Filtering and control of networked systems over unreliable channels

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Filtering and Control of Networked Systems over Unreliable Channels

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in fulfillment of the requirement for the degree of
Doctor of Philosophy

2011

Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Date

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Xiao Nan

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Abstract

Networked control systems have attracted recurring research interests from the control community due to their wide applications in intelligent transportation, industrial automation, advanced manufacturing, and national defense. This thesis summarizes research results on filtering and control of networked systems over unreliable channels with emphasis on two typical channel models: lossy channel model and fading channel model.

In the first place, it is assumed that a lossy network is placed in the networked system, where the packet-loss process is driven by a Markov chain. We start by considering the Kalman filtering of a linear time-invariant plant over a lossy channel with bounded Markovian packet dropouts. If the observability index of the plant is one, then the peak error covariance matrix of a remote Kalman filter is always bounded from above in the mean sense. For plants with observability index greater than one, a sufficient condition is provided for the stability of the peak error covariance matrix in the mean sense, which is specified in terms of the dynamics of the plant and the transition probability of the Markov chain that describes the packet-loss process.

The quantized stabilization over a lossy channel is then studied by modeling the networked system into a Markov jump linear system. Given a measure of quantization coarseness, it is shown that a logarithmic quantizer and a linear state feedback are optimal for the mean square quadratic stabilization of a single-input Markov jump linear system. The optimal co-design of quantizer and controller are addressed

via a linear matrix inequality approach. A recursive algorithm is presented to estimate the unknown mode process of the underlying Markov chain at the controller side.

Next, we suppose that a fading network is put into the networked system, and the channel fading is modeled by a white sequence. We begin with studying the Kalman filtering of a linear time-invariant plant across fading channels. Necessary and sufficient conditions for the stability of the mean error covariance matrix of the remote Kalman filter are derived in terms of the unstable poles of the plant. Lower and upper bounds for the mean error covariance matrix are provided in the form of a modified Lyapunov iteration and a modified Riccati iteration, respectively.

The mean square stabilization of networked systems over fading channels is then addressed. For the case of state feedback, the minimal overall mean square capacity of the network for mean square stabilizability is given in terms of the Mahler measure of the plant under the parallel transmission strategy and the assumption on capacity allocation. Under the serial transmission strategy, the minimal capacity for stabilizability in general can only be computed by optimization. For the case of dynamic output feedback, a tight lower bound on the capacity requirement for stabilization of single-input single-output plants is given in terms of the anti-stable poles, nonminimum phase zeros and relative degree of the plant. Sufficient and necessary conditions are also derived for stabilization of triangularly decoupled multi-input multi-output plants.

The effect of channel processing, channel feedback, and channel input power bound is further analyzed. We show that the channel feedback plays a key role in reducing the network requirement for stabilizability caused by the nonminimum phase zeros and high relative degree of the plant, and the network requirement for stabilizability becomes more stringent when a power bound is applied on the channel input. A suboptimal algorithm is proposed for performance design of networked systems over fading channels. Finally, the results on control of discrete-time networked systems over fading channels are extended to the continuous-time case.

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Notations and Acronyms

The following list of notations and acronyms will be used throughout this thesis.

Notations

$\triangleq$	defined as
$\equiv$	always equal to
$\square$	end of proof
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{Z}$	field of integers
$\mathbb{N}$	field of positive integers
$\mathbb{R}_+$ ($\mathbb{Z}_+$)	field of nonnegative real numbers (integers)
$x(t)$	value of signal x at time t
$x_{t_1}^{t_2}$	set $\{x(t_1), x(t_1 + 1), \dots, x(t_2)\}$ for $t_1 \leq t_2$
$\bar{\alpha}$	complex conjugate of $\alpha \in \mathbb{C}$
$ \alpha $	magnitude of $\alpha \in \mathbb{C}$
δ_{ij}	Kronecker's or Dirac's delta
$\inf$ (min, sup, max)	infimum (minimum, supremum, maximum)
$\arg$	argument

$X > Y$ ($X \geq Y$)	$X - Y$ is positive definite (semidefinite)
I_n (I)	identity matrix with dimension $n \times n$ (compatible dimension)
$[a_{ij}]$	matrix with a_{ij} as its i -th row and j -th column element
$\text{diag}(a_1, a_2, \dots, a_n)$	$n \times n$ diagonal matrix with a_i as its i -th diagonal element
$[A]_{ij}$	entry that lies in the i -th row and j -th column of A
A'	transpose of A
A^H	complex conjugate transpose of A
A^{-1}	inverse of A
$\text{rank}(A)$	rank of A
$\det(A)$	determinant of A
$\text{tr}(A)$	trace of A
$\rho(A)$	spectral radius of A
$\ A\ $	Euclidean norm of A
$\otimes$	Kronecker product
$\odot$	Hadamard product
$\text{col}(x, y)$	column vector with entries x and y
$\text{vec}(\cdot)$	column stacking operator
$\Pr\{\cdot\}$	probability operator
$\mathcal{E}\{\cdot\}$	expectation operator
$\mathcal{R}$	set of all real rational transfer functions
$\mathcal{R}_p$	subset of $\mathcal{R}$ containing all proper transfer functions
$\mathcal{R}_{sp}$	subset of $\mathcal{R}$ containing all strictly proper transfer functions
$\mathcal{RH}_\infty$	subset of $\mathcal{R}_p$ containing all stable transfer functions
$\mathcal{U}_\infty$	subset of $\mathcal{RH}_\infty$ containing all square transfer functions with inverses in $\mathcal{RH}_\infty$
$\mathcal{RH}_2$	subset of $\mathcal{R}_{sp}$ containing all stable transfer functions
$\ \cdot\ _2$	$\mathcal{H}_2$ norm
$\ \cdot\ _\infty$	$\mathcal{H}_\infty$ norm

Acronyms

CMO	current mode observation
CSI	channel side information
i.i.d.	independent and identically distributed
LMI	linear matrix inequality
LQG	linear quadratic Gaussian
LTl	linear time invariant
MARO	modified algebraic Riccati operator
MIMO	multi-input multi-output
MJLS	Markov jump linear system
MLO	modified Lyapunov operator
MMSE	minimum mean square error
MSQ	mean square quadratic/quadratically
NCS	networked control system
NMP	nonminimum phase
OSDMO	one-step-delayed mode observation
PTS	parallel transmission strategy
SISO	single-input single-output
SNR	signal-to-noise ratio
STS	serial transmission strategy

Chapter 1

Introduction

1.1 Motivation and Objectives

In recent years, there has been a significant body of research in the area of networked systems, which can be categorized into three major fields [1]: (1) control over networks, (2) control of networks, and (3) multi-agent systems. The Panel on Future Directions in Control, Dynamics, and Systems formed in April 2000 reached a consensus that control over networks should be one of the key future directions [2]. The present thesis fits within the study of control over networks which is mainly concerned with the design of feedback strategies adapted to networked control systems (NCSs) in which control/measurement data is exchanged through unreliable sensing/communication networks, as shown in Figure 1.1.

Estimation and control are two major functions of almost every engineering and non-engineering control systems. A long standing assumption in classical control theory is that the signal transmission among the plant(s), controller(s) and filter(s) is perfect, which has been demonstrated to be valid in a large number of small-scale applications with wired point-to-point connections. In the past few years, large-scale and spatially distributed NCSs have attracted growing interests due to their wide applications in intelligent transportation, advanced manufacturing, industrial

Figure 1.1: General configuration of an NCS.

automation, and national defense. On the one hand, NCSs have numerous advantages over classical feedback control systems, including low cost, high flexibility, easy installation, and simple maintenance. On the other hand, due to the nature of the unreliable sensing/communication networks (e.g., packet loss, data quantization, channel fading, communication delay, asynchronous sampling, ...), the assumption on perfect transmission becomes impractical in control over networks, especially wireless networks. NCSs thus require new tools for analysis and control synthesis. A systematic theory on analysis and design of general NCSs is not available at present, and the existing results (see Chapter 2 for a detailed literature review) focus on simplified channel models that highlight certain aspects of the unreliable networks.

In NCSs, control and communication are tightly coupled and thus it is not wise to address them separately. Specific questions arising from control over networks can be posed as follows. Is it possible and how (if possible) to stabilize/track an unstable system or to achieve overall system performance when the network condition is given? What is the critical network requirement (e.g., the minimal amount of packet arrival rate, data rate or capacity) such that the stability or performance of the networked system can be realized? Motivated by the above observations, this thesis mainly focuses on control and filtering of networked systems across unreliable channels with emphasis on the interplay between communication constraints and the stability and performance of the networked system.

1.2 Overview of the Thesis

It has been pointed out in the previous section that the primary objective of this thesis is to study control and filtering problems over unreliable channels. The framework of the networked systems under investigation is depicted in Figure 1.2.

Figure 1.2: Architecture of the networked systems considered in this thesis.

Next, we briefly comment on the plant model, the channel models and the control/filtering strategies to be adopted, and summarize the problems to be studied.

This thesis is mainly focused on the discrete-time case¹. The plant is assumed to be a finite-dimensional linear time-invariant (LTI) system in both Figures 1.2(a) and 1.2(b). Since the dominant factor of channel unreliability may vary depending on applications, different channel models have been proposed by communication and control engineers. In the simplest situation, the model of unreliable channel(s) used in this thesis is given in the following memoryless multiplicative form:

$$\begin{aligned} z(t) &= \xi(t)y(t), \quad \text{in Figure 1.2(a),} \\ u(t) &= \xi(t)v(t), \quad \text{in Figure 1.2(b),} \end{aligned} \tag{1.1}$$

where $t \in \mathbb{Z}_+$ is the time index. Specifically, the next two typical channel models will be considered.

(i). Lossy channel model: $\xi(t)$ in (1.1) indicates the loss ($\xi(t) = 0$) or arrival ($\xi(t) = 1$) of the transmitted signal.

¹Chapter 8 is the only exception, where the continuous-time case is studied.

(ii). **Fading channel model:** $\xi(t)$ in (1.1) represents the channel fading experienced by the transmitted signal.

The exact value of $\xi(t)$ is called the channel side information (CSI) in communications [3]. In Figure 1.2(a), the current CSI is assumed to be available at the filter, and the filter is chosen to be a linear time-varying Kalman filter which is optimal in the minimum mean square error (MMSE) sense. In Figure 1.2(b), not only one-step-delayed CSI can be used at the controller side, and the controller is assumed to be causal.

Remark 1.2.1. *If $\xi(t)$ models the packet-loss process, then the one-step-delayed CSI amounts to the packet acknowledgment in the TCP-like protocol [4]. For a fading channel with channel reciprocity², the one-step-delayed CSI can be realized by sending powerful pilot signals to the transmitter, which allows the transmitter to compute the channel fading [5].*

In this context, we are interested not only in providing explicit control/filtering strategies for achieving certain control objectives with a predefined network constraint, but also in deriving conditions on the network for the stability of the remote Kalman filter in Figure 1.2(a) and for stabilizability or closed-loop performance of the NCS in Figure 1.2(b).

1.3 Organization and Main Contributions of the Thesis

The thesis consists of eight chapters and one appendix. A list of the notations and acronyms used throughout this thesis is included at the beginning of the thesis for ease of reference. The whole thesis sheds light into the existing results in the liter-

²Namely, the channel from the transmitter to the receiver is identical to the channel from the receiver to the transmitter.

ature and also provides answers to several open questions in this area. Specifically, the contents and contributions of each chapter are summarized as follows.

Chapter 1: This introductory chapter provides the motivation, objectives, overview, organization and contributions of the thesis.

Chapter 2: This chapter gives a literature review for historical and state-of-the-art results on filtering and control over unreliable networks.

Chapter 3: In this chapter, we consider the stability of a remote Kalman filter over a lossy network with bounded Markovian dropouts, which can be classified into the framework of Figure 1.2(a). The number of consecutive packet losses is assumed to be finite, and the packet-loss process is driven by a Markov chain. First of all, a method is given to estimate the upper bound for the solution of the standard Riccati recursion. Then, we provide the joint distribution of the sojourn times that include both the loss cycle (a period of successive packet dropouts) and the normal cycle (a period of successive packet receipts). The condition for peak covariance stability of the remote Kalman filter is presented as the main result of the chapter. At last, we give two numerical examples to demonstrate the results. The main contributions of this chapter can be divided into the following two parts.

- The proposed approach for estimating the upper bound of the Riccati-recursion solution is less conservative than the existing results [6, 7].
- The stability issue of time-varying Kalman filtering with finite consecutive packet losses driven by a Markov chain is studied for the first time. If the observability index of the plant is one, then the Kalman filter is always peak covariance stable. For plants with observability index greater than one, a sufficient condition is given for peak covariance stability of the remote Kalman filter, which is specified in terms of the dynamics of the plant and the transition probability of the Markov chain that describes the packet-loss process.

Chapter 4: This chapter aims at addressing the quantized stabilization of the NCS in Figure 1.2(b), where the channel input is quantized before transmission over a lossy network. In the first place, the above problem is posed in a more general form as the stabilization of Markov jump linear systems (MJLSs) via mode-dependent quantized feedback. Given a single-input MJLS and a measure of quantization coarseness, we analyze the optimality of the logarithmic quantizer and the linear state feedback law. In virtue of the sector bound approach, the co-design of the controller and the quantizer is proposed to achieve optimal coarseness for mean square quadratic (MSQ) stabilization. Next, a mode estimation algorithm is given when the mode process is not directly observed by the controller, followed by a numerical example. The above results are applied to the quantized stabilization of NCSs over lossy channels, and an extension to the output feedback case is also included. The main contributions of this chapter are summarized as follows.

- It is shown that, for a single-input MJLS, a mode-dependent logarithmic quantizer and a mode-dependent linear state feedback law can achieve optimal coarseness for MSQ stabilization. The sector bound approach is proved to be non-conservative in studying the corresponding quantized state feedback stabilization problem, and a method of optimal quantizer and controller design is presented via a linear matrix inequality (LMI) approach.
- A mode estimation algorithm obtained by maximizing a probability criterion is presented to deal with the situation that the mode process is unknown at the controller and quantizer side.
- The results on quantized stabilization of MJLSs are applied to the quantized stabilization of NCSs over lossy channels with binary or bounded packet losses. It contains the result in [8] on quantized control with independent and identically distributed (i.i.d.) Bernoulli losses as a special case.

Chapter 5: In this chapter, we reconsider the stability of the remote Kalman filter as shown in Figure 1.2(a), where the unreliable network experiences both fading and additive noise described by white noise processes. The channel fading contains transmission failure and signal fluctuation simultaneously. First, several properties of a class of modified algebraic Riccati operators (MAROs) and modified Lyapunov operators (MLOs) are presented as the preliminary results. Then, we study the network requirement for the stability of the mean error covariance matrix of a remote Kalman filter and provide bounds for the mean error covariance matrix. At last, the results of this chapter are illustrated by a numerical example. The main contributions can be summarized as follows.

- Necessary and sufficient conditions on the network are given for the stability of the mean error covariance matrix of a remote Kalman filter over fading channels subject to both transmission failure and signal fluctuation. The conditions are specified in terms of the unstable poles of the plant for the first time.
- Lower and upper bounds for the mean error covariance matrix are provided in the form of a modified Lyapunov iteration and a modified Riccati iteration, respectively.
- The results in this chapter generalize the existing results on pure signal fluctuation [5] and pure transmission failure [9].

Chapter 6: This chapter is concerned with the stabilization problem for NCSs over fading channels, which fits within the framework of Figure 1.2(b). For the case of state feedback, necessary and sufficient conditions on the network are derived for mean square stabilizability. Both parallel transmission strategy (PTS) and serial transmission strategy (STS) are discussed. For the case of dynamic output feedback, we provide conditions for stabilization of single-input single-output (SISO) plants and multi-input multi-output (MIMO) plants separately. Finally, we present the extension to the case with output fading

channels and the application of the results to vehicle platooning. The main contributions of this chapter are divided into the following parts.

- The stability criterion for systems with independent multiplicative noises presented in [10] and Chapter 9 of [11] is extended to that with possibly correlated multiplicative noises.
- It is shown that under PTS and the assumption on capacity allocation, the minimal overall mean square capacity of the network for mean square stabilizability is given in terms of the Mahler measure of the plant in the case of state feedback. The minimal capacity is also given for stabilizability via state feedback under STS, which in general can only be computed by optimization.
- For SISO plants, the corresponding minimal mean square capacity for stabilizability via dynamic output feedback is given in terms of the anti-stable poles, nonminimum phase (NMP) zeros and relative degree of the plant. For triangularly decoupled MIMO plants, necessary and sufficient conditions are also provided for the dynamic output feedback case.
- The results are extended to the case with output fading channels and applied to vehicle platooning where the basic issue is to move a platoon of closely spaced vehicles from one place to another.

Chapter 7: This chapter generalizes the results in Chapter 6 to the case with additional components such as channel processing, channel feedback, and with a power constraint on the channel input. First of all, the effect of pre- and post-channel processing and channel feedback is discussed in terms of the network requirement for stabilizability. Then, we consider the feedback stabilization and performance design of an SISO plant controlled over a single channel, where the channel undergoes both fading and additive noise and its input power is bounded by a predefined level. Finally, a numerical example is provided to illustrate the validity of the results. We summarize the main contributions of this chapter as follows.

- It is shown that in the case of dynamic output feedback, the channel feedback plays a key role in eliminating the limitation on stabilization induced by the NMP zeros and high relative degree of the plant.
- A minimal requirement on the network over which an SISO unstable plant can be stabilized via dynamic output feedback is presented in terms of the statistics of channel fading and channel additive noise, the power bound of channel input, and the Mahler measure of the plant. It is shown that a more stringent network requirement is needed for stabilizability when a power bound is applied on the channel input. The above results contain the existing results on pure multiplicative noise channel [10] and pure additive noise channel [12] as special cases.
- Under the assumption on perfect reconstruction of the network, a suboptimal algorithm is proposed for performance design of the NCS.

Chapter 8: In this chapter, we provides a continuous-time counterpart of the results on control of discrete-time networked systems over fading channels presented in Chapter 6. The network requirement for stabilizability is given under PTS via both state feedback and output feedback. The main contributions can be summarized as follows.

- In the case of state feedback, the network requirement for stabilizability is given in terms of the unstable poles of the plant.
- For dynamic output feedback, additional requirement on the network is needed for stabilizability when the plant has NMP zeros.

Chapter 9: This chapter concludes the thesis and provides possible directions for future research.

Appendix A: This appendix presents several preliminary results on matrix analysis for ease of reference.

Chapter 2

Literature Review

In this chapter, we summarize the established results on NCSs and also point out several open questions that trigger our research interests in this area. A brief review on basics of communications and information theory is presented in Section 2.1. Section 2.2 introduces the concept of Mahler measure. Section 2.3 provides a literature review for filtering and control over unreliable channels, where the issues of packet dropout, limited data rate and constrained capacity are surveyed. A set of open questions that will be addressed in the following chapters are collected in Section 2.4.

2.1 Basics of Communications and Information Theory

In 1948, communications and information theory was founded by Claude Shannon in his seminal work, “A mathematical theory of communication” [13]. The central problem in this theory is how to transmit information over noisy channels reliably, and a general communication system is depicted in Figure 2.1.

Entropy and mutual information are two basic notions in information theory, which arise as natural answers to several questions in communications. Entropy is a

Figure 2.1: General configuration of a communication system.

measure of the average uncertainty in a random variable. For example, the entropy of a discrete random variable X with distribution function $p(x)$ and sample space $\mathcal{X}$ is defined as¹

$$\mathcal{H}(X) \triangleq - \sum_{x \in \mathcal{X}} p(x) \ln p(x). \quad (2.1)$$

On the other hand, mutual information is a measure of the dependence between two random variables. For two discrete random variables X, Y with distribution functions $p(x), p(y)$ and sample spaces $\mathcal{X}, \mathcal{Y}$, the mutual information is given by

$$\mathcal{I}(X; Y) \triangleq \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \ln \frac{p(x, y)}{p(x)p(y)}, \quad (2.2)$$

where $p(x, y)$ denotes the joint distribution function of X and Y . In particular, we have $\mathcal{I}(X; Y) = 0$ when X and Y are independent, and $\mathcal{I}(X; X) = \mathcal{H}(X)$.

The most fundamental results of communications and information theory are Shannon's source coding theorem and Shannon's channel coding theorem. The former states that the average number of bits (denoted by $\mathcal{R}$) required to represent the result of a random event is given by its entropy (denoted by $\mathcal{H}$), i.e., $\mathcal{R} > \mathcal{H}$, while the latter establishes that the probability of error could be made nearly zero for all communication rates (denoted by $\mathcal{R}$) below channel capacity (denoted by $\mathcal{C}$), i.e., $\mathcal{R} < \mathcal{C}$. The channel capacity is defined as

$$\mathcal{C} \triangleq \max_{p(x)} \mathcal{I}(X; Y), \quad (2.3)$$

¹Throughout this thesis, the base of logarithm is chosen to be e and thus information is measured in nats.

where X and Y represent the channel input and channel output, respectively. The maximum in (2.3) is taken over all possible input distributions $p(x)$. Similar to the well-known principle of separation of estimation and control in control theory, the source-channel separation theorem further combines the source coding theorem and channel coding theorem. It is shown that if a source with entropy $\mathcal{H}$ satisfies the asymptotic equipartition property² and $\mathcal{H} < \mathcal{C}$, then this source can be sent reliably over a discrete memoryless channel with capacity $\mathcal{C}$ [14].

The achievable channel capacity can be used to characterize diverse communication constraints depending on the underlying channel model and information pattern. Next, we highlight several practical issues of communication systems from information and communication point of view.

First of all, quantization and delay are unavoidable in every digital communication system. In information theory, quantizers are considered as information encoders and thus as an integral part of the whole system [14]. In order to achieve Shannon's capacity, the classic information theory allows for arbitrarily long sequences in coding, which results in significant time delays. Both quantization and delay are deemed to be necessary in standard information theory rather than undesirable.

Secondly, transmitted signal is usually corrupted by channel additive noise. The additive noise channel model is one of the simplest yet typical models for a communication link. The capacity constraint arises when a power bound on the channel input is introduced in order to avoid the interference to other communication users and/or due to the hardware limitations of the transmitter [14]. For an additive white Gaussian noise channel, the capacity can be computed simply from the noise characteristics of the channel as [14]

$$\mathcal{C} = \frac{1}{2} \ln(1 + \gamma), \quad (2.4)$$

where γ represents the signal-to-noise ratio (SNR) of the channel.

²Intuitively, the asymptotic equipartition property indicates that “almost all events are almost equally surprising”; see, e.g., Chapter 3 of [14] for more details.

Thirdly, data loss or erasure is another common issue in general communication systems. It may result from transmission failure as a result of errors in wireless physical links, or from buffer overflows caused by congestion, or from packet reordering due to long transmission delays where the outdated signal is discarded in real-time applications [15]. The capacity of a binary erasure channel is [14]

$$\mathcal{C} = 1 - \alpha, \quad (2.5)$$

where α denotes the fraction of the bits that are erased³, and the receiver is assumed to know which bits have been lost.

Fourthly, the issue of fading phenomenon becomes increasingly important in wireless communication systems and has attracted recurring research interests from the communication community [3, 16, 17]. Fading describes the fluctuation experienced by transmitted signals due to the effects of multipath and shadowing in wireless channels. A precise mathematical description of fading is either unavailable at present or too complex to deal with. Therefore, depending on the particular propagation environment and communication scenario, a range of simple and accurate statistical models have been proposed for fading channels (e.g., Rayleigh, Nakagami, Rician) [3]. Moreover, for noncoherent and ideal coherent modulations over fading channels, only the fading envelope statistics is required for analysis and design [17]. Results on the capacity of a fading channel can be found in, e.g., [3, 18–20]. For example, if the CSI is known at the receiver, then the capacity of a fading channel is [3]

$$\mathcal{C} = \int_0^\infty \frac{1}{2} \ln(1 + \gamma) p(\gamma) d\gamma, \quad (2.6)$$

where γ denotes the instantaneous SNR, and $p(\gamma)$ is the distribution function of γ .

Finally, MIMO communication system is needed in large-scale applications. The network information theory deals with the simultaneous rates of communication from multiple transmitters to multiple receivers in the presence of interference and

³We note that, for the erasure channel model considered in the control community, e.g., [10], the data is lost in the unit of packet (a collection of bits) rather than bit.

noise [21], however a systematic theory is yet unavailable at the present time.

Shannon's capacity provided in standard information theory is a universal upper bound for all communication schemes, and it is achievable under the following restricted assumptions.

- The capacity-achieving code is sufficiently long.
- The causality is not taken into consideration.
- There is no constraint on coding complexity.

Therefore, it is expected that one may be unable to achieve Shannon's capacity in practice, due to the constraint on coding complexity and the requirement on real-time and causality. Although some progress has been made on causal coding [22, 23], the existing results in communications and information theory still cannot provide an immediate answer to control over networks in which the issues such as delay and causality are of fundamental importance.

2.2 Mahler Measure

It is now well known that the Mahler measure [24] or the degree of instability [25] or the topological entropy [26] of the plant to be controlled/estimated poses a fundamental limitation on the NCS. The mathematician Kurt Mahler first introduced his measure to polynomials [24]. The Mahler measure of a monic polynomial $a(z) = \prod_{i=1}^n (z - a_i)$ is defined as

$$\mathcal{M}(a) \triangleq \prod_{i=1}^n \max\{|a_i|, 1\}. \quad (2.7)$$

The Mahler measure of a square matrix $A \in \mathbb{R}^{n \times n}$ is given by that of its characteristic polynomial:

$$\mathcal{M}(A) \triangleq \mathcal{M}(\det(zI - A)) = \prod_{i=1}^n \max\{|\lambda_i(A)|, 1\}, \quad (2.8)$$

where $\lambda_i(A)$, $i = 1, \dots, n$, are the eigenvalues of A . Then, the Mahler measure of an LTI plant with any detectable and stabilizable realization (A, B, C, D) can be defined as the Mahler measure of the system matrix A .

The degree of instability of a square matrix A is defined in the same way as in (2.8) [25]. The topological entropy is introduced in [26] as a measure of the information rate of a continuous map on a compact topological space. The topological entropy of a square matrix A can be interpreted as a measure of the rate at which the open-loop plant $x(t+1) = Ax(t)$ generates information about its initial state [27, 28], which is simply the logarithm of the Mahler measure of A .

2.3 Related Results on Filtering and Control over Unreliable Channels

As witnessed by the survey papers [1, 29–31], special issues [32–37], and research monographs [28, 38], there have been fruitful achievements in the field of networked control and networked estimation during the past few years. The research on NCSs has also spawned a large wave of publications in both technical journals and conference proceedings; see, e.g., [4, 6, 7, 9, 10, 39–63] on packet loss, [8, 64–91] on quantization and data-rate constraint, [5, 10, 12, 82, 92–100] on capacity constraint for additive and multiplicative noise channels, [101–104] on random time delays, [105–107] on scheduling. In this section, we are not planning to cover all the above topics but focus on the issues of packet dropout, limited data rate and constrained capacity.

2.3.1 Packet Dropouts

The packet-drop behavior of a typical communication channel is an important issue in NCSs, as it induces information loss and consequently affects the performance or even stability of the closed-loop system. A large number of papers can

be found in the control literature dealing with the packet-loss issue under zero-quantization-error assumption. Packet losses can be modeled either as stochastic or deterministic phenomena. Deterministic models for dropouts are generally given in terms of time averages [29] or in terms of the bound on the number of consecutive dropouts [39–41]. Stochastic models for dropouts can be roughly classified into binary i.i.d. dropouts model [4, 9, 10, 42–51], binary Markovian dropouts model [6, 7, 52–58], bounded dropouts model [59–63], Poisson dropouts model [108], etc. A comprehensive review on the packet-loss issue can be found in [4].

The binary i.i.d. dropouts model assumes that packet losses are driven by an i.i.d. Bernoulli process, where the transmitted signal is either perfectly received at the receiver with a given probability, or is completely lost. By applying MJLS theory, Seiler and Sengupta [42] studied the dynamic output feedback control problem with a binary i.i.d. lossy channel located at the path from the plant to the controller, where a zero-order hold is used at the controller side. In [43], the stability robustness of NCSs with binary i.i.d. losses is addressed in terms of the critical loss rate below which the closed-loop is mean square stable. The static state feedback gain is assumed to be independent of the loss rate, and the so-called zero-control strategy is adopted, i.e., the control input becomes zero when the data is lost. The critical loss rate can be numerically computed for any fixed controller, however the authors are unable to characterize the critical loss rate when the controller is designable. Elia [10] considered the binary i.i.d. lossy channel (i.e., the i.i.d. erasure channel) as a special case of a fading channel (see Section 2.3.3 for more details). Different from [43], the packet loss rate is assumed to be known and incorporated into the controller design in [10]. One of the interesting discoveries in [10] is that a necessary and sufficient condition for stabilizing a single-input LTI plant in the mean square sense via state feedback is

$$\alpha < \frac{1}{\mathcal{M}(A)^2}, \quad (2.9)$$

where α is the packet-loss rate (i.e., probability of erasure), and $\mathcal{M}(A)$ is the Mahler measure of the system matrix A . Sinopoli *et al.* [9] addressed the Kalman filtering

over an output channel with binary i.i.d. losses. When the plant to be observed is LTI and the transmission of measurements is perfect, it is well known that the Kalman filter converges to a steady-state filter under very mild hypotheses in terms of detectability and stabilizability of the plant [109, 110]. When the Kalman filter is implemented over a lossy channel, both the time and measurement updates are performed when packet arrives, and otherwise only the time update is performed. It is shown in [9] that the time-varying Kalman filter is optimal in the MMSE sense, and a critical value for the arrival rate (or the loss rate) exists such that the mean estimation error covariance matrix is bounded for any arrival rate greater than (or the loss rate smaller than) that critical value. The lower and upper bounds for the critical value are provided in [9], and [44] further refines the results of [9] under the assumption that the system observation matrix restricted to the observable subspace is invertible. The work [45] extends the results in [9] to the case with two output channels. A probabilistic approach is presented in [46] for Kalman filtering with binary i.i.d. packet losses. Imer *et al.* [47] investigated the optimal control problem for an NCS with two binary i.i.d. lossy channels located in both the controller-to-actuator link and the sensor-to-controller link. Two types of protocols are considered. For the TCP-like protocol, there exist acknowledgments that testify successful transmissions or not at the transmitter side, while for UDP-like protocol, no acknowledgment is available. Schenato *et al.* [4] extended the results in [47] with noiseless measurements to the case with noisy measurements. One of the main discoveries in [4, 47] is that the separation principle still holds under the TCP-like protocol while it does not hold under the UDP-like protocol.

The assumption that the packets are dropped independently is not true when the quality of a channel is temporally correlated. A more general model for dropouts would assume that the packet-loss sequence is governed by a binary Markovian process [52]. Seiler and Sengupta [53] extended their results on i.i.d. packet losses [42] to the Markovian case still based on the theory of MJLS. Xie and Xie [54] presented stability and stabilization results for noise-free networked systems with binary Markovian packet losses. A series of sufficient and necessary conditions based

on MJLSs and random sampled data systems are given. Huang and Dey [6, 55] addressed the stability analysis of Kalman filtering with binary Markovian packet losses, where the notion of peak covariance stability is first introduced in [55]. In [6, 55], the given sufficient condition for peak covariance stability involves an estimate of the upper bound for the solution of the standard Riccati recursion. Following [6], Xie and Xie [7] further studied the peak covariance stability of the random Riccati equation arising from Kalman filtering with binary Markovian packet losses. In [56], a local Kalman filter is implemented at the sensor side, the output of which instead of the measured output is transmitted over the network. Smith and Seiler [57] introduced a suboptimal estimator to replace the optimal time-varying Kalman filter, which reduces the computational complexity. A “powered up” Markov chain is used to calculate the estimator gains offline.

In the binary i.i.d. or binary Markovian dropouts model, the number of consecutive packet losses can be infinite. For real networked systems, fault detection and fault tolerant systems can be implemented [111], which can effectively avoid the performance degradation and instability of the whole system due to excessive consecutive packet losses. The above observation motivates the bounded model for dropouts. Sun *et al.* [59, 60] introduced a multiple dropouts model with bounded consecutive losses and solved the corresponding optimal filtering problem. Xiong and Lam [61] considered the NCS with bounded Markovian packet dropouts. The packet-loss process is driven by a time-homogeneous Markov chain, and the state of the Markov chain denotes the number of consecutive dropouts which is assumed to be finite. The stability and stabilization results are given based on MJLS theory again. Other random models for bounded dropouts can be found in [62, 63].

2.3.2 Limited Data Rate

In the control community, quantization has been considered to be undesirable at the very beginning and thereby many works have been done in mitigating the effect of quantization error [112]. Elia and Mitter [67] first brought the idea from

information theory to control theory and indicated that quantization is “useful, if not essential, instead of undesirable”. In modern wireless NCSs, one always seeks to quantize the control/measurement signal as coarsely as possible in order to save the channel usage and power consumption, while maintaining the stability and/or performance of the closed-loop system. In this subsection, we first review results on quantization without considering packet dropouts [64–72, 74–87], and then review results that combine the effects of packet losses and quantization [8, 88–90]. An overview on feedback control under data rate constraint can be found in [91].

Delchamps [64] addressed the problem of stabilizing an unstable LTI system via state feedback when the state measurements are quantized using a static quantizer with a finite number of levels. It is found that there is no admissible control strategy that can stabilize the system asymptotically. Then, Brockett and Liberzon [65] proved that the asymptotic stability of the closed loop is achievable if the quantizer is assumed to be dynamic and time-varying and the scaling policy of the quantizer is properly designed. Nair and Evans [66] presented a necessary and sufficient condition for stabilizing a noisy LTI plant over an error-free digital channel as

$$\mathcal{R} > \ln \mathcal{M}(A), \quad (2.10)$$

where $\mathcal{R}$ denotes the average data rate, and $\mathcal{M}(A)$ is the Mahler measure of the system matrix A . Time-varying and nonlinear encoding and decoding algorithms assumed only to be causal are used to establish the data rate bound. Elia and Mitter [67] proved that the coarsest quantizer is logarithmic in the sense of quadratic stabilization of single-input LTI systems, where the optimal quantization density is given as a function of $\mathcal{M}(A)$ as well. However, the results in [67] are hard to be extended to the MIMO case. Fu and Xie [68] established that under quadratic stability, quantized stabilization of a single-input system is equivalent to the robust stabilization of an associated system with sector-bounded uncertainty, and extended the results of [67] to MIMO systems and output feedback control in virtue of the sector bound approach. Based on the result in [68], quantized stabilization is consid-

ered in [69] by adopting a quantization-error-dependent Lyapunov function which offers a less conservative design. Motivated by logarithmic quantization [68], Gu and Qiu [70] considered the state feedback stabilization problem for a multi-input LTI plant with sector-bounded time-varying uncertainties in parallel input channels. They showed that a tight lower bound on the product of sector sizes which measure quantization densities of the input channels can be quantified in terms of the Mahler measure of the plant if the product can be allocated among the channels. Coutinho *et al.* [71] investigated simultaneous input and output quantization for SISO LTI systems. They showed that the quadratic stabilization can be addressed with no conservativeness by means of the sector bound approach, similarly to the single quantizer case [68]. The quantized stabilization with finite-level logarithmic quantizer is considered in [72, 73]. Qi and Su [74] studied the optimal tracking performance of an LTI system with a quantized control. The quantization error is assumed to be a product of the original signal and an i.i.d. random variable with a uniform distribution. The aforementioned papers are on the topic of quantized control, and remarkable papers on quantized filtering include [75–77], etc.

As quantization and packet drops may coexist in an NCS, it is natural and reasonable to take them into consideration simultaneously. Gao and Chen [88] investigated the problem of robust $\mathcal{H}_\infty$ estimation for systems subject to parameter uncertainty, measurement quantization, bounded transmission delay, and bounded packet dropout. The authors aim at designing a desirable $\mathcal{H}_\infty$ filter by directly adopting a logarithmic quantizer within the system, instead of seeking for the coarsest quantization strategy. The stabilization of a single-input system over a channel subject to both quantization and binary i.i.d. packet losses is addressed in [8]. It is shown that the optimal quantization density is given in terms of the packet-loss rate and the Mahler measure of the LTI plant. Following [66, 78], the recent papers [89, 90] discuss the minimal data rate problem for stabilizing an unstable LTI plant over a lossy channel.

2.3.3 Constrained Capacity

As pointed out in Section 2.1, (Shannon's) channel capacity denotes the tightest upper bound on the amount of information that can be reliably transmitted over a communication channel, which can be used to describe diverse communication constraints. Tatikonda and Mitter [82] studied the observability and stabilizability of an LTI plant observed/controlled over different types of communication channels: a lossy channel with a given loss rate, an error-free limited-data-rate channel, an additive noise channel with a power constraint, and a delayed transmission channel. By applying information theoretic tools of source coding and channel coding, a general necessary condition for almost sure asymptotic observability and stabilizability is

$$\mathcal{C} > \ln \mathcal{M}(A), \quad (2.11)$$

where $\mathcal{C}$ is Shannon's capacity, and $\mathcal{M}(A)$ is the Mahler measure of the system matrix A . The authors also studied the sufficiency of (2.11) for observability and stabilizability in the case of lossy channels. Unfortunately, the above sufficiency is shown to be incorrect [113]. In this subsection, we are not planning to explore all possible situations in NCSs where the notion of capacity is applicable but mainly focus on the additive noise channel model with SNR constraints [12, 92–98] and the fading channel model [5, 10, 99, 100].

