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Wang, Shenling

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CONTRIBUTIONS TO DEGREE STRUCTURES

WANG SHENLING

School of Physical and Mathematical Sciences

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Contents

Abstract

The investigation of computably enumerable degrees has led to the deep understanding of degree structures and the development of various construction techniques. This thesis is mainly concerned with the cupping and capping properties of computably enumerable degrees.

In Chapter 1, we give an introduction to the fundamentals of computability theory, and notations used through the thesis. In Chapter 2, we study the only-high cuppable degrees, which was recently found by Greenberg, Ng and Wu, we prove that such degrees can be plus-cupping. This result refutes a claim of Li and Y. Wang, which says that every plus-cupping degree is 3-plus-cupping.

In Chapter 3, we study the locally noncappable degrees, and we prove that for any nonzero incomplete c.e. degree $\mathbf{a}$, there exist two incomparable c.e. degrees $\mathbf{c}$, $\mathbf{d}$ $> \mathbf{a}$ witnessing that $\mathbf{a}$ is locally noncappable, and $\mathbf{c} \vee \mathbf{d}$ is high. This result implies that both classes of the plus-cuppping degrees and the nonbounding c.e. degrees do not form an ideal, which was proved by Li and Zhao by two separate constructions.

Chapter 4 is devoted to the study of the infima of n-c.e. degrees. Kaddah proved that there are n-c.e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and an (n+1)-c.e. degree $\mathbf{x}$ such that $\mathbf{a}$ is the infimum of $\mathbf{b}$ and $\mathbf{c}$ in the n-c.e. degrees, but not in the (n+1)-c.e. degrees, as $\mathbf{a} < \mathbf{x} < \mathbf{b}, \mathbf{c}$. We will prove that such 4-tuples occur densely in the c.e. degrees. This result immediately implies that the isolated (n+1)-c.e. degrees are dense in the c.e. degrees, which was first proved by LaForte.

Chapter 1

INTRODUCTION

People have been investigated calculations and algorithms for several centuries. A typical algorithm in mathematics is the Euclidean algorithm, an efficient method computing the greatest common divisors of natural numbers. Another famous algorithm is the sieve of Eratosthenes, an algorithm calculating all prime numbers up to a given integer. These algorithms share some common properties, such as a calculation proceeds deterministically, and stops after finitely many steps.

It was until 1930s when several formal definitions of computations were proposed. The formalization of the class of algorithmically computable functions (i.e. effectively calculable functions) began with attempts to solve some specific problems posed by David Hilbert, such as Hilbert's tenth problem and the Entscheidungsproblem. In Hilbert's tenth problem, Hilbert asked for a general algorithm to determine whether a given Diophantine equation has integer solutions. The Entscheidungsproblem asks for an algorithm to decide which formulas of first order logic are valid. Nowadays, it is known that both of them have negative answers — no such general algorithms exist. To answer these questions, one had to first formally characterize the informal class of effectively calculable functions. The research of such characterization work began at the beginning of 1930s, when the class of λ-definable functions was first studied by Church and then developed by Kleene. In his proof of Incompleteness Theorem, Gödel proposed in [25] the notion of primitive recursive functions, and then extended this notion to the (general) recursive functions, to capture more effectively calculable functions. Kleene obtained the μ-recursive functions from the primitive recursive functions and the μ-operator. Church and Kleene proved that the classes of (general) recursive functions, of λ-definable functions, and of μ-recursive functions, coincide. Based on this, Church published his famous proposition, now known as *Church's Thesis*, saying that the effectively calculable functions are μ-recursive.

Say a function on the natural numbers partial if the domain of this function is a subset of natural numbers, and a function is total if the domain is exactly the set of natural numbers. Turing proposed a simple computing device — Turing machines. A partial function on the natural numbers is Turing computable if it can be computed by a Turing machine. Turing proved that the classes of Turing computable functions and the (general) recursive functions coincide.

A set of natural numbers is computable if its characteristic function is computable. In addition to the study of computable sets and functions, we are interested in relative computability, a notion capturing sets being computable relative to another one. Given two sets of natural numbers A and B, say that A is Turing reducible to B (or A is computable relative to B), denoted by $A \leq_T B$, if the membership of A can be effectively computed from the information of B and B is known as *oracle*. Note that Turing reduction $\leq_T$ is reflexive and transitive and so it induces an equivalence relation $\equiv_T$ on the power set of the set of natural numbers. The corresponding equivalence classes are called *Turing degrees*.

A set A of natural numbers is *computably enumerable* (c.e. for short) if the elements of A can be listed in an effective way. It is easy to see that a set A is computable if and only if both A itself and its complement $\bar{A}$ are computably enumerable. The Halting problem, $K = \{e \in \omega : \varphi_e(e) \text{ converges}\}$, is a c.e. set, but incomputable. Here $\varphi_e(x)$ is a computation when the eth Turing program runs on the input x, and we say that a computation $\varphi_e(x)$ converges if this computation halts, and diverges otherwise. We use $\varphi_e(x)\downarrow$ to denote that $\varphi_e(x)$ converges and $\varphi_e(x)\uparrow$ for diverges.

For any sets $A, B \subseteq \omega$, $A \oplus B$ is defined as $\{2x : x \in A\} \cup \{2x+1 : x \in B\}$. The supremum of the Turing degrees $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \vee \mathbf{b}$, is the Turing degree of $A \oplus B$, where $A \in \mathbf{a}, B \in \mathbf{b}$. The definition of $\mathbf{a} \vee \mathbf{b}$ does not depend on the choices of the selected sets A and B. A Turing degree is c.e. if it contains a c.e. set. Let $\mathbf{R}$ be the class of all the c.e. degrees, and $\leq$ be the partial order relation on the c.e. degrees induced by the Turing reduction $\leq_T$. $\mathbf{R}$ is closed under join " $\vee$ " with respect to $\leq$, with the least element $\mathbf{0}$, the Turing degree of computable sets, and the greatest element $\mathbf{0}'$, the Turing degree of K. Post asked whether there exists any c.e. degree strictly between $\mathbf{0}$ and $\mathbf{0}'$. Friedberg [21], and Muchnik [49] independently, solved Post's problem by showing that there are intermediate c.e. degrees. In their proofs, they introduced the finite injury argument, which has turned out to be a powerful technique in modern computability theory. Shoenfield [52], and Sacks [56] independently, invented the infinite injury arguments. In [54], Sacks developed infinite injury argument to show that the c.e. degrees are dense. Another version of the infinite injury argument was later introduced by Yates [66] in his study of index sets. In [35], Lachlan introduced an even more powerful technique, called $\mathbf{0}'''$-priority argument to prove his nonsplitting theorem. This argument can be viewed as a finite injury argument on the top of an infinite injury argument, and was referred as the "monster method" in the 1980s because of the great complexity of constructions. This argument has been widely used in modern computability theory, in the constructions of Harrington plus-cupping degrees [28], Lachlan's nonbounding degrees [34] and Slaman triples [61], etc.

A c.e. degree $\mathbf{a}$ is called *cuppable* if there is an incomplete c.e. degree $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$, and *noncuppable* otherwise. Sacks' splitting theorem [55] implies the existence of incomplete cuppable degrees, and Yates (unpublished) and Cooper [6] proved the existence of noncuppable degrees (this is the well-known *Cooper-Yates noncupping theorem*). The cupping/noncupping properties were used by Harrington and Shelah [30] to show that the first-order theory of the structure ($\mathbf{R}$, $\leq$) is undecidable.

Let K^A be the halting problem relative to a set A, i.e. $K^A = \{e \in \omega : \Phi_e^A(e)\downarrow\}$. K^A is called the Turing jump of A and is denoted by A'. Let $\mathbf{a}' = \deg(A')$ for $A \in \mathbf{a}$, which is the Turing jump of $\mathbf{a}$. Using Turing jump, a hierarchy of the c.e. degrees was defined as follows: A c.e. degree $\mathbf{a}$ is called *low*$_n$(*high*$_n$) if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$ ($\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$ respectively), where $\mathbf{a}^{(0)} = \mathbf{a}$ and $\mathbf{a}^{(n+1)}$ is the Turing jump of $\mathbf{a}^{(n)}$. Let $\mathbf{H}_n$ and $\mathbf{L}_n$ be the sets of high$_n$ and low$_n$ c.e. degrees respectively. When $n = 1$, degrees in $\mathbf{H}_1$ and $\mathbf{L}_1$ are also called *high* degrees and *low* degrees respectively. Basic results on such hierarchy include that all the inclusions $\mathbf{L}_n \subset \mathbf{L}_{n+1}$ and $\mathbf{H}_n \subset \mathbf{H}_{n+1}$ are strict, and that there are c.e. degrees $\mathbf{a}$ such that $\mathbf{a}$ is not in $\bigcup_n(\mathbf{H}_n \cup \mathbf{L}_n)$.

Extending the Cooper-Yates noncupping theorem, Harrington [27] considered the relation between the high/low hierarchy and the cupping property: for each high c.e. degree $\mathbf{h}$, there exists a high c.e. degree $\mathbf{a} < \mathbf{h}$ such that for all c.e. degrees $\mathbf{x}$, if $\mathbf{h} \leq \mathbf{a} \vee \mathbf{x}$, then $\mathbf{h} \leq \mathbf{x}$. This is called the Harrington noncupping theorem. From this theorem, we know that every high c.e. degree bounds a high noncuppable degree. Let **NCUP** be the set of all noncuppable c.e. degrees. Obviously, **NCUP** is an ideal of $\mathbf{R}$.

In contrast to the noncuppable degrees, Harrington [28] proposed a much stronger notion of cupping — plus-cupping degrees, where a nonzero c.e. degree $\mathbf{a}$ is plus-cupping if for any c.e. degrees $\mathbf{b}, \mathbf{d}$ with $\mathbf{0} < \mathbf{b} \leq \mathbf{a} \leq \mathbf{d}$, there is a c.e. degree $\mathbf{e} < \mathbf{d}$ such that $\mathbf{b} \vee \mathbf{e} = \mathbf{d}$, and proved the existence of such plus-cupping degrees. Li [45] showed that the Harrington's plus-cupping degrees can be high$_2$. This result shows that the high c.e. degrees are not elementarily equivalent to the high$_n$ c.e. degrees for each $n > 1$. In [58], Shore proved that the low$_n$ and low$_m$ c.e. degrees are not elementarily equivalent for all $n > m > 1$.

Fejer and Soare [22] considered a special case of Harrington's plus-cupping degrees by restricting $\mathbf{d}$ to $\mathbf{0}'$.

Definition 1.1. *A nonzero c.e. degree* $\mathbf{a}$ *is called* plus-cupping *if every nonzero c.e. degree below* $\mathbf{a}$ *is cuppable, i.e. if* $\mathbf{0} < \mathbf{b} \leq \mathbf{a}$*, then there is an incomplete c.e. degree* $\mathbf{c}$ *such that* $\mathbf{c} \vee \mathbf{b} = \mathbf{0}'$

The construction of Fejer and Soare's plus-cupping degrees involves a standard gap-cogap argument. In the rest of this thesis, the plus-cupping degrees are those in the sense of Fejer and Soare.

In 1965, Shoenfield [53] conjectured that for any finite upper semi-lattices $P \subseteq Q$, with the least element 0 and the greatest element 1, any embedding of P into the upper semi-lattice $\mathbf{R}$ can be extended to an embedding of Q into $\mathbf{R}$. Shoenfield's conjecture would imply that the structure $\mathbf{R}$ is homogeneous. Shoenfield also listed two immediate consequences of this conjecture:

C1. For any incomparable c.e. degrees $\mathbf{a}$, $\mathbf{b}$, the infimum $\mathbf{a} \wedge \mathbf{b}$ does not exist.

C2. For any c.e. degrees $\mathbf{0} < \mathbf{b} < \mathbf{c}$, there is a c.e. degree $\mathbf{a} < \mathbf{c}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$.

Historically, this conjecture was first refuted by Lachlan [33] and Yates [65] by showing the existence of minimal pairs, where a pair $(\mathbf{a}, \mathbf{b})$ is called a *minimal pair* if $\mathbf{a}$ and $\mathbf{b}$ are nonzero c.e. degrees with infimum $\mathbf{0}$. Note that the existence of noncuppable degrees shows that C2 is not true.

A c.e. degree $\mathbf{a}$ is *cappable* if $\mathbf{a}$ is $\mathbf{0}$ or a half of a minimal pair. $\mathbf{a}$ is *noncappable*, if it is not cappable. Yates proved in [66] the existence of noncappable degrees. Let $\mathbf{M}$ and $\mathbf{NC}$ be the sets of all cappable and noncappable c.e. degrees, respectively. $\mathbf{M}$ is an ideal of $\mathbf{R}$ and $\mathbf{NC}$ is a filter of $\mathbf{R}$. $\mathbf{M}$ and $\mathbf{NC}$ form a nontrivial partition of the c.e. degrees. In [48], Maass introduces the notion of promptly simple sets: A coinfinite c.e. set A is *promptly simple* if there is a partial computable function p and a computable enumeration $\{A_s\}_{s \in \omega}$ of A such that for every e, if W_e is infinite, then there are s and x such that $x \in W_{e,\ at\ s} \cap A_{p(s)}$, where $W_{e,\ at\ s}$ is the set of numbers enumerated into W_e at stage s. A c.e. degree $\mathbf{a}$ is called promptly simple, if it contains a promptly simple set. Let $\mathbf{PS}$ be the set of all promptly simple degrees and $\mathbf{LC}$ be the set of all low cuppable c.e. degrees. The following theorem gives an important characterization of noncappable degrees.

Theorem 1.2. *(Ambos-Spies, Jockusch, Shore and Soare [1]) A c.e. degree is noncappable if and only if it contains a promptly simple set if and only if it is low cuppable. That is,* $\mathbf{NC} = \mathbf{PS} = \mathbf{LC}$.

Based on the high/low hierarchy, Li, Wu and Zhang [43] proposed a hierarchy of cuppable c.e. degrees $\mathbf{LC}_1 \subseteq \mathbf{LC}_2 \subseteq \mathbf{LC}_3 \subseteq \cdots$, where for each $n \geq 1$, $\mathbf{LC}_n$ denotes the class of low$_n$-cuppable degrees. Here, a c.e. degree $\mathbf{a}$ is called *low$_n$-cuppable* if there is a low$_n$ c.e. degree $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$. Li, Wu and Zhang [43] showed that $\mathbf{LC}_1$ is a proper subset of $\mathbf{LC}_2$ by constructing a cappable and low$_2$-cuppable c.e. degree. Recently, Greenberg, Ng and Wu proved in [23] that there is an incomplete cuppable degree which can only be cupped to $\mathbf{0}'$ by high degrees.

