A state space approach to the analysis of frames and wavelets

Chen, Xiaofang

2011

Chen, X. (2011). A state space approach to the analysis of frames and wavelets. Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/46299>

<https://doi.org/10.32657/10356/46299>

A State Space Approach to the Analysis of Frames and Wavelets

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A thesis submitted to the Nanyang Technological University
in fulfillment of the requirements for the degree of
Doctor of Philosophy

2011

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Date

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Chen Xiaofang

To my family.

Acknowledgement

I would like to take this chance to thank those who made this thesis possible. First of all, I owe my deepest gratitude to Professor Zhang Cishen for his encouragement, guidance and support. His impressive enthusiasm and attitude guide me throughout all these years in completing PhD project.

It is my pleasure to thank Professor Xie Lihua and Professor Zhang Jingxin for their opinions and suggestions during these years. I would also like to thank Assistant Professor Su Rong for his excellent supervision and help.

I would like to thank my friends and colleges Chen Ming and Liang Hongzhu for their sincere help and encouragement to my research work.

It is an honor of me to thank my lab technician Yang Xin and Yap Sing Long, and all the other EEE staffs who never hesitate to give me warm assistance.

I am grateful for the understanding and encourage from my husband Li Zengxiang, my parents and parents-in-law. I also would like to thank my angelical daughter. She brings me the courage and enthusiasm to complete my work.

Lastly, I offer my regards and blessings to all who supported me in any respect during the completion of the project.

Summary

Frames exist in source coding, robust transmission and signal reconstruction, etc. Frames introduce redundancies, yielding more flexibility in designs. Mathematically, a frame of a vector space with finite inner product allows each element in the vector space to be written as a linear combination of elements in the frame. The frame elements may be dependent or redundant. From linear system theory point of view, frames can be represented by linear operators, which may be associated with state space representations. Such frames may be modeled as linear time-invariant systems or linear time-varying systems with state space representations.

Frames are often associated with wavelet frames, which possess certain structures. One typical wavelet frame is realized in a discrete wavelet transform (DWT). The DWT partitions an input signal into several bands, where each band is in certain vector space. An inverse DWT (IDWT) reconstructs the input signal from signals in all bands. The IDWT system is an inverse or pseudo-inverse system of the DWT system. Typically the DWT employs an analysis tree-structure multirate filter bank (FB), which may be regarded as the adjoint of the pre-frame operator of the underlying frame. The IDWT may be regarded as the pre-frame operator (synthesis operator) of the underlying dual frame. Some IDWTs of certain wavelets such as Butterworth wavelets, Haar wavelets, Daubechies wavelets, Spline wavelets, employ synthesis tree-structure multirate FBs. State space realizations of such wavelet frames and dual frames can be represented in terms of state space matrices of the

elementary FB block. Another class of wavelet frames is given in an analysis tree-structure FB, but the dual wavelet frame is not realized in a synthesis tree-structure FB, for example, the approximation of Mexican hat wavelet. The dual frame is found by a pseudo-inverse of the given analysis FB system and is represented in terms of state space matrices only.

In studies of frames, frame bounds and frame bound ratio are very important indices characterizing the robustness and numerical performance of frame systems. In general, the smaller the ratio β/α , the better the numerical properties of the frame system, where β and α are respectively the optimal upper and lower frame bounds. For given perturbation energy, the frame bounds α and β provide lower and upper bounds on the resulting reconstruction error energy. The reconstruction error energy is minimized by making α and β as close to each other as possible. The upper frame bound is equivalent to the inverse of the lower frame bound of the dual frame system, and the lower frame bound is equivalent to the inverse of the upper frame bound of the dual frame system.

This thesis presents the author's research work on a state space approach to analyze frames and wavelets. The author develops a more efficient approach to compute frame bounds of some frames that can be modeled as mixed causal-anticausal linear systems, which include linear time-invariant (LTI) case and linear time-varying (LTV) case. As most of the existing techniques are suitable for causal LTI systems only, this thesis shows some techniques on how to convert a stable causal-anticausal realization into an unstable causal realization, and vice versa. A direct state space approach to obtain frame bounds of a mixed causal-anticausal LTI system is developed. The advantage of the proposed approach is that it avoids converting the mixed causal-anticausal representation into an unstable causal realization, hence saving many computation steps. A direct state space approach to obtain the frame bounds of frames modeled as LTV systems is developed similar to the LTI case.

The canonical dual frame system refers to a pseudo-inverse system. The computation of the pseudo-inverse system involves inner-coprime or inner-outer factorizations. Classical factorizations are suitable for causal LTI systems only. Hence stable mixed causal-anticausal realizations of LTI systems have to be converted to unstable causal realizations before factorizations. Causality properties of cascaded causal-anticausal LTI systems are known results. In this thesis, causality properties of cascaded causal-anticausal LTV systems are investigated. The causality properties of cascaded causal-anticausal LTV systems are obtained by making use of time-varying Sylvester equations. These properties can be applied to inner-coprime factorizations. Several algorithms have been presented to obtain the inner-coprime factorizations of strictly causal LTV systems and strictly anticausal LTV systems.

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Symbols and Acronyms

Symbols

$\mathbb{R}$: the set of real numbers

$\mathbb{R}^{q \times p}$: the set of real vectors with size $q \times p$

$\mathbb{C}$: the set of complex numbers

$\mathbb{C}^{q \times p}$: the set of complex vectors with size $q \times p$

$\mathbb{N}$: the set of nature numbers

$\mathbb{Z}$: the set of integer numbers

$\mathbb{H}$: Hilbert space

$L^2(\mathbb{R})$: space of complex-valued functions

$l^2(\mathbb{Z})$: space of square summable scalar sequence with a countable index set $\mathbb{Z}$

$\langle, \rangle$: inner product

$|\cdot|$: magnitude

$\|\cdot\|$: l^2 norm (Euclidean norm)

$\|\cdot\|_\infty$: infinity norm

$(\cdot)^T$: transpose of a matrix or vector

$(.)^*$: Hermitian transpose of a matrix or vector or a function

$(.)^{-1}$: inverse of $(.)$

$(.)^\dagger$: pseudo-inverse of $(.)$

Acronyms

CWT: continuous wavelet transform

DWT: discrete wavelet transform

FB: filter bank

FT: fourier transform

IDWT: inverse discrete wavelet transform

LMI: linear matrix inequality

LPTV: linear periodic time-varying

LTV: linear time-varying

LTI: linear time-invariant

MIMO: multi-input multi-output

PR: perfect reconstruction

STFT: short time fourier transform

Chapter 1

Introduction

1.1 Introduction to frames and frame bounds

A signal is a measurable function, which may be equipped by l^2 norm defined over either a finite or infinite time interval. Hence, these signals form a vector space equipped with l^2 norm. A linear system is a linear mapping from one signal space (input space) to another signal space (output space).

In the study of vector spaces or signal spaces, one of the most important concepts is basis. Each element in a vector space can be written as a linear combination of elements in the basis of the space, where the basis elements are linear independent. Frame is a more flexible and robust tool compared with basis. Each element in the vector space can also be written as a linear combination of elements in the frame. However, the frame elements may be linear dependent. Hence, frames are redundant. A given signal can be represented in a basis system, where its characteristics are more readily apparent in terms of transform coefficients. No redundancy appears in these representations, hence it is serious that if any corruption or loss of transform coefficients happen. If the given signal is represented in a frame system, which

typically has some degree of redundancy, the problems can be avoided.

Frame, as a mathematical theory was introduced by Duffin and Schaeffer [116] in 1950s. Some particular classes of frames have been extensively studied, for example, Gabor frames, which are also called Weyl-Heisenberg frames described in [21] and [57], and wavelet frames, which were introduced in [74], [36], and [37]. Frames have played an important role in signal processing since 1986, see [74] written by Daubechies, Grossman, and Meyer. Frames can also be found in pyramid coding [15]; source coding [78]; denoising [44]; robust transmission [9]; CDMA systems; multiantenna code design; segmentation; classification; restoration and enhancement; signal reconstruction; and so on. [17] and [18] presented the fundamental of frames and more detail descriptions on the frame applications.

In signal processing, the conventional signal analysis tool is the Fourier transform (FT), see [6] [31], which is designed for stationary signals. They could only provide frequency representations of stationary signals. For non-stationary signals with time varying frequencies, the short time Fourier transform (STFT) is employed, see [127]. In the STFT, the signal is divided into small segments where each segment signal is assumed stationary. Longer length of the segment gives better frequency resolution and poorer time resolution; shorter length of the segment gives poorer frequency resolution and better time resolution. The wavelet transform is another signal analysis tool. It is designed to provide time and frequency localizations simultaneously, see [36] and [127]. A wavelet is a wave-like oscillation with amplitude that starts at zero, increases, and then decreases back to zero. The wavelet transform employs a series of dilated version and translated version of the mother wavelet. If the dilation parameter and the translation parameter are parameterized in discrete steps, the wavelet transform is known as the discrete wavelet transform (DWT). The DWT often provides a very redundant description of the original function or signal.

The wavelet transform was first proposed for the analysis of seismic data, see [100]

[69] [70] [60] and [98]. Wavelet-based methods were used to examine electroencephalogram data [8], model distant galaxies [3]. Wavelets have some applications in mathematics area such as estimating densities and modeling time series in statistics [138] and solving partial differential equations [35]. DWTs have been widely used in signal processing such as signal denoising [43] [142] [138] and image processing: image compression [14], edge detection [93] [87], image morphing [67] and digital watermarking [146] [45], etc. More applications of wavelets and wavelet transforms are demonstrated in [99] and [54].

Wavelets and their adaptive versions are known as wavelet packets. A more flexible way of exploiting the time-varying nature of signals is by using time-varying transforms. The properties of the transformation are changed to match the short term properties of the signal. The time-varying wavelets and wavelet packets were introduced in [123] [122] [72] [59] and [117]. Time-varying wavelet packets were used in flexible tree-structured signal expansion shown in [149]. Time-varying wavelet transforms can also be used in image compression [104]. The time-varying DWTs are realized by time-varying filter banks (FBs), which are generally mixed causal-anticausal linear time-varying (LTV) systems.

In the analysis of wavelet transforms, the set of dilated and translated versions of the mother wavelet may construct a frame because of the existing of redundancy. In other words a class of frames has wavelet structures. Some wavelet frames are constructed by DWTs. The DWT gives a hierarchical and fast scheme for the computation of the wavelet coefficients of a given function. DWTs are closely related to multilevel tree-structure multirate FBs see [103]. This relationship leads to a computationally efficient implementation of DWTs and establishes the practical significance of wavelet transforms, especially in audio and image coding and in general signal analysis. The tree-structure multirate filter bank (FB) can be viewed as an application of the non-uniform FB which is defined by an elementary uniform

FB block. Typically the DWT employs an analysis tree-structure multirate FB, which may be regarded as the adjoint of the pre-frame operator (synthesis operator) of the underlying frame. The inverse DWT (IDWT) may employ a synthesis tree-structure multirate FB or may be represented by a state space model, which can be regarded as the synthesis operator of the underlying dual frame.

In signal processing, oversampled FBs have received much attention due to their noise reduction properties and the increased design freedom, that is, there exist more than one perfect reconstruction (PR) synthesis FBs for a given analysis FB. The input signal of an oversampled FB is analyzed with a redundant set of functions, typically obtained by means of a redundant FB. The transmitted signal can be recovered by means of a synthesis FB, which implements the pseudo-inverse of the analysis operator. The theory of frames is a powerful means for the analysis and design of oversampled uniform FBs. In [26], a vector space framework for linear time-invariant (LTI) FB is introduced. In particular, it is true that synthesis functions of a perfect reconstruction (PR) FB form a frame for l^2 space. [137] and [16] studied the properties of oversampled FBs. Necessary and sufficient conditions on a FB to implement a frame or a tight frame decomposition in $l^2(\mathbb{Z})$ were given in terms of the properties of the corresponding polyphase analysis matrix. The frame-theoretic analysis was based on the fact that the polyphase matrices provide a matrix representation of the frame operator, i.e. $S(z) = E^*(z)E(z)$, which was presented in [11]. Recently [19] presented a direct computational method for the frame-theory-based analysis and design of oversampled FBs, which employed state space representations of the polyphase matrices. [19] provided state-space parameterizations of all perfect reconstruction synthesis FB frames for a given analysis FB frame and presented explicit and numerically efficient formulas to compute the optimal frame bounds and obtain the dual FB frame. [19] also constructed a tight FB frame from a given non-tight FB frame.

Frame bounds in signal processing

A frame with lower and upper frame bounds of a vector space is a set of elements in the space, see the definition given in Section 3.2.

The upper frame bound is equivalent to the square of the operator norm of the linear system, which is defined in Section 3.2. If the system is causal and stable, the operator norm is also known as the H_∞ norm. H_∞ norm is an important concept in control area. It indicates the worst case "norm" of the linear system. Hence frames are related to the H_∞ control problems. The lower frame bound is the inverse of the upper frame bound of the canonical dual frame, which corresponds to the pseudo-inverse system. Hence the lower frame bound determines the error amplification in the presence of transmission errors, see [9].

It is pointed out in [106], [11], [105] and [96] that in general, the smaller the ratio β/α , the better the numerical properties of the FB, where β and α are respectively the optimal upper and lower frame bounds. For given perturbation energy, the frame bounds α and β provide lower and upper bounds on the resulting reconstruction error energy. The reconstruction error energy is minimized by making α and β as close to each other as possible. For $\alpha \approx \beta$ or equivalently $\beta/\alpha \approx 1$, which is snug frame, small perturbations of the subband signals yield small reconstruction error. The frame bounds ratio β/α can be related to the oversampling factor N/M (N-channel, subband decimated by M, $N > M$). For a paraunitary FB case, where $\alpha = \beta = N/M$, the reconstruction error variance is inversely proportional to the oversampling factor N/M , which means that more oversampling entails more noise reduction. [33], [37], [102] and [61] have presented such observations.

In the studies of frames, the frame bounds and frame bound ratio are very important indices characterizing the robustness and numerical performance of frame systems. In subband coding and telecommunications, the lower frame bound determines the sensitivity (amplification) of the decoder/receiver to subband/channel noises and

erasures. The quality of the reconstructed signal depends on the lower frame bound, which determines the amount of amplification of the quantization error in a frame-based coding scheme, see [9]. The frame bounds affect directly artifacts in the resulting image in watermarking shown in [94]. In pMRI, the frame bounds indicate the sensitivity of image reconstruction algorithm to the uncertainties of receiver coils and affect directly the quality of reconstructed images [152]. The problem of frame bound ratio minimization for oversampled perfect reconstruction filter banks frames has been investigated in [88].

1.2 Motivations

The traditional frame bounds computation is in the frequency domain. Let $E(z)$ denote the analysis polyphase matrix of an oversampled FB, the frame operator is defined as $S(z) = E^*(z)E(z)$. The upper frame bound β and the lower frame bound α are the essential supremum and essential infimum, respectively, of the eigenvalues of the frame operator. This traditional frequency approach is shown in Section 2.1. The frequency approach is an approximation method in the frequency domain which samples the polyphase matrix of the frame operator over the frequency range $\omega \in [0, 2\pi)$ and then performs eigenanalysis on the sampled matrices. Such sampling approach can be very tedious when the frequency grid is dense and the polyphase matrix is nondiagonal and of infinite impulse response. Moreover, the error due to the frequency-domain sampling of the polyphase matrix cannot be precisely quantified and predicted by the density of the frequency grid for generic oversampled FBs.

Since a proper function $E(z)$ can be represented by the state space representation, [20] presented an alternate approach to compute the frame bounds, which employed the state space representation in the time-domain via linear matrix inequality (LMI)

optimization technique. LMI techniques can be found in [12] and [55]. The state space approach can be found in Section 2.1. This state space approach is based on the time-frequency relationship presented by the KYP lemma, see [114]. This method avoids the frequency-domain sampling and approximation. However, the current state space approach employing LMI optimization technique to obtain the frame bounds is only designed for causal realizations. If the frame is modeled as a mixed causal-anticausal realization, the current state space approach cannot be directly applied.

The classic method to estimate the wavelet frame bounds is shown in [37], which is in the frequency domain, see Section 2.1. This approach requires that the explicitly expression of the wavelet in the frequency domain must be known. For some wavelet frames which are realized by DWTs, the explicitly expressions of the wavelets in the frequency domain are hard to obtain or even not exist, then the classic method to obtain the wavelet frame bounds is not applicable. Furthermore, there is no literature on obtaining wavelet frame bounds in the time domain. Therefore, one may ask if there is a time domain approach to compute the wavelet frame bounds.

In the robust transmission, which is realized by oversampled FBs, the transmission error may be resulted from coefficients missing. It is possible that coefficients in one channel are lost after certain time instance. In this case, the FB can then be viewed as a time-varying FB. The time-varying FB constitutes a frame if there exist upper and lower frame bounds. Similar to the LTI case, the frame bounds are important numerical characteristics in designing LTV systems. [109] has shown that the PR synthesis time-varying FB formed a compactly supported discrete-time frame for l^2 space. The dual frame was given by the analysis time-varying FB. However, [109] did not attempt to find the frame bounds. [47] moved one step further. It showed the equivalence of the H_∞ norm and the corresponding LMIs problem for stable causal LTV systems. The computation of the frame bounds of

general mixed causal-anticausal LTV is still an open question.

The study of causality properties of cascaded mixed causal-anticausal linear systems is an interesting subject. The causality properties of cascaded causal-anticausal LTI systems have been explicitly illustrated in [121]. They can be applied to translate a causal unstable realization into a stable mixed causal-anticausal realization or vice versa. The computation of the canonical dual frame (pseudo-inverse system) usually involves inner-coprime or outer factorizations, where the causality properties of cascaded causal-anticausal systems play an important role. However, the causality properties of cascaded causal-anticausal LTV systems are more complex and retain less attentions.

1.3 Objectives

Motivated by constraints and limitations of the current techniques stated in the above section, the author's work is intended to achieve these following objectives.

- Compute the frame bounds of frames that are modeled as mixed causal-anticausal LTI systems, such as oversampled FBs with bi-infinite dimensional impulse responses.
- Compute the frame bounds of frames that are modeled as mixed causal-anticausal LTV systems, such as switching FBs.
- Analysis the causality properties of cascaded causal-anticausal LTV systems and apply them to inner-coprime/outer factorizations since the computation of the pseudo-inverse system involves inner-coprime factorizations.
- Compute the wavelet frame bounds of one class of wavelet frames, which is realized by the DWT, via the proposed state space approach. Since usually the

wavelets do not have rational function representations, the first challenge is to obtain the state space representations of the wavelet frames. The resulting state space representation should be mixed causal-anticausal if the mother wavelet has bi-directional infinite impulse responses.

1.4 Major contributions

Throughout these years research, the author has successfully achieved the following aspects.

- A state space approach employing LMI optimization technique has been derived to obtain the frame bounds of frames that are modeled as mixed causal-anticausal linear systems including LTI systems and LTV systems, which have state space representations.
- The state space representation of one class of wavelet frames, which is implemented by the DWT, has been illustrated. With the resulting mixed causal-anticausal state space representation, the wavelet frame bounds are found by the proposed state space approach.
- Realization transforms, which includes stable causal-anticausal realizations transformed into unstable causal realizations or vice versa have been presented by making use of the causality properties of cascaded LTI systems.
- The causality properties of cascaded causal-anticausal LTV systems have been studied, which take an important role in obtaining the canonical dual frame. Time-varying Sylvester equations are derived to show the causality properties of cascaded causal-anticausal LTV systems. The inner-coprime factorizations or inner-outer factorizations of LTV systems are shown by applying the causality properties of cascaded causal-anticausal LTV systems.

1.5 Outlines

The rest of the thesis is organized as follows.

Chapter 2 is on literature reviews of frame bounds computation, wavelet frame bounds computation, and linear systems factorizations. The traditional computation methods to get the frame bounds of FB frames, or wavelet frames are in the frequency domain. Another computation approach of frame bounds of FB frames employs LMI optimization technique, but it applies to causal realizations only.

Chapter 3 firstly introduces the notations which are used throughout the whole thesis. Then the fundamental of frames and wavelets are given, followed by several representations of linear systems such as the linear operator representation, the state space representation, and the transfer function representation. Some useful matrix computations are given in this chapter.

Chapter 4 presents the realization transformation, such as transformations from stable causal-anticausal realizations to unstable causal realizations, by using the causality properties of cascaded LTI systems, and illustrates the causality properties of cascaded causal-anticausal LTV systems. The causality properties are important in the computation of dual frames. They can be applied to inner-coprime factorizations, which are essential in the computation of dual frame or pseudo-inverse system.

Chapter 5 shows that a frame, which is formed by the bi-directional impulse response of a linear system, can be represented by a state space representation. The state space approaches employing LMI optimization technique to compute the frame bounds of frames that are modeled as mixed causal-anticausal linear systems have been derived in this chapter. Two classes of frames are considered in this chapter: one class of frame is modeled as mixed causal-anticausal LTI systems and another class of frame is modeled as mixed causal-anticausal LTV systems.

Chapter 6 considers one class of wavelet frames, which is realized by the DWT. The DWT is realized by the analysis tree-structure multirate FB. The dual wavelet frame may be realized in a synthesis tree-structure multirate FB, or may be realized in a black box with a state space representation. The state space representations of such wavelet frames have been generated. Hence the wavelet frame bounds can be obtained via the proposed state space approaches in the time-domain.

Chapter 7 concludes the thesis and recommends some future research works. Several theorem proofs and RQ factorization matlab code are given in Appendix A and B.

Chapter 2

Literature Review

2.1 State-of-the-art of the frame bounds computation

This section will present the state-of-the-art of the computation of the frame bounds of FB frames and wavelet frames. The unit-directional frame is defined as

Definition 2.1.1. *A sequence $\{f_m\}_{m=1}^{\infty}$ of elements in $\mathbb{H}$ is a frame for $\mathbb{H}$ if there exist constants $\alpha, \beta > 0$ such that*

$$\alpha\|f\|^2 \leq \sum_{m=1}^{\infty} |\langle f, f_m \rangle|^2 \leq \beta\|f\|^2, \forall f \in \mathbb{H}.$$

The numbers α, β are called lower and upper frame bounds respectively.

2.1.1 Filter bank frame bounds computation approaches

The classical method to obtain the frame bounds of FB frames is searching the supremum and infimum of eigenvalues of the frame operator, see for example [106],

[11], [19]. [124] presented the polyphase matrix representation of the corresponding frame operator for general oversampled wavelet-type filter banks. [7] stated the frame bounds of an iterated FB. But [7] did not compute the frame bounds directly. Instead [7] made use of the wavelet frame bounds computed in the frequency domain to get the FB frame bounds. All these presented methods are in the frequency domain, see Lemma 2.1.1.

Lemma 2.1.1. *The (tightest possible) frame bounds α and β of a FB frame providing a uniform FB expansion are given by the essential infimum and supremum, respectively, of the eigenvalues $\lambda_n(\omega)$ of the uniform FB matrix $S(e^{j\omega}) = E^*(e^{j\omega})E(e^{j\omega})$:*

$$\begin{aligned}\alpha &= \text{ess inf}_{\omega \in [0, 2\pi), n=0,1,\dots,M-1} \lambda_n(\omega), \\ \beta &= \text{ess sup}_{\omega \in [0, 2\pi), n=0,1,\dots,M-1} \lambda_n(\omega),\end{aligned}$$

where M is the decimation factor of each band.

It follows from Lemma 2.1.1 that the computation of the optimal frame bounds α and β is equivalent to: minimizing β and maximizing α over $\omega \in [0, 2\pi)$ subject to the following constraints:

$$E^*(e^{j\omega})E(e^{j\omega}) \leq \beta I, E^*(e^{j\omega})E(e^{j\omega}) \geq \alpha I.$$

This problem involves infinite-dimensional optimization over $\omega \in [0, 2\pi)$. It can be formulated as a finite dimensional convex optimization problem in terms of LMIs. [20] presented a state space approach employing LMI optimization technique to obtain the frame bounds in the time-domain, which was an application of the KYP-lemma stated in [114]. This is a numeric approach making use of state space matrices in the time domain, see Lemma 2.1.2.

Lemma 2.1.2. *Given $E(z) = D + C(zI - A)^{-1}B$, the problem of minimizing β and*

maximizing α over $\omega \in [0, 2\pi)$ is equivalent to

$$\min_P \beta$$

subject to

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & \beta I \end{bmatrix} \leq 0,$$

$P = P^T$, $\beta > 0$. and

$$\max_Q \alpha$$

subject to

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} Q & 0 \\ 0 & -\alpha I \end{bmatrix} \leq 0,$$

$Q = Q^T$, $\alpha > 0$.

The KYP lemma is one of the most basic tools of system theory. It shows that the frequency dependent matrix inequalities can be solved by frequency independent matrix inequalities. [143], [133] and [134] demonstrated the close relationship between the KYP lemma and problems of linear quadratic optimal control. [114] presented a new elementary proof for the KYP lemma.

Lemma 2.1.3. (*KYP lemma*) Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$ and $\mathbf{M} \in \mathbb{R}^{(n+p) \times (n+p)}$ with (A, B) being controllable and $\det(e^{j\omega}I - A) \neq 0$ for $\omega \in [0, 2\pi)$, the following two statements are equivalent:

$$(i) \begin{bmatrix} (e^{j\omega}I - A)^{-1}B \\ I \end{bmatrix}^* \mathbf{M} \begin{bmatrix} (e^{j\omega}I - A)^{-1}B \\ I \end{bmatrix} \leq 0$$

$\forall \omega \in \mathbb{R}$.

(ii) There exists a symmetric matrix $P \in \mathbb{R}^{n \times n}$ such that $P = P^T$ and

$$\mathbf{M} + \begin{bmatrix} A^T P A - P & A^T P B \\ B^T P A & B^T P B \end{bmatrix} \leq 0.$$

The corresponding equivalence for strict inequalities holds even if (A, B) is not controllable.

If the system $E(z)$ is causal and stable, the upper frame bound is the square of the H_∞ norm. Future more, if this infinite norm is less than 1, there exists a positive definite matrix P such that $E^*(z)E(z) \leq I$ shown in Lemma 2.1.4.

Lemma 2.1.4. (*[147]*) Suppose that the system $E(z) = D + C(zI - A)^{-1}B$ is stable, then, $\|E(z)\|_\infty < 1$ if and only if there exists a matrix $P = P^T > 0$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} < 0,$$

where I is the identical matrix with appropriate dimensions.

It is considered as the Schur complement of the algebraic Riccati inequation:

$$A^T P A - P + (A^T P B + C^T D)(I - D^T D - B^T P B)^{-1}(B^T P A + D^T C) + C^T C < 0.$$

The same result can also be found in [85] chapter 21.

However, the computation approach employing LMI optimization technique stated in [20] is limited for causal realizations only. For a more general frame formed by the bi-directional infinite impulse response of a LTI system, such as wavelet frame, the existing LMI optimization technique cannot be directly applied since these frames have mixed causal-anticausal realizations.

2.1.2 Wavelet frame bounds computation approach

Daubechies constructed the wavelet frame in [74] and proposed the frequency approach to obtain the wavelet frame bounds in [36] [37]. Ole Christensen illustrated the relationship of frames, Riesz bases and wavelets in [29] [30]. The traditional frequency approach to obtain the wavelet frame bounds in the frequency domain stated in [37], [30] is presented in Lemma 2.1.5.

Lemma 2.1.5. *Let $a > 1$, $b > 0$ and $\psi \in L^2(\mathbb{R})$ is given in (6.1). Suppose that*

$$\beta := \frac{1}{b} \sup_{|\gamma| \in [1, a]} \sum_{j, k \in \mathbb{Z}} |\Psi(a^j \gamma) \Psi(a^j \gamma + k/b)| < \infty.$$

Then $\{a^{j/2} \psi(a^j x - kb)\}_{j, k \in \mathbb{Z}}$ is a Bessel sequence with bound β , and for all functions $f \in L^2(\mathbb{R})$,

$$\begin{aligned} \sum_{j, k \in \mathbb{Z}} |\langle f, \psi_{j, k} \rangle|^2 &= \frac{1}{b} \int_{-\infty}^{\infty} |F(\gamma)|^2 \sum_{j \in \mathbb{Z}} |\Psi(a^j \gamma)|^2 d\gamma \\ &\quad + \frac{1}{b} \sum_{k \neq 0} \sum_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} F(\gamma) \overline{F(\gamma - a^j k/b)} \overline{\Psi(a^{-j} \gamma)} \Psi(a^{-j} \gamma - k/b) d\gamma. \end{aligned}$$

If furthermore

$$\alpha := \frac{1}{b} \inf_{|\gamma| \in [1, a]} \left(\sum_{j \in \mathbb{Z}} |\Psi(a^j \gamma)|^2 - \sum_{k \neq 0} \sum_{j \in \mathbb{Z}} |\Psi(a^j \gamma) \Psi(a^j \gamma + k/b)| \right) > 0,$$

then $\{a^{j/2} \psi(a^j x - kb)\}_{j, k \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds α, β .

In Lemma 2.1.5, $F(\gamma)$ and $\Psi(\gamma)$ are the Fourier transformations of f and ψ respectively.

This frequency approach requires that there exists an explicitly expression of the wavelet in the frequency domain.

2.2 State space representations of tree-structure filter banks

The existing literature shows that there are two approaches to obtain the state space representation of tree-structure FB. The first approach treats the tree-structure FB as a nonuniform FB and then performs the lifting technique to obtain the polyphase matrix of the nonuniform FB. The polyphase matrix of nonuniform filter banks can be found in [22], [23], and [25]. With the polyphase matrix found, the state space representation of the polyphase matrix can be obtained. The problem of this approach is that if the FB is a mixed causal-anticausal IIR system, it will be difficult to convert the polyphase matrix representation into the state space representation.

The second approach also treats the tree-structure FB as a nonuniform FB then obtains the transfer function matrix of each band. Each channel has a state space representation, and the corresponding block state space representation for each channel can be obtained making use of the blocking technique. The resulting integrated block state space model can then be found, see [28]. This approach applies to both FIR systems and IIR systems. However, it cannot obtain the state space representation of the $L + 1$ level FB from the state space representation of the L level FB. In other words, to compute the state space representation of the $L + 1$ level FB, one has to start from the transfer function matrix of each channel again.

2.3 Reviews on linear systems

A linear system is a linear mapping from one signal space (the input space) to another signal space (the output space). It satisfies the superposition and scaling properties. Linear systems can be classified into linear time-invariant systems and linear time-varying systems. Time-invariant means that the output of the system

does not depend on the particular time when the input is applied. That is, if an input $x(t)$ produces an output $y(t)$, then the input $x(t - \tau)$ produces the output $y(t - \tau)$. The output due to the input $x(t - \tau)$ does not equal to $y(t - \tau)$ for time-varying case.

Linear systems can be realized causally or noncausally. A causal system is a system where the output depends on the past or current inputs but not future inputs. A system that has some dependence on the future input values is termed as a noncausal system. A system that depends solely on future and current input values is an anticausal system. A mixed causal-anticausal system denotes a system that consists of a causal subsystem and an anticausal subsystem, where the subsystems have same input and the outputs of the subsystems are summed together.

2.3.1 Noncausal linear systems: causality properties

Noncausal systems arise in various practical applications such as speech enhancement, image modeling, image restoration, audio coding, inverse problems and the biomedical application of identifying the human joint dynamics [32] and wavelets. The goal of speech enhancement is to improve the performance of speech communication systems in noisy environment. The background noise is typically suppressed by various forms of noncausal filtering or spectral subtraction schemes [118], [95], [89], [51]. An algorithm to estimate the anticausal component in speech was presented in [13]. The image restoration problem becomes the inverse problem without considering the additive noise. Various image restoration approaches were studied in [83], [77]. In [5], the noncausal system was studied for image modeling and is decomposed into two subsystems: the stable forward (causal) subsystem and stable backward (anticausal) subsystem. Efficient audio coding using perfect reconstruction noncausal IIR FBs was presented in [34]. [110] and [111] investigated the role of the anticausal inverse in multirate FBs. The anticausal inversion for time-

varying FBs was treated in [27]. [4] developed a novel method, which was based on a noncausal recursive allpass filter operating in reversed time, to deal with the nonminimum phase channel equalization. Possible solutions for noncausal filter implementation were considered in [42]. A subspace model identification of the mixed causal, anticausal LTI systems were described in [135]. The key numerical problem in solving this identification problem is the separation of the extended observability matrix of the causal part from that of the anticausal part when a mixture of both is determined from the input-output data. The state space representation of two cascaded LTI systems, causal or noncausal systems was presented in [121]. A new general state-space representation of causal and noncausal LTV systems was introduced in [81]. This state-space merging of both causal and anticausal parts allowed decomposition and recombination manipulations often arising in systems and network theory problems such as realization, filtering, model reduction and approximation, optimal control and estimation. Robust noncausal filtering was studied in [50].

Some wavelets are mixed causal-anticausal systems such as continuous wavelets: Mexican hat wavelets, Meyer wavelets; and discrete wavelets: Butterworth wavelets, coiflets wavelets, splines wavelets, variations on splines wavelets, etc shown in [37], [99], [54].