The control community has studied the additive noise model in the early work on mitigating quantization effects [112], and a brief review for recent results on networked control using additive noise model methodology can be found in [92]. In both continuous and discrete-time settings, Braslavsky *et al.* [94] showed that there are limitations on the ability to stabilize an unstable SISO LTI plant over an SNR constrained channel using LTI feedback. For the discrete-time case, a necessary and sufficient condition for stabilizability via static state feedback is given by

$$\gamma > \mathcal{M}(A)^2 - 1, \quad (2.12)$$

where γ is the SNR of the channel, and $\mathcal{M}(A)$ is the Mahler measure of the system matrix A . The authors also provided a connection between their results and the data rate results in [66, 78]. Moreover, additional SNR would be required for stabilizing an NMP plant when dynamic output feedback is concerned [94]. A series of results, e.g., [12, 95–98] follow the same line of research in [93, 94]. The thesis [95] analyzes the effect of colored channel noise and bandwidth limitations, and it is shown that those constraints further increase the SNR requirement for stabilizability. The performance issue in terms of the sensitivity function is also studied in [95]. In [96], a minimum phase plant with relative degree one and a single unstable pole is assumed to be controlled over a first order moving average Gaussian channel. It is shown in [96] that the stabilization is possible precisely when the feedback capacity defined in [114] is greater than the topological entropy of the plant. The work [97] introduces a scaling factor to the network and addresses the output variance minimization problem for a minimum phase plant with relative degree one by using linear quadratic Gaussian (LQG) theory. The networked architecture considered in [12] includes an LTI encoder-decoder pair and a channel feedback. It is proved in [12] that the availability of channel feedback plays a key role in reducing the minimal SNR for stabilizability. The tracking performance defined as the stationary variance of the tracking error is also investigated in [12]. The aforementioned results [12, 93–97] all focus on the SISO case except for [12] where an extension to a simple two-by-two MIMO system is considered. The authors of [98] studied the best achievable tracking performance of an MIMO LTI plant over parallel output SNR constrained channels, where the plant is assumed to be minimum phase.

From control point of view, the fading envelope can be considered as a random multiplicative gain that affects the transmitted signal. Elia [10] considered the mean square stabilization of an LTI plant over fading channels in the framework of robust control, where the randomness of the fading is interpreted as a stochastic model uncertainty. The additive noise of the channel is ignored in [10], since the main concern is the stabilization issue and the additive noise would not affect the results especially when there is no power constraint on the channel input. In this case, the

controller design that allows the maximum variance of the channel uncertainty is posed as a nonconvex optimization problem and the D-K iteration procedure [115] can be used to solve the problem. In addition, it is proved in [10] that under stochastic channel fading, a necessary and sufficient condition for stabilizing a single-input unstable plant via state feedback in the mean square sense is

$$\mathcal{C}_{\text{MS}} > \ln \mathcal{M}(A), \quad (2.13)$$

where $\mathcal{C}_{\text{MS}}$ denotes the mean square capacity defined in terms of the fading envelope statistics, and $\mathcal{M}(A)$ is the Mahler measure of the system matrix A . Note that the result on the binary i.i.d. dropouts case as shown in (2.9) is a special case of (2.13). On the other hand, the authors of [5, 99, 100] studied the Kalman filtering with faded observations. In [99, 100], the Kalman filtering over fading channels is studied under the assumption that the receiver can decide whether to accept or to reject each noisy packet. In the presence of information on instantaneous SNR at the receiver side, they showed that keeping all the packets will minimize the mean error covariance, whereas in the absence of such information, packet drop should be designed to balance information loss and communication noise in order to optimize the performance. The work [5] considers the case where single or multiple sensors transmit their measurements to a remote Kalman filter across noisy fading channels. The exact value of channel fading (CSI) is assumed to be known at the filter. Following the results in [116, 117] on Kalman filtering with random coefficients and assuming the probability of transmission failure to be zero, they proved that the mean error covariance matrix of the remote Kalman filter is bounded from above and convergent. Upper bounds for the mean error covariance matrix are provided when the fading distribution is Rayleigh or Nakagami.

2.4 Open Questions

Given the previous literature review, we have identified the following open questions in the area of control over networks.

- (1) The stability of Kalman filter has been studied in [6, 7, 9, 55] for the case with either binary i.i.d. or binary Markovian dropouts, and the stabilization problem of noise-free NCSs with bounded Markovian dropouts has been addressed in [61]. However, the stability issue for Kalman filtering over a lossy network with bounded Markovian dropouts remains open.
- (2) As we can see from the results in, to name a few, [39, 42, 53, 61], a large class of NCSs with random packet dropouts can be modeled as MJLSs, for which there have been many existing results on stability, optimal control and robust control [118]. Further motivated by the results on quantized stabilization of LTI systems in [67, 68], it would be interesting to study whether logarithmic quantization is still optimal for MJLSs and whether the sector bound approach is still non-conservative in dealing with quantized stabilization of MJLSs. In addition, if the answers to the aforementioned two questions are positive, then how to design the optimal quantizer and controller jointly?
- (3) If the exact knowledge of packet loss or channel fading is available at the filter, then the time-varying Kalman filter is proved to be optimal in the MMSE sense [5, 9]. The authors of [5] limited their attention to the signal fluctuation issue and assume that there is no transmission failure in the channel. In wireless networks, the coexistence of transmission failure and signal fluctuation is natural and reasonable, which raises the question as to what is the stability condition for Kalman filtering over a fading network subject to both transmission failure and signal fluctuation. In this case, we note that the results in [116, 117] cannot be directly applied or easily extended to solve the problem.
- (4) It has been shown in [10] that the minimal network requirement for stabi-

lizability through controller design is nonconvex and has no explicit solution in general. Only the minimal capacity for state feedback stabilization of a single-input plant over a single fading channel is given in [10]. The network requirement for both state feedback and output feedback stabilization of MIMO plants over multiple fading channels remains open.

- (5) Is there any benefit when additional components such as pre- and post-channel processing and channel feedback are introduced to the NCS with fading channels? What is the effect of the channel input power bound on the network requirement for stabilizability?
- (6) What is the network requirement for stabilizability when a continuous-time plant is controlled over fading channels?

We will address the above questions in the following chapters of this thesis.

Chapter 3

Filtering over Lossy Channels

This chapter addresses the stability of Kalman filtering over a lossy channel with bounded Markovian dropouts. As in [6,7,55], we study the stability of the peak estimation error covariance matrix of the remote Kalman filter in the mean sense. The term “peak covariance stability” is used for the consistency of the present thesis with the aforementioned papers. First, the filtering problem of an LTI plant with lossy observations is formulated in Section 3.1, where the number of consecutive dropouts is assumed to be finite and the packet-loss process is driven by a Markov chain. In Section 3.2, a less conservative method compared to [6,7] is given to estimate the upper bound for the solution of the standard Riccati recursion, followed by the joint distribution for the sojourn times of loss and normal cycles. The condition for peak covariance stability of the remote Kalman filter is provided as the main result of the chapter. Section 3.3 gives two numerical examples, and Section 3.4 concludes this chapter.

3.1 Problem Formulation

Consider a networked system as shown in Figure 3.1, where a remote filter is connected to the plant through a lossy channel.

Figure 3.1: Filtering over an output lossy channel.

Assume that the plant is LTI and has a state-space realization:

$$\begin{aligned} x(t+1) &= Ax(t) + w(t), \\ y(t) &= Cx(t) + v(t), \end{aligned} \quad (3.1)$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ are respectively the state and the measured output of the plant. The process noise $w(t) \in \mathbb{R}^n$ and the measurement noise $v(t) \in \mathbb{R}^p$ are vector-valued zero-mean white processes. The initial state $x(0)$ is a random variable with zero mean and covariance matrix $\Sigma_0 \geq 0$ and is independent of $\{w(t)\}_{t \geq 0}, \{v(t)\}_{t \geq 0}$. The above assumptions can be compactly expressed as

$$\mathcal{E} \left\{ \begin{bmatrix} w(i) \\ v(i) \\ x(0) \end{bmatrix} \begin{bmatrix} w(j) \\ v(j) \\ x(0) \\ 1 \end{bmatrix}' \right\} = \begin{bmatrix} Q\delta_{ij} & S\delta_{ij} & 0 & 0 \\ S'\delta_{ij} & R\delta_{ij} & 0 & 0 \\ 0 & 0 & \Sigma_0 & 0 \end{bmatrix}, \quad Q \geq 0, \quad R > 0, \quad \forall i, j \geq 0. \quad (3.2)$$

It is assumed that $\{w(t)\}_{t \geq 0}$ and $\{v(t)\}_{t \geq 0}$ are uncorrelated with each other, i.e., $S = 0$ in (3.2), $(A, Q^{1/2})$ is stabilizable, and (A, C) is observable with observability index $\mathcal{I}_o$. The observability index is defined as the smallest integer such that

$$\text{rank} [C' \quad A'C' \cdots (A')^{\mathcal{I}_o-1}C'] = n. \quad (3.3)$$

At each time t , the output data $y(t)$ is lumped into one packet and transmitted over the network without considering the quantization error and transmission delay. The lossy channel is assumed to undergo bounded packet losses as in [61]. Without loss of generality, let $i_0 = 0$ and $\{i_k\}_{k \geq 0}$ be a strictly monotonically increasing sequence representing the set of time instants at which the data packets are suc-

cessfully transmitted from the plant to the filter. In this situation, we define the packet-loss process as

$$\eta(i_k) \triangleq i_{k+1} - i_k - 1, \quad k \geq 0, \quad (3.4)$$

which is driven by a time-homogeneous Markov chain with a finite-state range set $\Theta \triangleq \{0, 1, \dots, N\}$. Observe that $\eta(i_k) = 0$ indicates that there is no packet dropout during the time interval $(i_k, i_{k+1}]$ and N represents the maximum number of successive packet losses. Denote by $\hat{\Theta} = \Theta \setminus \{0\}$ the packet-loss subset of Θ . The initial distribution $\pi = [\pi_0 \ \pi_1 \ \dots \ \pi_N]$ and the transition probability matrix $\Pi = [\pi_{ij}]_{i,j \in \Theta}$ of the underlying Markov chain are defined as

$$\pi_i \triangleq \Pr(\eta(0) = i), \quad \pi_{ij} \triangleq \Pr(\eta(i_{k+1}) = j | \eta(i_k) = i), \quad (3.5)$$

where $\sum_{j=0}^N \pi_{ij} = 1$ for all $i \in \Theta$. It is assumed that $\{\eta(i_k)\}_{k \geq 0}$ is independent of $\{w(t)\}_{t \geq 0}$, $\{v(t)\}_{t \geq 0}$ and $x(0)$.

If we use $\xi(t) \in \{0, 1\}$ to represent the loss (with $\xi(t) = 0$) or arrival (with $\xi(t) = 1$) of the packet at time t , then the network can be modeled as

$$z(t) = \xi(t)y(t). \quad (3.6)$$

As mentioned in Section 1.2, the exact value of $\xi(t)$ is assumed to be available at the filter at time t . Then, it is easy to prove that the time-varying Kalman filter is optimal in the MMSE sense. The state prediction error covariance matrix $P(t)$ is defined as¹

$$P(t) \triangleq \mathcal{E} \left[(x(t) - \hat{x}(t|t-1))(x(t) - \hat{x}(t|t-1))' | z_0^{t-1}, \xi_0^{t-1} \right],$$

where $\hat{x}(t|t-1)$ stands for the optimal estimate of the state at time t . Intuitively speaking, at each time step, the time-varying Kalman filter performs both the time and measurement updates when packet arrives, and otherwise it performs the time

¹Note that, throughout this thesis, we use $x_{t_1}^{t_2}$ to denote the set $\{x(t_1), x(t_1+1), \dots, x(t_2)\}$ for any signal x and $t_1 \leq t_2$.

update only. In this situation, we can easily derive the following random Riccati iteration:

$$P(t+1) = AP(t)A' + Q - \xi(t)AP(t)C'(CP(t)C' + R)^{-1}CP(t)A' \quad (3.7)$$

with initial condition $P(0) = \Sigma_0$.

Remark 3.1.1. *It is worth mentioning that $\{\xi(t)\}_{t \geq 0}$ driven by the packet-loss process $\{\eta(i_k)\}_{k \geq 0}$ is neither an i.i.d. Bernoulli process as in [9] nor a binary Markovian process as in [6]. In addition, although $\{\xi(t)\}_{t \geq 0}$ can be determined for any given realization of $\{\eta(i_k)\}_{k \geq 0}$, its stochastic properties are hard to characterize.*

Set $\beta_0 = 0$ and define

$$\begin{aligned} \alpha_1 &\triangleq \inf\{i_k + 1 : i_k \geq \beta_0, \eta(i_k) \geq 1\}, & \beta_1 &\triangleq \inf\{i_{k+1} : i_{k+1} > \alpha_1, \eta(i_k) \geq 1\}, \\ \vdots & & \vdots & \\ \alpha_l &\triangleq \inf\{i_k + 1 : i_k \geq \beta_{l-1}, \eta(i_k) \geq 1\}, & \beta_l &\triangleq \inf\{i_{k+1} : i_{k+1} > \alpha_l, \eta(i_k) \geq 1\}, \\ \vdots & & \vdots & \end{aligned}$$

Thus, $\alpha_l, \beta_l \geq 1$ are two stopping times of the Markov chain $\{\xi(t)\}_{t \geq 0}$, which describe the random jump time of $\{\xi(t)\}_{t \geq 0}$ from 1 to 0 and from 0 to 1, respectively. In this case, the sojourn times defined by

$$\alpha_l^* \triangleq \alpha_l - \beta_{l-1}, \quad \beta_l^* \triangleq \beta_l - \alpha_l, \quad l \geq 1, \quad (3.8)$$

are respectively the lengths of the l -th excursion to the states 1 and 0. The next example illustrates the notions of $\xi(t), i_k, \eta(i_k), \alpha_l, \beta_l, \alpha_l^*, \beta_l^*$.

Example 3.1.1. *Suppose that there is one sample path of $\xi(t)$ as shown in Figure 3.2. The corresponding sequence $\{i_k\}_{k \geq 0}$ is given by*

$$\{i_0, i_1, \dots, i_{14}\} = \{0, 1, 4, 5, 6, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20\}.$$

Figure 3.2: The path of $\xi(t)$ for one sample of simulation.

According to (3.4), we can get the packet-loss process as

$$\{\eta(i_0), \eta(i_1), \dots, \eta(i_{13})\} = \{0, 2, 0, 0, 3, 0, 0, 0, 0, 0, 1, 0, 0, 0\}.$$

Based on the stopping times and sojourn times defined above, it is easy to obtain that

$$\{\alpha_1, \alpha_2, \alpha_3\} = \{2, 7, 15\}, \quad \{\beta_0, \beta_1, \beta_2, \beta_3\} = \{0, 4, 10, 16\},$$

$$\{\alpha_1^*, \alpha_2^*, \alpha_3^*\} = \{2, 3, 5\}, \quad \{\beta_1^*, \beta_2^*, \beta_3^*\} = \{2, 3, 1\}.$$

In this chapter, we adopt the following definition of peak covariance stability introduced by Huang and Dey [6, 55].

Definition 3.1.1. *The Kalman filter with the packet-loss process (3.4) or the random Riccati iteration (3.7) is said to be peak covariance stable if $\sup_{l \geq 0} \mathcal{E} \{\|P(\beta_l)\|\} < \infty$, where the expectation is taken with respect to the packet-loss process.*

Remark 3.1.2. *The sequence $\{P(\beta_l)\}_{l \geq 1}$ is a collection of the prediction error covariance matrix after every sequence of consecutive losses (see, e.g., Figure 3.2). In particular, as pointed out in [6], $\{P(\beta_l)\}_{l \geq 0}$ gives the upper envelop of the prediction error covariance when the plant is a scalar system. The peak covariance stability*

given in Definition 3.1.1 captures the boundedness of the sequence $\{\mathcal{E}[\|P(\beta_l)\|]\}_{l \geq 0}$, which is reasonable in the stochastic sense. In addition, any possible $P(t)$ would be bounded if $\rho(A) < 1$ [7]. Therefore, we consider the nontrivial case $\rho(A) \geq 1$ in this chapter.

3.2 Main Results

3.2.1 Preliminaries

This subsection gives some preliminary results that will be used in the later developments.

Define an operator $\Psi(\cdot)$ as

$$\Psi(P, A, B, C, Q, R, S) \triangleq APA' + BQB' - (APC' + BS)(CPC' + R)^{-1}(APC' + BS)'.$$

The standard Lyapunov recursion $P(t+1) = G(P(t))$ and the standard Riccati recursion $P(t+1) = F(P(t))$ with initial condition $P(0)$ can be generated by

$$G(P) \triangleq \lim_{\sigma^2 \rightarrow \infty} \Psi(P, A, I, C, Q, \sigma^2 I, 0) = APA' + Q, \quad P \in \mathbb{S}, \quad (3.9)$$

$$F(P) \triangleq \Psi(P, A, I, C, Q, R, 0) = APA' + Q - APC'(CPC' + R)^{-1}CPA', \quad P \in \mathbb{S}, \quad (3.10)$$

where $\mathbb{S}$ denotes the field of positive-semidefinite matrices. The following lemma presents several properties of $G(\cdot)$ and $F(\cdot)$, and its proof can be found in [6, 119, 120].

Lemma 3.2.1. *For $G(\cdot)$ and $F(\cdot)$ defined in (3.9) and (3.10), the following statements are true.*

- (i) *Both $G(\cdot)$ and $F(\cdot)$ are monotonically increasing, i.e., for any $P_1 \geq P_2 \geq 0$, we have*

$$G(P_1) \geq G(P_2) \geq Q, \quad F(P_1) \geq F(P_2) \geq Q. \quad (3.11)$$

(ii) *There exists a scalar $d > 0$ such that for any $P \in \mathbb{S}_0$, $F^k(P) \leq dI_n$ for all $k \geq \mathcal{I}_o$, where $\mathbb{S}_0$ is a convex subset of $\mathbb{S}$ defined as*

$$\mathbb{S}_0 \triangleq \{P : 0 \leq P \leq G(\bar{P}), \text{ for some } \bar{P} \in \mathbb{S}\},$$

and $F^i(P)$ denotes the result of the i -th recursion of $F(\cdot)$ on P .

(iii) *There exists a scalar $d > 0$ such that for any $P \in \mathbb{S}$, $F^{k+1}(P) \leq dI_n$ for all $k \geq \mathcal{I}_o$.*

(iv) *If $\mathcal{I}_o = 1$, i.e., C is invertible, then*

$$F(P) \leq AC^{-1}R(C^{-1})'A', \quad (3.12)$$

i.e., there exists a scalar $d_1^{(0)} > 0$ such that

$$\|F(P)\| \leq d_1^{(0)}. \quad (3.13)$$

(v) *If $\mathcal{I}_o \geq 2$, then for $1 \leq i \leq (\mathcal{I}_o - 1)$, there exist positive scalars $d_i^{(0)}$ and $d_i^{(1)}$ such that*

$$\|F^i(P)\| \leq d_i^{(1)}\|P\| + d_i^{(0)}. \quad (3.14)$$

It is indispensable to derive a systematic method for estimating $d_i^{(1)}$ in (3.14), due to its importance in checking the peak covariance stability as we shall see in Section 3.2.2. To this end, we derive the next lemma based on the so-called global approach [109, p. 539].

Lemma 3.2.2. *The t -th recursion of the standard Riccati recursion (3.10) can be alternatively evaluated by*

$$F^t(P(0)) = \Psi(P(0), A^{(t)}, B^{(t)}, C^{(t)}, Q^{(t)}, D^{(t)}Q^{(t)}(D^{(t)})' + R^{(t)}, Q^{(t)}(D^{(t)})') \quad (3.15)$$

in one step, where

$$A^{(t)} = A^t, \quad B^{(t)} = [A^{t-1} \quad A^{t-2} \cdots I], \quad C^{(t)} = \text{col}\{C, CA, \dots, CA^{t-1}\},$$

$$D^{(1)} = 0, \text{ for } t = 1; \quad D^{(t)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{t-2} & CA^{t-3} & \cdots & 0 \end{bmatrix}, \text{ for } t \geq 2,$$

$$Q^{(t)} = \text{diag}(\underbrace{Q, Q, \dots, Q}_{t \text{ times}}), \quad R^{(t)} = \text{diag}(\underbrace{R, R, \dots, R}_{t \text{ times}}).$$

Proof. Observe that $F^t(P(0))$ corresponds to the prediction error covariance matrix of a Kalman filter with perfect transmissions, i.e., $\xi(k) \equiv 1$ for every $k \in [0, t-1]$. In this case, we can put the state-space model (3.1) into the following global form:

$$x(t) = A^{(t)}x(0) + B^{(t)}\mathbf{w}^{(t)},$$

$$\mathbf{y}^{(t)} = C^{(t)}x(0) + D^{(t)}\mathbf{w}^{(t)} + \mathbf{v}^{(t)} = C^{(t)}x(0) + [D^{(t)} \quad I] \begin{bmatrix} \mathbf{w}^{(t)} \\ \mathbf{v}^{(t)} \end{bmatrix}, \quad (3.16)$$

where

$$\mathbf{w}^{(t)} = \text{col}\{w(0), w(1), \dots, w(t-1)\}, \quad \mathbf{y}^{(t)} = \text{col}\{y(0), y(1), \dots, y(t-1)\},$$

$$\mathbf{v}^{(t)} = \text{col}\{v(0), v(1), \dots, v(t-1)\},$$

and

$$\mathcal{E} \left\{ \begin{bmatrix} \mathbf{w}^{(t)} \\ \mathbf{v}^{(t)} \\ x(0) \end{bmatrix} \begin{bmatrix} \mathbf{w}^{(t)} \\ \mathbf{v}^{(t)} \\ x(0) \end{bmatrix}' \right\} = \begin{bmatrix} Q^{(t)} & 0 & 0 \\ 0 & R^{(t)} & 0 \\ 0 & 0 & \Sigma_0 \end{bmatrix}.$$

The equality (3.15) follows from the optimality of Kalman filter. The only nontrivial thing is the correlation between the process noise term and the measurement noise term in the global model (3.16), which can be handled by introducing a non-zero S into $\Psi(\cdot)$. $\square$

Based on the above lemma, we can study the properties of $\{P(t)\}_{t \geq 0}$ without resorting to the standard Riccati recursion (3.10) step by step explicitly.

Lemma 3.2.3. *For $\mathcal{I}_o \geq 2$ and any $i \in [1, \mathcal{I}_o - 1]$, the positive scalar $d_i^{(1)}$ in (3.14) is bounded from above by $\underline{\lambda}_i$, and $\underline{\lambda}_i$ is the solution to the following convex optimization:*

$$\underline{\lambda}_i = \min_{L^{(i)}} \lambda_i \quad (3.17)$$

subject to the LMI constraint

$$\begin{bmatrix} \lambda_i I & (A^{(i)} + L^{(i)}C^{(i)})' \\ A^{(i)} + L^{(i)}C^{(i)} & I \end{bmatrix} \geq 0,$$

where $A^{(i)}$ and $C^{(i)}$ are as defined in Lemma 3.2.2.

Proof. First consider the case $i = 1$. Let

$$\psi(L, P) \triangleq (A + LC)P(A + LC)' + LRL' + Q.$$

By taking partial derivative $\partial\psi(L, P)/\partial L$, it is easy to get that

$$L = L_P \triangleq -APC'(CPC' + R)^{-1}$$

minimizes $\psi(L, P)$ and $\psi(L_P, P) = F(P)$, which further implies that

$$\|F(P)\| = \|\psi(L_P, P)\| \leq \|\psi(L, P)\| \leq \|A + LC\|^2 \|P\| + \|LRL' + Q\|.$$

By comparing the above inequality with (3.14), we can conclude that $d_1^{(1)}$ can be bounded by $\|A + LC\|^2$ for any L , which further implies that

$$d_1^{(1)} \leq \min_L \{\|A + LC\|^2\}.$$

We also have

$$\|A + LC\|^2 = \min\{\lambda_1 | \lambda_1 I - (A + LC)'(A + LC) \geq 0\},$$

and $\underline{\lambda}_1 = \min_L \{\|A + LC\|^2\}$ can be easily obtained through (3.17) with $i = 1$.

For $i \in (1, \mathcal{I}_o - 1]$, it follows from Lemma 3.2.2 that

$$F^i(P) = \Psi(P, A^{(i)}, B^{(i)}, C^{(i)}, Q^{(i)}, D^{(i)}Q^{(i)}(D^{(i)})' + R^{(i)}, Q^{(i)}(D^{(i)})').$$

Further denote

$$\bar{R}^{(i)} = D^{(i)}Q^{(i)}(D^{(i)})' + R^{(i)}, \quad S^{(i)} = Q^{(i)}(D^{(i)})',$$

and define

$$\bar{\psi}(L^{(i)}, P) \triangleq (A^{(i)} + L^{(i)}C^{(i)})P(A^{(i)} + L^{(i)}C^{(i)})' + [B^{(i)} \ L^{(i)}] \begin{bmatrix} Q^{(i)} & S^{(i)} \\ (S^{(i)})' & \bar{R}^{(i)} \end{bmatrix} \begin{bmatrix} (B^{(i)})' \\ (L^{(i)})' \end{bmatrix}.$$

Then, $L^{(i)} = L_P^{(i)} \triangleq -(A^{(i)}P(C^{(i)})' + B^{(i)}S^{(i)})(C^{(i)}P(C^{(i)})' + \bar{R}^{(i)})^{-1}$ minimizes $\bar{\psi}(L^{(i)}, P)$ and $\bar{\psi}(L_P^{(i)}, P) = F^i(P)$. It follows that

$$\begin{aligned} \|F^i(P)\| &= \|\bar{\psi}(L_P^{(i)}, P)\| \leq \|\bar{\psi}(L^{(i)}, P)\| \\ &\leq \|A^{(i)} + L^{(i)}C^{(i)}\|^2 \|P\| + \left\| [B^{(i)} \ L^{(i)}] \begin{bmatrix} Q^{(i)} & S^{(i)} \\ (S^{(i)})' & \bar{R}^{(i)} \end{bmatrix} \begin{bmatrix} (B^{(i)})' \\ (L^{(i)})' \end{bmatrix} \right\|. \end{aligned}$$

Similarly, after comparing the above inequality with (3.14), we have

$$d_i^{(1)} \leq \underline{\lambda}_i = \min_{L^{(i)}} \{\|A^{(i)} + L^{(i)}C^{(i)}\|^2\},$$

and (3.17) again offers a way to compute $\underline{\lambda}_i$ for $i \in (1, \mathcal{I}_o - 1]$. This completes the proof of this lemma. $\square$

From the proof of Lemma 3.2.3, there always exists $L^{(i)}$ such that $d_i^{(1)} = 0$ for

$i \geq \mathcal{I}_o$, since (A, C) is assumed to be observable and thus $C^{(i)}$ has full-column rank when $i \geq \mathcal{I}_o$.

Remark 3.2.1. *In the simulation example of [6], $\|F(P)\| \leq \|AA'\| \|P\|$ is used to estimate $d_1^{(1)}$. In [55], the same authors applied a suboptimal estimator to aid stability check and adopted a numerical method to search for a better estimate of $d_1^{(1)}$ for a two-dimensional system. The method in [7] does not involve estimating $d_1^{(1)}$ and has been shown to be less conservative than [6, 55] in some cases, however it approximates $F^i(P)$ by $G^i(P)$ when $i \in [1, (\mathcal{I}_o - 1)]$ and $\mathcal{I}_o \geq 2$, which is still conservative. Compared to the aforementioned papers, Lemma 3.2.3 presents an easy and efficient way to estimate $d_i^{(1)}$ (see Example 3.3.1).*

When $\{\xi(t)\}_{t \geq 0}$ is driven by a binary Markovian process, the sojourn times are proved to be i.i.d.; see Lemma 2 in [6]. In the case of bounded Markovian packet losses, the joint distribution of the sojourn times is given in the following lemma.

Lemma 3.2.4. *Denote the joint distribution of sojourn times by*

$$\text{Prob}(a_1^l, b_1^l) \triangleq \Pr(\alpha_1^* = a_1, \beta_1^* = b_1, \alpha_2^* = a_2, \beta_2^* = b_2, \dots, \alpha_l^* = a_l, \beta_l^* = b_l)$$

with $a_i \in \mathcal{N}$, $b_i \in \hat{\Theta}$ for all $i = 1, 2, \dots, l$. Then, $\text{Prob}(a_1^l, b_1^l)$ can be computed recursively as

$$\begin{aligned} \text{Prob}(a_1, b_1) &= \begin{cases} \pi_{b_1}, & \text{if } a_1 = 1; \\ \pi_0(\pi_{00})^{a_1-2}\pi_{0b_1}, & \text{if } a_1 \geq 2, \end{cases} \\ \text{Prob}(a_1^{l+1}, b_1^{l+1}) &= \begin{cases} \pi_{b_l b_{l+1}} \text{Prob}(a_1^l, b_1^l), & \text{if } a_{l+1} = 1; \\ \pi_{b_l 0}(\pi_{00})^{a_{l+1}-2}\pi_{0b_{l+1}} \text{Prob}(a_1^l, b_1^l), & \text{if } a_{l+1} \geq 2. \end{cases} \end{aligned}$$

Proof. The result can be obtained based on the Markov property of the packet-loss process $\{\eta(i_k)\}_{k \geq 0}$. □

3.2.2 Peak Covariance Stability

Now, we are in the position to present our main results. We start by considering the peak covariance stability for the special case $\mathcal{I}_o = 1$.

Theorem 3.2.1. *If the observability index of the plant is one, then the Kalman filter with bounded Markovian packet losses (3.4) is always peak covariance stable.*

Proof. First of all, $\mathcal{E} \{\|P(\beta_0)\|\} = \|P(0)\|$ since $\beta_0 = 0$ and $P(0) \in \mathbb{S}$ is given. Let $\{a_l, b_l\}_{l \geq 1}$ with $a_l \in \mathcal{N}$, $b_l \in \hat{\Theta}$, denote one realization of the random process $\{\alpha_l^*, \beta_l^*\}_{l \geq 1}$. Since $P(\beta_1)$ is uniquely determined for any fixed α_1^*, β_1^* and $P(0)$, we have

$$\begin{aligned} \mathcal{E} \{\|P(\beta_1)\|\} &= \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \|P(\beta_1)\| \text{Prob}(a_1, b_1) \\ &= \sum_{a_1=2}^{\infty} \sum_{b_1=1}^s \pi_0 (\pi_{00})^{a_1-2} \pi_{0b_1} \|G^{b_1}(F^{a_1}(P(0)))\| + \sum_{b_1=1}^s \pi_{b_1} \|G^{b_1}(F(P(0)))\|, \end{aligned} \quad (3.18)$$

where Lemma 3.2.4 is applied to obtain (3.18). The first part of (3.18) describes all the possible trajectories starting from initial condition $\eta_0 = 0$ and reaching $\beta_1^* = b_1$ before the first peak time β_1 , and the second part corresponds to $\eta_0 = b_1$. Then, it follows from Lemma 3.2.1 that

$$\begin{aligned} &\mathcal{E} \{\|P(\beta_1)\|\} \\ &\leq \sum_{b_1=1}^s \sum_{a_1=2}^{\infty} \pi_0 (\pi_{00})^{a_1-2} \pi_{0b_1} \|G^{b_1}(dI_n)\| + \sum_{b_1=1}^s \pi_{b_1} (d_1^{(1)} \|A^{b_1}\|^2 \|P(0)\| + d_1^{(0)} \|A^{b_1}\|^2 + c_1) \\ &\leq \sum_{b_1=1}^s \sum_{a_1=2}^{\infty} c_2 \pi_0 \pi_{0b_1} (\pi_{00})^{a_1-2} + \sum_{b_1=1}^s \pi_{b_1} (d_1^{(1)} \|A^{b_1}\|^2 \|P(0)\| + d_1^{(0)} \|A^{b_1}\|^2 + c_1) \quad (3.19) \\ &\leq c_3, \end{aligned}$$

where c_1, c_2 are scalars, and c_3 is a scalar-valued function of $P(0)$. Note that the first part of (3.19) is 0 when $\pi_{00} = 1$ since $\pi_{0b_1} = 0$, and it is convergent for $\pi_{00} < 1$.

Secondly, for any $l \geq 1$, we can obtain that

$$\begin{aligned}
& \mathcal{E} \{ \|P(\beta_{l+1})\| \} \\
&= \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{a_{l+1}=1}^{\infty} \sum_{b_{l+1}=1}^s \|P(\beta_{l+1})\| \text{Prob}(a_1^{l+1}, b_1^{l+1}) \\
&= \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{a_{l+1}=2}^{\infty} \sum_{b_{l+1}=1}^s \pi_{b_l 0} (\pi_{00})^{a_{l+1}-2} \pi_{0b_{l+1}} \|G^{b_{l+1}}(F^{a_{l+1}}(P(\beta_l)))\| \text{Prob}(a_1^l, b_1^l) \\
&\quad + \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{b_{l+1}=1}^s \pi_{b_l b_{l+1}} \|G^{b_{l+1}}(F(P(\beta_l)))\| \text{Prob}(a_1^l, b_1^l).
\end{aligned}$$

It follows from Lemma 3.2.1 again that

$$\begin{aligned}
& \mathcal{E} \{ \|P(\beta_{l+1})\| \} \\
&\leq \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{a_{l+1}=2}^{\infty} \sum_{b_{l+1}=1}^s \pi_{b_l 0} (\pi_{00})^{a_{l+1}-2} \pi_{0b_{l+1}} \|G^{b_{l+1}}(dI_n)\| \text{Prob}(a_1^l, b_1^l) \\
&\quad + \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{b_{l+1}=1}^s \pi_{b_l b_{l+1}} \|G^{b_{l+1}}(dI_n)\| \text{Prob}(a_1^l, b_1^l) \tag{3.20} \\
&\leq \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{a_{l+1}=2}^{\infty} \sum_{b_{l+1}=1}^s c_4 \pi_{b_l 0} \pi_{0b_{l+1}} (\pi_{00})^{a_{l+1}-2} \text{Prob}(a_1^l, b_1^l) \\
&\quad + \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{b_{l+1}=1}^s c_5 \pi_{b_l b_{l+1}} \text{Prob}(a_1^l, b_1^l) \\
&\leq \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{a_l=1}^{\infty} \sum_{b_l=1}^s c_6 \text{Prob}(a_1^l, b_1^l) + \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{a_l=1}^{\infty} \sum_{b_l=1}^s c_7 \text{Prob}(a_1^l, b_1^l) \\
&\leq c_8,
\end{aligned}$$

where $c_4, c_5, \dots, c_8$ are scalars, and (3.20) follows from the fact that any possible value of $P(\beta_l)$ belongs to $\mathbb{S}_0$ for $l \geq 1$. The inequalities $\mathcal{E} \{ \|P(\beta_1)\| \} \leq c_3$ and $\mathcal{E} \{ \|P(\beta_{l+1})\| \} \leq c_8$ imply the peak covariance stability of the remote Kalman filter, which complete the proof. $\square$

Remark 3.2.2. According to Theorem 3.2.1, we can conclude that every Kalman filter with bounded Markovian packet losses is peak covariance stable for observable

scalar systems, since their observability index is always equal to 1.

The next theorem provides a sufficient condition for peak covariance stability of the remote Kalman filter when $\mathcal{I}_o \geq 2$.