Theorem 1.3. *(Greenberg, Ng and Wu [23]) There is a cuppable degree* $\mathbf{a}$ *such that for any c.e. degree* $\mathbf{w}$*, if* $\mathbf{a} \vee \mathbf{w} = \mathbf{0}'$*, then* $\mathbf{w}$ *is high.*

Theorem 1.3 shows that $\cup_n \mathbf{LC}_n$ does not contain all of the cuppable degrees. This refutes a claim of Li [44] that all cuppable degrees are low$_3$-cuppable. We call the cuppable degrees constructed by Greenberg, Ng and Wu [23] *only-high cuppable degrees.*

Extending the notion of low$_n$-cuppability, Li and Wang [64] defined a notion of n-plus-cupping degrees. A nonzero c.e. degree $\mathbf{a}$ is called *n-plus-cupping* if every c.e. degree $\mathbf{x}$ with $\mathbf{0} < \mathbf{x} \leq \mathbf{a}$ is low$_n$-cuppable. The class of n-plus-cupping degrees is denoted by $\mathbf{PC}_n$. This gives rise to a hierarchy for plus-cupping degrees

$$\mathbf{PC}_1 \subseteq \mathbf{PC}_2 \subseteq \mathbf{PC}_3 \subseteq \cdots .$$

Note that $\mathbf{PC}_1 = \emptyset$, as low-cuppable degrees are noncappable, while plus-cupping degrees are cappable. Li and Wang proved the existence of a 2-plus-cupping degree, and hence $\mathbf{PC}_1$ is a proper subset of $\mathbf{PC}_2$. Li and Wang [64] also claimed that all plus-cupping degrees are 3-plus-cupping, and therefore $\mathbf{PC}_3 = \mathbf{PC}$, where $\mathbf{PC}$ is the class of plus-cupping degrees. Unfortunately, this claim is false, as we will show in Chapter 2.

It is easy to see that the join of an only-high cuppable degree and a noncuppable degree is again only-high cuppable. Thus, only-high cuppable degrees can be high. A natural question is whether there is an only-high cuppable degree which bounds no noncuppable degrees. In Chapter 2, we answer this question affirmatively. Actually, we extend the result of Greenberg, Ng and Wu by showing that there is a plus-cupping degree which is cupped to $\mathbf{0}'$ by high degrees only.

Theorem 1. *(Wang and Wu) There is a plus-cupping degree* $\mathbf{a}$ *such that for any c.e. degree* $\mathbf{w}$*, if* $\mathbf{a} \vee \mathbf{w} = \mathbf{0}'$*, then* $\mathbf{w}$ *is high.*

Theorem 1 implies the existence of a plus-cupping degree $\mathbf{a}$ such that any nonzero c.e. degree $\mathbf{b} \leq \mathbf{a}$ cannot be cupped to $\mathbf{0}'$ by a low$_n$ c.e. degree for any n. Hence, this refutes Li-Wang's claim mentioned above.

Chapter 3 is devoted to the study of locally noncappable degrees, which was first proposed by Seetapun in his thesis. Harrington and Soare [29] proved that there are no maximal cappable degrees. Seetapun's result of locally noncappable degrees implies that there are no maximal nonbounding degrees.

Theorem 1.4. *(Harrington and Soare [29]) If* $\mathbf{a}$ *and* $\mathbf{b}$ *form a minimal pair, then there is a c.e. degree* $\mathbf{c}$ *above* $\mathbf{a}$ *such that* $\mathbf{b}$ *and* $\mathbf{c}$ *also form a minimal pair.*

Seetapun [57] improved Theorem 1.4 to a much stronger version:

Theorem 1.5. *(Seetapun [57]) For any nonzero incomplete c.e. degree* $\mathbf{b}$*, there is a c.e. degree* $\mathbf{d}$ *above* $\mathbf{b}$ *such that for any c.e. degree* $\mathbf{c}$*, if* $\mathbf{c}$ *and* $\mathbf{b}$ *form a minimal pair, then* $\mathbf{c}$ *and* $\mathbf{d}$ *also form a minimal pair.*

Even though any nonzero c.e. degree bounds a cappable degree, Lachlan's Nonbounding Theorem says that not every c.e. degree bounds a minimal pair.

Theorem 1.6. *(Lachlan [34]) There is a nonzero c.e. degree* $\mathbf{a}$ *such that no nonzero c.e. degrees* $\mathbf{b}$*,* $\mathbf{c}$ *below* $\mathbf{a}$ *form a minimal pair.* $\mathbf{a}$ *is called a nonbounding degree.*

Note that every noncappable c.e. degree bounds a minimal pair, so all nonbounding c.e. degrees are cappable. Cooper [9] showed that every high c.e. degree bounds a minimal pair, and hence nonbounding c.e. degrees can not be high. Downey, Lempp and Shore [15] proved that nonbounding c.e. degrees can be high$_2$.

Definition 1.7. *A nonzero c.e. degree* $\mathbf{a}$ *is called* locally noncappable *if there is a c.e. degree* $\mathbf{c}$ *above* $\mathbf{a}$ *such that no nonzero c.e. degree* $\mathbf{w}$ *below* $\mathbf{c}$ *forms a minimal pair with* $\mathbf{a}$*. We say that* $\mathbf{c}$ *witnesses that* $\mathbf{a}$ *is locally noncappable.*

Seetapun proved in his thesis [57] that every nonzero incomplete c.e. degree is locally noncappable. Giorgi published Seetapun's result in [24], but with a Σ_3 outcome missing in his writing, so Giorgi's construction is actually not a complete one. Recently, Stephan and Wu [59] improved Seetapun's result by showing that such witnesses can be high$_2$ degrees.

Theorem 1.8. *(Stephan and Wu [59]) Given a nonzero incomplete c.e. degree* $\mathbf{a}$*, there exists a high*$_2$ *c.e. degree* $\mathbf{c} > \mathbf{a}$ *witnessing that* $\mathbf{a}$ *is locally noncappable.*

The proof of Theorem 1.8 combines Seetapun's construction and the high$_2$ strategy developed in Lerman [37] and Downey, Lempp and Shore [15]. The construction in the proof of Theorem 1.8 contains some new features. That is, after a gap is closed unsuccessfully, it can be reopened again. Theorem 1.8 is so strong that it has several well-known results as its corollaries, such as Downey, Lempp and Shore's result [15] that there is a high$_2$ nonbounding c.e. degree, Li's result [45] that there is a high$_2$ (Harrington) plus-cupping degree, and it can further imply that there are no maximal nonbounding c.e. degrees (first proved by Seetapun) and no maximal (Harrington) plus-cupping degrees (a new result). In Chapter 4, we show that such witnesses can have join high.

Theorem 2. *(Fang, Wang and Wu) For any nonzero incomplete c.e. degree* $\mathbf{a}$*, there exist two incomparable c.e. degrees* $\mathbf{c}$*,* $\mathbf{d} > \mathbf{a}$ *witnessing that* $\mathbf{a}$ *is locally noncappable, and* $\mathbf{c} \vee \mathbf{d}$ *is high.*

Theorem 2 implies that both classes of the plus-cuppping degrees and the nonbounding c.e. degrees do not form ideals, which was proved by Li and Zhao in [46], by using two separate constructions.

In 1965, Putnam [51] and Gold [26] introduced the notion of n-c.e. sets as a generalization of the c.e. sets:

Definition 1.9. *Let* n *be a positive natural number. A set* A *is n-c.e. if there is a computable function* f *such that for every* $x \in \omega$*,*

$$f(x, 0) = 0,$$

$$A(x) = \lim_s f(x, s),$$

$$|\{s : f(x, s) \neq f(x, s+1)\}| \leq n.$$

The intuition is that we can change our guesses about the membership of x in A at most n many times.

Obviously, the 1-c.e. sets are just the c.e. sets. The 2-c.e. sets are also known as the d.c.e. sets since they can be expressed as differences of two c.e. sets. Similarly, the n-c.e. sets are those given by starting with c.e. sets and alternating the Boolean operations of difference and union. These sets were first extensively studied (and extended to the α-c.e. sets for computable ordinals α) by Ershov [18] and are now known as the *Ershov hierarchy.*

A Turing degree is d.c.e. if it contains a d.c.e. set. The structure of d.c.e. degrees were first studied by Cooper [7] and Lachlan who showed that there is a *properly d.c.e. degree* (a d.c.e. degree that does not contain a c.e. set), and that every incomputable d.c.e. degree bounds an incomputable c.e. degree, respectively. The latter result established the downward density of the d.c.e. degrees. Let $\mathbf{D}_n$ be the class of all the n-c.e. degrees (Turing degrees of n-c.e. sets), so $\mathbf{R} = \mathbf{D}_1$. The main interest in the n-c.e. degrees lies in the comparison of structures $\mathbf{D}_m$, $\mathbf{D}_n$, when $m \neq n$. The first two structural differences between $\mathbf{D}_2$ and $\mathbf{R}$ were obtained by Arslanov and Downey in the 1980s. Arslanov [2] showed that every nonzero d.c.e degree is cuppable in $\mathbf{D}_2$, and this statement fails in $\mathbf{R}$ by the well-known Cooper-Yates noncupping theorem. Downey [14] showed that the diamond lattice can be embedded into the d.c.e. degrees preserving $\mathbf{0}$ and $\mathbf{1}$, and this embedding can not be done in $\mathbf{R}$ by Lachlan's Nondiamond Theorem [33].

The question that whether the d.c.e. degrees are dense or not motivated a lot of interest among computability theorists. Lachlan observed that the d.c.e. degrees are downward dense, and Cooper, Lempp and Watson [10] proved that the properly d.c.e. degrees are dense in the c.e. degrees. Arslanov, Cooper and Li [5] showed that there is no maximal low d.c.e. degrees, and Cooper [8] proved the density of the low$_2$ d.c.e. degrees. The general density problem was finally solved by Cooper, Harrington, Lachlan, Lempp and Soare [11].

Theorem 1.10. *(Cooper, Harrington, Lachlan, Lempp and Soare [11]) There is a maximal incomplete d.c.e. degree.*

This nondensity theorem gives another elementary difference between $\mathbf{D}_2$ and $\mathbf{R}$.

However, the elementary differences between the n-c.e. degrees for various $n > 1$ seemed hard to find. Downey [14] even conjectured that

Conjecture 1.11. *(Downey [14]) For any $m, n > 1$ with $m \neq n$, the degree structures* *of the m-c.e. and the n-c.e. degrees are elementarily equivalent.*

This conjecture was refuted recently by Arslanov, Kalimullin and Lempp in [3]:

Theorem 1.12. *(Arslanov, Kalimullin and Lempp [3]) The degree structures of the d.c.e. and the 3-c.e. degrees are not elementarily equivalent.*

Recently, Cai, Shore and Slaman [13] proved that the theories of $\mathbf{D}_n$ are all undecidable for every n. Note that the theories of $\mathbf{R}$, $\mathbf{D}(\leq \mathbf{0}')$, $\mathbf{D}$ are undecidable by Harrington and Shelah [30]; Epstein [19] and Lerman [38]; and Lachlan [36], respectively.

The notion of isolation was proposed by Cooper and Yi in 1995. A d.c.e. degree $\mathbf{d}$ is isolated by a c.e. degree $\mathbf{a}$, if $\mathbf{a} < \mathbf{d}$ and $\mathbf{a}$ is the greatest c.e. degree below $\mathbf{d}$. $(\mathbf{a}, \mathbf{d})$ is called an isolation pair. $\mathbf{d}$ is *isolated* if there is a c.e. degree $\mathbf{a}$ such that $\mathbf{d}$ is isolated by $\mathbf{a}$, and *nonisolated* otherwise. It is well-known that both the isolated and the nonisolated d.c.e. degrees are dense in the c.e. degrees.

Theorem 1.13. *(Ding and Qian [17]; LaForte [40]) The isolated d.c.e. degrees are dense in the c.e. degrees.*

Theorem 1.14. *(Arslanov, Lempp and Shore [4]) The nonisolated d.c.e. degrees are dense in the c.e. degrees.*

The following theorem shows that two degrees in an isolation pair can be far from each other in terms of the high/low hierarchy.

Theorem 1.15. *(Ishmukhametov and Wu [63]) There is an isolation pair* $(\mathbf{a}, \mathbf{d})$ *such that* $\mathbf{a}$ *is low and* $\mathbf{d}$ *is high.*

In 2002, Wu [62] gave another proof of Downey's diamond embedding into the d.c.e. degrees, where an isolation pair was used to separate the cupping and the capping into two separate phases, avoiding interactions between the cupping and capping parts. By developing this idea, Downey, Li and Wu proved in [16] that any cappable c.e. degree has a complement in the d.c.e. degrees. In 2010, Liu and Wu [41] proved that there exists an isolation pair $(\mathbf{a}, \mathbf{d})$ such that all c.e. degrees that can not cup $\mathbf{d}$ to $\mathbf{0}'$ are bounded by $\mathbf{a}$. As $\mathbf{d}$ cups all the c.e. degrees not below it to $\mathbf{0}'$, $\mathbf{d}$ is said to have almost universal cupping property. Recently, Fang, Liu and Wu use this structural phenomenon to prove a fairly strong cupping theorem.

Theorem 1.16. *(Fang, Liu and Wu [20]) For any nonzero cappable c.e. degree* $\mathbf{c}$*, there exists an isolation pair* $(\mathbf{a}, \mathbf{d})$ *such that* $\mathbf{c} \vee \mathbf{d} = \mathbf{0}'$*,* $\mathbf{c} \wedge \mathbf{a} = \mathbf{0}$*, and* $\mathbf{d}$ *has almost universal cupping property. Note that* $\mathbf{0}, \mathbf{c}, \mathbf{d}, \mathbf{0}'$ *form a diamond.*

This theorem has many known results as direct corollaries, including Arslanov's cupping theorem, Downey's diamond theorem, Downey, Li, and Wu's complementation theoremm, and also Li and Yi's cupping theorem [47].

Kleene and Post proved in [32] that the infimum of Turing degrees $\mathbf{a}$ and $\mathbf{b}$ may not exist. Lachlan proved in [33] that $\mathbf{a}$ and $\mathbf{b}$ can be c.e.. Lachlan also pointed out in this paper that for any c.e. degrees $\mathbf{a}$ and $\mathbf{b}$, the infimum of $\mathbf{a}$ and $\mathbf{b}$, considered in the c.e. degrees, if exists, is the same as the one when considered in the Δ^0_2 degrees. One natural question is whether such a coincidence is true when $\mathbf{a}$ and $\mathbf{b}$ are not c.e.. Kaddah proved in [31] that there are d.c.e. degrees $\mathbf{b}, \mathbf{c}$ such that their infimum in the d.c.e. degrees is different from their infimum in the 3-c.e. degrees. In particular, she proved in [31] that there are d.c.e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and a 3-c.e. degree $\mathbf{x}$ such that $\mathbf{a}$ is the infimum of $\mathbf{b}, \mathbf{c}$ in the d.c.e. degrees, but not in the 3-c.e. degrees, as $\mathbf{a} < \mathbf{x} < \mathbf{b}, \mathbf{c}$. In [42], we extended Kaddah's result by showing that such a structural difference occurs densely in the c.e. degrees.