The causality properties of cascaded causal-anticausal LTI systems are important in the design and analysis of noncausal systems. [121] stated the causality properties of cascaded causal-anticausal LTI systems. The main result of [121] is shown in Lemma 2.3.1.

Lemma 2.3.1. *Assuming the compatibility of the operators, the state space matrices*

of the cascaded causal-anticausal discrete time systems are shown as

$$\begin{aligned} \left[\begin{array}{c|c} A'_2 & B'_2 \\ \hline C'_2 & D'_2 \end{array} \right]_{ac} \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right]_c &= \left[\begin{array}{c|c} A_1 & B_1 \\ \hline D'_2 C_1 + C'_2 X A_1 & D'_2 D_1 + C'_2 X B_1 \end{array} \right]_c \\ &+ \left[\begin{array}{c|c} A'_2 & B'_2 D_1 + A'_2 X B_1 \\ \hline C'_2 & 0 \end{array} \right]_{ac}, \\ \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right]_c \left[\begin{array}{c|c} A'_1 & B'_1 \\ \hline C'_1 & D'_1 \end{array} \right]_{ac} &= \left[\begin{array}{c|c} A_2 & B_2 D'_1 + A_2 Y B'_1 \\ \hline C_2 & D_2 D'_1 + C_2 Y B'_1 \end{array} \right]_c + \left[\begin{array}{c|c} A'_1 & B'_1 \\ \hline D_2 C'_1 + C_2 Y A'_1 & 0 \end{array} \right]_{ac}, \end{aligned}$$

where X, Y are given by the Sylvester equations:

$$\begin{aligned} A'_2 X A_1 - X + B'_2 C_1 &= 0, \\ A_2 Y A'_1 - Y + B_2 C'_1 &= 0. \end{aligned}$$

2.3.2 LTI system inner-coprime factorizations

The lower bound of a frame is the inverse of the upper bound of its canonical dual frame. The dual frame computation involves the inner-coprime or inner-outer factorization. Inner-outer and spectral factorizations play an important role in signal processing [64], control [53] and communication such as blind channel estimation [126] [101] and inverse problems [130] [145]. The factorizations have been extensively studied for proper transfer function matrix, i.e. $G(s) = D + C(sI - A)^{-1}B$ or $G(z) = D + C(zI - A)^{-1}B$, see [62] [141] [113] [75] [91] [63], and rational transfer function matrix corresponding to descriptor systems, i.e. $G(z) = D + C(sE - A)^{-1}B$ or $G(z) = D + C(zE - A)^{-1}B$, [24] [80] [140] [148] [129] [2] [1].

The inner-coprime factorization is a standard tool for causal system with state space matrices A, B, C, D , which is shown in Lemma 2.3.2.

Lemma 2.3.2. Let $E(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c$ be a stabilizable and detectable realization.

Choose F and L such that $A+BF$ and $A+LC$ are both stable. Let $U, V, \tilde{U}, \tilde{V}, N, M, \tilde{N}$, and $\tilde{M}$ be given as follows

$$\begin{aligned} \begin{bmatrix} M \\ N \end{bmatrix} &:= \left[\begin{array}{c|c} A+BF & BZ_r \\ \hline F & Z_r \\ \hline C+DF & DZ_r \end{array} \right] \\ \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} &:= \left[\begin{array}{c|c} A+BF & LZ_l^{-1} \\ \hline F & 0 \\ \hline -(C+DF) & Z_l^{-1} \end{array} \right] \\ \begin{bmatrix} \tilde{M} \\ \tilde{N} \end{bmatrix} &:= \left[\begin{array}{c|cc} A+LC & L & B+LD \\ \hline Z_l C & Z_l & Z_l D \end{array} \right] \\ \begin{bmatrix} U \\ V \end{bmatrix} &:= \left[\begin{array}{c|cc} A+LC & L & -(B+LD) \\ \hline Z_l^{-1} F & 0 & Z_l^{-1} \end{array} \right] \end{aligned}$$

where Z_r and Z_l are any nonsingular matrices. Then $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ are right coprime factorization and left coprime factorization, respectively. And the equation below is satisfied:

$$\begin{bmatrix} V & U \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{U} \\ N & \tilde{V} \end{bmatrix} = I.$$

2.3.3 LTV systems and inner-coprime factorizations

LTV systems have widely been used in signal processing such as signal extrapolation [41], signal estimation [73], data resampling [66], image coding [79] [65], audio coding [49] [107] [120], wireless time-varying multi-path channel [56], 3G WCDMA [125] and etc.

One important class of LTV systems is the linear periodic time-varying (LPTV)

system. LPTV systems and their inverse systems have been extensively studied, e.g. [136] [84] [90] [139] [153] [132], and applied in digital processing and telecommunications such as modeling and designing of transmultiplexing systems [112], speech scrambling [82] and etc. In the study of the periodic and periodic inverse systems, the lifting technique can be adopted which reformulates the periodic time-varying system into a lifted time invariant framework, see [97] [119]. The interrelationships between an LPTV system and the equivalent LTI multi-input multi-output (MIMO) system were given in [38] [39]. [131] computed transfer function matrices of periodic systems. Another representation of LPTV systems, based on the time-varying z -transformation, was presented in [76].

A transfer-function approach to describe LTV discrete systems was developed in [52], later a polyphase representation for LTV FBs was introduced in [108]. In 2004, a new method to estimate reliable time-varying transfer functions and time-varying impulse response functions was presented in [154]. [109] introduced a time-varying polyphase approach to study some basic properties of LTV FBs. It was shown that the synthesis (or analysis) functions of a perfect reconstruction (PR) LTV FB form a frame for l^2 space and become a basis for l^2 only if the synthesis polyphase matrix is invertible.

Another approach to describe LTV systems is using state space representations. An unilateral shift operator Z is used in analyzing LTV systems. With this operator, the state space matrices can be replaced by block-diagonal operators hence the LTV systems look formally equivalent to LTI systems. [86] described the transition behavior of two stationary FBs using the block-diagonal state-space representation. Dullerud presented a new approach for the H_∞ analysis and synthesis problem for LTV systems in [46] and [47]. Lemma 2.3.3 stated below (can be found in [47]) shows the equivalence of the H_∞ norm and the corresponding LMIs for stable causal LTV systems. Similar to LTI systems, LMI optimization technique is an important tool

in H_∞ control of LTV systems, see for example [115], [144].

Lemma 2.3.3. *The following conditions are equivalent.*

(i) $\|C(I - ZA)^{-1}ZB + D\| < 1$ and $1 \notin \text{spec}(ZA)$.

(ii) *There exists $X \in \mathcal{X}$ such that*

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0,$$

where $\mathcal{X}$ is defined as the set that consists of positive definite self-adjoint operator X of the form:

$$X = \begin{bmatrix} X_0 & & & 0 \\ & X_1 & & \\ & & X_2 & \\ 0 & & & \ddots \end{bmatrix} > 0.$$

Patrick and Allen [40] illustrated all the main concepts and problems of LTV system theory. One main topic is the inner-outer, inner-coprime factorizations and spectral factorizations. They are crucial in the pseudo-inverse of LTV systems. [128] considered a class of infinite systems in which the linear operator was represented by a discrete time-varying dynamical system at each time point k , and then reduced to time invariant systems for time point $k \rightarrow \pm\infty$. Inner-outer factorizations were illustrated for both standard LTI case and the case of a finite set of linear equations. [48] considered the causal-anticausal LTV systems inversion. The first step was to produce the minimal external factorization (causally realization). The outer-inner factorizations are then taken. With the outer factor obtained, the inverse of the causal-anticausal LTV system can be found. They considers the LTV systems with

state space equations shown below

$$\begin{aligned}x_{k+1} &= x_k A_k + u_k B_k, \\x'_{k-1} &= x'_k A'_k + u_k B'_k, \\y_k &= x_k C_k + x'_k C'_k + u_k D_k,\end{aligned}$$

with appropriate dimensions.

Chapter 3

Preliminaries

3.1 Notations

$\mathbb{R}(\mathbb{C})$ denotes the set of real (complex) numbers, $\mathbb{R}^{q \times p}(\mathbb{C}^{q \times p})$ denotes the set of real (complex) matrix with size $q \times p$. Let $\mathbb{N}$ denote the set of nature numbers and $\mathbb{Z}$ denote the set of integer numbers. $\mathbb{H}$ denotes the Hilbert space, which is a real or complex inner product space. Let $(\cdot)^T$ denote the transpose of a matrix or vector, and $(\cdot)^*$ denote the Hermitian transpose of a matrix or vector or function, which is also known as the adjoint of $(\cdot)$. The inverse of $(\cdot)$ is denoted as $(\cdot)^{-1}$ and $(\cdot)^\dagger$ is the pseudo-inverse of $(\cdot)$.

$L^2(\mathbb{R})$ is the space of complex-valued functions defined as

$$L^2(\mathbb{R}) := \left\{ f : \mathbb{R} \rightarrow \mathbb{C} \mid f \text{ is measurable and } \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \right\}.$$

$L^2(\mathbb{R})$ is a Hilbert space with respect to the inner product:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g^*(x)dx, f, g \in L^2(\mathbb{R}).$$

$l^2(\mathbb{Z})$ is the space of square summable scalar sequence with a index set $\mathbb{Z}$ defined as

$$l^2(\mathbb{Z}) := \left\{ x_k \in \mathbb{C}, k \in \mathbb{Z} \mid \sum_{k \in \mathbb{Z}} |x_k|^2 < \infty \right\}.$$

The l^2 norm is a vector norm defined for a complex column vector $x = \begin{bmatrix} \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{bmatrix}$ by

$$\|x\| = \sqrt{x^* x}.$$

The $l^2(\mathbb{Z})$ space is a Hilbert space with respect to the inner product

$$\langle x, y \rangle = x^* y = y^* x.$$

Let l_p^2 denote the space consisting of square summable vector sequences such that

$$l_p^2 := \left\{ x_m \in \mathbb{C}^p, m \in \mathbb{Z} \mid \sum_{m \in \mathbb{Z}} \|x_m\|^2 < \infty \right\},$$

in which

$$\|x_m\|^2 = \sum_{k=1}^p |x_m^*(k) x_m(k)|.$$

The Fourier transform of f is defined by:

$$F(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \gamma} dx, \gamma \in \mathbb{R}.$$

The inner product norm obtained in the frequency domain is equivalent to that

obtained in the time domain, which is stated in the Plancherel's equation:

$$\langle F, G \rangle = \langle f, g \rangle, \forall f, g \in L^2(\mathbb{R}), \text{ and } \|F\| = \|f\|,$$

where F and G are the Fourier transforms of f and g .

3.2 Fundamentals of frames and wavelets

A vector space can be represented in terms of frames. All the elements in such vector space can be written as a linear combination of frame elements. This thesis considers a bi-infinite dimensional frame, which is defined as follows.

Definition 3.2.1. *A sequence $\{\hat{f}_i \in \mathbb{R}^{\infty \times 1}\}_{i \in \mathbb{Z}}$ of elements in l^2 is a frame for l^2 if there exist constants $\alpha, \beta > 0$ such that*

$$\alpha \|\hat{v}\|^2 \leq \sum_{i \in \mathbb{Z}} |\langle \hat{v}, \hat{f}_i \rangle|^2 \leq \beta \|\hat{v}\|^2, \forall \hat{v} \in l^2.$$

The constants α and β are called frame bounds.

The optimal lower frame bound is the supremum over all possible lower frame bounds, and the optimal upper frame bound is the infimum over all possible upper frame bounds. Note that the optimal frame bounds are called the frame bounds in short in the rest of the thesis. If the frame bounds $\alpha = \beta$, the frame is called a tight frame.

For a frame $\{\hat{f}_i\}_{i \in \mathbb{Z}}$ in l^2 , the pre-frame operator or the synthesis operator is given by

$$T : l^2 \rightarrow l^2, T\{c_i\}_{i \in \mathbb{Z}} = \sum_{i \in \mathbb{Z}} c_i \hat{f}_i.$$

The adjoint of the pre-frame operator is given by

$$T^* : l^2 \rightarrow l^2, T^* \hat{v} = \{ \langle \hat{v}, \hat{f}_i \rangle \}_{i \in \mathbb{Z}}.$$

By composing T with its adjoint T^* , we obtain the frame operator

$$\hat{S} : l^2 \rightarrow l^2, \hat{S} \hat{v} = TT^* \hat{v} = \sum_{i \in \mathbb{Z}} \langle \hat{v}, \hat{f}_i \rangle \hat{f}_i.$$

The frame operator $\hat{S}$ is bounded by the frame bounds α and β , invertible, self-adjoint, and positive.

If $\{\hat{f}_i\}_{i \in \mathbb{Z}}$ is a frame for l^2 , the pre-frame operator can be shown as an infinite dimensional matrix

$$T = \begin{bmatrix} \cdots & \hat{f}_{-1} & \hat{f}_0 & \hat{f}_1 & \hat{f}_2 & \cdots & \hat{f}_i & \cdots \end{bmatrix}, \quad (3.1)$$

i.e. the infinite dimensional matrix has the vectors $\hat{f}_i$ as columns. The adjoint of the pre-frame operator can be shown as

$$T^* = \begin{bmatrix} \vdots \\ \hat{f}_{-1}^T \\ \hat{f}_0^T \\ \hat{f}_1^T \\ \hat{f}_2^T \\ \vdots \\ \hat{f}_i^T \\ \vdots \end{bmatrix}, \quad (3.2)$$

i.e. the infinite dimensional matrix T^* has the vectors $\hat{f}_i^T$ as rows.

The frame $\{\hat{g}_i\}_{i \in \mathbb{Z}} = \{\hat{S}^{-1} \hat{f}_i\}_{i \in \mathbb{Z}}$ is called the canonical dual frame of $\{\hat{f}_i\}_{i \in \mathbb{Z}}$, which

satisfies

$$\hat{v} = \sum_{i \in \mathbb{Z}} \langle \hat{v}, \hat{g}_i \rangle \hat{f}_i, \forall \hat{v} \in l^2.$$

If α, β are the optimal frame bounds for $\{\hat{f}_i\}_{i \in \mathbb{Z}}$, then β^{-1}, α^{-1} are the optimal frame bounds for the canonical dual frame $\{\hat{S}^{-1}\hat{f}_i\}_{i \in \mathbb{Z}}$. Hence, the lower bound of a frame is equivalent to the inverse of the upper bound of the canonical dual frame.

Define the blocking operator $\mathcal{B}_p : l^2 \rightarrow l_p^2$ as follows:

$$\mathcal{B}_p \hat{v} = v = \{\cdots, v_{-1}, v_0, v_1, \cdots\},$$

where

$$v_k = \begin{bmatrix} \hat{v}_{kp} \\ \hat{v}_{kp+1} \\ \vdots \\ \hat{v}_{kp+p-1}(k) \end{bmatrix}, \hat{v}_k \in \mathbb{R}, k \in \mathbb{Z}. \quad (3.3)$$

The corresponding deblocking operator is $\mathcal{B}_p^{-1} : l_p^2 \rightarrow l^2$ such that $\mathcal{B}_p^{-1}v = \hat{v}$. It is clear that $\|v\| = \|\hat{v}\|$ and $\mathcal{B}_p$ and $\mathcal{B}_p^{-1}$ satisfies $\|\mathcal{B}_p\| = \|\mathcal{B}_p^{-1}\|$. For $v, f \in l_p^2$, define the inner product between v and f as

$$\langle v, f \rangle = \sum_k \langle v_k, f_k \rangle. \quad (3.4)$$

It follows that $\langle v, f \rangle = \langle \hat{v}, \hat{f} \rangle$.

Applying the blocking operator to the frame $\{\hat{f}_i\}$ yields

$$f_i = \mathcal{B}_p \hat{f}_i, i \in \mathbb{Z}. \quad (3.5)$$

If $\{\hat{f}_i\}$ is a frame for l^2 space, the sequence $\{f_i\}$ is a frame for l_p^2 space in the sense

$$\alpha\|v\|^2 \leq \sum_{i \in \mathbb{Z}} |\langle v, f_i \rangle|^2 \leq \beta\|v\|^2, \forall v \in l_p^2.$$

Following from the equivalence between $\{\hat{f}_i\}$ and $\{f_i\}$, it is straightforward that $\{f_i\}$ is a tight frame for l_p^2 if $\{\hat{f}_i\}$ is a tight frame for l^2 . $\{g_i\}$ is a dual frame of $\{f_i\}$ satisfying

$$v = \sum_{i \in \mathbb{Z}} \langle v, g_i \rangle f_i, \forall v \in l_p^2, \quad (3.6)$$

iff $\{\hat{g}_i\}$ is a dual frame of $\{\hat{f}_i\}$, and $\{g_i\}$ with $g_i = S^{-1}f_i, i \in \mathbb{Z}$ is a canonical dual frame of $\{f_i\}$ if $\{\hat{g}_i\}$ is a canonical dual frame of $\{\hat{f}_i\}$, where $S : l_p^2 \rightarrow l_p^2$ is the frame operator of $\{f_i\}$ given by

$$Sv = \sum_{i \in \mathbb{Z}} \langle v, f_i \rangle f_i \quad (3.7)$$

One special class of frames has wavelet structure. The continuous wavelet is written as:

$$\psi_{a,b}(x) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right),$$

where $a, b \in \mathbb{R}$ and $a \neq 0$, and a is the dilation parameter, b is the translation parameter. The continuous wavelet transform (CWT) with respect to this wavelet family is written as:

$$\langle f, \psi_{a,b} \rangle = \int f(x) |a|^{-1/2} \psi^*\left(\frac{x-b}{a}\right) dx.$$

If the dilation and translation parameters are in discrete steps, the wavelet transform becomes the discrete wavelet transform. The corresponding discretely labeled wavelets are therefore shown as:

$$\psi_{m,n}(x) = a_0^{-m/2} \psi(a_0^{-m}x - nb_0). \quad (3.8)$$

As DWTs employ redundant set of functions, the family of wavelets may constitute a frame. The wavelet frames are defined as follows.

Definition 3.2.2. *Let $a_0 > 1$, $b_0 > 0$ and $\psi \in L^2(\mathbb{R})$. A frame for $L^2(\mathbb{R})$ of the form $\{a_0^{-m/2}\psi(a_0^{-m}x - nb_0)\}_{m,n \in \mathbb{Z}}$ is called a wavelet frame.*

3.3 Linear system representations

Linear systems can be represented by linear operators. The LTI system has rational transfer function matrix representation in the frequency domain. A convenient and useful representation of linear systems is the state space representation. In this section, all these several representations of linear systems are presented.

3.3.1 Linear time-invariant systems representations

The state space equations of causal-anticausal LTI systems are given as

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ x'_{k-1} &= A'x'_k + B'u_k, \\ y_k &= Cx_k + C'x'_k + (D + D')u_k, \end{aligned} \tag{3.9}$$

where $x_k \in \mathbb{R}^{n_c}$ is the forward state, $x'_k \in \mathbb{R}^{n_{ac}}$ is the backward state, $u_k \in \mathbb{R}^p$ is the system input, $y_k \in \mathbb{R}^q$ is the system output, respectively, $A \in \mathbb{R}^{n_c \times n_c}$, $B \in \mathbb{R}^{n_c \times p}$, $C \in \mathbb{R}^{q \times n_c}$, $D \in \mathbb{R}^{q \times p}$, $A' \in \mathbb{R}^{n_{ac} \times n_{ac}}$, $B' \in \mathbb{R}^{n_{ac} \times p}$, $C' \in \mathbb{R}^{q \times n_{ac}}$, $D' \in \mathbb{R}^{q \times p}$ are the state space matrices of the system. Let

$$E_c(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c = D + C(zI - A)^{-1}B$$

represent the causal LTI system and

$$E_{ac}(z) = \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac} = D' + C'(z^{-1}I - A')^{-1}B'$$

represent the anticausal LTI system. Hence the mixed causal-anticausal LTI system can be represented by

$$\begin{aligned} E(z) &= E_c(z) + E_{ac}(z) \\ &= D + D' + C(zI - A)^{-1}B + C'(z^{-1}I - A')^{-1}B' \\ &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac}. \end{aligned}$$

Let

$$u = \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{bmatrix}, \quad y = \begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix}. \quad (3.10)$$

To make the consistency of the symbols, the operator of the mixed causal-anticausal LTI system (3.9) is denoted as E such that

$$y = Eu,$$

where

$$E = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & D + D' & C'B' & C'A'B' & \cdots \\ \cdots & CB & D + D' & C'B' & \cdots \\ \cdots & CAB & CB & D + D' & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (3.11)$$

The entries of the system operator E (i^{th} row and j^{th} column, $i, j \in \mathbb{Z}$) can be shown as:

$$E_{ij} = \begin{cases} CA^{i-j-1}B, & i > j \\ D + D', & i = j \\ C'(A')^{j-i-1}B', & i < j. \end{cases}$$

An LTI system is causal if the system operator E is left lower-triangular matrix and anticausal if E is right upper-triangular matrix.

The mixed causal-anticausal LTI system (3.9) is stable if and only if

$$|\rho(A)| < 1, \text{ and } |\rho(A')| < 1,$$

where $\rho(\cdot)$ indicates the largest eigenvalue of $(\cdot)$, which means that $E(z)$ has no poles on the unit circle.

The system operator is a linear map from one normed vector space $\mathcal{M}$ to another normed vector space $\mathcal{N}$, shown as $E : \mathcal{M} \rightarrow \mathcal{N}$. Mathematically speaking, the operator norm β is a means to measure the 'size' of the linear operator, which satisfies:

$$\|Eu\|^2 \leq \beta\|u\|^2, \forall u \in \mathcal{M}.$$

In this thesis, the input and output spaces are denoted as l^2 -normed vector space $\mathcal{M}$ and $\mathcal{N}$ respectively. The l^2 -normed vector spaces $\mathcal{M}$ and $\mathcal{N}$ are defined as:

$$\mathcal{M} := \left\{ \{u_k\}_{k \in \mathbb{Z}} \subseteq \mathbb{R}^p \mid \sum_{k \in \mathbb{Z}} |u_k|^2 < \infty \right\},$$

$$\mathcal{N} := \left\{ \{y_k\}_{k \in \mathbb{Z}} \subseteq \mathbb{R}^q \mid \sum_{k \in \mathbb{Z}} |y_k|^2 < \infty \right\}.$$

3.3.2 Linear time-varying system representations

LTV systems do not have proper transfer function representations in the frequency domain. The classic representations of LTV systems are state space representations given as

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k, \\ x'_{k-1} &= A'_k x'_k + B'_k u_k, \\ y_k &= C_k x_k + C'_k x'_k + (D_k + D'_k) u_k, \end{aligned} \quad (3.12)$$

where $x_k \in \mathbb{R}^{n_c}$ is the forward state, $x'_k \in \mathbb{R}^{n_{ac}}$ is the backward state, $u_k \in \mathbb{R}^p$ is the system input, $y_k \in \mathbb{R}^q$ is the system output. $A_k \in \mathbb{R}^{n_c \times n_c}$, $B_k \in \mathbb{R}^{n_c \times p}$, $C_k \in \mathbb{R}^{q \times n_c}$, $D_k \in \mathbb{R}^{q \times p}$, $A'_k \in \mathbb{R}^{n_{ac} \times n_{ac}}$, $B'_k \in \mathbb{R}^{n_{ac} \times p}$, $C'_k \in \mathbb{R}^{q \times n_{ac}}$ and $D'_k \in \mathbb{R}^{q \times p}$ are the state matrices of the system.

The mixed causal-anticausal LTV system (3.12) is exponentially stable if its state matrices satisfy

$$\begin{aligned} l_A &= \lim_{k \rightarrow \infty} \sup_i \|A_{i+k} \cdots A_i\|^{1/k} < 1, \\ l_{A'} &= \lim_{k \rightarrow \infty} \sup_i \|A'_{i+k} \cdots A'_i\|^{1/k} < 1. \end{aligned} \quad (3.13)$$

Similar to the LTI case, u, y follows (3.10). The mixed causal-anticausal LTV system (3.12) can be represented in the following infinite dimensional matrix equation

$$y = Eu, \quad (3.14)$$

where

$$E = \begin{bmatrix} \ddots & & & & \\ \cdots & E_{11} & E_{12} & E_{13} & \\ \cdots & E_{21} & E_{22} & E_{23} & \\ \cdots & E_{31} & E_{32} & E_{33} & \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (3.15)$$

The entries of the system operator E (i^{th} row and j^{th} column, $i, j \in \mathbb{Z}$) can be shown

as:

$$E_{ij} = \begin{cases} C_i A_{i-1} \cdots A_{j+1} B_j, & i > j, \\ D_i + D'_i, & i = j, \\ C'_i A'_{i+1} \cdots A'_{j-1} B'_j, & i < j. \end{cases}$$

An LTV system is causal if its system operator in the form E is left lower-triangular matrix and anticausal if its system matrix in the form E is right upper-triangular matrix.

Following from the notation in [47], a shift operator Z is introduced operating on a signal vector $s = [\cdots, s_{-1}, \underline{s_0}, s_1, \cdots]^T$, such that

$$Z^{-1}s = [\cdots, s_0, \underline{s_1}, s_2, \cdots]^T,$$

or

$$Zs = [\cdots, s_{-2}, \underline{s_{-1}}, s_0, \cdots]^T.$$

The matrix representation of Z is given by

$$Z = \begin{bmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & I & 0 & & & \\ & & I & 0 & & \\ & & & I & 0 & \\ & & & & \ddots & \ddots \end{bmatrix} \quad (3.16)$$

with appropriate dimensions. Z is unitary on l^2 . It satisfies $ZZ^* = I$ and $Z^*Z = I$, such that $Z^{-1} = Z^*$.

Let the states follow

$$A = \begin{bmatrix} \ddots & & \mathbf{0} \\ & A_k & \\ \mathbf{0} & & \ddots \end{bmatrix}, \quad B = \begin{bmatrix} \ddots & & \mathbf{0} \\ & B_k & \\ \mathbf{0} & & \ddots \end{bmatrix},$$

$$C = \begin{bmatrix} \ddots & & \mathbf{0} \\ & C_k & \\ \mathbf{0} & & \ddots \end{bmatrix}, \quad D = \begin{bmatrix} \ddots & & \mathbf{0} \\ & D_k & \\ \mathbf{0} & & \ddots \end{bmatrix},$$

and A', B', C', D' follows the same definitions. Let

$$x = [\cdots, x_{-1}, x_0, x_1, \cdots]^T,$$

$$x' = [\cdots, x'_{-1}, x'_0, x'_1, \cdots]^T,$$

and u, y defined in (3.10), the causal-anticausal system (3.12) can be represented as

$$\begin{aligned} Z^{-1}x &= Ax + Bu, \\ Zx' &= A'x' + B'u, \\ y &= Cx + C'x' + (D + D')u. \end{aligned}$$

Let

$$E_c = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c = D + C(I - ZA)^{-1}ZB$$

denote causal subsystem and

$$E_{ac} = \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac} = D' + (ZI - A')^{-1}B'$$

denote anticausal subsystem. Hence the mixed causal-anticausal LTV system can

be represented as

$$E = E_c + E_{ac} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac}.$$

The k^{th} time instant realization is given as:

$$E_{c,k} = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array} \right]_c, \quad E_{ac,k} = \left[\begin{array}{c|c} A'_k & B'_k \\ \hline C'_k & D'_k \end{array} \right]_{ac}.$$

For both stable LTI and LTV systems, the output y can be represented by the linear operator E such that

$$y = Eu.$$

The operator norm (induced l^2 norm) of the linear system is defined as

$$\|E\| = \sup_{u \in l^2(\mathbb{Z}), u \neq 0} \frac{\|Eu\|}{\|u\|}.$$

3.4 Useful matrix computations

Schur complement

In linear algebra and the theory of matrices, the Schur complement of a matrix block (i.e., a submatrix within a larger matrix) is defined as follows. Suppose A , B , C , D are respectively $\mathbb{R}^{p \times p}$, $\mathbb{R}^{p \times q}$, $\mathbb{R}^{q \times p}$ and $\mathbb{R}^{q \times q}$ matrices, and D is invertible. Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

so that M is a $\mathbb{R}^{(p+q) \times (p+q)}$ matrix.

Then the Schur complement of the block D of the matrix M is the $\mathbb{R}^{p \times p}$ matrix

$$A - BD^{-1}C.$$

Schur decomposition

The Schur decomposition is defined as follows: if $A \in \mathbb{C}^{n \times n}$, then A can be expressed as

$$A = QUQ^{-1}$$

where Q is a unitary matrix, and U is an upper triangular matrix, which is called a Schur form of A . The matrix Q is unitary matrix, yielding $Q^{-1} = Q^*$.

Matrix factorizations: QR, RQ, QL, LQ

Any real square matrix A may be decomposed as

$$A = QR,$$

where Q is an orthogonal matrix (its columns are orthogonal unit vectors meaning $Q^T Q = I$) and R is an upper triangular matrix (also called right triangular matrix). This generalizes to a complex square matrix A and a unitary matrix Q . If A is nonsingular, then the factorization is unique if the diagonal elements of R are positive.

The LQ factorization can be yielded by the QR factorization. $A = LQ$ is equivalent to $A^T = Q^T L^T$, where Q^T is the orthogonal matrix and L^T is the upper triangular matrix.

The RQ factorization of A is such that $A = RQ$. The thesis adopts the matlab code for RQ factorization written by Bruno Luong, which is shown in Appendix B.

The QL factorization can be obtained by the RQ factorization. $A = QL$ is equivalent

to $A^T = L^T Q^T$, where L^T is the upper triangular matrix and Q^T is the orthogonal matrix.

Chapter 4

The causality properties of cascaded causal-anticausal linear systems

4.1 Introduction

The causality properties of cascaded causal-anticausal linear systems are essential in the analysis of linear systems. First of all, they can be used in realization transformations, such as from a stable causal-anticausal realization to an unstable causal realization, or vice versa. Secondly, the causality properties can be employed in the frame analysis. For example, given a frame that can be modeled as a causal LTI system with transfer function $E(z)$, the frame operator is represented as $S(z) = E^*(z)E(z)$. In this case $E(z)$ is a causal realization, hence $E^*(z)$ is an anticausal realization. This may result in a causal realization or an anticausal realization or a mixed causal-anticausal realization of the frame operator $S(z)$. The causality properties are also important in looking for dual frame systems, which refer the pseudo-inverse systems. The reason is that the causality properties can be applied

to inner-coprime/outer factorizations of LTV systems, which take an important role in computing the pseudo-inverse system.

One of the contributions of the current thesis is to achieve realization conversion of LTI systems by using the causality properties of cascaded LTI systems. And another one of the contributions of the current thesis is to present the causality properties of cascaded LTV systems and apply them to inner-outer factorizations. This chapter will illustrate these two contributions achieved by the author.

4.2 The causality properties of cascaded causal-anticausal LTI systems

The causality properties of cascaded causal-anticausal LTI systems are shown in [121]. The main results are presented in lemma 2.3.1. These causality properties can be applied in computing unstable causal realizations from variance realizations.

4.2.1 Unstable causal realizations of stable causal-anticausal LTI systems

Assume a mixed causal-anticausal stable LTI system is in the state space representation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac}.$$

Applying the similarity transform to the anticausal subsystem, the new state space matrices of the anticausal subsystem are obtained and shown as

$$\begin{aligned}\hat{A}' &= K^{-1}A'K = \begin{bmatrix} \hat{A}'_1 & 0 \\ 0 & \hat{A}'_2 \end{bmatrix}, & \hat{B}' &= K^{-1}B' = \begin{bmatrix} \hat{B}'_1 \\ \hat{B}'_2 \end{bmatrix}, \\ \hat{C}' &= C'K = \begin{bmatrix} \hat{C}'_1 & \hat{C}'_2 \end{bmatrix}, & \hat{D}' &= D',\end{aligned}$$

in which K is an invertible matrix. And $\hat{A}'_1 \in \mathbb{R}^{n_d \times n_d}$, $\hat{A}'_2 \in \mathbb{R}^{(n_{ac}-n_d) \times (n_{ac}-n_d)}$, $\hat{B}'_1 \in \mathbb{R}^{n_d \times p}$, $\hat{B}'_2 \in \mathbb{R}^{(n_{ac}-n_d) \times p}$, $\hat{C}'_1 \in \mathbb{R}^{q \times n_d}$, $\hat{C}'_2 \in \mathbb{R}^{q \times (n_{ac}-n_d)}$ and $\hat{D}' \in \mathbb{R}^{q \times p}$.

Let

$$\left[\begin{array}{c|c} \hat{A}'_1 & \hat{B}'_1 \\ \hline \hat{C}'_1 & \hat{D}'_1 = 0 \end{array} \right]_{ac} \quad (4.1)$$

present the anticausal FIR system, where $\hat{A}'_1$ retains all zero eigenvalues of A' , and let

$$\left[\begin{array}{c|c} \hat{A}'_2 & \hat{B}'_2 \\ \hline \hat{C}'_2 & \hat{D}'_2 = D' \end{array} \right]_{ac} \quad (4.2)$$

present the anticausal IIR system, where $\hat{A}'_2$ is invertible, i.e. all the eigenvalues are nonzero.