Theorem 3.2.2. *If the observability index of the plant is greater than one, then the Kalman filter with bounded Markovian packet losses (3.4) is peak covariance stable, if $\rho(H) < 1$, where*

$$[H]_{ij} = \left(\pi_{ij} d_1^{(1)} + \sum_{m=2}^{\mathcal{I}_o-1} \pi_{i0} (\pi_{00})^{m-2} \pi_{0j} d_m^{(1)} \right) \|A^j\|^2, \quad i, j = 0, 1, \dots, N. \quad (3.21)$$

Proof. Following the similar argument as in Theorem 3.2.1, we have

$$\begin{aligned} \mathcal{E} \{ \|P(\beta_1)\| \} &= \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \|P(\beta_1)\| \text{Prob}(a_1, b_1) \\ &= \sum_{b_1=1}^s \pi_{b_1} \|G^{b_1}(F(P(0)))\| + \sum_{b_1=1}^s \sum_{a_1=\mathcal{I}_o+1}^{\infty} \pi_0 (\pi_{00})^{a_1-2} \pi_{0b_1} \|G^{b_1}(F^{a_1}(P(0)))\| \\ &\quad + \sum_{b_1=1}^s \sum_{a_1=2}^{\mathcal{I}_o} \pi_0 (\pi_{00})^{a_1-2} \pi_{0b_1} \|G^{b_1}(F^{a_1}(P(0)))\|. \end{aligned}$$

From Lemma 3.2.1, it holds that

$$\begin{aligned} &\mathcal{E} \{ \|P(\beta_1)\| \} \\ &\leq \sum_{b_1=1}^s \pi_{b_1} (d_1^{(1)} \|A^{b_1}\|^2 \|P(0)\| + d_1^{(0)} \|A^{b_1}\|^2 + e_2) + \sum_{b_1=1}^s \sum_{a_1=\mathcal{I}_o+1}^{\infty} \pi_0 (\pi_{00})^{a_1-2} \pi_{0b_1} \|G^{b_1}(dI_n)\| \\ &\quad + \sum_{b_1=1}^s \sum_{a_1=2}^{\mathcal{I}_o} \pi_0 (\pi_{00})^{a_1-2} \pi_{0b_1} (d_{a_1}^{(1)} \|A^{b_1}\|^2 \|P(0)\| + d_{a_1}^{(0)} \|A^{b_1}\|^2 + e_1) \\ &\leq e_3 + \sum_{b_1=1}^s \left(\sum_{a_1=2}^{\mathcal{I}_o} \pi_0 (\pi_{00})^{a_1-2} \pi_{0b_1} d_{a_1}^{(1)} + \pi_{b_1} d_1^{(1)} \right) \|A^{b_1}\|^2 \|P(0)\| = e_3 + \Phi \Omega, \end{aligned}$$

where e_1, e_2, e_3 are scalars, $\Omega = [1 \ 1 \cdots 1]' \in \mathbb{R}^{N \times 1}$, $\Phi = [\phi_1 \ \phi_2 \cdots \phi_N] \in \mathbb{R}^{1 \times N}$ with

$$\phi_i = \left(\pi_i d_1^{(1)} + \sum_{m=2}^{\mathcal{I}_o} \pi_0 (\pi_{00})^{m-2} \pi_{0i} d_m^{(1)} \right) \|A^i\|^2 \|P(0)\|, \quad i = 1, 2, \dots, N.$$

Next, for any $l \geq 1$, we have

$$\begin{aligned}
& \mathcal{E} \{ \|P(\beta_{l+1})\| \} \\
&= \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{a_{l+1}=1}^{\infty} \sum_{b_{l+1}=1}^s \|P(\beta_{l+1})\| \text{Prob}(a_1^{l+1}, b_1^{l+1}) \\
&= \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{b_{l+1}=1}^s \sum_{a_{l+1}=\mathcal{I}_o}^{\infty} \pi_{b_l 0} (\pi_{00})^{a_{l+1}-2} \pi_{0b_{l+1}} \|G^{b_{l+1}}(F^{a_{l+1}}(P(\beta_l)))\| \text{Prob}(a_1^l, b_1^l) \\
&\quad + \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{b_{l+1}=1}^s \sum_{a_{l+1}=2}^{\mathcal{I}_o-1} \pi_{b_l 0} (\pi_{00})^{a_{l+1}-2} \pi_{0b_{l+1}} \|G^{b_{l+1}}(F^{a_{l+1}}(P(\beta_l)))\| \text{Prob}(a_1^l, b_1^l) \\
&\quad + \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{b_{l+1}=1}^s \pi_{b_l b_{l+1}} \|G^{b_{l+1}}(F(P(\beta_l)))\| \text{Prob}(a_1^l, b_1^l) \\
&\triangleq \Gamma_1 + \Gamma_2 + \Gamma_3.
\end{aligned}$$

It is easy to derive that

$$\Gamma_1 \leq \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{b_{l+1}=1}^s \sum_{a_{l+1}=\mathcal{I}_o}^{\infty} \pi_{b_l 0} (\pi_{00})^{a_{l+1}-2} \pi_{0b_{l+1}} \|G^{b_{l+1}}(dI_n)\| \text{Prob}(a_1^l, b_1^l) \leq e_4, \quad (3.22)$$

$$\begin{aligned}
\Gamma_2 \leq & \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{b_{l+1}=1}^s \sum_{a_{l+1}=2}^{\mathcal{I}_o-1} \pi_{b_l 0} (\pi_{00})^{a_{l+1}-2} \pi_{0b_{l+1}} \left(d_{a_{l+1}}^{(1)} \|A^{b_{l+1}}\|^2 \|P(\beta_l)\| \right. \\
& \left. + d_{a_{l+1}}^{(0)} \|A^{b_{l+1}}\|^2 + e_5 \right) \text{Prob}(a_1^l, b_1^l), \quad (3.23)
\end{aligned}$$

$$\Gamma_3 \leq \sum_{a_1=1}^{\infty} \sum_{b_1=1}^s \cdots \sum_{b_{l+1}=1}^s \pi_{b_l b_{l+1}} (d_1^{(1)} \|A^{b_{l+1}}\|^2 \|P(\beta_l)\| + d_1^{(0)} \|A^{b_{l+1}}\|^2 + e_6) \text{Prob}(a_1^l, b_1^l) \quad (3.24)$$

with scalars e_4, e_5, e_6 . By induction, it holds that

$$\mathcal{E} \{ \|P(\beta_{l+1})\| \} \leq \Phi H^l \Omega + e_7, \quad (3.25)$$

where e_7 is a scalar, and H is given in (3.21). According to (3.25), if $\rho(H) < 1$, then $\mathcal{E} \{ \|P(\beta_{l+1})\| \}$ is bounded from above, which completes the proof. $\square$

Remark 3.2.3. When $\mathcal{I}_o = 2$, the second term in the parenthesis of (3.21) vanishes. It can be further observed from (3.21) that if the maximum number of consecutive packet dropouts, i.e., N , is large but transition probabilities to high consecutive packet losses, i.e., π_{ij} with $i = 0, 1, \dots, N$ and relatively large j , are small and the dynamics of the plant does not grow fast, then $[H]_{ij}$ is closed to 0 for relatively large j and thus the stability condition of Theorem 3.2.2 would be dominated by the part of the matrix H corresponding to low consecutive packet dropouts.

3.3 Numerical Examples

Next, we give two simulations to illustrate the results in this chapter.

Example 3.3.1. Consider a second-order plant as [6, 7]

$$A = \begin{bmatrix} 1.3 & 0.3 \\ 0 & 1.2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The covariances of $w(t)$ and $v(t)$ are chosen as $Q = I_2$ and $R = 1$.

First, we consider the network model in [6, 7] where the packet loss at each time step is driven by a binary Markovian process, which implies that the number of consecutive packet losses may be infinite. We need to estimate $d_1^{(1)}$ in (3.14) since $\mathcal{I}_o = 2$. From Lemma 3.2.3, we have $d_1^{(1)} = 1.2200$, which is smaller than the estimate: 2.00813 in [6]. Moreover, if we apply Lemma 3.2.3 of this chapter to Theorem 3.1 in [7], the condition therein can be rewritten as

$$\sum_{i=1}^{\infty} \|A^i\|^2 (1-q)^{i-1} q \sum_{j=1}^{\mathcal{I}_o-1} d_j^{(1)} (1-p)^{j-1} p < 1.$$

For the recovery rate $q = 0.65$, we can easily get that the maximum failure rate p is 0.1960, which also shows that Lemma 3.2.3 is less conservative than the result in [7] where the maximum failure rate p is 0.1191 for the same recovery rate.

Next, we consider the bounded Markovian packet losses case. Suppose that the number of maximum successive packet losses is 2 and the transition probability matrix is given by:

$$\Pi_1 = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.8 & 0.1 & 0.1 \\ 0.8 & 0.1 & 0.1 \end{bmatrix}.$$

From Theorem 3.2.2, we have $\rho(H) = 0.7352 < 1$, thus the Kalman filter is peak covariance stable. Figure 3.3 shows the paths of $\xi(t)$ and $\|P(t)\|$ for one sample of simulation.

Figure 3.3: The paths of $\xi(t)$ (above) and $\|P(t)\|$ (below) for one sample of simulation in Example 3.3.1.

Example 3.3.2. Consider the following third-order plant:

$$A = \begin{bmatrix} 0 & 0.1 & 1.2 \\ 0 & 0.5 & 0 \\ 1.3 & 0.1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

with $Q = I_3$, $R = 1$. The eigenvalues of A are $1.2490, -1.2490, 0.5000$, and the observability index $\mathcal{I}_o$ is 3. Suppose that the number of maximum consecutive packet

losses is 5 and

$$\Pi_2 = \begin{bmatrix} 0.5 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.4 & 0 & 0.3 & 0 & 0.2 \\ 0.1 & 0.3 & 0.1 & 0.3 & 0 & 0.2 \\ 0.1 & 0.4 & 0 & 0.3 & 0 & 0.2 \\ 0.1 & 0.3 & 0 & 0.3 & 0.1 & 0.2 \\ 0.1 & 0.4 & 0 & 0.3 & 0 & 0.2 \end{bmatrix}.$$

After obtaining $d_1^{(1)} = 1.4522, d_2^{(1)} = 0.0933$ from Lemma 3.2.3, we have $\rho(H) = 5.7932 > 1$ in Theorem 3.2.2, and Figure 3.4 shows that a sample path of $\|P(t)\|$ can reach a relatively high level even before 100 time steps. Note that the rapid drop of $\|P(t)\|$ around the 40-th time step is due to the successive packet receipts. Furthermore, if $C = I_3$, then $\mathcal{I}_o = 1$ and no additional condition is required for peak covariance stability according to Theorem 3.2.1. Figure 3.5 gives one possible sample path of $\xi(t)$ and $\|P(t)\|$ with Π_2 .

Figure 3.4: The paths of $\xi(t)$ (above) and $\|P(t)\|$ (below) for one sample of simulation in Example 3.3.2 with $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

Figure 3.5: The paths of $\xi(t)$ (above) and $\|P(t)\|$ (below) for one sample of simulation in Example 3.3.2 with $C = I_3$.

3.4 Summary

The Kalman filtering problem over a lossy channel with bounded Markovian dropouts has been investigated in this chapter. It has been shown that, if the observability index of the underlying discrete-time LTI system is one, then the Kalman filter is always peak covariance stable. For systems with observability index greater than one, a sufficient condition for peak covariance stability has been given in terms of the system dynamics and the transition probability of the Markov chain that describes the packet-loss process. Numerical examples have been provided to demonstrate the validity of the results.

Chapter 4

Quantized Control over Lossy Channels

This chapter aims at stabilizing an unstable plant across a lossy channel via quantized feedback. As discussed in Chapter 2, a large class of NCSs with random packet dropouts can be modeled as MJLSs. Therefore, we first consider the quantized stabilization problem for a single-input MJLS in Section 4.1. Given a measure of quantization coarseness, a mode-dependent logarithmic quantizer and a mode-dependent linear state feedback law can achieve optimal coarseness for MSQ stabilization of an MJLS. The sector bound approach is shown to be non-conservative in investigating the corresponding quantized state feedback problem, and then a method of optimal quantizer and controller design is presented in terms of LMIs. Moreover, when the mode process is not directly observed by the controller, we give a mode estimation algorithm by maximizing a probability criterion. In Section 4.2, the results presented in Section 4.1 are applied to the quantized stabilization of NCSs over lossy channels with binary or bounded packet losses. Both TCP-like and UDP-like protocols are discussed, and an extension to output feedback under TCP-like protocol is also included. Section 4.3 summarizes the chapter.

4.1 Stabilization of MJLSs Using Quantized State Feedback

As we can see from Figure 4.1, a quantized feedback control system generally comprises three parts: a system to be controlled, a controller and a quantizer.

Figure 4.1: Typical quantized feedback control system.

Assume that the system is described by a discrete-time single-input MJLS as

$$x(t+1) = A_{\theta(t)}x(t) + B_{\theta(t)}z(t) + w(t), \quad (4.1)$$

where $x(t) \in \mathbb{R}^n$ is the state with $x(0)$ being a second-order random variable, $z(t) \in \mathbb{R}$ is the quantized control input, $w(t) \in \mathbb{R}^n$ is a second-order process noise with zero mean and covariance matrix $\Sigma_{\theta(t)} > 0$, and $\theta(t) \in \Theta \triangleq \{0, 1, \dots, N\}$ is the system mode governed by a time-homogeneous Markov chain with initial distribution $\pi = [\pi_0 \ \pi_1 \ \dots \ \pi_N]$ and transition probability matrix $\Pi = [\pi_{ij}]_{i,j \in \Theta}$, where

$$\pi_i \triangleq \Pr(\theta(0) = i), \quad \pi_{ij} \triangleq \Pr(\theta(t+1) = j | \theta(t) = i). \quad (4.2)$$

For all $t \geq 0$, $x(0)$, θ_0^t , w_0^t are assumed to be independent of each other. Suppose that $x(t)$ is available at both the controller and the quantizer, and the static quantized state feedback is denoted by

$$v(t) = g(x(t), \gamma(t)), \quad (4.3)$$

$$z(t) = f(v(t), \gamma(t)), \quad (4.4)$$

where $\gamma(t) \in \Theta$ is a direct observation or an estimate of system mode $\theta(t-d)$

at the controller/quantizer side at time step t with $d \in \mathbb{Z}_+$ the constant mode observation/estimation delay. For every $t \in [0, d - 1]$, the initial $\gamma(t)$ is chosen arbitrarily from the mode set Θ .

The closed-loop system of (4.1), (4.3) and (4.4) is given by

$$x(t+1) = A_{\theta(t)}x(t) + B_{\theta(t)}f(g(x(t), \gamma(t)), \gamma(t)) + w(t). \quad (4.5)$$

It is worth mentioning that (4.5) is generally nonlinear, since the control signal $z(t)$ can be a nonlinear function of $v(t)$ in (4.4) due to quantization. We adopt the following definitions of mean square stability and MSQ stability.

Definition 4.1.1. For $w(t) \equiv 0$ and any initial condition of $x(0), \theta(0), \gamma(0)$, the equilibrium point at the origin of (4.5) is mean square stable if

$$\lim_{t \rightarrow \infty} \mathcal{E}[\|x(t)\|^2 | x(0), \theta(0), \gamma(0)] = 0; \quad (4.6)$$

it is mean square quadratically (MSQ) stable, if, for every $\gamma(t) \in \Theta$, there exist a function

$$V(x(t), \gamma(t)) \triangleq x(t)' P_{\gamma(t)} x(t), \quad P_{\gamma(t)} > 0, \quad (4.7)$$

and a positive-definite matrix $Q_{\gamma(t)}$ such that, for all $t \geq d$,

$$\begin{aligned} \nabla V(x(t), \gamma(t)) &\triangleq \mathcal{E}[V(x(t+1), \gamma(t+1)) - V(x(t), \gamma(t)) | x_0^t, \gamma_0^t] \\ &= \mathcal{E}[V(x(t+1), \gamma(t+1)) | x_0^t, \gamma_0^t] - V(x(t), \gamma(t)) \\ &< -x(t)' Q_{\gamma(t)} x(t), \quad \forall x(t) \in \mathbb{R}^n, x(t) \neq 0. \end{aligned} \quad (4.8)$$

Remark 4.1.1. Following a similar line of arguments as in the proof of Theorem 1 in [121], we can prove that the MSQ stability of the equilibrium point at the origin of (4.5) always implies the mean square stability. When $d \geq 1$, after taking $z(t) \equiv 0$ for all $t < d$, it is easy to see that x_d is still a second-order random variable, and thus we can consider $t = d$ as the starting point in (4.8) without loss of generality.

Remark 4.1.2. *Imposing the condition (4.8) in every system mode introduces some degree of conservativeness but has the following advantages: 1) under the notion of MSQ stability, we can prove the optimality of the logarithmic quantizer defined in the next section; 2) it makes existing well-established results in robust control of MJLSs applicable in quantized feedback control.*

Assume that $f(\cdot, \cdot)$ in (4.4) is an odd function of $v(t)$, i.e.,

$$f(-v(t), \gamma(t)) = -f(v(t), \gamma(t)). \quad (4.9)$$

We define the mode quantization density with respect to mode i , $i \in \Theta$, similarly to the quantization density in the LTI case [67] as

$$\eta_f(i) \triangleq \limsup_{\epsilon \rightarrow 0} \frac{\#l[\epsilon, i]}{-\ln \epsilon}, \quad (4.10)$$

where $\#l[\epsilon, i]$ is the number of quantization levels in the interval $[\epsilon, 1/\epsilon]$ for the quantizer $f(\cdot, i)$. Evidently, the mode quantization density is reduced to the quantization density defined in [67] when $N = 0$. For $N \neq 0$, there is a set of mode quantization densities $\eta_f(i)$, $i = 0, 1, \dots, N$. Motivated by the Pareto optimality of vector-valued criterion [122], we introduce the overall coarseness for a mode-dependent quantizer as follows.

Definition 4.1.2. *The overall coarseness of a mode-dependent quantizer (4.4) is defined as*

$$C_f \triangleq e(\eta_f(0), \eta_f(1), \dots, \eta_f(N)), \quad (4.11)$$

where e is a scalar-valued function of $\eta_f(i)$, $i = 0, 1, \dots, N$, satisfying the following property: if $\eta_{f1}(i) \leq \eta_{f2}(i)$ for all $i \in \Theta$, then

$$e(\eta_{f1}(0), \eta_{f1}(1), \dots, \eta_{f1}(N)) \leq e(\eta_{f2}(0), \eta_{f2}(1), \dots, \eta_{f2}(N)). \quad (4.12)$$

The property (4.12) reveals that the overall coarseness should always be non-

decreasing when any one of the mode quantization densities is increasing and all the others are fixed. Note that the smaller the value of C_f , the coarser the quantizer. The form of e in (4.11) can be chosen according to physical constraints or performance requirements of the quantizer. It is easy to see that the set of $\eta_f(i)$, $i = 0, 1, \dots, N$, corresponding to the globally optimal C_f may not be unique. The main purpose of this section is to find one possible combination of the controller (4.3) and the quantizer (4.4) with the optimal C_f such that the closed-loop system is MSQ stable.

4.1.1 Main Results

A mode-dependent quantizer is said to be logarithmic, if, for any $\gamma(t) \in \Theta$, the corresponding set of quantization levels $\mathcal{Z}_{\gamma(t)}$ has the following form:

$$\mathcal{Z}_{\gamma(t)} = \{\pm z_l(\gamma(t)) : z_l(\gamma(t)) = \rho^l(\gamma(t))z_0, z_0 > 0, \text{ for } l \in \pm 1, \pm 2, \dots\} \cup \{\pm z_0\} \cup \{0\}, \quad (4.13)$$

where

$$\rho(\gamma(t)) = \frac{1 - \delta(\gamma(t))}{1 + \delta(\gamma(t))}, \quad (4.14)$$

and $\delta(\gamma(t))$ denotes the sector bound as shown in Figure 4.2. Specifically, the associated logarithmic quantizer is defined as follows.

Figure 4.2: Mode-dependent logarithmic quantizer.

For the given $\gamma(t)$:

- if $\delta(\gamma(t)) = 0$, then

$$f(v(t), \gamma(t)) = v(t); \quad (4.15)$$

- if $0 < \delta(\gamma(t)) < 1$, then

$$f(v(t), \gamma(t)) = \begin{cases} z_l(\gamma(t)), & \text{if } \frac{1}{1+\delta(\gamma(t))} z_l(\gamma(t)) < v(t) \leq \frac{1}{1-\delta(\gamma(t))} z_l(\gamma(t)), \\ 0, & \text{if } v(t) = 0, \\ -f(-v(t), \gamma(t)), & \text{if } v(t) < 0; \end{cases} \quad (4.16)$$

- if $\delta(\gamma(t)) = 1$, then

$$f(v(t), \gamma(t)) = \begin{cases} z_0, & \text{if } v(t) > \frac{1}{2}z_0, \\ 0, & \text{if } 0 \leq v(t) \leq \frac{1}{2}z_0, \\ -f(-v(t), \gamma(t)), & \text{if } v(t) < 0. \end{cases} \quad (4.17)$$

There is no loss of generality by choosing the same z_0 for every $\gamma(t) \in \Theta$; see Lemma 2.1 in [67]. For a logarithmic quantizer, it is easy to verify that

$$\eta_f(i) = -2/\ln \rho(i) \quad (4.18)$$

for every $i \in \Theta$. Thus, the coarser the quantizer for mode i , the smaller the $\eta_f(i) \in \mathbb{R}_+ \cup \{\infty\}$ and $\rho(i) \in [0, 1]$, or equivalently the larger the $\delta(i) \in [0, 1]$.

Since we mainly focus on global stabilization in this subsection, we let $w(t) \equiv 0$ without loss of generality. The next assumption is essential to the existence of an optimal memoryless quantization strategy in the MSQ stability sense.

Assumption 4.1.1. (a). *The system (4.1) is not mean square stable with $z(t) \equiv 0$ but can be mean square stabilized via a linear state feedback law:*

$$z(t) = \bar{K}_{\gamma(t)} x(t). \quad (4.19)$$

(b). For any $i_1, i_2, i_3 \in \Theta$ and $t \geq d$,

$$\Pr\{\theta(t) = i_1, \gamma(t+1) = i_2 | x_0^t, \gamma_0^{t-1}, \gamma(t) = i_3\} = \Pr\{\theta(t) = i_1, \gamma(t+1) = i_2 | \gamma(t) = i_3\}. \quad (4.20)$$

Moreover, the conditional probability on the right-hand side of (4.20), denoted by $q_{i_1 i_2 i_3}$, is constant over time and known to the controller/quantizer.

Remark 4.1.3. Assumption 4.1.1(a) clearly avoids triviality and imposes a necessary restriction for ensuring the solvability of the stabilization problem. A systematic way to find a stabilizing state feedback law for an MJLS can be found in [118]. Assumption 4.1.1(b) facilitates an explicit evaluation of (4.8) and covers several situations as collected in Section 4.1.2.

As the first result of this chapter, it will be shown that for a fixed set of $P_i > 0, Q_i > 0, i = 0, 1, \dots, N$, the coarsest quantization in the sense of MSQ stability can be approached by a linear state feedback law and a logarithmic quantizer. To this end, for every $i \in \Theta$, let us define the row vector $\mathbf{a}_i$, the matrix F_i , and two scalars $b_i, \delta_m(i)$ as

$$\mathbf{a}_i \triangleq \sum_{i_1 \in \Theta, i_2 \in \Theta} [q_{i_1 i_2 i} B_{i_1}' P_{i_2} A_{i_1}], \quad (4.21)$$

$$F_i \triangleq \sum_{i_1 \in \Theta, i_2 \in \Theta} [q_{i_1 i_2 i} A_{i_1}' P_{i_2} A_{i_1}] \geq 0, \quad (4.22)$$

$$b_i \triangleq \sum_{i_1 \in \Theta, i_2 \in \Theta} [q_{i_1 i_2 i} B_{i_1}' P_{i_2} B_{i_1}] \geq 0, \quad (4.23)$$

$$\delta_m(i) \triangleq \begin{cases} \infty, & \text{if } b_i = 0, \\ \frac{1}{\sqrt{K_{mi} M_i^{-1} K_{mi}'}}, & \text{otherwise,} \end{cases} \quad (4.24)$$

where

$$K_{mi} \triangleq -\frac{\mathbf{a}_i}{b_i}, \quad (4.25)$$

$$M_i \triangleq \frac{\mathbf{a}_i' \mathbf{a}_i}{b_i^2} - \frac{F_i - P_i + Q_i}{b_i}. \quad (4.26)$$

Theorem 4.1.1. *Consider the MSQ stabilization with a given set of $P_i > 0, Q_i > 0, i = 0, 1, \dots, N$ in (4.8) for the system (4.1) using quantized state feedback (4.3)-(4.4). Then, under Assumption 4.1.1, the smallest C_f defined in (4.11) can be approached by a linear state feedback law $v(t) = K_{\gamma(t)}x(t)$ and a logarithmic quantizer (4.15)-(4.17) with controller and quantizer parameters chosen as below:*

$$K_i = \begin{cases} 0, & \text{if } \delta_m(i) > 1, \\ K_{mi}, & \text{otherwise,} \end{cases} \quad \delta(i) = \begin{cases} 1, & \text{if } \delta_m(i) > 1, \\ \delta_m(i), & \text{otherwise.} \end{cases}$$

Proof. Suppose that $\gamma(t) = i$ for any $i \in \Theta$, and drop the time index t when no confusion is caused. Then, for $t \geq d$ and the system (4.1) with $w(t) \equiv 0$, we have

$$\begin{aligned} & \nabla V(x, i) \\ &= \sum_{i_1 \in \Theta, i_2 \in \Theta} [q_{i_1 i_2 i} (A_{i_1} x + B_{i_1} z)' P_{i_2} (A_{i_1} x + B_{i_1} z)] - x' P_i x \\ &= b_i u^2 + 2\mathbf{a}_i x u + x' (F_i - P_i) x. \end{aligned} \tag{4.27}$$

For Case 1: $b_i = 0$. Based on the definition of b_i , it is direct to get $q_{i_1 i_2 i} B_{i_1}' = 0$ for any $i_1, i_2 \in \Theta$ since $P_{i_2} > 0$, which further implies that $\mathbf{a}_i = 0$. The MSQ stabilization guarantees that $F_i - P_i + Q_i < 0$, and thus $K_i = 0$, i.e., $z = 0$ can be adopted, which renders $\eta_f(i) = 0$. In this situation, we can set $\delta_m(i) = \infty$ without loss of generality.

For Case 2: $b_i \neq 0$. From (4.27), it holds that

$$\nabla V(x, i) + x' Q_i x = \{-x' M_i x + (z - K_{mi} x)^2\} b_i,$$

and therefore the MSQ stabilization ensures that $M_i > 0$. Then, $\nabla V(x, i) < -x' Q_i x, \forall x \neq 0$, if and only if $z = f(v, i) \in (z_1(i), z_2(i))$, where

$$z_1(i) = K_{mi} x - \sqrt{x' M_i x}, \quad z_2(i) = K_{mi} x + \sqrt{x' M_i x}.$$

By applying the orthogonal decomposition method, $M_i^{1/2}x$ can be decomposed into

$$M_i^{1/2}x = \alpha(i)M_i^{-1/2}K'_{mi} + \beta(i), \quad (4.28)$$

where $\alpha(i)$ is a scalar, and the vector $\beta(i)$ is orthogonal to $M_i^{-1/2}K'_{mi}$. Therefore, $z_1(i)$, $z_2(i)$ can be rewritten with respect to the new coordinate system (4.28) as

$$\begin{aligned} z_1(i) &= \frac{\alpha(i)}{\delta_m(i)^2} - \sqrt{\frac{\alpha(i)^2}{\delta_m(i)^2} + \beta'(i)\beta(i)}, \\ z_2(i) &= \frac{\alpha(i)}{\delta_m(i)^2} + \sqrt{\frac{\alpha(i)^2}{\delta_m(i)^2} + \beta'(i)\beta(i)}. \end{aligned}$$

Moreover, if $\delta_m(i) > 1$, then we can again choose $z = 0$ similarly to Case 1, since $z = 0$ belongs to the interval $(z_1(i), z_2(i))$; if $\delta_m(i) \leq 1$, then it can be proved that the optimal quantization strategy with the smallest $\eta_f(i)$ for mode i is logarithmic, as shown in (4.16)-(4.17), with $\delta(i) = \delta_m(i)$ [67].

By combining the above two cases and taking note of the property (4.12), we can conclude that the logarithmic quantizer stated in this theorem can achieve the smallest C_f for a given set of $P_i > 0, Q_i > 0, i = 0, 1, \dots, N$. The technique in the proof of Lemma 2.1 in [68] can still be used to prove that a linear state feedback law $v(t) = K_{\gamma(t)}x(t)$ is sufficient to obtain the coarsest quantization for Case 2 with $\delta_m(i) \leq 1$, while, for Case 2 with $\delta_m(i) > 1$ and Case 1, the argument is trivial, since $K_i = 0$ is adopted. This completes the proof. $\square$

The quantization error of a logarithmic quantizer is given by

$$e(t) = z(t) - v(t) = f(v(t), \gamma(t)) - v(t) = \Delta(v(t), \gamma(t))v(t), \quad (4.29)$$

where $\Delta(v(t), \gamma(t)) \in [-\delta(\gamma(t)), \delta(\gamma(t))]$. The closed-loop quantized feedback system with $v(t) = K_{\gamma(t)}x(t)$ becomes the following uncertain MJLS:

$$x(t+1) = A_{\theta(t)}x(t) + B_{\theta(t)}(1 + \Delta(K_{\gamma(t)}x(t), \gamma(t)))K_{\gamma(t)}x(t). \quad (4.30)$$

Before optimizing the overall coarseness with respect to all possible $P_i > 0$, $Q_i > 0$, $i \in \Theta$ such that (4.30) is MSQ stable in part (b) of the theorem that follows, we note that the uncertainty in (4.30) is a nonlinear function of $K_{\gamma(t)}x(t)$, which cannot be handled directly. The validity of the sector bound approach proved in part (a) of the next theorem shows that quantized stabilization is equivalent to the robust MSQ stabilization of an uncertain system with time-varying uncertainties.

Theorem 4.1.2. (a). *Given a logarithmic quantizer (4.15)-(4.17) with a set of fixed $\delta(i) \in [0, 1]$, $i = 0, 1, \dots, N$, the system (4.1) under Assumption 4.1.1 is MSQ stabilizable via quantized linear state feedback if and only if the following uncertain system*

$$x(t+1) = A_{\theta(t)}x(t) + B_{\theta(t)}(1 + \Delta(\gamma(t)))v(t) \quad (4.31)$$

is robustly MSQ stabilizable for uncertainty $\Delta(\gamma(t)) \in [-\delta(\gamma(t)), \delta(\gamma(t))]$ via a linear state feedback law $v(t) = K_{\gamma(t)}x(t)$.

(b). *Under Assumption 4.1.1, the optimal overall coarseness for the system (4.1) to be MSQ stabilizable via quantized linear state feedback can be obtained by the following optimization:*

$$\underline{C}_f \triangleq \min_{S_i > 0, W_i > 0, Y_i, \tau(i) > 0, \forall i \in \Theta} C_f$$

subject to the constraint

$$\begin{bmatrix} -S_i & S_i & Y'_i & \Phi_{0i} & \Phi_{1i} & \cdots & \Phi_{Ni} \\ * & -W_i & 0 & 0 & 0 & \cdots & 0 \\ * & * & -\tau(i) & 0 & 0 & \cdots & 0 \\ * & * & * & \Xi_{0i} & 0 & \cdots & 0 \\ * & * & * & * & \Xi_{1i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & * & * & \Xi_{Ni} \end{bmatrix} < 0, \quad (4.32)$$

where

$$\begin{aligned} \Phi_{ji} &= [\sqrt{q_{j0i}}(S_i A'_j + Y'_i B'_j) \quad \sqrt{q_{j1i}}(S_i A'_j + Y'_i B'_j) \quad \cdots \quad \sqrt{q_{jNi}}(S_i A'_j + Y'_i B'_j)], \\ \Xi_{ji} &= \begin{bmatrix} -S_0 + \tau(i)\delta(i)^2 q_{j0i} B_j B'_j & \tau(i)\delta(i)^2 \sqrt{q_{j0i} q_{j1i}} B_j B'_j & \cdots \\ * & -S_1 + \tau(i)\delta(i)^2 q_{j1i} B_j B'_j & \cdots \\ \vdots & \vdots & \ddots \\ * & * & \cdots \\ \tau(i)\delta(i)^2 \sqrt{q_{j0i} q_{jNi}} B_j B'_j \\ \tau(i)\delta(i)^2 \sqrt{q_{j1i} q_{jNi}} B_j B'_j \\ \vdots \\ -S_N + \tau(i)\delta(i)^2 q_{jNi} B_j B'_j \end{bmatrix}, \end{aligned}$$

for all $i, j = 0, 1, \dots, N$. Moreover, a logarithmic quantizer (4.15)-(4.17) and a linear state feedback law $v(t) = K_{\gamma(t)} x(t)$ are sufficient to achieve the $\underline{C}_f$, and a set of suitable state feedback gains is given by $K_i = Y_i S_i^{-1}$, $i = 0, 1, \dots, N$.

Proof. (a). Again, suppose that $\gamma(t) = i$, $i \in \Theta$. For the system (4.30), we have

$$\begin{aligned} &\nabla V(x, i) \\ &= \sum_{i_1 \in \Theta, i_2 \in \Theta} [q_{i_1 i_2 i} (A_{i_1} x + B_{i_1} (1 + \Delta(K_i x, i)) K_i x)' P_{i_2} (A_{i_1} x + B_{i_1} (1 + \Delta(K_i x, i)) K_i x)] \\ &\quad - x' P_i x. \end{aligned} \tag{4.33}$$

Following a similar proof of Lemma 2.2 in [68], it can be shown that, for all $x \neq 0$, the inequality $\nabla V(x, i) < -x' Q_i x$ is equivalent to

$$\begin{aligned} &x' P_i x - x' Q_i x > \\ &\sum_{i_1 \in \Theta, i_2 \in \Theta} [q_{i_1 i_2 i} (A_{i_1} x + B_{i_1} (1 + \Delta(i)) K_i x)' P_{i_2} (A_{i_1} x + B_{i_1} (1 + \Delta(i)) K_i x)], \end{aligned} \tag{4.34}$$

where $\Delta(i)$ is defined as in (4.31) with $\gamma(t) = i$. This kind of equivalence is true for any $i \in \Theta$, and thus, by Definition 4.1.1, (4.34) is the condition for the robust MSQ

stabilization of (4.31).

(b). The constraint (4.32) is obtained by using Schur's complement documented in Lemma A.0.1 on the inequality (4.34) and taking

$$S_i = P_i^{-1}, \quad W_i = Q_i^{-1}, \quad Y_i = K_i S_i, \quad (4.35)$$

where $\tau(i) > 0$ is the scaling variable. From the proof in part (a), we see that the quantized stabilization for (4.30) and the robust stabilization for (4.31) can share the same set of P_i , Q_i , $i = 0, 1, \dots, N$, as well as the same set of feedback gains. The result then follows directly from Theorem 4.1.1. $\square$

As we can see from (4.14) and (4.18), the overall coarseness C_f of a logarithmic quantizer can also be defined in terms of the set of $\delta(i)$ or $\rho(i)$, $i = 0, 1, \dots, N$. For example, one possible choice is

$$C_{f1} \triangleq -\min_{i \in \Theta} \{\delta(i)\}, \quad (4.36)$$

which captures the worst-case mode with the smallest sector bound (equivalently the largest mode quantization density) among all system modes. In this case, the optimization in part (b) of Theorem 4.1.2 becomes $\max_{S_i, W_i, Y_i, \tau(i)} \delta$ over (4.32) with $\delta(i) = \delta$ for every $i \in \Theta$. Moreover, suppose that $\theta(t)$ is driven by an ergodic Markov chain which admits a limiting probability distribution $\{\bar{\pi}_i; \bar{\pi}_i > 0, i \in \Theta\}$. Then, another choice for the overall coarseness can be

$$C_{f2} \triangleq -\sqrt{\sum_{i=0}^N \bar{\pi}_i \delta(i)^2}, \quad (4.37)$$

which characterizes the weighted average quantization performance. Since, for any fixed set of $\delta(i)$, (4.32) is convex in S_i, W_i, Y_i and $\tau(i)$, $\underline{C}_f$ can be obtained by searching the space of $\delta(i)$, $i = 0, 1, \dots, N$. Note that such a method may be time-consuming especially when the number of system modes is large.

4.1.2 Special Schemes

The results presented in Section 4.1.1 can be applied to different scenarios as follows.

- Scheme I [Current mode observation (CMO)]: $\gamma(t) = \theta(t)$. In this situation,

$$q_{i_1 i_2 i_3} = \begin{cases} \pi_{i_3 i_2}, & \text{if } i_1 = i_3, \\ 0, & \text{otherwise.} \end{cases}$$

- Scheme II [One-step-delayed mode observation (OSDMO)]: $\gamma(t) = \theta(t-1)$. In this situation,

$$q_{i_1 i_2 i_3} = \begin{cases} \pi_{i_3 i_2}, & \text{if } i_1 = i_2, \\ 0, & \text{otherwise.} \end{cases}$$

- Scheme III [Mode-independent manner]: $\gamma(t) \equiv \phi$ with ϕ representing a void signal. Assumption 4.1.1(b) is reduced to that

$$\Pr\{\theta(t) = i_1 | x_0^t\} = \Pr\{\theta(t) = i_1\}, \quad \forall i_1 \in \Theta,$$

is constant over time and known to the controller/quantizer, which is true if the underlying Markov chain is an i.i.d. process, i.e., $\pi_{ij} = \bar{\pi}_j$, for every $i, j \in \Theta$ [123], or the Markov chain is ergodic and the initial distribution π is equal to its limiting distribution.

4.1.3 Mode Estimation

When the system mode is not directly observed at the controller/quantizer, one can adopt the mode-independent strategy (Scheme III) in Section 4.1.2. However, Scheme III may be conservative since no mode information is taken into consideration. Alternatively, we can try to estimate the mode process at the controller. First of all, a special case of mode estimation is given.

- Scheme IV [Mode estimation without process noise]: $w(t) \equiv 0$ and for any $x \neq 0, i_1, i_2, i_3 \in \Theta, i_1 \neq i_2$,

$$A_{i_1}x + B_{i_1}f(g(x, i_3), i_3) \neq A_{i_2}x + B_{i_2}f(g(x, i_3), i_3). \quad (4.38)$$

In this situation, the next algorithm:

$$\gamma(t) = \hat{\theta}(t-1) = \arg \min_{i \in \Theta} \|x(t) - A_i x(t-1) - B_i f(g(x(t-1), \gamma(t-1)), \gamma(t-1))\|^2$$

with arbitrary $\gamma(0) \in \Theta$ can ensure $\gamma(t) = \theta(t-1)$ for all $t \geq 1$. Thus, the result on OSDMO (Scheme II) can be applied directly.

With nonzero process noise $w(t)$, one can still estimate the previous mode $\theta(t-1)$ at time t based on x_0^t, γ_0^{t-1} and the closed-loop system model (4.5). Assume that $x(0)$ is white Gaussian and $w(t)$ is zero-mean white Gaussian. Denote $\Omega(x, \mu, \Sigma)$ as the vector-valued Gaussian probability density function with mean vector μ and covariance matrix Σ . Suppose that the initial distribution π and the transition probability matrix Π of the underlying Markov process as well as the set of covariance matrices $\Sigma_i, i = 0, 1, \dots, N$, of the process noise $w(t)$ are exactly known to the controller/quantizer. The next algorithm gives an estimate of $\theta(t-1)$ by maximizing the probability

$$L(\theta(t-1)) \triangleq \Pr\{\theta(t-1) | x_0^t, \gamma_0^{t-1}\} \quad (4.39)$$

with respect to $\theta(t-1) \in \Theta$.