Theorem 1.17. *(Liu, Wang and Wu [42]) Given c.e. degrees* $\mathbf{u} < \mathbf{v}$*, there are d.c.e. degrees* $\mathbf{a}, \mathbf{b_1}, \mathbf{b_2}$ *and a 3-c.e. degree* $\mathbf{x}$ *between* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathbf{a} < \mathbf{x} < \mathbf{b_1}, \mathbf{b_2}$ *and* $\mathbf{b_1}$ *and* $\mathbf{b_2}$ *have infimum* $\mathbf{a}$ *in the d.c.e. degrees.*

Actually, Kaddah also pointed out in [31] that for all $n > 1$, there are n-c.e. degrees $\mathbf{b}, \mathbf{c}$ such that their infimum in the n-c.e. degrees is different from their infimum in the (n+1)-c.e. degrees. In Chapter 4, we prove that this generalization occurs densely in the c.e. degrees.

Theorem 3. *(Liu, Wang and Wu) For* $n > 1$ *and c.e. degrees* $\mathbf{u} < \mathbf{v}$*, there are n-c.e. degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and an (n+1)-c.e. degree* $\mathbf{x}$ *such that* $\mathbf{u} < \mathbf{a} < \mathbf{x} < \mathbf{b}, \mathbf{c} < \mathbf{v}$ *and* $\mathbf{b}$ *and* $\mathbf{c}$ *have infimum* $\mathbf{a}$ *in the n-c.e. degrees.*

Theorem 3 implies that the isolated (n+1)-c.e. degrees are dense in the c.e. degrees since there is no n-c.e. degree between **a** and **x**, which was first proved by LaForte [39].

Our notation and terminology are quite standard and generally follow Soare [60] or Odifreddi [50]. Basic knowledge of tree constructions in computability theory is assumed. We use capital Greek letters to denote partial computable (p.c. for short) functionals, with associated uses denoted by the corresponding lowercase Greek letters. In addition, during the course of a construction, a parameter p is defined to be fresh at a stage s means that $p > s$ and p is the least number not mentioned so far in the construction.

Chapter 2

ON A HIERARCHY OF PLUS-CUPPING DEGREES

2.1 Introduction

A c.e. degree $\mathbf{a}$ is *cuppable* if there is an incomplete c.e. degree $\mathbf{b}$ such that $\mathbf{a}\vee\mathbf{b} = \mathbf{0}'$, and *noncuppable* otherwise. Extending the cupping property, Harrington [28], and Fejer and Soare [22] proposed a much stronger notion — plus-cupping degrees. A nonzero c.e. degree $\mathbf{a}$ is *plus-cupping* if every nonzero c.e. degree below $\mathbf{a}$ is cuppable, i.e. if $\mathbf{0} < \mathbf{b} \leq \mathbf{a}$, then there is an incomplete c.e. degree $\mathbf{c}$ cupping $\mathbf{b}$ to $\mathbf{0}'$.

Recently, Greenberg, Ng and Wu [23] proved the existence of cuppable degrees which can be cupped to $\mathbf{0}'$ by high degrees only. We call such cuppable degrees *only-high cuppable degrees*. Note that the join of any only-high cuppable degree and any noncuppable degree is again only-high cuppable (suppose that $\mathbf{a}$ is an only-high cuppable degree, and $\mathbf{b}$ is a noncuppable degree. Obviously, the join $\mathbf{a}\vee\mathbf{b}$ is cuppable and incomplete. For any c.e. degree $\mathbf{w}$, if $(\mathbf{a}\vee\mathbf{b})\vee\mathbf{w} = \mathbf{0}'$, then $\mathbf{b}\vee(\mathbf{a}\vee\mathbf{w}) = \mathbf{0}'$. Since $\mathbf{b}$ is noncuppable, $\mathbf{a}\vee\mathbf{w} = \mathbf{0}'$. Hence, $\mathbf{w}$ is high as $\mathbf{a}$ is an only-high cuppable degree. This shows that $\mathbf{a}\vee\mathbf{b}$ is only-high cuppable). Moreover, only-high cuppable degrees can be high (take $\mathbf{b}$ to be a high, noncuppable degree in the proof stated above, then $\mathbf{a}\vee\mathbf{b}$ is only-high cuppable and high), and hence bounds noncuppable degrees.

A natural question is whether there is an only-high cuppable degree which bounds no noncuppable degrees. In this chapter, we answer this question affirmatively. Actually, we extend the result of Greenberg, Ng and Wu by showing that such only-high cuppable degrees can be plus-cupping. This refutes Li-Wang's claim that every plus-cupping degree is 3-plus-cupping, where a nonzero c.e. degree $\mathbf{a}$ is n-plus-cupping if for every c.e. degree $\mathbf{x}$ with $\mathbf{0} < \mathbf{x} \leq \mathbf{a}$, there is a low$_n$ c.e. degree $\mathbf{y}$ such that $\mathbf{x} \vee \mathbf{y} = \mathbf{0}'$.

Theorem 1. *(Wang and Wu) There is a plus-cupping degree* $\mathbf{a}$ *such that for any c.e. degree* $\mathbf{w}$*, if* $\mathbf{a} \vee \mathbf{w} = \mathbf{0}'$*, then* $\mathbf{w}$ *is high.*

2.2 Requirements and strategies

To prove Theorem 1, we will construct a c.e. set A and an auxiliary c.e. set P satisfying the following requirements:

$\mathcal{P}_e : A \neq \varphi_e$,

$\mathcal{Q}_e : P = \Phi_e^{A,V_e} \Rightarrow$ there exists a partial computable functional Δ_e such that for every i,

$$Tot(i) = \lim_x \Delta_e^{V_e}(i, x),$$

$\mathcal{R}_e : W_e = \Phi_e^A \Rightarrow W_e$ is computable or there are C_e, Γ_e such that C_e is an incomplete c.e. set, Γ_e is a partial computable functional, and $K = \Gamma_e^{C_e, W_e}$.

Here $\{(W_e, V_e) : e \in \omega\}$ is a fixed effective list of pairs of c.e. sets. For each e, Γ_e, Δ_e are partial computable functionals built by us and $Tot = \{i : \varphi_i \text{ is total}\}$ is a Π_2^0-complete set.

Let $\mathbf{a}$ be the degree of A. By the $\mathcal{P}$-requirements, $\mathbf{a}$ is nonzero. By the $\mathcal{R}$-requirements, $\mathbf{a}$ is a plus-cupping degree. The $\mathcal{Q}$-requirements ensure that for any c.e. set V_e, if $K \leq_T A \oplus V_e$ then V_e has high degree, as in this case, the $\mathcal{Q}$-requirements ensure that $Tot \leq_T V_e'$. Therefore, satisfying all the requirements will be enough to prove Theorem 1.

2.2.1 A $\mathcal{P}_e$ strategy

A $\mathcal{P}_e$-strategy α is a standard diagonalization strategy. That is, we choose a witness x and then wait for a stage at which $\varphi_e(x) \downarrow = 0$. If there is no such stage, then $\mathcal{P}_e$ is satisfied obviously. Otherwise, we put x into A and so $A(x) = 1 \neq 0 = \varphi_e(x)$, $\mathcal{P}_e$ is also satisfied.

2.2.2 A $\mathcal{Q}_e$ strategy

Let ξ be a $\mathcal{Q}_e$-strategy. We define the length of agreement function $l(\xi, s)$ at stage s as

$$l(\xi, s) = \max\{x < s : (\forall y < x)[P(y)[s] = \Phi_e^{A,V_e}(y)[s]]\},$$

and the maximum length of agreement function at stage s as

$$m(\xi, s) = \max\{l(\xi, t) : t < s \text{ and } t \text{ is a } \xi\text{-stage}\}.$$

Say that a stage s is a ξ-expansionary stage if $s = 0$ or s is a ξ-stage with $l(\xi, s) > m(\xi, s)$.

If $P = \Phi_e^{A,V_e}$, then there are infinitely many ξ-expansionary stages, and at these expansionary stages, we will construct a p.c. functional Δ_e to ensure that for any i,

$$Tot(i) = \lim_x \Delta_e^{V_e}(i, x).$$

The strategy of ξ has two outcomes $\infty <_L f$, where ∞ denotes that ξ has infinitely many expansionary stages and f denotes that ξ has only finitely many expansionary stages. Below the ∞ outcome of ξ, we will ensure that Δ_e satisfies the following subrequirements

$$\mathcal{T}_{e,i} : Tot(i) = \lim_x \Delta_e^{V_e}(i, x)$$

for all i.

2.2.3 A $\mathcal{T}_{e,i}$ strategy

Let ζ be a $\mathcal{T}_{e,i}$-strategy below the ∞ outcome of ξ. We define the length of convergence function $l(\zeta, s)$ at stage s as

$$l(\zeta, s) = \max\{x < s : (\forall y < x)[\varphi_i(y)[s] \downarrow]\},$$

and the maximum length of convergence function at stage s as

$$m(\zeta, s) = \max\{l(\zeta, t) : t < s \text{ and } t \text{ is a } \zeta\text{-stage}\}.$$

Say that a stage s is a ζ-expansionary stage if $s = 0$ or s is a ζ-stage with $l(\zeta, s) > m(\zeta, s)$. ζ has two outcomes $\infty <_L f$, where ∞ denotes the outcome that φ_i is total, in which we will define $\Delta_e^{V_e}(i, x) = 1$ for almost all x, and f denotes the outcome that φ_i is not total, in which case we will define $\Delta_e^{V_e}(i, x) = 0$ for almost all x.

Suppose that we define $\Delta_e^{V_e}(i, x) = 0$ under the outcome f at a previous stage, and now ζ changes its outcome to ∞, so we want to redefine $\Delta_e^{V_e}(i, x) = 1$, which requires V_e to have a corresponding change to first make $\Delta_e^{V_e}(i, x)$ undefined. For this purpose, before we define $\Delta_e^{V_e}(i, x)$ as 0, we first pick a big number p_ζ which is not in P, and wait for a ξ-expansionary stage, s_0 say, such that $l(\xi, s_0) > p_\zeta$, i.e. we see $\Phi_e^{A,V_e}(p_\zeta) \downarrow= 0$ at stage s_0, then we define $\Delta_e^{V_e}(i, x)[s_0] = 0$ with use $\delta_e(i, x)[s_0] > \varphi_e(p_\zeta)[s_0]$. At the next ζ-expansionary stage $s_1 > s_0$, i.e. ζ has outcome ∞, we first put p_ζ into P to force V_e to have a change below $\varphi_e(p_\zeta)[s_0]$ while restrain A on $\varphi_e(p_\zeta)[s_0]$ (if there is no such a change, then P and $\Phi_e^{A,V_e}(p_\zeta)$ will differ at p_ζ, a global win for ξ), and hence below $\delta_e(i, x)[s_0]$. This change undefines $\Delta_e^{V_e}(i, x)$, and as a consequence, we can redefine it.

We now consider the interactions between two $\mathcal{T}$-strategies. We describe a potential problem: if there are two or more $\mathcal{T}$-strategies working below the ∞ outcome of ξ, ζ_1, ζ_2, with $\xi^\frown\langle\infty\rangle \subseteq \zeta_1^\frown\langle\infty\rangle \subseteq \zeta_2$. Then the action of enumerating p_{ζ_1} into P for ζ_1 as above can always force V_e to have a change to undefine $\Delta_e^{V_e}(i(\zeta_1), x)$, but this change can also undefine $\Delta_e^{V_e}(i(\zeta_2), x)$. Since ζ_2 assumes that ζ_1 has outcome ∞, it knows that p_{ζ_1} will be enumerated into P, and will be updated infinitely often. As a consequence, if ζ_2 has outcome f, then the action of ζ_1 may lead $\Delta_e^{V_e}(i(\zeta_2), x)$ to diverge eventually.

To avoid this, if $\Delta_e^{V_e}(i(\zeta_2), x)$ is undefined by a V_e-change, at the next ζ_2-stage, if ζ_2 has outcome f, we redefine $\Delta_e^{V_e}(i(\zeta_2), x) = 0$, but with use the same as before if the computation $\Phi_e^{A,V_e}(p_{\zeta_2})$ is the same as before. Of course, if the computation $\Phi_e^{A,V_e}(p_{\zeta_2})$ changes, then we define $\Delta_e^{V_e}(i(\zeta_2), x) = 0$ with use bigger than the current use $\varphi_e(p_{\zeta_2})$. Therefore, if $P = \Phi_e^{A,V_e}$, and ζ_2 has outcome f, then the parameter p_{ζ_2} will be updated finitely often and hence it has a final value. As a consequence, the use $\varphi_e(p_{\zeta_2})$ will have a fixed value, which ensures that $\delta_e(i(\zeta_2), x)$-use can be lifted only finitely often. This ensures that ζ_1's definition has little effect on ζ_2's definition.

A $\mathcal{P}$-strategy and a $\mathcal{Q}$-strategy

We now consider the interactions between one $\mathcal{P}_{e'}$-strategy α and one $\mathcal{Q}_e$-strategy ξ with $\xi^\frown\langle\infty\rangle \subset \alpha$. Fix a witness y for α, when we see $\varphi_{e'}(y) \downarrow= 0$, we want to put y into A directly to satisfy $\mathcal{P}_{e'}$, but this action may injure ξ, which has higher priority.

Consider the following scenario: Let ζ be a $\mathcal{T}_{e,i}$-strategy between ξ and α with $\xi^\frown\langle\infty\rangle \subseteq \zeta^\frown\langle f\rangle \subseteq \alpha$ and ζ defines $\Delta_e^{V_e}(i, x) = 0$ with use $\delta_e(i, x) > \varphi_e(p_\zeta)$ under the outcome f, but now α puts y into A, and this enumeration can change the computation $\Phi_e^{A,V_e}(p_\zeta)$ and lead to a new use $\varphi_e(p_\zeta)$ bigger than $\delta_e(i, x)$. Now if later, at a ζ-expansionary stage, we put p_ζ into P, V_e may change below the new use $\varphi_e(p_\zeta)$, but not below $\delta_e(i, x)$, and this V_e-change cannot make $\Delta_e^{V_e}(i, x)$ undefined as wanted.

To avoid this problem, when α selects y as a number for diagonalization, α also selects a number $a_{\alpha,\xi}$. When we want to put a number y into A, we put $a_{\alpha,\xi}$ into P first to force a V_e-change to undefine $\Delta_e^{V_e}(i, x)$ defined by ζ. We put y into A only after we see such a V_e-change. Such a process delays the diagonalization, but it does not affect the satisfaction of α, since once $\varphi_{e'}(y) \downarrow= 0$, it converges to 0 forever.

The interactions between one $\mathcal{P}$-strategy and several $\mathcal{Q}$-strategies is a direct generalization of the simple case discussed above.