The anticausal FIR subsystem with the state space representation $\left[\begin{array}{c|c} \hat{A}'_1 & \hat{B}'_1 \\ \hline \hat{C}'_1 & \hat{D}'_1 \end{array} \right]_{ac}$ can be realized causally with delays, where the n_d actually decides the delay step z^{-n_d} . The state equations of z^{-n_d} are shown as:

$$\begin{aligned}x_{d,k+1} &= A_d x_{d,k} + B_d y_k, \\ y_{d,k} &= C_d x_k + D_d y_k,\end{aligned} \quad (4.3)$$

where $A_d \in \mathbb{R}^{n_d \times n_d}$, $B_d \in \mathbb{R}^{n_d \times q}$, $C_d \in \mathbb{R}^{q \times n_d}$ and $D_d \in \mathbb{R}^{q \times q}$. Figure 4.1 (a) realizes the causal-anticausal LTI system (3.9). And the equivalent causal-anticausal LTI system realization is shown in Figure 4.1 (b).

Figure 4.1: Causal-anticausal LTI system realization

The following theorem computes the state space representation of the delay system $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array} \right]_c$, and the resulting causal realization of the anticausal FIR system with delay. This theorem employs the causality properties of cascaded LTI systems.

Theorem 4.2.1. *Consider an anticausal FIR system with state equations (4.1), the computation of the delay system (4.3) is shown by:*

$$\begin{bmatrix} Y \hat{A}_1' & I & Y \hat{B}_1' \\ \hat{C}_1' & 0 & \hat{D}_1' \end{bmatrix} = \begin{bmatrix} A_d^T & C_d^T \\ B_d^T & D_d^T \end{bmatrix} \begin{bmatrix} Y & A_r & B_r \\ 0 & C_r & D_r \end{bmatrix}, \quad (4.4)$$

where

$$\hat{A}_1'^T X \hat{A}_1' + \hat{C}_1'^T \hat{C}_1' = X.$$

and $Y^T Y = X$. The computation involves QR factorization.

Proof: The anticausal FIR system $\left[\begin{array}{c|c} \hat{A}'_1 & \hat{B}'_1 \\ \hline \hat{C}'_1 & \hat{D}'_1 \end{array} \right]_{ac}$ is cascaded with the delay system $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array} \right]_c$, yielding a causal delayed system such that

$$\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array} \right]_c \left[\begin{array}{c|c} \hat{A}'_1 & \hat{B}'_1 \\ \hline \hat{C}'_1 & \hat{D}'_1 \end{array} \right]_{ac} = \left[\begin{array}{c|c} A_d & B_d \hat{D}'_1 + A_d Y \hat{B}'_1 \\ \hline C_d & D_d \hat{D}'_1 + C_d Y \hat{B}'_1 \end{array} \right]_c,$$

where

$$\begin{aligned} A_d Y \hat{A}'_1 - Y + B_d \hat{C}'_1 &= 0, \\ D_d \hat{C}'_1 + C_d Y \hat{A}'_1 &= 0. \end{aligned} \tag{4.5}$$

This result is yielded by lemma 2.3.1. Let $\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array} \right]_c = \left[\begin{array}{c|c} A_d & B_d \hat{D}'_1 + A_d Y \hat{B}'_1 \\ \hline C_d & D_d \hat{D}'_1 + C_d Y \hat{B}'_1 \end{array} \right]_c$.

The matrix formed by A_r, B_r, C_r, D_r can be rewritten as:

$$\left[\begin{array}{cc} A_r & B_r \\ C_r & D_r \end{array} \right] = \left[\begin{array}{cc} A_d & B_d \\ C_d & D_d \end{array} \right] \left[\begin{array}{cc} I & Y \hat{B}'_1 \\ 0 & \hat{D}'_1 \end{array} \right] \tag{4.6}$$

The equations in (4.5) yield:

$$\begin{bmatrix} Y \\ 0 \end{bmatrix} = \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} \begin{bmatrix} Y \hat{A}'_1 \\ \hat{C}'_1 \end{bmatrix}. \tag{4.7}$$

Thus the equations (4.6) and (4.7) together form equation (4.4).

Pre-multiplying both sides of equation (4.7) with its transpose results in

$$\hat{A}'_1{}^T X \hat{A}'_1 + \hat{C}'_1{}^T \hat{C}'_1 = X.$$

and $Y^T Y = X$. $\square$

While the anticausal FIR subsystem is realized causally with delays, the anticausal IIR subsystem is realized causally by the following theorem.

Theorem 4.2.2. *An anticausal stable IIR system $\left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac}$ has a causal unstable realization:*

$$\left[\begin{array}{c|c} (A')^{-1} & -(A')^{-1}B' \\ \hline C'(A')^{-1} & D' - C'(A')^{-1}B' \end{array} \right]_c.$$

Proof: The anticausal stable IIR system has the transfer function $E_{ac}(z)$ as

$$E_{ac}(z) = \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac} = D' + C'(z^{-1}I - A')^{-1}B'.$$

The system is anticausal stable and is IIR system, the state matrix A' is invertible and all the eigenvalues of it are inside the unit disk. Thus, $E_{ac}(z)$ can be rewritten as

$$\begin{aligned} E_{ac}(z) &= D' + C'(z^{-1}I - A')^{-1}B' \\ &= D' + C'(z^{-1}(A')^{-1} - I)^{-1}(A')^{-1}B' \\ &= D' - C'(I - z^{-1}(A')^{-1})^{-1}(A')^{-1}B' \\ &= D' - C'(I + z^{-1}(A')^{-1}(I - z^{-1}(A')^{-1})^{-1})(A')^{-1}B' \\ &= D' - C'(A')^{-1}B' - C'(A')^{-1}(zI - (A')^{-1})^{-1}(A')^{-1}B' \\ &= \left[\begin{array}{c|c} (A')^{-1} & -(A')^{-1}B' \\ \hline C'(A')^{-1} & D' - C'(A')^{-1}B' \end{array} \right]_c. \end{aligned}$$

□

The state space representation of the original causal-anticausal system $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c +$

$\left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac}$ cascaded with the delay system $\left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array} \right]_c$ is given by:

$$\begin{aligned} & \left[\begin{array}{cc|c} A_d & B_d C & B_d D \\ \hline 0 & A & B \\ \hline C_d & D_d C & D_d D \end{array} \right]_c + \left[\begin{array}{c|c} A_d & B_d \hat{D}'_1 + A_d Y_1 \hat{B}'_1 \\ \hline C_d & D_d \hat{D}'_1 + C_d Y_1 \hat{B}'_1 \end{array} \right]_c \\ & + \left[\begin{array}{c|c} A_d & B_d \hat{D}'_2 + A_d Y_2 \hat{B}'_2 \\ \hline C_d & D_d \hat{D}'_2 + C_d Y_2 \hat{B}'_2 \end{array} \right]_c + \left[\begin{array}{c|c} \hat{A}'_2 & \hat{B}'_2 \\ \hline D_d \hat{C}'_2 + C_d Y_2 \hat{A}'_2 & 0 \end{array} \right]_{ac}. \end{aligned}$$

The anticausal subsystem has a causal realization shown as

$$\left[\begin{array}{c|c} \hat{A}'_2 & \hat{B}'_2 \\ \hline D_d \hat{C}'_2 + C_d Y_2 \hat{A}'_2 & 0 \end{array} \right]_{ac} = \left[\begin{array}{c|c} (\hat{A}'_2)^{-1} & -(\hat{A}'_2)^{-1} \hat{B}'_2 \\ \hline (D_d \hat{C}'_2 + C_d Y_2 \hat{A}'_2)(\hat{A}'_2)^{-1} & -(D_d \hat{C}'_2 + C_d Y_2 \hat{A}'_2)(\hat{A}'_2)^{-1} \hat{B}'_2 \end{array} \right]_c,$$

where $(\hat{A}'_2)^{-1}$ indicates the inverse of $\hat{A}'_2$. And Y_1, Y_2 satisfy the following conditions:

$$\begin{aligned} Y_1 &= A_d Y_1 \hat{A}'_1 + B_d \hat{C}'_1, \\ Y_2 &= A_d Y_2 \hat{A}'_2 + B_d \hat{C}'_2. \end{aligned}$$

4.2.2 Stable causal-anticausal realizations of unstable causal LTI systems

If the given unstable system is represented by $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac}$ or

$\left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c$, where A and A' have all the eigenvalues inside the unit disk, lemma 2.3.1 is applied directly to obtain the stable causal-anticausal realization. If the unstable system is given in causal realization, Theorem 4.2.3 is applied to obtain the causal-anticausal stable realization.

Theorem 4.2.3. *Given an unstable causal system $E = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c$, there is a stable causal-anticausal realization such that*

$$E = \hat{E}_c + \hat{E}_{ac} = \left[\begin{array}{c|c} \hat{A}_1 & \hat{B}_1 \\ \hline \hat{C}_1 & D \end{array} \right]_c + \left[\begin{array}{c|c} (\hat{A}_2)^{-1} & -(\hat{A}_2)^{-1}\hat{B}_2 \\ \hline \hat{C}_2(\hat{A}_2)^{-1} & -\hat{C}_2(\hat{A}_2)^{-1}\hat{B}_2 \end{array} \right]_{ac},$$

where

$$\begin{aligned} \hat{A} &= K^{-1}AK = \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{bmatrix}, \quad \hat{B} = K^{-1}B = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \\ \hat{C} &= CK = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}, \quad \hat{D} = D, \end{aligned}$$

in which K is invertible and $\hat{A}_2$ has all the eigenvalues outside the unit disk.

Proof: The similarity transform realization of the unstable causal state space realization is given by

$$\begin{aligned} \hat{A} &= K^{-1}AK = \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{bmatrix}, \quad \hat{B} = K^{-1}B = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \\ \hat{C} &= CK = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix}, \quad \hat{D} = D, \end{aligned}$$

where K is invertible and $\hat{A}_2$ has all the eigenvalues outside the unit disk. The strictly unstable realization

$$\left[\begin{array}{c|c} \hat{A}_2 & \hat{B}_2 \\ \hline \hat{C}_2 & 0 \end{array} \right]_c$$

can be realized anticausally:

$$\begin{aligned}
\left[\begin{array}{c|c} \hat{A}_2 & \hat{B}_2 \\ \hline \hat{C}_2 & 0 \end{array} \right]_c &= \hat{C}_2(zI - \hat{A}_2)^{-1}\hat{B}_2 \\
&= \hat{C}_2(z(\hat{A}_2)^{-1} - I)^{-1}(\hat{A}_2)^{-1}\hat{B}_2 \\
&= -\hat{C}_2(I - z(\hat{A}_2)^{-1})^{-1}(\hat{A}_2)^{-1}\hat{B}_2 \\
&= -\hat{C}_2(I + z(\hat{A}_2)^{-1}(I - z(\hat{A}_2)^{-1})^{-1})(\hat{A}_2)^{-1}\hat{B}_2 \\
&= -\hat{C}_2(\hat{A}_2)^{-1}\hat{B}_2 - \hat{C}_2(\hat{A}_2)^{-1}(z^{-1}I - (\hat{A}_2)^{-1})^{-1}(\hat{A}_2)^{-1}\hat{B}_2 \\
&= \left[\begin{array}{c|c} (\hat{A}_2)^{-1} & -(\hat{A}_2)^{-1}\hat{B}_2 \\ \hline \hat{C}_2(\hat{A}_2)^{-1} & -\hat{C}_2(\hat{A}_2)^{-1}\hat{B}_2 \end{array} \right]_{ac}
\end{aligned}$$

□

4.2.3 Discussion on other cases to obtain causal realizations

The existing state space approach to get the frame bounds via LMI optimization technique applies to causal realizations only. It applies to both stable or unstable cases. Hence, if given systems are realized other than causal realizations, they have to be converted to causal realizations in order to apply the existing state space approach. This section discusses on other cases to get causal realizations.

Case 1: Stable anticausal realizations

This is a special case of the stable causal-anticausal LTI system with anticausal part only. Hence, this case follows Section 4.2.1.

Case 2: Unstable anticausal realizations

Apply the similarity transform to the unstable anticausal realization to separate the stable anticausal part and strictly unstable anticausal part, in which the strictly unstable anticausal part contains all the poles outside the unit circle. Hence the

strictly unstable anticausal part can be realized causally and stable. This can be concluded in Theorem 4.2.4. The stable anticausal part can be realized causally following case 1 result.

Theorem 4.2.4. *Given an unstable anticausal system $E = \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac}$, there is stable causal-anticausal realization such that*

$$E = \hat{E}_{ac} + \hat{E}_c = \left[\begin{array}{c|c} \hat{A}'_1 & \hat{B}'_1 \\ \hline \hat{C}'_1 & D' \end{array} \right]_{ac} + \left[\begin{array}{c|c} (\hat{A}'_2)^{-1} & -(\hat{A}'_2)^{-1}\hat{B}'_2 \\ \hline \hat{C}'_2(\hat{A}'_2)^{-1} & -\hat{C}'_2(\hat{A}'_2)^{-1}\hat{B}'_2 \end{array} \right]_c,$$

where

$$\begin{aligned} \hat{A}' &= K^{-1}A'K = \begin{bmatrix} \hat{A}'_1 & 0 \\ 0 & \hat{A}'_2 \end{bmatrix}, \quad \hat{B}' = K^{-1}B' = \begin{bmatrix} \hat{B}'_1 \\ \hat{B}'_2 \end{bmatrix}, \\ \hat{C}' &= C'K = \begin{bmatrix} \hat{C}'_1 & \hat{C}'_2 \end{bmatrix}, \quad \hat{D}' = D', \end{aligned}$$

in which K is invertible and $\hat{A}'_2$ has all the eigenvalues outside the unit disk.

Proof: The similarity transform realization of the unstable causal state space realization is given by

$$\begin{aligned} \hat{A}' &= K^{-1}A'K = \begin{bmatrix} \hat{A}'_1 & 0 \\ 0 & \hat{A}'_2 \end{bmatrix}, \quad \hat{B}' = K^{-1}B' = \begin{bmatrix} \hat{B}'_1 \\ \hat{B}'_2 \end{bmatrix}, \\ \hat{C}' &= C'K = \begin{bmatrix} \hat{C}'_1 & \hat{C}'_2 \end{bmatrix}, \quad \hat{D}' = D', \end{aligned}$$

where K is invertible and $\hat{A}'_2$ has all the eigenvalues outside the unit disk. Hence the strictly unstable anticausal realization

$$\left[\begin{array}{c|c} \hat{A}'_2 & \hat{B}'_2 \\ \hline \hat{C}'_2 & 0 \end{array} \right]_{ac}$$

can be realized causally:

$$\begin{aligned}
\left[\begin{array}{c|c} \hat{A}'_2 & \hat{B}'_2 \\ \hline \hat{C}'_2 & 0 \end{array} \right]_{ac} &= \hat{C}'_2(z^{-1}I - \hat{A}'_2)^{-1}\hat{B}'_2 \\
&= \hat{C}'_2(z^{-1}(\hat{A}'_2)^{-1} - I)^{-1}(\hat{A}'_2)^{-1}\hat{B}'_2 \\
&= -\hat{C}'_2(I - z^{-1}(\hat{A}'_2)^{-1})^{-1}(\hat{A}'_2)^{-1}\hat{B}'_2 \\
&= -\hat{C}'_2(I + z^{-1}(\hat{A}'_2)^{-1}(I - z^{-1}(\hat{A}'_2)^{-1})^{-1})(\hat{A}'_2)^{-1}\hat{B}'_2 \\
&= -\hat{C}'_2(\hat{A}'_2)^{-1}\hat{B}'_2 - \hat{C}'_2(\hat{A}'_2)^{-1}(zI - (\hat{A}'_2)^{-1})^{-1}(\hat{A}'_2)^{-1}\hat{B}'_2 \\
&= \left[\begin{array}{c|c} (\hat{A}'_2)^{-1} & -(\hat{A}'_2)^{-1}\hat{B}'_2 \\ \hline \hat{C}'_2(\hat{A}'_2)^{-1} & -\hat{C}'_2(\hat{A}'_2)^{-1}\hat{B}'_2 \end{array} \right]_c
\end{aligned}$$

□

Case 3: mixed causal-anticausal realizations with stable anticausal part

This case is studied in Section 4.2.1.

Case 4: mixed causal-anticausal realizations with unstable anticausal part

The first step is applying the similarity transform to the unstable anticausal part to separate it into stable anticausal subsystem and strictly unstable anticausal subsystem following case 2 result. The resulting stable mixed causal-anticausal realization is then converted to the unstable causal realization following the analysis in Section 4.2.1.

4.3 The causality properties of cascaded causal-anticausal LTV systems

The causality properties of cascaded LTI systems are related to the Sylvester equation of the form $A'XA - X + B'C = 0$ or $AYA' - Y + BC' = 0$, where A, B, C, D

and A', B', C', D' are state matrices of the causal and anticausal systems, respectively. Similarly, the structure and causality properties of cascaded LTV systems are related to a time-varying Sylvester equation. In order to derive the time-varying Sylvester equation, cascaded LTV systems are considered, in which one LTV system is causal with block-diagonal representation $E_c = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c$ and another LTV system is anticausal with $E_{ac} = \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac}$. Hence $E_{ac}E_c$ denote the cascaded system operator, and let $(E_{ac}E_c)_{i,j}$ be block entries of the cascaded system operator. The matrix entry $(E_{ac}E_c)_{i,j}$ (i^{th} row and j^{th} column) can be written as

$$(E_{ac}E_c)_{i,j} = \begin{cases} D'_i D_i + \sum_{k=i+1}^{\infty} C'_i A'_{i+1} A'_{i+2} \cdots A'_{k-1} B'_k C_k A_{k-1} \cdots A_{i+2} A_{i+1}, & i = j, \\ (D'_i C_i + C_i \sum_{k=i+1}^{\infty} A'_{i+1} A'_{i+2} \cdots A'_{k-1} B'_k C_k A_{k-1} \cdots A_{i+1} A_i) A_{i-1} \cdots A_{j+1} B_j, & i > j, \\ C'_i A'_{i+1} \cdots A'_{j-1} (B'_j D_j + \sum_{k=j+1}^{\infty} A'_j A'_{j+1} \cdots A'_{k-1} B'_k C_k A_{k-1} \cdots A_{j+1} B_j), & i < j. \end{cases}$$

Define

$$X_i = \sum_{k=i}^{\infty} A'_i A'_{i+1} \cdots A'_{k-1} B'_k C_k A_{k-1} \cdots A_{i+1} A_i. \quad (4.8)$$

A sufficient condition for the existence of X_i is that the causal and anticausal systems (3.12) are both exponentially stable, i.e. $l_A < 1$ and $l_{A'} < 1$. From the definition of the matrix X_i in equation (4.8), it is routine to obtain the time-varying Sylvester equation:

$$A'_i X_{i+1} A_i - X_i + B'_i C_i = 0. \quad (4.9)$$

The block diagonal form of equation (4.9) is represented by:

$$A' X^{(+1)} A - X + B' C = 0, \quad (4.10)$$

where $X^{(+1)} = Z^* X Z$.

Consider a causal LTV system E_c cascaded with an anticausal LTV system E_{ac} . Using the time-varying Sylvester equation (4.9) or (4.10), the causality properties of cascaded systems are shown in the following theorem.

Theorem 4.3.1. *The cascaded LTV system $E_{ac}E_c$ can be represented as a causal system in parallel with a strictly anticausal system and the state space realizations of these systems are,*

$$E_{ac}E_c = \bar{E}_c + \bar{E}_{ac},$$

where the k^{th} time instant state space matrices are given as:

$$\begin{aligned} \bar{E}_{c,k} &= \left[\begin{array}{c|c} A_k & B_k \\ \hline D'_k C_k + C'_k X_{k+1} A_k & D'_k D_k + C'_k X_{k+1} B_k \end{array} \right]_c \\ \bar{E}_{ac,k} &= \left[\begin{array}{c|c} A'_k & B'_k D_k + A'_k X_{k+1} B_k \\ \hline C'_k & 0 \end{array} \right]_{ac}, \end{aligned} \quad (4.11)$$

in which X_k satisfies equation (4.9).

Proof: The cascaded system $E_{ac}E_c$ can be represented by:

$$\begin{aligned} E_{ac}E_c &= (D' + C'(ZI - A')^{-1}B')(D + C(I - ZA)^{-1}ZB) \\ &= D'D + C'(ZI - A')^{-1}B'D + D'C(I - ZA)'ZB \\ &\quad + C'(ZI - A')^{-1}B'C(I - ZA)'ZB \end{aligned} \quad (4.12)$$

Substituting equation (4.10) into equation (4.12) yields

$$\begin{aligned}
E_{ac}E_c &= D'D + C'(ZI - A')^{-1}B'D + D'C(I - ZA)'ZB \\
&\quad + C'(ZI - A')^{-1}B'C(I - ZA)^{-1}ZB \\
&= D'D + C'(ZI - A')^{-1}B'D + D'C(I - ZA)'ZB + C'(ZI - A')^{-1} \\
&\quad (X - A'X^{(+1)}A)(I - ZA)^{-1}ZB \\
&= D'D + C'(ZI - A')^{-1}B'D + D'C(I - ZA)'ZB + C'(ZI - A')^{-1} \\
&\quad [ZX^{(+1)}Z^* - A'X^{(+1)}A + A'X^{(+1)}Z^*(I - ZA) \\
&\quad - A'X^{(+1)}Z^*(I - ZA)](I - ZA)^{-1}ZB \\
&= D'D + C'(ZI - A')^{-1}B'D + D'C(I - ZA)^{-1}ZB + C'(ZI - A')^{-1} \\
&\quad [(ZI - A')X^{(+1)}Z^* + A'X^{(+1)}(Z^*I - A)](I - ZA)^{-1}ZB \\
&= D'D + C'X^{(+1)}B + C'(ZI - A')^{-1}(B'D + A'X^{(+1)}B) + (D'C + \\
&\quad C'X^{(+1)}A)(I - ZA)^{-1}ZB.
\end{aligned}$$

The causal subsystem is given by

$$\begin{aligned}
&D'D + C'X^{(+1)}B + (D'C + C'X^{(+1)}A)(I - ZA)^{-1}ZB \\
&= \left[\begin{array}{c|c} A & B \\ \hline D'C + C'X^{(+1)}A & D'D + C'X^{(+1)}B \end{array} \right]_c,
\end{aligned}$$

and the anticausal subsystem is given by

$$C'(ZI - A')^{-1}(B'D + A'X^{(+1)}B) = \left[\begin{array}{c|c} A' & B'D + A'X^{(+1)}B \\ \hline C' & 0 \end{array} \right]_{ac}.$$

The diagonal entries of the causal and anticausal matrices directly produce the state matrices $\bar{E}_{c,k}$ and $\bar{E}_{ac,k}$ of the theorem. $\square$

It follows directly from the result of Theorem 4.3.1 that, provided X with block diagonal elements exists, $E_{ac}E_c$ is causal if $B'D + A'X^{(+1)}B = 0$ and is strictly causal if, in addition, $D'D + C'X^{(+1)}B = 0$. And $E_{ac}E_c$ is anticausal if $D'C + C'X^{(+1)}A = 0$

and is strictly anticausal if, in addition, $D'D + C'X^{(+1)}B = 0$.

The computing of X_i follows the backward recursion shown in equation (4.9). The exact initial point of X_i can be obtained in most cases of interest with the following steps.

- For systems which are time invariant after some point in time ($k = L$, say), an exact initial value can be computed analytically from the time-invariant algebraic equation that holds after time $k = L$:

$$A'_L X_{ini} A_L - X_{ini} + B'_L C_L = 0.$$

- If the cascaded system is stable such that l_A and $l_{A'}$ as given in (3.13) satisfy $l_A < 1$ and $l_{A'} < 1$, the time-varying Sylvester equation (4.9) is strongly convergent. In this case, X_i at some time point i is independent of the precise initialization of the recursion at $i \approx \infty$. For an interested finite time-interval $[1, L]$, it is possible to obtain arbitrarily accurate initial values by performing a finite backward recursion on data outside the interval. For the backward time-varying Sylvester recursion example,

$$\begin{aligned} X_i &= A'_i X_{i+1} A_i + B'_i C_i \\ &= A'_i A'_{i+1} X_{i+2} A_{i+1} A_i + A'_i B'_{i+1} C_{i+1} A_i + B'_i C_i \\ &\quad \vdots \\ &= A'_i A'_{i+1} \cdots A'_{i+n-1} X_{i+n} A_{i+n-1} \cdots A_{i+1} A_i + \\ &\quad (B'_i C_i + \sum_{j=1}^n A'_i \cdots A'_{i+j-1} B'_{i+j} C_{i+j} A_{i+j-1} \cdots A_i) \\ &\approx B'_i C_i + \sum_{j=1}^n A'_i \cdots A'_{i+j-1} B'_{i+j} C_{i+j} A_{i+j-1} \cdots A_i \end{aligned}$$

For a stable system, $\|A'_i A'_{i+1} \cdots A'_{i+n-1}\|$ and $\|A_{i+n-1} \cdots A_{i+1} A_i\|$ can be made arbitrarily small by choosing n large enough. The term $A'_i A'_{i+1} \cdots A'_{i+n-1} X_{i+n} A_{i+n-1} \cdots A_{i+1} A_i$ is approximating to zero. Neglecting this term gives an

approximation of X_i . The same approximation would have been obtained by choosing any X_{i+n} if n is large enough.

An analogy of Theorem 4.3.1 is the cascaded realization of $E_c E_{ac}$. It involves another matrix defined as

$$Y_i = \sum_{-\infty}^{k=i-1} A_{i-1} \cdots A_{k+1} B_k C'_k A'_{k+1} \cdots A'_{i-1}.$$

The associated time-varying Sylvester equation is given as

$$A_i Y_i A'_i + B_i C'_i - Y_{i+1} = 0. \quad (4.13)$$

And its block diagonal form is shown as

$$A Y A' - Y^{(+1)} + B C' = 0, \quad (4.14)$$

where $Y^{(+1)} = Z^* Y Z$. The following result can be obtained following similar steps.

Theorem 4.3.2. *The cascaded system $E_c E_{ac}$ can be decomposed into a causal subsystem in parallel with a strictly anticausal subsystem and the state matrices of these subsystems are given by*

$$E_c E_{ac} = \bar{E}_c + \bar{E}_{ac}$$

where the k^{th} entry has state space realizations as

$$\begin{aligned} \bar{E}_{c,k} &= \left[\begin{array}{c|c} A_k & B_k D'_k + A_k Y_k B'_k \\ \hline C_k & D_k D'_k + C_k Y_k B'_k \end{array} \right]_c \\ \bar{E}_{ac,k} &= \left[\begin{array}{c|c} A'_k & B'_k \\ \hline D_k C'_k + C_k Y_k A'_k & 0 \end{array} \right]_{ac}, \end{aligned}$$

in which Y_k satisfies equation (4.13).

The proof is given in Appendix A.

It follows immediately that, provided Y with block diagonal elements in equation (4.14) exists, the cascaded system $E_c E_{ac}$ is causal if $DC' + CYA' = 0$ and is strictly causal if, in addition, $DD' + CYB' = 0$. The cascaded system $E_c E_{ac}$ is anticausal if $BD' + AYB' = 0$ and is strictly anticausal if, in addition, $DD' + CYB' = 0$.

Now consider the general case of two mixed causal-anticausal LTV systems in cascade. Let $E_1 = E_{c,1} + E_{ac,1} = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right]_c + \left[\begin{array}{c|c} A'_1 & B'_1 \\ \hline C'_1 & D'_1 \end{array} \right]_{ac}$ and $E_2 = E_{c,2} + E_{ac,2} = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right]_c + \left[\begin{array}{c|c} A'_2 & B'_2 \\ \hline C'_2 & D'_2 \end{array} \right]_{ac}$ be two mixed causal-anticausal LTV systems. The cascaded LTV system, with the causal subsystem $\hat{E}_c$ and anticausal subsystem $\hat{E}_{ac}$, is represented as follows:

$$E_2 E_1 = (E_{c,2} + E_{ac,2})(E_{c,1} + E_{ac,1}) = E_{c,2}E_{c,1} + E_{ac,2}E_{c,1} + E_{c,2}E_{ac,1} + E_{ac,2}E_{ac,1}. \quad (4.15)$$

Let state space representations of $E_{c,i} + E_{ac,i}$, for $i = 1, 2$, be

$$E_{c,i,k} = \left[\begin{array}{c|c} A_{i,k} & B_{i,k} \\ \hline C_{i,k} & D_{i,k} \end{array} \right]_c, \quad E_{ac,i,k} = \left[\begin{array}{c|c} A'_{i,k} & B'_{i,k} \\ \hline C'_{i,k} & D'_{i,k} \end{array} \right]_{ac}, \quad (4.16)$$

where $E_{c,i,k}$ and $E_{ac,i,k}$ represent the k^{th} time instance realizations of $E_{c,i}$, $E_{ac,i}$, respectively.

In view of the cascaded system (4.15), its first term $E_{c,2}E_{c,1}$ is a cascaded system of two causal systems $E_{c,1}$ and $E_{c,2}$ so is an causal system with the state space realization

$$\bar{E}_{c,1,k} = \left[\begin{array}{cc|c} A_{2,k} & B_{2,k}C_{1,k} & B_{2,k}D_{1,k} \\ 0 & A_{1,k} & B_{1,k} \\ \hline C_{2,k} & D_{2,k}C_{1,k} & D_{2,k}D_{1,k} \end{array} \right]_c.$$

The fourth term $E_{ac,2}E_{ac,1}$ of the cascaded system (4.15) is a cascaded system of two anticausal systems $E_{ac,1}$ and $E_{ac,2}$ so is an anticausal system with the state space realization

$$\bar{E}_{ac,4,k} = \left[\begin{array}{cc|c} A'_{2,k} & B'_{2,k}C'_{1,k} & B'_{2,k}D'_{1,k} \\ 0 & A'_{1,k} & B'_{1,k} \\ \hline C'_{2,k} & D'_{2,k}C'_{1,k} & D'_{2,k}D'_{1,k} \end{array} \right]_{ac}.$$

The second term $E_{ac,2}E_{c,1}$ of the cascaded system (4.15) is a cascaded system of a causal and an anticausal systems $E_{c,1}$ and $E_{ac,2}$. It follows the Theorem 4.3.1 so that $E_{ac,2}E_{c,1}$ is, in general, a mixed causal and anticausal system with a causal realization $\bar{E}_{c,2,k}$ and an anticausal realization $\bar{E}_{ac,2,k}$ written, respectively, as

$$\bar{E}_{c,2,k} = \left[\begin{array}{c|c} A_{1,k} & B_{1,k} \\ \hline D'_{2,k}C_{1,k} + C'_{2,k}X_{k+1}A_{1,k} & D'_{2,k}D_{1,k} + C'_{2,k}X_{k+1}B_{1,k} \end{array} \right]_c,$$

$$\bar{E}_{ac,2,k} = \left[\begin{array}{c|c} A'_{2,k} & B'_{2,k}D_{1,k} + A'_{2,k}X_{k+1}B_{1,k} \\ \hline C'_{2,k} & 0 \end{array} \right]_{ac}.$$

Similarly, by Theorem 4.3.2, the third term $E_{c,2}E_{ac,1}$ of the cascaded system (4.15) is also a mixed causal and anticausal system with a causal realization $\bar{E}_{c,3,k}$ and an anticausal realization $\bar{E}_{ac,3,k}$ written, respectively, as

$$\bar{E}_{c,3,k} = \left[\begin{array}{c|c} A_{2,k} & B_{2,k}D'_{1,k} + A_{2,k}Y_kB'_{1,k} \\ \hline C_{2,k} & D_{2,k}D'_{1,k} + C_{2,k}Y_kB'_{1,k} \end{array} \right]_c,$$

$$\bar{E}_{ac,3,k} = \left[\begin{array}{c|c} A'_{1,k} & B'_{1,k} \\ \hline D_{2,k}C'_{1,k} + C_{2,k}Y_kA'_{1,k} & 0 \end{array} \right]_{ac}.$$

As a result, the cascaded system E_2E_1 is a mixed causal-anticausal system. Its causal state space realization is the parallel combination of $\bar{E}_{c,i,k}$, $i = 1, 2, 3$, and its

anticausal state space realization is the parallel combination of $\bar{E}_{ac,i,k}$, $i = 2, 3, 4$. Its overall state space realization can be summarized in the following theorem.

