Adaptive backstepping control of uncertain systems with actuator failures and subsystem interactions

Wang, Wei

2011

Wang, W. (2011). Adaptive backstepping control of uncertain systems with actuator failures and subsystem interactions. Doctoral thesis, Nanyang Technological University, Singapore.

https://hdl.handle.net/10356/46284

https://doi.org/10.32657/10356/46284

ADAPTIVE BACKSTEPPING CONTROL OF UNCERTAIN SYSTEMS WITH ACTUATOR FAILURES AND SUBSYSTEM INTERACTIONS

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A thesis submitted to the Nanyang Technological University

in fulfillment of the requirements for the degree of

Doctor of Philosophy

2011

Acknowledgement

This is a great opportunity to express my sincere gratitude to all who have supported me during the completion of this thesis.

First and foremost, I am heartily thankful to my supervisor, Professor Changyun Wen, for his constant guidance and inspirational encouragement. This thesis would not have been possible without his knowledge and valuable advices. I am also grateful to him for his genuine concern about my well-being and future plans.

I am specially grateful to Dr. Jing Zhou for her generous support which is highly beneficial to the related research. Some of her bright thoughts are very helpful in shaping up my ideas.

I would also like to extend my appreciation to Professor Guang-Hong Yang for his useful discussion about actuator redundancy. Appreciation continues to Professor Gang Tao for his constructive comments in improving the quality of this thesis, to Processors Romeo Ortega and David John Hill for their advisory comments and encouragement.

Furthermore, it is a great pleasure to thank my colleagues, Guoqi Li, Yan Chen and Jiangshuai Huang, for hours of productive discussions.

I would like to convey special acknowledgement to my close friends, Yue Tian, Jianglei Qin, Xi Wang, Ling Huang and Zhaolei Wang. I am extraordinarily fortunate to have them always by my side, giving me faith, sharing my joy and bitterness.

I feel also greatly indebted to Yu Zhao, Jiayin Zhou, Ping Wang, Tingting Gao, Yong Zhou, Song Cui and Dan Luo for their patience, assistance and encouragement throughout these years.

Last but not least, I owe my deepest gratitude to my beloved family for their care, understanding and constant encouragement, although they are in miles away. My parents, Xiaolie Wang and Minna Suo, my husband, Qiang Wu, my dear grandma, Jinglun Zhao and my pretty cousin, Yingping Suo, have never lost confidence in me. Their unconditional love is always the source of strength to me, when I am in the face of obstacles.

Wei Wang

January 2011

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Summary

In this thesis, new methodologies of designing adaptive controllers for uncertain systems in the presence of actuator failures and subsystem interactions are investigated. The main contributions are summarized in the following two parts.

Part I. Adaptive actuator failure compensation

- Since the actuators redundant for one another may not necessarily share similar characteristics for different reasons, the relative degrees with respect to the system inputs corresponding to these actuators are sometimes different. To stabilize such a class of systems in the presence of total loss of effectiveness (TLOE) type of actuator failures, modified output-feedback control schemes are proposed by introducing prefilters before the actuators. We start with linear systems and consider the output regulation problem firstly. It is shown that the effects due to the failures can be compensated for without explicit failure detection and isolation. Global boundedness of all closed-loop signals is maintained and system output regulation is ensured. The results are then extended to nonlinear systems with tracking problem being considered.

- There are few results available in investigating the transient performance of the adaptive system in failure cases, although it is of great importance for the control problems. It is analyzed in this thesis that, the transient performance of the system in the presence of uncertain actuator failures cannot be adjusted when a basic

adaptive backstepping control scheme is adopted, which can be regarded as a representative of currently available adaptive failure compensation results. A new design scheme incorporating a prescribed performance bound (PPB) is then proposed. By guaranteeing that the tracking error satisfies the PPB all the time no matter when the actuator failures occur, a prescribed transient performance of the tracking error is ensured in both failure and failure-free cases. Moreover, the transient performance in terms of the convergence rate and maximum overshoot of the tracking error can be improved by tuning certain parameters in characterizing the PPB.

- In most of the existing results on adaptive control of systems with actuator failures, only the cases with finite number of failures are considered. However, it is possible that some actuator failures occur intermittently in practice. Thus the actuators may unawarely change from a failure mode to a normally working mode or another failure mode infinitely many times. To address the problem of compensating for infinite number of actuator failures, we propose a new adaptive control scheme based on modular backstepping design. It is proved that the boundedness of all closed-loop signals is ensured in the case with infinite number of failures, as long as the time interval between two successive changes of failure pattern is bounded below by an arbitrary positive number. The performance of the tracking error in the mean square sense with respect to the frequency of failure pattern changes is also established. Furthermore, asymptotic tracking can be achieved with the proposed scheme when the number of failures is finite.

Part II. Decentralized adaptive stabilization

- In practice, an interconnected system unavoidably has dynamic interactions depending on both subsystem inputs and outputs. Because of the difficulties in handling the input variables and their derivatives during recursive design steps, the results on decentralized adaptive control of uncertain systems with interactions involving subsystem inputs based on backstepping technique are quite limited. In this

thesis, we propose a decentralized control scheme by using the standard adaptive backstepping technique without any modification to stabilize a class of linear interconnected systems with dynamic input and output interactions. Global asymptotic stabilization of the system is shown. The transient system performance in terms of the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms of the outputs are established as the functions of design parameters. The results are then extended to nonlinear interconnected systems.

- Besides, the reliability of our proposed decentralized adaptive control approach for linear interconnected systems in the presence of some subsystems breaking down is also analyzed.

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List of Symbols

$\in$	belongs to
$\notin$	does not belong to
$\subset$	subset of
$\cap$	intersection
$\triangleq$	defined as
$\forall$	for all
$\rightarrow$	tends to
$\Rightarrow$	implies
$\Re$	the space of real numbers
$\Re_+$	the space of non-negative real numbers
$\Re^n$	the space of real-valued vectors of dimension n
$\Re^{m\times n}$	the space of real-valued $m \times n$ matrices
$f : S_1 \longrightarrow S_2$	a function f mapping a space S_1 into a space S_2
$\|a\|$	the absolute value of a scalar a
$\\|x\\|(\\|A\\|)$	the Euclidean norm of a vector x or the induced $2-$norm of a matrix A
$x^T(A^T)$	the transpose of a vector x or a matrix A

$\|A\|_F$	the Frobenius norm of the matrix A
$\mathrm{Tr}\{A\}$	the trace of the matrix A
sup	supremum, the least upper bound
$\|x\|_p$	$\mathcal{L}_p$ norm for $p \in [1, \infty)$ that $\|x\|_p = (\int_0^\infty \|x(\tau)\|^p d\tau)^{1/p}$
$\|x\|_\infty$	$\mathcal{L}_\infty$ norm $\|x\|_\infty = \sup_{t \geq 0} \|x(t)\|$
$\mathcal{L}_1$	the signal space that $\mathcal{L}_1 = \{x(t) \in \Re^n \mid \|x\|_1 < \infty\}$
$\mathcal{L}_2$	the signal space that $\mathcal{L}_2 = \{x(t) \in \Re^n \mid \|x\|_2 < \infty\}$
$\mathcal{L}_\infty$	the signal space that $\mathcal{L}_\infty = \{x(t) \in \Re^n \mid \|x\|_\infty < \infty\}$
$\dot{y}$	the first derivative of y with respect to time
$\ddot{y}$	the second derivative of y with respect to time
$y^{(i)}$	the ith derivative of y with respect to time
I_n	the identity matrix of dimension $n \times n$
$\lambda_{\max}(A)$	the maximum eigenvalue of a matrix A
$\lambda_{\min}(A)$	the minimum eigenvalue of a matrix A
$\mathrm{sgn}(x)$	the signum function that $\mathrm{sgn}(x)$ is 1, 0, or -1 respectively, according as $x > 0, x = 0, x < 0$
$\mathrm{diag}\{a_1, \ldots, a_n\}$	a diagonal matrix with diagonal elements a_1 to a_n
Z^+	the set of positive integers: $\{1, 2, \ldots\}$
$\square$	designation of the end of proofs

Chapter 1

Introduction

To stabilize a system and achieve other desired objectives by using adaptive control methodology, a controller is normally constructed to involve adjustable parameters generated by a parameter estimator. Both the controller and parameter estimator are designed on the basis of the mathematical representation of the plant. Adaptive control is one of the most promising techniques to handle uncertainties on system parameters, structures, external disturbances and so on. Since the backstepping technique was proposed and utilized in designing adaptive controllers, numerous results on adaptive control of linear systems had been extended to certain classes of nonlinear systems not based solely on feedback linearization. In contrast to traditional adaptive control design methods, adaptive backstepping control can easily remove relative degree limitations and provide improved transient performance by tuning the design parameters. Although there have been a large number of results developed in the area of adaptive backstepping control, some open issues still exist such as investigating the transient performance of the plant in the presence of actuator failures and stabilizing large scale systems with dynamic interactions depending on subsystem inputs when adaptive backstepping control is applied.

In this chapter, we firstly give a brief overview of adaptive control and adaptive backstepping control. Literature review and the motivation of our work are then elaborated in two parts. After that, the objectives and major contributions of the thesis are presented followed by a preview of the remaining chapters.

1.1 Adaptive Control

Adaptive control is a design idea of self-tuning the control parameters based on the performance error related information to better fit the environment. Thus a variety of objectives such as system stability, desired signal regulation, steady-state and transient tracking performance can be achieved. Since it was conceived in the early 1950s, it has been a research area of great theoretical and practical significance. The design of autopilots for high performance aircraft was one of the primary motivations for active research in adaptive control [5]. During nearly six decades of its development, a good number of adaptive control design approaches have been proposed for different classes of systems to solve various problems. Model reference adaptive control (MRAC) [6–8], system and parameter identification based schemes [9, 10], adaptive pole placement control [11, 12] are some commonly used conventional adaptive control methods. In 1980s, several modification techniques such as normalization [13,14], dead-zone [15,16], switching $\sigma-$modification [17] and parameter projection [18–20] were developed to improve the robustness of the adaptive controllers against unmodeled dynamics, disturbances or other modeling errors. In the early 1990s, adaptive backstepping control [21] was presented to control certain classes of nonlinear plants with unknown parameters. The tuning functions concept provides improved transient performance of the adaptive control system. The results listed above are only a part of remarkable breakthroughs in the development of adaptive control, more detailed literature reviews of conventional adaptive control can be found in [5, 22–25] and other related textbooks or survey papers.

The prominent feature of adaptive control in handling systems with unknown and/or time varying parameters constitutes one of the reasons for the rapid development of this technique. An adaptive controller is normally designed by combining parameter update law and control law. The former one is also known as parameter estimator providing the adaptation law for the adjustable parameters of the controller at each time instant [25].

Adaptive control techniques used to be classified into direct and indirect ones according to the procedure of obtaining the controller parameters. The methods of computing the controller parameters based on the estimated system parameters are referred as indirect adaptive control, while the the controller parameters are estimated (directly) without intermediate calculation in direct adaptive control. The common principle of conventional adaptive control techniques, no matter direct or indirect, is certainty equivalence principle. This means the controller structure is designed as if all estimated parameters were true, to achieve desired performances.

1.2 Adaptive Backstepping Control

Adaptive control approaches can also be classified into Lyapunov-based and estimation-based ones according to the type of parameter update law and the corresponding proof of stability. In the former design procedure, the adaptive law and the synthesis of the control law are carried out simultaneously based on Lyapunov stability theory. However in estimation-based design, the construction of adaptive law and control law are treated as separate modules. The adaptive law can be chosen by following gradient, least-squares or other optimization algorithms.

To deal with linear systems, traditional Lyapunov-based adaptive control is only applicable to the plants with relative degree no more than two. Such relative degree limitation is translated to another structure obstacle on the "level of uncertainty" in the nonlinear parametric state-feedback case, where the "level of uncertainty" refers to the number of integrators between the control input and the unknown parameter [26]. The structure restrictions in linear and nonlinear cases can be removed by a recursive design procedure known as backstepping. The technique is comprehensively addressed in [21], where a brief review of its development can also be found. Tuning functions and modular design are the two main design approaches presented in the book. The former approach is proposed to solve an over-parameterization problem existed in previous results on Lyapunove-based adaptive backstepping control. It can keep the number of parameter estimates be equal to the number of unknown parameters and help simplify the implementation. In the latter design approach, the estimation-based type adaptive laws can also be selected to update controller parameters by synthesizing a controller with the aid of nonlinear damping terms to achieve input-to-state stability properties of the error system. Such an approach is known as modular design since a significant level of modularity of the controller-estimator pair is achieved.

Both tuning functions and modular design approaches can provide a systematic procedure to design the stabilizing controllers and parameter estimators. Moreover, the adaptive backstepping control technique has other advantages such as avoiding cancelation of useful nonlinearities, and improving transient performance of the system by tuning the design parameters. Although a number of results using this technique have been reported [27–38], there are still some open issues such as improving transient performance in the presence of uncertain actuator failures, accommodating unknown dynamic interactions depending on subsystem inputs.

1.3 Literature Review and Motivation

1.3.1 Actuator Failure Compensation

In a control system, an actuator is a mechanism representing the link between the controller and a system or a process to be controlled (which is often referred to as a plant). It performs the control command generated from the controller on the plant, for the purposes of stabilizing the closed-loop system and achieving other desired objectives. In practice, an actuator is not guaranteed to work normally all the time. Instead, it may undergo certain failures which will influence its effectiveness in executing the control law. These failures may cause deteriorated performance or even instability of the system. Accommodating such failures is important to ensure the safety of the systems, especially for life-critical systems such as aircrafts, spacecrafts, nuclear power plants and so on. Recently, increasing demands for safety and reliability in modern industrial systems with large complexity have motivated more and more researchers to concentrate on the investigation of proposing control design methods to tolerant actuator failures and related areas.

Several effective control design approaches have been developed to address the actuator failure accommodation problem for both linear [39–49] and nonlinear systems [50–61]. They can be roughly classified into two categories, i.e. passive and active approaches. Typical passive approaches aim at achieving insensitivity of the system to certain presumed failures by adopting robust control techniques, see for instance in [40, 42, 47, 48, 61, 62]. Since fixed controllers are used throughout failure/failure-free cases and failure detection/diagnostic (FDD) is not required in these results, the design methods are computationally attractive. However, they have the drawback that the designed controllers are often conservative for large failure pattern changes. This is because the achieved system performance based on worst-case failures may not be satisfactory for each failure scenario. In contrast to the passive methods, the structures and/or the parameters of the controllers are adjustable in real time when active design approaches are utilized. Furthermore, FDD is often required in active approaches and provide the estimated failure information to the controller design. Therefore the adverse effects brought by the actuator failures, even if large failure pattern changes are involved, can be compensated for and the system stability is maintained. A number of active schemes have been presented, such as pseudo-inverse method [63], eigenstructure assignment [41, 64], multiple model [43, 44, 59, 65], model predictive control [66], neural networks/fuzzy logic based scheme [50,51,53,57] and sliding mode control based scheme [49]. Different from the ideas of redesigning the nominal controllers for the post-failure plants in these schemes, virtual actuator method [67, 68] hides the effects of the failures from the nominal controller to preserve the nominal controller in the loop.

Apart from these, adaptive control is also an active method well suited for actuator failure compensation [39,52,69,70] because of its prominent adapting ability to the structural, parametric uncertainties and variations in the systems. As opposed to most of the active approaches, many adaptive control design schemes can be applied with neither control restructuring nor FDD processing. Moreover, not only the uncertainties caused by the failures, but also the unknown system parameters are estimated online for updating the controller parameters adaptively. In [45,46], Tao *et al.* proposed a class of adaptive control methods for linear systems with TLOE type of actuator failures. It is known that backstepping technique [21] has been widely used to design adaptive controllers for uncertain nonlinear systems due to its advantages. The results in [45, 46] have been successfully extended to nonlinear systems in [54–56, 71] by adopting the backstepping technique. In [72], a robust adaptive output feedback controller was designed based on the backstepping technique to stabilize nonlinear systems with uncertain TLOE failures involving parameterizable and unparameterizable time varying terms. In fact, adaptive control also serves as an assisting tool for other methods as in [39,43,44,50,51,53,57,59,73]. For example, a reconfigurable controller is designed by combining neural networks and adaptive backstepping technique to accommodate the incipient actuator failures for a class of single-input single-output (SISO) nonlinear systems in [53]. In [73], the actuator failure tolerance for linear systems with known system parameters is achieved by proposing a control scheme combining linear matrix inequality (LMI) and adaptive control.

In addition to the actuators, unexpected failures may occur on other components such as the sensors in control systems. The research area of accommodating these failures to improve the system reliability is also referred to as fault tolerant control (FTC). More complete survey of the concepts and methods in fault tolerant control could be found in [74–79].

Although fruitful results have been reported in control of systems in the presence of actuator failures, there are a number of challenging problems remained unsolved in this area. Some of the open issues drawing our attention are presented as follows.

- A common structural condition exists in [45, 46, 54–56]. That is, only two actuators, to which the corresponding relative degrees with respect to the inputs are the same, can be redundant for each other. The condition is restrictive in many practical situations such as to control a system with two rolling carts connected by a spring and a damper for the purpose of stability and regulating one of the carts at a specified position. Suppose that there are two motors generating external forces for distinct carts, respectively. One of them can be considered to be redundant for the other in case that it is blocked with the output stuck at an unknown value. The relative degrees corresponding to the two actuators are different. Moreover, an elevator and a stabilizer may compensate for each other in an aircraft control system, of which the relative degree condition is also hard to be satisfied. Thus the relaxation of such structural condition is of significant importance.

- It is well known that the backstepping technique [21] can provide a promising way to improve the transient performance of adaptive systems in terms of $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms of the tracking error in failure-free case if certain trajectory initialization can be performed. Some adaptive backstepping based failure compensation methods have been developed [54–56, 71, 72]. Nevertheless, there are few results available on characterizing and improving the transient performance of the systems in the presence of uncertain actuator failures. This is mainly because the trajectory initialization is difficult to perform when the failures are uncertain in time, pattern and value. Therefore, it is interesting to develop a new adaptive backstepping based design scheme with which the transient performance of the tracking error can still be established in failure cases.

- In most of the existing results on adaptive control of systems with actuator failures, only the cases with finite number of failures are considered. It is assumed that one actuator may only fail once and the failure mode does not change afterwards. This implies that there exists a finite time T_r such that no further failure occurs on the system after T_r. However, it is possible that some actuator failures occur intermittently in practice. Thus the actuators may unawarely change from a failure mode to a normally working mode or another different failure mode infinitely many times. For example, poor electrical contact can cause repeated unknown breaking down failures on the actuators in some control systems. Clearly, the actuator failures cannot be restricted to occur only before a finite time in such a case. Moveover, the idea of stability analysis based on Lyapunov function for the case with finite number of failures cannot be directly extended to the case with infinite number of failures, because the possible increase of the Lyapunov function cannot be ensured bounded automatically when the parameters may experience infinite number of jumps. It is thus of both practical and theoretical significance to address the problem of compensating for infinite number of failures.

1.3.2 Decentralized Adaptive Control

Nowadays, interconnected systems quite commonly exist in practice. Power networks, urban traffic networks, digital communication networks, ecological processes and economic systems are some typical examples of such systems. They normally consist of a number of subsystems which are separated geographically. Due to the lack of centralized information and computing capability, decentralized control strategy was proposed and has been proved effective in stabilizing these systems. Even though the local controllers are designed independently for each subsystem by using only the local available signals in a perfectly decentralized control scheme, to stabilize such large scale systems or achieve individual tracking objectives for each subsystem cannot be straightforwardly extended from the results for the single loop systems. This is because the subsystems are often interconnected and the interactions between any two subsystems may be difficult to be identified or measured. Sometimes, the uncertain interactions can be roughly modeled as static functions of signals from other subsystems and the bounding information is known. However, such bounding information is unknown to local designers more often and the interactions may also appear as dynamic processes. Moreover, external disturbances and unmodeled dynamics may also exists after modeling subsystems. In such cases, the problem of compensating for the effects from the uncertain interactions and other variety of uncertainties is quite complicated.

Adaptive control is one of the most promising tools to accommodate uncertainties, it is also widely adopted in developing decentralized control methods. Based on conventional adaptive approach, several results on global stability and steady state tracking were reported, see for examples [80–86]. In [80], a class of linear interconnected system with bounded external disturbances, unmodeled interactions and singular perturbations are considered. A direct MRAC based decentralized control scheme is proposed with the fixed $\sigma-$modification performed on the adaptive laws. Sufficient conditions are obtained which guarantee the existence of a region of attraction for boundedness and exponential convergence of the state errors to a small residual set. The related extension work could be found in [17] where nonlinearities are included. The relative degree corresponding to the decoupled subsystems are constrained no more than two due to the use of Kalman-Yakubovich (KY) lemma. An indirect pole assignment based decentralized adaptive control approach is developed to control a class of linear discrete-time interconnected systems in [85]. The minimum phase and relative degree assumptions in [80, 81] are not required. By using the projection operation technique in constructing the gradient parameter estimator, the parameter estimates can be constrained in a known convex compact region. Global boundedness of all states in the closed adaptive system for any bounded initial conditions, set points and external disturbances are ensured if unmodeled dynamics and interactions are sufficiently weak. The results are extended to continuous-time interconnected systems in [87].

The backstepping technique was firstly adopted in decentralize adaptive control by Wen in [28], where a class of linear interconnected systems involving nonlinear interactions were considered. In contrast to previous results by utilizing conventional direct adaptive control based methods, the restrictions on subsystem relative degrees were removed by following a step-by-step algorithm. Thus the interconnected system to be regulated consists of N subsystems, each of which can have arbitrarily relative degrees. By using the backstepping technique, more results have been reported on decentralized adaptive control [29, 32, 88–93]. Compared to [28], more general class of systems with the consideration of unmodeled dynamics is studied in [29, 32]. In [88, 89], nonlinear interconnected systems are addressed. In [89, 90], decentralized adaptive stabilization for nonlinear systems with dynamic interactions depending on subsystem outputs or unmodeled dynamics is studied. In [91], systems with non-smooth hysteresis nonlinearities and higher order nonlinear interactions are considered and in [92] results for stochastic nonlinear systems are established. A result on backstepping adaptive tracking is established in [93]. However, except for [29, 32, 89], all the results are only applicable to systems with interaction effects bounded by static functions of subsystem outputs. This is restrictive as it is a kind of matching condition in the sense that the effects of all the unmodeled interactions to a local subsystem must be in the range space of the output of this subsystem.

- In practice, an interconnected system unavoidably has dynamic interactions involving both subsystem inputs and outputs. Especially, dynamic interactions directly depending on subsystem inputs commonly exist. For example, the non-zero off-diagonal elements of a transfer function matrix represent such interactions. So far, there is few result reported to control systems with interactions directly depending on subsystem inputs even for the case of static input interactions by using the backstepping technique. This is due to the challenge of handling the input variables and their derivatives of all subsystems during the recursive design steps.

- Besides, even fewer result is available on investigating the reliability of such controlled systems in the presence of failures.

1.4 Objectives

Motivated by the open problems which were discussed in previous section, the main objectives of our research are as follows.

▷ To propose adaptive compensation control methods based on backstepping technique such that the following three issues can be addressed separately.

- The effects brought by the failed actuators can be compensated for with the remaining actuators, although the relative degrees corresponding to them may be different.

- In addition to system stability and steady state performance, the transient performance of the adaptive systems can also be guaranteed in failure cases.

- Infinite number of actuator failures can be accommodated and system stability can be maintained.

▷ To propose decentralized adaptive control methods based on backstepping technique such that,

- a class of interconnected systems with dynamic interactions directly depending on subsystem inputs can be stabilized;

- when some subsystems break down, the stability of the closed-loop system can still be ensured.

▷ To recommend some interesting topics which are worthy to be explored.

1.5 Major contributions of the Thesis

In achieving the objectives, some results have been obtained and will be presented in Chapters 3-7 in the thesis. The major contributions of the thesis are summarized in the following two parts.

Part I. Adaptive actuator failure compensation

- In Chapter 3, by introducing a pre-filter before each actuator in designing output-feedback controllers for the systems with TLOE type of failures, the relative degree restriction corresponding to the redundant actuators will be relaxed. Linear systems will be considered firstly, where the boundedness of all closed-loop signals and output regulation will be shown. The results will then be extended to nonlinear systems with asymptotic tracking to be achieved. These results have been published, see [7] and [10] in Author's Publications.

- In Chapter 4, transient performance of the adaptive systems in failure cases, when the existing backstepping based compensation control method is utilized, will be analyzed. A new adaptive backstepping based failure compensation scheme will be proposed to guarantee a prescribed transient performance of the tracking error, no matter when actuator failures occur. These results have been published, see [4] in Author's Publications.

- In Chapter 5, an adaptive backstepping based modular design strategy will be presented with the aid of projection operation technique to ensure system stability in the presence of infinite number of actuator failures. It will be shown that the tracking error can be small in the mean square sense when the failure pattern changes are infrequent and asymptotic tracking in the case with finite number of failures can be ensured. These results have been reported in the paper recently accepted by *Automatica* as a *regular* paper, see [1] in Author's Publications.

Part II. Decentralized adaptive stabilization

- In Chapter 6, a decentralized control method, by using the standard adaptive backstepping technique without any modification, will be proposed for a class of interconnected systems with dynamic interconnections and unmodeled dynamics depending on subsystem inputs as well as outputs. It will be shown that the overall interconnected system can be globally stabilized and the output regulation of each subsystem can be achieved. The relationship between the transient performance of the adaptive system and the design parameters will also be established. The results on linear interconnected systems will be presented firstly and then be extended to nonlinear interconnected systems. These results have been reported on *Automatica* as a *regular* paper, see [6] in Author's Publications.

- In Chapter 7, the reliability of the proposed decentralized controllers against some subsystems breaking down will be investigated. The conditions to ensure the global stability of the closed-loop system will be given. These results have been published, see [5] in Author's Publications.

1.6 Organization of the Thesis

The thesis is composed of 8 chapters.

In Chapter 1, an overview of the thesis is provided by illustrating the motivation, research objectives and main contributions achieved.

In Chapter 2, the concepts of adaptive backstepping technique, which is the basic tool in the thesis, will be given. By considering a class of parametric strict-feedback nonlinear systems, backstepping based tuning functions and modular design schemes will be introduced separately, where the procedures of both designing controllers and stability analysis will be presented.

In Chapters 3-5, new adaptive compensation methods based on backstepping technique will be proposed to handle uncertain actuator failures. In Chapters 6 and 7, decentralized adaptive backstepping stabilization for interconnected systems with dynamic interconnections and unmodeled dynamics depending on subsystem inputs will be investigated. Detailed contributions achieved in these chapters have been presented in previous section.

In Chapter 8, the thesis will be concluded. Furthermore, some interesting topics which are worthy to be further investigated in the areas of both adaptive failure compensation and decentralized adaptive control will be recommended.

Chapter 2

Adaptive Backstepping Control

The concepts of adaptive backstepping technique will be reviewed in this chapter based on [21, 37] to provide the underpinning framework for the new development in the remainder of the thesis.

The backstepping technique is a powerful design tool to stabilize nonlinear systems, which may not be completely linearizable. It was proposed in the early 1990s and was comprehensively discussed by Krstic, Kanellakopoulos and Kokotovic in [21]. "Backstepping" vividly describes a step by step procedure to generate control input to achieve system stabilization and tracking, which are the original control objectives. At each step, a scalar system is to be stabilized. One or more of the state variables are considered as a virtual control, for which a stabilizing function is chosen as if it was the final stage. At the last step, the control law of the actual input is obtained.

In the cases with unknown system parameters, adaptive backstepping controllers are designed by incorporating the estimated parameters. Parameter estimators can be constructed at the same time with the adaptive controllers based on the Lyapunov functions augmented by the squared terms of estimation errors. By combining tuning functions technique, the over-parametrization problem is solved and the cost for implementing the control scheme is reduced. Moreover, in adaptive backstepping based modular design approaches, parameter estimators can be generated separately from the controllers and formed as gradient or least-squares types.

In this chapter, the concepts of integrator backstepping and adaptive backstepping control will be firstly introduced. The procedures to design adaptive controllers by incorporating the tuning functions and modular design schemes, are then presented. In the second part, a class of parametric strict-feedback nonlinear systems is considered and stability analysis for the two schemes are also provided briefly.

2.1 Backstepping Concepts

2.1.1 Integrator Backstepping

Consider the system

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0, \tag{2.1}$$

where $x \in \Re^n$ and $u \in \Re$ are the state and control input respectively. To illustrate the concept of integrator backstepping, an assumption on (2.1) is firstly made.

Assumption 2.1.1. *There exists a continuously differentiable feedback control law*

$$u = \alpha(x), \quad \alpha(0) = 0 \tag{2.2}$$

and a smooth, positive definite, radially unbounded function $V\colon \Re^n \to \Re_+$ *such that*

$$\frac{\partial V}{\partial x}(x)[f(x) + g(x)\alpha(x)] \leq -W(x) \leq 0, \quad \forall x \in \Re^n, \tag{2.3}$$

where $W\colon \Re^n \to \Re$ *is positive semidefinite.*

We then consider a system that is (2.1) augmented by an integrator,

$$\dot{x} = f(x) + g(x)\xi \tag{2.4}$$

$$\dot{\xi} = u, \tag{2.5}$$

where $u \in \Re$ is the control input. Based on Assumption 2.1.1, the control law for u will be generated in two steps.

In the first step, we stabilize (2.4) by treating ξ as a virtual control variable. According to Assumption 2.1.1, $\alpha(x)$ is a "desired value" of the virtual control. We define an error variable z as the difference between the "desired value" $\alpha(x)$ and the actual value of ξ, i.e.

$$z = \xi - \alpha(x). \tag{2.6}$$

Rewrite the system (2.4) by considering the definition of z and differentiate z with respect to time,

$$\dot{x} = f(x) + g(x)(\alpha(x) + z) \tag{2.7}$$

$$\dot{z} = \dot{\xi} - \dot{\alpha}(x) = u - \frac{\partial \alpha(x)}{\partial x}[f(x) + g(x)(\alpha(x) + z)]. \tag{2.8}$$

In the second step, we define a positive definite function $V_a(x, z)$ by augmenting $V(x)$ in Assumption 2.1.1 as

$$V_a(x, z) = V(x) + \frac{1}{2}z^2. \tag{2.9}$$

Computing the time derivative of $V_a(x, z)$ along with (2.3), (2.7) and (2.8), we have

$$\begin{aligned}
\dot{V}_a(x, z) &= \dot{V}(x) + z\dot{z} \\
&= \frac{\partial V}{\partial x}(f + g\alpha + gz) + z\left(u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z))\right) \\
&= \frac{\partial V}{\partial x}(f + g\alpha) + z\left(u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x}g\right)
\end{aligned}$$

$$\leq \quad -W(x) + z\left(u - \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) + \frac{\partial V}{\partial x} g\right). \tag{2.10}$$

By observing (2.10), we choose u as

$$u = -cz + \frac{\partial \alpha}{\partial x}(f + g(\alpha + z)) - \frac{\partial V}{\partial x} g, \tag{2.11}$$

where c is a positive constant. Thus

$$\dot{V}_a \leq -W(x) - cz^2 \triangleq -W_a(x, z). \tag{2.12}$$

Thus global boundedness of all signals can be ensured. If $W(x)$ is positive definite, W_a can also be rendered positive definite. According to LaSalle-Yoshizawa theorem given in Appendix A, the globally asymptotic stability of $x = 0$, $z = 0$ is guaranteed. From (2.6) and $\alpha(0) = 0$, the equilibrium $x = 0$, $\xi = 0$ of (2.4)-(2.5) is also globally asymptotically stable.

The idea of integrator backstepping is further illustrated by the following example.

Example 2.1.1. Consider the following second order system

$$\dot{x} \quad = \quad x^2 + x\xi \tag{2.13}$$

$$\dot{\xi} \quad = \quad u. \tag{2.14}$$

Comparing with (2.4)-(2.5), we see that $x \in \Re$, $f(x) = x^2$ and $g(x) = x$. To stabilize (2.13) with ξ as the input, we define $V(x) = \frac{1}{2}x^2$. By choosing the desired value of ξ as

$$\alpha(x) = -x - 1, \tag{2.15}$$

we have

$$\dot{V} = x(x^2 + x\alpha) = -x^2, \tag{2.16}$$

which is positive definite. Thus the error variable is

$$z = \xi - \alpha = \xi + x + 1. \tag{2.17}$$

Substituting $\xi = z - x - 1$ into (2.13) and computing the derivative of z, we obtain

$$\dot{x} = xz - x \tag{2.18}$$

$$\dot{z} = u + xz - x. \tag{2.19}$$

The derivative of $V_a = \frac{1}{2}x^2 + \frac{1}{2}z^2$ is

$$\dot{V}_a = -x^2 + x^2 z + z(u + xz - x). \tag{2.20}$$

Thus the control

$$u = -z - xz + x - x^2 \tag{2.21}$$

can render $\dot{V}_a = -x^2 - z^2 < 0$. From LaSalle-Yoshizawa theorem, global uniform boundedness of x, z is achieved and $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$. From (2.15), $\xi = z - x - 1$ and (2.21), we have α, ξ and the control u are also globally bounded.

$\triangle$

2.1.2 Adaptive Backstepping Control

To illustrate the idea of adaptive backstepping control, we consider the following second order system as an example, in which the parametric uncertainty enters the system one integrator before the control u does.

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi^T(x_1)\theta & (2.22) \\ \dot{x}_2 &= u, & (2.23) \end{aligned}$$

where the states x_1, x_2 are measurable, $\varphi(x_1) \in \Re^p$ is a known vector of nonlinear functions and $\theta \in \Re^p$ is an unknown constant vector. The control objective is to stabilize the system and regulate x_1 to zero asymptotically.

We firstly present the design procedure of controller if θ is known. Introduce the change of coordinates as

$$\begin{aligned} z_1 &= x_1 & (2.24) \\ z_2 &= x_2 - \alpha_1, & (2.25) \end{aligned}$$

where α_1 is a function designed as a "desired value" of the virtual control x_2 to stabilize (2.22) and

$$\alpha_1 = -c_1 x_1 - \varphi^T \theta, \quad c_1 > 0. \tag{2.26}$$

Define the control Lyapunov function of system (2.22)-(2.23) as

$$V = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2, \tag{2.27}$$

whose derivative is computed as

$$\begin{aligned} \dot{V} &= z_1(z_2 - c_1 z_1) + z_2 \left(u - \frac{\partial \alpha_1}{\partial x_1} \left(x_2 + \varphi^T \theta \right) \right) \\ &= -c_1 z_1^2 + z_2 \left(u + z_1 - \frac{\partial \alpha_1}{\partial x_1} \left(x_2 + \varphi^T \theta \right) \right). & (2.28) \end{aligned}$$

By choosing the control input as

$$u = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} \left(x_2 + \varphi^T \theta \right), \quad c_2 > 0 \tag{2.29}$$

(2.28) becomes

$$\dot{V} = -c_1 z_1^2 - c_2 z_2^2 < 0. \tag{2.30}$$

From LaSalle-Yoshizawa theorem, z_1 and z_2 are ensured globally asymptotically stable. Since $x_1 = z_1$, we obtain that $\lim_{t\to\infty} x_1(t) = 0$. From (2.26) and (2.25), we have α_1, x_2 are also globally bounded. From (2.29), we conclude that the control u is also bounded.

However, θ is actually unknown. Thus the stabilizing function α_1 in (2.26) needs to be modified by using the estimated values of θ instead. Based on this, the design procedure is elaborated as the following.

Step 1. α_1 is now changed to

$$\alpha_1 = -c_1 x_1 - \varphi^T \hat{\theta}_1, \quad c_1 > 0 \tag{2.31}$$

where $\hat{\theta}_1$ is a estimated vector of θ. Keeping the definitions of z_1 and z_2 as in (2.24) and (2.25), we compute the time derivative of z_1 according to the new constructed α_1,

$$\dot{z}_1 = -c_1 z_1 - \varphi^T \hat{\theta}_1 + z_2 + \varphi^T \theta = -c_1 z_1 + z_2 + \varphi^T \tilde{\theta}_1, \tag{2.32}$$

where $\tilde{\theta}_1 = \theta - \hat{\theta}_1$.

We then define a Lyapunov function V_1 for this step as

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\theta}_1^T \Gamma_1^{-1} \tilde{\theta}_1, \tag{2.33}$$

where Γ_1 is a positive definite matrix. From (2.32), the time derivative of V_1 is computed as

$$\dot{V}_1 = z_1 \dot{z}_1 - \tilde{\theta}_1^T \Gamma_1^{-1} \dot{\hat{\theta}}_1 = z_1(-c_1 z_1 + z_2 + \varphi^T \tilde{\theta}_1) - \tilde{\theta}_1^T \Gamma_1^{-1} \dot{\hat{\theta}}_1. \tag{2.34}$$

By choosing the adaptive law of $\hat{\theta}_1$ as

$$\dot{\hat{\theta}}_1 = \Gamma_1 \varphi z_1, \tag{2.35}$$

we have

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2. \tag{2.36}$$

Step 2. Taking the time derivative of z_2, we obtain

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_1}{\partial \hat{\theta}_1}\dot{\hat{\theta}}_1 \\ &= u - \frac{\partial \alpha_1}{\partial x_1}(x_2 + \varphi^T \theta) - \frac{\partial \alpha_1}{\partial \hat{\theta}_1}\Gamma_1 \varphi z_1 \end{aligned} \tag{2.37}$$

Define a Lyapunov function V_2 for this step as

$$V_2 = V_1 + \frac{1}{2} z_2^2. \tag{2.38}$$

The time derivative of V_2 is computed as

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + z_2 \dot{z}_2 \\ &= -c_1 z_1^2 + z_2 \left(u + z_1 - \frac{\partial \alpha_1}{\partial x_1} x_2 - \frac{\partial \alpha_1}{\partial \hat{\theta}_1}\Gamma_1 \varphi z_1 - \frac{\partial \alpha_1}{\partial x_1}\varphi^T \theta \right). \end{aligned} \tag{2.39}$$

If u can be chosen as

$$u = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}_1}\Gamma_1 \varphi z_1 + \frac{\partial \alpha_1}{\partial x_1}\varphi^T \theta, \quad c_2 > 0, \tag{2.40}$$

the time derivative of V_2 becomes

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2. \tag{2.41}$$

However, θ is unknown. Therefore, it cannot be used in constructing u. It may be replaced by the estimated value $\hat{\theta}_1$ obtained in the last step. The time derivative of V_2 is then changed to

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 - z_2 \frac{\partial \alpha_1}{\partial x_1} \varphi^T \tilde{\theta}_1, \tag{2.42}$$

where the last term cannot be canceled. To eliminate this term, we replace θ in (2.40) with a new estimate $\hat{\theta}_2$:

$$u = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}_1} \Gamma_1 \varphi z_1 + \frac{\partial \alpha_1}{\partial x_1} \varphi^T \hat{\theta}_2, \quad c_2 > 0. \tag{2.43}$$

With this choice, (2.37) becomes

$$\dot{z}_2 \;=\; -z_1 - c_2 z_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi^T \tilde{\theta}_2, \tag{2.44}$$

where $\tilde{\theta}_2 = \theta - \hat{\theta}_2$. To stabilize the z system consisting of (2.32) and (2.44), the control Lyapunov function defined in (2.38) is augmented by including the quadratic term of $\tilde{\theta}_2$, i.e.,

$$V_2 = V_1 + \frac{1}{2} z_2^2 + \frac{1}{2} \tilde{\theta}_2^T \Gamma_2^{-1} \tilde{\theta}_2, \tag{2.45}$$

where Γ_2 is also a positive definite matrix. Taking the derivative of V_2 along with (2.36) and (2.44), we obtain

$$\dot{V}_2 = -c_1 z_1^2 + z_2 \left(-c_2 z_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi^T \tilde{\theta}_2 \right) + \tilde{\theta}_2^T \Gamma_2^{-1} \left(-\dot{\hat{\theta}}_2 \right). \tag{2.46}$$

Choose the update law for $\dot{\hat{\theta}}_2$ as

$$\dot{\hat{\theta}}_2 = -\Gamma_2 \frac{\partial \alpha_1}{\partial x_1} \varphi z_2, \tag{2.47}$$

we have

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2. \tag{2.48}$$

Therefore, global boundedness of $z_1(= x_1), z_2, \hat{\theta}_1, \hat{\theta}_2$ is ensured. We also have $\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} z_2(t) = 0$. From the boundedness of α_1 defined in (2.31) and the fact that $x_2 = z_2 + \alpha_1$ if follows that x_2 is also bounded. We can conclude that the control u is bounded based on (2.43).

2.2 Two Backstepping Based Design Schemes

In this section, two backstepping based adaptive controller design schemes to achieve system stabilization and desired tracking performance will be introduced. We consider a class of nonlinear systems as follows,

$$\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1^T(x_1)\theta \\
\dot{x}_2 &= x_3 + \varphi_2^T(x_1, x_2)\theta \\
\vdots \quad & \qquad \vdots \\
\dot{x}_{n-1} &= x_n + \varphi_{n-1}^T(x_1, \ldots, x_{n-1})\theta \\
\dot{x}_n &= \varphi_0(x) + \varphi_n^T(x)\theta + \beta(x)u \\
y &= x_1,
\end{aligned} \tag{2.49}$$

where $x = [x_1, \ldots, x_n]^T \in \Re^n$, $u \in \Re$ and $y \in \Re$ are the state, input and output of the system respectively. $\theta \in \Re^p$ is an unknown constant vector, $\varphi_0 \in \Re$, $\varphi_i \in \Re^p$ for $i = 1, \ldots, n$, β are known smooth nonlinear functions. Note that the class of nonlinear systems in the form of (2.49) are known as parametric strict-feedback systems since the nonlinearities depend only on variables which are “fed back” [21].

The control objective is to force the system output to asymptotically track a reference signal $y_r(t)$ while ensuring system stability. To achieve the objective, the following assumptions are required.

Assumption 2.2.1. *The reference signal $y_r(t)$ and its first n derivatives $y_r^{(i)}$, $i = 1, \ldots, n$ are known, bounded, and piecewise continuous.*

Assumption 2.2.2. $\beta(x) \neq 0, \forall x \in \Re^n$.

2.2.1 Tuning Functions Design

Observing the design procedure presented in Section 2.1.2, global stabilization and output regulation are ensured. However, there is a drawback that two estimates ($\hat{\theta}_1$ and $\hat{\theta}_2$) have been generated for only one vector of unknown parameters (θ). The dynamic order of the adaptive controller exceeds the number of unknown parameters. Such a problem is known as over-parametrization. It can be solved by adopting the tuning functions design scheme, in which a tuning function is determined recursively at each step. At the last step, the parameter update law is constructed based on the final tuning function and the control law is designed. Thus the dynamic order of the adaptive controller can be reduced to its minimum.

Different from the procedures in handling the second order system (2.22)-(2.23), n steps are required to determine the control signal for the system in (2.49). The design procedure is elaborated as follows.

Step 1. Introduce the first two error variables

$$z_1 = y - y_r \tag{2.50}$$

$$z_2 = x_2 - \dot{y}_r - \alpha_1, \tag{2.51}$$

where z_1 is the tracking error, of which the convergence of z_1, i.e. $\lim_{t\to\infty} z_1(t) = 0$ is to be achieved. The z_1 dynamics is derived as

$$\begin{aligned}\dot{z}_1 &= \dot{y} - \dot{y}_r \\ &= x_2 + \varphi_1^T\theta - \dot{y}_r \\ &= z_2 + \alpha_1 + \varphi_1^T\theta. \end{aligned} \tag{2.52}$$

α_1 is the first stabilizing function designed as

$$\alpha_1 = -c_1 z_1 - \varphi_1^T\hat{\theta}, \tag{2.53}$$

where c_1 is a positive constant and $\hat{\theta}$ is an estimate of θ. In fact, α_1 is the "desired value" of x_2 to stabilize $\dot{z}_1$ system (2.52) if $\dot{y}_r = 0$. Thus z_2 is the error between the actual and "desired" values of x_2 augmented by the term $-\dot{y}_r$.

Similar to (2.33), a Lyapunov function is defined at this step.

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}, \tag{2.54}$$

where Γ is a positive definite matrix and $\tilde{\theta}$ is the estimation error that $\tilde{\theta} = \theta - \hat{\theta}$. From (2.52) and (2.53), the derivative of V_1 is derived as

$$\begin{aligned}\dot{V}_1 &= z_1\left(-c_1 z_1 + z_2 + \varphi_1^T\tilde{\theta}\right) - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \\ &= z_1(-c_1 z_1 + z_2) - \tilde{\theta}^T\left(\Gamma^{-1}\dot{\hat{\theta}} - \varphi_1 z_1\right). \end{aligned} \tag{2.55}$$

Instead of determining the parameter update law as $\dot{\hat{\theta}} = \Gamma\varphi_1 z_1$ in Section 2.1.2 to eliminate the second term $\tilde{\theta}^T(\Gamma^{-1}\dot{\hat{\theta}} - \varphi_1 z_1)$ in (2.55), we define the first tuning function as

$$\tau_1 = \varphi_1 z_1. \tag{2.56}$$

Substituting (2.56) into (2.55), we obtain that

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 - \tilde{\theta}^T\left(\Gamma^{-1}\dot{\hat{\theta}} - \tau_1\right). \tag{2.57}$$

Step 2. We now treat the second equation of (2.49) by considering x_3 as the control variable. Introduce an error variable

$$z_3 = x_3 - \ddot{y}_r - \alpha_2. \tag{2.58}$$

Taking the derivative of z_2, we have

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \ddot{y}_r - \dot{\alpha}_1 \\ &= z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} x_2 + \left(\varphi_2 - \frac{\partial \alpha_1}{\partial x_1}\varphi_1\right)^T \theta - \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}, \end{aligned} \tag{2.59}$$

where α_2 is the second stabilizing function designed at this step to stabilize (z_1, z_2) system (2.52) and (2.59). We select α_2 as

$$\alpha_2 = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 - \left(\varphi_2 - \frac{\partial \alpha_1}{\partial x_1}\varphi_1\right)^T \hat{\theta} + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\tau_2, \tag{2.60}$$

where c_2 is a positive constant and τ_2 is the second tuning function designed based on τ_1 that

$$\tau_2 = \tau_1 + \left(\varphi_2 - \frac{\partial \alpha_1}{\partial x_1}\varphi_1\right) z_2. \tag{2.61}$$

We now define a Lyapunov function V_2 as

$$V_2 = V_1 + \frac{1}{2} z_2^2. \tag{2.62}$$

From (2.57), (2.59)-(2.61), the derivative of V_2 is computed as

$$\begin{aligned} \dot{V}_2 &= -c_1 z_1^2 + z_1 z_2 - \tilde{\theta}^T \left(\Gamma^{-1}\dot{\hat{\theta}} - \tau_1\right) + z_2(-z_1 - c_2 z_2 + z_3) \\ &\quad + z_2 \left(\varphi_2 - \frac{\partial \alpha_1}{\partial x_1}\varphi_1\right)^T \tilde{\theta} + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} \left(\Gamma\tau_2 - \dot{\hat{\theta}}\right) \\ &= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + \tilde{\theta}^T \left(\tau_2 - \Gamma^{-1}\dot{\hat{\theta}}\right) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} \left(\Gamma\tau_2 - \dot{\hat{\theta}}\right). \end{aligned} \tag{2.63}$$

Note that if x_3 were the actual control, we have $z_3 = 0$. If the parameter update law were chosen as $\dot{\hat{\theta}} = \Gamma\tau_2$, $\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2$ is rendered negative definite and the $(z_1, z_2)-$system can be stabilized. However, x_3 is not the actual control. Similar to $z_1 z_2$ canceled at this step, the term $z_2 z_3$ will be canceled at the next step. Moreover, the discrepancy between $\Gamma\tau_2$ and $\dot{\hat{\theta}}$ will be compensated partly by defining another tuning function τ_3 at the next step.

Step 3. We proceed to treat the third equation of (2.49). Introduce that

$$z_4 = x_4 - y_r^{(3)} - \alpha_3. \tag{2.64}$$

Computing the derivative of z_3, we have

$$\begin{aligned}\dot{z}_3 &= z_4 + \alpha_3 - \frac{\partial\alpha_2}{\partial x_1}x_2 - \frac{\partial\alpha_2}{\partial x_2}x_3 + \left(\varphi_3 - \frac{\partial\alpha_2}{\partial x_1}\varphi_1 - \frac{\partial\alpha_2}{\partial x_2}\varphi_2\right)^T\theta - \frac{\partial\alpha_2}{\partial y_r}\dot{y}_r - \frac{\partial\alpha_2}{\partial \dot{y}_r}\ddot{y}_r \\ &\quad -\frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}},\end{aligned} \tag{2.65}$$

where the fact that α_2 are a function of x_1, x_2, y_r, $\dot{y}_r$ and $\hat{\theta}$ is utilized. We then select α_3 as

$$\begin{aligned}\alpha_3 &= -z_2 - c_3 z_3 + \frac{\partial\alpha_2}{\partial x_1}x_2 + \frac{\partial\alpha_2}{\partial x_2}x_3 - \left(\varphi_3 - \frac{\partial\alpha_2}{\partial x_1}\varphi_1 - \frac{\partial\alpha_2}{\partial x_2}\varphi_2\right)^T\hat{\theta} + \frac{\partial\alpha_2}{\partial y_r}\dot{y}_r \\ &\quad + \frac{\partial\alpha_2}{\partial \dot{y}_r}\ddot{y}_r + \frac{\partial\alpha_2}{\partial\hat{\theta}}\Gamma\tau_3 + z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\left(\varphi_3 - \frac{\partial\alpha_2}{\partial x_1}\varphi_1 - \frac{\partial\alpha_2}{\partial x_2}\varphi_2\right),\end{aligned} \tag{2.66}$$

where c_3 is a positive constant and τ_3 is the third tuning function designed based on τ_2 that

$$\tau_3 = \tau_2 + \left(\varphi_3 - \frac{\partial\alpha_2}{\partial x_1}\varphi_1 - \frac{\partial\alpha_2}{\partial x_2}\varphi_2\right)z_3. \tag{2.67}$$

The $(z_1, z_2, z_3)-$system (2.52), (2.59), (2.65) is stabilized with respect to the Lyapunov function

$$V_3 = V_2 + \frac{1}{2}z_3^2, \tag{2.68}$$

whose derivative is

$$\begin{aligned}\dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 z_4 + \tilde{\theta}^T\left(\tau_3 - \Gamma^{-1}\dot{\hat{\theta}}\right) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}\left(\Gamma\tau_2 - \dot{\hat{\theta}}\right) \\ &\quad + z_3 \frac{\partial \alpha_2}{\partial \hat{\theta}}\left(\Gamma\tau_3 - \dot{\hat{\theta}}\right) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\left(\varphi_3 - \frac{\partial \alpha_2}{\partial x_1}\varphi_1 - \frac{\partial \alpha_2}{\partial x_2}\varphi_2\right) z_3. \end{aligned} \tag{2.69}$$

Note that

$$\begin{aligned} z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}\left(\Gamma\tau_2 - \dot{\hat{\theta}}\right) &= z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}\left(\Gamma\tau_3 - \dot{\hat{\theta}}\right) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}(\Gamma\tau_2 - \Gamma\tau_3) \\ &= z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}\left(\Gamma\tau_3 - \dot{\hat{\theta}}\right) - z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\left(\varphi_3 - \frac{\partial \alpha_2}{\partial x_1}\varphi_1 - \frac{\partial \alpha_2}{\partial x_2}\varphi_2\right) z_3. \end{aligned} \tag{2.70}$$

Substituting (2.70) into (2.69), we obtain

$$\begin{aligned}\dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 z_4 + \tilde{\theta}^T\left(\tau_3 - \Gamma^{-1}\dot{\hat{\theta}}\right) + \left(z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} + z_3 \frac{\partial \alpha_2}{\partial \hat{\theta}}\right) \\ &\quad \times \left(\Gamma\tau_3 - \dot{\hat{\theta}}\right). \end{aligned} \tag{2.71}$$

From the discussion above, we can see that the last term of the designed α_3 in (2.66) is important to cancel the term $z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}(\Gamma\tau_2 - \Gamma\tau_3)$ in rewriting the term $z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}(\Gamma\tau_2 - \dot{\hat{\theta}})$ as in (2.70).

Step i ($i = 4, \ldots, n-1$). Introduce the error variable

$$z_i = x_i - y_r^{(i-1)} - \alpha_{i-1} \tag{2.72}$$

Derive the dynamics of z_i

$$\dot{z}_i = z_{i+1} + \alpha_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \left(\varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\varphi_k\right)^T \theta - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}} y_r^{(k)} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}. \tag{2.73}$$

The stabilization function α_i is chosen as

$$\begin{aligned}\alpha_i &= -z_{i-1} - c_i z_i + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} - \left(\varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\varphi_k\right)^T \hat{\theta} + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \\ &\quad + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\Gamma\tau_i + \sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\left(\varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\varphi_j\right), \qquad (2.74)\end{aligned}$$

where c_i is a positive constant and τ_i is the ith tuning function defined as

$$\tau_i = \tau_{i-1} + \left(\varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\varphi_k\right) z_i. \qquad (2.75)$$

The $(z_1, \ldots, z_i)-$system is stabilized with respect to the Lyapunov function defined as

$$V_i = V_{i-1} + \frac{1}{2} z_i^2, \qquad (2.76)$$

whose derivatives is

$$\begin{aligned}\dot{V}_i &= -\sum_{k=1}^{i} c_k z_k^2 + z_i z_{i+1} + \tilde{\theta}^T \left(\tau_i - \Gamma^{-1}\dot{\hat{\theta}}\right) + \left(\sum_{k=2}^{i} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\right) \\ &\quad \times \left(\Gamma\tau_i - \dot{\hat{\theta}}\right). \qquad (2.77)\end{aligned}$$

Step n. We introduce

$$z_n = x_n - y_r^{(n-1)} - \alpha_{n-1}. \qquad (2.78)$$

The derivative of z_n is

$$\begin{aligned}\dot{z}_n &= \varphi_0 + \beta u - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} + \left(\varphi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}\varphi_k\right)^T \theta - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \\ &\quad - y_r^{(n)} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}. \qquad (2.79)\end{aligned}$$

The control input u is designed as

$$u = \frac{1}{\beta}\left(\alpha_n + y_r^{(n)}\right), \tag{2.80}$$

with

$$\begin{aligned}\alpha_n &= -z_{n-1} - c_n z_n - \varphi_0 + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} x_{k+1} - \left(\varphi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}\varphi_k\right)^T \hat{\theta} \\ &\quad + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_r^{(k-1)}} y_r^{(k)} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\Gamma\tau_n + \sum_{k=2}^{n-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma\left(\varphi_n - \sum_{j=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_j}\varphi_j\right),\end{aligned} \tag{2.81}$$

where c_n is a positive constant and τ_n is

$$\tau_n = \tau_{n-1} + \left(\varphi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}\varphi_k\right). \tag{2.82}$$

Define the Lyapunov function as

$$V_n = V_{n-1} + \frac{1}{2} z_n^2, \tag{2.83}$$

whose derivative is computed as

$$\dot{V}_n = -\sum_{k=1}^{n} c_k z_k^2 + \tilde{\theta}^T\left(\tau_n - \Gamma^{-1}\dot{\hat{\theta}}\right) + \left(\sum_{k=2}^{n} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\right)\left(\Gamma\tau_n - \dot{\hat{\theta}}\right). \tag{2.84}$$

By determining the parameter update law as

$$\dot{\hat{\theta}} = \Gamma\tau_n, \tag{2.85}$$

$\dot{V}_n$ is rendered negative definite that

$$\dot{V}_n = -\sum_{k=1}^{n} c_k z_k^2. \tag{2.86}$$

From the definition of V_n and (2.86), it follows that z, $\tilde{\theta}$ are bounded. Since $\hat{\theta} = \theta - \tilde{\theta}$, $\hat{\theta}$ is also bounded. From (2.50) and Assumption 2.2.1, y is bounded. From (2.53) and smoothness of $\varphi_1(x_1)$, α_1 is bounded. Combining with the definition of z_2 in (2.51) and the boundedness of $\dot{y}_r$, it follows that x_2 is bounded. By following similar procedure, the boundedness of α_i for $i = 2, \ldots, n$, x_i for $i = 3, \ldots, n$ is also ensured. From (2.80), we can conclude that the control signal u is bounded. Thus the boundedness of all the signals in the closed-loop adaptive system is guaranteed. Furthermore, we define $z = [z_1, \ldots, z_n]^T$. From the LaSalle-Yoshizawa theorem, $\lim_{t\to\infty} z(t) = 0$. This implies that asymptotic tracking is also achieved, i.e. $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$. The above facts are formally stated in the following theorem.

Theorem 2.2.1. *Consider the plant (2.49) under Assumptions 2.2.1-2.2.2. The controller (2.80) and the parameter update law (2.85) guarantee the global boundedness of all signals in the closed-loop adaptive system and the asymptotic tracking is achieved, i.e.* $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$.

2.2.2 Modular Design

From (2.74), we see that the terms $\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i + \sum_{k=2}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma(\varphi - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j)$ are crucial to form the $(\Gamma \tau_i - \dot{\hat{\theta}})$ related terms in deriving $\dot{V}_i$ at the ith step. The effects of $\dot{\hat{\theta}}$ are canceled by defining the parameter update law as $\dot{\hat{\theta}} = \Gamma \tau_n$ at the final step. Thus the adaptive controller and the parameter update law are constructed simultaneously with respect to a Lyapunov function encompassing all the states in the $(z, \tilde{\theta})-$system, when the tuning function design scheme is applied. In contrast to these, the parameter estimator can also be determined independently of the controller. By doing this, certain boundedness properties of $(\tilde{\theta}, \dot{\hat{\theta}})$ are guaranteed. The boundedness of z is thus ensured by establishing input-to-state stable properties with $(\tilde{\theta}, \dot{\hat{\theta}})$ as the inputs in controller design module. Since the modularity of the controller-estimator pair is achieved, such a design method is known as modular adaptive design.

The detailed procedure in generating the control law and the parameter update law for the system in (2.49) by using the backstepping based modular adaptive design scheme is presented as the following.

A. Design of Control Law

Similar to the tuning functions design, we introduce the change of coordinates firstly.

$$z_i = x_i - y_r^{(i-1)} - \alpha_{i-1}, \quad i = 1, \ldots, n \tag{2.87}$$

α_i is now designed to guarantee the boundedness of z_i whenever the signals $\tilde{\theta}, \dot{\hat{\theta}}$ are bounded.

Step 1. The derivative of z_1 is

$$\dot{z}_1 = z_2 + \alpha_1 + \varphi_1^T \theta. \tag{2.88}$$

We choose α_1 as

$$\alpha_1 = -c_1 z_1 - \varphi_1^T \hat{\theta} - \kappa_1 \|\varphi_1\|^2 z_1, \tag{2.89}$$

where c_1, κ_1 are positive constants. Substituting (2.89) into (2.88), we have

$$\dot{z}_1 = -c_1 z_1 + z_2 + \varphi_1^T \tilde{\theta} - \kappa_1 \|\varphi_1\|^2 z_1, \tag{2.90}$$

where $\tilde{\theta} = \theta - \hat{\theta}$. Define that

$$V_1 = \frac{1}{2} z_1^2. \tag{2.91}$$

$\dot{V}_1$ is then computed as

$$
\begin{aligned}
\dot{V}_1 &= -c_1 z_1^2 + z_1 z_2 + z_1 \varphi_1^T \tilde{\theta} - \kappa_1 \|\varphi_1\|^2 z_1^2 \\
&= -c_1 z_1^2 + z_1 z_2 - \kappa_1 \left\| \varphi_1 z_1 - \frac{1}{2\kappa_1}\tilde{\theta} \right\|^2 + \frac{1}{4\kappa_1}\|\tilde{\theta}\|^2 \\
&\leq -c_1 z_1^2 + z_1 z_2 + \frac{1}{4\kappa_1}\|\tilde{\theta}\|^2. \qquad (2.92)
\end{aligned}
$$

If z_2 were zero, z_1 is bounded whenever $\tilde{\theta}$ is bounded. By comparing (2.89) with (2.53), we see that the term $-\kappa_1\|\varphi\|^2 z_1$ is crucial to render $\dot{V}_1$ negative outside a compact region if $z_2 = 0$. Such a term is referred to as "nonlinear damping term" in [21].

Step 2. We proceed to the second equation of (2.49). Since α_1 (2.89) is a function of x_1, y_r and $\hat{\theta}$, the derivative of z_2 is

$$
\begin{aligned}
\dot{z}_2 &= x_3 + \varphi_2^T \theta - \ddot{y}_r - \frac{\partial \alpha_1}{\partial x_1}(x_2 + \varphi_1^T \theta) - \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} \\
&= z_3 + \alpha_2 + \left(\varphi_2 - \frac{\partial \alpha_1}{\partial x_1}\varphi_1 \right)^T \theta - \left(\frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r \right) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}. \qquad (2.93)
\end{aligned}
$$

Choose α_2 as

$$
\begin{aligned}
\alpha_2 &= -z_1 - c_2 z_2 - \left(\varphi_2 - \frac{\partial \alpha_1}{\partial x_1}\varphi_1 \right)^T \hat{\theta} + \left(\frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r \right) \\
&\quad -\kappa_2 \left\| \varphi_2 - \frac{\partial \alpha_1}{\partial x_1}\varphi_1 \right\|^2 z_2 - g_2 \left\| \frac{\partial \alpha_1}{\partial \hat{\theta}}^T \right\|^2 z_2, \qquad (2.94)
\end{aligned}
$$

where c_2, κ_2 and g_2 are positive constants. From (2.92), the derivative of $V_2 = V_1 + \frac{1}{2}z_2^2$ is

$$
\begin{aligned}
\dot{V}_2 &\leq -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 - \kappa_2 \left\| \left(\varphi_2 - \frac{\partial \alpha_1}{\partial x_1}\varphi_1 \right) z_2 - \frac{1}{2\kappa_2}\tilde{\theta} \right\|^2 + \sum_{i=1}^{2} \frac{1}{4\kappa_i}\|\tilde{\theta}\|^2 \\
&\quad -g_2 \left\| \frac{\partial \alpha_1}{\partial \hat{\theta}}^T z_2 + \frac{1}{2g_2}\dot{\hat{\theta}} \right\|^2 + \frac{1}{4g_2}\|\dot{\hat{\theta}}\|^2 \\
&\leq -\sum_{i=1}^{2} c_i z_i^2 + z_2 z_3 + \sum_{i=1}^{2} \frac{1}{4\kappa_i}\|\tilde{\theta}\|^2 + \frac{1}{4g_2}\|\dot{\hat{\theta}}\|^2. \qquad (2.95)
\end{aligned}
$$

If z_3 were zero, (z_1, z_2) is bounded whenever $\tilde{\theta}$ and $\dot{\hat{\theta}}$ are bounded. The last two terms in (2.94) are designed nonlinear damping terms at this step.

