometry. The drawback of GNSS-based systems is that they demand continuous signal availability. This is not always possible, because of trees, buildings and other sources of obstruction. High-end inertial-based systems, on the other hand, are likely to still be too expensive for mass-market applications [4].

Although impressive results have been attained for autonomous vehicle guidance in well-structured and open environments like motorways, the situation is different in densely populated areas like city dwellings. The effort to design applications meant for use in such areas has been fraught with difficulty – the environments are simply far too complex for current technologies. In contrast, semi-structured environments with low traffic density such as university campuses, large factory compounds or new housing estates make the prospects more viable, especially if the neighbourhoods have well defined road networks, speed limitations and controlled vehicle flow conditions. While the challenges are likely to be considerable, localisation solutions for vehicle guidance in real time can in all likelihood be successfully deployed in these environments.

Researchers stand to gain valuable insight through such an exercise and this will serve them well when they tackle systems for use in cities and other more complex environments.

This thesis will explore the localisation of ground vehicles for the purposes of vehicle guidance in what is largely considered a neighbourhood environment. Because of their historical development, complex road layouts are typical of older cities. However, Figure 1.2 illustrates scenes of neighbourhoods common to younger, more modern cities such as Singapore. Modern urban design principles are usually adopted in the plans for such neighbourhoods and they tend to be well structured and organised

Figure 1.2 Neighbourhood environments

1.2 The Vehicle Localisation Problem

The importance of vehicle localisation in urban environments has been recognised for some time and in the past decades much effort has been put into developing it for

vehicle navigation. Different approaches have been proposed to determine the vehicle's location based on data gathered by the sensory devices of both Proprioceptive and Exteroceptive types. The first type of sensory device measures signals generated by the vehicle about its ego-state e.g. rotation sensors and shaft encoders. The second measures signals from the immediate environment e.g. landmarks. Together, the signals can be used to estimate the vehicle's pose, but it is imperative that all necessary information is input in a continuous manner.

In the next section, we review several important solutions that have been forwarded to address this vehicle localisation problem before turning to the solution currently being proposed.

1.2.1 Dead Reckoning

In principle, localisation can be accomplished using only proprioceptive sensors, which include encoders, odometers and inertial measurement units (IMU). These sensors provide relative displacement measurements that allow the vehicle's location to be estimated from its initial position. This approach is known as dead reckoning and is used mainly in vehicle localisation systems. It provides good short-term accuracy at high sampling rates. However, the exact quality of these estimates depends largely on the bias and systematic errors of the sensors used. Because the approach relies on the integration of incremental motion information over time, the accumulation of errors is inevitable and the imprecision of the estimates grows [5] with travelled distance. To address this problem, much of the research in this field has centred on the development of high fidelity sensors, good error models and estimation algorithms [6], [7].

The use of such accurate inertial measurement devices with low drift and good bias models can reduce these accumulative errors. Unfortunately, these sensors tend to come at a prohibitive price. However, even when they are indeed deployed, low drift becomes an additional, significant source of estimation error [7]. The error is introduced over a long period because the sensors only provide relative measurements of the vehicle pose. The small offset errors can integrate over long periods of time to cause a large estimation error, which in turn, can be aggravated by errors arising from inaccurate vehicle models. For these reasons, dead reckoning alone is not sufficient for the localisation of vehicles over extended environments.

Despite these limitations, researchers and practitioners agree that dead reckoning approaches are important for vehicle navigation when used with other methods to overcome the errors [8] [9]. Approaches that circumvent the problem of error in dead reckoning approaches are discussed in the next section. These are based on the use of external references to correct estimation errors.

1.2.2 Active Beacon Based Localisation

In sea and aerial navigation, active beacons are the most commonly used methods for localisation [10]. These provide accurate positioning information with minimal processing. However, the high sampling rates and reliability they allow come together with high installation and maintenance costs.

Currently, the best known localisation system that uses active beacons is based on a network of satellites. The Global Navigation Satellite Systems (GNSS) uses satellites to emulate beacons. In principle, signals can be received outdoors anywhere in the globe

and the satellites allow the device that holds the GNSS receiver to estimate its absolute position [11], [12] by estimating the distances between the position of the satellites and the receiver. The exact location of the receiver is then determined using triangulation techniques. However, error gets introduced from several sources and accuracy is largely determined by the quality of the received signal. The accuracy of the measures may be poor – not only because of the physics of the whole process, but because signal availability may be affected by blockages in the form of buildings, trees and tunnels. Another common problem in built-up areas comes from the multiple paths of the GNSS signals.

Because GNSS alone clearly does not provide a reliable continuous localisation solution for vehicle guidance applications, researchers have been led to devise approaches that combine dead reckoning or inertial sensors with GNSS improving performances. The two complement each other. One provides relative position estimations that are at high frequencies, the other provides absolute estimations but at a lower frequency [13]. In recent years, the reductions in the cost of IMUs and the increases in computational power have resulted in the development of multiple solutions based on IMUs and GPS measurements [14]. It is possible to attain greater accuracy and robustness by maintaining a link with the ground through the use of odometric measurements in addition to the IMU/GPS tandem [15], [16]. Although these fusion-based approaches have resulted in significant improvements in vehicle localisation capabilities, they still rely on the availability of GNSS signals to provide a reliable continuous localisation solution.

1.2.3 Map-aided Localisation

An alternative approach is to use spatial information known *a priori* in the form of the positions of known landmarks to geometrically localise the vehicle [17]. The use of artificial landmarks is preferred over natural ones when operating in a structured environment. While wall corners and doorways are potentially useful landmarks for indoor localisation, man-made objects like signposts and lampposts of regular geometrical shapes can be similarly used as suitable markers in urban/neighbourhood environments. With knowledge of the position of these landmarks, the vehicle's position can be inferred by straightforward triangulation [18] [19]. However, the recognition of landmarks amongst the clutter of a city street is difficult. Data association in the presence of such "noise" remains a formidable challenge.

A commonly used sensor for landmark detection is vision, which in contrast to sparse sampled data like LADARS or narrow beams like RADAR [20] provides both large and rich fields of view. However, understanding scenes and identifying landmarks from one observation to the next is difficult because of illumination changes, differences between the viewing angles across samples and so on. Consequently, there are several efforts being made to address these issues for localisation-based applications [21], [22], [23]. The physics of most sensors limits the amount of information that can be extracted from the data they capture. Limitations are presented also by the complexities of the operating environment. By combining multiple sensor fusion methods, it is possible to identify and track landmarks with a good degree of robustness [24]. An alternative is to add artificial beacons and cars extensively to infrastructure communications systems. Realistically however, this approach is expensive and difficult to apply in a large context.

Other researchers have investigated the use of *a priori* information in the form of digital road maps. The availability of a map of the environment traversed by the vehicle can simplify the localisation problem significantly [25], [26], [27]. By making the assumption that under normal circumstances, all vehicles are constrained to the road, a correlation can be made between a vehicle's movement pattern and digitally mapped roads or graph representations. For example, if a relative direction change can be detected when the vehicle turns at a known intersection, the vehicle's dead reckoning position estimate can be updated by utilising the coordinates of that particular intersection as retrieved from the digital map database. Forssell *et. al.* [28] have implemented a particle filter to localise a vehicle under the constraints of the roadmap network. The preliminary results appear to be promising; however, sufficiently sampling the vehicle position requires intensive computational capabilities. The uncertainty of the vehicle position is further exacerbated when no direction change is made for an extended period. The vehicle position estimates obtained by dead reckoning techniques (like the use of odometry and inertial means) can be corrected or updated using the map only if a significant change of direction is detected. On long straight roads, longitudinal position errors are unobservable, and the accumulative effects of vehicle motion error cause the estimation to be unbounded. Therefore, if there are no recognisable curvatures, errors will accumulate to such a degree that they result in the vehicle being "lost" [26].

In addition, digital road maps sometimes lack detailed information. They may be outdated by the time they reach the end user, or may be inaccurate because of the complexity of the road network. For these reasons, it may be more viable for the system to have the ability to learn or to build maps of the environment itself, in an incremental

fashion as the vehicle moves. This would allow for the use of more updated information. This approach is known as Concurrent Mapping and Localisation (CML)

1.2.4 Concurrent Localisation and Mapping (CML)

The robotics research community has addressed mapping and localisation as a joint problem. It is considered fundamental in the field for autonomous robots to be able to evolve in unknown environments. The approach of Concurrent Mapping and Localisation (CML) is also often referred to as Simultaneous Localisation and Mapping (SLAM) [29], [30]. The reason for compounding the two problems is that mapping involves both an estimation of the vehicle's position relative to the map and the generation of a map using the data from the sensory devices and estimates of the vehicle's position.

The CML approach was first introduced in the last decade by Smith *et. al.* [29] and has attracted much attention within the research community. Several techniques have been proposed to solve inherent problems in CML such as computation complexity, data association and large loop closure. Several researchers have concentrated on solving the problem of the intensive computational cost. This is incurred in the theoretic estimation framework (Extended Kalman Filter) [29], which an increasingly huge covariance matrix $n \times n$ that describes the correlation of the estimates has to be maintained at the cost $o(n^2)$ in order to guarantee the consistency of the filtered quantities. To reduce or postpone the computation updates and to achieve near real-time systems, researchers have envisaged sub-mapping approaches.

Guivant *et. al.* made an early attempt with the Compressed Extended Kalman Filter (CEKF) method [31]. This is a modified Extended Kalman Filter (EKF) that allows the accumulation of measurements in a local region with k landmarks at cost $o(k^2)$ independent of the overall map size. When the robot leaves the region, the accumulated result must be propagated to the full EKF at the cost $o(kn^2)$. Although the computation requirement seems to be reduced at that instant, the method is computationally still very intensive. Many methods are similar in that they either postpone the computational needs [32] or reduce the optimality of the solution in order to lessen computation complexity [33]. Researchers have found that the CML problem carries optimality and computation trade-offs. Another outstanding issue is data association. The task of relating sensor observations with elements included in the map becomes fundamental and incorrect assignments can cause the estimation to be inconsistent and to eventually diverge. A common method of solving this problem involves the use of the gated nearest neighbour (NN) algorithm, which is a classical technique for tracking problems [34]. Although, this solution is conceptually simple and less computationally intensive, it appears too permissive under the individual innovation compatibility test. The approach neglects the correlations between innovations when different observations are obtained from the same robot pose. The easy acceptance of the hypotheses formed by inconsistent pairings will lead to divergences in state estimation. Hence, to solve the data association problem in CML, Neira *et. al.* [35] have proposed the use of the Joint Compatibility test together with the Branch and Bound searching algorithm (JCBB). Although, this algorithm improves the full tree search, it is again very computationally intensive [36]. Moreover,

the problem may not be resolved to a unique solution because of the great level of uncertainty involved in estimating the robot's position.

The quest for a match between observations and mapped features has led to the revisiting problem or what is known as the problem of "Loop Closure". When the robot returns to an area to which it has been before, it is unable to consistently match the new observations with the mapped features and assumes that new features are being observed. Castellanos *et.al.* [37] found that this inconsistency is due mainly to limitations in the EKF technique used. The approximation of the non-linear CML problem (which incurs linearisation errors) has led to inconsistent estimates. Thus, combined methods are proposed, such as *local map joining* and *robocentric mapping*. *Local map joining*, like the name suggests, maps and joins independent local maps and minimises linearisation errors. *Robocentric mapping* then improves the consistency of the joined map. Although these methods effectively reduce the effect of the linearisation error, they also diminish the optimality of the solution, making it distinctly possible for the solution to be incorrect.

To attain the optimal solution and to prevent linearisation errors, several researchers apply a batch optimisation algorithm in solving the CML problem. In an earlier attempt, Thrun *et. al* used the Expectation Maximisation (EM) algorithm for robot mapping [38]. EM is a statistical algorithm developed in the context of maximum likelihood (ML) estimation with latent variables, Dempster *et. al.* [39]. It iterates between the E-step (expectation step) and the M-step (maximisation step). The E-step estimates the robot's path over the current map and the M-step estimates the map over the robot's current path. The technique is promising in that it is able to search for an optimum

solution. However, it requires good initial estimates in order to avoid falling into a local optimum. This is a common problem with local optimisation techniques.

Frese *et. al.* [40] also used a local method called Gauss-Seidel relaxation, which is an iterative procedure for solving the linear equation system in CML. Although, the algorithm is recursive and less computational intensive, Frese *et. al.* have stated in [40] that their method still requires an additional global means of searching for the optimum solution. Therefore, it is clear that in order to overcome the deficiencies of local optimisation, other methods are needed that include global search mechanisms.

The development of several optimisation techniques has been reported in the literature, but few researchers have tried to use them to solve the CML problem. Only recently, the use of Genetic Algorithms (GA) was proposed by Duckett [41] and this marked the formalisation of CML as a global optimisation problem. The method uses two heuristic metrics – consistency and compactness – to define the fitness level of each hypothesis or string. It produces a grid map using a laser range finder that defines the cell status as being either occupied or empty. The consistency criterion is derived from such a map to measure the degree of incongruence between laser readings. This is done by counting the corresponding cells. By contrast, the compactness metric constrains the size of the grid map and avoids maps with low conflict based on the wrong map configuration. This is essentially a heuristic approach as it is taken without specific theoretical or empirical analysis.

The review provided above raises the question of whether or not CML is a well-posed or solvable problem. The issues invoked by the reviewed approaches show that an optimum global solution is not obtainable without additional information in the form of

either heuristic or metric constraints in the estimation. Although significant methods have been proposed throughout the development of the CML problem, few of these methods have fundamentally addressed the theoretical limitation of the CML problem itself. The focus has mainly been on solving the computational complexity and data association problems with little emphasis on the fundamental limitation of the framework.

Using the EKF framework Dissanayake *et. al.*[42] first ratified and demonstrated objectively that the estimated map converges, but their study is limited to a relative map. In addition, they only prove that the absolute accuracy of the map reaches a lower bound as defined by the initial level of uncertainty. This is valid only when the vehicle is stationary. With these strong assumptions, many researchers have assumed that a solution to the CML problem is obtainable. However, a counter example given by Julier and Uhlmann [43] has shown experimentally that maps resulting from the application of EKF in CML are inconsistent after several hundred time steps. The inherent approximations lead to divergence of the EKF due to linearisation errors (Bar-Shalom *et. al.* [34]).

Recently, Andrade-Cetto *et. al.*[44] demonstrated the theoretical limitations of the CML estimation framework by applying observability tools on a linearised CML system model. According to the authors, a full correlation of the map model (even one-dimensional KF-SLAM) hinders the full observability of the state estimate. Although they concluded that the conventional 2D CML problem was unobservable, the analysis lacked theoretical correctness. This lack of accuracy has contributed in some cases to the obtainment of incorrect observability results by Andrade-Cetto *et. al.*[44]. CML is an inherently highly non-linear and coupled problem and the control inputs of such a system play a pivotal role in its observability. Methods based on linearised system

models completely ignore the effect of the inputs. Although, inputs do not affect observability analysis in linear systems, this is not quite the case for highly non-linear and coupled systems. This difference renders observability analysis based on linearised models inappropriate. For this reason, the notion of local observability introduced by Hermann and Krener [45] would be more suitable for addressing the system observability property. Its advantage is that it takes into account non-linearities in formulating the observability matrix.

Bicchi *et. al.* [46] emphasised that the typical model used in the CML state system is referred to as *world-centric* because the state estimates refer to a global coordinate frame. The observability of the system is not achievable, unless known geographic markers or absolute onboard sensors (from a magnetic compass, GPS, etc.) are present. Martinelli *et. al.* [47] have represented the problem as an estimator based on relative mapping using a *robot-centric* model. Their approach decouples the vehicle position estimation and proposes that to transform the resultant relative map to an absolute one, *a priori* spatial information is required. That is, at least three known feature positions are required in order to obtain an invariant rotation and shift optimal solution. It shows that the CML solution in the *world-centric* form will require additional absolute information.

Several elegant methods have been reported in the literature; nevertheless, more progress is needed to address the problem in a proper observable manner. The estimation of the system state must be formulated as a *world-centric* model, which requires absolute information in order to derive distinctive solutions. A unique solution detailing the characteristics of the map and vehicle pose will facilitate a less ambiguous and more efficient navigation system. While absolute information regarding the vehicle states is

clearly required in order to fulfil the observability condition, these are not always available. GPS signals are often blocked by tree canopies and building structures [11]. Similarly, the magnetic compasses used will likely suffer from electromagnetic interference that can result in inaccuracies in the data [8]. *A priori* information in the form of known landmarks or feature points give spatial information which can then be used to estimate the vehicle states [17]. This will be possible, if the sensors used are able to observe the known landmarks. This information is not very easy to obtain and becomes impractical for large scaled operations. Other alternatives for representing absolute spatial information exist and these can be used to estimate the vehicle states [48] [49].

This thesis investigates the observability properties of different vehicle localisation approaches using a means of analysis that is commonly applied for non-linear systems. What is proposed is an observable approach to solve the problem of vehicle localisation in a large-scale neighbourhood environment such as a campus road (See Figure 1.3). The approach involves the aid of road maps and a built-in capability to maintain spatial relationships between the static environment (as indicated by a feature map) and the moving vehicle. The method is shown later in this thesis to be observable. Experiments were carried out to validate the approach in a university campus at with the vehicle travelling at 15km/h. An efficient and robust algorithm is implemented and its efficiency is demonstrated experimentally using a vehicle test bed, as shown in Figure 1.4.

Figure 1.3 Test site (Campus road)

Figure 1.4 Vehicle test-bed

1.3 Thesis Statement

The fundamental premise of this study is that localising a ground vehicle in areas where GNSS signals are only partially available, such as neighbourhood environments, requires more information than what is available solely from a vehicle's proprioceptive sensors. This study claims, however, that it is possible to localise the vehicle with a good level of certainty over long trajectories if data from the sensors is combined with *a priori* information in the form of digital road maps. The resulting feature maps enable the vehicle to determine the spatial relationships between itself and the static environment, thus helping it improve its localisation capability. Given this premise, the thesis formulates an observable localisation system that integrates and associates data from proprioceptive sensors, *a priori* information from digital maps and relative observations of unknown features from the environment using an exteroceptive sensor. The

theoretical formulation is developed first, following which extensive field experiments are conducted to test the proposed approach.

The objectives of this thesis are:

- To analyse localisation approaches theoretically using methods applied to non-linear systems.
- To theoretically determine the level of *a priori* spatial information required in order to achieve the condition of observability.
- To establish a mathematical framework that integrates the efforts to solve the problem of localisation using a road map together with the CML mechanism.
- To demonstrate theoretically that the CML estimation problem becomes observable through the introduction of a *a priori* spatial information in the form of a road map.
- To prove the feasibility of the proposed approach through extensive field experiments in real conditions.

1.4 Major Contributions

The main contributions of this thesis can be summarised as follows:

- Theoretically analyse the observability problem of different localisation approaches that may or may not incorporate *a priori* spatial information and determines the minimum amount of spatial information required to achieve the observability condition.

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- Introduce a re-formulated observable approach for solving the localisation problem using *a priori* spatial information in the form of a road network map to constrain estimation and using the CML mechanism which provides the vehicle with the ability to maintain spatial relationships with the static environment in terms of a feature map as it moves to improve the localisation estimation.
 - Demonstrate the feasibility of the proposed solutions through the use of simulations and an experiment conducted in a large-scale neighbourhood environment.

1.5 Thesis Overview

Chapter 2 introduces the concept of observability as applied to non-linear systems and reviews the mathematical formulation of the localisation problem. The latter refers to the problem of determining the pose of a vehicle with respect to a global reference frame. The probabilistic approach to the problem is developed by identifying the pose of the vehicle as a state that needs to be estimated. Further, two common types of algorithm used to approximate the estimation process, the Kalman filter and Particle filter, as well as other extension algorithms, such as the Rao-Blackwellisation Particle filter, which are used to solve the Concurrent Mapping and Localisation (CML) problem, are reviewed.

Chapter 3 presents the theoretical limitations of using the observability matrix to analyse different localisation approaches. These include the Feature-Based, Path Constrained and Concurrent Mapping and Localisation approaches. The analysis is then extended to determine the minimum number of *a priori* features or amount of spatial

information required for the observability problem to be circumvented. The chapter also uses simulations to validate the observability results of each individual approach.

Chapter 4 describes how observability is obtained by using alternative *a priori* information in the CML formulation. Digital road maps, in the form of graph-like diagrams, are used to constrain estimates of the vehicle's motion. Next, a formulation of a Bayesian CML framework is presented that integrates the road map's observations. Simulations are then run to validate the observability results and as well to prove the feasibility of the proposed approach.

Chapter 5 presents the experimental component of this research. The approach that has been formulated is optimised by implementing an adaptive Rao-Blackwellisation particle filter. Finally, the results from the experiments and the findings are discussed in detail. A comparison with other approaches such as FastSLAM is also included.

Chapter 6 concludes by summarising the major findings, critiquing the results and providing recommendations for future research in this domain.

Chapter 2

Estimation Theoretic Probabilistic Vehicle Localisation

2.1 Introduction

In this chapter, the concepts of observability as applied to non-linear systems are first reviewed. The study of observability in a localisation problem is important as the analysis allows for the correct formulation of a state system that would guarantee the achievement of theoretical bounds on the estimation error. In most localisation approaches, issues of observability and their estimation error implications have been neglected or taken for granted. In the cases in which such an analysis has indeed been carried out, the methods and tools applied to linearised models have often been inappropriate or incorrect. Still, while in linear systems, the effects of inputs on observability can be ignored; this is not the case of coupled non-linear systems. In order to correctly analyse observability in non-linear systems, one needs to apply tools that can accommodate the coupling effects and nonlinearities of the system.

Once the appropriate analysis proves the problem formulation is indeed observable, a probabilistic mathematical solution is established. In this chapter, we seek to determine a vehicle's pose based on observations of its relationship with objects in its environment. However, the measurements provided by sensors and obtained from *a priori* information sources (such as maps) inherently contain a certain degree of noises and uncertainty. Because of this ambiguity, we seek instead to form a probabilistic

estimate of the pose. If successful, such an approach should allow the uncertainty to be handled in a way that is both mathematically elegant and consistent. At the same time, it will provide valuable insight into how the vehicle localisation problem can be efficiently and effectively solved.

In the section that follows, concepts of system observability are introduced and analysed as applied to vehicle localisation. Section 2.3 reviews the Bayesian filtering approach to state estimation for vehicle localisation, along with two common Bayes filter types: the Kalman filter (KF) and Sequential Monte Carlo Method (SMC). We discuss also the common extensions or variants of these filters and their applications, paying close attention to the Rao-Blackwellisation Particle filter and its common implementation. Section 2.4 concludes the chapter with a summary of our key findings.

2.2 System Observability

In general, the vehicle localisation problem is both highly non-linear and coupled. This renders any observability analysis method that is based on linear methods both inappropriate and inaccurate. Hermann and Krener [45] relate observability for such non-linear systems to the concept of “indistinguishability” of states. In other words, they emphasise that the observability of non-linear systems depends largely on the control inputs – This is not the case in linear systems. Consequently, linear methods of analysis cannot be applied directly without accounting for inputs that act on the non-linear system.

The observability of a non-linear system can be characterised from a differential geometric point of view. Consider the system Σ :

$$\Sigma \left\{ \begin{array}{l} \frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, \mathbf{u}) = \begin{bmatrix} f_1(\mathbf{X}, \mathbf{u}) \\ f_2(\mathbf{X}, \mathbf{u}) \\ \vdots \\ f_n(\mathbf{X}, \mathbf{u}) \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n \quad \mathbf{u} \in R^e \\ \mathbf{Z} = \mathbf{H}(\mathbf{X}) = \begin{bmatrix} h_1(\mathbf{X}) \\ h_2(\mathbf{X}) \\ \vdots \\ h_m(\mathbf{X}) \end{bmatrix} \quad \mathbf{Z} \in R^m \end{array} \right. \quad (2.1)$$

where $\mathbf{X}$, $\mathbf{u}$ and $\mathbf{Z}$ are the state, input, and the measurable output vectors of dimensions n , e and m respectively. Hermann and Krener proposed a rank condition test for what they termed “local weak observability” of a non-linear system [45]. The system Σ is said to be **observable** at $\mathbf{X}(0)$ (or alternatively, at time t_0) if the state vector $\mathbf{X}(t_0)$ can be determined from the observation of $\mathbf{Z}(t)$ over an infinite time interval, $t \rightarrow \infty$. The notion of “local” observability is a stronger condition than observability. Local observability indicates that only the local state space is required to distinguish between states. That means if the system Σ is **locally observable**, then the state vector $\mathbf{X}(t_0)$ can be determined from the observation of $\mathbf{Z}(t)$ over the finite time interval $t_0 \leq t \leq t_1$. The notion of “weak” observability indicates a weakening of the observability condition such that a given state needs only to be distinguished from its neighbours rather than from the entire manifold Ξ . The system Σ is said to be **weakly observable**, if the state vector $\mathbf{X}(t_0)$ can be distinguished not among all possible states but among states in a given neighbourhood of $\mathbf{X}(t_0)$ from the observation of $\mathbf{Z}(t)$ over an infinite time interval, $t \rightarrow \infty$. Weak observability is a necessary condition for observability. Now the system Σ is said to be **locally weakly observable**, if the state vector $\mathbf{X}(t_0)$ can be distinguished among states in

a given neighbourhood of $\mathbf{X}(t_0)$ from the observation of $\mathbf{Z}(t)$ over a finite time interval $t_0 \leq t \leq t_1$. This intuitively means that if the system is locally weakly observable then “one can instantaneously distinguish each point (state) from its neighbours.”

Suppose that the system Σ can be transformed to its control affine form:

$$\begin{aligned} \frac{d\mathbf{X}}{dt} &= \mathbf{F}(\mathbf{X}, \mathbf{u}) = \mathbf{F}_0(\mathbf{X}) + \sum_{i=1}^e \mathbf{F}_i(\mathbf{X}) u_i \\ \mathbf{Z} &= \mathbf{H}(\mathbf{X}) \end{aligned} \quad (2.2)$$

where e is the total number of inputs to the system (dimension of input vector $\mathbf{u}$). The local observability of the system (equation (2.2)) can then be determined using a matrix that is analogous to the linear observability matrix. It can be shown that the non-linear system is locally weakly observable at $\mathbf{X}(0)$ if the $n \times n$ observability matrix

$$O_j = \begin{bmatrix} dL_{\mathbf{F}}^0 h_j(\mathbf{X}) \\ dL_{\mathbf{F}}^1 h_j(\mathbf{X}) \\ \vdots \\ dL_{\mathbf{F}}^{n-1} h_j(\mathbf{X}) \end{bmatrix} \text{ for } 1 \leq j \leq m \quad (2.3)$$

or for any combination of n Lie derivatives $L_{\mathbf{F}}^D h_j(\mathbf{X})$ that form a square matrix of dimension n i.e. of full rank [45]. The total number of entries is equals to $n * m$. The elements of these matrices, $L_{\mathbf{F}}^D h_j(\mathbf{X})$, are the D^{th} repeated Lie derivatives of the j^{th} component of $d\mathbf{H}(\mathbf{X})$ with respect to $\mathbf{F}(\mathbf{X}, \mathbf{u})$. Specifically, the Lie derivative of a scalar $h_j(\mathbf{X})$ with respect to a vector $\mathbf{F}(\mathbf{X}, \mathbf{u})$ is defined by

$$L_{\mathbf{F}}^1 h_j(\mathbf{X}) = \frac{\partial h_j(\mathbf{X})}{\partial \mathbf{X}} \mathbf{F}(\mathbf{X}, \mathbf{u}) \quad (2.4)$$

Similarly the Lie derivative of $dh_j(\mathbf{X})$ with respect to $\mathbf{F}(\mathbf{X}, \mathbf{u})$ is a vector field defined by

$$L_{\mathbf{F}}^1 dh_j(\mathbf{X}) = d(L_{\mathbf{F}}^1 h_j(\mathbf{X})) = \frac{\partial h(\mathbf{X})}{\partial \mathbf{X}} \frac{\partial \mathbf{F}(\mathbf{X}, \mathbf{u})}{\partial \mathbf{X}} + \left[\frac{\partial}{\partial \mathbf{X}} \left(\frac{\partial h_j(\mathbf{X})}{\partial \mathbf{X}} \right)^T \mathbf{F}(\mathbf{X}, \mathbf{u}) \right]^T \quad (2.5)$$

The superscript indicates repeated Lie derivatives, which are defined recursively as follows:

$$\begin{aligned} L_{\mathbf{F}}^0 dh_j(\mathbf{X}) &= \frac{\partial h_j(\mathbf{X})}{\partial \mathbf{X}} \\ L_{\mathbf{F}}^D dh_j(\mathbf{X}) &= L_{\mathbf{F}}^{D-1} dh_j(\mathbf{X}) \frac{\partial \mathbf{F}(\mathbf{X}, \mathbf{u})}{\partial \mathbf{X}} + \left[\frac{\partial}{\partial \mathbf{X}} (L_{\mathbf{F}}^{D-1} dh_j(\mathbf{X}))^T \mathbf{F}(\mathbf{X}, \mathbf{u}) \right]^T \end{aligned} \quad (2.6)$$

It is important to note that in a localisation efforts using *a priori* spatial information, the structure parameters of the observation model may change when the vehicle moves from one area to another. In this study, the switching behaviour of the observation model and its parameters are accounted for in establishing the observability matrix (2.3). Let, $\Gamma \in R^g$ describe a vector of parameters of the observation model, $\mathbf{H}(\mathbf{X})$ in a particular time segment. Note that the parameter vector Γ is constant in a time segment, but changes its values from one time segment to another. In order to adequately represent this change of parameter values at different time segments, the observation model is re-expressed as:

$$\begin{aligned} \mathbf{Z} = \mathbf{H}(\mathbf{X}, \Gamma_i) &= \begin{bmatrix} h_1(\mathbf{X}, \Gamma_i) \\ h_2(\mathbf{X}, \Gamma_i) \\ \vdots \\ h_j(\mathbf{X}, \Gamma_i) \\ \vdots \\ h_m(\mathbf{X}, \Gamma_i) \end{bmatrix} & \mathbf{Z} \in R^m \\ \Gamma_i &= [\Gamma_1 \quad \Gamma_2 \quad \dots \quad \Gamma_g]^T & \Gamma_i \in R^g \end{aligned} \quad (2.7)$$

where Γ_i denotes the parameter vector of the modified observation model $\mathbf{H}(\mathbf{X}, \Gamma_i)$ during the time segment i (see Figure 2.1). Assuming that we know the sequence of $\{\Gamma_1 \ \Gamma_2 \ \dots \ \Gamma_i \ \dots\}$, the Lie derivatives of this system can be derived from,

$$\begin{aligned} L_{\mathbf{F}}^0 dh_j(\mathbf{X}(t_{i-1}), \Gamma_i) &= \frac{\partial h_j(\mathbf{X}(t_{i-1}), \Gamma_i)}{\partial \mathbf{X}(t_0)} \\ L_{\mathbf{F}}^D dh_j(\mathbf{X}(t_{i-1}), \Gamma_i) &= L_{\mathbf{F}}^{D-1} dh_j(\mathbf{X}(t_{i-1}), \Gamma_i) \frac{\partial \mathbf{F}(\mathbf{X}(t_{i-1}), \mathbf{u})}{\partial \mathbf{X}(t_0)} \\ &+ \left[\frac{\partial}{\partial \mathbf{X}(t_0)} \left(L_{\mathbf{F}}^{D-1} dh_j(\mathbf{X}(t_{i-1}), \Gamma_i) \right)^T \mathbf{F}(\mathbf{X}(t_{i-1}), \mathbf{u}) \right]^T \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} \mathbf{X}(t_1) &= \mathbf{X}(t_0) + \mathbf{F}(\mathbf{X}(t_0), \mathbf{u}) \\ \mathbf{X}(t_2) &= \mathbf{X}(t_1) + \mathbf{F}(\mathbf{X}(t_1), \mathbf{u}) \\ &\vdots \\ \mathbf{X}(t_i) &= \mathbf{X}(t_{i-1}) + \mathbf{F}(\mathbf{X}(t_{i-1}), \mathbf{u}) \\ &\vdots \end{aligned} \quad (2.9)$$

$\mathbf{X}(t_i)$ is the state vector at time t_i , which is the start of the time segment $i+1$ (see Figure 2.1). These Lie derivatives can now be used to formulate an observability matrix that corresponds to the switched observation model.