Theorem 4.3.3. *Given the state space realizations of $E_1 = E_{c,1} + E_{ac,1}$ and $E_2 = E_{c,2} + E_{ac,2}$, the cascaded system E_2E_1 can be represented as a causal-anticausal system such that $E_2E_1 = \hat{E}_c + \hat{E}_{ac}$. The state space realizations of the causal and anticausal subsystems, denoted by $\hat{E}_{c,k}$ and $\hat{E}_{ac,k}$, respectively, are given, by*

$$\hat{E}_{c,k} = \left[\begin{array}{c|c} \hat{A}_k & \hat{B}_k \\ \hline \hat{C}_k & \hat{D}_k \end{array} \right]_c, \quad \hat{E}_{ac,k} = \left[\begin{array}{c|c} \hat{A}'_k & \hat{B}'_k \\ \hline \hat{C}'_k & \hat{D}'_k \end{array} \right]_{ac},$$

in which

$$\begin{aligned} \hat{A}_k &= \begin{bmatrix} A_{2,k} & B_{2,k}C_{1,k} & 0 & 0 \\ 0 & A_{1,k} & 0 & 0 \\ 0 & 0 & A_{1,k} & 0 \\ 0 & 0 & 0 & A_{2,k} \end{bmatrix}, & \hat{B}_k &= \begin{bmatrix} B_{2,k}D_{1,k} \\ B_{1,k} \\ B_{1,k} \\ B_{2,k}D'_{1,k} + A_{2,k}Y_kB'_{1,k} \end{bmatrix}, \\ \hat{C}_k &= \begin{bmatrix} C_{2,k}^T \\ (D_{2,k}C_{1,k})^T \\ (D'_{2,k}C_{1,k} + C'_{2,k}X_{k+1}A_{1,k})^T \\ C_{2,k}^T \end{bmatrix}^T, & \hat{D}_k &= D_{2,k}D_{1,k} + D'_{2,k}D_{1,k} + C'_{2,k}X_{k+1}B_{1,k} \\ & & & + D_{2,k}D'_{1,k} + C_{2,k}Y_kB'_{1,k}, \\ \hat{A}'_k &= \begin{bmatrix} A'_{2,k} & B'_{2,k}C'_{1,k} & 0 & 0 \\ 0 & A'_{1,k} & 0 & 0 \\ 0 & 0 & A'_{2,k} & 0 \\ 0 & 0 & 0 & A'_{1,k} \end{bmatrix}, & \hat{B}'_k &= \begin{bmatrix} B'_{2,k}D'_{1,k} \\ B'_{1,k} \\ B'_{2,k}D_{1,k} + A'_{2,k}X_{k+1}B_{1,k} \\ B'_{1,k} \end{bmatrix}, \\ \hat{C}'_k &= \begin{bmatrix} (C'_{2,k})^T \\ (D'_{2,k}C'_{1,k})^T \\ (C'_{2,k})^T \\ (D_{2,k}C'_{1,k} + C_{2,k}Y_kA'_{1,k})^T \end{bmatrix}^T, & \hat{D}'_k &= D'_{2,k}D'_{1,k}, \end{aligned}$$

and X_k, Y_k satisfy

$$\begin{aligned} A'_{2,k}X_{k+1}A_{1,k} - X_k + B'_{2,k}C_{1,k} &= 0, \\ A_{2,k}Y_kA'_{1,k} - Y_{k+1} + B_{2,k}C'_{1,k} &= 0, \end{aligned}$$

respectively.

4.3.1 Example

Consider a causal LTV system E_c with the following state matrices:

for $k=1:M$

$$\begin{aligned} A_1 &= \begin{bmatrix} -1.4 & -1 \\ 0.5 & 0 \end{bmatrix}, & B_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 0.7 & 0.4 \end{bmatrix}, & D_1 &= 1, \end{aligned}$$

for $k=(M+1):L$

$$\begin{aligned} A_2 &= \begin{bmatrix} -0.4 & 0 \\ 0.7 & -0.9 \end{bmatrix}, & B_2 &= \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 1 & 0.4 \end{bmatrix}, & D_2 &= 1. \end{aligned}$$

The output of the causal LTV system E_c passes into an anticausal LTV system E_{ac} with state matrices:

for $k=1:N$

$$\begin{aligned} A'_1 &= \begin{bmatrix} -1.4 & -0.7 \\ 0.7 & 0 \end{bmatrix}, & B'_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C'_1 &= \begin{bmatrix} 0.7 & 0.3 \end{bmatrix}, & D'_1 &= 1 \end{aligned}$$

for $k=(N+1):L$

$$\begin{aligned} A'_2 &= \begin{bmatrix} -0.5 & 0 \\ 0.548 & -0.5 \end{bmatrix}, & B'_2 &= \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \\ C'_2 &= \begin{bmatrix} 1 & 0.5 \end{bmatrix}, & D'_2 &= 1. \end{aligned}$$

For simulation purpose, $L=400$ samples of input and output signal are collected. A

Figure 4.2: Unit Input Signal

unit input starts from point 101, and ends at point 300 shown in Figure 4.2. M and N are the time-varying point of causal system and anticausal system respectively. Set $M = 200$, and $N = 250$. It is expected that the signal changes at M and N points. Figure 4.3 shows the output signal of the original cascaded system. From Figure 4.3, several transient parts caused by the overall cascaded mixed causal-anticausal system are observed. The cascaded system $E_{ac}E_c$ can be separated into a causal system $\hat{E}_c$ in parallel with an anticausal system $\hat{E}_{ac}$. Applying Theorem 4.3.1, the parallel system state matrices are found.

The causal system $\hat{E}_c$ has state matrices

$$\left[\begin{array}{c|c} \hat{A}_k & \hat{B}_k \\ \hline \hat{C}_k & \hat{D}_k \end{array} \right]_c = \left[\begin{array}{c|c} A_k & B_k \\ \hline \hat{C}_k & \hat{D}_k \end{array} \right]_c.$$

The anticausal system $\hat{E}_{ac}$ has state matrices

$$\left[\begin{array}{c|c} \hat{A}'_k & \hat{B}'_k \\ \hline \hat{C}'_k & 0 \end{array} \right]_{ac} = \left[\begin{array}{c|c} A'_k & \hat{B}'_k \\ \hline C'_k & 0 \end{array} \right]_{ac}.$$

Figure 4.3: Original cascaded system output signal

Thus $\hat{E}_c$ has the same state matrices $\{A_k, B_k\}$ as E , and $\hat{E}_{ac}$ has the same state matrices $\{A'_k, C'_k\}$ with E_{ac} . Some plots are given to illustrate the time-variant values of $\hat{C}_k, \hat{D}_k$ and $\hat{B}'_k$. For each k , $\hat{C}_k \in \mathbb{R}^{1 \times 2}$, $\hat{D}_k \in \mathbb{R}^{1 \times 1}$, and $\hat{B}'_k \in \mathbb{R}^{2 \times 1}$. Figure 4.4 shows the first and second entry of each $\hat{C}_k$, Figure 4.5 represents the time-variant values of $\hat{D}_k$ and Figure 4.6 gives the first and second entry of each $\hat{B}'_k$. From these three figures, transient values at M and N points are observed. The values converge to constant values after a few more computational steps.

4.4 LTV system factorizations

The causality properties of cascaded LTV systems have been studied in the previous section. In this section, the causality properties are applied in system factorizations, which are important in computing the pseudo-inverse of the LTV system.

The transfer operator of a causal LTV system E_c can be represented in terms of the

Figure 4.4: The time-variant first and second entry of $\hat{C}_k$ Figure 4.5: The time-variant value $\hat{D}_k$

Figure 4.6: The time-variant first and second entry of $\hat{B}'_k$

shift operator Z such that

$$E_c = D + C(I - ZA)^{-1}ZB = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c,$$

E_c^* is the adjoint of E_c , which is shown as

$$E_c^* = D^* + B^*(ZI - A^*)^{-1}C^* = \left[\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array} \right]_{ac}.$$

In short, E_c is causal and E_c^* is anticausal.

An operator $V : \mathcal{M} \rightarrow \mathcal{N}$ is called an isometry if $VV^* = I$, a (co-)isometry if $V^*V = I$, and unitary if both $VV^* = I$ and $V^*V = I$, or $V^{-1} = V^*$. Equivalently, an operator is an isometry if its domain and range are closed subspaces in l^2 and if its inner products are conserved: for $F, G \in l^2$, $\langle FV, GV \rangle = \langle F, G \rangle$. An operator is inner if it is unitary and lower. Systems described by isometry or inner

operators satisfy an energy conservation property, let $U, Y \in l^2$,

- if $VV^* = I$ then $Y = V^*U \Rightarrow \|Y\| = \|U\|$
- if $V^*V = I$ then $Y = VU \Rightarrow \|Y\| = \|U\|$

Let $\left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array} \right]_c$ be a realization of V . The realization is called unitary if

$$\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} \begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix}^* = I,$$

and

$$\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix}^* \begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} = I.$$

Theorem 4.4.1. Let $V = \left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array} \right]_c$ be a state realization of a locally finite operator V . If the matrix $\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix}$ is unitary, then V is inner.

Proof: From $\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix}^* \begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} = I$, it is easy to get:

$$A_v^* A_v + C_v^* C_v = I,$$

$$B_v^* B_v + D_v^* D_v = I,$$

$$B_v^* A_v = -D_v^* C_v.$$

The first term implies $l_{A_v} < 1$, thus $(I - ZA_v)^{-1}$ is bounded, or V is bounded. Hence

$$\begin{aligned}
V^*V &= [D_v^* + B_v^*Z^*(I - A_v^*Z^*)^{-1}C_v^*][D_v + C_v(I - ZA_v)^{-1}ZB_v] \\
&= D_v^*D_v + B_v^*Z^*(I - A_v^*Z^*)^{-1}C_v^*D_v + D_v^*C_v(I - ZA_v)^{-1}ZB_v \\
&\quad + B_v^*Z^*(I - A_v^*Z^*)^{-1}C_v^*C_v(I - ZA_v)^{-1}ZB_v \\
&= D_v^*D_v - B_v^*Z^*(I - A_v^*Z^*)^{-1}A_v^*B_v - B_v^*A_v(I - ZA_v)^{-1}ZB_v \\
&\quad + B_v^*Z^*(I - A_v^*Z^*)^{-1}(I - A_v^*A_v)(I - ZA_v)^{-1}ZB_v \\
&= D_v^*D_v + B_v^*Z^*(I - A_v^*Z^*)^{-1}[I - A_v^*A_v - A_v^*Z^*(I - ZA_v) - \\
&\quad (I - A_v^*Z^*)ZA_v](I - ZA_v)^{-1}ZB_v \\
&= D_v^*D_v + B_v^*Z^*(I - A_v^*Z^*)^{-1}(I - A_v^*Z^*)(I - ZA_v)(I - ZA_v)^{-1}ZB_v \\
&= D_v^*D_v + B_v^*B_v \\
&= I.
\end{aligned}$$

$VV^* = I$ is verified by an analogous procedure. $\square$

Let E_c be some causal LTV transfer operator. It has a factorization of the form

$$E_c = V\Delta^*,$$

where $\Delta = E_c^*V$ is causal and V is an inner operator. If the factorization is such that V is an inner transfer operator of smallest possible local degree and $\Delta = E_c^*V$ is lower, then the factorization is inner-coprime. If E_c has a locally finite state space and $l_A < 1$, then such factorizations exist.

Inner-coprime factorization for LTV systems can be obtained by making use of the causality properties of cascaded LTV systems.

Theorem 4.4.2. (a) Suppose that a causal system E_c which has an exponentially stable finite dimensional state realization at k^{th} entry of the block-diagonal form such

that $E_{c,k} = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array} \right]_c$, with appropriate dimensions. Then E_c has a factorization $E_c = V\Delta^*$, where V is inner.

(b) Denote realizations of V and Δ by $V_k := \left[\begin{array}{c|c} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right]_c$, and $\Delta_k := \left[\begin{array}{c|c} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right]_c$.

Then V_k and Δ_k follow recursively from the QR factorization:

$$\left[\begin{array}{c|cc} R_{k+1}A_k & I & R_{k+1}B_k \\ \hline C_k & 0 & D_k \end{array} \right] = \left[\begin{array}{cc} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right] \left[\begin{array}{c|cc} R_k & A_{\Delta k}^* & C_{\Delta k}^* \\ \hline 0 & B_{\Delta k}^* & D_{\Delta k}^* \end{array} \right] \quad (4.17)$$

The Lyapunov equations associated to $\{A_k, C_k\}$ are given by

$$A_k^* Q_{k+1} A_k + C_k^* C_k = Q_k.$$

R_k is the square root of Q_k such that $Q_k = R_k^* R_k$.

Proof: $E_c = V\Delta^*$ results in $\Delta = E_c^* V$, where E_c^* is the anticausal system with state space matrices: $\left[\begin{array}{c|c} A_k^* & C_k^* \\ \hline B_k^* & D_k^* \end{array} \right]_{ac}$. V is causal and has state space matrices

$$\left[\begin{array}{c|c} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right]_c \text{ and } \Delta \text{ is causal with state space matrices } \left[\begin{array}{c|c} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right]_c.$$

Applying Theorem 4.3.1 gives

$$\left[\begin{array}{c|c} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right]_c = \left[\begin{array}{c|c} A_{vk} & B_{vk} \\ \hline B_k^* X_{k+1} A_{vk} + D_k^* C_{vk} & B_k^* X_{k+1} B_{vk} + D_k^* D_{vk} \end{array} \right]_c, \quad (4.18)$$

where X_k satisfies:

$$A_k^* X_{k+1} A_{vk} - X_k + C_k^* C_{vk} = 0, \quad (4.19)$$

and

$$C_k^* D_{vk} + A_k^* X_{k+1} B_{vk} = 0. \quad (4.20)$$

Equation (4.18) yields:

$$\begin{bmatrix} A_{\Delta k} & B_{\Delta k} \\ C_{\Delta k} & D_{\Delta k} \end{bmatrix} = \begin{bmatrix} I & 0 \\ B_k^* X_{k+1} & D_k^* \end{bmatrix} \begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix}. \quad (4.21)$$

and equation (4.19) together with equation (4.20) gives:

$$\begin{bmatrix} X_k & 0 \end{bmatrix} = \begin{bmatrix} A_k^* X_{k+1} & C_k^* \end{bmatrix} \begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix} \quad (4.22)$$

The above two equations (4.21) and (4.22) yield:

$$\left[\begin{array}{c|cc} X_{k+1}^* A_k & I & X_{k+1}^* B_k \\ \hline C_k & 0 & D_k \end{array} \right] = \begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix} \left[\begin{array}{c|cc} X_k^* & A_{\Delta k}^* & C_{\Delta k}^* \\ \hline 0 & B_{\Delta k}^* & D_{\Delta k}^* \end{array} \right]$$

Post-multiplying (4.22) by each terms' adjoint gives:

$$A_k^* X_{k+1} X_{k+1}^* A_k + C_k^* C_k = X_k X_k^*,$$

since $\begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix}$ is unitary. Let $R_k = X_k^*$, and $Q_k = X_k X_k^*$, this theorem is proved. $\square$

Dually, the coprime-inner factorization of a causal LTV system can be derived in the following theorem.

Theorem 4.4.3. (a) Suppose that a causal system E_c has an exponentially stable finite dimensional state realization $\left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array} \right]_c$, with appropriate dimensions. Then E_c has a factorization $E_c = \Delta^* V$, where V is inner.

(b) Denote realizations of V and Δ by $V_k := \left[\begin{array}{c|c} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right]_c$, and $\Delta_k := \left[\begin{array}{c|c} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right]_c$.

Then V_k and Δ_k follow recursively from the LQ factorization:

$$\left[\begin{array}{cc} A_k R_k & B_k \\ I & 0 \\ C_k R_k & D_k \end{array} \right] = \left[\begin{array}{cc} R_{k+1} & 0 \\ A_{\Delta k}^* & C_{\Delta k}^* \\ B_{\Delta k}^* & D_{\Delta k}^* \end{array} \right] \left[\begin{array}{cc} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{array} \right]$$

The Lyapunov equations associated to $\{A_k, B_k\}$:

$$A_k Q_k A_k^* + B_k B_k^* = Q_{k+1}$$

R_k is the square root of Q_k such that $Q_k = R_k R_k^*$.

The proof is similar with that of Theorem 4.4.2 and is given in Appendix A.

Now considering the inner-coprime factorization of an anticausal LTV system E_{ac} , similar results are derived as that of a causal LTV system E_c by applying Theorem 4.3.2.

Theorem 4.4.4. (a) Suppose that an anticausal system E_{ac} has an exponentially stable finite dimensional state realization $E_{ac,k} = \left[\begin{array}{c|c} A'_k & B'_k \\ \hline C'_k & D'_k \end{array} \right]$, with appropriate dimensions. Then E_{ac} has a factorization $E_{ac} = V^* \Delta$, where V is inner.

(b) Denote realizations of V and Δ by $V_k := \left[\begin{array}{c|c} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right]_c$, and $\Delta_k := \left[\begin{array}{c|c} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right]_c$. Then V_k and Δ_k follow recursively from the QR factorization:

$$\left[\begin{array}{c|cc} R_k A'_k & I & R_k B'_k \\ \hline C'_k & 0 & D'_k \end{array} \right] = \left[\begin{array}{cc} A_{vk}^* & C_{vk}^* \\ B_{vk}^* & D_{vk}^* \end{array} \right] \left[\begin{array}{c|cc} R_{k+1} & A_{\Delta k} & B_{\Delta k} \\ \hline 0 & C_{\Delta k} & D_{\Delta k} \end{array} \right].$$

The Lyapunov equations associated to $\{A'_k, C'_k\}$:

$$A_k'^* Q_k A_k' + C_k'^* C_k' = Q_{k+1}$$

R_k is the square root of Q_k such that $Q_k = R_k^* R_k$.

The proof is given in Appendix A.

Theorem 4.4.5. (a) Suppose that an anticausal system E_{ac} has an exponentially stable finite dimensional state realization $E_{ac,k} = \left[\begin{array}{c|c} A'_k & B'_k \\ \hline C'_k & D'_k \end{array} \right]_c$, with appropriate dimensions. Then E_{ac} has a factorization $E_{ac} = \Delta V^*$, where V is inner.

(b) Denote realizations of V and Δ by $V_k := \left[\begin{array}{c|c} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right]_c$, and $\Delta_k := \left[\begin{array}{c|c} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right]_c$.

Then V_k and Δ_k follow recursively from the LQ factorization:

$$\left[\begin{array}{cc} A'_k R_{k+1} & B'_k \\ \hline I & 0 \\ \hline C'_k R_{k+1} & D'_k \end{array} \right] = \left[\begin{array}{cc} R_k & 0 \\ \hline A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right] \left[\begin{array}{cc} A_{vk}^* & C_{vk}^* \\ \hline B_{vk}^* & D_{vk}^* \end{array} \right]$$

The Lyapunov equations associated to $\{A'_k, B'_k\}$:

$$A'_k Q_{k+1} A'^*_{k+1} + B'_k B'^*_{k+1} = Q_k$$

R_k is the square root of Q_k such that $Q_k = R_k R_k^*$.

The proof is given in Appendix A.

A mixed causal-anticausal system $E = E_c + E_{ac}$ is then considered. Applying Theorem 4.4.5 to the anticausal operator E_{ac} , the mixed causal-anticausal system can be decomposed into a causal system cascaded with an anticausal system such that

$$E = E_c + E_{ac} = E_c + Q^* R = Q^* (R + Q E_c) = Q^* \Delta.$$

Δ has causal realization and Q^* is inner and has anticausal realization. The state-

space realizations of causal and anticausal parts are represented

$$\Delta_k = \left[\begin{array}{ccc|c} A_{Rk} & 0 & 0 & B_{Rk} \\ 0 & A_{Qk} & B_{Qk}C_k & B_{Qk}D_k \\ 0 & 0 & A_k & B_k \\ \hline C_{Rk} & C_{Qk} & D_{Qk}C_k & D_{Qk}D_k \end{array} \right]_c,$$

$$\mathbf{Q}_k^* = \left[\begin{array}{c|c} A_{Qk}^* & C_{Qk}^* \\ \hline B_{Qk}^* & D_{Qk}^* \end{array} \right]_{ac},$$

respectively.

Δ can be future factorized using inner-coprime factorization shown in Theorem 4.4.3.

4.4.1 Examples

Example 1: Consider a causal LTV system E_c with the following state matrices:

for $k=1:M$

$$A_1 = \begin{bmatrix} -1.4 & -1 \\ 0.5 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.7 & 0.4 \end{bmatrix}, \quad D_1 = 1,$$

for $k=(M+1):L$

$$A_2 = \begin{bmatrix} -0.4 & 0 \\ 0.7 & -0.9 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0.4 \end{bmatrix}, \quad D_2 = 1.$$

The output of the causal LTV system E_c passes to an anticausal LTV system E_{ac} with state matrices:

for $k=1:N$

$$\begin{aligned} A'_1 &= \begin{bmatrix} -1.4 & -0.7 \\ 0.7 & 0 \end{bmatrix}, & B'_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C'_1 &= \begin{bmatrix} 0.7 & 0.3 \end{bmatrix}, & D'_1 &= 1 \end{aligned}$$

for $k=(N+1):L$

$$\begin{aligned} A'_2 &= \begin{bmatrix} -0.5 & 0 \\ 0.548 & -0.5 \end{bmatrix}, & B'_2 &= \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \\ C'_2 &= \begin{bmatrix} 1 & 0.5 \end{bmatrix}, & D'_2 &= 1. \end{aligned}$$

For simulation purpose, $L=400$ samples of input and output signal are collected. In this example, a unit input starts from point 101, and ends at point 300 shown in Figure 4.2. M and N are the time-varying point of causal system and anticausal system respectively. Set $M = 200$, and $N = 250$. It is expected that the signal changes at M and N points. Figure 4.3 shows the output signal of the original cascaded system. From Figure 4.3, several transient parts caused by the overall cascaded mix-causal system are observed. The cascaded system $E_{ac}E_c$ can be separated into a causal system in parallel with an anticausal system. Applying the Theorem 4.3.1, the parallel system output is obtained. The error between the parallel system output and original cascaded system output is shown in Figure 4.7. There are some small errors at M and N points due to non-exact initial state values.

Example 2: Consider a causal LTV system E_c with the following state matrices:

for $k=1:M$

$$\begin{aligned} A_1 &= \begin{bmatrix} -1.4 & -1 \\ 0.5 & 0 \end{bmatrix}, & B_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 0.7 & 0.4 \end{bmatrix}, & D_1 &= 1 \end{aligned}$$

Figure 4.7: Error between the parallel system output and the original cascaded system output

for $k=(M+1):L$

$$A_2 = \begin{bmatrix} -0.4 & 0 \\ 0.7 & -0.9 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0.4 \end{bmatrix}, \quad D_2 = 1.$$

The causal system is in parallel with an anticausal LTV system E_{ac} with state matrices: for

$k=1:N$

$$A'_1 = \begin{bmatrix} -1.4 & -0.7 \\ 0.7 & 0 \end{bmatrix}, \quad B'_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C'_1 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix}, \quad D'_1 = 1$$

for $k=(N+1):L$

$$A'_2 = \begin{bmatrix} -0.5 & 0 \\ 0.548 & -0.5 \end{bmatrix}, \quad B'_2 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},$$

$$C'_2 = \begin{bmatrix} 1 & 0.5 \end{bmatrix}, \quad D'_2 = 1.$$

The simulation setting is similar to that in previous example. The input signal is

Figure 4.8: Original parallel system output signal

shown in Figure 4.2. The total number of collected points is 400. In this example, the time-varying point M and N is set to be 150 and 250 respectively. It is expected that signal changes in point 150 and 250. The original output signal is presented in Figure 4.8. The parallel system $E_{ac} + E_c$ can be decomposed into a causal system with operator Δ cascaded with an anticausal system with operator Q^* . The error between the cascaded system output and original parallel system output is shown in Figure 4.9, which is very small.

4.5 Decomposition of unstable causal LPTV system into strictly stable causal subsystem and strictly unstable causal subsystem

Given an unstable LPTV system $\mathcal{G} = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array} \right]_c$, where $A_k = A_{k+N}$, $B_k = B_{k+N}$, $C_k = C_{k+N}$, $D_k = D_{k+N}$, in which N is the period. It is possible to decompose it

Figure 4.9: Error between the cascaded system output and the original parallel system output

into

$$\mathcal{G} = G^s + G^u,$$

where $G^s = \left[\begin{array}{c|c} A_k^s & B_k^s \\ \hline C_k^s & D_k^s \end{array} \right]_c$ represents the strictly stable subsystem, and $G^u = \left[\begin{array}{c|c} A_k^u & B_k^u \\ \hline C_k^u & D_k^u \end{array} \right]_c$ represents the strictly unstable causal subsystem.

The periodic Schur decomposition introduced in [10] was used to deduce the cyclic Schur form of A_k which is in the upper triangular form with the absolute value of diagonal entries in descending order. However, it requires that $A_2, A_3, \dots, A_N$ are nonsingular. For A_N being singular, a deflation procedure is required and a deflation procedure for the 2-periodic case is presented in [68]. In the following, the deflation procedure to a more general case is generated. It will enable us to present a corresponding solution for the orthogonal matrices Q_k for the N-periodic case.

Lemma 4.5.1. *If $A_N \in \mathbb{R}^{n \times n}$ is a singular matrix, then there exist orthogonal matrices $Q_k \in \mathbb{R}^{n \times n}$, $k = 1, 2, \dots, N$, such that $Q_2^* A_1 Q_1$ is quasi-upper triangular,*

and $Q_1^* A_N Q_N$ and $Q_{k+1}^* A_k Q_k$, $k = 2, 3, \dots, N$, are upper triangular.

Proof: For a singular matrix A_N , there exists a QR factorization with column pivoting [58] such that $A_N = U_N T_N P_N^*$, where U_N is an orthogonal matrix, P_N is an orthogonal permutation matrix and T_N is an upper triangular matrix of the form:

$$T_N = \left[\begin{array}{c|c} T_{11N} & T_{12N} \\ \hline 0 & 0 \end{array} \right],$$

with T_{11N} being nonsingular. T_k and P_k^* are obtained by RQ factorization of $P_{k+1}^* A_k$, such that $P_{k+1}^* A_k = T_k P_k^*$ for $k = N-1, N-2, \dots, 2$, where T_k is an upper triangular matrix and P_k is an orthogonal matrix. Further, the matrix $T_1 = P_2^* A_1 U_N$ can be found such that the product $T_N T_{N-1} \dots T_1$ is orthogonally similar to $A_N A_{N-1} \dots A_1$.

By QR decomposition, there exists an orthogonal matrix V_1 and an upper triangular matrix M_1 such that $T_1 = V_1 M_1$. Hence V_k and M_k are found by QR factorization of $T_k V_{k-1}$, such that $T_k V_{k-1} = V_k M_k$ for $k = 2, 3, \dots, N-1$, and $M_N = T_N V_{N-1}$ such that $M_N M_{N-1} \dots M_1$ is orthogonally similar to $A_N A_{N-1} \dots A_1$. These matrices are of the form:

$$M_N = \left[\begin{array}{c|c} M_{11N} & M_{12N} \\ \hline 0 & 0 \end{array} \right], \quad M_i = \left[\begin{array}{c|c} M_{11i} & M_{12i} \\ \hline 0 & M_{22i} \end{array} \right],$$

where M_{11N} is nonsingular, and $k = 1, 2, \dots, N-1$.

Now the nonsingular matrices $M_{111}, M_{112}, \dots, M_{11N}$ are considered. By applying the periodic Schur decomposition with constant state dimension, orthogonal matrices $z_1, z_2, \dots, z_N$ are found to reduce the system $M_{11N} \dots M_{111}$ to the cyclic Schur form. Accumulating all of the orthogonal matrices above produces the matrices

$Q_1, Q_2, \dots, Q_N$ that reduce $A_1, A_2, \dots, A_N$ to cyclic Schur form. Let

$$Z_k = \begin{bmatrix} z_k & 0 \\ 0 & I \end{bmatrix}.$$

Q_k is found as follows:

$$Q_1 = U_N Z_1,$$

and

$$Q_k = P_k V_{k-1} Z_k,$$

for $k = 2, 3, \dots, N$. □

Lemma 4.5.1 deals with the case when A_N is singular. In the case that A_N is nonsingular but some A_i with $1 < i < N$ is singular, Q_i is found starting from $S^{(i+1)} = A_i A_{i-1} \dots A_1 A_N A_{N-1} \dots A_{i+1}$. Then follow the procedures above to find all other matrices Q_k for $k = 1, \dots, i-1, i+1, \dots, N$.

Applying the periodic Schur decomposition procedure, the matrices Q_k for $k = 1, 2, \dots, N$ are obtained, which are unitary matrices, i.e. $Q_k^{-1} = Q_k^*$. Note that Q_1 is chosen such that the diagonal elements of $\hat{S}^1$ are in descending order if the eigenvalues of S^1 are real. If some of the eigenvalues of S^1 are complex, $\hat{S}^1$ is quasi-upper diagonal with diagonal elements as blocks. Then Q_1 is chosen such that the absolute value of eigenvalues of the diagonal blocks are in descending order.

With the obtained Q_1 , the Schur form of the lifted state matrix $S^1 \in \mathbb{R}^{n \times n}$ can be partitioned as:

$$\hat{S}^1 = Q_1^* S^1 Q_1 = \left[\begin{array}{c|c} \hat{S}_{11}^1 & \hat{S}_{12}^1 \\ \hline 0 & \hat{S}_{22}^1 \end{array} \right], \quad (4.23)$$

such that $|\lambda(\hat{S}_{11}^1)| > 1$, and $|\lambda(\hat{S}_{22}^1)| < 1$, where $\lambda(X)$ denotes the eigenvalues of X .

Notice that $\hat{S}_{11}^1 \in \mathbb{R}^{n_1 \times n_1}$, and $\hat{S}_{22}^1 \in \mathbb{R}^{n_2 \times n_2}$ with $n = n_1 + n_2$.

Substituting Q_k obtained, an equivalent system $\bar{\mathcal{G}} = \left[\begin{array}{c|c} \bar{A}_k & \bar{B}_k \\ \hline \bar{C}_k & \bar{D}_k \end{array} \right]_c$ is found. Then equivalent system matrices are shown as

$$\begin{aligned} \bar{A}_k &= \left[\begin{array}{c|c} \bar{A}_{k,11} & \bar{A}_{k,12} \\ \hline 0 & \bar{A}_{k,22} \end{array} \right], \\ \bar{B}_k &= \left[\begin{array}{c} \bar{B}_{k,1} \\ \bar{B}_{k,2} \end{array} \right], \\ \bar{C}_k &= \left[\bar{C}_{k,1} \mid \bar{C}_{k,2} \right], \\ \bar{D}_k &= D_k, \end{aligned} \quad (4.24)$$

with $\bar{A}_{k,11} \in \mathbb{R}^{n_1 \times n_1}$, $\bar{A}_{k,12} \in \mathbb{R}^{n_1 \times n_2}$, $\bar{A}_{k,22} \in \mathbb{R}^{n_2 \times n_2}$, $\bar{B}_{k,1} \in \mathbb{R}^{n_1 \times 1}$, $\bar{B}_{k,2} \in \mathbb{R}^{n_2 \times 1}$, $\bar{C}_{k,1} \in \mathbb{R}^{1 \times n_1}$, $\bar{C}_{k,2} \in \mathbb{R}^{1 \times n_2}$, and $\bar{D}_k \in \mathbb{R}$.

The equivalent system $\bar{\mathcal{G}}$ is applied to decompose the LPTV system in the following theorem.

Theorem 4.5.1. *Given an LPTV system $\mathcal{G}$ in terms of the state realization matrices in (4.24), there exist subsystems $\mathcal{G}^s$ and $\mathcal{G}^u$ such that:*

$$\mathcal{G} = \mathcal{G}^s + \mathcal{G}^u, \quad (4.25)$$

where $\mathcal{G}^s$ represents the stable subsystems with system matrices $\{A_k^s, B_k^s, C_k^s, D_k^s\}$, and $\mathcal{G}^u$ represents the unstable subsystem with system matrices $\{A_k^u, B_k^u, C_k^u, D_k^u\}$.

The system realization matrices are given as:

$$\begin{aligned} A_k^u &= \bar{A}_{k,11}, & A_k^s &= \bar{A}_{k,22}, \\ B_k^u &= \bar{B}_{k,1} - W_{k+1}\bar{B}_{k,2}, & B_k^s &= \bar{B}_{k,2}, \\ C_k^u &= \bar{C}_{k,1}, & C_k^s &= \bar{C}_{k,1}W_k + \bar{C}_{k,2}, \\ D_k^u &= 0, & D_k^s &= \bar{D}_k, \end{aligned} \quad (4.26)$$

where W_1 is the Sylvester solution of

$$\hat{S}_{11}^1 W_1 - W_1 \hat{S}_{22}^1 + \hat{S}_{12}^1 = 0,$$

W_N is given by

$$W_N = (\bar{A}_{N,11})^{-1}(W_1 \bar{A}_{N,22} - \bar{A}_{N,12}),$$

and W_k for $k = 2, 3, \dots, N-1$ is given by

$$W_k = (\bar{A}_{k,11})^{-1}(W_{k+1} \bar{A}_{k,22} - \bar{A}_{k,12}).$$

where $\bar{A}_{k,11}, \bar{A}_{k,22}, \bar{B}_{k,1}, \bar{B}_{k,2}, \bar{C}_{k,1}, \bar{C}_{k,2}, \bar{D}_k$ are state matrices of the equivalent system $\bar{\mathcal{G}}$ given in (4.24).