Step i $(i = 3, \ldots, n-1)$. α_{i-1} is a function of $x_1, \ldots, x_{i-1}, y_r, \ldots, y_r^{(i-2)}, \hat{\theta}$, thus the ith equation in (2.49) yields

$$\begin{aligned} \dot{z}_i &= z_{i+1} + \alpha_i + \left(\varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\varphi_k\right)^T \theta - \sum_{k=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_k}x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}}y_r^{(k)}\right) \\ &\quad -\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}. \end{aligned} \tag{2.96}$$

We choose that

$$\begin{aligned} \alpha_i &= -z_{i-1} - c_i z_i - \left(\varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\varphi_k\right)^T \hat{\theta} + \sum_{k=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_k}x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}}y_r^{(k)}\right) \\ &\quad -\kappa_i \left\| \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\varphi_k \right\|^2 z_i - g_i \left\| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}^T \right\|^2 z_i, \end{aligned} \tag{2.97}$$

where c_i, κ_i and g_i are positive constants.

Using completion of the squares as in (2.92) and (2.95), we obtain the derivative of $V_i = V_{i-1} + \frac{1}{2}z_i^2$

$$\begin{aligned} \dot{V}_i &\leq -\sum_{k=1}^{i} c_k z_k^2 + z_i z_{i+1} - \kappa_i \left\| \left(\varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\varphi_k\right) z_i - \frac{1}{2\kappa_i}\tilde{\theta} \right\|^2 + \sum_{k=1}^{i} \frac{1}{4\kappa_k}\|\tilde{\theta}\|^2 \\ &\quad -g_i \left\| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}^T z_i + \frac{1}{2g_i}\dot{\hat{\theta}} \right\|^2 + \sum_{k=2}^{i} \frac{1}{4g_k}\|\dot{\hat{\theta}}\|^2 \\ &\leq -\sum_{k=1}^{i} c_k z_k^2 + z_i z_{i+1} + \sum_{k=1}^{i} \frac{1}{4\kappa_k}\|\tilde{\theta}\|^2 + \sum_{k=2}^{i} \frac{1}{4g_k}\|\dot{\hat{\theta}}\|^2. \end{aligned} \tag{2.98}$$

Step n. We have

$$\begin{aligned} \dot{z}_n &= \varphi_0 + \beta u + \left(\varphi_n - \sum_{k=1}^{n-2} \frac{\partial \alpha_{n-1}}{\partial x_k}\varphi_k\right)^T \theta - \sum_{k=1}^{n-1}\left(\frac{\partial \alpha_{n-1}}{\partial x_k}x_{k+1} + \frac{\partial \alpha_{n-1}}{\partial y_r^{(k-1)}}y_r^{(k)}\right) \\ &\quad -y_r^{(n)} - \frac{\alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}. \end{aligned} \tag{2.99}$$

The control input u is designed as

$$u = \frac{1}{\beta}(\alpha_n + y_r^{(n)}). \tag{2.100}$$

α_n is chosen as

$$\begin{aligned}\alpha_n &= -z_{n-1} - c_n z_n - \varphi_0 - \left(\varphi_n - \sum_{k=1}^{n-2}\frac{\partial \alpha_{n-1}}{\partial x_k}\varphi_k\right)^T \hat{\theta} + \sum_{k=1}^{n-1}\left(\frac{\partial \alpha_{n-1}}{\partial x_k}x_{k+1}\right. \\ &\quad \left.+\frac{\partial \alpha_{n-1}}{\partial y_r^{(k-1)}}y_r^{(k)}\right) - \kappa_n \left\|\varphi_n - \sum_{k=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_k}\varphi_k\right\|^2 z_n - g_n \left\|\frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}^T\right\|^2 z_n, \end{aligned} \tag{2.101}$$

where c_n, κ_n and g_n are positive constants. Define V_n as

$$V_n = V_{n-1} + \frac{1}{2}z_n^2. \tag{2.102}$$

By following similar procedure in (2.98), we have

$$\dot{V}_n \leq -\sum_{i=1}^{n} c_i z_i^2 + \sum_{i=1}^{n}\frac{1}{4\kappa_i}\|\tilde{\theta}\|^2 + \sum_{i=2}^{n}\frac{1}{4g_i}\|\dot{\hat{\theta}}\|^2. \tag{2.103}$$

Based on (2.103), we can establish the input-to-state properties for the $z-$system with respect to $\tilde{\theta}, \dot{\hat{\theta}}$ as the inputs, where $z = [z_1, \ldots, z_n]^T$.

Lemma 2.2.1. *For the $z-$system, the following input-to-state properties hold:*

(i) If $\tilde{\theta}, \dot{\hat{\theta}} \in \mathcal{L}_\infty$, then $z \in \mathcal{L}_\infty$, and

$$\|z(t)\| \leq \frac{1}{2\sqrt{c_0}}\left(\frac{1}{\kappa_0} \parallel \tilde{\theta} \parallel_\infty^2 + \frac{1}{g_0} \parallel \dot{\hat{\theta}} \parallel_\infty^2\right)^{\frac{1}{2}} + \|z(0)\|e^{-c_0 t}. \tag{2.104}$$

(ii) If $\tilde{\theta} \in \mathcal{L}_\infty$ and $\dot{\hat{\theta}} \in \mathcal{L}_2$, then $z \in \mathcal{L}_\infty$, and

$$\|z(t)\| \leq \left(\frac{1}{4c_0\kappa_0} \parallel \tilde{\theta} \parallel_\infty^2 + \frac{1}{2g_0} \parallel \dot{\hat{\theta}} \parallel_2^2\right)^{\frac{1}{2}} + \|z(0)\|e^{-c_0 t}. \tag{2.105}$$

c_0, κ_0 and g_0 are defined as

$$c_0 = \min_{1\le i\le n} c_i, \qquad \kappa_0 = \left(\sum_{i=1}^{n}\frac{1}{\kappa_i}\right)^{-1}, \qquad g_0 = \left(\sum_{i=2}^{n}\frac{1}{g_i}\right)^{-1} \tag{2.106}$$

Proof: From the definition of V_i for $i = 1, \dots, n$ and (2.103), it follows that

$$\frac{d}{dt}\left(\frac{1}{2}\|z\|^2\right) \le -c_0\|z\|^2 + \frac{1}{4}\left(\frac{1}{\kappa_0}\|\tilde{\theta}\|^2 + \frac{1}{g_0}\|\dot{\hat{\theta}}\|^2\right). \tag{2.107}$$

(i) Multiplying both sides of (2.107) by two, we have

$$\frac{d}{dt}\left(\|z(t)\|^2\right) = -2c_0\|z(t)\|^2 + \frac{1}{2}\left(\frac{1}{\kappa_0}\|\tilde{\theta}\|^2 + \frac{1}{g_0}\|\dot{\hat{\theta}}\|^2\right). \tag{2.108}$$

Solving (2.108), we have

$$\begin{aligned}
\|z(t)\|^2 &= \|z(0)\|^2 e^{-2c_0 t} + \frac{1}{2}\int_0^t e^{-2c_0(t-\tau)}\left(\frac{1}{\kappa_0}\|\tilde{\theta}(\tau)\|^2 + \frac{1}{g_0}\|\dot{\hat{\theta}}(\tau)\|^2\right) d\tau \\
&\le \|z(0)\|^2 e^{-2c_0 t} + \frac{1}{2}\sup_{\tau\in[0,t]}\left\{\frac{1}{\kappa_0}\|\tilde{\theta}(\tau)\|^2 + \frac{1}{g_0}\|\dot{\hat{\theta}}(\tau)\|^2\right\}\int_0^t e^{-2c_0(t-\tau)} d\tau \\
&\le \|z(0)\|^2 e^{-2c_0 t} + \frac{1}{2}\left(\frac{1}{\kappa_0}\parallel \tilde{\theta} \parallel_\infty^2 + \frac{1}{g_0}\parallel \dot{\hat{\theta}} \parallel_\infty^2\right)\frac{1}{2c_0}(1 - e^{-2c_0 t}) \\
&\le \|z(0)\|^2 e^{-2c_0 t} + \frac{1}{4c_0}\left(\frac{1}{\kappa_0}\parallel \tilde{\theta} \parallel_\infty^2 + \frac{1}{g_0}\parallel \dot{\hat{\theta}} \parallel_\infty^2\right).
\end{aligned} \tag{2.109}$$

Thus if $\tilde{\theta}, \dot{\hat{\theta}} \in \mathcal{L}_\infty$, $z \in \mathcal{L}_\infty$. (2.104) is achieved by using the fact that $\sqrt{a^2+b^2} \le a+b$ for $a, b \ge 0$.

(ii) From (2.109), it follows that

$$\begin{aligned}
\|z(t)\|^2 &= \|z(0)\|^2 e^{-2c_0 t} + \frac{1}{2}\left(\int_0^t \frac{1}{\kappa_0}\|\tilde{\theta}(\tau)\|^2 e^{-2c_0(t-\tau)} d\tau \right. \\
&\quad \left. + \int_0^t \frac{1}{g_0}\|\dot{\hat{\theta}}(\tau)\|^2 e^{-2c_0(t-\tau)} d\tau\right) \\
&\le \|z(0)\|^2 e^{-2c_0 t} + \frac{1}{4c_0\kappa_0}\parallel \tilde{\theta} \parallel_\infty^2 + \frac{1}{2g_0}\sup_{\tau\in[0,t]}\{e^{-2c_0(t-\tau)}\}\int_0^t \|\dot{\hat{\theta}}(\tau)\|^2 d\tau \\
&\le \|z(0)\|^2 e^{-2c_0 t} + \frac{1}{4c_0\kappa_0}\parallel \tilde{\theta} \parallel_\infty^2 + \frac{1}{2g_0}\parallel \dot{\hat{\theta}} \parallel_2^2.
\end{aligned} \tag{2.110}$$

Thus if $\tilde{\theta} \in \mathcal{L}_\infty$ and $\dot{\hat{\theta}} \in \mathcal{L}_2$, $z \in \mathcal{L}_\infty$. (2.105) is also proved. □

B. Design of Parameter Update Law

According to Lemme 2.2.1, the boundedness of z is achieved if the boundedness of $\tilde{\theta}$ and $\dot{\hat{\theta}}$ is guaranteed. We present a x-swapping scheme to design the parameter estimator at this position. The properties of the parameter estimator will also be given.

Rewrite (2.49) in a parametric x-form firstly that

$$\dot{x} = f(x,u) + F^T(x,u)\theta, \tag{2.111}$$

where

$$f(x,u) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ \varphi_0 + \beta u \end{bmatrix}, \quad F^T(x,u) = \begin{bmatrix} \varphi_1^T \\ \vdots \\ \varphi_{n-1}^T \\ \varphi_n^T \end{bmatrix}. \tag{2.112}$$

Two filters are then introduced that

$$\begin{aligned} \dot{\Omega}^T &= A(x,t)\Omega^T + F^T(x,u) && (2.113) \\ \dot{\Omega}_0 &= A(x,t)(\Omega_0 + x) - f(x,u), && (2.114) \end{aligned}$$

where

$$A(x,t) = A_0 - \gamma F^T(x,u)F(x,u)P, \quad P = P^T > 0 \tag{2.115}$$

γ is a positive constant and A_0 is an arbitrary constant matrix such that $PA_0 + A_0^T P = -I$. Similar to the proof of Theorem 4.10 in [94], it can be shown that

$A(x,t)$ is exponentially stable for each x continuous in t. Combining (2.111) and (2.114), we define $\mathcal{Y} = \Omega_0 + x$, whose derivative is

$$\dot{\mathcal{Y}} = A(x,t)\mathcal{Y} + F^T(x,u)\theta. \tag{2.116}$$

For an $\varepsilon \triangleq \mathcal{Y} - \Omega^T\theta$, the derivative is computed as

$$\dot{\varepsilon} = \dot{\mathcal{Y}} - \dot{\Omega}^T\theta = A(x,t)\varepsilon. \tag{2.117}$$

Introducing the "prediction" of $\mathcal{Y}$ as $\hat{\mathcal{Y}} = \Omega^T\hat{\theta}$, the "prediction error" $\epsilon \triangleq \mathcal{Y} - \hat{\mathcal{Y}}$ is then written as

$$\epsilon = \varepsilon + \Omega^T\theta - \Omega^T\hat{\mathcal{Y}} = \varepsilon + \Omega^T\tilde{\theta}. \tag{2.118}$$

Based on (2.118), we choose the parameter update law by employing the unnormalized gradient algorithm [5]

$$\dot{\hat{\theta}} = \Gamma\Omega\epsilon, \tag{2.119}$$

where Γ is a positive definite matrix.

Lemma 2.2.2. *The design of parameter estimator encompassing the filters (2.113)-(2.114), the regressor form (2.118) and adaptive law (2.119), guarantee the following properties:*

$$\textit{(i)} \quad \tilde{\theta} \in \mathcal{L}_\infty, \quad \textit{(ii)} \quad \epsilon \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \quad \textit{(iii)} \quad \dot{\hat{\theta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$$

Proof: *(i)* Let us consider the positive definite function

$$V = \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \varepsilon^T P\varepsilon. \tag{2.120}$$

Along with (2.117), (2.119) and $PA + A^TP \leq -I$, the derivative of V is computed as

$$\begin{aligned}\dot{V} &\leq \tilde{\theta}^T\Gamma^{-1}(\Gamma\Omega\epsilon) - \varepsilon^T\varepsilon = -(\epsilon-\varepsilon)^T\epsilon - \varepsilon^T\varepsilon \\ &\leq -\frac{3}{4}\epsilon^T\epsilon - \left\|\frac{1}{2}\epsilon - \varepsilon\right\|^2 \\ &\leq -\frac{3}{4}\|\epsilon\|^2. \end{aligned} \tag{2.121}$$

The nonpositivity of $\dot{V}$ proves that $\tilde{\theta} \in \mathcal{L}_\infty$.

(ii) Integrating (2.121), we have

$$\int_0^\infty \|\epsilon(\tau)\|^2 d\tau \leq -\frac{3}{4}\int_0^\infty \dot{V} d\tau \leq \frac{3}{4}[V(0) - V(\infty)]. \tag{2.122}$$

Since $V(t)$ is nonnegative and $\dot{V}$ is nonpositive, we have

$$\int_0^\infty \|\epsilon(\tau)\|^2 d\tau \leq \frac{3}{4}V_0 < \infty. \tag{2.123}$$

Thus $\epsilon \in \mathcal{L}_2$. We now prove the boundedness of $\Omega \in \Re^{p\times n}$. Compute that

$$\begin{aligned}\frac{d}{dt}\mathrm{tr}\{\Omega P \Omega^T\} &= -\|\Omega\|_F^2 - 2\gamma\left\|FP\Omega^T - \frac{1}{2\gamma}I_p\right\|_F^2 + \frac{1}{2\gamma}\mathrm{tr}\{I_p\} \\ &\leq -\|\Omega\|_F^2 + \frac{p}{2\gamma}. \end{aligned} \tag{2.124}$$

From $\lambda_{\min}(P)\|\Omega\|_F^2 \leq \mathrm{tr}\{\Omega P\Omega^T\}$, it follows that $\Omega \in \mathcal{L}_\infty$. Combining with $\tilde{\theta} \in \mathcal{L}_\infty$, $\epsilon = \varepsilon + \Omega^T\tilde{\theta}$ and ε in (2.117) is exponentially decaying, we conclude $\epsilon \in \mathcal{L}_\infty$. Thus $\epsilon \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ is proved.

(iii) From (2.119), $\dot{\hat{\theta}}$ is bounded. By utilizing Hölder's inequality given in Appendix B, we obtain that

$$\int_0^\infty \dot{\hat{\theta}}^T\dot{\hat{\theta}} d\tau \leq \lambda_{\max}(\Gamma)^2 \left\|\|\Omega\|_F^2\right\|_\infty \int_0^\infty \epsilon^T\epsilon d\tau. \tag{2.125}$$

Since $\epsilon \in \mathcal{L}_2$ and $\Omega \in \mathcal{L}_\infty$, we conclude that $\dot{\hat{\theta}} \in \mathcal{L}_2$. Thus $\dot{\hat{\theta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. □

According to Lemma 2.2.1 and Lemma 2.2.2, the following result can be obtained.

Theorem 2.2.2. *Consider the plant (2.49) under Assumptions 2.2.1-2.2.2. The controller (2.100) and the parameter update law (2.119) ensure that*

(i) all signals in the closed-loop adaptive system are bounded;

(ii) asymptotic tracking is achieved, i.e. $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$.

Proof:

(i) According to Lemma 2.2.2, the boundedness of $\tilde{\theta}$ and $\dot{\hat{\theta}}$ is ensured. Thus from the *(i)* in Lemma 2.2.1, z is bounded. Since $\tilde{\theta}$ is bounded, $\hat{\theta}$ is also bounded. From the change of coordinates in (2.96), the boundedness of α_i, x_i for $i = 1, \ldots, n$ is guaranteed recursively as in Section 2.2.1. Similarly from (2.100), $u \in \mathcal{L}_\infty$. From (2.113) and the proof of Lemma (2.2.2), Ω, Ω_0 and ϵ are all bounded. Therefore, the boundedness of all signals in the closed-loop adaptive system is ensured.

(ii) From (2.88), (2.93), (2.96) and (2.99), the dynamics of z can be rewritten as

$$\dot{z} = A_z(z, \hat{\theta}, t)z + W^T(z, \hat{\theta}, t)\tilde{\theta} + Q^T(z, \hat{\theta}, t)\dot{\hat{\theta}}, \tag{2.126}$$

where

$$A_z = \begin{bmatrix} -c_1 - \kappa_1\|\varphi_1\|^2 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 - \kappa_2\left\|\varphi_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi_1\right\|^2 - g_2\left\|\frac{\partial\alpha_1^T}{\partial\hat{\theta}}\right\|^2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & -c_n - \kappa_n\left\|\varphi_n - \sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}\varphi_k\right\|^2 - g_n\left\|\frac{\partial\alpha_{n-1}^T}{\partial\hat{\theta}}\right\|^2 \end{bmatrix} \tag{2.127}$$

$$W^T = \begin{bmatrix} \varphi_1^T \\ \varphi_2^T - \frac{\partial\alpha_1}{\partial x_1}\varphi_1^T \\ \vdots \\ \varphi_n^T - \sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}\varphi_k^T \end{bmatrix}, \quad Q = \begin{bmatrix} 0 \\ -\frac{\partial\alpha_1}{\partial\hat{\theta}} \\ \vdots \\ -\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}} \end{bmatrix}. \tag{2.128}$$

From the proof of (i), it follows that $\dot{z} \in \mathcal{L}_\infty$. Moreover, for a time varying system

$$\dot{\zeta} = A_z(z(t), \hat{\theta}(t), t)\zeta. \tag{2.129}$$

By defining a positive definite function $V = \zeta^T\zeta$ and computing that $\dot{V} \leq -2c_0\zeta^T\zeta$, we have that the state transition matrix $\Phi_{A_z}(t, t_0)$ satisfies $\|\Phi_{A_z}(t, t_0)\| \leq ke^{-r(t-t_0)}$, $k, r > 0$. If $z \in \mathcal{L}_2$ is also achieved, $\lim_{t\to\infty} z(t) = 0$ can be ensured by Barbalat lemma and its corollary given in Appendix A, which implies the result of asymptotic tracking.

From Lemma 2.2.2, $\epsilon \in \mathcal{L}_2$. From (2.115) and (2.117), it follows that

$$\frac{d}{dt}(\varepsilon^T P\varepsilon) \leq -\varepsilon^T\varepsilon. \tag{2.130}$$

Integrating both sides of (2.130), we get $\varepsilon \in \mathcal{L}_2$. Thus $\Omega^T\tilde{\theta} = \epsilon - \varepsilon \in \mathcal{L}_2$.

Introduce a filter that

$$\dot{\chi}^T = A_z\chi^T + W^T. \tag{2.131}$$

We now prove that $\varsigma = z - \chi^T\tilde{\theta} \in \mathcal{L}_2$. From (2.126) and (2.131), we have

$$\begin{aligned} \dot{\varsigma} &= A_z z + W^T\tilde{\theta} + Q^T\dot{\hat{\theta}} - (A_z\chi^T + W^T)\tilde{\theta} + \chi^T\dot{\hat{\theta}} \\ &= A_z\varsigma + (Q^T + \chi^T)\dot{\hat{\theta}}. \end{aligned} \tag{2.132}$$

The solution of (2.132) is

$$\varsigma(t) = \Phi_{A_z}(t, 0)\varsigma(0) + \int_0^t \Phi_{A_z}(t, \tau)(Q + \chi)^T\dot{\hat{\theta}} \tag{2.133}$$

From the proof of *(i)*, we obtain that Q and W are bounded. From (2.131) and A_z is exponentially stable, it follows that χ is also bounded. Then

$$\begin{aligned}\|\varsigma(t)\| &\leq ke^{-rt}\|\varsigma(0)\| + k\|Q+\chi\|_\infty \int_0^t e^{-r(t-\tau)}\|\dot{\hat{\theta}}\|d\tau \\ &\leq ke^{-rt}\|\varsigma(0)\| + k\|Q+\chi\|_\infty \left(\int_0^t e^{-r(t-\tau)}d\tau\right)^{\frac{1}{2}} \left(\int_0^t e^{-r(t-\tau)}\|\dot{\hat{\theta}}\|^2 d\tau\right)^{\frac{1}{2}} \\ &\leq ke^{-rt}\|\varsigma(0)\| + k\|Q+\chi\|_\infty \frac{1}{\sqrt{r}} \left(\int_0^t e^{-r(t-\tau)}\|\dot{\hat{\theta}}\|^2 d\tau\right)^{\frac{1}{2}}, \qquad (2.134)\end{aligned}$$

where the second inequality is obtained by using the Schwartz inequality as given in Appendix B. By squaring (2.134) and integrating over $[0, t]$, we obtain that

$$\int_0^t \|\varsigma(\tau)\|^2 d\tau \leq \frac{k^2}{2r}\|\varsigma(0)\|^2 + \frac{k^2}{r}\|Q+\chi\|_\infty^2 \int_0^t \left(\int_0^\tau e^{-r(\tau-s)}\|\dot{\hat{\theta}}\|^2 ds\right) d\tau. \qquad (2.135)$$

Changing the sequence of integration, (2.135) becomes

$$\begin{aligned}\int_0^t \|\varsigma(\tau)\|^2 d\tau &\leq \frac{k^2}{2r}\|\varsigma(0)\|^2 + \frac{k^2}{r}\|Q+\chi\|_\infty^2 \int_0^t e^{rs}\|\dot{\hat{\theta}}\|^2 \left(\int_s^t e^{-r\tau}d\tau\right) ds \\ &\leq \frac{k^2}{2r}\|\varsigma(0)\|^2 + \frac{k^2}{r}\|Q+\chi\|_\infty^2 \int_0^t e^{rs}\|\dot{\hat{\theta}}\|^2 \frac{1}{r}e^{-rs} ds \\ &= \frac{k^2}{2r}\|\varsigma(0)\|^2 + \frac{k^2}{r^2}\|Q+\chi\|_\infty^2 \int_0^t \|\dot{\hat{\theta}}\|^2 ds, \qquad (2.136)\end{aligned}$$

where $\int_s^t e^{-r\tau}d\tau \leq \frac{1}{r}e^{-rs}$ is used. Since $\dot{\hat{\theta}} \in \mathcal{L}_2$, $\varsigma \in \mathcal{L}_2$ is concluded.

We then show that $\Omega^T\tilde{\theta} \in \mathcal{L}_2$ implies that $\chi^T\tilde{\theta} \in \mathcal{L}_2$. Introduce two filters

$$\begin{aligned}\dot{\zeta}_1 &= A\zeta_1 + F^T\tilde{\theta} \qquad (2.137) \\ \dot{\zeta}_2 &= A_z\zeta_2 + W^T\tilde{\theta}. \qquad (2.138)\end{aligned}$$

From (2.112) and (2.128), we note that

$$W^T(z,\hat{\theta},t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{\partial \alpha_1}{\partial x_1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\frac{\partial \alpha_{n-1}}{\partial x_1} & \cdots & -\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} & 1 \end{bmatrix} F^T(x) \triangleq M(z,\hat{\theta},t)F^T(x). \quad (2.139)$$

Based on this, (2.138) can be rewritten to be

$$\dot{\zeta}_2 = A_z\zeta_2 + MF^T\tilde{\theta}. \quad (2.140)$$

By following similar procedures in proving $\varsigma \in \mathcal{L}_2$, it can be shown that $\zeta_1 - \Omega^T\tilde{\theta} \in \mathcal{L}_2$ and $\zeta_2 - \chi^T\tilde{\theta} \in \mathcal{L}_2$. From $\Omega^T\tilde{\theta} \in \mathcal{L}_2$, it follows that $\zeta_1 \in \mathcal{L}_2$. The solution of (2.140) is computed as

$$\begin{aligned} \zeta_2 &= \Phi_{A_z}(t,0)\zeta_2(0) + \int_0^t \Phi_{A_z}(t,\tau)M(\tau)F^T(\tau)\tilde{\theta}(\tau)d\tau \\ &= \Phi_{A_z}(t,0)\zeta_2(0) + \int_0^t \Phi_{A_z}(t,\tau)M(\tau)(\dot{\zeta}_1 - A\zeta_1))d\tau \\ &= \Phi_{A_z}(t,0)\zeta_2(0) + M(t)\zeta_1(t) - \Phi_{A_z}(t,0)M(0)\zeta_1(0) \\ &\quad - \int_0^t \Phi_{A_z}(t,\tau)(\dot{M}(\tau) + A_z(\tau)M(\tau) + M(\tau)A(\tau))\zeta_1(\tau)d\tau. \end{aligned} \quad (2.141)$$

From (2.89), (2.94), (2.97), (2.101), (2.139) and the smoothness of $F^T(x)$, we see that the terms $\frac{\partial \alpha_i}{\partial x_j}$ are continuous functions of z, $\hat{\theta}$ and bounded functions of t. Thus M is bounded. Similarly, we can show that $\frac{\partial M}{\partial z}$, $\frac{\partial M}{\partial \hat{\theta}}$ and $\frac{\partial M}{\partial t}$ are bounded. Since $\dot{z}$ and $\dot{\hat{\theta}}$ are bounded in view of (2.126) and (2.119), $\dot{M} = \frac{\partial M}{\partial z}\dot{z} + \frac{\partial M}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial M}{\partial t}$ is bounded. Thus we have

$$\begin{aligned} &\left\| \int_0^t \Phi_{A_z}(t,\tau)(\dot{M}(\tau) + A_z(\tau)M(\tau) + M(\tau)A(\tau))\zeta_1(\tau)d\tau \right\|^2 \\ \le\ &\|\dot{M} + A_zM + MA\|_\infty^2 k^2 \int_0^t e^{-2r(t-\tau)}\|\zeta_1(\tau)\|^2 d\tau. \end{aligned} \quad (2.142)$$

By following similar procedures in (2.134)-(2.135), we can conclude that $\int_0^t \Phi_{A_z}(t,\tau)$

$(\dot{M}(\tau) + A_z(\tau)M(\tau) + M(\tau)A(\tau))\zeta_1(\tau)d\tau \in \mathcal{L}_2$. Further more, $\Phi_{A_z}(t,0)\zeta_2(0) + M(t)\zeta_1(t) - \Phi_{A_z}(t,0)M(0)\zeta_1(0) \in \mathcal{L}_2$ because $\Phi_{A_z}(t,0)$ is exponentially decaying, M is bounded and $\zeta_1 \in \mathcal{L}_2$. Hence, $\zeta_2 \in \mathcal{L}_2$ and $\chi^T\tilde{\theta} \in \mathcal{L}_2$. Consequently, $z \in \mathcal{L}_2$. Combining with $\dot{z} \in \mathcal{L}_\infty$, it is concluded that $\lim_{t\to\infty} z(t) = 0$. □

This section gives standard procedures to design adaptive backstepping controllers, with tuning function and modular design schemes respectively. In the corresponding analysis parts, system stability and tracking performance are investigated. It should be noted that the designed controllers in this chapter are known as full "state-feedback" controllers. That is because the results are obtained under the assumption that the full state of the system is measurable. However for many realistic problems, only a part of the state or the plant output is available for measurement. To address these problems, state observers are often needed to provide the estimates of unmeasurable states.

As basic design ideas and related analysis of adaptive backstepping technique are only introduced here as preliminary knowledge for the remainder of the thesis, the procedures of extending the full state-feedback results to partial state-feedback and output-feedback problems will not be included in this chapter. Interested readers can refer to [21] and [37] for more details.

Based on backstepping technique, some new developments in adaptive control of uncertain systems with actuator failures and subsystem interactions will be presented in the remaining part of the thesis.

Part I

Adaptive Actuator Failure Compensation

Chapter 3

Adaptive Failure Compensation with Relative Degree Condition Relaxed

In this chapter, we aim to develop adaptive output-feedback controllers for a class of uncertain systems with multiple inputs and single output (MISO). In achieving satisfactory output regulation and the boundedness of all closed-loop signals, the actuators corresponding to the inputs are redundant for one another if the output of it is stuck at some unknown constant. The considered class of systems has a characteristic that the relative degrees with respect to the inputs are not necessarily the same. To deal with these inputs using backstepping technique, we introduce a pre-filter before each actuator such that its output is the input to the actuator. The orders of the pre-filters are chosen properly to ensure all their inputs can be designed at the same step in the systematic design. To illustrate our design idea, we will firstly consider set-point regulation problem for linear systems and then extend the results to nonlinear systems with asymptotic tracking performance to be achieved.

3.1 Background

We consider TLOE type of actuator failures in this chapter, which is characterized by the output of a failed actuator being stuck at some unknown values. As the failed actuator cannot respond to the control inputs in this scenario, it loses the effectiveness completely in manipulating the variables of the system by executing the control commands. To stabilize the system and maintain desired performances in the presence of such failures, actuation redundancy has been widely employed. For example, in an aircraft control system as shown in Figure 3.1, the orientation of the aircraft can be achieved through deflecting appropriate control surfaces including left (right) aileron, left (right) elevator and rudder. The control surfaces are

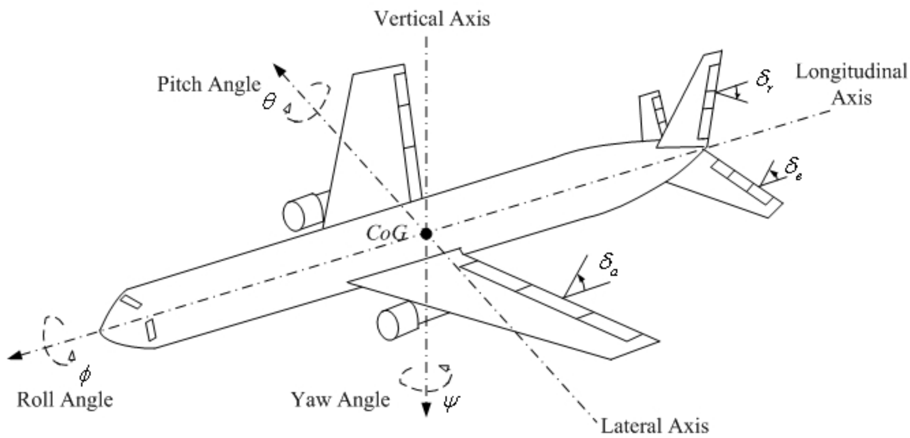


Figure 3.1: An aircraft control system [1]

the actuators of the system and often divided into several individually segments. Thus if some of the segments are icing up and stuck at some fixed positions, the remaining functional segments can still be properly controlled to guarantee system performances satisfied by compensating for the effects of the failed ones.

There are also some other examples of actuation redundancy in improving the system reliability with actuator failures. In [2], a dual-actuator ball-beam system is described as in Figure 3.2. The system involves two driving motors, one at each

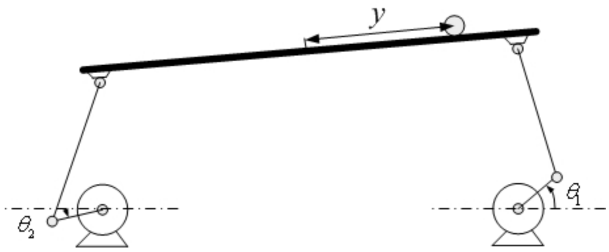


Figure 3.2: A dual-motor beam-ball system [2]

end. The two motors take responsibilities of moving the beam at the two ends up and down for balancing the ball at a desired position, in which any one can be considered as redundant if the other is blocked and of which the angular position is fixed. In [3], a hexapod robot system is studied as plotted in Figure 3.3. To pre-

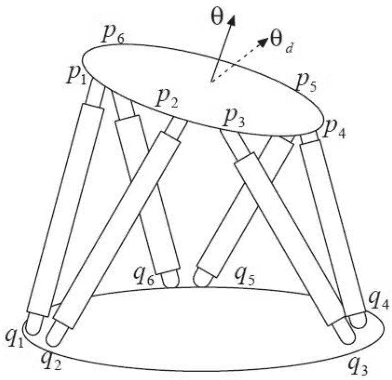


Figure 3.3: A hexapod robot system [3]

cisely regulate the angular positions of the object on the platform at some desired values, only three degree of freedom (DOF) are required. However, there are six struts whose length can be controlled. The extra three DOF can thus be adopted as a built-in redundancy in control designs with actuator failures. A three-tank system in Figure 3.4 is considered in [4] to develop a failure tolerant control design scheme.

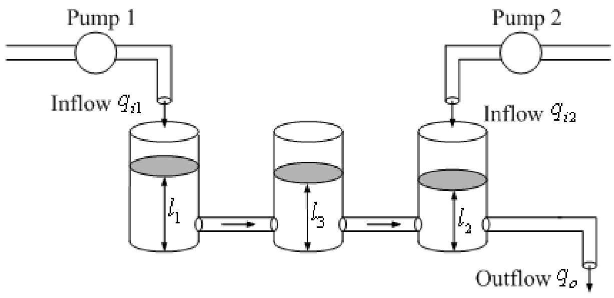


Figure 3.4: A three-tank system [4]

The system has three cylindrical tanks with identical cross section. The tanks are coupled by two connecting cylindrical pipes and the nominal outflow is located at the tank 2. Two pumps driven by DC motors supply required liquids to the tanks 1 and 2. If one of the pumps is blocked and the inflow of which is stuck at a fixed value, the other can still be adjusted accordingly to maintain the liquid level in tank 2.

As discussed in Chapter 1, actuator failures are often uncertain in time, value and pattern. Because of its prominent feature in handling uncertainties, adaptive control has been proved as a desirable tool to accommodate actuator failures for both linear systems and nonlinear systems [43, 45, 54, 55, 69, 70, 73, 95]. In [45], a MRAC based actuator failure compensation method is proposed to solve tracking problem for linear system with actuator failures. Unknown system parameters are considered and handled simultaneously with the large uncertain structural and parametric changes caused by the failures in control design, where the available actuator redundancy is utilized and explicit failure detection and diagnostic is not required. The class of failure compensation control schemes combining these features is referred by Tao *et al.* as "direct" adaptive solutions. Backstepping technique has been widely used to design adaptive controllers for nonlinear systems with uncertainties. Based on that, adaptive state feedback and output-feedback controllers are designed for nonlinear systems with actuator failures in [54] and [95] respectively. The results are extended to nonlinear multi-input and multi-output (MIMO) system in [55]. Unknown nonlinearities are treated in [96] by adopting adaptive fuzzy approximation approach.

3.1.1 A Motivating Example

In [45, 54, 95], a common condition exists that the relative degrees with respect to each control inputs to the system output are identical. In [55], it is also indicated that only the actuators, corresponding to which the relative degrees with respect to the inputs are the same, can be designed to compensate for one another. However, in some control systems, such a condition on the relative degrees may not be satisfied.

For example, in a system with two rolling carts connected by a spring and a damper as shown in Figure 3.5, two external forces u_1, u_2 located at distinct carts are generated by two motors respectively. Other variables of interest are noted on

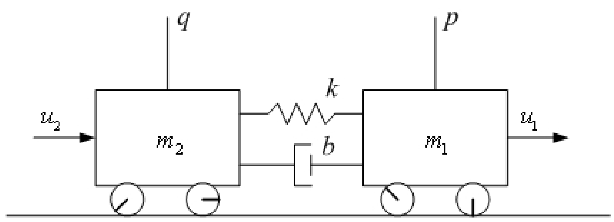


Figure 3.5: Two rolling carts attached with spring and damper

the figure and defined as: m_1, m_2 = mass of carts, p, q = positions of two carts, k = spring constant and b = damping coefficient. We assume that the carts have negligible rolling friction. The control objective is to regulate cart 1 to a desired position while maintaining the boundedness of all signals in the presence of one motor failing.

We now determine the dynamic model of such a control system. Define $\bar{x}_1 = p$, $\bar{x}_2 = q$, $\bar{x}_3 = \dot{p}$, $\bar{x}_4 = \dot{q}$, where $\dot{p}, \dot{q}$ denote the velocity of m_1, m_2. By using Newton's second law, i.e. sum of the forces equaling mass of the object multiplied by its acceleration, the state space model of the system can be obtained as follows,

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + B_1 u_1 + B_2 u_2 \\ y &= C\bar{x}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & -\frac{b}{m_1} & \frac{b}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} & \frac{b}{m_2} & -\frac{b}{m_2} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} \\ C &= [1, 0, 0, 0]. \end{aligned} \tag{3.2}$$

If the observability matrix

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \tag{3.3}$$

of the system (3.1) has full rank, O^{-1} exists and system (3.1) is observable. We define $O_4 = O^{-1}e_4$, $P = [O_4, AO_4, A^2O_4, A^3O_4]$, $T = [e_4, e_3, e_2, e_1]P^{-1}$, where e_i denotes the ith coordinate vector in $\Re^4$. Under transformation $x = T\bar{x}$, (3.1) can be transformed to the observable canonical form of (3.6) as

$$\begin{aligned} \dot{x} &= Ax - ya + \begin{bmatrix} 0 \\ b_1 \end{bmatrix} u_1 + \begin{bmatrix} 0_2 \\ b_2 \end{bmatrix} u_2 \\ y &= e_1^T x, \end{aligned} \tag{3.4}$$

where $0_2 \in \Re^2$, $a = [a_3, a_2, a_1, a_0]^T$, $b_1 = [b_{12}, b_{11}, b_{10}]^T$, $b_2 = [b_{21}, b_{20}]^T$. Either u_1 or u_2 can be properly designed to accommodate the stuck failure of the other. However, observing from (3.4), the relative degrees with respect to u_1 and u_2 are 2 and 3 respectively.

Note that the relative degree condition is relaxed in [97] where failure accommodation is performed on the basis of accurate failure detection and isolation. In this chapter, we focus on a "direct" adaptive solution to the actuator failure compensation problem with different relative degrees. To achieve this, a pre-filter is introduced before each actuator such that its output is the input to the actuator. The order of the filter is properly chosen so that all their inputs can be designed at the same step. We will start with set-point regulation for linear systems and extend the results to nonlinear systems by considering tracking problem.

3.1.2 Modeling of Actuator Failures

Suppose there are m inputs in the system. The block diagram of a single loop consisting of the plant preceded by the jth actuator and a feedback controller is given in Figure 3.6. u_{cj} denotes the input of the jth actuator, which is the control

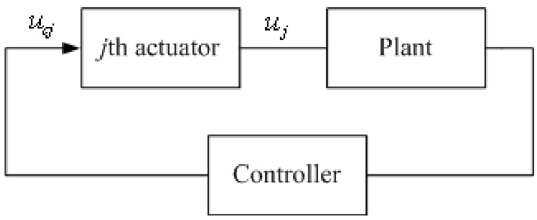


Figure 3.6: The block diagram of a single loop.

signal generated by the designed controller. If the internal dynamics of an actuator is neglected, it is regarded as a failure-free actuator when $u_j = u_{cj}$. The considered TLOE type of failures in this chapter, which occur on the jth actuator are modeled as follows,

$$u_j(t) = u_{kj}, \quad t \geq t_{jF}, \quad j \in \{1, 2, \ldots, m\} \tag{3.5}$$

where u_{kj} is a constant and t_{jF} is the time instant at which the jth actuator fails. Eqn. (3.5) describes that from time t_{jF}, the jth actuator is stuck at some fixed value and can no longer respond to the input u_{cj}. Both u_{kj} and t_{jF} are unknown.

To solve the actuator failure compensation problem for the systems with m inputs and single output in this chapter, a common assumption is imposed.

Assumption 3.1.1. *Up to $m-1$ actuators may suffer from the actuator failures modeled as in (3.5) simultaneously so that the remaining actuators can still achieve a desired control objective.*

Remark 3.1.1.

• Observing (3.5), the uniqueness of t_{jF} indicates that a failure occurs only once on the jth actuator. The failure case is unidirectional, which is commonly encountered in practice since fault repairing is sometimes hardly implemented such as during the flight of an apparatus. This implies that there exists a finite T_r denoting the time instant of the last failure and the total number of failures along the time scale $[0, +\infty)$ is finite. Similar assumptions could be found in many pervious results, such as in [43, 45, 54, 55, 95].

• As discussed in [98], Assumption 3.1.1 is a basic condition to ensure the controllability of the plant and existence of a nominal solution to actuator failure compensation problem with known failure pattern and system parameters.

3.2 Set-Point Regulation for Linear Systems

In this section, the control problem is firstly formulated. The designs of pre-filters and control laws are elaborated with the relative degree condition corresponding to redundant actuators relaxed. It will be shown that the effects due to actuator failures can be compensated for with the designed controllers. The boundedness of the closed-loop signals can be ensured. Further, the system output can also be regulated to a specific value. The effectiveness of the proposed approach is evaluated through the application to the mass-spring-damper system in Figure 3.5.

3.2.1 Problem Formulation

Similar to [97], we consider a class of linear systems described as

$$y = \sum_{j=1}^{m} G_j(p) u_j, \tag{3.6}$$

where $u_j \in \Re$, $j = 1, \ldots, m$ and $y \in \Re$ are the inputs and output respectively, p denotes the differential operator $\frac{d}{dt}$, $G_j(p)$, $j = 1, \ldots, m$ are rational functions of p. With p replaced by s, the corresponding $G_j(s)$ is the transfer function

$$G_j(s) = \frac{b_j(s)}{a(s)} = \frac{b_{j\bar{n}_j} s^{\bar{n}_j} + \cdots + b_{j1} s + b_{j0}}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \tag{3.7}$$

An assumption on $G_j(s)$ is made as follows,

Assumption 3.2.1. *For each $G_j(s)$, $a_k, k = 0, \ldots, n-1$ and $b_{jk}, k = 0, \ldots, \bar{n}_j$ are unknown constants, $b_{j\bar{n}_j} \neq 0$. The order n, the sign of $b_{j\bar{n}_j}$, i.e. $sgn(b_{j\bar{n}_j})$ and the relative degrees $\rho_j (= n - \bar{n}_j)$ are known.*

The design objective is to regulate the output y of the system as described in (3.6) to a specific value y_s while maintaining boundedness of all closed-loop signals by designing output-feedback controllers, despite the presence of actuator failures as modeled in (3.5).

3.2.2 Preliminary Designs

A. Design of Pre-filters

Observed from (3.7), the relative degree ρ_j of the transfer function with respect to each system input u_j, $j = 1, \ldots, m$ may not be identical. To overcome the difficulties when the backstepping technique is applied, we firstly introduce a pre-filter before each actuator as suggested in Figure 3.7.

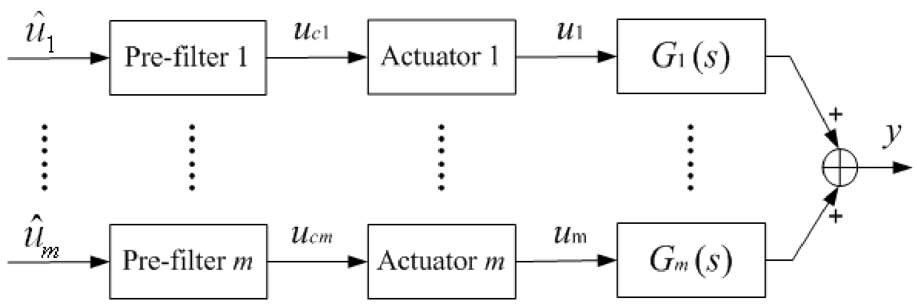


Figure 3.7: Design of pre-filters before each actuator

For the jth pre-filter, it is designed that

$$u_{cj} = \frac{1}{(p+\delta)^{\bar{n}_j+\rho-n}}\hat{u}_j, \tag{3.8}$$

where $\rho = \max\{\rho_j\}$ for $j = 1, \ldots, m$, $\delta > 0$ is to be chosen. $\hat{u}_j$ is the input of the jth pre-filter. Note that for those u_j with $\rho_j = \rho$, $u_{cj} = \hat{u}_j$. As indicated in Section 3.1.2, $u_j = u_{cj}$ for a failure-free actuator. Based on this, (3.6) can be rewritten with $\hat{u}_j$ as the jth input in failure-free case.

1) *Failure-free Case*: In this case, all of the actuators are 100% effective in exe-

cuting their inputs. Thus by substituting (3.8) into (3.6), we obtain

$$\begin{aligned} y &= \sum_{j=1}^{m} G_j(p) \frac{1}{(p+\delta)^{\bar{n}_j+\rho-n}} \hat{u}_j \\ &= \sum_{j=1}^{m} \frac{b_j(p)}{a(p)} \frac{(p+\delta)^{\bar{n}-\bar{n}_j}}{(p+\delta)^{\bar{n}+\rho-n}} \hat{u}_j \\ &= \sum_{j=1}^{m} \frac{\bar{b}_{j\bar{n}} p^{\bar{n}} + \cdots + \bar{b}_{j1} p + \bar{b}_{j0}}{p^{\bar{n}+\rho} + \bar{a}_{\bar{n}+\rho-1} p^{\bar{n}+\rho-1} + \cdots + \bar{a}_1 + \bar{a}_0} \hat{u}_j, \end{aligned} \tag{3.9}$$

where $\bar{n} = \max\{\bar{n}_j\}$ for $j = 1, \ldots, m$, $\bar{b}_{j\bar{n}} = b_{j\bar{n}_j}$. From (3.9), we see that the relative degrees with respect to each $\hat{u}_j$ are all equal to ρ. We can represent (3.9) in the observer canonical form

$$\begin{aligned} \dot{x} &= Ax - y\bar{a} + \sum_{j=1}^{m} \begin{bmatrix} 0_{\rho-1} \\ \bar{b}_j \end{bmatrix} \hat{u}_j \\ y &= e_{\bar{n}+\rho,1}^T x, \end{aligned} \tag{3.10}$$

where

$$A = \begin{bmatrix} 0_{\bar{n}+\rho-1} & I_{\bar{n}+\rho-1} \\ 0 & 0_{\bar{n}+\rho-1}^T \end{bmatrix}, \quad \bar{a} = \begin{bmatrix} \bar{a}_{\bar{n}+\rho-1} \\ \vdots \\ \bar{a}_0 \end{bmatrix}, \quad \bar{b}_j = \begin{bmatrix} \bar{b}_{j\bar{n}} \\ \vdots \\ \bar{b}_{j0} \end{bmatrix}. \tag{3.11}$$

$0_i \in \Re^i$ and $e_{i,j}$ denotes the jth coordinate vector in $\Re^i$.

2) *Failure Case*: Suppose that there are a finite number of time instants $T_1, T_2, \ldots,$ $T_r (T_1 < T_2 < \cdots < T_r \ll +\infty)$ and only at which some of the m actuators fail. During the time interval $[T_{k-1}, T_k)$, where $k = 1, \ldots, r$ and $T_{r+1} = \infty$, there are g_k failed actuators, i.e. $u_j(t) = u_{kj}$ for $j = j_i,\ i = 1, 2, \cdots, g_k$. Then (3.9) is changed

to

$$\begin{aligned} y &= \sum_{j\neq j_1,\cdots,j_{g_k}} G_j(p)\frac{1}{(p+\delta)^{\bar{n}_j+\rho-n}}\hat{u}_j + \sum_{j=j_1,\cdots,j_{g_k}} G_j(p)\frac{(p+\delta)^{\bar{n}+\rho-n}}{(p+\delta)^{\bar{n}+\rho-n}}u_{kj} \\ &= \sum_{j\neq j_1,\cdots,j_{g_k}} \frac{\bar{b}_{j\bar{n}}p^{\bar{n}}+\cdots+\bar{b}_{j1}p+\bar{b}_{j0}}{p^{\bar{n}+\rho}+\bar{a}_{\bar{n}+\rho-1}p^{\bar{n}+\rho-1}+\cdots+\bar{a}_1+\bar{a}_0}\hat{u}_j \\ &\quad + \sum_{j=j_1,\cdots,j_{g_k}} \frac{\underline{b}_{j(\bar{n}+\rho-\rho_j)}p^{\bar{n}+\rho-\rho_j}+\cdots+\underline{b}_{j1}p+\underline{b}_{j0}}{p^{\bar{n}+\rho}+\bar{a}_{\bar{n}+\rho-1}p^{\bar{n}+\rho-1}+\cdots+\bar{a}_1+\bar{a}_0}u_{kj}, \end{aligned} \tag{3.12}$$

where $\underline{b}_{j(\bar{n}+\rho-\rho_j)} = b_{j\bar{n}_j}$ for $j = j_1, \ldots, j_{g_k}$. We define $h = \min\{\rho_j\}$ for $j = 1, \ldots, m$. Similar to (3.10), (3.12) can be represented in the following state space form

$$\begin{aligned} \dot{x}_1 &= x_2 - \bar{a}_{\bar{n}+\rho-1}y \\ \vdots &\qquad \vdots \\ \dot{x}_h &= x_{h+1} - \bar{a}_{\bar{n}+\rho-h}y + \bar{u}_{\bar{n}+\rho-h} \\ \vdots &\qquad \vdots \\ \dot{x}_\rho &= x_{\rho+1} - \bar{a}_{\bar{n}}y + \sum_{j\neq j_1,\cdots,j_{g_k}} \bar{b}_{j\bar{n}}\hat{u}_j + \bar{u}_{\bar{n}} \\ \vdots &\qquad \vdots \\ \dot{x}_{\bar{n}+\rho} &= -\bar{a}_0 y + \sum_{j\neq j_1,\cdots,j_{g_k}} \bar{b}_{j0}\hat{u}_j + \bar{u}_0, \end{aligned} \tag{3.13}$$

where $\bar{u}_q = \sum_{j=j_1,\ldots,j_{g_k}} \underline{b}_{jq}u_{kj}$ for $q = 0, \ldots, \bar{n}+\rho-h$ are unknown constants to be identified together with unknown system parameters. (3.13) can be rewritten as

$$\begin{aligned} \dot{x} &= Ax - y\bar{a} + \sum_{j\neq j_1,\cdots,j_{g_i}} \begin{bmatrix} 0_{\rho-1} \\ \bar{b}_j \end{bmatrix} \hat{u}_j + \begin{bmatrix} 0_{h-1} \\ \bar{u} \end{bmatrix} \\ y &= e_{\bar{n}+\rho,1}^T x, \end{aligned} \tag{3.14}$$

where $\bar{u} = [\bar{u}_{\bar{n}+\rho-h}, \ldots, \bar{u}_0]^T$.

B. Design of $\hat{u}_j$

For the inputs of each pre-filters, $\hat{u}_j$ is designed as

$$\hat{u}_j = \mathrm{sgn}(b_{j\bar{n}_j})u_0 \tag{3.15}$$

where u_0 is the actual control signal to be generated by performing the backstepping technique. By substituting (3.15) into (3.10) in failure-free case and (3.14) in failure-free case respectively, the controlled plant can be expressed in the following unified form

$$\begin{aligned} \dot{x} &= Ax - y\bar{a} + \sum_{j\neq j_1,\ldots,j_{g_k}} \begin{bmatrix} 0_{\rho-1} \\ \bar{\bar{b}}_j \end{bmatrix} u_0 + \begin{bmatrix} 0_{h-1} \\ \bar{u} \end{bmatrix} \\ y &= e^T_{\bar{n}+\rho,1}x, \end{aligned} \tag{3.16}$$

where $\bar{\bar{b}}_j = [|b_{j\bar{n}_j}|, \mathrm{sgn}(b_{j\bar{n}_j})\bar{b}_{j\bar{n}-1}, \ldots, \mathrm{sgn}(b_{j\bar{n}_j})\bar{b}_{j0}]^T$. $\bar{u}$ can be considered as a piecewise constant disturbance. In failure-free case, $\sum_{j\neq j_1,\ldots,j_{g_k}} \bar{\bar{b}}_j$ actually includes $\bar{\bar{b}}_j$ for all $j = 1, \ldots, m$ and $\bar{u} = 0$.

Remark 3.2.1. It is important to note that the unknown vectors $\sum_{j\neq j_1,\ldots,j_{g_k}} \bar{\bar{b}}_j$ and $\bar{u}$ depend on the system parameters $b_{j0}, \ldots, b_{j\bar{n}_j}$ as well as the the knowledge of the actuator failures. Jumpings on $\sum_{j\neq j_1,\ldots,j_{g_k}} \bar{\bar{b}}_j$ and $\bar{u}$ will occur whenever the actuator failure pattern changes. They are actually piecewise constant vectors, which will be identified together with $\bar{a}$. By doing this, the effects due to failed actuators can be compensated for.

C. State Estimation Filters

It should be noted that the full states of system are not measurable. Thus we introduce the following filters to estimate the unmeasurable states x in (3.16), as similarly discussed in [21, 37],

$$\dot{\eta} = A_0\eta + e_{\bar{n}+\rho,\bar{n}+\rho}y \tag{3.17}$$

$$\dot{\lambda} = A_0\lambda + e_{\bar{n}+\rho,\bar{n}+\rho}u_0 \tag{3.18}$$

$$\dot{\Phi} = A_0\Phi + e_{\bar{n}+\rho,\bar{n}+\rho} \tag{3.19}$$

All states of the filters in (3.17) and (3.19) are available for feedback. We define

$$\mu_k = A_0^k\lambda, \quad k = 0, \ldots, \bar{n} \tag{3.20}$$

$$\Psi_k = A_0^k\Phi, \quad k = 0, \ldots, \bar{n} + \rho - h \tag{3.21}$$

where $A_0 = A - le^T_{\bar{n}+\rho,1}$, the vector $l = [l_1, \ldots, l_{\bar{n}+\rho}]^T$ is chosen that the matrix A_0 is Hurwitz. Hence there exists a matrix P such that $PA_0 + A_0^TP = -I, P = P^T > 0$. With these designed filters x can be estimated by

$$\hat{x} = \xi + \Omega^T\theta, \tag{3.22}$$

where

$$\xi = -A_0^{\bar{n}+\rho}\eta \tag{3.23}$$

$$\Omega^T = [\mu_{\bar{n}}, \ldots, \mu_1, \mu_0, \Xi, \Psi_{\bar{n}+\rho-h}, \ldots, \Psi_0] \tag{3.24}$$

$$\Xi = -[A_0^{\bar{n}+\rho-1}\eta, \ldots, A_0\eta, \eta] \tag{3.25}$$

$$\theta = [\sum_{j\neq j_1,\ldots,j_{g_k}} \bar{\bar{b}}_j^T, \bar{a}^T, \bar{u}^T]^T \in \Re^{3\bar{n}+2\rho-h+2}. \tag{3.26}$$

The state estimation error $\epsilon = x - \hat{x}$ satisfies

$$\dot{\epsilon} = A_0\epsilon. \tag{3.27}$$

Thus, system (3.16) can be expressed in the following form

$$\dot{y} = \sum_{j \neq j_1, \ldots, j_{g_k}} |b_{j\bar{n}_j}| \mu_{\bar{n},2} + \xi_2 + \bar{\omega}^T \theta + \epsilon_2 \tag{3.28}$$

$$\dot{\mu}_{\bar{n},q} = \mu_{\bar{n},q+1} - l_q \mu_{\bar{n},1}, \quad q = 2, \ldots, \rho - 1 \tag{3.29}$$

$$\dot{\mu}_{\bar{n},\rho} = \mu_{\bar{n},\rho+1} - l_\rho \mu_{\bar{n},1} + u_0, \tag{3.30}$$

where

$$\bar{\omega}^T = [0, \mu_{\bar{n}-1,2}, \ldots, \mu_{0,2}, \Xi_2 - y e^T_{\bar{n}+\rho,1}, \Psi_{\bar{n}+\rho-h,2}, \ldots, \Psi_{0,2}] \tag{3.31}$$

and $\mu_{k,2}$ for $k = 0, \ldots, \bar{n}$, $\Psi_{k,2}$ for $k = 0, \ldots, \bar{n} + \rho - h$, ξ_2, Ξ_2 denote the second entries of μ_k, Ψ_k, ξ, Ξ respectively.

3.2.3 Design of u_0 and Parameter Update Laws

Performing standard backstepping procedures in [21,37], u_0 can be generated at the ρth step as summarized below.

The change of coordinates are:

$$z_1 = y - y_s \tag{3.32}$$

$$z_q = \mu_{\bar{n},q} - \alpha_{q-1}, \quad q = 2, 3, \ldots, \rho \tag{3.33}$$

Design u_0 as:

$$u_0 = \alpha_\rho - \mu_{\bar{n},\rho+1}, \tag{3.34}$$

where

$$\alpha_1 = \hat{\varrho} \bar{\alpha}_1 \tag{3.35}$$

$$\bar{\alpha}_1 = -c_1 z_1 - d_1 z_1 - \xi_2 - \bar{\omega}^T \hat{\theta} \tag{3.36}$$

$$\alpha_2 = -e^T_{3\bar{n}+2\rho-h+2,1} \hat{\theta} z_1 - \left[c_2 + d_2 \left(\frac{\partial \alpha_1}{\partial y} \right)^2 \right] z_2 + \bar{B}_2 + \frac{\partial \alpha_1}{\partial \hat{\varrho}} \dot{\hat{\varrho}} + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2 \tag{3.37}$$

$$\begin{aligned}
\alpha_q &= -z_{q-1} - \left[c_q + d_q \left(\frac{\partial \alpha_{q-1}}{\partial y}\right)^2\right] z_q + \bar{B}_q + \frac{\partial \alpha_{q-1}}{\partial \hat{\varrho}}\dot{\hat{\varrho}} + \frac{\partial \alpha_{q-1}}{\partial \hat{\theta}}\Gamma\tau_q \\
&\quad -\sum_{k=2}^{q-1} \frac{\partial \alpha_{q-1}}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_{q-1}}{\partial y}\delta z_q, \quad q = 3, \ldots, \rho \qquad (3.38) \\
\bar{B}_q &= \frac{\partial \alpha_{q-1}}{\partial y}(\xi_2 + \omega^T\hat{\theta}) + \frac{\partial \alpha_{q-1}}{\partial \eta}(A_0\eta + e_{\bar{n}+\rho,\bar{n}+\rho}y) + l_q\mu_{\bar{n},1} \\
&\quad + \sum_{k=1}^{\bar{n}+q-1} \frac{\partial \alpha_{q-1}}{\partial \lambda_k}(-l_k\lambda_1 + \lambda_{k+1}) + \frac{\partial \alpha_{q-1}}{\partial \Phi}(A_0\Phi + e_{\bar{n}+\rho,\bar{n}+\rho}), \quad q = 2, \ldots, \rho \\
&\qquad (3.39) \\
\tau_1 &= (\omega - \hat{\varrho}\bar{\alpha}_1 e_{2\bar{n}+2\rho-h+2,1})z_1 \qquad (3.40) \\
\tau_q &= \tau_{q-1} - \frac{\partial \alpha_{q-1}}{\partial y}\omega z_q, \quad q = 2, \ldots, \rho \qquad (3.41) \\
\omega &= [\mu_{\bar{n},2}, \mu_{\bar{n}-1,2}, \ldots, \mu_{0,2}, \Xi_2 - ye^T_{\bar{n}+\rho,1}, \Psi_{\bar{n}+\rho-h,2}, \ldots, \Psi_{0,2}]^T. \qquad (3.42)
\end{aligned}$$

$\hat{\varrho}$ is an estimate of $1/\sum_{j\neq j_1,\ldots,j_{g_i}} |b_{j\bar{n}_j}|$, $\hat{\theta}$ is an estimate of θ and c_q, d_q for $q = 1, \ldots, \rho$ are positive constants.

Parameter Update Laws are given by

$$\begin{aligned}
\dot{\hat{\varrho}} &= -\gamma\bar{\alpha}_1 z_1 \qquad (3.43) \\
\dot{\hat{\theta}} &= \Gamma\tau_\rho, \qquad (3.44)
\end{aligned}$$

where γ is positive constant and Γ is a positive definite matrix.

To better illustrate the structure of designed adaptive controllers, a block diagram is given in Figure 3.8.

3.2.4 Stability Analysis

To prove the boundedness of all the closed-loop signals, the following assumption is required. Suppose that there are r time instants, from which some of the actuators

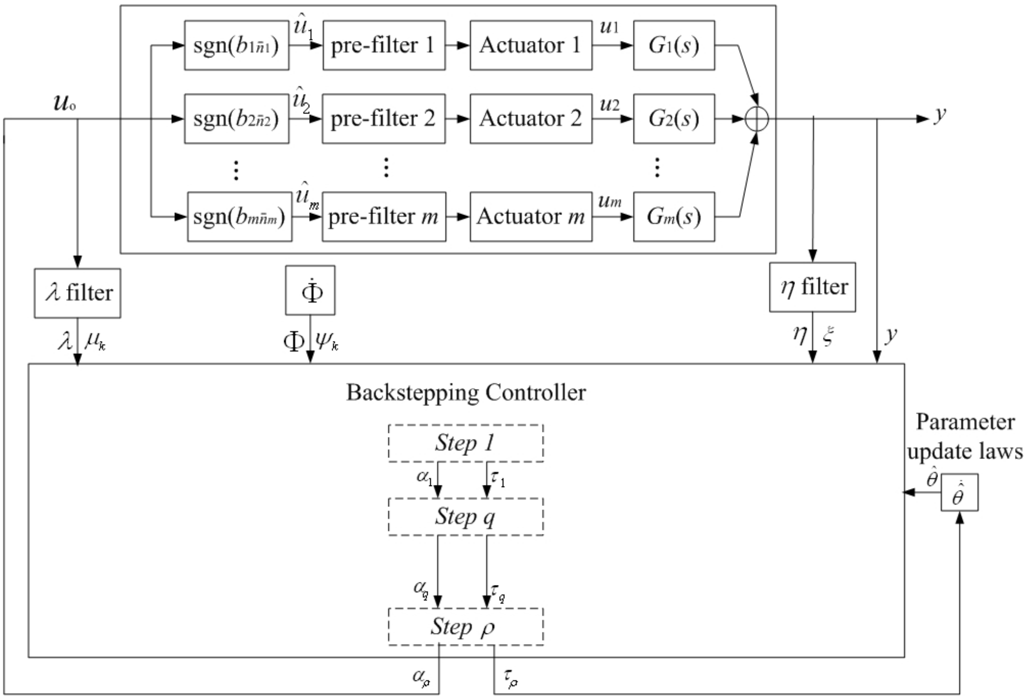


Figure 3.8: Control block diagram

fail. During the time intervals $[T_{k-1}, T_k)$, where $k = 1, \ldots, r+1$, $T_0 = 0$ and $T_{r+1} = \infty$, the failure pattern is fixed and there are g_k failed actuators indexed by j_i, for $i = 1, \ldots, g_k$.

Assumption 3.2.2.