Figure 2.1 A set of sequences in the piecewise constant system

Now consider the observation model (2.7). Here, the system observability matrix (2.3) is re-expressed as,

$$O_j = \begin{bmatrix} dL_{\mathbf{F}}^0 h_j(\mathbf{X}, \Gamma_i) \\ dL_{\mathbf{F}}^1 h_j(\mathbf{X}, \Gamma_i) \\ \vdots \\ dL_{\mathbf{F}}^{n-1} h_j(\mathbf{X}, \Gamma_i) \end{bmatrix} \text{ for } 1 \leq j \leq m \quad (2.10)$$

2.3 Bayesian Filtering for Vehicle Localisation Estimation

Bayesian filtering has been a topic of interest in the statistical community for many years now [50]. Using this technique, researchers have attempted to sequentially estimate the state of a dynamic system using a sequence of noisy measurements. This effort is of particular significance in the quest to create reliable autonomous ground vehicles because the pose or state of a vehicle often has to be estimated in the presence of large doses of measurement noise. Bayes filter techniques provide a powerful tool to help manage measurement uncertainty (and noise) and combine informational inputs from different types of sensors (multi-sensor fusion). Their statistical nature makes Bayes filters applicable to arbitrary sensor types. In the following section, we explore such Bayes filters and the variants that are commonly applied to localisation problems.

2.3.1 Bayes Filter

Estimation techniques for vehicle localisation are usually determined probabilistically. The vehicle state $\mathbf{X}(t)$ at time t evolves according to the following discrete-time model:

$$\mathbf{X}(t) = \Phi(\mathbf{X}(t-1), \mathbf{u}(t); \mathbf{w}(t-1), \gamma(t-1)) \quad (2.11)$$

where Φ is a non-linear discrete-time function of the state $\mathbf{X}(t-1)$, $\mathbf{w}(t-1)$ is referred to as a process noise sequence and $\gamma(t-1)$ is input noise sequence. The process noise element of the equation caters for the effects of less than precise modelling and unforeseen disturbances in the vehicle's motion. The input noise caters for the effects of disturbances in the input $\mathbf{u}(t)$. The measurement $\mathbf{Z}(t)$ is related to the vehicle state via the observation equation:

$$\mathbf{Z}(t) = \mathbf{H}(\mathbf{X}(t); \mathbf{v}(t)) \quad (2.12)$$

where $\mathbf{H}$ is a non-linear function and $\mathbf{v}(t)$ is the measurement noise sequence. In particular, we seek filtered estimates of $\mathbf{X}(t)$ based on the set of all available measurements $\mathbf{Z}_t \triangleq \{\mathbf{Z}(1), \mathbf{Z}(2), \dots, \mathbf{Z}(t)\}$ and inputs $\mathbf{U}_t \triangleq \{\mathbf{u}(1), \mathbf{u}(2), \dots, \mathbf{u}(t)\}$ up to time t .

From a Bayesian aspect, the localisation problem is to recursively calculate degree of belief in the state $\mathbf{X}(t)$ at time t , given the data $\mathbf{Z}_t$ and $\mathbf{U}_t$ up to time t . Thus, it is required to determine the probability distribution function, as

$$p(\mathbf{X}(t) | \mathbf{U}_t, \mathbf{Z}_t) \quad (2.13)$$

Assuming the initial probability distribution function or prior $p(\mathbf{X}(0) | \mathbf{U}_0, \mathbf{Z}_0)$ is available ($\mathbf{Z}_0$ and $\mathbf{U}_0$ being the set of no measurements), the probability distribution function in principle can be obtained recursively in two stages: prediction and update.

With the required probability distribution function $p(\mathbf{X}(t-1) | \mathbf{U}_{t-1}, \mathbf{Z}_{t-1})$ at time $t-1$, the prediction stage involves using the vehicle model (2.11) and input $\mathbf{u}(t)$ to obtain the

prior probability distribution function of the state $\mathbf{X}(t)$ at time t via the Chapman-Kolmogorov equation

$$p(\mathbf{X}(t) | \mathbf{U}_t, \mathbf{Z}_{t-1}) = \underbrace{\int p(\mathbf{X}(t) | \mathbf{X}(t-1), \mathbf{u}(t)) p(\mathbf{X}(t-1) | \mathbf{U}_{t-1}, \mathbf{Z}_{t-1}) d\mathbf{X}(t-1)}_{\text{Prediction Stage}} \quad (2.14)$$

Note that equation (2.11) describes a Markov process of order one; thus $p(\mathbf{X}(t) | \mathbf{X}(t-1), \mathbf{U}_t) = p(\mathbf{X}(t) | \mathbf{X}(t-1), \mathbf{u}(t))$ has been derived in equation (2.14). The probabilistic model of the state evolution $p(\mathbf{X}(t) | \mathbf{X}(t-1), \mathbf{u}(t))$ is defined by the vehicle model (2.11) as well as by the known statistics of $\mathbf{w}(t-1)$ and $\gamma(t-1)$.

At time t , a measurement $\mathbf{Z}(t)$ becomes available and this may be used to update the prior $p(\mathbf{X}(t) | \mathbf{U}_t, \mathbf{Z}_{t-1})$ via Bayes' rule (update stage)

$$p(\mathbf{X}(t) | \mathbf{U}_t, \mathbf{Z}_t) = \underbrace{\eta p(\mathbf{Z}(t) | \mathbf{X}(t))}_{\text{Update Stage}} p(\mathbf{X}(t) | \mathbf{U}_t, \mathbf{Z}_{t-1}) \quad (2.15)$$

where the normalising constant η

$$\eta = \frac{1}{p(\mathbf{Z}(t) | \mathbf{U}_t, \mathbf{Z}_{t-1})} = \frac{1}{\int p(\mathbf{Z}(t) | \mathbf{X}(t)) p(\mathbf{X}(t) | \mathbf{U}_t, \mathbf{Z}_{t-1}) d\mathbf{X}(t)} \quad (2.16)$$

depends on the likelihood function $p(\mathbf{Z}(t) | \mathbf{X}(t))$ defined by the observation model (2.12) and known statistics of $\mathbf{v}(t)$. In the update stage, where equation (2.15) is used, measurement $\mathbf{Z}(t)$ is used to alter the prior probability distribution to obtain the required posterior probability distribution of the current state $\mathbf{X}(t)$.

The recursive propagation of the posterior probability distribution function given by (2.15) is only a conceptual solution; that is, in general, it cannot be determined analytically. Since the analytic solution of equation (2.15) is intractable in most practical

situations, one has to use approximations or suboptimal Bayesian algorithms instead. In the following section, conventional and more advanced approximations from the literature are described.

2.3.2 Linear and Non-Linear Approximation

Kalman Filter:

In the process of obtaining the optimal algorithm for recursive Bayesian state estimation, the functional recursion of equation (2.15) becomes the Kalman Filter – but only if the system is linear and the probability distribution function is Gaussian. The Kalman Filter has become the most widely used variant of recursive estimation methods [52]. It assumes that the process and observation models are linear. Therefore, equations (2.11) and (2.12) can be rewritten as

$$\begin{aligned}\mathbf{X}(t) &= \mathbf{A}\mathbf{X}(t-1) + \mathbf{B}\mathbf{u}(t) + \mathbf{B}\boldsymbol{\gamma}(t-1) + \mathbf{w}(t-1) \\ \mathbf{Z}(t) &= \mathbf{H}\mathbf{X}(t) + \mathbf{v}(t)\end{aligned}\tag{2.17}$$

where the random variables $\mathbf{w}(t-1)$ represent the process noise, $\boldsymbol{\gamma}(t-1)$ represent the input noise and $\mathbf{v}(t)$ represents the measurement noise which is assumed to be independent and to have a Gaussian probability distribution

$$\begin{aligned}p(\mathbf{w}) &\sim \mathcal{N}(0, \mathbf{W}) \\ p(\boldsymbol{\gamma}) &\sim \mathcal{N}(0, \boldsymbol{\Upsilon}) \\ p(\mathbf{v}) &\sim \mathcal{N}(0, \boldsymbol{\mathbf{V}})\end{aligned}\tag{2.18}$$

$\mathbf{W}$ is the process noise covariance matrix, $\boldsymbol{\Upsilon}$ is the input noise covariance matrix and $\boldsymbol{\mathbf{V}}$ is the measurement noise covariance matrix.

The Kalman filter algorithm, which was derived using equation (2.14) and (2.15), can then be understood in the following manner:

$$\begin{aligned} p(\mathbf{X}(t-1) | \mathbf{U}_{t-1}, \mathbf{Z}_{t-1}) &= \mathcal{N}(\mathbf{X}(t-1); \hat{\mathbf{X}}(t-1), \Omega_{t-1}) \\ p(\mathbf{X}(t) | \mathbf{U}_t, \mathbf{Z}_{t-1}) &= \mathcal{N}(\mathbf{X}(t); \hat{\mathbf{X}}(t)^-, \Omega_t^-) \\ p(\mathbf{X}(t) | \mathbf{U}_t, \mathbf{Z}_t) &= \mathcal{N}(\mathbf{X}(t); \hat{\mathbf{X}}(t), \Omega_t) \end{aligned} \quad (2.19)$$

where

$$\begin{aligned} \hat{\mathbf{X}}(t)^- &= A\hat{\mathbf{X}}(t-1) + B\mathbf{u}(t) \\ \Omega_t^- &= A\Omega_{t-1}A^T + B\mathbf{\Upsilon}B^T + \mathbf{w} \\ \hat{\mathbf{X}}(t) &= \hat{\mathbf{X}}(t)^- + \mathbf{K}_t(\mathbf{Z}(t) - \mathbf{H}\hat{\mathbf{X}}(t)^-) \\ \Omega_t &= (I - \mathbf{K}_t\mathbf{H})\Omega_t^- \end{aligned} \quad (2.20)$$

and where $\mathcal{N}(\mathbf{X}; \hat{\mathbf{X}}, \Omega)$ is a Gaussian distribution with argument $\mathbf{X}$, mean $\hat{\mathbf{X}}$ and covariance Ω and

$$\mathbf{K}_t = \Omega_t^- \mathbf{H}^T (\mathbf{H}\Omega_t^- \mathbf{H}^T + \mathbf{v})^{-1} \quad (2.21)$$

is the Kalman gain (the transpose of a matrix $\mathbf{H}$ is denoted by $\mathbf{H}^T$). Through recursive iterations over time using the “predict” and “update” steps, the mean $\hat{\mathbf{X}}$ and covariance Ω of the probability distribution will be the optimal solution of the vehicle pose.

Extended Kalman Filter:

Several extensions and variants have been proposed for the Kalman Filter due to the non-linearity of process and observation models. One extension that has been put forward to solve such non-linear estimation problems involves the approximation of the system model using Talyor series expansions up to the second order term. This method is known as the Extended Kalman Filter (EKF), and has been successfully applied to estimation problems involving non-linear systems [53], including the Concurrent

Mapping and Localisation (CML) problem [54]. Using the technique of augmenting unknown parameters into the state vector, the EKF algorithm allows the vehicle's pose to be estimated along with the position of other unknown environmental features.

Unscented Kalman Filter:

One of the main drawbacks of the EKF is that complex Jacobian matrices are derived and these tend to cause implementation difficulties. If the time step intervals are not sufficiently small, the linearisation can lead to inconsistencies in the filter. In order to address these limitations, Julier and Uhlmann developed the unscented Kalman filter (UKF) [55]. Instead of linearising using Jacobian matrices, the ideal arrangement would be to use a set of deterministically chosen sample points to capture the mean and covariance of the estimates. The UKF has been shown to be a superior alternative to the EKF in a variety of applications including parameter estimation for time series modelling [56] and visual contour tracking [57].

Sequential Monte Carlo Method:

Both EKF and UKF rely on the use of approximations to ensure mathematical tractability and computational efficiency. These are the main advantages of these filters but they come at the cost of restricted representational power. Both the EKF and UKF can only represent unimodal distributions and that too only when the uncertainty and nonlinearity in the system process model is not very high. Recently, simulation-based techniques known as Sequential Monte Carlo (SMC) have been used to solve the problem of non-unimodal distributions estimation [58]. Such techniques have proven to be successful and their use has spread widely. The SMC method has several names and has also been referred to in other studies as the bootstrap filtering method [59], the

condensation algorithm method [60], particle filtering method [61], interacting particle approximation method [62] and the survival of the fittest method [63].

The SMC technique implements recursive Bayesian filters using Monte Carlo simulation methods (a stochastic approach). A key characteristic of this approach is that the required probability distribution function (2.13) is represented by a set of random samples with associated weights. Estimates are computed based on these samples and weights. As the number of samples becomes very large, this Monte Carlo characterisation becomes an equivalent representation to the usual functional description of the posterior probability distribution [58].

Sequential Importance Sampling (SIS) is the basic principle or framework used by most Sequential Monte Carlo filters developed in recent decades [64]. This algorithm can be described in detail as follows: let $\{\mathbf{X}(t)^{\{i\}}, \omega_t^{\{i\}}\}_{i=1}^{N_s}$ be a random measure that characterises the posterior probability distribution function $p(\mathbf{X}(t)|\mathbf{U}_t, \mathbf{Z}_t)$, where $\{\mathbf{X}(t)^{\{i\}}, i=1, \dots, N_s\}$ is a set of support points (samples or particles) with associated weights $\{\omega_t^{\{i\}}, i=1, \dots, N_s\}$. The weights are normalised such that $\sum_{i=1}^{N_s} \omega_t^{\{i\}} = 1$. The probability distribution at t can then be approximated as

$$p(\mathbf{X}(t)|\mathbf{U}_t, \mathbf{Z}_t) \approx \sum_{i=1}^{N_s} \omega_t^{\{i\}} \delta(\mathbf{X}(t) - \mathbf{X}(t)^{\{i\}}) \quad (2.22)$$

This is a discrete, weighted approximation to the true posterior, $p(\mathbf{X}(t)|\mathbf{U}_t, \mathbf{Z}_t)$.

$p(\mathbf{X}(t)|\mathbf{U}_t, \mathbf{Z}_t)$ is typically difficult to draw samples from but can, however, be evaluated.

Thus, the samples $\mathbf{X}(t)^{\{i\}}$ are normally drawn from a *proposal distribution* $q(\mathbf{X}(t)|\mathbf{U}_t, \mathbf{Z}_t)$,

which is easier to generate. This is also termed an *importance density*. Based on the *importance sampling* principle [64], the weights $\omega_t^{\{i\}}$ in equation (2.22) can be defined as

$$\begin{aligned}\omega_t^{\{i\}} &\propto \frac{p(\mathbf{X}(t)^{\{i\}} | \mathbf{U}_t, \mathbf{Z}_t)}{q(\mathbf{X}(t)^{\{i\}} | \mathbf{U}_t, \mathbf{Z}_t)} \\ &= \omega_{t-1}^{\{i\}} \frac{p(\mathbf{Z}(t) | \mathbf{X}(t)^{\{i\}}) p(\mathbf{X}(t)^{\{i\}} | \mathbf{X}(t-1)^{\{i\}}, \mathbf{u}(t))}{q(\mathbf{X}(t)^{\{i\}} | \mathbf{X}(t-1)^{\{i\}}, \mathbf{u}(t), \mathbf{Z}(t))}\end{aligned}\quad (2.23)$$

A pseudo-code description of the SIS algorithm is given by [65] where the recursive propagation of weights and support points is sequentially presented as each measurement is received.

A common problem presented by the SIS is that of degeneracy. This phenomenon appears after a few iterations, when all but one particle has negligible weight. This degeneracy implies that a great degree of computational effort is devoted to updating particles whose contribution to the approximation to $p(\mathbf{X}(t) | \mathbf{U}_t, \mathbf{Z}_t)$ is almost zero. An estimate of this degeneracy can be obtained by

$$\hat{N}_{eff} = \frac{1}{\sum_{i=1}^{N_s} (\omega_t^{\{i\}})^2} \quad (2.24)$$

where $\omega_t^{\{i\}}$ is the weight obtained using equation (2.23). It is posited that if the effective particle size $\hat{N}_{eff}$ is significantly smaller than the particle size N_s , $\hat{N}_{eff} \ll N_s$. The small N_{eff} indicates severe degeneracy. To reduce the degeneracy problem, the SIS algorithm relies on two methods: the choice of Proposal distribution and the use of re-sampling techniques [65].

A number of variants of the SIS algorithm have been proposed that include different proposal distribution choices and re-sampling techniques. One of the most significant SIS variants is the Sampling Importance Re-sampling (SIR) filter proposed by Gordon [59]. The SIR filter is derived from the SIS algorithm by applying re-sampling step and by an appropriate choice of the *proposal distribution* where $q(\mathbf{X}(t)^{\{i\}} | \mathbf{X}(t-1)^{\{i\}}, \mathbf{u}(t), \mathbf{Z}(t))$ is chosen to be the probability distribution $p(\mathbf{X}(t)^{\{i\}} | \mathbf{X}(t-1)^{\{i\}}, \mathbf{u}(t))$. The choice of the *proposal distribution* implies that a sample $\mathbf{X}(t)^{\{i\}} \sim p(\mathbf{X}(t)^{\{i\}} | \mathbf{X}(t-1)^{\{i\}}, \mathbf{u}(t))$ is generated by first creating a process noise sample $\mathbf{w}_{t-1}^{\{i\}} \sim p_{\mathbf{w}}(\mathbf{w}_{t-1})$ and an input noise sample $\gamma_{t-1}^{\{i\}} \sim p_{\gamma}(\gamma_{t-1})$ and setting $\mathbf{X}(t)^{\{i\}} = \Phi(\mathbf{X}(t-1)^{\{i\}}, \mathbf{u}(t); \mathbf{w}_{t-1}^{\{i\}}, \gamma_{t-1}^{\{i\}})$ (vehicle model), where $p_{\mathbf{w}}(\cdot)$ and $p_{\gamma}(\cdot)$ is the normal distribution of $\mathbf{w}_{t-1}^{\{i\}}$ and $\gamma_{t-1}^{\{i\}}$ respectively. From the proposal distribution choice, it follows that $q(\mathbf{X}(t)^{\{i\}} | \mathbf{X}(t-1)^{\{i\}}, \mathbf{u}(t), \mathbf{Z}(t))$ equals to $p(\mathbf{X}(t)^{\{i\}} | \mathbf{X}(t-1)^{\{i\}}, \mathbf{u}(t))$. The weights from (2.23) are then redefined as

$$\omega_t^{\{i\}} \propto \omega_{t-1}^{\{i\}} p(\mathbf{Z}(t) | \mathbf{X}(t)^{\{i\}}) \quad (2.25)$$

The weights given by the proportionality in equation (2.25) are normalised before the re-sampling stage. Re-sampling is carried out to reduce degeneracy [65]. The basic idea of re-sampling is to eliminate samples that have small weights and to concentrate on samples with larger weights. The re-sampling step involves generating a new set $\{\mathbf{X}(t)^{\{i\}}\}_{i=1}^{N_s}$ by re-sampling (with replacement) N_s times from an approximate discrete representation of $p(\mathbf{X}(t) | \mathbf{u}_t, \mathbf{Z}_t)$. The resulting sample is a set of samples with equal

weights from the discrete distribution $p(\mathbf{X}(t)|\mathbf{U}_t, \mathbf{Z}_t)$. Several re-sampling techniques have been developed as discussed in [66]. In this thesis, Systematic re-sampling [67] is adopted for its computational efficiency and effectiveness in minimising the sample variation. The pseudo code of the systemic re-sampling algorithm is described in [66].

To further reduce the computation expense of re-sampling, a threshold N_{thres} is commonly applied. If the condition $\hat{N}_{eff} > N_{thres}$ is met, re-sampling will not be done. After re-sampling, the weights will be reset to $\frac{1}{N_s}$ and if re-sampling is avoided, the weights will remain the same. At the end of the sequence, the weights and samples will be reused with the time update in the next iteration.

This filter can demonstrate that without re-sampling, the importance weight variance increases over time, giving rise to degeneracy. However, the re-sampling step should only be done when the effective particle size is small so as to avoid unnecessary computations. Details of the SIR algorithm are described in [65].

A successful example of the use of Sequential Monte Carlo filters is in map-aided localisation [28]. This application addresses the problem of estimating a vehicle's pose relative to a road map. The key advantage of SMC filters here is their ability to represent arbitrary probability distributions. This makes them a viable alternative when faced with problems to which the Kalman Filters are not well suited. Compared to other multi-modal approaches, particle filters are very efficient since they automatically focus their resources (particles) on the regions in the state space that have high probability. However, because the worst-case complexity of these methods grows exponentially with

the dimension of state space, special care is needed when applying them to high-dimensional estimation problems.

Rao-Blackwellisation Particle Filter:

Rao-Blackwellised Particle Filters [68] combine a particle filter with multiple Kalman filters. Recently, these have been used successfully to track the locations and identities of multiple objects [69]. Thrun *et. al.* have also applied these filters to solve the CML problem in an approach known as the FastSLAM [70]. The CML estimation problem is described as a problem of jointly estimating a time-variant vehicle pose $\mathbf{X}(t)$ and time-invariant position of K unknown features denoted by $\mathbf{M} \triangleq \{\mathbf{m}_1, \mathbf{m}_2 \dots \mathbf{m}_k \dots \mathbf{m}_K\}$:

$$p(\mathbf{X}(t), \mathbf{M} | \mathbf{Z}_t, \mathbf{U}_t) = \eta p(\mathbf{Z}(t) | \mathbf{X}(t), \mathbf{M}) p(\mathbf{X}(t) | \mathbf{Z}_{t-1}, \mathbf{U}_t) \quad (2.26)$$

Note that the Chapman-Kolmogorov equation $p(\mathbf{X}(t) | \mathbf{Z}_{t-1}, \mathbf{U}_t)$ (2.26) involves only the vehicle pose and not the features. By exploiting the independence property found in the structure of the CML problem [70], the knowledge of the path of the vehicle renders the individual's unknown feature positions conditionally independent. This important conditional independence property leads to the formulation of a more efficient translation of equation (2.26) – one that estimates the posterior probability distribution over vehicle paths $\mathbf{X}_t$ along with the unknown feature positions $\mathbf{m}_k$:

$$p(\mathbf{X}_t, \mathbf{M} | \mathbf{Z}_t, \mathbf{U}_t) = \underbrace{p(\mathbf{X}_t | \mathbf{Z}_t, \mathbf{U}_t)}_{\text{Vehicle path estimator}} \underbrace{\prod_k p(\mathbf{m}_k | \mathbf{X}_t, \mathbf{Z}_t, \mathbf{U}_t)}_{\text{Feature estimators}} \quad (2.27)$$

where $\mathbf{X}_t \triangleq \{\mathbf{X}(1), \mathbf{X}(2), \dots, \mathbf{X}(t)\}$ represents the vehicle path up to time t . The posterior probability distribution $p(\mathbf{X}_t | \mathbf{Z}_t, \mathbf{U}_t)$ in equation (2.27) is implemented by a particle filter, with each particle being represented as a possible path. The resulting feature estimators

$p(\mathbf{m}_k | \mathbf{X}_t, \mathbf{Z}_t, \mathbf{U}_t)$ are conditioned on individual particles representing vehicle path posterior probability distribution $p(\mathbf{X}_t | \mathbf{Z}_t, \mathbf{U}_t)$. However, since feature posterior probability distributions are conditionally independent given the vehicle path, the joint posterior probability distribution $p(\mathbf{X}_t, \mathbf{M} | \mathbf{Z}_t, \mathbf{U}_t)$ over the features can be decomposed into separate estimators for each feature $p(\mathbf{m}_k | \mathbf{X}_t, \mathbf{Z}_t, \mathbf{U}_t)$, $k=1, \dots, K$. In the FastSLAM algorithm [70], these feature posterior probability distributions are implemented by EKFs, one for each feature. Thus, the filter is a combination of a particle filter and Kalman filters.

Each particle $\mathbf{X}_t^{[i]}$ will contain a set of feature estimations $\mathbf{M} = \{\mathbf{m}_1, \mathbf{m}_2 \dots \mathbf{m}_k \dots \mathbf{m}_K\}$. These estimations reduce the computational requirements of maintaining the correlation matrix, as in the case of the EKF. Intensive computation is required from the EKF system in order to arrive at the full correlation matrix of the optimum estimates. However, a sufficient number of samples are required for the conditional independent properties to be valid in Rao-Blackwellisation particle filters. The vehicle path posterior probability distribution may then require a large sample size, which will again render the filter computationally intensive. An adaptive scheme can be devised to postulate the true representation of the evolving distribution, $p(\mathbf{X}_t | \mathbf{Z}_t, \mathbf{U}_t)$ while avoiding the use of large sample sizes that result in an increase in computational intensity.

Fox *et. al.* [71] have recently devised a statistical approach that increases the efficiency of the particle filter by adapting the sample size during the estimation process. This adaptive scheme is known as the KLD (Kullback-Leibler Distance) Sampling method. The KLD serves to estimate, maintain or bound the approximation error introduced by the sample-based representation of the estimates' uncertainty. A small

number of samples will be chosen if the density is focused on a small part of the state space and a large sample will be used if the state is highly uncertain. Through extensive experiments, this approach has been shown to drastically improve particle filters with fixed sample set sizes.

The adaptation approach becomes crucial in the implementation of the Rao-Blackwellised particle filter because it maintains the validity of the conditional independence properties while preventing the filter from being overly confident neglecting potential vehicle trajectory and corresponding feature maps. The Rao-Blackwellised Particle Filter also allows a good degree of computational efficiency and was chosen for the purposes of this thesis because it is capable of achieving the optimality solution when implemented in real time with KLD-sampling. More specifically the Rao-Blackwellised Particle Filter is expected to aid the implementation of the proposed Path Constrained CML framework, and solve the large-scale localisation problem. Extensive field experiments will be conducted to demonstrate the drastic improvements this method promises to facilitate over other prominent estimation techniques that have been adopted to solve the localisation problem.

2.4 Conclusions

Observability analysis is a way to show analytically, whether the designed state system guarantees a lower limit on estimation errors regardless of model uncertainties and measurement noise. Observability is an important concern in any state estimation problem and this chapter introduced an appropriate non-linear framework for its analysis. The non-linear analysis framework employing Lie derivatives accommodates the effects

of inputs and the significant coupling and non-linear effects of the system. In this regard, the analysis is very different from the approaches taken for linear systems under which the effects tend to be ignored.

Over the past few decades, researches have established many estimation algorithms. This thesis surveys the probabilistic approaches that are most often used in localisation problems. The basic mathematical formulation of the Bayes filter was presented, together with two variants – the Kalman filter (KF) and the Sequential Monte Carlo (SMC) method. The Kalman filter is used mainly for linear and Gaussian systems where the estimates are represented by the means and the uncertainties are modelled through the covariance matrices. Its extension, the Extended Kalman filter (EKF), approximates non-linear systems by using Jacobian matrices. In contrast, the SMC method uses a set of random points (samples or particles) to represent the probability distribution. The advantage of this is that it allows the SMC to model non-Gaussian uncertainties in the system. Unfortunately, for high dimensional systems, the number of particles required to maintain the probability distribution can make the computational demands of the SMC overly intense.

Recently, the Rao-Blackwellisation Particle filter was introduced to solve the problem of high-dimensionality systems. Using conditionally independent properties, the filter is a combination of an SMC and several EKFs. The Rao-Blackwellisation Particle filter has been applied successfully to many tracking problems, as well as to CML problems. However, for the conditional independent properties to be valid and for them to truly represent the actual posterior probability distribution, again a sufficiently large number of samples is needed in the SMC. To avoid the computational intensity that arises

with large sample sizes, this chapter reviewed an adaptive scheme that has been found to efficiently carry out the computations required.

In summary, this chapter has surveyed the theoretical framework for analysing the observability properties of non-linear systems and reviewed the probabilistic approaches to solving the localisation problem. The next chapter will analyse the observability properties of several localisation approaches, such as feature-based localisation, path-constrained localisation and CML. Prior to this, these approaches have either been neglected or investigated with inappropriate tools. The chapter ends with a verification of the observability results obtained from simulations carried out based on the EKF algorithms.

Chapter 3

Observability Analysis of Vehicle Localisation

3.1 Introduction

Although, many advances have been made in ground vehicle localisation systems, their fundamental limitations have not been rigorously addressed. No systematic studies or theoretical analyses have been carried out to fully appreciate the limitations of such techniques or to understand the properties of the estimation errors that result from their use. In particular, issues of observability have not been rigorously investigated or analysed. In this chapter, detailed analysis of the observability of related vehicle localisation problem are analysed using an appropriate non-linear observability analysis framework. The framework was presented in Chapter 2 and theoretical analysis was demonstrated to be a very effective [45] method of studying system observability.

The current chapter is organised as follows: Section 3.2 analysed the observability of an *a priori* known Feature-based localisation approach. Simulations are carried out and the data obtained is presented in order to verify the theoretical observability results established. Section 3.3 introduces the Path Constrained localisation problem and describes how *a priori* spatial information available from the road maps can be used in vehicle localisation. Observability properties of this Path Constrained localisation system are studied and verified through simulations. In Section 3.4, the observability analysis for a typical CML system is presented and the observability results are verified through

simplified simulations. The chapter is summarised and concluding remarks are made in Section 3.5.

3.2 Observability of Feature-based Localisation

For a vehicle moving on a flat horizontal plane, the system state, $\mathbf{x}_r$, can be represented as:

$$\mathbf{x}_r = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix} \quad \mathbf{x}_r \in R^3 \quad (3.1)$$

where, (x, y) and ψ describes the vehicle's position and orientation with respect to a global reference frame as shown in Figure 3.1.

Process Model:

The process model describes the evolution of state vector $\mathbf{x}_r$ over time. The vehicle is assumed to be four-wheeled and steered by the front wheels. Figure 3.1 provides a plan of the vehicle showing relevant kinematics parameters and coordinated frames. The kinematics of the vehicle corresponds to that of a simple bicycle model [72]:

$$\frac{d\mathbf{x}_r}{dt} = \frac{d}{dt} \begin{bmatrix} x \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} \cos \psi \\ \sin \psi \\ 0 \end{bmatrix} v + \begin{bmatrix} -a \sin \psi \\ a \cos \psi \\ 1 \end{bmatrix} v \delta \quad (3.2)$$

where, a is the distance between the rear axle and exteroceptive sensor (LADAR) as shown in Figure 3.1, $\delta = \frac{\tan(\alpha)}{L}$ is the effective steering input (α is the steering angle, and L is the wheel base) and v is the vehicle velocity. The input to the kinematics model is represented as $u = [\delta \ v]^T$. The kinematics model of the vehicle is designed to represent

the trajectory of the centre of the exteroceptive sensor (LADAR) mounted in front of the vehicle as shown in Figure 3.1. It is apparent that the process model (equation 3.2) is highly non-linear and coupled.