Proof: Firstly the stability property of the subsystems is shown. Using the lifting technique, the LPTV system is lifted to the block time-invariant system with system realization matrices $\{A^{lift}, B^{lift}, C^{lift}, D^{lift}\}$, which are shown as:

$$\begin{aligned} A^{lift} &= \phi_A(N+1, 1) = A_N A_{N-1} \cdots A_1, \\ B^{lift} &= [\phi_A(N+1, 2) B_1 \cdots B_N], \\ C^{lift} &= \begin{bmatrix} C_1 \\ C_2 \phi_A(2, 1) \\ \vdots \\ C_N \phi_A(N, 1) \end{bmatrix}, \\ D^{lift} &= \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ l_{2,1} & D_2 & \ddots & 0 \\ \vdots & & \ddots & \vdots \\ l_{N,1} & l_{N,2} & \cdots & D_N \end{bmatrix}, \\ l_{i,j} &= C_i \phi_A(i, j+1) B_j, (i > j). \end{aligned}$$

It is obvious that S^1 defined above is exactly the lifted state matrix A^{lift} . The Schur form of the lifted state matrix $\hat{S}^1$ is shown in equation (4.23). The submatrix $\hat{S}_{11}^1$ contains all the eigenvalues of the lifted state matrix that lie outside the unit disk, and the submatrix $\hat{S}_{22}^1$ contains all the eigenvalues of the lifted state matrix within the unit disk. For the LPTV system, the eigenvalues of the lifted state matrix within the unit disk contribute the stable parts, and the eigenvalues of the lifted state matrix that lie outside the unit disk contribute the unstable parts. Thus $\hat{S}_{11}^1$ represents the strictly unstable subsystem, and $\hat{S}_{22}^1$ represents the strictly stable subsystem.

Moreover, the Schur form $\hat{S}^1$ can be expressed as:

$$\hat{S}^1 = \bar{A}_N \bar{A}_{N-1} \cdots \bar{A}_2 \bar{A}_1.$$

The matrices $\bar{A}_k$, for $k = 2, 3, \dots, N$, are upper-triangular matrix, and the matrices $\bar{A}_1, \hat{S}^1$ are upper-triangular or quasi-upper triangular matrices. It is not difficult to find that:

$$\begin{aligned} \hat{S}_{11}^1 &= \bar{A}_{N,11} \bar{A}_{N-1,11} \cdots \bar{A}_{1,11}, \\ \hat{S}_{22}^1 &= \bar{A}_{N,22} \bar{A}_{N-1,22} \cdots \bar{A}_{1,22}. \end{aligned}$$

Therefore, $\bar{A}_{k,11}$ contributes the strictly unstable part, and $\bar{A}_{k,22}$ contributes the strictly stable part.

Next, how to obtain matrices W_k for $k = 1, 2, \dots, N$ is shown. The system $\mathcal{G} = \mathcal{G}^s + \mathcal{G}^u$ in (4.25) represents a parallel connection of $\mathcal{G}^s$ and $\mathcal{G}^u$ and can be denoted by $\mathcal{G} = \{A_k^{par}, B_k^{par}, C_k^{par}, D_k^{par}\}$ with

$$A_k^{par} = \begin{bmatrix} A_k^u & 0 \\ 0 & A_k^s \end{bmatrix}, \quad B_k^{par} = \begin{bmatrix} B_k^u \\ B_k^s \end{bmatrix}, \quad (4.27)$$

$$C_k^{par} = \begin{bmatrix} C_k^u & C_k^s \end{bmatrix}, \quad D_k^{par} = D_k.$$

In order to get the equivalent system realization matrices given in (4.27), the matrices X_k and Y_k are introduced such that

$$X_k = \begin{bmatrix} I_u & -W_i \\ 0 & I_s \end{bmatrix}, \quad Y_k = \begin{bmatrix} I_u & W_i \\ 0 & I_s \end{bmatrix},$$

where both I_u and I_s are identity matrices with appropriate dimensions, and $k = 1, 2, \dots, N$. It is obvious that Y_k is an inverse matrix of X_k , i.e. $Y_k X_k = I$, and $X_k Y_k = I$.

Since the Schur form of S^1 and the equivalent state matrix $\bar{A}_k$ are in upper triangular or quasi-upper triangular form, the following equations exist:

$$X_1 \hat{S}^1 Y_1 = \begin{bmatrix} \hat{S}_{11}^1 & \hat{S}_{11}^1 W_1 - W_1 \hat{S}_{22}^1 + \hat{S}_{12}^1 \\ 0 & \hat{S}_{22}^1 \end{bmatrix},$$

$$X_1 \bar{A}_N Y_N = \begin{bmatrix} \bar{A}_{N,11} & \bar{A}_{N,11} W_N - W_1 \bar{A}_{N,22} + \bar{A}_{N,12} \\ 0 & \bar{A}_{N,22} \end{bmatrix},$$

and

$$X_{k+1} \bar{A}_k Y_k = \begin{bmatrix} \bar{A}_{k,11} & \bar{A}_{k,11} W_k - W_{k+1} \bar{A}_{k,22} + \bar{A}_{k,12} \\ 0 & \bar{A}_{k,22} \end{bmatrix},$$

for $k = N - 1, \dots, 2$.

W_1 is the solution of the Sylvester equation $\hat{S}_{11}^1 W_1 - W_1 \hat{S}_{22}^1 + \hat{S}_{12}^1 = 0$. Hence, X_1 and Y_1 can be found respectively.

By setting

$$\bar{A}_{N,11} W_N - W_1 \bar{A}_{N,22} + \bar{A}_{N,12} = 0,$$

and

$$\bar{A}_{k,11} W_k - W_{k+1} \bar{A}_{k,22} + \bar{A}_{k,12} = 0,$$

W_k for $k = 1, 2, \dots, N$, are given by

$$W_N = (\bar{A}_{N,11})^{-1}(W_1 \bar{A}_{N,22} - \bar{A}_{N,12}),$$

and

$$W_k = (\bar{A}_{k,11})^{-1}(W_{k+1} \bar{A}_{k,22} - \bar{A}_{k,12}).$$

With the obtained W_k for $k = 1, 2, \dots, N$, the equivalent state realization matrices developed are shown as follows:

$$\begin{aligned} A_k^{par} &= X_{k+1} \bar{A}_k Y_k = \left[\begin{array}{c|c} \bar{A}_{k,11} & 0 \\ \hline 0 & \bar{A}_{k,22} \end{array} \right], \\ B_k^{par} &= X_{k+1} \bar{B}_k = \left[\begin{array}{c} \bar{B}_{k,1} - W_{k+1} \bar{B}_{k,2} \\ \hline \bar{B}_{k,2} \end{array} \right], \\ C_k^{par} &= \bar{C}_k Y_k = \left[\begin{array}{c|c} \bar{C}_{k,1} & \bar{C}_{k,1} W_k + \bar{C}_{k,2} \end{array} \right], \\ D_k^{par} &= \bar{D}_k. \end{aligned}$$

Obviously A_k^{par} for $k = N, N-1, \dots, 2$ is in block diagonal form, A_1^{par} is proved in block diagonal form too. Here is the very simple proof.

$$\begin{aligned} X_1 \hat{S}^1 Y_1 &= X_1 \bar{A}_N \cdots \bar{A}_1 Y_1 \\ &= X_1 \bar{A}_N Y_N X_N \bar{A}_{N-1} Y_{N-1} \cdots \bar{A}_2 Y_2 X_2 \bar{A}_1 Y_1 \\ &= A_N^{par} A_{N-1}^{par} \cdots A_2^{par} X_2 \bar{A}_1 Y_1 \end{aligned}$$

Since $X_1 \hat{S}^1 Y_1$ is in block diagonal form, and A_k^{par} for $k = N, N-1, \dots, 2$ are in block diagonal form, $A_1^{par} = X_2 \bar{A}_1 Y_1$ must be in the block diagonal form. $\square$

4.6 Summary

The causality properties of cascaded causal-anticausal linear systems take an important role in the analysis of linear systems. This chapter presents the application of the causality properties of cascaded causal-anticausal LTI systems, such as transforming the stable causal-anticausal realizations into unstable realizations and vice versa. The causality properties of cascaded LTV systems are investigated in this chapter. The roles of the causality properties in the inner-coprime factorizations are presented for the LTV system (3.12).

Chapter 5

Frame analysis using linear systems methods

5.1 Introduction

One class of frames has its elements as the bi-directional infinite impulse responses of a linear system, which may be an LTI system or an LTV system. For those frames modeled as causal LTI systems with transfer function representations, the frame bounds can be computed either in the frequency domain or in the time domain. The frequency approach to compute the frame bounds, see for example [106] [11] [9], is an approximation method that depends on the frequency grid density. The frame bounds can also be obtained by using state space matrices in the time domain via LMI optimization technique, see [20], which is an application of the KYP lemma stated in [114]. This method avoids the frequency-domain sampling and approximation. However, the computation approach via LMI optimization technique is applicable for causal realizations only. For those frames modeled as mixed causal-anticausal LTI systems, the existing state space approach employing LMI optimization technique cannot be directly applied. One class of frames can be modeled as

LTV systems, which have mixed causal-anticausal state space representations. Such frames do not have proper transfer function representation in the frequency domain, hence the computation of frame bounds can not be achieved by frequency approach. One of the contributions of the current thesis is to develop a state space approach which can derive the bounds of frames modeled as mixed causal-anticausal LTV systems, and has never been proposed before.

5.2 Frames and linear systems

Let $\{h(l, k) \in \mathbb{R}^{q \times p}\}$ be the impulse response of a stable causal-anticausal LTV system E in the form of (3.12) such that

$$y = Eu,$$

where $u_k \in l_p^2, y_k \in l_q^2$, and u, y follows (3.10). The system operator E satisfies (3.15), and $h(l, k)$ can be represented by:

$$h(l, k) = \begin{cases} C_l A_{l-1} \cdots A_{k+1} B_k, & l > k, \\ D_l + D'_l, & l = k, \\ C'_l A'_{l+1} \cdots A'_{k-1} B'_k, & l < k. \end{cases} \quad (5.1)$$

Each $h(l, k) \in \mathbb{R}^{q \times p}$ can be partitioned as

$$\begin{aligned}
 h(l, k) &= \begin{bmatrix} h_0^T(l, k) \\ h_1^T(l, k) \\ \vdots \\ h_{q-1}^T(l, k) \end{bmatrix} \\
 &= \begin{bmatrix} h_{0,0}(l, k) & h_{0,1}(l, k) & \cdots & h_{0,p-1}(l, k) \\ h_{1,0}(l, k) & h_{1,1}(l, k) & \cdots & h_{1,p-1}(l, k) \\ \vdots & \vdots & \vdots & \vdots \\ h_{q-1,0}(l, k) & h_{q-1,1}(l, k) & \cdots & h_{q-1,p-1}(l, k) \end{bmatrix}, \tag{5.2}
 \end{aligned}$$

where

$$h_{j,m}(l, k) = \begin{cases} (C_l)_j A_{l-1} \cdots A_{k+1} (B_k)_m, & l > k, \\ (D_l)_{j,m} + (D'_l)_{j,m}, & l = k, \\ (C'_l)_j A'_{l+1} \cdots A'_{k-1} (B'_k)_m, & l < k, \end{cases} \tag{5.3}$$

for $j = 0, 1, \dots, q-1, m = 0, 1, \dots, p-1$, $(C_l)_j$ and $(C'_l)_j$ are respectively the j th row of C_l and C'_l , $(B_k)_m$ and $(B'_k)_m$ are respectively the m th column of B_k and B'_k , $(D_l)_{j,m}$ and $(D'_l)_{j,m}$ are the entry of D_l and D'_l at j th row and m th column.

For a given blocked input sequence $u = \{u_k\}_{k \in \mathbb{Z}} \in l_p^2$, the system produces a blocked sequence output $y = \{y_k\}_{k \in \mathbb{Z}} \in l_q^2$ by the following convolution:

$$y = \begin{bmatrix} h_0^T \\ h_1^T \\ \vdots \\ h_{q-1}^T \end{bmatrix} * u, \tag{5.4}$$

where

$$h_j^T := \left\{ \cdots \quad h_j^T(-1) \quad h_j^T(0) \quad h_j^T(1) \quad \cdots \right\}, \tag{5.5}$$

$j = 0, 1, \dots, q-1$. Equation (5.4) is equivalent to

$$\begin{aligned} y_l &= \sum_{k \in \mathbb{Z}} \begin{bmatrix} h_0^T(l, k) \\ h_1^T(l, k) \\ \vdots \\ h_{q-1}^T(l, k) \end{bmatrix} u_k \\ &= \sum_{k \in \mathbb{Z}} \begin{bmatrix} \langle h_0(l, k), u_k \rangle \\ \langle h_1(l, k), u_k \rangle \\ \vdots \\ \langle h_{q-1}(l, k), u_k \rangle \end{bmatrix}. \end{aligned} \quad (5.6)$$

Define $h_j(l, k) = f_{lq+j}(k)$ and $f_{lq+j} = \left\{ \dots f_{lq+j}(-1) \ f_{lq+j}(0) \ f_{lq+j}(1) \ \dots \right\}$ for $l \in \mathbb{Z}$ and $j = 0, 1, \dots, q-1$. Equation (5.6) can be written as

$$y_l = \sum_{k \in \mathbb{Z}} \begin{bmatrix} \langle u_k, f_{lq}(k) \rangle \\ \langle u_k, f_{lq+1}(k) \rangle \\ \vdots \\ \langle u_k, f_{lq+q-1}(k) \rangle \end{bmatrix} = \begin{bmatrix} \langle u, f_{lq} \rangle \\ \langle u, f_{lq+1} \rangle \\ \vdots \\ \langle u, f_{lq+q-1} \rangle \end{bmatrix}. \quad (5.7)$$

As a result, the system output satisfies

$$\|y\|^2 = \sum_{k \in \mathbb{Z}} y_k^* y_k = \sum_{i \in \mathbb{Z}} |\langle u, f_i \rangle|^2$$

where $i = lq+j$ for $l \in \mathbb{Z}$ and $j = 0, 1, \dots, q-1$, $f_i = \{\dots, f_i(-1), f_i(0), f_i(1), \dots\} \in l_p^2$.

It follows from (5.3) that

$$\begin{aligned}
f_i^T(k) &= f_{lq+j}^T(k) \\
&= h_j^T(l, k) \\
&= \begin{cases} (C_l)_j A_{l-1} \cdots A_{k+1} B_k, & l > k \\ (D_l)_j + (D'_l)_j, & l = k \\ (C'_l)_j A'_{l+1} \cdots A'_{k-1} B'_k, & l < k \end{cases} \quad (5.8)
\end{aligned}$$

Thus the stable causal-anticausal system (3.12) yields a set of l_p^2 sequences $\{f_i\}$ consisting of the system impulse response matrices that constitutes a frame on l_p^2 , providing that $E(e^{j\omega}) \neq 0, \omega \in [0, 2\pi)$. Further, if $u_k \in l_p^2$ is formed by the blocking operation $u_k = \begin{bmatrix} \hat{u}_{kp} & \hat{u}_{kp+1} & \cdots & \hat{u}_{kp+p-1} \end{bmatrix}^T, \hat{u}_k \in l^2$, then $\{f_i\}_{i \in \mathbb{Z}}$ is a frame on l^2 .

The LTI system is one special case of LTV systems. The impulse response of the LTI system can be represented by:

$$h(k) = \begin{cases} CA^{k-1}B, & k > 0, \\ D + D', & k = 0, \\ C'(A')^{-k-1}B', & k < 0, \end{cases} \quad (5.9)$$

yielding the frame elements

$$\begin{aligned}
f_i^T(k) &= f_{lq+j}^T(k) \\
&= h_j^T(l - k) \\
&= \begin{cases} C_j A^{l-k-1} B, & k - l > 0, \\ D_j + D'_j, & k - l = 0, \\ C'_j (A')^{k-l-1} B', & k - l < 0. \end{cases} \quad (5.10)
\end{aligned}$$

Theorem 5.2.1. *Let $\{f_m\}_{m \in \mathbb{Z}}(\{h_m^T\}_{m \in \mathbb{Z}}$ be a frame implemented by the impulse response $\{h_m^T\}_{m \in \mathbb{Z}}$ of a stable mixed causal-anticausal LTV system E . Then its canonical dual frame is given by the impulse response of $E^\dagger$, the pseudo inverse*

system of E . The upper frame bound of $E^\dagger$ equals the lower frame bound of $\{f_m\}_{m \in \mathbb{Z}}$.

Proof: There exists a dual frame $\{g_m\}_{m \in \mathbb{Z}}$ for given frame $\{f_m\}_{m \in \mathbb{Z}}$ for which

$$u = \sum_{m=-\infty}^{\infty} \langle u, g_m \rangle f_m = \sum_{m=-\infty}^{\infty} \langle u, f_m \rangle g_m. \quad (5.11)$$

The frame element represents the bi-directional infinite impulse responses of a linear system, hence $f_m = h_m^T$, yields $E = T^*$, where T^* is the adjoint of the synthesis operator of the frame $\{f_m\}_{m \in \mathbb{Z}}$. The linear system operator E can be factorized into:

$$E = Q^* U R_{lr} V$$

where Q is inner, $U^*U = I$ and $VV^* = I$ and R_{lr} are the left and right outer. More details of the inner factor and the outer factor can be found in [40]. E has a Moore-Penrose (pseudo-) inverse

$$E^\dagger = V^* R_{lr}^{-1} U^* Q.$$

The pre-frame operator for $\{g_m\}_{m=-\infty}^{\infty}$ is denoted by $\tilde{T}$. The canonical dual frame is obtained by

$$\{g_m\}_{m=-\infty}^{\infty} = \{S^{-1}f_m\}_{m=-\infty}^{\infty},$$

which implies

$$\tilde{T} = S^{-1}T = (TT^*)^{-1}T,$$

hence $\tilde{T}T^* = I$. The canonical dual frame has the minimum l^2 norm among the dual frames.

Since $E = T^*$, the pre-frame operator of the canonical dual frame can be obtained

by:

$$\begin{aligned}
\tilde{T} &= (TT^*)^{-1}T \\
&= (V^*R_{lr}^*U^*QQ^*UR_{lr}V)^{-1}V^*R_{lr}^*U^*Q \\
&= V^*R_{lr}^{-1}U^*QQ^*U(R_{lr}^*)^{-1}VV^*R_{lr}^{-1}U^*Q \\
&= V^*R_{lr}^{-1}U^*Q.
\end{aligned} \tag{5.12}$$

We can see that

$$\tilde{T} = E^\dagger.$$

Thus it satisfies $\tilde{T}T^* = I$ and $E^\dagger E = I$. As a result, the canonical dual frame is corresponding to the pseudo-inverse system. Hence the lower bound of the frame is equivalent to the inverse of the upper bound of the pseudo-inverse system. $\square$

5.3 Frame bounds of frames modeled as mixed causal-anticausal linear systems

There is no transfer function matrix representation with constant parameters for the causal LTV system, thus the frame bounds of frames modeled as LTV systems cannot be solved via the transfer function matrix. Instead of the transfer function matrix, the mixed causal-anticausal LTV system can be represented by state space equations as in (3.12) with state matrices $D'_k = 0$, and it can also be represented by the transfer operator as in (3.15). In this section, a state space approach is presented to obtain the frame bounds of frames that modeled as LTV systems in terms of state matrices via LMI optimization technique.

Theorem 5.3.1. *Given a frame that can be modeled as a stable mixed causal-anticausal LTV system E (3.12) with $D'_k = 0$, the optimal upper frame bound is*

the infimum (minimum) of all possible β satisfying

$$\|Eu\|^2 \leq \beta \|u\|^2.$$

The problem of finding the minimum β is equivalent to the following optimization problem:

$$\min_{P_k, Q_k, X_k} \beta$$

such that

$$\begin{bmatrix} A_k^T P_{k+1} A_k - P_k + C_k^T C_k & A_k^T X_k - X_k A'_k + C_k^T C'_k \\ X_k^T A_k - A_k'^T X_k^T + C_k'^T C_k & A_k'^T Q_k A'_k - Q_{k+1} + C_k'^T C'_k \\ D_k^T C_k + B_k^T P_{k+1} A_k - B_k'^T X_k^T & B_k'^T Q_k A'_k + D_k^T C'_k + B_k^T X_k \\ C_k^T D_k + A_k^T P_{k+1} B_k - X_k B'_k \\ A_k'^T Q_k B'_k + C_k'^T D_k + X_k^T B_k \\ D_k^T D_k + B_k^T P_{k+1} B_k + B_k'^T Q_k B'_k - \beta I \end{bmatrix} \leq 0, \quad (5.13)$$

where $P_k = P_k^T > 0$, $Q_k = Q_k^T > 0$ and X_k are general matrices.

Proof: The class of frames modeled by the mixed causal-anticausal LTV system can be represented by

$$E = D + C(I - ZA)^{-1}ZB + C'(ZI - A')^{-1}B' = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac},$$

and $\|Eu\|^2 \leq \beta\|u\|^2$ implies $E^*E \leq \beta I$. Hence

$$\begin{aligned} E^*E &= \left(\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \right) \left(\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \right) \\ &= \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \\ &\quad + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac}. \end{aligned}$$

There are four terms in the above expression, each term is a cascaded system. By using theorem 4.3.1, the first term is written as:

$$\begin{aligned} &\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c \\ &= \left[\begin{array}{c|c} A & B \\ \hline D^T C + B^T P^{(+1)} A & D^T D + B^T P^{(+1)} B \end{array} \right]_c + \left[\begin{array}{c|c} A^T & C^T D + A^T P^{(+1)} B \\ \hline B^T & 0 \end{array} \right]_{ac} \end{aligned}$$

where $A^T P^{(+1)} A - P + C^T C = 0$, yielding $B^T Z^*(I - A^T Z^*)^{-1}(A^T P^{(+1)} A - P + C^T C)(I - ZA)^{-1}ZB = 0$. The second term can follow Theorem 4.3.2 and can be rewritten as:

$$\left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} = \left[\begin{array}{c|c} A'^T & A'^T Q B' \\ \hline B'^T & B'^T Q B' \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline B'^T Q A' & 0 \end{array} \right]_{ac}$$

where $A'^T Q A' - Q^{(+1)} + C'^T C' = 0$, yielding $B'^T (Z^* I - A'^T)^{-1}(A'^T Q A' - Q^{(+1)} +$

$C'^T C')(ZI - A')^{-1} B' = 0$. The third term can be rewritten as:

$$\begin{aligned} \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c &= \left[\begin{array}{cc|c} A & 0 & B \\ C'^T C & A'^T & C'^T D \\ \hline 0 & B'^T & 0 \end{array} \right]_c \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A'^T & C'^T D + UB \\ \hline -B'^T U & B'^T & 0 \end{array} \right]_c, \end{aligned}$$

where $UA - A'^T U + C'^T C = 0$, yielding $B'^T (Z^* I - A'^T)^{-1} (UA - A'^T U + C'^T C) (I - ZA)^{-1} ZB = 0$. The fourth term can be rewritten as:

$$\begin{aligned} \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} &= \left[\begin{array}{cc|c} A^T & C^T C' & 0 \\ 0 & A' & B' \\ \hline B^T & D^T C' & 0 \end{array} \right]_{ac} \\ &= \left[\begin{array}{cc|c} A^T & 0 & -VB' \\ 0 & A' & B' \\ \hline B^T & D^T C' + B^T V & 0 \end{array} \right]_{ac}, \end{aligned}$$

where $A^T V - V A' + C^T C' = 0$, yielding $B^T Z^* (I - A^T Z^*)^{-1} (A^T V - V A' + C^T C') (ZI - A')^{-1} B' = 0$. The third term condition $UA - A'^T U + C'^T C = 0$ and fourth term condition $A^T V - V A' + C^T C' = 0$ result in $V = U^T, U = V^T$.

Hence $E^*E - \beta I$ can be rewritten as

$$\begin{aligned}
& E^*E - \beta I \\
&= \begin{bmatrix} B^T Z^* (I - A^T Z^*)^{-1} & B'^T (Z^* I - A'^T)^{-1} & I \end{bmatrix} \\
&\times \begin{bmatrix} A^T P^{(+1)} A - P + C^T C & A^T V - V A' + C^T C' \\ U A - A'^T U + C'^T C & A'^T Q A' - Q^{(+1)} + C'^T C' \\ D^T C + B^T P^{(+1)} A - B'^T U & B'^T Q A' + D^T C' + B^T V \\ C^T D + A^T P^{(+1)} B - V B' & \\ A'^T Q B' + C'^T D + U B & \\ D^T D + B^T P^{(+1)} B + B'^T Q B' - \beta I & \end{bmatrix} \begin{bmatrix} (I - Z A)^{-1} Z B \\ (Z I - A')^{-1} B' \\ I \end{bmatrix} \leq 0.
\end{aligned}$$

Thus the matrix on the k^{th} time instance

$$\begin{bmatrix} A_k^T P_{k+1} A_k - P_k + C_k^T C_k & A_k^T V_k - V_k A'_k + C_k^T C'_k \\ U_k A_k - A_k'^T U_k + C_k'^T C_k & A_k'^T Q_k A'_k - Q_{k+1} + C_k'^T C'_k \\ D_k^T C_k + B_k^T P_{k+1} A_k - B_k'^T U_k & B_k'^T Q_k A'_k + D_k^T C'_k + B_k^T V_k \\ C_k^T D_k + A_k^T P_{k+1} B_k - V_k B'_k & \\ A_k'^T Q_k B'_k + C_k'^T D_k + U_k B_k & \\ D_k^T D_k + B_k^T P_{k+1} B_k + B_k'^T Q_k B'_k - \beta I & \end{bmatrix}$$

is a negative definite matrix. Let $X_k = V_k$, $X_k^T = U_k$, this theorem is proved. $\square$

The LMIs to compute the lower frame bounds of mixed causal-anticausal LTV systems can be derived analogically.

Theorem 5.3.2. *Given a frame that can be modeled as a stable mixed causal-anticausal LTV system E (3.12) with $D'_k = 0$, the optimal lower frame bound is the supremum (maximum) of all possible α satisfying*

$$\alpha \|u\|^2 \leq \|Eu\|^2.$$

The problem of finding the maximum α is equivalent to the following optimization problem:

$$\min_{P_k, Q_k, X_k} -\alpha$$

such that

$$\begin{bmatrix} A_k^T P_{k+1} A_k - P_k + C_k^T C_k & A_k^T X_k - X_k A'_k + C_k^T C'_k \\ X_k^T A_k - A'^T_k X_k^T + C'^T_k C_k & A'^T_k Q_k A'_k - Q_{k+1} + C'^T_k C'_k \\ D_k^T C_k + B_k^T P_{k+1} A_k - B'^T_k X_k^T & B'^T_k Q_k A'_k + D_k^T C'_k + B_k^T X_k \\ C_k^T D_k + A_k^T P_{k+1} B_k - X_k B'_k \\ A'^T_k Q_k B'_k + C'^T_k D_k + X_k^T B_k \\ D_k^T D_k + B_k^T P_{k+1} B_k + B'^T_k Q_k B'_k - \alpha I \end{bmatrix} \geq 0 \quad (5.14)$$

where $P_k = P_k^T > 0$, $Q_k = Q_k^T > 0$ and X_k are general matrices.

The proof is given in Appendix A.

Immediately, the upper bound and lower bound of causal LTV systems are obtained by setting $A'_k = 0, B'_k = 0, C'_k = 0, D'_k = 0$.

5.3.1 Frame bounds of frames modeled as mixed causal-anticausal LTI systems

The LTI system is one special case of LTV systems such that all the state space matrices are time-invariant. The theorem below can be directly derived from Theorem 5.3.1.

Theorem 5.3.3. *Given a frame that can be modeled as a stable mixed causal-anticausal LTI system E (3.9) with $E(e^{j\omega})$ being full column rank on $\omega \in [0, 2\pi)$,*

the optimal upper frame bound is the infimum (minimum) of all possible β satisfying

$$\|Eu\|^2 \leq \beta \|u\|^2.$$

The problem of finding the minimum β is equivalent to the following optimization problem:

$$\min_{P,Q,X} \beta$$

subject to:

$$\begin{bmatrix} A^T P A - P + C^T C & A^T X - X A' + C^T C' & A^T P B + C^T D - X B' \\ X^T A - A'^T X^T + C'^T C & A'^T Q A' - Q + C'^T C' & A'^T Q B' + C'^T D + X^T B \\ B^T P A + D^T C - B'^T X^T & B'^T Q A' + D^T C' + B^T X & B^T P B + B'^T Q B' + D^T D - \beta I \end{bmatrix} \leq 0,$$

where $P = P^T > 0$, $Q = Q^T > 0$ and X is a general matrix.

The proof can be found in Appendix A.

Theorem 5.3.3 can be used to obtain the upper bound of the causal LTI system by setting $A' = 0, B' = 0, C' = 0, D' = 0$, resulting $X = 0$. In this case, Theorem 5.3.3 is equivalent to Lemma 2.1.2. This theorem can also be used to obtain the upper bound of the anticausal LTI system by setting $A = 0, B = 0, C = 0, D = 0$, resulting $X = 0$.

Similarly, the lower bound is computed by the following theorem directly derived from Theorem 5.3.2.

Theorem 5.3.4. *Given a frame that can be modeled as a stable mixed causal-anticausal LTI system E (3.9) with $E(e^{j\omega})$ being full column rank on $\omega \in [0, 2\pi)$, the optimal infinite norm (operator norm) is the supremum (maximum) of all possible*

α satisfying

$$\alpha \|u\|^2 \leq \|Eu\|^2.$$

The problem of finding the maximum α is equivalent to the following optimization problem

$$\min_{P,Q,X} -\alpha$$

subject to:

$$\begin{bmatrix} A^T P A - P + C^T C & A^T X - X A' + C^T C' & A^T P B + C^T D - X B' \\ X^T A - A'^T X^T + C'^T C & A'^T Q A' - Q + C'^T C' & A'^T Q B' + C'^T D + X^T B \\ B^T P A + D^T C - B'^T X^T & B'^T Q A' + D^T C' + B^T X & B^T P B + B'^T Q B' + D^T D - \alpha I \end{bmatrix} \geq 0,$$

where $P = P^T > 0$, $Q = Q^T > 0$ and X is a general matrix.

The proof is given in Appendix A.

The inverse of the lower frame bound is actually the upper bound of the canonical dual frame. Hence, another approach to get the lower frame bound is obtaining the pseudo-inverse of the system, since the pseudo-inverse system is the canonical dual frame as shown in Theorem 5.2.1.