Without loss of generality, suppose that, above a $\mathcal{P}_e$-strategy α, there are n many $\mathcal{Q}$-strategies, $\xi_1, \xi_2, \cdots, \xi_n$ say, with $\xi_1^\frown\langle\infty\rangle \subset \xi_2^\frown\langle\infty\rangle \subset \cdots \subset \xi_n^\frown\langle\infty\rangle \subset \alpha$. We will associate α with parameters $a_{\alpha,\xi_1} < \cdots < a_{\alpha,\xi_n}$ for those $\mathcal{Q}$-strategies. After α sees that $\varphi_e(y) \downarrow= 0$ at some stage s_0, where y is a candidate for the $\mathcal{P}_e$-strategy α, it first creates a link between α and ξ_n and puts the number a_{α,ξ_n} into P simultaneously. Thus, at the next ξ_n-expansionary stage s_1, a $V_{e(\xi_n)}$-change appears, and this $V_{e(\xi_n)}$-change will undefine those $\Delta_{e(\xi_n)}^{V_{e(\xi_n)}}(i,x)$ defined by ζ-strategies between α and ξ_n. So if we now put y into A, this enumeration will not cause incorrectness of $\Delta_{e(\xi_n)}^{V_{e(\xi_n)}}$. At stage s_1, the previous link between α and ξ_n is cancelled, we create a new link between α and ξ_{n-1} and put the number $a_{\alpha,\xi_{n-1}}$ into P simultaneously (note that once $\varphi_e(y) \downarrow= 0$, it converges to 0 forever). At the next ξ_{n-1}-expansionary stage s_2, a $V_{e(\xi_{n-1})}$-change appears, and this $V_{e(\xi_{n-1})}$-change will undefine those $\Delta_{e(\xi_{n-1})}^{V_{e(\xi_{n-1})}}(i,x)$ defined by ζ-strategies between α and ξ_{n-1}. We repeat this process for all the other such $\mathcal{Q}$-strategies. After we cancel the link between α and ξ_1, then α can perform the diagonalization by putting y into A, this enumeration will not cause incorrectness of those $\Delta_{e(\xi_i)}^{V_{e(\xi_i)}}$ for all $1 \leq i \leq n$, and hence will not injure the $\mathcal{Q}$-strategies with higher priority.

2.2.4 An $\mathcal{R}_e$ strategy

Assume that a strategy β works on an $\mathcal{R}_e$-requirement. We define the length of agreement function $l(\beta,s)$ at stage s as

$$l(\beta,s) = \max\{x < s : (\forall y < x)[W_e(y)[s] = \Phi_e^A(y)[s]]\},$$

and the maximum length of agreement function at stage s as

$$m(\beta,s) = \max\{l(\beta,t) : t < s \text{ and } t \text{ is a } \beta\text{-stage}\}.$$

Say that a stage s is a β-expansionary stage if $s = 0$ or s is a β-stage with $l(\beta,s) > m(\beta,s)$. Here, a stage s is a β-stage means that β is visited at stage s.

If $W_e = \Phi_e^A$, then there are infinitely many β-expansionary stages. At β-expansionary stages, we will construct an incomplete c.e. set C_e and a p.c. functional Γ_e such that either $K = \Gamma_e^{C_e,W_e}$, or W_e is computable.

The p.c. functional Γ_e will be built as follows.

1. (Rectification) If $\Gamma_e^{C_e,W_e}(x)\downarrow= 0 \neq 1 = K(x)$, then we put $\gamma_e(x)$ into C_e to undefine $\Gamma_e^{C_e,W_e}(x)$.

2. (Extension) Let k be the least x such that $\Gamma_e^{C_e,W_e}(x)\uparrow$, then we define $\Gamma_e^{C_e,W_e}(k)\downarrow= K(k)$ with use $\gamma_e(k)$ a fresh number.

We will ensure that if $x < y$, then $\gamma_e(x) \le \gamma_e(y)$ and thus, we have: if $\Gamma_e^{C_e,W_e}(x)$ is undefined, then for all $y \ge x$, $\Gamma_e^{C_e,W_e}(y)$ will be undefined automatically.

β has two outcomes $\infty <_L f$, where ∞ denotes that β has infinitely many expansionary stages, and f denotes that β has only finitely many expansionary stages. Below outcome ∞, we will construct a c.e. set C_e and ensure that it is incomplete by constructing an auxiliary c.e. set E to satisfy the following subrequirements

$$\mathcal{S}_{e,i} : E \neq \Phi_i^{C_e}.$$

2.2.5 An $\mathcal{S}_{e,i}$ strategy

Let η be an $\mathcal{S}_{e,i}$-strategy below β's outcome ∞. A single η-strategy is a standard Friedberg-Muchnik strategy, which works as follows:

(1) Pick x as a fresh number.

(2) Wait for a stage s such that $\Phi_i^{C_e}(x)[s]\downarrow= 0$.

(3) Put x into E and preserve $C_e \upharpoonright \varphi_{i,s}(x)$.

But note that, after η performs the diagonalization, to rectify the definition of $\Gamma_e^{C_e,W_e}$, β needs to enumerate γ_e-uses into C_e, and this enumeration may injure the computation $\Phi_i^{C_e}(x)$. To ensure that a computation $\Phi_i^{C_e}(x)$ is clear of the γ_e-uses, we apply the gap-cogap argument to ensure that if η fails to protect a computation $\Phi_i^{C_e}(x)$, then the corresponding W_e is computable.

We first fix a number $k(\eta)$, as the *threshold* of η. In the construction, if $y < k(\eta)$ enters K, we put $\gamma_e(y)$ into C_e immediately to rectify $\Gamma_e^{C_e,W_e}(y)$, and we will *reset* η

by undefining the associated parameters, but with $k(\eta)$ unchanged. As $k(\eta)$ is kept unchanged, after a stage large enough , K has no change below $k(\eta)$, so η can be reset at most finitely often.

In the construction, when we see $\Phi_i^{C_e}(x)\downarrow= 0$ at an η-stage s, we do not perform the diagonalization immediately. Instead, we open a gap for A to change (and hence expecting a W_e-change, which can be small enough for us to lift the γ_e-uses) and create a link between β and η. At the next β-expansionary stage s', we check whether W_e changes below $\gamma_e(k(\eta))[s]$ between stage s and s'. If W_e changes, then we close the gap successfully, that is, we do the diagonalization by putting x into E, this W_e-change can lift γ_e-use and hence the computation $\Phi_i^{C_e}(x)$ is clear of the γ_e-uses, and can be preserved forever. Cancel the link. Otherwise, we close the gap unsuccessfully, and put $\gamma_e(k(\eta))[s]$ into C_e to lift $\gamma_e(k(\eta))$ to a big number. Cancel the link. We will define a partial computable function h_η such that if η opens infinitely many gaps, then h_η is defined as a total function, and computes W_e correctly. This shows that W_e is computable. We write this idea formally as follows:

Case a. (Close the gap successfully)

If $W_{e,s'}\restriction(\gamma_e(k(\eta))[s]+1)\neq W_{e,s}\restriction(\gamma_e(k(\eta))[s]+1)$, then we travel the link and put x into E to satisfy η. Cancel the link.

In this case, this W_e-change lifts $\gamma_e(y)$ use, for all $y\geq k(\eta)$, to big numbers, and hence the computation $\Phi_i^{C_e}(x)$ is clear of the γ_e-uses, and can be preserved forever.

Case b. (Close the gap unsuccessfully)

If $W_{e,s'}\restriction(\gamma_e(k(\eta))[s]+1)=W_{e,s}\restriction(\gamma_e(k(\eta))[s]+1)$, then we put $\gamma_e(k(\eta))[s]$ into C_e to lift $\gamma_e(k(\eta))$ to a big number. Cancel the link.

In this case, for all $x\leq\gamma_e(k(\eta))[s]$, if $h_\eta(x)\uparrow$, then we define $h_\eta(x)=W_{e,s}(x)$. After stage s', we prevent A from changing to preserve $W_{e,s}\restriction(\gamma_e(k(\eta))[s]+1)$ till the next β-expansionary stage at which η opens a gap again.

The η-strategy has three outcomes $g <_L w <_L d$, the outcome g denotes η opens gaps infinitely often, the outcome w denotes that η waits for $\Phi_i^{C_e}(x) \downarrow = 0$ forever, and the outcome d denotes that η successfully performs diagonalization. Note that if η has outcome g, then η shows that W_e is computable as $W_e = h_\eta$, β is satisfied at η, and there is no need to satisfy β's substrategies at the nodes extending $\eta^\frown g$. In this case, the use $\gamma_e(k(\eta))$ approaches to infinity, and η shows that W_e is computable.

We now consider the interaction between more $\mathcal{R}$-strategies. Suppose that we also have an $\mathcal{R}_{e'}$-strategy β' and its substrategy η' with $\beta^\frown\langle\infty\rangle \subseteq \beta'^\frown\langle\infty\rangle \subseteq \eta^\frown\langle g\rangle \subseteq \eta'$, where η is a substrategy of β. Then a link is created between β' and η' (η' opens a gap) only when η opens a gap (a link is created between β and η). These two links are crossing which needs to be avoided in a gap-cogap argument (we can see the necessity of this concern in the construction of a high$_2$ nonbounding degree in Downey, Lempp and Shore's paper [15]). With this in mind, on the construction tree, when η has outcome g, we will say that β' becomes inactive, or β' is injured by η, and we need to arrange a back-up strategy β' below this outcome.

We assume that readers have some basic ideas of the framework of $0'''$-priority argument. The construction tree will be labelled in a way where a single $\mathcal{R}$-requirement might be allocated along a single path several times, corresponding to the "injuries" mentioned above.

For a given $\mathcal{R}$-requirement, as only finitely many $\mathcal{R}$-requirements can have higher priority, and only substrategies of these $\mathcal{R}$-strategies with higher priority can injure a strategy of the given $\mathcal{R}$-requirement. By induction, we can see that on any path of the construction tree, for each $\mathcal{R}$-requirement, there is an $\mathcal{R}$-strategy on the path such that no $\mathcal{S}$-strategy of other $\mathcal{R}$-strategies can injure it.

2.2.6 Interaction between more strategies

We now consider the interaction between $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$. Suppose that a $\mathcal{Q}_e$-strategy ξ works between an $\mathcal{R}_{e'}$-strategy β and an $\mathcal{S}_{e',i}$-strategy η with $\beta^\frown\langle\infty\rangle \subseteq \xi \subset \xi^\frown\langle\infty\rangle \subseteq \eta \subset \eta^\frown\langle g\rangle \subset \alpha$, where α is a $\mathcal{P}$-strategy and $0 \leq e' \leq e$. Then we may have that η opens a gap and creates a link between β and η and α creates a link between α and

ξ at the same stage, s_0 say. That is, we have two crossed links (β, η) and (ξ, α) at stage s_0. At the next β-expansionary stage s_1, suppose η closes the gap unsuccessfully and cancels the link (β, η), so η will impose an A-restraint after stage s_1 till the next η-stage, s_2 say, at which η opens another gap. But, before stage s_2, i.e. during the cogap of η-strategy, we may travel the link (ξ, α) which was created at stage s_0 and may create another link (ξ', α) if there is a $\mathcal{Q}$-strategy ξ' with $\xi'^\frown\langle\infty\rangle \subset \xi$. In this case, α will perform diagonalization before stage s_2. However we cannot put a small number ($\leq s_1$) into A before stage s_2. To avoid this problem, we use a backup strategy to deal with this. That is, we will put a backup strategy $\hat{\xi}$ below $\eta^\frown\langle g\rangle$ to satisfy the $\mathcal{Q}_e$-requirement. In this case, ξ has no substrategy working below $\eta^\frown\langle g\rangle$. This will avoid the crossing of links as above. That is, we will have two links (β, η) and $(\hat{\xi}, \alpha)$ at the same stage such that $\beta^\frown\langle\infty\rangle \subseteq \eta \subset \eta^\frown\langle g\rangle \subseteq \hat{\xi} \subset \hat{\xi}^\frown\langle\infty\rangle \subset \alpha$.

We may have two nested links (β, η) and (ξ'', α) at the same stage for some $\mathcal{Q}$-strategy ξ'' with higher priority such that $\xi''^\frown\langle\infty\rangle \subseteq \beta \subset \beta^\frown\langle\infty\rangle \subseteq \eta \subset \eta^\frown\langle g\rangle \subset \alpha$. Since η-gaps are never closed until the outer link (ξ'', α) is travelled, α will perform the diagonalization when an η-gap is open.

In the construction, we put a backup $\mathcal{Q}_e$-strategy below the $\mathcal{S}_{e',i}$-strategy η with g outcome, if $e' \leq e$. Therefore, for a fixed e, there are at most finitely many backup $\mathcal{Q}_e$-strategies on any path of the priority tree, and the longest node assigned a $\mathcal{Q}_e$-requirement is responsible for satisfying the requirement.

2.3 Construction

Before we give the full construction, we first define the priority tree T effectively.

Definition 1 (1) Define *the priority ranking of the requirements* as follows:

$$\mathcal{P}_0 < \mathcal{R}_0 < \mathcal{S}_{0,0} < \mathcal{Q}_0 < \mathcal{T}_{0,0} < \mathcal{P}_1 < \mathcal{R}_1 < \mathcal{S}_{0,1} < \mathcal{S}_{1,0} < \mathcal{S}_{1,1} < \mathcal{Q}_1 < \mathcal{T}_{0,1} < \mathcal{T}_{1,0} < \mathcal{T}_{1,1} < \cdots < \mathcal{P}_n < \mathcal{R}_n < \mathcal{S}_{0,n} < \cdots < \mathcal{S}_{n,n} < \mathcal{Q}_n < \mathcal{T}_{0,n} < \cdots < \mathcal{T}_{n,n} < \mathcal{P}_{n+1} < \cdots,$$

where $\mathcal{X} < \mathcal{Y}$ means that $\mathcal{X}$ has higher priority than $\mathcal{Y}$.

(2) A $\mathcal{P}$-strategy has two possible outcomes $d <_L w$.

An $\mathcal{R}$-, $\mathcal{Q}$-, or $\mathcal{T}$-strategy has two possible outcomes $\infty <_L f$.

An $\mathcal{S}$-strategy has three possible outcomes $g <_L w <_L d$.

Definition 2 Given $\tau \in T$.

(1) A requirement $\mathcal{R}_e$ is *satisfied at* τ if there is an $\mathcal{R}_e$-strategy β with $\beta^\frown\langle f\rangle \subset \tau$, or there is an $\mathcal{R}_e$-strategy β and an $\mathcal{S}_{e,i}$-strategy η for some i with the following properties:

- $\beta^\frown\langle\infty\rangle \subseteq \eta \subset \eta^\frown\langle g\rangle \subset \tau$.
- there is no $\mathcal{S}_{e',i'}$-strategy η' such that $\beta^\frown\langle\infty\rangle \subseteq \eta' \subset \eta'^\frown\langle g\rangle \subset \eta$ for any $e' < e$ and any i'.

In the latter case, β has a Σ_3-outcome g at η, and under this outcome, all the strategies between β and η are said to be injured at η. When a strategy is injured, then all its substrategies are injured.

(2) A requirement $\mathcal{R}_e$ is *active at* τ *via* β if $\mathcal{R}_e$ is not satisfied at τ and there is an $\mathcal{R}_e$-strategy β such that

- $\beta^\frown\langle\infty\rangle \subset \tau$,
- there is no $\mathcal{S}_{e',i'}$-strategy η' such that $\beta^\frown\langle\infty\rangle \subseteq \eta' \subset \eta'^\frown\langle g\rangle \subset \tau$ for any $e' < e$ and any i'.

(3) A requirement $\mathcal{S}_{e,i}$ is *satisfied at* τ if either $\mathcal{R}_e$ is satisfied at τ, or $\mathcal{R}_e$ is active at τ via β and there is an $\mathcal{S}_{e,i}$-strategy η with $\beta^\frown\langle\infty\rangle \subset \eta \subset \tau$.