Algorithm 4.1.1. *A recursive procedure to compute $\gamma(t) = \hat{\theta}(t-1)$ at time $t \geq 1$ for the quantized system (4.5), such that L defined in (4.39) is maximized, is stated as follows.*

- Choose $\gamma(0)$ as an arbitrary element in Θ and set $z(0) = 0$.
- For $t = 1$, $\gamma(1) = \arg \max_{i \in \Theta} [a(i, 1)]$ with

$$a(i, 1) = \pi_i \Omega(x(1), A_i x(0), \Sigma_i).$$

(c). For $t \geq 2$, $\gamma(t) = \arg \max_{i \in \Theta} [a(i, t)]$, where $a(i, t)$ can be computed iteratively as

$$a(i, t) = \sum_{j \in \Theta} a(j, t-1) \pi_{ji} \Omega(x(t), A_i x(t-1) + B_i f(g(x(t-1), \gamma(t-1)), \gamma(t-1)), \Sigma_i).$$

The above algorithm is modified from the well-known Viterbi algorithm [124, 125]. The optimality criterion of the standard Viterbi algorithm [124], different from (4.39), is to find the single best mode sequence. Moreover, the maximum likelihood estimation can be used to iteratively update the parameters such as π, Π, Σ_i , if some or all of them are unknown to the controller/quantizer. For more complicated cases, e.g., partial state observation with corrupted noise, algorithms for mode estimation may be constructed based on more sophisticated hidden Markov model; see, e.g., [126, 127].

Remark 4.1.4. Note that, for direct mode observation $\gamma(t) = \theta(t-d)$ with $d \geq 2$, and general cases of Algorithm 4.1.1, the probability on the left-hand side of equation (4.20) becomes a function of state x and thus is dynamic, which renders an optimal memoryless quantization strategy impossible. In this situation, dynamic or state-dependent quantization strategy would be required for an optimal design.

The next numerical example demonstrates the usefulness of Algorithm 4.1.1.

Example 4.1.1. Consider an MJLS (4.1) with $A_0 = 1.2$, $A_1 = -1.2$, $B_0 = B_1 = 1$, and transition probability matrix

$$\Pi_1 = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}.$$

First, suppose that direct mode observation is available at the controller/quantizer. Then, for CMO (Scheme I), the smallest allowable C_{f1} defined in (4.36) is -0.8333 with $K_0 = -1.2$, $K_1 = 1.2$; for OSDMO (Scheme II), the smallest achievable C_{f1} is -0.7229 with $K_0 = 0.9600$, $K_1 = -0.9600$.

Second, if the system mode is not observed at the controller/quantizer, then we can easily verify that the mode-independent strategy (Scheme III) cannot stabilize the system. Furthermore, assume that the covariance of $w(t)$ is given by $W_0 = W_1 = 1$ and the initial state $x(0)$ is Gaussian distributed with mean 20 and variance 10. Then, the first 30 mode estimates for one sample of simulation using Algorithm 4.1.1 are shown in Figure 4.3. The parameters of the controller and quantizer are chosen as in OSDMO: $K_0 = 0.9600$, $K_1 = -0.9600$, $\delta(0) = \delta(1) = 0.7229$. Figure 4.4 further gives the empirical norm of state by averaging 10000 Monte Carlo simulations. As we can see from Figures 4.3 and 4.4, there exist some mode estimation errors, but the error rate is low, and the empirical norm of state by applying Algorithm 4.1.1 is convergent.

Figure 4.3: Original and estimated mode processes for one sample of simulation using Algorithm 4.1.1.

4.2 Application to Quantized Stabilization over Lossy Channels

We refer to TCP-like channel if there exist acknowledgments that testify successful transmissions or not at the transmitter side, and to UDP-like channel otherwise. As

Figure 4.4: The empirical norm of state by averaging 10000 Monte Carlo simulations using Algorithm 4.1.1.

mentioned in Remark 1.2.1, the packet acknowledgment is assumed to be available with one step delay. See [4,47] for more details on TCP-like and UDP-like protocols.

Consider a quantized feedback NCS in Figure 4.5, where the LTI plant is described in a discrete-time form as

$$x(t+1) = Ax(t) + Bu(t), \quad (4.40)$$

which may be obtained through discretization of a continuous-time system. The pair (A, B) is assumed to be stabilizable. We note that the TCP-like protocol falls into the OSDMO scheme, whereas the UDP-like protocol falls into the mode-independent pattern. Next, two stochastic models for the lossy channel will be discussed.

Figure 4.5: Quantized control over an input lossy channel.

4.2.1 Binary Dropouts Model

Suppose that a zero-control strategy is adopted in dealing with the binary packet losses over the network. The network is modeled by

$$u(t) = \xi(t)z(t), \quad (4.41)$$

where $\xi(t) \in \Theta = \{0, 1\}$ represents the loss (with $\xi(t) = 0$) or arrival (with $\xi(t) = 1$) of the packet at time t . The system as a combination of the network and the plant can be modeled as an MJLS (4.1) with

$$A_0 = A_1 = A, \quad B_0 = 0, \quad B_1 = B. \quad (4.42)$$

For TCP-like channel with $\gamma(t) = \xi(t - 1)$, we assume that $\xi(t)$ is driven by a Markov chain with transition probability matrix

$$\Pi = \begin{bmatrix} 1 - q & q \\ p & 1 - p \end{bmatrix}. \quad (4.43)$$

In this situation, the OSDMO result (Scheme II) is applicable. Note that the CMO result (Scheme I) is of theoretical importance in the quantization of MJLSs but may not be practical in the NCS depicted in Figure 4.5, since it is unrealistic for the quantizer to know whether the current packet will be lost or not before the packet is sent over the network.

For the UDP-like protocol, we can easily verify that the inequality (4.38) is true, and thus Scheme IV can be used directly when $w(t) \equiv 0$. If $\xi(t)$ is assumed to be an i.i.d. random variable with

$$\Pr(\xi(t) = 0) = \alpha, \quad \Pr(\xi(t) = 1) = 1 - \alpha, \quad (4.44)$$

i.e., the NCS adopts an unreliable network with packet-loss rate α , then Scheme III

is applicable, and the inequality (4.32) is reduced to the following modified Riccati inequality:

$$A'PA - P + Q - (1 - \alpha)(1 - \delta^2)A'PB(B'PB)^{-1}B'PA < 0. \quad (4.45)$$

Based on Lemma 5.4 in [4], the condition

$$(1 - \alpha)(1 - \delta^2) > 1 - \frac{1}{\mathcal{M}(A)^2}$$

can ensure the existence of $P > 0$ to (4.45), where $\mathcal{M}(A)$ denotes the Mahler measure of A . It is easy to check that the above result is consistent with Theorem 2.1 of [8], which can be seen as a special case of Theorem 4.1.2 (b).

Example 4.2.1. Consider the NCS in Figure 4.5, where the plant (4.40) is borrowed from [8] with

$$A = \begin{bmatrix} 0 & 1 \\ 1.8 & -0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (4.46)$$

Suppose that the transition probability matrix (4.43) is given by

$$\Pi_2 = \begin{bmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{bmatrix}. \quad (4.47)$$

For TCP-like channel, the smallest allowable C_{f1} defined in (4.36) is -0.2474 with

$$K_0 = K_1 = [-1.8000 \quad 0.5991]$$

according to Scheme II. The limiting transition probability of (4.47) is given by

$$\Pi_3 = \lim_{t \rightarrow \infty} \Pi_2^t = \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & 0.75 \end{bmatrix}. \quad (4.48)$$

Therefore, for UDP-like channel with initial distribution $\pi_1 = [0.25 \quad 0.75]$, the smallest achievable C_{f1} is -0.2796 with $K_\phi = [-1.8000 \quad 0.6750]$ based on Scheme III.

4.2.2 Bounded Dropouts Model

Suppose that the lossy channel undergoes bounded Markovian dropouts as in Section 3.1, and a zero-order hold is used in dealing with the bounded packet losses. The model of the network is given by

$$u(t) = \xi(t)z(t) + (1 - \xi(t))z(t-1), \quad \xi(t) \in \{0, 1\}.$$

Then, from Theorem 9 of [61], the system as a combination of the network and the plant in Figure 4.5 can be modeled as an MJLS (4.1) with

$$A_i = A^{i+1}, \quad B_i = \sum_{r=0}^i A^r B, \quad i \in \Theta = \{0, 1, \dots, N\}, \quad (4.49)$$

where N is the number of maximum successive packet losses.

Example 4.2.2. Let the plant in Figure 4.5 be given by (4.40) with [61]

$$A = \begin{bmatrix} 0.6065 & 0 & -0.2258 \\ 0.3445 & 0.7788 & -0.0536 \\ 0 & 0 & 1.2840 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0582 \\ -0.0093 \\ 0.5681 \end{bmatrix}. \quad (4.50)$$

The number of maximum consecutive packet dropouts of the underlying network is 4 and the transition probability matrix is given by:

$$\Pi_4 = \begin{bmatrix} 0.5 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.5 & 0.3 & 0 & 0 \\ 0 & 0.2 & 0.5 & 0.3 & 0 \\ 0 & 0 & 0.2 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.1 & 0.2 & 0.5 \end{bmatrix}. \quad (4.51)$$

For TCP-like channel, the smallest allowable C_{f1} is -0.3470 with

$$K_0 = [0 \quad 0 \quad -0.7783], \quad K_1 = [0 \quad 0 \quad -1.0614], \quad K_2 = [0 \quad 0 \quad -0.8321],$$

$$K_3 = [0 \ 0 \ -0.7452], \quad K_4 = [0 \ 0 \ -0.7201].$$

For UDP-like channel with initial distribution

$$\pi_2 = [0.1101 \ 0.1878 \ 0.2718 \ 0.2552 \ 0.1751],$$

the smallest achievable C_{f1} is -0.3700 with $K_\phi = [0 \ 0 \ -0.7936]$.

4.2.3 Extension to Output Feedback

If the plant state is not fully observed at the controller in Figure 4.5, then output feedback should be adopted. Assume that the plant is described as

$$\begin{aligned} x(t+1) &= Ax(t) + Bz(t), \\ y(t) &= Cx(t), \end{aligned} \tag{4.52}$$

and the measured output $y(t) \in \mathbb{R}^p$ is available at the controller. The proposition that follows can be proved similarly to Theorem 3.1 in [68].

Proposition 4.2.1. *Consider the plant (4.52) to be controlled over a lossy channel via quantized output feedback. Assume that the pair (A, C) is observable. Under TCP-like protocol, the optimal overall coarseness for MSQ stabilization by state feedback can be achieved by output feedback. Moreover, one possible deadbeat-observer-based controller can be constructed as*

$$\begin{aligned} \hat{x}(t+1) &= A\hat{x}(t) + Bz(t) + \hat{L}(y(t) - C\hat{x}(t)), \\ v(t) &= K_{\gamma(t)}x(t), \\ z(t) &= f(v(t), \gamma(t)), \end{aligned} \tag{4.53}$$

where $K_{\gamma(t)}$ and $f(\cdot, \cdot)$ are chosen according to the OSDMO scheme, and $\hat{L}$ is any deadbeat observer gain.

4.3 Summary

Motivated by quantized feedback control over lossy channels, the quantized stabilization problem for MJLSs has been investigated in this chapter. It has been shown that, for a single-input linear system with Markovian jump parameters, a mode-dependent logarithmic quantizer is still optimal in the MSQ stability sense, and the sector bound approach again provides a non-conservative way for studying the corresponding quantized state feedback stabilization problem. In addition, a recursive algorithm has been presented to estimate the unknown mode process at the controller side. The above results have been applied in the quantized stabilization of NCSs over lossy channels under either TCP-like or UDP-like protocol.

Chapter 5

Filtering over Fading Channels

This chapter focuses on the network requirement for ensuring the stability of a remote Kalman filter with faded measurements, where the fading channels undergo transmission failure and signal fluctuation simultaneously. The networked estimation problem is formulated in Section 5.1. In Section 5.2, some preliminary results on MAROs and MLOs are derived for later developments. After that, necessary and sufficient conditions on the network for the stability of the mean error covariance matrix of the remote Kalman filter are presented in terms of the unstable poles of the plant. Lower and upper bounds for the mean error covariance matrix are provided in the form of a modified Lyapunov iteration and a modified Riccati iteration, respectively. A simulation is given in Section 5.3 to demonstrate the results, and Section 5.4 concludes the chapter.

5.1 Problem Formulation

Consider a discrete-time networked system as shown in Figure 5.1, where an unreliable network with fading channel(s) is placed between the plant and the filter.

Figure 5.1: Filtering over output fading channel(s).

The plant is assumed to be linear with a state-space realization:

$$\begin{aligned} x(t+1) &= Ax(t) + w(t), \\ y(t) &= Cx(t) + v(t), \end{aligned} \quad (5.1)$$

where $x(t) \in \mathbb{R}^n$ is the state, and $y(t) \in \mathbb{R}^p$ is the measured output. The process noise $w(t) \in \mathbb{R}^n$ and the measurement noise $v(t) \in \mathbb{R}^p$ are white process with zero means and covariance matrices $Q > 0$ and $R_v > 0$, respectively. Without loss of generality, assume that A is unstable, (A, C) is detectable, and $C = [C'_1 \ C'_2 \ \cdots \ C'_p]'$ has full-row rank.

The model of fading channel(s) is given by

$$z(t) = \xi(t)y(t) + n(t), \quad (5.2)$$

where $n(t) \in \mathbb{R}^p$ denotes the channel additive noise, which is white with zero mean and covariance matrix $R_n > 0$, and $\xi(t) \in \mathbb{R}^{p \times p}$ represents the channel fading, which is assumed to have the diagonal structure:

$$\xi(t) = \text{diag}\{\xi_1(t), \xi_2(t), \dots, \xi_p(t)\}. \quad (5.3)$$

Remark 5.1.1. *Different from the general MIMO communication system with multiple receiving and transmitting antennas [128], the input and the output of the network (5.2)-(5.3) are assumed to have the same dimension since our main focus is to investigate the effect of fading on the stability of the remote filter rather than recovery of the transmitted information.*

It is assumed that the network experiences both transmission failure and signal

fluctuation as

$$\xi_i(t) = \Omega_i(t)\Upsilon_i(t), \quad (5.4)$$

where $\Omega_i(t)$ is 0/1 valued (0 for “failure”, 1 for “success”), white and identically distributed with probability distribution

$$\Pr\{\Omega_i(t) = 0\} = \alpha_i, \Pr\{\Omega_i(t) = 1\} = 1 - \alpha_i, \quad 0 \leq \alpha_i < 1, \quad (5.5)$$

and $\Upsilon_i(t)$ is nonzero, white and identically distributed with a given continuous distribution function such as Nakagami. Furthermore, let

$$\begin{aligned} \Sigma_\xi &\triangleq [\mathcal{E}\{\xi_i(t)\xi_j(t)\} - \mathcal{E}\{\xi_i(t)\}\mathcal{E}\{\xi_j(t)\}]_{i,j=1,2,\dots,p}, \\ \Pi_\xi &\triangleq \mathcal{E}\{\text{diag}\{\xi_1(t), \xi_2(t), \dots, \xi_p(t)\}\}, \\ \Omega(t) &= \text{diag}\{\Omega_1(t), \Omega_2(t), \dots, \Omega_p(t)\}, \\ \Upsilon(t) &= \text{diag}\{\Upsilon_1(t), \Upsilon_2(t), \dots, \Upsilon_p(t)\}, \end{aligned}$$

and $\Sigma_\Omega, \Sigma_\Upsilon, \Pi_\Omega, \Pi_\Upsilon$ are defined similarly to Σ_ξ, Π_ξ . Note that the fading experienced by different channels may or may not be correlated depending on whether a non-orthogonal or an orthogonal access scheme [3] is adopted. Further assume that the distribution of $\xi(t)$ is known and $\{w(t)\}_{t \geq 0}, \{v(t)\}_{t \geq 0}, \{n(t)\}_{t \geq 0}, \{\Omega(t)\}_{t \geq 0}, \{\Upsilon(t)\}_{t \geq 0}$ are uncorrelated with each other.

Remark 5.1.2. For Kalman filtering with faded measurements studied in [5], in order to apply the results on Kalman filtering with random parameters presented in [116, 117], one of the technical assumptions is that the channel fading $\xi_i(t)$ is positive almost surely for all time $t \geq 0$ and every channel $i = 1, 2, \dots, p$, which excludes the existence of possible transmission failure. The model (5.2)-(5.4) in this chapter considers transmission failure and signal fluctuation at the same time, for which the results in [116, 117] cannot be applied.

As mentioned in Section 1.2, we assume that the exact value of $\xi(t)$ is known at the filter at time t . In this situation, it is easy to prove that the time-varying

Kalman filter is optimal in the MMSE sense. The plant together with the network can be rewritten as

$$\begin{aligned} x(t+1) &= Ax(t) + w(t), \\ z(t) &= \xi(t)Cx(t) + \xi(t)v(t) + n(t). \end{aligned} \quad (5.6)$$

Denote $\tilde{C}(t) = \xi(t)C$ and $\tilde{v}(t) = \xi(t)v(t) + n(t)$. The state prediction error covariance matrix $P(t)$ can be computed by the following random Riccati iteration:

$$P(t+1) = AP(t)A' + Q - AP(t)\tilde{C}(t)' \left[\tilde{C}(t)P(t)\tilde{C}(t)' + \tilde{R}(t) \right]^{-1} \tilde{C}(t)P(t)A' \quad (5.7)$$

with $\tilde{R}(t) = \xi(t)R_v\xi(t) + R_n$ representing the covariance matrix of $\tilde{v}(t)$. Since $P(t+1)$ is a function of $\xi(t)$, it is random in nature and thus can only be computed for a given realization of ξ_0^t . Therefore, we turn to consider the stability of its expectation defined as follows.

Definition 5.1.1. *The Kalman filter over the fading channel(s) (5.2) or the random Riccati iteration (5.7) is said to be mean covariance stable if $\sup_{t \geq 0} \mathcal{E}\{P(t)\} < \infty$, where the expectation is taken over the channel fading sequence ξ_0^t .*

5.2 Main Results

5.2.1 Preliminaries

Define an MARO and an MLO as below:

$$g(X) \triangleq AXA' + Q - AXC'\Pi[\Pi CXC'\Pi + \Sigma \odot (CXC') + R]^{-1} \Pi CXA', \quad (5.8)$$

$$h(X) \triangleq \alpha AXA' + Q, \quad (5.9)$$

where $\Sigma \geq 0$, $R > 0$, $\Pi > 0$ is a diagonal matrix, and $\alpha \geq 0$ is a scalar. The next two lemmas present several properties of the MARO (5.8) and the MLO (5.9).

Lemma 5.2.1. *Let*

$$\begin{aligned} g_1(X) &\triangleq AXA' - AXC'\Pi[\Pi CXC'\Pi + \Sigma \odot (CXC')]^{-1} \Pi CXA' \\ \phi_1(L, X) &\triangleq (A + L\Pi C)X(A + L\Pi C)' + L[\Sigma \odot (CXC')]L', \\ \phi(L, X) &\triangleq \phi_1(L, X) + LRL' + Q. \end{aligned}$$

Then, the following properties hold.

(i) *For any $b \in \mathbb{R}$, $\phi_1(L, bX) = b\phi_1(L, X)$.*

(ii) *For any $X \geq 0$ and L , we have $h(X) \geq Q$, and*

$$0 \leq g_1(X) = \phi_1(L_1^*(X), X) \leq \phi_1(L, X), \quad (5.10)$$

$$Q \leq g(X) = \phi(L^*(X), X) \leq \phi(L, X), \quad (5.11)$$

with

$$L_1^*(X) = -AXC'\Pi[\Sigma \odot (CXC')]^{-1}, L^*(X) = -AXC'\Pi[\Sigma \odot (CXC') + R]^{-1}.$$

(iii) *For any $X_2 \geq X_1 \geq 0$, we have*

$$h(X_2) \geq h(X_1), \quad g_1(X_2) \geq g_1(X_1), \quad g(X_2) \geq g(X_1).$$

Proof. (i): Straightforward.

(ii): The inequality $h(X) \geq Q$ is obvious. It follows directly from the expressions of $L_1^*(X)$ and $L^*(X)$ that $g_1(X) = \phi(L_1^*(X), X)$ and $g(X) = \phi(L^*(X), X)$. Based on the property of Hadamard product in Lemma A.0.5, we have $\Sigma \odot (CXC') \geq 0$ since both Σ and CXC' are positive semidefinite. Thus, $\phi_1(L, X) \geq 0$ and $\phi(L, X) \geq Q$ for any $X \geq 0$ and L . Considering the partial derivatives $\partial\phi_1(L, X)/\partial L$ and $\partial\phi(L, X)/\partial L$, we can conclude that $\phi_1(L_1^*(X), X) \leq \phi_1(L, X)$ and $\phi(L^*(X), X) \leq \phi(L, X)$.

(iii): Observe that $h(X)$, $\phi_1(L, X)$ and $\phi(L, X)$ are affine in X . It is direct to see that $h(X_2) \geq h(X_1)$. We also have

$$\begin{aligned} g_1(X_1) &= \phi_1(L_1^*(X_1), X_1) \leq \phi_1(L_1^*(X_2), X_1) \leq \phi_1(L_1^*(X_2), X_2) = g_1(X_2), \\ g(X_1) &= \phi(L^*(X_1), X_1) \leq \phi(L^*(X_2), X_1) \leq \phi(L^*(X_2), X_2) = g(X_2), \end{aligned}$$

which completes the proof. $\square$

Lemma 5.2.2. *The following statements are equivalent.*

- i) *There exists $X > 0$ such that $X > g_1(X)$.*
- ii) *There exist $X > 0$ and L such that $X > \phi_1(L, X)$.*
- iii) *There exists $X > 0$ such that $X > g(X)$.*
- iv) *There exist $X > 0$ and L such that $X > \phi(L, X)$.*

Moreover, if any of the conditions i)-iv) holds, then the claim as below is true.

- v) *The sequence $\{X(t)\}_{t \geq 0}$ computed by $X(t+1) = g(X(t))$ with any initial condition $X(0) \geq 0$ is bounded from above and convergent as t approaches ∞ , i.e., $\lim_{t \rightarrow \infty} X(t) = \tilde{X}$, where $\tilde{X}$ is the unique positive-semidefinite fixed point of the MARO (5.8), i.e., $\tilde{X} = g(\tilde{X})$, $\tilde{X} \geq 0$.*

Proof. i) $\Leftrightarrow$ ii): If i) holds, then we can derive that $X > \phi_1(L, X)$ by setting $L = L_1^*(X)$. On the other hand, when ii) is true, we have $X > \phi_1(L, X) \geq \phi_1(L_1^*(X), X) = g_1(X)$.

i) $\Leftarrow$ iii): It follows directly by choosing the same X .

i) $\Rightarrow$ iii): Suppose that i) is true. By continuity, it holds that

$$X > AXA' - AXC'\Pi[\Pi CXC'\Pi + \Sigma \odot (CXC') + \gamma I]^{-1} \Pi CXA' + \beta I$$

for sufficiently small scalars $\beta > 0$ and $\gamma > 0$. Therefore, for any scalar $\delta > 0$, we have

$$\begin{aligned} \delta X &> \delta A X A' - \delta A X C' \Pi [\Pi C X C' \Pi + \Sigma \odot (C X C') + \gamma I]^{-1} \Pi C X A' + \delta \beta I \\ &= A(\delta X) A' - A(\delta X) C' \Pi [\Pi C(\delta X) C' \Pi + \Sigma \odot (C(\delta X) C') + \delta \gamma I]^{-1} \Pi C(\delta X) A' \\ &\quad + \delta \beta I. \end{aligned}$$

By selecting $\delta > 0$ such that $\delta \gamma I \geq R$ and $\delta \beta I \geq Q$ and letting $\hat{X} = \delta X > 0$, we obtain

$$\begin{aligned} \hat{X} &> A \hat{X} A' - A \hat{X} C' \Pi [\Pi C \hat{X} C' \Pi + \Sigma \odot (C \hat{X} C') + \delta \gamma I]^{-1} \Pi C \hat{X} A' + \delta \beta I \\ &\geq A \hat{X} A' - A \hat{X} C' \Pi [\Pi C \hat{X} C' \Pi + \Sigma \odot (C \hat{X} C') + R]^{-1} \Pi C \hat{X} A' + Q \\ &= g(\hat{X}), \end{aligned}$$

which completes the proof.

iii) $\Leftrightarrow$ iv): It can be proved similarly to the proof of the equivalence between i) and ii).

i) $\Rightarrow$ v): Assume that i) holds. In this situation, we can always choose $b_1 \in [0, 1)$, $b_2 \in (0, \infty)$ such that $g_1(X) \leq b_1 X$, $X(0) \leq b_2 X$, and $L_1^*(X) R L_1^*(X)' + Q \leq b_2 X$. It follows from Lemma 5.2.1 that

$$\begin{aligned} X(1) &= g(X(0)) = \phi(L^*(X(0)), X(0)) \\ &\leq \phi(L_1^*(X), X(0)) \\ &\leq \phi(L_1^*(X), b_2 X) \\ &= \phi_1(L_1^*(X), b_2 X) + L_1^*(X) R L_1^*(X)' + Q \\ &= b_2 g_1(X) + L_1^*(X) R L_1^*(X)' + Q \\ &\leq (b_1 b_2 + b_2) X, \\ X(2) &= g(X(1)) \end{aligned}$$

$$\begin{aligned}
&\leq g((b_1 b_2 + b_2)X) \\
&\leq (b_1^2 b_2 + b_1 b_2 + b_2)X.
\end{aligned}$$

By induction, it can be shown that

$$X(t) = \sum_{i=0}^t b_1^i b_2 X \leq \frac{b_2}{1-b_1} X, \quad \forall t \geq 0,$$

which implies that the sequence $\{X(t)\}_{t \geq 0}$ is bounded from above for any initial $X(0) \geq 0$.

Next, the existence of the limit of $\{X(t)\}_{t \geq 0}$ will be shown by using its boundedness proved above and the monotonicity property shown in Lemma 5.2.1 (iii), which extends the proof of [9, Theorem 1] for the packet-loss case to the general fading case. For Case 1: $X(0) = 0$. It is easy to see by induction that $X(t+1) \geq X(t)$ for all $t \geq 0$. In this situation, since $\{X(t)\}_{t \geq 0}$ is monotonically nondecreasing and bounded from above, it is convergent to $\tilde{X}$, which is positive semidefinite by continuity and is also the fixed point of the MARO (5.8). For Case 2: $X(0) \geq \tilde{X}$. We can derive that $X(t) \geq \tilde{X}$ for all $t \geq 0$ following an inductive argument. Note that

$$\begin{aligned}
X(t+1) - \tilde{X} &= g(X(t)) - g(\tilde{X}) \\
&= \phi(L^*(X(t)), X(t)) - \phi(L^*(\tilde{X}), \tilde{X}) \\
&\leq \phi(L^*(\tilde{X}), X(t)) - \phi(L^*(\tilde{X}), \tilde{X}) \\
&= \phi_1(L^*(\tilde{X}), X(t) - \tilde{X}) \\
&= \phi_1^{t+1}(L^*(\tilde{X}), X(0) - \tilde{X}).
\end{aligned}$$

It follows from $Q > 0$ that $\tilde{X} = \phi(L^*(\tilde{X}), \tilde{X}) > \phi_1(L^*(\tilde{X}), \tilde{X}) \geq 0$, which further implies that $\lim_{t \rightarrow \infty} \phi_1^{t+1}(L^*(\tilde{X}), X(0) - \tilde{X}) = 0$ for $X(0) \geq \tilde{X}$, i.e., the sequence $\{X(t)\}_{t \geq 0}$ is also convergent to $\tilde{X}$ for Case 2. For any possible $X(0)$, we always have $0 \leq X(0) \leq X(0) + \tilde{X}$. Then, the convergence of $\{X(t)\}_{t \geq 0}$ follows from Cases 1 and 2 and the squeeze theorem [129, p. 64]. The uniqueness of $\tilde{X}$ can be easily

shown by contradiction. $\square$

Note that the solution $\tilde{X} \geq 0$ to $\tilde{X} = g(\tilde{X})$ can be obtained by solving the optimization [9, Theorem 6]

$$\tilde{X} = \arg \max_{X \geq 0} \text{tr}(X) \quad (5.12)$$

subject to the LMI constraint

$$\begin{bmatrix} X - AXA' - Q & AXC'\Pi \\ \Pi CXA' & -\Pi CXC'\Pi - \Sigma \odot (CXC') - R \end{bmatrix} \leq 0.$$

The next theorem provides explicit conditions in terms of the Mahler measure of the plant for ensuring the existence of X to $X > g_1(X)$. To this end, suppose that the pair (A, C) has the following Wonham decomposition [130]:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ \star & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \cdots & \star & A_p \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ \star & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \cdots & \star & c_p \end{bmatrix}, \quad (5.13)$$

where $\star$ represents terms that will not be used in the derivation, $A_i \in \mathbb{R}^{n_i \times n_i}$, $c_i \in \mathbb{R}^{1 \times n_i}$, $\sum_{i=1}^p n_i = n$, and each pair (A_i, c_i) is detectable.

Theorem 5.2.1. *There exists $X > 0$ such that $X > g_1(X)$ if*

$$1 + \frac{[\Pi]_{ii}^2}{[\Sigma_1]_{ii}} > \mathcal{M}(A_i)^2, \quad \forall i = 1, 2, \dots, p, \quad (5.14)$$

and only if

$$\prod_{i=1}^p \left(1 + \frac{[\Pi]_{ii}^2}{[\Sigma_2]_{ii}} \right) > \mathcal{M}(A)^2, \quad (5.15)$$

where Σ_1 and Σ_2 are any positive-semidefinite diagonal matrices satisfying $0 \leq \Sigma_2 \leq \Sigma \leq \Sigma_1$.

The next result (see [4, Lemma 5.4]) is needed in the proof of Theorem 5.2.1.

Lemma 5.2.3. *Suppose that A is unstable, (A, c) is detectable for a row vector $c \neq 0$. Then, there exists $X > 0$ such that*

$$X > AXA' - \nu AXc'(cXc')^{-1}cXA' \quad (5.16)$$

if and only if

$$\nu > 1 - \frac{1}{\mathcal{M}(A)^2}. \quad (5.17)$$

Proof of Theorem 5.2.1. First of all, it is easy to see that the existence of $X > 0$ to $X > g_1(X)$ is invariant under similarity transformations on the pair (A, C) . Without loss of generality, let (A, C) have the form

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \quad C = [C_s \quad C_u],$$

after a similarity transformation, where A_s is stable, all the eigenvalues of A_u are either on or outside the unit circle, and (A_u, C_u) is observable. In this case, we can prove that the existence of $X > 0$ to $X > g_1(X)$ is further equivalent to the existence of $X_u > 0$ to

$$X_u > A_u X_u A_u' - A_u X_u C_u' \Pi [\Pi C_u X_u C_u' \Pi + \Sigma \odot (C_u X_u C_u')]^{-1} \Pi C_u X_u A_u'.$$

Therefore, in the sequel we assume that all the eigenvalues of A are either on or outside the unit circle.

First, we will prove the necessity of the condition (5.15). Suppose that there exists $X > 0$ such that $X > g_1(X)$. Based on the property of Hadamard product documented in Lemma A.0.5, we have $\Sigma \odot (CXC') \geq \Sigma_2 \odot (CXC')$. The condition (5.15) holds when any $[\Sigma_2]_{ii} = 0$, otherwise it follows from $X > g_1(X)$ that

$$\det(X) > \det(AXA' - AXc'\Pi[\Pi CXC'\Pi + \Sigma \odot (CXC')]^{-1}\Pi CXA')$$

$$\begin{aligned}
&= \det(A)^2 \det(X) \det \left(I - [\Pi C X C' \Pi + \Sigma \odot (C X C')]^{-1} \Pi C X C' \Pi \right) \\
&= \det(A)^2 \det(X) \left(\det(I + \Pi C X C' \Pi [\Sigma \odot (C X C')]^{-1}) \right)^{-1} \\
&= \det(A)^2 \det(X) \left(\det(I + X^{1/2} C' \Pi [\Sigma \odot (C X C')]^{-1} \Pi C X^{1/2}) \right)^{-1} \\
&\geq \det(A)^2 \det(X) \left(\det(I + X^{1/2} C' \Pi [\Sigma_2 \odot (C X C')]^{-1} \Pi C X^{1/2}) \right)^{-1} \\
&= \det(A)^2 \det(X) \times \\
&\quad \left(\det(I + [\Sigma_2 \odot (C X C')]^{-1/2} \Pi C X C' \Pi [\Sigma_2 \odot (C X C')]^{-1/2}) \right)^{-1} \\
&\geq \mathcal{M}(A)^2 \det(X) \prod_{i=1}^p \left(\frac{[\Sigma_2]_{ii}}{[\Sigma_2]_{ii} + [\Pi]_{ii}^2} \right), \tag{5.18}
\end{aligned}$$

where the last inequality follows from Hadamard's inequality; see Lemma A.0.2. Thus, the proof of necessity is completed since (5.18) implies (5.15).

Next, the sufficiency of the condition (5.14) will be shown. According to Lemma 5.2.3 and letting $\nu = \frac{[\Pi]_{ii}^2}{[\Pi]_{ii}^2 + [\Sigma_1]_{ii}}$, if (5.14) is true, then there exists $X_i > 0$ such that

$$X_i > A_i X_i A_i' - A_i X_i c_i' [\Pi]_{ii} [[\Pi]_{ii} c_i X_i c_i' [\Pi]_{ii} + [\Sigma_1]_{ii} \odot (c_i X_i c_i')]^{-1} [\Pi]_{ii} c_i X_i A_i',$$

for every $i = 1, 2, \dots, p$, which, based on Lemma 5.2.2, is also equivalent to the existence of $X_i > 0$ and l_i such that

$$X_i > (A_i + l_i [\Pi]_{ii} c_i) X_i (A_i + l_i [\Pi]_{ii} c_i)' + l_i ([\Sigma_1]_{ii} \odot (c_i X_i c_i')) l_i'. \tag{5.19}$$

The sufficiency for the case $p = 1$ is straightforward. We turn to consider the case $p = 2$. Since the existence of $X > 0$ to $X > g_1(X)$ is invariant under similarity transformations on the pair (A, C) , we let the pair (A, C) have the form given in (5.13), namely

$$A = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & 0 \\ c_{21} & c_2 \end{bmatrix}.$$

Based on (5.19), we can always choose a sufficiently large γ_1 such that

$$\gamma_1 X_2 > (A_2 + l_2 [\Pi]_{22} c_2) \gamma_1 X_2 (A_2 + l_2 [\Pi]_{22} c_2)' + l_2 ([\Sigma_1]_{22} \odot (c_2 \gamma_1 X_2 c_2')) l_2' + F(X_1),$$

where

$$\begin{aligned} F(X_1) = & l_2 ([\Sigma_1]_{22} \odot (c_{21} X_1 c'_{21})) l'_2 + (A_{21} + l_2 [\Pi]_{22} c_{21}) X_1 (A_1 + l_1 [\Pi]_{11} c_1)' \times \\ & [X_1 - (A_1 + l_1 [\Pi]_{11} c_1) X_1 (A_1 + l_1 [\Pi]_{11} c_1)' - l_1 ([\Sigma_1]_{11} \odot (c_1 X_1 c'_1)) l'_1]^{-1} \times \\ & (A_1 + l_1 [\Pi]_{11} c_1) X_1 (A_{21} + l_2 [\Pi]_{22} c_{21})'. \end{aligned}$$

It then follows from Schur's complement in Lemma A.0.1 that there exist $X = \text{diag}\{X_1, \gamma_1 X_2\}$ and $L = \text{diag}\{l_1, l_2\}$ such that

$$X > \phi_1(L, X). \quad (5.20)$$

By induction, for the case $p > 2$, there exist $X = \text{diag}\{X_1, \gamma_1 X_2, \dots, \gamma_{p-1} X_p\}$ and $L = \text{diag}\{l_1, l_2, \dots, l_p\}$ with sufficiently large $\gamma_1, \gamma_2, \dots, \gamma_{p-1}$ such that (5.20) holds. It follows from Lemma 5.2.2 and $\Sigma_1 \odot (CXC') \geq \Sigma \odot (CXC')$ that $X > g_1(X)$, which completes the proof. $\square$

Remark 5.2.1. *The condition (5.14) depends on the specific Wonham decomposition in (5.13), and both (5.14) and (5.15) are related to the structure of Σ . Obviously, we can put $\Sigma_1 = \Sigma_2 = \Sigma$ when Σ is diagonal, and the bounds in (5.14) and (5.15) are the same for the single-output case ($p = 1$).*

5.2.2 Mean Covariance Stability

Now, we are in the position to present the main result of this chapter that provides necessary and sufficient conditions on the network for mean covariance stability of a remote Kalman filter with faded observations along with lower and upper bounds for the mean error covariance matrix.

Theorem 5.2.2. *The Kalman filter over the fading channel(s) (5.2) is mean covariance stable if*

$$1 + \frac{[\Pi_\xi]_{ii}^2}{[\Sigma_{\xi 1}]_{ii}} > \mathcal{M}(A_i)^2, \quad \forall i = 1, 2, \dots, p, \quad (5.21)$$

and only if

$$\frac{1}{\alpha} > \rho(A)^2, \quad (5.22)$$

where Σ_{ξ_1} is any positive-semidefinite matrix satisfying $\Sigma_{\xi_1} \geq \Sigma_{\xi}$, and

$$\alpha = \Pr\{\xi_1(t) = \xi_2(t) = \cdots = \xi_p(t) = 0\}. \quad (5.23)$$

Moreover, it holds that $h^t(P(0)) \leq \mathcal{E}\{P(t)\} \leq g^t(P(0))$ for all $P(0) \geq 0$ and $t \geq 0$, where $g(\cdot)$ and $h(\cdot)$ are defined in (5.8) and (5.9) with

$$\Pi = \Pi_{\xi}, \quad \Sigma = \Sigma_{\xi}, \quad R = \Pi_{\xi} R_v \Pi_{\xi} + \Sigma_{\xi} \odot R_v + R_n, \quad (5.24)$$

and α as defined in (5.23). If (5.21) is true, then $\lim_{t \rightarrow \infty} g^t(P(0)) = \tilde{P}_{g1} \geq 0$ with $g(\tilde{P}_{g1}) = \tilde{P}_{g1}$. If (5.22) is true, then $\lim_{t \rightarrow \infty} h^t(P(0)) = \tilde{P}_h \geq 0$ with $h(\tilde{P}_h) = \tilde{P}_h$.