The polynomials $\sum_{j \neq j_1, \cdots, j_{g_k}} sgn(b_{j\bar{n}_j})(\bar{b}_{j\bar{n}} p^{\bar{n}} + \cdots + \bar{b}_{j1} p + \bar{b}_{j0})$ *are Hurwitz.*

Remark 3.2.2. Similar to [54, 95], Assumption 3.2.2 refers to the minimum phase condition for the controlled systems (3.10), (3.14) in the failure-free case and all possible failure cases. It should be noted that if the order of the original plant (3.6) is $n = 2$, all the polynomials $b_j(p)$ for $j = 1, \ldots, m$ being Hurwitz is sufficient to satisfy Assumption 3.2.2. For a third order plant, the coefficients $b_{j\bar{n}_j}, \ldots, b_{j0}$ in $b_j(p)$ having the same signs for $j = 1, \ldots, m$ respectively can also meet the assumption. Nevertheless, further investigations are still needed to determine how this assumption be justified for higher order system.

We now define a positive definite function $V_{k-1}(t)$ for $t \in [T_{k-1}, T_k)$ for $k = 1, \dots, r+1$ with T_r denoting the time instant of the last failure.

$$V_{k-1}(t) = \frac{1}{2}z^T z + \frac{1}{2}\sum_{q=1}^{\rho} \frac{1}{4d_q}\epsilon^T P\epsilon + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \frac{\sum_{j\neq j_1,\dots,j_{g_k}} |b_{j\bar{n}_j}|}{2\gamma}\tilde{\varrho}^2, \quad (3.45)$$

where $z = [z_1, \dots, z_\rho]^T$, $\tilde{\theta} = \theta - \hat{\theta}$ and $\tilde{\varrho} = \varrho - \hat{\varrho}$. With the designed adaptive controllers, the time derivative of $V_{k-1}(t)$ can be rendered negative definite.

$$\dot{V}_{k-1}(t) \leq -\sum_{q=1}^{\rho} c_q z_q^2, \quad t \in [T_{k-1}, T_k) \quad (3.46)$$

We define $V_{k-1}(T_k^-) = \lim_{\triangle t\to 0^-} V_{k-1}(T_k + \triangle t)$ and $V_{k-1}(T_{k-1}^+) = \lim_{\triangle t\to 0^+} V_{k-1}(T_{k-1} + \triangle t) = V_{k-1}(T_{k-1})$. If we let $V(t) = V_{k-1}(t)$ for $t \in [T_{k-1}, T_k)$ where $i = 1, \dots, r+1$, $V(t)$ is a piece-wise continuous function. From (3.46), we have $V_{k-1}(T_k^-) \leq V_{k-1}(T_{k-1}^+)$. At each T_k, parameter jumpings occur on $\sum_{j\neq j_1,\dots,j_{g_k}} \bar{\bar{b}}_j$ and $\bar{u}$, due to new actuators' failing, will result in changes on the last two terms in (3.45) by comparing $V_k(T_k^+)$ with $V_{k-1}(T_k^-)$. It can be shown that $V_k(T_k^+) \leq 2V_{k-1}(T_k^-) + \triangle V_k$. We illustrate an example to explain such boundedness. For simplicity of presentation, choose $\Gamma = I_{(3\bar{n}+2\rho-h+1)\times(3\bar{n}+2\rho-h+1)}$, $\gamma = 1$. We have

$$\begin{aligned} \tilde{\theta}(T_k^+)^T\tilde{\theta}(T_k^+) &= (\theta(T_k^+) - \hat{\theta}(T_k))^T(\theta(T_k^+) - \hat{\theta}(T_k)) \\ &\leq 2(\theta(T_k^-) - \hat{\theta}(T_k))^T(\theta(T_k^-) - \hat{\theta}(T_k)) \\ &\quad +2(\theta(T_k^+) - \theta(T_k^-))^T(\theta(T_k^+) - \theta(T_k^-)), \end{aligned} \quad (3.47)$$

where the fact $(a+b)^2 \leq 2a^2 + 2b^2$ is used. Suppose that there are p_1 failed actuators $(\hbar_1, \dots, \hbar_{p_1})$ before time T_k, while $p_2 - p_1$ actuators fail at time T_k. Hence we have $\varrho(T_k^-) = 1/\sum_{j\neq \hbar_1,\dots,\hbar_{p_1}} |b_{j\bar{n}_j}|$ while $\varrho(T_k^+) = 1/\sum_{j\neq \hbar_1,\dots,\hbar_{p_2}} |b_{j\bar{n}_j}|$. Define

$\varsigma(T_k) = \frac{1}{\varrho(T_k)}$, similar to (3.47), we obtain that

$$\begin{aligned} \varsigma(T_k^+)\tilde{\varrho}^2(T_k^+) &= \varsigma(T_k^+)(\varrho(T_k^+) - \hat{\varrho}(T_k))^2 \\ &\leq \varsigma(T_k^-)(\varrho(T_k^+) - \hat{\varrho}(T_k))^2 \\ &\leq \varsigma(T_k^-)\big[2(\varrho(T_k^-) - \hat{\varrho}(T_k))^2 + 2(\varrho(T_k^+) - \hat{\varrho}(T_k^-))^2\big]. \end{aligned} \quad (3.48)$$

Note that $0 \leq \varsigma(T_k^+) \leq \varsigma(T_k^-)$. From (3.47) and (3.48), we have $V_k(T_k^+) \leq 2V_{k-1}(T_k^-) + \triangle V_k$ where $\triangle V_k$ is bounded. Hence $V_r(T_r^+) \leq 2V_{r-1}(T_r^-) + \triangle V_r \leq 2V_{r-1}(T_{r-1}^+) + \triangle V_r \leq 4V_{r-2}(T_{r-1}^-) + 2\triangle V_{r-1} + \triangle V_r$. By proceeding to such iterative procedures, $V_r(t) \leq \Lambda V_0(0) + \Upsilon$ for $t \in [T_r, \infty)$ will be achieved, where $\Lambda > 0$ and $\Upsilon > 0$ denote generic positive constants. Therefore z, ϵ, $\hat{\theta}$, $\hat{\varrho}$ are bounded since $V_0(0)$ is bounded. From (3.32), y is also bounded. From (3.17), we conclude that η is bounded. From (3.16) and (3.18), we have

$$\lambda_i = \frac{p^{i-1} + l_1 p^{i-2} + \ldots + l_{i-1}}{L(p)\sum_{j\neq j_1,\ldots,j_{g_k}} \mathrm{sgn}(b_{j\bar{n}_j})(\bar{b}_{j\bar{n}}p^{\bar{n}} + \cdots + \bar{b}_{j1}p + \bar{b}_{j0})}(p^{\bar{n}+\rho} + \bar{a}_{\bar{n}+\rho-1}p^{\bar{n}+\rho-1} + \bar{a}_0)y, \quad (3.49)$$

where $L(p) = p^{\bar{n}+\rho} + l_1 p^{\bar{n}+\rho-1} + \ldots + l_{\bar{n}+\rho}$. From the boundedness of y and Assumption 3.2.2, it follows that $\lambda_1, \ldots, \lambda_{\bar{n}+1}$ are bounded. The coordinate change (3.33) gives that $\mu_{\bar{n},2} = z_2 + \alpha_1$. Since α_1 is the function of $y, \eta, \lambda_1, \ldots, \lambda_{\bar{n}+1}, \Phi$ and the boundedness of all the arguments and z_2, we conclude that $\mu_{\bar{n},2}$ is bounded. From (3.20), $\mu_{\bar{n},2} = [*, \ldots, *, 1][\lambda_1, \ldots, \lambda_{\bar{n}+2}]^T$. Thus $\lambda_{\bar{n}+2}$ is bounded. By repeating the similar procedures, λ being bounded can be established. From (3.22), $x = \epsilon + \hat{x}$, (3.24), (3.25), (3.20) and the boundedness of $\eta, \lambda, \Psi, \epsilon$, we conclude that x is bounded. u_0 is bounded based on (3.34). From (3.15), the boundedness of $\hat{u}_j$ is then ensured. Since $\delta > 0$ in (3.8), u_{cj} is bounded. Thus all the signals in the closed-loop adaptive system are bounded. From (3.46), $z(t) \in \mathcal{L}_2$. Noting $\dot{z} \in \mathcal{L}_\infty$, it follows that $\lim_{t\to\infty} z(t) = 0$, which implies that $\lim_{t\to\infty} y(t) = y_s$. The above results is formally stated in the following theorem.

Theorem 3.2.1. *Consider the closed-loop adaptive system consisting of the plant (3.6), pre-filters (3.8), the controllers (3.15), (3.34), the parameter estimators (3.107), (3.108) and the state estimation filters (3.17)-(3.19) in the presence of actuator failures as modeled in (3.5) under Assumptions 3.1.1-3.2.2. All the closed-loop signals are bounded and the system output can be regulated to* y_s*, i.e.* $\lim_{t\to\infty} y(t) = y_s$.

3.2.5 Application to The Mass-spring-damper System

We consider the mass-spring-damper system as shown in Figure 3.5. The control objective is to regulate the position of m_1 to $p = 2m$ while maintaining the boundedness of all signals in the presence of actuator failures. In simulation, the variables are chosen as $m_1 = 1$ kg, $m_2 = 2$ kg, $k = 10$ N/sec, $b = 20$ N·sec/m, which are unknown in control design. As discussed in Section 3.1.1, the controlled plant can be expressed as in (3.4). Suppose that the only information known in simulation is that b_{12}, b_{11}, b_{10} and b_{21}, b_{20} are all positive constants. Then according to Remark 3.2.2, Assumption 3.2.2 is satisfied. Since the relative degrees with respect to u_1 and u_2 are 2 and 3 respectively. Thus the pre-filters for u_1 and u_2 are designed as $u_1 = \frac{1}{p+\delta}\hat{u}_1$ and $u_2 = \hat{u}_2$. We choose $\delta = 1$. In simulation, all the initial values are set as 0 except for $q(0) = -1m$. Other design parameters are chosen as $l = [10, 40, 80, 80, 32]^T$, $c_1 = c_2 = c_3 = 3, d_1 = d_2 = d_3 = 0.01$, $\gamma = 0.1$, $\Gamma = 0.1 \times I$.

Two failure cases are considered respectively,

- Case 1: The output of actuator u_1 is stuck at $u_{k1} = 2$ from $t = 5$ seconds.
- Case 2: The output of actuator u_2 is stuck at $u_{k2} = 2$ from $t = 5$ seconds.

The error $e = y - y_s$ as well as control inputs for both cases are given in Fig. 3.9-3.12. It is observed that the system output can still be regulated to $y_s = 2$ in both failure cases despite of a degradation of performance.

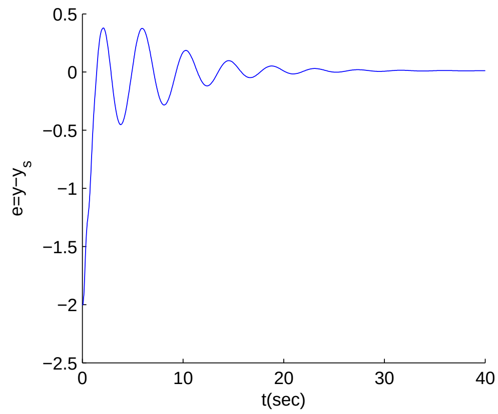


Figure 3.9: Error $y - y_s$ in failure case 1

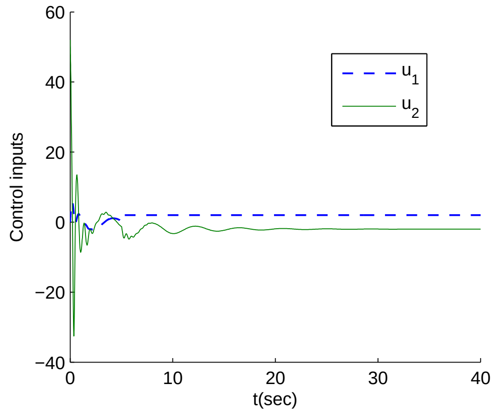


Figure 3.10: Controller inputs in failure case 1

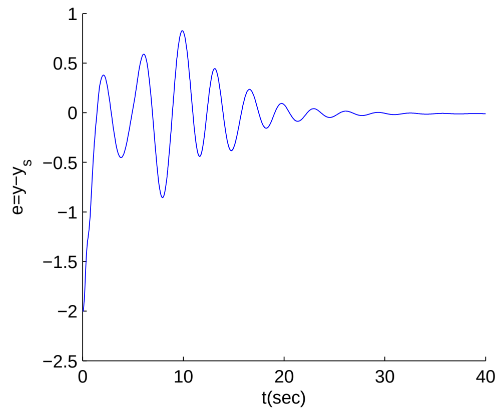


Figure 3.11: Error $y - y_s$ in failure case 2

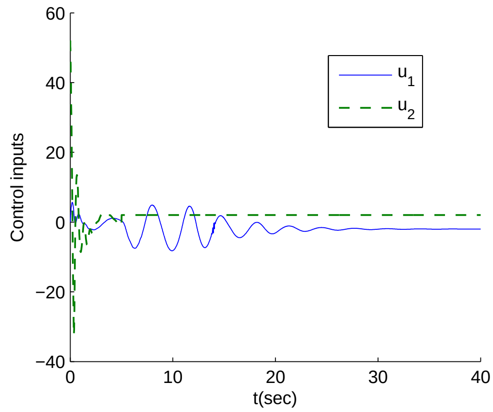


Figure 3.12: Controller inputs in failure case 2

3.3 Tracking for Nonlinear Systems

In this section, we will design adaptive output-feedback controllers for a class of nonlinear MISO systems with unknown parameters and uncertain actuator failures to force the system output asymptotically tracking a given reference signal. In the previous section, the state space model of the controlled linear system is established on the basis of arithmetic operations of polynomials with respect to p as performed in (3.9), (3.12). In contrast to this, we will establish the state space model of the nonlinear system consisting of the original plant and the designed pre-filters through defining new states equation by equation in this section.

3.3.1 Problem Formulation

Extending from the observable canonical form of state space model for the linear systems (3.6) by including output dependent nonlinearities, we consider a class of nonlinear MISO systems described as follows,

$$\dot{x} = \mathscr{A}x + \phi(y) + \bar{\Phi}(y)a + \sum_{j=1}^{m} \begin{bmatrix} 0 \\ b_j \end{bmatrix} \sigma_j(y)u_j \tag{3.50}$$

$$y = x_1, \tag{3.51}$$

where $x = [x_1, \ldots, x_n]^T \in \Re^n$ is the state, $u_j \in \Re$ for $j = 1, 2, \ldots, m$ are the m inputs of the system, i.e. the outputs of the m actuators, $y \in \Re$ is the system output.

$$\mathscr{A} = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ 0 & 0_{n-1}^T \end{bmatrix}, \ \phi(y) = \begin{bmatrix} \phi_1(y) \\ \vdots \\ \phi_n(y) \end{bmatrix} \tag{3.52}$$

$$\bar{\Phi}(y) = \begin{bmatrix} \bar{\Phi}_1(y) \\ \vdots \\ \bar{\Phi}_n(y) \end{bmatrix} = \begin{bmatrix} \varphi_{1,1}(y) & \cdots & \varphi_{q,1}(y) \\ \vdots & \ddots & \vdots \\ \varphi_{1,n}(y) & \cdots & \varphi_{q,n}(y) \end{bmatrix}. \tag{3.53}$$

$\phi_i(y)$ for $i = 1, \ldots, n$, $\varphi_{i,k}$ for $i = 1, \ldots, q$, $k = 1, \ldots, n$ and $\sigma_j(y)$ for $j = 1, \ldots, m$ are known smooth nonlinear functions, $a = [a_1, \ldots, a_q]^T \in \Re^q$, $b_j = [b_{j\bar{n}_j}, \ldots, b_{j0}]^T \in \Re^{\bar{n}_j+1}$ for $j = 1, \ldots, m$ are vectors of unknown constant parameters.

The control objective is to design adaptive output-feedback controllers such that the effects of the actuator failures can be compensated for. Thus the boundedness of all closed-loop signals is achieved and the system output $y(t)$ asymptotically tracks a given reference signal $y_r(t)$.

Similar to Assumption 3.2.1 for the considered linear systems, the following assumption is imposed.

Assumption 3.3.1. *The sign of $b_{j\bar{n}_j}$, i.e. $sgn(b_{j\bar{n}_j})$, for $j = 1, \ldots, m$ is known. $b_{j\bar{n}_j} \neq 0$ and $\sigma_j(y) \neq 0$, $\forall y \in \Re$. The plant order n and relative degree with respect to each input $\rho_j = n - \bar{n}_j$ are known.*

In addition, the following assumption is also required to achieve the control objectives.

Assumption 3.3.2. *The reference signal y_r and its first ρth order derivatives, where $\rho = \max_{1 \leq j \leq m} \rho_j$, are known and bounded, and piecewise continuous.*

3.3.2 Preliminary Designs

Without loss of generality, we assume that in (3.50), $\bar{n}_1 \geq \bar{n}_2 \geq \cdots \geq \bar{n}_m$. Thus we have $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_m$, $\rho = \rho_m$ based on the definition of ρ in Assumption 3.3.2.

A. Design of Pre-filters

Design a pre-filter for the jth actuator as

$$u_{cj} = \mathrm{sgn}(b_{j\bar{n}_j})\hat{u}_j/\sigma_j(y), \quad j = 1, \ldots, m \tag{3.54}$$

$$\hat{u}_j = \frac{u_0}{(p+\delta)^{\rho-\rho_j}}, \tag{3.55}$$

where p denotes the differential operator $\frac{d}{dt}$, $\delta > 0$ is to be chosen. u_0 is the input of the pre-filter, which is the actual control variable to be generated by performing backstepping technique. Note that for those u_j with $\rho_j = \rho$, $\hat{u}_j$ is designed as $\hat{u}_j = u_0$. The state space model of (3.55) is

$$\dot{\varsigma}_{j,k} = -\delta\varsigma_{j,k} + \varsigma_{j,k+1}, \quad k = 1, \ldots, \rho - \rho_j - 1 \tag{3.56}$$

$$\dot{\varsigma}_{j,\rho-\rho_j} = -\delta\varsigma_{j,\rho-\rho_j} + u_0. \tag{3.57}$$

Let $\hat{u}_j = \varsigma_{j,1}$ and

$$\varsigma_{j,k} = u_0/(p+\delta)^{\rho-\rho_j-k+1}, \ k = 2, \ldots, \rho - \rho_j \tag{3.58}$$

B. Construction of A New Plant

At this point, we construct a new plant based on the designed pre-filters (3.54)-(3.55). The state space models of the newly constructed plant will be derived under failure-free and failure cases respectively. To the end, a unified state space model will be established for both cases.

1) *Failure-free Case*: Note that the newly constructed plant is a $(n + \rho - \rho_1)$th-order system.

◇ For the case that $n = 1$, we have that $\bar{n}_j = 0$ and $\rho_j = 1$ for all inputs. This implies that all u_j appear firstly at the equation of $\dot{y}$. The model in this case is quite straightforward.

◇ For the case that $n = 2$, suppose we have some inputs with $\rho_j = 1$ and the rest of the inputs with $\rho_j = 2$. Obviously, $\rho = 2$. We now suppose that $\rho_j = 1$ for $j = 1, 2, \ldots, j_1$ and $\rho_j = 2$ for $j = j_1 + 1, \ldots, m$. Define that $\varkappa_1 = x_1$. From (3.50) and the fact that $\hat{u}_j = \varsigma_{j,1}$, it is obtained that

$$\dot{\varkappa}_1 = \varkappa_2 + \phi_1 + \bar{\Phi}_1 a, \tag{3.59}$$

where $\varkappa_2$ is defined as

$$\varkappa_2 = x_2 + \sum_{j=1}^{j_1} |b_{j1}| \varsigma_{j,1}. \tag{3.60}$$

As it is designed that $\hat{u}_j = u_0/(p+\delta)$ for $j = 1, \ldots, j_1$, $\hat{u}_j = u_0$ for $j = j_1+1, \ldots, m$, from (3.50) and (3.56), the time derivative of $\varkappa_2$ is computed as

$$\begin{aligned} \dot{\varkappa}_2 &= \dot{x}_2 + \sum_{j=1}^{j_1} |b_{j\bar{n}_j}| \dot{\varsigma}_{j,1} \\ &= \varkappa_3 + \phi_2 + \bar{\Phi}_2 a + \mathfrak{b}_1 u_0, \end{aligned} \tag{3.61}$$

where $\varkappa_3$ is defined as

$$\varkappa_3 = \sum_{j=1}^{j_1} \left[\operatorname{sgn}(b_{j1}) b_{j0} + |b_{j1}|(-\delta)\right] \varsigma_{j,1} \tag{3.62}$$

and $\mathfrak{b}_1 = \sum_{j=1}^{m} |b_{j\bar{n}_j}|$. The derivative of $\varkappa_3$ is

$$\dot{\varkappa}_3 = -\delta \varkappa_3 + \mathfrak{b}_0 u_0, \tag{3.63}$$

where $\mathfrak{b}_0 = \sum_{j=1}^{j_1} [\operatorname{sgn}(b_{j1}) b_{j0} + |b_{j1}|(-\delta)]$.

$\diamond$ We consider the case that $n > 2$.

If $\rho = 1$, $\rho_j = 1$ for all the inputs. This is similar to the case that $n = 1$.

If $\rho = 2$, suppose that $\rho_j = 1$ for $j = 1, \ldots, j_1$ and $\rho_j = 2$ for $j = j_1 + 1, \ldots, m$. Similar to the case that $n = 2$, by introducing new states $\varkappa_{i+1}$ that include the original states x_{i+1} for $i = 1, \ldots, n$ with $x_{n+1} = 0$ and all the terms with respect to the states $\varsigma_{j,k}$ in (3.56)-(3.57), the first n equations of the state space model are derived as follows,

$$\dot{\varkappa}_1 = \varkappa_2 + \phi_1 + \bar{\Phi}_2 a \tag{3.64}$$

$$\dot{\varkappa}_i = \varkappa_{i+1} + \phi_i + \bar{\Phi}_i a + \mathfrak{b}_{n-i+1} u_0, \quad i = 2, \ldots, n \tag{3.65}$$

where $\mathfrak{b}_{n-1} = \sum_{j=1}^{r} |b_{j\bar{n}_j}|$. If we define $f(\delta, q)$ as a qth order polynomial of δ, $f(\delta, q) = \nu_q \delta^q + \nu_{q-1}\delta^{q-1} + \cdots + \nu_0$ with ν_i for $i = 0, \ldots, q$ representing constants, one can show that $\varkappa_n$ is expressed as

$$\begin{aligned} \varkappa_n &= x_n + \sum_{j=1}^{j_1} (f(\delta, n-2)\varsigma_{j,1} + f(\delta, n-3)\varsigma_{j,2} + \cdots + f(\delta, 0)\varsigma_{j,n-1}) \\ &+ \sum_{j=j_1+1}^{m} (f(\delta, n-3)\varsigma_{j,1} + \cdots + f(\delta, n-4)\varsigma_{j,2} + \cdots + f(\delta, 0)\varsigma_{j,n-2}) \end{aligned} \tag{3.66}$$

and $\varkappa_{n+1}$ consists only the terms with respect to the states $\varsigma_{j,1}$. The derivative of $\varkappa_{n+1}$ is thus computed as

$$\dot{\varkappa}_{n+1} = -\delta\varkappa_{n+1} + \mathfrak{b}_0 u_0. \tag{3.67}$$

If $\rho > 2$, the first n equations are changed to

$$\dot{\varkappa}_i = \varkappa_{i+1} + \phi_i + \bar{\Phi}_i a, \; i = 1, \ldots, \rho - 1 \tag{3.68}$$

$$\dot{\varkappa}_q = \varkappa_{q+1} + \mathfrak{b}_{n+\rho-\rho_1-q} u_0, \; q = \rho, \ldots, n \tag{3.69}$$

The derivatives of $\varkappa_i$ for $i > n$ are computed as

$$\dot{\varkappa}_k = -\delta\varkappa_k + \varkappa_{k+1} + \mathfrak{b}_{n+\rho-\rho_1-k} u_0, \quad k = n+1, \ldots, \bar{n} + \rho - 1 \tag{3.70}$$

$$\dot{\varkappa}_{\bar{n}+\rho} = -\delta\varkappa_{n+\rho-\rho_1} + \mathfrak{b}_0 u_0, \tag{3.71}$$

where $\bar{n} = n - \rho_1 = \bar{n}_1$.

In summary, the state space model of the newly constructed plant under failure-free case can be written as follows,

$$\dot{\varkappa} = A\varkappa + \begin{bmatrix} \phi(y) \\ 0_{\rho-\rho_1} \end{bmatrix} + \begin{bmatrix} \bar{\Phi}(y) \\ 0_{(\rho-\rho_1)\times q} \end{bmatrix} a + \begin{bmatrix} 0_{\rho-1} \\ \mathfrak{b} \end{bmatrix} u_0 \tag{3.72}$$

$$y = \varkappa_1, \tag{3.73}$$

where the new states $\varkappa \in \Re^{\bar{n}+\rho}$,

$$A = \begin{array}{c} \left[\begin{array}{cccccccc} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & -\delta & 1 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & -\delta & 1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & -\delta \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} n \\ \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} \rho-\rho_1 \end{array} \end{array} \tag{3.74}$$

and $\mathfrak{b} = [\mathfrak{b}_{\bar{n}}, \mathfrak{b}_{\bar{n}-1}, \ldots, \mathfrak{b}_0]^T \in \Re^{\bar{n}+1}$ with $\mathfrak{b}_{\bar{n}} = \sum_{j=1}^{m} |b_{j\bar{n}_j}| > 0$.

2) *Faulty Case*: Suppose that there are a finite number of time instants $T_1, T_2, \ldots,$ $T_r (T_1 < T_2 < \cdots < T_r \ll \infty)$ and only at the time instants T_k, $k = 1, 2, \ldots, r$, some of the r actuators fail. During the time interval (T_{k-1}, T_k), for $k = 1, 2, \ldots, r$ with $T_{r+1} = \infty$, there are g_k failed actuators' outputs are stuck at u_{kj} for $j = j_{g_1}, j_{g_2}, \ldots, j_{g_k}$. Then due to the effects from failed actuators, the state space model in (3.72)-(3.73) is changed to

$$\begin{aligned} \dot{\varkappa} &= A\varkappa + \begin{bmatrix} \phi \\ 0_{\rho-\rho_1} \end{bmatrix} + \begin{bmatrix} \bar{\Phi} \\ 0_{(\rho-\rho_1)\times q} \end{bmatrix} a + \sum_{j=j_{g_1},\ldots,j_{g_k}} \begin{bmatrix} 0_{\rho_j-1} \\ b_j u_{k_j} \\ 0_{\rho-\rho_1} \end{bmatrix} \sigma_j \\ &\quad + \begin{bmatrix} 0_{\rho-1} \\ \mathfrak{b} \end{bmatrix} u_0 && (3.75) \\ y &= \varkappa_1, && (3.76) \end{aligned}$$

where A is defined in (3.74), $\mathfrak{b}_{\bar{n}} = \sum_{j \neq j_{g_1},\ldots,j_{g_k}} |b_{j\bar{n}_j}| > 0$. Note that (3.75)-(3.76) are also applicable to the case when all the actuators with which $\rho_j = \rho_1$ for $j =$

$1, 2, \ldots, j_1$ fail with $\varkappa_i = 0$ for $i = n + \rho - \rho_2 + 1, \ldots, \bar{n} + \rho$ and $\mathfrak{b}_i = 0$ for $i = 0, \ldots, \rho_2 - \rho_1 - 1$.

From the models derived under both failure-free and faulty cases, i.e. (3.72) and (3.73), (3.75) and (3.76), the controlled plant can be expressed in the following unified form

$$\begin{aligned} \dot{\varkappa} &= A\varkappa + \begin{bmatrix} \phi \\ 0_{\rho-\rho_1} \end{bmatrix} + \begin{bmatrix} \bar{\Phi} \\ 0_{(\rho-\rho_1)\times q} \end{bmatrix} a + \sum_{j=1}^{m} \begin{bmatrix} 0_{\rho_j-1} \\ K_j \end{bmatrix} \sigma_j(y) \\ &\quad + \begin{bmatrix} 0_{\rho-1} \\ \mathfrak{b} \end{bmatrix} u_0 \qquad (3.77) \\ y &= \varkappa_1, \qquad (3.78) \end{aligned}$$

where $K_j \in \Re^{\bar{n}_j+1+\rho-\rho_1}$.

C. State Estimation Filters

The unmeasured state $\varkappa$ can be estimated by introducing filters as follows:

$$\dot{\xi} = A_0\xi + ly + \begin{bmatrix} \phi(y) \\ 0 \end{bmatrix} \qquad (3.79)$$

$$\dot{\Xi} = A_0\Xi + \begin{bmatrix} \bar{\Phi}(y) \\ 0 \end{bmatrix} \qquad (3.80)$$

$$\dot{\lambda} = A_0\lambda + e_{\bar{n}+\rho,\bar{n}+\rho}u_0 \qquad (3.81)$$

$$\dot{\eta}_j = A_0\eta_i + e_{\bar{n}+\rho,\bar{n}+\rho}\sigma_j(y), \quad j = 1, \ldots, m \qquad (3.82)$$

where $A_0 = A - le^T_{\bar{n}+\rho,1}$ with $l = [l_1, \ldots, l_{\bar{n}+\rho}]^T$ and is chosen to be Hurwitz, and $e_{i,j}$ denotes the jth coordinate vector in $\Re^i$. Hence there exist a P such that $PA_0 + A_0^T P = -I$, $P = P^T > 0$.

Remark 3.3.1. It can be shown that $\det(sI - A_0) = \mathfrak{L}(s, l, \delta)$ where

$$\mathfrak{L}(s,l,\delta) \quad = \quad (s+l_1)s^{n-1-\rho+\rho_1}(s+\delta)^{\rho-\rho_1} + l_2 s^{n-2-\rho+\rho_1}(s+\delta)^{\rho-\rho_1} + \cdots + l_{\bar{n}}. \tag{3.83}$$

From (3.83), we know that l can be computed based on $\bar{l}$ and δ, where $\bar{l} = [\bar{l}_1, \ldots, \bar{l}_{\bar{n}+\rho}]$ is the normal vector chosen as in previous section such that $\mathscr{A}_0 = \mathscr{A} - \bar{l}e^T_{\bar{n}+\rho,1}$ is Hurwitz, where $\mathscr{A} \in \Re^{(\bar{n}+\rho)\times(\bar{n}+\rho)}$ is of the same form as in (3.52).

We now define

$$v_k \quad = \quad A_0^k \lambda, \quad k = 0, \ldots, \bar{n}_1 \tag{3.84}$$

$$\mu_{jk} \quad = \quad A_0^k \eta_j, \quad j = 1, \ldots, m, \quad k = 0, \ldots, \bar{n}_j + \rho - \rho_1 \tag{3.85}$$

One can show that

$$A_0^k e_{\bar{n}+\rho,\bar{n}+\rho} = \begin{bmatrix} 0_{\bar{n}+\rho-k-1} \\ 1 \\ * \end{bmatrix}, \quad k = 0, \ldots, \bar{n}+\rho-1 \tag{3.86}$$

where $* \in \Re^k$ is a constant vector. Hence we have

$$\begin{bmatrix} 0 \\ \mathfrak{b} \end{bmatrix} \quad = \quad A_0^{\bar{n}} e_{\bar{n}+\rho,\bar{n}+\rho}\bar{\mathfrak{b}}_{\bar{n}} + \cdots + e_{\bar{n}+\rho,\bar{n}+\rho}\bar{\mathfrak{b}}_0 \tag{3.87}$$

$$K_j \quad = \quad A_0^{\bar{n}_j+\rho-\rho_1} e_{\bar{n}+\rho,\bar{n}+\rho}\bar{K}_{j\bar{n}_j+\rho-\rho_1} + \cdots + e_{\bar{n}+\rho,\bar{n}+\rho}\bar{K}_{j0}, j = 1, \ldots, m \tag{3.88}$$

With the designed filters (3.79)-(3.82), the unmeasured states in (3.77) can be estimated by

$$\begin{aligned}\hat{\varkappa} &= \xi + \Xi a + [v_{\bar{n}}, \ldots, v_0]\bar{\mathfrak{b}} + [\mu_{1\bar{n}_1+\rho-\rho_1}, \ldots, \mu_{10}, \mu_{2\bar{n}_2+\rho-\rho_1}, \ldots, \mu_{20}, \ldots, \\ &\quad \mu_{m\bar{n}_m+\rho-\rho_1}, \ldots, \mu_{m0}]\bar{K}, \qquad (3.89)\end{aligned}$$

where $\bar{\mathfrak{b}} = [\bar{\mathfrak{b}}_{\bar{n}}, \ldots, \bar{\mathfrak{b}}_0]^T$, $\bar{K} = [\bar{K}_{1\bar{n}_1+\rho-\rho_1}, \ldots, \bar{K}_{10}, \bar{K}_{2\bar{n}_2+\rho-\rho_1}, \ldots, \bar{K}_{20}, \ldots, \bar{K}_{m\bar{n}_m+\rho-\rho_1}, \ldots, \bar{K}_{m0}]^T$ are constant vectors. $\bar{\mathfrak{b}}_{m_1} > 0, \forall t > 0$.
The state estimation error $\epsilon = \varkappa - \hat{\varkappa}$ is readily shown to satisfy

$$\dot{\epsilon} = A_0\epsilon. \qquad (3.90)$$

Defining

$$\begin{aligned}\theta^T &= [\bar{\mathfrak{b}}^T, a^T, \bar{K}^T] \qquad (3.91)\\ \omega^T &= [v_{\bar{n},2}, v_{\bar{n}-1,2}, \ldots, v_{0,2}, \Xi_{(2)} + \Phi_1, \mu_{1\bar{n}_1+\rho-\rho_1,2}, \ldots, \mu_{10,2}, \mu_{2\bar{n}_2+\rho-\rho_1,2}, \ldots, \\ &\quad \mu_{20,2}, \ldots, \mu_{m\bar{n}_m+\rho-\rho_1,2}, \ldots, \mu_{m0,2}] \qquad (3.92)\\ \bar{\omega}^T &= [0, v_{\bar{n}-1,2}, \ldots, v_{0,2}, \Xi_{(2)} + \Phi_1, \mu_{1\bar{n}_1+\rho-\rho_1,2}, \ldots, \mu_{10,2}, \mu_{2\bar{n}_2+\rho-\rho_1,2}, \ldots, \\ &\quad \mu_{20,2}, \ldots, \mu_{m\bar{n}_m+\rho-\rho_1,2}, \ldots, \mu_{m0,2}], \qquad (3.93)\end{aligned}$$

where $v_{i,2}$ for $i = 0, \ldots, \bar{n}$, $\Xi_{(2)}$, $\mu_{jk,2}$ for $j = 1, \ldots, m$ $k = 0, \ldots, \bar{n}_j + \rho - \rho_1$ denote the second entries of v_i, Ξ and μ_{jk} respectively.

Then system (3.77)-(3.78) can expressed as follows, to which we will apply backstepping technique.

$$\begin{aligned}\dot{y} &= v_{\bar{n},2}\bar{\mathfrak{b}}_{\bar{n}} + \xi_2 + \bar{\omega}^T\theta + \phi_1 + \epsilon_2 \qquad (3.94)\\ \dot{v}_{\bar{n},i} &= -l_i v_{\bar{n},1} + v_{\bar{n},i+1}, \quad i = 2, \ldots, \rho - 1 \qquad (3.95)\\ \dot{v}_{\bar{n},\rho} &= -l_\rho v_{\bar{n},1} + v_{\bar{n},\rho+1} + u_0 \qquad (3.96)\end{aligned}$$

3.3.3 Design of u_0 and Parameter Update Laws

Define a change of coordinates

$$z_1 = y - y_r \tag{3.97}$$

$$z_q = v_{\bar{n},q} - y_r^{(q-1)} - \alpha_{q-1}, \quad q = 2, \ldots, \rho \tag{3.98}$$

Design u_0 as

$$u_0 = \alpha_\rho - v_{\bar{n},\rho+1} \tag{3.99}$$

with

$$\alpha_1 = \hat{\varrho}\bar{\alpha}_1 \tag{3.100}$$

$$\bar{\alpha}_1 = -c_1 z_1 - d_1 z_1 - \xi_2 - \bar{\omega}^T\hat{\theta} - \phi_1 \tag{3.101}$$

$$\alpha_2 = -c_2 z_2 - d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_2 - \hat{\bar{\mathfrak{b}}}_{\bar{n}} z_1 + l_2 v_{\bar{n},1}\bar{B}_2 + \frac{\partial\alpha_1}{\partial\hat{\varrho}}\dot{\hat{\varrho}} + \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\tau_2 \tag{3.102}$$

$$\alpha_q = -c_q z_q - d_q\left(\frac{\partial\alpha_{q-1}}{\partial y}\right)^2 z_q - z_{q-1} + \bar{B}_q + \frac{\partial\alpha_{q-1}}{\partial\hat{\varrho}}\dot{\hat{\varrho}} + \frac{\partial\alpha_{q-1}}{\partial\hat{\theta}}\Gamma\tau_q, \quad q = 3, \ldots, \rho \tag{3.103}$$

$$\begin{aligned}\bar{B}_q = {} & l_q v_{\bar{n},1} + \frac{\partial\alpha_{q-1}}{\partial y}(\xi_2 + \phi_1 + \omega^T\hat{\theta}) + \sum_{k=1}^{q-1}\frac{\partial\alpha_{q-1}}{\partial y_r^{(k-1)}}y_r^{(k)} + \frac{\partial\alpha_{q-1}}{\partial\xi}\Bigg(A_0\xi + ly \\ & + \begin{bmatrix}\phi \\ 0\end{bmatrix}\Bigg) + \frac{\partial\alpha_{i-1}}{\partial\Xi}\left(A_0\Xi + \begin{bmatrix}\Phi \\ 0\end{bmatrix}\right) + \sum_{j=1}^{m}\frac{\partial\alpha_{q-1}}{\partial\eta_j}(A_0\eta_j + e_{\bar{n}+\rho,\bar{n}+\rho}\sigma_j) \\ & + \sum_{k=1}^{\bar{n}+q-1}\frac{\partial\alpha_{q-1}}{\partial\lambda_k}(-l_k\lambda_1 + \lambda_{k+1}), \quad q = 2, \ldots, \rho\end{aligned} \tag{3.104}$$

$$\tau_1 = \omega z_1 - \hat{\varrho}\bar{\alpha}_1 e_{n^*,1} z_1 \tag{3.105}$$

$$\tau_q = \tau_{q-1} - \frac{\partial\alpha_{q-1}}{\partial y}\omega z_q, \quad q = 2, \ldots, \rho \tag{3.106}$$

where $n^* = \bar{n}_1 + 1 + \sum_{j=1}^{m} \bar{n}_j + m + m(\rho - \rho_1) + q$. The design parameters c_q, d_q, γ are positive constants and Γ is a positive definite matrix of dimension $n^* \times n^*$. $\hat{\varrho}$, $\hat{\theta}$, $\hat{\bar{\mathfrak{b}}}_{\bar{n}}$ are the estimates of $\varrho = \bar{\mathfrak{b}}_{m_1}^{-1}$, θ and $\bar{\mathfrak{b}}_{\bar{n}}$ respectively.

Parameter update laws are chosen as

$$\dot{\hat{\varrho}} = -\gamma \bar{\alpha}_1 z_1 \tag{3.107}$$

$$\dot{\hat{\theta}} = \Gamma \tau_\rho \tag{3.108}$$

3.3.4 Stability Analysis

Similarly to the previous section, one more assumption related to minimum phase condition is required to prove the boundedness of closed-loop signals. Suppose there are g_k failed actuators ($j = j_{g_1}, \ldots, j_{g_k}$) and the failure pattern is fixed during the time interval (T_{k-1}, T_k), for $k = 1, \ldots, r+1$. T_r denotes the time instant at which the last failure occur. $T_{r+1} = \infty$.

Assumption 3.3.3. *The polynomials* $\sum_{j \neq j_{g_1}, \ldots, j_{g_k}} sgn(b_{j\bar{n}_j}) \mathfrak{B}_j(p)(p+\delta)^{\bar{n}-\bar{n}_j}$, $\forall \{j_{g_1}, \ldots, j_{g_k}\} \subset \{1, \ldots, m\}$ *are Hurwitz, where*

$$\mathfrak{B}_j(p) = b_{j\bar{n}_j} p^{\bar{n}_j} + \ldots, + b_{j1} p + b_{j0}. \tag{3.109}$$

For the adaptive scheme developed in the previous section, we establish the following result.

Theorem 3.3.1. *Consider the closed-loop adaptive system consisting of the plant (3.50)-(3.51), the pre-filters (3.54)-(3.55), the controller law (3.99), the parameter update laws (3.107), (3.108) and the state estimation filters (3.79)-(3.82) under Assumption 3.1.1 and Assumptions 3.3.1-3.3.2, all the signals in the closed-loop* *system are bounded and asymptotic tracking is achieved, i.e.* $\lim_{t\to\infty}[y(t)-y_r(t)] = 0$.

Proof: A mathematical model for the error system $\dot{z} = [\dot{z}_1, \ldots, \dot{z}_\rho]^T$ is derived from (3.97)-(3.106).

$$\dot{z} \;=\; A_z z + W_\epsilon \epsilon_2 + W_\theta^T \tilde{\theta} - \bar{\mathfrak{b}}_{m_1} \bar{\alpha}_1 e_{\rho,1} \tilde{\varrho}, \tag{3.110}$$

where $\tilde{\theta} = \theta - \hat{\theta}$, $\tilde{\varrho} = \varrho - \hat{\varrho}$, A_z is the matrix having the same structure as given in [21] and W_ϵ and W_θ are defined as

$$W_\epsilon \;=\; \left[1, -\frac{\partial \alpha_1}{\partial y}, \cdots, -\frac{\partial \alpha_{\rho-1}}{\partial y}\right]^T \in \Re^\rho \tag{3.111}$$

$$W_\theta \;=\; W_\epsilon \omega^T - \hat{\varrho} \bar{\alpha}_1 e_{\rho,1} e_{n^*,1}^T \in \Re^{\rho \times n^*}. \tag{3.112}$$

From (3.105), (3.106) and (3.108) and $\dot{\tilde{\varrho}} = -\dot{\hat{\varrho}}$, we obtain that

$$\dot{\tilde{\theta}} = -\Gamma W_\theta z. \tag{3.113}$$

We define a candidate Lyapunov function V_{k-1} as

$$V_{k-1} = \frac{1}{2} z^T z + \sum_{q=1}^{\rho} \frac{1}{4d_q} \epsilon^T P \epsilon + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \frac{1}{2\gamma} \bar{\mathfrak{b}}_{m_1} \tilde{\varrho}^2. \tag{3.114}$$

From (3.90), (3.110), (3.107), and the fact that $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$, $PA_0 + A_0^T P = -I$, the derivative of V_{k-1} can be computed as

$$\begin{aligned} \dot{V}_{k-1} \;&=\; \frac{1}{2} z^T (A_z + A_z^T) z + z^T W_\epsilon \epsilon_2 + z^T W_\theta^T \tilde{\theta} - z^T \bar{\mathfrak{b}}_{m_1} \bar{\alpha}_1 e_{\rho,1} \tilde{\varrho} - \sum_{q=1}^{\rho} \frac{1}{4d_q} \epsilon^T \epsilon \\ &\quad - \tilde{\theta}^T W_\theta z + \tilde{\varrho} \bar{\mathfrak{b}}_{m_1} \bar{\alpha}_1 e_{\rho,1}^T z \\ &\leq\; -\sum_{q=1}^{\rho} c_q z_q^2. \end{aligned} \tag{3.115}$$

Starting from the first time interval $[T_0, T_1)$ with $T_0 = 0$, we can conclude that $z, \hat{\theta}, \hat{\varrho}$ and ϵ are bounded for $t \in [T_0, T_1)$ based on (3.114) and (3.115) and $V_0(0)$ being bounded. Since z_1 and y_r are bounded, y is also bounded. From (3.79), (3.80) and (3.82), we conclude that ξ, Ξ and η_j, σ_{y_j} for $j = 1, \ldots, m$ are bounded.

We now prove the boundedness of λ. The input filter (3.81) gives

$$\lambda_i = \frac{s^{i-1} + \tilde{l}_1 s^{i-2} + \ldots, + \tilde{l}_{i-1}}{\mathfrak{L}(s, l, \delta)} u_0, \quad i = 1, \ldots, \bar{n} + \rho, \tag{3.116}$$

where $\mathfrak{L}(s, l, \delta)$ is defined in (3.83), $\tilde{l}_i$ is a bounded function of l_i and δ. Since no failures have occurred on any of the m actuators before time T_1, we can show that for the plant (3.50)-(3.51) with pre-filters (3.54)-(3.55),

$$\frac{d^n y}{dt^n} - \sum_{i=1}^{n} \frac{d^{n-i}}{dt^{n-i}} [\phi_i(y) + \bar{\Phi}_i(y) a] = \sum_{j=1}^{m} \sum_{i=0}^{\bar{n}_j} \mathrm{sgn}(b_j \bar{n}_j) b_{ji} \frac{d^i}{dt^i} \frac{(\frac{d}{dt} + \delta)^{\bar{n} - \bar{n}_j} u_0}{(\frac{d}{dt} + \delta)^{\rho - \rho_1}} \tag{3.117}$$

Substituting (3.117) into (3.116), we get

$$\begin{aligned} \lambda_i \;\; = \;\; & \frac{(s^{i-1} + \bar{l}_1 s^{i-2} + \ldots, + \bar{l}_{i-1})(\frac{d}{dt} + \delta)^{\rho - \rho_1}}{\mathfrak{L}(s, l, \delta) \sum_{j=1}^{m} \mathrm{sgn}(b_{j\bar{n}_j}) \mathfrak{B}_j(s) (\frac{d}{dt} + \delta)^{\bar{n} - \bar{n}_j}} \\ & \times \left\{ \frac{d^n y}{dt^n} - \sum_{i=1}^{n} \frac{d^{n-i}}{dt_{n-i}} [\phi_i(y) + \bar{\Phi}_i(y) a] \right\}, \quad i = 1, \ldots, \bar{n} + \rho. \end{aligned} \tag{3.118}$$

If the polynomial $\sum_{j=1}^{m} \mathrm{sgn}(b_{j\bar{n}_j}) \mathfrak{B}_j(s) (\frac{d}{dt} + \delta)^{\bar{n} - \bar{n}_j}$ is stable, the boundedness of y, the smoothness of $\phi(y)$, $\bar{\Phi}(y)$, and (3.118) imply that $\lambda_1, \ldots, \lambda_{\bar{n}+1}$ are bounded. From (3.98), the boundedness of $\lambda_1, \ldots, \lambda_{\bar{n}+1}$ and the fact that α_{i-1} is the function of y, $\bar{y}_r^{(i-2)}$, ξ, Ξ, η_j and σ_j for $j = 1, \ldots, m$, $\bar{\lambda}_{\bar{n}+i-1}$, $\hat{\varrho}$, $\hat{\theta}$ where $\bar{y}_r^{(i-2)} = (y_r, y_r^{(1)}, \ldots, y_r^{(i-2)}$, $\bar{\lambda}_{\bar{n}+i-1} = (\lambda_1, \ldots, \lambda_{\bar{n}+i-1})$, $v_{\bar{n},2}$ is bounded. Then from $v_{\bar{n},i} = [*, \ldots, *, 1][\lambda_1, \ldots, \lambda_{\bar{n}+i}]^T$, it follows that $\lambda_{\bar{n}+2}$ is bounded. By repeating the similar procedures, λ being bounded can be established. From (3.89), (3.85) and the boundedness of ξ, Ξ, η_j for $j = 1, \ldots, m$, λ, $\hat{\varkappa}$ is then bounded. Since $\varkappa = \epsilon + \hat{\varkappa}$

and ϵ is bounded, the boundedness of $\varkappa$ is proven. u_0 is bounded from (3.99). From (3.55) and $\delta > 0$, $\hat{u}_j$ for $j = 1, \dots, m$ is bounded. Since $\sigma_j(y)$ is bounded away from zero and u_{cj} is designed as (3.54), u_{cj} for $j = 1, \dots, m$ are bounded. From $\hat{u}_j = \varsigma_{j,1}$ and (3.58), the states $\varsigma_{j,i}$ for $j = 1, \dots, m$, $i = 1, \dots, \rho - \rho_j$ are all bounded. From $\varkappa_1 = x_1$ and fact that $\varkappa_i$ for $i = 2, \dots, n$ are linear expansions of x_i and states $\varsigma_{j,k}$, like in (3.60), (3.62) and (3.66), we can conclude that x is bounded. Thus, we obtain the boundedness of all closed-loop signals for $t \in [T_0, T_1)$. At T_1, parameter jumpings occurring on $\bar{\mathfrak{b}}$, $\bar{K}$ as well as the states $\varsigma_{j,k}$ in constructing $\varkappa$ due to actuator failures are also bounded. Thus we have $V_1(T_1) < V_0(T_1) + \triangle V_0 \in \mathcal{L}_\infty$ where $\triangle V_0$ is bounded. From (3.115) for $k = 2$, we get that $V_1(T_2)$ is bounded. The boundedness of all the signals can be proved by following the similar procedures above. However, (3.117) is changed to

$$\frac{d^n y}{dt^n} - \sum_{i=1}^{n} \frac{d^{n-i}}{dt^{n-i}}[\phi_i(y) + \bar{\Phi}_i(y)a] - \sum_{j=j_{g_1},\dots,j_{g_k}} \sum_{i=1}^{\bar{n}_j+1} b_{ji} u_{k_j} \frac{d^{\bar{n}_j+1-i}}{dt^{\bar{n}_j+1-i}} \sigma_j(y)$$
$$= \sum_{j \neq j_{g_1},\dots,j_{g_k}} \sum_{i=0}^{\bar{n}_j} \mathrm{sgn}(b_j \bar{n}_j) b_{ji} \frac{d^i}{dt^i} \frac{(\frac{d}{dt} + \delta)^{\bar{n} - \bar{n}_j} u_0}{(\frac{d}{dt} + \delta)^{\rho - \rho_1}} \tag{3.119}$$

By noting the finite times of actuator failures, the boundedness of all the signals in the system is achieved. Further, from (3.115) for $t \in [T_r, \infty)$ and LaSalle-Yoshizawa theorem, it follows that $\lim_{t\to\infty} z(t) = 0$, which implies that $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$. $\square$

3.3.5 An Illustrated Example

A second-order system with dual actuators is considered,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} y^2 \\ \sin(y) \end{bmatrix} + \begin{bmatrix} y^3 \\ \cos(y) \end{bmatrix} a + \begin{bmatrix} b_{11} \\ b_{10} \end{bmatrix} (y^2+1)u_1 + \begin{bmatrix} 0 \\ b_{20} \end{bmatrix} (e^y+1)u_2 \quad (3.120)$$

$$y = x_1, \quad (3.121)$$

where the system parameters $a = 2$, $b_{11} = b_{10} = b_{20} = 1$ are unknown. However we know that b_{11} and b_{20} are positive. Obviously, the polynomials $\mathfrak{B}_1(p) = b_{11}p + b_{10}$ and $\mathfrak{B}_2(p) = b_{20}$ are both stable. It can be easily shown that Assumption 3.3.3 is satisfied in failure-free case and all possible failure cases with arbitrary positive δ is chosen. Observing from (3.121), we get $\rho = 2$, $\rho_1 = 1$. The pre-filters for u_1 and u_2 are designed as $u_1 = \frac{\hat{u}_1}{y^2+1}$, $u_2 = \frac{\hat{u}_2}{e^y+1}$ where $\hat{u}_1 = \frac{u_0}{p+\delta}$ with $\hat{u}_1(0) = 0$ and $\hat{u}_2 = u_0$. The reference signal is $y_r = \sin(0.01t)$ and all the initials are set as 0. u_1 is stuck at $u_1 = 0.2$ from $t = 20$s. The design parameters are chosen as $\delta = 2$, $l = [12, 48, 64]^T$, $c_1 = c_2 = 5$, $d_1 = d_2 = 1$, $\gamma = 1$ and $\Gamma = 2 \times I_8$. The tracking error $y(t) - y_r(t)$ and control inputs u_1, u_2 are given in Fig. 3.13-3.14. It is observed that the asymptotic tracking can still be achieved in failure case despite a degradation of performance.

3.4 Conclusion

In this chapter, a "direct" adaptive output-feedback control scheme by introducing pre-filters is proposed to stabilize the uncertain systems in the presence of stuck type actuator failures. With the proposed failure compensation scheme, the condition existing in the previous results that the relative degrees corresponding to the redundant actuators with respect to the system inputs being identical is relaxed. The design for linear systems is firstly considered and the results are extended to

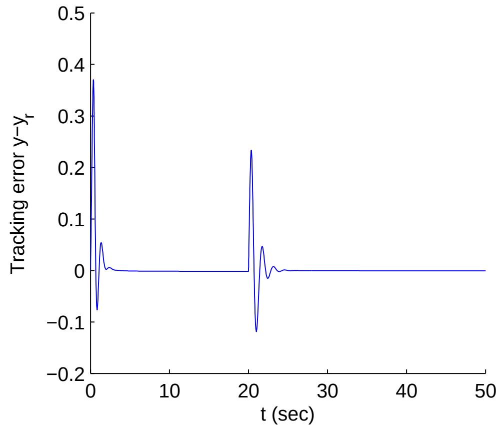


Figure 3.13: The tracking error $y(t) - y_r(t)$.

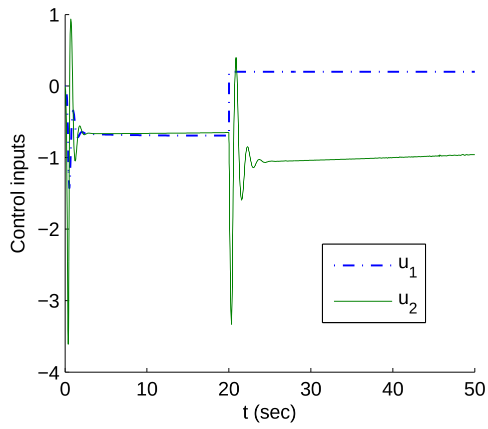


Figure 3.14: Control inputs.

nonlinear systems. It is shown that the boundedness of all the signals in the closed-loop system is ensured. Moreover, the set-point regulation and asymptotic tracking of the system output is achieved for linear systems and nonlinear systems, respectively.

Chapter 4

Adaptive Failure Compensation with Guaranteed Transient Performance

In this chapter, we propose two adaptive backstepping control schemes for parametric strict feedback systems with uncertain actuator failures. Firstly a basic design scheme on the basis of existing approaches is considered. It is analyzed that, when actuator failures occur, transient performance of the adaptive system cannot be adjusted through changing controller design parameters. Then we propose a new controller design scheme based on a prescribed performance bound (PPB) which characterizes the convergence rate and maximum overshoot of the tracking error. It is shown that the tracking error satisfies the prescribed performance bound all the time. Simulation studies also verify the established theoretical results that the PPB based scheme can improve transient performance compared with the basic scheme, while both ensure stability and asymptotic tracking with zero steady state error in the presence of uncertain actuator failures.

4.1 Introduction

As discussed in Chapter 1, many effective approaches have been developed to address the problem of accommodating actuator failures. They can be roughly classified into two categories: passive [40, 42, 47, 48, 61] and active ones [39, 41, 43–46, 49–51, 53–55, 57, 59, 70, 95, 99, 100]. Passive approaches use unchangeable controllers throughout failure-free and all possible failure cases. Since neither the structure reconfigurable nor the parameter adjustment is involved, the designed controllers are easy to be implemented. However, they are often conservative for changes of failure pattern or values. Among the numerous active approaches, adaptive control designs [39, 43, 44, 50, 51, 53, 57, 59, 70, 100] form a class of methods that handel the large uncertain structural and parametric variation caused by failures with the aid of adaptation mechanisms. Moreover, the adaptive design schemes proposed in [45, 46, 54, 55, 95] have been proved effective in accommodating the uncertainties in both system dynamics and actuator failures without explicit failure detection/diagnostic. However to the best knowledge of authors, very few results in adaptive control are available on investigating how to guarantee the transient performance of the system, besides showing system stability and steady state tracking performance. Note that multiple model adaptive control, switching and tuning (MMST) approaches, such as in [43] may offer improved transient behaviors, but the bounds of failure magnitudes and the unknown parameters associated with failures are often needed in advance to construct a finite set of models which can cover the state space. Besides, a safe switching rule is required as mentioned in [101] since an MMST closed loop is not intrinsically stable.

In this chapter, we shall deal with the problem of guaranteeing transient performance in adaptive control of uncertain parametric strict feedback systems in the presence of actuator failures. To accommodate the effects due to actuator failures, we propose two adaptive backstepping control schemes for parametric strict feedback systems. Firstly a design scheme based on an existing approach in [54] is considered. It is shown that the scheme can ensure both stability and asymptotic tracking as in [54] and we name it as a basic scheme. Note that the backstepping technique [21] provides a promising way to improve the transient performance of adaptive systems in terms of $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms of the tracking error. However, the transient performance is tunable only if certain trajectory initialization can be performed, see for example [21,36]. Apparently, such trajectory initializations involving state-resetting actions are difficult at the time instants when actuator failures occur, because they are uncertain in occurrence time, pattern and value. Therefore, transient performance of the adaptive system cannot be adjusted through changing controller design parameters with the basic scheme. By employing prescribed performance bounds (PPB) originally presented in [102], we propose a new controller design scheme. A prescribed performance bound can characterize the convergence rate and maximum overshoot of the tracking error. With certain transformation techniques, a new transformed system is obtained by incorporating the prescribed performance bound into the original nonlinear system. An adaptive controller, named as PPB based controller, is designed for the transformed system. It is established that the tracking error can be guaranteed within the prescribed error bound all the time as long as the stability of the transformed error system is ensured, without resetting system states no matter whether actuator failures occur or not. Thus the transient performance is ensured and can be improved by varying certain design parameters. It is also shown that, with suitable modifications on the prescribed performance bound in [102], the tracking error can converge to zero asymptotically.

4.2 Plant Models and Problem Formulation

Similar to [54], we consider a class of nonlinear MISO systems as follows,

$$\dot{\chi} = f_0(\chi) + \sum_{l=1}^{p} \theta_l f_l(\chi) + \sum_{j=1}^{m} b_j g_j(\chi) u_j \tag{4.1}$$

$$y = h(\chi), \tag{4.2}$$

where $\chi \in \Re^n$, $y \in \Re$ are the state and the output, $u_j \in \Re$ for $j = 1, 2, \ldots, m$ is the jth input of the system, i.e. the output of the jth actuator, $f_l(\chi) \in \Re^n$ for $l = 0, 1, \ldots, p$, $g_j(\chi) \in \Re^n$ for $j = 1, 2, \ldots, m$ and $h(\chi)$ are known smooth nonlinear functions, θ_l for $l = 1, 2, \ldots, p$ and b_j for $j = 1, \ldots, m$ are unknown parameters and control coefficients.

4.2.1 Model of Actuator Failures

We denote u_{cj} as the input of the jth ($j = 1, 2, \ldots, m$) actuator. Similar to Chapter 3, an actuator with its input equal to its output, i.e. $u_j = u_{cj}$, is regarded as a failure-free actuator. The type of actuator failures considered in this chapter, which may take place on the jth actuator, can be modeled as follows,

$$u_j = \rho_j u_{cj} + u_{kj}, \quad \forall t \geq t_{jF} \tag{4.3}$$

$$\rho_j u_{kj} = 0, \quad j = 1, 2, \ldots, m \tag{4.4}$$

where $\rho_j \in [0, 1)$, u_{kj} and t_{jF} are all unknown constants. (4.3) shows that the jth actuator fails suddenly from time t_{jF}. (4.4) implies the following three cases, in which two typical types of failures (TLOE and PLOE) are included.

1) $\rho_j \neq 0$ and $u_{kj} = 0$.

In this case, $u_j = \rho_j u_{cj}$, where $0 < \rho_j < 1$. This indicates partial loss of effectiveness (PLOE). For example, $\rho_j = 70\%$ means that the jth actuator loses 30% of its effectiveness.

2) $\rho_j = 0$.

$\rho_j = 0$ indicates that u_j can no longer be influenced by the control inputs u_{cj}. The fact that u_j is stuck at an unknown value u_{kj} is known as total loss of effectiveness (TLOE). Such a failure type is also considered in Chapter 3.

Remark 4.2.1.

- Note that actuators working in failure-free case can also be represented as (4.3) with $\rho_i = 1$, $u_{ki} = 0$ for $t \geq 0$.
- Similar to Chapter 3, possible changes from normal case to any one of the failure cases are assumed unidirectional here. That is, the values of ρ_j can change only from $\rho_j = 1$ to $\rho_j = 0$ or some values with $0 < \rho_j < 1$). The uniqueness of t_{jF} indicates that a failure occurs only once on the jth actuator. Hence there exists a finite T_r denoting the time instant of the last failure. Such an assumption on the finite number of actuator failures can be found in many previous results, such as [43, 45, 46, 54, 55, 95].

4.2.2 Control Objectives and Assumptions

The control objects in this chapter are as follows,

- The effects of considered types of actuator failures can be compensated so that the global stability of the closed-loop system is ensured and asymptotic tracking can be achieved.
- Tracking error $e(t) = y(t) - y_r(t)$ can be preserved within certain given prescribed performance bounds (PPB). In addition, transient performance in terms of the convergence rate and maximum overshoot of $e(t)$ can be improved by tuning

design parameters.

To achieve the control objectives, the following assumptions are applied.

Assumption 4.2.1. *The plant (4.1)-(4.2) is so constructed that for any TLOE type of actuator failures up to $m-1$, the remaining actuators can still achieve a desired control objective.*

Assumption 4.2.2. $g_j(\chi) \in span\{g_0(\chi)\}$*,* $g_0(\chi) \in \Re^n$*, for* $i = j, 2, \ldots, m$ *and the nominal system*

$$\dot{\chi} = f_0(\chi) + F(\chi)\theta + g_0(\chi)u_0, \quad y = h(\chi) \tag{4.5}$$

with $u_0 \in \Re$*, is transformable into the parametric-strict-feedback form with relative degree* ϱ*, where* $F(\chi) = [f_1(\chi), f_2(\chi), \ldots, f_p(\chi)] \in \Re^{n\times p}$*,* $\theta = [\theta_1, \theta_2, \ldots, \theta_p]^T \in \Re^p$*.*

Remark 4.2.2.

• As discussed in [45,46,54,95,98,103] and Chapter 3, Assumption 4.2.1 is a basic assumption to ensure the controllability of the plant and the existence of a nominal solution for the actuator failure compensation problem. Nevertheless, all actuators are allowed to suffer from PLOE type of actuator failures simultaneously.

• Assumption 4.2.2 corresponds to the first actuator structure condition in [54] that the nonlinear actuator functions $g_j(\chi)$ for $j = 1, 2, \ldots, m$ have similar structures.