(4) A requirement $\mathcal{Q}_e$ is *satisfied at* τ if there is a $\mathcal{Q}_e$-strategy ξ with $\xi^\frown\langle f\rangle \subset \tau$, and ξ is not injured at τ.

(5) A requirement $\mathcal{Q}_e$ is *active at* τ *via* ξ if there is a $\mathcal{Q}_e$-strategy ξ such that $\xi^\frown\langle\infty\rangle \subset \tau$, and ξ is not injured at τ.

(6) A requirement $\mathcal{T}_{e,i}$ is *satisfied at* τ if either $\mathcal{Q}_e$ is satisfied at τ, or $\mathcal{Q}_e$ is active at τ via ξ and there is a $\mathcal{T}_{e,i}$-strategy ζ with $\xi^\frown\langle\infty\rangle \subset \zeta \subset \tau$.

(7) A requirement $\mathcal{P}_e$ is *satisfied at* τ if there is a $\mathcal{P}_e$-strategy α with $\alpha \subset \tau$, and α is not injured up to τ.

Now we define the priority tree T as follows.

Definition 3 (1) Define the root node λ as a $\mathcal{P}_0$-strategy.

(2) The immediate successors of a node are the possible outcomes of the corresponding strategy.

(3) For $\tau \in T$, τ works for the highest priority requirement which has neither been satisfied, nor been active at τ.

(4) Continuing the inductive steps above, the priority tree T is built.

Definition 4 Given an $\mathcal{S}_{e,i}$-strategy η, we define the *mother node* of η as the longest $\mathcal{R}_e$-strategy β such that $\beta^\frown\langle\infty\rangle \subset \eta$, we use $\mathrm{top}(\eta)$ to denote the mother node of η.

Similarly, for a given $\mathcal{T}_{e,i}$-strategy ζ, we define the *mother node* $\mathrm{top}(\zeta)$ of ζ as the longest $\mathcal{Q}_e$-strategy ξ such that $\xi^\frown\langle\infty\rangle \subset \zeta$.

In the construction, a $\mathcal{P}$-strategy α has several parameters: one is $x(\alpha)$, a candidate for the diagonalization, and the others are numbers $a_{\alpha,\xi}$, which are associated to those $\mathcal{Q}$-strategies ξ with higher priority which are active at α.

An $\mathcal{S}_{e,i}$-strategy η has two parameters: one is the threshold $k(\eta)$, and the other one is $x(\eta)$, a candidate for the diagonalization.

For a $\mathcal{T}_{e,i}$-strategy ζ, except for the parameter p_ζ, it has another parameter u_ζ with $u_\zeta < p_\zeta$. The parameter u_ζ is designed to ensure that ζ's work in defining $\Delta_e^{V_e}$ can be undone whenever ζ is initialized. When ζ is initialized, u_ζ is enumerated into P automatically and if no number is enumerated into A (which is guaranteed in the construction), then this will force a V_e-change to undefine all $\Delta_e^{V_e}(i,x) = 0$ defined by ζ. Note that p_ζ can be updated many times, but u_ζ will be kept the same unless ζ is initialized.

In the construction, when a strategy τ is initialized, then all the strategies with lower priority will be also initialized automatically, and all the parameters of τ will be cancelled. For an $\mathcal{S}$-strategy η, let $\beta = \text{top}(\eta)$, and if there is a $k < k(\eta)$ such that $\gamma_\beta(k)$ is enumerated into C_β, then η is reset automatically. When an $\mathcal{S}$-strategy η is reset, then all the strategies with lower priority will be also initialized automatically.

The full construction is as follows.

Stage 0: Initialize all nodes on T, and let $A_0 = P_0 = \emptyset$.

Stage $s > 0$: This stage has two phases.

Phase I. (finding σ_s)

Substage 0: Let $\sigma_s(0)$ be the root node.

Substage t**:** Given $\tau = \sigma_s \upharpoonright t$.

If $t = s$ then define $\sigma_s = \tau$ and initialize all the nodes with lower priority than σ_s. Go to Phase II.

If $t < s$, then take action for τ and define $\sigma_s(t)$ as follows:

Case 1 $\tau = \alpha$ is a $\mathcal{P}_e$-strategy. There are four subcases.

(α**1**) If $x(\alpha)\uparrow$, then define $x(\alpha)$ to be a fresh number and choose fresh numbers for parameters $a_{\alpha,\xi}$ with $x(\alpha) < a_{\alpha,\xi}$, for all $\mathcal{Q}$-strategies ξ with higher priority which are active at α, and $a_{\alpha,\xi_1} < a_{\alpha,\xi_2}$ if ξ_1 has higher priority than ξ_2. Request that a later stage s' is a ξ-expansionary stage if s' is ξ-expansionary in the standard sense, and also $l(\xi, s')$ is greater than $a_{\alpha,\xi}$. Let $\sigma_s = \alpha^\frown\langle w\rangle$ and initialize all the nodes with priority lower than σ_s. Go to Phase II.

(α**2**) If $x(\alpha)\downarrow$, and $\varphi_e(x(\alpha))[s]\downarrow= 0$, then among those $\mathcal{Q}$-strategies active at α if any, choose ξ with the lowest priority, create a link between α and ξ, and put $a_{\alpha,\xi}$ into P. Let $\sigma_s = \alpha^\frown\langle w\rangle$ and initialize all the nodes with priority lower than σ_s. Go to Phase II.

If there is no such $\mathcal{Q}$-strategies active at α, then put $x(\alpha)$ into A. Let $\sigma_s = \alpha^\frown\langle d\rangle$ and initialize all the nodes with priority lower than σ_s. Go to Phase II. We say that α receives attention at stage s.

(α3) If α is satisfied (i.e. α has already received attention), then let $\sigma_s(t) = \alpha^\frown\langle d\rangle$, and go to the next substage.

(α4) Otherwise, let $\sigma_s(t) = \alpha^\frown\langle w\rangle$, and go to the next substage.

Case 2 $\tau = \xi$ is a $\mathcal{Q}_e$ strategy. There are three subcases.

(ξ1) If s is not a ξ-expansionary stage (it may happen that s is ξ-expansionary in the standard sense but $l(\xi, s)$ is still less than a number requested by a $\mathcal{P}$-strategy or $\mathcal{T}_{e,i}$-strategy below $\xi^\frown\langle\infty\rangle$, in this case, we still treat this stage not ξ-expansionary stage), then let $\sigma_s(t) = \xi^\frown\langle f\rangle$, and go to the next substage.

(ξ2) If s is a ξ-expansionary stage, and no link between ξ and a $\mathcal{P}$-strategy α below $\xi^\frown\langle\infty\rangle$ exists, then let $\sigma_s(t) = \xi^\frown\langle\infty\rangle$, and go to the next substage.

(ξ3) If s is a ξ-expansionary stage, and a link between ξ and a $\mathcal{P}$-strategy α below $\xi^\frown\langle\infty\rangle$ exists, then cancel this link, and check whether there is an active $\mathcal{Q}$-strategy ξ' with $\xi'^\frown\langle\infty\rangle \subset \xi$.

If there is such a ξ', then choose ξ' with the lowest priority, and create a link between α and ξ', put $a_{\alpha,\xi'}$ into P. Let $\sigma_s = \alpha^\frown\langle w\rangle$ and initialize all the nodes with priority lower than σ_s. Go to Phase II.

If there is no such a ξ', then put $x(\alpha)$ into A. Let $\sigma_s = \alpha^\frown\langle d\rangle$ and initialize all the nodes with priority lower than σ_s. Go to Phase II. We say that α receives attention at stage s.

Case 3 $\tau = \zeta$ is a $\mathcal{T}_{e,i}$ strategy.

If u_ζ and p_ζ are not defined, then define them as two fresh numbers with $u_\zeta < p_\zeta$, request that a later stage s' is ξ-expansionary stage, where $\xi = \mathrm{top}(\zeta)$, then $l(\xi, s')$ must be greater than u_ζ and p_ζ. Let $\sigma_s = \zeta$ and initialize all the nodes with priority lower than σ_s. Go to Phase II.

If u_ζ and p_ζ are defined, then check whether s is a ζ-expansionary stage.

(ζ1) If s is a ζ-expansionary stage, then let $\sigma_s(t) = \zeta^\frown\langle\infty\rangle$, and put p_ζ into P, go to the next substage.

(ζ2) If s is not a ζ-expansionary stage, then let $\sigma_s(t) = \zeta^\frown\langle f\rangle$, and go to the next substage.

Case 4 $\tau = \beta$ is an $\mathcal{R}_e$ strategy.

(β1) If s is not a β-expansionary stage, then let $\sigma_s(t) = \beta^\frown\langle f\rangle$, and go to the next substage.

(β2) If s is a β-expansionary stage, then

(2.1) If there is a link between β and some η which was created and has not been cancelled (such an $\mathcal{S}_{e,i}$ strategy η is unique), then let $v < s$ be the stage at which this link was created.

(2.1.1) (Close the gap successfully)
If $W_{e,s} \upharpoonright (\gamma_\beta(k(\eta))[v]+1) \neq W_{e,v} \upharpoonright (\gamma_\beta(k(\eta))[v]+1)$, then put $x(\eta)$ into E_e, for each $y \geq k(\eta)$, set $\Gamma_\beta^{C_e,W_e}(y)[s]$ to be undefined if it is defined. Let $\sigma_s = \eta^\frown\langle d\rangle$ and initialize all the nodes with priority lower than σ_s. Cancel this link, we say that η is satisfied at stage s. Go to Phase II.

(2.1.2) (Close the gap unsuccessfully)
If $W_{e,s} \upharpoonright (\gamma_\beta(k(\eta))[v]+1) = W_{e,v} \upharpoonright (\gamma_\beta(k(\eta))[v]+1)$, then put $\gamma_\beta(k(\eta))[v]$ into C_e. For all $x \leq \gamma_\beta(k(\eta))[v]$, if $h_\eta(x)\uparrow$, then define $h_\eta(x) = W_{e,s}(x)$. Cancel this link. Let $\sigma_s = \eta^\frown\langle g\rangle$ and initialize all the nodes $>_L \eta^\frown\langle g\rangle$. Go to Phase II.

(2.2) If (2.1) fails, then there are two subcases.

• If there is some x with $\Gamma_\beta^{C_e,W_e}(x)[s]\downarrow = 0 \neq 1 = K(x)[s]$, let k be the least such one, then put $\gamma_\beta(k)[s]$ into C_e. For any substrategy ($\mathcal{S}_{e,i}$-strategy) η of β, if $k(\eta)$ is defined and $k < k(\eta)$, then we reset η. Let $\sigma_s = \beta^\frown\langle\infty\rangle$, and initialize all strategies $>_L \beta^\frown\langle\infty\rangle$. Go to Phase II.

• Otherwise, find the least x such that $\Gamma_\beta^{C_e,W_e}(x)[s]\uparrow$, define $\Gamma_\beta^{C_e,W_e}(x)[s] = K(x)[s]$ with a fresh use $\gamma_\beta(x)[s]$.

Let $\sigma_s(t) = \beta^\frown\langle\infty\rangle$, and go to the next substage.

Case 5 $\tau = \eta$ is an $\mathcal{S}_{e,i}$ strategy. Let β= top(η).

(η1) If $k(\eta)\uparrow$, then define $k(\eta)$ to be a fresh number. Let $\sigma_s = \eta$ and initialize all the nodes with priority lower than σ_s. Go to Phase II.

(η2) If $k(\eta)\downarrow$, but $x(\eta)\uparrow$, then choose a fresh number as $x(\eta)$. Let $\sigma_s = \eta^\frown\langle w\rangle$ and initialize all the nodes with priority lower than σ_s. Go to Phase II.

(η3) If $k(\eta)\downarrow$, $x(\eta)\downarrow$, and $\Phi_i^{C_e}(x(\eta))[s]$ does not converge to 0, then let $\sigma_s(t) = \eta^\frown\langle w\rangle$, and go to the next substage.

(η4) If $k(\eta)\downarrow$, $x(\eta)\downarrow$, and $\Phi_i^{C_e}(x(\eta))[s]\downarrow= 0$ and $x(\eta)\notin E_e[s]$ and no link between β and η exists, then, if $\Gamma_\beta^{C_e,W_e}(k(\eta))[s]\downarrow$ and $l(\beta, s) > \gamma_\beta(k(\eta))[s]$, we create a link between β and η, define $\sigma_s(t) = \eta^\frown\langle g\rangle$ and go to the next substage; otherwise, let $\sigma_s(t) = \eta^\frown\langle w\rangle$, and go to the next substage.

(η5) If $k(\eta)\downarrow$, $x(\eta)\downarrow$, and $\Phi_i^{C_e}(x(\eta))[s]\downarrow= 0$ and $x(\eta)\in E_e[s]$, then let $\sigma_s(t) = \eta^\frown\langle d\rangle$, and go to the next substage.

Phase II. Having σ_s, for $\tau\subset\sigma_s$, do as follows, and then go to the next stage. Recall that for the $\mathcal{T}$ strategies, ζ say, being initialized at this stage, u_ζ is put into P automatically.

If $\tau = \zeta$ is a $\mathcal{T}_{e,i}$ strategy, and p_ζ is enumerated into P during Phase I (i.e. s is ζ-expansionary), then assign a fresh number to p_ζ.

(1) If s is a ζ-expansionary stage, then extend the definition of $\Delta_e^{V_e}$ to all arguments (i, x) with $x < l(\zeta, s)$ and $\Delta_e^{V_e}(i, x)$ is not defined yet, defining $\Delta_e^{V_e}(i, x) = 1$ with use -1.

(2) If s is not a ζ-expansionary stage, then extend the definition of $\Delta_e^{V_e}$ to all arguments (i, x) with $x < s$ such that if $\Delta_e^{V_e}(i, x)$ is not defined yet. See whether $\Delta_e^{V_e}(i, x)$ has been defined so far, after the current p_ζ is defined.

If no, then define $\Delta_e^{V_e}(i, x) = 0$ with use $\delta_e(i, x) = s$.

If yes, then check whether the computation $\Phi_e^{A,V_e}(p_\zeta)$ has changed from the stage when $\Delta_e^{V_e}(i, x)$ was defined last time. If the computation keeps the same, then define $\Delta_e^{V_e}(i,x) = 0$ with use the same as before. Otherwise, define $\Delta_e^{V_e}(i,x) = 0$ with use $\delta_e(i,x) = s$.

This completes the construction.

2.4 Verification

Define the true path of the construction to be $\sigma = \liminf_s \sigma_s$, i.e. the leftmost path of the construction. The following lemma implies that the true path σ is infinite.

Lemma 4. *Let τ be any node on the true path σ. Then*

(1) *τ can be initialized or reset at most finitely often.*

(2) *τ has an outcome $\mathcal{O}$ such that $\tau^\frown\mathcal{O}$ is on σ.*

(3) *τ can initialize the node $\tau^\frown\mathcal{O}$ at most finitely often.*

Proof. We prove the lemma by induction on the length of τ.