5.3.2 Frame bounds computation algorithms for some special LTV systems

Case 1: simple switching system

For $k < 0$, the state matrices are given as $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right]_c + \left[\begin{array}{c|c} A'_1 & B'_1 \\ \hline C'_1 & 0 \end{array} \right]_{ac}$, and for

$k \geq 0$, the system matrices are given as $\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right]_c + \left[\begin{array}{c|c} A'_2 & B'_2 \\ \hline C'_2 & 0 \end{array} \right]_{ac}$. Theorem 5.3.1 yields the following 3 LMIs:

$$\begin{bmatrix} A_1^T P_1 A_1 - P_1 + C_1^T C_1 & A_1^T X_1 - X_1 A'_1 + C_1^T C'_1 & C_1^T D_1 + A_1^T P_1 B_1 - X_1 B'_1 \\ X_1^T A_1 - A_1'^T X_1^T + C_1'^T C_1 & A_1'^T Q_1 A'_1 - Q_1 + C_1'^T C'_1 & A_1'^T Q_1 B'_1 + C_1'^T D_1 + X_1^T B_1 \\ D_1^T C_1 + B_1^T P_1 A_1 - B_1'^T X_1^T & B_1'^T Q_1 A'_1 + D_1^T C'_1 + B_1^T X_1 & D_1^T D_1 + B_1^T P_1 B_1 + B_1'^T Q_1 B'_1 - \beta I \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} A_1^T P_2 A_1 - P_1 + C_1^T C_1 & A_1^T X_1 - X_1 A'_1 + C_1^T C'_1 & C_1^T D_1 + A_1^T P_2 B_1 - X_1 B'_1 \\ X_1^T A_1 - A_1'^T X_1^T + C_1'^T C_1 & A_1'^T Q_1 A'_1 - Q_2 + C_1'^T C'_1 & A_1'^T Q_1 B'_1 + C_1'^T D_1 + X_1^T B_1 \\ D_1^T C_1 + B_1^T P_2 A_1 - B_1'^T X_1^T & B_1'^T Q_1 A'_1 + D_1^T C'_1 + B_1^T X_1 & D_1^T D_1 + B_1^T P_2 B_1 + B_1'^T Q_1 B'_1 - \beta I \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} A_2^T P_2 A_2 - P_2 + C_2^T C_2 & A_2^T X_2 - X_2 A'_2 + C_2^T C'_2 & C_2^T D_2 + A_2^T P_2 B_2 - X_2 B'_2 \\ X_2^T A_2 - A_2'^T X_2^T + C_2'^T C_2 & A_2'^T Q_2 A'_2 - Q_2 + C_2'^T C'_2 & A_2'^T Q_2 B'_2 + C_2'^T D_2 + X_2^T B_2 \\ D_2^T C_2 + B_2^T P_2 A_2 - B_2'^T X_2^T & B_2'^T Q_2 A'_2 + D_2^T C'_2 + B_2^T X_2 & D_2^T D_2 + B_2^T P_2 B_2 + B_2'^T Q_2 B'_2 - \beta I \end{bmatrix} \leq 0,$$

Case 2: linear periodic time-varying (LPTV) system

For an LPTV system with state matrices $\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array} \right]_c + \left[\begin{array}{c|c} A'_i & B'_i \\ \hline C'_i & 0 \end{array} \right]_{ac}$ $i = 1, 2, \dots, N$, Theorem 5.3.1 yields the following N LMIs:

$$\begin{bmatrix} A_1^T P_2 A_1 - P_1 + C_1^T C_1 & A_1^T X_1 - X_1 A'_1 + C_1^T C'_1 & C_1^T D_1 + A_1^T P_2 B_1 - X_1 B'_1 \\ X_1^T A_1 - A_1'^T X_1^T + C_1'^T C_1 & A_1'^T Q_1 A'_1 - Q_2 + C_1'^T C'_1 & A_1'^T Q_1 B'_1 + C_1'^T D_1 + X_1^T B_1 \\ D_1^T C_1 + B_1^T P_2 A_1 - B_1'^T X_1^T & B_1'^T Q_1 A'_1 + D_1^T C'_1 + B_1^T X_1 & D_1^T D_1 + B_1^T P_2 B_1 + B_1'^T Q_1 B'_1 - \beta I \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} A_2^T P_2 A_2 - P_2 + C_2^T C_2 & A_2^T X_2 - X_2 A'_2 + C_2^T C'_2 & C_2^T D_2 + A_2^T P_3 B_2 - X_2 B'_2 \\ X_2^T A_2 - A_2'^T X_2^T + C_2'^T C_2 & A_2'^T Q_2 A'_2 - Q_3 + C_2'^T C'_2 & A_2'^T Q_2 B'_2 + C_2'^T D_2 + X_2^T B_2 \\ D_2^T C_2 + B_2^T P_3 A_2 - B_2'^T X_2^T & B_2'^T Q_2 A'_2 + D_2^T C'_2 + B_2^T X_2 & D_2^T D_2 + B_2^T P_3 B_2 + B_2'^T Q_2 B'_2 - \beta I \end{bmatrix} \leq 0,$$

$\vdots$

$$\begin{bmatrix} A_N^T P_1 A_N - P_N + C_N^T C_N & A_N^T X_N - X_N A'_N + C_N^T C'_N & C_N^T D_N + A_N^T P_1 B_N - X_N B'_N \\ X_N^T A_N - A_N'^T X_N^T + C_N'^T C_N & A_N'^T Q_N A'_N - Q_1 + C_N'^T C'_N & A_N'^T Q_N B'_N + C_N'^T D_N + X_N^T B_N \\ D_N^T C_N + B_N^T P_1 A_N - B_N'^T X_N^T & B_N'^T Q_N A'_N + D_N^T C'_N + B_N^T X_N & D_N^T D_N + B_N^T P_1 B_N + B_N'^T Q_N B'_N - \beta I \end{bmatrix} \leq 0.$$

Similar analysis apply to the lower bound computation.

5.4 Examples

Example 1: Simple switching filter banks

Consider a time-varying IIR filter bank, which consists of a low pass filter $H(z)$ and a high pass filter $G(z)$.

For $k < 0$, the low pass filter is given as

$$H(z) = \frac{0.2445 + 0.4980z^{-1} + 0.2445z^{-2}}{1 - 0.2094z^{-1} + 0.3388z^{-2}},$$

and the high pass filter is given as

$$G(z) = \frac{0.3352 - 0.6704z^{-1} + 0.3352z^{-2}}{1 - 0.0294z^{-1} + 0.3388z^{-2}};$$

for $k \geq 0$, the low pass filter are switched to the low pass filter given as

$$H(z) = \frac{1 + 7z^{-1} + 21z^{-2} + 35z^{-3} + 35z^{-4} + 21z^{-5} + 7z^{-6} + z^{-7}}{\sqrt{2}(1 + 21z^{-2} + 35z^{-4} + 7z^{-6})},$$

and the high pass filter are switched to the high pass filter given as

$$G(z) = z^{-1} \frac{1 - 7z^1 + 21z^2 - 35z^3 + 35z^4 - 21z^5 + 7z^6 - z^7}{\sqrt{2}(1 + 21z^2 + 35z^4 + 7z^6)}.$$

The switching IIR filter bank can be represented by the state space representations.

For $k < 0$, the IIR filter bank is causal system with state space matrices given as:

$$A_1 = \begin{bmatrix} 0.2094 & -0.3388 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.2094 & -0.3388 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.5402 & 0.1617 & 0 & 0 \\ 0 & 0 & -0.6002 & 0.2216 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.2445 \\ 0.3352 \end{bmatrix},$$

for $k \geq 0$, the IIR filter bank is switched to the mixed causal-anticausal system with state space matrices given as:

$$A_2 = \begin{bmatrix} 0 & -0.2319 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.6881 & 0 & -0.0331 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$C_2 = \begin{bmatrix} 0.3499 & 0 & 0.0234 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7071 & -0.5431 & 0.4865 & -0.1524 & 0.0234 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.7071 & -0.3499 \end{bmatrix}^T,$$

$$\begin{aligned}
A'_2 &= \begin{bmatrix} 0 & -0.6881 & 0 & -0.0331 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.2319 \\ 0 & 0 & 0 & 0 & 1.001 & 0 \end{bmatrix}, \\
B'_2 &= \begin{bmatrix} 0 & -6.6787 & 0 & 16.3918 & -0.0404 & 0 \end{bmatrix}^T, \\
C'_2 &= \begin{bmatrix} 0 & 0 & 0 & 0.0331 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.001 & 1.4273 \end{bmatrix}, \\
D'_2 &= \begin{bmatrix} 0 & 0 \end{bmatrix}^T.
\end{aligned}$$

The time-varying IIR filter bank constructs a frame with upper frame bound: 1.5079, and the lower frame bound: 0.5543.

Example 2: 2-periodic time-varying filter bank

Now consider a 2-periodic time-varying filter bank. For $k = \text{odd}$ number, the low pass filter is given as:

$$H(z) = \frac{0.2445 + 0.4980z^{-1} + 0.2445z^{-2}}{1 - 0.2094z^{-1} + 0.3388z^{-2}},$$

and the high pass filter is given as:

$$G(z) = \frac{0.3352 - 0.6704z^{-1} + 0.3352z^{-2}}{1 - 0.0294z^{-1} + 0.3388z^{-2}};$$

for $k = \text{even}$ number, the low pass filter are switched to the low pass filter given as:

$$H(z) = \frac{1 + 7z^{-1} + 21z^{-2} + 35z^{-3} + 35z^{-4} + 21z^{-5} + 7z^{-6} + z^{-7}}{\sqrt{2}(1 + 21z^{-2} + 35z^{-4} + 7z^{-6})},$$

and the high pass filter are switched to the high pass filter given as:

$$G(z) = z^{-1} \frac{1 - 7z^1 + 21z^2 - 35z^3 + 35z^4 - 21z^5 + 7z^6 - z^7}{\sqrt{2}(1 + 21z^2 + 35z^4 + 7z^6)}.$$

The 2-periodic time-varying filter bank can be modeled as state space model. For $k = \text{odd}$ number, the state space matrices are given as A_1, B_1, C_1, D_1 ; for $k = \text{even}$ number, the state space matrices are given as $A_2, B_2, C_2, D_2, A'_2, B'_2, C'_2, D'_2$. The numerical values of A_1, B_1, C_1, D_1 and $A_2, B_2, C_2, D_2, A'_2, B'_2, C'_2, D'_2$ are given in Example 1.

The 2-periodic time-varying filter bank constructs a frame with upper frame bound: 1.3986, and the lower frame bound: 0.5554.

Example 3: An 2×2 PR TVFB

Consider a 2×2 LTV FB system (from [109]) with the system equation

$$y_n = [I - v_n v_n^\dagger] u_n + v_n v_{n-1}^\dagger u_{n-1}$$

where the 2×1 vector v_n satisfies $v_n^\dagger v_n = 1$ for all n . The output of the system can be described as

$$y_n = \begin{cases} u_n, & n < 0, \\ (I - v_0 v_0^\dagger) u_0, & n = 0, \\ (I - v_n v_n^\dagger) u_n + v_n v_{n-1}^\dagger u_{n-1}, & n > 0. \end{cases}$$

The inverse system that achieves perfect reconstruction is given as follows

$$\hat{u}_n = \begin{cases} y_n, & n < 0, \\ (I - v_n v_n^\dagger) y_n + v_n v_{n+1}^\dagger y_{n+1}, & n \geq 0, \end{cases}$$

which is an anticausal LTV system.

Let $v_n = 0$ for $n < 0$ and $v_n = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, yielding $v_n^\dagger = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ for $n \geq 0$.

The state space matrices of the LTV FB system (causal system) are shown as: for $n < 0$:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

for $n = 0$:

$$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix},$$

for $n > 0$:

$$A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}.$$

The state space matrices of the inverse system (anticausal system) that gives perfect reconstruction are shown as: for $n < 0$:

$$\hat{A}'_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B}'_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\hat{C}'_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{D}'_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

for $n \geq 0$:

$$\begin{aligned}\hat{A}'_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \hat{B}'_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \hat{C}'_2 &= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, & \hat{D}'_2 &= \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}.\end{aligned}$$

Applying the proposed approach, we can find the upper frame bound is 1 and the lower frame bound is 1.

5.5 Summary

This chapter presents state space approaches to obtain the frame bounds of those frames which can be modeled as LTI systems and LTV systems. The existing state space approach employing LMI optimization technique to compute the frame bounds applies to causal realizations only. The mixed causal-anticausal realizations have to be transformed into unstable causal realizations in order to apply the existing technique. The proposed approach avoids the transformation of realizations, hence saving a lot of computation.

Chapter 6

A state space approach to computation of wavelet frame bounds

6.1 Introduction

The existing literature shows that wavelet frame bounds are computed in the frequency domain see for example [37] and [30]. The frequency approach to approximate the wavelet frame bounds is complex and tedious since it involves very dense frequency grids. It requires that the explicitly expressions of the wavelets in the frequency domain are known. For some wavelet frames realized by DWTs, the explicitly expressions of the wavelets in the frequency domain may be hard to obtain or even not exists, for example Butterworth wavelet, then the classic frequency approach is not applicable.

The DWT employs an analysis tree-structure multirate FB, which may be regarded as the adjoint of the pre-frame (synthesis) operator of the underlying frame.

The tree-structure multirate FB is an application of a non-uniform filter banks, defined by an elementary uniform FB. The inverse DWT (IDWT) may be regarded as the synthesis operator of the underlying dual frame. The IDWT may have synthesis tree-structure multirate FB realization or maybe a black box with state space representations only. The L level DWT consists of L wavelets, whose state space realizations are derived in this chapter. The L level IDWT consists of L dual wavelets, whose state space realizations are also derived in this chapter. As L is large enough, the L dual wavelets will reconstruct the original signal, hence the L dual wavelets may be considered as the dual wavelet frame while the L wavelets in the DWT constitute a wavelet frame. The upper bound of the wavelet frame is computed with the proposed state space approach. The lower bound of the wavelet frame is equivalent to the inverse of the upper bound of the L dual wavelets system.

This chapter will present a state space approach to compute the wavelet frames for one class of wavelet frames, which is realized by the DWT. The state space representation of the tree-structure FB is derived in this chapter, which is one of the contributions of the current thesis. As introduced in Section 2.2, there are two approaches to obtain the state space representation of a tree-structure FB by treating it as a nonuniform FB. The first approach is achieved by looking for the polyphase matrix representation of the nonuniform FB and the second approach is based on the transfer function representations of each channel of the nonuniform FB. The proposed approach is different from both approaches. It starts from the state space representation of the elementary FB and then recursively compute the state space representation of the multilevel tree-structure FB. Hence it is easy to derive the state space representation of the $(L + 1)$ level FB from the state space representation of L level FB.

In order to get the state space representation of the tree-structure FB, the state space representations of cascaded LTI systems and parallel systems are found by

using lemma 6.1.1 and 6.1.2 respectively. The causal-anticausal LTI system has representations

$$E_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right]_c + \left[\begin{array}{c|c} A'_1 & B'_1 \\ \hline C'_1 & D'_1 \end{array} \right]_{ac},$$

and

$$E_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right]_c + \left[\begin{array}{c|c} A'_2 & B'_2 \\ \hline C'_2 & D'_2 \end{array} \right]_{ac}.$$

Lemma 6.1.1. *The system E_1 is cascaded with E_2 . The state space representation of the cascaded system is shown as*

$$\left[\begin{array}{cccc|cccc} A_2 & B_2C_1 & 0 & 0 & & & & B_2D_1 \\ 0 & A_1 & 0 & 0 & & & & B_1 \\ 0 & 0 & A_1 & 0 & & & & B_1 \\ 0 & 0 & 0 & A_2 & & & & B_2D'_1 + A_2YB'_1 \\ \hline C_2 & D_2C_1 & D'_2C_1 + C'_2XA_1 & C_2 & D_2D_1 + D'_2D_1 + C'_2XB_1 + D_2D'_1 + C_2YB'_1 & & & \end{array} \right]_c$$

$$+ \left[\begin{array}{cccc|cccc} A'_2 & B'_2C'_1 & 0 & 0 & & & & B'_2D'_1 \\ 0 & A'_1 & 0 & 0 & & & & B'_1 \\ 0 & 0 & A'_2 & 0 & & & & B'_2D_1 + A'_2XB_1 \\ 0 & 0 & 0 & A'_1 & & & & B'_1 \\ \hline C'_2 & D'_2C'_1 & C'_2 & D_2C'_1 + C_2YA'_1 & & & & D'_2D'_1 \end{array} \right]_{ac}$$

where X, Y are given by the Sylvester equations:

$$A'_2XA_1 - X + B'_2C_1 = 0,$$

$$A_2YA'_1 - Y + B_2C'_1 = 0.$$

Lemma 6.1.1 is a general case extended from [121] and is a direct result from Theorem 4.3.3 for the restricted LTI case.

Lemma 6.1.2. *The system E_1 is in parallel with E_2 . Assume the compatibility*

of the state space matrices. If the two subsystems have same input but different outputs, the state space representation of the parallel system is given as:

$$\left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & 0 & D_1 \\ 0 & C_2 & D_2 \end{array} \right]_c + \left[\begin{array}{cc|c} A'_1 & 0 & B'_1 \\ 0 & A'_2 & B'_2 \\ \hline C'_1 & 0 & D'_1 \\ 0 & C'_2 & D'_2 \end{array} \right]_{ac}.$$

If the two subsystems have different inputs and the outputs are summed together, the state space representation of the parallel system is shown as:

$$\left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline C_1 & C_2 & D_1 & D_2 \end{array} \right]_c + \left[\begin{array}{cc|cc} A'_1 & 0 & B'_1 & 0 \\ 0 & A'_2 & 0 & B'_2 \\ \hline C'_1 & C'_2 & D'_1 & D'_2 \end{array} \right]_{ac}.$$

6.2 Frame bounds of wavelet frames

The DWTs employ the wavelets in the form:

$$\psi_{m,n}(t) = a_0^{-m/2} \psi(a_0^{-m}t - nb_0). \quad (6.1)$$

If the set $\{\psi_{m,n}\}$ defined by (6.1) is a complete set in $L^2(\mathbb{R})$ for some choice of ψ , a_0 , and b_0 , then the set is called an affine wavelet. Such complete sets are called frames.

It can be shown that

$$\alpha \|v\|^2 \leq \sum_{m,n} |\langle v, \psi_{m,n} \rangle|^2 \leq \beta \|v\|^2 \quad (6.2)$$

for all $f \in L^2(\mathbb{R})$.

For computational efficiency, the typical DWT has $a_0 = 2$ and $b_0 = 1$. In the DWT

analysis, there exist a scaling function $\phi(x)$ and a mother wavelet or just wavelet $\psi(x)$ such that:

$$\begin{aligned}\phi(x) &= \sqrt{2} \sum_n h_n \phi(2x - n), \\ \psi(x) &= \sqrt{2} \sum_n g_n \phi(2x - n),\end{aligned}$$

where $h_n \in \mathbb{R}$ and $g_n \in \mathbb{R}$. Hence:

$$\begin{aligned}\langle v, \phi_{m,n} \rangle &= \sum_k h_{k-2n} \langle v, \phi_{m-1,k} \rangle, \\ \langle v, \psi_{m,n} \rangle &= \sum_k g_{k-2n} \langle v, \phi_{m-1,k} \rangle.\end{aligned}$$

Let a signal $v(t) = \sum_n c_n^0 \psi_{0,n}$. Assume the label of the fine scale is $j = 0$. The wavelet coefficients in this scale are given by

$$c_n^0 = \langle v, \phi_{0,n} \rangle.$$

Let

$$c_n^j = \langle v, \phi_{j,n} \rangle, \quad d_n^j = \langle v, \psi_{j,n} \rangle.$$

DWTs can be computed by the pyramidal algorithm

$$\begin{aligned}c_n^0 &= c_n^0, \\ c_n^j &= \sum_k h_{k-2n} c_k^{j-1}, j = 1, 2, \dots \\ d_n^j &= \sum_k g_{k-2n} c_k^{j-1}, j = 1, 2, \dots\end{aligned}$$

Figure 6.1 (a) shows the DWT of a signal. The signal is firstly filtered by a pre-filter, giving the wavelet coefficients $c^0 = [\dots c_{-1}^0 \ c_0^0 \ c_1^0 \ \dots]$. The wavelet coefficients then follow the L -level tree-structure multirate FB, resulting $c^L, d^L, d^{L-1}, \dots, d^1$. The input signal is reconstructed by the IDWT followed by a post-filter, see Figure 6.1 (b). The inverse DWT may not in tree-structure, instead it has a state space representation, see Figure 6.1 (c).

Figure 6.1: The discrete wavelet transform and inverse discrete wavelet transform

There are two types of approximations used in wavelet applications. The first is the approximation of the original signal using its projection in a multiresolution subspace. The other is the approximation of the true wavelet coefficients of a given signal in a scale subspace by computationally efficient sampling or prefiltering. In the DWT application, the first task is to determine a scale-limited subspace onto which the projection of the signal $f(t)$ is the best approximation. Secondly, the projection is calculated. Recently, the second step has been investigated using the idea of prefiltering, and numerous FIR prefilters have been designed, see for example [151], [92], [150]. A special FIR prefilter is the scaling function $\psi(t)$ sampled at the integer points. In the following analysis, the pre-filter and post-filter are set to be all pass filters with magnitude 1, in this case, the wavelet frames bounds are equivalent to the multirate FB frame bounds.

Theorem 6.2.1. *If c^L can be neglected for sufficient large L , the system E consisting of L wavelets in the L level DWT form a wavelet frame with upper bound β such*

that

$$\|Ev\|^2 = \sum_{k=1}^L \|d^k\|^2 \leq \beta \|v\|^2.$$

The lower bound α of the wavelet frame is given as

$$\alpha \|v\|^2 \leq \sum_{k=1}^L \|d^k\|^2 = \|Ev\|^2.$$

Proof: The upper bound follows the definition of wavelet frames shown in (6.2) and $d^k = \{d_n^k\}_{n \in \mathbb{Z}}$. As $L \rightarrow \infty$, the original signal v is decomposed into a set of frequency range containing the details of it, i.e. $d^L, \dots, d^1$ and the lowest frequency (or 'smooth') part of v , i.e. c^L . For sufficient large L , $c^L \rightarrow 0$, that is c^L can be neglected. As a result, the original signal v can be perfectly reconstructed from $d^L, \dots, d^1$. The lower wavelet frame bound is equivalent to the inverse of the upper bound of the dual systems. $\square$

The state space representations of the wavelet frames which are realized in DWTs, can be shown in terms of the state space matrices of the elementary uniform FB with scaling filter $H(z)$ and wavelet filter $G(z)$. Upsampling process and downsampling process are involved in the computation. Lemma 6.2.1 and 6.2.2 present the state space representations of the system together with the downsampler or upsampler. Let $D^M : l^2 \rightarrow l^2$ and $U^L : l^2 \rightarrow l^2$ denote the decimator with a factor of M and interpolator with a factor L respectively, and $E = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac}$ denote the causal-anticausal LTI system.

Lemma 6.2.1. *If the decimation process (with a factor of M) is performed before the causal-anticausal LTI system E , then the state space representation of the multirate system is given as:*

$$ED^M = \left[\begin{array}{c|c|c} A & B & 0_1 \\ \hline C & D & 0_2 \end{array} \right]_c + \left[\begin{array}{c|c|c} A' & B' & 0_3 \\ \hline C' & 0 & 0_4 \end{array} \right]_{ac},$$

where $0_1 - 0_4$ have $M - 1$ number of columns and the number of rows depends on the number of rows of B, D, B' and $D' = 0$ respectively.

If the decimation process is performed after the causal-anticausal LTI system, then the state space representation of the multirate system is given as:

$$D^M E = \left[\begin{array}{c|ccc} A^M & A^{M-1}B & \dots & AB & B \\ \hline CA^{M-1} & CA^{M-2}B & \dots & CB & D \end{array} \right]_c + \left[\begin{array}{c|cccc} (A')^M & B' & A'B' & \dots & (A')^{M-1}B' \\ \hline C' & 0 & & & \end{array} \right]_{ac}.$$

The proof of Lemma 6.2.1 can be found in Appendix A.

Lemma 6.2.2. *If the interpolation process (with a factor of L) is performed before the causal-anticausal system E , the state space representation of the multirate system is given as:*

$$EU^L = \left[\begin{array}{c|c} A^L & A^{L-1}B \\ \hline C & D \\ CA^1 & CB \\ \vdots & \vdots \\ CA^{L-1} & CA^{L-2}B \end{array} \right]_c + \left[\begin{array}{c|c} (A')^L & B' \\ \hline C'(A')^{L-1} & 0 \\ \vdots & \\ C'A' & \\ C' & \end{array} \right]_{ac}.$$

If the interpolation process is performed after the causal-anticausal system, the state space representation of the state space representation of the multirate system is given as:

$$U^L E = \left[\begin{array}{c|c} A & B \\ \hline C & D \\ 0_1 & 0_2 \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \\ 0_3 & 0_4 \end{array} \right]_{ac},$$

where $0_1 - 0_4$ have $(L - 1)$ number of rows and the number of columns depend on the number of columns of C, D, C' and $D' = 0$ respectively.

The proof of Lemma 6.2.2 is analog with that of Lemma 6.2.1 and can be found in

Appendix A.

6.2.1 Upper bound of DWTs via state space approach

The 1st level DWT employs a scaling filter $H(z)$ and a wavelet filter $G(z)$ with impulse response h_{-k} and g_{-k} , $k \in \mathbb{Z}$ respectively, followed by a downsampler with a factor of 2. The scaling filter $H(z)$ and wavelet filter $G(z)$ can be realized using state space equations, in which the state space matrices satisfy:

$$H(z) = \sum_{k \in \mathbb{Z}} h_{-k} z^{-k} = \left[\begin{array}{c|c} A_h & B_h \\ \hline C_h & D_h \end{array} \right]_c + \left[\begin{array}{c|c} A'_h & B'_h \\ \hline C'_h & D'_h \end{array} \right]_{ac},$$

and

$$G(z) = \sum_{k \in \mathbb{Z}} g_{-k} z^{-k} = \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array} \right]_c + \left[\begin{array}{c|c} A'_g & B'_g \\ \hline C'_g & D'_g \end{array} \right]_{ac}.$$

The corresponding linear operators are denoted as H and G respectively.

Thus one-level DWT gives :

$$c^j = D^2 H c^{j-1}, \quad d^j = D^2 G c^{j-1},$$

where $D^2 H$ is the linear operator mapping from c^{j-1} to c^j , and $D^2 G$ is the linear operator mapping from c^{j-1} to d^j . $D^2 H$ and $D^2 G$ have state space representations represented in terms of the state space matrices of $H(z)$ and $G(z)$ following Lemma 6.2.1.

The two-level DWT gives:

$$c^{j+1} = D^2 H c^j = D^2 H D^2 H c^{j-1}, \quad d^{j+1} = D^2 G D^2 H c^{j-1},$$

where the computation of the state space matrices of $D^2 H D^2 H$ ($D^2 G D^2 H$) can

be separated to two steps based on the state matrices of H (G). The first step is to compute the state space representation of the cascaded system HD^2H (GD^2H) following Lemma 6.1.1. The second step is to obtain the state space matrices of D^2HD^2H (D^2GD^2H) by down sampling the output of HD^2H (GD^2H) by a factor of 2 following Lemma 6.2.1.

These two-level wavelets can be shown as $\begin{bmatrix} D^2GD^2H \\ D^2G \end{bmatrix}$, which are multirate parallel systems. The sampling periods of these two systems are not the same. Hence, the state space representations cannot follow Lemma 6.1.2 directly. In order to make the sampling period to be the same, the output from D^2G can be upsampled by a factor of 2 first, followed by a downsampler with a factor of 2. Assume the input sampling period is T_s . By inserting one zero in between the successive samples of the $\bar{G}$ outputs. The time difference of two new successive samples (including zeros inserted) now is still T_s . The new output sequence is then downsampled by a factor of 2, hence the sampling period becomes $2T_s$. That is $\begin{bmatrix} D^2GD^2H \\ D^2G \end{bmatrix} = \begin{bmatrix} D^2GD^2H \\ D^2U^2D^2G \end{bmatrix}$. Thus Lemma 6.1.2 can be applied to get the state space representations of $\begin{bmatrix} D^2GD^2H \\ D^2U^2D^2G \end{bmatrix}$.

Similarly, the state space representation of L -level wavelets can be recursively found. The upper bound can be computed by Theorem 5.3.3, which is the upper bound of the the wavelet frame.

6.2.2 Lower bound of DWTs via state space approach

The IDWTs are the pseudo-inverse systems of DWTs which give the reconstruction process of c^0 . The reconstructed signal $\hat{c}^0$ is exactly the same as c^0 if the IDWT provides perfect reconstruction. Otherwise, there is an error between $\hat{c}^0$ and c^0 . The first level IDWT employs the dual scaling filter and dual wavelet filter with impulse

response $\tilde{h}_{-k}$ and $\tilde{g}_{-k}$, $k \in \mathbb{Z}$ respectively. They can be realized using state space equations, where state space matrices are given as:

$$\tilde{H}(z) = \sum_{k=-\infty}^{\infty} \tilde{h}_{-k} z^{-k} = \left[\begin{array}{c|c} \tilde{A}_h & \tilde{B}_h \\ \hline \tilde{C}_h & \tilde{D}_h \end{array} \right]_c + \left[\begin{array}{c|c} \tilde{A}'_h & \tilde{B}'_h \\ \hline \tilde{C}'_h & 0 \end{array} \right]_{ac}$$

and

$$\tilde{G}(z) = \sum_{k=-\infty}^{\infty} \tilde{g}_{-k} z^{-k} = \left[\begin{array}{c|c} \tilde{A}_g & \tilde{B}_g \\ \hline \tilde{C}_g & \tilde{D}_g \end{array} \right]_c + \left[\begin{array}{c|c} \tilde{A}'_g & \tilde{B}'_g \\ \hline \tilde{C}'_g & 0 \end{array} \right]_{ac}.$$

c^{j-1} can be reconstructed from c^j and d^j such that

$$c_n^{j-1} = \sum_k \tilde{h}_{n-2k} c_k^j + \sum_k \tilde{g}_{n-2k} d_k^j$$

yielding

$$c^{j-1} = \tilde{H}U^2 c^j + \tilde{G}U^2 d^j.$$

where the state space matrices of $\tilde{H}U^2$ and $\tilde{G}U^2$ are represented in terms of the state space matrices of $\tilde{H}(z)$ and $\tilde{G}(z)$ following Lemma 6.2.2 with an interpolation factor of 2.

The two-level IDWT reconstructs c^{j-2} from c^j, d^j and d^{j-1} :

$$c^{j-2} = \tilde{H}U^2(\tilde{H}U^2 c^j + \tilde{G}U^2 d^j) + \tilde{G}U^2 d^{j-1},$$

where the computation of the state space representations of $\tilde{H}U^2 \tilde{H}U^2$ ($\tilde{H}U^2 \tilde{G}U^2$) can be separated to two steps based on the state space matrices of $\tilde{H}$ ($\tilde{G}$). The first step is to compute the state space representation of the cascaded system $\tilde{H}U^2 \tilde{H}$ ($\tilde{H}U^2 \tilde{G}$) following Lemma 6.1.1. The second step is to obtain the state space matrices of $\tilde{H}U^2 \tilde{H}U^2$ ($\tilde{H}U^2 \tilde{G}U^2$) by up sampling the input of $\tilde{H}U^2 \tilde{H}$ ($\tilde{G}U^2 \tilde{H}$) by a factor of 2 following Lemma 6.2.2

The two-level dual wavelets are the multirate parallel system $\begin{bmatrix} \tilde{H}U^2\tilde{G}U^2 & \tilde{G}U^2 \end{bmatrix}$, with different sampling period. Hence, the state space representations cannot follow Lemma 6.1.2. Similar to the DWT case, the input of $\tilde{G}U^2$ can be upsampled by a factor of 2 first, followed by a downsampler with a factor of 2 to make the sampling time to be the same. Thus Lemma 6.1.2 can be applied to get the state space representations of $\begin{bmatrix} \tilde{H}U^2\tilde{G}U^2 & \tilde{G}U^2D^2U^2 \end{bmatrix}$.

Similarly the state space representation of L level dual wavelets can be found. As $L \rightarrow \infty$, the L level dual wavelets will reconstruct the input, hence the upper bound of the dual wavelets system is the inverse of the lower bound of the original wavelets, resulting a wavelet frames.

6.2.3 General state space representations of wavelet frames

Assume the state space model for the L level multirate FB are found and represented as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac},$$

the state space equations are then shown in (3.9), with input

$$u_k = c_k^0,$$

and the output:

$$y_k = \begin{bmatrix} c_k^L \\ d_k^L \\ \vdots \\ d_k^1 \end{bmatrix},$$

The state space representation of the L level wavelets is partial set of the state space representation of the multirate FB such that:

$$\left[\begin{array}{c|c} A & B \\ \hline C(2:L,:) & D(2:L,:) \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C'(2:L,:) & D'(2:L,:) \end{array} \right]_{ac}, \quad (6.3)$$

where $C(2:L,:)$ discards the first row of C and retains the remaining, same with $D(2:L,:), C'(2:L,:), D'(2:L,:)$. Hence the output of the L level wavelets are the collection of $d^L, \dots, d^1$ with input c^0 . Hence the proposed approach can be applied to obtain the upper wavelet frame bound.

The state space representation of the pseudo-inverse system is shown as:

$$\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right]_c + \left[\begin{array}{c|c} \hat{A}' & \hat{B}' \\ \hline \hat{C}' & \hat{D}' \end{array} \right]_{ac}.$$

The state space equations of the pseudo-inverse system are given as:

$$\begin{aligned} \hat{x}_{k+1} &= \hat{A}\hat{x}_k + \hat{B}\hat{u}_k, \\ \hat{x}'_{k-1} &= \hat{A}'\hat{x}'_k + \hat{B}'\hat{u}_k, \\ \hat{y}_k &= \hat{C}\hat{x}_k + \hat{C}'\hat{x}'_k + (\hat{D} + \hat{D}')\hat{u}_k, \end{aligned}$$

where the input

$$\hat{u}_k = \begin{bmatrix} c_k^L \\ d_k^L \\ \vdots \\ d_k^1 \end{bmatrix},$$

and the output

$$\hat{y}_k = \hat{c}_k^0.$$

The L level wavelets system is presented by the state space representation (6.3). If

the pseudo-inverse system has stable mixed causal-anticausal realization, the inverse of the upper bound of the pseudo-inverse system of the L level wavelets system (6.3) is equivalent to the inverse of the lower bound of the L level wavelets system, hence constructs a wavelet frame.

Another case is that the pseudo-inverse system of the L level wavelets system is unstable, or oscillated. In this case, the computation of upper bound of the pseudo-inverse system is infeasible, resulting an unknown lower wavelet frame bound. Instead of looking for the pseudo-inverse system of the L level wavelets system directly, an approximate dual system is considered, which is the partial set of the pseudo-inverse of the given tree-structure FB. The reconstruction process is done by a state space model found as the pseudo-inverse of the given tree-structure FB with $A, B, C, D, A', B', C', D'$.