Proof. We will first prove the sufficiency of the condition (5.21). It follows from (5.7) that, for any scalar $\varepsilon > 0$,

$$\begin{bmatrix} P(t+1) - AP(t)A' - Q - \varepsilon I & AP(t)\tilde{C}(t)' \\ \tilde{C}(t)P(t)A' & -\tilde{C}(t)P(t)\tilde{C}(t)' - \tilde{R}(t) \end{bmatrix} < 0, \quad (5.25)$$

which is affine in both $P(t)$ and $P(t+1)$. By noting the independence between $P(t)$ and $\xi(t)$, taking the expectation on both sides of (5.25) and denoting $\hat{P}(t) \triangleq \mathcal{E}\{P(t)\}$, we have

$$\begin{bmatrix} \hat{P}(t+1) - A\hat{P}(t)A' - Q - \varepsilon I & A\hat{P}(t)C'\Pi_{\xi} \\ \Pi_{\xi}C\hat{P}(t)A' & -\Pi_{\xi}C\hat{P}(t)C'\Pi_{\xi} - \Sigma_{\xi} \odot (C\hat{P}(t)C') - R \end{bmatrix} < 0,$$

which further implies that

$$\begin{aligned} \hat{P}(t+1) &< -A\hat{P}(t)C'\Pi_{\xi} \left[\Pi_{\xi}C\hat{P}(t)C'\Pi_{\xi} + \Sigma_{\xi} \odot (C\hat{P}(t)C') + R \right]^{-1} \Pi_{\xi}C\hat{P}(t)A' \\ &\quad + A\hat{P}(t)A' + Q + \varepsilon I. \end{aligned} \quad (5.26)$$

Using the continuity arguments with respect to ε , (5.26) yields $\hat{P}(t+1) \leq g(\hat{P}(t))$ by letting ε approach 0. By taking $X(0) = P(0)$, $X(t+1) = g(X(t))$ and noting that $\hat{P}(t) \leq X(t)$ for all t , we can easily see from Theorem 5.2.1 and Lemma 5.2.2 that the sequence $\{\hat{P}(t)\}_{t \geq 0}$ is bounded from above, i.e., the remote Kalman filter is mean covariance stable. In addition, $\lim_{t \rightarrow \infty} g^t(P(0)) = \tilde{P}_{g1} = g(\tilde{P}_{g1}) \geq 0$ follows from Lemma 5.2.2.

Next, we will consider the necessity of the condition (5.22). Based on the definition of mathematical expectation, we can derive that $\hat{P}(0) = P(0)$,

$$\begin{aligned} \hat{P}(1) &= \mathcal{E}\{P(1)\} \\ &= \mathcal{E}\{AP(0)A' + Q - AP(0)\tilde{C}(0)' [\tilde{C}(0)P(0)\tilde{C}(0)' + \tilde{R}(0)]^{-1} \tilde{C}(0)P(0)A'\} \\ &\geq \Pr\{\xi_1(t) = \xi_2(t) = \cdots = \xi_p(t) = 0\} \mathcal{E}\{AP(0)A'\} + Q \\ &= h(\hat{P}(0)), \end{aligned}$$

and

$$\hat{P}(t+1) \geq h(\hat{P}(t)) \geq h^{t+1}(\hat{P}(0)) = h^{t+1}(P(0)).$$

Since $(A, Q^{1/2})$ is controllable, there exists $\tilde{P}_h > 0$ such that $\tilde{P}_h = h(\tilde{P}_h)$ if and only if $\rho(\sqrt{\alpha}A) < 1$, i.e., the condition (5.22) holds. We will show the necessity by contradiction. Assume that $P(0) = 0$ and (5.22) does not hold. In this case, $\tilde{P}_h$ does not exist. Based on Lemma 5.2.1 (iii), we can easily show that $\{h^t(0)\}_{t \geq 0}$ is monotonically nondecreasing and thus unbounded, which contradicts the mean covariance stability of the remote Kalman filter. Thus, the proof of the necessity of (5.22) is completed. Furthermore, if (5.22) is true, then, following a similar line of proof as in Lemma 5.2.2, we obtain $\lim_{t \rightarrow \infty} h^t(P(0)) = \tilde{P}_h$ for any $P(0)$. $\square$

Note that a tighter sufficient condition than (5.21) can be obtained under the following technical assumption.

Assumption 5.2.1. *There exists a matrix $\tilde{R}_n \geq 0$ such that $\mathcal{E}\{\Upsilon(t)^{-1}R_n\Upsilon(t)^{-1}\} \leq \tilde{R}_n$.*

Remark 5.2.2. *The term $\Upsilon(t)^{-1}R_n\Upsilon(t)^{-1}$ is nonlinear in $\Upsilon(t)$, and thus its expectation can only be evaluated case by case for the given distribution of $\Upsilon(t)$.*

Proposition 5.2.1. *Under Assumption 5.2.1, the Kalman filter over the fading channel(s) (5.2) is mean covariance stable if*

$$\frac{1}{\alpha_i} > \mathcal{M}(A_i)^2, \quad \forall i = 1, 2, \dots, p, \quad (5.27)$$

where α_i is as defined in (5.5). Moreover, we have

$$\mathcal{E}\{P(t)\} \leq g^t(P(0)), \quad \forall P(0) \geq 0, \quad t \geq 0,$$

where $g(\cdot)$ is defined in (5.8) with

$$\Pi = \Pi_\Omega, \quad \Sigma = \Sigma_\Omega, \quad R = \Pi_\Omega R_v \Pi_\Omega + \Sigma_\Omega \odot R_v + \tilde{R}_n. \quad (5.28)$$

If (5.27) is true, then $\lim_{t \rightarrow \infty} g^t(P(0)) = \tilde{P}_{g2} \geq 0$ with $g(\tilde{P}_{g2}) = \tilde{P}_{g2}$.

Proof. It follows from (5.25) that

$$\begin{bmatrix} P(t+1) - AP(t)A' - Q - \varepsilon I & AP(t)C'\Omega(t) \\ \Omega(t)CP(t)A' & -\Omega(t)(CP(t)C' + R_v)\Omega(t) - \Upsilon(t)^{-1}R_n\Upsilon(t)^{-1} \end{bmatrix} < 0. \quad (5.29)$$

Under Assumption 5.2.1, we can derive that $\hat{P}(t+1) \leq g(\hat{P}(t))$ with Π, Σ, R given in (5.28). The rest of the proof follows analogously to the proof of Theorem 5.2.2. $\square$

Remark 5.2.3. *As we can see from Proposition 5.2.1, the signal fluctuation does not affect the mean covariance stability of the remote Kalman filter under Assumption 5.2.1, however, it would influence the upper bound for $\mathcal{E}\{P(t)\}$ through the term $\tilde{R}_n$. Under Assumption 5.2.1, the condition (5.22) is still necessary for mean covariance stability and $h^t(P(0))$ provides a lower bound for $\mathcal{E}\{P(t)\}$. It is also direct to see that (5.21) always implies (5.27), and they are consistent if there is no signal*

fluctuation, i.e., $\Omega_i(t) \equiv 1$ in (5.4). The optimization (5.12) can be used to compute $\tilde{P}_{g1}$ and $\tilde{P}_{g2}$, while $\tilde{P}_h$ can be obtained by solving a Lyapunov equation.

5.3 Numerical Example

The numerical example that follows shows the validity of the results in this chapter.

Example 5.3.1. *Reconsider the plant given in Example 3.3.2 with $p = 1$ and*

$$A = \begin{bmatrix} 0 & 0.1 & 1.2 \\ 0 & 0.5 & 0 \\ 1.3 & 0.1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

and let $Q = I_3, R_v = R_n = 1$. We have $\rho(A) = 1.2490, \mathcal{M}(A) = 1.5600$. Suppose that $\Upsilon_1(t)$ is Nakagami distributed with the mean channel power gain $\varpi_1 = 2$ and the severity of fading $\varrho_1 = 2$. In this case, $\mathcal{E}\{\Upsilon(t)^{-1}R_n\Upsilon(t)^{-1}\} = 1$, and we can choose $\tilde{R}_n = 1$ in Assumption 5.2.1. Based on Theorem 5.2.2, the sufficient condition for the mean covariance stability of the remote Kalman filter is $\alpha_1 < 0.2050$, while the corresponding necessary condition is $\alpha = \alpha_1 < 0.6410$. According to Proposition 5.2.1, the sufficient condition for the mean covariance stability of the remote Kalman filter becomes $\alpha_1 < 0.4109$, which is less conservative than $\alpha_1 < 0.2050$ obtained from Theorem 5.2.2.

Next, assume that $\alpha_1 = 0.2$. Figure 5.2 shows the path of $\xi_1(t)$ for one sample of simulation and the empirical norm of $P(t)$ by averaging 10000 Monte Carlo simulations. As we can see, the Kalman filter with faded observations in this situation is mean covariance stable. In addition, we can compute that

$$\tilde{P}_{g1} = \begin{bmatrix} 29.9731 & 0.1248 & 0.0753 \\ 0.1248 & 1.3332 & 0.0972 \\ 0.0753 & 0.0972 & 20.0949 \end{bmatrix}, \quad \tilde{P}_{g2} = \begin{bmatrix} 12.0830 & 0.1218 & 0.0600 \\ 0.1218 & 1.3331 & 0.0924 \\ 0.0600 & 0.0924 & 7.6722 \end{bmatrix},$$

Figure 5.2: The path of $\xi_1(t)$ for one sample of simulation (above) and the empirical norm of $P(t)$ by averaging 10000 Monte Carlo simulations (below) when $\alpha_1 = 0.2$.

$$\tilde{P}_h = \begin{bmatrix} 1.4307 & 0.0120 & 0.0039 \\ 0.0120 & 1.0526 & 0.0121 \\ 0.0039 & 0.0121 & 1.4863 \end{bmatrix}.$$

When $\alpha_1 = 0.7$, the norm of $P(t)$ can reach a relatively high value for one sample of simulation as shown in Figure 5.3.

5.4 Summary

In this chapter, necessary and sufficient conditions on the network for guaranteeing the mean covariance stability of Kalman filtering over fading channels subject to both transmission failure and signal fluctuation have been given in terms of the unstable poles of the plant. It has been shown that the mean error covariance matrix of the remote Kalman filter is bounded from above by a modified Riccati iteration and from below by a modified Lyapunov iteration.

Figure 5.3: The paths of $\xi_1(t)$ (above) and $\|P(t)\|$ (below) for one sample of simulation when $\alpha_1 = 0.7$.

Chapter 6

Control over Fading Channels

In this chapter, we focus on the mean square stabilization problem of MIMO plants over multiple fading channels via both state and output feedback. The mean square stabilization problem for an NCS over an input network with fading channel(s) is formulated in Section 6.1, followed by a preliminary lemma on stabilization over predefined fading channels. Section 6.2 presents the network requirement for stabilizability via state feedback, where necessary and sufficient conditions on the network are given as the main results of this chapter. The minimum mean square capacity is provided for stabilizability under the STS and the PTS, respectively. Section 6.3 deals with the network requirement for stabilizability via output feedback. Both SISO and triangularly decoupled MIMO cases are considered. The extension to stabilization over output fading channels is given in Section 6.4, followed by an application in multi-vehicle platooning. Section 6.5 summarizes the chapter.

6.1 Problem Formulation

Consider a discrete-time NCS as depicted in Figure 6.1, where an unreliable network with fading channel(s) is placed in the path from the controller to the plant.

Figure 6.1: Control over input fading channel(s).

Assume that the plant $G(z)$ is strictly proper and has a state-space representation:

$$\begin{aligned} x_G(t+1) &= Ax_G(t) + Bu(t), \\ y(t) &= Cx_G(t), \end{aligned} \quad (6.1)$$

where $x_G(t) \in \mathbb{R}^n$ is the plant state, $u(t) \in \mathbb{R}^m$ is the plant input, and $y(t) \in \mathbb{R}^p$ is the measured output. Without loss of generality, assume that A is unstable, $B = [B_1 \ B_2 \ \cdots \ B_m]$ has full-column rank, $C = [C'_1 \ C'_2 \ \cdots \ C'_p]'$ has full-row rank, and the triple (A, B, C) is stabilizable and detectable.

The controller is assumed to be time invariant, and has a proper transfer function $K(z)$ and a state-space realization:

$$\begin{aligned} x_K(t+1) &= A_K x_K(t) + B_K y(t), \\ v(t) &= C_K x_K(t) + D_K y(t), \end{aligned} \quad (6.2)$$

where $v(t) \in \mathbb{R}^m$ is the controller output. The dimension of the controller state $x_K(t)$ is not specified *a priori*.

The model of the fading channel(s) is given in the following memoryless multiplicative form:

$$u(t) = \xi(t)v(t), \quad (6.3)$$

where $u(t)$ is directly applied to the plant¹, and $\xi(t) \in \mathbb{R}^{m \times m}$ represents the channel fading and has the diagonal structure:

$$\xi(t) = \text{diag} \{ \xi_1(t), \xi_2(t), \dots, \xi_m(t) \}. \quad (6.4)$$

¹The NCS with pre- and post-channel filters will be further discussed in Section 7.1.

It is assumed that $\xi_i(t)$, $i = 1, 2, \dots, m$, are scalar-valued white noise processes with

$$\mu_i \triangleq \mathcal{E}\{\xi_i(t)\}, \quad \sigma_{ij} \triangleq \mathcal{E}\{(\xi_i(t) - \mu_i)(\xi_j(t) - \mu_j)\}, \quad (6.5)$$

satisfying $\mu_i \neq 0$, $\sigma_{ii} > 0$, and $\sigma_{ij} = \sigma_{ji}$, for all $i, j = 1, 2, \dots, m$. We further denote $\sigma_i \triangleq \sqrt{\sigma_{ii}}$ and

$$\Sigma \triangleq [\sigma_{ij}]_{i,j=1,2,\dots,m}, \quad \Pi \triangleq \text{diag}\{\mu_1, \mu_2, \dots, \mu_m\}, \quad \Lambda \triangleq \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_m\}. \quad (6.6)$$

It is easy to see that Σ is positive semidefinite.

Remark 6.1.1. *In model (6.3)-(6.4), if the i -th and j -th components of $v(t)$ are sent over the same fading channel, whose channel fading is constant over each time step of the underlying discrete-time system², then we can set $\xi_i(t) = \xi_j(t)$. The additive noise is not included in (6.3), since the main focus of this chapter is the stabilization issue. In the next chapter, we shall revisit the effect of channel additive noise under the assumption that the channel input power is bounded by a predefined level.*

Denote the overall system state by $x(t) = [x_G(t)' \ x_K(t)']'$, then the closed loop of the NCS in Figure 6.1 can be written into

$$x(t+1) = [\tilde{A} + \tilde{B}\xi(t)\tilde{C}]x(t), \quad (6.7)$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ B_K C & A_K \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} = [D_K C \ C_K]. \quad (6.8)$$

Throughout this chapter, we shall concentrate on the mean square stabilization defined next.

²This is a valid assumption when the fading is coherent over the sampling interval of the system [16].

Definition 6.1.1 ([10]). *Consider the NCS in Figure 6.1, and denote $\Xi(t) \triangleq \mathcal{E}[x(t)x'(t)]$. The plant (6.1) is stabilized in the mean square sense by the controller (6.2), if, for any initial state $x(0)$, $\Xi(t)$ is well defined for all $t \geq 0$ and $\lim_{t \rightarrow \infty} \Xi(t) = 0$.*

The model (6.3)-(6.4) of the fading channel(s) is said to be predefined if μ_i, σ_{ij} in (6.5) are fixed and known for all $i, j = 1, 2, \dots, m$. Let $\tilde{B} = [\tilde{B}_1 \ \tilde{B}_2 \ \dots \ \tilde{B}_m]$ and $\tilde{C} = [\tilde{C}'_1 \ \tilde{C}'_2 \ \dots \ \tilde{C}'_m]'$. The next lemma will be used in the later developments, which extends the stability criterion for systems with independent multiplicative noises presented in [10] and Chapter 9 of [11] to that with possibly correlated multiplicative noises.

Lemma 6.1.1. *Consider the NCS in Figure 6.1 with predefined fading channel(s) modeled by (6.3)-(6.4). The following statements are equivalent.*

- (i) *The plant (6.1) or the triple (A, B, C) can be stabilized in the mean square sense via controller of the form (6.2).*
- (ii) *There exists a set of A_K, B_K, C_K, D_K such that the sequence $\{\Xi(t)\}_{t \geq 0}$ computed by*

$$\Xi(t+1) = (\tilde{A} + \tilde{B}\Pi\tilde{C})\Xi(t)(\tilde{A} + \tilde{B}\Pi\tilde{C})' + \sum_{i=1}^m \sum_{j=1}^m \sigma_{ij} \tilde{B}_j \tilde{C}_j \Xi(t) \tilde{C}'_i \tilde{B}'_i \quad (6.9)$$

with any $\Xi(0) \geq 0$ is convergent to 0 as t approaches ∞ .

- (iii) *There exists a set of A_K, B_K, C_K, D_K such that $\rho(\Psi) < 1$, where*

$$\Psi = (\tilde{A} + \tilde{B}\Pi\tilde{C}) \otimes (\tilde{A} + \tilde{B}\Pi\tilde{C}) + \sum_{i=1}^m \sum_{j=1}^m \sigma_{ij} (\tilde{B}_i \tilde{C}_i) \otimes (\tilde{B}_j \tilde{C}_j). \quad (6.10)$$

- (iv) *There exist a set of A_K, B_K, C_K, D_K and $P > 0$ such that*

$$P > (\tilde{A} + \tilde{B}\Pi\tilde{C})' P (\tilde{A} + \tilde{B}\Pi\tilde{C}) + \tilde{C}' J \tilde{C}, \quad (6.11)$$

where $J = \Sigma \odot (\tilde{B}'P\tilde{B}) > 0$.

(v) There exist a set of A_K, B_K, C_K, D_K and $W > 0$ such that

$$W > (\tilde{A} + \tilde{B}\Pi\tilde{C})W(\tilde{A} + \tilde{B}\Pi\tilde{C})' + \tilde{B}H\tilde{B}', \quad (6.12)$$

where $H = \Sigma \odot (\tilde{C}W\tilde{C}')$.

Proof. (i) $\Leftrightarrow$ (ii): It follows directly from Definition 6.1.1.

(ii) $\Leftrightarrow$ (iii): Based on Lemma A.0.4, the recursion (6.9) can be written into

$$\text{vec}(\Xi(t+1)) = \Psi \text{vec}(\Xi(t)) = \Psi^{t+1} \text{vec}(\Xi(0)).$$

It is easy to see that $\lim_{t \rightarrow \infty} \Xi(t) = 0$ is equivalent to the existence of A_K, B_K, C_K, D_K such that $\rho(\Psi) < 1$.

(i) $\Leftrightarrow$ (iv): First, suppose that (iv) is true. Introduce a Lyapunov function $V(\Xi(t)) = \text{tr}(\Xi(t)P)$, then

$$\begin{aligned} V(\Xi(t+1)) &= \text{tr} \left\{ \left((\tilde{A} + \tilde{B}\Pi\tilde{C})\Xi(t)(\tilde{A} + \tilde{B}\Pi\tilde{C})' + \sum_{i=1}^m \sum_{j=1}^m \sigma_{ij} \tilde{B}_j \tilde{C}_j \Xi(t) \tilde{C}_i' \tilde{B}_i' \right) P \right\} \\ &= \text{tr} \left\{ \Xi(t) \left((\tilde{A} + \tilde{B}\Pi\tilde{C})' P (\tilde{A} + \tilde{B}\Pi\tilde{C}) + \tilde{C}' J \tilde{C} \right) \right\} \\ &< \text{tr}(\Xi(t)P) = V(\Xi(t)). \end{aligned}$$

It follows from Lyapunov theory that $\lim_{t \rightarrow \infty} \Xi(t) = 0$. On the other hand, assume that the closed loop is mean square stable. Then, $\rho(\Psi) < 1$ based on (iii), which is equivalent to $\rho(\Psi') < 1$. Therefore, the sequence $\{\hat{\Xi}(t)\}_{t \geq 0}$ computed by

$$\hat{\Xi}(t+1) = (\tilde{A} + \tilde{B}\Pi\tilde{C})'\hat{\Xi}(t)(\tilde{A} + \tilde{B}\Pi\tilde{C}) + \sum_{i=1}^m \sum_{j=1}^m \sigma_{ij} \tilde{C}_i' \tilde{B}_i' \hat{\Xi}(t) \tilde{B}_j \tilde{C}_j \quad (6.13)$$

is convergent to zero as t approaches ∞ for any $\hat{\Xi}(0) \geq 0$. Let $\hat{\Xi}(0) > 0$ and

$P(t) = \sum_{s=0}^t \hat{\Xi}(s)$, then we have

$$\begin{aligned} P(t+1) &= \hat{\Xi}(0) + (\tilde{A} + \tilde{B}\Pi\tilde{C})'P(t)(\tilde{A} + \tilde{B}\Pi\tilde{C}) + \sum_{i=1}^m \sum_{j=1}^m \sigma_{ij} \tilde{C}_i' \tilde{B}_i' P(t) \tilde{B}_j \tilde{C}_j \\ &> (\tilde{A} + \tilde{B}\Pi\tilde{C})'P(t)(\tilde{A} + \tilde{B}\Pi\tilde{C}) + \sum_{i=1}^m \sum_{j=1}^m \sigma_{ij} \tilde{C}_i' \tilde{B}_i' P(t) \tilde{B}_j \tilde{C}_j. \end{aligned} \quad (6.14)$$

It follows from the convergence of $\{\hat{\Xi}(t)\}_{t \geq 0}$ and (6.14) that $P = \lim_{t \rightarrow \infty} P(t)$ exists and satisfies (6.11) in (iv). It is easy to see that $P > 0$ since $\hat{\Xi}(0) > 0$. In addition, according to Lemma A.0.5, we have $J = \Sigma \odot (B'PB) > 0$ since $\Sigma \geq 0$, $B'PB > 0$, and Σ has no diagonal entry that is equal to zero.

(iv) $\Leftrightarrow$ (v): Assume that (v) holds. By selecting a Lyapunov function $V(\hat{\Xi}(t)) = \text{tr}(\hat{\Xi}(t)W)$, we can easily prove that (v) implies the convergence of $\{\hat{\Xi}(t)\}_{t \geq 0}$ computed by (6.13), which further implies (iv). The converse follows by setting $W = \lim_{t \rightarrow \infty} \sum_{s=0}^t \Xi(s)$ and $\Xi(0) > 0$. $\square$

Remark 6.1.2. *It is easy to show that the mean square stabilizability is invariant under similarity transformations on the triple (A, B, C) .*

6.2 Network Requirement for Stabilizability via State Feedback

Consider the state feedback case with $C = I$. In view of Remark 6.1.2, we take (A, B) to be of the form:

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \quad B = \begin{bmatrix} B_s \\ B_u \end{bmatrix}, \quad (6.15)$$

where A_s is stable, all the eigenvalues of A_u are either on or outside the unit circle, and (A_u, B_u) is controllable. We can derive the next lemma.

Lemma 6.2.1. *Consider the NCS in Figure 6.1 with state feedback and predefined fading channel(s) modeled by (6.3)-(6.4). The following statements are equivalent.*

- i) *The plant (6.1) or the pair (A, B) can be stabilized in the mean square sense via static state feedback.*
- ii) *The pair (A, B) can be stabilized in the mean square sense via dynamic state feedback.*
- iii) *There exists $P > 0$ such that*

$$P > A'PA - A'PB\Pi(J + \Pi B'PB\Pi)^{-1}\Pi B'PA, \quad (6.16)$$

where J is as defined in (6.11) with $\tilde{B} = B$.

- iv) *For any $R > 0$, there exists $P > 0$ such that*

$$P = A'PA - A'PB\Pi(J + \Pi B'PB\Pi)^{-1}\Pi B'PA + R. \quad (6.17)$$

- v) *The pair (A_u, B_u) can be stabilized in the mean square sense via static or dynamic state feedback.*

Furthermore, if any of the conditions i)-v) is true, then a state feedback gain ensuring the mean square stability of the closed loop is given by

$$K = -(J + \Pi B'PB\Pi)^{-1}\Pi B'PA, \quad (6.18)$$

where $P > 0$ is any solution to (6.16) or (6.17).

Proof. i) $\Leftrightarrow$ ii): For static state feedback, $\tilde{A}, \tilde{B}, \tilde{C}$ in (6.8) is reduced to $\tilde{A} = A, \tilde{B} = B, \tilde{C} = K$. Obviously, the stabilizability via static state feedback always implies the stabilizability via dynamic state feedback. In what follows, we will show the converse is also true. Suppose the closed loop is mean square stable under a dynamic state

feedback controller, i.e., there exist a set of A_K, B_K, C_K, D_K and $W > 0$ such that (6.12) in Lemma 6.1.1 is true for $C = I$. Partition W in accordance with $\tilde{A}$ in (6.8) as $W = \begin{bmatrix} W_{11} & W_{12} \\ W'_{12} & W_{22} \end{bmatrix}$. In this case, the 1×1 block of the inequality (6.12) yields

$$\begin{aligned} W_{11} &> (A + B\Pi D_K)W_{11}(A + B\Pi D_K)' + (A + B\Pi D_K)W_{12}C'_K\Pi B' \\ &\quad + B(\Sigma \odot (D_K W_{11}D'_K + C_K W'_{12}D'_K + D_K W_{12}C'_K + C_K W_{22}C'_K))B' \quad (6.19) \\ &\quad + B\Pi C_K W_{22}C'_K \Pi B' + B\Pi C_K W'_{12}(A + B\Pi D_K)'. \end{aligned}$$

By setting $K = D_K + C_K W'_{12} W_{11}^{-1}$, it follows from (6.19), the property of Hadamard product in Lemma A.0.5, and $W_{22} - W'_{12} W_{11}^{-1} W_{12} > 0$ that

$$W_{11} > (A + B\Pi K)W_{11}(A + B\Pi K)' + B(\Sigma \odot (K W_{11} K'))B'. \quad (6.20)$$

From Lemma 6.1.1 (v), the inequality (6.20) is equivalent to the existence of a state feedback gain K such that the closed-loop system is mean square stable.

i) $\Leftarrow$ iii): By substituting $\tilde{A} = A, \tilde{B} = B, \tilde{C} = K$ into (6.11), we can obtain that

$$P > (A + B\Pi K)'P(A + B\Pi K) + K'JK. \quad (6.21)$$

The inequality (6.16) implies (6.21) by setting K as in (6.18).

i) $\Rightarrow$ iii): By taking derivative with respect to u , we can conclude that $u = Kx$ with K defined in (6.18) minimizes the function $f(u) = -x'Px + (Ax + B\Pi u)'P(Ax + B\Pi u) + u'Ju$.

iii) $\Leftrightarrow$ iv): By choosing the same P , iv) obviously implies iii). Next, suppose that iii) holds. Define the operators

$$\begin{aligned} \mathcal{L}_1(Z) &= A'ZA - A'ZB\Pi(\Sigma \odot (B'ZB) + \Pi B'ZB\Pi)^{-1}\Pi B'ZA, \\ \mathcal{L}_2(Z, K) &= (A + B\Pi K)'Z(A + B\Pi K) + K'(\Sigma \odot (B'ZB))K. \end{aligned}$$

It follows that $\mathcal{L}_2(Z, K)$ is affine in Z , and for any $Z \geq 0$ and K ,

$$0 \leq \mathcal{L}_1(Z) = \mathcal{L}_2(Z, K^*(Z)) \leq \mathcal{L}_2(Z, K)$$

with $K^*(Z) = -(\Sigma \odot (B'ZB) + \Pi B'ZB\Pi)^{-1} \Pi B'ZA$. For any $Z_2 \geq Z_1 \geq 0$, we have

$$\mathcal{L}_1(Z_1) = \mathcal{L}_2(Z_1, K^*(Z_1)) \leq \mathcal{L}_2(Z_1, K^*(Z_2)) \leq \mathcal{L}_2(Z_2, K^*(Z_2)) = \mathcal{L}_1(Z_2).$$

Let the sequence $\{Z(t)\}_{t \geq 0}$ computed by $Z(t+1) = \mathcal{L}_1(Z(t)) + R$ with any initial condition $Z(0) \geq 0$. The condition iii) can be written into $P > \mathcal{L}_1(P)$ with $P > 0$. In this case, we can always choose $c_1 \in [0, 1)$ and $c_2 \in (0, \infty)$ such that $\mathcal{L}_1(P) \leq c_1 P$, $Z(0) \leq c_2 P$, $R \leq c_2 P$. Since $\mathcal{L}_1(c_2 P) = c_2 \mathcal{L}_1(P)$, we have

$$\begin{aligned} Z(1) &= \mathcal{L}_1(Z(0)) + R \leq c_1 c_2 P + c_2 P, \\ Z(2) &= \mathcal{L}_1(Z(1)) + R \leq c_1^2 c_2 P + c_1 c_2 P + c_2 P. \end{aligned}$$

By mathematical induction, we have $Z(t) \leq \sum_{s=0}^t c_1^s c_2 P \leq \frac{c_2}{1-c_1} P$, which implies that the sequence $\{Z(t)\}_{t \geq 0}$ is bounded. Let $Z(0) = 0$, then it follows from the monotonicity and boundedness of $\{Z(t)\}_{t \geq 0}$ that $Z = \lim_{t \rightarrow \infty} Z(t)$ exists and satisfies $Z = \mathcal{L}_1(Z) + R$. Note that $Z > 0$ since $R > 0$ and $\mathcal{L}_1(Z) \geq 0$, which completes the proof.

i) $\Leftarrow$ v): In view of the equivalence between i) and ii), we limit our attention to the static state feedback case in the proof. Let (A, B) be of the form (6.15). Since A_s in (6.15) is stable, there exists $P_1 > 0$ such that $P_1 - A_s' P_1 A_s > 0$. The pair (A_u, B_u) is mean square stabilizable, thus based on Lemma 6.1.1 (iv), there exist $P_2 > 0$ and K_u such that $P_2 > (A_u + B_u \Pi K_u)' P_2 (A_u + B_u \Pi K_u) + K_u' J_u K_u$, where $J_u = \Sigma \odot (B_u' P_2 B_u)$. It follows that, for some $\beta > 0$, the inequality

$$P_2 > \beta I + (A_u + B_u \Pi K_u)' P_2 (A_u + B_u \Pi K_u) + K_u' J_u K_u$$

is true. Let $J_s = \Sigma \odot (B_s' P_1 B_s)$. By choosing $P = \text{diag}\{P_1, \gamma P_2\}$, $K = [0 \ K_u]$ with

a sufficiently large $\gamma > 0$ such that

$$\gamma\beta I > K'_u \Pi B'_s P_1 B_s \Pi K_u + K'_u J_s K_u + K'_u \Pi B'_s P_1 A_s (P_1 - A'_s P_1 A_s)^{-1} A'_s P_1 B_s \Pi K_u,$$

and in view of $J = J_s + J_u$, we have (6.21) holds, i.e., the pair (A, B) is mean square stabilizable.

i) $\Rightarrow$ v): According to the condition iii), the mean square stabilizability of (A, B) in the form of (6.15) ensures that there exists $P = \begin{bmatrix} P_{11} & P_{12} \\ P'_{12} & P_{22} \end{bmatrix} > 0$ such that (6.16) is true. After applying the linear coordinate transformation matrix $\begin{bmatrix} I & -P_{11}^{-1}P_{12} \\ 0 & I \end{bmatrix}$, the 2×2 block of the inequality (6.16) implies that

$$P_2 > A'_u P_2 A_u - A'_u P_2 B_u \Pi (J_u + \Pi B'_u P_2 B_u \Pi)^{-1} \Pi B'_u P_2 A_u,$$

where $P_2 = P_{22} - P'_{12} P_{11}^{-1} P_{12} > 0$ and $J_u = \Sigma \odot (B'_u P_2 B_u)$. Therefore, the pair (A_u, B_u) is mean square stabilizable. $\square$

Remark 6.2.1. By adopting similar ideas in the proof of Theorem 6 in [9], one solution $P > 0$ to (6.17) can be obtained by solving the following optimization:

$$P = \arg \max_{\hat{P} > 0} \text{tr}(\hat{P}) \quad (6.22)$$

subject to the LMI constraint

$$\begin{bmatrix} \hat{P} - A' \hat{P} A - R & A' \hat{P} B \Pi \\ \Pi B' \hat{P} A & -\Sigma \odot (B' \hat{P} B) - \Pi B' \hat{P} B \Pi \end{bmatrix} \leq 0.$$

However, the optimization in (6.22), despite of the ease of computation, offers no explicit condition on μ_i and σ_{ij} even when a solution $P > 0$ exists. Therefore, we turn to investigate the requirement on the network over which an unstable plant can be stabilized.

In view of Lemma 6.2.1, we limit our attention to the static state feedback case in the rest part of this section and assume that all the eigenvalues of A are either on or outside the unit circle without loss of generality. Based on Remark 6.1.2, the pair (A, B) is further assumed to have the following Wonham decomposition [130]:

$$A = \begin{bmatrix} A_1 & \star & \cdots & \star \\ 0 & A_2 & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & \star & \cdots & \star \\ 0 & b_2 & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_m \end{bmatrix}, \quad (6.23)$$

where $\star$ represents terms that will not be used in the derivation, $A_i \in \mathbb{R}^{n_i \times n_i}$, $b_i \in \mathbb{R}^{n_i \times 1}$, $\sum_{i=1}^m n_i = n$, and each pair (A_i, b_i) is controllable. The next theorem parallel to Theorem 5.2.1 in the previous chapter provides necessary and sufficient conditions on the network for stabilizability via state feedback.

Theorem 6.2.1. *The NCS in Figure 6.1 is mean square stabilizable via state feedback if*

$$1 + \frac{\mu_i^2}{[\Sigma_1]_{ii}} > \mathcal{M}(A_i)^2, \quad \forall i = 1, 2, \dots, m, \quad (6.24)$$

and only if

$$\prod_{i=1}^m \left(1 + \frac{\mu_i^2}{[\Sigma_2]_{ii}} \right) > \mathcal{M}(A)^2, \quad (6.25)$$

where Σ_1 and Σ_2 are arbitrary positive-semidefinite diagonal matrices satisfying $0 \leq \Sigma_2 \leq \Sigma \leq \Sigma_1$.

Proof. We will first prove the necessity of the condition (6.25). Suppose that the closed loop is mean square stable under a static state feedback controller. Based on the property of Hadamard product in Lemma A.0.5, we have $J \geq \hat{J}$ with $\hat{J} = \Sigma_2 \odot (B'PB)$. The condition (6.25) obviously holds when any $[\Sigma_2]_{ii} = 0$, otherwise it follows from (6.16) that

$$\begin{aligned} \det(P) &> \det(A') \det(I - PB\Pi(J + \Pi B'PB\Pi)^{-1}\Pi B') \det(P) \det(A) \\ &= \det(A)^2 \det(P) \det(I - (J + \Pi B'PB\Pi)^{-1}\Pi B'PB\Pi) \end{aligned}$$

$$\begin{aligned}
&= \det(A)^2 \det(P) \left(\det(I + \Pi B' P B \Pi J^{-1}) \right)^{-1} \\
&= \det(A)^2 \det(P) \left(\det(I + P^{1/2} B \Pi J^{-1} \Pi B' P^{1/2}) \right)^{-1} \\
&\geq \det(A)^2 \det(P) \left(\det(I + P^{1/2} B \Pi \hat{J}^{-1} \Pi B P^{1/2}) \right)^{-1} \\
&= \det(A)^2 \det(P) \left(\det(I + \hat{J}^{-1/2} \Pi B' P B \Pi \hat{J}^{-1/2}) \right)^{-1} \\
&\geq \mathcal{M}(A)^2 \det(P) \prod_{i=1}^m \left(\frac{[\Sigma_2]_{ii}}{[\Sigma_2]_{ii} + \mu_i^2} \right). \tag{6.26}
\end{aligned}$$

The inequality (6.26) follows from Hadamard's inequality; see Lemma A.0.2. The proof of necessity is completed since (6.26) implies (6.25).

Next, the sufficiency of the condition (6.24) will be shown. According to Corollary 8.4 in [10] and Lemma 6.1.1, if (6.24) is true, then there exist $P_i > 0$ and $k_i \in \mathbb{R}^{1 \times n_i}$ such that $P_i > (A_i + b_i \mu_i k_i)' P_i (A_i + b_i \mu_i k_i) + [\Sigma_1]_{ii} k_i' b_i' P_i b_i k_i$. It is direct to show that the closed loop is mean square stable for $m = 1$. Next, we consider the case $m = 2$. Adopting the similar technique in the proof of the equivalence between i) and v) in Lemma 6.2.1, we can show that there exist $P = \text{diag}\{P_1, \gamma_1 P_2\}$ and $K = \text{diag}\{k_1, k_2\}$ with sufficiently large γ_1 such that

$$P > (A + B \Pi K)' P (A + B \Pi K) + K' (\Sigma_1 \odot (B' P B)) K, \tag{6.27}$$

where the pair (A, B) has the form given in (6.23). By induction, for the case $m > 2$, there exist $P = \text{diag}\{P_1, \gamma_1 P_2, \dots, \gamma_{m-1} P_m\}$ and $K = \text{diag}\{k_1, k_2, \dots, k_m\}$ with sufficiently large $\gamma_1, \gamma_2, \dots, \gamma_{m-1}$ such that (6.27) holds. It follows from $\Sigma_1 \odot (B' P B) \geq J$ that (6.11) holds with $\tilde{A} = A, \tilde{B} = B, \tilde{C} = K$. Therefore, the closed-loop NCS is mean square stable under a static state feedback controller. $\square$

Remark 6.2.2. Note that for the single-input case, the bounds presented in (6.24) and (6.25) are consistent by taking $\Sigma_1 = \Sigma_2 = \Sigma$, and Theorem 6.2.1 is reduced to Corollary 8.4 of [10]. For multi-input case, the tightness of (6.24) and (6.25) is affected by the selection of Σ_1, Σ_2 and thus depends on the structure of Σ . Moreover, (6.24) is also related to the particular Wonham decomposition in (6.23) which is generally not unique.