Figure 3.1 Coordinates frame and kinematics model of the vehicle

It may be noted that the process model or kinematics model (3.2) is not in the input affine non-linear function form (equation 3.2). However, equation (3.2) can be manipulated to fit the required form as shown in Chapter 2 (see equation 2.2) through a time-scale transformation technique [73]. To do this, a new time-scale, ρ , is inserted that denotes the distance along the vehicle path. Note that that $\frac{d\rho}{dt}$ is the speed of the vehicle v . Multiplying equation (3.2) by v^{-1} , the kinematics model can be transformed to the required input affine form:

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{f}_0(\mathbf{x}_r) + \mathbf{f}_1(\mathbf{x}_r)u_1 \\ \frac{d}{d\rho} \begin{bmatrix} x \\ y \\ \psi \end{bmatrix} &= \begin{bmatrix} \cos \psi \\ \sin \psi \\ 0 \end{bmatrix} + \begin{bmatrix} -a \sin \psi \\ a \cos \psi \\ 1 \end{bmatrix} \delta \end{aligned} \quad (3.3)$$

Equation (3.3) also represents a non-stationary model subject to a single steer input, $\delta (= u_1)$. This time-scale technique can be applied to any model (i.e 6 degree-of-freedom model) that is structurally similar to the process model given in (3.2).

Observation model:

In an *a priori* known feature-based localisation system, an exteroceptive sensor mounted on the vehicle is used to take range and bearing observations of the known environmental features. The observation model with one known feature point can be represent as,

$$\begin{aligned} \mathbf{Z} &= \begin{bmatrix} \hat{\rho}_i \\ \hat{\phi}_i \end{bmatrix} = \mathbf{H}(\mathbf{x}_r, \mathbf{\Gamma}_i) & \mathbf{Z} \in R^2, \\ &= \begin{bmatrix} h_1(\mathbf{x}_r, \mathbf{\Gamma}_i) \\ h_2(\mathbf{x}_r, \mathbf{\Gamma}_i) \end{bmatrix} = \begin{bmatrix} \sqrt{(\xi_i - x)^2 + (\zeta_i - y)^2} \\ \tan^{-1} \left(\frac{(\zeta_i - y)}{(\xi_i - x)} \right) - \psi + \frac{\pi}{2} \end{bmatrix} & (3.4) \\ \mathbf{\Gamma}_i &= [\Gamma_1 \quad \Gamma_2]^T = [\xi_i \quad \zeta_i]^T & \mathbf{\Gamma}_i \in R^2 \end{aligned}$$

where $\mathbf{Z} = [\hat{\rho}_i \quad \hat{\phi}_i]^T$ is the expected sensor measurement of range ρ_i and bearing ϕ_i of the i^{th} known feature whose position vector is $\mathbf{\Gamma}_i = [\xi_i \quad \zeta_i]^T$.

Analysis:

By applying the localisation system process model (3.3) and the observation model (3.4) with constant parameters, input δ and one known feature $\mathbf{\Gamma}_1 = [\xi_1 \quad \zeta_1]^T$, the

repeated Lie derivatives (the elements of the observability matrix) are obtained as follows,

$$\begin{aligned} L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r, \Gamma_1) &= \frac{\partial h(\mathbf{x}_r, \Gamma_1)}{\partial \mathbf{x}_r} = \begin{bmatrix} \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(\xi_1 - x)}{\Delta} & -\frac{(\zeta_1 - y)}{\Delta} & 0 \end{bmatrix} \end{aligned} \quad (3.5)$$

$$\begin{aligned} L_{\mathbf{F}}^D dh_1(\mathbf{x}_r, \Gamma_1) &= L_{\mathbf{F}}^{D-1} dh_1(\mathbf{x}_r, \Gamma_1) \frac{\partial \mathbf{F}(\mathbf{x}_r, \mathbf{u})}{\partial \mathbf{x}_r} + \left[\frac{\partial}{\partial \mathbf{x}_r} (L_{\mathbf{F}}^{D-1} dh_1(\mathbf{x}_r, \Gamma_1))^T \mathbf{F}(\mathbf{x}_r, \mathbf{u}) \right]^T \\ &= \begin{bmatrix} \mathbf{A}_1^{D1} & \mathbf{A}_1^{D2} & \mathbf{A}_1^{D3} \end{bmatrix} \\ D &= 1, 2 \end{aligned} \quad (3.6)$$

$$\begin{aligned} L_{\mathbf{F}}^0 dh_2(\mathbf{x}_r, \Gamma_1) &= \frac{\partial h(\mathbf{x}_r, \Gamma_1)}{\partial \mathbf{x}_r} = \begin{bmatrix} \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} \end{bmatrix} \\ &= \begin{bmatrix} \frac{(\zeta_1 - y)}{\Delta^2} & -\frac{(\xi_1 - x)}{\Delta^2} & -1 \end{bmatrix} \end{aligned} \quad (3.7)$$

$$\begin{aligned} L_{\mathbf{F}}^D dh_2(\mathbf{x}_r, \Gamma_1) &= L_{\mathbf{F}}^{D-1} dh_2(\mathbf{x}_r, \Gamma_1) \frac{\partial \mathbf{F}(\mathbf{x}_r, \mathbf{u})}{\partial \mathbf{x}_r} + \left[\frac{\partial}{\partial \mathbf{x}_r} (L_{\mathbf{F}}^{D-1} dh_2(\mathbf{x}_r, \Gamma_1))^T \mathbf{F}(\mathbf{x}_r, \mathbf{u}) \right]^T \\ &= \begin{bmatrix} \mathbf{B}_1^{D1} & \mathbf{B}_1^{D2} & \mathbf{B}_1^{D3} \end{bmatrix} \\ D &= 1, 2 \end{aligned} \quad (3.8)$$

where

$$\Delta = \sqrt{(\xi_1 - x)^2 + (\zeta_1 - y)^2} \quad (3.9)$$

$\begin{bmatrix} \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} \end{bmatrix}$ are the corresponding first order Lie derivatives of $h_1(\mathbf{x}_r, \Gamma_1)$ and $h_2(\mathbf{x}_r, \Gamma_2)$. The first digit of the superscript indicates the order of the Lie derivatives whilst the second digit indicates the component (i.e. 01 indicates the zero order Lie derivative with respect to the first component of the state). The subscript

denotes the index of the parameter value used, Γ_i (i.e. $i=1$). By combining the above elements, possible observability matrices, $O_{1,2}$, can be constructed as follows,

$$O_{1,2} = \begin{bmatrix} L_{\mathbf{r}}^0 dh_1(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{r}}^0 dh_2(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{r}}^1 dh_1(\mathbf{x}_r, \Gamma_1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} \\ \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} \\ \mathbf{A}_1^{11} & \mathbf{A}_1^{12} & \mathbf{A}_1^{13} \end{bmatrix} \quad (3.10)$$

$$O_{1,2} = \begin{bmatrix} L_{\mathbf{r}}^0 dh_1(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{r}}^0 dh_2(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{r}}^1 dh_2(\mathbf{x}_r, \Gamma_1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} \\ \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} \\ \mathbf{B}_1^{11} & \mathbf{B}_1^{12} & \mathbf{B}_1^{13} \end{bmatrix} \quad (3.11)$$

$$O_{1,2} = \begin{bmatrix} L_{\mathbf{r}}^0 dh_1(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{r}}^0 dh_2(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{r}}^2 dh_2(\mathbf{x}_r, \Gamma_1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} \\ \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} \\ \mathbf{A}_1^{21} & \mathbf{A}_1^{22} & \mathbf{A}_1^{23} \end{bmatrix} \quad (3.12)$$

$$O_{1,2} = \begin{bmatrix} L_{\mathbf{r}}^0 dh_1(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{r}}^0 dh_2(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{r}}^2 dh_1(\mathbf{x}_r, \Gamma_1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} \\ \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} \\ \mathbf{B}_1^{21} & \mathbf{B}_1^{22} & \mathbf{B}_1^{23} \end{bmatrix} \quad (3.13)$$

where the first order Lie derivatives can be expressed as,

$$\begin{bmatrix} \mathbf{A}_1^{11} \\ \mathbf{A}_1^{12} \\ \mathbf{A}_1^{13} \end{bmatrix}^T = \Theta_1 \begin{bmatrix} \mathbf{B}_1^{01} \\ \mathbf{B}_1^{02} \\ \mathbf{B}_1^{03} \end{bmatrix}^T \quad (3.14)$$

$$\begin{bmatrix} \mathbf{A}_1^{21} \\ \mathbf{A}_1^{22} \\ \mathbf{A}_1^{23} \end{bmatrix}^T = \Theta_2 \begin{bmatrix} \mathbf{B}_1^{01} \\ \mathbf{B}_1^{02} \\ \mathbf{B}_1^{03} \end{bmatrix}^T + \Theta_3 \begin{bmatrix} \mathbf{A}_1^{01} \\ \mathbf{A}_1^{02} \\ \mathbf{A}_1^{03} \end{bmatrix}^T \quad (3.15)$$

$$\begin{bmatrix} \mathbf{B}_1^{11} \\ \mathbf{B}_1^{12} \\ \mathbf{B}_1^{13} \end{bmatrix}^T = \Theta_4 \begin{bmatrix} \mathbf{B}_1^{01} \\ \mathbf{B}_1^{02} \\ \mathbf{B}_1^{03} \end{bmatrix}^T + \Theta_5 \begin{bmatrix} \mathbf{A}_1^{01} \\ \mathbf{A}_1^{02} \\ \mathbf{A}_1^{03} \end{bmatrix}^T \quad (3.16)$$

$$\begin{bmatrix} \mathbf{B}_1^{21} \\ \mathbf{B}_1^{22} \\ \mathbf{B}_1^{23} \end{bmatrix}^T = \Theta_6 \begin{bmatrix} \mathbf{B}_1^{01} \\ \mathbf{B}_1^{02} \\ \mathbf{B}_1^{03} \end{bmatrix}^T + \Theta_7 \begin{bmatrix} \mathbf{A}_1^{01} \\ \mathbf{A}_1^{02} \\ \mathbf{A}_1^{03} \end{bmatrix}^T \quad (3.17)$$

$\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6$ and Θ_7 are the common factors found in equation (3.14), (3.15), (3.16) and (3.17) respectively. The values of these factors depend on the state values. Detail expressions of these factors are listed in Appendix A. Note that equations (3.14), (3.15), (3.16) and (3.17) can be represented in terms of their zero order Lie derivatives. The Lie derivatives of higher orders can be similarly expressed and these are listed in Appendix A of this thesis. None of the combinations of Lie derivatives used in the formation of the observability matrix (as described in Section 2.2), has full rank 3, making it clear that localisation from the observation of just one known feature is not feasible.

*A feature-based localisation approach is **unobservable** when used with a process model that describes the evolution of 2D vehicle pose over time and an observation model that describes **ranges and bearings of a known feature point** with respect to the moving vehicle.*

Moving on to the case of two known features, the modified observation model incorporating two *a priori* known feature points, $\Gamma_1 = [\xi_1 \quad \zeta_1]^T$ and $\Gamma_2 = [\xi_2 \quad \zeta_2]^T$ is:

$$\mathbf{Z} = \begin{bmatrix} \hat{\rho}_1 \\ \hat{\phi}_1 \\ \hat{\rho}_2 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} h_1(\mathbf{x}_r, \Gamma_1) \\ h_2(\mathbf{x}_r, \Gamma_1) \\ h_3(\mathbf{x}_r, \Gamma_2) \\ h_4(\mathbf{x}_r, \Gamma_2) \end{bmatrix} = \begin{bmatrix} \sqrt{(\xi_1 - x)^2 + (\zeta_1 - y)^2} \\ \tan^{-1}\left(\frac{\zeta_1 - y}{\xi_1 - x}\right) - \psi + \frac{\pi}{2} \\ \sqrt{(\xi_2 - x)^2 + (\zeta_2 - y)^2} \\ \tan^{-1}\left(\frac{\zeta_2 - y}{\xi_2 - x}\right) - \psi + \frac{\pi}{2} \end{bmatrix} \quad (3.18)$$

where $\begin{bmatrix} \hat{\rho}_2 & \hat{\phi}_2 \end{bmatrix}^T$ is the expected range and bearing measurement of the known feature point $\Gamma_2 = [\xi_2 \ \zeta_2]^T$. The repeated Lie derivatives for the second known feature observation are derived similarly by equations (3.5), (3.6), (3.7) and (3.8), except that the parameter values are $[\xi_2 \ \zeta_2]^T$. The following observability matrix can be formed using one possible combination of Lie derivatives:

$$O_{1,2,3} = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{F}}^0 dh_2(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{F}}^0 dh_3(\mathbf{x}_r, \Gamma_2) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} \\ \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} \\ \mathbf{A}_2^{01} & \mathbf{A}_2^{02} & \mathbf{A}_2^{03} \end{bmatrix} \quad (3.19)$$

$\begin{bmatrix} \mathbf{A}_2^{01} & \mathbf{A}_2^{02} & \mathbf{A}_2^{03} \end{bmatrix}$ are terms involving the zero order Lie derivatives of the known feature $\Gamma_2 = [\xi_2 \ \zeta_2]^T$ and can be obtained using equation (3.5). This observability matrix (3.19) can satisfy the full rank condition (rank=3), provided that the following conditions are satisfied:

$$\begin{aligned} \Gamma_1 &\neq \Gamma_2 \\ [\xi_1 \ \zeta_1]^T &\neq [\xi_2 \ \zeta_2]^T \end{aligned} \quad (3.20)$$

Thus, it appears that the localisation system is only locally weakly observable when there are observations of two distinct feature points.

A feature-based localisation approach is only **weakly observable locally** when used with a 2D vehicle process model and an observation model that describes **ranges and bearings of two distinct known feature points** with respect to the moving vehicle.

The analysis is extended to consecutive observations of different known feature points. Using the same observation function $\mathbf{H}(\mathbf{x}_r, \Gamma_i)$ of equation (3.4), the system first observes one known feature of $\Gamma_1 = [\xi_1 \ \zeta_1]^T$ on the first vehicle path segment before the observation is switched to a known feature of $\Gamma_2 = [\xi_2 \ \zeta_2]^T$ on the second path segment.

The observation model takes the following form:

$$\mathbf{Z} = \begin{cases} \begin{bmatrix} h_1(\mathbf{x}_r(t_0), \Gamma_1) \\ h_2(\mathbf{x}_r(t_0), \Gamma_1) \end{bmatrix} & \text{for } t_0 \leq t < t_1 \\ \begin{bmatrix} h_1(\mathbf{x}_r(t_1), \Gamma_2) \\ h_2(\mathbf{x}_r(t_1), \Gamma_2) \end{bmatrix} & \text{for } t_1 \leq t < t_2 \end{cases} \quad (3.21)$$

where $h_1(\mathbf{x}_r(t_0), \Gamma_1)$ and $h_2(\mathbf{x}_r(t_0), \Gamma_1)$ are the observation functions for the known feature, $\Gamma_1 = [\xi_1 \ \zeta_1]^T$ from time t_0 to t_1 . $h_1(\mathbf{x}_r(t_1), \Gamma_2)$ and $h_2(\mathbf{x}_r(t_1), \Gamma_2)$ are the observation functions for the known feature of $\Gamma_2 = [\xi_2 \ \zeta_2]^T$ from time t_1 to t_2 . Using Lie derivatives, one can derive:

$$O_{1,2} = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r(t_0), \Gamma_1) \\ L_{\mathbf{F}}^0 dh_2(\mathbf{x}_r(t_0), \Gamma_1) \\ L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r(t_1), \Gamma_2) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} \\ \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} \\ \mathbf{A}_2^{01} & \mathbf{A}_2^{02} & \mathbf{A}_2^{03} \end{bmatrix} \quad (3.22)$$

$\begin{bmatrix} A_2^{01} & A_2^{02} & A_2^{03} \end{bmatrix}$ are terms involving the zero order Lie derivatives of the known feature $\Gamma_2 = [\xi_2 \quad \zeta_2]^T$ as observed during the second path segment. It can be verified that this observability matrix also achieves full rank ($=3$), if the following conditions are satisfied:

$$\begin{aligned} \Gamma_1 &\neq \Gamma_2 \\ [\xi_1 \quad \zeta_1]^T &\neq [\xi_2 \quad \zeta_2]^T \end{aligned} \quad (3.23)$$

In general, it is shown that the feature-based localisation system is locally weakly observable for consecutive observation of distinct known feature points,

*A feature-based localisation approach is **locally weakly observable** when used with the 2D vehicle process model and the observation model that describes **consecutive ranges and bearings of a distinct known feature point** with respect to the moving vehicle.*

Simulations are run to verify these observability results, and their outcomes are shown in the following section.

3.3 Simulation Results of Feature-based Localisation

The system process model (3.2) is used to determine the evolution of the vehicle's state. Complete estimation processes are carried out with two, one and no observations of features known *a priori* using the observation models provided by equations (3.18), and (3.4). The simulated path in Figure 3.2 shows the results of these vehicle localisation estimations.

The simulation data for the vehicle path is generated using the kinematics model, which is also defined in equation (3.2). The vehicle is assumed to start at the origin and

travel along a straight path with a 0° orientation for $100m$ as shown in Figure 3.2. The data generated includes the vehicle's speeds and yaw angles, as well as the corresponding ranges and bearings of two feature points located at $(110,20)$ and $(120,10)$. The data is corrupted with the noise parameters shown in Table 3.1.

Parameters		Values
Initial offset	x	$-2m$
	y	$-2m$
	ψ	2°
Vehicle's wheel base	L	$1.6m$
Distance between LADAR and rear axle	a	$1.8m$
Velocity	v	$3m/s$
Control noise	Velocity, σ_v	$2m/s$
	Steer, σ_α	2°
Measurement noise	Range, σ_r	$0.5m$
	Bearing, σ_θ	1°

Table 3.1 Parameter used in the Simulation

Note that the simulation is conducted assuming that there is an initial pose error. It is used to demonstrate whether the system is able to correct these errors with the subsequent input and output measurements. The rationale for introducing this initial error is that in actual practice, regardless of whether one starts with zero pose error the estimation error builds due to the inaccuracies in the process and measurement models. Thus, even if one assumes that the vehicle starts with zero error, at the next state there will be an error in its current position. Now if the system is unobservable it is not possible to infer this state (or correct the estimation error) regardless of how many measurements that one makes subsequently. That is the simulation attempts to demonstrate the observability of the system and its practical implications. The fact that the estimator is unable to correct this initial error verifies the un-observability of the system. Thus even if

one assumes that there is no error in the initial position (or that the initial position is known absolutely) there is no guarantee that the estimator will be able to provide the unique pose estimates given the input and output data.

An Extended Kalman Filter (EKF) algorithm described in Section 2.3.2 is used to estimate the vehicle pose and the results obtained are shown in Figures 3.2, 3.3 and 3.4. Figures 3.3 and 3.4 show also the simulation results for the absolute position error and heading error. The absolute position error is defined as the Euclidean distance between the actual and estimated positions. The heading error is the orientation difference between the actual and the estimated heading.

The results show that in localisation system with no feature observations, the absolute position error does not converge to a minimum when the heading error exhibits an offset with a nonzero mean error. When one *a priori* known feature is observed using the observation model (3.4), the absolute position error decreases to a minimum (See Figure 3.3). However, the heading error (shown in Figure 3.4) still exhibits a constant bias or offset, with nonzero mean error.

Figure 3.2 Simulations for localisation estimation using two, one and no *a priori* features.

These simulation results substantiate the earlier theoretical conclusion that one *a priori* known feature point is inadequate to observe the full vehicle state because the initial orientation error remains throughout the estimation process. It can therefore be concluded that the system is unobservable when only one *a priori* known feature point is observed.

Using the observation model (3.18), when two known feature points exist, the absolute position error and the heading error converge to a minimum. Figures 3.3 and 3.4 show the results of simulations involving two, one or no *a priori* known features. These results clearly verify the findings from the theoretical analysis and lend weight to the conclusion that for the localisation system to be observable, there has to be at least two *a priori* known feature point observations.

Figure 3.3 Absolute position errors using two, one and no *a priori* known features for localisation estimation

Figure 3.4 Heading errors using two, one and no *a priori* known features for localisation estimation

Now consider the observation model (3.21). Here, a different feature point is observed at a particular vehicle path interval and is then used for the EKF updates. Two known features located at (50,10) and (110,10) are used to generate this scenario. The vehicle starts by observing the first feature, but as it moves to the middle of the path, the observation switches to the next feature located at (110,10). The vehicle's trajectory is shown in Figure 3.5.

Figure 3.5 System localisation simulation results for no observation and observation of *a priori* known feature. Results from the sequential observation of one *a priori* known feature are also demonstrated here.

When there is one feature, the results of the simulation that is run to determine the absolute position error converge to a minimum. When no features exist, the localisation estimation diverges, as shown in Figure 3.6. However, for the heading error shown in Figure 3.7, there is an offset error during the first path interval (observing the first feature $(50,10)$), when the system observes the new *a priori* known feature $(110,10)$, in the second path interval, this offset is corrected, producing a zero mean error. The system is shown to be observable when consecutive measurements of different *a priori* known feature points are received. This further verifies the results obtained from the theoretical analysis.

Figure 3.6 Absolute position errors for localisation based on no observation and observations of a known feature. Error registered in the sequential observation of one known feature is also shown here.

Figure 3.7 Heading errors for localisation based on no observation and observations of a known feature. Heading error registered during the sequential observation of one known feature is also indicated here.

3.4 Observability of Path constrained Localisation

Similar to Section 3.2, the problem faced in this section is the determination of the vehicle's pose in terms of a traversable path in the environment given *a priori* spatial information.

Observation model:

In path-constrained localisation, *a priori* information on a vehicle's traversable path is used to localise the vehicle. The vehicle model of the system is defined in equation (3.2) and the vehicle's traversable path is derived by the use of a road map. A road with arbitrary curvature is represented as a sequence of connected segments or links. These are straight and their specific orientations are shown in Figure 3.8. The entire road network can be represented by a connected graph, with each link representing a linear

road segment $G_i(x_l)$. $J = (j_1, j_2, \dots, j_i)$ is the set of nodes (Cartesian coordinates) connecting the links.

Figure 3.8 Piecewise straight road model $G_i(x_l)$ of the true road path.

Such representations are already being used on road networks for car navigation applications by Navtech [74] and Teleatlas [75] among others. The digital roadmap can be used to improve the position estimates of the vehicle at any given time, provided that the vehicle is on a specific segment i and is travelling nominally in the direction, γ_i . The vehicle estimates will be confined within the lateral breadth of the segment.

With these assumptions, the relevant measurement for the road segment i where the vehicle is plying is obtained, as follows

$$\ell_i = \begin{bmatrix} \sin \gamma_i & -\cos \gamma_i \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \quad (3.24)$$

where ℓ_i is the distance from the road segment i to its parallel straight line through the origin as shown in Figure 3.9. $\begin{bmatrix} \sin \gamma_i & -\cos \gamma_i \end{bmatrix}$ is the vector that projects $\begin{bmatrix} x_i & y_i \end{bmatrix}^T$ to a coordinate that its axis is perpendicular to the parallel straight line through the origin.

The observation model for the measurement ℓ_i can then be expressed as:

$$\begin{aligned}
\mathbf{Z} &= [\hat{\ell}_i] = \mathbf{H}(\mathbf{x}_r, \Gamma_i) & , \mathbf{Z} \in R^1 \\
&= [h_1(\mathbf{X}, \Gamma_i)] = [\sin \gamma_i \quad -\cos \gamma_i \quad 0] \begin{bmatrix} x \\ y \\ \psi \end{bmatrix} & (3.25) \\
\Gamma_i &= [\Gamma_1] = [\gamma_i] & , \Gamma_i \in R^1
\end{aligned}$$

where $\hat{\ell}_i$ is the expected observation output of ℓ_i . ℓ_i is the distance between the vehicle's current position and the straight line parallel to the corresponding road segment i through the origin as shown in Figure 3.9. An assumption is made that the vehicle is always travelling on a specific road segment and this constrains the vehicle pose estimation.

Figure 3.9 The measurement ℓ_i of road segment i and its expected measurement output $\hat{\ell}_i$ given the vehicle's predicted pose.

For the purposes of this thesis, the uncertainty of the straight-line model road segment can be probabilistically modelled using a Gaussian likelihood function. This is similar to the Path Constrained localisation approach used in [28]. Given that a road

segment is defined using nodes j_1 and j_2 , its likelihood function is modelled as a Gaussian distribution, which is cylindrically extended along the direction of the segment (See Figure 3.10). Figure 3.11(a) shows the probability distribution along a road segment and Figure 3.11(b) shows the probability distribution at a T-intersection.

The use of cylindrically extended Gaussian distributions with their peaks at the centres of the road is mathematically convenient and also quite popular with other researches [28]. However one could use other density functions to reflect the nature of road and applicable traffic rules. For example, if it is known that the vehicle can be on any part of the road (one way) we could use a uniform density over the road segment.

Figure 3.10 Likelihood function of a particular road segment

Figure 3.11 (a) Likelihood function of a road segment
(b) The likelihood representation of a T-intersection

Note that ℓ_i can be regarded as a virtual measurement. We could express the likelihood of the measurement ℓ_i on the road segment i and predicted robot pose $\mathbf{x}_r$ is:

$$p(\ell_i(t) | \mathbf{x}_r, i(t)) = \frac{1}{\sqrt{2\pi C}} \exp \left\{ -\frac{1}{2} (\ell_i(t) - h_1(\mathbf{x}_r, \Gamma_i))^T C_i^{-1} (\ell_i(t) - h_1(\mathbf{x}_r, \Gamma_i)) \right\} \quad (3.26)$$

where $\ell_i(t)$ is the measurement defined and obtained from equation (3.24). C_i is the variance of this lateral measurement $\ell_i(t)$, which consists of both the uncertainties of the road model and the road width. $h_1(\mathbf{x}_r, \Gamma_i)$ is a function of the observation model defined in equation (3.25).

At every update it is important to know (observe) the road segment where the vehicle is likely to be plying. This information is then used to update the predicted vehicle pose. A simple strategy would be to use a maximum likelihood estimator as follows:

$$\hat{i}(t) = \arg \max_{i(t)} p(\ell_i(t) | \mathbf{x}_r, i(t)) \quad (3.27)$$

The map index $i(t)$ of the current road segment is selected specifically to yield the minimum negative log likelihood value or *Mahalanobis distance*. Any other suitable means can also be used to determine which particular road segment the vehicle is on. A common method is matching the vision camera's view of landmarks such as buildings or signboards against an existing database of landmarks. Such a method is effective because it is only necessary for the system to know which road segment the vehicle is on, without regards to the absolute position of the vehicle within the road segment.

Analysis:

When the vehicle is on road segment 1 ($i=1$), the repeated Lie derivatives of the Path constrained localisation system defined by equations (3.3) and (3.25) are:

$$\begin{aligned} L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r, \Gamma_1) &= \frac{\partial h(\mathbf{x}_r, \Gamma_1)}{\partial \mathbf{x}_r} = [\mathbf{A}_1^{01} \quad \mathbf{A}_1^{02} \quad \mathbf{A}_1^{03}] \\ &= [\sin \gamma_1 \quad -\cos \gamma_1 \quad 0] \end{aligned} \quad (3.28)$$

$$\begin{aligned} L_{\mathbf{F}}^1 dh_1(\mathbf{x}_r, \Gamma_1) &= \frac{\partial h(\mathbf{x}_r, \Gamma_1)}{\partial \mathbf{x}_r} \frac{\partial \mathbf{F}(\mathbf{x}_r, \mathbf{u})}{\partial \mathbf{x}_r} + \left[\frac{\partial}{\partial \mathbf{x}_r} \left(\frac{\partial h(\mathbf{x}_r, \Gamma_1)}{\partial \mathbf{x}_r} \right)^T \mathbf{F}(\mathbf{x}_r, \mathbf{u}) \right]^T \\ &= [0 \quad 0 \quad \mathbf{A}_1^{13}] \end{aligned} \quad (3.29)$$

$$\begin{aligned} L_{\mathbf{F}}^2 dh_1(\mathbf{x}_r, \Gamma_1) &= L_{\mathbf{F}}^1 dh_1(\mathbf{x}_r, \Gamma_1) \frac{\partial \mathbf{F}(\mathbf{x}_r, \mathbf{u})}{\partial \mathbf{x}_r} + \left[\frac{\partial}{\partial \mathbf{X}} (L_{\mathbf{F}}^1 dh_1(\mathbf{x}_r, \Gamma_1))^T \mathbf{F}(\mathbf{x}_r, \mathbf{u}) \right]^T \\ &= [0 \quad 0 \quad \mathbf{A}_1^{23}] \end{aligned} \quad (3.30)$$

where

$$\begin{aligned} \mathbf{A}_1^{13} &= -\sin \gamma_1 (\sin \psi + a\delta \cos \psi) - \cos \gamma_1 (\cos \psi - a\delta \sin \psi) \\ \mathbf{A}_1^{23} &= -\delta \sin \gamma_1 (\cos \psi - a\delta \sin \psi) + \delta \cos \gamma_1 (\sin \psi + a\delta \cos \psi) \end{aligned} \quad (3.31)$$

Hence the observability matrix for the system is:

$$O_1 = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{F}}^1 dh_1(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{F}}^2 dh_1(\mathbf{x}_r, \Gamma_1) \end{bmatrix} = \begin{bmatrix} \sin \gamma_1 & -\cos \gamma_1 & 0 \\ 0 & 0 & \mathbf{A}_1^{13} \\ 0 & 0 & \mathbf{A}_1^{23} \end{bmatrix} \quad (3.32)$$

Note that this is the only observability matrix (3.32) for this system and for the orientation of road segment γ_1 . For local weak observability, matrix (3.32) must register a full rank (the rank must be 3). However, it is clear that the rank of matrix (3.32) is 2. Hence, it is concluded that the fact that the system knows that the vehicle is on a specific segment does not guarantee observability.

The path constrained localisation approach is **unobservable** when it involves a 2D vehicle process model and an observation model that describes lateral offsets of the vehicle position based on **single road segment**.