Hence the approximate dual system of the L level wavelets system is realized by the partial set of the pseudo-inverse system state space representation presented as:

$$\left[\begin{array}{c|c} \hat{A} & \hat{B}(:, 2:L) \\ \hline \hat{C} & \hat{D}(:, 2:L) \end{array} \right]_c + \left[\begin{array}{c|c} \hat{A}' & \hat{B}'(:, 2:L) \\ \hline \hat{C}' & \hat{D}'(:, 2:L) \end{array} \right]_{ac},$$

where $\hat{B}(:, 2:L)$ discards the first column of $\hat{B}$ and retains the remaining, same with $\hat{D}(:, 2:L), \hat{B}'(:, 2:L), \hat{D}'(:, 2:L)$. Hence the input of the approximate dual wavelet frame is the collection of $d^L, \dots, d^1$ and the output is the reconstructed signal $\hat{c}^0$, which is approximate equal to c^0 .

As L becomes large enough, the approximate dual system will reconstruct the original input signal, hence the upper bound of the approximate dual system equals to the inverse of the lower bound of the L level wavelets system, resulting a wavelet frame.

6.3 Examples

In this section, examples are given in which the DWT and IDWT are both realized in tree-structure multirate FBs. The pre-filter and post-filter are set to be all pass filters with magnitude equals to 1.

Example 1: Butterworth wavelets (IIR) [71] gives the scaling filter $H(z)$ and wavelet filter $G(z)$ such that:

$$\begin{aligned} \langle f, \phi_{m,n} \rangle &= \sum_k h_{k-2n} \langle f, \phi_{m-1,k} \rangle, \\ \langle f, \psi_{m,n} \rangle &= \sum_k g_{k-2n} \langle f, \phi_{m-1,k} \rangle. \end{aligned}$$

in which $H(z) = \sum_{k \in \mathbb{Z}} h_{-k} z^{-k} = \frac{\sum_{k=0}^N \binom{N}{k} z^{-k}}{\sqrt{2} \sum_{l=0}^{(N-1)/2} \binom{N}{2l} z^{-2l}}$ where N is the order of the filter and $G(z) = \sum_{k \in \mathbb{Z}} g_{-k} z^{-k} = z^{2n-1} H(-z^{-1})$. The dual wavelets have functions:

$$\tilde{H}(z) = H(z^{-1}), \tilde{G}(z) = G(z^{-1})$$

This example makes use of the example stated in [71].

$$\begin{aligned} H(z) &= \frac{1+7z^{-1}+21z^{-2}+35z^{-3}+35z^{-4}+21z^{-5}+7z^{-6}+z^{-7}}{\sqrt{2}(1+21z^{-2}+35z^{-4}+7z^{-6})} \\ G(z) &= z^{-1} \frac{1-7z^1+21z^2-35z^3+35z^4-21z^5+7z^6-z^7}{\sqrt{2}(1+21z^2+35z^4+7z^6)} \\ \tilde{H}(z) &= H(z^{-1}) \\ \tilde{G}(z) &= G(z^{-1}) \end{aligned}$$

The causal-anticausal state space representations the scaling filter $H(z)$ and wavelet

filter $G(z)$ are found as:

$$H(z) = \left[\begin{array}{ccc|c} 0 & -0.2319 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0.3499 & 0 & 0.0234 & 0.7071 \end{array} \right]_c + \left[\begin{array}{cccc|c} 0 & -0.6881 & 0 & -0.0331 & 0 \\ 1 & 0 & 0 & 0 & -6.6787 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 16.3918 \\ \hline 0 & 0 & 0 & 0.0331 & 0 \end{array} \right]_{ac},$$

and

$$G(z) = \left[\begin{array}{ccccc|c} 0 & -0.6881 & 0 & -0.0331 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0.7071 & -0.5431 & 0.4865 & -0.1524 & 0.0234 & -0.3499 \end{array} \right]_c + \left[\begin{array}{cc|c} 0 & -0.2319 & -0.0404 \\ 1.001 & 0 & 0 \\ \hline -0.001 & -1.4273 & 0 \end{array} \right]_{ac}.$$

Hence the dual scaling filter $\tilde{H}(z)$ and dual wavelet filter $\tilde{G}(z)$ have causal-anticausal state space representations shown as:

$$\tilde{H}(z) = \left[\begin{array}{ccc|c} 0 & -0.2319 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0.3499 & 0 & 0.0234 & 0 \end{array} \right]_{ac} + \left[\begin{array}{cccc|c} 0 & -0.6881 & 0 & -0.0331 & 0 \\ 1 & 0 & 0 & 0 & -6.6787 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 16.3918 \\ \hline 0 & 0 & 0 & 0.0331 & 0.7071 \end{array} \right]_c,$$

Level	1	2	3
$\sqrt{\beta}$	1.0003	1.0002	1.0002
$\sqrt{1/\beta_{dual}}$	0.9997	0.9998	0.9998

Table 6.1: Butterworth wavelet frame bounds

and

$$\tilde{G}(z) = \left[\begin{array}{ccccc|c} 0 & -0.6881 & 0 & -0.0331 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0.7071 & -0.5431 & 0.4865 & -0.1524 & 0.0234 & 0 \end{array} \right]_{ac} + \left[\begin{array}{cc|c} 0 & -0.2319 & -0.0404 \\ 1.001 & 0 & 0 \\ \hline -0.001 & -1.4273 & -0.3499 \end{array} \right]_c.$$

The state space representations L -level tree structure DWTs and IDwTs are obtained following sections 6.2.1 and 6.2.2. With the causal-anticausal state space representations obtained, Theorem 5.3.3 is applied to find the upper bound of analysis filter banks and the upper bounds of the synthesis filter banks, which is the lower bounds of the analysis filter banks. The norms computed are given in Table 6.1.

β indicates the upper bound of the L level wavelets, and β_{dual} indicates the upper bound of the L level dual wavelets. If L is large enough, the dual wavelets will reconstruct the input signal, hence the upper bound of the dual wavelets β_{dual} becomes the inverse of the lower bound of the wavelets, yielding a wavelet frame.

Example 2: Suppose the scaling filter and wavelet filter are given as:

$$\begin{aligned} H(z) &= -\frac{1}{16}z + \frac{1}{4} - \frac{1}{16}z^{-1}, \\ G(z) &= z - 4 + 6z^{-1} - 4z^{-2} + z^{-3}, \end{aligned}$$

and the dual scaling filter and wavelet filter are given as:

$$\begin{aligned}\tilde{H}(z) &= z^2 + 4z + 6 + 4z^{-1} + z^{-2}, \\ \tilde{G}(z) &= \frac{1}{16}z^2 + \frac{1}{4}z + \frac{1}{16},\end{aligned}$$

The causal-anticausal state space representations the scaling filter $H(z)$ and wavelet filter $G(z)$ are found as:

$$H(z) = \left[\begin{array}{ccc|c} 0 & & & 0.25 \\ \hline & & & 0.25 \\ -0.25 & & & \end{array} \right]_c + \left[\begin{array}{ccc|c} 0 & & & 0.25 \\ \hline & & & 0 \\ -0.25 & & & \end{array} \right]_{ac},$$

and

$$G(z) = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1.5 & -1 & 0.25 & -4 \end{array} \right]_c + \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array} \right]_{ac}.$$

Hence the dual scaling filter $\tilde{H}(z)$ and dual wavelet filter $\tilde{G}(z)$ have causal-anticausal state space representations shown as:

$$\tilde{H}(z) = \left[\begin{array}{ccc|c} 0 & 0 & & 2 \\ 1 & 0 & & 0 \\ \hline 2 & 0.5 & & 6 \end{array} \right]_c + \left[\begin{array}{ccc|c} 0 & 0 & & 2 \\ 1 & 0 & & 0 \\ \hline 2 & 0.5 & & 0 \end{array} \right]_{ac},$$

and

$$\tilde{G}(z) = \left[\begin{array}{ccc|c} 0 & 0 & & 1 \\ 1 & 0 & & 0 \\ \hline 0 & 0 & & 0.0625 \end{array} \right]_c + \left[\begin{array}{ccc|c} 0 & 0 & & 0.5 \\ 1 & 0 & & 0 \\ \hline 0.5 & 0.125 & & 0 \end{array} \right]_{ac}.$$

The wavelets constitute a wavelet frame with frame bounds shown in table 6.2.

β indicates the upper bound of the L level wavelets, and β_{dual} indicates the upper

Level	1	2	3
$\sqrt{\beta}$	11.3137	8.4853	8.3800
$\sqrt{1/\beta_{dual}}$	3.5778	0.3304	0.0293

Table 6.2: FIR wavelet frame bounds

Level		1	2	3	4	5
Haar	$\sqrt{1/\beta_{dual}}$	1	1	1	1	1
	$\sqrt{\beta}$	1	1	1	1	1
Daube4	$\sqrt{1/\beta_{dual}}$	1	1	1	1	1
	$\sqrt{\beta}$	1	1	1	1	1
SP(2,4)	$\sqrt{1/\beta_{dual}}$	0.7071	0.8157	0.8400	0.8407	0.8407
	$\sqrt{\beta}$	1	1.4142	1.6334	1.7091	1.7338
VSP(4,4)	$\sqrt{1/\beta_{dual}}$	0.9547	0.8599	0.8790	0.8952	0.9056
	$\sqrt{\beta}$	1.1496	1.2025	1.2360	1.2466	1.2494
COR(2)	$\sqrt{1/\beta_{dual}}$	0.9999	0.9849	0.9823	0.9817	0.9816
	$\sqrt{\beta}$	1.0102	1.0142	1.0117	1.0116	1.0112

Table 6.3: Multilevel parallel wavelets frame bounds

bound of the L level dual wavelets. If L is large enough, the dual wavelets will reconstruct the input signal, hence the upper bound of the dual wavelets β_{dual} becomes the inverse of the lower bound of the wavelets, yielding a wavelet frame.

The wavelet frames constructed from some DWTs such as Haar DWTs, Daubechies DWTs, etc, have the frame bounds shown in table 6.3. β indicates the upper bound of the L level wavelets, and β_{dual} indicates the upper bound of the L level dual wavelets. If L is large enough, the dual wavelets will reconstruct the input signal, hence the upper bound of the dual wavelets β_{dual} becomes the inverse of the lower bound of the wavelets, yielding a wavelet frame. The scaling function and wavelet function of these DWTs and IDWTs can be found in Appendix C.

6.4 Frame bounds of Mexican hat wavelet approximation

[37] illustrated one practical implementation of Mexican hat wavelets, which employed one auxiliary function ϕ to achieve hierarchy algorithm.

If $a_0 = 2$ and $b_0 = 0.5$, then

$$\langle f, \psi_{m,n} \rangle = \sum_k d_k \langle f, \phi_{m,n+k} \rangle;$$

and

$$\langle f, \phi_{m,n} \rangle = \frac{1}{\sqrt{2}} \sum_k c_k \langle f, \phi_{m-1,n+k} \rangle,$$

so that the $\langle f, \psi_{m,n} \rangle, \langle f, \phi_{m,n} \rangle$ can be computed recursively, working from the smallest scale to the largest scale.

The scaling filter and wavelet filter are given as:

$$\begin{aligned} H(z) &= \frac{1}{\sqrt{2}} \left(\frac{1}{8}z^2 + \frac{1}{2}z + \frac{3}{4} + \frac{1}{2}z^{-1} + \frac{1}{8}z^{-2} \right), \\ G(z) &= 6\sqrt{\frac{70}{1313}} \left(-\frac{1}{2}z + 1 - \frac{1}{2}z^{-1} \right). \end{aligned}$$

Let the pre-filter and post-filter to be 1. The computation of upper bound follows Section 6.2.1.

In order to obtain the lower bound, the dual or pseudo-inverse of the analysis FB is obtained. The causal-anticausal state space representations can be transformed into causally unstable realization following Section 4.2.1. Inner-coprime factorization is then applied, see Lemma 2.3.2. The left-inverse or right-inverse can be obtained in terms of the inner and coprime factors. The dual obtained is in state space representations, which is not in tree-structure realization. The analysis follows Section 6.2.3.

Level	1	2	3
$\sqrt{\beta}$	2.7708	2.7708	2.7708
$\sqrt{1/\beta_{dual}}$	1.3630	1.2111	1.1559

Table 6.4: Mexican hat wavelet frame bounds without pre-filter

Level	1	2	3
$\sqrt{\beta}$	1.0390	1.0438	1.0438
$\sqrt{1/\beta_{dual}}$	1.0402	0.9777	1.0957

Table 6.5: Mexican hat wavelet frame bounds with pre-filter

The frame bounds are given in table 6.4.

If the pre-filter is defined as:

$$\hat{\phi}(z) = 0.1667z + 0.6667 + 0.1667z^{-1},$$

the frame bounds computed are given in table 6.5.

6.5 Summary

This chapter present a state space approach to compute the wavelet frames for one class of wavelet frames, which are realized by the DWTs. The state space representation of the wavelet frame is derived from the state space representation of the elementary FB. The upper bound of the wavelet frame are computed with the proposed state space approach. The lower bound of the wavelet frame is equivalent to the inverse of the upper bound of the L dual wavelets system, which gives PR of the input signal.

Chapter 7

Conclusions and Recommendation of Future Work

7.1 Conclusions

For a frame that can be modeled as a causal LTI system with transfer function $E(z)$, there are two approaches to obtain the frame bounds. The first approach is the frequency approach, which employs eigenanalysis of the frame operator $S(z) = E^*(z)E(z)$. The upper frame bound β and lower frame bound α are the essential supremum and essential infimum, respectively, of the eigenvalues of the frame operator. The frequency approach is an approximation approach that depends on the frequency grid. The transfer function $E(z)$ is a proper function, hence it can be represented by a state space representation. Thus another frame bounds computation approach is the state space approach in the time domain, which is achieved by using LMI optimization technique.

This thesis considers a class of frames modeled as mixed causal-anticausal LTI realizations. In order to apply the existing state space approach employing LMI op-

timization technique to obtain the frame bounds, the stable mixed causal-anticausal realizations must be converted into a unstable causal realization, which is very tedious. In this thesis, some techniques are shown to convert a stable mixed causal-anticausal realization into an unstable causal realization, and vice versa. One of the major contributions of this thesis is that a direct state space approach to obtain the frame bounds of the frame modeled as mixed causal-anticausal LTI systems is developed. This state space approach employs LMI optimization technique. The advantage of this approach is that it avoids converting the mixed causal-anticausal representation into unstable causal realization, hence saving many computation steps.

For frames that can be modeled as LTV systems, there are no proper transfer function representations. Instead, these frames are represented by the state space representation with state matrices $A_k, B_k, C_k, D_k, A'_k, B'_k, C'_k$ and D'_k . Similar to the LTI case, a state space approach is derived to obtain the frame bounds. This is another major contribution of this thesis.

The inverse of the lower frame bound is actually the upper bound of the canonical dual frame, which is constructed by the impulse response of the pseudo-inverse system. Hence another method to compute the lower frame bound is achieved by looking for the operator norm of the pseudo-inverse system. The computation of the pseudo-inverse system involves inner-coprime or outer factorizations. However, the traditional factorizations are suitable for causal LTI systems only. Hence the mixed causal-anticausal realizations have to be converted to the unstable causal realizations before factorization. The causality properties of cascaded causal-anticausal LTI systems are known result. The realization transformations have been presented by using the causality properties of cascaded causal-anticausal LTI systems. In this thesis, the causality properties of cascaded causal-anticausal LTV systems are investigated, which are obtained by making use of the time-varying Sylvester equa-

tions. The causality properties can be applied to inner-coprime factorizations. In this thesis, several algorithms have been presented to obtain the inner-coprime factorizations of strictly causal LTV systems and strictly anticausal LTV systems. And such analysis is one major contribution of this thesis.

The classical computation approach of wavelet frame bounds is a frequency domain approach, which gives the approximation of the wavelet frame bounds. The frequency approach requires that the explicitly expressions of the wavelets in the frequency domain are known. For some wavelet frames that realized in DWTs, the explicitly expressions of the wavelets in the frequency domain are hard to obtain or even not exist. Hence the wavelet frame bounds cannot be obtained via the classical frequency approach. The proposed state space computation approach can be applied to obtain the upper bound of the class of wavelet frames, which are realized by analysis tree-structure FBs, for example, IIR wavelets (Butterworth wavelets), FIR wavelets (Haar wavelets, Daubechies wavelets, Spline wavelets, etc). The lower bounds of such class of wavelet frames are the inverse of the upper bounds of the dual systems, which may be realized in synthesis tree-structure FBs or in state space models. The state space realizations of such class wavelet frames and dual systems with tree-structure realization can be represented in terms of the state space matrices of the elementary FB block. With the mixed causal-anticausal state space representation, the direct computation approach presented in this thesis is applied to get the upper bounds. Another class of wavelet frames is also given in analysis tree-structure FBs, but the dual systems are not in synthesis tree-structure FBs, for example, Mexican hat wavelet approximation. The dual system is found by the pseudo-inverse of the given analysis FBs and is represented in terms of state space matrices only. In this thesis, these two classes of wavelet frames have been considered and the wavelet frame bounds can be obtained by using the proposed state space approach. This novel state space approach to obtain the wavelet frame bounds is one of the major contributions of this thesis.

7.2 Recommendations of Future Work

To continue on this research work on the state space approach to the frames is recommended in the end of this thesis.

1. This thesis has presented the current works on the state space approach to the class of wavelet frames, which are realized in the DWT. In the future work, one may derive the state space representation of other class of wavelet frames if they exist so that the proposed state space approaches can be applied to obtain the frame bounds.
2. Some frames may be modeled as LTI systems with rational transfer function matrix. In this case, state space representations are known. In the future work, one may investigate on the state space representation of fractional linear systems, which may lead to a frame.

Author's Publications

1. Xiaofang Chen, Cishen Zhang and Jingxin Zhang, Error Analysis of the Anti-Causal Realization of Periodic Inverse Systems, *Circuits, Systems, and Signal Processing*, vol. 27, No. 4, pp.519-536, 2008.
2. Xiaofang Chen, Cishen Zhang and Jingxin Zhang, A State Space Approach to the Inverse of LPTV Filters, *IEEE International Conference on Acoustics Speech and Signal Processing*, May 2006, Toulouse France.
3. Xiaofang Chen, Cishen Zhang and Jingxin Zhang, Decomposition and Non-causal Realization of Unstable LPTV System, *IEEE International Conference on Control Automation Robotics and Vision*, Dec 2006, pp.1-6, Singapore
4. Xiaofang Chen, Cishen Zhang and Jingxin Zhang, Computation of Inverse Systems Using Inner-outer Factorization, *IEEE International Conference on Control and Automation*, May 30 2007-June 1 2007, pp.3059-3064, Guangzhou.
5. Xiaofang Chen, Cishen Zhang and Jingxin Zhang, Stability and error analysis of the anti-causal realization of periodic inverse systems, *Chinese Control and Decision Conference*, 2-4 July 2008, pp. 4797-4802, Yantai, Shandong.
6. Xiaofang Chen, Cishen Zhang and Jingxin Zhang, Realizations of cascaded LTV mixed causal and anticausal systems, *IEEE International Conference on Control and Automation*, 9-11 June 2010, pp. 1305-1310, Xiamen.
7. Xiaofang Chen, Cishen Zhang and Jingxin Zhang, Error modeling and reduction of inverse LPTV systems, *IEEE Conference on Industrial Electronics and Applications*, 2011. Accepted.
8. Xiaofang Chen, Cishen Zhang and Jingxin Zhang, Analysis and computation of optimal bounds of bi-directional frames, *International Conference on Informatics in Control Automation and Robotics*, 2011. Accepted.

9. Xiaofang Chen, Cishen Zhang and Jingxin Zhang, A state space approach to the analysis of frames with applications to wavelet frames, to be submitted.
10. Xiaofang Chen, Cishen Zhang and Jingxin Zhang, Performance Analysis and Evaluation of A Class of Frames, to be submitted.

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Appendix A

Proof of Theorems and Lemmas

Proof of Theorem 4.3.2

The cascaded LTV system $E_c E_{ac}$ can be represented by:

$$\begin{aligned}
 E_c E_{ac} &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right]_{ac} \\
 &= (D + C(I - ZA)^{-1}ZB)(D' + C'(ZI - A')^{-1}B') \\
 &= DD' + C(I - ZA)^{-1}ZBD' + DC'(ZI - A')^{-1}B' \\
 &\quad + C(I - ZA)^{-1}ZBC'(ZI - A')^{-1}B',
 \end{aligned}$$

in which $BC' = Y^{(+1)} - AY A' = Z^*YZ - AY A'$. Thus $E_c E_{ac}$ can be rewritten as:

$$\begin{aligned}
 E_c E_{ac} &= DD' + C(I - ZA)^{-1}ZBD' + DC'(ZI - A')^{-1}B' \\
 &\quad + C(I - ZA)^{-1}Z(Z^*YZ - AY A')(ZI - A')^{-1}B' \\
 &= DD' + C(I - ZA)^{-1}ZBD' + DC'(ZI - A')^{-1}B' \\
 &\quad + C(I - ZA)^{-1}(YZ - ZAY A' - YA' + YA')(ZI - A')^{-1}B' \\
 &= DD' + C(I - ZA)^{-1}ZBD' + DC'(ZI - A')^{-1}B' \\
 &\quad + C(I - ZA)^{-1}YB' + CYA'(ZI - A')^{-1}B' \\
 &= DD' + C(I - ZA)^{-1}ZBD' + DC'(ZI - A')^{-1}B' \\
 &\quad + C(I + (I - ZA)^{-1}ZA)YB' + CYA'(ZI - A')^{-1}B' \\
 &= \left[\begin{array}{c|c} A & BD' + AYB' \\ \hline C & DD' + CYB' \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline DC' + CYA' & 0 \end{array} \right]_{ac}.
 \end{aligned}$$

Hence

$$E_c E_{ac} = \bar{E}_c + \bar{E}_{ac},$$

where the diagonal entries are shown as:

$$\begin{aligned}
 \bar{E}_{c,k} &= \left[\begin{array}{c|c} A_k & B_k D'_k + A_k Y_k B'_k \\ \hline C_k & D_k D'_k + C_k Y_k B'_k \end{array} \right]_c \\
 \bar{E}_{ac,k} &= \left[\begin{array}{c|c} A'_k & B'_k \\ \hline D_k C'_k + C_k Y_k A'_k & 0 \end{array} \right]_{ac}.
 \end{aligned}$$

Proof of Theorem 4.4.3

$E_c = \Delta^* V$ results in $\Delta = V E_c^*$, where E_c^* is the anti-causal system with state space matrices: $\left[\begin{array}{c|c} A_k^* & C_k^* \\ \hline B_k^* & D_k^* \end{array} \right]_{ac}$. V is causal and has state space matrices $\left[\begin{array}{c|c} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right]_c$

and Δ is causal and its state space matrices are $\left[\begin{array}{c|c} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right]_c$.

Applying Theorem 4.3.2, we can get

$$\begin{aligned} \left[\begin{array}{c|c} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right]_c &= \left[\begin{array}{c|c} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right]_c \left[\begin{array}{c|c} A_k^* & C_k^* \\ \hline B_k^* & D_k^* \end{array} \right]_{ac} \\ &= \left[\begin{array}{c|c} A_{vk} & B_{vk}D_k^* + A_{vk}Y_kC_k^* \\ \hline C_{vk} & D_{vk}D_k^* + C_{vk}Y_kC_k^* \end{array} \right]_c \end{aligned} \quad (\text{A.1})$$

where Y_k satisfies:

$$A_{vk}Y_kA_k^* + B_{vk}B_k^* - Y_{k+1} = 0, \quad (\text{A.2})$$

and

$$D_{vk}B_k^* + C_{vk}Y_kA_k^* = 0. \quad (\text{A.3})$$

(A.1) yields:

$$\left[\begin{array}{cc} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right] = \left[\begin{array}{cc} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right] \left[\begin{array}{cc} I & Y_kC_k^* \\ \hline 0 & D_k^* \end{array} \right]. \quad (\text{A.4})$$

and (A.2) together with (A.3) gives:

$$\left[\begin{array}{cc} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right] \left[\begin{array}{c} Y_kA_k^* \\ B_k^* \end{array} \right] = \left[\begin{array}{c} Y_{k+1} \\ 0 \end{array} \right] \quad (\text{A.5})$$

The above two equations (A.4) and (A.5) yield:

$$\left[\begin{array}{cc} A_kY_k^* & B_k \\ \hline I & 0 \\ \hline C_kY_k^* & D_k \end{array} \right] = \left[\begin{array}{cc} Y_{k+1}^* & 0 \\ \hline A_{\Delta k}^* & C_{\Delta k}^* \\ \hline B_{\Delta k}^* & D_{\Delta k}^* \end{array} \right] \left[\begin{array}{cc} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right].$$

Pre-multiplying (A.5) by each terms' adjoint gives:

$$A_kY_k^*Y_kA_k^* + B_kB_k^* = Y_{k+1}^*Y_{k+1},$$

since $\begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix}$ is unitary. Let $R_k = Y_k^*$, and $Q_k = Y_k^* Y_k$, we proof this theorem.

Proof of Theorem 4.4.4

$E_{ac} = V^* \Delta$ results in $\Delta = V E_{ac}$, where E_{ac} is the anti-causal system with state space matrices: $\begin{bmatrix} A'_k & B'_k \\ C'_k & D'_k \end{bmatrix}_{ac}$. V is causal and its state space matrices $\begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix}_c$

and Δ is causal and the state space matrices are $\begin{bmatrix} A_{\Delta k} & B_{\Delta k} \\ C_{\Delta k} & D_{\Delta k} \end{bmatrix}_c$.

Applying Theorem 4.3.2, we can get

$$\begin{aligned} \begin{bmatrix} A_{\Delta k} & B_{\Delta k} \\ C_{\Delta k} & D_{\Delta k} \end{bmatrix}_c &= \begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix}_c \begin{bmatrix} A'_k & B'_k \\ C'_k & D'_k \end{bmatrix}_{ac} \\ &= \begin{bmatrix} A_{vk} & B_{vk} D'_k + A_{vk} Y_k B'_k \\ C_{vk} & D_{vk} D'_k + C_{vk} Y_k B'_k \end{bmatrix}_c \end{aligned} \quad (\text{A.6})$$

where Y_k satisfies:

$$A_{vk} Y_k A'_k + B_{vk} C'_k - Y_{k+1} = 0, \quad (\text{A.7})$$

and

$$D_{vk} C'_k + C_{vk} Y_k A'_k = 0. \quad (\text{A.8})$$

(A.6) yields:

$$\begin{bmatrix} A_{\Delta k} & B_{\Delta k} \\ C_{\Delta k} & D_{\Delta k} \end{bmatrix} = \begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix} \begin{bmatrix} I & Y_k B'_k \\ 0 & D'_k \end{bmatrix}. \quad (\text{A.9})$$

and (A.7) together with (A.8) gives:

$$\begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix} \begin{bmatrix} Y_k A'_k \\ C'_k \end{bmatrix} = \begin{bmatrix} Y_{k+1} \\ 0 \end{bmatrix} \quad (\text{A.10})$$

The above two equations (A.9) and (A.10) yield:

$$\left[\begin{array}{c|cc} Y_k A'_k & I & Y_k B'_k \\ \hline C'_k & 0 & D'_k \end{array} \right] = \left[\begin{array}{cc} A_{vk}^* & C_{vk}^* \\ B_{vk}^* & D_{vk}^* \end{array} \right] \left[\begin{array}{c|cc} Y_{k+1} & A_{\Delta k} & B_{\Delta k} \\ \hline 0 & C_{\Delta k} & D_{\Delta k} \end{array} \right].$$

Pre-multiplying (A.10) by each terms' adjoint gives:

$$A_k'^* Y_k^* Y_k A_k' + C_k'^* C_k' = Y_{k+1}^* Y_{k+1},$$

since $\left[\begin{array}{cc} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{array} \right]$ is unitary. Let $R_k = Y_k$, and $Q_k = Y_k^* Y_k$, we proof this theorem.

Proof of Theorem 4.4.5

$E_{ac} = \Delta V^*$ results in $\Delta = E_{ac} V$, where E_{ac} is the anti-causal system with state space matrices: $\left[\begin{array}{c|c} A'_k & C'_k \\ \hline B'_k & D'_k \end{array} \right]_{ac}$. V is causal and has state space matrices $\left[\begin{array}{c|c} A_{vk} & B_{vk} \\ \hline C_{vk} & D_{vk} \end{array} \right]_c$

and Δ is causal and its state space matrices are $\left[\begin{array}{c|c} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right]_c$.

Applying Theorem 4.3.1, we can get

$$\left[\begin{array}{c|c} A_{\Delta k} & B_{\Delta k} \\ \hline C_{\Delta k} & D_{\Delta k} \end{array} \right]_c = \left[\begin{array}{c|c} A_{vk} & B_{vk} \\ \hline C'_k X_{k+1} A_{vk} + D'_k C_{vk} & C'_k X_{k+1} B_{vk} + D'_k D_{vk} \end{array} \right]_c, \quad (\text{A.11})$$

where X_k satisfies:

$$A_k' X_{k+1} A_{vk} - X_k + B_k' C_{vk} = 0, \quad (\text{A.12})$$

and

$$B_k' D_{vk} + A_k' X_{k+1} B_{vk} = 0. \quad (\text{A.13})$$

(A.11) yields:

$$\begin{bmatrix} A_{\Delta k} & B_{\Delta k} \\ C_{\Delta k} & D_{\Delta k} \end{bmatrix} = \begin{bmatrix} I & 0 \\ C'_k X_{k+1} & D'_k \end{bmatrix} \begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix}. \quad (\text{A.14})$$

and (A.12) together with (A.13) gives:

$$\begin{bmatrix} X_k & 0 \end{bmatrix} = \begin{bmatrix} A'_k X_{k+1} & B'_k \end{bmatrix} \begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix} \quad (\text{A.15})$$

The above two equations (A.14) and (A.15) yield:

$$\begin{bmatrix} A'_k X_{k+1} & B'_k \\ I & 0 \\ C'_k X_{k+1} & D'_k \end{bmatrix} = \begin{bmatrix} X_k & 0 \\ A_{\Delta k} & B_{\Delta k} \\ C_{\Delta k} & D_{\Delta k} \end{bmatrix} \begin{bmatrix} A_{vk}^* & C_{vk}^* \\ B_{vk}^* & D_{vk}^* \end{bmatrix}$$

Post-multiplying (A.15) by each terms' adjoint gives:

$$A'_k X_{k+1} X_{k+1}^* A_k'^* + B'_k B_k'^* = X_k X_k^*,$$

since $\begin{bmatrix} A_{vk} & B_{vk} \\ C_{vk} & D_{vk} \end{bmatrix}$ is unitary. Let $R_k = X_k$, and $Q_k = X_k X_k^*$, we proof this theorem.

Proof of Theorem 5.3.3

The class of frames modeled by the mixed causal-anticausal LTI system has rational transfer function as

$$E(z) = D + C(zI - A)^{-1}B + C'(z^{-1}I - A')^{-1}B' = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac},$$

and $\|Eu\|^2 \leq \beta\|u\|^2$ implies $E^*(z)E(z) \leq \beta I$. Hence

$$\begin{aligned} E^*(z)E(z) &= \left(\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \right) \left(\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \right) \\ &= \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \\ &\quad + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac}. \end{aligned}$$

There are four terms in the above expression, each term is a cascaded system. By using lemma 2.3.1, we can represent the first term as:

$$\begin{aligned} &\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c \\ &= \left[\begin{array}{c|c} A & B \\ \hline D^T C + B^T P A & D^T D + B^T P B \end{array} \right]_c + \left[\begin{array}{c|c} A^T & C^T D + A^T P B \\ \hline B^T & 0 \end{array} \right]_{ac} \end{aligned}$$

where $A^T P A - P + C^T C = 0$, yielding $B^T(z^{-1}I - A^T)^{-1}(A^T P A - P + C^T C)(zI - A)^{-1}B = 0$. The second term can be rewritten as:

$$\left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} = \left[\begin{array}{c|c} A'^T & A'^T Q B' \\ \hline B'^T & B'^T Q B' \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline B'^T Q A' & 0 \end{array} \right]_{ac}$$

where $A'^T Q A' - Q + C'^T C' = 0$, yielding $B'^T(zI - A'^T)^{-1}(A'^T Q A' - Q + C'^T C')(z^{-1}I -$

$A')^{-1}B' = 0$. The third term can be rewritten as:

$$\begin{aligned} \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c &= \left[\begin{array}{c|c} A & 0 \\ \hline C'^T C & A'^T \\ \hline 0 & B'^T \end{array} \middle| \begin{array}{c} B \\ C'^T D \end{array} \right]_c \\ &= \left[\begin{array}{c|c} A & 0 \\ \hline 0 & A'^T \\ \hline -B'^T U & B'^T \end{array} \middle| \begin{array}{c} B \\ C'^T D + UB \\ 0 \end{array} \right]_c \end{aligned}$$

where $UA - A'^T U + C'^T C = 0$, yielding $B'^T(zI - A'^T)^{-1}(UA - A'^T U + C'^T C)(zI - A)^{-1}B = 0$. The fourth term can be rewritten as:

$$\begin{aligned} \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} &= \left[\begin{array}{c|c} A^T & C^T C' \\ \hline 0 & A' \\ \hline B^T & D^T C' \end{array} \middle| \begin{array}{c} 0 \\ B' \\ 0 \end{array} \right]_{ac} \\ &= \left[\begin{array}{c|c} A^T & 0 \\ \hline 0 & A' \\ \hline B^T & D^T C' + B^T V \end{array} \middle| \begin{array}{c} -VB' \\ B' \\ 0 \end{array} \right]_{ac} \end{aligned}$$

where $A^T V - V A' + C^T C' = 0$, yielding $B^T(z^{-1}I - A^T)^{-1}(A^T V - V A' + C^T C')(z^{-1}I - A')^{-1}B' = 0$. The third term condition $UA - A'^T U + C'^T C = 0$ and fourth term condition $A^T V - V A' + C^T C' = 0$ result in $V = U^T, U = V^T$.