When $\tau = \lambda$, the root node of the priority tree T, i.e. $\mathcal{P}_0$-strategy. Note that λ can never be initialized or reset, so (1) is clearly true. By our construction, after λ defines the witness $x(\lambda)$, $x(\lambda)$ will never be cancelled. If there is a stage s such that $\varphi_0(x(\lambda))[s]\downarrow= 0$, then we will put $x(\lambda)$ into A and hence we have $A(x(\lambda)) = 1 \neq 0 = \varphi_0(x(\lambda))$, so $\mathcal{P}_0$ is satisfied at any stage after s and $\lambda^\frown\langle d\rangle \subset \sigma$, thus (2) is true for λ. After stage s, λ will not initialize other strategies, and so (3) is also true for λ. If there is no such a stage s with $\varphi_0(x(\lambda))[s]\downarrow= 0$, then we will have $A(x(\lambda)) = 0 \neq \varphi_0(x(\lambda))$ (i.e. $\mathcal{P}_0$ is satisfied) and $\lambda^\frown\langle w\rangle \subset \sigma$, and λ will not initialize other strategies. Thus, (2), (3) are true for λ obviously.

Now suppose the lemma is true for all $\tau' \subset \tau$, we now show that the lemma is also true for τ. Let τ^- be the immediate predecessor of τ. By the induction hypothesis, τ^- can be initialized or reset at most finitely often and τ^- has a true outcome $\mathcal{O}$ on σ, so we can fix a stage s_0 after which τ^- can not be initialized or reset and the nodes on the left of $\tau = \tau^{-\frown}\mathcal{O}$ can never be visited. Also by the induction hypothesis again, τ^- can initialize the node $\tau = \tau^{-\frown}\mathcal{O}$ at most finitely often. Thus, we can fix some (least) stage $s_1 \geq s_0$ after which τ can never be initialized by higher priority strategies. If τ is an $\mathcal{S}_{e,i}$-strategy, then after stage s_1, once the threshold $k(\tau)$ is defined, it can never be cancelled. Thus, τ can be reset at most $k(\tau)$ times more, and so (1) is true for τ.

Now we show that (2) and (3) are true for τ.

$\tau = \alpha$ is a $\mathcal{P}_e$-strategy. Apply the same argument as for $\mathcal{P}_0$-strategy λ to α. Note that if there are $\mathcal{Q}_e$-strategies ξ active at α, then once α sees $\varphi_e(x(\alpha)) \downarrow= 0$, it does not do the diagonalization immediately. It first enumerates numbers $a_{\alpha,\xi}$ into P one by one for all $\mathcal{Q}_e$-strategies ξ active at α (there are finitely many such ξ), as these $\mathcal{Q}_e$-strategies ξ have infinitary outcome, all the links being created will eventually be cancelled, and $x(\alpha)$ is put into A, making $A(x(\alpha)) = 1 \neq 0 = \varphi_e(x(\alpha))$. $\mathcal{P}_e$ is satisfied via this $x(\alpha)$, and hence α will take no further actions. So (2) and (3) are true for τ obviously.

$\tau = \beta$ is an $\mathcal{R}_e$-strategy. By our construction, if there are infinitely many β-expansionary stages then we have that $\beta^\frown\langle\infty\rangle \subset \sigma$, and otherwise $\beta^\frown\langle f\rangle \subset \sigma$. Note that β never initializes the strategies with lower priority. So (2), (3) are true for τ.

$\tau = \eta$ is an $\mathcal{S}_{e,i}$-strategy. Let $s_2 > s_1$ be the least stage after which η can not be reset. Then $x(\eta)$ will be defined at a stage $s_3 > s_2$, and it will never be cancelled once defined. If there is an η-stage $s_4 > s_3$ such that $\Phi_i^{C_e}(x(\eta))[s_4] \downarrow= 0$, then we will create a link between η and its mother node, β say, at this stage. At the next β-expansionary stage $s_5 > s_4$, if $W_{e,s_5} \upharpoonright (\gamma_e(k(\eta))[s_4]+1) \neq W_{e,s_4} \upharpoonright (\gamma_e(k(\eta))[s_4]+1)$, then we put $x(\eta)$ into E_e, and so we have $E_e(x(\eta)) = 1 \neq 0 = \Phi_i^{C_e}(x(\eta)) \downarrow= 0$ since the computation $\Phi_i^{C_e}(x(\eta))[s_4] \downarrow= 0$ is clear of the γ_e-uses (by the choice of s_2 and the above W_e-change will lift the $\gamma_e(y)$-use for all $y \geq k(\eta)$, so there is no γ_e-use less than $\varphi_i(x(\eta))[s_4]$ will be enumerated into C_e after stage s_5). In this case, $\eta^\frown\langle d\rangle$ will be on the true path, and η will take no further actions after stage s_5, hence (2), (3) are true for η. So we assume that $x(\eta)$ is not enumerated into E_e in the construction. Without loss of generality, we assume that there are infinitely many η-stages at which $\Phi_i^{C_e}(x(\eta)) \downarrow= 0$ (since otherwise, we have that $\eta^\frown\langle w\rangle$ is on the true path, and (2),

(3) are true for η obviously). In this case, by our construction, η will open infinitely many gaps and we do not get W_e-changes during gaps. So we have $\eta^\frown\langle g\rangle$ is on the true path, and note that η will never initialize the strategies below $\eta^\frown\langle g\rangle$. Thus, (2), (3) are true for η.

$\tau = \xi$ is a $\mathcal{Q}_e$-strategy. We have $\xi^\frown\langle\infty\rangle \subset \sigma$ if there are infinitely many ξ-expansionary stages, and $\xi^\frown\langle f\rangle \subset \sigma$ otherwise. Note that, a strategy below $\xi^\frown\langle\infty\rangle$ can request that a stage s is ξ-expansionary only when s is ξ-expansionary in the standard sense and also $l(\xi, s)$ is greater than the associated parameters, this kind of requests do not affect the outcome of ξ on the true path. So (2) is true for τ. Furthermore, ξ never initializes the strategies with lower priority in the construction. (3) is also true for τ.

$\tau = \zeta$ is a $\mathcal{T}_{e,i}$-strategy. We have that $\zeta^\frown\langle\infty\rangle \subset \sigma$ if there are infinitely many ζ-expansionary stages; and $\zeta^\frown\langle f\rangle \subset \sigma$ otherwise. Note that ζ never initializes the strategies with lower priority in the construction. So (2), (3) are true for τ.

This completes the proof of Lemma 4.

$\square$

Lemma 5. *For any $e \in \omega$, let α be the longest $\mathcal{P}_e$-strategy on the true path σ. Then $\mathcal{P}_e$ is satisfied via α.*

Proof. Follows the proof of Lemma 4 immediately. $\square$

Lemma 6. *For any $e \in \omega$, let β be the longest $\mathcal{R}_e$-strategy on the true path σ. Then β, together with its substrategies, satisfies the $\mathcal{R}_e$ requirement.*

Proof. Fix e. Let β be the longest $\mathcal{R}_e$-strategy on the true path σ. By Lemma 4, β can be initialized at most finitely often. Let s_0 be the least stage after which β can never be initialized. We assume that $W_e = \Phi_e^A$. Then $\beta^\frown\langle\infty\rangle \subset \sigma$ and below outcome ∞, we need to construct an incomplete c.e. set C_e and a p.c. functional Γ_e at β-expansionary stages such that either $K = \Gamma_e^{C_e, W_e}$ or W_e is computable.

By our construction, β can either have only finitely many substrategies or have infinitely many substrategies on the true path σ.

Case 1 β has only finitely many substrategies on the true path σ.

In this case, there must be some (the last one) substrategy η which works on $\mathcal{S}_{e,i}$ with $\beta^\frown\langle\infty\rangle \subset \eta^\frown\langle g\rangle \subset \sigma$ for some i. That is, η opens infinitely many gaps, all these gaps are closed unsuccessfully. $\gamma_e(k(\eta))$ is put into C_e infinitely often and hence $\lim_s \gamma_e(k(\eta))[s] = \infty$. Thus, h_η is defined to be a total computable function over the course of the construction. We will show that $h_\eta = W_e$ and hence W_e is computable.

Now let $s_1 > s_0$ be the least stage after which η can never be initialized or reset. Given an $x \in \omega$, if $h_\eta(x)$ is defined for the first time at a stage $v_1 > s_1$, then we will prove that for all stages $s \geq v_1$, $h_\eta(x) \downarrow= W_e(x)$ holds at stage s by induction. Suppose that $t_1 < t_2 < \cdots$ are the stages $> s_1$ at which a link (β, η) is created, and that $v_1 < v_2 < \cdots$ are the stages at which the link (β, η) created at stages $t_1 < t_2 < \cdots$ are cancelled, respectively. By the choice of v_1, we know that $W_{e,v_1}(x) = W_{e,t_1}(x)$ as otherwise, the gap is closed successfully. After stage v_1, we will impose A-restraint to preserve W_e till $t_2 > v_1$ at which a new link (β, η) is created. That is, $h_\eta(x) \downarrow= W_e(x)$ holds at the beginning of stage t_2. Suppose by induction that $h_\eta(x) \downarrow= W_e(x)$ holds at the beginning of stage t_n for some $n \geq 2$. Then a gap is open at stage t_n and this gap is closed unsuccessfully at stage v_n (this link is cancelled), and hence $h_\eta(x) = W_{e,v_n}(x)$. Again, at stage v_n, all strategies on the right of $\eta^\frown\langle g\rangle$ are initialized and hence an A-restraint is imposed, and $W_e(x)$ is preserved till t_{n+1}, at which a new gap is open. That is, $h_\eta(x) \downarrow= W_e(x)$ holds at the beginning of stage t_{n+1}. This induction argument implies that, for each n, $h_\eta(x) \downarrow= W_{e,t_n}(x)$ holds. Therefore, $h_\eta = W_e$ and W_e is computable.

Case 2 β has infinitely many substrategies on the true path σ.

In this case, there are infinitely many substrategies η with $\beta^\frown\langle\infty\rangle \subseteq \eta \subset \sigma$ with top(η)=β, and each such η will not have outcome g on the true path, and so $\gamma(k(\eta))$ is enumerated into C_e only finitely often as η opens and closes gaps only finitely often. By the definition of thresholds, for each k, there is at most one substrategy η with top(η)=β which chooses k to be the threshold $k(\eta)$ in the construction, in which case, if η is on the left of the true path, then η puts

$\gamma(k)$ into C_e only finitely many times; if η is on the right of the true path, then k will be cancelled eventually and so η puts $\gamma(k)$ into C_e only finitely many times. Note that, for those k which are not chosen to be the threshold, β puts $\gamma(k)$ into C_e only when k enters K. Thus, for each k, $\Gamma_e^{C_e,W_e}(k)\downarrow$ eventually, i.e. $\Gamma_e^{C_e,W_e}$ is total. Moreover, by the properties of γ-use in the $\mathcal{R}_e$-strategy, whenever $\Gamma_e^{C_e,W_e}(x)\downarrow= 0 \neq 1 = K(x)$, we will put $\gamma(x)$ into C_e to rectify the definition of $\Gamma_e^{C_e,W_e}$. Thus, $\Gamma_e^{C_e,W_e}$ is defined correctly and hence $K = \Gamma_e^{C_e,W_e}$.

Now, we show that all the subrequirements $\mathcal{S}_{e,i}$ are satisfied. For each i, let η be an $\mathcal{S}_{e,i}$-strategy with $\beta^\frown\langle\infty\rangle \subseteq \eta \subset \sigma$. We will show that $\mathcal{S}_{e,i}$ is satisfied by η. Let $s_1 > s_0$ be the least stage after which η can never be initialized or reset. Then $x(\eta)$ will be defined at a stage $s_2 > s_1$, and this $x(\eta)$ can never be cancelled. Assume that there are infinitely many η-stages s at which $\Phi_i^{C_e}(x(\eta))\downarrow= 0$ (otherwise, we have $\eta^\frown\langle w\rangle \subset \sigma$ and $\mathcal{S}_{e,i}$ is satisfied since $E_e(x(\eta)) = 0 \neq \Phi_i^{C_e}(x(\eta))$).

Because η cannot open infinitely many gaps in Case 2, eventually, η will close the last gap successfully, which implies that $E_e(x(\eta)) = 1 \neq 0 = \Phi_i^{C_e}(x(\eta))$. Here, the computation $\Phi_i^{C_e}(x(\eta))$ is preserved, as W_e has changes below $\gamma_e(k(\eta))$, which undefined $\Gamma_e^{C_e,W_e}(y)$ for each $y \geq k(\eta)$. The new value of $\gamma_e(y)$ will be defined as big numbers, in particular, $\gamma_e(k(\eta))$ will be bigger than $\varphi_i(x(\eta))$. The enumeration of these numbers into C_e will not change the computation $\Phi_i^{C_e}(x(\eta))$. Thus, $\mathcal{S}_{e,i}$ is satisfied by η.

$\square$

Lemma 7. *For any $e \in \omega$, let ξ be the longest $\mathcal{Q}_e$-strategy on the true path σ. Then ξ, together with its substrategies, satisfies the $\mathcal{Q}_e$ requirement.*

Proof. Fix e. Let ξ be the longest $\mathcal{Q}_e$-strategy on the true path σ. We assume that $P = \Phi_e^{A,V_e}$. Then there are infinitely many ξ-expansionary stages, so $\xi^\frown\langle\infty\rangle \subset \sigma$. Below outcome ∞, ξ has infinitely many substrategies $\mathcal{T}_{e,i}$, we need to show that all its substrategies $\mathcal{T}_{e,i}$ on the true path work together to define a p.c. functional $\Delta_e^{V_e}$ such that $Tot(i) = \lim_x \Delta_e^{V_e}(i,x)$, to show that V_e has high degree.

Let ζ be a $\mathcal{T}_{e,i}$-strategy with $\xi^\frown\langle\infty\rangle \subseteq \zeta \subset \sigma$. We will show that ζ defines $\Delta_e^{V_e}(i,x)$ for almost all x to ensure that $Tot(i) = \lim_x \Delta_e^{V_e}(i,x)$. Note that when a $\mathcal{T}_{e,i}$-strategy ζ' is initialized, then the associated initialization parameter $u_{\zeta'}$ will be enumerated into P, if no small number is enumerated into A, then this will force a V_e-change at the next ξ-expansionary stage, and this V_e-change can undefine all $\Delta_e^{V_e}(i,x)$ defined by ζ'. This means that ζ, the $\mathcal{T}_{e,i}$-strategy on the true path σ, will define $\Delta_e^{V_e}(i,x)$ for almost all x. So we need to show that there is no small number being enumerated into A between the stage when $u_{\zeta'}$ is enumerated into P and the next ξ-expansionary stage. Without loss of generality, we assume that there is a small number being enumerated into A at the next ξ-expansionary stage. In this case, when ζ' is initialized, a link between ξ and some $\mathcal{P}$-strategy α is created, $a_{\alpha,\xi}$ is enumerated into P, and no number is enumerated into A at this stage. Note that the next ξ-expansionary stage is required that its length of agreement is greater than $u_{\zeta'}$, and before the next ξ-expansionary stage appears, every ξ-stage is not expansionary and hence no small number, in particular, no number less than $\varphi_e(u_{\zeta'})$, is enumerated into A. This means that when the next ξ-expansionary stage appears, V_e must change below $\varphi_e(u_{\zeta'})$. This V_e-change can undefine all $\Delta_e^{V_e}(i,x)$ defined by ζ'.