Next, we will further refine Theorem 6.2.1 under the following two typical transmission policies for the network (6.3)-(6.4).

- **Parallel transmission strategy (PTS):** Each element of $v(t)$ is sent across an individual fading channel, and $\xi_1(t), \xi_2(t), \dots, \xi_m(t)$ are uncorrelated with each other.
- **Serial transmission strategy (STS):** All elements of $v(t)$ are transmitted over the same fading channel one after another, and $\xi_1(t) = \xi_2(t) = \dots = \xi_m(t)$.

Remark 6.2.3. *In some industrial and military applications where a group of ground, aerial, and/or underwater vehicles/robots are geographically separated from each other and from the remote controller, the control signal to each vehicle/robot is sent across an individual channel, which motivates the consideration of PTS. As mentioned in Remark 6.1.1, $\xi_1(t), \xi_2(t), \dots, \xi_m(t)$ can be made to be uncorrelated with each other in PTS by adopting an orthogonal access scheme. For STS, the condition $\xi_1(t) = \xi_2(t) = \dots = \xi_m(t)$ is valid in applications when the fading is coherent over the sampling interval. Note that the network model with parallel channels or a single channel has been widely used in the literature on networked estimation and networked control; see, e.g., [9, 43, 70, 98, 131].*

Under PTS or STS, define the overall mean square capacity of the network as

$$\mathcal{C}_{\text{MS}} \triangleq \sum_{i=1}^m \frac{1}{2} \ln \left(1 + \frac{\mu_i^2}{\sigma_i^2} \right), \quad (6.28)$$

which is reduced to the mean square capacity presented in [10] when $m = 1$. Further denote $\mathcal{C}_{\text{MS}i} \triangleq \frac{1}{2} \ln \left(1 + \frac{\mu_i^2}{\sigma_i^2} \right)$, $g_i \triangleq 1 + \frac{\mu_i^2}{\sigma_i^2}$ and $g \triangleq \prod_{i=1}^m g_i$, then it follows that $\mathcal{C}_{\text{MS}i} = \frac{1}{2} \ln g_i$, $\mathcal{C}_{\text{MS}} = \frac{1}{2} \ln g$, and the larger the g_i , the larger the $\mathcal{C}_{\text{MS}i}$. Note that under STS, $\mathcal{C}_{\text{MS}} = m\mathcal{C}_{\text{MS}1}$. Next, we will present a tight lower bound on the overall mean square capacity of the network for mean square stabilizability under PTS and STS, respectively.

6.2.1 PTS

Under PTS, the overall mean square capacity is considered as the network resource and assumed to satisfy the following assumption.

Assumption 6.2.1. *The overall mean square capacity of the network is fixed and can be allocated among the parallel channels.*

Remark 6.2.4. *It is worth mentioning that the resource (e.g., power, code, time and frequency) allocation technique has been employed extensively for studying capacity maximization in communications [19, 132], and recently it has attracted the interests from the control community as well [70, 133].*

A relationship between the overall mean square capacity and the Mahler measure of the plant for ensuring the mean square stabilizability via state feedback under PTS is characterized by the next theorem.

Theorem 6.2.2. *Under PTS and Assumption 6.2.1, the NCS in Figure 6.1 is mean square stabilizable via state feedback if and only if*

$$\mathcal{C}_{\text{MS}} > \ln \mathcal{M}(A). \quad (6.29)$$

Proof. Note that under PTS, $\sigma_{ij} = 0$ for all $i \neq j$, i.e., Σ is diagonal. The necessity of (6.29) for mean square stabilizability via state feedback follows from (6.25) in Theorem 6.2.1 by taking $\Sigma_2 = \Sigma$. On the other hand, under Assumption 6.2.1 on capacity allocation and $\mathcal{C}_{\text{MS}} > \ln \mathcal{M}(A)$, i.e., $g > \mathcal{M}(A)^2 = \prod_{i=1}^m \mathcal{M}(A_i)^2$, we can always choose $g_i > \mathcal{M}(A_i)^2$ such that the condition (6.24) in Theorem 6.2.1 is true by letting $\Sigma_1 = \Sigma$. Therefore, there exist a stabilizing state feedback gain $K = \text{diag}\{k_1, k_2, \dots, k_m\}$ as in Theorem 6.2.1 and an allocation $\{\mathcal{C}_{\text{MS}1}, \mathcal{C}_{\text{MS}2}, \dots, \mathcal{C}_{\text{MS}m}\}$ satisfying $\mathcal{C}_{\text{MS}} = \sum_{i=1}^m \mathcal{C}_{\text{MS}i}$ such that the closed loop is mean square stable. $\square$

Remark 6.2.5. *Similar to the well-known data rate theorem [66] which quantifies the minimal data rate necessary for stabilizing an unstable system, Theorem 6.2.2*

characterizes the minimal overall mean square capacity of the network in order to stabilize an unstable system in the mean square sense. The result also pinpoints a relationship between the minimal overall mean square capacity and the unstable eigenvalues of the system matrix A .

6.2.2 STS

Based on Schur's complement in Lemma A.0.1, the inequality (6.11) in Lemma 6.1.1 (iv) under STS with $\tilde{A} = A, \tilde{B} = B, \tilde{C} = K$ is equivalent to

$$\begin{bmatrix} -S & (AS + BY)' & (\sqrt{\frac{1}{g_1-1}}BY)' \\ AS + BY & -S & 0 \\ \sqrt{\frac{1}{g_1-1}}BY & 0 & -S \end{bmatrix} < 0 \quad (6.30)$$

with $S = P^{-1}/\mu$, $Y = KP^{-1}$. Then, we can deduce the following result.

Proposition 6.2.1. *Under STS, the NCS in Figure 6.1 is mean square stabilizable via state feedback if and only if*

$$C_{\text{MS}} > \frac{m}{2} \ln g_{1c}, \quad (6.31)$$

where

$$g_{1c} = \begin{cases} \mathcal{M}(A)^2, & \text{if } m = 1; \\ \rho(A)^2, & \text{if } m = n; \\ \inf_{S>0, Y} g_1, \text{ subject to (6.30),} & \text{otherwise.} \end{cases} \quad (6.32)$$

Furthermore, it holds that

$$\frac{m}{2} \ln g_{1c} \geq \ln \mathcal{M}(A). \quad (6.33)$$

Proof. First, we will show the expression for g_{1c} in (6.32). When $m = 1$, the STS is the same as the PTS, thus the critical value g_{1c} for mean square stabilizability follows directly from Theorem 6.2.2. Note that (6.16) in Lemma 6.2.1 iii) under

STS becomes

$$P > A'PA - \frac{\mu_1^2}{\sigma_1^2 + \mu_1^2} A'PB(B'PB)^{-1}B'PA. \quad (6.34)$$

When $m = n$, the inequality (6.34) is further reduced to

$$P > \left(g_1^{-\frac{1}{2}}A\right)' P \left(g_1^{-\frac{1}{2}}A\right). \quad (6.35)$$

In this case, the mean square stabilizability is equivalent to $\rho(g_1^{-\frac{1}{2}}A) < 1$, and thus the critical value g_{1c} is given by $\rho(A)^2$. For general cases, we can use (6.30) as the necessary and sufficient condition for mean square stabilizability. Observe that if (6.30) is true for some $g_1 = g_{1a} > 1$, then it holds for all $g_1 \geq g_{1a}$. Therefore, the critical value g_{1c} can be obtained via the minimization of g_1 over (6.30).

Next, we will prove (6.33) for an arbitrary m . Since all the eigenvalues of A are assumed to be on or outside the unit circle, it follows from (6.34) that

$$\begin{aligned} \det(P) &> \det(A) \det\left(I - \frac{\mu_1^2}{\sigma_1^2 + \mu_1^2} PB(B'PB)^{-1}B'\right) \det(P) \det(A') \\ &= \det(A)^2 \det(P) \det\left(I_m - \frac{\mu_1^2}{\sigma_1^2 + \mu_1^2} I_m\right) \\ &= \mathcal{M}(A)^2 \det(P) g_1^{-m}, \end{aligned}$$

which gives (6.33). In particular, the equality in (6.33) holds for $m = 1$, while for $m = n$, we have $\frac{m}{2} \ln g_{1c} = \ln \rho(A)^n \geq \ln \mathcal{M}(A)$. $\square$

Remark 6.2.6. Note that Proposition 6.2.1 is consistent with Lemma 5.4 of [4] where the packet-loss issue is considered. Except for some special cases, the critical value g_{1c} is, in general, not connected with the system matrix A explicitly. From Theorem 6.2.2 and Proposition 6.2.1, we can conclude that if the overall mean square capacity is allocatable among fading channels, i.e., additional flexibility of designing the communication component is added, then the optimization in (6.32) is avoided, and the requirement on overall mean square capacity would be reduced when $m > 1$.

6.3 Network Requirement for Stabilizability via Output Feedback

In the case of output feedback, we will base our analysis on both the state-space and stable coprime factorization approaches. The frequency variable z will be omitted whenever no confusion is caused. In view of the framework of fading channels in [10], the network (6.3)-(6.4) under PTS is equivalent to the structure shown in Figure 6.2, where $z(t) = v(t)$, $w(t) = \tilde{\Delta}(t)z(t)$, and the uncertainty block $\tilde{\Delta}(t)$ is defined as $\tilde{\Delta}(t) \triangleq \Lambda^{-1}(\xi(t) - \Pi)$.

Figure 6.2: Equivalent description of the network (6.3)-(6.4) under PTS.

It is easy to verify that the mean square stability of the NCS in Figure 6.1 with the network (6.3)-(6.4) always implies the internal stability of the corresponding closed-loop system with $w(t) = 0$. Denote the set of all proper controllers achieving the above internal stability by $\mathcal{K}$. The transfer function of the closed loop from $w(t)$ to $z(t)$, without considering the uncertainty block $\tilde{\Delta}(t)$, is given by

$$T(z) = (I - K(z)G(z)\Pi)^{-1}K(z)G(z)\Lambda. \quad (6.36)$$

The mean square norm of $G(z)$ with dimension $p \times m$, if exists, is defined as

$$\|G(z)\|_{\text{MS}} \triangleq \sqrt{\max_{i=1,2,\dots,p} \frac{1}{2\pi} \int_{-\pi}^{\pi} [G(e^{j\omega})G^H(e^{j\omega})]_{ii} d\omega}. \quad (6.37)$$

As proved in [10], under PTS with given Π and Λ in (6.6), the mean square stabi-

lization of the NCS in Figure 6.1 is equivalent to that

$$\inf_{K(z) \in \mathcal{K}, \Theta > 0, \text{diag}} \|\Theta^{-1}T(z)\Theta\|_{\text{MS}}^2 < 1, \quad (6.38)$$

where $T(z)$ is given in (6.36) and $\Theta \in \mathbb{R}^{m \times m}$ is a positive-definite diagonal matrix. Similarly to other robust control problems, the search for the optimal controller $K(z)$ on the left-hand side of (6.38) is generally nonconvex in Θ , and the minimal overall capacity for stabilizability has no explicit solution in general.

Let a pair of right and left coprime factorizations of $G(z)$ be given in the familiar way, namely

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \quad (6.39)$$

where $N, M, \tilde{N}, \tilde{M} \in \mathcal{RH}_\infty$, and satisfy the double Bezout identity

$$\begin{bmatrix} \tilde{Y} & -\tilde{X} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & X \\ N & Y \end{bmatrix} = I$$

for some $X, Y, \tilde{X}, \tilde{Y} \in \mathcal{RH}_\infty$. Then, the set of all proper controllers achieving the internal stability of the closed loop in Figure 6.1 with the network in Figure 6.2 can be parameterized as [115, p. 228]

$$\begin{aligned} \mathcal{K} &= \{K(z) : K(z) = \Pi^{-1}(X + MQ)(Y + NQ)^{-1} \\ &= \Pi^{-1}(\tilde{Y} + Q\tilde{N})^{-1}(\tilde{X} + Q\tilde{M}), Q \in \mathcal{RH}_\infty\}. \end{aligned} \quad (6.40)$$

In this situation, the transfer function $T(z)$ in (6.36) becomes

$$T(z) = \Pi^{-1}(X\tilde{N} + MQ\tilde{N})\Lambda, \quad (6.41)$$

where Π and Σ are given in (6.6).

6.3.1 SISO Plants

For SISO plants, we have $m = p = 1$, and $\xi(t) = \xi_1(t)$ is a scalar-valued white noise signal. In this case, the STS and the PTS are the same.

Consider an SISO plant $G(z)$ with n_ϕ anti-stable poles $\phi_1, \phi_2, \dots, \phi_{n_\phi}$, n_ζ distinct³ NMP zeros $\zeta_1, \zeta_2, \dots, \zeta_{n_\zeta}$ and relative degree $r \geq 1$. It is easy to derive that

$$\mathcal{M}(G) = \begin{cases} \prod_{i=1}^{n_\phi} |\phi_i|, & \text{if } n_\phi \geq 1; \\ 1, & \text{if } n_\phi = 0, \end{cases} \quad (6.42)$$

which is equal to $\mathcal{M}(A)$ with $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ being any detectable and stabilizable realization of $G(z)$. The Blaschke product with respect to the anti-stable poles of $G(z)$ is given by [94]

$$B_\phi(z) \triangleq \prod_{i=1}^{n_\phi} \frac{z - \phi_i}{1 - z\bar{\phi}_i}. \quad (6.43)$$

Since $B_\phi(z)$ is inner [135, p. 66], we can derive that $B_\phi(z^{-1}) = \sum_{k=0}^{\infty} \beta_k z^{-k}$ with $\beta_k \triangleq \frac{1}{k!} \frac{d^k}{dz^k} B_\phi(z)|_{z=0}$. Further denote [136]

$$\eta(G) \triangleq \sum_{l=1}^{n_\zeta} \sum_{k=1}^{n_\zeta} \frac{\bar{\gamma}_k \gamma_l}{\bar{\zeta}_k \zeta_l - 1}, \quad (6.44)$$

$$\varphi(G) \triangleq \sum_{l=1}^{r-1} (|\beta_l|^2 + |\psi_l|^2), \quad (6.45)$$

where $\psi_l \triangleq \sum_{k=1}^{n_\zeta} \gamma_k \zeta_k^{l-1}$ and

$$\gamma_k \triangleq (1 - |\zeta_k|^2) \left(B_\phi(\zeta_k^{-1}) - \sum_{i=0}^{r-1} \beta_i \zeta_k^{-i} \right) \prod_{i=1, i \neq k}^{n_\zeta} \frac{1 - \bar{\zeta}_i \zeta_k}{\zeta_k - \zeta_i}.$$

Proposition 6.3.1. *Assume that $G(z)$ is SISO and has no or distinct (if any) NMP zeros. The NCS in Figure 6.1 is mean square stabilizable via dynamic output*

³The assumption on distinct NMP zeros simplifies the subsequent analysis and may be relaxed at the expense of more complex expressions [134].

feedback if and only if

$$\mathcal{C}_{\text{MS}} > \frac{1}{2} \ln \{ \mathcal{M}(G)^2 + \eta(G) + \varphi(G) \}, \quad (6.46)$$

where $\mathcal{M}(G), \eta(G), \varphi(G)$ are defined in (6.42), (6.44) and (6.45), respectively.

Proof. For $m = 1$, we have $\|\Theta^{-1}T(z)\Theta\|_{\text{MS}}^2 = \|T(z)\|_2^2$ with $T(z) = (X\tilde{N} + MQ\tilde{N})(g - 1)^{-\frac{1}{2}}$. It follows that (6.38) is equivalent to

$$g > \inf_{Q \in \mathcal{RH}_\infty} \|X\tilde{N} + MQ\tilde{N}\|_2^2 + 1. \quad (6.47)$$

The evaluation of the right-hand side of (6.47) follows from [94, 136]. $\square$

6.3.2 Triangularly Decoupled Plants

Motivated by the observation that Wonham decomposition plays an important role in establishing the sufficiency part of Theorem 6.2.1, we adopt the next definition of triangular decoupling.

Definition 6.3.1 ([137]). $G(z)$ is said to be triangularly decoupled by $K(z) \in \mathcal{K}$, if $T(z)$ in (6.41) is either lower or upper triangular.

For $G = NM^{-1}$ with $M, N \in \mathcal{RH}_\infty$ and $m \leq p$, there always exist suitable unimodular matrices U_M, U_N such that

$$L_M = MU_M, \quad \begin{bmatrix} L_N \\ 0 \end{bmatrix} = U_N NU_M, \quad U_N G = \begin{bmatrix} L_N L_M^{-1} \\ 0 \end{bmatrix}, \quad (6.48)$$

where L_M, L_N are respectively square and lower (or upper) triangular with compatible dimensions. We refer to $L_G \triangleq L_N L_M^{-1}$ as the left triangular structure of $G(z)$.

Remark 6.3.1. As shown in [137], (i) L_M and L_N are unique up to postmultiplication and/or premultiplication by any lower (or upper) triangular unimodular

matrix, and are independent of the particular factorization of $G(z)$ in (6.39); (ii) a necessary and sufficient condition for the solvability of triangular decoupling is the corprimeness between each diagonal entry of L_M and the corresponding diagonal entry of L_N .

We sum up the assumptions as follows.

Assumption 6.3.1. *i). The plant $G(z)$ can be triangularly decoupled and satisfies $m \leq p$.*

ii). The i -th diagonal entry of the left triangular structure, $[L_G]_{ii}$, has no or distinct (if any) NMP zeros for every $i = 1, 2, \dots, m$.

Note that $[L_G]_{ii}$ has relative degree $r_i \geq 1$, since $U_N(z)$ in (6.48) is unimodular and $G(z)$ is strictly proper. Suppose that $U_N(z)$ has a minimal realization $\left[\begin{array}{c|c} A_U & B_U \\ \hline C_U & D_U \end{array} \right]$, then $\rho(A_U) < 1$ since $U_N(z)$ is unimodular. Under Assumption 6.3.1 i),

$$\left[\begin{array}{cc|c} A & 0 & B \\ B_U C & A_U & 0 \\ \hline D_U C & C_U & 0 \end{array} \right]$$

is a stabilizable and detectable realization for $U_N(z)G(z)$. It follows that

$$\prod_{i=1}^m \mathcal{M}([L_G]_{ii}) = \mathcal{M}(U_N G) = \mathcal{M} \left(\left[\begin{array}{cc} A & 0 \\ B_U C & A_U \end{array} \right] \right) = \mathcal{M}(A) = \mathcal{M}(G). \quad (6.49)$$

Theorem 6.3.1. *Under PTS and Assumptions 6.2.1 and 6.3.1, the NCS in Figure 6.1 is mean square stabilizable via dynamic output feedback if*

$$\mathcal{C}_{\text{MS}} > \frac{1}{2} \sum_{i=1}^m \ln \{ \mathcal{M}([L_G]_{ii})^2 + \eta([L_G]_{ii}) + \varphi([L_G]_{ii}) \}, \quad (6.50)$$

and only if

$$\mathcal{C}_{\text{MS}} > \ln \mathcal{M}(A). \quad (6.51)$$

Proof. We will first show the necessity. Assume that there exists $K(z)$ such that the closed loop is mean square stable. Since any dynamic output feedback controller can always be constructed from a dynamic state feedback controller, the condition (6.51) follows directly from the equivalence between i) and ii) in Lemma 6.2.1 and Theorem 6.2.2.

Next, the sufficiency of (6.50) will be shown. Suppose that L_G is in a lower triangular form. Let a left stable coprime factorization of L_G be $L_{\tilde{M}}^{-1}L_{\tilde{N}}$. According to Assumption 6.3.1 i) and Remark 6.3.1, $[L_M]_{ii}$ and $[L_N]_{ii}$ are coprime, and thus there always exist lower triangular $L_X, L_Y, L_{\tilde{X}}, L_{\tilde{Y}} \in \mathcal{RH}_\infty$ such that

$$\begin{bmatrix} L_{\tilde{Y}} & -L_{\tilde{X}} \\ -L_{\tilde{N}} & L_{\tilde{M}} \end{bmatrix} \begin{bmatrix} L_M & L_X \\ L_N & L_Y \end{bmatrix} = I.$$

It is easy to see that $[L_G]_{ii} = [L_N]_{ii}[L_M]_{ii}^{-1} = [L_{\tilde{M}}]_{ii}^{-1}[L_{\tilde{N}}]_{ii}$, and

$$\begin{bmatrix} [L_{\tilde{Y}}]_{ii} & -[L_{\tilde{X}}]_{ii} \\ -[L_{\tilde{N}}]_{ii} & [L_{\tilde{M}}]_{ii} \end{bmatrix} \begin{bmatrix} [L_M]_{ii} & [L_X]_{ii} \\ [L_N]_{ii} & [L_Y]_{ii} \end{bmatrix} = I.$$

If we construct right and left coprime factorizations of $U_N G$ as

$$U_N G = \begin{bmatrix} L_N \\ 0 \end{bmatrix} L_M^{-1} = \begin{bmatrix} L_{\tilde{M}} & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} L_{\tilde{N}} \\ 0 \end{bmatrix}, \quad (6.52)$$

then the set of all proper controllers achieving triangular decoupling can be parameterized as

$$\begin{aligned} \mathcal{K}_L &\triangleq \{[L_K \star]U_N \in \mathcal{K} : L_K = \Pi^{-1}(L_X + L_M L_Q)(L_Y + L_N L_Q)^{-1} \\ &= \Pi^{-1}(L_{\tilde{Y}} + L_Q L_{\tilde{N}})^{-1}(L_{\tilde{X}} + L_Q L_{\tilde{M}}), \text{ triangular } L_Q \in \mathcal{RH}_\infty\}, \end{aligned} \quad (6.53)$$

where $\star$ denotes the part that will not be used in the proof. Adopt one of those

triangular decoupling controllers $K(z) \in \mathcal{K}_L$, then $T(z)$ in (6.36) becomes

$$\begin{aligned}
T(z) &= K(I - G\Pi K)^{-1}G\Lambda \\
&= [L_K \star]U_N(I - G\Pi[L_K \star]U_N)^{-1}G\Lambda \\
&= [L_K \star](I - U_N G\Pi[L_K \star])^{-1}U_N G\Lambda \\
&= [\Pi^{-1}(L_X + L_M L_Q)(L_Y + L_N L_Q)^{-1} \star] \times \\
&\quad \left(I - \begin{bmatrix} L_{\tilde{M}} & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} L_{\tilde{N}} \\ 0 \end{bmatrix} \Pi[\Pi^{-1}(L_X + L_M L_Q)(L_Y + L_N L_Q)^{-1} \star] \right)^{-1} \times \\
&\quad \begin{bmatrix} L_{\tilde{M}} & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} L_{\tilde{N}} \\ 0 \end{bmatrix} \Lambda \\
&= \Pi^{-1}(L_X L_{\tilde{N}} + L_M L_Q L_{\tilde{N}})\Lambda.
\end{aligned}$$

By choosing $\Theta_1 \triangleq \Pi\Theta = \text{diag}\{\epsilon^{m-1}, \epsilon^{m-2}, \dots, 1\}$ with a small real number $\epsilon > 0$, it follows that

$$\begin{aligned}
\Theta^{-1}T(z)\Theta &= \Theta^{-1}\Pi^{-1}(L_X L_{\tilde{N}} + L_M L_Q L_{\tilde{N}})\Lambda\Theta \\
&= \Theta_1^{-1}(L_X L_{\tilde{N}} + L_M L_Q L_{\tilde{N}})\Lambda\Pi^{-1}\Theta_1 \\
&= \text{diag} \left\{ T_1(z)\sqrt{\frac{1}{g_1-1}}, T_2(z)\sqrt{\frac{1}{g_2-1}}, \dots, T_m(z)\sqrt{\frac{1}{g_m-1}} \right\} + o_z(\epsilon),
\end{aligned}$$

where $T_i(z) = [L_N]_{ii}[L_{\tilde{X}}]_{ii} + [L_M]_{ii}[L_Q]_{ii}[L_{\tilde{N}}]_{ii}$ with $[L_Q]_{ii} \in \mathcal{RH}_\infty$. According to the proof of Proposition 6.3.1, it follows that $\mathcal{C}_{\text{MSi}} > \frac{1}{2} \ln \{\mathcal{M}([L_G]_{ii})^2 + \eta([L_G]_{ii}) + \varphi([L_G]_{ii})\}$ is sufficient to ensure $\|T_i(z)\|_2^2/(g_i - 1) < 1$ under Assumption 6.3.1 ii). Moreover, given the condition (6.50) and Assumption 6.2.1, we have $\|\Theta^{-1}T(z)\Theta\|_{\text{MS}}^2 < 1$ by choosing a sufficiently small ϵ , i.e., the mean square stability of the closed-loop system is guaranteed. The result follows in a similar way by setting $\Theta_1 = \text{diag}\{1, \epsilon, \dots, \epsilon^{m-1}\}$ when L_G is in an upper triangular form. $\square$

Remark 6.3.2. Note that the proof of the necessity does not rely on Assumptions 6.2.1 and 6.3.1, thus (6.51) actually provides a uniform lower bound on the overall capacity for stabilizability via output feedback under PTS over all proper sta-

bilizing controllers. In view of (6.49), the gap between (6.50) and (6.51) shrinks to zero if $[L_G]_{ii}$ is minimum phase with relative degree 1 for all $i = 1, 2, \dots, m$.

6.4 Extension and Application

6.4.1 Extension on Stabilization over Output Fading Channels

Now, consider an NCS with an output network as depicted in Figure 6.3, where

$$y = G(z)u, \quad u = K(z)v, \quad v(t) = \xi(t)y(t). \quad (6.54)$$

Motivated by applications where a sensor network is used to monitor a phenomenon

Figure 6.3: Control over output fading channel(s).

or object of interest (e.g., a moving vehicle to be tracked) and the sensors are geographically distributed and are not collocated with the fusion center and/or the remote controller, we let the fading gain $\xi(t)$ in (6.54) and the overall mean square capacity of the network be given by $\xi(t) = \text{diag} \{\xi_1(t), \xi_2(t), \dots, \xi_p(t)\}$ and $\mathcal{C}_{\text{MS}} \triangleq \sum_{i=1}^p \mathcal{C}_{\text{MS}i}$, respectively. In this subsection, we assume that the controller only knows the first and second moments but not the exact value of $\xi(t)$.

Note that for $G = \tilde{M}^{-1}\tilde{N}$ with $\tilde{M}, \tilde{N} \in \mathcal{RH}_\infty$ and $p \leq m$, there exist suitable unimodular matrices $U_{\tilde{M}}, U_{\tilde{N}}$ such that $L_{\tilde{M}} = U_{\tilde{M}}\tilde{M}, [L_{\tilde{N}} \ 0] = U_{\tilde{M}}\tilde{N}U_{\tilde{N}}, GU_{\tilde{N}} = \begin{bmatrix} L_{\tilde{M}}^{-1}L_{\tilde{N}} & 0 \end{bmatrix}$, where $L_{\tilde{M}}, L_{\tilde{N}}$ are respectively square and lower (or upper) triangular

with compatible dimensions. We refer to $L_{\tilde{G}} \triangleq L_{\tilde{M}}^{-1} L_{\tilde{N}}$ as the right triangular structure of $G(z)$.

The proposition that follows extends the previous results on the NCS in Figure 6.1.

Proposition 6.4.1. *Consider the NCS in Figure 6.3.*

a) Assume that $B = I$. Under PTS and Assumption 6.2.1, the networked system is mean square stabilizable via static output feedback if and only if

$$\mathcal{C}_{\text{MS}} > \ln \mathcal{M}(A).$$

b) Assume that $G(z)$ is SISO and has no or distinct (if any) NMP zeros. The networked system is mean square stabilizable via dynamic output feedback if and only if

$$\mathcal{C}_{\text{MS}} > \frac{1}{2} \ln \{ \mathcal{M}(G)^2 + \eta(G) + \varphi(G) \}.$$

c) Assume that $G(z)$ can be triangularly decoupled, $p \leq m$, and $[L_{\tilde{G}}]_{ii}$ has no or distinct (if any) NMP zeros. Under PTS and Assumption 6.2.1, the networked system is mean square stabilizable via dynamic output feedback if

$$\mathcal{C}_{\text{MS}} > \frac{1}{2} \sum_{i=1}^p \ln \{ \mathcal{M}([L_{\tilde{G}}]_{ii})^2 + \eta([L_{\tilde{G}}]_{ii}) + \varphi([L_{\tilde{G}}]_{ii}) \},$$

and only if

$$\mathcal{C}_{\text{MS}} > \ln \mathcal{M}(A).$$

Proof. The proof follows analogously to those of Theorem 6.2.2, Proposition 6.3.1 and Theorem 6.3.1. $\square$

6.4.2 Application to Stabilization of a Finite Platoon

In automated highway systems, one basic issue is to move a platoon of closely spaced vehicles from one place to another [138, 139]. Suppose that there are $p + 1$

vehicles in a platoon, and the leader, i.e., vehicle 0, generates a position trajectory $\{x_0(t)\}_{t \geq 0}$ with $x_0(0) = 0$ according to its local reference signal. Each follower, i.e., vehicle i , for all $i = 1, 2, \dots, p$, uses its ranging sensor and local controller to keep a fixed distance behind the preceding vehicle.

Given any time-domain signal $x(t)$, denote its z -transform by $X(z)$. Let $x_i(t)$, $u_i(t)$, $G_i(z)$ be the position, the input and the dynamics of the i -th vehicle, and model the i -th vehicle starting from rest by

$$X_i(z) = G_i(z)U_i(z) + \frac{zx_i(0)}{z-1}, \quad i = 1, \dots, p. \quad (6.55)$$

We assume that $G_i(z)$ is SISO and strictly proper. The vehicle separations $e_i(t)$, $i = 1, 2, \dots, p$, are defined as $e_i(t) = x_i(t) - x_{i-1}(t) + \tau$, where $\tau > 0$ is a constant target separation for all followers. Suppose that the initial position of the i -th vehicle, $x_i(0)$, is $-i\tau$ indicating that the platoon starts with zero spacing errors. It follows that $E(z) = G(z)U(z) - [1 \ 0 \ \dots \ 0]'X_0(z)$ with $E(z) = [E_1(z) \ E_2(z) \ \dots \ E_p(z)]'$, $U(z) = [U_1(z) \ U_2(z) \ \dots \ U_p(z)]'$ and

$$G(z) = \begin{bmatrix} G_1(z) & 0 & 0 & \dots & 0 \\ -G_1(z) & G_2(z) & 0 & \dots & 0 \\ 0 & -G_2(z) & G_3(z) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -G_{p-1}(z) & G_p(z) \end{bmatrix}. \quad (6.56)$$

If we take the separation $e_i(t)$ as the output of vehicle i and let its ranging sensor obtain $e_i(t)$ through the i -th channel as $v_i(t) = \xi_i(t)e_i(t)$, then without considering $X_0(z)$, the platoon system fits in the model of Figure 6.3 with $m = p$, $y(t) = e(t)$. Since the local controller on vehicle i only uses $v_i(t)$, the distributed controller in Figure 6.3 has the form

$$U(z) = \text{diag}\{K_1(z), K_2(z), \dots, K_p(z)\}V(z), \quad (6.57)$$

where $V(z) = [V_1(z) \ V_2(z) \ \cdots \ V_p(z)]'$ is the output vector of parallel channels. As in the model (5.4), each channel experiences transmission failure and Nakagami fading [3] simultaneously:

$$\xi_i(t) = \Omega_i(t)\Upsilon_i(t), \quad i = 1, 2, \dots, p, \quad (6.58)$$

and $\xi_1(t), \xi_2(t), \dots, \xi_p(t)$ are uncorrelated with each other. In (6.58), $\Omega_i(t)$ is 0/1-valued (0 for “failure”, 1 for “success”) with probability distribution

$$\Pr\{\Omega_i(t) = 0\} = \alpha_i, \quad \Pr\{\Omega_i(t) = 1\} = 1 - \alpha_i, \quad 0 \leq \alpha_i < 1,$$

and $\Upsilon_i(t)$ is Nakagami distributed with ϖ_i denoting the mean channel power gain and $\varrho_i \in [\frac{1}{2}, \infty)$ describing the severity of fading (i.e., the severity of fading decreases as ϱ_i increases) [5]. Further assume that $\Omega_i(t), \Upsilon_i(t)$ are uncorrelated, then it is easy to derive that

$$g_i = \frac{\varrho_i \Gamma(\varrho_i)^2}{\varrho_i \Gamma(\varrho_i)^2 - (1 - \alpha_i) \Gamma(\varrho_i + \frac{1}{2})^2}, \quad (6.59)$$

where $\Gamma(\cdot)$ is the gamma function. Based on the expression of g_i in (6.59) and $\mathcal{C}_{\text{MS}i} = \frac{1}{2} \ln g_i$, it is intuitive to see that the smaller the α_i or the larger the ϱ_i , the larger the g_i and $\mathcal{C}_{\text{MS}i}$. The effect of transmission failure and Nakagami fading disappears as $\alpha_i \rightarrow 0$ and $\varrho_i \rightarrow \infty$, respectively. In particular, we have $\lim_{\varrho_i \rightarrow \infty} g_i = \frac{1}{\alpha_i}$ since $\lim_{\varrho_i \rightarrow \infty} \frac{\varrho_i \Gamma(\varrho_i)^2}{\Gamma(\varrho_i + \frac{1}{2})^2} = 1$, which is reduced to the scenario considered in Corollary 5.1 of [140].

We have the next corollary on stabilization of a finite platoon.

Corollary 6.4.1. *Assume that $G_i(z)$, for all $i = 1, 2, \dots, p$, is SISO, strictly proper and has no or distinct (if any) NMP zeros. Then, the finite platoon system described above with $p + 1$ vehicles is mean square stabilizable via dynamic output feedback over the network (6.58) if and only if $\mathcal{C}_{\text{MS}i} > \frac{1}{2} \ln \{\mathcal{M}(G_i)^2 + \eta(G_i) + \varphi(G_i)\}$, $i = 1, 2, \dots, p$, where $\mathcal{C}_{\text{MS}i} = \frac{1}{2} \ln g_i$, and g_i is given in (6.59).*

Proof. Based on the diagonal structure of $K(z)$ in (6.57) and the lower triangular

structure of $G(z)$ in (6.56), without considering both $X_0(z)$ and the uncertainty block, we have

$$\begin{aligned}
 T(z) &= G(z)(I - K(z)\Pi G(z))^{-1}K(z)\Lambda \\
 &= \begin{bmatrix} T_1 & 0 & 0 & \cdots & 0 \\ -T_1S_2 & T_2 & 0 & \cdots & 0 \\ T_1T_2S_3 & -T_2S_3 & T_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \prod_{i=1}^{p-1}(-T_i)S_p & \cdots & T_{p-2}T_{p-1}S_p & -T_{p-1}S_p & T_p \end{bmatrix} \Pi^{-1}\Lambda, \quad (6.60)
 \end{aligned}$$

where $S_i = (1 - K_i\mu_i G_i)^{-1}$ and $T_i = G_i(1 - K_i\mu_i G_i)^{-1}K_i\mu_i$. Denote the set of all proper controllers achieving the internal stability by $\mathcal{K}$. It follows from the proofs of Propositions 6.4.1 b) and 6.3.1 that

$$\inf_{K(z) \in \mathcal{K}} \|T_i(z)\|_2^2 = \mathcal{M}(G_i)^2 + \eta(G_i) + \varphi(G_i) - 1. \quad (6.61)$$

For any diagonal scaling matrix Θ , the condition (6.38) implies that $g_i > \mathcal{M}(G_i)^2 + \eta(G_i) + \varphi(G_i)$, which shows the necessity.

For the sufficiency, a constructive proof will be given. By choosing $K_i(z)$ based on (6.61), we can ensure that $G_i(1 - K_i\mu_i G_i)^{-1}$, $(1 - K_i\mu_i G_i)^{-1}K_i\mu_i$ and S_i are all stable and proper. It is then easy to verify that $K(z) = \text{diag}\{K_1(z), K_2(z), \dots, K_p(z)\} \in \mathcal{K}$ by invoking the special structures of $G(z)$ and $K(z)$, i.e., the internal stability of the overall closed loop in Figure 6.3 is guaranteed. Thus, by setting $\Theta = \text{diag}\{\epsilon^{p-1}, \epsilon^{p-2}, \dots, 1\}$ with sufficiently small ϵ , we have (6.38) is true. This completes the proof. $\square$

The numerical example as below demonstrates the usefulness of Corollary 6.4.1.

Example 6.4.1. Consider a homogeneous platoon with 5 vehicles and

$$G_i(z) = \frac{1}{(z-1)^2}, \quad i = 1, 2, 3, 4.$$

Suppose that the leader has a constant speed: 1 meter/second. It is easy to check that the infimum in (6.61) is 0 which is not achievable by any stabilizing controller due to the double marginally stable poles of $G_i(z)$. If we approximate $z = 1$ by $z = 1 - \varepsilon$ with a small real number $\varepsilon > 0$, then the design procedure in [136] yields the controller:

$$K_i = \frac{2\varepsilon - (2\varepsilon + \varepsilon^2)z}{(z + 2\varepsilon)\mu_i}, \quad (6.62)$$

which can approach the infimum in (6.61) with any desired accuracy by choosing a sufficiently small ε .