When the vehicle moves from one segment to another, the model of segment 1 has to be replaced by that of segment 2. In such cases, the observation model is:

$$\mathbf{Z} = \begin{cases} \left[h_1(\mathbf{x}_r(t_0), \Gamma_1) \right] & \text{for } t_0 \leq t < t_1 \\ \left[h_1(\mathbf{x}_r(t_1), \Gamma_2) \right] & \text{for } t_1 \leq t < t_2 \end{cases} \quad (3.33)$$

where $h_1(\mathbf{x}_r(t_0), \Gamma_1)$ and $h_1(\mathbf{x}_r(t_1), \Gamma_2)$ are the observation functions for the road segment of $\Gamma_1 = [\gamma_1]$ from time t_0 to t_1 and $\Gamma_2 = [\gamma_2]$ from time t_1 to t_2 respectively. Thus, the observability matrix is shown to be any of the following:

$$O_1 = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r(t_0), \Gamma_1) \\ L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r(t_1), \Gamma_2) \\ L_{\mathbf{F}}^1 dh_1(\mathbf{x}_r(t_0), \Gamma_1) \end{bmatrix} = \begin{bmatrix} \sin \gamma_1 & -\cos \gamma_1 & 0 \\ \sin \gamma_2 & -\cos \gamma_2 & 0 \\ 0 & 0 & \mathbf{A}_1^{13} \end{bmatrix} \quad (3.34)$$

$$O_1 = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r(t_0), \Gamma_1) \\ L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r(t_1), \Gamma_2) \\ L_{\mathbf{F}}^2 dh_1(\mathbf{x}_r(t_0), \Gamma_1) \end{bmatrix} = \begin{bmatrix} \sin \gamma_1 & -\cos \gamma_1 & 0 \\ \sin \gamma_2 & -\cos \gamma_2 & 0 \\ 0 & 0 & \mathbf{A}_1^{23} \end{bmatrix} \quad (3.35)$$

$$O_1 = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r(t_0), \Gamma_1) \\ L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r(t_1), \Gamma_2) \\ L_{\mathbf{F}}^1 dh_1(\mathbf{x}_r(t_1), \Gamma_2) \end{bmatrix} = \begin{bmatrix} \sin \gamma_1 & -\cos \gamma_1 & 0 \\ \sin \gamma_2 & -\cos \gamma_2 & 0 \\ 0 & 0 & \mathbf{A}_2^{13} \end{bmatrix} \quad (3.36)$$

$$O_1 = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r(t_0), \Gamma_1) \\ L_{\mathbf{F}}^0 dh_1(\mathbf{x}_r(t_1), \Gamma_2) \\ L_{\mathbf{F}}^2 dh_1(\mathbf{x}_r(t_1), \Gamma_2) \end{bmatrix} = \begin{bmatrix} \sin \gamma_1 & -\cos \gamma_1 & 0 \\ \sin \gamma_2 & -\cos \gamma_2 & 0 \\ 0 & 0 & \mathbf{A}_2^{23} \end{bmatrix} \quad (3.37)$$

where

$$\begin{aligned} A_2^{13} &= -\sin \gamma_2 (\sin \psi + a\delta \cos \psi) - \cos \gamma_2 (\cos \psi - a\delta \sin \psi) \\ A_2^{23} &= -\delta \sin \gamma_2 (\cos \psi - a\delta \sin \psi) + \delta \cos \gamma_2 (\sin \psi + a\delta \cos \psi) \end{aligned} \quad (3.38)$$

Although there are several other combinations that of the observability matrices that can be constituted, the matrices in equations (3.34), (3.35), (3.36) and (3.37) are the only ones that satisfy the full rank condition (rank=3). However, the following conditions have to be satisfied:

$$\sin \gamma_1 \neq \sin \gamma_2 \quad (3.39)$$

What this condition means physically is that the road segments are having different orientation. The observability analysis thus shows that the Path Constrained localisation system is locally weakly observable when a transition is made with a change in orientation, γ from one segment to another.

*The path constrained localisation approach is **locally weakly observable** when used in association with the 2D vehicle process model and an observation model that describes **consecutive lateral offsets of vehicle position based on road segment with different orientation.***

3.5 Simulation Results of Path constrained Localisation

In this section, the observability results derived above are verified through stimulations. The vehicle path simulation data is generated using the kinematics model defined in equation (3.2). The vehicle is assumed to start at the origin and travel along a straight road segment for 50m at the orientation of 0° before turning into a road segment

at 10° and travelling a further $50m$ (See Figure 3.12). The data generated includes the vehicle's speeds and yaw angles, as well as the noise parameters with which it is corrupted (see Table 3.2). It is assumed that the observation model appropriate for the transition from one road segment to another is known. An Extended Kalman Filter (EKF) algorithm is used to estimate the vehicle pose and the results obtained are shown in Figures 3.12, 3.13 and 3.14.

Figure 3.12 shows the results of localisation, with and without the application of the path constraints. As shown in Figure 3.13, when the vehicle is moving along a specific segment and the observation model (3.25) is used to estimate the vehicle pose, the actual absolute position error (the Euclidean distance between actual and estimated) does not converge. As theoretically proven in Section 3.3.2, the observation that the vehicle is moving along a path segment does not yield an observable system. Thus, the absolute position error does not converge despite the application of the single path constraint. The initial position error would be corrected if it were observable.

Parameters		Values
Initial offset	x	$-2m$
	y	$-2m$
	ψ	2°
Vehicle's wheel base	L	$1.6m$
Distance between LADAR and rear axle	a	$1.8m$
Velocity	v	$3m/s$
Control noise	Velocity, σ_v	$2m/s$
	Steer, σ_α	2°
Measurement noise	Path constraint, σ_ℓ	$2m$

Table 3.2 Parameter used in the Simulation

However, when the vehicle makes a transition onto the second path segment, the absolute position error is corrected (See Figure 3.13). The correction of absolute position error when the vehicle moves from one segment to another is as a result of the application of the observation model (3.33), which guarantees observability. It may also be noted that with a different path constraint, the heading error converges to a zero mean error after the transition (See Figure 3.14) whereas without the constraint, there is still an offset.

The simulations verify that the observability matrix is of a full rank when changes in orientation are applied which result in consecutive observations of different road segments. The switching of the observation parameter in the observation model makes the system locally weakly observable.

Nevertheless, the simulation results after the vehicle's transit to the second path constraint show that the absolute position error starts to accumulate to a higher value than the initial error posted. The localisation approach using a single path constraint will be unobservable along the direction of travel, until an orientation transition is made. If there is no variation in the vehicle direction, the error may increase to a degree that finding the corresponding path constraint, that is, ascertaining where the vehicle is located at a road segment, can become a challenging task. The vehicle could in fact end up getting "lost".

Figure 3.12 Localisation simulation results with and without path constraints.

Figure 3.13 Absolute position error on localisation with and without path constraints

Figure 3.14 Heading error on localisation with and without path constraints

3.6 Observability of Concurrent Mapping and Localisation

Now consider the CML problem, where the system state, $\mathbf{X}$ can be represented as:

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_r \\ \mathbf{M} \end{bmatrix} \quad \mathbf{X} \in R^{3+(2 \times K)} \quad (3.40)$$

The state $\mathbf{x}_r$ of the vehicle pose is augmented with state vector $\mathbf{M}$, where $\mathbf{M} = [\mathbf{m}_1, \dots, \mathbf{m}_k, \dots, \mathbf{m}_K]^T$ is the vector of the observed point features referenced to a global reference frame. $\mathbf{m}_k = [x_k \ y_k]^T$ is the k^{th} feature and K is the total number of features observed at the instant.

Process Model:

Assuming that the environmental features are static, the state evolution of the feature vector is,

$$\frac{d\mathbf{M}}{dt} = \frac{d}{dt} \begin{bmatrix} \mathbf{m}_1 \\ \vdots \\ \mathbf{m}_k \\ \vdots \\ \mathbf{m}_K \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ x_k \\ y_k \\ \vdots \\ x_K \\ y_K \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad (3.41)$$

Combining the vehicle model defined by equations (3.2) and (3.40), the CML process model evaluates to:

$$\dot{\mathbf{X}} = \frac{d}{dt} \begin{bmatrix} x \\ y \\ \psi \\ x_1 \\ y_1 \\ \vdots \\ x_k \\ y_k \\ \vdots \\ x_K \\ y_K \end{bmatrix} = \begin{bmatrix} \cos \psi \\ \sin \psi \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} v + \begin{bmatrix} -a \sin \psi \\ a \cos \psi \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} v \delta \quad (3.42)$$

Using a method similar to that described in Section 3.2.1, the input affine process model required for the observability analysis of CML problem can be represented as

$$\dot{\mathbf{X}} = \mathbf{f}_0(\mathbf{X}) + \mathbf{f}_1(\mathbf{X}) u_1$$

$$\frac{d}{d\rho} \begin{bmatrix} x \\ y \\ \psi \\ x_1 \\ y_1 \\ \vdots \\ x_k \\ y_k \\ \vdots \\ x_K \\ y_K \end{bmatrix} = \begin{bmatrix} \cos \psi \\ \sin \psi \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -a \sin \psi \\ a \cos \psi \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \delta \quad (3.43)$$

Note that equation (3.43) represents a non-stationary model subjected to a single steer input, $\delta (= u_1)$.

Observation Model:

In a typical CML problem, an exteroceptive sensor is used to take range and bearing observations of the unknown point features in the environment. Given the current

vehicle pose $\mathbf{x}_r$ and the observed k^{th} feature position in the state vector, the observation of the range r_k and bearing θ_k can be modelled as

$$\begin{aligned} \mathbf{Z} &= \begin{bmatrix} \hat{r}_k \\ \hat{\theta}_k \end{bmatrix} = \mathbf{H}(\mathbf{X}) \\ &= \begin{bmatrix} h_1(\mathbf{X}) \\ h_2(\mathbf{X}) \end{bmatrix} = \begin{bmatrix} \sqrt{(x_k - x)^2 + (y_k - y)^2} \\ \tan^{-1}\left(\frac{(y_k - y)}{(x_k - x)}\right) - \psi + \frac{\pi}{2} \end{bmatrix}, \mathbf{Z} \in R^2 \end{aligned} \quad (3.44)$$

where $\mathbf{Z} = [\hat{r}_k \ \hat{\theta}_k]^T$ is the expected sensor measurement of the k^{th} feature in the state vector with range r and bearing θ in respect to the sensor reference frame.

Analysis:

Consider the CML process model (3.43) and the observation model (3.44) with constant parameters, input δ and one unknown feature, i.e. $k=1$. The repeated Lie derivatives (the elements of the observability matrix) can be obtained as follows,

$$\begin{aligned} L_{\mathbf{F}}^0 dh_1(\mathbf{X}) &= \frac{\partial h_1(\mathbf{X})}{\partial \mathbf{X}} = \begin{bmatrix} \mathbf{A}^{01} & \mathbf{A}^{02} & \mathbf{A}^{03} & -\mathbf{A}^{01} & -\mathbf{A}^{02} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(x_1 - x)}{\Delta} & -\frac{(y_1 - y)}{\Delta} & 0 & \frac{(x_1 - x)}{\Delta} & \frac{(y_1 - y)}{\Delta} \end{bmatrix} \end{aligned} \quad (3.45)$$

$$\begin{aligned} L_{\mathbf{F}}^D dh_1(\mathbf{X}) &= L_{\mathbf{F}}^{D-1} dh_1(\mathbf{X}) \frac{\partial \mathbf{F}(\mathbf{X}, \mathbf{u})}{\partial \mathbf{X}} + \left[\frac{\partial}{\partial \mathbf{X}} (L_{\mathbf{F}}^{D-1} dh_1(\mathbf{X}))^T \mathbf{F}(\mathbf{X}, \mathbf{u}) \right]^T \\ &= \begin{bmatrix} \mathbf{A}^{D1} & \mathbf{A}^{D2} & \mathbf{A}^{D3} & -\mathbf{A}^{D1} & -\mathbf{A}^{D2} \end{bmatrix} \\ D &= 1, 2, 3, 4 \end{aligned} \quad (3.46)$$

$$\begin{aligned} L_{\mathbf{F}}^0 dh_2(\mathbf{X}) &= \frac{\partial h_2(\mathbf{X})}{\partial \mathbf{X}} = \begin{bmatrix} \mathbf{B}^{01} & \mathbf{B}^{02} & \mathbf{B}^{03} & -\mathbf{B}^{01} & -\mathbf{B}^{02} \end{bmatrix} \\ &= \begin{bmatrix} \frac{(y_1 - y)}{\Delta^2} & -\frac{(x_1 - x)}{\Delta^2} & -1 & -\frac{(y_1 - y)}{\Delta^2} & \frac{(x_1 - x)}{\Delta^2} \end{bmatrix} \end{aligned} \quad (3.47)$$

$$\begin{aligned}
L_{\mathbf{F}}^D dh_2(\mathbf{X}) &= L_{\mathbf{F}}^{D-1} dh_2(\mathbf{X}) \frac{\partial \mathbf{F}(\mathbf{X}, \mathbf{u})}{\partial \mathbf{X}} + \left[\frac{\partial}{\partial \mathbf{X}} \left(L_{\mathbf{F}}^{D-1} dh_2(\mathbf{X}) \right)^T \mathbf{F}(\mathbf{X}, \mathbf{u}) \right]^T \\
&= \begin{bmatrix} \mathbf{B}^{D1} & \mathbf{B}^{D2} & \mathbf{B}^{D3} & -\mathbf{B}^{D1} & -\mathbf{B}^{D2} \end{bmatrix} \\
D &= 1, 2, 3, 4
\end{aligned} \tag{3.48}$$

where

$$\Delta = \sqrt{(x_1 - x)^2 + (y_1 - y)^2} \tag{3.49}$$

By combining the above elements, an observability matrix $O_{1,2}$ can be constructed as follows,

$$O_{1,2} = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{X}) \\ L_{\mathbf{F}}^0 dh_2(\mathbf{X}) \\ L_{\mathbf{F}}^1 dh_1(\mathbf{X}) \\ L_{\mathbf{F}}^1 dh_2(\mathbf{X}) \\ L_{\mathbf{F}}^2 dh_1(\mathbf{X}) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{01} & \mathbf{A}^{02} & \mathbf{A}^{03} & -\mathbf{A}^{01} & -\mathbf{A}^{02} \\ \mathbf{B}^{01} & \mathbf{B}^{02} & \mathbf{B}^{03} & -\mathbf{B}^{01} & -\mathbf{B}^{02} \\ \mathbf{A}^{11} & \mathbf{A}^{12} & \mathbf{A}^{13} & -\mathbf{A}^{11} & -\mathbf{A}^{12} \\ \mathbf{B}^{11} & \mathbf{B}^{12} & \mathbf{B}^{13} & -\mathbf{B}^{11} & -\mathbf{B}^{12} \\ \mathbf{A}^{21} & \mathbf{A}^{22} & \mathbf{A}^{23} & -\mathbf{A}^{21} & -\mathbf{A}^{22} \end{bmatrix} \tag{3.50}$$

where the first order Lie derivatives can be expressed as,

$$\begin{bmatrix} \mathbf{A}^{11} \\ \mathbf{A}^{12} \\ \mathbf{A}^{13} \\ -\mathbf{A}^{11} \\ -\mathbf{A}^{12} \end{bmatrix}^T = \Upsilon_1 \begin{bmatrix} \mathbf{B}^{01} \\ \mathbf{B}^{02} \\ \mathbf{B}^{03} \\ -\mathbf{B}^{01} \\ -\mathbf{B}^{02} \end{bmatrix}^T \tag{3.51}$$

$$\begin{bmatrix} \mathbf{B}^{11} \\ \mathbf{B}^{12} \\ \mathbf{B}^{13} \\ -\mathbf{B}^{11} \\ -\mathbf{B}^{12} \end{bmatrix}^T = \Upsilon_4 \begin{bmatrix} \mathbf{B}^{01} \\ \mathbf{B}^{02} \\ \mathbf{B}^{03} \\ -\mathbf{B}^{01} \\ -\mathbf{B}^{02} \end{bmatrix}^T + \Upsilon_5 \begin{bmatrix} \mathbf{A}^{01} \\ \mathbf{A}^{02} \\ \mathbf{A}^{03} \\ -\mathbf{A}^{01} \\ -\mathbf{A}^{02} \end{bmatrix}^T \tag{3.52}$$

Υ_1 , Υ_4 and Υ_5 are the common factors found in equation (3.51) and (3.52) respectively.

Detailed expressions of these common factors are listed in Appendix A. Note that equations (3.51) and (3.52) can be represented in terms of their zero order Lie derivatives. The rest of the higher order Lie derivatives can also be similarly expressed

and their deviations are listed in Appendix A for easy reference. It can be shown that none of the combinations of the Lie derivatives used in the formation of the observability matrix as described in Section 3.2, achieve the full rank of 5. This means that the CML formulation is unobservable when observing one unknown feature. The result can be easily generalised to the observation of as many unknown features or landmarks as required. Andrade-Cetto *et. al* [44] arrived at the same conclusion using linearised models of the CML problem. However, as will be shown next, the linearised analysis that they used in [44] is inappropriate for use in the analysis of observability in the CML problem.

*The CML approach is **unobservable** when used with the process model that describes both the evolution of 2D vehicle pose and feature position over time and an observation model that describes **ranges and bearings of unknown feature points** with respect to the moving vehicle.*

In the following paragraphs, we investigate the observability of the CML framework in detail. An observation involving a single known feature is first considered, along with that involving an unknown feature. The modified observation model incorporating the *a priori* known feature point, equation (3.4), $i=1$ with the unknown feature point $k=1$ is:

$$\mathbf{Z} = \begin{bmatrix} \hat{r}_1 \\ \hat{\theta}_1 \\ \hat{\rho}_1 \\ \hat{\phi}_1 \end{bmatrix} = \begin{bmatrix} h_1(\mathbf{X}) \\ h_2(\mathbf{X}) \\ h_3(\mathbf{x}_r, \Gamma_1) \\ h_4(\mathbf{x}_r, \Gamma_1) \end{bmatrix} = \begin{bmatrix} \sqrt{(x_1 - x)^2 + (y_1 - y)^2} \\ \tan^{-1}\left(\frac{y_1 - y}{x_1 - x}\right) - \psi + \frac{\pi}{2} \\ \sqrt{(\xi_1 - x)^2 + (\zeta_1 - y)^2} \\ \tan^{-1}\left(\frac{\zeta_1 - y}{\xi_1 - x}\right) - \psi + \frac{\pi}{2} \end{bmatrix} \quad (3.53)$$

where $[\hat{\rho}_1 \ \hat{\phi}_1]^T$ is the corresponding expected range and bearing of the known feature point $\Gamma_1 = [\xi_1 \ \zeta_1]^T$. The repeated Lie derivatives of the unknown feature observation are given by equations (3.45), (3.46), (3.47) and (3.48). The repeated Lie derivative for the known feature point is as follows:

$$\begin{aligned} L_{\mathbf{F}}^0 dh_3(\mathbf{x}_r, \Gamma_1) &= \frac{\partial h_3(\mathbf{x}_r, \Gamma_1)}{\partial \mathbf{X}} = \begin{bmatrix} \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(\xi_1 - x)}{\Delta} & -\frac{(\zeta_1 - y)}{\Delta} & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (3.54)$$

$$\begin{aligned} L_{\mathbf{F}}^D dh_3(\mathbf{x}_r, \Gamma_1) &= L_{\mathbf{F}}^{D-1} dh_3(\mathbf{x}_r, \Gamma_1) \frac{\partial \mathbf{F}(\mathbf{X}, \mathbf{u})}{\partial \mathbf{X}} + \left[\frac{\partial}{\partial \mathbf{X}} (L_{\mathbf{F}}^{D-1} dh_3(\mathbf{x}_r, \Gamma_1))^T \mathbf{F}(\mathbf{X}, \mathbf{u}) \right]^T \\ &= \begin{bmatrix} \mathbf{A}_1^{D1} & \mathbf{A}_1^{D2} & \mathbf{A}_1^{D3} & 0 & 0 \end{bmatrix} \\ D &= 1, 2, 3, 4 \end{aligned} \quad (3.55)$$

$$\begin{aligned} L_{\mathbf{F}}^0 dh_4(\mathbf{x}_r, \Gamma_1) &= \frac{\partial h_4(\mathbf{x}_r, \Gamma_1)}{\partial \mathbf{X}} = \begin{bmatrix} \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(\zeta_1 - y)}{\Delta^2} & -\frac{(\xi_1 - x)}{\Delta^2} & -1 & 0 & 0 \end{bmatrix} \end{aligned} \quad (3.56)$$

$$\begin{aligned} L_{\mathbf{F}}^D dh_4(\mathbf{x}_r, \Gamma_1) &= L_{\mathbf{F}}^{D-1} dh_4(\mathbf{x}_r, \Gamma_1) \frac{\partial \mathbf{F}(\mathbf{X}, \mathbf{u})}{\partial \mathbf{X}} + \left[\frac{\partial}{\partial \mathbf{X}} (L_{\mathbf{F}}^{D-1} dh_4(\mathbf{x}_r, \Gamma_1))^T \mathbf{F}(\mathbf{X}, \mathbf{u}) \right]^T \\ &= \begin{bmatrix} \mathbf{B}_1^{D1} & \mathbf{B}_1^{D2} & \mathbf{B}_1^{D3} & 0 & 0 \end{bmatrix} \\ D &= 1, 2, 3, 4 \end{aligned} \quad (3.57)$$

One of the possible observability matrices that can be formed using a particular combination of the Lie derivatives is:

$$O_{1,2,3,4} = \begin{bmatrix} L_F^0 dh_1(\mathbf{X}) \\ L_F^0 dh_2(\mathbf{X}) \\ L_F^0 dh_3(\mathbf{x}_r, \Gamma_1) \\ L_F^0 dh_4(\mathbf{x}_r, \Gamma_1) \\ L_F^1 dh_3(\mathbf{x}_r, \Gamma_1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{01} & \mathbf{A}^{02} & \mathbf{A}^{03} & -\mathbf{A}^{01} & -\mathbf{A}^{02} \\ \mathbf{B}^{01} & \mathbf{B}^{02} & \mathbf{B}^{03} & -\mathbf{B}^{01} & -\mathbf{B}^{02} \\ \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} & 0 & 0 \\ \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} & 0 & 0 \\ \mathbf{A}_1^{11} & \mathbf{A}_1^{12} & \mathbf{A}_1^{13} & 0 & 0 \end{bmatrix} \quad (3.58)$$

$\begin{bmatrix} \mathbf{A}_1^{11} & \mathbf{A}_1^{12} & \mathbf{A}_1^{13} & 0 & 0 \end{bmatrix}$ are terms involving the first order Lie derivatives that can be obtained using equation (3.55). As described for the case of unknown features, the higher order Lie derivatives of the known feature observation can also be represented in terms of their zero order Lie derivatives. Although the rank of the observability matrix has increased to 4 by incorporating a known feature, it is still rank deficient. Using a similar analysis one can show that the same result holds when there is one known feature and an arbitrary number of unknown feature observations. In other words, CML with one known feature is unobservable. This finding contradicts the results reported by Andrade-Cetto *et al.* [44].

*The CML approach is **unobservable** when used with the 2D vehicle and feature process model and an observation model that describes **ranges and bearings of unknown feature points plus a known feature point** with respect to the moving vehicle.*

In order to have a system with a full rank matrix, the analysis is extended to include two known feature points. By proceeding in a manner similar to that used to

analyse a system that considers only known feature point, the observation model for the case of two known and one unknown feature point is:

$$\mathbf{Z} = \begin{bmatrix} \hat{r}_1 \\ \hat{\theta}_1 \\ \hat{\rho}_1 \\ \hat{\phi}_1 \\ \hat{\rho}_2 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} h_1(\mathbf{X}) \\ h_2(\mathbf{X}) \\ h_3(\mathbf{x}_r, \Gamma_1) \\ h_4(\mathbf{x}_r, \Gamma_1) \\ h_5(\mathbf{x}_r, \Gamma_2) \\ h_6(\mathbf{x}_r, \Gamma_2) \end{bmatrix} \quad (3.59)$$

where $h_1(\mathbf{X})$ and $h_2(\mathbf{X})$ are the observation functions for the unknown feature. $h_3(\mathbf{x}_r, \Gamma_1)$ and $h_4(\mathbf{x}_r, \Gamma_1)$, and $h_5(\mathbf{x}_r, \Gamma_2)$ and $h_6(\mathbf{x}_r, \Gamma_2)$, are the observation functions for the known features $\Gamma_1 = [\xi_1 \ \zeta_1]^T$ and $\Gamma_2 = [\xi_2 \ \zeta_2]^T$ respectively. Using Lie derivatives, one can derive an observability $O_{1,2,3,4,5}$:

$$O_{1,2,3,4,5} = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{X}) \\ L_{\mathbf{F}}^0 dh_2(\mathbf{X}) \\ L_{\mathbf{F}}^0 dh_3(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{F}}^0 dh_4(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{F}}^1 dh_5(\mathbf{x}_r, \Gamma_2) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{01} & \mathbf{A}^{02} & \mathbf{A}^{03} & -\mathbf{A}^{01} & -\mathbf{A}^{02} \\ \mathbf{B}^{01} & \mathbf{B}^{02} & \mathbf{B}^{03} & -\mathbf{B}^{01} & -\mathbf{B}^{02} \\ \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} & 0 & 0 \\ \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} & 0 & 0 \\ \mathbf{A}_2^{01} & \mathbf{A}_2^{02} & \mathbf{A}_2^{03} & 0 & 0 \end{bmatrix} \quad (3.60)$$

where $[\mathbf{A}^{01} \ \mathbf{A}^{02} \ \mathbf{A}^{03} \ -\mathbf{A}^{01} \ -\mathbf{A}^{02}]$ and $[\mathbf{B}^{01} \ \mathbf{B}^{02} \ \mathbf{B}^{03} \ -\mathbf{B}^{01} \ -\mathbf{B}^{02}]$ are the zero order Lie derivatives for the unknown feature. $[\mathbf{A}_1^{01} \ \mathbf{A}_1^{02} \ \mathbf{A}_1^{03} \ 0 \ 0]$, $[\mathbf{B}_1^{01} \ \mathbf{B}_1^{02} \ \mathbf{B}_1^{03} \ 0 \ 0]$ and $[\mathbf{A}_2^{01} \ \mathbf{A}_2^{02} \ \mathbf{A}_2^{03} \ 0 \ 0]$ are the zero order Lie derivatives for the two known features, $\Gamma_1 = [\xi_1 \ \zeta_1]^T$ and $\Gamma_2 = [\xi_2 \ \zeta_2]^T$. This observability matrix has a full rank of 5. In general, it can be shown that for observing two known feature points and an arbitrary number of unknown features, CML is locally weakly observable.

The observability matrix (3.60) could satisfy the full rank condition (rank=5), provided that the following conditions are satisfied:

$$\begin{aligned} \Gamma_1 &\neq \Gamma_2 \\ [\xi_1 \quad \zeta_1]^T &\neq [\xi_2 \quad \zeta_2]^T \end{aligned} \quad (3.61)$$

In general, it can be shown that CML is locally weakly observable when there are two distinct known feature points and an arbitrary number of unknown features.

*The CML approach is **locally weakly observable** when used with the 2D vehicle and feature process model and an observation model that describes **ranges and bearings of unknown feature points plus two distinct known feature points** with respect to the moving vehicle.*

The analysis is extended to investigate the CML system when different known feature points are consecutively observed. It found to be locally weakly observable. Using the same observation function $\mathbf{H}(\mathbf{X}, \Gamma_i)$ of equation (3.48), the system is to observe one unknown feature $m_1 = [x_1 \quad y_1]^T$ along with the known feature $\Gamma_1 = [\xi_1 \quad \zeta_1]^T$ on the first vehicle path segment. Thereafter, the observation is switched to a known feature $\Gamma_2 = [\xi_2 \quad \zeta_2]^T$ on the second path segment. The observation model takes the form of

$$\mathbf{Z} = \begin{cases} \begin{bmatrix} \hat{r}_1 \\ \hat{\theta}_1 \\ \hat{\rho}_1 \\ \hat{\phi}_1 \end{bmatrix} = \begin{bmatrix} h_1(\mathbf{X}(t_0)) \\ h_2(\mathbf{X}(t_0)) \\ h_3(\mathbf{x}_r(t_0), \Gamma_1) \\ h_4(\mathbf{x}_r(t_0), \Gamma_1) \end{bmatrix} & \text{for } t_0 \leq t < t_1 \\ \begin{bmatrix} \hat{r}_1 \\ \hat{\theta}_1 \\ \hat{\rho}_2 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} h_1(\mathbf{X}(t_1)) \\ h_2(\mathbf{X}(t_1)) \\ h_3(\mathbf{x}_r(t_1), \Gamma_2) \\ h_4(\mathbf{x}_r(t_1), \Gamma_2) \end{bmatrix} & \text{for } t_1 \leq t < t_2 \end{cases} \quad (3.62)$$

where $h_1(\mathbf{X}(t_0))$ and $h_2(\mathbf{X}(t_0))$ are the observation functions for the unknown feature. $h_3(\mathbf{x}_r(t_0), \Gamma_1)$ and $h_4(\mathbf{x}_r(t_0), \Gamma_1)$, and $h_3(\mathbf{x}_r(t_1), \Gamma_2)$ and $h_4(\mathbf{x}_r(t_1), \Gamma_2)$, are the observation functions for the known feature of $\Gamma_1 = [\xi_1 \ \zeta_1]^T$ from time t_0 to t_1 and $\Gamma_2 = [\xi_2 \ \zeta_2]^T$ from time t_1 to t_2 respectively. Using Lie derivatives one can derive observability $O_{1,2,3,4}$ such that:

$$O_{1,2,3,4} = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{X}(t_0)) \\ L_{\mathbf{F}}^0 dh_2(\mathbf{X}(t_0)) \\ L_{\mathbf{F}}^0 dh_3(\mathbf{x}_r(t_0), \Gamma_1) \\ L_{\mathbf{F}}^0 dh_4(\mathbf{x}_r(t_0), \Gamma_1) \\ L_{\mathbf{F}}^0 dh_3(\mathbf{x}_r(t_1), \Gamma_2) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{01} & \mathbf{A}^{02} & \mathbf{A}^{03} & -\mathbf{A}^{01} & -\mathbf{A}^{02} \\ \mathbf{B}^{01} & \mathbf{B}^{02} & \mathbf{B}^{03} & -\mathbf{B}^{01} & -\mathbf{B}^{02} \\ \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & \mathbf{A}_1^{03} & 0 & 0 \\ \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & \mathbf{B}_1^{03} & 0 & 0 \\ \mathbf{A}_2^{01} & \mathbf{A}_2^{02} & \mathbf{A}_2^{03} & 0 & 0 \end{bmatrix} \quad (3.63)$$

where $[\mathbf{A}^{01} \ \mathbf{A}^{02} \ \mathbf{A}^{03} \ -\mathbf{A}^{01} \ -\mathbf{A}^{02}]$ and $[\mathbf{B}^{01} \ \mathbf{B}^{02} \ \mathbf{B}^{03} \ -\mathbf{B}^{01} \ -\mathbf{B}^{02}]$ are the zero order Lie derivatives for the unknown feature. $[\mathbf{A}_1^{01} \ \mathbf{A}_1^{02} \ \mathbf{A}_1^{03} \ 0 \ 0]$ and $[\mathbf{B}_1^{01} \ \mathbf{B}_1^{02} \ \mathbf{B}_1^{03} \ 0 \ 0]$ are the zero order Lie derivatives for the known feature, $\Gamma_1 = [\xi_1 \ \zeta_1]^T$ and $[\mathbf{A}_2^{01} \ \mathbf{A}_2^{02} \ \mathbf{A}_2^{03} \ 0 \ 0]$ is the zero order Lie derivatives for the known feature, $\Gamma_2 = [\xi_2 \ \zeta_2]^T$ observed during the second path segment. This observability matrix achieves the full rank of 5, if the following conditions are satisfied:

$$\begin{aligned} \Gamma_1 &\neq \Gamma_2 \\ [\xi_1 \ \zeta_1]^T &\neq [\xi_2 \ \zeta_2]^T \end{aligned} \quad (3.64)$$

In general, it can be shown that CML with an arbitrary number of unknown feature observations is also locally weakly observable when at least a distinct known feature point is consecutively observed.