Hence we can represent $E^*(z)E(z) - \beta I$ in transfer function matrix:

$$\begin{aligned}
& E^*(z)E(z) - \beta I \\
&= (D^T D + B^T P B + B'^T Q B' - \beta I) + (D^T C + B^T P A)(zI - A)^{-1} B + \\
& B^T (z^{-1} I - A^T)^{-1} (C^T D + A^T P B) \\
&+ B^T (z^{-1} I - A^T)^{-1} (A^T P A - P + C^T C)(zI - A)^{-1} B + B'^T (zI - A'^T)^{-1} (A'^T Q B') \\
&+ (B'^T Q A')(z^{-1} I - A')^{-1} B' + B'^T (zI - A'^T)^{-1} (A'^T Q A' - Q + C'^T C') \\
&(z^{-1} I - A')^{-1} B' - B'^T U (zI - A)^{-1} B + B'^T (zI - A'^T)^{-1} (C'^T D + U B) \\
&+ B'^T (zI - A'^T)^{-1} (U A - A'^T U + C'^T C)(zI - A)^{-1} B \\
&- B^T (z^{-1} I - A^T)^{-1} V B' + (D^T C' + B^T V)(z^{-1} I - A')^{-1} B' \\
&+ B^T (z^{-1} I - A^T)^{-1} (A^T V - V A' + C^T C')(z^{-1} I - A')^{-1} B' \\
&= \begin{bmatrix} B^T (z^{-1} I - A^T)^{-1} & B'^T (zI - A'^T)^{-1} & I \end{bmatrix} \\
& \begin{bmatrix} A^T P A - P + C^T C & A^T V - V A' + C^T C' & C^T D + A^T P B - V B' \\ U A - A'^T U + C'^T C & A'^T Q A' - Q + C'^T C' & A'^T Q B' + C'^T D + U B \\ D^T C + B^T P A - B'^T U & B'^T Q A' + D^T C' + B^T V & D^T D + B^T P B + B'^T Q B' - \beta I \end{bmatrix} \\
& \begin{bmatrix} (zI - A)^{-1} B \\ (z^{-1} I - A')^{-1} B' \\ I \end{bmatrix} \leq 0.
\end{aligned}$$

Thus the matrix

$$\begin{bmatrix} A^T P A - P + C^T C & A^T V - V A' + C^T C' & C^T D + A^T P B - V B' \\ U A - A'^T U + C'^T C & A'^T Q A' - Q + C'^T C' & A'^T Q B' + C'^T D + U B \\ D^T C + B^T P A - B'^T U & B'^T Q A' + D^T C' + B^T V & D^T D + B^T P B + B'^T Q B' - \beta I \end{bmatrix}$$

is a negative definite matrix. Let $X = V$, $X^T = U$, we proof this theorem.

Proof of Theorem 5.3.4

$$E(z) = D + C(zI - A)^{-1}B + C'(z^{-1}I - A')^{-1}B' = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac}.$$

The lower bound refers to the value α such that $\alpha I \leq E^*(z)E(z)$.

$$\begin{aligned} E^*(z)E(z) &= \left(\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \right) \left(\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \right) \\ &= \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \\ &\quad + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \end{aligned}$$

By using Lemma 2.3.1, we can represent the first term as:

$$\begin{aligned} &\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c \\ &= \left[\begin{array}{c|c} A & B \\ \hline D^T C + B^T P A & D^T D + B^T P B \end{array} \right]_c + \left[\begin{array}{c|c} A^T & C^T D + A^T P B \\ \hline B^T & 0 \end{array} \right]_{ac} \end{aligned}$$

where $A^T P A - P + C^T C = 0$, yielding $B^T(z^{-1}I - A^T)^{-1}(A^T P A - P + C^T C)(zI - A)^{-1}B = 0$. The second term can be rewritten as:

$$\left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} = \left[\begin{array}{c|c} A'^T & A'^T Q B' \\ \hline B'^T & B'^T Q B' \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline B'^T Q A' & 0 \end{array} \right]_{ac}$$

where $A'^T Q A' - Q + C'^T C' = 0$, yielding $B'^T(zI - A'^T)^{-1}(A'^T Q A' - Q + C'^T C')(z^{-1}I -$

$A')^{-1}B' = 0$. The third term can be rewritten as:

$$\begin{aligned} & \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ C'^T C & A'^T & C'^T D \\ \hline 0 & B'^T & 0 \end{array} \right]_c = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A'^T & C'^T D + UB \\ \hline -B'^T U & B'^T & 0 \end{array} \right]_c \end{aligned}$$

where $UA - A'^T U + C'^T C = 0$, yielding $B'^T(zI - A'^T)^{-1}(UA - A'^T U + C'^T C)(zI - A)^{-1}B = 0$. The fourth term can be rewritten as:

$$\begin{aligned} & \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \\ &= \left[\begin{array}{cc|c} A^T & C^T C' & 0 \\ 0 & A' & B' \\ \hline B^T & D^T C' & 0 \end{array} \right]_{ac} \\ &= \left[\begin{array}{cc|c} A^T & 0 & -VB' \\ 0 & A' & B' \\ \hline B^T & D^T C' + B^T V & 0 \end{array} \right]_{ac} \end{aligned}$$

where $A^T V - V A' + C^T C' = 0$, yielding $B^T(z^{-1}I - A^T)^{-1}(A^T V - V A' + C^T C')(z^{-1}I - A')^{-1}B' = 0$. The third term condition $UA - A'^T U + C'^T C = 0$ and fourth term condition $A^T V - V A' + C^T C' = 0$ result in $V = U^T, U = V^T$.

Hence we can represent $E^*(z)E(z) - \alpha I$ in transfer function matrix:

$$\begin{aligned}
& E^*(z)E(z) - \alpha I \\
&= (D^T D + B^T P B + B'^T Q B' - \alpha I) + (D^T C + B^T P A)(zI - A)^{-1} B + \\
& B^T (z^{-1} I - A^T)^{-1} (C^T D + A^T P B) \\
&+ B^T (z^{-1} I - A^T)^{-1} (A^T P A - P + C^T C)(zI - A)^{-1} B + B'^T (zI - A'^T)^{-1} (A'^T Q B') \\
&+ (B'^T Q A')(z^{-1} I - A')^{-1} B' + B'^T (zI - A'^T)^{-1} (A'^T Q A' - Q + C'^T C') \\
&(z^{-1} I - A')^{-1} B' - B'^T U (zI - A)^{-1} B + B'^T (zI - A'^T)^{-1} (C'^T D + U B) \\
&+ B'^T (zI - A'^T)^{-1} (U A - A'^T U + C'^T C)(zI - A)^{-1} B \\
&- B^T (z^{-1} I - A^T)^{-1} V B' + (D^T C' + B^T V)(z^{-1} I - A')^{-1} B' \\
&+ B^T (z^{-1} I - A^T)^{-1} (A^T V - V A' + C^T C')(z^{-1} I - A')^{-1} B' \\
&= \begin{bmatrix} B^T (z^{-1} I - A^T)^{-1} & B'^T (zI - A'^T)^{-1} & I \end{bmatrix} \\
&\begin{bmatrix} A^T P A - P + C^T C & A^T V - V A' + C^T C' & C^T D + A^T P B - V B' \\ U A - A'^T U + C'^T C & A'^T Q A' - Q + C'^T C' & A'^T Q B' + C'^T D + U B \\ D^T C + B^T P A - B'^T U & B'^T Q A' + D^T C' + B^T V & D^T D + B^T P B + B'^T Q B' - \alpha I \end{bmatrix} \\
&\begin{bmatrix} (zI - A)^{-1} B \\ (z^{-1} I - A')^{-1} B' \\ I \end{bmatrix} \geq 0
\end{aligned}$$

Thus the matrix

$$\begin{bmatrix} A^T P A - P + C^T C & A^T V - V A' + C^T C' & C^T D + A^T P B - V B' \\ U A - A'^T U + C'^T C & A'^T Q A' - Q + C'^T C' & A'^T Q B' + C'^T D + U B \\ D^T C + B^T P A - B'^T U & B'^T Q A' + D^T C' + B^T V & D^T D + B^T P B + B'^T Q B' - \alpha I \end{bmatrix}$$

is a positive definite matrix. Let $X = V$, $X^T = U$, we proof this theorem.

Proof of Theorem 5.3.2

LTV systems can be represented by the transfer operators such that:

$$E = D + C(I - ZA)^{-1}ZB + C'(ZI - A')^{-1}B' = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac},$$

and $\alpha\|u\|^2 \leq \|Eu\|^2$ implies $\alpha I \leq E^*E$. Hence

$$\begin{aligned} E^*E &= \left(\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \right) \left(\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \right) \\ &= \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \\ &\quad + \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c + \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \end{aligned}$$

By using Theorem 4.3.1, we can represent the first term as:

$$\begin{aligned} &\left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c \\ &= \left[\begin{array}{c|c} A & B \\ \hline D^T C + B^T P^{(+1)} A & D^T D + B^T P^{(+1)} B \end{array} \right]_c + \left[\begin{array}{c|c} A^T & C^T D + A^T P^{(+1)} B \\ \hline B^T & 0 \end{array} \right]_{ac} \end{aligned}$$

where $A^T P^{(+1)} A - P + C^T C = 0$, yielding $B^T Z^*(I - A^T Z^*)^{-1}(A^T P^{(+1)} A - P + C^T C)(I - ZA)^{-1}ZB = 0$. The second term can follow 4.3.2 and can be rewritten as:

$$\left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} = \left[\begin{array}{c|c} A'^T & A'^T Q B' \\ \hline B'^T & B'^T Q B' \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline B'^T Q A' & 0 \end{array} \right]_{ac}$$

where $A'^T Q A' - Q^{(+1)} + C'^T C' = 0$, yielding $B'^T (Z^* I - A'^T)^{-1} (A'^T Q A' - Q^{(+1)} +$

$C'^T C')(ZI - A')^{-1} B' = 0$. The third term can be rewritten as:

$$\begin{aligned} & \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B'^T & 0 \end{array} \right]_c \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]_c \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ C'^T C & A'^T & C'^T D \\ \hline 0 & B'^T & 0 \end{array} \right]_c = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A'^T & C'^T D + UB \\ \hline -B'^T U & B'^T & 0 \end{array} \right]_c \end{aligned}$$

where $UA - A'^T U + C'^T C = 0$, yielding $B'^T (Z^* I - A'^T)^{-1} (UA - A'^T U + C'^T C) (I - ZA)^{-1} ZB = 0$. The fourth term can be rewritten as:

$$\begin{aligned} & \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right]_{ac} \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right]_{ac} \\ &= \left[\begin{array}{cc|c} A^T & C^T C' & 0 \\ 0 & A' & B' \\ \hline B^T & D^T C' & 0 \end{array} \right]_{ac} \\ &= \left[\begin{array}{cc|c} A^T & 0 & -VB' \\ 0 & A' & B' \\ \hline B^T & D^T C' + B^T V & 0 \end{array} \right]_{ac} \end{aligned}$$

where $A^T V - V A' + C^T C' = 0$, yielding $B^T Z^* (I - A^T Z^*)^{-1} (A^T V - V A' + C^T C') (ZI - A')^{-1} B' = 0$. The third term condition $UA - A'^T U + C'^T C = 0$ and fourth term condition $A^T V - V A' + C^T C' = 0$ result in $V = U^T, U = V^T$.

Hence we can represent $E^* E - \alpha I$ in transfer function matrix:

$$\begin{aligned} & E^* E - \alpha I \\ &= \begin{bmatrix} B^T Z^* (I - A^T Z^*)^{-1} & B'^T (Z^* I - A'^T)^{-1} & I \end{bmatrix} \\ & \begin{bmatrix} A^T P^{(+1)} A - P + C^T C & A^T V - V A' + C^T C' & C^T D + A^T P^{(+1)} B - V B' \\ UA - A'^T U + C'^T C & A'^T Q A' - Q^{(+1)} + C'^T C' & A'^T Q B' + C'^T D + UB \\ D^T C + B^T P^{(+1)} A - B'^T U & B'^T Q A' + D^T C' + B^T V & D^T D + B^T P^{(+1)} B + B'^T Q B' - \alpha I \end{bmatrix} \\ & \begin{bmatrix} (I - ZA)^{-1} ZB \\ (ZI - A')^{-1} B' \\ I \end{bmatrix} \geq 0 \end{aligned}$$

Thus the matrix

$$\begin{bmatrix} A_k^T P_{k+1} A_k - P_k + C_k^T C_k & A_k^T V_k - V_k A_k' + C_k^T C_k' & C_k^T D_k + A_k^T P_{k+1} B_k - V_k B_k' \\ U_k A_k - A_k'^T U_k + C_k'^T C_k & A_k'^T Q_k A_k' - Q_{k+1} + C_k'^T C_k' & A_k'^T Q_k B_k' + C_k'^T D_k + U_k B_k \\ D_k^T C_k + B_k^T P_{k+1} A_k - B_k'^T U_k & B_k'^T Q_k A_k' + D_k^T C_k' + B_k^T V_k & D_k^T D_k + B_k^T P_{k+1} B_k + B_k'^T Q_k B_k' - \alpha I \end{bmatrix}$$

is a positive definite matrix. Let $X_k = V_k$, $X_k^T = U_k$, we proof this theorem.

Proof of Lemma 6.2.1

The mixed causal-anticausal LTI system E has state space representation given as:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ x_{k-1}' &= A'x_k' + B'u_k, \\ y_k &= Cx_k + C'x_k' + Du_k. \end{aligned} \tag{A.16}$$

Consider the decimation process (with a factor of M) that is performed before the causal-anticausal LTI system E . Let s_k denote the input signals for this system. The decimation process is performed before the system E , which means that we retain every M th input samples and discard other input samples, i.e.

$$s_{Mk} = u_k, k \in \mathbb{Z}.$$

Thus we have the following state equations to represent the decimated system:

$$\begin{aligned}
 x_{k+1} &= Ax_k + \begin{bmatrix} B & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} s_{Mk} \\ s_{Mk+1} \\ \vdots \\ s_{Mk+M-1} \end{bmatrix} \\
 x'_{k+1} &= A'x'_k + \begin{bmatrix} B' & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} s_{Mk} \\ s_{Mk+1} \\ \vdots \\ s_{Mk+M-1} \end{bmatrix}, \\
 y_k &= Cx_k + C'x'_k + \begin{bmatrix} D & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} s_{Mk} \\ s_{Mk+1} \\ \vdots \\ s_{Mk+M-1} \end{bmatrix}.
 \end{aligned}$$

Hence the state space representation of the decimated system is given as:

$$ED^M = \left[\begin{array}{c|cc} A & B & 0_1 \\ \hline C & D & 0_2 \end{array} \right]_c + \left[\begin{array}{c|cc} A' & B' & 0_3 \\ \hline C' & 0 & 0_4 \end{array} \right]_{ac},$$

where $0_1 - 0_4$ have $M - 1$ number of columns and the number of rows depends on the number of rows of B, D, B' and $D' = 0$ respectively.

Now consider the decimation process that is performed after the causal-anticausal LTI system. The input signal is u_k and the output signal is y_k for the mixed causal-anticausal LTI system. The output signal is then passed a decimator with a factor of M , hence the output from the decimator becomes y_{Mk} , i.e., retaining every M th value.

Recursively apply the state equations (A.16), we have

$$\begin{aligned}
 y_{Mk} &= Cx_{Mk} + c' x'_{Mk} + Du_{Mk} \\
 &= C(Ax_{Mk-1} + Bu_{Mk-1}) + Du_{Mk} + c' x'_{Mk} \\
 &= CAx_{Mk-1} + \begin{bmatrix} CB & D \end{bmatrix} \begin{bmatrix} u_{Mk-1} \\ u_{Mk} \end{bmatrix} + c' x'_{Mk} \\
 &\vdots \\
 &= CA^{M-1}x_{Mk-M+1} + \begin{bmatrix} CA^{M-2} & \dots & CB & B \end{bmatrix} \begin{bmatrix} u_{Mk-M+1} \\ \vdots \\ u_{Mk-1} \\ u_{Mk} \end{bmatrix} + c' x'_{Mk},
 \end{aligned}$$

$$\begin{aligned}
 x_{Mk+1} &= Ax_{Mk} + Bu_{Mk} \\
 &= A^2x_{Mk-1} + \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u_{Mk-1} \\ u_{Mk} \end{bmatrix} \\
 &\vdots \\
 &= A^Mx_{Mk-M+1} + \begin{bmatrix} A^{M-1}B & \dots & AB & B \end{bmatrix} \begin{bmatrix} u_{Mk-M+1} \\ \vdots \\ u_{Mk-1} \\ u_{Mk} \end{bmatrix},
 \end{aligned}$$

and

$$\begin{aligned}
x'_{M(k-1)} &= A' x'_{Mk-M+1} + B' u_{Mk-M+1} \\
&= (A')^2 x'_{Mk-M+2} + \begin{bmatrix} B' & A' B' \end{bmatrix} \begin{bmatrix} u_{Mk-M+1} \\ u_{Mk-M+2} \end{bmatrix} \\
&\vdots \\
&= (A')^M x'_{Mk} + \begin{bmatrix} B' & A' B' & \dots & (A')^{M-1} B' \end{bmatrix} \begin{bmatrix} u_{Mk-M+1} \\ u_{Mk-M+2} \\ \vdots \\ u_{Mk} \end{bmatrix}.
\end{aligned}$$

Redefine the forward state as $\bar{x}_k = x_{Mk-M+1}$, yielding $\bar{x}_{k+1} = x_{Mk+1}$, the backward

state as $\bar{x}'_k = x'_{Mk}$, resulting in $\bar{x}'_{k-1} = x'_{Mk-M}$. Let $\bar{u}_k = \begin{bmatrix} u_{Mk-M+1} \\ u_{Mk-M+2} \\ \vdots \\ u_{Mk} \end{bmatrix}$, and output

$z_k = y_{Mk}$, we have the state equations representing the multirate system:

$$\begin{aligned}
\bar{x}_{k+1} &= A^M \bar{x}_k + \begin{bmatrix} A^{M-1} B & A^{M-2} B & \dots & B \end{bmatrix} \bar{u}_k \\
\bar{x}'_{k-1} &= (A')^M \bar{x}'_k + \begin{bmatrix} B' & A' B' & \dots & (A')^{M-1} B' \end{bmatrix} \bar{u}_k, \\
z_k &= C A^{M-1} \bar{x}_k + C' \bar{x}'_k + \begin{bmatrix} C A^{M-2} B & \dots & C B & D \end{bmatrix} \bar{u}_k.
\end{aligned}$$

Hence the state space representation of the multirate system is given as:

$$D^M E = \left[\begin{array}{c|ccc} A^M & A^{M-1} B & A^{M-2} B & \dots & B \\ \hline C A^{M-1} & C A^{M-2} B & \dots & C B & D \end{array} \right]_c + \left[\begin{array}{c|cccc} (A')^M & B' & A' B' & \dots & (A')^{M-1} B' \\ \hline C' & 0 & & & \end{array} \right]_{ac}.$$

Proof of Lemma 6.2.2

The mixed causal-anticausal LTI system E has state space representation given in (A.16). The proof for Lemma 6.2.2 is analog to the proof of Lemma 6.2.1.

If the interpolation process (with a factor of L) is performed before the mixed causal-anticausal system E . Let s_k denote the input signals for this system. The interpolation process is performed before the system E , which means that we insert $L - 1$ zeros in the successive input samples, i.e.

$$s_k = u_{Lk}, k \in \mathbb{Z},$$

and $u_{Lk+i} = 0$ for $i = 1, 2, \dots, L - 1$. Recursively apply the state equations (A.16), we have

$$\begin{aligned} y_{Lk} &= Cx_{Lk} + C'x'_{Lk} + Du_{Lk} \\ &= Cx_{Lk} + Du_{Lk} + C'A'x'_{Lk+1} \\ &\vdots \\ &= Cx_{Lk} + Du_{Lk} + C'(A')^{L-1}x'_{Lk+L-1}, \\ y_{Lk+1} &= Cx_{Lk+1} + C'x'_{Lk+1} + Du_{Lk+1} \\ &= CAx_{Lk} + CBu_{Lk} + C'(A')x'_{Lk+2} \\ &\vdots \\ &= CAx_{Lk} + CBu_{Lk} + C'(A')^{L-2}x'_{Lk+L-1}, \\ y_{Lk+2} &= Cx_{Lk+2} + C'x'_{Lk+2} + Du_{Lk+2} \\ &= CAx_{Lk+1} + C'(A')x'_{Lk+3} \\ &= CA^2x_{Lk} + CABu_{Lk} + C'(A')^2x'_{Lk+4} \\ &\vdots \\ &= CA^2x_{Lk} + CABu_{Lk} + C'(A')^{L-3}x'_{Lk+L-1}, \\ &\vdots \end{aligned}$$

$$\begin{aligned}
y_{Lk+L-1} &= Cx_{Lk+L-1} + C'x'_{Lk+L-1} + Du_{Lk+L-1} \\
&= CAx_{Lk+L-2} + C'x'_{Lk+L-1} \\
&= CA^2x_{Lk+L-3} + CABu_{Lk} + C'x'_{Lk+L-1} \\
&\vdots \\
&= CA^{L-1}x_{Lk} + CA^{L-2}Bu_{Lk} + C'x'_{Lk+L-1},
\end{aligned}$$

hence we have

$$\begin{bmatrix} y_{Lk} \\ y_{Lk+1} \\ \vdots \\ y_{Lk+L-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{L-1} \end{bmatrix} x_{Lk} + \begin{bmatrix} D \\ CB \\ \vdots \\ CA^{L-2} \end{bmatrix} u_{Lk} + \begin{bmatrix} C'(A')^{L-1} \\ \vdots \\ C'A' \\ C' \end{bmatrix} x'_{Lk+L-1}.$$

$$\begin{aligned}
x_{Lk+L} &= Ax_{Lk+L-1} + Bu_{Lk+L-1} \\
&= A^2x_{Lk+L-1} \\
&\vdots \\
&= A^Lx_{Lk} + A^{L-1}Bu_{Lk},
\end{aligned}$$

and

$$\begin{aligned}
x'_{Lk-1} &= A'x'_{Lk} + B'u_{Lk} \\
&= (A')^2x'_{Lk+1} + B'u_{Lk} \\
&\vdots \\
&= (A')^Lx'_{Lk+L-1} + B'u_{Lk}.
\end{aligned}$$

Redefine the forward state as $\bar{x}_k = x_{Lk}$, resulting in $\bar{x}_{k+1} = x_{Lk+L}$, the backward

state as $\bar{x}'_k = x'_{Lk+L-1}$, yielding $\bar{x}'_{k-1} = x'_{Lk-1}$. Let $z_k = \begin{bmatrix} y_{Lk} \\ y_{Lk+1} \\ \vdots \\ y_{Lk+L-1} \end{bmatrix}$, and $s_k = u_{Lk}$,

we have the following state equations:

$$\begin{aligned}
 \bar{x}_{k+1} &= A^M \bar{x}_k + A^{L-1} B s_k \\
 \bar{x}'_{k-1} &= (A')^M \bar{x}'_k + B' s_k, \\
 z_k &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{L-1} \end{bmatrix} \bar{x}_k + \begin{bmatrix} C'(A')^{L-1} \\ \vdots \\ C'A' \\ C' \end{bmatrix} \bar{x}'_k + \begin{bmatrix} D \\ CB \\ \vdots \\ CA^{L-2}B \end{bmatrix} s_k.
 \end{aligned}$$

Hence the state space representation of the multirate system is given as:

$$EU^L = \left[\begin{array}{c|c} A^L & A^{L-1}B \\ \hline C & D \\ CA^1 & CB \\ \vdots & \vdots \\ CA^{L-1} & CA^{L-2}B \end{array} \right]_c + \left[\begin{array}{c|c} (A')^L & B' \\ \hline C'(A')^{L-1} & 0 \\ \vdots & \\ C'A' & \\ C' & \end{array} \right]_{ac}.$$

Now consider the interpolation process is performed after the causal-anticausal LTI system. The output signal is then passed a interpolator with a factor of L , i.e. inserting $L - 1$ zeros between the successive output values. Thus we have the following state equations to represent the interpolated system:

$$\begin{aligned}
 x_{k+1} &= Ax_k + Bu_k \\
 x'_{k-1} &= A'x'_k + B'u_k, \\
 \begin{bmatrix} y_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} &= \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_k + \begin{bmatrix} C' \\ 0 \\ \vdots \\ 0 \end{bmatrix} x'_k + \begin{bmatrix} D \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_k.
 \end{aligned}$$

Hence the state space representation of the interpolated system is given by:

$$U^L E = \left[\begin{array}{c|c} A & B \\ \hline C & D \\ \hline 0_1 & 0_2 \end{array} \right]_c + \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \\ \hline 0_3 & 0_4 \end{array} \right]_{ac},$$

where $0_1 - 0_4$ have $(L - 1)$ number of rows and the number of columns depend on the number of columns of C, D, C' and $D' = 0$ respectively.

Appendix B

RQ factorization matlab code

RQ factorization matlab code by Bruno Luong 05/Oct/2008

```
function [R Q]=rq(A, varargin)

[m n]=size(A);

[Q R]=qr(flipud(A).');

R=flipud(R. ');

Q=Q. ';

if m > n

warning('RQ:DimensionBizarre',...

'RQ: number of rows is larger the number of columns');

R(end,m)=0;

Q(m,end)=0;

end

R(:,1:m)=R(:,m:-1:1);
```

```
Q(1:m,:)=Q(m:-1:1,:);

last=strcmpi(getoption("",varargin:),'last');

if xor(m > n, last)

R=circshift(R,[0 n-m]);

Q=circshift(Q,[n-m 0]);

end

end

function res=getoption(default, option)

if nargin < 2||isempty(option)

res=default;

else

res=option;

end

end
```


Appendix C

Some DWTs and IDWTs examples

Scaling function and Wavelet function of some DWTs and IDWTs

Haar DWT:

$$H(z) = \frac{1}{\sqrt{2}}(1 + z^{-1});$$

$$G(z) = \frac{1}{\sqrt{2}}(1 - z^{-1}).$$

Daub4 DWT:

$$H(z) = \frac{1+\sqrt{3}}{4\sqrt{2}} + \frac{3+\sqrt{3}}{4\sqrt{2}}z^{-1} + \frac{3-\sqrt{3}}{4\sqrt{2}}z^{-2} + \frac{1-\sqrt{3}}{4\sqrt{2}}z^{-3};$$

$$G(z) = \frac{3-\sqrt{3}}{4\sqrt{2}} + \frac{-3+\sqrt{3}}{4\sqrt{2}}z^{-1} + \frac{3+\sqrt{3}}{4\sqrt{2}}z^{-2} + \frac{-1-\sqrt{3}}{4\sqrt{2}}z^{-3};$$

SP(2,4) DWT:

$$H(z) = \sqrt{2}\left(\frac{3}{128}z^4 - \frac{3}{64}z^3 - \frac{1}{8}z^2 + \frac{19}{64}z + \frac{45}{64} + \frac{19}{64}z^{-1} - \frac{1}{8}z^{-2} - \frac{3}{64}z^{-3} + \frac{3}{128}z^{-4}\right);$$

$$G(z) = \frac{\sqrt{2}}{4}(-1 + 2z^{-1} - z^{-2}).$$

VSP(4,4) DWT:

$$\begin{aligned}
 H(z) &= \sqrt{2}(0.045636z^3 - 0.028772z^2 + 0.295636z + 0.557544 + 0.295636z^{-1} \\
 &\quad - 0.028772z^{-2} - 0.045636z^{-3}); \\
 G(z) &= \sqrt{2}(0.026749z^3 + 0.016864z^2 - 0.078223z - 0.266864 + 0.602949z^{-1} \\
 &\quad - 0.266864z^{-2} - 0.078223z^{-3} + 0.016864z^{-4} + 0.026749z^{-5}).
 \end{aligned}$$

COR(2) DWT:

$$\begin{aligned}
 H(z) &= \sqrt{2}(0.0125z^4 - 0.03125z^3 - 0.05z^2 + 0.28125z + 0.575 + 0.28125z^{-1} \\
 &\quad - 0.05z^{-2} - 0.03125z^{-3} + 0.0125z^{-4}); \\
 G(z) &= \sqrt{2}(0.000507z^6 - 0.001266z^5 - 0.003838z^4 + 0.015925z^3 + 0.039724z^2 \\
 &\quad - 0.052305z - 0.286393 + 0.575292z^{-1} - 0.286393z^{-2} - 0.052305z^{-3} \\
 &\quad + 0.039724z^{-4} + 0.015925z^{-5} - 0.003838z^{-6} - 0.001266z^{-7} \\
 &\quad + 0.000507z^{-8}).
 \end{aligned}$$

Dual of Haar DWT:

$$\begin{aligned}
 \tilde{H}(z) &= \frac{1}{\sqrt{2}}(1 + z^{-1}); \\
 \tilde{G}(z) &= \frac{1}{\sqrt{2}}(1 - z^{-1}).
 \end{aligned}$$

Dual of Daub4 DWT:

$$\begin{aligned}
 \tilde{H}(z) &= \frac{1+\sqrt{3}}{4\sqrt{2}} + \frac{3+\sqrt{3}}{4\sqrt{2}}z^{-1} + \frac{3-\sqrt{3}}{4\sqrt{2}}z^{-2} + \frac{1-\sqrt{3}}{4\sqrt{2}}z^{-3}; \\
 \tilde{G}(z) &= \frac{3-\sqrt{3}}{4\sqrt{2}} + \frac{-3+\sqrt{3}}{4\sqrt{2}}z^{-1} + \frac{3+\sqrt{3}}{4\sqrt{2}}z^{-2} + \frac{-1-\sqrt{3}}{4\sqrt{2}}z^{-3};
 \end{aligned}$$

Dual of SP(2,4) DWT:

$$\begin{aligned}
 \tilde{H}(z) &= \frac{\sqrt{2}}{4}(z + 2 + z^{-1}); \\
 \tilde{G}(z) &= \sqrt{2}\left(\frac{3}{128}z^3 + \frac{3}{64}z^2 - \frac{1}{8}z - \frac{19}{64} + \frac{45}{64}z^{-1} - \frac{19}{64}z^{-2} - \frac{1}{8}z^{-3} + \frac{3}{64}z^{-4} + \frac{3}{128}z^{-5}\right).
 \end{aligned}$$

Dual of VSP(4,4) DWT:

$$\begin{aligned}\tilde{H}(z) &= \sqrt{2}(0.026749z^4 - 0.016864z^3 - 0.078223z^2 + 0.266864z + 0.602949 \\ &\quad + 0.266864z^{-1} - 0.078223z^{-2} - 0.016864z^{-3} + 0.026749z^{-4}); \\ \tilde{G}(z) &= \sqrt{2}(0.045636z^2 - 0.028772z - 0.295636 + 0.557544z^{-1} - 0.295636z^{-2} \\ &\quad - 0.028772z^{-3} + 0.045636z^{-4}).\end{aligned}$$

Dual of COR(2) DWT:

$$\begin{aligned}\tilde{H}(z) &= \sqrt{2}(-0.000507z^7 - 0.001266z^6 + 0.003838z^5 + 0.015925z^4 - 0.039724z^3 \\ &\quad - 0.052305z^2 + 0.286393z + 0.575292 + 0.286393z^{-1} - 0.052305z^{-2} \\ &\quad - 0.039724z^{-3} + 0.015925z^{-4} + 0.003838z^{-5} - 0.001266z^{-6} \\ &\quad - 0.000507z^{-7}); \\ \tilde{G}(z) &= \sqrt{2}(0.0125z^3 + 0.03125z^2 - 0.05z - 0.28125 + 0.575z^{-1} - 0.28125z^{-2} \\ &\quad - 0.05z^{-3} + 0.03125z^{-4} + 0.0125z^{-5}).\end{aligned}$$