Let $\alpha_i = 0.2$, $\varpi_i = 2$, $\varrho_i = 2$ for the i -th channel, then g_i in (6.59) is 3.4113. By setting $\varepsilon = 0.1$ in (6.62), we have

$$\|T_i(z)\|_2^2 = \|G_i(1 - K_i\mu_i G_i)^{-1}K_i\mu_i\|_2^2 = 0.1489.$$

Since $\|T_i(z)\|_2^2/(g_i - 1) = 0.0436 < 1$ for all $i = 1, 2, 3, 4$, Corollary 6.4.1 implies that the platoon can be stabilized in the mean square sense by the controller (6.62) with $\varepsilon = 0.1$. The trajectories of the vehicles in the platoon for one sample of simulation are shown in Figure 6.4.

Figure 6.4: Vehicle trajectories in the platoon for one sample of simulation.

6.5 Summary

It has been shown in this chapter that for a discrete-time LTI plant with LTI feedback over input fading channels under PTS and the assumption on capacity allocation, the minimal overall mean square capacity of the network for mean square stabilizability can be given in terms of the Mahler measure of the plant in the case of state feedback. The minimal capacity for stabilizability via state feedback under STS in general can only be computed by optimization. We have also investigated the mean square stabilization of SISO plants or MIMO plants via dynamic output feedback. For SISO plants, the corresponding minimal mean square capacity for stabilizability is given in terms of the anti-stable poles, NMP zeros and relative degree of the plant. For triangularly decoupled MIMO plants, necessary and sufficient conditions have been provided. In addition, the results have been extended to the case with output fading channels and applied to vehicle platooning.

Chapter 7

Control over Fading Channels: Additional Degree of Freedom and Power Constraint

This is a complementary chapter of Chapter 6. In Section 7.1, we analyze the possible benefits of introducing pre- and post-channel filters and channel feedback to the NCS in Figure 6.1. Section 7.2 considers the feedback stabilization and performance design of an SISO NCS with a stochastic channel contaminated by both multiplicative and additive noises. We provide a necessary and sufficient condition on the network for mean square stabilizability, a suboptimal algorithm for performance design along with a numerical example. Some concluding remarks are given in Section 7.3.

7.1 The Effect of Channel Processing and Channel Feedback

As pointed out in Remark 6.2.6, additional degree of freedom can benefit the NCS in reducing the capacity requirement for stabilizability. Here, we further analyze the

possible advantages of introducing pre- and post-channel filters and channel feedback into the NCS in Figure 6.1. To this end, we consider the NCS in Figure 7.1.

Figure 7.1: Control over input fading channel(s) with channel processing and channel feedback.

The system interconnections are described as follows:

$$y = G(z)u, \quad v = K(z)y, \quad s = F_{1v}(z)v + F_{1h}(z)h, \quad u = F_2(z)h, \quad h(t) = \xi(t)s(t), \quad (7.1)$$

where $F_1(z) = [F_{1v}(z) \ F_{1h}(z)]$, $s(t) \in \mathbb{R}^m$ is the channel input, $h(t) \in \mathbb{R}^m$ is the channel output, and $\xi(t)$ has the form as in (6.4). We assume that the channel feedbacks h back to the pre-filter with one-step delay, therefore $F_{1h}(z)$ is assumed to be strictly proper. As mentioned in Remark 1.2.1, the channel feedback in Figure 7.1 is equivalent to sending the CSI back to the pre-filter. We can derive the following result.

Theorem 7.1.1. *Consider the NCS in Figure 7.1, where $K(z)$, $F_1(z)$, $F_2(z)$ are to be designed. Under PTS and Assumption 6.2.1, the networked system is mean square stabilizable via state feedback or dynamic output feedback if and only if*

$$\mathcal{C}_{\text{MS}} > \ln \mathcal{M}(A).$$

Proof. First, we will show the necessity for the dynamic state feedback case, which also implies the necessity for the static state feedback and dynamic output feedback cases. Note that Lemma 6.1.1 is still true with the overall system state $x(t) = [x_G(t)' \ x_K(t)' \ x_{F_1}(t)' \ x_{F_2}(t)']'$, where $x_{F_1}(t)$ and $x_{F_2}(t)$ are the states of the pre-filter and the post-filter, respectively. Suppose that $F_2(z)$ has a state-space realization

$\left[\begin{array}{c|c} A_{F_2} & B_{F_2} \\ \hline C_{F_2} & D_{F_2} \end{array} \right]$, then the interconnection between the plant and the post-filter yields

$$G(z)F_2(z) = \left[\begin{array}{cc|c} A & BC_{F_2} & BD_{F_2} \\ 0 & A_{F_2} & B_{F_2} \\ \hline C & 0 & 0 \end{array} \right].$$

Considering $G(z)F_2(z)$ as an augmented plant and following similar lines of the proof of Theorem 6.3.1, we have

$$\prod_{i=1}^m \left(1 + \frac{\mu_i^2}{\sigma_i^2} \right) > \mathcal{M} \left(\left[\begin{array}{cc} A & BC_{F_2} \\ 0 & A_{F_2} \end{array} \right] \right)^2 = \mathcal{M}(A)^2 \mathcal{M}(A_{F_2})^2 \geq \mathcal{M}(A)^2,$$

i.e., $\mathcal{C}_{MS} > \ln \mathcal{M}(A)$.

To show the sufficiency, a constructive proof will be given. Let $F_{1v}(z) = I$, $F_2(z) = I$, then we have $s = K(z)y + F_{1h}(z)u$. Construct an observer-based realization for $[K(z) \ F_{1h}(z)]$ as

$$\begin{aligned} \hat{x}_G(t+1) &= A\hat{x}_G(t) + L(y(t) - C\hat{x}_G(t)) + Bu(t), \\ s(t) &= K\hat{x}_G(t), \end{aligned} \tag{7.2}$$

where L and K are respectively the observer gain and the state feedback gain to be designed. Then, $F_{1h}(z)$ is strictly proper, and the closed loop with $x(t) = [x_G(t)' \ \hat{x}_G(t)']'$ is given by

$$\begin{aligned} x(t+1) &= \left(\left[\begin{array}{cc} A & 0 \\ LC & A - LC \end{array} \right] + \left[\begin{array}{c} B \\ B \end{array} \right] \xi(t) [0 \ K] \right) x(t) \\ &= \left[\begin{array}{cc} I & 0 \\ I & I \end{array} \right] \left(\left[\begin{array}{cc} A & 0 \\ 0 & A - LC \end{array} \right] + \left[\begin{array}{c} B \\ 0 \end{array} \right] \xi(t) [K \ K] \right) \left[\begin{array}{cc} I & 0 \\ I & I \end{array} \right]^{-1} x(t). \end{aligned}$$

By using the similar technique in the proof of the equivalence between i) and v) in Lemma 6.2.1, we can show that the mean square stability of the closed loop is

ensured by selecting L such that $\rho(A - LC) < 1$ and K as in the sufficiency part of Proposition 6.2.2. $\square$

Remark 7.1.1. *As we can see from Proposition 6.2.2 and Theorem 7.1.1, when the full state is available to the controller, both the filtering and the channel feedback of the network are redundant in stabilization. By comparing Proposition 6.3.1 and Theorem 6.3.1 with Theorem 7.1.1 and in view of the proof of Theorem 7.1.1, we can conclude that, in the case of dynamic output feedback, 1) $F_{1v}(z)$ and $F_2(z)$ are redundant parameters in stabilizing the NCS in Figure 7.1; 2) the channel feedback and $F_{1h}(z)$ can completely eliminate the limitation on stabilization of the NCS induced by the NMP zeros and high relative degree of the plant; 3) the post-filter should be chosen to be stable otherwise the network requirement would be more stringent; 4) the assumptions on the triangular decoupling of the plant and the distinctness of NMP zeros are not needed in Theorem 7.1.1.*

7.2 The Effect of Power Constraint

In this section, we further analyze the effect of channel additive noise and channel input power bound on stability and performance of the NCS. In the following, a random variable or process is said to be uncorrelated, without specifying with respect to which variables or processes, if it is uncorrelated with any other random variable or process. If a scalar-valued discrete-time stochastic process $x = \{x(t)\}_{t=0}^{\infty}$ is bounded and convergent in the mean square sense, then the power norm of x is defined as $\|x\|_{\mathcal{P}} \triangleq \sqrt{\lim_{t \rightarrow \infty} \mathcal{E}\{x(t)^2\}}$. Both the time index t and the frequency variable z will be omitted whenever no confusion is caused.

We limit our attention to the SISO LTI case and consider the NCS as depicted in Figure 7.2. The SISO plant has a transfer function $G(z)$, and the measured output $y \in \mathbb{R}$ is given by

$$y = G(z)u + d, \quad (7.3)$$

Figure 7.2: Control over an input fading channel with channel additive noise and network scaling.

where $u \in \mathbb{R}$ is the plant input, and $d \in \mathbb{R}$ is the output disturbance. The controller and the pre-filter have the form

$$v = L_r(z)r + L_y(z)y, \quad (7.4)$$

$$g = L_v(z)v + L_h(z)h, \quad (7.5)$$

where $g \in \mathbb{R}$, $r \in \mathbb{R}$, $h \in \mathbb{R}$ are respectively the output of the pre-filter, the reference signal, and the input of the post-filter. The post-filter has a transfer function $F(z)$ as

$$u = F(z)h. \quad (7.6)$$

Let the pre-scaling and post-scaling in Figure 7.2 be α^{-1} and α respectively, then we have

$$\hat{g} = \alpha^{-1}g, \quad h = \alpha\hat{h}, \quad (7.7)$$

and α is called the scaling factor of the network. The model of the single fading channel is described as

$$\hat{h}(t) = \xi(t)\hat{g}(t) + n(t), \quad (7.8)$$

where $\xi(t) \in \mathbb{R}$ denotes the channel fading, and $n(t) \in \mathbb{R}$ represents the channel additive noise.

Denote the state of the overall system in Figure 7.2 by $x(t)$. The assumption that

follows will be adopted throughout this section.

Assumption 7.2.1. *Consider the NCS described by Figure 7.2 and equations (7.3)-(7.8).*

- (i) *[Constraints on LTI components] The plant transfer function $G(z)$ is unstable and strictly proper. The transfer functions $L_h(z) \in \mathcal{R}_{\text{sp}}$, $L_r(z)$, $L_y(z)$, $L_v(z)$, $F(z) \in \mathcal{R}_{\text{p}}$ are to be designed. The scaling factor of the network satisfies $0 < \alpha < \infty$.*
- (ii) *[Constraints on the randomness] The initial state of the whole system, $x(0)$, is an uncorrelated second-order random variable with mean $\mu_x(0)$ and second-order moment matrix $\Xi(0)$. The exogenous signals r, d, n are uncorrelated white noise processes with zero means and bounded variances $\sigma_r^2, \sigma_d^2, \sigma_n^2$. The channel fading $\xi(t)$ is an uncorrelated white noise process with mean $\mu_\xi \neq 0$ and bounded variance σ_ξ^2 .*
- (iii) *[Constraint on the power of channel input] The channel input $\hat{g}$ satisfies*

$$\|\hat{g}\|_{\mathcal{P}}^2 < \mathcal{P}, \quad (7.9)$$

and $\mathcal{P} > 0$ is the predefined power level.

Remark 7.2.1. *Note that the strict properness of $G(z)$ and the constraints on the randomness lead to simpler formulae and can be relaxed at the expense of more involved expressions. For communication systems, a power bound on the channel input as in (7.9) is usually introduced in order to avoid the interference to other communication users and/or due to the hardware limitations of the transmitter.*

7.2.1 Feedback Stabilization

The concept of stability used in this section is presented as follows.

Definition 7.2.1. Consider the NCS described by Figure 7.2 and equations (7.3)-(7.8). The closed loop is said to be mean square stable, if, for every $x(0), r, d, n, \xi$ satisfying Assumption 7.2.1 (ii), there exists a bounded matrix $\Xi_x \geq 0$ independent of $x(0)$ such that $\Xi(t) \triangleq \mathcal{E}\{x(t)x(t)'\}$ is well defined for every $t \geq 0$, and $\lim_{t \rightarrow \infty} \Xi(t) = \Xi_x$.

Since the controller is collocated with the pre-filter, thus we can lump (7.4) and (7.5) together as

$$g = L_v(z)L_r(z)r + L_v(z)L_y(z)y + L_h(z)h. \quad (7.10)$$

Under Assumption 7.2.1, the NCS in Figure 7.2 with equations (7.3)-(7.8) is equivalent to the NCS in Figure 7.3, where $z(t) = \hat{g}(t)$, $w(t) = \Delta(t)z(t)$, and

$$L(z) \triangleq [L_v(z)L_r(z) \ L_v(z)L_y(z) \ L_h(z)],$$

$$\Delta(t) \triangleq \sigma_\xi^{-1} (\xi(t) - \mu_\xi).$$

It is easy to verify that the mean square stability of the NCS in Figure 7.3 always

Figure 7.3: Representation of the NCS in Figure 7.2 with channel model (7.8) under Assumption 7.2.1.

implies the internal stability of the corresponding closed-loop system in the absence of r, d, n, w . Let

$$K(z) \triangleq \begin{bmatrix} L_v(z)L_r(z) & L_v(z)L_y(z) & L_h(z) \\ 0 & 0 & F(z) \end{bmatrix}, \quad (7.11)$$

and denote the set of $K(z)$ satisfying Assumption 7.2.1 (i) and achieving the above

internal stability by $\mathcal{K}$.

The theorem that follows presents a necessary and sufficient condition for mean square stabilizability.

Theorem 7.2.1. *Consider the NCS described by Figure 7.2 and equations (7.3)-(7.8). Under Assumption 7.2.1, the plant $G(z)$ can be mean square stabilized by designing $K(z)$ and α if and only if*

$$\frac{\mu_\xi^2 \mathcal{P}}{\sigma_\xi^2 \mathcal{P} + \sigma_n^2} + 1 > \mathcal{M}(G)^2. \quad (7.12)$$

Proof. To show the necessity of (7.12), we first assume that the closed loop is mean square stable. In this case, all signals that appear in the loop of Figure 7.3 are convergent in the mean square sense. Therefore, without loss of generality we can assume that $x(0) = 0$ and consider directly the limiting second-order moments of the relevant signals. When an SISO plant is controlled over a single fading channel, we have $\|T_{w\hat{g}}(z)\|_2^2 < 1$ from Theorem 6.4 in [10], where $T_{w\hat{g}}(z)$ denotes the transfer function of the closed loop from w to $\hat{g}$ without considering the uncertainty block Δ . It is easy to see from Figure 7.3 that

$$\hat{g} = T_{r\hat{g}}r + T_{d\hat{g}}d + T_{n\hat{g}}n + T_{w\hat{g}}w, \quad (7.13)$$

where

$$T_{r\hat{g}} = \alpha^{-1}(1 - L_v L_y G F \mu_\xi - L_h \mu_\xi)^{-1} L_v L_r, \quad (7.14)$$

$$T_{d\hat{g}} = \alpha^{-1}(1 - L_v L_y G F \mu_\xi - L_h \mu_\xi)^{-1} L_v L_y, \quad (7.15)$$

$$T_{n\hat{g}} = (1 - L_v L_y G F \mu_\xi - L_h \mu_\xi)^{-1} (L_v L_y G F + L_h), \quad (7.16)$$

$$T_{w\hat{g}} = T_{n\hat{g}} \sigma_\xi. \quad (7.17)$$

In view of Assumption 7.2.1 (iii), (7.13), $w = \Delta \hat{g}$ and $1 - \|T_{w\hat{g}}(z)\|_2^2 > 0$, we have

$$\mathcal{P} > \|\hat{g}\|_{\mathcal{P}}^2 = \|T_{r\hat{g}}\|_2^2 \sigma_r^2 + \|T_{d\hat{g}}\|_2^2 \sigma_d^2 + \|T_{w\hat{g}}\|_2^2 \left(\frac{\sigma_n^2}{\sigma_\xi^2} + \|\hat{g}\|_{\mathcal{P}}^2 \right)$$

$$= \frac{\|T_{r\hat{g}}\|_2^2 \sigma_r^2 + \|T_{d\hat{g}}\|_2^2 \sigma_d^2 + \|T_{w\hat{g}}\|_2^2 \frac{\sigma_n^2}{\sigma_\xi^2}}{1 - \|T_{w\hat{g}}\|_2^2} \quad (7.18)$$

$$\geq \frac{\|T_{w\hat{g}}\|_2^2 \frac{\sigma_n^2}{\sigma_\xi^2}}{1 - \|T_{w\hat{g}}\|_2^2}, \quad (7.19)$$

i.e.,

$$\frac{\sigma_\xi^2 \mathcal{P}}{\sigma_\xi^2 \mathcal{P} + \sigma_n^2} > \|T_{w\hat{g}}\|_2^2. \quad (7.20)$$

Rewrite $T_{w\hat{g}}$ in (7.17) as

$$T_{w\hat{g}} = (1 - K_1 G - K_2)^{-1} (K_1 G + K_2) \mu_\xi^{-1} \sigma_\xi$$

with

$$K_1 = L_v L_y F \mu_\xi, \quad K_2 = L_h \mu_\xi. \quad (7.21)$$

Let

$$\check{G} = \begin{bmatrix} G \\ z^{-1} \end{bmatrix}, \quad \check{K} = [K_1 \quad K_2 z].$$

Under Assumption 7.2.1 (i), we have

$$\begin{aligned} \inf_{K \in \mathcal{K}, \alpha \in (0, \infty)} \|T_{w\hat{g}}\|_2^2 &= \inf_{\text{IS}, K_1 \in \mathcal{R}_p, K_2 \in \mathcal{R}_{sp}} \left\| (1 - K_1 G - K_2)^{-1} (K_1 G + K_2) \frac{\sigma_\xi}{\mu_\xi} \right\|_2^2 \\ &= \frac{\sigma_\xi^2}{\mu_\xi^2} \inf_{\text{IS}, \check{K} \in \mathcal{R}_p} \|(1 - \check{K} \check{G})^{-1} \check{K} \check{G}\|_2^2 \end{aligned} \quad (7.22)$$

$$= \frac{\sigma_\xi^2}{\mu_\xi^2} (\mathcal{M}(G)^2 - 1), \quad (7.23)$$

where the abbreviation ‘IS’ stands for the constraint on the internal stability of the closed loop, and (7.23) follows from $\mathcal{H}_2$ optimization and the Residue theorem; see, e.g., Theorem 4.1 in [12]. It follows from (7.23) that

$$\|T_{w\hat{g}}\|_2^2 \geq \frac{\sigma_\xi^2}{\mu_\xi^2} (\mathcal{M}(G)^2 - 1). \quad (7.24)$$

By combining the inequalities (7.20) and (7.24), we can conclude that (7.12) is true.

To show the converse, we will prove that if (7.12) is true, then there exists a set of $K(z)$ and α satisfying Assumption 7.2.1 such that the closed-loop system is mean square stable. Note that it is always possible to select $\check{K}$ in (7.22) appropriately such that

$$\|T_{w\hat{g}}\|_2^2 \leq \frac{\sigma_\xi^2}{\mu_\xi^2}(\mathcal{M}(G)^2 - 1) + \epsilon_1 \quad (7.25)$$

for any small positive real number ϵ_1 . Choose $\epsilon_1 \leq \frac{\sigma_n^2}{\mu_\xi^2 \mathcal{P}}$, then $\|T_{w\hat{g}}\|_2^2 < 1$ is true under the condition (7.12). It is direct to derive a set of $L_v \in \mathcal{R}_p, L_y \in \mathcal{R}_p, F \in \mathcal{R}_p, L_h \in \mathcal{R}_{sp}$ according to

$$\check{K} = [L_v L_y F \mu_\xi \quad L_h \mu_\xi z] \quad (7.26)$$

for the given $\check{K} \in \mathcal{R}_p$, and the internal stability of the closed loop in Figure 7.3 is ensured after choosing a proper and stable L_r (e.g., $L_r = 0$). The mean square stability of the closed loop follows from Theorem 6.4 in [10]. The only thing left to be shown is that the power constraint $\|\hat{g}\|_{\mathcal{P}}^2 < \mathcal{P}$ can be satisfied along with the above mean square stability. Since the closed loop has been mean square stabilized, in the following we can merely focus on the limiting second-order moments of relevant signals. According to the expressions of $T_{r\hat{g}}, T_{d\hat{g}}$ in (7.14) and (7.15), we can ensure that, for any small positive real number ϵ_2 ,

$$\|T_{r\hat{g}}\|_2^2 \sigma_r^2 + \|T_{d\hat{g}}\|_2^2 \sigma_d^2 \leq \epsilon_2 \quad (7.27)$$

by choosing a sufficiently large α . It follows from (7.25), (7.27) and (7.18) that

$$\|\hat{g}\|_{\mathcal{P}}^2 \leq \frac{\epsilon_2 + \left[\frac{\sigma_\xi^2}{\mu_\xi^2}(\mathcal{M}(G)^2 - 1) + \epsilon_1 \right] \frac{\sigma_n^2}{\sigma_\xi^2}}{1 - \left[\frac{\sigma_\xi^2}{\mu_\xi^2}(\mathcal{M}(G)^2 - 1) + \epsilon_1 \right]}. \quad (7.28)$$

Thus, $\mathcal{P} > \|\hat{g}\|_{\mathcal{P}}^2$ if

$$\frac{\mu_\xi^2 \mathcal{P}}{\sigma_\xi^2 \mathcal{P} + \sigma_n^2} \geq \mathcal{M}(G)^2 - 1 + \frac{\epsilon_1 (\frac{\mu_\xi^2}{\sigma_\xi^2} \sigma_n^2 + \mu_\xi^2 \mathcal{P}) + \epsilon_2 \mu_\xi^2}{\sigma_\xi^2 \mathcal{P} + \sigma_n^2}. \quad (7.29)$$

We note that (7.12) implies (7.29) if both ϵ_1 and ϵ_2 are made to be sufficiently small. Therefore, $\|\hat{g}\|_{\mathcal{P}}^2 < \mathcal{P}$ can be satisfied by designing K and α , which completes the proof of sufficiency. $\square$

Remark 7.2.2. *Theorem 7.2.1 recovers the scenario with a pure additive noise channel by setting $\mu_\xi = 1$ and $\sigma_\xi^2 = 0$, and the inequality (7.12) comes to be*

$$\frac{\mathcal{P}}{\sigma_n^2} + 1 > \mathcal{M}(G)^2, \text{ i.e., } \frac{1}{2} \ln \left(\frac{\mathcal{P}}{\sigma_n^2} + 1 \right) > \ln \mathcal{M}(G),$$

as identified in Theorem 4.1 of [12], where $\frac{1}{2} \ln \left(\frac{\mathcal{P}}{\sigma_n^2} + 1 \right)$ is the communication capacity of an additive white Gaussian noise channel. Theorem 7.2.1 also recovers the scenario with a pure multiplicative noise channel by letting $\sigma_n^2 = 0$ or $\mathcal{P} \rightarrow \infty$, where α becomes redundant and can be chosen arbitrarily from $(0, \infty)$ (e.g., $\alpha = 1$), and the condition (7.12) is reduced to

$$\frac{\mu_\xi^2}{\sigma_\xi^2} + 1 > \mathcal{M}(G)^2, \text{ i.e., } \frac{1}{2} \ln \left(\frac{\mu_\xi^2}{\sigma_\xi^2} + 1 \right) > \ln \mathcal{M}(G), \quad (7.30)$$

where $\frac{1}{2} \ln \left(\frac{\mu_\xi^2}{\sigma_\xi^2} + 1 \right)$ is the mean square capacity of a multiplicative noise channel as defined in (6.28) in the previous chapter. As we can see from (7.12) and (7.30), when the power of channel input is limited, a more stringent requirement on the quality of the channel is needed for mean square stabilizability.

Remark 7.2.3. *Note that L_v, L_y, F play the same role in Theorem 7.2.1, and it is thus sufficient to use any of them in stabilizing the NCS. For example, for any $\check{K} \in \mathcal{R}_p$, (7.26) can be easily satisfied by designing L_y and selecting $L_v, F \in \mathcal{U}_\infty$ (e.g., $L_v = F = 1$) without affecting the degree of freedom and the internal stability. In addition, the scaling factor of the network α is an essential design parameter in satisfying the power constraint on the channel input.*

Different from the model (7.8), another channel model is given by

$$\hat{h}(t) = \xi(t)(\hat{g}(t) + n(t)), \quad (7.31)$$

where $\xi(t)$ and $n(t)$ are as defined in (7.8). The model (7.31) is suitable for describing the digital erasure channel, where $n(t)$ is used to model uncertainties arising from quantization, and $\xi(t)$ is 0-1 binary valued and represents the packet-loss process. In this case, the NCS in Figure 7.2 with equations (7.3)-(7.7) and (7.31) can be rewritten into the NCS in Figure 7.4, and Theorem 7.2.1 can be readily extended as follows.

Figure 7.4: Representation of the NCS in Figure 7.2 with channel model (7.31) under Assumption 7.2.1.

Corollary 7.2.1. *Consider the NCS described by Figure 7.4 and equations (7.3)-(7.7) and (7.31). Under Assumption 7.2.1, the plant $G(z)$ can be mean square stabilized by designing $K(z)$ and α if and only if*

$$\frac{\mu_\xi^2 \mathcal{P}}{\sigma_\xi^2 \mathcal{P} + (\mu_\xi^2 + \sigma_\xi^2) \sigma_n^2} + 1 > \mathcal{M}(G)^2. \quad (7.32)$$

7.2.2 Performance Design

Denote the tracking error by $e = r - y$, and use its power norm $J \triangleq \|e\|_{\mathcal{P}}^2$ as the performance index to be minimized. First of all, we have the following result.

Proposition 7.2.1. *Consider the NCS described by Figure 7.2 and equations (7.3)-(7.8). Under Assumption 7.2.1, J is bounded if $G(z)$ is mean square stabilized by $K(z)$ and α .*

Proof. If the closed-loop system is mean square stable, then we can easily deduce

that

$$e = r - (T_{ry}r + T_{dy}d + T_{ny}n + T_{wy}w), \quad (7.33)$$

where

$$\begin{aligned} T_{ry} &= GF(1 - \mu_\xi L_v L_y GF - \mu_\xi L_h)^{-1} \mu_\xi L_v L_r, \\ T_{dy} &= GF(1 - \mu_\xi L_v L_y GF - \mu_\xi L_h)^{-1} \mu_\xi L_v L_y + 1, \\ T_{wy} &= GF(1 - \mu_\xi L_v L_y GF - \mu_\xi L_h)^{-1} \alpha \sigma_\xi, \\ T_{ny} &= T_{wy} \sigma_\xi^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} J &= \|T_{wy}\|_2^2 \left(\frac{\sigma_n^2}{\sigma_\xi^2} + \|\hat{g}\|_{\mathcal{P}}^2 \right) + \|1 - T_{ry}\|_2^2 \sigma_r^2 + \|T_{dy}\|_2^2 \sigma_d^2 \\ &= \|T_{wy}\|_2^2 \left(\frac{\sigma_n^2}{\sigma_\xi^2} + \frac{\|T_{r\hat{g}}\|_2^2 \sigma_r^2 + \|T_{d\hat{g}}\|_2^2 \sigma_d^2 + \|T_{w\hat{g}}\|_2^2 \frac{\sigma_n^2}{\sigma_\xi^2}}{1 - \|T_{w\hat{g}}\|_2^2} \right) + \|1 - T_{ry}\|_2^2 \sigma_r^2 + \|T_{dy}\|_2^2 \sigma_d^2, \end{aligned} \quad (7.34)$$

where the expression for $\|\hat{g}\|_{\mathcal{P}}^2$ is given in (7.18). Since the mean square stability of the closed-loop system always implies the internal stability and the inequality $\|T_{w\hat{g}}\|_2^2 < 1$, the performance index J in (7.34) is bounded. $\square$

After introducing the notation:

$$\begin{aligned} \beta_1 &\triangleq \|T\mu_\xi^{-1}\|_2^2, \\ \beta_2 &\triangleq \|1 - GFS\mu_\xi L_v L_r\|_2^2 \sigma_r^2 + \|GFS\mu_\xi L_v L_y + 1\|_2^2 \sigma_d^2, \\ \beta_3 &\triangleq \|GFS\|_2^2, \\ \beta_4 &\triangleq \|SL_v L_r\|_2^2 \sigma_r^2 + \|SL_v L_y\|_2^2 \sigma_d^2 \end{aligned}$$

with $S = (1 - \mu_\xi L_v L_y G F - \mu_\xi L_h)^{-1}$ and $T = 1 - S$, we have

$$\|\hat{g}\|_{\mathcal{P}}^2 = \frac{\beta_1 \sigma_n^2 + \alpha^{-2} \beta_4}{1 - \beta_1 \sigma_\xi^2}, \quad (7.35)$$

$$\begin{aligned} J &= \beta_2 + \alpha^2 \beta_3 \sigma_\xi^2 \left(\frac{\sigma_n^2}{\sigma_\xi^2} + \frac{\beta_1 \sigma_n^2 + \alpha^{-2} \beta_4}{1 - \beta_1 \sigma_\xi^2} \right) \\ &= \beta_2 + \frac{\beta_3 \beta_4}{1 - \beta_1 \sigma_\xi^2} \sigma_\xi^2 + \alpha^2 \frac{\beta_3 \sigma_n^2}{1 - \beta_1 \sigma_\xi^2}. \end{aligned} \quad (7.36)$$

For a given set of $G(z), \sigma_r^2, \sigma_d^2, \sigma_n^2, \sigma_\xi^2, \mu_\xi, \mathcal{P}$ satisfying (7.12), the globally optimal solution to

$$(K, \alpha) = \arg \inf_{K \in \mathcal{K}, \alpha \in (0, \infty)} J$$

is hard to obtain due to the coupling among the design parameters. Therefore, we turn to a suboptimal design and adopt the perfect reconstruction constraint (see, e.g., [12, 141]) on the network.

Assumption 7.2.2. *The network in Figure 7.2 has a unit transfer function from v to u in the absence of n and w , i.e.,*

$$F \mu_\xi (1 - L_h \mu_\xi)^{-1} L_v = 1. \quad (7.37)$$

Under Assumption 7.2.2, we have

$$\begin{aligned} S &= (1 - L_y G)^{-1} (1 - L_h \mu_\xi)^{-1}, \\ \beta_2 &= \|1 - G(1 - L_y G)^{-1} L_r\|_2^2 \sigma_r^2 + \|(1 - L_y G)^{-1}\|_2^2 \sigma_d^2. \end{aligned}$$

From (7.36), J is an increasing function in $\beta_1, \beta_2, \beta_3$ and β_4 . First of all, we can choose $L_h \mu_\xi$ and $L_v L_y F \mu_\xi$ according to (7.26) and let $L_r \in \mathcal{RH}_\infty$, such that the mean square stability of the closed loop is ensured and β_1 is minimized. In this situation, L_h and L_y are determined under Assumption 7.2.2. Secondly, $L_r \in \mathcal{RH}_\infty$ can be designed to minimize $\|1 - G(1 - L_y G)^{-1} L_r\|_2^2$, which amounts to minimize β_2 since $\|(1 - L_y G)^{-1}\|_2^2$ is fixed with a given L_y . In the last step, the design of both

α and the pair F, L_v is given in the theorem that follows.

Theorem 7.2.2. *Consider the NCS described by Figure 7.2 and equations (7.3)-(7.8). Further assume that (7.12) is true, and L_y, L_h, L_r are chosen as above. Under Assumptions 7.2.1 and 7.2.2,*

$$J_{opt} \triangleq \inf_{L_v \in \mathcal{U}_\infty, F \in \mathcal{U}_\infty, \alpha \in (0, \infty)} J$$

$$= \beta_2 + \frac{\beta_5}{1 - \beta_1 \sigma_\xi^2} \left\{ \sigma_\xi^2 + \frac{\sigma_n^2}{(1 - \beta_1 \sigma_\xi^2) \mathcal{P} - \beta_1 \sigma_n^2} \right\}$$

with

$$\beta_5 \triangleq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{j\omega}) S(e^{j\omega})|^2 (1 - L_h(e^{j\omega}) \mu_\xi) \mu_\xi^{-1} \sqrt{|L_r(e^{j\omega}) \sigma_r|^2 + |L_y(e^{j\omega}) \sigma_d|^2} d\omega \right)^2.$$

Moreover, J_{opt} can be approached by choosing $L_v \in \mathcal{U}_\infty$ such that, for any $\omega \in [-\pi, \pi]$,

$$|L_v(e^{j\omega})|^4 \approx \gamma \frac{|(1 - L_h(e^{j\omega}) \mu_\xi) G(e^{j\omega})|^2}{\mu_\xi^2 (|L_r(e^{j\omega}) \sigma_r|^2 + |L_y(e^{j\omega}) \sigma_d|^2)}, \quad (7.38)$$

$F = (1 - L_h \mu_\xi) L_v^{-1} \mu_\xi^{-1}$, and

$$\alpha = \sqrt{\frac{\beta_4}{(1 - \beta_1 \sigma_\xi^2) \mathcal{P} - \beta_1 \sigma_n^2}} + \epsilon_3,$$

where $\gamma > 0$ is arbitrary, and $\epsilon_3 > 0$ is sufficiently small.

Proof. Based on the expression (7.35), the power constraint $\|\hat{g}\|_{\mathcal{P}}^2 < \mathcal{P}$ indicates that

$$\alpha > \sqrt{\frac{\beta_4}{(1 - \beta_1 \sigma_\xi^2) \mathcal{P} - \beta_1 \sigma_n^2}}. \quad (7.39)$$

By combining (7.36) and (7.39), we have

$$J > \beta_2 + \frac{\beta_3 \beta_4}{1 - \beta_1 \sigma_\xi^2} \left\{ \sigma_\xi^2 + \frac{\sigma_n^2}{(1 - \beta_1 \sigma_\xi^2) \mathcal{P} - \beta_1 \sigma_n^2} \right\}.$$

It follows from the Cauchy-Schwartz inequality that

$$\begin{aligned}
\beta_3\beta_4 &= \|GFS\|_2^2 \|SL_v[L_r\sigma_r \quad L_y\sigma_d]\|_2^2 \\
&\geq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{j\omega})F(e^{j\omega})S(e^{j\omega})^2 L_v(e^{j\omega})| \sqrt{|L_r(e^{j\omega})\sigma_r|^2 + |L_y(e^{j\omega})\sigma_d|^2} d\omega \right)^2 \\
&= \beta_5,
\end{aligned} \tag{7.40}$$

where the last equality follows from the perfect reconstruction condition (7.37), and (7.40) can approach the equality with any desired accuracy by choosing L_v according to (7.38); see, e.g., [12, 142]. The remaining proof is straightforward. $\square$

To sum up, we propose the next algorithm for a suboptimal performance design of the NCS.

Algorithm 7.2.1. *Consider the NCS described by Figure 7.2 and equations (7.3)-(7.8). Assume that (7.12) is true, and Assumptions 7.2.1 and 7.2.2 hold.*

- a) *Choose L_h and L_y according to (7.26) in Theorem 7.2.1 with $L_h = K_2\mu_\xi^{-1}$ and $L_y = (1 - L_h\mu_\xi)^{-1}K_1$ such that β_1 is minimized.*
- b) *Choose L_r according to*

$$L_r = \arg \inf_{L_r \in \mathcal{RH}_\infty} \|1 - G(1 - L_y G)^{-1} L_r\|_2^2.$$

- c) *Choose L_v, F and α based on Theorem 7.2.2.*

7.2.3 Numerical Example

The following numerical example illustrates the results of this section.

Example 7.2.1. *Consider the NCS in Figure 7.2 with channel model (7.8), where $G(z) = 1/(z - 2)$, and $\sigma_r^2 = \sigma_d^2 = \sigma_n^2 = 1$. Suppose that Assumptions 7.2.1 and*

7.2.2 hold. Based on Theorem 7.2.1, we can obtain the stabilizability regions as shown in Figure 7.5. After applying Algorithm 7.2.1 with $\mu_\xi = 2$ and $\sigma_\xi^2 = 0.5$, the relationship between J_{opt} and $\mathcal{P}$ is given in Figure 7.6.

Figure 7.5: The stabilizability regions on the $\mathcal{P}$ - σ_ξ^2 plane with $\mu_\xi = 2$ (left) and the $\mathcal{P}$ - μ_ξ plane with $\sigma_\xi^2 = 0.5$ (right).

Figure 7.6: The relationship between J_{opt} and $\mathcal{P}$ using Algorithm 7.2.1.

7.3 Summary

It has been shown that the channel feedback plays a key role in eliminating the limitation on stabilization induced by the NMP zeros and high relative degree of the plant. For an SISO LTI plant with LTI feedback over a fading channel with channel feedback, the minimal requirement on the network for mean square stabilizability can be exactly characterized in terms of the statistics of channel fading and channel additive noise, the power bound of channel input, and the Mahler measure of the plant. When a power bound is applied on the channel input, the network requirement for stabilizability becomes more stringent, and the scaling of the network is an essential parameter in satisfying the power constraint. Under the assumption on perfect reconstruction of the network, a suboptimal method has been given for performance design of the NCS.

Chapter 8

Control over Fading Channels: Continuous-Time Case

Nowadays, the classical analog control systems are less frequently used compared to their digital counterparts. However, a comprehensive study of control over networks should consider the continuous-time case which also provides insights on the fundamental issues of the NCS. This chapter studies the stabilization of continuous-time LTI plants over fading channels. The problem is formulated in Section 8.1. Section 8.2 extends the state feedback result for the discrete-time NCS presented in Chapter 6 to the continuous-time case. In Section 8.3, we address the output feedback stabilization of single-input plants over a single channel or a class of essentially triangular plants over multiple channels. Concluding remarks are given in the last section.

8.1 Problem Formulation

Consider an NCS as shown in Figure 8.1, where a continuous-time LTI plant is controlled over fading channel(s).

Figure 8.1: Control over input fading channel(s): continuous-time case.