*The CML approach is **locally weakly observable** when used with the 2D vehicle and feature process model and the observation model that describes **ranges and bearings of unknown feature points plus consecutive ranges and bearings of a distinct known feature point** with respect to the moving vehicle.*

3.7 Simulation Results of Concurrent Mapping and Localisation

In CML simulation estimation was carried out using an EKF algorithm. The system process model is defined by equation (3.43) and the typical observation model (3.44) was used to estimate the vehicle's pose and an unknown feature position. The simulated observations were also made of the CML system's running given one and two known features along with one unknown feature using observation models (3.53) and (3.59). The vehicle with input $\delta=0$ is travelling in a straight line whilst observing both the unknown feature located at (130,20) and the known features located at (130,10) and (120,30). The simulation data generated using the kinematics model defined in equation (3.43) shows the speed and yaw angle of the moving vehicle as well as the corresponding ranges and bearings of three feature points. These are corrupted with the noise parameters shown in Table 3.1.

The results for the estimations involving two, one and no known features are shown in Figure 3.15. Using the typical observation model (3.44) for the case of no known features, it appears that the absolute position error of the vehicle does not converge to a minimum (see Figure 3.16). The heading error also exhibits a constant bias, with a nonzero mean error. This simulation demonstrates that the CML system is unobservable without known features. Consequently, the initial system errors are not

corrected and the vehicle's true pose as well as the position of landmarks cannot be estimated even if the presence of noise is negligible.

Figure 3.15 CML simulation for one unknown and two known landmark observations

Using the observation model (3.53), this time for a CML system with one known feature observation, it appears that an initial error in the estimation results in an instantaneous decrease in absolute position error (See Figure 3.16). However, the heading errors also show a constant bias, nonzero mean error (See Figure 3.17). Based on this simulation results, it can be inferred that a CML system is still unobservable if the system registers the presence of only one known feature. The heading errors have a constant bias, which the system is unable to correct. Intuitively it can be stated that the relative measurement from one *a priori* known point will not be sufficient to correct any error occurring in the vehicle's orientation. Therefore, one known feature point does not provide the minimum level of spatial information required in order to yield an observable CML system.

Figure 3.16 Absolute position errors for CML using one unknown and two known feature observations

Figure 3.17 Heading errors for CML using one unknown and two known feature observations

When the CML system registers two known features using the observation model (3.59), the absolute position error converges to a minimum, as shown in Figure 3.16. At the same time, the system heading in Figure 3.17 shows a zero mean error. The results of this simulation are consistent with that of the rank tests described previously, which demonstrated that the system is locally weakly observable when two known features are observed.

Now consider the case when the CML system observes different known features consecutively. Simulations are run using the observation model (3.62) (See Figure 3.18). When the vehicle starts and moves to the middle of the path, it observes both the known feature point located at (50,10) and the unknown feature located at (130,20). Following this, the vehicle moves onto the second half of the simulated path and the vehicle observation switches to the second known feature point located at (130,10) while still observing the same unknown feature. In this simulation, the EKF estimation process obtains measurements regarding the two known feature points in a sequential form, thus realistically representing characteristics of large-scale localisation operations.

The simulation results shown in Figure 3.19 indicate that the absolute position error converges to a minimum. However, as shown in Figure 3.20, the heading error shows a bias during the first vehicle path interval. It is found that the heading error is only corrected when the vehicle observes the second *a priori* known feature point. The system can be said to be locally weakly observable when a different *a priori* feature point is sequentially observed.

Figure 3.18 CML with the observation of zero and one known feature, and the sequential observation of one known feature

Figure 3.19 Absolute vehicle position error for the CML approach with zero and one feature, and the sequential observation of the one known feature

Figure 3.20 Heading Error for CML with zero and one known feature, and the sequential observation of the one known feature

3.8 Conclusions

Because of the large number of uncertainties in the vehicle's motion, a localisation system is generally unable to generate usable estimates of the vehicle's subsequent poses without a constant absolute observation of the vehicle's pose with respect to the global reference frame. It is necessary to analyse the observability properties of the system used to formulate the localisation problem. This is to determine whether the system is observable that guarantees the convergence of state estimates. A study of system observability can assist in the formulation of a theoretically viable solution to the localisation problem. By applying non-linear observability analysis onto representations of localisation systems, the minimum spatial information required to devise an observable localisation system can be determined.

In this chapter, the observability properties of Feature-based localisation, of Path constrained localisation, and of the CML system were analysed. The results are summarised in the followings:

A feature-based localisation approach for a 2D vehicle process model

- *is unobservable when it perceives only ranges and bearings of a known feature point with respect to the moving vehicle.*
- *is locally weakly observable when it perceives ranges and bearings of two distinct known feature points with respect to the moving vehicle.*
- *is also locally weakly observable when it consecutively perceives ranges and bearings of a distinct known feature point with respect to the moving vehicle.*

The path constrained localisation approach for a 2D vehicle process model

- *is unobservable when it perceives only lateral offsets of the vehicle position based on single road segment.*
- *is locally weakly observable when it perceives consecutive lateral offsets of the vehicle position based on road segment with different orientation.*

The typical CML approach with the 2D vehicle and feature process model

- *is unobservable when it perceives only ranges and bearings of unknown feature points with respect to the moving vehicle.*
- *is unobservable when it perceives ranges and bearings of unknown feature points plus a known feature point with respect to the moving vehicle.*

-
- *is locally weakly observable when it perceives ranges and bearings of unknown feature points plus two distinct known feature points with respect to the moving vehicle.*
 - *is locally weakly observable when it perceives ranges and bearings of unknown feature points plus consecutive ranges and bearings of a distinct known feature point with respect to the moving vehicle.*

In this chapter, the Feature-based, Path Constrained localisation and CML formulations have been theoretically demonstrated to require a minimum amount of *a priori* spatial information in order to be observed by the system and to become observable. In the next chapter, the CML system formulated with path constraints will be studied for its observability properties. Chapter 4 will also demonstrate that the proposed Path Constraint CML approach is more feasible for large-scale localisation problems than any of the approaches mentioned in this chapter.

Chapter 4

Path Constrained Concurrent Mapping and Localisation in Neighbourhood Environments

4.1 Introduction

An approach is proposed in this chapter to solve the localisation problem presented in the quest to navigate a vehicle through roads in neighbourhood-like environments. The proposed approach is guaranteed to be observable and follows from the theoretical analyses and demonstrations presented in Chapter 3.

The problem of localisation is formulated as a Path Constrained Concurrent Mapping and Localisation (CML) problem, where path constraints are derived from a hybrid metric and topological road map representation. The road map provides valuable metric information in the form of the length, orientation and width of the road segments. Unlike conventional CML, the approach adopted here relies on two types of maps: an *a priori* hybrid approximate road map and a stochastic feature map that is built on the fly from data acquired by the vehicle's onboard sensors. The CML approach allows for the use of such exteroceptive sensors to build an incremental map of features of a neighbourhood environment involving as tree trunks, lampposts, building corners and edges as well as other natural and artificial landmarks when these are within the sensor's field of view. Using these features, an incremental feature map can be constructed of the environment in which the vehicle operates.

The chapter is organised as follows: Section 4.2 presents the system model used in the proposed path constrained CML approach. Section 4.3 analyses the system's observability properties using the non-linear observability test. Section 4.4 establishes the Bayesian formulation of the approach. Section 4.5 presents simulation results that verify the obtained observability and proves the proposed approach's viability for use in solving the large-scale localisation problem. Section 4.6 concludes the chapter and provides a summary of its key contributions.

4.2 Problem Definition

As outlined above, the state estimation theoretic CML framework is extended to incorporate path constraints derived on the fly through road map matching. This Path Constrained CML system is described using the process model defined by equation (3.38) in Section 3.6,

$$\frac{d}{d\rho} \begin{bmatrix} x \\ y \\ \psi \\ x_1 \\ y_1 \\ \vdots \\ x_k \\ y_k \\ \vdots \\ x_K \\ y_K \end{bmatrix} = \begin{bmatrix} \cos \psi \\ \sin \psi \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -a \sin \psi \\ a \cos \psi \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \delta \quad (4.1)$$

and the observation model is the combination of the observation models of unknown features as defined by equation (3.39) and the path constraints as defined by equation (3.20). This new observation model is evaluated as follows:

$$\mathbf{Z} = \begin{bmatrix} \hat{r}_k \\ \hat{\theta}_k \\ \hat{\ell}_i \end{bmatrix} = \mathbf{H}(\mathbf{X}, \Gamma_i) = \begin{bmatrix} h_1(\mathbf{X}) \\ h_2(\mathbf{X}) \\ h_3(\mathbf{x}_r, \Gamma_i) \end{bmatrix} = \begin{bmatrix} \sqrt{(x_k - x)^2 + (y_k - y)^2} \\ \tan^{-1}\left(\frac{y_k - y}{x_k - x}\right) - \psi + \frac{\pi}{2} \\ x \sin \gamma_i - y \cos \gamma_i \end{bmatrix} \quad (4.2)$$

where $\begin{bmatrix} \hat{r}_k & \hat{\theta}_k \end{bmatrix}^T$ is the expected sensor measurement of the k^{th} feature in the state vector with range $\hat{r}$ and bearing $\hat{\theta}$ with respect to the sensor reference frame. $\hat{\ell}_i$ is the expected observation output of ℓ_i , which is the distance from the vehicle's predicted position to the straight line parallel to the corresponding road segment i passing through the origin (See section 3.4). A basic assumption is made that the vehicle is always travelling on the road (this principle has been described in Section 3.4).

4.3 Observability of Path Constrained CML

To analyse the observability properties of the proposed Path Constrained CML approach, the process model (4.1) and the observation model (4.2) are considered for one unknown feature $k=1$ and a road segment $i=1$. The repeated Lie derivatives of the unknown feature observation can be obtained using equations (3.40), (3.41), (3.42) and (3.43) (See Section 3.6). A path constraint of $i=1$, results in:

$$\begin{aligned} L_{\mathbf{F}}^0 dh_3(\mathbf{x}_r(t_0), \Gamma_1) &= \frac{\partial h_3(\mathbf{x}_r(t_0), \Gamma_1)}{\partial \mathbf{X}(t_0)} = \begin{bmatrix} \mathbf{C}_1^{01} & \mathbf{C}_1^{02} & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \sin \gamma_1 & -\cos \gamma_1 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (4.3)$$

$$\begin{aligned}
L_{\mathbf{F}}^D dh_3(\mathbf{x}_r, \Gamma_1) &= L_{\mathbf{F}}^{D-1} dh_3(\mathbf{x}_r, \Gamma_1) \frac{\partial \mathbf{F}(\mathbf{X}, \mathbf{u})}{\partial \mathbf{X}} + \left[\frac{\partial}{\partial \mathbf{X}} \left(L_{\mathbf{F}}^{D-1} dh_3(\mathbf{x}_r, \Gamma_1) \right)^T \mathbf{F}(\mathbf{X}, \mathbf{u}) \right]^T \\
&= \begin{bmatrix} 0 & 0 & \mathbf{C}_1^{D3} & 0 & 0 \end{bmatrix} \\
D &= 1, 2, 3, 4
\end{aligned} \tag{4.4}$$

where $\begin{bmatrix} \mathbf{C}_1^{01} & \mathbf{C}_1^{02} & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & \mathbf{C}_1^{D3} & 0 & 0 \end{bmatrix}$ are the corresponding Lie derivatives of $h_3(\mathbf{x}_r, \Gamma_1)$. The first digit of the superscript indicates the order of the Lie derivatives whilst the second digit indicates the component of the Lie derivatives (i.e. 01 indicates the zero order Lie derivative of the first component). The subscript denotes the index of the parameter value Γ_i used (i.e. $i=1$). By combining these elements, one possible observability matrix, $O_{1,2,3}$, can be constructed as follows,

$$O_{1,2,3} = \begin{bmatrix} L_{\mathbf{F}}^0 dh_1(\mathbf{X}) \\ L_{\mathbf{F}}^0 dh_2(\mathbf{X}) \\ L_{\mathbf{F}}^0 dh_3(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{F}}^1 dh_3(\mathbf{x}_r, \Gamma_1) \\ L_{\mathbf{F}}^2 dh_3(\mathbf{x}_r, \Gamma_1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{01} & \mathbf{A}^{02} & \mathbf{A}^{03} & -\mathbf{A}^{01} & -\mathbf{A}^{02} \\ \mathbf{B}^{01} & \mathbf{B}^{02} & \mathbf{B}^{03} & -\mathbf{B}^{01} & -\mathbf{B}^{02} \\ \mathbf{C}_1^{01} & \mathbf{C}_1^{02} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{C}_1^{13} & 0 & 0 \\ 0 & 0 & \mathbf{C}_1^{23} & 0 & 0 \end{bmatrix} \tag{4.5}$$

where

$$\begin{aligned}
\mathbf{C}_1^{13} &= -\sin \gamma_1 (\sin \psi + a\delta \cos \psi) - \cos \gamma_1 (\cos \psi - a\delta \sin \psi) \\
\mathbf{C}_1^{23} &= -\delta \sin \gamma_1 (\cos \psi - a\delta \sin \psi) + \delta \cos \gamma_1 (\sin \psi + a\delta \cos \psi)
\end{aligned} \tag{4.6}$$

It is noted that the second and the remaining higher order Lie derivatives of the path constraint observation model can be expressed in terms of the first order Lie derivative. Any combination of the Lie derivatives used in the formation of the observability matrix can thus be shown to be rank deficient (rank of 4). Clearly then, the CML is unobservable when just one road segment is observed.

*The Path Constrained CML approach is **unobservable** when used with the process model that describes both the evolution of 2D vehicle pose and feature position over time and the observation that describes **ranges and bearings of unknown feature points** with respect to the moving vehicle **plus** lateral offsets of the vehicle position based on **single road segment**.*

The analysis is extended to a CML system, which sequentially observes different road segments. Using the same observation function $\mathbf{H}(\mathbf{X}, \Gamma_i)$ of equation (4.2), the system observes one unknown feature $\mathbf{m}_1 = [x_1 \ y_1]^T$ and the road segment, $\Gamma_1 = [\gamma_1]$ on the first path interval before observing the second road segment $\Gamma_2 = [\gamma_2]$ on the second path interval. In this case, the observation model is expressed in the form of

$$\mathbf{Z} = \begin{cases} \begin{bmatrix} \hat{r}_1 \\ \hat{\theta}_1 \\ \hat{\ell}_1 \end{bmatrix} = \begin{bmatrix} h_1(\mathbf{X}(t_0)) \\ h_2(\mathbf{X}(t_0)) \\ h_3(\mathbf{x}_r(t_0), \Gamma_1) \end{bmatrix} & \text{for } t_0 \leq t < t_1 \\ \begin{bmatrix} \hat{r}_1 \\ \hat{\theta}_1 \\ \hat{\ell}_2 \end{bmatrix} = \begin{bmatrix} h_1(\mathbf{X}(t_1)) \\ h_2(\mathbf{X}(t_1)) \\ h_3(\mathbf{x}_r(t_1), \Gamma_2) \end{bmatrix} & \text{for } t_1 \leq t < t_2 \end{cases} \quad (4.7)$$

where $h_1(\mathbf{X}(t_0))$ and $h_2(\mathbf{X}(t_0))$ are the observation functions for the unknown feature. $h_3(\mathbf{x}_r(t_0), \Gamma_1)$ and $h_3(\mathbf{x}_r(t_1), \Gamma_2)$ are the observation functions for the road segments $\Gamma_1 = [\gamma_1]$ from time t_0 to t_1 and $\Gamma_2 = [\gamma_2]$ from time t_1 to t_2 respectively. Using the Lie derivatives, the observability, $O_{1,2,3}$, can be derived as:

$$O_{1,2,3} = \begin{bmatrix} L_F^0 dh_1(\mathbf{X}) \\ L_F^0 dh_2(\mathbf{X}) \\ L_F^0 dh_3(\mathbf{x}_r(t_0), \Gamma_1) \\ L_F^1 dh_3(\mathbf{x}_r(t_0), \Gamma_1) \\ L_F^0 dh_3(\mathbf{x}_r(t_1), \Gamma_2) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{01} & \mathbf{A}^{02} & \mathbf{A}^{03} & -\mathbf{A}^{01} & -\mathbf{A}^{02} \\ \mathbf{B}^{01} & \mathbf{B}^{02} & \mathbf{B}^{03} & -\mathbf{B}^{01} & -\mathbf{B}^{02} \\ \mathbf{C}_1^{01} & \mathbf{C}_1^{02} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{C}_1^{13} & 0 & 0 \\ \mathbf{C}_2^{01} & \mathbf{C}_2^{02} & 0 & 0 & 0 \end{bmatrix} \quad (4.8)$$

where $[\mathbf{A}^{01} \ \mathbf{A}^{02} \ \mathbf{A}^{03} \ -\mathbf{A}^{01} \ -\mathbf{A}^{02}]$ and $[\mathbf{B}^{01} \ \mathbf{B}^{02} \ \mathbf{B}^{03} \ -\mathbf{B}^{01} \ -\mathbf{B}^{02}]$ are the zero order Lie derivatives for the unknown feature. $[\mathbf{C}_1^{01} \ \mathbf{C}_1^{02} \ 0 \ 0 \ 0]$ and $[0 \ 0 \ \mathbf{C}_1^{13} \ 0 \ 0]$ are the Lie derivatives for the path constraint provided by the observation model with $\Gamma_1 = [\gamma_1]$ and $[\mathbf{C}_2^{01} \ \mathbf{C}_2^{02} \ 0 \ 0 \ 0]$ is the Lie derivative for the second path constraint provided by the observation model with $\Gamma_2 = [\gamma_2]$. It is now easy to verify if the observability can achieve rank (equal to 5) when the following conditions are satisfied:

$$\sin \gamma_1 \neq \sin \gamma_2 \quad (4.9)$$

Physically, what this condition means is that the road segments are of different orientations.

In general, it can be shown that CML is locally weakly observable when a different road segment is observed consecutively together with an arbitrary number of unknown features.

*The Path Constrained CML approach is **locally weakly observable** when used with the 2D vehicle and feature process model and the observation model that describes **ranges and bearings of unknown feature points** with respect to the moving vehicle **plus consecutive lateral offsets of vehicle position based on road segment with different orientation**.*

In the next section, we show that this observable path constrained CML approach can be effectively solved using a Bayesian formulation.

4.4 Bayesian Solution of Path Constrained CML

The Bayesian formulation of the CML problem has been established in [76] and in this section, the CML formulation is extended to include path constraints provided by the road maps. Within the Bayes estimation theoretical framework, the aim is to obtain the posterior probability distribution $p(\mathbf{X}_t, \mathbf{M} | \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)$ over the vehicle path $\mathbf{X}_t = \{\mathbf{x}_t(0), \mathbf{x}_t(1), \dots, \mathbf{x}_t(t)\}$ and set of features $\mathbf{M} = \{\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k\}$. This effort is conditioned by all the corresponding measurements $\mathbf{Z}_t = \{\mathbf{z}(0), \mathbf{z}(1), \dots, \mathbf{z}(t)\}$, control inputs $\mathbf{U}_t = \{\mathbf{u}(0), \mathbf{u}(1), \dots, \mathbf{u}(t)\}$ and *a priori* map information $\mathbf{L}_t = \{\ell_i(0), \ell_i(1), \dots, \ell_i(t)\}$ with a correspondence index sequence $\mathbf{I}_t = \{i(0), i(1), \dots, i(t)\}$ up to time t . The recursive solution for Path Constrained CML can be derived based on Bayes' theorem with the Markov assumption. Under the conditional independence property similar to that stipulated in FastSLAM [70]:

$$p(\mathbf{X}_t, \mathbf{M} | \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t) \quad (4.10)$$

$$= \underbrace{p(\mathbf{X}_t | \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)}_{\text{Vehicle path estimator}} \underbrace{\prod_k p(\mathbf{m}_k | \mathbf{X}_t, \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)}_{\text{Feature estimators}} \quad (4.11)$$

$$= \underbrace{\eta p(\ell_i(t) | \mathbf{X}_t, i(t)) p(\mathbf{X}_t | \mathbf{Z}_{t-1}, \mathbf{U}_t, \mathbf{L}_{t-1}, \mathbf{I}_{t-1})}_{\text{Road map constrained Localisation}} \underbrace{\prod_k p(\mathbf{m}_k | \mathbf{X}_t, \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)}_{\text{Feature estimators}} \quad (4.11)$$

Road Constrained CML

It may be noted that each feature $\mathbf{m}_k$ is characterised by its (x, y) location coordinates as defined by equation (3.36) in Section 3.4.1. $\mathbf{z}(t)$ is the bearing and range of the

corresponding feature with respect to the vehicle reference frame. $\ell_i(t)$ is the measurement of the path constraint provided by the road map, as described by equation (3.19) in Section 3.4. $\mathbf{u}(t)$ is the input to the vehicle, which can be measured by dead reckoning sensors and η is a normalising constant factor.

Equation (4.11) can be used with measurements from any sensors together with *a priori* information about the vehicle to update the predicted posterior probability distribution. This is done by multiplying the measurements with the likelihood function value of the sensors and the *a priori* information available at the predicted vehicle pose. If multiple measurements or measurements from multiple sensors and *a priori* information sources are available, the update can be performed simultaneously as a group or sequentially by applying equation (4.11) iteratively.

Using the estimates and corresponding distributions of the vehicle's pose and the feature map at time $t-1$ (as shown in Figure 4.1), the posterior probability distribution can be represented as $p(\mathbf{X}_{t-1}, \mathbf{M} | \mathbf{Z}_{t-1}, \mathbf{U}_{t-1}, \mathbf{L}_{t-1}, \mathbf{I}_{t-1})$ in equation (4.11) at time $t-1$. In Figure 4.2, the vehicle is shown to have moved as a result of the control input $\mathbf{u}(t)$. The estimate and distribution of the vehicle's next pose can then be predicted by using the vehicle motion model and the control input. The motion model is calculated according to the vehicle's kinematics as defined by equation (3.2) in Section 3.2. This is the prediction stage represented by the Chapman-Kolmogorov equation $p(\mathbf{X}_t | \mathbf{Z}_{t-1}, \mathbf{U}_t, \mathbf{L}_{t-1}, \mathbf{I}_{t-1})$ as defined in equation (4.11).

Figure 4.3 shows that the vehicle takes a measurement $\ell_1(t)$ at the new location $\mathbf{x}_r(t)$ and associates this with the nearest road segment $i(t)=1$. This is the data

association stage of the path constrained localisation problem. The *Mahalanobis distance* method described in Section 3.4 is the technique used to determine whether the measurement $\ell_i(t)$ originates from $i(t)$. Besides the measurement $\ell_1(t)$, the vehicle also makes new measurements $\mathbf{z}_1(t)$ and $\mathbf{z}_2(t)$ at the location $\mathbf{x}_r(t)$ and associates them with the feature maps $\mathbf{m}_1$ and $\mathbf{m}_2$. This describes the data association part of the CML problem.

Figure 4.4 shows the update stage that is represented by the $p(\ell_1(t) | \mathbf{X}_t, i(t)=1)$, $p(\mathbf{m}_1 | \mathbf{X}_t, \mathbf{Z}_t)$, $p(\mathbf{m}_2 | \mathbf{X}_t, \mathbf{Z}_t)$ and $p(\mathbf{X}_t | \mathbf{Z}_{t-1}, \mathbf{U}_t, \mathbf{L}_{t-1}, \mathbf{I}_{t-1})$ components of equation (4.11). At this stage, the uncertainty regarding the states of the vehicle and the environmental features will be reduced to give a compact probability distribution. In Figure 4.4, the estimates of the vehicle pose and the feature map and their corresponding distributions at time t , $p(\mathbf{X}_t, \mathbf{M} | \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)$ are updated. The state of the vehicle along with the path constraints and the updates of the feature estimates will help bound the initial predictions of the vehicle's pose and provide a more compact probability distribution.

Using the Bayesian framework formulation defined in equation (4.11), it becomes clear that an appropriate estimation technique can indeed be chosen to effectively solve the localisation problem in large-scale neighbourhood environments. A simulation using an EKF algorithm is carried out to verify the Path Constrained CML approach's observability and viability, and the results are described in the following section.

Figure 4.1 Vehicle pose and corresponding distribution at time $t-1$

Figure 4.2 Representation of the uncertainties the vehicle faces as it moves

Figure 4.3 Vehicle pose associated with the nearest road segment at time t

Figure 4.4 Depiction of vehicle sensing features and associating them at time t Figure 4.5 Updated vehicle pose and feature map together with the associated uncertainties at time t

4.5 Simulation Results of Path Constrained CML

Simplified simulation tests are run both with and without path constraints using the parameters summarised in Table 4.1. The simulation data for the vehicle path is generated using the kinematics model defined by equation (3.2) in Section 3.2. The results of the simulation are shown in Figure 4.6.

Under the simulation, the vehicle starts at the origin and travels in a straight line with an orientation angle of zero degrees. When the vehicle reaches the 50m mark, it changes its orientation counter clockwise by 10° and travels in a straight line for a further 50m. The data generated as a result of the simulation includes the vehicle's speeds and yaw angles. In addition, data is also generated regarding the range and bearing of a feature located at (120,5). Two road segments are defined – the first from 0 to 50m with an orientation of 0° and the second from 50 to 100m with an orientation of 10° . These speed and heading related data are corrupted with noise using the uncertainties included in Table 4.1. The vehicle is also made to start with an initial pose error in order to make it

Parameters		Values
Initial offset	x	$-2m$
	y	$-2m$
	ψ	2°
Vehicle's wheel base	L	$1.6m$
Distance between LADAR and rear axle	a	$1.8m$
Velocity	v	$3m/s$
Control noise	Velocity, σ_v	$2m/s$
	Steer, σ_α	2°
Measurement noise	Range, σ_r	$0.5m$
	Bearing, σ_θ	1°
	Path constraint, σ_ℓ	$2m$

Table 4.1 Parameters used in the simulation for Path constrained CML approach

apparent if the approach is capable of converging the estimates to the truth vehicle pose. The road segment on which the vehicle is travelling is to be identified using the *Mahalanobis distance* method described in Section 3.3.1. Using the observation model of

equation (4.2) and the process model (4.1), an EKF algorithm described in Section 2.3.2 estimates the vehicle pose and the position of the unknown feature. The results of the simulation are shown in Figures 4.6, 4.7 and 4.8.

Figure 4.6 A CML observation scene with and without constraints using *a priori* known spatial vehicle path.

The simulation results obtained when working with and without path constraints are shown in Figures 4.7 and 4.8. When there are no path constraints, the results indicate that, all errors have offsets throughout their vehicle trajectory. This demonstrates again that typical CML estimations cannot converge to the true state. In the case of path constrained CML systems, although they have a mean heading error of zero, their absolute position error continues to have a bias in the first path interval (a result of the application of the observation model (4.2)). This bias is corrected when the vehicle traverses to the second path. The correction of the absolute position error that occurs when the vehicle moves from one segment to another is a result of the application of the observation model (4.7) – a model that guarantees observability.

Figure 4.7 The effects of absolute vehicle position errors on the CML with and without path constraint

Figure 4.8 The effects of vehicle heading errors as applied to the CML with and without path constraint

The analysis in Section 4.3 and the results of the simulation of this scenario verify that a full rank observability matrix can be obtained, if a different path constraint is required along with measurements of a path's sequence of constraint observations.

By comparing the absolute position error results obtained for the Path Constrained localisation and the Path Constrained CML system shown in Figure 4.9, it can be observed that the error for the CML estimations is bounded for each path interval. In contrast, error for the Path Constrained localisation system is cumulative. This shows that

Figure 4.9 Vehicle absolute position error results for Path Constrained localisation and Path Constrained CML systems.

with the incorporation of relative observations of unknown features, the system is unable to correct the errors but is able to prevent them from accumulating. Thus, an improvement can be inferred to have taken place in the estimation and this quantifiably demonstrates that the path constrained CML system suffers fewer ambiguities than the

Path Constrained localisation system in its association of the spatial information (road segments) observed by the system. This will be significant when a simple data association method such as *Mahalanobis distance* is used to find the right road segment that the vehicle is plying on.

In addition, the simulation results demonstrated above appear to converge with the theoretical understandings of the link between system observability and error. The Path Constrained CML approach is therefore theoretically proven to be more viable for solving large-scale localisation problems.

4.6 Summary and Conclusions

A general framework for constrained CML has been proposed in this chapter that is based on the standard CML approach. It uses *a priori* information in the form of a digital road map represented by nodes and links to circumvent the observability issues encountered in the standard CML approach. The central belief is that knowledge of the vehicle's presence on a known road segment can be used to bound the vehicle's pose estimates and their refinements through simultaneously mapping and tracking unknown features.

A Bayesian formulation of the proposed approach, Path Constrained CML, has been established to solve the estimation problem. Simulations were made using an EKF algorithm to verify system observability and the viability of this novel approach. By using *a priori* known spatial information (a road map) that represents attributes of accessible areas, observations can be derived of the vehicle's path constraint. This enables the convergence of CML estimations. The simulation results show that the road

segment on which the vehicle is operating assist in constraining the lateral and orientation errors, and the longitudinal errors when the vehicle turns into another road segment. Along straight roads however, it has been demonstrated that path constraints are insufficient to prevent the vehicle from accumulating longitudinal position errors. The CML framework used here helped bound this accumulation of longitudinal error. Judging from the results of the simulation, the effort was justified. The bounded errors allow for the effective implementation of the *Mahalanobis distance* method, which matched the expected vehicle path with *a priori* road information stored on a database (road map matching). This approach works only when the vehicle is in a road segment. Techniques to determine road boundaries will be required if vehicles are to be used in actual autonomous operations.

An important characteristic of the proposed framework is that it paves the way for improved vehicle localisation accuracy by fusing of *a priori* information available in different forms with data gathered from the vehicle onboard sensors. In the next chapter, results of extensive experiments are presented to verify the feasibility of this proposed Path Constrained CML approach.

Chapter 5

Implementation of Path Constrained CML and Analysis of Field Experimental Results

5.1 Introduction

Several novel techniques have been presented in the earlier chapters to improve the performance of the CML algorithm. What remains to be seen, however, is how far these techniques can be adopted in a real world setting. One of the goals of this thesis is to evaluate the proposed approach using an actual vehicle operating in a neighbourhood that realistically represents the environment and conditions in which the system may eventually be deployed.

As of now, enabling a vehicle navigation system to adequately interpret its environment remains a daunting prospect. Because of the vast distances between objects and the limitations of the sensors, the data yield is often of poor quality. GPS has often been heralded as the solution, but while many ground vehicles have indeed used it to provide accurate information on their changing positions, the semi-urban canopy of obstructing buildings and trees prevents GPS signals from being received effectively in most road networks. The creation of a more workable approach demands the further development of some of engineering's most advanced technologies.

Among the alternative approaches, the use of *a priori* spatial road maps has been found to be very promising in aiding navigation in neighbourhood environments. The

maps are used in this study to constrain the CML algorithm by providing information for both absolute localisation and for building feature maps on the fly. This chapter presents the results of an application of the observable Path Constrained CML approach described earlier in this thesis.

Section 5.2 begins by showing the manner in which the proposed Path Constrained CML approach is implemented using the adaptive Rao-Blackwellised particle filter. Section 5.3 describes the experimental setup and the environment in which the vehicle is deployed. A brief description is also provided of the vehicle and the sensors used. Section 5.5 presents a report of the trials performed on the campus road including an analysis of the data collected during the deployment. The approaches of previous researchers are tested in comparison with the Path Constrained CML system and the results are presented before concluding remarks are made. Section 5.6 summarises the chapter.