As presented in [54], based on Assumption 4.2.2, there exists a diffeomorphism $[x, \xi]^T = T(\chi)$ where $x = [x_1, \ldots, x_\varrho] \in \Re^\varrho$, $\xi \in \Re^{n-\varrho}$ such that the nominal system (4.5) can be transformed to the following canonical parametric-strict-feedback form

$$\begin{aligned}
\dot{x}_i &= x_{i+1} + \varphi_i^T(x_1, \ldots, x_i)\theta, \quad i = 1, 2, \ldots, \varrho - 1, \\
\dot{x}_\varrho &= \varphi_0(x, \xi) + \varphi_\varrho^T(x, \xi)\theta + \beta_0(x, \xi)u_0, \\
\dot{\xi} &= \Psi(x, \xi) + \Phi(x, \xi)\theta, \\
y &= x_1,
\end{aligned} \tag{4.6}$$

where the definitions of φ_i, for $i = 0, 1, \ldots, \varrho$, β_0 and Ψ, Φ can be found in [54, Sec. 3.1]. With the same diffeomorphism, the plant (4.1)-(4.2) can be transformed to the following form by incorporating the actuator failure model (4.3).

$$\begin{aligned}
\dot{x}_i &= x_{i+1} + \varphi_i^T(x_1, \ldots, x_i)\theta, \quad i = 1, 2, \ldots, \varrho - 1, \\
\dot{x}_\varrho &= \varphi_0(x, \xi) + \varphi_\varrho^T(x, \xi)\theta + \sum_{j=1}^{m} b_j \beta_j(x, \xi)(\rho_j u_{cj} + u_{kj}), \\
\dot{\xi} &= \Psi(x, \xi) + \Phi(x, \xi)\theta, \\
y &= x_1, \qquad (4.7)
\end{aligned}$$

Note that the transformed system (4.7) is the plant to be stabilized. Three additional assumptions are required.

Assumption 4.2.3. *The reference signal $y_r(t)$ and its first ϱth order derivatives $y_r^{(q)}(q = 1, \ldots, \varrho)$ are known, bounded, and piecewise continuous.*

Assumption 4.2.4. *$\beta_j(x, \xi) \neq 0$, the signs of b_j, i.e. sgn(b_j), for $j = 1, \ldots, m$ are known.*

Assumption 4.2.5. *The nominal system (4.6) is minimum phase, that is, the subsystem $\dot{\xi} = \Psi(x, \xi) + \Phi(x, \xi)\theta$ is input-to-state stable with respect to x as the input.*

Detailed discussions about Assumption 4.2.5 could be found in [54].

4.3 Basic Control Design

The main purpose of designing basic controllers is to carry out comparisons with our prescribed performance bounds (PPB) based controllers to be proposed later. It will be noted that a basic controller, from its design approaches and performances, can be considered as a representative of currently available adaptive failure compensation controllers.

The design of u_{cj} is generated by following the procedures in [54, Sec. 3.1] with slight modifications. Thus only some important steps are presented. Meanwhile, stability analysis will be sketched briefly.

4.3.1 Design of Controllers

We firstly design u_0 to stabilize the nominal system (4.6) by utilizing the tuning functions design scheme summarized in Chapter 2. Introducing ϱ error variables

$$z_1 = y - y_r \tag{4.8}$$

$$z_q = x_q - \alpha_{q-1} - y_r^{(q-1)} \quad \text{for } q = 2, \ldots, \varrho \tag{4.9}$$

where α_q is the stabilizing function determined at the qth step that

$$\begin{aligned} \alpha_q &= -z_{q-1} - c_q z_q - \omega_q^T \hat{\theta} + \sum_{k=1}^{q-1} \left(\frac{\partial \alpha_{q-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{q-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) + \frac{\partial \alpha_{q-1}}{\partial \hat{\theta}} \Gamma \tau_q \\ &\quad + \sum_{k=2}^{q-1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \omega_q z_k, \quad \text{for } q = 1, \ldots, \varrho - 1 \end{aligned} \tag{4.10}$$

$$\begin{aligned} \alpha_\varrho &= -z_{\varrho-1} - c_\varrho z_\varrho - \varphi_0 - \omega_\varrho^T \hat{\theta} + \sum_{k=1}^{\varrho-1} \left(\frac{\partial \alpha_{\varrho-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{\varrho-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) + \frac{\partial \alpha_{\varrho-1}}{\partial \hat{\theta}} \Gamma \tau_\varrho \\ &\quad + \sum_{k=2}^{\varrho-1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \omega_\varrho z_k, \end{aligned} \tag{4.11}$$

where c_q for $q = 1, 2, \ldots, \varrho$ are positive constants, Γ is a positive definite matrix, and

$$\tau_1 = \omega_1 z_1 \tag{4.12}$$

$$\tau_q = \tau_{q-1} + \omega_q z_q, \quad \text{for } q = 2, \ldots, \varrho \tag{4.13}$$

$$\omega_q = \varphi_q - \sum_{k=1}^{q-1} \frac{\partial \alpha_{q-1}}{\partial x_k} \varphi_k, \quad \text{for } q = 1, \ldots, \varrho \tag{4.14}$$

Design u_0 as

$$u_0 = \frac{v_0}{\beta_0}, \quad v_0 = \alpha_\varrho + y_r^{(\varrho)} \tag{4.15}$$

Parameter update law is chosen as

$$\dot{\hat{\theta}} = \Gamma \tau_\varrho \tag{4.16}$$

Based on these, we now determine the design of u_{cj} for $j = 1, \ldots, m$. Comparing (4.7) with (4.6), the difference consists in the ϱth equation. Suppose there are q_{tot} actuators $j_1, j_2, \ldots, j_{q_{tot}}$ suffer from TLOE. The rest of actuators are either normal with $\rho_j = 1$ or undergoing PLOE with $0 < \rho_j < 1$. The dynamics of x_ϱ in (4.7) is changed to

$$\dot{x}_\varrho = \varphi_0 + \varphi_\varrho^T \theta + \sum_{j \neq j_1, \ldots, j_{q_{tot}}} b_j \rho_j \beta_j u_{cj} + \sum_{j = j_1, \ldots, j_{q_{tot}}} b_j u_{kj} \beta_j \tag{4.17}$$

If b_j for $j = 1, \ldots, m$ and all the failure information are known, we can design u_{cj} as

$$u_{cj} = \text{sgn}(b_j) \frac{1}{\beta_j} \kappa^T w, \quad \text{for } j = 1, 2, \ldots, m \tag{4.18}$$

where

$$\kappa = [\kappa_1, \kappa_2^T]^T, \quad \kappa_2 = [\kappa_{2,1}, \kappa_{2,2}, \ldots, \kappa_{2,m}]^T \tag{4.19}$$

$$w = [v_0, \beta^T]^T \tag{4.20}$$

$$\beta = [\beta_1, \beta_2, \ldots, \beta_m]^T, \tag{4.21}$$

such that the effects due to actuator failures can be compensated and the sum of the last two terms in (4.17) will be equal to v_0 as designed in (4.15). The details of κ will be given in later discussions. However, b_j and the failure information are actually unknown. Therefore, the estimate of κ ($\hat{\kappa}$) is adopted instead in determining u_{cj}, i.e.

$$u_{cj} = \mathrm{sgn}(b_j)\frac{1}{\beta_j}\hat{\kappa}^T w, \quad \text{for } j = 1, 2, \ldots, m. \tag{4.22}$$

The adaptive law of $\hat{\kappa}$ is designed as

$$\dot{\hat{\kappa}} = -\Gamma_\kappa w z_\varrho. \tag{4.23}$$

The controllers designed are named as basic controllers since they can only ensure system stability and a tracking property similar to those in [54], as analyzed below.

4.3.2 Stability Analysis

For the basic controllers developed, we establish the following result.

Theorem 4.3.1. *Consider the closed-loop adaptive system consisting of the plant (4.1)-(4.2), the controller (4.22), the parameter update laws (4.16), (4.23) in the presence of possible actuator failures (4.3)-(4.4) under Assumptions 4.2.1-4.2.5. The boundedness of all the signals are ensured and the asymptotic tracking is achieved, i.e.* $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$.

Proof: As presented in Remark 4.2.1, there are a finite number of time instants T_k for $k = 1, 2, \ldots, r$ ($r \leq m$) at which one or more of the actuators fail. T_r is referred as the last time of failure in Remark 4.2.1. Suppose during time interval $[T_{k-1}, T_k)$, where $k = 1, \ldots, r+1$, $T_0 = 0$, $T_{r+1} = \infty$, there are p_k ($p_k \geq 1$) failed actuators $j_1, j_2, \ldots, j_{p_k}$ and the failure pattern will not change until time T_k. Among these p_k failed actuators, q_{tot_k} actuators $j_{1,1}, j_{1,2}, \ldots, j_{1,q_{tot_k}}$ suffer from

TLOE and q_{par_k} actuators $j_{2,1}, j_{2,2}, \ldots, j_{2,q_{par_k}}$ undergo PLOE. We define a set $P_k = \{j_1, j_2, \ldots, j_{p_k}\}$ and two subsets of P_k that $Q_{tot_k} = \{j_{1,1}, j_{1,2}, \ldots, j_{1,q_{tot_k}}\}$ and $Q_{par_k} = \{j_{2,1}, j_{2,2}, \ldots, j_{2,q_{par_k}}\} = P_k \backslash Q_{tot_k}$. We define a positive definite function V_{k-1} during $[T_{k-1}, T_k)$ as

$$V_{k-1} = \frac{1}{2} z^T z + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \sum_{j=1, j \notin Q_{tot_k}}^{m} \frac{\rho_j |b_j|}{2} \tilde{\kappa}^T \Gamma_\kappa^{-1} \tilde{\kappa}, \tag{4.24}$$

where $z = [z_1, z_2, \ldots, z_\varrho]^T, \tilde{\theta} = \theta - \hat{\theta}$ and $\tilde{\kappa} = \kappa - \hat{\kappa}$. If b_j, ρ_j and u_{kh} for $j = 1, 2, \ldots, m$, $h \in Q_{tot_k}$ are known, κ is a desired constant vector which can be chosen to satisfy that

$$\begin{aligned} & \sum_{j=1, j \notin Q_{tot_k}}^{m} |b_j| \rho_j \kappa^T w = v_0 - \sum_{h \in Q_{tot_k}} b_h \beta_h u_{kh} \\ \Rightarrow \quad & \kappa_1 = \frac{1}{\sum_{j=1, j \notin Q_{tot_k}}^{m} |b_j| \rho_j}, \ \kappa_{2,h} = \frac{-b_h u_{kh}}{\sum_{j=1, j \notin Q_{tot_k}}^{m} |b_j| \rho_j}, \\ & \text{for} \ \ h \in Q_{tot_k} \text{ and } \kappa_{2,h} = 0, \ h \in \{1, 2, \ldots, m\} \backslash Q_{tot_k}. \end{aligned} \tag{4.25}$$

From the design through (4.8)-(4.23), the time derivative of V_{k-1} is computed as

$$\dot{V}_{k-1} = -\sum_{q=1}^{\varrho} c_q z_q^2, \quad k = 1, 2, \ldots, r+1. \tag{4.26}$$

We define $V_{k-1}(T_k^-) = \lim_{\Delta t \to 0^-} V_{k-1}(T_k + \Delta t)$ and $V_{k-1}(T_{k-1}^+) = \lim_{\Delta t \to 0^+} V_{k-1}(T_{k-1} + \Delta t) = V_{k-1}(T_{k-1})$. If we let a function $V(t) = V_{k-1}(t)$, for $t \in [T_{k-1}, T_k)$, $k = 1, \ldots, r+1$, $V(t)$ is thus a piecewise continuous function. From (4.26), we have V_{k-1} is non-increasing during the time interval $[T_{k-1}, T_k)$ and $V_{k-1}(T_k^-) \leq V_{k-1}(T_{k-1}^+)$. When $k = 1$, $V_0(t) \leq V_0(0)$ for $t \in [0, T_1)$, the boundedness of $z(t)$, $\tilde{\theta}(t)$ and $\tilde{\kappa}(t)$ for $t \in [0, T_1)$ is ensured since the initial value $V_0(0)$ is finite. $V_0(T_1^-) \leq V_0(0)$. When $k > 1$, $V_{k-1}(t)$ is bounded if $V_{k-1}(T_{k-1}^+)$ is bounded. Observing (4.24), at the time instant $t = T_k$, $V_{k-1}(T_k^-)$ is changed to $V_k(T_k^+) = V_{k-1}(T_k^-) + \Delta V_k$, where ΔV_k is due to the changes on the coefficients in front of $\kappa^T\Gamma_\kappa\kappa$ and possible jumpings on κ and ΔV_k is finite. This implies that the initial value $V_k(T_k^+)$ for $[T_k, T_{k+1})$ is bounded if the final value $V_{k-1}(T_k^-)$ for $[T_{k-1}, T_k)$ is bounded. The above facts conclude the boundedness of $z(t)$, $\tilde{\theta}(t)$, $\tilde{\kappa}(t)$ for $t \in [0, \infty)$ and $z(t) \in \mathcal{L}_2$. From (4.22), control signals u_{cj} for $j = 1, 2, \ldots, m$ are also bounded. From (4.8)-(4.9) and Assumption 4.2.3, $x(t)$ is bounded. From Assumption 4.2.5, $\xi(t)$ is bounded with respect to $x(t)$ as the input. The closed-loop stability is then established. Noting $\dot{z} \in \mathcal{L}_\infty$, it follows that $\lim_{t\to\infty} z(t) = 0$. From (4.8), the asymptotic tracking is achieved, i.e. $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$. □

4.3.3 Transient Performance Analysis

We firstly define two norms $\mathcal{L}_{2[a,b]}$ and $\mathcal{L}_{\infty[a,b]}$ as follows.

$$\begin{aligned} \| x(t) \|_{2[a,b]} &= \left(\int_a^b \|x(t)\|^2 dt\right)^{1/2} && (4.27)\\ \| x(t) \|_{\infty[a,b]} &= \sup_{t\in[a,b]} \|x(t)\| && (4.28)\end{aligned}$$

We then derive the bounds for the tracking error $z_1(t)$ in terms of both $\mathcal{L}_{2[T_{k-1},t_k]}$ and $\mathcal{L}_{\infty[T_{k-1},t_k]}$ norms, where $k = 1, \ldots, r+1$, $t_k \in (T_{k-1}, T_k)$ with $T_0 = 0$, $T_{r+1} = \infty$. From (4.26), we have

$$\dot{V}_{k-1} \leq -c_1 z_1^2 \leq 0. \tag{4.29}$$

It follows that

$$\begin{aligned} \| z_1(t) \|^2_{2[T_{k-1},t_k]} &= \int_{T_{k-1}}^{t_k} z_1(t)^2 dt \leq -\frac{1}{c_1}\int_{T_{k-1}}^{t_k} \dot{V}_{k-1}(t)dt \\ &= -\frac{1}{c_1}\left[V_{k-1}(T_{k-1}) - V_{k-1}(t_k)\right] \leq \frac{1}{c_1}V_{k-1}(T_{k-1}) && (4.30)\end{aligned}$$

and

$$z_1(t)^2 \leq 2V_{k-1}(t) \leq 2V_{k-1}(T_{k-1}), \quad t \in [T_{k-1}, T_k). \tag{4.31}$$

Define that $\| \tilde{\theta}(T_{k-1}) \|^2_{\Gamma^{-1}} = \tilde{\theta}^T(T_{k-1})\Gamma^{-1}\tilde{\theta}(T_{k-1})$ and $\| \tilde{\kappa}(T_{k-1}) \|^2_{\Gamma_\kappa^{-1}} = \tilde{\kappa}^T(T_{k-1})\Gamma_\kappa^{-1} \cdot \tilde{\kappa}(T_{k-1})$. From (4.30) and (4.31), we have

$$\begin{aligned}
\| z_1(t) \|_{2[T_{k-1},t_k]} &\leq \frac{1}{\sqrt{2c_1}} \Bigg[z^T z(T_{k-1}) + \| \tilde{\theta}(T_{k-1}) \|^2_{\Gamma^{-1}} + \sum_{j=1, j \notin Q_{tot_k}}^{m} \rho_j |b_j| \\
&\quad \times \| \tilde{\kappa}(T_{k-1}) \|^2_{\Gamma_\kappa^{-1}} \Bigg]^{\frac{1}{2}} && (4.32) \\
\| z_1(t) \|_{\infty[T_{k-1},t_k]} &\leq \Bigg[z^T z(T_{k-1})^2 + \| \tilde{\theta}(T_{k-1}) \|^2_{\Gamma^{-1}} + \sum_{j=1, j \notin Q_{tot_k}}^{m} \rho_i |b_i| \\
&\quad \times \| \tilde{\kappa}(T_{k-1}) \|^2_{\Gamma_\kappa^{-1}} \Bigg]^{\frac{1}{2}}. && (4.33)
\end{aligned}$$

Based on these results, we have the following discussions.

1) When $k = 1$, (4.32)-(4.33) gives the bounds of the $\mathcal{L}_{2[0,t_1]}$ and $\mathcal{L}_{\infty[0,t_1]}$ norms ($t_1 < T_1$) for the tracking error $z_1(t)$ before the first failure occurs. From the definition in (4.9), the initial value $z(0)$ may increase by increasing c_1, Γ, Γ_κ. By performing trajectory initialization, i.e. setting $z(0) = 0$ (see for instance [21, 36]), the transient performance of $z_1(t)$ in the sense of these two norms during $[0, T_1)$ can be improved by increasing c_1 and/or Γ, Γ_κ.

2) However, it is impossible to perform trajectory initialization at each T_{k-1} for $k > 1$ because the failure time, type and value are all unknown. Thus the initial value $V_{k-1}(T_{k-1})$ during $[T_{k-1}, T_k)$ for $k > 1$ may be increased by increasing c_1, Γ, Γ_κ. Moreover, it cannot be guaranteed from 1) that the final value $V_0(T_1^-)$ during $[0, T_1)$ is smaller with larger c_1, Γ, Γ_κ. Hence a larger $V_0(T_1^-)$ may result in a larger initial value $V_1(T_1)$ for the next interval. Therefore, the conclusion on improving transient performance in terms of either the $\mathcal{L}_{2[T_{k-1},t_k]}$ or $\mathcal{L}_{\infty[T_{k-1},t_k]}$ norm by adjust-

ing c_1, Γ, Γ_κ cannot be drawn for $z_1(t)$ with $t \geq T_1$.

To guarantee transient performance of the tracking error, especially when failures take place, an alternative approach based on prescribed performance bounds proposed in [102] is employed to design adaptive compensation controllers.

4.4 Prescribed Performance Bounds (PPB) based Control Design

The objective in this section is to ensure the transient performance in the sense that the tracking error $e(t) = y(t) - y_r(t)$ is preserved within a specified PPB all the time no matter when actuator failures occur, in addition to stability and steady state tracking properties. Similar to [102], the characterization of a prescribed performance bound is required. To do this, a decreasing smooth function $\eta(t)$: $\Re_+ \rightarrow \Re_+ \backslash \{0\}$ with $\lim_{t\rightarrow\infty} \eta(t) = \eta_\infty > 0$ is firstly chosen as a performance function. For example, $\eta(t) = (\eta_0 - \eta_\infty)e^{-at} + \eta_\infty$ where $\eta_0 > \eta_\infty$ and $a > 0$. Then by satisfying the condition that

$$-\underline{\delta}\eta(t) < e(t) < \bar{\delta}\eta(t), \quad \forall t \geq 0 \tag{4.34}$$

where $0 < \underline{\delta}, \bar{\delta} \leq 1$ are prescribed scalars, the objective of guaranteeing transient performance can be achieved.

Remark 4.4.1.

- As shown in Figure 4.1, $\bar{\delta}\eta(0)$ and $-\underline{\delta}\eta(0)$ serve as the upper bound of the maximum overshoot and lower bound of the undershoot (i.e. negative overshoot) of $e(t)$, respectively. The decreasing rate of $\eta(t)$ introduces a lower bound on the convergence speed of $e(t)$.

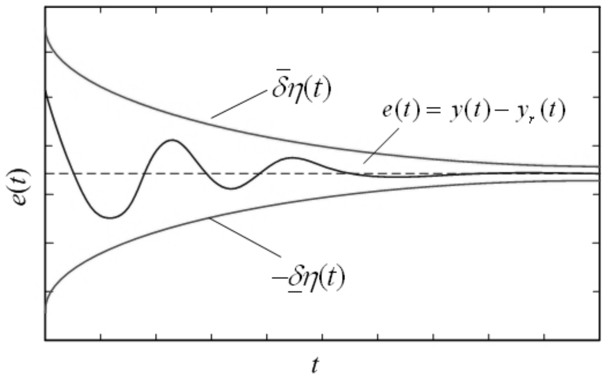


Figure 4.1: Tracking error $e(t)$ constrained within a prescribed performance bound.

- If an actuator failure occurs when $\eta(t)$ approaches to η_∞ closely enough, $-\underline{\delta}(\eta_\infty + \epsilon) < e(t) < \bar{\delta}(\eta_\infty + \epsilon)$ will be satisfied, where $\epsilon > 0$ is sufficiently small. This implies that there will be no occurrence of unacceptable large overshooting due to such an actuator failure.
- No trajectory initialization action is required, hence the transient performance of the system can be guaranteed without a priori knowledge of the failure time, type and value. In fact, by changing the design parameters of function $\eta(t)$ and the positive scalars $\underline{\delta}$, $\bar{\delta}$, the transient performance in terms of the convergence rate and maximum overshoot of tracking error $e(t)$ can be improved.

4.4.1 Transformed System

Solving the control problem satisfying the "constrained" error condition (4.34) can be transformed to solving a problem with boundedness of signals as the only requirements. Moreover, to achieve asymptotic tracking, asymptotic stabilization of the transformed system to be constructed is essential. To do these, we design a smooth and strictly increasing function $S(\nu)$ with the following properties:

(i)

$$-\underline{\delta} < S(\nu) < \bar{\delta} \tag{4.35}$$

(ii)
$$\lim_{\nu\to+\infty} S(\nu) = \bar{\delta}, \lim_{\nu\to-\infty} S(\nu) = -\underline{\delta} \tag{4.36}$$

(iii)
$$S(0) = 0 \tag{4.37}$$

From properties (i) and (ii) of $S(\nu)$, performance condition (4.34) can be expressed as

$$e(t) = \eta(t)S(\nu) \tag{4.38}$$

Because of the strict monotonicity of $S(\nu)$ and the fact that $\eta(t) \neq 0$, the inverse function

$$\nu = S^{-1}\left(\frac{e(t)}{\eta(t)}\right) \tag{4.39}$$

exists. We call ν as a transformed error. If $-\underline{\delta}\eta(0) < e(0) < \bar{\delta}\eta(0)$, and $\nu(t)$ is ensured bounded for $t \geq 0$ by our designed controller, we will have that $-\underline{\delta} < \frac{e(t)}{\eta(t)} < \bar{\delta}$. Furthermore, from property (iii) of $S(\nu)$, asymptotic tracking (i.e. $\lim_{t\to\infty} e(t) = 0$) can be achieved if $\lim_{t\to\infty} \nu(t) = 0$ is followed.

In this chapter, we design $S(\nu)$ as

$$S(\nu) = \frac{\bar{\delta}e^{(\nu+r)} - \underline{\delta}e^{-(\nu+r)}}{e^{(\nu+r)} + e^{-(\nu+r)}}, \tag{4.40}$$

where $r = \frac{\ln(\underline{\delta}/\bar{\delta})}{2}$. It can be easily shown that $S(\nu)$ has the properties (i)-(iii). The transformed error $\nu(t)$ is solved as

$$\nu = S^{-1}(\lambda(t)) = \frac{1}{2}\ln(\bar{\delta}\lambda(t) + \bar{\delta}\underline{\delta}) - \frac{1}{2}\ln(\underline{\delta}\bar{\delta} - \underline{\delta}\lambda(t)) \tag{4.41}$$

where $\lambda(t) = e(t)/\eta(t)$. We compute the time derivative of ν as

$$\begin{aligned} \dot{\nu} &= \frac{\partial S^{-1}}{\partial \lambda}\dot{\lambda} = \frac{1}{2}\left[\frac{1}{\lambda + \underline{\delta}} - \frac{1}{\lambda - \bar{\delta}}\right]\left(\frac{\dot{e}}{\eta} - \frac{e\dot{\eta}}{\eta^2}\right) \\ &= \zeta\left(\dot{e} - \frac{e\dot{\eta}}{\eta}\right) = \zeta\left(\dot{y} - \dot{y}_r - \frac{e\dot{\eta}}{\eta}\right), \end{aligned} \tag{4.42}$$

where ζ is defined as

$$\zeta = \frac{1}{2\eta}\left[\frac{1}{\lambda+\underline{\delta}} - \frac{1}{\lambda-\bar{\delta}}\right]. \tag{4.43}$$

Owing to the property (i) of $S(\nu)$ and (4.38), ζ is well defined and $\zeta \neq 0$. We now incorporate the prescribed performance bound into the original nonlinear system (4.7). By replacing the equation of $\dot{x}_1$ with $\dot{\nu}$, (4.7) can be transformed to

$$\begin{aligned}
\dot{\nu} &= \zeta(x_2 + \varphi_1^T\theta - \dot{y}_r - \frac{e\dot{\eta}}{\eta}) && (4.44)\\
\dot{x}_i &= x_{i+1} + \varphi_i^T\theta, \quad i = 2, \ldots, \varrho - 1 && (4.45)\\
\dot{x}_\varrho &= \varphi_0 + \varphi_\varrho^T\theta + \sum_{j=1}^{m} b_j\beta_j(\rho_j u_{cj} + u_{kj}) && (4.46)\\
\dot{\xi} &= \Psi(x,\xi) + \Phi(x,\xi)\theta && (4.47)
\end{aligned}$$

4.4.2 Design of Controllers

Compared with the basic design, the major difference lies in the first two steps in performing the backstepping procedure. Thus the details of Step 1 and Step 2 are elaborated. Define

$$\begin{aligned}
z_1 &= \nu && (4.48)\\
z_q &= x_q - \alpha_{q-1} - y_r^{(q-1)}, \quad q = 2, \ldots, \varrho && (4.49)
\end{aligned}$$

Step 1. From (4.44), (4.48) and the definition of z_2 in (4.49), we have

$$\dot{z}_1 = \zeta(z_2 + \alpha_1 + \varphi_1^T\theta - \frac{e\dot{\eta}}{\eta}). \tag{4.50}$$

To stabilize (4.50), α_1 is designed as

$$\alpha_1 = -\frac{c_1 z_1}{\zeta} - \varphi_1^T\hat{\theta} + \frac{e\dot{\eta}}{\eta} \tag{4.51}$$

where c_1 is a positive constant and $\hat{\theta}$ is an estimate of θ. We define a positive definite function $\bar{V}_1$ as

$$\bar{V}_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}, \tag{4.52}$$

where $\tilde{\theta} = \theta - \hat{\theta}$, Γ is a positive definite design matrix. Then

$$\dot{\bar{V}}_1 = -c_1 z_1^2 + \zeta z_1 z_2 + \tilde{\theta}^T\Gamma^{-1}(\Gamma\varphi_1 z_1\zeta - \dot{\hat{\theta}}) \tag{4.53}$$

We choose the first tuning function τ_1 as

$$\tau_1 = \varphi_1 z_1 \zeta \tag{4.54}$$

It follows that

$$\dot{\bar{V}}_1 = -c_1 z_1^2 + \zeta z_1 z_2 + \tilde{\theta}^T\Gamma^{-1}(\Gamma\tau_1 - \dot{\hat{\theta}}) \tag{4.55}$$

Step 2. We firstly clarify the arguments of the function α_1. By examining (4.51) along with (4.41), (4.43), we see that α_1 is a function of x_1, y_r, η, $\dot{\eta}$ and $\hat{\theta}$. Differentiating (4.49) for $q = 2$, with the help of (4.45) and the definition that $z_3 = x_3 - \alpha_2 - \ddot{y}_r$, we obtain

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 - \ddot{y}_r \\ &= z_3 + \alpha_2 + \varphi_2^T\theta - \frac{\partial\alpha_1}{\partial x_1}(x_2 + \varphi_1^T\theta) - \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial\alpha_1}{\partial\eta}\dot{\eta} - \frac{\partial\alpha_1}{\partial\dot{\eta}}\ddot{\eta} - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} \end{aligned} \tag{4.56}$$

With the second tuning function τ_2 chosen as

$$\tau_2 = \tau_1 + \omega_2 z_2, \tag{4.57}$$

where

$$\omega_2 = \varphi_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi_1. \tag{4.58}$$

The second stabilization function α_2, if $z_3 = 0$, is designed as

$$\begin{aligned}\alpha_2 &= -\zeta z_1 - c_2 z_2 - \left(\varphi_2 - \frac{\partial \alpha_1}{\partial x_1}\varphi_1\right)^T \hat{\theta} + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \sum_{k=1}^{2} \frac{\partial \alpha_1}{\partial \eta^{(k-1)}}\eta^{(k)} \\ &\quad + \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma \tau_2. \end{aligned} \tag{4.59}$$

Denote $\bar{x}_q = (x_1, \ldots, x_q)$, $\bar{\eta}^{(q)} = (\eta, \dot{\eta}, \ldots, \eta^{(q)})$ and $\bar{y}_r^{(q-1)} = (y_r, \dot{y}_r, \ldots, y_r^{(q-1)})$. Note that in the backstepping procedure, α_q for $q \geq 2$, is a function of $\bar{x}_q$, $\bar{\eta}^{(q)}$, $\bar{y}_r^{(q-1)}$, $\hat{\theta}$. Define a positive definite function at this step as

$$\bar{V}_2 = \bar{V}_1 + \frac{1}{2} z_2^2. \tag{4.60}$$

From (4.55), (4.56) and (4.59), the time derivative of V_2 can be computed as

$$\dot{\bar{V}}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + \tilde{\theta}^T \Gamma^{-1}(\Gamma \tau_2 - \dot{\hat{\theta}}) - \frac{\partial \alpha_1}{\partial \hat{\theta}}(\dot{\hat{\theta}} - \Gamma \tau_2) z_2. \tag{4.61}$$

Step q where $q = 3, \ldots, \varrho$.

$$\begin{aligned}\alpha_q &= -z_{q-1} - c_q z_q - \omega_j^T \hat{\theta} + \sum_{k=1}^{q} \frac{\partial \alpha_{q-1}}{\partial \eta^{(k-1)}}\eta^{(k)} + \frac{\partial \alpha_{q-1}}{\partial \hat{\theta}}\Gamma \tau_q + \sum_{k=2}^{q-1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma \omega_q z_k \\ &\quad + \sum_{k=1}^{q-1}\left(\frac{\partial \alpha_{q-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{q-1}}{\partial y_r^{(k-1)}} y_r^{(k)}\right), \quad q = 3, \ldots, \varrho - 1 \end{aligned} \tag{4.62}$$

$$\begin{aligned}\alpha_\varrho &= -z_{\varrho-1} - c_\varrho z_\varrho - \varphi_0 - \omega_\varrho^T \hat{\theta} + \sum_{k=1}^{\varrho-1}\left(\frac{\partial \alpha_{\varrho-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{\varrho-1}}{\partial y_r^{(k-1)}} y_r^{(k)}\right) \\ &\quad + \sum_{k=1}^{\varrho} \frac{\partial \alpha_{\varrho-1}}{\partial \eta^{(k-1)}}\eta^{(k)} + \frac{\partial \alpha_{\varrho-1}}{\partial \hat{\theta}}\Gamma \tau_\varrho + \sum_{k=2}^{\varrho-1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\Gamma \omega_\varrho z_k \end{aligned} \tag{4.63}$$

$$v_0 = \alpha_\varrho + y_r^{(\varrho)} \tag{4.64}$$

$$\tau_q = \tau_{q-1} + \omega_q z_q \tag{4.65}$$

$$\omega_q = \varphi_q - \sum_{k=1}^{q-1} \frac{\partial \alpha_{q-1}}{\partial x_k}\varphi_k, \quad q = 3, \ldots, \varrho \tag{4.66}$$

Control laws and parameter update laws are determined at the ϱth step as

$$u_{cj} = \operatorname{sgn}(b_j)\frac{1}{\beta_j}\hat{\kappa}^T w, \quad \text{for } j = 1, \ldots, m \tag{4.67}$$

$$\dot{\hat{\theta}} = \Gamma \tau_\varrho \tag{4.68}$$

$$\dot{\hat{\kappa}} = -\Gamma_\kappa w z_\varrho \tag{4.69}$$

Note that u_{cj}, $\dot{\hat{\theta}}$ and $\dot{\hat{\kappa}}$ are designed in the same form as in (4.22)-(4.23) with the signals v_0, τ_ϱ and constructed $w = [v_0, \beta^T]^T$ changed appropriately.

4.4.3 Stability Analysis

For an arbitrary initial tracking error $e(0)$, we can select $\eta(0)$, $\bar{\delta}$ and $\underline{\delta}$ to satisfy that $-\underline{\delta}\eta(0) < e(0) < \bar{\delta}\eta(0)$. As discussed in Remark 4.4.1, the transient performance of $e(t)$ can be improved by tuning the design parameters $\bar{\delta}$, $\underline{\delta}$ and parameters of $\eta(t)$ including its speed of convergence, η_∞ at a steady state as long as $e(t)$ is preserved within a specified PPB as described in (4.34). Observing the generated transformed error $\nu = S^{-1}\left(\frac{e(t)}{\eta(t)}\right)$ and the injective property of $S(\nu)$, we conclude that (4.34) is satisfied if $\nu(t) \in \mathcal{L}_\infty$ with the designed controllers in the previous subsection. Moreover, $\lim_{t\to\infty} \nu(t) = 0$ is essential to achieve asymptotic tracking. Therefore, the asymptotic stabilization of the transformed system (4.44)-(4.47) is sufficient to attain the control objectives. The main results of PPB based control design are established in the following theorem.

Theorem 4.4.1. *Consider the closed-loop adaptive system consisting of the plant (4.1)-(4.2), the PPB based controller (4.67) with the parameter update laws (4.68)-(4.69) in the presence of possible actuator failures (4.3) and (4.4) under Assumptions 4.2.1-4.2.5. The boundedness of all the signals and tracking error* $e(t) =$

$y(t) - y_r(t)$ asymptotically approaching zero are ensured. Furthermore, the transient performance of the system in the sense that $e(t)$ is preserved within a specified PPB all the time, i.e. $-\underline{\delta}\eta(t) < e(t) < \bar{\delta}\eta(t)$ with $t \geq 0$ is guaranteed.

Proof: From (4.50) and (4.51), it is obtained that

$$\dot{z}_1 = -c_1 z_1 + \zeta z_2 + \zeta \varphi_1^T \tilde{\theta}. \tag{4.70}$$

From (4.56), (4.59), (4.65) and (4.66), we have

$$\begin{aligned} \dot{z}_2 &= -c_2 z_2 - \zeta z_1 + z_3 + \omega_2^T \tilde{\theta} + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma (\tau_2 - \tau_\varrho) \\ &= -c_2 z_2 - \zeta z_1 + z_3 + \omega_2^T \tilde{\theta} - \sum_{k=3}^{\varrho} \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \omega_k z_k. \end{aligned} \tag{4.71}$$

From the design along (4.62)-(4.66) for $q = 3, \ldots, \varrho - 1$, it can be shown that

$$\dot{z}_q = -c_q z_q - z_{q-1} + z_{q+1} + \omega_q^T \tilde{\theta} + \sum_{k=2}^{q-1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \omega_q z_k - \sum_{k=q+1}^{\varrho} \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \omega_k z_k. \tag{4.72}$$

Similar to the proof of Theorem 4.3.1, suppose that there are $(r+1)$ time intervals $[T_{k-1}, T_k)$ $(k = 1, \ldots, r+1)$ along $[0, \infty)$. $T_0 = 0$, T_1 and T_r refer to the first and last time that failures occur respectively, $T_{r+1} = \infty$. During $[0, T_1)$, from (4.46), (4.49), (4.63) and (4.67), the derivative of z_ϱ is computed as

$$\begin{aligned} \dot{z}_\varrho &= \varphi_0 + \varphi_\varrho^T \theta + \sum_{j=1}^{m} |b_j| \hat{\kappa}^T w - \dot{\alpha}_{\varrho-1} - y_r^{(\varrho)} \\ &= \varphi_0 + \varphi_\varrho^T \theta + \sum_{j=1}^{m} |b_j| (\kappa - \tilde{\kappa})^T w - \dot{\alpha}_{\varrho-1} - y_r^{(\varrho)}, \end{aligned} \tag{4.73}$$

where $\tilde{\kappa} = \kappa - \hat{\kappa}$. If b_j is known, κ is a desired constant vector which can be chosen to satisfy

$$\sum_{j=1}^{m} |b_j|\kappa^T w = v_0 \;\Rightarrow\; \kappa_1 = \frac{1}{\sum_{j=1}^{m} |b_j|},\; \kappa_{2,k} = 0 \text{ for } k = 1, \ldots, m \tag{4.74}$$

Substituting (4.74) into (4.73), we have

$$\dot{z}_\varrho = -c_\varrho z_\varrho - z_{\varrho-1} + \omega_\varrho^T \tilde{\theta} + \sum_{k=2}^{\varrho-1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \omega_\varrho z_k + \sum_{j=1}^{m} |b_j| \tilde{\kappa}^T w. \tag{4.75}$$

We define the error vector $z(t) = [z_1, z_2, \ldots, z_\varrho]^T$, $\omega_1 = \zeta\varphi_1$. From (4.70)-(4.72), (4.75), the derivative of $z(t)$ during $[0, T_1)$ is summarized as

$$\dot{z} = A_z z + \Omega^T \tilde{\theta} - \begin{bmatrix} 0_{(\varrho-1)\times 1} \\ \sum_{j=1}^{m} |b_j| \tilde{\kappa}^T w \end{bmatrix}, \tag{4.76}$$

where

$$A_z = \begin{bmatrix} -c_1 & \zeta & 0 & \cdots & 0 \\ -\zeta & -c_2 & 1+\sigma_{2,3} & \cdots & \sigma_{2,\varrho} \\ 0 & -1-\sigma_{2,3} & -c_3 & \cdots & \sigma_{3,\varrho} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\sigma_{2,\varrho} & \cdots & -1-\sigma_{\varrho-1,\varrho} & -c_\varrho \end{bmatrix} \tag{4.77}$$

$$\sigma_{q,k} = -\frac{\partial \alpha_{q-1}}{\partial \hat{\theta}} \Gamma \omega_k \tag{4.78}$$

$$\Omega = [\omega_1, \omega_2, \ldots, \omega_n] \tag{4.79}$$

It can be shown that $A_z + A_z^T = -2\text{diag}\{c_1, c_2, \ldots, c_\varrho\}$. Define a positive definite $V_0(t)$ for $t \in [0, T_1)$ as

$$V_0 = \frac{1}{2} z^T z + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \sum_{j=1}^{m} \frac{|b_j|}{2} \tilde{\kappa}^T \Gamma_\kappa^{-1} \tilde{\kappa}. \tag{4.80}$$

Differentiating V_0, we obtain

$$\dot{V}_0 = -\sum_{q=1}^{\varrho} c_q z_q^2 \tag{4.81}$$

Thus we have $V_0(T_1^-) \leq V_0(0)$, where $V_0(T_1^-)$ is defined as the same as in Section 4.3.2. Assume also that during the time interval $[T_{k-1}, T_k)$ with $k = 2, \ldots, r$, subsets Q_{tot_k} and Q_{par_k} are correspond to the actuators undergoing TLOE and PLOE respectively. The derivative of $z(t)$ during $[T_{k-1}, T_k)$ can then be written as

$$\dot{z} = A_z z + \Omega^T \tilde{\theta} - \begin{bmatrix} 0_{(\varrho-1)\times 1} \\ \sum_{i=1, i\notin Q_{tot_k}}^{m} \rho_j |b_j| w^T \tilde{\kappa} \end{bmatrix}. \tag{4.82}$$

Define V_{k-1} during $[T_{k-1}, T_k)$ in the same form of (4.24). $\dot{V}_{k-1} = -\sum_{q=1}^{\varrho} c_q z_q^2$ can also be achieved. Then by following the similar procedure in Section 4.3.2, it can be shown that z, $\tilde{\theta}$, $\tilde{\kappa}$, $x(t)$ and u_{cj} are bounded and $z(t) \in \mathcal{L}_2$. From the fact that $\nu = z_1$, $\nu(t)$ is bounded. ζ is bounded from (4.43) and (4.34) is thus satisfied. The closed-loop stability is then established. Noting $\dot{z} \in \mathcal{L}_\infty$, it follows that $\lim_{t\to\infty} z(t) = 0$. From (4.37), $\lim_{t\to\infty} e(t) = 0$ which implies that asymptotic tracking can still be retained. □

4.5 Simulation Studies

To compare the PPB based control scheme with the basic control method, we use the same twin otter aircraft longitudinal nonlinear dynamics model as in [54].

$$\begin{aligned} \dot{V} &= \frac{F_x \cos(\alpha) + F_z \sin(\alpha)}{m} \\ \dot{\alpha} &= q + \frac{-F_x \sin(\alpha) + F_z \cos(\alpha)}{mV} \\ \dot{\theta} &= q \\ \dot{q} &= \frac{M}{I_y}, \end{aligned} \tag{4.83}$$

where

$$\begin{aligned} F_x &= \bar{q}SC_x + T_x - mg\sin(\theta) \\ F_z &= \bar{q}SC_z + T_z + mg\cos(\theta) \\ M &= \bar{q}cSC_m \end{aligned} \tag{4.84}$$

and $\bar{q} = \frac{1}{2}\rho V^2$, C_x, C_z and C_m are polynomial functions

$$\begin{aligned} C_x &= C_{x1}\alpha + C_{x2}\alpha^2 + C_{x3} + C_{x4}(d_1\delta_{e1} + d_2\delta_{e2}) \\ C_z &= C_{z1}\alpha + C_{z2}\alpha^2 + C_{z3} + C_{z4}(d_1\delta_{e1} + d_2\delta_{e2}) + C_{z5}q \\ C_m &= C_{m1}\alpha + C_{m2}\alpha^2 + C_{m3} + C_{m4}(d_1\delta_{e1} + d_2\delta_{e2}) + C_{m5}q. \end{aligned} \tag{4.85}$$

In (4.83), V is the velocity, α is the attack angle, θ is the pitch angle and q is the pitch rate. They are chosen as states $\chi_1, \chi_2, \chi_3, \chi_4$ respectively. In (4.85), δ_{e1}, δ_{e2} are the elevator angles of an augmented two-piece elevator chosen as two actuators u_1 and u_2. The rest of the notations are illustrated in the following table.

m	the mass
I_y	the moment of inertia
ρ	the air density
S	the wing area
c	the mean chord
T_x	The components of the thrust along the body x
T_z	The components of the thrust along the body z

The control objective is to ensure that the closed-loop system is stable and the pitch angle $y = \chi_3$ can asymptotically track a given signal y_r in the presence of actuator failures with guaranteed transient performance of $e(t) = y(t) - y_r(t)$. As explained in [54], there exists a diffeomorphism $[\xi, x]^T = T(\chi) = [T_1(\chi), T_2(\chi), \chi_3, \chi_4]$ that

(4.83) can be transformed into the parametric-strict-feedback form as in (4.7).

$$
\begin{aligned}
\dot{\chi}_3 &= \chi_4 \\
\dot{\chi}_4 &= \varphi(\chi)^T\vartheta + \sum_{i=1}^{2} b_i\chi_1^2(\rho_i u_{ci} + u_{ki}) \\
\dot{\xi} &= \Psi(\xi, x) + \Phi(\xi, x)\vartheta,
\end{aligned} \tag{4.86}
$$

where $\vartheta \in R^4$ is an unknown constant vector and $\varphi(\chi) = [\chi_1^2\chi_2, \chi_1^2\chi_2^2, \chi_1^2, \chi_1^2\chi_4]^T$, $x = [\chi_3, \chi_4]^T$. Input-to-state stability of zero dynamics subsystem is shown in [54]. Relative degree $\varrho = 2$. The aircraft parameters in the simulation study are set based on the data sheet in [104]: $m = 4600kg$, $I_y = 31027kg \cdot m^2$, $S = 39.02m^2$, $c = 1.98m$, $T_x = 4864N$, $T_z = 212N$, $\rho = 0.7377kg/m^3$ at the altitude of 5000 m, and for the 0° flap setting. In addition, $d_1 = 0.6, d_2 = 0.4$, $C_{x1} = 0.39, C_{x2} = 2.9099, C_{x3} = -0.0758, C_{x4} = 0.0961$, $C_{z1} = -7.0186, C_{z2} = 4.1109, C_{z3} = -0.3112, C_{z4} = -0.2340$, $C_{z5} = -0.1023$, $C_{m1} = -0.8789, C_{m2} = -3.852, C_{m3} = -0.0108, C_{m4} = -1.8987$, $C_{m5} = -0.6266$ are unknown constants. The reference signal y_r is set as $y_r = e^{-0.05t}\sin(0.2t)$. The initial states and estimates are set as $\chi(0) = [75, 0, 0.15, 0]^T$, $\hat{\vartheta}(0) = [0, 0, -0.04, 0]$.

Design the control inputs with PPB through the procedures as given in Section 4.4.2. By noting that in (4.67) β_1 and β_2 are the same as χ_1^2, the control laws are designed as $u_{ci} = \mathrm{sgn}(b_i)\frac{1}{\chi_1^2}\hat{\kappa}[\alpha_2, \chi_1^2]$, for $i = 1, 2$. A prescribed performance bound (PPB) is given by choosing $\eta(t) = 0.4e^{-2t} + 0.01$, $\underline{\delta} = 0.1$ and $\bar{\delta} = 1$. Other design parameters are chosen as $c_1 = c_2 = 1$, $\Gamma = 0.005I$ and $\Gamma_\kappa = [1, 0; 0, 0.01]$. The initial value of $\hat{\kappa}$ are set as $\hat{\kappa}(0) = [-1.2, 0]$. Three failure cases are considered respectively,

- Case 1: actuator u_1 loses 90% of its effectiveness from $t = 10$ second, thus undergoes a PLOE type of failure.

The tracking error $e(t) = y(t) - y_r(t)$ is plotted in Fig. 4.2. To show the im-

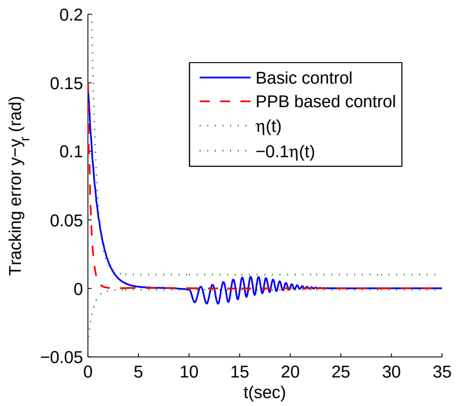


Figure 4.2: Tracking errors $e(t)$ in failure case 1.

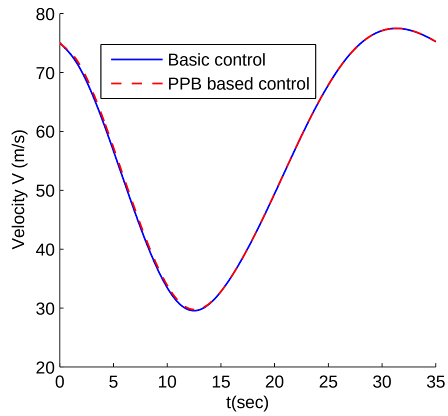


Figure 4.3: Velocity V in failure case 1.

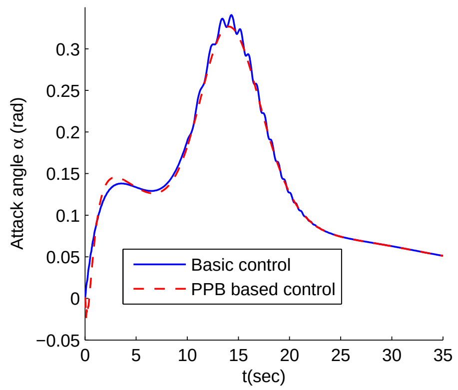


Figure 4.4: Attach angle α in failure case 1.

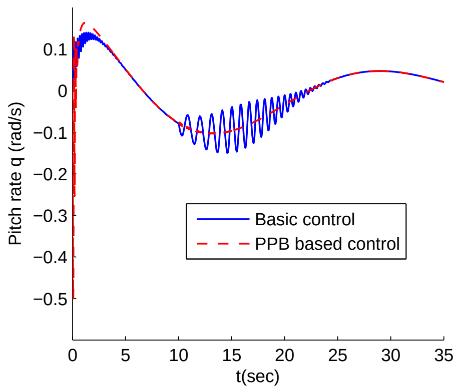


Figure 4.5: Pitch rate $\bar{q}$ in failure case 1.

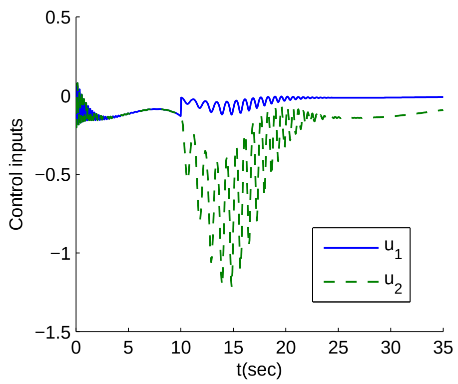


Figure 4.6: Control inputs with basic design method in failure case 1.

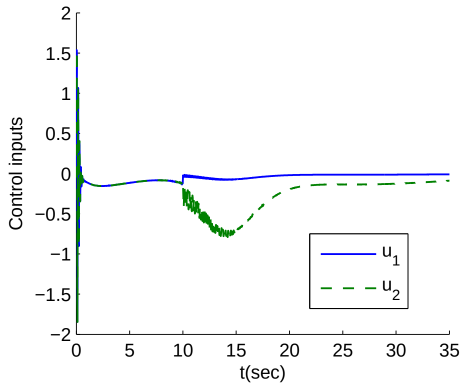


Figure 4.7: Control inputs with PPB based control method in failure case 1.

proved transient performance with PPB based proposed scheme, the tracking error performance using the basic design method with the same design parameters is also plotted for comparison. The comparisons on the performances of velocity, attack angle, pitch rate as well as control inputs using the PPB based control scheme and the basic design method are given in Fig. 4.3-Fig. 4.7.

- Case 2: actuator u_2 is stuck at $u_1 = 4$ from $t = 10$ second, thus undergoes a TLOE type of failure.

The comparisons on the performances of tracking error, velocity, attack angle, pitch rate and control inputs are given in Fig. 4.8-Fig. 4.13, respectively.

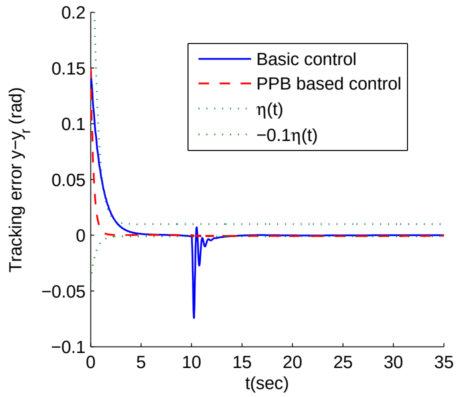


Figure 4.8: Tracking errors $e(t)$ in failure case 2.

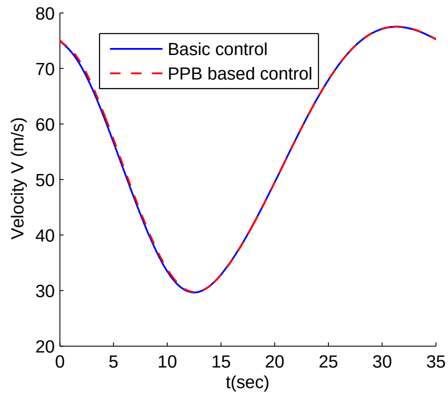


Figure 4.9: Velocity V in failure case 2.

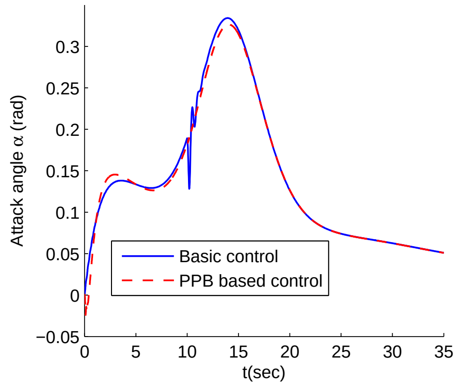


Figure 4.10: Attach angle α in failure case 2.

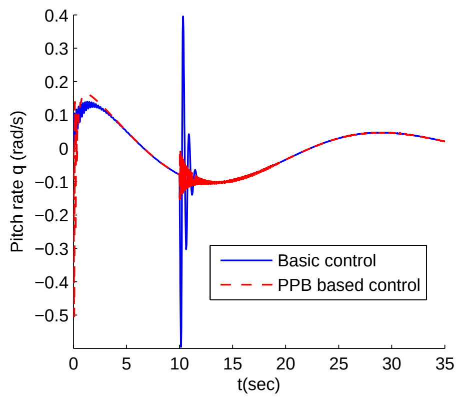


Figure 4.11: Pitch rate $\bar{q}$ in failure case 2.

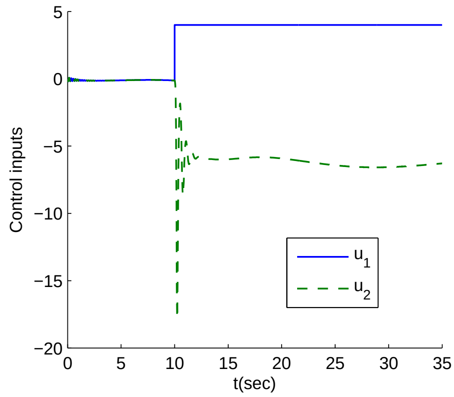


Figure 4.12: Control inputs with basic design method in failure case 2.

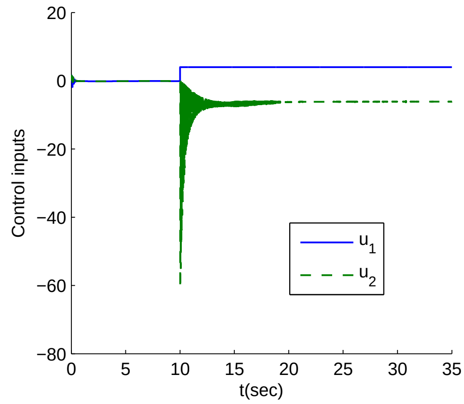


Figure 4.13: Control inputs with PPB based control method in failure case 2.

- Case 3: actuator u_1 loses its 50% of its effectiveness from $t = 10$ second. and actuator u_2 is stuck at $u_2 = 2$ from $t = 25$ second.

The comparisons on the performances of tracking error, velocity, attack angle, pitch rate and control inputs are given in Fig. 4.14-Fig. 4.19, respectively.

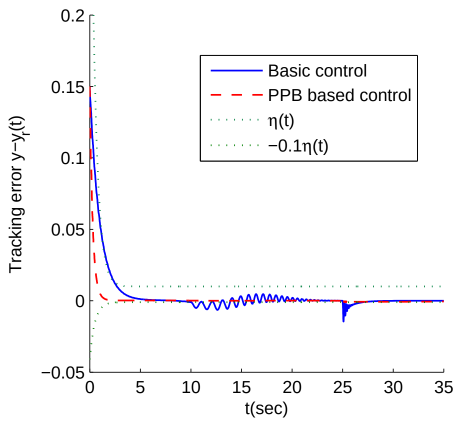


Figure 4.14: Tracking errors $e(t)$ in failure case 3.

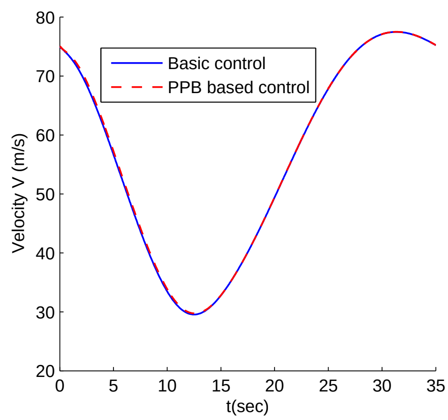


Figure 4.15: Velocity V in failure case 3.

It can be seen that all signals are bounded and asymptotic tracking can be ensured under all three cases. From Fig. 4.2, Fig. 4.8 and Fig. 4.14, the tracking error is

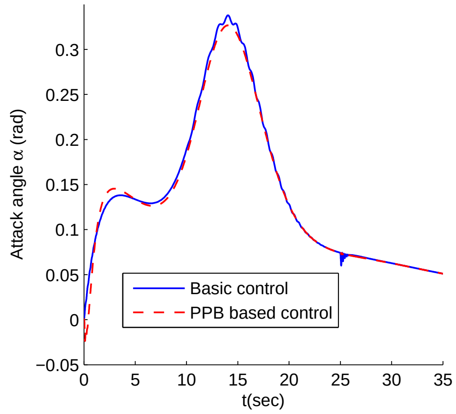


Figure 4.16: Attach angle α in failure case 3.

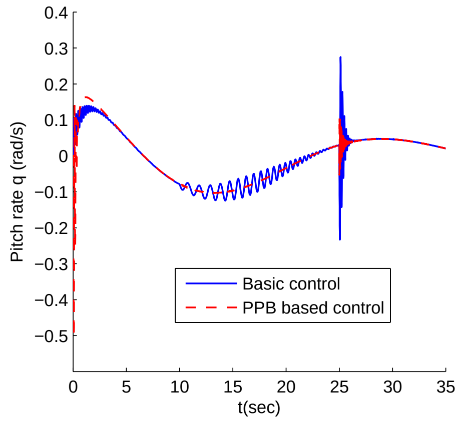


Figure 4.17: Pitch rate $\bar{q}$ in failure case 3.

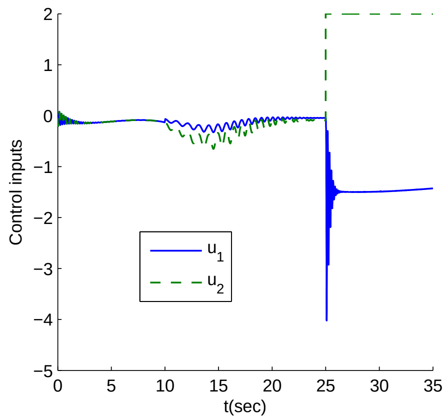


Figure 4.18: Control inputs with basic design method in failure case 3.

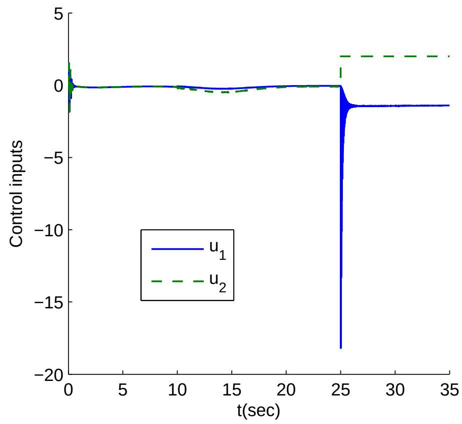


Figure 4.19: Control inputs with PPB based control method in failure case 3.

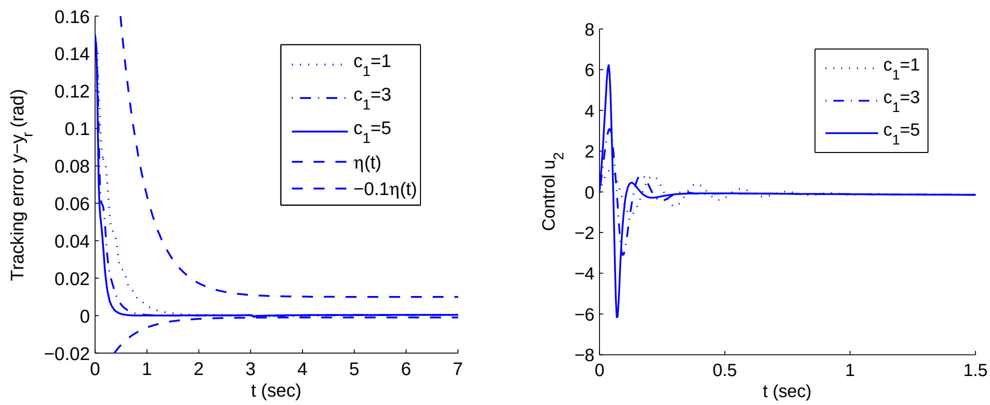
shown to convergent at a faster rate in the initial phase before failures occur using PPB based control method. At the time instant when failures occur, the large overshoot on tracking error with basic design method can be reduced by preserving the tracking error within a prescribed bound with PPB based control method.

Remark 4.5.1.

- From (4.43) and (4.44), it can be seen that the term $1/\eta$ is involved in the derivative of ν. Thus a small η could make the signal ν as well as the tracking error $e(t)$ less smooth. Although decreasing η_0 and η_∞ can improve the transient performance of $e(t)$ in terms of the maximum overshoot as discussed in Remark 4.4.1, there is a compromise in choosing these two parameters.

- About the issue on how to choose the free design parameters c_j, Γ and Γ_κ, there is still no quantitative measure in terms of certain cost functions when the PPB based control method is utilized. Also no explicit relationship between the performance of tracking error and these parameters has been obtained in the failure case. However, we may choose these parameters by following the well established rule of the basic design scheme in the failure free case, as in [21,37], etc. According to the discussions in Section 4.3.3, with the basic design method, the transient performance of the tracking error in the sense of both $\mathcal{L}_{2[0,t_1]}$ and $\mathcal{L}_{\infty[0,t_1]}$ norms ($t_1 < T_1$, where T_1 denotes the time instant when the first failure takes place) can be improved by increasing c_1, Γ, Γ_κ. However, their increases may increase the magnitudes of the control signals. Thus a compromise might be reached.

For the choice of these free parameters with PPB based control, we now use an example to illustrate how the choice of c_1 affects the $\mathcal{L}_2$ performance of the tracking error. Consider the same plant as in (4.83)-(4.85) in the failure case that



(a) Tracking errors with different c_1.

(b) Control u_2 with different c_1 for the first 1.5 seconds.

Figure 4.20: Comparisons of tracking errors and control u_2 with different c_1.

actuator u_1 loses 90% of its effectiveness from $t = 3$ second. All parameters and the initial states are the same as those given above, except for c_1, Γ and Γ_κ. We change c_1 by setting $c_1 = 1, 3$ and 5 respectively with Γ and Γ_κ being fixed at $\Gamma = 0.01 \times I(4)$ and $\Gamma_\kappa = 0.01 \times I(2)$, the tracking error $y - y_r$ with different c_1 are compared in Fig. 4.20(a). Obviously, the $\mathcal{L}_{2[0,t_1]}$ norms of the tracking error decrease as c_1 increases especially before the failure occur. We also present control u_2 with different c_1 for the first 1.5 seconds in Fig. 4.20(b). It can be seen that the magnitude of u_2 increases with increased c_1. Similar results would be followed if we change Γ and Γ_κ with a fixed c_1. The results once again show that a compromise may be reached in choosing novel free design parameters.

4.6 Conclusion

Two adaptive backstepping control schemes for parametric strict feedback systems in the presence of unknown actuator failures are presented in this chapter. The actuator failures under consideration include TLOE and PLOE types. System stability and asymptotic tracking are shown to be maintained with both schemes. It is analyzed that transient performance of the adaptive system is not adjustable with the first control scheme proposed on the basis of an existing adaptive failure compensation approach. However, the transient performance can be improved and adjusted by preserving the tracking error within a prescribed performance bound (PPB) by the second control scheme. Simulation studies also verify the theoretical results.

As discussed in Remark 4.2.1, the assumption that there are only finite number of actuator failures was commonly imposed in many existing results on adaptive actuator failure compensation. Our main task of the next chapter is to propose a new adaptive solution with this assumption removed.