Let the plant have a strictly proper transfer function $G(s)$ and state equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t),\end{aligned}\tag{8.1}$$

where $x(t) \in \mathbb{R}^n$ is the plant state, $u(t) \in \mathbb{R}^m$ is the plant input, and $y(t) \in \mathbb{R}^q$ is the measured output. Without loss of generality, assume that A is unstable, $B = [B_1 \ B_2 \ \cdots \ B_m]$ has full-column rank, $C = [C'_1 \ C'_2 \ \cdots \ C'_q]'$ has full-row rank, (A, B) is stabilizable and (A, C) is detectable. The Mahler measure of A defined in (2.8) is mostly used in the discrete-time case and can be generalized to the continuous-time case as

$$\mathcal{M}_1(A) \triangleq \sum_{i=1}^n \max\{\Re(\lambda_i(A)), 0\},\tag{8.2}$$

where $\Re(\lambda_i(A))$ stands for the real part of the i -th eigenvalue of A . The mean square norm of $G(s)$, if exists, is defined as

$$\|G(s)\|_{\text{MS}} \triangleq \sqrt{\max_{i=1,2,\dots,q} \frac{1}{2\pi} \int_{-\infty}^{\infty} [G(j\omega)G^H(j\omega)]_{ii} d\omega}.\tag{8.3}$$

The controller has a proper transfer function $K(s)$ and a realization

$$\begin{aligned}\dot{x}_K(t) &= A_K x_K(t) + B_K y(t), \\ v(t) &= C_K x_K(t) + D_K y(t),\end{aligned}\tag{8.4}$$

where $v(t) \in \mathbb{R}^m$ is the controller output and the dimension of controller state $x_K(t)$

is unspecified in advance.

Inspired by practical situations where the actuators are geographically separated from each other and from the controller, the PTS is adopted throughout this chapter, i.e., each element of the controller output $v(t)$ is separately sent across an independent channel to the corresponding actuator. The network output $u(t)$ is directly applied to the plant thereafter. The above communication scheme is modeled by

$$u(t) = \xi(t)v(t), \quad (8.5)$$

where $\xi(t)$ has a diagonal structure as

$$\xi(t) = \text{diag} \{ \xi_1(t), \xi_2(t), \dots, \xi_m(t) \}, \quad (8.6)$$

and $\xi_1(t), \xi_2(t), \dots, \xi_m(t)$ are assumed to be scalar-valued white noise processes satisfying

$$\mathcal{E}\{\xi_i(t)\} = \mu_i \neq 0, \quad \mathcal{E}\{(\xi_i(s) - \mu_i)(\xi_j(t) - \mu_j)\} = \sigma_i^2 \delta_{ij} \delta_{st}.$$

Definition 8.1.1. *The quality of service (QoS) of the i -th communication channel is defined by $g_i \triangleq \frac{\mu_i^2}{\sigma_i^2}$ and the overall QoS by $g \triangleq \sum_{i=1}^m g_i$.*

Assumption 8.1.1. *The overall QoS can be allocated among the parallel channels.*

Again, the network can be reformulated as in Figure 6.2. In this case, the transfer function from $w(t)$ to $z(t)$ without considering the uncertainty block is given by

$$T(s) = K(s)(I - G_{\Pi}(s)K(s))^{-1}G_{\Pi}(s)\Phi, \quad (8.7)$$

where $G_{\Pi}(s) = G(s)\Pi$, $\Phi = \Pi^{-1}\Lambda$ with Π and Λ defined in (6.6). Denote $\hat{x}(t) \triangleq [x'(t) \ x'_K(t)]'$, and recall the definition of mean square stabilization as follows.

Definition 8.1.2 ([118]). *The plant (8.1) is mean square stabilized by the controller (8.4) over the network (8.5)-(8.6), if for any given initial state $\hat{x}(0)$, $\Xi(t) \triangleq \mathcal{E}[\hat{x}(t)\hat{x}'(t)]$ is well-defined for any $t \geq 0$, and $\lim_{t \rightarrow \infty} \Xi(t) = 0$.*

For a given set of $\{g_1, g_2, \dots, g_m\}$, we can prove that the mean square stabilization of the NCS in Figure 8.1 is equivalent to

$$\inf_{K(s) \in \mathcal{K}, \Theta > 0, \text{diag}} \|\Theta^{-1}T(s)\Theta\|_{\text{MS}}^2 < 1, \quad (8.8)$$

where $\Theta \in \mathbb{R}^{m \times m}$ is a positive-definite diagonal scaling matrix and $K(s)$ can be chosen from the set of all the proper stabilizing LTI controllers $\mathcal{K}$. The above result is a continuous-time counterpart of Theorem 6.4 in [10]. Similarly to other robust control problems, the search for the optimal controller $K(s)$ on the left-hand side of (8.8) is generally nonconvex in Θ . However, by introducing Assumption 8.1.1, the above nonconvex optimization problem is analytically solved in the case of state feedback, as we shall see in the next section.

8.2 Mean Square Stabilization via State Feedback

In this section, $C = I$ and $K(s) = K$ is a static state feedback gain. Then, $T(s)$ defined in (8.7) is simplified into $T(s) = K(sI - A - B_{\Pi}K)^{-1}B_{\Pi}\Phi$ with $B_{\Pi} \triangleq B\Pi$. For any stabilizable pair (A, B) , we can always find a linear coordinate transformation matrix L_1 such that

$$\tilde{A} = L_1^{-1}AL_1 = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \quad \tilde{B} = L_1^{-1}B = \begin{bmatrix} B_s \\ B_u \end{bmatrix}, \quad (8.9)$$

where A_s is stable, all the eigenvalues of A_u are either unstable or marginally stable, and (A_u, B_u) is controllable. The closed-loop system

$$\dot{x}(t) = Ax(t) + B\xi(t)Kx(t) \quad (8.10)$$

can be rewritten into the Itô form as below

$$dx(t) = (A + B_{\Pi}K)x(t)dt + \sum_{i=1}^m \sigma_i B_i K_i x(t) dw_i(t), \quad (8.11)$$

where $dw_i(t) = \sigma_i^{-1}(\xi_i(t) - \mu_i)dt$, and $w_i(t)$ is a standard scalar Wiener process for every $i = 1, 2, \dots, m$.

The relationship between the overall QoS and $\mathcal{M}_1(A)$ defined in (8.2) for ensuring mean square stability of (8.10) via state feedback is characterized by the next theorem.

Theorem 8.2.1. *Under Assumption 8.1.1, the plant (8.1) is mean square stabilizable via state feedback over the network (8.5)-(8.6) if and only if*

$$g > 2\mathcal{M}_1(A). \quad (8.12)$$

The next lemma is needed for the proof of Theorem 8.2.1.

Lemma 8.2.1. *For a given set of $\{g_1, g_2, \dots, g_m\}$, the following statements are equivalent.*

- (i) *The plant (8.1) or the pair (A, B) can be stabilized in the mean square sense via static state feedback over the network (8.5)-(8.6).*
- (ii) *There exists $K = [K'_1 \ K'_2 \ \dots \ K'_m]'$ such that Ψ is Hurwitz, i.e., $\max_i \Re\{\lambda_i^u(\Psi)\} < 0$, where*

$$\Psi = (A + B_\Pi K) \otimes I + I \otimes (A + B_\Pi K) + \sum_{i=1}^m \sigma_i^2 B_i K_i \otimes B_i K_i.$$

- (iii) *There exist $P > 0$ and $K = [K'_1 \ K'_2 \ \dots \ K'_m]'$ such that*

$$0 > (A + B_\Pi K)'P + P(A + B_\Pi K) + \sum_{i=1}^m \sigma_i^2 K'_i B'_i P B_i K_i. \quad (8.13)$$

- (iv) *There exists $P > 0$ such that*

$$0 > A'P + PA - PB_\Pi J_c^{-1} B'_\Pi P, \quad (8.14)$$

where

$$J_c = \text{diag} \{ \sigma_1^2 B_1' P B_1, \sigma_2^2 B_2' P B_2, \dots, \sigma_m^2 B_m' P B_m \}.$$

(v) The pair (A_u, B_u) can be stabilized in the mean square sense via static state feedback over the network (8.5)-(8.6).

Proof. (i) $\Leftrightarrow$ (ii): From Definition 8.1.2 with $\hat{x}(t) = x(t)$ and Itô's formula, we have

$$\dot{\Xi}(t) = (A + B_\Pi K)\Xi(t) + \Xi(t)(A + B_\Pi K)' + \sum_{i=1}^m \sigma_i^2 B_i K_i \Xi(t) K_i' B_i',$$

i.e.,

$$\text{vec}(\dot{\Xi}(t)) = \Psi \text{vec}(\Xi(t)).$$

Thus, the mean square stabilization of (8.10) via static state feedback is equivalent to the existence of K such that Ψ is Hurwitz [143].

(i) $\Leftarrow$ (iii): Introduce a linear Lyapunov function $V(\Xi(t)) \triangleq \text{tr}\{\Xi(t)P\}$. It holds that

$$\begin{aligned} \dot{V}(\Xi(t)) &= \text{tr} \left\{ (A + B_\Pi K)\Xi(t)P + \Xi(t)(A + B_\Pi K)'P + \sum_{i=1}^m \sigma_i^2 B_i K_i \Xi(t) K_i' B_i' P \right\} \\ &= \text{tr} \left\{ \Xi(t)P(A + B_\Pi K) + \Xi(t)(A + B_\Pi K)'P + \Xi(t) \sum_{i=1}^m \sigma_i^2 K_i' B_i' P B_i K_i \right\}. \end{aligned}$$

In view of Lemma A.0.4 on the property of trace, we can prove that (iii) implies that $\dot{V}(\Xi(t)) < 0$. Thus, Lyapunov theory tells us that the convergence condition in Definition 8.1.2 is satisfied.

(ii) $\Rightarrow$ (iii): Note that $\max_i \Re\{\lambda_i^u(\Psi)\} < 0$ is equivalent to $\max_i \Re\{\lambda_i^u(\Psi')\} < 0$. Thus, the following linear dynamic system

$$\dot{\Omega}(t) = (A + B_\Pi K)' \Omega(t) + \Omega(t)(A + B_\Pi K) + \sum_{i=1}^m \sigma_i^2 K_i' B_i' \Omega(t) B_i K_i$$

is stable for any initial condition $\Omega(0) > 0$, and $\text{vec}(\Omega(t))$ converges to zero expo-

nentially as t approaches ∞ . In this case, $P(t) \triangleq \int_0^t \Omega(r) dr$ is well-defined for any t and has a finite limit denoted by $P > 0$. Note that

$$\begin{aligned} \int_0^t \dot{\Omega}(s) ds &= \Omega(t) - \Omega(0) \\ &= (A + B_{\Pi}K)'P(t) + P(t)(A + B_{\Pi}K) + \sum_{i=1}^m \sigma_i^2 K_i' B_i' P(t) B_i K_i. \end{aligned}$$

By taking the limit, we can conclude that P satisfies (8.13).

The rest of the results can be proved similarly to Lemma 6.2.1. $\square$

Proof of Theorem 8.2.1. In view of the equivalence between (i) and (v) in Lemma 8.2.1, we assume that all the eigenvalues of A are in the closed-right half of the complex plane without loss of generality.

First, we will prove that the mean square stabilizability implies (8.12). The inequality (8.14) is equivalent to

$$0 > P^{-\frac{1}{2}} A' P^{\frac{1}{2}} + P^{\frac{1}{2}} A P^{-\frac{1}{2}} - P^{\frac{1}{2}} B_{\Pi} J_c^{-1} B_{\Pi}' P^{\frac{1}{2}}. \quad (8.15)$$

Taking the trace of both sides of (8.15), we have

$$\begin{aligned} 0 &> \text{tr}(P^{-\frac{1}{2}} A' P^{\frac{1}{2}}) + \text{tr}(P^{\frac{1}{2}} A P^{-\frac{1}{2}}) - \text{tr}(P^{\frac{1}{2}} B_{\Pi} J_c^{-1} B_{\Pi}' P^{\frac{1}{2}}) \\ &= \text{tr}(A') + \text{tr}(A) - \text{tr}(B_{\Pi}' P B_{\Pi} J_c^{-1}) \\ &= 2\mathcal{M}_1(A) - g. \end{aligned}$$

Thus, we can conclude that $g > 2\mathcal{M}_1(A)$.

Next, a constructive proof will be given to prove the converse. Choose

$$\Theta = \text{diag}\{1, \epsilon, \dots, \epsilon^{m-1}\}, \quad S = \text{diag}\{I_{n_1}, \epsilon I_{n_2}, \dots, \epsilon^{m-1} I_{n_m}\}$$

with a small real number $\epsilon > 0$. The stabilizability of (A, B) guarantees that of (A, B_{Π}) , since $\mu_i \neq 0$ for every $i = 1, 2, \dots, m$. For a stabilizable pair (A, B_{Π}) , we

have the following Wonham decomposition [130]:

$$\check{A} = \begin{bmatrix} A_1 & \star & \cdots & \star \\ 0 & A_2 & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad \check{B}_\Pi = \begin{bmatrix} b_1 & \star & \cdots & \star \\ 0 & b_2 & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_m \end{bmatrix} \quad (8.16)$$

with $\star$ representing terms that will not be used in the derivation and

$$A_i \in \mathbb{R}^{n_i \times n_i}, \quad b_i \in \mathbb{R}^{n_i \times 1}, \quad \sum_{i=1}^m n_i = n,$$

where $\check{A} = L_2^{-1} A L_2$, $\check{B}_\Pi = L_2^{-1} B_\Pi$ with L_2 being a similarity transformation matrix and each pair (A_i, b_i) stabilizable.

Now, we denote $\hat{K} \triangleq K L_2$ and let $\hat{K}$ have the block-diagonal form

$$\hat{K} = \text{diag}\{k_1, k_2, \dots, k_m\}, \quad k_i \in \mathbb{R}^{1 \times n_i},$$

such that $A_i + b_i k_i$ is stable. Based on Theorem II.1 of [94], we have

$$\inf_{k_i \in \mathcal{K}(A_i)} \|k_i(sI - A_i - b_i k_i)^{-1} b_i\|_2^2 = 2\mathcal{M}_1(A_i). \quad (8.17)$$

Since

$$\begin{aligned} \hat{A} &= S^{-1} \check{A} S = \begin{bmatrix} A_1 & o(\epsilon) & \cdots & o(\epsilon) \\ 0 & A_2 & \cdots & o(\epsilon) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad \hat{B}_\Pi = S^{-1} \check{B}_\Pi \Theta = \begin{bmatrix} b_1 & o(\epsilon) & \cdots & o(\epsilon) \\ 0 & b_2 & \cdots & o(\epsilon) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_m \end{bmatrix}, \\ \hat{K} &= \Theta^{-1} K L_2 S \end{aligned}$$

with $\lim_{\epsilon \rightarrow 0} o(\epsilon) = 0$, we have

$$\Theta^{-1} T(s) \Theta = \Theta^{-1} K (sI - A - B_\Pi \Theta \Theta^{-1} K)^{-1} B_\Pi \Theta$$

$$\begin{aligned}
&= \Theta^{-1} K L_2 S (sI - S^{-1} \check{A} S - S^{-1} \check{B}_\Pi \Theta \Theta^{-1} K L_2 S)^{-1} S^{-1} \check{B}_\Pi \Theta \Phi \\
&= \hat{K} (sI - \hat{A} - \hat{B}_\Pi \hat{K})^{-1} \hat{B}_\Pi \Phi \\
&= \text{diag} \left\{ T_1(s) \sqrt{\frac{1}{g_1}}, T_2(s) \sqrt{\frac{1}{g_2}}, \dots, T_m(s) \sqrt{\frac{1}{g_m}} \right\} + o_s(\epsilon),
\end{aligned}$$

where $T_i(s) = k_i(sI - A_i - b_i k_i)^{-1} b_i$, and $o_s(\epsilon)$ is a function of ϵ and s that satisfies $\lim_{\epsilon \rightarrow 0} o_s(\epsilon) = 0$. For $g > 2\mathcal{M}_1(A) = 2 \sum_{i=1}^m \mathcal{M}_1(A_i)$, we can always choose $g_i > 2\mathcal{M}_1(A_i)$ and k_i according to (8.17), such that

$$\frac{\|T_i(s)\|_2^2}{g_i} < 1, \quad \forall i = 1, 2, \dots, m.$$

It follows that $\|\Theta^{-1} T(s) \Theta\|_{\text{MS}}^2 < 1$ for sufficiently small ϵ , i.e., there exist a positive diagonal matrix Θ , a stabilizing state feedback gain $K = \hat{K} L_2^{-1}$ and an allocation $\{g_1, g_2, \dots, g_m\}$ for $g = \sum_{i=1}^m g_i$ such that the closed-loop system (8.10) is mean square stable. $\square$

8.3 Mean Square Stabilization via Output Feedback

Let a pair of right and left coprime factorizations of $G(s)$ be given in the familiar way, namely

$$G = NM^{-1} = \tilde{M}^{-1} \tilde{N}, \quad (8.18)$$

where $N, M, \tilde{N}, \tilde{M} \in \mathcal{RH}_\infty$, and satisfy the double Bezout identity

$$\begin{bmatrix} \tilde{Y} & -\tilde{X} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & X \\ N & Y \end{bmatrix} = I$$

for some $X, Y, \tilde{X}, \tilde{Y} \in \mathcal{RH}_\infty$. It follows that

$$G_\Pi = N(\Pi^{-1} M)^{-1} = \tilde{M}^{-1}(\tilde{N} \Pi). \quad (8.19)$$

Then, the set of all proper controllers achieving internal stability of the closed loop, with respect to G_Π factorized in (8.19), can be parameterized as [115]

$$\begin{aligned}\mathcal{K}(G_\Pi) &\triangleq \{K : K = \Pi^{-1}(X + MQ)(Y + NQ)^{-1} \\ &= \Pi^{-1}(\tilde{Y} + Q\tilde{N})^{-1}(\tilde{X} + Q\tilde{M}), \quad Q \in \mathcal{RH}_\infty\}.\end{aligned}$$

In this situation, $T(s)$ in (8.7) becomes

$$T(s) = \Pi^{-1}(M\tilde{Y} + MQ\tilde{N} - I)\Lambda = \Pi^{-1}(X\tilde{N} + MQ\tilde{N})\Lambda. \quad (8.20)$$

Suppose $G(s)$ has n_p unstable poles $p_1, p_2, \dots, p_{n_p}$ and n_z distinct NMP zeros $z_1, z_2, \dots, z_{n_z}$, where the poles and zeros are defined as in [115]. Let $B_p(s)$, $B_z(s)$ denote the Blaschke products with respect to the unstable poles and the NMP zeros of $G(s)$ respectively, i.e.,

$$B_p(s) \triangleq \prod_{i=1}^{n_p} \frac{s - p_i}{s + \bar{p}_i}, \quad B_z(s) \triangleq \prod_{i=1}^{n_z} \frac{s - z_i}{s + \bar{z}_i}. \quad (8.21)$$

We further define $\mathcal{M}_1(G) \triangleq \sum_{i=1}^{n_p} \Re\{p_i\}$, which is equal to $\mathcal{M}_1(A)$ with (A, B, C) being any detectable and stabilizable realization of $G(s)$, and denote [94]

$$\eta(G) \triangleq \sum_{l=1}^{n_z} \sum_{k=1}^{n_z} \frac{\bar{\gamma}_k \gamma_l}{\bar{z}_k + z_l}, \quad (8.22)$$

where

$$\gamma_k \triangleq 2\Re(z_k)(1 - B_p^{-1}(z_k)) \prod_{i=1, i \neq k}^{n_z} \frac{z_k + \bar{z}_i}{z_k - z_i}.$$

8.3.1 Stabilization over a Single Channel

First, we consider the case $m = 1$, where $\xi(t)$ in (8.5) is scalar-valued.

Proposition 8.3.1. *Assume that $m = 1$ and the NMP zeros of $G(s)$ are distinct. Then, the plant (8.1) is mean square stabilizable by an output feedback controller*

(8.4) over the network (8.5) if and only if

$$g = \frac{\mu_1^2}{\sigma_1^2} > 2\mathcal{M}_1(G) + \eta(G). \quad (8.23)$$

Proof. For $m = 1$, we can obtain that

$$\|\Theta^{-1}T(s)\Theta\|_{\text{MS}}^2 = \|T(s)\|_{\text{MS}}^2 = \|T(s)\|_2^2$$

with $T(s) = (M\tilde{Y} + MQ\tilde{N} - 1)\sqrt{g^{-1}}$.

After factorizing $M = B_p M_m$, $\tilde{N} = \tilde{N}_m B_z$ with allpass B_p, B_z in (8.21) and minimum phase $M_m, \tilde{N}_m$, and decomposing $M_m \tilde{Y} B_z^{-1} - B_z^{-1} = \Gamma + \Gamma^\perp$ with $\Gamma \in \mathcal{H}_2$, $\Gamma^\perp \in \mathcal{H}_2^\perp$, we have

$$\begin{aligned} \|T(s)\|_2^2 &= g^{-1} \|B_p M_m \tilde{Y} + B_p M_m Q \tilde{N}_m B_z - 1\|_2^2 \\ &= g^{-1} (\|1 - B_p^{-1}\|_2^2 + \|M_m \tilde{Y} + M_m Q \tilde{N}_m B_z - 1\|_2^2) \\ &= g^{-1} (\|1 - B_p^{-1}\|_2^2 + \|M_m \tilde{Y} B_z^{-1} - B_z^{-1} + M_m Q \tilde{N}_m\|_2^2) \\ &= g^{-1} (\|1 - B_p^{-1}\|_2^2 + \|\Gamma^\perp\|_2^2 + \|\Gamma + M_m Q \tilde{N}_m\|_2^2). \end{aligned}$$

Since $\tilde{N}_m$ has full-column rank, we can choose Q such that

$$\inf_{Q \in \mathcal{RH}_\infty} \|\Gamma + M_m Q \tilde{N}_m\|_2^2 = 0.$$

Thus, the inequality (8.8) is equivalent to

$$g > \|1 - B_p^{-1}\|_2^2 + \|\Gamma^\perp\|_2^2. \quad (8.24)$$

The evaluation of the right hand side of (8.24) follows from the proof of Theorem II.2 in [94]. $\square$

8.3.2 Stabilization of Essentially Triangular Plants over Multiple Channels

Definition 8.3.1. A transfer function matrix $G(s)$ with $m \leq q$ is said to be essentially triangular, if there exists a pair of permutation matrices W_1, W_2 such that $G_\Delta(s) \triangleq W_1 G(s) W_2$ has either of the following two forms:

$$\begin{bmatrix} G_1(s) & \star & \cdots & \star \\ 0 & G_2(s) & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_m(s) \end{bmatrix}, \text{ or } \begin{bmatrix} G_1(s) & 0 & \cdots & 0 \\ \star & G_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & G_m(s) \end{bmatrix}, \quad (8.25)$$

where $G_i(s)$ has dimension $q_i \times 1$ and $\sum_{i=1}^m q_i = q$.

A permutation matrix is a square matrix that has exactly one entry 1 in each row and each column and 0's elsewhere. Note that the set of permutation matrices belongs to the set of allpass factors. In this subsection, we make the following assumption.

Assumption 8.3.1. (a). $m \leq q$, and $G(s)$ is essentially triangular.

(b). $G_i(s)$ in (8.25) has distinct NMP zeros for every $i = 1, 2, \dots, m$.

Theorem 8.3.1. Under Assumptions 8.1.1 and 8.3.1, the plant (8.1) is mean square stabilizable via output feedback over the network (8.5)-(8.6) if

$$g > 2\mathcal{M}_1(G) + \sum_{i=1}^m \eta(G_i), \quad (8.26)$$

where $G_i(s)$ is given in (8.25).

The next two lemmas will be used in the proof of Theorem 8.3.1.

Lemma 8.3.1. For any permutation matrix W , allpass factor $Z(s)$ and transfer function matrix $G(s)$ with compatible dimensions, the mean square norm of $G(s)$

is invariant after premultiplying W and postmultiplying $Z(s)$, i.e., $\|G(s)\|_{\text{MS}} = \|WG(s)Z(s)\|_{\text{MS}}$.

Proof. It follows from the definition of mean square norm given in (8.3). $\square$

Lemma 8.3.2. *For any diagonal matrix Θ_1 and permutation matrix W , the matrix $\Theta_2 = W\Theta_1W'$ is a diagonal matrix reordering the diagonal elements of Θ_1 .*

Proof. $W\Theta_1W'$ permutes the rows and columns of Θ_1 in the same way, and thus this transformation amounts to reordering the diagonal elements of Θ_1 . $\square$

Proof of Theorem 8.3.1. It is easy to see that

$$\begin{aligned}\Theta^{-1}T(s)\Theta &= \Theta^{-1}K(s)(I - G_{\Pi}(s)K(s))^{-1}G_{\Pi}(s)\Phi\Theta \\ &= \Theta_1^{-1}K_{\Pi}(s)(I - G(s)K_{\Pi}(s))^{-1}G(s)\Phi\Theta_1,\end{aligned}$$

where $\Theta_1 = \Pi\Theta$ is a diagonal scaling matrix, and $K_{\Pi}(s) = \Pi K(s)$.

By using Lemma 8.3.1 and $G_{\Delta}(s) = W_1G(s)W_2$, we have

$$\begin{aligned}\|\Theta^{-1}T(s)\Theta\|_{\text{MS}}^2 &= \|W_2'\Theta_1^{-1}K_{\Pi}(s)(I - G(s)K_{\Pi}(s))^{-1}G(s)\Phi\Theta_1W_2\|_{\text{MS}}^2 \\ &= \|\Theta_2^{-1}K_{\Delta\Pi}(s)(I - G_{\Delta}(s)K_{\Delta\Pi}(s))^{-1}G_{\Delta}(s)\Phi_{\Delta}\Theta_2\|_{\text{MS}}^2,\end{aligned}$$

where $\Theta_2 = W_2'\Theta_1W_2$, $\Phi_{\Delta} = W_2'\Phi W_2$ are diagonal according to Lemma 8.3.2, and $K_{\Delta\Pi}(s) = W_2'K_{\Pi}(s)W_1'$.

When $G_{\Delta}(s)$ is in a block upper triangular form, choose $\Theta_2 = \text{diag}\{1, \epsilon, \dots, \epsilon^{m-1}\}$ with a small real number $\epsilon > 0$, and let $K_{\Delta\Pi}(s)$ have the block-diagonal form $K_{\Delta\Pi}(s) = \text{diag}\{K_1(s), K_2(s), \dots, K_m(s)\}$ and $K_i(s)$ have dimension $1 \times q_i$. Then,

$$\|\Theta^{-1}T(s)\Theta\|_{\text{MS}}^2 = \text{diag}\left\{T_1(s)\sqrt{\frac{1}{g_{j_1}}}, T_2(s)\sqrt{\frac{1}{g_{j_2}}}, \dots, T_m(s)\sqrt{\frac{1}{g_{j_m}}}\right\} + o_s(\epsilon),$$

where $T_i(s) = K_i(s)(I - G_i(s)K_i(s))^{-1}G_i(s)$, and $j_1, j_2, \dots, j_m$ is a reordering of $1, 2, \dots, m$. Based on Proposition 8.3.1, it holds that

$$\inf_{K_i(s) \in \mathcal{K}(G_i(s))} \|T_i(s)\|_2^2 = 2\mathcal{M}_1(G_i) + \eta(G_i).$$

Following the similar argument as in Theorem 8.2.1, $g = \sum_{j_i=1}^m g_{j_i} > 2\mathcal{M}_1(G) + \sum_{i=1}^m \eta(G_i)$ is sufficient to ensure the mean square stability of the closed-loop system.

If $G_\Delta(s)$ is in a block lower triangular form, then the result follows in a similar way by setting $\Theta_2 = \text{diag}\{\epsilon^{m-1}, \epsilon^{m-2}, \dots, 1\}$. $\square$

8.4 Summary

It has been shown in this chapter that for a continuous-time LTI plant with LTI feedback over input fading channels, the minimal requirement on the overall QoS of the network for mean square stabilizability under PTS can be given in terms of the inherent properties of the plant, e.g., the unstable poles of the plant in the case of state feedback. For dynamic output feedback, it has been shown that additional requirement is needed if the plant is NMP.

Chapter 9

Conclusion and Recommendations for Future Research

9.1 Conclusion

With the development and integration of theories of information, communication and control, NCSs have attracted increasing interests of researchers and engineers. Filtering and control of networked systems over unreliable channels have been studied in this thesis, and extra emphasis has been placed on two typical channel models: lossy channel model and fading channel model. The contributions of the thesis can be summarized in the following aspects.

- We have presented a method to estimate the upper bound for the solution of the standard Riccati recursion and obtained less conservative results compared to the existing ones [6, 7]. It has been shown that, if the observability index of the plant is one, then the Kalman filter is always peak covariance stable. For plants with observability index greater than one, a sufficient condition for peak covariance stability of the remote Kalman filter has been given in terms of the dynamics of the plant and the transition probability of the Markov chain that describes the packet-loss process.

- We have shown that, for a single-input MJLS, a mode-dependent logarithmic quantizer and a mode-dependent linear state feedback law can achieve optimal coarseness for MSQ stabilization, and the sector bound approach provides a non-conservative way for studying the corresponding quantized state feedback stabilization problem. We have addressed the optimal co-design of quantizer and controller in virtue of the LMI approach. In addition, we have presented a recursive algorithm to estimate the unknown mode process of the underlying Markov chain at the controller side. The above results on quantized stabilization of MJLSs can be applied to the quantized stabilization of NCSs with binary or bounded packet losses.
- We have derived necessary and sufficient conditions on the network for the mean covariance stability of a remote Kalman filter implemented over fading channels subject to both transmission failure and signal fluctuation. The conditions are specified in terms of the unstable poles of the plant. We have also provided lower and upper bounds for the mean error covariance matrix in the form of a modified Lyapunov iteration and a modified Riccati iteration, respectively.
- We have extended the stability criterion for systems with independent multiplicative noises presented in [10] and Chapter 9 of [11] to that with possibly correlated multiplicative noises. We have shown that under PTS and the assumption on capacity allocation, the minimal overall mean square capacity of the network for mean square stabilizability is given in terms of the Mahler measure of the plant in the case of state feedback. The minimal capacity for stabilizability via state feedback under STS has also been presented, which in general can only be computed by optimization. For SISO plants, the corresponding minimal mean square capacity for stabilizability via dynamic output feedback is given in terms of the anti-stable poles, NMP zeros and relative degree of the plant. For triangularly decoupled MIMO plants, necessary and sufficient conditions have also been provided for the dynamic output feedback

case. We have extended the results to the case with output fading channels and applied them to vehicle platooning.

- We have shown that in the case of dynamic output feedback, the channel feedback plays a key role in eliminating the limitation on stabilization induced by the NMP zeros and high relative degree of the plant. A minimal requirement on the network over which an SISO unstable plant can be stabilized via dynamic output feedback has been presented, which relates the statistics of channel fading and channel additive noise as well as the power bound of channel input with the Mahler measure of the plant. It has been shown that when a power bound is applied on the channel input, the network requirement for stabilizability becomes more stringent. At last, we have proposed a sub-optimal algorithm for performance design of the NCS under the assumption on perfect reconstruction of the network.
- We have extended the results on control of discrete-time networked systems over fading channels to the continuous-time case. In the case of static state feedback, the network requirement for stabilizability under PTS is given in terms of the unstable poles of the plant. For dynamic output feedback, additional requirement on the network is needed for stabilizability when the plant has NMP zeros.

9.2 Recommendations for Future Research

The following directions are envisaged for future research.

- The sufficient condition on the peak covariance stability of a remote Kalman filter with bounded Markovian dropouts provided in Theorem 3.2.2 still contains some conservativeness, and weaker stability conditions deserve further studies. Other interesting topics include conditions for covariance stability in

a usual sense (see, e.g., Definition 5.1.1) and the relationship between peak covariance stability and usual covariance stability.

- Static quantization strategy and an MJLS approach have been employed in Chapter 4. Possible future work includes dynamic quantization strategy and mode-dependent quantized stabilization via a switching system approach.
- In Chapter 4 and Section 7.2, only the single-input case has been addressed. One future research topic is the generalization of the results to the multi-input case.
- In Chapter 5, we have only addressed the stability issue for a remote Kalman filter with faded observations. The network resource allocation in optimizing the performance of a remote Kalman filter can be an interesting topic.
- The results in Chapter 6 on control over fading channels may be further extended to the stability robustness with respect to the fading envelope statistics and the string instability analysis of an infinite platoon over fading channels.
- Throughout this thesis (see Figure 1.2), the network has been placed in either the sensor-controller link or the controller-actuator link. The control problem with both unreliable input channel(s) and unreliable output channel(s) is one of the potential directions.
- As pointed out in Section 2.1, Shannon's capacity provided in standard information theory is not suitable for the control-oriented design of NCSs. On the other hand, there is a lack of physical interpretation of the results on networked control in the context of information theory. Therefore, one possible research direction in NCSs is to fill in the gap between information theory and control theory. Instead of Shannon's capacity, new notions of capacity are needed in the area of control over communication networks, which should serve purposes of both control and communication. The anytime capacity proposed by Sahai and Mitter [144] seems to be promising along this research line.

Appendix A

On Matrices

This appendix collects several results on matrix analysis that are repeatedly used throughout the thesis. The proofs are omitted and can be found in, e.g., [145, 146].

Lemma A.0.1 (Schur's complement [145]). *Suppose that a Hermitian matrix is partitioned as $\begin{bmatrix} A & B \\ B^H & C \end{bmatrix}$, where A and C are square. This matrix is positive definite if and only if A is positive definite and $C > B^H A^{-1} B$. Furthermore, this condition is equivalent to having $\rho(B^H A^{-1} B C^{-1}) < 1$.*

Lemma A.0.2 (Hadamard's inequality [145]). *If $A \in \mathbb{C}^{n \times n}$ is positive semidefinite, then $\det(A) \leq \prod_{i=1}^n [A]_{ii}$. Furthermore, when A is positive definite, then equality holds if and only if A is diagonal.*

Lemma A.0.3 (Matrix inversion lemma [145]). *Suppose that $Y = X + MQN$, and X, Y, Q are nonsingular. It holds that*

$$Y^{-1} = X^{-1} - X^{-1}M(Q^{-1} + NX^{-1}M)^{-1}NX^{-1}.$$

Lemma A.0.4 (Properties of trace, determinant and Kronecker product [145, 146]). *For X, Y, M, N with compatible dimensions, the following properties hold.*

$$(1). \quad \text{tr}(M+N) = \text{tr}(M) + \text{tr}(N), \quad \text{tr}(MN) = \text{tr}(NM), \quad \det(MN) = \det(M)\det(N).$$

(2). If $M \geq 0, N > 0$, then

$$\left(\min_i \lambda_i(N)\right) \text{tr}(M) \leq \text{tr}(MN) \leq \left(\max_i \lambda_i(N)\right) \text{tr}(M),$$

where $\lambda_i(N)$ is the i -th eigenvalue of N .

(3). $\det(I + MN) = \det(I + NM)$.

(4). The matrix equation $Y = MXN$ can be written into $\text{vec}(Y) = (N' \otimes M) \text{vec}(X)$.

Lemma A.0.5 (Property of Hadamard product [146]). *If $A \geq B \geq 0$ and $C \geq D \geq 0$, then $A \odot C \geq B \odot D$.*

Lemma A.0.6 (Shur product theorem [146]). *If A, B are positive semidefinite, then so is $A \odot B$. If, in addition, B is positive definite and A has no diagonal entry equals to 0, then $A \odot B$ is positive definite. In particular, if A, B are positive definite, then so is $A \odot B$.*

Author's Publications

Journal Papers

- 1) Nan Xiao, Lihua Xie, and Li Qiu, “Feedback stabilization of discrete-time networked systems over fading channels,” *conditionally accepted by IEEE Transactions on Automatic Control as a regular paper*, 2011.
- 2) Nan Xiao, Lihua Xie, and Minyue Fu, “Stabilization of Markov jump linear systems using quantized state feedback,” *Automatica*, vol. 46, no. 10, pp. 1696-1702, 2010.
- 3) Nan Xiao, Lihua Xie, and Minyue Fu, “Kalman filtering over unreliable communication networks with bounded Markovian packet dropouts,” *International Journal of Robust and Nonlinear Control*, vol. 19, no. 16, pp. 1770-1786, 2009.
- 4) Shuli Sun, Lihua Xie, Wendong Xiao, and Nan Xiao, “Optimal filtering for systems with multiple packet dropouts,” *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 55, no. 7, pp. 695-699, 2008.

Conference Papers

- 1) Nan Xiao, Lihua Xie, and Carlos E. de Souza, “Kalman filtering over fading channels with both transmission failure and signal fluctuation,” accepted by

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- the 9th IEEE International Conference on Control and Automation (ICCA), 2011.
- 2) Nan Xiao and Lihua Xie, "Analysis and design of discrete-time networked systems over fading channels," Proceedings of the 30th Chinese Control Conference (CCC), 2011, pp. 6562-6567.
 - 3) Nan Xiao and Lihua Xie, "Feedback stabilization over stochastic multiplicative input channels: continuous-time case," Proceedings of the 11th International Conference on Control, Automation, Robotics and Vision (ICARCV), 2010, pp. 543-548.
 - 4) Nan Xiao, Lihua Xie, and Li Qiu, "Mean square stabilization of multi-input systems over stochastic multiplicative channels," Proceedings of the 48th IEEE Conference on Decision and Control (CDC), 2009, pp. 6893-6898.
 - 5) Nan Xiao, Lihua Xie, and Minyue Fu, "Quantized stabilization of Markov jump linear systems via state feedback," Proceedings of American Control Conference (ACC), 2009, pp. 4020-4025.
 - 6) Nan Xiao and Lihua Xie, "Peak covariance stability of Kalman filter with bounded Markovian packet losses," Proceedings of the 7th World Congress on Intelligent Control and Automation (WCICA), 2008, pp. 335-340.

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