5.2 An Adaptive FastSLAM Algorithm

A Rao-Blackwellised particle filter [68], [70] is used to implement the proposed Path Constrained CML and to address the non-linearity and computational complexity of the CML framework. With such a filter, the posterior probability distribution of the vehicle's path is represented by a sample-based distribution. The posterior probability distribution of the feature map is represented by a close-form solution that is conditioned on the aforementioned vehicle path. Under the appropriate conditioning, cross-correlations should not be maintained between the features. The vehicle path's posterior is estimated using a particle filter, and the remaining k conditional feature posteriors are estimated using extended Kalman Filters (EKF). Each EKF tracks a single feature

position, conditioned on the vehicle path which is represented by one of the particles in the particle filter. Individual particles have their own set of EKF. This algorithm is known as the FastSLAM approach [70].

For the purposes of this thesis, the FastSLAM implemented with the Path Constrained CML is extended to incorporate an adaptive particle scheme proposed by Fox *et. al.* [71]. This makes the computation of the vehicle's posterior probability distribution more efficient and is a crucial step that allows posteriors of the conditioned feature to be correctly estimated (See Section 2.3.2.) Computing the posterior probability distribution involves four basic steps as represented by equation (4.11) in Section 4.5. :

1) *Posterior Path Sampling*: For reasons described in equation (4.11) of Section 4.4 the vehicle path estimator $p(\mathbf{X}_t | \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)$ employs an adaptive particle filter. Let $\mathcal{S}_t$ represent the particle set for the posterior $p(\mathbf{X}_t | \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)$ at time t .

$$\begin{aligned} \mathcal{S}_t &= \{ \mathbf{X}_t^{\{1\}}, \dots, \mathbf{X}_t^{\{m\}}, \dots \} \\ &= \left\{ \left\langle \mathbf{x}_r^{\{1\}}(0), \mathbf{x}_r^{\{1\}}(1), \dots, \mathbf{x}_r^{\{1\}}(t) \right\rangle, \dots, \left\langle \mathbf{x}_r^{\{m\}}(0), \mathbf{x}_r^{\{m\}}(1), \dots, \mathbf{x}_r^{\{m\}}(t) \right\rangle, \dots \right\} \end{aligned} \quad (5.1)$$

where, m is used to refer to the m^{th} particle in the set. Each particle $\mathbf{x}_t^{\{m\}} \in \mathcal{S}_t$ thus represents a possible vehicle path. The particle set $\mathcal{S}_t$ is calculated from the set $\mathcal{S}_{t-1}$ at time $t-1$, the vehicle dead reckoning sensors measurement is denoted by $\mathbf{u}(t)$, the range and bearing of feature measurements are indicated by $\mathbf{Z}_t$ and road map information is represented by $\mathbf{L}_t$ with index sequence $\mathbf{I}_t$. First, each particle $\mathbf{x}_{t-1}^{\{m\}}$ in $\mathcal{S}_{t-1}$ is used to generate the vehicle pose at time t using the vehicle motion model as defined by equation (3.2) in Section 3.2. Since the model and the inputs are inherently uncertain and noisy,

Gaussian noise is added to the expected motion, creating the probabilistic model that describes a posterior probability distribution over possible successors $\mathbf{x}_r^{\{m\}}(t)$,

$$\mathbf{x}_r^{\{m\}}(t) \sim p(\mathbf{x}_r(t) | \mathbf{x}_r^{\{m\}}(t-1), \mathbf{u}(t)) \quad (5.2)$$

The set of independent identically distributed (i.i.d.) particles in $\mathcal{S}_{t-1}$ is distributed according to $p(\mathbf{X}_{t-1} | \mathbf{Z}_{t-1}, \mathbf{U}_{t-1}, \mathbf{L}_{t-1}, \mathbf{I}_{t-1})$ and the new (i.i.d.) particle is distributed according to $p(\mathbf{X}_t | \mathbf{Z}_{t-1}, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)$. This is commonly referred to as the proposal distribution [65] in particle filtering. Using a fixed number of samples to approximate the probability distribution $p(\mathbf{X}_t | \mathbf{Z}_{t-1}, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)$ can cause sample deficiencies when there is a large degree of uncertainty in the state estimation process. Even when the uncertainty is reduced, the fixed number of samples may give rise to an unnecessarily heavy computational burden. Thus, an adaptable sample size scheme is essential to effectively represent the probability distribution $p(\mathbf{X}_t | \mathbf{Z}_{t-1}, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)$.

2) *Sample Size Adaptation:* The sample-based representation of the posterior probability distribution requires a sufficient number of samples to be effective and efficient at every instant. This is an important requirement in the fulfilment of the independence property of the CML problem as it is exploited in the FastSLAM. Using more than the required number of samples to represent the posterior probability distribution would increase the computational burden. Hence, the particle sample size is adapted in a manner similar to that in [71]. If the density is focused on a small part of the state space, the scheme chooses a small number of samples. If the state uncertainty is high, it chooses a larger number of samples. An approximated error is devised to postulate the desired sample

density. This approximation error represents the distance (measured by the Kullback-Leibler distance, KL-distance) between the resultant sample distribution $\hat{p}$ of the estimates and the desired sample density q . The sample density q is in turn represented by a discrete, piecewise constant distribution such as a discrete density tree. With a pre-specified probability β and a threshold error ε , the number of samples necessary to represent $\hat{p}$ is such that the KL-distance or error between $\hat{p}$ and q is less than ε with a probability of β .

Adapting the sample size as such ensures that there is a sufficient number of particles representing the true posterior of the vehicle path at all times. The deficiencies in the sample of the given localisation framework are resolved and the computational efficiency of the particle filter is improved. If the uncertainty of the estimates is always increasing however, the adaptation scheme can become highly computationally intensive. This is because a large sample size is required to maintain the desired density of the posterior distribution. Observation measurement updates are important because they bound and reduce the incremental errors brought over from previous estimates. The feature estimation updates are discussed in the next section.

3) Feature Estimation Updates: The conditional feature estimates $p(\mathbf{m}_k | \mathbf{X}_t, \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)$ described in equation (4.11) in Section 4.4 are calculated using Extended Kalman Filter (EKF) updates. This is done in a manner similar to that in a typical FastSLAM algorithm. As mentioned earlier, the estimate is conditioned on the vehicle path; hence each particle representing the possible paths is attached with the number of observed features

$\mathbf{M}^{[m]} = \{\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k\}^{[m]}$. According to Bayes theorem and the Markov assumption, the posterior can be derived and expressed as:

$$\begin{aligned} p(\mathbf{m}_k^{[m]} | \mathbf{X}_t, \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t) &\propto \\ p(\mathbf{z}(t) | \mathbf{m}_k^{[m]}, \mathbf{x}_r(t)) &\sim N(\mathbf{z}(t); h_{1,2}(\mathbf{x}_r(t), \mathbf{m}_k^{[m]}) + J_\theta(\mathbf{m}_k^{[m]} - \mu_k^{[m]}(t-1)), \mathbf{V}) \\ \times p(\mathbf{m}_k^{[m]} | \mathbf{X}_{t-1}, \mathbf{Z}_{t-1}) &\sim N(\mathbf{m}_k^{[m]}; \mu_k^{[m]}(t-1), \Omega_k^{[m]}(t-1)) \end{aligned} \quad (5.3)$$

$\mathbf{V}$ is the measurement noise covariance and J_θ is the Jacobian measurement matrix of function $h_{1,2}(\mathbf{x}_r(t), \mathbf{m}_k^{[m]})$ for the range and bearing defined by equation (4.2) in Section 4.2. The observation estimation is approximated as a linear Gaussian model and each feature processes its EKF update. The update of the measurements of the k^{th} feature $\mathbf{m}_k^{[m]}$ is represented by the mean $\mu_k^{[m]}(t)$, feature location $\in R^2$ and covariance $\Omega_k^{[m]}(t)$ of the Gaussian distribution. The following are the equations used in the update:

$$\mathbf{S}_k^{[m]}(t) = \mathbf{V} + J_\theta \Omega_k^{[m]}(t-1) J_\theta^T \quad (5.4)$$

$$\mathbf{K}_k^{[m]}(t) = \Omega_k^{[m]}(t-1) J_\theta^T \mathbf{S}_k^{[m]}(t) \quad (5.5)$$

$$\mu_k^{[m]}(t) = \mu_k^{[m]}(t-1) + \mathbf{K}_k^{[m]}(t) (\mathbf{z}_k(t) - h_{1,2}(\mathbf{x}_r, \mathbf{m}_k^{[m]})) \quad (5.6)$$

$$\Omega_k^{[m]}(t) = \Omega_k^{[m]}(t-1) - \mathbf{K}_k^{[m]}(t) J_\theta \Omega_k^{[m]}(t-1) \quad (5.7)$$

where $\mathbf{S}_k^{[m]}(t)$ is the innovation covariance matrix, and $\mathbf{K}_k^{[m]}(t)$ is the Kalman gain of the feature estimate conditioned on a particular vehicle path which in this case, is particle $\mathbf{X}_t^{[m]}$. In this implementation, the association of the feature measurements is carried out using the per-particle Maximum Likelihood method as described in [70].

When the features are conditioned on a particular vehicle path that does not receive any observation measurement $\mathbf{z}(t)$, their estimates are not updated and they

remain unchanged. To initialise a new feature, a single measurement is sufficient if the observation function $h_{1,2}(\mathbf{x}_r, \mathbf{m}_k^{\{m\}})$ is invertible. In this implementation, $h_{1,2}(\mathbf{x}_r, \mathbf{m}_k^{\{m\}})$ is an invertible observation and so a new feature is created with a single measurement in the following manner:

$$\mu_k^{\{m\}}(t) = h_{1,2}^{-1}(\mathbf{x}_r^{\{m\}}, \mathbf{z}^{\{m\}}(t)) \quad (5.8)$$

$$\Omega_k^{\{m\}}(t) = (J_\theta^T \mathbf{V}^{-1} J_\theta)^{-1} \quad (5.9)$$

4) *Importance sampling*: Samples drawn from the system model $p(\mathbf{X}_t, \mathbf{M} | \mathbf{Z}_{t-1}, \mathbf{U}_t, \mathbf{L}_{t-1}, \mathbf{I}_t)$ do not match the desired posterior $p(\mathbf{X}_t, \mathbf{M} | \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)$. To correct this difference, a process called importance sampling is used to compute the importance weights or probabilities of the particles. This computation is influenced by both the observation of features and the constraints provided by the observation of the road segment on which the vehicle is travelling. Thus the importance weights $\omega_t^{\{m\}}$ are calculated as follows:

$$\omega_t^{\{m\}} = \frac{\text{target distribution}}{\text{proposal distribution}} = \frac{p(\mathbf{X}_t^{\{m\}}, \mathbf{M}^{\{m\}} | \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)}{p(\mathbf{X}_t^{\{m\}}, \mathbf{M}^{\{m\}} | \mathbf{Z}_{t-1}, \mathbf{U}_t, \mathbf{L}_{t-1}, \mathbf{I}_t)} \quad (5.10)$$

$$\begin{aligned} \omega_t^{\{m\}} &\propto p(\mathbf{z}(t), \ell_i(t) | \mathbf{X}_t^{\{m\}}, \mathbf{M}^{\{m\}}, \mathbf{Z}_{t-1}, \mathbf{U}_t, \mathbf{L}_{t-1}, \mathbf{I}_t) \\ &\propto \int p(\mathbf{z}(t), \ell_i(t) | \mathbf{m}_k^{\{m\}}, \mathbf{X}_t^{\{m\}}, \mathbf{I}_t) p(\mathbf{m}_k^{\{m\}}) d\mathbf{m}_k^{\{m\}} \\ &\propto \underbrace{p(\ell_i(t) | \mathbf{X}_t^{\{m\}}, i(t))}_{\text{Road constrained weight}} \times \underbrace{\int p(\mathbf{z}(t) | \mathbf{m}_k^{\{m\}}, \mathbf{X}_t^{\{m\}}) p(\mathbf{m}_k^{\{m\}}) d\mathbf{m}_k^{\{m\}}}_{\text{Feature observed weight}} \end{aligned} \quad (5.11)$$

where the importance weights represent the difference or ratio between the target posterior $p(\mathbf{X}_t, \mathbf{M} | \mathbf{Z}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_t)$ and the proposal distribution $p(\mathbf{X}_t, \mathbf{M} | \mathbf{Z}_{t-1}, \mathbf{U}_t, \mathbf{L}_{t-1}, \mathbf{I}_t)$. By expanding the numerator of equation (5.10) using the Byes rule, the ratio can be reduced

to equation (5.11). The equation represents both the feature's observation likelihood and the path constraint estimates.

Since the observation models for feature observation and path constraint are assumed to be Gaussian distributed, the weights can be written as:

$$\omega_i^{\{m\}} = \underbrace{\frac{1}{\sqrt{2\pi C}} \exp \left\{ -\frac{1}{2} \left(\ell_i(t) - h_3(\mathbf{x}_r^{\{m\}}, \mathbf{\Gamma}_i) \right)^T C^{-1} \left(\ell_i(t) - h_3(\mathbf{x}_r^{\{m\}}, \mathbf{\Gamma}_i) \right) \right\}}_{\text{Road constrained weight}} \times \underbrace{\frac{1}{\sqrt{|2\pi S|}} \exp \left\{ -\frac{1}{2} \left[\left(\mathbf{z}(t) - h_{1,2}(\mathbf{x}_r^{\{m\}}, \mathbf{M}^{\{m\}}) \right)^T S^{-1} \left(\mathbf{z}(t) - h_{1,2}(\mathbf{x}_r^{\{m\}}, \mathbf{M}^{\{m\}}) \right) \right] \right\}}_{\text{Feature observed weight}} \quad (5.12)$$

A re-sampling technique is required to re-draw a new set of samples from this existing set with probabilities in proportion to the importance weights [66]. A systematic sampling algorithm is chosen from the vast variety of re-sampling techniques available, in order to ensure the simplicity, viability and accuracy of the implementation.

The particles with additional feature estimates are expressed in the following form

$$\Lambda_i^{\{m\}} = \left\langle \mathbf{X}_i^{\{m\}}, \mu_1^{\{m\}}(t), \Omega_1^{\{m\}}(t), \dots, \mu_k^{\{m\}}(t), \Omega_k^{\{m\}}(t), \dots \right\rangle \quad (5.13)$$

where each $\Lambda_i^{\{m\}}$ consist of the mean, $\mu_k^{\{m\}}(t)$ and covariance, $\Omega_k^{\{m\}}(t)$ representing each feature location condition on the vehicle path $\mathbf{X}_i^{\{m\}}$. To summarise the algorithm, Figure 5.1 shows the overall structure and implementation of the algorithm proposed for the Path Constrained Adaptive FastSLAM.

Inputs: $\Lambda_{i-1} = \left\{ \dots, \left\langle \mathbf{X}_{i-1}^{\{m\}}, \mu_1^{\{m\}}(t-1), \Omega_1^{\{m\}}(t-1), \dots, \mu_k^{\{m\}}(t-1), \Omega_k^{\{m\}}(t-1), \dots \right\rangle, \dots \right\}$ representing posterior distribution $p(\mathbf{X}_{i-1}, \mathbf{M})$, set bounds ε and β , cell size, minimum number of

samples $n_{\min}$, road segment $G(x_i)$.

- $b = b_{\min}, n = n_{\min}, m = 1$ // Initialise
- **do** // Generate samples
- **draw** $\mathbf{x}_r^{\{m\}}(t) \sim p(\mathbf{x}_r(t) | \mathbf{x}_r^{\{m\}}(t-1), \mathbf{u}(t))$ // Sampling: Predict next state
- **if** ($\mathbf{x}_r^{\{m\}}(t)$ falls into empty cell)
- $b = b + 1$ // Update number of cells with support
- $n = \frac{b-1}{2\varepsilon} \left\{ 1 - \frac{2}{9(b-1)} + \sqrt{\frac{2}{9(b-1)}} Z_\beta \right\}^3$ // Update number of desired samples
- **end if**
- $m = m + 1$ // Update number of generated samples
- **while**($m < n$) // Until KL bound is reached
- **for** $i = 1$ to n // Loop over all particles
- $\hat{i}(t) = \arg \max_{i(t)} p(\ell_i(t) | \mathbf{X}_t, \mathbf{U}_t, \mathbf{L}_t, \mathbf{I}_{t-1}, i(t))$ // Search for closest road segment
- $\ell_i = [\sin \gamma_i \quad -\cos \gamma_i] \begin{bmatrix} x_i \\ y_i \end{bmatrix}$ // Derive the path constraint measurement
- **if** $\mathbf{z}(t)$ associated to a mapped feature
- $\mathbf{S}_k^{\{m\}}(t) = \mathbf{Q} + J_\theta \mathbf{\Omega}_k^{\{m\}}(t-1) J_\theta^T$
- $\mathbf{K}_k^{\{m\}}(t) = \mathbf{\Omega}_k^{\{m\}}(t-1) J_\theta^T \mathbf{S}_k^{\{m\}}(t)$
- $\mu_k^{\{m\}}(t) = \mu_k^{\{m\}}(t-1) + \mathbf{K}_k^{\{m\}}(t) (\mathbf{z}_k(t) - h(\mathbf{x}_r, \mathbf{m}_k^{\{m\}}))$
- $\mathbf{\Omega}_k^{\{m\}}(t) = \mathbf{\Omega}_k^{\{m\}}(t-1) - \mathbf{K}_k^{\{m\}}(t) J_\theta \mathbf{\Omega}_k^{\{m\}}(t-1)$
- $\omega_t^{\{m\}} = \underbrace{\frac{1}{\sqrt{2\pi C}} \exp \left\{ -\frac{1}{2} \left(\ell_i(t) - h_3(\mathbf{x}_r^{\{m\}}, \mathbf{\Gamma}_i) \right)^T C^{-1} \left(\ell_i(t) - h_3(\mathbf{x}_r^{\{m\}}, \mathbf{\Gamma}_i) \right) \right\}}_{\text{Road constrained weight}} \times$
- $\underbrace{\frac{1}{\sqrt{|2\pi \mathbf{S}|}} \exp \left\{ -\frac{1}{2} \left[\left(\mathbf{z}(t) - h_{1,2}(\mathbf{x}_r^{\{m\}}, \mathbf{M}^{\{m\}}) \right)^T \mathbf{S}^{-1} \left(\mathbf{z}(t) - h_{1,2}(\mathbf{x}_r^{\{m\}}, \mathbf{M}^{\{m\}}) \right) \right] \right\}}_{\text{Feature observed weight}}$

Figure 5.1 Algorithm Proposed For The Path Constrained Adaptive FastSLAM

5.3 Experimental Setup

This section describes the site in which the test were conducted and the vehicle and sensors that were used to implement the proposed approach.

5.3.1 Vehicle Platform

The experimental platform used in this thesis was a Carryall 1 golf cart constructed into a mobile robot. We named this platform the GenOME (Generic Outdoor Mobile Explorer) (See Figure 5.2). The vehicle is equipped with two laser range finders,

Figure 5.2 GenOME – A mobile robot specially constructed from a golf cart mounted with several sensors including a Gyroscope, Camera and an encoder.

a colour CCD camera, a fibre optic gyroscope, a three axis inertial sensor, a magneto-inductive compass and a Differential GPS (DGPS) receiver. Three industrial personal computers were installed in the vehicle to collect data from all the sensors at near real-time and the data are annotated with time-stamps to allow for systematic evaluations. This vehicle was built primarily as a research platform on which to test novel sensing strategies and control methods. Only the sensors that are used in the implementation of the proposed approach will be discussed in the next sections.

5.3.2 Sensors

The exteroceptive sensor used in the implementation of the Path Constrained CML approach is a commercial laser range finder. This sensor scans a semicircular area ahead of it at the rate of about 10Hz and returns the range of the closest laser reflection programmable at 0.5 degrees intervals. A typical output of the scan is shown in Figure 5.3(b) along with a photograph of the actual environment (Figure 5.3(a)). The poles/tree trunks and other such structures reflect the laser beam and appear visibly on the scan. In all such scans, point features appear as small clusters when the objects are at a distance. The range and bearing of these small clusters has to be extracted in order for the sensor data to be used to build a feature map. The development of an algorithm for extraction is not central to the thesis and a simple algorithm [54] is chosen from existing selections in order to accomplish the task.

While the use of exteroceptive sensor (Laser rangefinder) is outlined above, the implementation also includes the use of several proprioceptive sensors, which include encoders, a fibre optic gyroscope and a magnetic compass.

Four sets of incremental mechanical encoders were fitted to all four wheels of the mobile robot (GenOME) through the gears as shown in Figure 5.4. These encoders measure the speeds of each individual wheel.

A Hitachi Fiber Optic Gyroscope was used in order to measure the yaw rate of the vehicle. The sensor's output is an analog voltage proportional to the angular rate about the vehicle's vertical axis. This voltage is a result of the Sagnac Effect [77] which states that the difference in the optical path lengths of two beams travelling in opposite directions along the same closed optical circuit, is proportional to the angular velocity of

the circuit in inertial coordinates. The gyroscope is fastened to a horizontal plane housed inside the cabin situated at the centre of the vehicle (See Figure 5.5). It has negligible drift errors, and because of this, the time taken to conduct the experiments did not cause significant bias. The angular velocity range of the gyro is from -60 to 60 *deg/sec* and the data rate is $15ms$.

Figure 5.3(a) A photograph showing the vehicle's environment (as taken by a camera mounted on the laser scanner in the front of the vehicle).

(b) Data corresponding to the view photographed in Figure 5.5(a) as registered by the laser scanner. The two straight lines indicate the camera's field of view (FOV) with respect to the laser scanner's 180-degree span.

Figure 5.4 Mechanical incremental wheel encoders

A magneto-inductive compass with an integrated two-axis tilt sensor is also attached to the vehicle. The compass provides independent measurements of the vehicle's orientation. However, as the sensor is highly sensitive to interference caused by the vehicle's motion, the sensor is only used to provide the initial orientation of the vehicle. Changes of vehicle's yaw angle are therefore estimated using the yaw rate of the gyroscope instead.

Figure 5.5 The mounting of inertial sensors in the vehicle's cabin.

5.3.3 Test Site

The Path Constrained CML approach developed here works on the assumption that the environment consists of multiple point features.

Roads on the Nanyang Technological University's campus were chosen as the experimental site and Figure 5.6 provides a photographic view of the site. The intersection points representing the digital road map were obtained using a differential GPS (See Figure 5.7). The accuracy of these measures is within $\pm 0.5m$ in both x and y and the points were connected in order to form a skeletal road map. The GenOME (See Figure 5.2) was driven around chosen sections of the campus as illustrated in Figure 5.8. The roadsides are dotted with lampposts and tree trunks that are thin enough to be considered point features on the two-dimensional road map.

Figure 5.6 GenOME moving off from the starting point.

5.3 Experimental Setup

Figure 5.7 The test site within the larger NTU road network

Figure 5.8 The GenOME's experimental route

5.4 Results and Analysis

The Path Constrained CML algorithm was tested in the NTU campus described earlier. The vehicle was deployed on a section of road flanked by typical natural and man-made features such as tree trunks and lampposts and driven at approximately 4m/s . It started its run at Point A as depicted in Figure 5.9, and was driven 1.1 km along the closed path A-B-C- back to the starting point A. The large loop it made covered a surface of area $350\text{m} \times 350\text{m}$. The blue line in Figure 5.9 indicates the vehicle's path around the student accommodation halls. Care was taken to drive the vehicle approximately at the centre of the road at all times so as to assess how closely the estimated path corresponds to the skeletal map of the road network created using GPS measurements of the road's intersection points.

Figure 5.9 The vehicle path is represented by the blue line. The black crosses indicate the positions of the posts set up to indicate the vehicle's true ground position at any instant of the trial.

To evaluate the accuracy of the results, the vehicle's true ground position is acquired from a set of ten control points (made up of posts) along the vehicle's path. These posts or points are surveyed and mapped to the global reference frame. They provide information on the vehicle's actual position on ground truth. The black crosses depicted in Figure 5.9 indicate the control points (posts) on the test site and an image of one such post is shown in Figure 5.10(a). The vehicle is made to drive past each post at close proximity so as to reduce the offset error produced when the laser scanner mounted on the vehicle detects their signals. The scanner registers that the vehicle has indeed reached each post-marked position and their returns are hand filtered from the rest of the data. The vehicle's true ground position can be accurately obtained (with an error rate less than $\pm 0.5m$) by examining the range and bearing offsets of the laser returns in relation to the known position of each post. The true ground position registered by the vehicle in relation to the posts provides adequate means to evaluate the performance of the proposed approach.

In the implementation, the vehicle is modelled as a rigid body operating in a two-dimensional world. The elevation of the environment is assumed to be insignificant. Although, this may not be an entirely accurate reflection of the motion of the vehicle in the real world, it adequately allows for the development and testing of required algorithms. A kinematics bicycle model described by equation (3.2) in Section 3.2 is used for the purpose of estimating the vehicle's state transitions. The fiber-optic gyroscope measures the vehicle's yaw rate and is used to deduce the relative orientation estimates. The vehicle velocity v is taken from the average of the left and right wheel speeds as this helps approximate the velocity of the centre rear axle where the gyroscope is mounted.

Figure 5.10 (a) The posts used to determine the vehicle's true position during the run.
 (b) The measures obtained from the vehicle-mounted scanner in relation to the posts provide an effective means to evaluate the ground truth of the run.

The uncertainty of the vehicle process model is represented by, the variances σ_x^2 , σ_y^2 in x , y and σ_ψ^2 orientation. These noise parameters are given in Table 5.1. The random noise disturbances are included in the generation of the particle samples for the vehicle prediction equation. The parameter values used for the sampling rate Δt and the vehicle parameters L and a are also shown in Table 5.1.

Feature observation is carried out using the LADAR sensor described in Section 5.3.2. The distance and bearing of a feature is measured in relation to the vehicle. The observations are assumed to be corrupted by the zero-mean and temporally uncorrelated white noise w_r and w_θ with variance σ_r^2 and σ_θ^2 respectively. The parameters are given in Table 5.1.

Sampling period	Δt	$0.1s$
Wheel base (distance between wheel axles)	L	$1.6m$
Distance between the rear axle and exteroceptive sensor (LADAR)	a	$1.53m$
Vehicle x process noise standard deviation	σ_x	$0.3m$
Vehicle y process noise standard deviation	σ_y	$0.3m$
Vehicle heading process noise standard deviation	σ_ψ	1.5°
Range measurement standard deviation	σ_r	$0.5m$
Bearing measurement standard deviation	σ_θ	1°
Road map lateral error standard deviation	σ_ℓ	$2m$

Table 5.1 Parameters of the vehicle and sensor model.

The uncertainty of the road map used is described by the uncertainties associated with each road segment. It is shown in Figure 5.11 with white representing areas with higher probabilities and grey indicating areas of lower probability such as buildings and other natural infrastructure. The probability map shows the vehicle's traversable areas.

Figure 5.11 The probability road map of the selected test site.

In the proposed approach, the estimates of the vehicle path are based on a particle filter with an adaptive scheme. This effectively varies the sample size to closely represent the true probability distribution of the vehicle's path. The adaptive scheme is set with a fixed bin size of $1m \times 1m \times 20^\circ$. The probability β is set at 0.99 and the threshold ϵ is set at 0.1. Based on these parameters, the sample distribution is maintained at a constant density level. At this level, drastic changes that arise out of sensor's measurement ambiguities will not cause insufficiency in the sample size from fully representing all possible vehicle paths.

The vehicle was made to register the satellite-transmitted signals while being driven through the campus course in order to demonstrate the difficulty of receiving good GPS signals at a ground level. The results are plotted as red stars as indicated in Figure 5.12 and as expected, they clearly show that GPS data does not provide the continuous stream of reliable information needed to determine a moving vehicle's position. Buildings and tree canopies cause the system to suffer signal blockages and multiple path effects. Thus, the decision to use an alternative approach to the localisation problem appears to be adequately justified.

The performance of the odometry computed from the encoders mounted on the vehicle's wheels was also evaluated in order to compare its efficacy to the method proposed in this thesis. By the end of the loop, the position estimates obtained using data from the odometers (see blue line in Figure 5.12) were found to register errors as large as $18m$ during a $1.1km$ run (error at loop closing).

In contrast, the Path Constrained localisation approach described in Section 3.4 registered an error of only $2m$ based on the calculated average mean of the sample pose

distribution. The performance of this localisation approach was rigorously evaluated using a particle filter with 1000 samples representing the probability distribution of the vehicle's path. Although, the results show significant improvement in terms of error size, the samples are too widely spread along the road segment to provide a precise estimate of the vehicle's pose. The resulting vehicle path is shown in Figure 5.12 indicated by the red line. In Figure 5.13, the samples are represented by the green dots and it is clear that they are too widely dispersed. To reliably maintain the estimated distribution, the number of samples has to be increased. However, increasing the sample size means a massive increase in the computational burden and this reduces the viability of the approach.

Figure 5.12 The blue line represents the vehicle's pose as estimated using only proprioceptive or odometry sensors. The red dots represent the vehicle's position as observed from GPS data collected during the trial run. The red line indicates the road map as well as the resulting path obtained from the path constrained localisation approach. The black crosses indicate the positions of the posts set up to indicate the vehicle's true ground position at any instant of the trial.

Figure 5.13 The particle distribution indicating stochastic estimates of the vehicle's pose. The longitudinal spread of the particle indicates the uncertainty of the vehicle pose along the road segment.

To devise a more feasible approach, CML mechanisms are incorporated that estimate the spatial relationships between various features in the environment. A solely CML based approach on the FastSLAM framework is also implemented in order to compare its efficacy with that of the proposed approach. Both use similar Rao-Blackwellisation particle filter techniques. Figure 5.14 shows the estimated vehicle path (indicated by a solid line) that results and its associated feature map, as well as the road map (dotted line) constructed using differential GPS. The estimated vehicle path does not match the *a priori* road map well. The dots indicating the position of features (tree trunks and lamps post along side of the roads) are also erroneously mapped along the estimated path. The final loop closing error averaged over 10 runs is approximately $\pm 15m$.

In contrast, the proposed Path Constrained Adaptive FastSLAM provides a much more precise estimate of the vehicle path and the mapped features (See Figure 5.15).

Here, the road network is again, shown as a dotted line, the estimated vehicle path is indicated by a solid line and the mapped features appear as dots. The loop closing error over the 10 runs is $\pm 1m$, indicating a high level of consistency. It is important to note that both algorithms had to make use of the same set of sparse features that resulted from the poor resolution of the laser scanner for smaller objects (lamp posts and tree trunks) over large distances. The results amply demonstrate the effectiveness of Path Constrained Adaptive CML (FastSLAM) over standard CML (FastSLAM), especially when applied in large spaces with sparsely distributed features.

Figure 5.16 shows that 310 features have been mapped and that the average variance is $0.5m$. The variation in sample size is shown in Figure 5.17 with an average of 90. The sudden changes in sample size are due to temporary loss in relative measurements of distinct features during that time interval. When there are large samples, a majority of the particles spread along the road segment, thus demonstrating the large longitudinal uncertainty of the vehicle pose. This is due to the deficiency in acquiring information or sensor measurements necessary to deduce the actual vehicle pose on the road segment.