Chapter 5

Adaptive Compensation for Infinite Number of Failures

It is both theoretically and practically important to investigate the problem of accommodating infinite number of actuator failures in controlling uncertain systems. However, there is still no result available in developing adaptive controllers to address this problem. In this chapter, a new adaptive failure compensation control scheme is proposed for parametric strict feedback nonlinear systems. The techniques of nonlinear damping and parameter projection are employed in the design of controllers and parameter estimators respectively. It is proved that the boundedness of all closed-loop signals can still be ensured in the case with infinite number of failures, provided that the time interval between two successive changes of failure pattern is bounded below by an arbitrary positive number. The performance of the tracking error in the mean square sense with respect to the frequency of failure pattern changes is also established. Moreover, asymptotic tracking can be achieved when the number of failures is finite.

5.1 Introduction

In most of the existing results on adaptive control of systems with actuator failures, such as [43, 45, 46, 54, 56, 59, 71, 72], only the cases with finite number of failures are considered. It is assumed that one actuator may only fail once and the failure mode does not change afterwards. As discussed in Remark 4.2.1, this implies that there exists a finite time T_r such that no further failure occurs on the system after T_r. In these cases, although some unknown parameters will experience jumps at the time instants when failures occur, the jumping sizes are bounded and the total number of jumps are finite. Thus the possible increase of the considered Lyapunov function, which includes the estimation errors of the unknown parameters, is bounded, which enables the stability of the closed loop to be established. However, we cannot show the system stability in the same way when the total number of failures is infinite, because the possible increase of the Lyapunov function mentioned earlier cannot be ensured bounded automatically when the parameters experience infinite number of jumps. This is indeed the main challenge to find an adaptive solution to the problem of compensating for infinite number of failures theoretically. On the other hand, it is possible that some actuator failures occur intermittently in practice. Thus the actuators may unawarely change from a failure mode to a normally working mode or another different failure mode infinitely many times. For example, poor electrical contact can cause repeated unknown breaking down failures on the actuators in some control systems. Although it is of both theoretical and practical importance to consider the case with infinite number of failures, there is still no solid result available in this area so far. In [55], the authors only conjectured that their proposed scheme could possibly be applied to this case. It was remarked that all the signals might still be ensured bounded as long as the time interval between two sequential changes of failure status is not too small. Nevertheless, to the best of our knowledge, no rigorous analysis has been reported by them.

In this chapter, we shall deal with the problem of compensating for possibly infinite number of actuator failures in controlling uncertain nonlinear systems based on adaptive backstepping technique. Through tremendous studies, we find that it is difficult to show the boundedness of all the signals using the tuning function design approaches as in [54–56, 71] and Chapter 4, mainly because the unbounded derivatives of the parameters caused by jumps need to be considered in computing the derivative of the Lyapunov function. In fact from our simulation studies, instability is observed when the tuning function scheme as summarized in Section 4.3 is utilized to compensate for infinite number of relatively frequent actuator failures. To overcome the difficulty, we propose a modular design scheme. Actually, so far there is also no result available by using backstepping based modular design scheme to compensate for actuator failures even for the case of finite number of failures. With compared to the existing tuning function methods, our designs have the following features. The control module and parameter estimator module are designed separately; nonlinear damping terms functions are introduced in the control design to establish an input-to-state property of an error system; impulses caused by failures are considered in computing the derivatives of the unknown parameters and these parameters are shown to satisfy a finite mean variation condition; the parameter update law involves projection operation to ensure the boundedness of estimation errors; the properties of the parameter estimator, which are useful for stability analysis, are also obtained. It is proved that the boundedness of all the closed-loop signals can be ensured with our scheme, provided that the time interval between two successive changes of failure pattern is bounded below by an arbitrary positive number. It is also established that the tracking error can be small in the mean square sense if the changes of failure pattern are infrequent. This shows that the less frequent the failure pattern changes, the better the tracking performance is. Moreover, asymptotic tracking can still be achieved with the proposed scheme in the case with finite number failures as the tuning function methods.

5.1.1 Notations

For a vector function $x(t) = [x_1, \ldots, x_n]^T \in \Re^n$,

- $x(t) \in S_1(\mu)$, if $\int_t^{t+T} \|x(\tau)\| d\tau \leq \bar{c}_1 \mu T + \bar{c}_2$ for $\mu \geq 0$, where $\bar{c}_1$, $\bar{c}_2$ are some positive constants, and $\bar{c}_1$ is independent of μ.
- $x(t) \in S_2(\mu)$, if $\int_t^{t+T} x(\tau)^T x(\tau) d\tau \leq (\bar{c}_1 \mu^2 + \bar{c}_3 \mu)T + \bar{c}_2$ for $\mu \geq 0$, where $\bar{c}_i$ for $i = 1, 2, 3$ are some positive constants, and $\bar{c}_1, \bar{c}_3$ are independent of μ. We say that x is of the order μ in the mean square sense if $x \in S_2(\mu)$.

5.2 Problem Formulation

We consider a class of multiple-input single-output nonlinear systems that are transformable into the following parametric strict feedback form.

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_i(\bar{x}_i)^T \theta, \qquad i = 1, 2, \ldots, \varrho - 1 \\ \dot{x}_\varrho &= \varphi_0(x, \xi) + \varphi_\varrho(x, \xi)^T \theta + \sum_{j=1}^{m} b_j \beta_j(x, \xi) u_j \\ \dot{\xi} &= \Psi(x, \xi) + \Phi(x, \xi)\theta \\ y &= x_1, \end{aligned} \tag{5.1}$$

where $x = [x_1, x_2, \ldots, x_\varrho]^T$, $\xi \in \Re^{n-\varrho}$ are the states, $y \in \Re$ is the output and $u_j \in \Re$ for $j = 1, 2, \ldots, m$ is the jth input of the system, i.e. the output of the jth actuator. $\beta_j(x, \xi)$, $\varphi_0(x, \xi) \in \Re$, $\varphi_\varrho(x, \xi), \varphi_i(\bar{x}_i) \in \Re^p$ for $i = 1, 2, \ldots, \varrho - 1$ are known smooth nonlinear functions with $\bar{x}_i = (x_1, x_2, \ldots, x_i)$. $\theta \in \Re^p$ is a vector of unknown parameters and b_j for $j = 1, \ldots, m$ are unknown control coefficients.

Remark 5.2.1. As presented in [54, Sec. 3.1] and Chapter 4, suppose there is a class of nonlinear systems modeled as,

$$\begin{aligned}\dot{\chi} &= f_0(\chi) + \sum_{l=1}^{p} \theta_l f_l(\chi) + \sum_{j=1}^{m} b_j g_j(\chi) u_j \\ y &= h(\chi), \end{aligned} \tag{5.2}$$

where $\chi \in \Re^n$, y, u_j for $j = 1, \ldots, m$ are the states, output and jth input of the system respectively, $f_l(\chi) \in \Re^n$ for $l = 0, 1, \ldots, p$, $g_j(\chi) \in \Re^n$ for $j = 1, \ldots, m$ and $h(\chi)$ are known smooth nonlinear functions, θ_l for $l = 1, \ldots, p$ and b_j are unknown parameters and control coefficients. If $g_j(\chi) \in \mathrm{span}\{g_0(\chi)\}$, $g_0(\chi) \in \Re^n$ and the nominal system $\dot{\chi} = f_0(\chi) + F(\chi)\theta + g_0(\chi)u_0$, $y = h(\chi)$, where $u_0 \in \Re$, $F(\chi) = [f_1(\chi), f_2(\chi), \ldots, f_p(\chi)] \in \Re^{n \times p}$, $\theta = [\theta_1, \theta_2, \ldots, \theta_p]^T \in R^p$, is transformable into the parametric-strict-feedback form with relative degree ϱ, the nonlinear plant (5.2) can be transformed to the form of (5.1).

5.2.1 Model of Actuator Failures

Suppose that the internal dynamics in actuators is negligible. We denote u_{cj} for $j = 1, \ldots, m$ as the input of the jth actuator, which is to be designed. An actuator with its input equal to its output, i.e. $u_j = u_{cj}$, is regarded as failure-free. The actuator failures of interest are modeled as follows,

$$u_j(t) = \rho_{jh} u_{cj} + u_{kj,h}, \quad t \in [t_{jh,s}, t_{jh,e}), \quad h \in Z^+ \tag{5.3}$$

$$\rho_{jh} u_{kj,h} = 0, \quad j = 1, \ldots, m, \tag{5.4}$$

where $\rho_{jh} \in [0, 1), u_{kj,h}, t_{jh,s}, t_{jh,e}$ are all unknown constants and $0 \leq t_{j1,s} < t_{j1,e} \leq t_{j2,s} < \cdots < t_{jh,e} \leq t_{j(h+1),s} < t_{j(h+1),e}$ and so forth. Equation (5.3) indicates that the jth actuator fails from time $t_{jh,s}$ till $t_{jh,e}$. $t_{j1,s}$ denotes the time instant when the first failure takes place on the jth actuator.

Similar to Section 4.2.1, (5.4) also includes two typical types of failures, i.e. PLOE and TLOE and the failure status for different ρ_{jh} and $u_{kj,h}$ can also be elaborated as follows.

1) $\rho_{jh} \neq 0$ and $u_{kj,h} = 0$.

In this case, $u_j = \rho_{jh} u_{cj}$, where $0 < \rho_{jh} < 1$. This indicates PLOE type of failures. For example, $\rho_{jh} = 70\%$ means that the jth actuator loses 30% of its effectiveness.

2) $\rho_{jh} = 0$.

This indicates that u_j can no longer be influenced by the control inputs u_{cj}. The fact that u_j is stuck at an unknown value $u_{kj,h}$ is usually referred to as a TLOE type of actuator failures.

It is important to be noted that actuators working in failure-free case can also be represented as (5.3) with $\rho_{jh} = 1$ and $u_{kj,h} = 0$. Therefore, the model in (5.3) is applicable to describe the output of an actuator no matter it fails or not.

Remark 5.2.2. By comparing (5.3)-(5.4) to the failure models considered in [43, 45, 46, 54, 56, 59, 71, 72], h is not restricted to be finite. This implies i) a failed actuator may operate normally again from time $t_{jh,e}$ till $t_{j(h+1),s}$ when the next failure occurs on the same actuator; ii) the failure values ρ_{jh} or $u_{kj,h}$ changes to a new one, i.e $\rho_{j(h+1)}$ or $u_{kj,h+1}$, from the time $t_{jh,e}(= t_{j(h+1),s})$.

5.2.2 Control Objectives and Assumptions

The control objectives in this chapter are as follows,

- The effects of considered types of actuator failures can be compensated for so that all the closed-loop signals are ensured bounded all the time.
- The tracking error $z_1(t) = y(t) - y_r(t)$ satisfies that $z_1(t) \in S_2(\mu)$, where $S_2(\mu)$ is defined in Section 5.1.1.
- Asymptotic tracking can still be achieved if the total number of failures is finite.

To achieve the control objectives, the following assumptions are imposed.

Assumption 5.2.1. *The plant (5.1) is so constructed that for any up to $m-1$ actuators suffering from TLOE type of actuator failures simultaneously, the remaining actuators can still achieve the desired control objectives.*

Assumption 5.2.2. *The reference signal $y_r(t)$ and its first ϱth order derivatives $y_r^{(i)}(i = 1, \ldots, \varrho)$ are known, bounded, and piecewise continuous.*

Assumption 5.2.3. *$\beta_j(x, \xi) \neq 0$, the signs of b_j, i.e. sgn(b_j), for $j = 1, \ldots, m$ are known.*

Assumption 5.2.4. $0 < \underline{b}_j \leq |b_j| \leq \bar{b}_j$, $|u_{kj,h}| \leq \bar{u}_{kj}$. *For the PLOE type of actuator failures, $\underline{\rho}_j \leq \rho_{jh} < 1$. There exists a convex compact set $\mathcal{C} \subset \Re^p$ such that $\exists \bar{\theta}$, θ_0, $\|\theta - \theta_0\| \leq \bar{\theta}$ for all $\theta \in \mathcal{C}$. Note that $\underline{b}_j, \bar{b}_j, \underline{\rho}_j, \bar{u}_{kj}, \theta_0, \bar{\theta}$ are all known finite positive constants.*

Assumption 5.2.5. *The subsystem $\dot{\xi} = \Psi(x, \xi) + \Phi(x, \xi)\theta$ is input-to-state stable with respect to x as the input.*

Remark 5.2.3. In Assumption 5.2.4, $\underline{\rho}_j$ denotes the lower bound of ρ_{jh} on the jth actuator in the case of PLOE failures. The knowledge of $\underline{\rho}_j$ will be used in designing the controllers and the estimators. The control objectives can be achieved with such designs no matter TLOE or PLOE failures occur.

5.3 Adaptive Control Design for Failure Compensation

Design u_{cj} in parallel as follows

$$u_{cj} = \frac{\mathrm{sgn}(b_j)}{\beta_j} u_0, \tag{5.5}$$

where u_0 will be generated by performing backstepping technique. Based on (5.5) and the considered failures modeled as in (5.3)-(5.4), the ϱth equation of the plant (5.1) has different forms in failure-free and failure cases.

1) Failure-free Case

$$\dot{x}_\varrho = \varphi_0 + \varphi_\varrho^T \theta + \sum_{j=1}^{m} |b_j| u_0. \tag{5.6}$$

2) Failure Case

We denote T_h for $h \in Z^+$ as the time instants at which the failure pattern of the plant changes. Suppose that during time interval (T_h, T_{h+1}), there are q_h $(1 \le q_h \le m-1)$ actuators $j_1, j_2, \ldots, j_{qh}$ undergoing TLOE type of failures and the failure pattern will be fixed until time T_{h+1}. We have

$$\dot{x}_\varrho = \varphi_0 + \varphi_\varrho^T \theta + \sum_{j \neq j_1, j_2, \ldots, j_{qh}} \rho_{jh} |b_j| u_0 + \sum_{j = j_1, j_2, \ldots, j_{qh}} b_j u_{kj,h} \beta_j. \tag{5.7}$$

From (5.1), (5.6) and (5.7), a unified model of $\dot{x}$ for both cases is constructed as

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_i^T\theta, \qquad i = 1, 2, \ldots, \varrho - 1 \\ \dot{x}_\varrho &= \varphi_0 + \varphi_\varrho^T\theta + bu_0 + \beta^T k, \end{aligned} \tag{5.8}$$

where

$$b = \begin{cases} \sum_{j=1}^{m} |b_j|, & \text{Failure free} \\ \sum_{j\neq j_1,\ldots,j_{q_h}} \rho_{jh}|b_j|, & \text{Failure} \end{cases} \tag{5.9}$$

$$\beta = [\beta_1, \ldots, \beta_m]^T \in R^m, \tag{5.10}$$

$$k = \begin{cases} [0, \cdots, 0]^T \in R^m, & \text{Failure free} \\ \left[0, \cdots, b_{j_1}u_{kj_1,h}, 0, \cdots, b_{j_{qh}}u_{kj_{qh},h}, 0, \cdots, 0\right]^T \in R^m. & \text{Failure} \end{cases} \tag{5.11}$$

Define that $\zeta = \min_{1\leq j\leq m}\{\underline{\rho}_j \underline{b}_j\}$, $k_j = e_{m,j}^T k$, where $e_{i,j}$ denotes the jth coordinate vector in $\Re^i$, with 1 for the jth entry and zero elsewhere. From Assumption 5.2.1, there is at least one actuator free from TLOE failures, we have $b \geq \zeta$. Note that b, k_j for $j = 1, \ldots, m$ are time varying parameters that may jump. We further define $\vartheta = [b, \theta^T, k^T]^T \in \Re^{p+m+1}$, the property of ϑ is established in the following lemma.

Lemma 5.3.1. *The derivative of $\vartheta(t)$ satisfies that $\dot{\vartheta}(t) \in S_1(\mu)$, where $S_1(\mu)$ is defined in Section 5.1.1, i.e.*

$$\int_t^{t+T} \|\dot{\vartheta}(\tau)\| d\tau \leq C_1\mu T + C_2, \qquad \forall t, T \tag{5.12}$$

with $C_1, C_2 > 0$, μ is defined as

$$\mu = \frac{1}{T^*}, \tag{5.13}$$

where T^* denotes the minimum value of time intervals between any successive changes *of failure pattern. C_1 is independent of μ.*

Proof: From Assumption 5.2.4, the upper bounds of the jumping sizes on b and k_j can be calculated. If b or k_j jumps at time instant t, we obtain that

$$|b(t^+) - b(t^-)| \leq \sum_{j=1}^{m} \bar{b}_j - \zeta, \tag{5.14}$$

$$|k_j(t^+) - k_j(t^-)| \leq 2\bar{b}_j \bar{u}_{kj}. \tag{5.15}$$

Define $\bar{K} = \max_{1 \leq j \leq m}\{\sum_{k=1}^{m} \bar{b}_k - \zeta, 2\bar{b}_j \bar{u}_{kj}\}$. Clearly, $\bar{K}$ is finite. Denote T_h, where $h \in Z^+$, as the time instant when the failure pattern changes. The failure pattern will be fixed during time interval (T_h, T_{h+1}). Because of the definition of T^*, $T_{h+1} - T_h \geq T^*$ is satisfied for all T_h, T_{h+1}. We know that $\|\dot{\vartheta}(t)\| \leq \bar{\bar{K}} \sum_h \delta(t - T_h)$, where $\delta(t - T_h)$ is the shifted unit impulse function and $\bar{\bar{K}} = \sqrt{p + m + 1}\bar{K}$. Consider the integral interval $t \sim t + T$ in the following cases:

◇ $T < T^*$ and $T_{h-1} < t \leq T_h \leq t + T < T_{h+1}$, which corresponds to the case that there is one and only one time of failure pattern change during $[t, t + T]$. Thus we have

$$\int_t^{t+T} \|\dot{\vartheta}(\tau)\| d\tau \leq \bar{\bar{K}}. \tag{5.16}$$

◇ $T < T^*$, $t > T_h$ and $t + T < T_{h+1}$, which corresponds to the case that the failure pattern is fixed during $[t, t + T]$. We have

$$\int_t^{t+T} \|\dot{\vartheta}(\tau)\| d\tau = 0. \tag{5.17}$$

◇ $T \geq T^*$, $t \leq T_h$ and $t + T \geq T_{h+N}$, where N is the largest integer that is less than or equal to T/T^*. This refers to the case that there are at most $N + 1$ times of failure pattern changes occurring during $[t, t + T]$. We then obtain

$$\int_t^{t+T} \|\dot{\vartheta}(\tau)\| d\tau = \bar{\bar{K}}(N+1) \leq \bar{\bar{K}}\frac{1}{T^*}T + \bar{\bar{K}}. \tag{5.18}$$

$\diamond$ $T \geq T^*$, $t \leq T_h$ and $t + T < T_{h+N}$, where N is the same as the above case. This refers to the case that there are at most N times of failure pattern changes occurring during $[t, t+T]$. We then have

$$\int_t^{t+T} \|\dot{\vartheta}(\tau)\| d\tau = \bar{\bar{K}}N \leq \bar{\bar{K}}\frac{1}{T^*}T. \tag{5.19}$$

Clearly, the above four cases include all the possibilities of t and $t+T$. From (5.16)-(5.19), if it is defined that $C_1, C_2 = \bar{\bar{K}}$, (5.12) follows and C_1 is independent of μ. Therefore $\dot{\vartheta} \in S_1(\mu)$. Note that μ decreases as T^* increases. $\square$

5.3.1 Design of u_0

This subsection is devoted to constructing u_0 by performing backstepping technique on the model (5.8). We introduce the error variables

$$z_i = x_i - y_r^{(i-1)} - \alpha_{i-1}, \quad i = 1, \ldots, \varrho \tag{5.20}$$

where $\alpha_0 = 0$ and α_i is the stabilizing function generated at the ith step given by,

$$\begin{aligned} \alpha_i &= -z_{i-1} - (c_i + s_i)z_i - w_i^T\hat{\theta} + \sum_{k=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_k}x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}}y_r^{(k)}\right), \\ &\qquad i = 1, \ldots, \varrho - 1 &(5.21)\\ \alpha_\varrho &= \frac{1}{\hat{b}}\bar{\alpha}_\varrho - \frac{1}{\zeta}(c_\varrho + s_\varrho)z_\varrho &(5.22)\\ \bar{\alpha}_\varrho &= -z_{\varrho-1} - \varphi_0 - w_\varrho^T\hat{\theta} - \beta^T\hat{k} + \sum_{k=1}^{\varrho-1}\left(\frac{\partial \alpha_{\varrho-1}}{\partial x_k}x_{k+1} + \frac{\partial \alpha_{\varrho-1}}{\partial y_r^{(k-1)}}y_r^{(k)}\right), &(5.23) \end{aligned}$$

where $\hat{b}$, $\hat{\theta}$ and $\hat{k}$ are the estimates of b, θ and k respectively. w_i and the nonlinear damping functions s_i are designed as

$$w_i = \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_i, \quad i = 1, \ldots, \varrho \tag{5.24}$$

$$s_i = \kappa_i \|w_i\|^2 + g_i \left\| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}^T \right\|^2, \quad i = 1, \ldots, \varrho - 1 \tag{5.25}$$

$$s_\varrho = \kappa_\varrho \left[\|w_\varrho\|^2 + \left| \frac{y_r^{(\varrho)} + \bar{\alpha}_\varrho}{\hat{b}} \right|^2 + \|\beta\|^2 \right] + g_\varrho \left\| \frac{\partial \alpha_{\varrho-1}}{\partial \hat{\theta}}^T \right\|^2 . \tag{5.26}$$

Remark 5.3.1. Similar to the designs in Section 2.2.2, the use of nonlinear damping functions here is to construct a controller such that an input-to-state property of an error system given later in (5.68) with respect to $\tilde{\vartheta}$ and $\dot{\hat{\theta}}$ as the inputs will be established in Section 5.4.

Finally, u_0 is designed as

$$u_0 = \alpha_\varrho + \frac{y_r^{(\varrho)}}{\hat{b}}. \tag{5.27}$$

5.3.2 Design of Parameter Update Law

In this subsection, preliminary design of certain filters is first presented and some boundedness properties of related signals are also established. Then the design of adaptive law involving the details of parameter projection design is provided. Further, the properties of the estimator which are useful in the analysis of system stability and the performance of tracking error in the mean square sense will also be shown.

A. Preliminary Design

Eqn. (5.8) can be written in parametric x-model as

$$\dot{x} = f(x) + F^T(x,u)\vartheta, \tag{5.28}$$

where $f(x) = [x_2, x_3, \ldots, x_\varrho, \varphi_0]^T$ and

$$F^T(x,u) = \begin{bmatrix} 0, & \varphi_1^T, & 0_{1\times m} \\ 0, & \varphi_2^T, & 0_{1\times m} \\ \vdots & \vdots & \vdots \\ u_0, & \varphi_\varrho^T, & \beta^T \end{bmatrix} \in \Re^{\varrho\times(p+m+1)}. \tag{5.29}$$

We introduce two filters

$$\dot{\Omega}^T = A(x,t)\Omega^T + F^T(x,u), \qquad \Omega \in \Re^{(p+m+1)\times\varrho} \tag{5.30}$$

$$\dot{\Omega}_0 = A(x,t)(\Omega_0 + x) - f(x), \qquad \Omega_0 \in \Re^{\varrho} \tag{5.31}$$

where $A(x,t)$ is chosen as

$$A(x,t) = A_0 - \gamma F^T(x,u)F(x,u)P, \tag{5.32}$$

with $\gamma > 0$ and A_0 is an arbitrary constant matrix such that $PA_0 + A_0^T P = -I$, $P = P^T > 0$. We now have the following lemmas.

Lemma 5.3.2. *For a time varying system $\dot{\psi} = A(x(t),t)\psi$, the state transition matrix $\Phi_A(t,t_0)$ satisfies that*

$$\|\Phi_A(t,t_0)\| \le \bar{\bar{k}}_0 e^{-r_0(t-t_0)}, \tag{5.33}$$

where $\bar{\bar{k}}_0$ and r_0 are some positive constants.

Proof: Defining a positive definite quadratic function $V = \psi^T P \psi$. It satisfies that $\dot{V} \leq -\psi^T\psi$ and $\lambda_{\max}(P)\psi^T\psi \leq V \leq \lambda_{\max}(P)\psi^T\psi$. Thus the equilibrium point $\psi = 0$ is exponentially stable from Theorem 4.10 in [94]. Moreover, $\|\Phi_A(t,t_0)\| \leq \bar{\bar{k}}_0 e^{-r_0(t-t_0)}$ for $\bar{\bar{k}}_0, r_0 > 0$ can be shown by following similar procedures in proving Theorem 4.11 in [94]. □

Lemma 5.3.3. *The state Ω of the filter (5.30) satisfies that $\|\Omega\|_\infty \leq C_3$ irrespectively of the boundedness of its input F^T, where C_3 is a positive constant given by*

$$C_3 = \sqrt{\varrho}\max\left\{\|\Omega(0)\|_F, \sqrt{\frac{p+m+1}{2\gamma}}\right\}. \tag{5.34}$$

Proof: Similar to (2.124) in the proof of Lemma 2.2.2, we obtain that

$$\begin{aligned} \frac{d}{dt}\mathrm{tr}\{\Omega P \Omega^T\} &= -\|\Omega\|_F^2 - 2\gamma\left\|FP\Omega^T - \frac{1}{2\gamma}I_{p+m+1}\right\|_F^2 + \frac{1}{2\gamma}\mathrm{tr}\{I_{p+m+1}\} \\ &\leq -\|\Omega\|_F^2 + \frac{p+m+1}{2\gamma}. \end{aligned} \tag{5.35}$$

From (5.35) and the fact that $\lambda_{min}(P)\|\Omega\|_F^2 \leq \mathrm{tr}\{\Omega P \Omega^T\}$, it follows that $\Omega \in \mathcal{L}_\infty$ and

$$\|\Omega\|_\infty \leq \sqrt{\varrho}\|\Omega\|_F \leq \sqrt{\varrho}\max\left\{\|\Omega(0)\|_F, \sqrt{\frac{p+m+1}{2\gamma}}\right\}. \tag{5.36}$$

□

Combining (5.28), (5.31), and defining $\mathcal{Y} = \Omega_0 + x$, we have

$$\dot{\mathcal{Y}} = A\mathcal{Y} + F^T\vartheta. \tag{5.37}$$

Introduce that $\varepsilon = \mathcal{Y} - \Omega^T\vartheta$. From (5.30) and (5.37), the derivative of ε is computed as

$$\begin{aligned}\dot{\varepsilon} &= A\mathcal{Y} + F^T\vartheta - (A\Omega^T + F^T)\vartheta - \Omega^T\dot{\vartheta} \\ &= A\varepsilon - \Omega^T\dot{\vartheta},\end{aligned} \tag{5.38}$$

Then, the following results are obtained.

Lemma 5.3.4.

(i) If μ is finite, ε is bounded;

(ii) $\varepsilon \in S_1(\mu)$ and $\varepsilon \in S_2(\mu)$.

Proof:

- Proof of *(i)*.

The solution of (5.38) is

$$\varepsilon(t) = \Phi_A\varepsilon(0) - \int_0^t \Phi_A(t,\tau)\Omega^T(\tau)\dot{\vartheta}(\tau)d\tau. \tag{5.39}$$

From Lemmas 5.3.2 and 5.3.3, we have

$$\begin{aligned}\|\varepsilon(t)\| &\le \bar{\bar{k}}_0 e^{-r_0 t}\|\varepsilon(0)\| + \bar{\bar{k}}_0\|\Omega\|_\infty \int_0^t e^{-r_0(t-\tau)}\|\dot{\vartheta}(\tau)\|d\tau \\ &= \varepsilon_1 + \varepsilon_2,\end{aligned} \tag{5.40}$$

where $\varepsilon_1 = \bar{\bar{k}}_0 e^{-r_0 t}\|\varepsilon(0)\|$ and $\varepsilon_2 = \bar{\bar{k}}_0\|\Omega\|_\infty \int_0^t e^{-r_0(t-\tau)}\|\dot{\vartheta}(\tau)\|d\tau$ respectively.

From Lemma 5.3.1 and the definition of ε_2, we obtain that

$$\begin{aligned}\varepsilon_2 &= \bar{\bar{k}}_0\|\Omega\|_\infty e^{-r_0 t}\int_0^t e^{r_0\tau}|\dot{\vartheta}(\tau)|d\tau \\ &\le \bar{\bar{k}}_0\|\Omega\|_\infty e^{-r_0 t}\sum_{i=0}^{N}\int_i^{i+1} e^{r_0\tau}|\dot{\vartheta}(\tau)|d\tau \\ &\le \bar{\bar{k}}_0\|\Omega\|_\infty e^{-r_0 t}\sum_{i=0}^{N} e^{r_0(i+1)}\int_i^{i+1}|\dot{\vartheta}(\tau)|d\tau \\ &\le \bar{\bar{k}}_0\|\Omega\|_\infty (C_1\mu + C_2)e^{-r_0 t}\frac{e^{r_0}(e^{r_0 N} - e^{-r_0})}{1 - e^{-r_0}}\end{aligned}$$

$$\leq \bar{\bar{k}}_0\|\Omega\|_\infty \frac{e^{r_0}(C_1\mu + C_2)}{1-e^{-r_0}} = C_4\mu + C_5, \tag{5.41}$$

where

$$C_4 = \frac{\bar{\bar{k}}_0 C_1\|\Omega\|_\infty e^{r_0}}{1-e^{-r_0}}, \quad C_5 = \frac{\bar{\bar{k}}_0 C_2\|\Omega\|_\infty e^{r_0}}{1-e^{-r_0}}, \tag{5.42}$$

Note that $N \leq t \leq N+1$ has been used with N as the largest integer that is less than or equal to t. From (5.41), which is similar to the procedures in proving that $\triangle(t, t_0) \leq c$ on Pages 84-85 in [5], we conclude that ε_2 is bounded provided that μ is finite. Consequently, ε is bounded.

- Proof of *(ii)*.

  By integrating (5.40) over $[t, t+T]$, we have

$$\begin{aligned}
\int_t^{t+T} \|\varepsilon(\tau)\|d\tau &\leq \int_t^{t+T} \bar{\bar{k}}_0 e^{-r_0\tau}\|\varepsilon(0)\|d\tau + \bar{\bar{k}}_0\|\Omega\|_\infty \int_t^{t+T}\int_0^\tau e^{-r_0(\tau-s)}\|\dot{\vartheta}(s)\|dsd\tau \\
&= \frac{\bar{\bar{k}}_0\|\varepsilon(0)\|}{r_0} + \bar{\bar{k}}_0\|\Omega\|_\infty \int_t^{t+T}\left(\int_0^t e^{-r_0(\tau-s)}\|\dot{\vartheta}(s)\|ds\right. \\
&\quad \left. + \int_t^\tau e^{-r_0(\tau-s)}\|\dot{\vartheta}(s)\|ds\right) d\tau \\
&= \frac{\bar{\bar{k}}_0\|\varepsilon(0)\|}{r_0} + \bar{\bar{k}}_0\|\Omega\|_\infty \int_t^{t+T} e^{-r_0\tau}\int_0^t e^{r_0 s}\|\dot{\vartheta}(s)\|dsd\tau \\
&\quad + \bar{\bar{k}}_0\|\Omega\|_\infty \int_t^{t+T} e^{-r_0\tau}\int_t^\tau e^{r_0 s}\|\dot{\vartheta}(s)\|dsd\tau \\
&\leq \frac{\bar{\bar{k}}_0\|\varepsilon(0)\|}{r_0} + \frac{\bar{\bar{k}}_0\|\Omega\|_\infty}{r_0}\int_0^t e^{-r_0(t-s)}\|\dot{\vartheta}(s)\|ds \\
&\quad + \bar{\bar{k}}_0\|\Omega\|_\infty \int_t^{t+T} e^{-r_0\tau}\int_t^\tau e^{r_0 s}\|\dot{\vartheta}(s)\|dsd\tau,
\end{aligned} \tag{5.43}$$

where the last inequality is obtained by using $e^{-r_0 t} - e^{-r_0(t+T)} \leq e^{-r_0 t}$.

From the Proof of *(i)*, we have

$$\int_t^{t+T} \|\varepsilon(\tau)\|d\tau \leq \frac{\bar{\bar{k}}_0\|\varepsilon(0)\| + C_4\mu + C_5}{r_0} + \bar{\bar{k}}_0\|\Omega\|_\infty \int_t^{t+T} e^{-r_0\tau}\int_t^\tau e^{r_0 s}\|\dot{\vartheta}(s)\|dsd\tau. \tag{5.44}$$

By changing the sequence of integration, (5.44) becomes

$$\begin{aligned}\int_t^{t+T} \|\varepsilon(\tau)\| d\tau &\leq \frac{\bar{\bar{k}}_0|\varepsilon(0)| + C_4\mu + C_5}{r_0} + \bar{\bar{k}}_0\|\Omega\|_\infty \int_t^{t+T} e^{r_0 s}\|\dot{\vartheta}(s)\| \int_s^{t+T} e^{-r_0\tau} d\tau ds \\ &\leq \frac{\bar{\bar{k}}_0\|\varepsilon(0)\| + C_4\mu + C_5}{r_0} + \frac{\bar{\bar{k}}_0\|\Omega\|_\infty}{r_0} \int_t^{t+T} \|\dot{\vartheta}(s)\| ds. \qquad (5.45)\end{aligned}$$

From Lemma 5.3.1, we obtain that

$$\int_t^{t+T} \|\varepsilon(\tau)\| d\tau \leq C_6\mu T + C_7, \qquad (5.46)$$

where $C_6 = \bar{\bar{k}}_0 C_1 \|\Omega\|_\infty / r_0$ and

$$C_7 = \frac{\bar{\bar{k}}_0\|\varepsilon(0)\| + C_4\mu + C_5 + \bar{\bar{k}}_0 C_2 \|\Omega\|_\infty}{r_0}. \qquad (5.47)$$

Therefore, $\varepsilon \in S_1(\mu)$.

From (5.40), it follows that $\|\varepsilon\|_\infty \leq \bar{\bar{k}}_0\|\varepsilon(0)\| + C_4\mu + C_5$. By utilizing Hölder's inequality, we obtain that

$$\begin{aligned}\int_t^{t+T} \varepsilon(\tau)^T \varepsilon(\tau) d\tau &\leq \|\varepsilon\|_\infty \int_t^{t+T} \|\varepsilon(\tau)\| d\tau \\ &= \|\varepsilon\|_\infty (C_6\mu T + C_7) \\ &= (C_8\mu^2 + C_9\mu)T + C_{10}, \qquad (5.48)\end{aligned}$$

where $C_8 = C_4C_6$, $C_9 = C_6(\bar{\bar{k}}_0\|\varepsilon(0)\| + C_5) + C_4C_7$, $C_{10} = C_7(\bar{\bar{k}}_0\|\varepsilon(0)\| + C_4\mu + C_5)$. Hence $\varepsilon \in S_2(\mu)$ is concluded. □

B. Design of Adaptive Law

Now we introduce the "prediction" of $\mathcal{Y}$ as $\hat{\mathcal{Y}} = \Omega^T\hat{\vartheta}$, where $\hat{\vartheta} = [\hat{b}, \hat{\theta}^T, \hat{k}^T]^T$. The "prediction error" $\epsilon = \mathcal{Y} - \hat{\mathcal{Y}}$ can be written as

$$\epsilon = \Omega^T \tilde{\vartheta} + \varepsilon, \tag{5.49}$$

where $\tilde{\vartheta} = \vartheta - \hat{\vartheta}$.

Design the update law for $\hat{\vartheta}$ by following standard parameter estimation algorithm [21] as

$$\dot{\hat{\vartheta}} = \mathrm{Proj}\left\{\Gamma\Omega\epsilon\right\}, \qquad \Gamma = \Gamma^T > 0 \tag{5.50}$$

where $\mathrm{Proj}\{\cdot\}$ is a smooth projection operator to ensure that

$$\hat{\vartheta}(t) = (\hat{\vartheta}_1, \ldots, \hat{\vartheta}_{p+m+1})^T \in \Pi_0, \qquad \forall t. \tag{5.51}$$

In (5.51), the set Π_0 is defined similarly as in Example 1 of [105], i.e.

$$\Pi_0 = \left\{ \hat{\vartheta} \left| \begin{array}{ll} |\hat{\vartheta}_i - \nu_i| < \sigma_i, & i = 1, p+2, \ldots, p+m+1 \\ \|\hat{\theta} - \theta_0\| < \bar{\theta}, & \hat{\theta} = [\hat{\vartheta}_2, \ldots, \hat{\vartheta}_{p+1}]^T \end{array} \right. \right\}. \tag{5.52}$$

Note that θ_0 and $\bar{\theta}$ are given in Assumption 5.2.4 and ν_i, σ_i are given as

$$\begin{aligned} \nu_1 &= (\zeta + \sum_{j=1}^{m} \bar{b}_j)/2, \\ \nu_i &= 0, \quad i = p+2, \ldots, p+m+1; \end{aligned} \tag{5.53}$$

$$\begin{aligned} \sigma_1 &= \nu_1 - \zeta, \\ \sigma_i &= \bar{b}_j \bar{u}_{k(i-p-1)}, \quad i = p+2, \ldots, p+m+1. \end{aligned} \tag{5.54}$$

By doing these, $\zeta \le \hat{b} \le \sum_{j=1}^{m} \bar{b}_j$, $|\hat{k}_j| \le \bar{b}_j \bar{u}_{kj}$ and $\hat{\theta} \in \mathcal{C}$ all the time. Based on [105] and [21], the detailed design of projection operator is given below.

Choosing a C^2 function $\mathcal{P}(\hat{\vartheta})$: $\Re^{p+m+1} \to \Re$ as

$$\mathcal{P}(\hat{\vartheta}) = \sum_{i=1,p+2,\ldots,p+m+1} \left| \frac{\hat{\vartheta}_i - \nu_i}{\sigma_i} \right|^q + \left(\frac{\|\hat{\theta} - \theta_0\|}{\bar{\theta}} \right)^q - 1 + \varsigma, \tag{5.55}$$

where $0 < \varsigma < 1$ and $q \geq 2$ are two real numbers. We then define the set Π as

$$\Pi = \left\{ \hat{\vartheta} \middle| \mathcal{P}(\hat{\vartheta}) \leq 0 \right\}. \tag{5.56}$$

Clearly, Π approaches Π_0 as ς decreases and q increases. Similar to $(E.3)$ in [21], we consider the following convex set

$$\Pi_\varsigma = \left\{ \hat{\vartheta} \middle| \mathcal{P}(\hat{\vartheta}) \leq \frac{\varsigma}{2} \right\}, \tag{5.57}$$

which contains Π for the purpose of constructing a smooth projection operator as

$$\mathrm{Proj}(\tau) = \begin{cases} \tau, & \mathcal{P}(\hat{\vartheta}) \leq 0 \ \text{ or } \ \frac{\partial \mathcal{P}}{\partial \vartheta}(\hat{\vartheta})\tau \leq 0 \\ \tau - c(\hat{\vartheta})\Gamma \frac{\frac{\partial \mathcal{P}}{\partial \hat{\vartheta}}(\hat{\vartheta}) \frac{\partial \mathcal{P}}{\partial \hat{\vartheta}}(\hat{\vartheta})^T}{\frac{\partial \mathcal{P}}{\partial \hat{\vartheta}}(\hat{\vartheta})^T \Gamma \frac{\partial \mathcal{P}}{\partial \hat{\vartheta}}(\hat{\vartheta})}\tau, & \text{if not} \end{cases} \tag{5.58}$$

where $\hat{\vartheta}(0) \in \Pi$ and

$$c(\hat{\vartheta}) = \min \left\{ 1, \frac{2\mathcal{P}(\hat{\vartheta})}{\varsigma} \right\}. \tag{5.59}$$

It is helpful to be noted that

$$c(\hat{\vartheta}) = \begin{cases} 0, & \mathcal{P}(\hat{\vartheta}) = 0 \\ 1, & \mathcal{P}(\hat{\vartheta}) = \frac{\varsigma}{2} \end{cases} \tag{5.60}$$

The properties of projection operator (5.58) are rendered in the following lemma.

Lemma 5.3.5.

(i) $\mathit{Proj}(\tau)^T \Gamma^{-1} \mathit{Proj}(\tau) \leq \tau^T \Gamma^{-1} \tau, \quad \forall \hat{\vartheta} \in \Pi_\varsigma$.

(ii) Let $\Gamma(t)$, $\tau(t)$ *be continuously differentiable and* $\dot{\hat{\vartheta}} = \mathit{Proj}(\tau)$, $\hat{\vartheta}(0) \in \Pi_\varsigma$. *Then on its domain of definition, the solution* $\hat{\vartheta}(t)$ *remains in* Π_ς.

(iii) $-\tilde{\vartheta}^T \Gamma^{-1} \mathit{Proj}(\tau) \leq -\tilde{\vartheta} \Gamma^{-1} \tau$, $\forall \hat{\vartheta} \in \Pi_\varsigma, \theta \in \Pi$.

Proof: The proof is similar to the proof of Lemma E.1 in [21]. □

Based on these, we have the following results, which will be useful in the analysis of system stability and the performance of tracking error in the mean square sense.

Lemma 5.3.6. *The estimator (5.50) has the following properties.*

(i) $\epsilon \in S_2(\mu)$*;*

(ii) $\dot{\hat{\vartheta}} \in S_2(\mu)$.

Proof: We define a positive definite function

$$V_\vartheta = \frac{1}{2}\tilde{\vartheta}^T\Gamma^{-1}\tilde{\vartheta}. \tag{5.61}$$

From Lemma 5.3.5 *(iii)* above, we have

$$\begin{aligned}
\dot{V}_\vartheta &= \tilde{\vartheta}^T\Gamma^{-1}(\dot{\vartheta} - \dot{\hat{\vartheta}}) \\
&\leq -\tilde{\vartheta}^T\Gamma^{-1}\left(\Gamma\Omega\epsilon\right) + \tilde{\vartheta}^T\Gamma^{-1}\dot{\vartheta} \\
&= -(\epsilon - \varepsilon)^T\epsilon + \tilde{\vartheta}^T\Gamma^{-1}\dot{\vartheta} \\
&\leq -\epsilon^T\epsilon + |\varepsilon^T\epsilon| + \tilde{\vartheta}^T\Gamma^{-1}\dot{\vartheta}.
\end{aligned} \tag{5.62}$$

- Proof of *(i)*.

By integrating both sides of (5.62) and using Hölder's inequality, we obtain

$$\begin{aligned}
\int_t^{t+T}\dot{V}_\vartheta d\tau &\leq -\int_t^{t+T}\epsilon^T\epsilon d\tau + \|\epsilon\|_\infty\int_t^{t+T}\|\varepsilon\|d\tau + \|\tilde{\vartheta}\|_\infty\|\Gamma^{-1}\|_\infty\int_t^{t+T}\|\dot{\vartheta}\|d\tau \\
&\leq -\int_t^{t+T}\epsilon^T\epsilon d\tau + \|\epsilon\|_\infty(C_6\mu T + C_7) + \|\tilde{\vartheta}\|_\infty\frac{1}{\lambda_{min}(\Gamma)}(C_1\mu T + C_2).
\end{aligned} \tag{5.63}$$

Thus

$$\begin{aligned}\int_t^{t+T} \epsilon(\tau)^T\epsilon(\tau)d\tau &\leq \frac{1}{2\lambda_{min}(\Gamma)}\Big(\tilde{\vartheta}(t)^T\tilde{\vartheta}(t) - \tilde{\vartheta}(t+T)^T\tilde{\vartheta}(t+T)\Big) \\ &\quad +\|\epsilon\|_\infty(C_6\mu T + C_7) + \frac{\|\tilde{\vartheta}\|_\infty}{\lambda_{min}(\Gamma)}(C_1\mu T + C_2) \\ &\leq (C_{11}\mu^2 + C_{12}\mu)T + C_{13}, \end{aligned} \tag{5.64}$$

where $C_{11} = C_8$ and

$$C_{12} = C_9 + \frac{C_1\|\tilde{\vartheta}\|_\infty}{\lambda_{min}(\Gamma)}, \quad C_{13} = C_{10} + \frac{\|\tilde{\vartheta}\|_\infty^2 + 2C_2\|\tilde{\vartheta}\|_\infty}{2\lambda_{min}(\Gamma)}. \tag{5.65}$$

From (iii) of Lemma 5.3.5, $\hat{\vartheta}(t)$ remains in Π_ς if $\hat{\vartheta}(0) \in \Pi_\varsigma$. From Assumption 5.2.4 and the definition of Π_ς, we know that $\vartheta \in \Pi_\varsigma$. Therefore $\tilde{\vartheta}$ is bounded by utilizing the projection operator, $\epsilon \in S_2(\mu)$.

• Proof of *(ii)*.

From *(i)* of Lemma 5.3.5 and Hölder's inequality, we have

$$\begin{aligned}\int_t^{t+T} \dot{\hat{\vartheta}}^T\dot{\hat{\vartheta}}d\tau &\leq \int_t^{t+T} \epsilon^T\Omega^T\Gamma^2\Omega\epsilon \\ &\leq \lambda_{max}(\Gamma)^2\left\|\|\Omega\|_F^2\right\|_\infty \int_t^{t+T}\epsilon^T\epsilon d\tau. \end{aligned} \tag{5.66}$$

Thus from (5.64),

$$\int_t^{t+T} \dot{\hat{\vartheta}}(\tau)^T\dot{\hat{\vartheta}}(\tau)d\tau \leq (C_{14}\mu^2 + C_{15}\mu)T + C_{16}, \tag{5.67}$$

where $C_{1i} = C_{1i-3}\lambda_{max}(\Gamma)^2\|\|\Omega\|_F^2\|_\infty$ for $i = 4, 5, 6$. Therefore, $\dot{\hat{\vartheta}} \in S_2(\mu)$ is concluded. □

5.4 Stability Analysis

We will first prove the input-to-state stability of an error system with $\tilde{\vartheta}$ and $\dot{\hat{\theta}}$ as the inputs. An error system obtained by applying the design procedure (5.20)-(5.27) to system (5.8) is given by

$$\dot{z} = A_z(z, \hat{\vartheta}, t)z + W_\vartheta(z, \hat{\vartheta}, t)^T \tilde{\vartheta} + Q_\theta(z, \hat{\vartheta}, t)^T \dot{\hat{\theta}}, \tag{5.68}$$

where

$$A_z = \begin{bmatrix} -(c_1+s_1) & 1 & 0 & \cdots & 0 \\ -1 & -(c_2+s_2) & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & -\frac{b}{\zeta}(c_\varrho + s_\varrho) \end{bmatrix}, \tag{5.69}$$

$$W_\vartheta^T = \begin{bmatrix} 0 & w_1^T & 0_{1\times m} \\ 0 & w_2^T & 0_{1\times m} \\ \vdots & \vdots & \vdots \\ \frac{\bar{\alpha}_\varrho + y_r^{(\varrho)}}{\hat{b}} & w_\varrho^T & \beta^T \end{bmatrix}, \tag{5.70}$$

$$Q_\theta^T = [0, -\frac{\partial \alpha_1}{\partial \hat{\theta}}, \ldots, \frac{-\partial \alpha_{\varrho-1}}{\partial \hat{\theta}}]^T. \tag{5.71}$$

For the error system (5.68)-(5.71), the following input-to-state property holds.

Lemma 5.4.1. *If $\tilde{\theta}, \tilde{b}, \tilde{k}, \dot{\hat{\theta}} \in \mathcal{L}_\infty$, then $z \in \mathcal{L}_\infty$ and*

$$\|z(t)\| \leq \frac{1}{2\sqrt{c_0}} \left[\frac{1}{\kappa_0}(\|\tilde{\theta}\|_\infty^2 + \|\tilde{b}\|_\infty^2 + \|\tilde{k}\|_\infty^2) + \frac{1}{g_0}\|\dot{\hat{\theta}}\|_\infty^2 \right]^{\frac{1}{2}} + \|z(0)\|e^{-c_0 t}, \tag{5.72}$$

where $\tilde{\theta} = \theta - \hat{\theta}$, $\tilde{b} = b - \hat{b}$ *and* $\tilde{k} = k - \hat{k}$ *and* c_0, κ_0 *and* g_0 *are defined as*

$$c_0 = \min_{1 \le i \le \varrho} c_i, \quad \kappa_0 = \left(\sum_{i=1}^{\varrho} \frac{1}{\kappa_i} \right)^{-1}, \quad g_0 = \left(\sum_{i=1}^{\varrho} \frac{1}{g_i} \right)^{-1}. \tag{5.73}$$

Proof: Along the solutions of (5.68), we compute

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|z\|^2\right) &\le -\sum_{i=1}^{\varrho} c_i z_i^2 - \sum_{i=1}^{\varrho} \left(\kappa_i \|w_i\|^2 + g_i \left\| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}^T \right\|^2 \right) z_i^2 \\
&\quad -\kappa_\varrho \left[\left| \frac{y_r^{(n)} + \bar{\alpha}_n}{\hat{b}} \right|^2 + \|\beta\|^2 \right] z_\varrho^2 + \sum_{i=1}^{\varrho} \left(w_i^T \tilde{\theta} - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} \right) z_i \\
&\quad + \left(\frac{y_r^{(\varrho)} + \bar{\alpha}_\varrho}{\hat{b}} \tilde{b} \right) z_\varrho + \beta^T \tilde{k} z_\varrho \\
&\le -c_0 \|z\|^2 - \sum_{i=1}^{\varrho} \kappa_i \left\| w_i z_i - \frac{1}{2\kappa_i} \tilde{\theta} \right\|^2 - \sum_{i=1}^{\varrho} g_i \left\| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}^T z_i \right. \\
&\quad \left. + \frac{1}{2g_i} \dot{\hat{\theta}} \right\|^2 - \kappa_\varrho \left[\left(\frac{y_r^{(\varrho)} + \bar{\alpha}_\varrho}{\hat{b}} \right) z_\varrho - \frac{1}{2\kappa_n} \tilde{b} \right]^2 \\
&\quad -\kappa_\varrho \left\| \beta z_\varrho - \frac{1}{2\kappa_\varrho} \tilde{k} \right\|^2 + \left(\sum_{i=1}^{n} \frac{1}{4\kappa_i} \right) \|\tilde{\theta}\|^2 \\
&\quad + \left(\sum_{i=1}^{n} \frac{1}{4g_i} \right) \|\dot{\hat{\theta}}\|^2 + \frac{1}{4\kappa_n} (\tilde{b}^2 + \|\tilde{k}\|^2) \\
&\le -c_0 |z|^2 + \frac{1}{4} \left[\frac{1}{\kappa_0} (\|\tilde{\theta}\|^2 + \tilde{b}^2 + |\tilde{k}|^2) + \frac{1}{g_0} \|\dot{\hat{\theta}}\|^2 \right].
\end{aligned} \tag{5.74}$$

Let $v = \|z\|^2$ and $\mathscr{L} = \left[\frac{1}{\kappa_0} (\|\tilde{\theta}\|^2 + \tilde{b}^2 + \|\tilde{k}\|^2) + \frac{1}{g_0} \|\dot{\hat{\theta}}\|^2 \right]^{1/2}$, it follows that

$$\dot{v} \le -2c_0 v + \frac{1}{2} \mathscr{L}^2. \tag{5.75}$$

If $\tilde{\theta}$, $\tilde{b}$, $\tilde{k}$ and $\dot{\hat{\theta}} \in \mathcal{L}_\infty$, $\mathscr{L} \in \mathcal{L}_\infty$, then $v \in \mathcal{L}_\infty$ and

$$\begin{aligned}
v(t) &\le v(0) e^{-2c_0 t} + \frac{1}{4c_0} \|\mathscr{L}\|_\infty^2 \\
&\le v(0) e^{-2c_0 t} + \frac{1}{4c_0} \left[\frac{1}{\kappa_0} (\|\tilde{\theta}\|_\infty^2 + \|\tilde{b}\|_\infty^2 + \|\tilde{k}\|_\infty^2) + \frac{1}{g_0} \|\dot{\hat{\theta}}\|_\infty^2 \right].
\end{aligned} \tag{5.76}$$

□

We are now at the position to present the main results of this chapter in the following theorem.

Theorem 5.4.1. *Consider the closed-loop adaptive system consisting of the nonlinear plant (5.1), the controller (5.5), (5.27), the parameter update law (5.50). Irrespective of actuator failures modeled in (5.3)-(5.4) under Assumptions 5.2.1-5.2.5, we have the following results.*

(i) All the signals of the closed-loop system are ensured bounded provided that μ is finite.

(ii) The tracking error $z_1 = y - y_r$ is small in the mean square sense that $z_1(t) \in S_2(\mu)$.

(iii) The asymptotic tracking can be achieved for a finite number of failures, i.e. $\lim_{t\to\infty} z_1(t) = 0$.

Proof:

- Proof of *(i)*.

$\tilde{\vartheta}$ is bounded by utilizing the projection operator in (5.50). From Lemma 5.3.3, Ω is bounded. From Lemma 5.3.4, ε is bounded as long as μ is finite. Thus from (5.49), ϵ is bounded and so is $\dot{\hat{\vartheta}}$. Thus all the conditions in Lemma 5.4.1 are satisfied, then $z(t) \in \mathcal{L}_\infty$. From Assumption 5.2.2, the definition of z_i in (5.20) and the design of α_i in (5.21)-(5.23), $x(t) \in \mathcal{L}_\infty$. From Assumption 5.2.5, ξ is bounded with respect to $x(t)$ as the input. α_ρ is then bounded. From (5.27) and (5.5), control signals u_{cj} for $j = 1, 2, \ldots, m$ are also bounded. The closed-loop stability is then established.

- Proof of *(ii)*.

Rewrite (5.68) as

$$\dot{z} = \bar{A}_z(z, \hat{\vartheta}, t)z + \bar{W}_\vartheta(z, \hat{\vartheta}, t)^T \tilde{\vartheta} + Q_\theta(z, \hat{\vartheta}, t)^T \dot{\hat{\theta}}, \tag{5.77}$$

where Q_θ is the same as in (5.71) and

$$\bar{A}_z = \begin{bmatrix} -(c_1+s_1) & 1 & 0 & \cdots & 0 \\ -1 & -(c_2+s_2) & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & -\frac{\hat{b}}{\zeta}(c_\varrho+s_\varrho) \end{bmatrix}, \tag{5.78}$$

$$\bar{W}_\vartheta^T = \begin{bmatrix} 0 & w_1^T & 0_{1\times m} \\ 0 & w_2^T & 0_{1\times m} \\ \vdots & \vdots & \vdots \\ u_0 & w_n^T & \beta^T \end{bmatrix}. \tag{5.79}$$

Introduce the state χ^T as

$$\dot{\chi}^T = \bar{A}_z \chi^T + \bar{W}_\vartheta^T. \tag{5.80}$$

Similarly to Lemma 5.3.2, we obtain that $\|\Phi_{\bar{A}_z}(t,t_0)\| \le \bar{\bar{k}}_1 e^{-r_1(t-t_0)}$ where $\bar{\bar{k}}_1, r_1$ are positive contants. Thus $\chi \in \mathcal{L}_\infty$ is shown from (5.80) and the boundedness of $\bar{W}_\vartheta$. By defining η as

$$\eta = z - \chi^T \tilde{\vartheta}, \tag{5.81}$$

we will show *(ii)* in two steps. In Step 1, $\eta \in S_2(\mu)$ will be proved. Then we will establish that $\chi^T\tilde{\vartheta} \in S_2(\mu)$ in Step 2. Thus from (5.81), $z(t) \in S_2(\mu)$ will be obtained.

Step 1.

Computing the derivative of η gives that

$$\begin{aligned}
\dot{\eta} &= \dot{z} - \dot{\chi}^T\tilde{\vartheta} - \chi^T(\dot{\vartheta} - \dot{\hat{\vartheta}}) \\
&= \bar{A}_z z + \bar{W}_\vartheta^T \vartheta + Q_\theta^T \dot{\hat{\theta}} - (\bar{A}_z \chi^T + \bar{W}_\vartheta^T)\tilde{\vartheta} - \chi^T(\dot{\vartheta} - \dot{\hat{\vartheta}}) \\
&= \bar{A}_z \eta + Q_\theta^T \dot{\hat{\theta}} + \chi^T \dot{\hat{\vartheta}} - \chi^T \dot{\vartheta}. \qquad (5.82)
\end{aligned}$$

The solution of (5.82) is

$$\begin{aligned}
\eta(t) &= \Phi_{\bar{A}_z}(t,0)\eta(0) + \int_0^t \Phi_{\bar{A}_z}(t,\tau) Q_\theta(z(\tau), \hat{\vartheta}(\tau), \tau)^T \dot{\hat{\theta}}(\tau) d\tau \\
&\quad + \int_0^t \Phi_{\bar{A}_z}(t,\tau)\chi(\tau)^T \dot{\hat{\vartheta}}(\tau) d\tau - \int_0^t \Phi_{\bar{A}_z}(t,\tau)\chi(\tau)^T \dot{\vartheta}(\tau) d\tau. \qquad (5.83)
\end{aligned}$$

Since Q_θ and χ are bounded, we have

$$\begin{aligned}
\|\eta(t)\| &\leq \bar{\bar{k}}_1 e^{-r_1 t}\|\eta(0)\| + \bar{\bar{k}}_1 \|Q_\theta\|_\infty \int_0^t e^{-r_1(t-\tau)} \|\dot{\hat{\theta}}(\tau)\| d\tau + \bar{\bar{k}}_1 \|\chi\|_\infty \\
&\quad \times \int_0^t e^{-r_1(t-\tau)} \|\dot{\hat{\vartheta}}(\tau)\| d\tau + \bar{\bar{k}}_1 \|\chi\|_\infty \int_0^t e^{-r_1(t-\tau)} \|\dot{\vartheta}\| d\tau \\
&= \eta_1 + \eta_2, \qquad (5.84)
\end{aligned}$$

where η_1 and η_2 are defined respectively as

$$\eta_1 = \bar{\bar{k}}_1 \left(\|Q_\theta\|_\infty \int_0^t e^{-r_1(t-\tau)} \|\dot{\hat{\theta}}(\tau)\| d\tau + \|\chi\|_\infty \int_0^t e^{-r_1(t-\tau)} \|\dot{\hat{\vartheta}}(\tau)\| d\tau \right) \qquad (5.85)$$

$$\eta_2 = \bar{\bar{k}}_1 \left(e^{-r_1 t}\|\eta(0)\| + \|\chi\|_\infty \int_0^t e^{-r_1(t-\tau)} |\dot{\vartheta}| d\tau \right). \qquad (5.86)$$

By following similar procedures to the proof of Lemma 5.3.4 *(ii)*, it can be shown that $\eta_2 \in S_2(\mu)$. Now we show that $\eta_1 \in S_2(\mu)$. Using Schwartz inequality, we obtain

$$\begin{aligned}
\eta_1 &\leq \bar{\bar{k}}_1 \left[\|Q_\theta\|_\infty \left(\int_0^t e^{-r_1(t-\tau)} d\tau \right)^{\frac{1}{2}} \left(\int_0^t e^{-r_1(t-\tau)} \|\dot{\hat{\theta}}(\tau)\|^2 d\tau \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \|\chi\|_\infty \left(\int_0^t e^{-r_1(t-\tau)} d\tau \right)^{\frac{1}{2}} \left(\int_0^t e^{-r_1(t-\tau)} \|\dot{\hat{\vartheta}}(\tau)\|^2 d\tau \right)^{\frac{1}{2}} \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\bar{\bar{k}}_1}{\sqrt{r_1}}\left[\|Q_\theta\|_\infty\left(\int_0^t e^{-r_1(t-\tau)}\|\dot{\hat{\theta}}(\tau)\|^2 d\tau\right)^{\frac{1}{2}}\right. \\
&\left.\quad +\|\chi\|_\infty^2\left(\int_0^t e^{-r_1(t-\tau)}\|\dot{\hat{\vartheta}}(\tau)\|^2 d\tau\right)^{\frac{1}{2}}\right] \qquad (5.87)
\end{aligned}
$$

By squaring both sides of (5.87) and integrating it over $[t, t+T]$, we have

$$
\begin{aligned}
\int_t^{t+T} \eta_1^2 d\tau \quad &\leq \quad \frac{2\bar{\bar{k}}_1^2}{r_1}\left[\|Q_\theta\|_\infty^2 \int_t^{t+T}\int_0^\tau e^{-r_1(\tau-s)}\|\dot{\hat{\theta}}(s)\|^2 ds d\tau\right. \\
&\left.\quad +\|\chi\|_\infty^2 \int_t^{t+T}\int_0^\tau e^{-r_1(\tau-s)}\|\dot{\hat{\vartheta}}(s)\|^2 ds d\tau\right] \qquad (5.88)
\end{aligned}
$$

Similar to the proof of Lemma 5.3.4, we obtain that

$$
\begin{aligned}
\int_t^{t+T} \eta_1^2 d\tau \quad &\leq \quad \frac{2\bar{\bar{k}}_1^2\|Q_\theta\|_\infty^2}{r_1}\left(\frac{1}{r_1}\int_0^t e^{-r_1(t-s)}\|\dot{\hat{\theta}}(s)\|^2 ds + \int_t^{t+T} e^{r_1 s}\right. \\
&\quad \left.\times\|\dot{\hat{\theta}}(s)\|^2 \int_s^{t+T} e^{-r_1\tau} d\tau ds\right) + \frac{2\bar{\bar{k}}_1^2\|\chi\|_\infty^2}{r_1} \\
&\quad \times\left(\frac{1}{r_1}\int_0^t e^{-r_1(t-s)}\|\dot{\hat{\vartheta}}(s)\|^2 ds + \int_t^{t+T} e^{r_1 s}\|\dot{\hat{\vartheta}}(s)\|^2 \int_s^{t+T} e^{-r_1\tau} d\tau ds\right) \\
&\leq \quad \frac{2\bar{\bar{k}}_1^2}{r_1^2}(\|Q_\theta\|_\infty^2 + \|\chi\|_\infty^2)\frac{e^{r_1}(C_{14}\mu^2 + C_{15}\mu + C_{16})}{1 - e^{-r_1}} \\
&\quad + \frac{2\bar{\bar{k}}_1^2\|Q_\theta\|_\infty^2}{r_1^2}\int_t^{t+T}\|\dot{\hat{\theta}}(s)\|^2 ds + \frac{2\bar{\bar{k}}_1^2\|\chi\|_\infty^2}{r_1^2}\int_t^{t+T}\|\dot{\hat{\vartheta}}(s)\|^2 ds. \qquad (5.89)
\end{aligned}
$$

From Lemma 5.3.6 *(ii)*, $\dot{\hat{\vartheta}} \in S_2(\mu)$, thus $\dot{\hat{\theta}} \in S_2(\mu)$ and $\eta_1 \in S_2(\mu)$. From (5.84), $\eta \in S_2(\mu)$ where we have used the fact that $\|\eta\|^2 \leq 2(\eta_1^2 + \eta_2^2)$.

Step 2.

From (5.49), Lemma 5.3.4 *(ii)* and Lemma 5.3.6 *(i)*, we have $\Omega^T\tilde{\vartheta} \in S_2(\mu)$. Thus our main task in this step is to show that $\Omega^T\tilde{\vartheta} \in S_2(\mu)$ implies $\chi^T\tilde{\vartheta} \in S_2(\mu)$. The procedures are quite similar to those in the proof of Theorem 2.2.2.