We have shown that ζ defines $\Delta_e^{V_e}(i,x)$ for almost all x in the above paragraph, so we can assume that all $\Delta_e^{V_e}(i,x)$ defined by those $\mathcal{T}_{e,i}$-strategies with lower priority are undefined automatically whenever ζ is visited. Since ζ is on the true path, by Lemma 4, let s_0 be the least stage after which ζ can never be initialized. So u_ζ and p_ζ will be defined at a stage $s_1 \geq s_0$ and u_ζ will be defined to be less than p_ζ. Note that u_ζ will be kept the same after stage s_1, but p_ζ can be updated many times after s_1. We require that if a stage $s_2 > s_1$ is a ξ-expansionary stage, then its length of agreement must be greater than u_ζ and p_ζ. Thus, from now on, in the construction, at a further ζ-stage (clearly, it is also a ξ-expansionary stage), if it is a ζ-expansionary stage, then we will put p_ζ into P to undefine all $\Delta_e^{V_e}(i,x)$ defined under the f outcome of ζ between the last ζ-expansionary stage and the current stage, so ζ can extend the definition of $\Delta_e^{V_e}(i,x)$ correctly at a later ζ-stage. We will update the value of p_ζ after p_ζ is enumerated into P.

Thus, if $\zeta^\frown\langle\infty\rangle \subset \sigma$, then the value of p_ζ will be updated infinitely many times and all $\Delta_e^{V_e}(i,x)$ defined after stage s_1 have value 1, and so $\lim_x \Delta_e^{V_e}(i,x) = 1$.

If $\zeta^\frown\langle f\rangle \subset \sigma$, then after some large enough stage $s_3 > s_1$, there is no more ζ-expansionary stage, and hence from then on p_ζ will remain the same, ζ will define $\Delta_e^{V_e}(i,x) = 0$ with use $\delta_\zeta(i,x) > \varphi_e(p_\zeta)$, and such a δ-use can be lifted only when V_e changes below $\varphi_e(p_\zeta)$. Thus, if $\Phi_e^{A,V_e}(p_\zeta)\downarrow$ eventually, then the V_e-part of the computation $\Phi_e^{A,V_e}(p_\zeta)$ will be fixed, and so the use $\delta_\zeta(i,x)$ can be lifted at most finitely often and $\Delta_e^{V_e}(i,x)$ will be defined eventually. In this case, we have that $\lim_x \Delta_e^{V_e}(i,x) = 0$. Note that, by our assumption of $P = \Phi_e^{A,V_e}$, $\Phi_e^{A,V_e}(p_\zeta)\downarrow$ eventually. Therefore, $Tot(i) = \lim_x \Delta_e^{V_e}(i,x)$, and $\mathcal{T}_{e,i}$ is satisfied by ζ. Moreover, $\mathcal{Q}_e$ is satisfied by ξ since all its substrategies $\mathcal{T}_{e,i}$ are satisfied, and all these substrategies ensure that $\Delta_e^{V_e}$ is well-defined and for each i, $Tot(i) = \lim_x \Delta_e^{V_e}(i,x)$. So $Tot \leq_T V_e'$, V_e has high degree.

$\square$

This completes the proof of Theorem 1.

Chapter 3

HIGHNESS AND LOCALLY NONCAPPABLE DEGREES

3.1 Introduction

Lachlan [33] and Yates [65] proved the existence of minimal pairs, where a pair $(\mathbf{a}, \mathbf{b})$ is called a *minimal pair* if $\mathbf{a}$ and $\mathbf{b}$ are nonzero c.e. degrees such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. A c.e. degree $\mathbf{a}$ is called *cappable* if $\mathbf{a}$ is $\mathbf{0}$ or a half of a minimal pair, and *noncappable* otherwise. Yates proved in [66] the existence of noncappable degrees. Let $\mathbf{M}$ and $\mathbf{NC}$ be the sets of all cappable and noncappable c.e. degrees, respectively. Ambos-Spies, Jockusch, Shore and Soare [1] showed that $\mathbf{M}$ is an ideal of $\mathbf{R}$, that $\mathbf{NC}$ is a filter of $\mathbf{R}$, and that $\mathbf{NC} = \mathbf{LC} = \mathbf{PS}$, where $\mathbf{LC}$ is the set of all low cuppable degrees and $\mathbf{PS}$ is the set of all promptly simple degrees.

A c.e. degree $\mathbf{a}$ is called *locally noncappable* if there is a c.e. degree $\mathbf{c}$ above $\mathbf{a}$ such that no nonzero c.e. degree $\mathbf{w}$ below $\mathbf{c}$ forms a minimal pair with $\mathbf{a}$. We say that $\mathbf{c}$ witnesses that $\mathbf{a}$ is locally noncappable.

Seetapun proved in his thesis [57] that every nonzero incomplete c.e. degree is locally noncappable. Giorgi published Seetapun's result in [24], but with one Σ_3 outcome missing, so Giorgi's construction is not complete. Recently, Stephan and Wu [59] improved Seetapun's result by showing that such witnesses can always be chosen as high$_2$ degrees, which implies the existence of Downey, Lempp and Shore's high$_2$ nonbounding degrees, Li's high$_2$ plus-cupping degrees. In this chapter, we prove another theorem on locally noncappable degrees, which also implies several known results as direct corollaries.

Theorem 2. *For any nonzero incomplete c.e. degree* $\mathbf{a}$*, there exist two incomparable c.e. degrees* $\mathbf{c}$*,* $\mathbf{d} > \mathbf{a}$ *witnessing that* $\mathbf{a}$ *is locally noncappable, and* $\mathbf{c} \vee \mathbf{d}$ *is high.*

Theorem 2 implies both classes of the plus-cupping degrees and the nonbounding c.e. degrees do not form ideals, which was proved by Li and Zhao in [46], by using two separate constructions.

3.2 Requirements and strategies

To prove Theorem 2, let A be an incomputable and incomplete c.e. set in $\mathbf{a}$. We will construct two c.e. sets C, D and a p.c. functional Λ satisfying the following requirements:

$$\mathcal{H}_e : Tot(e) = \lim_x \Lambda^{A\oplus C\oplus D}(e,x),$$
$$\mathcal{Q}_e^C : C \neq \Phi_e^{A\oplus D},$$
$$\mathcal{Q}_e^D : D \neq \Phi_e^{A\oplus C},$$
$$\mathcal{R}_e^C : W_e = \Phi_e^{A\oplus C} \Rightarrow W_e \leq_T A \ \text{ or } \ \exists X_e \leq_T A, W_e(X_e \ \text{ is incomputable}),$$
$$\mathcal{R}_e^D : V_e = \Psi_e^{A\oplus D} \Rightarrow V_e \leq_T A \ \text{ or } \ \exists Y_e \leq_T A, V_e(Y_e \ \text{ is incomputable}).$$

Here $\{(W_e, \Phi_e, V_e, \Psi_e) : e \in \omega\}$ is an effective enumeration of all quadruples (W, Φ, V, Ψ) of c.e. sets W, V and p.c. functionals Φ, Ψ. X_e, Y_e are c.e. sets built by us. $Tot = \{i : \varphi_i \text{ is total}\}$ is a Π_2^0-complete set. Let $\mathbf{c}$ be the degree of $A \oplus C$, and $\mathbf{d}$ be the degree of $A \oplus D$. By the $\mathcal{Q}$-requirements, $\mathbf{c}$ and $\mathbf{d}$ are two incomparable degrees above $\mathbf{a}$, by the $\mathcal{R}$-requirements, both $\mathbf{c}$ and $\mathbf{d}$ witness that $\mathbf{a}$ is locally noncappable. $\mathcal{H}$-requirements ensure that $\mathbf{c} \vee \mathbf{d}$ is high. Therefore, the above requirements are sufficient to prove Theorem 2.

3.2.1 A $\mathcal{Q}_e^C$ strategy

A $\mathcal{Q}_e^C$-strategy α is a Sacks coding strategy (here, a $\mathcal{Q}_e^D$-strategy is similar). That is, even though A is not in our control, we can still satisfy the $\mathcal{Q}_e^C$-requirement by the assumption that A is incomplete. α will run cycles j for $j \in \omega$, and all cycles of α define a functional Ξ_α jointly. Each cycle j tries to find a number x_j such that $C(x_j) \neq \Phi_e^{A \oplus D}(x_j)$, and if cycle j fails to make it, then this cycle will define $\Xi_\alpha^A(j) = K(j)$ successfully. If we fail to satisfy a $\mathcal{Q}_e^C$-strategy, then we will threaten $K = \Xi_\alpha^A$ to get a contradiction.

Cycle j proceeds as follows:

(1) Choose x_j as a fresh number.

(2) Wait for $\Phi_e^{A \oplus D}(x_j) \downarrow = 0$.

(3) Preserve $D \upharpoonright \varphi(x_j)$ from other strategies, and define $\Xi_\alpha^A(j) = K(j)$ with use $\xi_\alpha(j) = \varphi_e(x_j)$. Start cycle $j + 1$ simultaneously.

(4) Wait for $A \upharpoonright \varphi_e(x_j)$ or $K(j)$ to change.

   (a) If $A \upharpoonright \varphi_e(x_j)$ changes first, then cancel all cycles $j' > j$ and drop the D-restraint of cycle j to 0. Go back to step 2.

   (b) If $K(j)$ changes first, then stop cycles $j' > j$, and go to step 5.

(5) Put x_j into C. Wait for $A \upharpoonright \varphi_e(x_j)$ to change.

(6) Define $\Xi_\alpha^A(j) = K(j) = 1$ with use 0, and start cycle $j + 1$.

α has two sorts of outcomes:

(j, f) : Some cycle j waits forever at step 2 or 5.

(The $\mathcal{Q}_e^C$-strategy is satisfied via witness x_j in an obvious way.)

(j, ∞) : Some (least) cycle j runs infinitely often.

(It must be that cycle j returns from step 4 to 2 infinitely often. Thus $\Phi_e^{A \oplus D}(x_j)$ diverges. So the $\mathcal{Q}_e^C$-strategy is satisfied.)

Note that it is impossible that α runs infinitely many cycles and each cycle runs only finitely often, since otherwise Ξ_α^A is defined as a total function and $\Xi_\alpha^A = K$, a contradiction.

We introduce some notions for convenience when we deal with such cycles. Say that *cycle j acts* if it chooses a fresh number x_j as its attacker at step 1 or, it changes the value of $C(x_j)$ by enumerating x_j into C at step 5. Say that *cycle j is active* at stage s if at this stage, when α is visited, α is running cycle j, except the situation that cycle j is just started at stage s.

3.2.2 An $\mathcal{R}_e^C$ strategy

Assume that a strategy β works on $\mathcal{R}_e^C$, we define the length of agreement function $l(\beta, s)$ at stage s to be

$$l(\beta, s) = \max\{x < s : (\forall y < x)[W_e(y)[s] = \Phi_e^{A\oplus C}(y)[s]]\},$$

and the maximum length of agreement function at stage s is defined to be

$$m(\beta, s) = \max\{l(\beta, t) : t < s \text{ and } t \text{ is a } \beta\text{-stage}\}.$$

Say that a stage s is a β-expansionary stage if $s = 0$ or s is a β-stage with $l(\beta, s) > m(\beta, s)$.

At β-expansionary stages, we will construct a c.e. set X_e and two p.c. functionals Γ_β and Δ_β such that if $W_e = \Phi_e^{A\oplus C}$, then either $W_e \leq_T A$ or X_e is incomputable with $X_e = \Gamma_\beta^A, \Delta_\beta^{W_e}$.

β has two outcomes $\infty <_L f$. ∞ denotes that β has infinitely many expansionary stages, and f denotes that β has only finitely many expansionary stages. Below the ∞ outcome of β, to ensure that X_e is incomputable, we need to satisfy the following subrequirements

$$\mathcal{S}_{e,i}^C : X_e \neq \varphi_i$$

for all i.

In the construction, we always define the use $\delta_\beta(x) = x$, which will ensure that

$\Delta_\beta^{W_e}$ is totally defined. When $\Gamma_\beta^A(x)$ is defined at a β-expansionary stage s, we will define $\gamma_\beta(x)[s] > \varphi_e(x)[s]$, so Γ_β^A could be partial when $\Phi_e^{A\oplus C}$ is partial.

3.2.3 An $\mathcal{S}_{e,i}^C$ strategy

Assume that η is an $\mathcal{S}_{e,i}^C$-strategy below the ∞ outcome of an $\mathcal{R}_e^C$-strategy β, and η has two parameters x_η and z_η. The basic η-strategy is almost the same as the one given in [59]:

(1) Pick x_η as a fresh number.

(2) Wait for $\varphi_i(x_\eta) \downarrow = 0$.

(3) Assign $z_\eta = x_\eta$ and let $x_\eta \uparrow$. Go back to step 1, and wait for A to change below z_η. (If such a change occurs, then $\Gamma_\beta^A(z_\eta)$ is undefined.)

(4) (Open a gap)

Create a link between β and η, wait for W_e to change below z_η.

(5a) (Close the gap successfully)

If W_e has a change below z_η, then η performs the diagonalization by putting z_η into X_e. Cancel the link to close the gap. In this case, η is satisfied forever.

(5b) (Close the gap unsuccessfully)

If W_e does not have a change below z_η, then cancel the link to close the gap. Impose a restraint on C. Request that $\gamma_\beta(z_\eta)$ be defined greater than $\varphi_e(z_\eta)$.

We will define a p.c. functional Θ_η such that if η opens infinitely many different gaps (each gap associated with different value of z_η), then Θ_η^A is totally defined and computes W_e correctly, and hence η provides a global win for β.

In the construction, we define Θ_η in step (5b), i.e. when a gap for z_η is closed unsuccessfully, for all $x \leq z_\eta$, if $\Theta_\eta^A(x) \uparrow$, then we define $\Theta_\eta^A(x) = W_e(x)$ with $\theta_\eta(x) = \varphi_e(x)$.

The period between a stage at which a gap is closed unsuccessfully and the stage at which the next gap is open is called a cogap. Note that, when a gap is closed unsuccessfully, a restraint is imposed on C to prevent W_e from changing below z_η. But, as A is given, A can change below $\varphi_e(z_\eta)$ during the cogaps, which may change W_e below z_η. In other words, during a cogap for z_η, if W_e changes on some number $x \leq z_\eta$, then A must change below $\varphi_e(x)$ since C is protected, and hence $\Theta_\eta^A(x)$ is undefined by this A-change, so $\Theta_\eta^A(x)$ can be defined correctly when it is defined later.

An η-strategy has four outcomes $s <_L g <_L d <_L w$, the outcome s denotes the case that a gap is closed successfully; the outcome g denotes the case that η opens infinitely many different gaps (each gap associated with different value of z_η), we can show that if $W_e = \Phi_e^{A \oplus C}$ then $W_e = \Theta_\eta^A$, and hence we have a global win for β; the outcome d denotes the case that η opens only finitely many different gaps and for the last gap, A changes below the corresponding use $\varphi_e(z_\eta)$ infinitely often, and hence $\Phi_e^{A\oplus C}(z_\eta) \uparrow$, we have a global win for β; the outcome w denotes the case that η waits for $\varphi_i(x_\eta) \downarrow = 0$ forever for some x_η. We will reconsider $\mathcal{S}_{e,i}^C$-strategies after the $\mathcal{H}$-strategies are introduced.