As described earlier in this section, ten control points were setup along the road network and these were used to further compare the performance of the standard CML (FastSLAM), over the proposed Path Constrained CML (FastSLAM). Comparisons were made with two versions of the proposed approach – one with and one without adaptive sampling. The controls points represent the vehicle's true path as it travels on the ground. The time instances at which the vehicle travels past these control points is recorded. Figure 5.18, shows the error bars of the 95% confidence intervals corresponding to the

Figure 5.14 Conventional FastSLAM results

Figure 5.15 Path Constrained Adaptive CML results

Figure 5.16 Variations in sample size

Figure 5.17 The final map features 2-sigma variance

average norm x - y localisation error occurring at the ten control points against the particle sample size for the three different CML schemes. In the adaptive case, a parameter threshold ε is varied to obtain a different set of experimental results. Each experiment was iterated 10 times to obtain an average error.

The Path Constrained Adaptive FastSLAM outperformed the rest in these experiments by achieving an average accuracy rate of $1.4m$ with a sample size of just 90 particles. It is 10 times more accurate than the conventional FastSLAM and more efficient in terms of its particle sample size. Results also show that the larger the sample size, the better the estimation – up to a limit. The adaptive Rao-Blackwellised particle filter is indeed a beneficial addition to the proposed Path Constrained CML approach.

Figure 5.18 Localisation error for different average sample sizes

Although the gain in localisation accuracy is relatively small here in comparison to that of the Path Constrained non-adaptive FastSLAM, the reduced sample size does bring about a significant reduction in the computational overheads. This reduction allows for the proposed Path Constrained CML approach to be effectively implemented in real-time. Further, when there is a temporary loss of detectable features, the adaptive implementation has the ability to prevent the estimates from diverging by increasing the number of samples. This makes the Path Constrained Adaptive FastSLAM more robust and efficient than the implementation that uses a fixed sample size.

The comparative results obtained from the field experiments are summarised and tabulated as follows:

Methods	Number of Samples	End of the loop error
Dead reckoning	-	18m
Path Constrained Localisation	1000	2m
FastSLAM	100	15m
Path Constrained Adaptive FastSLAM	100	1m

Table 5.2 Comparison results on Dead reckoning, Path Constrained localisation, FastSLAM and Path Constrained Adaptive FastSLAM

5.5 Conclusions

The Path constrained CML approach is an integration of the CML mechanism with the Path constrained localisation approach. An actual application of the Path Constrained CML algorithm was demonstrated in this chapter using a mobile robot

(GenOME) deployed on campus roads. The results of the experiment show that the Path Constrained CML has outperformed the Dead reckoning, Path constrained localisation and CML approaches. The Path constrained localisation approach requires approximately 10 times more particles than the Path constrained CML in order to obtain the same level of accuracy. The approach forwarded in this thesis clearly provides more efficient vehicle path estimates than Path constrained localisation approaches alone.

The adaptive Rao-Blackwellisation particle filter is integral to the deployment of our approach. The adaptive scheme helps to maintain the size of the sample and both effectively and efficiently represents the stochastic distribution of vehicle's localisation estimates. Experimental results indicate that the proposed approach is 10 times more accurate than systems using the CML approach alone

Although the vehicle has been successfully localised in the particular test site, there are still several areas in which improvements can be made. Possibilities to explore include the use of exteroceptive sensors to devise more robust feature extraction algorithms. The extension of the localisation problem is also crucial to cover urban environments where there is heavy traffic of both humans and vehicles. The next section concludes the thesis and makes recommendations on the direction of future works.

Chapter 6

Conclusions and Recommendations

6.1 Conclusions

Over the years, a fairly substantial body of literature dealing with the involutions of the localisation problem has accumulated. As a result, many researchers have concluded somewhat prematurely that the localisation problem has been solved. In truth however, this effort has been thwarted by the use of relatively noisy sensors that fail to produce convergent estimates. In addition, while techniques involving more absolute measures (e.g. GPS) were earlier heralded, these systems have since been found to be highly unreliable for use beyond a targeted environment.

After intensively analysis and a thorough review of recommended techniques, Concurrent Mapping and Localisation framework is applied in this thesis to help solve the localisation problem. The extensive experimentation and theoretical analysis carried out has shown that the inclusion of two new techniques is highly effective.

A major contribution of this thesis is the theoretical proof provided on the limitations of a localisation system (in the context of CML) that uses spatial information only in terms of known feature points and path constraints. The theoretical analysis indicated the need for more absolute information and established a theoretical framework within which available *a priori* spatial information could be integrated. It also stipulated

the minimum amount of *a priori* spatial information necessary in order to render the localisation problem observable.

A new Path Constrained CML approach is proposed as a consequence of these theoretical findings. The localisation problem is solved using *a priori* spatial information in the form of a road network map that constrains the estimation of the vehicle as it moves. It does this by tracking the spatial relationships between it and any static objects present in the environment.

An efficient and robust algorithm is devised to implement the proposed approach and its effectiveness is amply demonstrated by the experimental results. The FastSLAM approach is also implemented to help benchmark the improvements. Results of this show that the estimates derived by using the Path Constrained CML are 10 times more accurate than those obtained when the conventional FastSLAM approach is used. Thus, the study shows that incorporating previously known information in the form of *a priori* road maps significantly improves the results.

6.2 Recommendations for Future Research

Even with the significant results produced by this study, certain other issues warrant further attention and they are outlined below.

6.2.1 Natural Terrain Features

One of the most difficult things the CML mechanism has to do is detect the environment's features and incorporate them into the estimation process. There are many instances in which point features do not provide a sufficient representation of the

environment in which the vehicle is operating. Also, because of the poor resolution of the sensors, objects at a distance are difficult to detect and the data generated does not allow for their geometrical form and placement to be sufficiently well described.

With such sensor-driven limitations, the only perceivable features are the terrain itself. The question of how the natural terrain should be represented and how this representation could be applied to the localisation problem when no underlying models exist remains unanswered. A considerable amount of work needs to be done before a system is created that can be effectively deployed in more complex environments.

6.2.2 Extension to 3-D Deployment

When the vehicle encountered uneven road surfaces, the laser sensors pitched rapidly and caused features to be falsely detected. Similar changes in the road gradient are to be expected in many sites, all of which will violate the flat road assumptions made in the 2-D derivations. 3-D sensors may possibly aid the effort to successfully deploy autonomous vehicles in more complex environments. It may be necessary, therefore, to extend the 2-D formulation to 3-D and to investigate the improvements that this brings about.

However, this effort will not be without its challenges. When applied in three dimensions, uncertainty estimation and analysis will undoubtedly be more complex. From a theoretical point of view, the 3-D localisation problem requires either additional *a priori* spatial information or the inclusion of additional absolute sensors to constrain the estimation process.

The analysis proposed in this paper also needs to be extended in order to verify observability in the 3-D CML framework. For practical reasons, perception and motion modelling methods should be modified in accordance with the capabilities of the sensors.

The richness of 3-D spatial information will certainly make data association easier and more robust. However, this robustness will inevitably come at the cost of the additional computational power required to process such large amounts of perception and motion data.

6.2.3 Sensor Fusion

Sensor fusion is crucial for the reliable maintenance of spatial relationships in a complex environment. From the experimental results presented in this work, the use of laser scanners, odometers and IMUs appears to be promising. Still, laser scanners may not be sufficient for the perception of complex environments. Important features such as traffic signs, lights and lanes are beyond the recognition of the scanners and they may fail to produce reliable measurements in situations where objects are at vast distances and the terrain is heavily undulated. Other informational sources need to be available concurrently in order to boost the reliability and integrity of the recognition system. One possible source is a camera. Camera images are rich in detail and may compensate for some of the disadvantages of laser scanners. Such images have been found to be very useful for the perception and understanding of complex scenes.

In addition, state-of-the-art algorithms can be integrated from the computer vision literature to further improve system performance. Such recognition algorithms can be used to integrate and confirm the validity of information obtained from the camera

images and laser scanners. Only portions of the image that are likely to be targeted by the laser scanner have to be processed. Range measurements from the laser scanners will provide valuable readings of scale, thus speeding up the recognition process for efficient real-time operation. With effective recognition capabilities, known landmarks or buildings can easily be integrated into existing spatial maps and used to provide the much-needed observability in the CML mechanism.

6.2.4 Performance Metrics

As yet, few studies have attempted to make quantifiable experimental comparisons of different localisation approaches. Quantitative metrics would undoubtedly make the evaluation of localisation algorithms more meaningful by providing valuable information on the capabilities and accuracy of different sensors types. The comparisons will serve to increase our understanding of the mechanisms that are crucial to the advancement of the state of the art of autonomous vehicle navigation.

Author's Publications

Journal papers:

1. K. W. Lee, W.S. Wijesoma and Ibanez-Guzman, J., A Constrained SLAM Approach to Robust and Accurate Localisation Autonomous Ground Vehicles. Submitted to Robotics and Automation Systems.

Conference papers:

2. Wijesoma, W. S., K. W. Lee and Ibanez-Guzman, J., "On the Observability of Path Constrained Vehicle Localisation", Proceedings of the 9th Annual International IEEE intelligent Transportation Systems Conference, 2006.
3. K. W. Lee, Wijesoma, W. S. and Ibanez-Guzman, J., "On the Observability and Observability Analysis of SLAM", Proceedings of the International Conference on Intelligent Robots and Systems, 2006.
4. Wijesoma, W. S., K. W. Lee and Ibanez-Guzman, J., "Motion Constrained Simultaneous Localization and Mapping in Neighbourhood Environments", Proceedings of the IEEE International Conference on Robotics and Automation (ICRA 2005), April 18-22, 2005, Barcelona, Spain.

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Appendix A

Derivation of Higher Order Lie derivatives

This appendix provides necessary derivation of the higher order Lie derivatives for the observability analysis in this thesis. Due to their mathematical complexity, the results with detail derivation are developed in this appendix.

A.1 Feature-based Localisation System

First order Lie derivative of range observation:

The general equation for deriving the Lie derivatives of the range observation of the Feature-based localisation system can be expressed as

$$\begin{aligned} L_{\mathbf{F}}^D dh_1(\mathbf{x}_r, \Gamma_1) &= L_{\mathbf{F}}^{D-1} dh_1(\mathbf{x}_r, \Gamma_1) \frac{\partial \mathbf{F}(\mathbf{x}_r, \mathbf{u})}{\partial \mathbf{x}_r} + \left[\frac{\partial}{\partial \mathbf{x}_r} (L_{\mathbf{F}}^{D-1} dh_1(\mathbf{x}_r, \Gamma_1))^T \mathbf{F}(\mathbf{x}_r, \mathbf{u}) \right]^T \\ &= \begin{bmatrix} \mathbf{A}_1^{D1} & \mathbf{A}_1^{D2} & \mathbf{A}_1^{D3} \end{bmatrix} \\ D &= 1, 2 \end{aligned} \quad (\text{A.1})$$

where

$$\frac{\partial \mathbf{F}(\mathbf{x}_r, \mathbf{u})}{\partial \mathbf{x}_r} = \begin{bmatrix} 0 & 0 & -\Phi_2 \\ 0 & 0 & \Phi_1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sin \psi - a\delta \cos \psi \\ 0 & 0 & \cos \psi - a\delta \sin \psi \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A.2})$$

$$\mathbf{F}(\mathbf{x}_r, \mathbf{u}) = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} \cos \psi - a\delta \sin \psi \\ \sin \psi + a\delta \cos \psi \\ \delta \end{bmatrix} \quad (\text{A.3})$$

By using equation (A.1), the first order Lie derivative can be in the form of

$$\begin{aligned}
L_{\mathbf{f}}^1 dh_1(\mathbf{x}_r, \Gamma_1) &= \begin{bmatrix} \mathbf{A}_1^{01} & \mathbf{A}_1^{02} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\Phi_2 \\ 0 & 0 & \Phi_1 \\ 0 & 0 & 0 \end{bmatrix} + \left[\frac{\partial}{\partial \mathbf{x}_r} \begin{bmatrix} \mathbf{A}_1^{01} \\ \mathbf{A}_1^{02} \\ 0 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} \right]^T \\
&= \begin{bmatrix} 0 & 0 & \mathbf{A}_1^{02} \Phi_1 - \mathbf{A}_1^{01} \Phi_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathbf{A}_1^{01}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}_1^{01}}{\partial y} \Phi_2 \\ \frac{\partial \mathbf{A}_1^{02}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}_1^{02}}{\partial y} \Phi_2 \\ 0 \end{bmatrix}^T \\
&= \begin{bmatrix} \mathbf{A}_1^{11} & \mathbf{A}_1^{12} & \mathbf{A}_1^{13} \end{bmatrix}
\end{aligned} \tag{A.4}$$

and subsequently, the expression (A.4) can be further simplified as

$$\begin{aligned}
\begin{bmatrix} \mathbf{A}_1^{11} \\ \mathbf{A}_1^{12} \\ \mathbf{A}_1^{13} \end{bmatrix}^T &= \begin{bmatrix} \frac{\partial \mathbf{A}_1^{01}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}_1^{01}}{\partial y} \Phi_2 \\ \frac{\partial \mathbf{A}_1^{02}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}_1^{02}}{\partial y} \Phi_2 \\ \mathbf{A}_1^{02} \Phi_1 - \mathbf{A}_1^{01} \Phi_2 \end{bmatrix}^T \\
&= \begin{bmatrix} -\mathbf{B}_1^{01} \mathbf{A}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \mathbf{A}_1^{01} \Phi_2 \\ -\mathbf{B}_1^{02} \mathbf{A}_1^{02} \Phi_1 + \mathbf{B}_1^{02} \mathbf{A}_1^{01} \Phi_2 \\ \mathbf{A}_1^{02} \Phi_1 - \mathbf{A}_1^{01} \Phi_2 \end{bmatrix}^T = \begin{bmatrix} \mathbf{B}_1^{01} (-\mathbf{A}_1^{02} \Phi_1 + \mathbf{A}_1^{01} \Phi_2) \\ \mathbf{B}_1^{02} (-\mathbf{A}_1^{02} \Phi_1 + \mathbf{A}_1^{01} \Phi_2) \\ \mathbf{B}_1^{03} (-\mathbf{A}_1^{02} \Phi_1 + \mathbf{A}_1^{01} \Phi_2) \end{bmatrix}^T
\end{aligned} \tag{A.5}$$

By representing the expression (A.5) in term of the common factor, the first order Lie derivative can be re-expressed as

$$\begin{bmatrix} \mathbf{A}_1^{11} \\ \mathbf{A}_1^{12} \\ \mathbf{A}_1^{13} \end{bmatrix}^T = \Theta_1 \begin{bmatrix} \mathbf{B}_1^{01} \\ \mathbf{B}_1^{02} \\ \mathbf{B}_1^{03} \end{bmatrix}^T \tag{A.6}$$

where

$$\begin{aligned}\Theta_1 &= (-\mathbf{A}_1^{02}\Phi_1 + \mathbf{A}_1^{01}\Phi_2) \\ &= \left(-\frac{(\xi_1 - x)}{\Delta}(\sin\psi + a\delta\cos\psi) + \frac{(\zeta_1 - y)}{\Delta}(\cos\psi - a\delta\sin\psi) \right)\end{aligned}\quad (\text{A.7})$$

The equation (A.6) shows that the first order Lie derivative of the range observation function is equal to the zero order Lie derivative of the bearing observation function multiplied with the factor Θ_1 . This implies that the first order Lie derivative of the range observation function is a dependent vector of the zero order Lie derivative of the bearing observation function.

Second order Lie derivative of range observation:

Now with the first order Lie derivative of the range observation function, its second order Lie derivative can be derived similarly using equation (A.1) and expressed as

$$\begin{aligned}L_{\mathbf{F}}^2 dh_1(\mathbf{x}_r, \Gamma_1) &= \begin{bmatrix} \mathbf{A}_1^{11} & \mathbf{A}_1^{12} & \mathbf{A}_1^{13} \end{bmatrix} \begin{bmatrix} 0 & 0 & -\Phi_2 \\ 0 & 0 & \Phi_1 \\ 0 & 0 & 0 \end{bmatrix} + \left[\frac{\partial}{\partial \mathbf{x}_r} \begin{bmatrix} \mathbf{A}_1^{11} \\ \mathbf{A}_1^{12} \\ \mathbf{A}_1^{13} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} \right]^T \\ &= \begin{bmatrix} 0 & 0 & \mathbf{A}_1^{12}\Phi_1 - \mathbf{A}_1^{11}\Phi_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathbf{A}_1^{11}}{\partial x}\Phi_1 + \frac{\partial \mathbf{A}_1^{11}}{\partial y}\Phi_2 + \frac{\partial \mathbf{A}_1^{11}}{\partial \psi}\Phi_3 \\ \frac{\partial \mathbf{A}_1^{12}}{\partial x}\Phi_1 + \frac{\partial \mathbf{A}_1^{12}}{\partial y}\Phi_2 + \frac{\partial \mathbf{A}_1^{12}}{\partial \psi}\Phi_3 \\ \frac{\partial \mathbf{A}_1^{13}}{\partial x}\Phi_1 + \frac{\partial \mathbf{A}_1^{13}}{\partial y}\Phi_2 + \frac{\partial \mathbf{A}_1^{13}}{\partial \psi}\Phi_3 \end{bmatrix}^T \\ &= \begin{bmatrix} \mathbf{A}_1^{21} & \mathbf{A}_1^{22} & \mathbf{A}_1^{23} \end{bmatrix}\end{aligned}\quad (\text{A.8})$$

The expression (A.8) is further simplified in the form of

$$\begin{bmatrix} \mathbf{A}_1^{21} \\ \mathbf{A}_1^{22} \\ \mathbf{A}_1^{23} \end{bmatrix}^T = \begin{bmatrix} \frac{\partial \mathbf{A}_1^{11}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}_1^{11}}{\partial y} \Phi_2 + \frac{\partial \mathbf{A}_1^{11}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{A}_1^{12}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}_1^{12}}{\partial y} \Phi_2 + \frac{\partial \mathbf{A}_1^{12}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{A}_1^{13}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}_1^{13}}{\partial y} \Phi_2 + \frac{\partial \mathbf{A}_1^{13}}{\partial \psi} \Phi_3 + \mathbf{A}_1^{12} \Phi_1 - \mathbf{A}_1^{11} \Phi_2 \end{bmatrix}^T \quad (\text{A.9})$$

and by representing the expression (A.9) in terms of common factors, the second order

Lie derivative can be re-expressed as

$$\begin{bmatrix} \mathbf{A}_1^{21} \\ \mathbf{A}_1^{22} \\ \mathbf{A}_1^{23} \end{bmatrix}^T = \Theta_2 \begin{bmatrix} \mathbf{B}_1^{01} \\ \mathbf{B}_1^{02} \\ \mathbf{B}_1^{03} \end{bmatrix}^T + \Theta_3 \begin{bmatrix} \mathbf{A}_1^{01} \\ \mathbf{A}_1^{02} \\ \mathbf{A}_1^{03} \end{bmatrix}^T \quad (\text{A.10})$$

where

$$\begin{aligned} \Theta_2 &= \begin{pmatrix} -2\mathbf{A}_1^{01}\mathbf{B}_1^{01}\Phi_1^2 + 2\mathbf{A}_1^{02}\mathbf{B}_1^{01}\Phi_1\Phi_2 - 2\mathbf{A}_1^{01}\mathbf{B}_1^{02}\Phi_1\Phi_2 \\ +2\mathbf{A}_1^{01}\mathbf{B}_1^{01}\Phi_2^2 + \mathbf{A}_1^{01}\Phi_2\Phi_3 + \mathbf{A}_1^{02}\Phi_1\Phi_3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2(\xi_1 - x)(\zeta_1 - y)}{\Delta^3} (\cos \psi - a\delta \sin \psi)^2 \\ + \frac{2(\zeta_1 - y)^2 - 2(\xi_1 - x)^2}{\Delta^3} (\cos \psi - a\delta \sin \psi)(\sin \psi + a\delta \cos \psi) \\ - \frac{2(\xi_1 - x)(\zeta_1 - y)}{\Delta^3} (\sin \psi + a\delta \cos \psi)^2 \\ - \frac{(\xi_1 - x)\delta}{\Delta} (\sin \psi + a\delta \cos \psi) - \frac{(\zeta_1 - y)\delta}{\Delta} (\cos \psi - a\delta \sin \psi) \end{pmatrix} \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} \Theta_3 &= \begin{pmatrix} -\mathbf{A}_1^{02}\mathbf{B}_1^{01}\Phi_1^2 + 2\mathbf{A}_1^{01}\mathbf{B}_1^{01}\Phi_1\Phi_2 + \mathbf{A}_1^{01}\mathbf{B}_1^{02}\Phi_2^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{(\zeta_1 - y)^2}{\Delta^3} (\cos \psi - a\delta \sin \psi)^2 \\ - \frac{2(\zeta_1 - y)(\xi_1 - x)}{\Delta^3} (\cos \psi - a\delta \sin \psi)(\sin \psi + a\delta \cos \psi) \\ + \frac{(\xi_1 - x)^2}{\Delta^3} (\sin \psi + a\delta \cos \psi)^2 \end{pmatrix} \end{aligned} \quad (\text{A.12})$$

The equation (A.10) shows that the second order Lie derivative of the range observation function is equal to the combination of both the zero order Lie derivatives of the bearing and range observation multiplied with their respective factor Θ_2 and Θ_3 . This implies that the second order Lie derivative of the range observation function is a dependent vector of the combination of zero order Lie derivative of the range and bearing observation function.

First order Lie derivative of bearing observation:

Now consider the derivation of Lie derivatives of the bearing observation function, the general equation can be expressed as

$$\begin{aligned} L_{\mathbf{F}}^D dh_2(\mathbf{x}_r, \Gamma_1) &= L_{\mathbf{F}}^{D-1} dh_2(\mathbf{x}_r, \Gamma_1) \frac{\partial \mathbf{F}(\mathbf{x}_r, \mathbf{u})}{\partial \mathbf{x}_r} + \left[\frac{\partial}{\partial \mathbf{X}} (L_{\mathbf{F}}^{D-1} dh_2(\mathbf{x}_r, \Gamma_1))^T \mathbf{F}(\mathbf{x}_r, \mathbf{u}) \right]^T \\ &= \begin{bmatrix} \mathbf{B}_1^{D1} & \mathbf{B}_1^{D2} & \mathbf{B}_1^{D3} \end{bmatrix} \\ D &= 1, 2 \end{aligned} \quad (\text{A.13})$$

By using equation (A.13), the first order Lie derivative can be in the form of

$$\begin{aligned} L_{\mathbf{F}}^1 dh_2(\mathbf{x}_r, \Gamma_1) &= \begin{bmatrix} \mathbf{B}_1^{01} & \mathbf{B}_1^{02} & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\Phi_2 \\ 0 & 0 & \Phi_1 \\ 0 & 0 & 0 \end{bmatrix} + \left[\frac{\partial}{\partial \mathbf{x}_r} \begin{bmatrix} \mathbf{B}_1^{01} \\ \mathbf{B}_1^{02} \\ -1 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} \right]^T \\ &= \begin{bmatrix} 0 & 0 & \mathbf{B}_1^{02} \Phi_1 - \mathbf{B}_1^{01} \Phi_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathbf{B}_1^{01}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}_1^{01}}{\partial y} \Phi_2 \\ \frac{\partial \mathbf{B}_1^{02}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}_1^{02}}{\partial y} \Phi_2 \\ 0 \end{bmatrix}^T \\ &= \begin{bmatrix} \mathbf{B}_1^{11} & \mathbf{B}_1^{12} & \mathbf{B}_1^{13} \end{bmatrix} \end{aligned} \quad (\text{A.14})$$

and subsequently, the expression (A.14) can be further simplified as

$$\begin{aligned}
\begin{bmatrix} \mathbf{B}_1^{11} \\ \mathbf{B}_1^{12} \\ \mathbf{B}_1^{13} \end{bmatrix}^T &= \begin{bmatrix} \frac{\partial \mathbf{B}_1^{01}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}_1^{01}}{\partial y} \Phi_2 \\ \frac{\partial \mathbf{B}_1^{02}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}_1^{02}}{\partial y} \Phi_2 \\ \mathbf{B}_1^{02} \Phi_1 - \mathbf{B}_1^{01} \Phi_2 \end{bmatrix}^T = \begin{bmatrix} \mathbf{B}_1^{01} (-\mathbf{B}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \Phi_2) - \mathbf{B}_1^{01} \mathbf{B}_1^{02} \Phi_1 - \mathbf{B}_1^{02} \mathbf{B}_1^{02} \Phi_2 \\ \mathbf{B}_1^{02} (-\mathbf{B}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \Phi_2) + \mathbf{B}_1^{01} \mathbf{B}_1^{01} \Phi_1 + \mathbf{B}_1^{02} \mathbf{B}_1^{01} \Phi_2 \\ -(-\mathbf{B}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \Phi_2) \end{bmatrix}^T \\
&= \begin{bmatrix} \mathbf{B}_1^{01} (-\mathbf{B}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \Phi_2) \\ \mathbf{B}_1^{02} (-\mathbf{B}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \Phi_2) \\ \mathbf{B}_1^{03} (-\mathbf{B}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \Phi_2) \end{bmatrix}^T + \begin{bmatrix} -\mathbf{B}_1^{02} (\mathbf{B}_1^{01} \Phi_1 + \mathbf{B}_1^{02} \Phi_2) \\ \mathbf{B}_1^{01} (\mathbf{B}_1^{01} \Phi_1 + \mathbf{B}_1^{02} \Phi_2) \\ 0 \end{bmatrix}^T \\
&= \begin{bmatrix} \mathbf{B}_1^{01} (-\mathbf{B}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \Phi_2) \\ \mathbf{B}_1^{02} (-\mathbf{B}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \Phi_2) \\ \mathbf{B}_1^{03} (-\mathbf{B}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \Phi_2) \end{bmatrix}^T + \begin{bmatrix} -\mathbf{A}_1^{01} \Delta^{-1} (\mathbf{B}_1^{01} \Phi_1 + \mathbf{B}_1^{02} \Phi_2) \\ -\mathbf{A}_1^{02} \Delta^{-1} (\mathbf{B}_1^{01} \Phi_1 + \mathbf{B}_1^{02} \Phi_2) \\ 0 \end{bmatrix}^T
\end{aligned} \tag{A.15}$$

By representing the expression (A.15) in terms of common factors, the first order Lie derivative can be re-expressed as

$$\begin{bmatrix} \mathbf{B}_1^{11} \\ \mathbf{B}_1^{12} \\ \mathbf{B}_1^{13} \end{bmatrix}^T = \Theta_4 \begin{bmatrix} \mathbf{B}_1^{01} \\ \mathbf{B}_1^{02} \\ \mathbf{B}_1^{03} \end{bmatrix}^T + \Theta_5 \begin{bmatrix} \mathbf{A}_1^{01} \\ \mathbf{A}_1^{02} \\ \mathbf{A}_1^{03} \end{bmatrix}^T \tag{A.16}$$

where

$$\begin{aligned}
\Theta_4 &= (-\mathbf{B}_1^{02} \Phi_1 + \mathbf{B}_1^{01} \Phi_2) \\
&= \left(\frac{(\xi_1 - x)}{\Delta^2} (\cos \psi - a\delta \sin \psi) + \frac{(\zeta_1 - y)}{\Delta^2} (\sin \psi + a\delta \cos \psi) \right)
\end{aligned} \tag{A.17}$$

$$\begin{aligned}
\Theta_5 &= \Delta^{-1} (\mathbf{B}_1^{01} \Phi_1 + \mathbf{B}_1^{02} \Phi_2) \\
&= \left(\frac{(\zeta_1 - y)}{\Delta^3} (\cos \psi - a\delta \sin \psi) - \frac{(\xi_1 - x)}{\Delta^3} (\sin \psi + a\delta \cos \psi) \right)
\end{aligned} \tag{A.18}$$

The equation (A.16) shows that the first order Lie derivative of the bearing observation function is equal to the combination of both the zero order Lie derivatives of the bearing

and range observation multiplied with their respective factor Θ_4 and Θ_5 . This implies that the first order Lie derivative of the bearing observation function is a dependent vector of the combination of zero order Lie derivative of the range and bearing observation function.

Second order Lie derivative of bearing observation:

Now with the first order derivative of the bearing observation function, its the second order Lie derivative can be derived using equation (A.13) and expressed as,

$$\begin{aligned}
 L_{\mathbf{F}}^2 dh_2(\mathbf{x}_r, \Gamma_1) &= [\mathbf{B}_1^{11} \quad \mathbf{B}_1^{12} \quad \mathbf{B}_1^{13}] \begin{bmatrix} 0 & 0 & -\Phi_2 \\ 0 & 0 & \Phi_1 \\ 0 & 0 & 0 \end{bmatrix} + \left[\frac{\partial}{\partial \mathbf{x}_r} \begin{bmatrix} \mathbf{B}_1^{11} \\ \mathbf{B}_1^{12} \\ \mathbf{B}_1^{13} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} \right]^T \\
 &= [0 \quad 0 \quad \mathbf{B}_1^{12}\Phi_1 - \mathbf{B}_1^{11}\Phi_2] + \begin{bmatrix} \frac{\partial \mathbf{B}_1^{11}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}_1^{11}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}_1^{11}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{B}_1^{12}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}_1^{12}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}_1^{12}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{B}_1^{13}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}_1^{13}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}_1^{13}}{\partial \psi} \Phi_3 \end{bmatrix}^T \\
 &= [\mathbf{B}_1^{21} \quad \mathbf{B}_1^{22} \quad \mathbf{B}_1^{23}]
 \end{aligned} \tag{A.19}$$

The expression can be further simplified in the form of

$$\begin{bmatrix} \mathbf{B}_1^{21} \\ \mathbf{B}_1^{22} \\ \mathbf{B}_1^{23} \end{bmatrix}^T = \begin{bmatrix} \frac{\partial \mathbf{B}_1^{11}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}_1^{11}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}_1^{11}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{B}_1^{12}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}_1^{12}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}_1^{12}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{B}_1^{13}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}_1^{13}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}_1^{13}}{\partial \psi} \Phi_3 + \mathbf{B}_1^{12}\Phi_1 - \mathbf{B}_1^{11}\Phi_2 \end{bmatrix}^T \tag{A.20}$$

and by representing the expression (A.20) in terms of common factors, the second order Lie derivative can be re-expressed as

$$\begin{bmatrix} \mathbf{B}_1^{21} \\ \mathbf{B}_1^{22} \\ \mathbf{B}_1^{23} \end{bmatrix}^T = \Theta_6 \begin{bmatrix} \mathbf{B}_1^{01} \\ \mathbf{B}_1^{02} \\ \mathbf{B}_1^{03} \end{bmatrix}^T + \Theta_7 \begin{bmatrix} \mathbf{A}_1^{01} \\ \mathbf{A}_1^{02} \\ \mathbf{A}_1^{03} \end{bmatrix}^T \quad (\text{A.21})$$

where

$$\begin{aligned} \Theta_6 &= \begin{pmatrix} 2\mathbf{B}_1^{02}\mathbf{B}_1^{02}\Phi_1^2 - 2\mathbf{B}_1^{01}\mathbf{B}_1^{01}\Phi_1^2 - 4\Delta^{-1}\mathbf{A}_1^{01}\mathbf{A}_1^{01}\mathbf{A}_1^{02}\Phi_1\Phi_2 \\ + 2\mathbf{B}_1^{01}\mathbf{B}_1^{01}\Phi_2^2 - 2\mathbf{B}_1^{02}\mathbf{B}_1^{02}\Phi_2^2 + \mathbf{B}_1^{02}\Phi_2\Phi_3 + \mathbf{B}_1^{01}\Phi_1\Phi_3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2(\xi_1 - x)^2 - 2(\zeta_1 - y)^2}{\Delta^4} (\cos\psi - a\delta\sin\psi)^2 \\ + \frac{4(\zeta_1 - y)(\xi_1 - x)^2}{\Delta^4} (\cos\psi - a\delta\sin\psi)(\sin\psi + a\delta\cos\psi) \\ + \frac{2(\zeta_1 - y) - 2(\xi_1 - x)}{\Delta^4} (\sin\psi + a\delta\cos\psi)^2 \\ - \frac{(\xi_1 - x)\delta}{\Delta^2} (\sin\psi + a\delta\cos\psi) + \frac{(\zeta_1 - y)\delta}{\Delta^2} (\cos\psi - a\delta\sin\psi) \end{pmatrix} \end{aligned} \quad (\text{A.22})$$

$$\begin{aligned} \Theta_7 &= \begin{pmatrix} -4\mathbf{B}_1^{02}\mathbf{B}_1^{01}\Phi_1^2 + 8\mathbf{B}_1^{01}\mathbf{B}_1^{01}\Phi_1\Phi_2 - 4\mathbf{B}_1^{02}\mathbf{B}_1^{02}\Phi_1\Phi_2 + \mathbf{B}_1^{02}\Phi_1\Phi_3 - \mathbf{B}_1^{01}\Phi_1\Phi_3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{4(\zeta_1 - y)(\xi_1 - x)}{\Delta^4} (\cos\psi - a\delta\sin\psi)^2 \\ + \frac{8(\zeta_1 - y)^2 - 4(\xi_1 - x)^2}{\Delta^4} (\cos\psi - a\delta\sin\psi)(\sin\psi + a\delta\cos\psi) \\ - \frac{4(\zeta_1 - y)(\xi_1 - x)}{\Delta^4} (\sin\psi + a\delta\cos\psi)^2 \\ - \frac{(\xi_1 - x)\delta}{\Delta^2} (\cos\psi - a\delta\sin\psi) - \frac{(\zeta_1 - y)\delta}{\Delta^2} (\sin\psi + a\delta\cos\psi) \end{pmatrix} \end{aligned} \quad (\text{A.23})$$

The equation (A.21) shows that the second order Lie derivative of the bearing observation function is equal to the combination of both the zero order Lie derivatives of the bearing and range observation function multiplied with their respective factor Θ_6 and Θ_7 . This implies that the second order Lie derivative of the bearing

observation function is a dependent vector of the combination of zero order Lie derivative of the range and bearing observation function.