For simplicity of presentation, we represent the following system by an operator

$T_{A_i}[\cdot]$,

$$\dot{\zeta}_i = A_i(t)\zeta_i + u, \tag{5.90}$$

where $A_i : R_+ \to R^{\varrho\times\varrho}$ is continuous, bounded, and exponentially stable. For example, $\zeta_1 = T_A[F^T\tilde{\vartheta}]$ if $\dot{\zeta}_1 = A\zeta_1 + F^T\tilde{\vartheta}$, where A is defined in (5.32).

Since the stability of the closed-loop system has been shown, F is bounded. Similarly to the proof of $\eta \in S_2(\mu)$, $\zeta_1 - \Omega^T\tilde{\vartheta} = T_A[F^T\tilde{\vartheta}] - T_A[F^T]\tilde{\vartheta} \in S_2(\mu)$ can also be shown. From $\Omega^T\tilde{\vartheta} \in S_2(\mu)$, it follows that $\zeta_1 \in S_2(\mu)$.

We now show that $\zeta_2 = T_{\bar{A}_z}[\bar{W}_\vartheta\tilde{\vartheta}] \in S_2(\mu)$, where $\bar{A}_z$ is the same as in (5.78). From (5.24), (5.29) and (5.79), we have

$$\bar{W}_\vartheta = MF^T, \tag{5.91}$$

where

$$M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{\partial\alpha_1}{\partial x_1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\frac{\partial\alpha_{\varrho-1}}{\partial x_1} & \cdots & -\frac{\partial\alpha_{\varrho-1}}{\partial x_{\varrho-1}} & 1 \end{bmatrix}. \tag{5.92}$$

Note that M has a similar form to that in (2.139). By following similar analysis to show that $\zeta_2 \in \mathcal{L}_2$ in the proof of Theorem 2.2.2, we can obtain that $\zeta_2 = T_{\bar{A}_z}[MF^T\tilde{\vartheta}] \in S_2(\mu)$. Moreover, $T_{\bar{A}_z}[MF^T\tilde{\vartheta}] - T_{\bar{A}_z}[MF^T]\tilde{\vartheta} \in S_2(\mu)$ can also be shown by following the similar procedures in the proof of $\eta \in S_2(\mu)$. We then obtain that $T_{\bar{A}_z}[MF^T]\tilde{\vartheta} \in S_2(\mu)$. Thus $T_{\bar{A}_z}[\bar{W}_\vartheta^T]\tilde{\vartheta} = \chi^T\tilde{\vartheta} \in S_2(\mu)$. From $z = \chi^T\tilde{\vartheta} + \eta$, $z \in S_2(\mu)$. Hence $z_1 \in S_2(\mu)$ follows.

From Lemma 5.3.1, we know that $\mu = \frac{1}{T^*}$ where T^* is the minimum time interval between two successive changes of failure pattern. Clearly, μ can be very small for a large T^*.

- Proof of *(iii)*.

For the case with a finite number of failures, the result that $\dot{\vartheta}(t) \in S_1(\mu)$ will be changed to that $\dot{\vartheta}(t) \in \mathcal{L}_1$. Through the similar procedures in the analysis above, $z(t) \in \mathcal{L}_2$ will be followed instead of $z(t) \in S_2(\mu)$. From (5.68), $\dot{z}(t) \in \mathcal{L}_\infty$. Together with the facts that $z(t) \in \mathcal{L}_\infty$, from the corollary of Barbalat lemma as provided in Appendix A, asymptotic tracking will be achieved, i.e. $\lim_{t\to\infty} z_1(t) = 0$. □

Remark 5.4.1. With our proposed scheme, all the closed-loop signals are ensured bounded even if there are infinite number of TLOE and PLOE actuator failures as long as the time interval between two successive changes of failure pattern is bounded below by an arbitrary positive number. Such a condition is less restrictive than that conjectured in [55]. Moreover, from the established tracking error performance in (ii) of Theorem 5.4.1, we see that the frequency of changing failure patterns will affect the tracking performance. In fact for a designed adaptive controller with a given set of design parameters and initial conditions, the less frequent the failure pattern changes, the better the tracking performance is.

Our results can also be extended to the following situations, even though they are not the focus of the chapter.

Remark 5.4.2.

- As far as the 'offline' repair situation (namely actuators may repeatedly fail, be removed from the loop and then put back into the loop after recovery) is concerned, stability result cannot be established by using the existing tuning function schemes. This is because when the actuators change only from a working mode to an 'offline' repairing mode infinitely many times, the parameter b in (5.8) will experience infinite number of jumps which will lead to instability if they are not carefully handled. However, system stability can be ensured with our proposed scheme if Assumption 5.2.1-5.2.5 are satisfied and the time intervals between two successive changes of failure pattern are bounded below by an arbitrary positive number.

- The results achieved in this chapter can also be applied to time varying systems. The derivatives of the unknown parameters are not required to be bounded like many other results on adaptive backstepping control of time varying systems such as [106–108]. On the other hand, the parameter μ being finite is the only condition to achieve the boundedness of all closed-loop signals in this chapter. In contrast to previous results on adaptive control of systems with possible jumping parameters such as in [34,109], μ is not required to satisfy that $\mu \in (0, \mu^*]$ where μ^* is a function of the bounds of unknown system parameters as well as design parameters. Thus the results here are more general than those in [34, 109].

- Similar to the comments in [56], more general failures modeled like $u_j(t) = u_{kj,h} + \sum_{i=1}^{n_j} d_{jh,i} \cdot f_{jh,i}(t)$ for $j = 1, 2, \ldots, m$, with smooth functions $f_{jh,i}(t)$ and unknown constants $u_{kj,h}$, $d_{jh,i}$ can also be handled with our proposed scheme. However different from [56], $f_{jh,i}(t)$ can be allowed unknown with our proposed scheme, as long as it varies in such a way that $\dot{\vartheta} \in S_1(\mu)$ is still satisfied.

5.5 Simulation Studies

5.5.1 A Numerical Example

In this subsection, a numerical example is considered to illustrate the ability of the proposed scheme in compensating for infinite number of relatively frequent actuator failures. To carry out a comparison, the results by using a tuning function scheme in Section 4.3, which can be regarded as a representative of currently available results in the area of adaptive failure compensation for nonlinear systems, are also presented.

We consider a system with dual actuators

$$\begin{aligned}
\dot{\chi} &= f_0(\chi) + f(\chi)\theta + \sum_{j=1}^{2} b_j g_j(\chi) u_j \\
y &= \chi_2,
\end{aligned} \tag{5.93}$$

where the state $\chi \in \Re^3$,

$$f_0 = \begin{bmatrix} -\chi_1 \\ \chi_3 \\ \chi_2\chi_3 \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ \chi_2^2 \\ \frac{1-e^{-\chi_3}}{1+e^{-\chi_3}} \end{bmatrix}, \tag{5.94}$$

and

$$g_1 = g_2 = \left[\frac{2+\chi_3^2}{1+\chi_3^2}, 0, 1\right]^T, \tag{5.95}$$

which is modeled similarly to Example 13.6 in [94]. As discussed in [94], to transform (5.93) into the form of (5.1), we choose $[\xi, x_1, x_2]^T = T(\chi) = [\phi(\chi), \chi_2, \chi_3]^T$ where $\phi(\chi) = -\chi_1 + \chi_3 + \tan^{-1}\chi_3$. We have $\phi(0) = 0$ and

$$\frac{\partial \phi}{\partial \chi} g_j(\chi) = \frac{\partial \phi}{\partial \chi_1} \cdot \frac{2+\chi_3^2}{1+\chi_3^2} + \frac{\partial \phi}{\partial \chi_3} = 0. \tag{5.96}$$

Since the equation $T(\chi) = s$ for any $s \in \Re^3$ has a unique solution, the mapping $T(\chi)$ is a global diffeomorphism. Thus, the transformed system below

$$\begin{aligned}
\dot{\xi} &= -\xi + x_2 + \tan^{-1} x_2 + \frac{2+x_2^2}{1+x_2^2}\left(x_1 x_2 + \frac{1-e^{-x_2}}{1+e^{-x_2}}\theta\right) \\
\dot{x}_1 &= x_2 + x_1^2\theta \\
\dot{x}_2 &= x_1 x_2 + \frac{1-e^{-x_2}}{1+e^{-x_2}}\theta + \sum_{j=1}^{2} b_j u_j
\end{aligned} \tag{5.97}$$

is defined globally. Because of the boundedness of functions $\tan^{-1}(x_2)$, $\frac{2+x_2^2}{1+x_2^2}$ and $\frac{1-e^{-x_2}}{1+e^{-x_2}}$, it is concluded that $\dot{\xi} = -\xi + \eta(x_1, x_2, \theta)$ is ISS where $\eta = x_2 + \tan^{-1} x_2 + \frac{2+x_2^2}{1+x_2^2}(x_1 x_2 + \frac{1-e^{-x_2}}{1+e^{-x_2}}\theta)$. Thus Assumption 5.2.5 is satisfied.

The considered failure case is modeled as

$$u_1(t) = u_{k1,h}, \quad t \in [hT^*, (h+1)T^*), \ h = 1, 3, \ldots, \tag{5.98}$$

which implies that the output of first actuator (u_1) is stuck at $u_1 = u_{k1,h}$ at every hT^* seconds and is back to normal operation at every $(h+1)T^*$ seconds until the next failure occurs.

The following information is unknown in the designs.

$$\theta = 2, \quad b_1 = 1, \quad b_2 = 1.1, \quad u_{k1,h} = 5, \quad T^* = 5. \tag{5.99}$$

However, we know that $b_1, b_2 > 0$ and

$$1 \le \theta \le 3, \quad 0.8 \le |b_1| \le 1.4, \quad 0.6 \le |b_2| \le 2 \tag{5.100}$$

$$0.5 \le \rho_{jh} \le 1, \quad |u_{kj,h}| \le 6, j = 1, 2. \tag{5.101}$$

The reference signal $y_r = \sin(t)$.

We firstly design the adaptive controllers following the procedures in Section 4.3. In simulation, the initial states and estimates are all set as 0 except that $\chi_2(0) = 0.4$ and $\hat{\theta}(0) = 1$. The design parameters are chosen as $c_1 = c_2 = 5$, $\Gamma = 3$, $\Gamma_\kappa = 3 \times I_3$. The performances of the tracking error and control inputs (u_1, u_2) versus time are given in Fig.5.1 and Fig. 5.2, respectively. It can be seen that after 150 seconds, the magnitudes of the error signal grows larger and larger. Growing phenomenon can also be observed from the control signal even more rapidly. It seems that the boundedness of the signals cannot be guaranteed in this case.

We then adopt the proposed modular scheme to redesign the adaptive con-

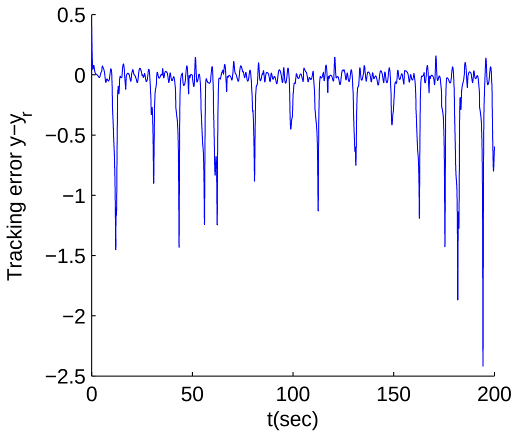


Figure 5.1: Tracking error $y(t)-y_r(t)$ with the scheme in Section 4.3 when $T^* = 5$ seconds.

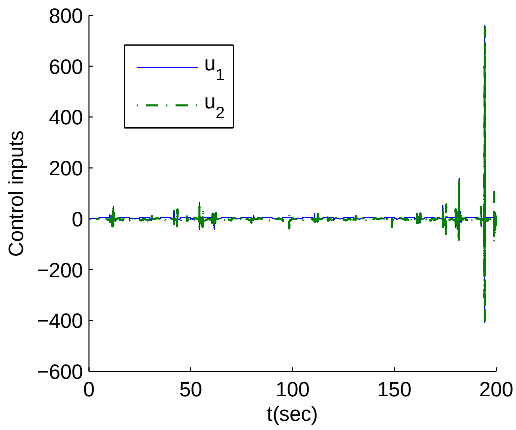


Figure 5.2: Control u_1 and u_2 with scheme in Section 4.3 when $T^* = 5$ seconds.

trollers. $\hat{b}(0) = 1.5$, the rest of the initial states and estimates are kept the same as in the tuning function design. The design parameters c_1, c_2 are fixed at $c_1 = c_2 = 5$, while other design parameters are chosen as

$$\zeta = 0.3, \quad \kappa_1 = \kappa_2 = 3, \quad g_2 = 3, \quad \Gamma = 40 \times I_4, \tag{5.102}$$

$$\nu_1 = \frac{0.3+3.4}{2}, \quad \nu_3 = \nu_4 = 0, \tag{5.103}$$

$$\sigma_1 = 3.4 - 2 = 1.4, \quad \sigma_3 = \sigma_4 = 12, \tag{5.104}$$

$$\theta_0 = \frac{1+3}{2} = 2, \quad \bar{\theta} = 2, \quad q = 40, \quad \varsigma = 0.01. \tag{5.105}$$

The performances of tracking error and control signals in this case are given in Fig. 5.3-5.4. Apart from these, the states χ_1 and χ_3, parameter estimates are also plotted in Fig. 5.5-5.6. Obviously, the boundedness of all the signals is now ensured.

To show how T^* affects the tracking performance when the proposed design scheme is utilized, we set $T^* = 25$ seconds. The performance of tracking error

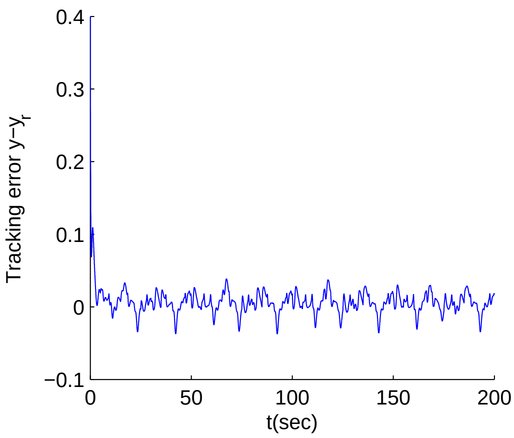


Figure 5.3: Tracking error $y(t) - y_r(t)$ with proposed scheme when $T^* = 5$ seconds.

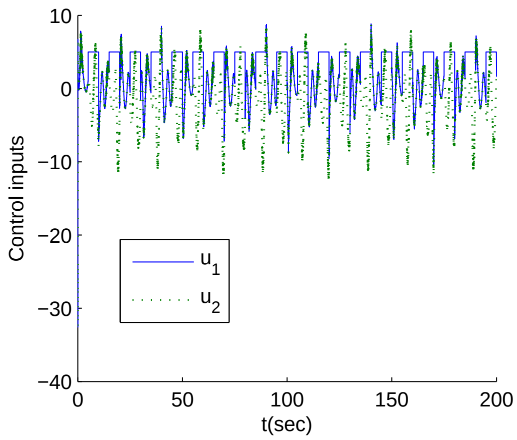


Figure 5.4: Control u_1 and u_2 with proposed scheme when $T^* = 5$ seconds.

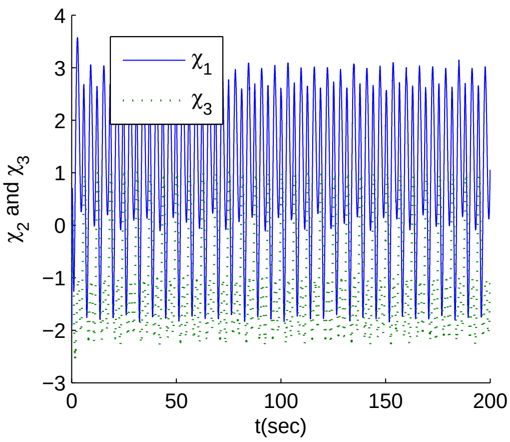


Figure 5.5: χ_1 and χ_3 with proposed scheme when $T^* = 5$ seconds.

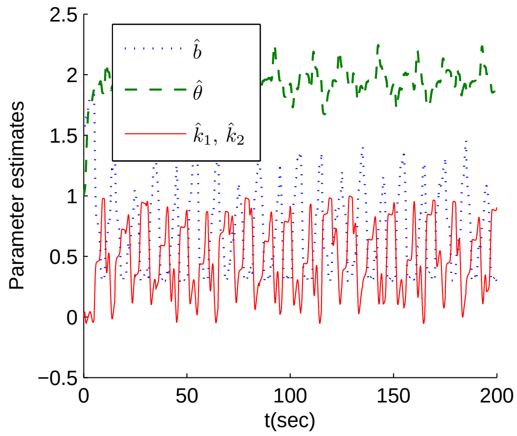


Figure 5.6: Parameter estimates with proposed scheme when $T^* = 5$ seconds.

is now shown in Fig. 5.7. Comparing Fig. 5.7 and Fig. 5.3, better tracking error performance in the mean square sense is observed.

Now we consider the case that there are finite number of failures by setting $T^* = 5$ seconds and there will be no failure for $t > 100$ seconds. The performance of tracking error with our proposed scheme is given in Fig. 5.8, which shows that the tracking error will converge to zero asymptotically in this case.

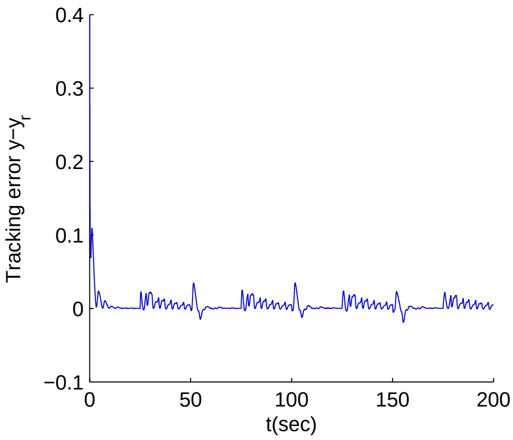


Figure 5.7: Tracking error $y(t)-y_r(t)$ with proposed scheme when $T^*=25$ seconds.

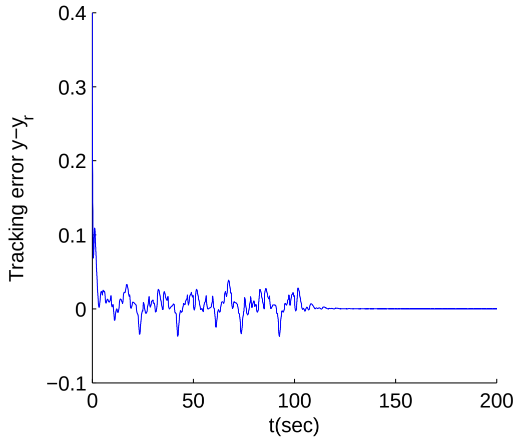


Figure 5.8: Tracking error $y(t)-y_r(t)$ with finite number of failures when the proposed scheme is applied.

5.5.2 Application to An Aircraft System

In this subsection, we apply the proposed scheme to accommodate infinite number of PLOE and TLOE actuator failures for the twin otter aircraft longitudinal nonlinear dynamics model as described in (4.83). In simulation, the aircraft parameters in use are set the same as in Section 4.5 except for $d_1 = 6$, $d_2 = 4$.

As discussed in [54], (4.83) can be transformed into the form of (5.1), i.e.

$$\begin{aligned} \dot{\chi}_3 &= \chi_4 \\ \dot{\chi}_4 &= \varphi(\chi)^T\bar{\bar{\theta}} + \sum_{i=1}^{2} b_i\chi_1^2 u_j \\ \dot{\xi} &= \Psi(\xi, x) + \Phi(\xi, x)\bar{\bar{\theta}} \end{aligned} \tag{5.106}$$

where $\bar{\bar{\theta}} \in R^4$ is an unknown constant vector, $\varphi(\chi) = [\chi_1^2\chi_2, \chi_1^2\chi_2^2, \chi_1^2, \chi_1^2\chi_4]^T$, $x = [\chi_3, \chi_4]^T$. The failure case considered in this example is modeled as

$$\begin{matrix} u_1(t) = u_{k1,h} \\ u_2 = \rho_{2h}u_{c2} \end{matrix}, \quad t \in [hT^*, (h+1)T^*), \quad h = 1, 3, \ldots, \tag{5.107}$$

which implies that at every hT^* seconds, the output of the 1st actuator (u_1) is stuck at $u_1 = u_{k1,h}$ and the 2nd actuator loses $(1-\rho_{2h})$ percent of its effectiveness. While at every $(h+1)T^*$ seconds, both actuators are back to normal operation until the next failure occurs.

In simulation, we choose that $u_{k1,h} = 0.4$, $\rho_{2h} = 30\%$ and $T^* = 10$ seconds, which and the parameters in (5.106) are all unknown in the designs. However, we know that b_1, b_2 in (5.106) are both negative and

$$\|\bar{\bar{\theta}}\| \leq 0.02,\ 0.01 \leq |b_1| \leq 0.02,\ 0.005 \leq |b_2| \leq 0.01, \tag{5.108}$$

$$0.2 \leq \rho_{jh} \leq 1,\ |u_{kj,h}| \leq 1. \tag{5.109}$$

The reference signal $y_r = 0.1\sin(0.05t)$. The initial states and estimates are all set as 0 except that $\chi(0) = [85, 0, 0.03, 0]^T$, $\hat{b}(0) = 0.01$. The design parameters are chosen as

$$\xi = 0.001,\ c_1 = c_2 = 1,\ \kappa_2 = 10^{-6},\ \Gamma = 0.1 \times I_7, \tag{5.110}$$

$$\nu_1 = \frac{0.03 + 0.001}{2},\quad \sigma_1 = 0.03 - 0.001, \tag{5.111}$$

$$\theta_0 = [0, 0, 0, 0]^T,\quad \bar{\theta} = 0.02, \tag{5.112}$$

$$\nu_6 = \nu_7 = 0,\quad \sigma_6 = 0.02,\quad \sigma_7 = 0.01, \tag{5.113}$$

$$q = 20,\quad \varsigma = 0.01 \tag{5.114}$$

The performances of tracking error, velocity, attack angle, pitch rate and control u_1, u_2 are given in Fig. 5.9-5.13, respectively. It can be seen that all the signals are bounded and the tracking error is small in the mean square sense.

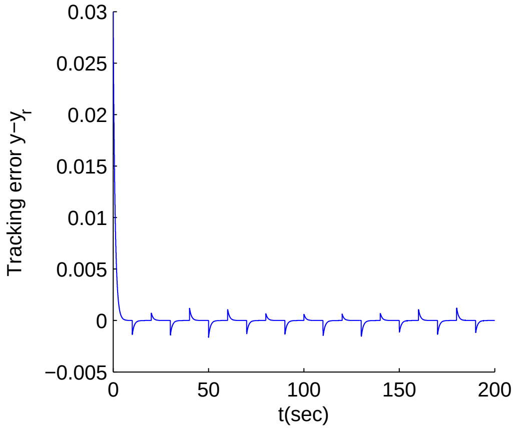


Figure 5.9: Tracking error $y(t) - y_r(t)$.

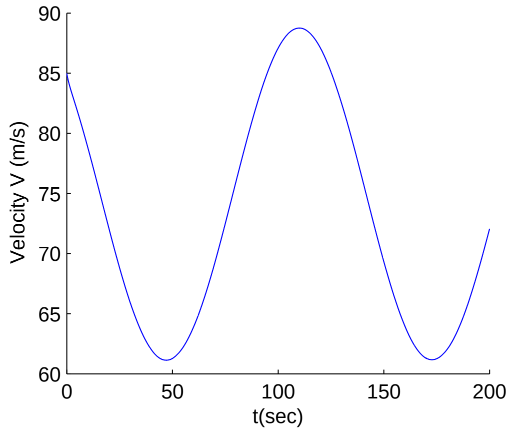


Figure 5.10: Velocity V.

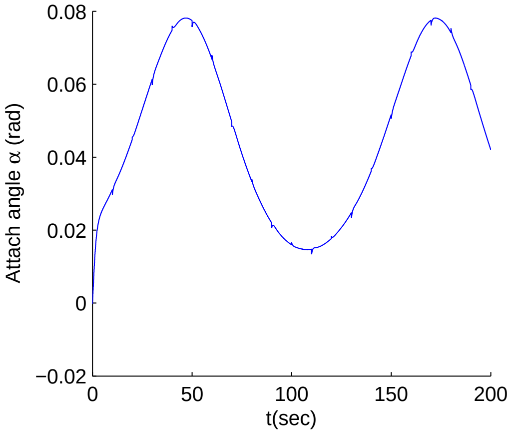


Figure 5.11: Attach angle α.

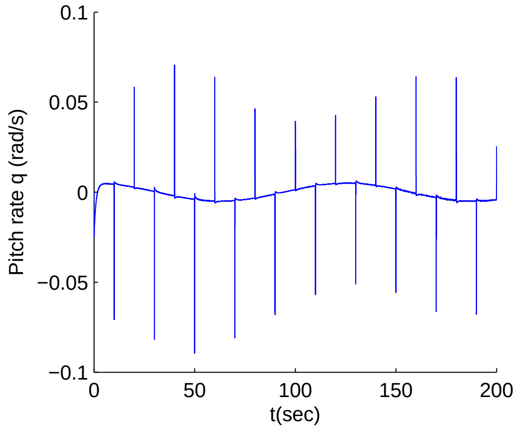


Figure 5.12: Pitch rate q.

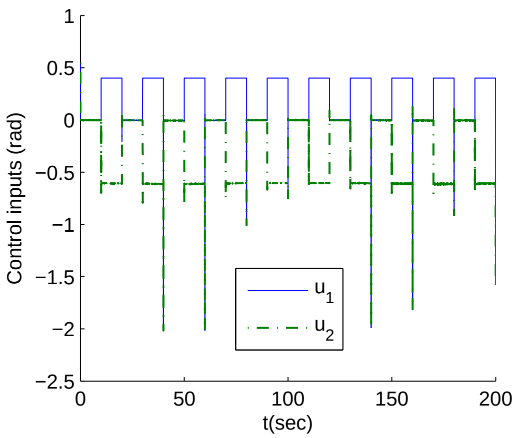


Figure 5.13: Control inputs (elevator angle (rad)) u_1 and u_2.

5.6 Conclusion

In this chapter, the problem of adaptive control of uncertain nonlinear systems in the presence of infinite number actuator failures are addressed. It has been proved that the boundedness of all closed-loop signals can be ensured by adopting the proposed scheme, provided that the time interval between two successive changes of failure pattern is bounded below by an arbitrary positive number. From the established performance of tracking error in the mean square sense, it is shown that the less frequent the failure pattern changes, the better the tracking performance is. Moreover, the tracking error can converge to zero asymptotically in the case with finite number of failures. In simulation studies, the ability of the proposed scheme to compensate for infinite number of relatively frequent failures is compared with a tuning function design scheme through a numerical example. The effectiveness of the proposed scheme is also shown on an aircraft system through simulation.

Part II

Decentralized Adaptive Stabilization

Chapter 6

Decentralized Adaptive Stabilization of Interconnected Systems

So far there is still no result available for backstepping based decentralized adaptive stabilization of unknown systems with interactions directly depending on subsystem inputs, even though such interactions commonly exist in practice. In this chapter, we provide solutions to this problem by considering both input and output dynamic interactions. Each local controller, designed simply based on the model of each subsystem by using the standard adaptive backstepping technique without any modification, only employs local information to generate control signals. It is shown that the designed decentralized adaptive backstepping controllers can globally stabilize the overall interconnected system asymptotically. The $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms of the system outputs are also established as functions of design parameters. This implies that the transient system performance can be adjusted by choosing suitable design parameters.

6.1 Introduction

In the control of uncertain complex interconnected systems, decentralized adaptive control technique is an efficient and practical strategy to be employed for many reasons such as ease of design, familiarity and so on. However, simplicity of the design makes the analysis of the overall designed system quite difficult. Thus the obtained results with rigorous analysis are still limited. Based on conventional adaptive approach, several results on global stability and steady state tracking were reported, see for examples [81,85–87,110,111]. However, transient performance is not ensured and non-adjustable by changing design parameters due to the methods used.

Since backstepping technique was proposed, it has been widely used to design adaptive controllers for uncertain systems [21]. This technique has a number of advantages over the conventional approaches such as providing a promising way to improve the transient performance of adaptive systems by tuning design parameters. Because of such advantages, research on decentralized adaptive control using backstepping technique has also received great attention. In [28], the first result on decentralized adaptive control using such a technique was reported without restriction on subsystem relative degrees. More general class of systems with the consideration of unmodeled dynamics was studied in [29, 32]. In [88, 89], nonlinear interconnected systems were addressed. In [90, 112], decentralized adaptive stabilization for nonlinear systems with dynamic interactions depending on subsystem outputs or unmodeled dynamics is studied. In [91], systems with non-smooth hysteresis nonlinearities and higher order nonlinear interactions were considered and in [92] results for stochastic nonlinear systems were established. More recently, a result on backstepping adaptive tracking was established in [93]. However, except for [29,32,112], all the results are only applicable to systems with interaction effects bounded by static functions of subsystem outputs. This is restrictive as it is a kind of matching condition in the sense that the effects of all the unmodeled interactions to a local subsystem must be in the range space of the output of this subsystem. Note that in [29], the transient performance of the adaptive systems is not established. In [32,112], the interactions are not directly depending on subsystem inputs.

In practice, an interconnected system unavoidably has dynamic interactions involving both subsystem inputs and outputs. Especially, dynamic interactions directly depending on subsystem inputs commonly exist. For example, the non-zero off-diagonal elements of a transfer function matrix represent such interactions. So far there is still no result reported to control systems with interactions directly depending on subsystem inputs even for the case of static input interactions by using the backstepping technique. This is due to the challenge of handling the input variables and their derivatives of all subsystems during the recursive design steps. In this chapter, we will use the backstepping design approach in [21] to design decentralized adaptive controllers for both linear and nonlinear systems having such interactions. It is shown that the designed controllers can globally stabilize the overall interconnected system asymptotically. This reveals that the standard backstepping controller offers an additional advantage to conventional adaptive controllers in term of its robustness against unmodeled dynamics and interactions. For conventional adaptive controllers without any modification, they are non-robust as shown by counter examples in [113]. Besides global stability, the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms of the system outputs are also shown to be bounded by functions of design parameters. Thus the transient system performance is tunable by adjusting design parameters. To achieve these results, two key techniques are used in our analysis. Firstly, we transform the dynamic interactions from subsystem inputs to dynamic interactions from subsystem states. Secondly, we introduce two dynamic systems associated with interaction dynamics. In this way, the effects of unmodeled interactions are bounded by static functions of the state variables of subsystems. To clearly illustrate our approach, we will start with linear systems involving block diagram manipulation. Then the obtained results are generalized to nonlinear systems.

6.2 Decentralized Adaptive Control of Linear Systems

6.2.1 Modeling of Linear Interconnected Systems

To show our ideas, we first consider linear systems consisting of N interconnected subsystems described in (6.1),

$$\begin{aligned} y(t) &= \begin{bmatrix} G_1(p)+\nu_{11}H_{11}(p) & \nu_{12}H_{12}(p) & \dots & \nu_{1N}H_{1N}(p) \\ \nu_{21}H_{21}(p) & G_2(p)+\nu_{22}H_{22}(p) & \dots & \nu_{2N}H_{2N}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{N1}H_{N1}(p) & \nu_{N2}H_{N2}(p) & \dots & G_N(p)+\nu_{NN}H_{NN}(p) \end{bmatrix} \\ &\cdot u(t) + \begin{bmatrix} \mu_{11}\Delta_{11}(p) & \mu_{12}\Delta_{12}(p) & \dots & \mu_{1N}\Delta_{1N}(p) \\ \mu_{21}\Delta_{21}(p) & \mu_{22}\Delta_{22}(p) & \dots & \mu_{2N}\Delta_{2N}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{N1}\Delta_{N1}(p) & \mu_{N2}\Delta_{N2}(p) & \dots & \mu_{NN}\Delta_{NN}(p) \end{bmatrix} y(t), \end{aligned} \tag{6.1}$$

where $u \in \Re^N$ and $y \in \Re^N$ are inputs and outputs respectively, p denotes the differential operator $\frac{d}{dt}$, $G_i(p)$, $H_{ij}(p)$ and $\Delta_{ij}(p)$, $i, j = 1, \dots, N$, are rational functions of p, ν_{ij} and μ_{ij} are positive scalars. With p replaced by s, the corresponding $G_i(s), H_{ij}(s)$ and $\Delta_{ij}(s)$ are the transfer functions of each local subsystem and interactions, respectively.

A block diagram including the ith and jth subsystems is shown in Figure 6.1.

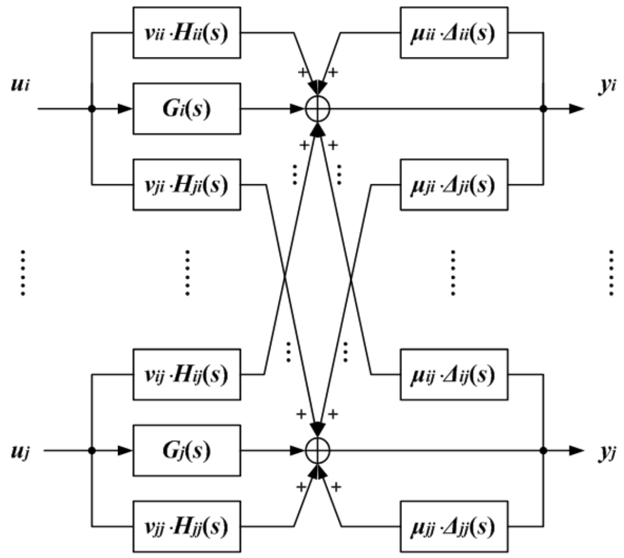


Figure 6.1: Block diagram including the ith and jth subsystems

Remark 6.2.1. $\nu_{ij}H_{ij}(p)u_j(t)$ and $\mu_{ij}\Delta_{ij}(p)y_j(t)$ denote the dynamic interactions from the input and output of the jth subsystem to the ith subsystem for $j \neq i$, or un-modelled dynamics of the ith subsystem for $j = i$ with ν_{ij} and μ_{ij} indicating the strength of the interactions or unmodeled dynamics. Such interactions are rather general. However there is no result on decentralized backstepping adaptive control applicable to interactions directly from the inputs when using the backstepping technique.

For the system, we have the following assumptions.

Assumption 6.2.1. *For each subsystem,*

$$G_i(s) = \frac{B_i(s)}{A_i(s)} = \frac{b_{i,m_i}s^{m_i} + \cdots + b_{i,1}s + b_{i,0}}{s^{n_i} + a_{i,n_i-1}s^{n_i-1} + \cdots + a_{i,1}s + a_{i,0}} \tag{6.2}$$

where $a_{i,j}, j = 0, \ldots, n_i - 1$ and $b_{i,k}, k = 0, \ldots, m_i$ are unknown constants, $B_i(s)$ is a Hurwitz polynomial. The order n_i, the sign of b_{i,m_i} and the relative degree $\rho_i(= n_i - m_i)$ are known;

Assumption 6.2.2. *For all $i, j = 1, \ldots, N$, $\Delta_{ij}(s)$ is stable, strictly proper and has a unity high frequency gain, and $H_{ij}(s)$ is stable with a unity high frequency gain and its relative degree is larger than ρ_j.*

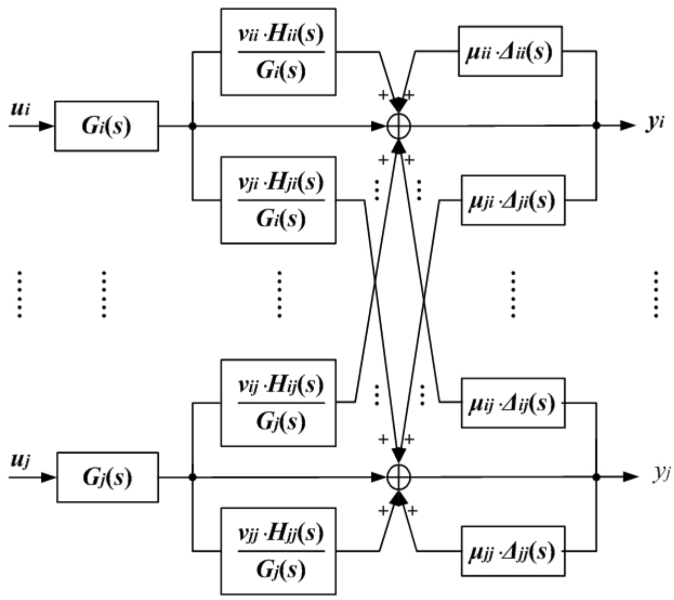


Figure 6.2: Transformed block diagram of Figure 6.1

The block diagram in Figure 6.1 can be transformed to Figure 6.2. Clearly, the ith subsystem has the following state space realization:

$$\dot{x}_i = A_i x_i - a_i x_{i,1} + \begin{bmatrix} 0 \\ b_i \end{bmatrix} u_i \tag{6.3}$$

$$y_i = x_{i,1} + \sum_{j=1}^{N} \nu_{ij} \frac{H_{ij}(p)}{G_j(p)} x_{j,1} + \sum_{j=1}^{N} \mu_{ij} \Delta_{ij}(p) y_j, \tag{6.4}$$

where

$$A_i = \begin{bmatrix} 0_{n_i-1} & I_{n_i-1} \\ 0 & 0^T_{n_i-1} \end{bmatrix},$$

$$a_i = [a_{i,n_i-1}, \ldots, a_{i,0}]^T, \; b_i = [b_{i,m_i}, \ldots, b_{i,0}]^T \tag{6.5}$$

and $0_{n_i-1} \in \Re^{(n_i-1)}$. In the design of a local controller for the ith subsystem, we only consider transfer function $G_i(s)$, i.e,

$$\begin{aligned} \dot{x}_i &= A_i x_i - a_i x_{i,1} + \begin{bmatrix} 0 \\ b_i \end{bmatrix} u_i & (6.6) \\ y_i &= x_{i,1}, \quad \text{for} \quad i = 1, \dots, N. & (6.7) \end{aligned}$$

But in analysis, we will also take into account the effects of the unmodeled interactions and subsystem unmodeled dynamics, i.e.

$$\sum_{j=1}^{N} \nu_{ij} \frac{H_{ij}(p)}{G_j(p)} x_{j,1} + \sum_{j=1}^{N} \mu_{ij} \Delta_{ij}(p) y_j. \tag{6.8}$$

Remark 6.2.2. It is clear that the effect of the dynamic interactions or unmodeled dynamics given in (6.8) cannot be bounded by functions of the outputs $y_j, j = 1, 2, \dots, N$, as assumed in the previous work. Instead, based on the given assumptions, it satisfies,

$$\begin{aligned} & \left| \sum_{j=1}^{N} \nu_{ij} \frac{H_{ij}(p)}{G_j(p)} x_{j,1} + \sum_{j=1}^{N} \mu_{ij} \Delta_{ij}(p) y_j \right| \\ \leq \quad & c_{0,i} + \sum_{j=1}^{N} c_{1,ij} \sup_{0 \leq \tau \leq t} |x_{j,1}(\tau)| + \sum_{j=1}^{N} c_{2,ij} \sup_{0 \leq \tau \leq t} |y_j(\tau)| \text{ for } i = 1, \dots, N \quad (6.9) \end{aligned}$$

for some constants $c_{0,i}, c_{1,ij}$, and $c_{2,ij}$. The above bound involves infinite memory of state $x_{j,1}$ depending on inputs u_j and outputs y_j, which makes the analysis of decentralized backstepping adaptive control systems difficult. This is the main reason why there is still no result available for such a class of systems, due to the requirement of changing coordinates and handling the input variables and their derivatives during the recursive design steps.

Note that in our analysis given in Section 6.2.4, bound (6.9) will not be used. Instead, we will consider signals generated from two dynamic systems related to interactions or unmodeled dynamics to bound the effect.

Our problem is formulated to design decentralized controllers only using local signals to ensure the stability of the overall interconnected system and regulate all the subsystem outputs to zeros. The system transient performance should also be adjustable by changing design parameters in certain sense.

6.2.2 Design of local filters

Since the full states of system are not measurable in our problem, the decentralized adaptive controllers are required to be designed based on output feedback. Note that we only present the decentralized adaptive controllers designed using the standard backstepping technique in [21], without giving the details. Firstly, a local filter using only local input and output is designed to estimate the states of each unknown local system as follows:

$$\dot{\lambda}_i = A_{i,0}\lambda_i + e_{n_i,n_i}u_i \tag{6.10}$$

$$\dot{\eta}_i = A_{i,0}\eta_i + e_{n_i,n_i}y_i \tag{6.11}$$

$$v_{i,k} = (A_{i,0})^k\lambda_i, \qquad k = 0, \ldots, m_i \tag{6.12}$$

$$\xi_{i,n_i} = -(A_{i,0})^{n_i}\eta_i \tag{6.13}$$

where $A_{i,0} = A_i - k_i(e_{n_i,1})^T$, the vector $k_i = [k_{i,1}, \ldots, k_{i,n_i}]^T$ is chosen so that the matrix $A_{i,0}$ is Hurwitz, and $e_{i,k}$ denotes the kth coordinate vector in $\Re^i$. Hence there exists a P_i such that $P_iA_{i,0} + A_{i,0}P_i^T = -I_{n_i}$, $P_i = P_i^T > 0$. With these designed filters our state estimate is given by

$$\hat{x}_i = \xi_{i,n_i} + \Omega_i^T \theta_i \tag{6.14}$$

where

$$\begin{aligned} \theta_i^T &= [b_i^T, a_i^T] & (6.15)\\ \Omega_i^T &= [v_{i,m_i}, \ldots, v_{i,1}, v_{i,0}, \Xi_i] & (6.16)\\ \Xi_i &= -[(A_{i,0})^{n_i-1}\eta_i, \ldots, A_{i,0}\eta_i, \eta_i] & (6.17) \end{aligned}$$

Note that

$$\begin{aligned} \dot{\xi}_{i,n_i} &= -(A_{i,0})^{n_i}(A_{i,0}\eta_i + e_{n_i,n_i}y_i) = A_{i,0}\xi_{i,n_i} + k_i y_i & (6.18)\\ \dot{\Xi}_i &= -[(A_{i,0})^{n_i-1}\dot{\eta}_i, \ldots, A_{i,0}\dot{\eta}_i, \dot{\eta}_i] = -[(A_{i,0})^{n_i-1}, \ldots, A_{i,0}, I_{n_i}](A_{i,0}\eta_i + e_{n_i,n_i}y_i) & \\ &= A_{i,0}\Xi_i - I_{n_i}y_i & (6.19)\\ \dot{v}_{i,k} &= A_{i,0}v_{i,k} + e_{n_i,n_i-k}u_i, \quad k = 0, \ldots, m_i & (6.20) \end{aligned}$$

Then from (6.14), the derivative of $\hat{x}_i$ is given as

$$\begin{aligned} \dot{\hat{x}}_i &= \dot{\xi}_{i,n_i} + \dot{\Omega}_i^T \theta_i \\ &= A_{i,0}\xi_{i,n_i} + k_i y_i + A_{i,0}[v_{i,m_i}, \ldots, v_{i,1}, v_{i,0}, \Xi_i]\theta_i - I_{n_i}y_i a_i + [0, b_i^T]^T u_i \\ &= A_{i,0}\hat{x}_i - (a_i - k_i)y_i + [0, b_i^T]^T u_i & (6.21) \end{aligned}$$

From (6.3) and (6.21), the state estimation error $\epsilon_i = x_i - \hat{x}_i$ satisfies

$$\dot{\epsilon}_i = A_{i,0}\epsilon_i + (a_i - k_i)\left(\sum_{j=1}^{N} \nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N} \mu_{ij}\Delta_{ij}(s)y_j\right) \tag{6.22}$$

Now we replace (6.3) with a new system, whose states depend on those of filters (6.10)-(6.13) and thus are available for control design, as follows:

$$\begin{aligned}
\dot{y}_i &= b_{i,m_i} v_{i,(m_i,2)} + \xi_{i,(n_i,2)} + \bar{\delta}_i^T \theta_i + \epsilon_{i,2} \\
&\quad + (s + a_{i,n_i-1}) \left(\sum_{j=1}^{N} \nu_{ij} \frac{H_{ij}(s)}{G_j(s)} x_{j,1} + \sum_{j=1}^{N} \mu_{ij} \Delta_{ij}(s) y_j \right) &(6.23) \\
\dot{v}_{i,(m_i,q)} &= v_{i,(m_i,q+1)} - k_{i,q} v_{i,(m_i,1)}, \quad q = 2, \ldots, \rho_i - 1 &(6.24) \\
\dot{v}_{i,(m_i,\rho_i)} &= v_{i,(m_i,\rho_i+1)} - k_{i,\rho_i} v_{i,(m_i,1)} + u_i &(6.25)
\end{aligned}$$

where

$$\begin{aligned}
\delta_i^T &= [v_{i,(m_i,2)}, v_{i,(m_i-1,2)}, \ldots, v_{i,(0,2)}, \Xi_{i,2} - y_i (e_{n_i,1})^T] &(6.26) \\
\bar{\delta}_i^T &= [0, v_{i,(m_i-1,2)}, \ldots, v_{i,(0,2)}, \Xi_{i,2} - y_i (e_{n_i,1})^T] &(6.27)
\end{aligned}$$

and $v_{i,(m_i,2)}, \epsilon_{i,2}, \xi_{i,(n_i,2)}, \Xi_{i,2}$ denote the second entries of $v_{i,m_i}, \epsilon_i, \xi_{i,n_i}, \Xi_i$ respectively.

Remark 6.2.3. The output signals $\lambda_i, \eta_i, v_{i,k}, \xi_{i,n_i}$ of filters (6.10)-(6.13) are available for feedback. They are also used to generate an estimate $\hat{x}_i$ of system states x_i in (6.14), with an estimation error given by (6.22). The error will converge to zero in the absence of interactions and unmodeled dynamics. However, the estimate $\hat{x}_i$ is not used in the controller design because it involves unknown parameter vector θ_i which is unavailable. But the state estimation error in (6.22) will be considered in system analysis, as it may not converge to zero unconditionally due to its dependence on interactions and unmodeled dynamics in our case. A block diagram is given in Figure 6.3 to show the signal flow of the filters to the controller of the ith subsystem.

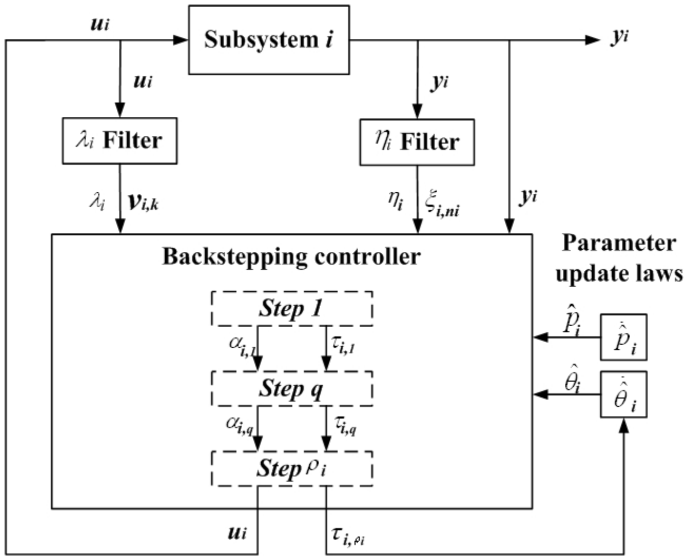


Figure 6.3: Control block diagram.

6.2.3 Design of Decentralized Adaptive Controllers

As usual in backstepping approach [21], the following change of coordinates is made.

$$z_{i,1} = y_i \tag{6.28}$$

$$z_{i,q} = v_{i,(m_i,q)} - \alpha_{i,q-1}, \quad q = 2, 3, \ldots, \rho_i \tag{6.29}$$

To illustrate the controller design procedures, we now give a brief description on the first step.

Step 1: From (6.23), (6.28) and (6.29), we have

$$\begin{aligned} \dot{z}_{i,1} &= b_{i,m_i}(z_{i,2} + \alpha_{i,1}) + \xi_{i,(n_i,2)} + \bar{\delta}_i^T \theta_i + \epsilon_{i,2} \\ &\quad + (s + a_{i,n_i-1}) \left(\sum_{j=1}^{N} \nu_{ij} \frac{H_{ij}(s)}{G_j(s)} x_{j,1} + \sum_{j=1}^{N} \mu_{ij} \Delta_{ij}(s) y_j \right) \end{aligned} \tag{6.30}$$

The virtual control law $\alpha_{i,1}$ is designed as

$$\alpha_{i,1} = \hat{p}_i \bar{\alpha}_{i,1} \quad (6.31)$$

$$\bar{\alpha}_{i,1} = -c_{i1} z_{i,1} - l_{i1} z_{i,1} - \xi_{i,(n_i,2)} - \bar{\delta}_i^T \hat{\theta}_i \quad (6.32)$$

where c_{i1}, l_{i1} are positive constants, $\hat{p}_i$ is an estimate of $p_i = 1/b_{i,m_i}$ and $\hat{\theta}_i$ is an estimate of θ_i. Note that

$$\begin{aligned} b_{i,m_i}\alpha_{i,1} &= b_{i,m_i}\hat{p}_i\bar{\alpha}_{i,1} \\ &= \bar{\alpha}_{i,1} - b_{i,m_i}\tilde{p}_i\bar{\alpha}_{i,1} \quad (6.33) \\ \bar{\delta}_i^T\tilde{\theta}_i + b_{i,m_i}z_{i,2} &= \bar{\delta}_i^T\tilde{\theta}_i + \tilde{b}_{i,m_i}z_{i,2} + \hat{b}_{i,m_i}z_{i,2} \\ &= \bar{\delta}_i^T\tilde{\theta}_i + (v_{i,(m_i,2)} - \alpha_{i,1})(e_{(n_i+m_i+1),1})^T\tilde{\theta}_i \\ &\quad + \hat{b}_{i,m_i}z_{i,2} \\ &= (\delta_i^T - \hat{p}_i\bar{\alpha}_{i,1}e_{n_i+m_i+1,1})^T\tilde{\theta}_i + \hat{b}_{i,m_i}z_{i,2} \quad (6.34) \end{aligned}$$

where $\hat{b}_{i,m_i}$ is an estimate of b_{i,m_i}, $\tilde{b}_{i,m_i} = b_{i,m_i} - \hat{b}_{i,m_i}$, $\tilde{p}_i = p_i - \hat{p}_i$ and $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$. Then we have

$$\begin{aligned} \dot{z}_{i,1} &= -c_{i1}z_{i,1} - l_{i1}z_{i,1} - b_{i,m_i}\tilde{p}_i\bar{\alpha}_{i,1} + \hat{b}_{i,m_i}z_{i,2} + \epsilon_{i,2} + (\delta_i - \hat{p}_i\bar{\alpha}_{i,1}e_{n_i+m_i+1,1})^T\tilde{\theta}_i \\ &\quad + (s + a_{i,n_i-1})\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)y_j\right) \quad (6.35) \end{aligned}$$

We now define a function V_{i1} as

$$V_{i1} = \frac{1}{2}z_{i,1}^2 + \frac{1}{l_{i1}}\epsilon_i{}^T P_i \epsilon_i + \frac{1}{2}\tilde{\theta}_i^T\Gamma_i^{-1}\tilde{\theta}_i + \frac{|b_{i,m_i}|}{2\gamma_i'}\tilde{p}_i^2 \quad (6.36)$$

where Γ_i is a positive definite design matrix and $\gamma_i^{'}$ is a positive design parameter. Then

$$\begin{aligned}\dot{V}_{i1} &= -c_{i,1}z_{i,1}^2 - \frac{l_{i1}}{2}(z_{i,1})^2 + \hat{b}_{i,m_i}z_{i,1}z_{i,2} - |b_{i,m_i}|\tilde{p}_i\frac{1}{\gamma_i^{'}}[\gamma_i^{'}\mathrm{sgn}(b_{i,m_i})\bar{\alpha}_{i,1}z_{i,1} + \dot{\hat{p}}_i] \\ &\quad +\tilde{\theta}_i^T\Gamma_i^{-1}[\Gamma_i(\delta_i - \hat{p}_i\bar{\alpha}_{i,1}e_{n_i+m_i+1,1})z_{i,1} - \dot{\hat{\theta}}_i] - \frac{l_{i1}}{2}(z_{i,1})^2 + \epsilon_{i,2}z_{i,1} - \frac{1}{l_{i1}}\|\epsilon_i\|^2 \\ &\quad +z_{i,1}(s + a_{i,n_i-1})\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)z_{j,1}\right) - \frac{2}{l_{i1}}(a_i - k_i)^T \\ &\quad \times P_i\epsilon_i\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)z_{j,1}\right) \qquad (6.37)\end{aligned}$$

To handle the unknown indefinite $\tilde{p}_i, \tilde{\theta}_i$-terms in (6.37), we choose the update law of $\hat{p}$ and a tuning function $\tau_{i,1}$ as

$$\dot{\hat{p}}_i = -\gamma_i^{'}\mathrm{sgn}(b_{i,m_i})\bar{\alpha}_{i,1}z_{i,1} \qquad (6.38)$$

$$\tau_{i,1} = (\delta_i - \hat{p}_i\bar{\alpha}_{i,1}e_{n_i+m_i+1,1})z_{i,1} \qquad (6.39)$$

It follows that

$$\begin{aligned}\dot{V}_{i1} &\leq -c_{i,1}(z_{i,1})^2 - \frac{l_{i1}}{2}(z_{i,1})^2 - \frac{1}{2l_{i1}}\|\epsilon_i\|^2 + \hat{b}_{i,m_i}z_{i,1}z_{i,2} + \tilde{\theta}_i^T\Gamma_i^{-1}[\Gamma_i\tau_{i,1} - \dot{\hat{\theta}}_i] \\ &\quad +z_{i,1}(s + a_{i,n_i-1})\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)z_{j,1}\right) \\ &\quad -\frac{2}{l_{i1}}(a_i - k_i)^T P_i\epsilon_i\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)z_{j,1}\right) \qquad (6.40)\end{aligned}$$

After going through design steps q for $q = 2, \ldots, \rho_i$ as in [21], we have the ith local controller

$$u_i = \alpha_{i,\rho_i} - v_{i,(m_i,\rho_i+1)} \qquad (6.41)$$

where $\alpha_{i,1}$ is designed in (6.31) and

$$\alpha_{i,2} = -\hat{b}_{i,m_i} z_{i,1} - \left[c_{i2} + l_{i2} \left(\frac{\partial \alpha_{i,1}}{\partial y_i} \right)^2 \right] z_{i,2} + \bar{B}_{i,2} + \frac{\partial \alpha_{i,1}}{\partial \hat{p}_i} \dot{\hat{p}}_i + \frac{\partial \alpha_{i,1}}{\partial \hat{\theta}_i} \Gamma_i \tau_{i,2} \tag{6.42}$$

$$\begin{aligned} \alpha_{i,q} &= -z_{i,(q-1)} - \left[c_{iq} + l_{iq} \left(\frac{\partial \alpha_{i,(q-1)}}{\partial y_i} \right)^2 \right] z_{i,q} + \bar{B}_{i,q} + \frac{\partial \alpha_{i,(q-1)}}{\partial \hat{p}_i} \dot{\hat{p}}_i \\ &\quad + \frac{\partial \alpha_{i,(q-1)}}{\partial \hat{\theta}_i} \Gamma_i \tau_{i,q} - \left(\sum_{k=2}^{q-1} z_{i,k} \frac{\partial \alpha_{i,(k-1)}}{\partial \hat{\theta}_i} \right) \Gamma_i \frac{\partial \alpha_{i,(q-1)}}{\partial y_i} \delta_i, \quad q = 3, \ldots, \rho_i \end{aligned} \tag{6.43}$$

$$\begin{aligned} \bar{B}_{i,q} &= \frac{\partial \alpha_{i,(q-1)}}{\partial y_i} (\xi_{i,(n_i,2)} + \delta_i^T \hat{\theta}_i) + \frac{\partial \alpha_{i,(q-1)}}{\partial \eta_i} (A_{i,0} \eta_i + e_{n_i,n_i} y_i) + k_{i,q} v_{i,(m_i,1)} \\ &\quad + \sum_{j=1}^{m_i+q-1} \frac{\partial \alpha_{i,(q-1)}}{\partial \lambda_{i,j}} (-k_{i,j} \lambda_{i,1} + \lambda_{i,(j+1)}), \quad q = 2, \ldots, \rho_i,\ i = 1, \ldots, N \end{aligned} \tag{6.44}$$

where c_{iq}, l_{iq} are positive constants. With $\tau_{i,1}$ in (6.39) , other tuning functions $\tau_{i,q}$ for $q = 2, \ldots, \rho_i$ are given as

$$\tau_{i,q} = \tau_{i,(q-1)} - \frac{\partial \alpha_{i,(q-1)}}{\partial y_i} \delta_i z_{i,q} \tag{6.45}$$

Then parameter update law $\hat{\theta}_i$ is designed to be

$$\dot{\hat{\theta}}_i = \Gamma_i \tau_{i,\rho_i} \tag{6.46}$$

Clearly, the designed controller for the ith subsystem only uses the local signals, as shown in its block diagram Figure 6.3.

6.2.4 Stability Analysis

In this section, the stability of the overall closed-loop system consisting of the interconnected system and decentralized adaptive controllers will be established.

We define $z_i(t) = [z_{i,1}, z_{i,2}, \ldots, z_{i,\rho_i}]^T$. The ith subsystem (6.3) and (6.4) subject to local controller (6.41) is characterized by

$$\begin{aligned} \dot{z}_i &= A_{zi}z_i + W_{\epsilon i}\epsilon_{i,2} + W_{\theta i}^T\tilde{\theta}_i - b_{i,m_i}\bar{\alpha}_{i,1}\tilde{p}_i e_{\rho_i,1} \\ &\quad + W_{\epsilon i}\left[(s + a_{i,n_i-1})\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)y_j\right)\right] \end{aligned} \tag{6.47}$$

where

$$A_{zi} = \begin{bmatrix} -c_{i1} - l_{i1} & \hat{b}_{i,m_i} & 0 & \cdots & 0 \\ -\hat{b}_{i,m_i} & -c_{i2} - l_{i2}\left(\frac{\partial\alpha_{i,1}}{\partial y_i}\right)^2 & 1 + \sigma_{i,(2,3)} & \cdots & \sigma_{i,(2,\rho_i)} \\ 0 & -1 - \sigma_{i,(2,3)} & -c_{i3} - l_{i3}\left(\frac{\partial\alpha_{i,2}}{\partial y_i}\right)^2 & \cdots & \sigma_{i,(3,\rho_i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\sigma_{i,(2,\rho_i)} & -\sigma_{i,(3,\rho_i)} & \cdots & -c_{i\rho_i} - l_{i\rho_i}\left(\frac{\partial\alpha_{i,(\rho_i-1)}}{\partial y_i}\right)^2 \end{bmatrix} \tag{6.48}$$

$$W_{\epsilon i} = \left[1, \ -\frac{\partial\alpha_{i,1}}{\partial y_i}, \ \ldots, -\frac{\partial\alpha_{i,(\rho_i-1)}}{\partial y_i}\right] \tag{6.49}$$

$$W_{\theta i}^T = W_{\epsilon i}\delta_i^T - \hat{p}_i\bar{\alpha}_{i,1}e_{\rho_i,1}{e_{n_i+m_i+1,1}}^T \tag{6.50}$$

where the terms $\sigma_{i,(k,q)}$ are due to the terms $\frac{\partial\alpha_{i,(k-1)}}{\partial\hat{\theta}_i}\Gamma_i(\tau_{i,q} - \tau_{i,(q-1)})$ in the $z_{i,q}$ equation.

With respect to (6.47), we consider a function V_{ρ_i} defined as:

$$V_{\rho_i} = \sum_{q=1}^{\rho_i}\left(\frac{1}{2}z_{i,q}^2 + \frac{1}{l_{iq}}\epsilon_i{}^T P_i \epsilon_i\right) + \frac{1}{2}\tilde{\theta}_i^T \Gamma_i^{-1}\tilde{\theta}_i + \frac{|b_{i,m_i}|}{2\gamma_i^{'}}\tilde{p}_i^2 \qquad (6.51)$$

From (6.22), (6.23) and the designed controller (6.41)-(6.46), it can be shown that the derivative of V_{ρ_i} satisfies

$$\begin{aligned}
\dot{V}_{\rho_i} &= \sum_{q=1}^{\rho_i} z_{i,q}\dot{z}_{i,q} - \tilde{\theta}_i^T\Gamma^{-1}\dot{\hat{\theta}}_i - \frac{|b_{i,m_i}|}{\gamma_i^{'}}\tilde{p}_i\dot{\hat{p}}_i - \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}\|\epsilon_i\|^2 - 2\sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}(a_i - k_i)^T P_i \epsilon_i \\
&\quad \times\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)y_j\right) \\
&\le -\sum_{q=1}^{\rho_i} c_{iq}z_{i,q}^2 - \sum_{q=2}^{\rho_i}\frac{l_{iq}}{2}\left(\frac{\partial\alpha_{i,(q-1)}}{\partial y_i}\right)^2 z_{i,q}^2 - \sum_{q=1}^{\rho_i}\frac{1}{2l_{iq}}\|\epsilon_i\|^2 - \sum_{q=2}^{\rho_i} z_{i,q}\frac{\partial\alpha_{i,(q-1)}}{\partial y_i}\epsilon_{i,2} \\
&\quad -\frac{l_{i1}}{2}z_{i,1}^2 + z_{i,1}(s + a_{i,n_i-1})\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)z_{j,1}\right) \\
&\quad -\sum_{q=2}^{\rho_i}\left[\frac{l_{iq}}{2}\left(\frac{\partial\alpha_{i,(q-1)}}{\partial y_i}\right)^2 z_{i,q}^2 + z_{i,q}\frac{\partial\alpha_{i,(q-1)}}{\partial y_i}(s + a_{i,n_i-1})\right. \\
&\quad \left.\times\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)z_{j,1}\right)\right] \\
&\quad -\sum_{q=1}^{\rho_i}\left[\frac{1}{2l_{iq}}\|\epsilon_i\|^2 + \Phi_i^T\epsilon_i\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)z_{j,1}\right)\right] \\
&\le -\sum_{q=1}^{\rho_i} c_{iq}z_{i,q}^2 + \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}(s + a_{i,n_i-1})^2 L_i - \sum_{q=1}^{\rho_i}\frac{1}{4l_{iq}}\|\epsilon_i\|^2 \\
&\quad + \sum_{q=1}^{\rho_i} 2\|\Phi_i\|^2 l_{iq} L_i
\end{aligned} \qquad (6.52)$$

where

$$\Phi_i^T = \frac{2}{l_{iq}}(a_i - k_i)^T P_i \qquad (6.53)$$

$$Ł_i = \left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1}\right)^2 + \left(\sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)z_{j,1}\right)^2 \qquad (6.54)$$

To deal with the dynamic interaction or unmodeled dynamics, we show that their effects can be bounded by static functions of system states, as given in Lemma 6.2.1 later. Let $h_{i,j}$ and $g_{i,j}$ be the state vectors of systems with transfer functions $H_{ij}(s)G_j^{-1}(s)$ and $\Delta_{ij}(s)$, respectively. They are given by

$$\begin{aligned} \dot{h}_{i,j} &= B_{hi,j}h_{i,j} + b_{hi,j}x_{j,1}, \quad H_{ij}(s)G_j^{-1}(s)x_{j,1} = (1,0,\ldots,0)h_{i,j} & (6.55) \\ \dot{g}_{i,j} &= A_{gi,j}g_{i,j} + b_{gi,j}y_j, \quad \Delta_{ij}(s)y_j = (1,0,\ldots,0)g_{i,j} & (6.56) \end{aligned}$$

where $A_{gi,j}$ and $B_{hi,j}$ are Hurwitz because $\Delta_{ij}(s)$, $H_{ij}(s)$ and $B_j^{-1}(s)$ are stable from Assumptions 6.2.1 and 6.2.2. It is obvious that

$$\begin{aligned} \|\Delta_{ij}(s)y_j\|^2 &\le \|\chi\|^2 & (6.57) \\ \left\|\sum_{j=1}^{N} H_{ij}(s)G_j^{-1}(s)x_{j,1}\right\|^2 &\le k_{i0}\|\chi\|^2 & (6.58) \end{aligned}$$

where $\chi = [\chi_1^T, \ldots, \chi_N^T]^T$ and $\chi_i = [z_i^T, \epsilon_i^T, \tilde{\eta}_i^T, \zeta_i^T, \, h_{i,1}{}^T \ldots, h_{i,N}{}^T, g_{i,1}{}^T, \ldots, g_{i,N}{}^T]^T$. We also have

$$\begin{aligned} & \left\|(s + a_{i,n_i-1})\sum_{j=1}^{N} H_{ij}(s)G_j^{-1}(s)x_{j,1}\right\|^2 \\ = & \ \|\sum_{j=1}^{N} sH_{ij}(s)G_j^{-1}(s)x_{j,1} + a_{i,n_i-1}\sum_{j=1}^{N} H_{ij}(s)G_j^{-1}(s)x_{j,1}\|^2 \\ = & \ \left\|\sum_{j=1}^{N}(1,0,\ldots,0)\dot{h}_{i,j} + a_{i,n_i-1}\sum_{j=1}^{N} H_{ij}(s)G_j^{-1}(s)x_{j,1}\right\|^2 \\ = & \ \left\|(\sum_{j=1}^{N}(1,0,\ldots,0)[B_{hi,j}h_{i,j} + b_{hi,j}x_{j,1}] + a_{i,n_i-1})\sum_{j=1}^{N} H_{ij}(s)G_j^{-1}(s)x_{j,1}\right\|^2 \\ \le & \ k_{i1}\sum_{j=1}^{N}\|x_{j,1}\|^2 + k_{i2}\|\chi\|^2 \qquad (6.59) \end{aligned}$$

$$
\begin{aligned}
& \left\| (s + a_{i,n_i-1}) \sum_{j=1}^{N} \Delta_{ij}(s) y_j \right\|^2 \\
= & \left\| \sum_{j=1}^{N} (1, 0, \ldots, 0)[A_{gi,j} g_{i,j} + b_{gi,j} y_j] + a_{i,n_i-1} \sum_{j=1}^{N} \Delta_{ij}(s) y_j \right\|^2 \\
\leq & \ k_{i3} \|\chi\|^2 \qquad (6.60)
\end{aligned}
$$

where k_{i0}, k_{i1}, k_{i2} and k_{i3} are some positive constants. It is clear from (6.4) and (6.28) that

$$
x_{i,1} = z_{i,1} - \sum_{j=1}^{N} \nu_{ij} \frac{H_{ij}(s)}{G_j(s)} x_{j,1} - \sum_{j=1}^{N} \mu_{ij} \Delta_{ij}(s) y_j \qquad (6.61)
$$

Thus

$$
\begin{aligned}
& \left\| (s + a_{i,n_i-1}) \sum_{j=1}^{N} H_{ij}(s) G_j^{-1}(s) x_{j,1} \right\|^2 \\
\leq & \left[k_{i4} + 2 \left(\max_{1 \leq, i, j \leq N} \{\nu_{ij}^2\} + \max_{1 \leq, i, j \leq N} \{\mu_{ij}^2\} \right) k_{i4} \right] \|\chi\|^2 \qquad (6.62)
\end{aligned}
$$

where $k_{i4} = \max\{k_{i2} + 2k_{i1}, 2k_{i1}, 2k_{i1}k_{i0}\}$ are constants and independent of μ_{ij} and ν_{ij}.

Then we can get the following lemma.

Lemma 6.2.1. *The effects of the interactions and unmodeled dynamics are bounded as follows*

$$
\begin{aligned}
\left\| \sum_{j=1}^{N} \Delta_{ij}(s) z_{j,1} \right\|^2 & \leq \|\chi\|^2 & (6.63) \\
\left\| \sum_{j=1}^{N} H_{ij}(s) G_j^{-1}(s) x_{j,1} \right\|^2 & \leq k_{i0} \|\chi\|^2 & (6.64) \\
\left\| (s + a_{i,n_i-1}) \sum_{j=1}^{N} \Delta_{ij}(s) z_{j,1} \right\|^2 & \leq k_{i3} \|\chi\|^2 & (6.65)
\end{aligned}
$$

$$\left\|(s+a_{i,n_i-1})\sum_{j=1}^{N}H_{ij}(s)G_j^{-1}(s)x_{j,1}\right\|^2 \le \left[k_{i4}+2\left(\max_{1\le,i,j\le N}\{\nu_{ij}^2\} + \max_{1\le,i,j\le N}\{\mu_{ij}^2\}\right)k_{i4}\right]\|\chi\|^2 \quad (6.66)$$

With these preliminaries established, we can obtain our first main result stated in the following theorem.

Theorem 6.2.1. *Consider the closed-loop adaptive system consisting of the plant (6.1) under Assumptions 6.2.1 and 6.2.2, the controller (6.41), the estimator (6.38), (6.46), and the filters (6.10)-(6.13). There exists a constant* μ^* *such that for all* $\nu_{ij} < \mu^*$ *and* $\mu_{ij} < \mu^*, i, j = 1, 2, \ldots, N$, *all the signals in the system are globally uniformly bounded and* $\lim_{t\to\infty} y_i(t) = 0$.

Proof: To show the stability of the overall system, the state variables of the filters in (6.11) and state vector ζ_i associated with the zero dynamics of ith subsystems should be considered. Under a similar transformation as in [28], these variables can be shown to satisfy

$$\dot{\zeta}_i = A_{i,b_i}\zeta_i + \bar{b}_i x_{i,1} \quad (6.67)$$

$$\dot{\tilde{\eta}}_i = A_{i,0}\tilde{\eta}_i + e_{n_i,n_i}z_{i,1} \quad (6.68)$$

$$\dot{\eta}_i^r = A_{i,0}\eta_i^r, \tilde{\eta}_i = \eta_i - \eta_i^r \quad (6.69)$$

where the eigenvalues of the $m_i \times m_i$ matrix A_{i,b_i} are the zeros of the Hurwitz polynomial $N_i(s), \bar{b}_i \in \Re^{m_i}$.

A Lyapunov function for the ith local system is defined as

$$V_i = V_{\rho_i} + \frac{1}{l_{\eta i}}\tilde{\eta}_i^T P_i\tilde{\eta}_i + \frac{1}{l_{\zeta i}}\zeta_i^T P_{i,b_i}\zeta_i + \sum_{j=1}^{N} l_{hij}{h_{i,j}}^T P_{hi,j}h_{i,j} + \sum_{j=1}^{N} l_{gij}{g_{i,j}}^T P_{gi,j}g_{i,j} \quad (6.70)$$

where $l_{\eta i}, l_{\zeta i}, l_{hij}, l_{gij}$ are positive constants, and P_{i,b_i}, $P_{hi,j}$ and $P_{gi,j}$ satisfy

$$P_{i,b_i}A_{i,b_i} + A_{i,b_i}^T P_{i,b_i} = -I_{m_i} \tag{6.71}$$

$$P_{hi,j}B_{hi,j} + B_{hi,j}^T P_{hi,j} = -I_{h_{ij}} \tag{6.72}$$

$$P_{gi,j}A_{gi,j} + A_{gi,j}^T P_{gi,j} = -I_{g_{ij}} \tag{6.73}$$

From equations (6.4), (6.52)-(6.56), (6.67)-(6.69) and (6.71)-(6.73), we get

$$\begin{aligned}
\dot{V}_i &= \dot{V}_{\rho_i} - \frac{1}{l_{\eta i}}\|\tilde{\eta}_i\|^2 + \frac{2}{l_{\eta i}}P_i\tilde{\eta}_i^T e_{n_i,n_i}z_{i,1} - \frac{1}{l_{\zeta i}}\|\zeta_i\|^2 + \frac{2}{l_{\zeta i}}\zeta_i^T P_{i,b_i}\bar{b}_i x_{i,1} - \sum_{j=1}^{N} l_{hij}\|h_{i,j}\|^2 \\
&\quad +2\sum_{j=1}^{N} l_{hij}h_{i,j}{}^T P_{hi,j}b_{hi,j}x_{j,1} - \sum_{j=1}^{N} l_{gij}\|g_{i,j}\|^2 + 2\sum_{j=1}^{N} l_{gij}g_{i,j}{}^T P_{gi,j}b_{gi,j}z_{j,1} \\
&\le -\frac{1}{2}c_{i1}z_{i,1}^2 - \sum_{q=2}^{\rho_i} c_{iq}(z_{i,q})^2 - \sum_{q=1}^{\rho_i}\frac{1}{4l_{iq}}\|\epsilon_i\|^2 - \frac{1}{2l_{\eta i}}\|\tilde{\eta}_i\|^2 - \frac{1}{2l_{\zeta i}}\|\zeta_i\|^2 \\
&\quad -\sum_{j=1}^{N}\frac{1}{2}l_{hij}\|h_{i,j}\|^2 - \sum_{j=1}^{N}\frac{1}{2}l_{gij}\|g_{i,j}\|^2 + \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}(s+a_{i,n_i-1})^2 L_i \\
&\quad +\sum_{q=1}^{\rho_i}2\|\Phi_i\|^2\frac{1}{l_{iq}}L_i - \frac{1}{4l_{\zeta i}}\|\zeta_i\|^2 - \frac{2}{l_{\zeta i}}\zeta_i^T P_{i,b_i}\bar{b}_i\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1}\right. \\
&\quad \left. +\sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)z_{j,1}\right) - \sum_{j=1}^{N}\left[\frac{l_{hij}}{4}\|h_{i,j}\|^2 + 2l_{hij}h_{i,j}{}^T P_{hi,j}b_{hi,j}\right. \\
&\quad \left.\times\left(\sum_{j=1}^{N}\nu_{ij}\frac{H_{ij}(s)}{G_j(s)}x_{j,1} + \sum_{j=1}^{N}\mu_{ij}\Delta_{ij}(s)z_{j,1}\right)\right] \\
&\quad -\frac{1}{8}c_{i1}(z_{i,1})^2 - \sum_{j=1}^{N}\frac{1}{2}l_{gij}\|g_{i,j}\|^2 + 2\sum_{j=1}^{N} l_{gij}g_{i,j}{}^T P_{gi,j}b_{gi,j}z_{j,1} \\
&\quad -\frac{1}{8}c_{i1}(z_{i,1})^2 - \sum_{j=1}^{N}\frac{l_{hij}}{4}\|h_{i,j}\|^2 + 2\sum_{j=1}^{N} l_{hij}h_{i,j}{}^T P_{hi,j}b_{hi,j}z_{j,1} \\
&\quad -\frac{1}{8}c_{i1}(z_{i,1})^2 - \frac{1}{2l_{\eta i}}\|\tilde{\eta}_i\|^2 + \frac{2}{l_{\eta i}}P_i\tilde{\eta}_i^T e_{n_i,n_i}z_{i,1} \\
&\quad -\frac{1}{8}c_{i1}(z_{i,1})^2 - \frac{1}{4l_{\zeta i}}\|\zeta_i\|^2 + \frac{2}{l_{\zeta i}}\zeta_i^T P_{i,b_i}\bar{b}_i z_{i,1}
\end{aligned} \tag{6.74}$$

Taking

$$l_{\eta i} \geq \frac{16\|P_i e_{n_i,n_i}\|^2}{c_{i1}}, \quad l_{\zeta i} \geq \frac{32\|P_{i,b_i}\bar{b}_i\|^2}{c_{i1}} \tag{6.75}$$

$$l_{hij} \leq \frac{c_{j1}}{32N\|P_{hi,j}b_{hi,j}\|^2}, \quad l_{gij} \leq \frac{c_{j1}}{16N\|P_{gi,j}b_{gi,j}\|^2} \tag{6.76}$$

we then obtain

$$\begin{aligned}
\dot{V}_i \leq\ & -\beta_i\|\chi_i\|^2 + \left[\sum_{q=1}^{\rho_i} 2\|\Phi_i\|^2 l_{iq} + \frac{8}{l_{\zeta i}}\|P_{i,b_i}\bar{b}_i\|^2 + 8\sum_{j=1}^{N} l_{hij}\|P_{hi,j}b_{hi,j}\|^2\right] L_i \\
& + \sum_{j=1}^{N}\frac{1}{4N}c_{j1}z_{j,1}^2 + \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}(s+a_{i,n_i-1})^2 L_i - \frac{1}{2}c_{i1}(z_{i,1})^2 \\
\leq\ & -\beta_i\|\chi_i\|^2 - \frac{1}{4}c_{i1}z_{i,1}^2 + \mu^2\left[k_{i6}\left(\left\|\sum_{j=1}^{N}\frac{H_{ij}(s)}{G_j(s)}x_{j,1}\right\|^2 + \left\|\sum_{j=1}^{N}\Delta_{ij}(s)z_{j,1}\right\|^2\right)\right. \\
& \left. + k_{i5}\left(\left\|(s+a_{i,n_i-1})\sum_{j=1}^{N}\frac{H_{ij}(s)}{G_j(s)}x_{j,1}\right\|^2 + \left\|(s+a_{i,n_i-1})\sum_{j=1}^{N}\Delta_{ij}(s)z_{j,1}\right\|^2\right)\right] \\
& - \left(\frac{1}{4}c_{i1}z_{i,1}^2 - \sum_{j=1}^{N}\frac{1}{4N}c_{j1}(z_{j,1})^2\right)
\end{aligned} \tag{6.77}$$

where

$$\beta_i = \min\left\{\frac{c_{i1}}{4}, c_{i2}, \ldots, c_{i\rho_i}, \sum_{q=1}^{\rho_i}\frac{1}{4l_{iq}}, \frac{1}{2l_{\eta i}}, \frac{1}{2l_{\zeta i}}, \min_{1\leq j\leq N}\left\{\frac{1}{2}l_{hij}, \frac{1}{2}l_{gij}\right\}\right\} \tag{6.78}$$

$$k_{i5} = \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}} \tag{6.79}$$

$$k_{i6} = \sum_{q=2}^{\rho_i} 2\|\Phi_i\|^2 l_{iq} + \frac{8}{l_{\zeta i}}\|P_{i,b_i}\bar{b}_i\|^2 + 8\sum_{j=1}^{N} l_{hij}\|P_{hi,j}b_{hi,j}\|^2 \tag{6.80}$$

$$\mu = \max_{1\leq i,j\leq N}\{\mu_{ij}, \nu_{ij}\} \tag{6.81}$$

Now we define a Lyapunov function for the overall decentralized adaptive control system as

$$V = \sum_{i=1}^{N} V_i \tag{6.82}$$

Using Lemma 6.2.1 and (6.77), we have

$$\begin{aligned}\dot{V} \quad \leq \quad & -\sum_{i=1}^{N}\left[\beta - \left((1+k_{i0})k_{i6} + (k_{i3}+k_{i4})k_{i5}\right)\mu^2 - k_{i4}k_{i5}\mu^4\right]\|\chi\|^2 \\ & -\frac{1}{4}\sum_{i=1}^{N} c_{i1} z_{i,1}^2 \end{aligned} \tag{6.83}$$

where

$$\beta = \frac{\min_{1\leq i\leq N}\beta_i}{N} \tag{6.84}$$

By taking μ^* as

$$\mu^* = \min_{1\leq i\leq N} \sqrt{\frac{\sqrt{((1+k_{i0})k_{i6} + (k_{i3}+k_{i4})k_{i5})^2 + 4k_{i4}k_{i5}\beta} + ((1+k_{i0})k_{i6} + (k_{i3}+k_{i4})k_{i5})}{2k_{i4}k_{i5}}} \tag{6.85}$$

we have $\dot{V} \leq -\frac{1}{4}\sum_{i=1}^{N} c_{i1} z_{i,1}^2$. This concludes the proof of Theorem 6.2.1 that all the signals in the system are globally uniformly bounded. By applying the LaSalle-Yoshizawa theorem, it further follows that $\lim_{t\to\infty} y_i(t) = 0$ for arbitrary initial $x_i(0)$. □

We now derive bounds for system output $y_i(t)$ on both $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms. Firstly, the following definitions are made.

$$d_i^0 = \sum_{q=1}^{\rho_i} \frac{1}{2l_{iq}} \tag{6.86}$$

As shown in (6.83), the derivative of V is given by

$$\dot{V} \leq -\sum_{i=1}^{N} \frac{1}{4} c_{i1} z_{i,1}^2 \tag{6.87}$$

Since V is non-increasing, we have

$$\|y_i(t)\|_2^2 = \int_0^\infty \|z_{i,1}(t)\|^2 dt \leq \frac{4}{c_{i1}} \left(V(0) - V(\infty)\right) \leq \frac{4}{c_{i1}} \left(V(0)\right) \tag{6.88}$$

$$\|y_i(t)\|_\infty \leq \sqrt{2V(0)} \tag{6.89}$$

From (6.69), we can set $\tilde{\eta}_i(0) = 0$ by selecting $\eta_i^r(0) = \eta_i(0)$. Consider the zero initial values

$$\tilde{\eta}_i(0) = 0,\ \zeta_i(0) = 0,\ h_{i,j}(0) = 0,\ g_{i,j}(0) = 0 \tag{6.90}$$

Note that the initial values $z_{i,q}(0)$ depends on $c_{i1}, \gamma_i^{'}, \Gamma_i$. We can set $z_{i,q}(0), q = 2, \ldots, \rho_i$ to zero by suitably initializing our designed filters (6.10)-(6.13) as follows:

$$v_{i,(m_i,q)}(0) = \alpha_{i,(q-1)}\left(y_i(0), \hat{\theta}_i(0), \hat{p}_i(0), \eta_i(0), \lambda_i(0), v_{i,(m_i,q-1)}(0)\right),\ q = 1, \ldots, \rho_i \tag{6.91}$$

By setting $\tilde{\eta}_i(0) = 0$, $\zeta_i(0) = 0$, $h_{i,j}(0) = 0$, $g_{i,j}(0) = 0$ and $z_{i,q}(0) = 0, q = 2, \ldots, \rho_i$, we have

$$V(0) = \sum_{i=1}^{N} \frac{1}{2}(y_i(0))^2 + d_i^0 \|\epsilon_i(0)\|_{P_i}^2 + \|\tilde{\theta}_i(0)\|_{\Gamma_i^{-1}}^2 + \frac{|b_{i,m_i}|}{\gamma_i^{'}} |\tilde{p}_i(0)|^2 \tag{6.92}$$

where $\|\epsilon_i\|_{P_i}^2 = \epsilon_i{}^T(0) P_i \epsilon_i(0)$, $\|\tilde{\theta}_i(0)\|_{\Gamma_i^{-1}}^2 = \tilde{\theta}_i^T(0)\Gamma_i^{-1}\tilde{\theta}_i(0)$. Thus the bounds for $y_i(t)$ is established and formally stated in the following theorem.

Theorem 6.2.2. *Consider the initial values* $z_{i,q}(0) = 0, q = 2, \ldots, \rho_i,\ \tilde{\eta}_i(0) = 0,$

$\zeta_i(0) = 0$, $h_{i,j}(0) = 0$ and $g_{i,j}(0) = 0$, the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms of output $y_i(t)$ are given by

$$\|y_i(t)\|_2 \leq \frac{2}{\sqrt{c_{i1}}}\left[\sum_{i=1}^{N}\frac{1}{2}(y_i(0))^2 + d_i^0\|\epsilon_i(0)\|_{P_i}^2 + \|\tilde{\theta}_i(0)\|_{\Gamma_i^{-1}}^2 + \frac{|b_{i,m_i}|}{\gamma_i^{'}}|\tilde{p}_i(0)|^2\right]^{1/2} \tag{6.93}$$

$$\|y_i(t)\|_\infty \leq \sqrt{2}\left[\sum_{i=1}^{N}\frac{1}{2}(y_i(0))^2 + d_i^0\|\epsilon_i(0)\|_{P_i}^2 + \|\tilde{\theta}_i(0)\|_{\Gamma_i^{-1}}^2 + \frac{|b_{i,m_i}|}{\gamma_i^{'}}|\tilde{p}_i(0)|^2\right]^{1/2} \tag{6.94}$$

Remark 6.2.4. Regarding the above bound, the following conclusions can be drawn by noting that $\tilde{\theta}_i(0)$,$\tilde{p}_i(0)$, $\epsilon_i(0)$ and $y_i(0)$ are independent of $c_{i1}, \Gamma_i, \gamma_i^{'}$.

- The $\mathcal{L}_2$ norm of output $y_i(t)$ given in (6.93) depends on the initial estimation errors $\tilde{\theta}_i(0), \tilde{p}_i(0)$ and $\epsilon_i(0)$. The closer the initial estimates to the true values, the better the transient tracking error performance. This bound can also be systematically reduced down to a lower bound by increasing $\Gamma_i, \gamma_i^{'}$ and c_{i1}.
- The $\mathcal{L}_\infty$ norm of output $y_i(t)$ given in (6.94) depends on the initial estimation errors $\tilde{\theta}_i(0), \tilde{p}_i(0)$ and $\epsilon_i(0)$ and design parameters $\Gamma_i, \gamma_i^{'}$.

6.3 Decentralized Control of Nonlinear Systems

In this section, we extend our results to a class of nonlinear interconnected systems.

6.3.1 Modeling of Nonlinear Interconnected Systems

On the basis of state space realization (6.3)-(6.4) for the ith linear subsystem and the modeling of interaction and unmodeled dynamics in (6.55) and (6.56), the class of nonlinear systems is described as

$$\begin{aligned}
\dot{x}_i &= A_i x_i + \Phi_i(y_i) a_i + \begin{bmatrix} 0 \\ b_i \end{bmatrix} \sigma_i(y_i) u_i & (6.95)\\
y_i &= x_{i,1} + \sum_{j=1}^{N} \nu_{ij} e_1^T h_{i,j}(x_{j,1}) + \sum_{j=1}^{N} \mu_{ij} e_1^T g_{i,j}(y_j), \quad \text{for } i = 1, \ldots, N & (6.96)
\end{aligned}$$

where A_i, a_i and b_i are defined in (6.5),

$$\Phi_i(y_i) = \begin{bmatrix} \varphi_{1,1}(y_i) & \cdots & \varphi_{n_i,1}(y_i) \\ \vdots & \ddots & \vdots \\ \varphi_{1,n_i}(y_i) & \cdots & \varphi_{n_i,n_i}(y_i) \end{bmatrix}. \tag{6.97}$$

$x_i \in \Re^{n_i}$, $u_i \in \Re$ and $y_i \in \Re$ are states, inputs and outputs respectively. $\varphi_{i,j} \in \Re$ for $j = 1, \ldots, n_i$ and $\sigma_i(y_i) \in \Re$ are known smooth nonlinear functions. ν_{ij} and μ_{ij} are positive scalars specifying the magnitudes of dynamic interactions ($i \neq j$) and unmodeled dynamics ($i = j$). $h_{i,j}$ and $g_{i,j}$ denote the state vectors of the dynamic systems associated with the dynamic interactions or unmodeled dynamics, i.e.

$$\begin{aligned}
\dot{h}_{i,j} &= f_{hi,j}(h_{i,j}, x_{j,1}) & (6.98)\\
\dot{g}_{i,j} &= f_{gi,j}(g_{i,j}, y_j) & (6.99)
\end{aligned}$$

Remark 6.3.1. From Remark 6.2.2, we can see that the effects of the dynamic interactions and unmodeled dynamics considered here are also depending on subsystem inputs and outputs.

For such a class of systems, we need the following assumptions.

Assumption 6.3.1. *For each subsystem, $a_{i,j}, j = 0, \ldots, n_i - 1$ and $b_{i,k}, k = 0, \ldots, m_i$ are unknown constants. The polynomial $B_i(s) = b_{i,m_i} s^{m_i} + \cdots + b_{i,1} s + b_{i,0}$ is Hurwitz. The order n_i, the sign of b_{i,m_i} and the relative degree $\rho_i (= n_i - m_i)$ are known.*

$\sigma_i(y_i) \neq 0$, $\forall y_i \in \Re$.

Assumption 6.3.2. *Functions $f_{hi,j}(h_{i,j}, x_{j,1})$ and $f_{gi,j}(g_{i,j}, y_j)$ are continuously differentiable nonlinear functions and globally Lipschitz in $x_{j,1}$ and y_j respectively. Also the following inequalities hold:*

$$\begin{aligned} \|f_{hi,j}(h_{i,j}, x_{j,1})\|^2 &\leq \varrho_{hij}\|h_{i,j}\|^2 + \bar{\varrho}_{hij}x_{j,1}^2 && (6.100) \\ \|f_{gi,j}(g_{i,j}, y_j)\|^2 &\leq \varrho_{gij}\|g_{i,j}\|^2 + \bar{\varrho}_{gij}y_j^2 && (6.101) \end{aligned}$$

where $\varrho_{hij}, \bar{\varrho}_{hij}, \varrho_{gij}$ and $\bar{\varrho}_{gij}$ are unknown positive constants.

Assumption 6.3.3. *There exist two smooth positive definite radially unbounded functions $V_{hi,j}$ and $V_{gi,j}$ such that the following inequations are satisfied:*

$$\begin{aligned} \frac{\partial V_{hi,j}}{\partial h_{i,j}} f_{hi,j}(h_{i,j}, 0) &\leq -d_{hij,1}\|h_{i,j}\|^2 && (6.102) \\ \left\| \frac{\partial V_{hi,j}}{\partial h_{i,j}} \right\| &\leq d_{hij,2}\|h_{i,j}\| && (6.103) \\ \frac{\partial V_{gi,j}}{\partial g_{i,j}} f_{gi,j}(g_{i,j}, 0) &\leq -d_{gij,1}\|g_{i,j}\|^2 && (6.104) \\ \left\| \frac{\partial V_{gi,j}}{\partial g_{i,j}} \right\| &\leq d_{gij,2}\|g_{i,j}\| && (6.105) \end{aligned}$$

where $d_{hij,1}$, $d_{hij,2}$, $d_{gij,1}$ and $d_{gij,2}$ are positive constants.

6.3.2 Design of local filters

Similar to the design for linear systems in Section 6.2, a local filter using only local input and output is firstly designed as follows:

$$\dot{\lambda}_i = A_{i,0}\lambda_i + e_{n_i,n_i}\sigma_i(y_i)u_i \tag{6.106}$$

$$\dot{\Xi}_i = A_{i,0}\Xi_i + \Phi_i(y_i) \tag{6.107}$$

$$v_{i,k} = (A_{i,0})^k\lambda_i, \qquad k = 0,\ldots,m_i \tag{6.108}$$

$$\dot{\xi}_{i,0} = A_{i,0}\xi_{i,0} + k_i y_i \tag{6.109}$$

where $A_{i,0}, e_{i,k}$ and k_i are defined in the same way as filters (6.10)-(6.13). With these designed filters our state estimate is given by

$$\hat{x}_i = \xi_{i,0} + \Omega_i^T\theta_i \tag{6.110}$$

where

$$\theta_i^T = [b_i^T, a_i^T] \tag{6.111}$$

$$\Omega_i^T = [v_{i,m_i},\ldots,v_{i,1},v_{i,0},\Xi_i] \tag{6.112}$$

The state estimation $\epsilon_i = x_i - \hat{x}_i$ satisfies

$$\dot{\epsilon}_i = A_{i,0}\epsilon_i - k_i\left(\sum_{j=1}^{N}\nu_{ij}e_1^T h_{i,j}(x_{j,1}) + \sum_{j=1}^{N}\mu_{ij}e_1^T g_{i,j}(y_j)\right) \tag{6.113}$$

Thus, system (6.95) can be expressed in the following form

$$\begin{aligned}\dot{y}_i &= b_{i,m_i}v_{i,(m_i,2)} + \xi_{i,(0,2)} + \bar{\delta}_i^T\theta_i + \epsilon_{i,2}\\ &\quad + \sum_{j=1}^{N}\nu_{ij}e_1^T f_{hi,j}(h_{i,j},x_{j,1}) + \sum_{j=1}^{N}\mu_{ij}e_1^T f_{gi,j}(g_{i,j},y_j)\end{aligned} \tag{6.114}$$

$$\dot{v}_{i,(m_i,q)} = v_{i,(m_i,q+1)} - k_{i,q}v_{i,(m_i,1)} \tag{6.115}$$

$$\dot{v}_{i,(m_i,\rho_i)} = v_{i,(m_i,\rho_i+1)} - k_{i,\rho_i}v_{i,(m_i,1)} + \sigma_i(y_i)u_i \tag{6.116}$$

where

$$\delta_i^T = [v_{i,(m_i,2)}, \ldots, v_{i,(0,2)}, \Xi_{i,2} + e_{n_i,1}^T \Phi_i] \tag{6.117}$$

$$\bar{\delta}_i^T = [0, v_{i,(m_i-1,2)}, \ldots, v_{i,(0,2)}, \Xi_{i,2} + e_{n_i,1}^T \Phi_i] \tag{6.118}$$

and $v_{i,(m_i,2)}, \epsilon_{i,2}, \xi_{i,(0,2)}, \Xi_{i,2}$ denote the second entries of $v_{i,m_i}, \epsilon_i, \xi_{i,0}, \Xi_i$ respectively. All states of the local filters in (6.106)-(6.109) are available for feedback.

Remark 6.3.2. Note that δ_i includes the vector of nonlinear functions $e_{n_i,1}^T \Phi_i$, which is from the dynamics $\dot{x}_{i,1} = x_{i,2} + e_{n_i,1}^T \Phi_i a_i$ in (6.95).

6.3.3 Design of Decentralized Adaptive Controllers

Performing similar backstepping procedures to linear systems, we can obtain local adaptive controllers summarized in (6.119)-(6.130) below.

Coordinate transformation:

$$z_{i,1} = y_i \tag{6.119}$$

$$z_{i,q} = v_{i,(m_i,q)} - \alpha_{i,q-1}, \ q = 2, 3, \ldots, \rho_i \tag{6.120}$$

Control Laws:

$$u_i = \frac{1}{\sigma_i(y_i)} \left(\alpha_{i,\rho_i} - v_{i,(m_i,\rho_i+1)} \right) \tag{6.121}$$

with

$$\alpha_{i,1} = \hat{p}_i \bar{\alpha}_{i,1} \tag{6.122}$$

$$\bar{\alpha}_{i,1} = -c_{i1} z_{i,1} - l_{i1} z_{i,1} - \xi_{i,(0,2)} - \bar{\delta}_i^T \hat{\theta}_i \tag{6.123}$$

$$\alpha_{i,2} = -\hat{b}_{i,m_i} z_{i,1} - \left[c_{i2} + l_{i2} \left(\frac{\partial \alpha_{i,1}}{\partial y_i} \right)^2 \right] z_{i,2} + \bar{B}_{i,2} + \frac{\partial \alpha_{i,1}}{\partial \hat{p}_i} \dot{\hat{p}}_i + \frac{\partial \alpha_{i,1}}{\partial \hat{\theta}_i} \Gamma_i \tau_{i,2} \tag{6.124}$$

$$\begin{aligned}\alpha_{i,q} &= -z_{i,(q-1)} - \left[c_{iq} + l_{iq}\left(\frac{\partial \alpha_{i,(q-1)}}{\partial y_i}\right)^2\right] z_{i,q} + \bar{B}_{i,q} + \frac{\partial \alpha_{i,(q-1)}}{\partial \hat{p}_i}\dot{\hat{p}}_i \\ &\quad + \frac{\partial \alpha_{i,(q-1)}}{\partial \hat{\theta}_i}\Gamma_i \tau_{i,q} - \left(\sum_{k=2}^{q-1} z_{i,k}\frac{\partial \alpha_{i,(k-1)}}{\partial \hat{\theta}_i}\right)\Gamma_i \frac{\partial \alpha_{i,(q-1)}}{\partial y_i}\delta_i, \quad q = 3, \ldots, \rho_i \end{aligned} \tag{6.125}$$

$$\begin{aligned}\bar{B}_{i,q} &= \frac{\partial \alpha_{i,(q-1)}}{\partial y_i}(\xi_{i,(0,2)} + \delta_i^T \hat{\theta}_i) + \frac{\partial \alpha_{i,(q-1)}}{\partial \Xi_i}(A_{i,0}\Xi_i + \Phi_i) + \frac{\partial \alpha_{i,(q-1)}}{\partial \xi_{i,0}}(A_{i,0}\xi_{i,0} \\ &\quad + k_i y_i) + k_{i,q} v_{i,(m_i,1)} + \sum_{j=1}^{m_i+q-1} \frac{\partial \alpha_{i,(q-1)}}{\partial \lambda_{i,j}}(-k_{i,j}\lambda_{i,1} + \lambda_{i,(j+1)}) \end{aligned} \tag{6.126}$$

$$\tau_{i,1} = (\delta_i - \hat{p}_i \bar{\alpha}_{i,1} e_{(n_i+m_i+1),1}) z_{i,1} \tag{6.127}$$

$$\tau_{i,q} = \tau_{i,(q-1)} - \frac{\partial \alpha_{i,(q-1)}}{\partial y_i}\delta_i z_{i,q}, \quad q = 2, \ldots, \rho_i, \; i = 1, \ldots, N \tag{6.128}$$

Parameter Update Laws:

$$\dot{\hat{p}}_i = -\gamma_i' \mathrm{sgn}(b_{i,m_i}) \bar{\alpha}_{i,1} z_{i,1} \tag{6.129}$$

$$\dot{\hat{\theta}}_i = \Gamma_i \tau_{i,\rho_i} \tag{6.130}$$

where $\hat{\theta}_i$, $\hat{p}_i$, Γ_i and $c_{iq}, l_{iq}, \gamma_i', q = 1, \ldots, \rho_i, i = 1, \ldots, N$ are defined in Section 6.2.3.

6.3.4 Stability Analysis

Similarly to Section 6.2.4, the purpose of this section is to prove that there exists a positive number μ^* such that the closed-loop system with the controller given by (6.121) is asymptotically stable for all ν_{ij}, $\mu_{ij} \in [0, \mu^*)$, $i, j = 1, \ldots, N$. To this end, the ith subsystem (6.95) and (6.96) subject to local controller (6.121) is characterized by

$$\begin{aligned}
\dot{z}_i &= A_{zi}z_i + W_{\epsilon i}\epsilon_{i,2} + W_{\theta i}^T\tilde{\theta}_i - b_{i,m_i}\bar{\alpha}_{i,1}\tilde{p}_i e_{\rho_i,1} + W_{\epsilon i}\left[\sum_{j=1}^{N}\nu_{ij}e_1^T f_{hi,j}(h_{i,j},x_{j,1})\right. \\
&\quad \left. + \sum_{j=1}^{N}\mu_{ij}e_1^T f_{gi,j}(g_{i,j},y_j)\right] \qquad (6.131)
\end{aligned}$$

where $z_i(t) = [z_{i,1}, z_{i,2}, \ldots, z_{i,\rho_i}]^T$, $A_{zi}, W_{\epsilon i}, W_{\theta i}$ are defined as the same form in (6.48)-(6.50).

To study (6.131), we consider a function V_{ρ_i} defined as:

$$V_{\rho_i} = \sum_{q=1}^{\rho_i}\left(\frac{1}{2}z_{i,q}^2 + \frac{1}{l_{iq}}\epsilon_i{}^T P_i\epsilon_i\right) + \frac{1}{2}\tilde{\theta}_i^T\Gamma_i^{-1}\tilde{\theta}_i + \frac{|b_{i,m_i}|}{2\gamma_i^{'}}\tilde{p}_i^2 \qquad (6.132)$$

Following similar procedures to (6.52), using (6.113), (6.114) and the designed controller (6.121)-(6.130), it can be shown that the derivative of V_{ρ_i} satisfies

$$\begin{aligned}
\dot{V}_{\rho_i} &= \sum_{q=1}^{\rho_i} z_{i,q}\dot{z}_{i,q} - \tilde{\theta}_i^T\Gamma^{-1}\dot{\hat{\theta}}_i - \frac{|b_{i,m_i}|}{\gamma_i^{'}}\tilde{p}_i\dot{\hat{p}}_i - \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}\|\epsilon_i\|^2 - 2\sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}k_i^T P_i\epsilon_i \\
&\quad \times\left(\sum_{j=1}^{N}\nu_{ij}e_1^T h_{i,j}(x_{j,1}) + \sum_{j=1}^{N}\mu_{ij}e_1^T g_{i,j}(y_j)\right) \\
&\le -\sum_{q=1}^{\rho_i}c_{iq}z_{i,q}^2 - \sum_{q=1}^{\rho_i}\frac{1}{4l_{iq}}\|\epsilon_i\|^2 + \sum_{q=1}^{\rho_i}\frac{8}{l_{iq}}\|k_i^T P_i\|^2 L_{1,i} + \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}L_{2,i} \qquad (6.133)
\end{aligned}$$

where we used the Young's Inequality as given in Appendix B and

$$L_{1,i} = \left(\sum_{j=1}^{N}\nu_{ij}e_1^T h_{i,j}\right)^2 + \left(\sum_{j=1}^{N}\mu_{ij}e_1^T g_{i,j}\right)^2 \qquad (6.134)$$

$$L_{2,i} = \left(\sum_{j=1}^{N}\nu_{ij}e_1^T f_{hi,j}(h_{i,j},x_{j,1})\right)^2 + \left(\sum_{j=1}^{N}\mu_{ij}e_1^T f_{gi,j}(g_{i,j},y_j)\right)^2 \qquad (6.135)$$

Similar to Lemma 6.2.1, we have the following useful lemma.

Lemma 6.3.1. *The effects of the interactions and unmodeled dynamics are bounded* *as follows*

$$\begin{aligned} L_{1,i} &\leq \left(\max_{1\leq,i,j\leq N}\{\nu_{ij}^2\} + \max_{1\leq,i,j\leq N}\{\mu_{ij}^2\}\right)\|\chi\|^2 && (6.136)\\ \left(\sum_{j=1}^{N} e_1^T f_{gi,j}(g_{i,j}, y_j)\right)^2 &\leq k_{i1}\|\chi\|^2 && (6.137)\\ \left(\sum_{j=1}^{N} e_1^T f_{hi,j}(h_{i,j}, x_{j,1})\right)^2 &\leq \left(k_{i2} + k_{i3}(\max_{1\leq,i,j\leq N}\{\nu_{ij}^2\} + \max_{1\leq,i,j\leq N}\{\mu_{ij}^2\})\right)\|\chi\|^2 && (6.138) \end{aligned}$$

where $\chi = [\chi_1^T, \ldots, \chi_N^T]^T$ *and* $\chi_i = [z_i^T, \epsilon_i^T, {h_{i,1}}^T, \ldots, {h_{i,N}}^T, {g_{i,1}}^T, \ldots, {g_{i,N}}^T]^T$, k_{i2}, k_{i2}, k_{i3} *are positive constants.*

Proof: By following similar analysis to Lemma 6.2.1, using Assumption 6.3.2 and (6.96), the result can be proved. □

Based on Lemma 6.3.1, it follows from (6.133) that

$$\begin{aligned} \dot{V}_{\rho_i} &\leq -\sum_{q=1}^{\rho_i} c_{iq}(z_{i,q})^2 - \sum_{q=1}^{\rho_i}\frac{1}{4l_{iq}}\|\epsilon_i\|^2 + \sum_{q=1}^{\rho_i}\frac{16}{l_{iq}}\|k_i^T P_i\|^2\mu^2\|\chi\|^2\\ &\quad + \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}((k_{i1}+k_{i2})\mu^2 + 2k_{i3}\mu^4)\|\chi\|^2 && (6.139) \end{aligned}$$

where

$$\mu = \max_{1\leq i,j\leq N}\{\mu_{ij}, \nu_{ij}\} \qquad (6.140)$$

As $f_{hi,j}$ is globally Lipschitz in $x_{j,1}$ according to Assumption 6.3.2, the derivative of

$V_{hi,j}$ with respect to $f_{hi,j}(h_{i,j}, x_{j,1})$ in Assumption 6.3.3 satisfies

$$\begin{aligned}\frac{\partial V_{hi,j}}{\partial h_{i,j}} f_{hi,j}(h_{i,j}, x_{j,1}) &= \frac{\partial V_{hi,j}}{\partial h_{i,j}} f_{hi,j}(h_{i,j}, 0) + \frac{\partial V_{hi,j}}{\partial h_{i,j}} [f_{hi,j}(h_{i,j}, x_{j,1}) - f_{hi,j}(h_{i,j}, 0)] \\ &\leq -d_{hij,1}\|h_{i,j}\|^2 + d_{hij,2}\|h_{i,j}\| L_{hij}|x_{j,1}| \qquad (6.141)\end{aligned}$$

where L_{hij} is a positive constant. Similarly, there exists a positive constant L_{gij} such that

$$\frac{\partial V_{gi,j}}{\partial g_{i,j}} f_{gi,j}(g_{i,j}, y_j) \leq -d_{gij,1}\|g_{i,j}\|^2 + d_{gij,2}\|g_{i,j}\| L_{gij}|y_j| \qquad (6.142)$$

We are now at the position to establish the following theorem on the stability of nonlinear systems.

Theorem 6.3.1. *Consider the closed-loop adaptive system consisting of the plant (6.95) under Assumptions 6.3.1 to 6.3.3, the controller (6.121), the estimator (6.129), (6.130) and the filters (6.106)- (6.109). There exists a constant* μ^* *such that for all* $\nu_{ij} < \mu^*$ *and* $\mu_{ij} < \mu^*, i, j = 1, 2, \ldots, N$, *all the signals in the system are globally uniformly bounded and* $\lim_{t\to\infty} y_i(t) = 0$.

Proof: We define a Lyapunov function for the ith local system

$$V_i = V_{\rho_i} + \sum_{j=1}^{N} l_{hij} V_{hi,j} + \sum_{j=1}^{N} l_{gij} V_{gi,j} \qquad (6.143)$$

where l_{hij} and l_{gij} are positive constants. Computing the time derivative of V_i and using (6.96), (6.139)-(6.142), we have

$$\begin{aligned}\dot{V}_i &= \dot{V}_{\rho_i} - \sum_{j=1}^{N} l_{hij} d_{hij,1}\|h_{i,j}\|^2 - \sum_{j=1}^{N} l_{gij} d_{gij,1}\|g_{i,j}\|^2 + \sum_{j=1}^{N} l_{hij} d_{hij,2}\|h_{i,j}\| L_{hij}|x_{j,1}| \\ &\quad + \sum_{j=1}^{N} l_{gij} d_{gij,2}\|g_{i,j}\| L_{gij}|y_j| \qquad (6.144)\end{aligned}$$

$$
\begin{aligned}
\leq\ & -\frac{1}{2}c_{i1}z_{i,1}^2 - \sum_{q=2}^{\rho_i} c_{iq}z_{i,q}^2 - \sum_{q=1}^{\rho_i}\frac{1}{4l_{iq}}\|\epsilon_i\|^2 + \sum_{q=1}^{\rho_i}\frac{16}{l_{iq}}\|k_i^T P_i\|^2\mu^2\|\chi\|^2 \\
& + \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}\left((k_{i1}+k_{i2})\mu^2 + 2k_{i3}\mu^4\right)\|\chi\|^2 - \sum_{j=1}^{N}(\frac{1}{2}l_{hij}d_{hij,1}\|h_{i,j}\|^2 \\
& + \frac{1}{2}l_{gij}d_{gij,1}\|g_{i,j}\|^2) - \sum_{j=1}^{N}\frac{1}{4}l_{hij}d_{hij,1}\|h_{i,j}\|^2 - \frac{1}{4}c_{i1}z_{i,1}^2 \\
& + \sum_{j=1}^{N} l_{hij}d_{hij,2}\|h_{i,j}\|L_{hij}|z_{j,1}| - \sum_{j=1}^{N}\left[\frac{1}{4}l_{hij}d_{hij,1}\|h_{i,j}\|^2 + l_{hij}d_{hij,2}\|h_{i,j}\|\right. \\
& \left. \times L_{hij}\left|\sum_{j=1}^{N}\nu_{ij}e_1^T h_{i,j} + \sum_{j=1}^{N}\mu_{ij}e_1^T g_{i,j}\right|\right] - \frac{1}{4}c_{i1}z_{i,1}^2 - \sum_{j=1}^{N}\frac{1}{2}l_{gij}d_{gij,1}\|g_{i,j}\|^2 \\
& + \sum_{j=1}^{N} l_{gij}d_{gij,2}\|g_{i,j}\|L_{gij}|z_{j,1}| \qquad (6.145)
\end{aligned}
$$

Taking

$$
l_{hij} \leq \frac{d_{hij,1}c_{j1}}{4Nd_{hij,2}^2L_{hij}^2}, \quad l_{gij} \leq \frac{d_{gij,1}c_{j1}}{2Nd_{gij,2}^2L_{gij}^2} \qquad (6.146)
$$

and using Young's inequality, we have

$$
\begin{aligned}
\dot{V}_i \leq\ & -\beta_i\|\chi_i\|^2 - \frac{1}{4}c_{i1}(z_{i,1})^2 + \left((k_{i4}(k_{i1}+k_{i2}) + k_{i5})\mu^2 + 2k_{i3}k_{i4}\mu^4\right)\|\chi\|^2 \\
& - \left(\frac{1}{4}c_{i1}(z_{i,1})^2 - \sum_{j=1}^{N}\frac{1}{4N}c_{j1}(z_{j,1})^2\right) \qquad (6.147)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_i &= \min\left\{\frac{c_{i1}}{4}, c_{i2}, \ldots, c_{i\rho_i}, \sum_{q=1}^{\rho_i}\frac{1}{4l_{iq}}, \min_{1\leq j\leq N}\left\{\frac{1}{2}l_{hij}d_{hij,1}, \frac{1}{2}l_{gij}d_{gij,1}\right\}\right\} \qquad (6.148) \\
k_{i4} &= \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}} \qquad (6.149) \\
k_{i5} &= \|k_i^T P_i\|^2\sum_{q=1}^{\rho_i}\frac{16}{l_{iq}} + \sum_{j=1}^{N}\frac{4l_{hij}d_{hij,2}^2L_{hij}^2}{d_{hij,1}} \qquad (6.150)
\end{aligned}
$$

Now we consider the Lyapunov function for the overall decentralized adaptive control system defined as

$$V = \sum_{i=1}^{N} V_i \tag{6.151}$$

From (6.147) and Lemma 6.3.1, the derivative of V is given by

$$\dot{V} \leq - \sum_{i=1}^{N} \left[\beta - \big(k_{i4}(k_{i1} + k_{i2}) + k_{i5}\big)\mu^2 - 2k_{i3}k_{i4}\mu^4\right] \|\chi\|^2 - \frac{1}{4}\sum_{i=1}^{N} c_{i1} z_{i,1}^2 \tag{6.152}$$

where

$$\beta = \frac{\min_{1\leq i\leq N} \beta_i}{N} \tag{6.153}$$

By taking μ^* as

$$\mu^* = \min_{1\leq i\leq N} \sqrt{\frac{\sqrt{(k_{i4}(k_{i1}+k_{i2})+k_{i5})^2 + 8k_{i3}k_{i4}\beta} + k_{i4}(k_{i1}+k_{i2}) + k_{i5}}{4k_{i3}k_{i4}}} \tag{6.154}$$

we have $\dot{V} \leq -\frac{1}{4}\sum_{i=1}^{N} c_{i1}(z_{i,1})^2$ for all $\nu_{ij} < \mu^*$ and $\mu_{ij} < \mu^*$. This implies that $z_i, \hat{p}_i, \hat{\theta}_i, \hat{\epsilon}_i$ are bounded. Because of the boundedness of y_i, variables $v_{i,k}, \xi_{i,0}$ and Ξ_i are bounded as $A_{i,0}$ is Hurewitz. Following similar analysis to the last section, states ζ_i associated with the zero dynamics of the ith subsystem are bounded. This concludes the proof of Theorem 6.3.1 that all the signals in the system are globally uniformly bounded. By applying the LaSalle-Yoshizawa theorem, it further follows that $\lim_{t\to\infty} y_i(t) = 0$ for arbitrary initial $x_i(0)$. □

Remark 6.3.3. The transient performance for system output $y_i(t)$ in terms of both $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms can also be obtained as in Theorem 6.2.2.

6.4 Illustrative Examples

6.4.1 Linear Systems

To verify our results by simulation, we consider interconnected system with two subsystems as described in (6.1) (i.e. N=2). The transfer function of each local subsystem is $G_i(s) = \frac{1}{s(s+a_i)}, i = 1, 2$. In the simulation, $a_1 = -1$ and $a_2 = 2$ which are considered to be unknown in controller design and hence require identification. The dynamic interactions are $H_{ij} = \frac{1}{(s+1)^3}$, $\Delta_{ij} = \frac{1}{(s+1)}$ for $i = 1, 2$ and $j = 1, 2$, respectively. As the high-frequency gain b_{i,m_i} is known, the additional parameter $\dot{\hat{p}}_i$ in equation (6.38) is no longer to be estimated. The initials of subsystem outputs are set as $y_1(0) = 1, y_2(0) = 0.4$.

6.4.2 Verification of Theorem 6.2.1

The design parameters are chosen as $k_i = [4, 4]^T, i = 1, 2$, $c_{11} = c_{12} = c_{21} = c_{22} = 1, l_{11} = l_{12} = 1_{21} = l_{22} = 0.001$. To see the effects of the proposed decentralized adaptive controllers, we also consider the case that the parameters of all local controllers are fixed without adaptation, i.e. $\Gamma_1 = \Gamma_2 = 0$. If constants $\nu_{ij} = \mu_{ij} = 0$ for $i = 1, 2$ and $j = 1, 2$, the two subsystems are totally decoupled. In this case, the two fixed-parameter local controllers can stabilize the two subsystems as shown from the responses given in Figures 6.4-6.7. However, when $\nu_{ij} = \mu_{ij} = 0.7$ for $i = 1, 2$ and $j = 1, 2$, the system outputs y_1, y_2 illustrated in Figures 6.8-6.9 show that these two local controllers can no longer stabilize the interconnected system, due to the presence of interactions and unmodeled dynamics.

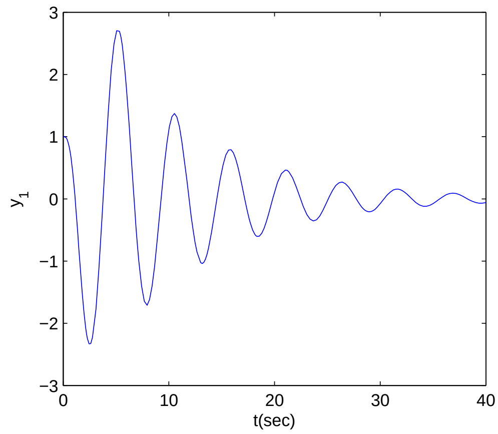


Figure 6.4: System output y_1 with fixed controllers (decoupled case with $\nu_{ij} = \mu_{ij} = 0$)

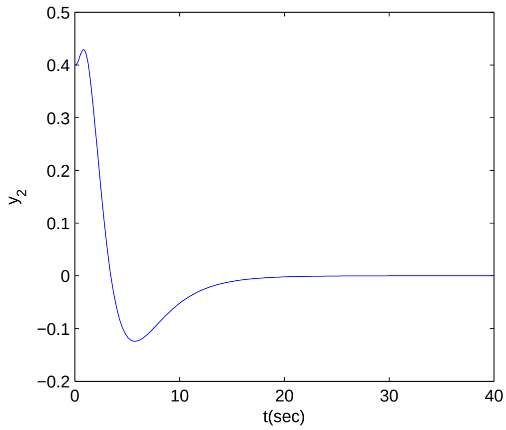


Figure 6.5: System output y_2 with fixed controllers (decoupled case with $\nu_{ij} = \mu_{ij} = 0$)

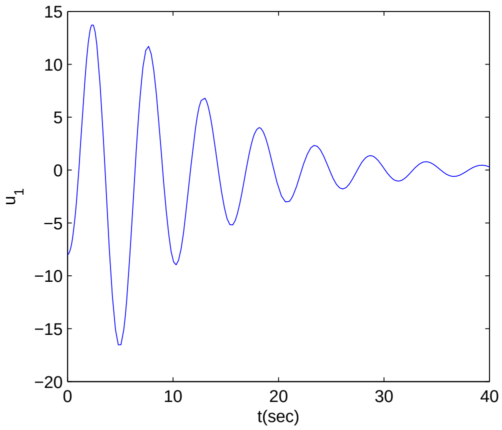


Figure 6.6: Control u_1 with fixed controllers (decoupled case with $\nu_{ij} = \mu_{ij} = 0$)

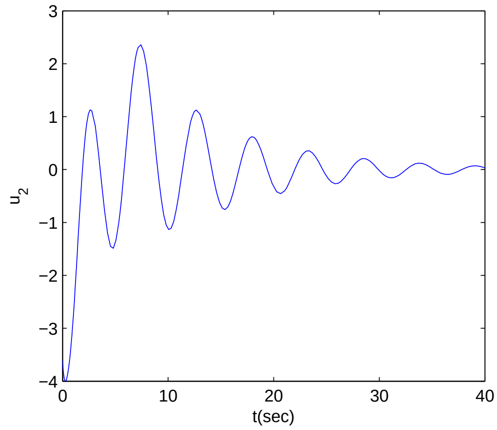


Figure 6.7: Control u_2 with fixed controllers (decoupled case with $\nu_{ij} = \mu_{ij} = 0$)

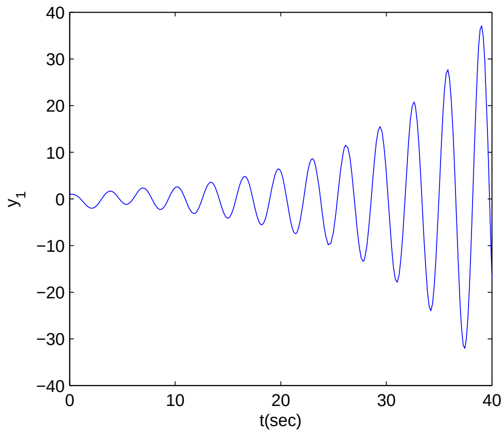


Figure 6.8: System output y_1 with fixed controllers (coupled case with $\nu_{ij} = \mu_{ij} = 0.7$)

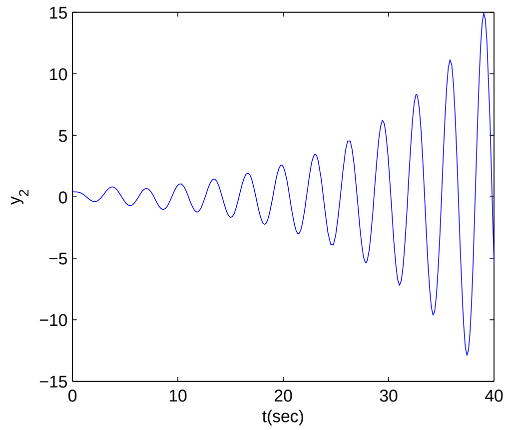


Figure 6.9: System output y_2 with fixed controllers (coupled case with $\nu_{ij} = \mu_{ij} = 0.7$)

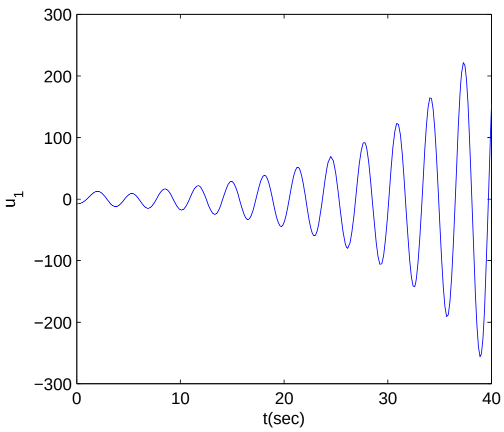


Figure 6.10: Control u_1 with fixed controllers (coupled case with $\nu_{ij} = \mu_{ij} = 0.7$)

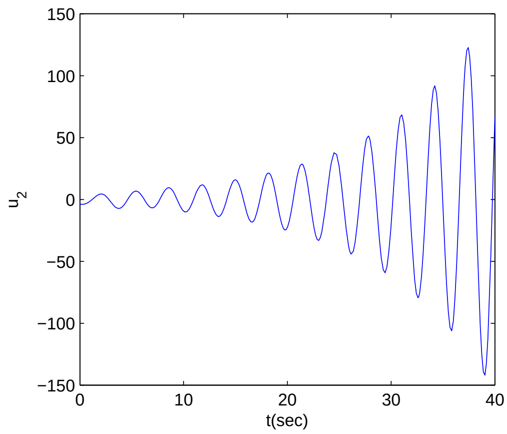


Figure 6.11: Control u_2 with fixed controllers (coupled case with $\nu_{ij} = \mu_{ij} = 0.7$)

With the presented adaptation mechanism on by choosing $\Gamma_1 = \Gamma_2 = 0.1$, the results are given in Figures 6.12-6.15. Clearly, the system is now stabilized and the outputs of both subsystems converge to zero. This verifies that the proposed scheme is effective in handling interactions and unmodeled dynamics as stated in Theorem 6.2.1.

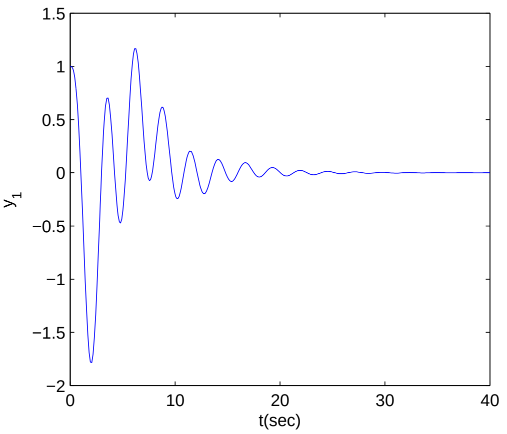


Figure 6.12: System output y_1 with adaptive controllers (coupled case with $\nu_{ij} = \mu_{ij} = 0.7$)

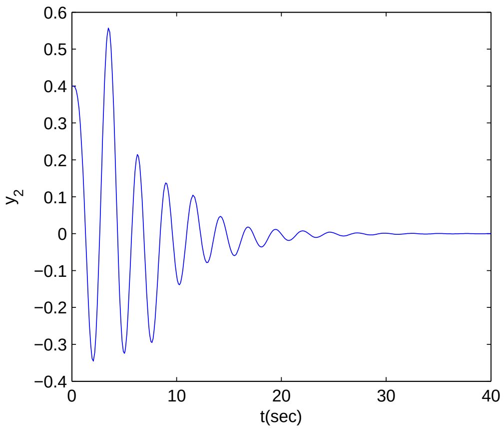


Figure 6.13: System output y_2 with adaptive controllers (coupled case with $\nu_{ij} = \mu_{ij} = 0.7$)

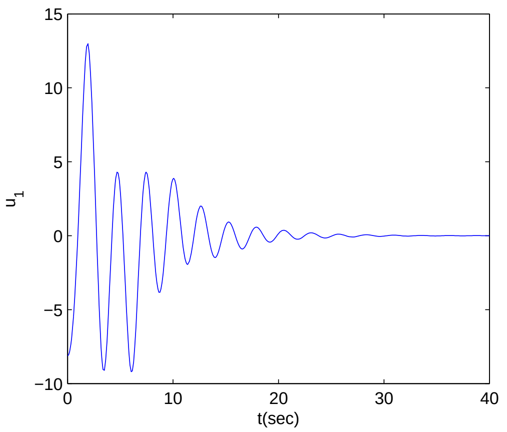


Figure 6.14: Control u_1 with adaptive controllers (coupled case with $\nu_{ij} = \mu_{ij} = 0.7$)

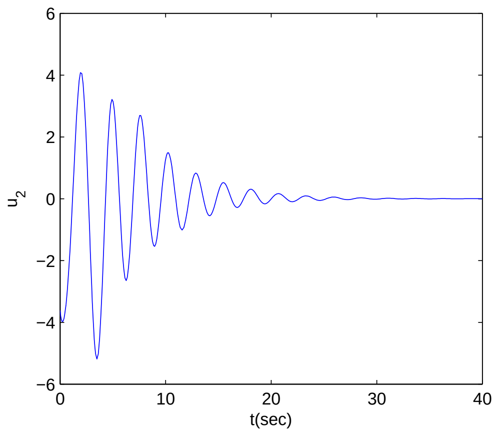


Figure 6.15: Control u_2 with adaptive controllers (coupled case with $\nu_{ij} = \mu_{ij} = 0.7$)

6.4.3 Verification of Theorem 6.2.2

We still consider the interconnected system with parameters given above. The initial values $z_{i,q}(0)$ for $i = 1, 2$ and $q = 2$ are set to 0 by properly initializing filters according to equation (6.91). In our case, $v_{i,(0,2)}(0) = \alpha_{i,(0,2)}(0)$ for $i = 1, 2$. The design parameters l_{ij} are fixed as 0.001 and $c_{12} = c_{22} = 1$, which are the same as the above. We now consider the following two cases:

(1) **Effects of Parameters** c_{i1}

The effects of changing design parameters c_{i1} stated in Theorem 6.2.2 are now verified by choosing $c_{11} = c_{21} = 1$ and 3 respectively. The corresponding initials $v_{i,(0,2)}(0)$ are selected as $v_{1,(0,2)}(0) = -1.001$, $v_{2,(0,2)}(0) = -0.4004$, and $v_{1,(0,2)}(0) = -3.001$, $v_{2,(0,2)}(0) = -1.2004$ for the two sets of choices of c_{i1}. In the verification, we fix $\Gamma_1 = \Gamma_2 = 0.1$. The outputs of the two subsystems y_1, y_2 are compared in Figures 6.16 and 6.17. Obviously, the $\mathcal{L}_2$ norms of the outputs decrease as c_{i1} for $i = 1, 2$ increase.