Remarks:

From these derivations, the higher order Lie derivatives of range and bearing observation functions are shown to be the combination of their zero order Lie derivatives of range and bearing observation function. Thus, this concludes that the observability matrix of Feature-based localisation system will only have two independent vectors (the zero order Lie derivative of the range and bearing observation function) or the rank of two which is deficient.

A.2 CML System

First order Lie derivative of range observation:

The general equation for deriving the Lie derivatives of the range observation of the CML system can be expressed as

$$\begin{aligned}
 L_{\mathbf{F}}^D dh_1(\mathbf{X}) &= L_{\mathbf{F}}^{D-1} dh_1(\mathbf{X}) \frac{\partial \mathbf{F}(\mathbf{X}, \mathbf{u})}{\partial \mathbf{X}} + \left[\frac{\partial}{\partial \mathbf{X}} (L_{\mathbf{F}}^{D-1} dh_1(\mathbf{X}))^T \mathbf{F}(\mathbf{X}, \mathbf{u}) \right]^T \\
 &= \begin{bmatrix} \mathbf{A}^{D1} & \mathbf{A}^{D2} & \mathbf{A}^{D3} & -\mathbf{A}^{D1} & -\mathbf{A}^{D2} \end{bmatrix} \\
 D &= 1, 2, 3, 4
 \end{aligned} \tag{A.24}$$

where

$$\mathbf{F}(\mathbf{X}, \mathbf{u}) = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \psi - a\delta \sin \psi \\ \sin \psi + a\delta \cos \psi \\ \delta \\ 0 \\ 0 \end{bmatrix} \tag{A.25}$$

$$\frac{\partial \mathbf{F}(\mathbf{X}, \mathbf{u})}{\partial \mathbf{X}} = \begin{bmatrix} 0 & 0 & -\Phi_2 & 0 & 0 \\ 0 & 0 & \Phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sin \psi - a\delta \cos \psi & 0 & 0 \\ 0 & 0 & \cos \psi - a\delta \sin \psi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{A.26})$$

By using equation (A.25), the first order Lie derivative can be derived as

$$\begin{aligned} L_{\mathbf{F}}^1 dh_1(\mathbf{X}) &= \begin{bmatrix} \mathbf{A}^{01} & \mathbf{A}^{02} & 0 & -\mathbf{A}^{01} & -\mathbf{A}^{02} \end{bmatrix} \begin{bmatrix} 0 & 0 & -\Phi_2 & 0 & 0 \\ 0 & 0 & \Phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\partial}{\partial \mathbf{X}} \begin{bmatrix} \mathbf{A}^{01} \\ \mathbf{A}^{02} \\ 0 \\ -\mathbf{A}^{01} \\ -\mathbf{A}^{02} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ 0 \\ 0 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 0 & \mathbf{A}^{02}\Phi_1 - \mathbf{A}^{01}\Phi_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathbf{A}^{01}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}^{01}}{\partial y} \Phi_2 \\ \frac{\partial \mathbf{A}^{02}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}^{02}}{\partial y} \Phi_2 \\ 0 \\ -\frac{\partial \mathbf{A}^{01}}{\partial x} \Phi_1 - \frac{\partial \mathbf{A}^{01}}{\partial y} \Phi_2 \\ -\frac{\partial \mathbf{A}^{02}}{\partial x} \Phi_1 - \frac{\partial \mathbf{A}^{02}}{\partial y} \Phi_2 \end{bmatrix}^T \\ &= \begin{bmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} & 0 & -\mathbf{A}^{11} & -\mathbf{A}^{12} \end{bmatrix} \end{aligned} \quad (\text{A.27})$$

The expression (A.27) can be further simplified in the form of

$$\begin{bmatrix} \mathbf{A}^{11} \\ \mathbf{A}^{12} \\ \mathbf{A}^{13} \\ -\mathbf{A}^{11} \\ -\mathbf{A}^{12} \end{bmatrix}^T = \begin{bmatrix} -\mathbf{B}^{01} \mathbf{A}^{02} \Phi_1 + \mathbf{B}^{01} \mathbf{A}^{01} \Phi_2 \\ -\mathbf{B}^{02} \mathbf{A}^{02} \Phi_1 + \mathbf{B}^{02} \mathbf{A}^{01} \Phi_2 \\ \mathbf{A}^{02} \Phi_1 - \mathbf{A}^{01} \Phi_2 \\ \mathbf{B}^{01} \mathbf{A}^{02} \Phi_1 - \mathbf{B}^{01} \mathbf{A}^{01} \Phi_2 \\ \mathbf{B}^{02} \mathbf{A}^{02} \Phi_1 - \mathbf{B}^{02} \mathbf{A}^{01} \Phi_2 \end{bmatrix}^T = \begin{bmatrix} \mathbf{B}^{01} (-\mathbf{A}^{02} \Phi_1 + \mathbf{A}^{01} \Phi_2) \\ \mathbf{B}^{02} (-\mathbf{A}^{02} \Phi_1 + \mathbf{A}^{01} \Phi_2) \\ \mathbf{B}^{03} (-\mathbf{A}^{02} \Phi_1 + \mathbf{A}^{01} \Phi_2) \\ -\mathbf{B}^{01} (-\mathbf{A}^{02} \Phi_1 + \mathbf{A}^{01} \Phi_2) \\ -\mathbf{B}^{02} (-\mathbf{A}^{02} \Phi_1 + \mathbf{A}^{01} \Phi_2) \end{bmatrix}^T \quad (\text{A.28})$$

and by representing the expression (A.28) in terms of common factors, the first order Lie derivative can be re-expressed as

$$\begin{bmatrix} \mathbf{A}^{11} \\ \mathbf{A}^{12} \\ \mathbf{A}^{13} \\ -\mathbf{A}^{11} \\ -\mathbf{A}^{12} \end{bmatrix}^T = \Upsilon_1 \begin{bmatrix} \mathbf{B}^{01} \\ \mathbf{B}^{02} \\ \mathbf{B}^{03} \\ -\mathbf{B}^{01} \\ -\mathbf{B}^{02} \end{bmatrix}^T \quad (\text{A.29})$$

where

$$\begin{aligned} \Upsilon_1 &= (-\mathbf{A}^{02}\Phi_1 + \mathbf{A}^{01}\Phi_2) \\ &= \left(-\frac{(x_1 - x)}{\Delta} (\sin \psi + a\delta \cos \psi) + \frac{(y_1 - y)}{\Delta} (\cos \psi - a\delta \sin \psi) \right) \end{aligned} \quad (\text{A.30})$$

The equation (A.29) shows that the first order Lie derivative of the range observation function is equal to the zero order Lie derivative of the bearing observation function multiplied with the factor Υ_1 . This implies that the first order Lie derivative of the range observation function is a dependent vector of the zero order Lie derivative of the bearing observation function.

Second order Lie derivative of range observation:

With the first order Lie derivative of the range observation function, its second order Lie derivative can be similarly derived using equation (A.24) and expressed as,

$$\begin{aligned} &L_{\mathbf{F}}^2 dh_1(\mathbf{X}) \\ &= \begin{bmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} & \mathbf{A}^{13} & -\mathbf{A}^{11} & -\mathbf{A}^{12} \end{bmatrix} \begin{bmatrix} 0 & 0 & -\Phi_2 & 0 & 0 \\ 0 & 0 & \Phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\partial}{\partial \mathbf{X}} \begin{bmatrix} \mathbf{A}^{11} \\ \mathbf{A}^{12} \\ \mathbf{A}^{13} \\ -\mathbf{A}^{11} \\ -\mathbf{A}^{12} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ 0 \\ 0 \end{bmatrix}^T \end{aligned} \quad (\text{A.30})$$

$$\begin{aligned}
& L_F^2 dh_1(\mathbf{X}) \\
& = \begin{bmatrix} 0 & 0 & \mathbf{A}^{12}\Phi_1 - \mathbf{A}^{11}\Phi_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathbf{A}^{11}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}^{11}}{\partial y} \Phi_2 + \frac{\partial \mathbf{A}^{11}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{A}^{12}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}^{12}}{\partial y} \Phi_2 + \frac{\partial \mathbf{A}^{12}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{A}^{13}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}^{13}}{\partial y} \Phi_2 + \frac{\partial \mathbf{A}^{13}}{\partial \psi} \Phi_3 \\ -\frac{\partial \mathbf{A}^{11}}{\partial x} \Phi_1 - \frac{\partial \mathbf{A}^{11}}{\partial y} \Phi_2 - \frac{\partial \mathbf{A}^{11}}{\partial \psi} \Phi_3 \\ -\frac{\partial \mathbf{A}^{12}}{\partial x} \Phi_1 - \frac{\partial \mathbf{A}^{12}}{\partial y} \Phi_2 - \frac{\partial \mathbf{A}^{12}}{\partial \psi} \Phi_3 \end{bmatrix}^T \\
& = \begin{bmatrix} \mathbf{A}^{21} & \mathbf{A}^{22} & \mathbf{A}^{23} & -\mathbf{A}^{21} & -\mathbf{A}^{22} \end{bmatrix}
\end{aligned} \tag{A.31}$$

The expression (A.31) can be further simplified in the form of

$$\begin{bmatrix} \mathbf{A}^{21} \\ \mathbf{A}^{22} \\ \mathbf{A}^{23} \\ -\mathbf{A}^{21} \\ -\mathbf{A}^{22} \end{bmatrix}^T = \begin{bmatrix} \frac{\partial \mathbf{A}^{11}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}^{11}}{\partial y} \Phi_2 + \frac{\partial \mathbf{A}^{11}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{A}^{12}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}^{12}}{\partial y} \Phi_2 + \frac{\partial \mathbf{A}^{12}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{A}^{13}}{\partial x} \Phi_1 + \frac{\partial \mathbf{A}^{13}}{\partial y} \Phi_2 + \frac{\partial \mathbf{A}^{13}}{\partial \psi} \Phi_3 + \mathbf{A}^{12}\Phi_1 - \mathbf{A}^{11}\Phi_2 \\ -\frac{\partial \mathbf{A}^{11}}{\partial x} \Phi_1 - \frac{\partial \mathbf{A}^{11}}{\partial y} \Phi_2 - \frac{\partial \mathbf{A}^{11}}{\partial \psi} \Phi_3 \\ -\frac{\partial \mathbf{A}^{12}}{\partial x} \Phi_1 - \frac{\partial \mathbf{A}^{12}}{\partial y} \Phi_2 - \frac{\partial \mathbf{A}^{12}}{\partial \psi} \Phi_3 \end{bmatrix}^T \tag{A.32}$$

and by representing the expression (A.32) in terms of common factors, the second order

Lie derivative can be re-expressed as

$$\begin{bmatrix} \mathbf{A}^{21} \\ \mathbf{A}^{22} \\ \mathbf{A}^{23} \\ -\mathbf{A}^{21} \\ -\mathbf{A}^{22} \end{bmatrix}^T = \Upsilon_2 \begin{bmatrix} \mathbf{B}^{01} \\ \mathbf{B}^{02} \\ \mathbf{B}^{03} \\ -\mathbf{B}^{01} \\ -\mathbf{B}^{02} \end{bmatrix}^T + \Upsilon_3 \begin{bmatrix} \mathbf{A}^{01} \\ \mathbf{A}^{02} \\ \mathbf{A}^{03} \\ -\mathbf{A}^{01} \\ -\mathbf{A}^{02} \end{bmatrix}^T \tag{A.33}$$

where

$$\begin{aligned} \Upsilon_2 = & \begin{pmatrix} -2\mathbf{A}^{01}\mathbf{B}^{01}\Phi_1^2 + 2\mathbf{A}^{02}\mathbf{B}^{01}\Phi_1\Phi_2 - 2\mathbf{A}^{01}\mathbf{B}^{02}\Phi_1\Phi_2 \\ + 2\mathbf{A}^{01}\mathbf{B}^{01}\Phi_2^2 + \mathbf{A}^{01}\Phi_2\Phi_3 + \mathbf{A}^{02}\Phi_1\Phi_3 \end{pmatrix} \\ = & \begin{pmatrix} \frac{2(x_1 - x)(y_1 - y)}{\Delta^3}(\cos\psi - a\delta\sin\psi)^2 \\ + \frac{2(y_1 - y)^2 - 2(x_1 - x)^2}{\Delta^3}(\cos\psi - a\delta\sin\psi)(\sin\psi + a\delta\cos\psi) \\ - \frac{2(x_1 - x)(y_1 - y)}{\Delta^3}(\sin\psi + a\delta\cos\psi)^2 \\ - \frac{(x_1 - x)\delta}{\Delta}(\sin\psi + a\delta\cos\psi) - \frac{(y_1 - y)\delta}{\Delta}(\cos\psi - a\delta\sin\psi) \end{pmatrix} \end{aligned} \quad (\text{A.34})$$

$$\begin{aligned} \Upsilon_3 = & \begin{pmatrix} -\mathbf{A}^{02}\mathbf{B}^{01}\Phi_1^2 + 2\mathbf{A}^{01}\mathbf{B}^{01}\Phi_1\Phi_2 + \mathbf{A}^{01}\mathbf{B}^{02}\Phi_2^2 \end{pmatrix} \\ = & \begin{pmatrix} \frac{(y_1 - y)^2}{\Delta^3}(\cos\psi - a\delta\sin\psi)^2 \\ - \frac{2(y_1 - y)(x_1 - x)}{\Delta^3}(\cos\psi - a\delta\sin\psi)(\sin\psi + a\delta\cos\psi) \\ + \frac{(x_1 - x)^2}{\Delta^3}(\sin\psi + a\delta\cos\psi)^2 \end{pmatrix} \end{aligned} \quad (\text{A.35})$$

The equation (A.33) shows that the second order Lie derivative of the range observation function is equal to the combination of both the zero order Lie derivatives of the bearing and range observation multiplied with their respective factor Υ_2 and Υ_3 . This implies that the second order Lie derivative of the range observation function is a dependent vector of the combination of zero order Lie derivative of the range and bearing observation function.

First order Lie derivative of bearing observation:

Now consider the derivation of Lie derivatives of the bearing observation function, the general equation can be expressed as

$$\begin{aligned}
 L_{\mathbf{F}}^D dh_2(\mathbf{X}) &= L_{\mathbf{F}}^{D-1} dh_2(\mathbf{X}) \frac{\partial \mathbf{F}(\mathbf{X}, \mathbf{u})}{\partial \mathbf{X}} + \left[\frac{\partial}{\partial \mathbf{X}} (L_{\mathbf{F}}^{D-1} dh_2(\mathbf{X}))^T \mathbf{F}(\mathbf{X}, \mathbf{u}) \right]^T \\
 &= [\mathbf{B}^{D1} \quad \mathbf{B}^{D2} \quad \mathbf{B}^{D3} \quad -\mathbf{B}^{D1} \quad -\mathbf{B}^{D2}] \\
 D &= 1, 2, 3, 4
 \end{aligned} \tag{A.36}$$

By using equation (A.36), the first order Lie derivative can be derived as

$$\begin{aligned}
 &L_{\mathbf{F}}^1 dh_2(\mathbf{X}) \\
 &= [\mathbf{B}^{01} \quad \mathbf{B}^{02} \quad -1 \quad -\mathbf{B}^{01} \quad -\mathbf{B}^{02}] \begin{bmatrix} 0 & 0 & -\Phi_2 & 0 & 0 \\ 0 & 0 & \Phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\partial}{\partial \mathbf{X}} \begin{bmatrix} \mathbf{B}^{01} \\ \mathbf{B}^{02} \\ -1 \\ -\mathbf{B}^{01} \\ -\mathbf{B}^{02} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ 0 \\ 0 \end{bmatrix}^T \\
 &= [0 \quad 0 \quad \mathbf{B}^{02}\Phi_1 - \mathbf{B}^{01}\Phi_2 \quad 0 \quad 0] + \begin{bmatrix} \frac{\partial \mathbf{B}^{01}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}^{01}}{\partial y} \Phi_2 \\ \frac{\partial \mathbf{B}^{02}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}^{02}}{\partial y} \Phi_2 \\ 0 \\ -\frac{\partial \mathbf{B}^{02}}{\partial x} \Phi_1 - \frac{\partial \mathbf{B}^{02}}{\partial y} \Phi_2 \\ -\frac{\partial \mathbf{B}^{02}}{\partial x} \Phi_1 - \frac{\partial \mathbf{B}^{02}}{\partial y} \Phi_2 \end{bmatrix}^T \\
 &= [\mathbf{B}^{11} \quad \mathbf{B}^{12} \quad 0 \quad -\mathbf{B}^{11} \quad -\mathbf{B}^{12}]
 \end{aligned} \tag{A.37}$$

The expression can be further simplified in form of

$$\begin{aligned}
\begin{bmatrix} \mathbf{B}^{11} \\ \mathbf{B}^{12} \\ \mathbf{B}^{13} \\ -\mathbf{B}^{11} \\ -\mathbf{B}^{12} \end{bmatrix}^T &= \begin{bmatrix} \mathbf{B}^{01}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) - \mathbf{B}^{01}\mathbf{B}^{02}\Phi_1 - \mathbf{B}^{02}\mathbf{B}^{02}\Phi_2 \\ \mathbf{B}^{02}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) + \mathbf{B}^{01}\mathbf{B}^{01}\Phi_1 + \mathbf{B}^{02}\mathbf{B}^{01}\Phi_2 \\ -(\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \\ -\mathbf{B}^{01}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) + \mathbf{B}^{01}\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{02}\mathbf{B}^{02}\Phi_2 \\ -\mathbf{B}^{02}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) - \mathbf{B}^{01}\mathbf{B}^{01}\Phi_1 - \mathbf{B}^{02}\mathbf{B}^{01}\Phi_2 \end{bmatrix}^T \\
&= \begin{bmatrix} \mathbf{B}^{01}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \\ \mathbf{B}^{02}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \\ \mathbf{B}^{03}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \\ -\mathbf{B}^{01}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \\ -\mathbf{B}^{02}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \end{bmatrix}^T + \begin{bmatrix} -\mathbf{B}^{02}(\mathbf{B}^{01}\Phi_1 + \mathbf{B}^{02}\Phi_2) \\ \mathbf{B}^{01}(\mathbf{B}^{01}\Phi_1 + \mathbf{B}^{02}\Phi_2) \\ 0 \\ \mathbf{B}^{02}(\mathbf{B}^{01}\Phi_1 + \mathbf{B}^{02}\Phi_2) \\ -\mathbf{B}^{01}(\mathbf{B}^{01}\Phi_1 + \mathbf{B}^{02}\Phi_2) \end{bmatrix}^T \quad (\text{A.38}) \\
&= \begin{bmatrix} \mathbf{B}^{01}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \\ \mathbf{B}^{02}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \\ \mathbf{B}^{03}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \\ -\mathbf{B}^{01}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \\ -\mathbf{B}^{02}(-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \end{bmatrix}^T + \begin{bmatrix} -\mathbf{A}^{01}\Delta^{-1}(\mathbf{B}^{01}\Phi_1 + \mathbf{B}^{02}\Phi_2) \\ -\mathbf{A}^{02}\Delta^{-1}(\mathbf{B}^{01}\Phi_1 + \mathbf{B}^{02}\Phi_2) \\ 0 \\ \mathbf{A}^{01}\Delta^{-1}(\mathbf{B}^{01}\Phi_1 + \mathbf{B}^{02}\Phi_2) \\ \mathbf{A}^{02}\Delta^{-1}(\mathbf{B}^{01}\Phi_1 + \mathbf{B}^{02}\Phi_2) \end{bmatrix}^T
\end{aligned}$$

and by representing the expression (A.38) in terms of common factors, the first order Lie derivative can be re-expressed as

$$\begin{bmatrix} \mathbf{B}^{11} \\ \mathbf{B}^{12} \\ \mathbf{B}^{13} \\ -\mathbf{B}^{11} \\ -\mathbf{B}^{12} \end{bmatrix}^T = \Upsilon_4 \begin{bmatrix} \mathbf{B}^{01} \\ \mathbf{B}^{02} \\ \mathbf{B}^{03} \\ -\mathbf{B}^{01} \\ -\mathbf{B}^{02} \end{bmatrix}^T + \Upsilon_5 \begin{bmatrix} \mathbf{A}^{01} \\ \mathbf{A}^{02} \\ \mathbf{A}^{03} \\ -\mathbf{A}^{01} \\ -\mathbf{A}^{02} \end{bmatrix}^T \quad (\text{A.39})$$

where

$$\begin{aligned}
\Upsilon_4 &= (-\mathbf{B}^{02}\Phi_1 + \mathbf{B}^{01}\Phi_2) \\
&= \left(\frac{(x_1 - x)}{\Delta^2} (\cos \psi - a\delta \sin \psi) + \frac{(y_1 - y)}{\Delta^2} (\sin \psi + a\delta \cos \psi) \right) \quad (\text{A.40})
\end{aligned}$$

$$\begin{aligned}\Upsilon_5 &= \Delta^{-1} (\mathbf{B}^{01} \Phi_1 + \mathbf{B}^{02} \Phi_2) \\ &= \left(\frac{(y_1 - y)}{\Delta^3} (\cos \psi - a\delta \sin \psi) - \frac{(x_1 - x)}{\Delta^3} (\sin \psi + a\delta \cos \psi) \right)\end{aligned}\quad (\text{A.41})$$

The equation (A.39) shows that the first order Lie derivative of the bearing observation function is equal to the combination of both the zero order Lie derivatives of the bearing and range observation multiplied with their respective factor Υ_4 and Υ_5 . This implies that the first order Lie derivative of the bearing observation function is a dependent vector of the combination of zero order Lie derivative of the range and bearing observation function.

Second order Lie derivative of bearing observation:

Now with the first order Lie derivative of the bearing observation function, its second order Lie derivative can be derived using equation (A.36) and expressed as,

$$\begin{aligned}L_{\mathbf{F}}^2 dh_2(\mathbf{X}) &= \begin{bmatrix} \mathbf{B}^{21} & \mathbf{B}^{22} & \mathbf{B}^{23} & -\mathbf{B}^{21} & -\mathbf{B}^{22} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{B}^{11} & \mathbf{B}^{12} & \mathbf{B}^{13} & -\mathbf{B}^{11} & -\mathbf{B}^{12} \end{bmatrix} \begin{bmatrix} 0 & 0 & -\Phi_2 & 0 & 0 \\ 0 & 0 & \Phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\partial}{\partial \mathbf{X}} \begin{bmatrix} \mathbf{B}^{11} \\ \mathbf{B}^{12} \\ \mathbf{B}^{13} \\ -\mathbf{B}^{11} \\ -\mathbf{B}^{12} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ 0 \\ 0 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 0 & \mathbf{B}^{12}\Phi_1 - \mathbf{B}^{11}\Phi_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathbf{B}^{11}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}^{11}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}^{11}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{B}^{12}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}^{12}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}^{12}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{B}^{13}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}^{13}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}^{13}}{\partial \psi} \Phi_3 \\ -\frac{\partial \mathbf{B}^{11}}{\partial x} \Phi_1 - \frac{\partial \mathbf{B}^{11}}{\partial y} \Phi_2 - \frac{\partial \mathbf{B}^{11}}{\partial \psi} \Phi_3 \\ -\frac{\partial \mathbf{B}^{12}}{\partial x} \Phi_1 - \frac{\partial \mathbf{B}^{12}}{\partial y} \Phi_2 - \frac{\partial \mathbf{B}^{12}}{\partial \psi} \Phi_3 \end{bmatrix}^T\end{aligned}\quad (\text{A.42})$$

The expression can be further simplified in form of

$$\begin{bmatrix} \mathbf{B}^{21} \\ \mathbf{B}^{22} \\ \mathbf{B}^{23} \\ -\mathbf{B}^{21} \\ -\mathbf{B}^{22} \end{bmatrix}^T = \begin{bmatrix} \frac{\partial \mathbf{B}^{11}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}^{11}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}^{11}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{B}^{12}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}^{12}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}^{12}}{\partial \psi} \Phi_3 \\ \frac{\partial \mathbf{B}^{13}}{\partial x} \Phi_1 + \frac{\partial \mathbf{B}^{13}}{\partial y} \Phi_2 + \frac{\partial \mathbf{B}^{13}}{\partial \psi} \Phi_3 + \mathbf{B}^{12} \Phi_1 - \mathbf{B}^{11} \Phi_2 \\ -\frac{\partial \mathbf{B}^{11}}{\partial x} \Phi_1 - \frac{\partial \mathbf{B}^{11}}{\partial y} \Phi_2 - \frac{\partial \mathbf{B}^{11}}{\partial \psi} \Phi_3 \\ -\frac{\partial \mathbf{B}^{12}}{\partial x} \Phi_1 - \frac{\partial \mathbf{B}^{12}}{\partial y} \Phi_2 - \frac{\partial \mathbf{B}^{12}}{\partial \psi} \Phi_3 \end{bmatrix}^T \quad (\text{A.43})$$

and by representing the expression (A.43) in terms of common factors, the second order

Lie derivative can be re-expressed as

$$\begin{bmatrix} \mathbf{B}^{21} \\ \mathbf{B}^{22} \\ \mathbf{B}^{23} \\ -\mathbf{B}^{21} \\ -\mathbf{B}^{22} \end{bmatrix}^T = \Upsilon_6 \begin{bmatrix} \mathbf{B}^{01} \\ \mathbf{B}^{02} \\ \mathbf{B}^{03} \\ -\mathbf{B}^{01} \\ -\mathbf{B}^{02} \end{bmatrix}^T + \Upsilon_7 \begin{bmatrix} \mathbf{A}^{01} \\ \mathbf{A}^{02} \\ \mathbf{A}^{03} \\ -\mathbf{A}^{01} \\ -\mathbf{A}^{02} \end{bmatrix}^T \quad (\text{A.44})$$

where

$$\begin{aligned} \Upsilon_6 &= \begin{pmatrix} 2\mathbf{B}^{02}\mathbf{B}^{02}\Phi_1^2 - 2\mathbf{B}^{01}\mathbf{B}^{01}\Phi_1^2 - 4\Delta^{-1}\mathbf{A}^{01}\mathbf{A}^{01}\mathbf{A}^{02}\Phi_1\Phi_2 \\ + 2\mathbf{B}^{01}\mathbf{B}^{01}\Phi_2^2 - 2\mathbf{B}^{02}\mathbf{B}^{02}\Phi_2^2 + \mathbf{B}^{02}\Phi_2\Phi_3 + \mathbf{B}^{01}\Phi_1\Phi_3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2(x_1 - x)^2 - 2(y_1 - y)^2}{\Delta^4} (\cos \psi - a\delta \sin \psi)^2 \\ + \frac{4(y_1 - y)(x_1 - x)^2}{\Delta^4} (\cos \psi - a\delta \sin \psi)(\sin \psi + a\delta \cos \psi) \\ + \frac{2(y_1 - y) - 2(x_1 - x)}{\Delta^4} (\sin \psi + a\delta \cos \psi)^2 \\ - \frac{(x_1 - x)\delta}{\Delta^2} (\sin \psi + a\delta \cos \psi) + \frac{(y_1 - y)\delta}{\Delta^2} (\cos \psi - a\delta \sin \psi) \end{pmatrix} \end{aligned} \quad (\text{A.45})$$

$$\begin{aligned}
\Upsilon_7 &= \left(-4\mathbf{B}^{02}\mathbf{B}^{01}\Phi_1^2 + 8\mathbf{B}^{01}\mathbf{B}^{01}\Phi_1\Phi_2 - 4\mathbf{B}^{02}\mathbf{B}^{02}\Phi_1\Phi_2 + \mathbf{B}^{02}\Phi_1\Phi_3 - \mathbf{B}^{01}\Phi_1\Phi_3 \right) \\
&= \begin{pmatrix} \frac{4(y_1 - y)(x_1 - x)}{\Delta^4} (\cos \psi - a\delta \sin \psi)^2 \\ + \frac{8(y_1 - y)^2 - 4(x_1 - x)^2}{\Delta^4} (\cos \psi - a\delta \sin \psi) (\sin \psi + a\delta \cos \psi) \\ - \frac{4(y_1 - y)(x_1 - x)}{\Delta^4} (\sin \psi + a\delta \cos \psi)^2 \\ - \frac{(x_1 - x)\delta}{\Delta^2} (\cos \psi - a\delta \sin \psi) - \frac{(y_1 - y)\delta}{\Delta^2} (\sin \psi + a\delta \cos \psi) \end{pmatrix} \quad (\text{A.46})
\end{aligned}$$

The equation (A.44) shows that the second order Lie derivative of the bearing observation function is equal to the combination of both the zero order Lie derivatives of the bearing and range observation function multiplied with their respective factor Υ_6 and Υ_7 . This implies that the second order Lie derivative of the bearing observation function is a dependent vector of the combination of zero order Lie derivative of the range and bearing observation function.

Remarks:

From these derivations, the higher order Lie derivatives of range and bearing observation functions are shown to be the combination of their zero order Lie derivatives of range and bearing observation function. The result can be easily generalised to observing as many unknown features or landmarks as possible. Thus, this concludes that the observability matrix of the CML system will only have two independent vectors (the zero order Lie derivative of the range and bearing observation function) or the rank equals to two, which is deficient.