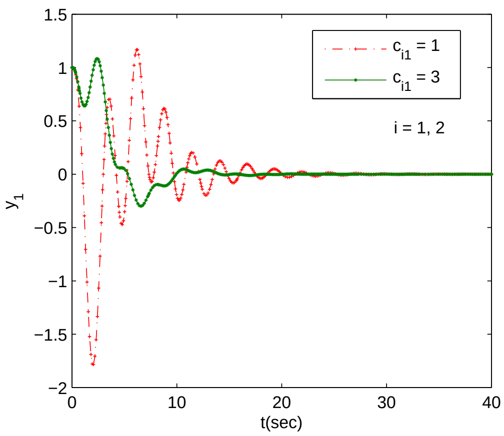


Figure 6.16: Comparison of system output y_1 with different c_{i1}

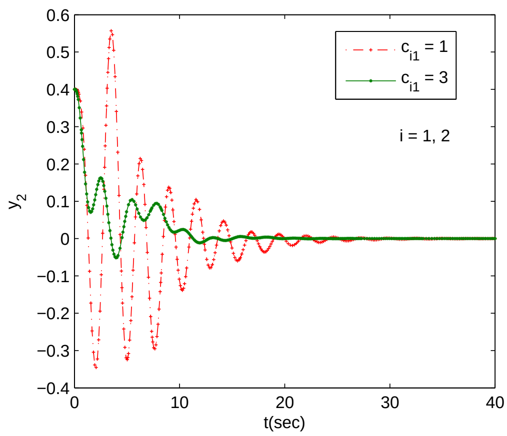


Figure 6.17: Comparison of system output y_2 with different c_{i1}

(2) **Effects of Parameters Γ_i**

We now fix c_{i1} at 1 for all $i = 1, 2$ and choose initials $v_{1,(0,2)}(0) = -1.001$ and $v_{2,(0,2)}(0) = -0.4004$. For comparison, Γ_i are set as 0.1 and 1, respectively for $i = 1, 2$. The subsystem outputs y_1, y_2 are compared in Figures 6.18 and 6.19. Clearly, the transient tracking performances are found significantly improved by increasing Γ_i.

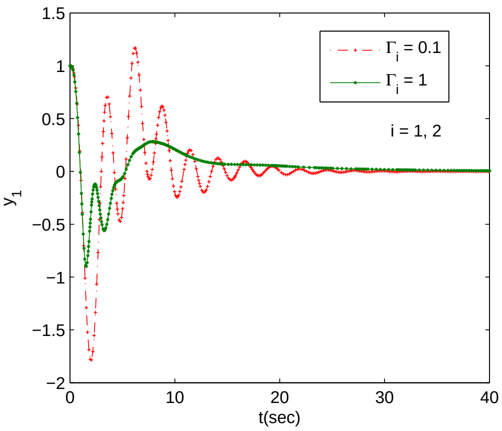


Figure 6.18: Comparison of system output y_1 with different Γ_i

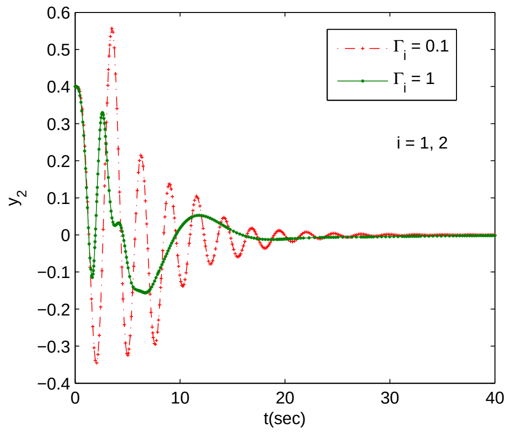


Figure 6.19: Comparison of system output y_2 with different Γ_i

6.4.4 Nonlinear Systems

To further verify the effectiveness of our proposed scheme applied to nonlinear interconnected systems, we consider two nonlinear interconnected subsystems with $n_i = 2$, for $i = 1, 2$ as described in (6.95)-(6.96), where $\Phi_1 = [0, (y_1)^2]^T$, $\Phi_2 = [0, (y_2)^2 + y_2]^T$, $\sigma_i(y_i) = 1$.

$$\dot{h}_{i,j} = \begin{bmatrix} -3 & 1 \\ -2.25 & 0 \end{bmatrix} h_{i,j} + \begin{bmatrix} \frac{1-e^{-h_{i,j}(1)}}{1+e^{-h_{i,j}(1)}} \\ \frac{1-e^{-h_{i,j}(2)}}{1+e^{-h_{i,j}(2)}} \end{bmatrix} + \begin{bmatrix} \sin(h_{i,j(1)}) \\ \sin(h_{i,j(2)}) \end{bmatrix} x_{j,1} \tag{6.155}$$

$$\dot{g}_{i,j} = \begin{bmatrix} -4 & 1 \\ -4 & 0 \end{bmatrix} g_{i,j} + \begin{bmatrix} \frac{1-e^{-g_{i,j}(1)}}{1+e^{-g_{i,j}(1)}} \\ \frac{1-e^{-g_{i,j}(2)}}{1+e^{-g_{i,j}(2)}} \end{bmatrix} + \begin{bmatrix} \frac{y_j}{|\ln y_j|+2} \\ y_j \end{bmatrix} \tag{6.156}$$

In simulation, $a_1 = -1, a_2 = 2, b_1 = 1, b_2 = 2$, $h_{i,j}$ and $g_{i,j}$ given in (6.155)-(6.156) below are all considered to be unknown in controller design. All the initials are set as 0 except that subsystem outputs $y_1(0) = 1$, $y_2(0) = 0.4$.

When $\nu_{ij} = \mu_{ij} = 0.01$ for $i = 1, 2$ and $j = 1, 2$, the design parameters are chosen as $k_i = [4, 4]^T, i = 1, 2$, $c_{11} = c_{12} = c_{21} = c_{22} = 0.5, l_{11} = l_{12} = 1_{21} = l_{22} = 0.001$. With the adaptation mechanism on by choosing $\gamma_1 = \gamma_2 = 1$; $\Gamma_1 = \Gamma_2 = 1 \times I_2$, the system outputs y_1, y_2 and the control inputs u_1, u_2 are illustrated in Figures 6.20-6.23. These results verify that the system can be stabilized and the outputs of both nonlinear subsystems converge to zero in the presence of interactions and unmodeled dynamics.

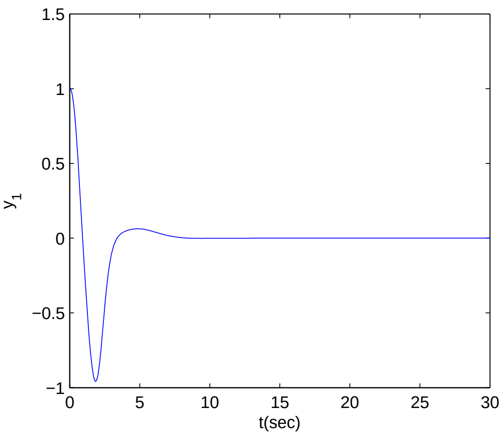


Figure 6.20: System output y_1 with adaptive controllers (Nonlinear case)

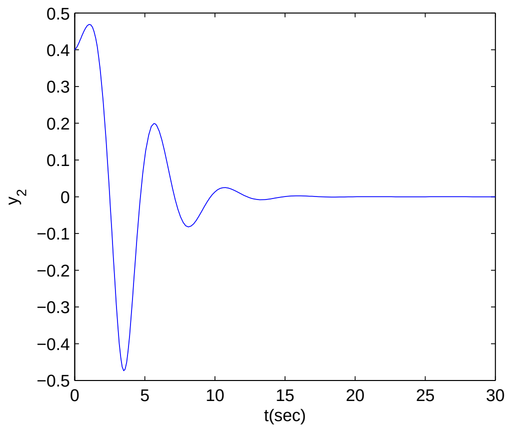


Figure 6.21: System output y_2 with adaptive controllers (Nonlinear case)

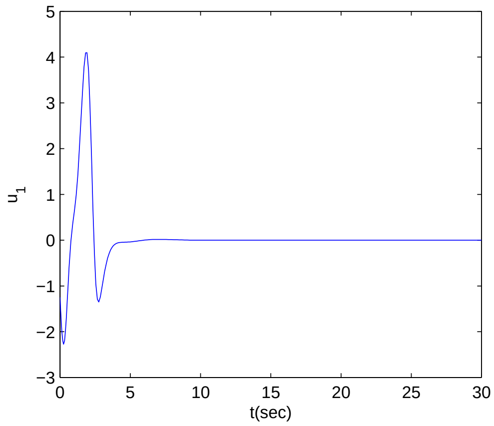


Figure 6.22: Control u_1 with adaptive controllers (Nonlinear case)

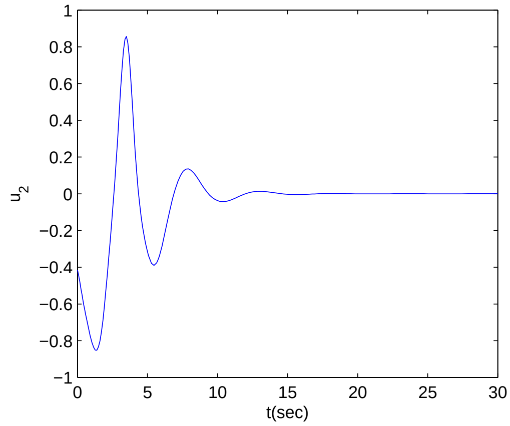


Figure 6.23: Control u_2 with adaptive controllers (Nonlinear case)

6.5 Conclusion

In this chapter, decentralized adaptive output feedback stabilization of interconnected systems with dynamic interactions depending on both subsystem inputs and outputs is considered. Especially, this chapter presents a solution to decentrally stabilize systems with interactions directly depending on subsystem inputs for the first time, when the backstepping technique is used. By using the standard backstepping technique, totally decentralized adaptive controllers are designed. In our design, there is no a priori information on parameters of subsystems and thus they can be allowed totally uncertain. It is established that the proposed decentralized controllers can ensure the overall system globally asymptotically stable. Furthermore, the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms of the system outputs are also shown to be bounded by functions of design parameters. This implies that the transient system performance can be adjusted by choosing suitable design parameters. Simulation results illustrate the effectiveness of our proposed scheme.

Chapter 7

Stability Analysis of Decentralized Adaptive Backstepping Control Systems with Actuator Failures

In this chapter, we analyze the stability of the class of linear interconnected systems in (6.1) in the presence of outage type of actuator failures when our proposed decentralized controllers in Section 6.2.3 are applied. It will be shown that global stability of the closed-loop system can still be ensured and the outputs are also regulated to zero when some subsystems break down due to the failures.

7.1 Introduction

Although a number of decentralized adaptive control schemes have been reported in which some are based on backstepping approach, there are few discussions on the stability of decentralized control systems with actuator failures. Actuator failures may lead to undesirable performances or even instability of the systems, which seem inevitable in practice especially in the control of a complex system. We expect that a control scheme can still ensure the system to continue its operation in spite of actuator failures. In [114], a result on fault tolerance of decentralized adaptive backstepping control was reported based on [28]. However, dynamics depending on subsystem inputs were not considered in [114].

In this chapter, we analyze the stability of interconnected systems with sufficiently weak unmodeled dynamics and dynamic interactions directly depending on subsystem inputs using the control scheme in the last chapter when some subsystems break down. It will be shown that adaptive stabilization of closed-loop system can still be achieved. A numerical simulation example is given to illustrate fault tolerance of the proposed decentralized control system.

7.2 Problem Formulation

As $G_i(s)$ is assumed minimum phase for the ith subsystem in Assumption 6.2.1, we can rewrite the model of the liner interconnected systems in (6.1) as follows.

$$y(t) = \bar{G}(p)\bar{H}(p)u(t) + \bar{\triangle}(p)y(t), \tag{7.1}$$

where

$$\bar{G}(p) = \begin{bmatrix} G_1(p) & 0 & \dots & 0 \\ 0 & G_2(p) & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & G_N(p) \end{bmatrix} \tag{7.2}$$

$$\bar{H}(p) = \begin{bmatrix} 1+\nu_{11}H_{11}(p) & \nu_{12}H_{12}(p) & \dots & \nu_{1N}H_{1N}(p) \\ \nu_{21}H_{21}(p) & 1+\nu_{22}H_{22}(p) & \dots & \nu_{2N}H_{2N}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{N1}H_{N1}(p) & \nu_{N2}H_{N2}(p) & \dots & 1+\nu_{NN}H_{NN}(p) \end{bmatrix} \tag{7.3}$$

$$\bar{\triangle}(p) = \begin{bmatrix} \mu_{11}\Delta_{11}(p) & \mu_{12}\Delta_{12}(p) & \dots & \mu_{1N}\Delta_{1N}(p) \\ \mu_{21}\Delta_{21}(p) & \mu_{22}\Delta_{22}(p) & \dots & \mu_{2N}\Delta_{2N}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{N1}\Delta_{N1}(p) & \mu_{N2}\Delta_{N2}(p) & \dots & \mu_{NN}\Delta_{NN}(p) \end{bmatrix}. \tag{7.4}$$

It should be noted that $u \in \Re^N$ are not only the control inputs of the system, but also the outputs of the actuators. Comparing (7.1)-(7.4) with (6.1), y, p, ν_{ij}, μ_{ij} and with p replaced by s, the corresponding $\Delta_{ij}(s)$, $G_i(s)$ have the same definitions as in Section 6.2.1. Nevertheless, $H_{ij}(s)$ in (7.3) actually denotes $\frac{H_{ij}(s)}{G_i(s)}$ in (6.1). Assumption 6.2.1 and Assumption 6.2.2 are still required, while the latter one needs to be modified accordingly since the definition of $H_{ij}(s)$ is changed.

Assumption 7.2.1. *For each subsystem,*

$$G_i(s) = \frac{B_i(s)}{A_i(s)} = \frac{b_{i,m_i}s^{m_i} + \cdots + b_{i,1}s + b_{i,0}}{s^{n_i} + a_{i,n_i-1}s^{n_i-1} + \cdots + a_{i,1}s + a_{i,0}} \tag{7.5}$$

where $a_{i,j}, j = 0, \dots, n_i - 1$ and $b_{i,k}, k = 0, \dots, m_i$ are unknown constants, $B_i(s)$ is a Hurwitz polynomial. The order n_i, the sign of b_{i,m_i} and the relative degree $\rho_i(= n_i - m_i)$ are known;

Assumption 7.2.2. *For all $i, j = 1, \dots, N$, $H_{ij}(s)$ and $\Delta_{ij}(s)$ are stable, strictly proper and have a unity high frequency gain.*

7.2.1 Model of Actuator Failures

The failure model considered in this chapter is a special outage case similarly as in [114], wherein the outputs of some failed actuators become zero and the corresponding subsystems break down.

$$u_k(t) = 0, \quad t \geq t_1, \quad k = k_1, k_2, \ldots, k_m \tag{7.6}$$

Eqn. (7.6) indicates that for $t \geq t_1$, the kth subsystem, for $k = k_1, k_2, \ldots, k_m$, breaks down and the local feedback loop is cut off. Without loss of generality, we assume that $k_1 < k_2 < \cdots < k_m$. Define a set K as $K = \{k_1, \ldots, k_m\}$. Clearly, $K \subset \{1, \ldots, N\}$.

We then divide the whole system into two sub-interconnected systems. One is composed of the m failed subsystems, while the other consists of the remaining subsystems which are still in operation. For the former one, it is an MIMO system with the outputs as $y_{k_1}, \ldots, y_{k_m}$ and the inputs as the interactions from the latter sub-interconnected system. We can easily derive that the transfer function matrix of this sub-interconnected system is

$$S_K = (I - \bar{\bar{\triangle}}(s))^{-1}, \quad S_K \in R^{m \times m} \tag{7.7}$$

where

$$\bar{\bar{\triangle}} = \begin{bmatrix} \bar{\triangle}_{k_1 k_1} & \cdots & \bar{\triangle}_{k_1 k_m} \\ \vdots & \ddots & \vdots \\ \bar{\triangle}_{k_m k_1} & \cdots & \bar{\triangle}_{k_m k_m} \end{bmatrix}, \tag{7.8}$$

and $\bar{\triangle}_{k_i k_j}$ denotes the (k_i, k_j)th entry of $\bar{\triangle}$. A further assumption on S_K is made as follows.

Assumption 7.2.3. *S_K is proper and the strengths of both interactions and unmodeled dynamics are sufficiently weak to ensure that S_K is stable.*

Remark 7.2.1. Assumption 7.2.3 implies the requirement that the failed subsystems are stable themselves. For instance, when only the kth subsystem breaks down since time T, it's difficult to be stabilized via the interactions from other subsystems. As sketched in Figure 7.1, S_K is composed of only one element with $K = k$ and

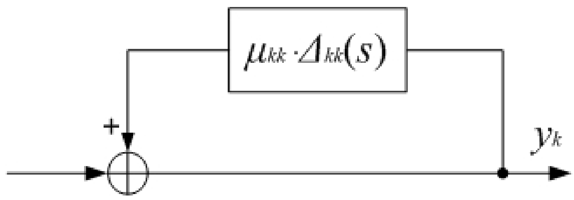


Figure 7.1: Block diagram of kth subsystem when it breaks down

$S_k(s) = (1-\mu_{kk}\Delta_{kk}(s))^{-1}$ is only related to the unmodeled dynamics from its output. If μ_{kk} is small enough to satisfy Small Gain Theorem [94], i.e. $\mu_{kk} < \frac{1}{\|\Delta_{kk}(s)\|_\infty}$ where $\|G(s)\|_\infty$ is defined as $\|G(s)\|_\infty = \sup_\omega \|G(j\omega)\|$, the kth subsystem's internal stability is ensured. Since $\|\Delta(s)\|$ is assumed to be stable and strictly proper, $\|\Delta_{kk}(s)\|_\infty \leq M_k$ where M_k is a positive constant. Thus, $\mu_{kk} < M_k^{-1}$ is required. Similarly, for the case that two or more subsystems break down due to the outage failures, S_K in (7.7) is assured to be stable with $\nu_{k_ik_j}$ and $\mu_{k_ik_j}$ bounded by a positive constant. This chapter only discusses the effectiveness controllers proposed in Chapter 6 in the presence of actuator failures. It's worthy to investigate the actuator compensation control method with Assumption 7.2.3 in future work.

7.2.2 Objective

Similar to (6.3)-(6.4), the ith subsystem in failure-free case has the following state space realization.

$$\dot{x}_i = A_i x_i + a_i x_{i,1} + \begin{bmatrix} 0 \\ b_i \end{bmatrix} u_i \tag{7.9}$$

$$y_i = x_{i,1} + \sum_{j=1}^{N} \nu_{ij} H_{ij}(p) x_{j,1} + \sum_{j=1}^{N} \mu_{ij} \Delta_{ij}(p) y_j, \qquad \text{for} \;\; i = 1, \ldots, N \tag{7.10}$$

where A_i, a_i and b_i are defined the same as in (6.5). With the last two terms in (7.10) eliminated, the remaining state space form corresponds to the local transfer function $G_i(s)$, i.e.,

$$\dot{x}_i = A_i x_i + a_i x_{i,1} + \begin{bmatrix} 0 \\ b_i \end{bmatrix} u_i \tag{7.11}$$

$$y_i = x_{i,1}, \qquad \text{for} \;\; i = 1, \ldots, N \tag{7.12}$$

As mentioned in Section 6.2.1, only the transfer function $G_i(s)$ is considered in the design of a local controller for the ith subsystem. Since the detailed procedures of controller design have been elaborated in Section 6.2.3, they are omitted here.

However, in stability analysis of the overall closed-loop system in failure-free case, the effects of the dynamic interactions and unmodeled dynamics should be considered, i.e.

$$\sum_{j=1}^{N} \nu_{ij} H_{ij}(p) x_{j,1} + \sum_{j=1}^{N} \mu_{ij} \Delta_{ij}(p) y_j, \qquad \text{for} \;\; i = 1, \ldots, N \tag{7.13}$$

The global asymptotical stability of the whole system and regulation of subsystem outputs is failure-free case have already been established, as stated in the Theorem 6.2.1.

Our objective in this chapter is to analyze the effectiveness of the decentralized adaptive control scheme proposed in Section 6.2.3 in the presence of the failures as modeled in (7.6). It should be noted that in failure case, the effects of unmodeled dynamics and dynamic interactions on the remaining system in (7.13) is derived as follows

$$\begin{aligned}
&\sum_{\substack{j=1\\ j\notin K}}^{N} \nu_{ij} H_{ij} x_{j,1} + \sum_{j=1}^{N} \mu_{ij}\Delta_{ij} y_j \\
= &\sum_{\substack{j=1\\ j\notin K}}^{N} \nu_{ij} H_{ij} x_{j,1} + \sum_{\substack{j=1\\ j\notin K}}^{N} \mu_{ij}\Delta_{ij} y_j + \sum_{\substack{j=1\\ j\notin K}}^{N} \sum_{k,l\in K} \mu_{ik}\Delta_{ik} S_{kl}(\nu_{lj} H_{lj} x_{j,1} + \mu_{lj}\Delta_{lj} y_j) \\
= &\sum_{\substack{j=1\\ j\notin K}}^{N} \left(\nu_{ij} H_{ij} + \sum_{k,l\in K} \mu_{ik}\nu_{lj}\Delta_{ik} S_{kl} H_{lj} \right) x_{j,1} \\
&+ \sum_{\substack{j=1\\ j\notin K}}^{N} \left(\mu_{ij}\Delta_{ij} + \sum_{k,l\in K} \mu_{ik}\mu_{lj}\Delta_{ik} S_{kl}\Delta_{lj} \right) y_j, \quad \text{for} \quad i = 1, \ldots, N\backslash K \qquad (7.14)
\end{aligned}$$

where S_{kl} is a (k,l)th entry denoting the transfer function from the input to the lth subsystem to the output of the kth subsystem in the sub-interconnected system composed of m failed subsystems.

7.3 Stability Analysis

By noting the main difference between faulty and normal cases as given in (7.13) and (7.14), (6.22) and (6.23) are changed to the following forms in failure case.

$$\begin{aligned}
\dot{\epsilon}_i = &A_{i,0}\epsilon_i + (a_i - k_i)\left(\sum_{\substack{j=1\\ j\notin K}}^{N} \left(\nu_{ij} H_{ij} + \sum_{k,l\in K} \mu_{ik}\nu_{lj}\Delta_{ik} S_{kl} H_{lj} \right) x_{j,1} \right. \\
&\left. + \sum_{\substack{j=1\\ j\notin K}}^{N} \left(\mu_{ij}\Delta_{ij} + \sum_{k,l\in K} \mu_{ik}\mu_{lj}\Delta_{ik} S_{kl}\Delta_{lj} \right) y_j \right) \qquad (7.15)
\end{aligned}$$

$$\begin{aligned}\dot{y}_i &= b_{i,m_i}v_{i,(m_i,2)} + \xi_{i,(n_i,2)} + \bar{\delta}_i^T\theta_i + \epsilon_{i,2} + (s + a_{i,n_i-1}) \\ &\quad \times \left(\sum_{\substack{j=1 \\ j\notin K}}^{N} \left(\nu_{ij}H_{ij} + \sum_{k,l\in K} \mu_{ik}\nu_{lj}\Delta_{ik}S_{kl}H_{lj} \right) x_{j,1} \right. \\ &\quad \left. + \sum_{\substack{j=1 \\ j\notin K}}^{N} \left(\mu_{ij}\Delta_{ij} + \sum_{k,l\in K} \mu_{ik}\mu_{lj}\Delta_{ik}S_{kl}\Delta_{lj} \right) y_j \right) \end{aligned} \tag{7.16}$$

$z_i(t)$ is still defined as$[z_{i,1}, z_{i,2}, \ldots, z_{i,\rho_i}]^T$. The transformed ith subsystem subject to local controller (6.41) is characterized by

$$\begin{aligned}\dot{z}_i &= A_{zi}z_i + W_{\epsilon i}\epsilon_{i,2} + W_{\theta i}^T\tilde{\theta}_i - b_i^{m_i}\bar{\alpha}_{i,1}\tilde{p}_i e_{\rho_i,1} \\ &\quad + W_{\epsilon i}\left[(s + a_{i,n_i-1}) \left(\sum_{\substack{j=1 \\ j\notin K}}^{N} \left(\nu_{ij}H_{ij} + \sum_{k,l\in K} \mu_{ik}\nu_{lj}\Delta_{ik}S_{kl}H_{lj} \right) x_{j,1} \right.\right. \\ &\quad \left.\left. + \sum_{\substack{j=1 \\ j\notin K}}^{N} \left(\mu_{ij}\Delta_{ij} + \sum_{k,l\in K} \mu_{ik}\mu_{lj}\Delta_{ik}S_{kl}\Delta_{lj} \right) y_j \right) \right] \end{aligned} \tag{7.17}$$

where $A_{zi}, W_{\epsilon i}$ and $W_{\theta i}$ are matrices with appropriate dimensions having the similar structures as in the scalar systems given in (6.48)-(6.50).

Define a Lyapunov function V_{ρ_i} as in (6.51), it can be shown that the derivative of V_{ρ_i} satisfies

$$\dot{V}_{\rho_i} \leq -\sum_{q=1}^{\rho_i} c_{iq}z_{i,q}^2 - \sum_{q=1}^{\rho_i} \frac{1}{4l_{iq}}\|\epsilon_i\|^2 + \sum_{q=1}^{\rho_i} \frac{1}{l_{iq}}(s + a_{i,n_i-1})^2 L_i + \sum_{q=1}^{\rho_i} 2\|\Phi_i\|^2 l_{iq}L_i \tag{7.18}$$

where

$$\Phi_i^T = \sum_{q=1}^{\rho_i} \frac{2}{l_{iq}}(a_i - k_i)^T P_i \tag{7.19}$$

$$\begin{aligned}
Ł_i &= \left(\sum_{\substack{j=1 \\ j\notin K}}^{N} \left(\nu_{ij} H_{ij} + \sum_{k,l\in K} \mu_{ik}\nu_{lj}\Delta_{ik}S_{kl}H_{lj} \right) x_{j,1} \right)^2 \\
&\quad + \left(\sum_{\substack{j=1 \\ j\notin K}}^{N} \left(\mu_{ij}\Delta_{ij} + \sum_{k,l\in K} \mu_{ik}\mu_{lj}\Delta_{ik}S_{kl}\Delta_{lj} \right) y_j \right)^2 \qquad (7.20)
\end{aligned}$$

To show the stability of the overall system, the variables of the filters in (6.11) and the zero dynamics of subsystems should be included in the Lyapunov function. Similarly as discussed in Section 6.2.4, the variables ζ_i associated with the zero dynamics of the ith subsystem can also be shown to satisfy (6.67)-(6.69).

To deal with the dynamic interaction or unmodeled dynamics, we let h_{ij} and g_{ij} be the state vectors of systems with transfer functions $H_{ij}(s)$ and $\Delta_{ij}(s)$, respectively. They are given by

$$\dot{h}_{ij} = A_{hij}h_{ij} + b_{hij}x_{j,1}, \quad H_{ij}(s)x_{j,1} = (1,0,\ldots,0)h_{ij} \qquad (7.21)$$

$$\dot{g}_{ij} = A_{gij}g_{ij} + b_{gij}y_j, \quad \Delta_{ij}(s)y_j = (1,0,\ldots,0)g_{ij} \qquad (7.22)$$

where A_{gij} and A_{hij} are Hurwitz because $\Delta_{ij}(s)$, $H_{ij}(s)$ are assumed stable. Similarly, we let $\bar{h}_{ij}$ and $\bar{g}_{ij}$ be the state vectors associated with $\sum_{k,l\in K} \Delta_{ik}S_{kl}H_{lj}(s)$ and $\sum_{k,l\in K} \Delta_{ik}S_{kl}\Delta_{lj}(s)$, i.e.

$$\dot{\bar{h}}_{ij} = \bar{A}_{hij}\bar{h}_{ij} + \bar{b}_{hij}x_{j,1}, \quad \sum_{k,l\in K} \Delta_{ik}S_{kl}H_{lj}(s)x_{j,1} = (1,0,\ldots,0)\bar{h}_{ij} \qquad (7.23)$$

$$\dot{\bar{g}}_{ij} = \bar{A}_{gij}g_{ij} + \bar{b}_{gij}y_j, \quad \sum_{k,l\in K} \Delta_{ik}S_{kl}\Delta_{lj}(s)y_j = (1,0,\ldots,0)\bar{g}_{ij} \qquad (7.24)$$

where $\bar{A}_{hij}$ and $\bar{A}_{gij}$ are Hurwitz from Assumptions 7.2.2 and 7.2.3. It is obvious that

$$\left\| \sum_{\substack{j=1\\ j\notin K}}^{N} \Delta_{ij}(s) y_j \right\|^2 \leq \|\chi\|^2 \tag{7.25}$$

$$\left\| \sum_{\substack{j=1\\ j\notin K}}^{N} H_{ij}(s) x_{j,1} \right\|^2 \leq \|\chi\|^2 \tag{7.26}$$

$$\left\| \sum_{\substack{j=1\\ j\notin K}}^{N} \left(\sum_{k,l\in K} \Delta_{ik} S_{kl} \Delta_{lj} \right) y_j \right\|^2 \leq \|\chi\|^2 \tag{7.27}$$

$$\left\| \sum_{\substack{j=1\\ j\notin K}}^{N} \left(\sum_{k,l\in K} \Delta_{ik} S_{kl} H_{lj} \right) x_{j,1} \right\|^2 \leq \|\chi\|^2 \tag{7.28}$$

where $\chi = [\chi_1^T, \ldots, \chi_N^T]^T$ and $\chi_i = [z_i^T, \epsilon_i^T, \tilde{\eta}_i^T, \zeta_i^T,\ {h_{i1}}^T, \ldots, {h_{iN}}^T, {g_{i1}}^T, \ldots, {g_{iN}}^T,$ $\bar{h}_{i1}^T, \ldots, \bar{h}_{iN}^T,\ \bar{g}_{i1}^T, \ldots, \bar{g}_{iN}^T]^T$ with $i = 1, \ldots, N\backslash K$.

We also have

$$\left\| (s + a_{i,(n_i-1)}) \sum_{\substack{j=1\\ j\notin K}}^{N} H_{ij}(s) x_{j,1} \right\|^2 \leq k_{i1} \sum_{\substack{j=1\\ j\notin K}}^{N} \|x_{j,1}\|^2 + k_{i2} \|\chi\|^2 \tag{7.29}$$

$$\left\| (s + a_{i,(n_i-1)}) \sum_{\substack{j=1\\ j\notin K}}^{N} \Delta_{ij}(s) y_j \right\|^2 \leq k_{i3} \|\chi\|^2 \tag{7.30}$$

$$\left\| (s + a_{i,(n_i-1)}) \sum_{\substack{j=1\\ j\notin K}}^{N} \left(\sum_{k,l\in K} \Delta_{ik} S_{kl} H_{lj} \right) x_{j,1} \right\|^2 \leq \bar{k}_{i1} \sum_{\substack{j=1\\ j\notin K}}^{N} \|x_{j,1}\|^2 + \bar{k}_{i2} \|\chi\|^2 \tag{7.31}$$

$$\left\| (s + a_{i,(n_i-1)}) \sum_{\substack{j=1\\ j\notin K}}^{N} \left(\sum_{k,l\in K} \Delta_{ik} S_{kl} \Delta_{lj} \right) y_j \right\|^2 \leq \bar{k}_{i3} \|\chi\|^2 \tag{7.32}$$

where k_{i1}, k_{i2}, k_{i3} and $\bar{k}_{i1}, \bar{k}_{i2}, \bar{k}_{i3}$ are positive constants. From

$$\begin{aligned} x_{i,1} &= z_{i,1} - \sum_{\substack{j=1 \\ j \notin K}}^{N} \left(\nu_{ij} H_{ij} + \sum_{k,l \in K} \mu_{ik} \nu_{lj} \Delta_{ik} S_{kl} H_{lj} \right) x_{j,1} \\ &\quad - \sum_{\substack{j=1 \\ j \notin K}}^{N} \left(\mu_{ij} \Delta_{ij} + \sum_{k,l \in K} \mu_{ik} \mu_{lj} \Delta_{ik} S_{kl} \Delta_{lj} \right) y_j, \quad \text{for } i = 1, \ldots, N \backslash K \end{aligned} \tag{7.33}$$

we obtain that

$$\begin{aligned} & \left\| (s + a_{i,n_i-1}) \sum_{\substack{j=1 \\ j \notin K}}^{N} H_{ij}(s) x_{j,1} \right\|^2 \\ \leq & \left[k_{i4} + 2 \left(\max_{\substack{1 \leq, i,j \leq N \\ i,j \notin K}} \{\nu_{ij}^2\} + \max_{\substack{1 \leq, i,j \leq N \\ i,j \notin K}} \{\mu_{ij}^2\} \right) k_{i4} + 2 \left(\max_{\substack{1 \leq, i,j \leq N \\ i,j \notin K, k,l \in K}} \{\mu_{ik}^2 \nu_{lj}^2\} \right.\right. \\ & \left.\left. + \max_{\substack{1 \leq, i,j \leq N \\ i,j \notin K, k,l \in K}} \{\mu_{ik}^2 \mu_{lj}^2\} \right) k_{i4} \right] \|\chi\|^2 \end{aligned} \tag{7.34}$$

$$\begin{aligned} & \left\| (s + a_{i,n_i-1}) \sum_{\substack{j=1 \\ j \notin K}}^{N} \Big(\sum_{k,l \in K} \Delta_{ik} S_{kl} H_{lj} \Big) x_{j,1} \right\|^2 \\ \leq & \left[\bar{k}_{i4} + 2 \left(\max_{\substack{1 \leq, i,j \leq N \\ i,j \notin K}} \{\nu_{ij}^2\} + \max_{\substack{1 \leq, i,j \leq N \\ i,j \notin K}} \{\mu_{ij}^2\} \right) \bar{k}_{i4} + 2 \left(\max_{\substack{1 \leq, i,j \leq N \\ i,j \notin K, k,l \in K}} \{\mu_{ik}^2 \nu_{lj}^2\} \right.\right. \\ & \left.\left. + \max_{\substack{1 \leq, i,j \leq N \\ i,j \notin K, k,l \in K}} \{\mu_{ik}^2 \mu_{lj}^2\} \right) \bar{k}_{i4} \right] \| \chi \|^2 \end{aligned} \tag{7.35}$$

where $k_{i4} = \max\{k_{i2} + 2k_{i1}, 2k_{i1}\}$ and $\bar{k}_{i4} = \max\{\bar{k}_{i2} + 2\bar{k}_{i1}, 2\bar{k}_{i1}\}$ are constants and independent of μ_{ij} and ν_{ij}.

We can now present the main result of this chapter as follows,

Theorem 7.3.1. *Consider the closed-loop adaptive system consisting of the plant*

(7.1) and the controller (6.41), the estimator (6.38), (6.46), and the filters (6.10)-(6.13). Suppose $k_1, \ldots, k_m$ *subsystems break down whose control inputs become zero as modeled in (7.6). Based on Assumption 7.2.1-7.2.3, there still exists a constant* μ^* *such that for all* $\nu_{ij} < \mu^*$ *and* $\mu_{ij} < \mu^*, i, j = 1, 2, \ldots, N$*, all the signals in the system are globally asymptotically stabilized for arbitrary initial* $x_i(0)$*.*

Proof: We define a Lyapunov function for the ith system as

$$\begin{aligned} V_i &= V_{\rho_i} + \frac{1}{l_{\eta i}}\tilde{\eta}_i^T P_i \tilde{\eta}_i + \frac{1}{l_{\zeta i}}\zeta_i^T P_{i,b_i}\zeta_i + \sum_{\substack{j=1\\ j\notin K}}^{N} l_{hij} {h_{ij}}^T P_{hij} h_{ij} + \sum_{\substack{j=1\\ j\notin K}}^{N} l_{gij} {g_{ij}}^T P_{gij} g_{ij} \\ &\quad + \sum_{\substack{j=1\\ j\notin K}}^{N} l_{\bar{h}ij} \bar{h}_{ij}^T P_{\bar{h}ij} \bar{h}_{ij} + \sum_{\substack{j=1\\ j\notin K}}^{N} l_{\bar{g}ij} \bar{g}_{ij}^T P_{\bar{g}ij} \bar{g}_{ij} \end{aligned} \tag{7.36}$$

where $l_{\zeta i}, l_{hij}, l_{gij}, l_{\bar{h}ij}, l_{\bar{g}ij}$ are positive constants, and P_{i,b_i}, P_{hij}, P_{gij}, $P_{\bar{h}ij}$ and $P_{\bar{g}ij}$ satisfy

$$P_{i,b_i}(A_{i,b_i}) + (A_{i,b_i})^T P_{i,b_i} = -I_{m_i} \tag{7.37}$$

$$P_{hij}A_{hij} + (A_{hij})^T P_{hij} = -I_{hij} \tag{7.38}$$

$$P_{gij}A_{gij} + (A_{gij})^T P_{gij} = -I_{gij} \tag{7.39}$$

$$P_{\bar{h}ij}\bar{A}_{hij} + (\bar{A}_{hij})^T P_{\bar{h}ij} = -I_{\bar{h}ij} \tag{7.40}$$

$$P_{\bar{g}ij}\bar{A}_{gij} + (\bar{A}_{gij})^T P_{\bar{g}ij} = -I_{\bar{g}ij} \tag{7.41}$$

By following similar procedures in Section 6.2.4, we compute the time derivative of V_i as follows,

$$\begin{aligned} \dot{V}_i &= \dot{V}_{\rho_i} - \frac{1}{l_{\eta i}}\|\tilde{\eta}_i\|^2 + \frac{2}{l_{\eta i}}\tilde{\eta}_i^T P_i e_{n_i,n_i} z_{i,1} - \frac{1}{l_{\zeta i}}\|\zeta_i\|^2 + \frac{2}{l_{\zeta i}}\zeta_i^T P_{i,b_i}\bar{b}_i x_{i,1} - \sum_{\substack{j=1\\ j\notin K}}^{N} l_{hij}\|h_{ij}\|^2 \\ &\quad +2\sum_{\substack{j=1\\ j\notin K}}^{N} l_{hij} h_{ij}^T P_{hij} b_{hij} x_{j,1} - \sum_{\substack{j=1\\ j\notin K}}^{N} l_{gij}\|g_{ij}\|^2 + 2\sum_{\substack{j=1\\ j\notin K}}^{N} l_{gij} g_{ij}^T P_{gij} b_{gij} y_j - \sum_{\substack{j=1\\ j\notin K}}^{N} l_{\bar{h}ij} \end{aligned}$$

$$
\begin{aligned}
& \times\|\bar{h}_{ij}\|^2 + 2\sum_{\substack{j=1\\ j\notin K}}^{N} l_{\bar{h}ij} h_{ij}^T P_{\bar{h}ij} \bar{b}_{hij} x_{j,1} - \sum_{\substack{j=1\\ j\notin K}}^{N} l_{\bar{g}ij}\|\bar{g}_{ij}\|^2 + 2\sum_{\substack{j=1\\ j\notin K}}^{N} l_{\bar{g}ij} g_{ij}^T P_{\bar{g}ij} \bar{b}_{gij} y_j \\
\leq\; & -\frac{1}{2}c_{i1}z_{i,1}^2 - \sum_{q=2}^{\rho_i} c_{i,q} z_{i,q}^2 - \sum_{q=1}^{\rho_i}\frac{1}{4l_{iq}}\|\epsilon_i\|^2 - \frac{1}{2l_{\eta i}}\|\tilde{\eta}_i\|^2 - \frac{1}{2l_{\zeta i}}\|\tilde{\zeta}_i\|^2 - \sum_{\substack{j=1\\ j\notin K}}^{N}\Bigg(\frac{1}{2}l_{hij} \\
& \times\|h_{ij}\|^2 + \frac{1}{2}l_{gij}\|g_{ij}\|^2 + \frac{1}{2}l_{\bar{h}ij}\|\bar{h}_{ij}\|^2 + \frac{1}{2}l_{\bar{g}ij}\|\bar{g}_{ij}\|^2\Bigg) + \sum_{q=1}^{\rho_i}\frac{1}{d_i^q}(s + a_{i,n_i-1})^2 L_i \\
& + \sum_{q=1}^{\rho_i} 2\|\Phi_i\|^2 l_{iq} L_i - \frac{1}{4l_{\zeta i}}\|\zeta_i\|^2 - \frac{2}{l_{\zeta i}}\zeta_i^T P_{i,b_i}\bar{b}_i\left(\sum_{\substack{j=1\\ j\notin K}}^{N}\Big(\nu_{ij}H_{ij}\right. \\
& \left. + \sum_{k,l\in K}\nu_{ik}\mu_{lj}\Delta_{ik}S_{kl}H_{lj}\Big) x_{j,1} + \sum_{\substack{j=1\\ j\notin K}}^{N}\Big(\mu_{ij}\Delta_{ij} + \sum_{k,l\in K}\mu_{ik}\mu_{lj}\Delta_{ik}S_{kl}\Delta_{lj}\Big) z_{j,1}\right) \\
& - \sum_{\substack{j=1\\ j\notin K}}^{N}\left[\frac{l_{hij}}{4}\|h_{ij}\|^2 + 2l_{hij}h_{ij}^T P_{hij} b_{hij}\left(\sum_{\substack{j=1\\ j\notin K}}^{N}\Big(\nu_{ij}H_{ij} + \sum_{k,l\in K}\nu_{ik}\mu_{lj}\Delta_{ik}S_{kl}H_{lj}\Big) x_{j,1}\right.\right. \\
& \left.\left. + \sum_{\substack{j=1\\ j\notin K}}^{N}\Big(\mu_{ij}\Delta_{ij} + \sum_{k,l\in K}\mu_{ik}\mu_{lj}\Delta_{ik}S_{kl}\Delta_{lj}\Big) y_j\right)\right] \\
& - \sum_{\substack{j=1\\ j\notin K}}^{N}\left[\frac{l_{\bar{h}ij}}{4}\|\bar{h}_{ij}\|^2 + 2l_{\bar{h}ij}\bar{h}_{ij}^T P_{\bar{h}ij} \bar{b}_{hij}\left(\sum_{\substack{j=1\\ j\notin K}}^{N}\Big(\nu_{ij}H_{ij} + \sum_{k,l\in K}\nu_{ik}\mu_{lj}\Delta_{ik}S_{kl}H_{lj}\Big) x_{j,1}\right.\right. \\
& \left.\left. + \sum_{\substack{j=1\\ j\notin K}}^{N}\Big(\mu_{ij}\Delta_{ij} + \sum_{k,l\in K}\mu_{ik}\mu_{lj}\Delta_{ik}S_{kl}\Delta_{lj}\Big) y_j\right)\right] \\
& - \frac{1}{16}c_{i1}z_{i,1}^2 - \sum_{\substack{j=1\\ j\notin K}}^{N}\frac{1}{2}l_{gij}\|g_{ij}\|^2 + 2\sum_{\substack{j=1\\ j\notin K}}^{N} l_{gij}{g_{ij}}^T P_{gij} b_{gij} z_{j,1} - \frac{1}{16}c_{i1}z_{i,1}^2 \\
& - \sum_{\substack{j=1\\ j\notin K}}^{N}\frac{1}{2}l_{\bar{g}ij}\|\bar{g}_{ij}\|^2 + 2\sum_{\substack{j=1\\ j\notin K}}^{N} l_{\bar{g}ij}\bar{g}_{ij}^T P_{\bar{g}ij}\bar{b}_{gij} z_{j,1} - \frac{1}{16}c_{i1}z_{i,1}^2 - \sum_{\substack{j=1\\ j\notin K}}^{N}\frac{l_{hij}}{4}\|h_{ij}\|^2
\end{aligned}
$$

$$+2\sum_{\substack{j=1\\ j\notin K}}^{N} l_{hij}{h_{ij}}^T P_{hij} b_{hij} z_{j,1} - \frac{1}{16}c_{i1}z_{i,1}^2 - \sum_{\substack{j=1\\ j\notin K}}^{N} \frac{l_{\bar{h}ij}}{4}\|\bar{h}_{ij}\|^2$$
$$+2\sum_{\substack{j=1\\ j\notin K}}^{N} l_{\bar{h}ij}\bar{h}_{ij}^T P_{\bar{h}ij}\bar{b}_{hij} z_{j,1} - \frac{1}{8}c_{i1}z_{i,1}^2 - \frac{1}{2l_{\eta i}}\|\tilde{\eta}_i\|^2 + \frac{2}{l_{\eta i}}P_i\tilde{\eta}_i^T e_{n_i,n_i} z_{i,1}$$
$$-\frac{1}{8}c_{i1}z_{i,1}^2 + -\frac{1}{4l_{\zeta i}}\|\zeta_i\|^2 \frac{2}{l_{\zeta i}}\zeta_i^T P_{i,b_i}\bar{b}_i z_{i,1} \qquad (7.42)$$

where (7.33) is used. By taking

$$l_{\eta i} \geq \frac{16 \parallel P_i e_{n_i,n_i} \parallel^2}{c_{i1}} \quad , \quad l_{\zeta i} \geq \frac{32 \parallel P_{i,b_i}\bar{b}_i \parallel^2}{c_{i1}} \qquad (7.43)$$
$$l_{hij} \leq \frac{c_{j1}}{64(N-m) \parallel P_{hij} b_{hij} \parallel^2} \quad , \quad l_{\bar{h}ij} \leq \frac{c_{j1}}{64(N-m) \parallel P_{\bar{h}ij}\bar{b}_{hij} \parallel^2} \qquad (7.44)$$
$$l_{gij} \leq \frac{c_{j1}}{32(N-m) \parallel P_{gij} b_{gij} \parallel^2} \quad , \quad l_{\bar{g}ij} \leq \frac{c_{j1}}{32(N-m) \parallel P_{gij}\bar{b}_{gij} \parallel^2} \qquad (7.45)$$

and using the Young's inequality, we have

$$\dot{V}_i \leq -\beta_i\|\chi_i\|^2 + \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}}(s+a_{i,n_i-1})^2 L_i + \sum_{q=1}^{\rho_i} 2\|\Phi_i\|^2 l_{iq} L_i + \frac{8}{l_{\zeta i}}\|P_{i,b_i}\bar{b}_i\|^2 L_i$$
$$+\sum_{\substack{j=1\\ j\notin K}}^{N} 8 l_{hij}\|P_{hij}b_{hij}\|^2 L_i + \sum_{\substack{j=1\\ j\notin K}}^{N} 8 l_{\bar{h}ij}\|P_{\bar{h}ij}\bar{b}_{hij}\|^2 L_i - \frac{1}{2}c_{i1}z_{i,1}^2$$
$$+\sum_{\substack{j=1\\ j\notin K}}^{N}\frac{1}{4(N-m)}c_{j1}z_{j,1}^2 \qquad (7.46)$$

where

$$\beta_i = \min\left\{\frac{c_{i1}}{4}, c_{i2}, \ldots, c_{i\rho_i}, \sum_{q=1}^{\rho_i}\frac{1}{4l_{iq}}, \frac{1}{2l_{\eta i}}, \frac{1}{2l_{\zeta i}}, \min_{\substack{1\leq i,j\leq N\\ i,j\notin K}}\left\{\frac{1}{2}l_{hij}, \frac{1}{2}l_{gij}\right\},\right.$$
$$\left.\min_{\substack{1\leq i,j\leq N\\ i,j\notin K}}\left\{\frac{1}{2}l_{\bar{h}ij}, \frac{1}{2}l_{\bar{g}ij}\right\}\right\} \qquad (7.47)$$

From (7.25)-(7.35), it can be shown that

$$L_i \leq 4(\mu^2+\mu^4)\|\chi\|^2 \tag{7.48}$$

$$\begin{aligned}(s+a_i^{n_i-1})^2 L_i \leq & \big(2(k_{i4}+k_{i3})\mu^2+(4k_{i4}+2\bar{k}_{i4}+\bar{k}_{i3})\mu^4+(4k_{i4}+4\bar{k}_{i4})\mu^6 \\ & +4\bar{k}_{i4}\mu^8\big)\|\chi\|^2\end{aligned} \tag{7.49}$$

Then the time derivative of $\dot{V}_i$ satisfies

$$\begin{aligned}\dot{V}_i \leq & -\beta_i\|\chi_i\|^2-\frac{1}{4}c_{i1}z_{i,1}^2+k_{i5}\big(2(k_{i4}+k_{i3})\mu^2+(4k_{i4}+2\bar{k}_{i4}+\bar{k}_{i3})\mu^4 \\ & +(4k_{i4}+4\bar{k}_{i4})\mu^6+4\bar{k}_{i4}\mu^8\big)\|\chi\|^2)+4k_{i6}(\mu^2+\mu^4)\|\chi\|^2 \\ & -\left(\frac{1}{4}c_{i1}z_{i,1}^2-\sum_{j=1}^{N}\frac{1}{4(N-m)}c_{j1}z_{j,1}^2\right)\end{aligned} \tag{7.50}$$

where

$$k_{i5} = \sum_{q=1}^{\rho_i}\frac{1}{l_{iq}} \tag{7.51}$$

$$k_{i6} = \sum_{q=1}^{\rho_i}2\|\Phi_i\|^2 l_{iq}+\frac{8}{l_{\zeta i}}\|P_{i,b_i}\bar{b}_i\|^2+\sum_{\substack{j=1\\ j\notin K}}^{N}8l_{hij}\|P_{hij}b_{hij}\|^2+\sum_{\substack{j=1\\ j\notin K}}^{N}8l_{\bar{h}ij}\|P_{\bar{h}ij}\bar{b}_{hij}\|^2 \tag{7.52}$$

Now we define a Lyapunov function of the overall decentralized adaptive control system as

$$V=\sum_{\substack{i=1\\ i\notin K}}^{N}V_i \tag{7.53}$$

From (7.50), the derivative of V gives that

$$\dot{V} \leq -\sum_{\substack{i=1\\ i\notin K}}^{N}\left[\beta-K_{i1}\mu^2-K_{i2}\mu^4-K_{i3}\mu^6-K_{i4}\mu^8\right]\|\chi\|^2-\frac{1}{4}\sum_{\substack{i=1\\ i\notin K}}^{N}c_{i1}z_{i,1}^2 \tag{7.54}$$

where K_{i1}, K_{i2}, K_{i3} and K_{i4} are positive constants and

$$\beta = \frac{\min_{\substack{1\leq i\leq N \\ i\notin K}} \beta_i}{N-m} > 0 \tag{7.55}$$

The existence of positive root to equation $-\beta+K_{i1}\mu^2+K_{i2}\mu^4+K_{i3}\mu^6+K_{i4}\mu^8 = 0$ can be easily shown. By taking μ^* as the smallest positive square root to the equation, we have $\dot{V} \leq -\frac{1}{4}\sum_{\substack{i=1 \\ i\notin K}}^{N} c_{i1}z_{i,1}^2$. This implies that the signals of the rest subsystems are globally uniformly bounded, and $\lim_{t\to\infty} y_i(t) = 0$ for arbitrary initial $x_i(0)$. Since the failed subsystems are assumed stable themselves, the overall stability of closed loop system being ensured is then concluded. □

7.4 An Illustrative Example

We consider an interconnected system composed of two subsystems as described in (7.1) with $G_i(s) = \frac{1}{s(s+a_i)}$ for $i = 1, 2$. Parameters a_1 and a_2 are unknown in controller design and require identification. In the simulation, we make the following choices for the interconnected system: $a_1 = -1, a_2 = -2$, $H_{1j} = \frac{s(s-1)}{(s+1)^3}$, $H_{2j} = \frac{s(s-2)}{(s+1)^3}$ for $j = 1, 2$, $\Delta_{ij} = \frac{1}{s+1}$ for $i = 1, 2$ and $j = 1, 2$ and $\nu_{ij} = \mu_{ij} = 0.6$ for $i = 1, 2$ and $j = 1, 2$. The initials of subsystem outputs are set as $y_1(0) = 1, y_2(0) = 0.4$. The local controller of the first subsystem u_1 breaks down at $t = 10$ sec. In this case, $S_k = (1-0.5\frac{1}{s+1})^{-1} = \frac{s+1}{s+0.5}$ for $k = 1$ is stable and proper as assumed in Assumption 7.2.3.

The design parameters are chosen as $k_i = [4, 4]^T, i = 1, 2$, $c_{11} = c_{21} = 2, c_{12} = c_{22} = 1, l_{11} = l_{12} = 1_{21} = l_{22} = 0.001$, $\Gamma_1 = \Gamma_2 = 0.1$. The outputs and control inputs of both subsystems for both cases with and without actuator failure are given in Figures 7.2-7.5. Clearly, global stability of the system is still be ensured

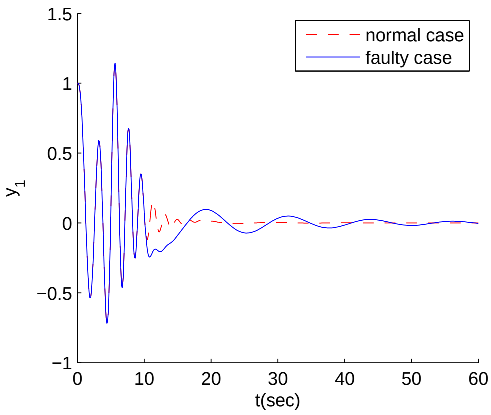


Figure 7.2: Subsystem output y_1 (normal and faulty cases)

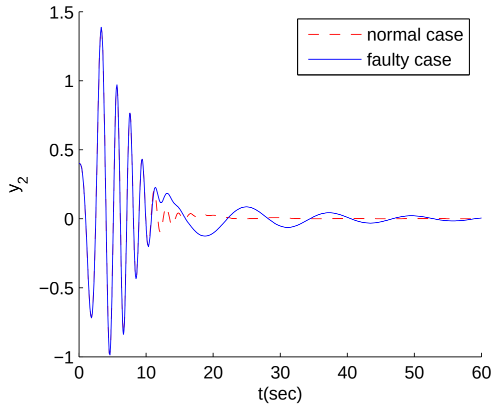


Figure 7.3: Subsystem output y_2 (normal and faulty cases)

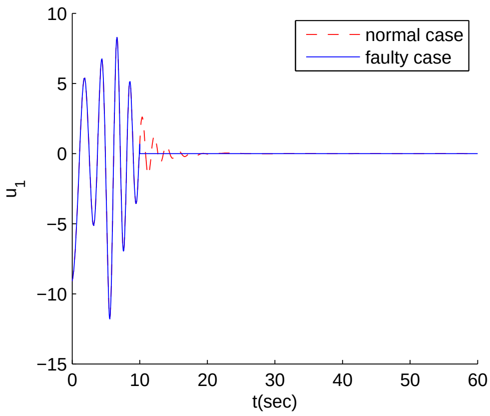


Figure 7.4: Control u_1 (normal and faulty cases)

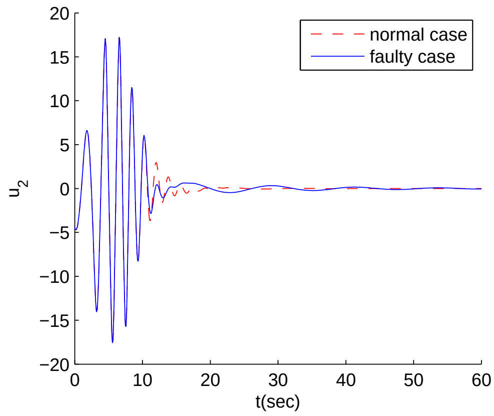


Figure 7.5: Control u_2 (normal and faulty cases)

and the outputs of both subsystems are also regulated to zero in faulty case despite a degradation of performance.

7.5 Conclusion

In this chapter, we studied the effectiveness of the decentralized adaptive backstepping controllers developed in Section 6.2.3 for the class of interconnected systems in in (6.1) with outage type of actuator failures. It is shown that the stability of the interconnected system can still be achieved and the outputs can be regulated as zero in the failure case.

Chapter 8

Conclusion and Recommendations

8.1 Conclusion

The research in this thesis is aimed at developing novel backstepping based approaches to design adaptive controllers for systems with not only unknown parameters, but also uncertain actuator failures and subsystem interactions. The main control objectives are to ensure the boundedness of all the closed-loop signals and achieve desired regulation of the system outputs. Compared to the existing design methods in the related areas, we have solved the following problems:

- relaxing the relative degree condition imposed on the redundant actuators;
- characterizing and improving the transient performance of the adaptive control systems in failure cases;
- compensating for infinite number of actuator failures;
- stabilizing the interconnected systems with unmodeled dynamics and dynamic interactions depending on subsystem inputs;

- analyzing the stability of the decentralized adaptive control systems in the presence of actuator failures.

According to the contributions we have made, the results are reported in Chapters 3-7 respectively. Chapters 3-5 focus on the problems of accommodating uncertain actuator failures, whereas Chapters 6 and 7 mainly discuss the decentralized stabilization in the presence of unmodeled dynamics and dynamic interactions. Apart from these, we further conclude the thesis in the followings aspects.

- **Tuning Functions vs. Modular Design**

Note that Chapter 5 can be separated from Chapters 3 and 4 as it employs adaptive backstepping based modular design method rather than tuning functions approach. As illustrated in Chapter 2, the design and analysis of these two design schemes are quite different. In contrast to the popularity of tuning functions methods, there is still no result available by using backstepping based modular design scheme to compensate for actuator failures even for the case of finite number of failures. Therefore, Chapter 5 can also be regarded as filling the gap that exists in adaptive backstepping based failure compensation approaches. In Chapters 3 and 4, the systems are shown stable in the sense that all the closed-loop signals are bounded and asymptotic tracking can be achieved if the number of failures is finite. Such results can also be obtained with the proposed modular design method, as shown in Chapter 5. In addition to that, Chapter 5 proves the effectiveness of the modular design method in maintaining the closed-loop boundedness with infinite number of failures and establishes the relationship between the frequency of failure pattern changes and the tracking error in the mean square sense.

- **State-feedback vs. Output-feedback**

Chapters 3-7 can also be classified as state-feedback control (Chapters 4 and 5) and output-feedback control (Chapters 3, 6 and 7). As full state measurement is absent in the latter class of control problems, nominal observers are needed to provide desired state estimates. In [21, Sec. 8], some filters are developed to construct the state estimate, with which the estimation error can converge to zero exponentially if the observer is implementable with known system parameters. Based on this, the state estimation filters designed in Chapter 3 are modified by considering also the effects caused by the uncertain actuator failures. It is shown that the estimation error can still vanish exponentially when the system parameters and actuator failures are known. On the other hand, the standard filters in [21] are adopted without any modification in Chapter 6 to estimate the local state variables. However, since the effects of the unmodeled dynamics and dynamic interactions are encompassed, the dynamics of the achieved state estimation error changes. This results in a more complicated process in stability analysis.

- **Transient Performances**

In Chapter 6, the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms of the system outputs are shown to be bounded by functions of design parameters including c_{i1} and adaptation gains. This implies that the transient performance can be adjusted by suitably choosing these parameters on the basis of trajectory initialization. In fact, providing a promising way to improve the transient performance of adaptive systems by tuning design parameters is one of the advantages of adaptive backstepping control over the conventional approaches, as stated in Chapters 1, 6 and references therein. However, it is analyzed in Chapter 4 that the transient performance cannot be guaranteed in the same way in the case with uncertain actuator failures. This is because the trajectory initializations involving state-resetting actions are difficult to perform without a priori knowledge of the failure time, type and value. Nevertheless, by employing a PPB technique to design adaptive backstepping controllers, the tracking error can be preserved within a prescribed performance bound. Therefore, the transient performance of the tracking error in terms of convergence rate and maximum overshoot can be improved by tuning the design parameters of the PPB.

- **Stabilization with both Actuator Failures and Interactions**

Furthermore, Chapter 7 can be regarded as an initial result on decentralized stabilization by comprehensively considering the effects of both actuator failures and subsystem interactions. It is proved that the proposed decentralized adaptive controllers without any modifications are reliable in the face of outage type actuator failures. Nevertheless, the research on developing an effective decentralized adaptive control method by incorporating proper compensation techniques, such that more general failure cases can be handled, will be of greater importance.

8.2 Recommendations for Further Research

Some open problems which are worthy to be explored in the areas of adaptive failure compensation and decentralized adaptive control are suggested as follows.

- Design and analysis of adaptive controllers by using tuning functions method to accommodate infinite number of actuator failures
- Guaranteeing the transient performance of adaptive control systems in the presence of uncertain actuator failures when modular design method is utilized
- Adaptive control of systems with more general type of actuator failures and other component failures including sensor failures
- Extension of our adaptive control design and analysis to a larger class of

systems in the presence of actuator failures, including output-feedback control of nonlinear systems with state dependent nonlinearities

- Adaptive failure compensation control of non-minimum phase systems
- Decentralized adaptive control of interconnected systems with other class of input unmodeled dynamics and dynamic interactions
- Decentralized adaptive control of interconnected systems in the presence of more general type of actuator failures and other component failures
- Decentralized adaptive control of interconnected systems with input time delay
- Possible application of the results in the thesis to flight control systems, marine control systems, chemical processes, etc.

Appendix A

A.1 LaSalle-Yoshizawa Theorem [21]

Consider the time-varying system

$$\dot{x} = f(x,t), \tag{A.1}$$

where $x \in \Re^n$ and $f : \Re^n \times \Re_+ \to \Re^n$ is piecewise continuous in t and locally Lipschitz in x.

Theorem A.1 *Let $x = 0$ be an equilibrium point of (A.1) and suppose f is locally Lipschitz in x uniformly in t. Let $V : \Re^n \to \Re_+$ be a continuouly differentiable, positive definite and radially unbounded function $V(x)$ such that*

$$\dot{V} = \frac{\partial V}{\partial x}(x) f(x,t) \leq -W(x) \leq 0, \ \forall t \geq 0, \ \forall x \in \Re^n, \tag{A.2}$$

where W is a continuous function. Then, all solutions of (A.1) are globally uniformly bounded and satisfy

$$\lim_{t\to\infty} W(x(t)) = 0. \tag{A.3}$$

In addition, if $W(x)$ is positive definite, then the equilibrium $x = 0$ is globally uniformly asymptotically stable.

A.2 Barbalat Lemma [21]

Lemma A.1 *Consider the function $\phi : \Re_+ \rightarrow \Re$. If ϕ is uniformly continuous and $\lim_{t\rightarrow\infty} \int_0^\infty \phi(\tau)d\tau$ exists and is finite, then*

$$\lim_{t\rightarrow\infty} \phi(t) = 0. \tag{A.4}$$

Corollary A.1 *Consider the function $\phi : \Re_+ \rightarrow \Re$. If $\phi, \dot{\phi} \in \mathcal{L}_\infty$, and $\phi \in \mathcal{L}_p$ for some $p \in [1, \infty)$, then*

$$\lim_{t\rightarrow\infty} \phi(t) = 0. \tag{A.5}$$

Appendix B

B.1 Some inequalities [5]

Hölder's Inequality *If* $p, q \in [1, \infty]$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then* $f \in \mathcal{L}_p$*,* $g \in \mathcal{L}_q$ *imply that* $fg \in \mathcal{L}_1$ *and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \tag{B.1}$$

Schwartz Inequality *When* $p = q = 2$*, the Hölder's inequality becomes the Schwartz inequality, i.e.,*

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2 \tag{B.2}$$

If we define the truncated function f_t as

$$f_t(\tau) \triangleq \begin{cases} f(\tau) & 0 \leq \tau \leq t \\ 0 & \tau > t \end{cases} \tag{B.3}$$

for all $t \in [0, \infty)$, then for any $p \in [1, \infty]$, $f \in \mathcal{L}_{pe}$ implies that $f_t \in \mathcal{L}_p$ for any finite t. The $\mathcal{L}_{pe}$ space is called the extended $\mathcal{L}_p$ space and is defined as the set of all functions f such that $f_t \in \mathcal{L}_p$.

The above lemmas also hold for the truncated functions f_t, g_t, respectively, provided that $f, g \in \mathcal{L}_{pe}$.

Young's Inequality *If* $p, q \in [1, \infty)$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then for any* $a, b \geq 0$*, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{B.4}$$

Typically, if $p = q = 2$, then we have $ab \leq \frac{a^2+b^2}{2}$.

Author's Publications

Journal Paper

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