An $\mathcal{R}_e^D$-strategy is similar to an $\mathcal{R}_e^C$-strategy and hence we have the following subrequirements

$$\mathcal{S}_{e,i}^D : Y_e \neq \varphi_i$$

for all i.

An $\mathcal{S}_{e,i}^D$-strategy is similar to an $\mathcal{S}_{e,i}^C$-strategy.

3.2.4 An $\mathcal{H}_e$ strategy

Assume τ works on $\mathcal{H}_e$. We define the length of convergence function $l(\tau, s)$ at stage s to be

$$l(\tau, s) = \max\{x < s : (\forall y < x)[\varphi_e(y)[s] \downarrow]\},$$

and the maximum length of convergence function at stage s is defined to be

$$m(\tau, s) = \max\{l(\tau, t) : t < s \text{ and } t \text{ is a } \tau\text{-stage}\}.$$

Say that a stage s is a τ-expansionary stage if $s = 0$ or s is a τ-stage with $l(\tau, s) > m(\tau, s)$. τ has two outcomes $\infty <_L f$. If τ has ∞ outcome, i.e. φ_e is total, then we will define $\Lambda^{A\oplus C\oplus D}(e, x) = 1$ for almost all x; if τ has f outcome, i.e. φ_e is not total, then we will define $\Lambda^{A\oplus C\oplus D}(e, x) = 0$ for almost all x.

Note that Λ is a global p.c. functional built by us through the whole construction. $\Lambda^{A\oplus C\oplus D}(e, x)$ is undefined automatically if some number $\leq \lambda(e, x)$ is enumerated into C or D. As a consequence, $\lambda(e, x)$ may be lifted when $\Lambda^{A\oplus C\oplus D}(e, x)$ is redefined later.

If we defined $\Lambda^{A\oplus C\oplus D}(e, x) = 0$ under the f outcome at a previous stage, and now we see φ_e converges on more arguments, i.e. τ changes its outcome from f to ∞, then we want to (re)define $\Lambda^{A\oplus C\oplus D}(e, x) = 1$, but first we need to undefine all the previous $\Lambda^{A\oplus C\oplus D}(e, x) = 0$. So generally, at a τ-expansionary stage, we will put the $\lambda(e, x)$ use into C to undefine $\Lambda^{A\oplus C\oplus D}(e, x) = 0$, and we (re)define $\Lambda^{A\oplus C\oplus D}(e, x) = 1$ with use $\lambda(e, x)$ as -1 at τ-expansionary stages since we never want to undefine it later. This means that we only care about the $\lambda(e, x)$ use defined under the f outcome.

Actually, in the construction, we need to consider the restraint imposed on τ, so we will define *boundary* $bd(\tau)$ of τ (playing a role of the restraint) as follows: when τ is visited for the first time, we define $bd(\tau)$ to be a fresh number, whenever τ is initialized, we will redefine $bd(\tau)$ as a fresh number. At a τ-expansionary stage s, we will put the $\lambda(e, x)$ use with $bd(\tau) < \lambda(e, x) \leq s$ into C to undefine $\Lambda^{A\oplus C\oplus D}(e, x) = 0$.

One may wonder why the $\mathcal{H}_e$-strategy τ enumerates the $\lambda(e, x)$ use only into C when it wants to undefine $\Lambda^{A\oplus C\oplus D}(e, x) = 0$. Actually, we first fix C just for simplicity, and in the construction, we may need to put the $\lambda(e, x)$ use into D sometimes.

3.2.5 Interaction between strategies

First, we consider the interaction between the high strategies and the gap-cogap argument used in $\mathcal{S}$-strategies.

Suppose that an $\mathcal{R}_e^C$-strategy β works below the ∞ outcome of an $\mathcal{H}$-strategy τ, i.e. $\tau^\frown\langle\infty\rangle \subset \beta$. Then, τ may put the λ-uses into C infinitely often and hence injure a computation $\Phi_e^{A\oplus C}(z_\eta)$ infinitely often, where η is a substrategy of β. Note that, when β closes a gap for η, a restraint is imposed on C to prevent W_e from changing below z_η, which will ensure that, during a cogap for η, if W_e changes below z_η then it must be due to the A-change below the corresponding use $\varphi_e(z_\eta)$. But, the enumerations of numbers into C by such $\mathcal{H}$-strategy τ of higher priority affect our idea in this gap-cogap argument. Thus, we need to consider believable computations for β in the construction.

Suppose that an $\mathcal{H}_{e_0}$-strategy τ works between β and η with $\beta^\frown\langle\infty\rangle \subset \tau \subset \tau^\frown\langle\infty\rangle \subset \eta$, where β is an $\mathcal{R}_e^C$-strategy and η is an $\mathcal{S}_{e,i}^C$-strategy. Then we may have that β opens a gap for η and creates a link between β and η at some β-expansionary stage, s_0 say. At the next β-expansionary stage s_1 (β-believable computations are considered, we will give the definition for it in our construction), suppose this gap is closed unsuccessfully by β, then we cancel the link (β, η) and let $\beta^\frown\langle\infty\rangle$ be accessible at stage s_1. After stage s_1, and before the next gap for η is open, i.e. during the cogap for η, we want to impose a C-restraint below s_1 to prevent W_e from changing. But, during the cogap for η, τ may be visited and enumerate some small $\lambda(e_0, x) \leq s_1$ into C. That is, during the cogap for η, the idea of imposing a C-restraint to prevent W_e from changing fails, which should be avoided. To solve this problem, we apply the method introduced in [46]: when β closes the gap for η unsuccessfully at stage s_1, it requires that all the $\mathcal{H}$-strategies with ∞ outcome between β and η enumerate the λ-uses into set D to lift λ-uses if needed. Thus, after stage s_1, all the numbers enumerated into C in the future by such $\mathcal{H}$-strategies are greater than s_1.

Similarly, if β is an $\mathcal{R}_e^D$-strategy and η is an $\mathcal{S}_{e,i}^D$-strategy as above, then when β closes a gap for η unsuccessfully, it requires that all the $\mathcal{H}$-strategies with ∞ outcome between β and η enumerate the λ-uses into set C to lift λ-uses if needed. Note that, for the interactions between $\mathcal{R}^C$ and $\mathcal{R}^D$ strategies, we will explain how to deal with these interactions later.

We now describe in detail about the $\mathcal{S}$-strategies together with the high strategies. Suppose that an $\mathcal{H}_{e_0}$-strategy τ works between β and η with $\beta^\frown\langle\infty\rangle \subset \tau \subset \tau^\frown\langle\infty\rangle \subset \eta$, where β is an $\mathcal{R}_e^C$-strategy and η is an $\mathcal{S}_{e,i}^C$-strategy. Assume that β opens a gap (for η) for a value of z_η and creates a link between β and η at some β-expansionary stage, s_0 say. That is, $\Gamma_\beta^A(z_\eta)$ is undefined at stage s_0.

At the next β-expansionary stage s_1, if W_e changes below z_η (i.e. $\Delta_\beta^{W_e}(z_\eta)$ is undefined), then η can close this gap successfully by putting z_η into X_e. Cancel the link (β, η), we say that η is satisfied at stage s_1. If W_e does not change below z_η, then β closes this gap for η unsuccessfully at stage s_1, cancel the link (β, η). Note that when β closes a gap, what we do is to extend the definitions of Γ_β^A and $\Delta_\beta^{W_e}$.

Suppose that β closes this gap for η unsuccessfully at stage s_1, then at the next β-expansionary stage s_2, if W_e changes below z_η, it must be due to the A-change below the corresponding use $\varphi_e(z_\eta)$ (as C is protected), so both $\Delta_\beta^{W_e}(z_\eta)$ and $\Gamma_\beta^A(z_\eta)$ are undefined, then η can perform the diagonalization by putting z_η into X_e in this cogap. If W_e does not change below z_η, but A changes below $\varphi_e(z_\eta)$. In this case, if we let β reopen this gap for W_e to change and create a link between β and η at stage s_2, and let η is visited immediately (i.e. the strategies between β and η are not visited at stage s_2), having outcome d as in [59]. Then this gap can be closed unsuccessfully and reopened by β infinitely often in this way, and hence it requires that all the $\mathcal{H}$-strategies with ∞ outcome between β and η enumerate the λ-uses into set D to lift λ-uses to protect C infinitely often. However, such $\mathcal{H}$-strategies may have outcome different from the one seen at η. That is, η guesses that τ has outcome ∞, but τ may actually have outcome f. As a consequence, it may require τ to enumerate the λ-uses into set D infinitely often (when this gap is closed unsuccessfully by β), but τ actually has outcome f. Thus, τ can not be satisfied since Λ can not be well-defined in this case. To avoid this problem, we will modify the method in [59] as follows: After a gap for z_η is closed unsuccessfully by β, at the next β-expansionary stage, if A changes below $\varphi_e(z_\eta)$, but W_e does not change below z_η, we will not let β reopen this gap immediately. Instead, if η is visited at this stage, then we let η reopen this gap. That is, in the construction, at an η-stage, if there is a gap which was closed unsuccessfully by β at a previous stage, and if A changes below the correspongding use $\varphi_e(z_\eta)$ at this η-stage, then we will let η reopen this gap. Note that, in the construction, when a gap for η is reopened, it must be reopened by η, not by β. This will avoid the problem above, since if a gap can be closed unsuccessfully and reopened infinitely often then the $\mathcal{H}$-strategies between β and η must have outcome the same with the one seen at η.

Note that, in a cogap for η, it may happen that we see W_e change below z_η at a β-expansionary stage, but $\Gamma_\beta^A(z_\eta)\downarrow$ at this stage. We now describe why this can happen in detail as follows. Suppose that a gap for z_η is closed unsuccessfully by β at a β-expansionary stage s_1, and we define $\Delta_\beta^{W_e}(z_\eta)$ and $\Gamma_\beta^A(z_\eta)$ at the end of stage s_1, with $\gamma_\beta(z_\eta)[s_1] > \varphi_e(z_\eta)[s_1]$. At the next β-expansionary stage s_2, assume that A changes below $\varphi_e(z_\eta)$, but W_e does not change below z_η, i.e. $\Gamma_\beta^A(z_\eta)\uparrow$, but $\Delta_\beta^{W_e}(z_\eta)\downarrow$. And assume that η is not visited at stage s_2 (and hence η can not reopen this gap at stage s_2), so we define $\Gamma_\beta^A(z_\eta)$ at the end of stage s_2, with $\gamma_\beta(z_\eta)[s_2] > \varphi_e(z_\eta)[s_2]$. Assume that $\varphi_e(z_\eta)[s_2] > \varphi_e(z_\eta)[s_1]$, and hence the $\mathcal{H}$-strategies between β and η may enumerate some small λ-uses into C, changing the computation $\Phi_e^{A\oplus C}(z_\eta)[s_2]$. Thus, at the next β-expansionary stage s_3, it may happen that $\Delta_\beta^{W_e}(z_\eta)\uparrow$, but $\Gamma_\beta^A(z_\eta)\downarrow$, i.e. W_e changes below z_η, but A does not change below $\varphi_e(z_\eta)$. In this case, we can not let η perform the diagonalization at stage s_3 even though W_e changes below z_η.

On the other hand, at an η-stage, if a gap (which was closed unsuccessfully by β at a previous stage) can not be reopened, i.e. A does not change below $\varphi_e(z_\eta)$, then we will consider a new value of z_η (if any) for η and check whether we can open a new gap. If we can open a new gap, then we say that the existing gap is closed completely.

Next, we consider the interaction between the high strategies and a $\mathcal{Q}^D$-strategy (the case of $\mathcal{Q}^C$-strategy is similar).

Suppose that a $\mathcal{Q}^D$-strategy α works below the ∞ outcome of an $\mathcal{H}$-strategy τ, i.e. $\tau^\frown\langle\infty\rangle\subset\alpha$. Then, τ may put the λ-use into C infinitely often and hence injure the computation $\Phi_e^{A\oplus C}(x_\alpha)$ after α performs diagonalization infinitely often. Thus, we need to consider believable computations for α-strategy in the construction.

Finally, we consider the interaction between an $\mathcal{R}^C$-strategy and an $\mathcal{R}^D$-strategy. Suppose that an $\mathcal{R}_e^D$-strategy β' works between an $\mathcal{R}_{e'}^C$-strategy β and an $\mathcal{S}_{e',i'}^C$-strategy η with $\beta^\frown\langle\infty\rangle\subset\beta'\subset\beta'^\frown\langle\infty\rangle\subset\eta\subset\eta^\frown\langle\mathcal{O}\rangle\subset\tau\subset\tau^\frown\langle\infty\rangle\subset\eta'$, where $\mathcal{O}\in\{g,d\}$, τ is an $\mathcal{H}$-strategy and η' is an $\mathcal{S}_{e,i}^D$-strategy and $0\le e'\le e$. Then we may have that a gap is open (or reopened) for η and creates a link between β and η and η' reopens a gap and creates a link between β' and η' at the same stage, s_0 say. That is, we have two crossed links (β, η) and (β', η') at stage s_0. At the next β-expansionary stage s_1, suppose that β closes the gap for η unsuccessfully and cancels the link (β, η), it requires that all the $\mathcal{H}$-strategies with ∞ outcome between β and η enumerate the λ-uses into set D to lift λ-uses if needed to protect C. And, η imposes a C-restraint after stage s_1 till the stage, s_2 say, at which the next gap for η is open. But, during this cogap for η, we will travel the link (β', η') created at stage s_0. Suppose that we travel the link (β', η') at stage t ($s_1 < t < s_2$) say, and β' closes the gap for η' unsuccessfully, thus it requires that all the $\mathcal{H}$-strategies with ∞ outcome between β' and η' enumerate the λ-uses into set C to lift λ-uses if needed to protect D. In particular, suppose that the $\mathcal{H}$-strategy τ is required to enumerate the λ-uses into set C to lift λ-uses at stage t, these λ-uses may be less than s_1. But, η does not allow small numbers ($\leq s_1$) to be enumerated into C before stage s_2. So η (i.e. β) injures the satisfaction of β'. Thus, such crossed links (β, η) and (β', η') should be avoided.

In the construction, we will use a backup strategy to deal with this. That is, we will put a backup strategy $\hat{\beta}$ below $\eta^\frown\langle\mathcal{O}\rangle$ to try to satisfy the $\mathcal{R}_e^D$-requirement. Therefore, in the construction, there will never be two crossed links (β, η) and (β', η') as above at the same stage.

Actually, in a gap-cogap argument, the crossed links need to be avoided generally. So, in our construction, suppose that we have $\beta^\frown\langle\infty\rangle \subset \beta' \subset \beta'^\frown\langle\infty\rangle \subset \eta$, where β can be an $\mathcal{R}_e^C$-strategy or $\mathcal{R}_e^D$-strategy, β' can be an $\mathcal{R}_{e'}^C$-strategy or $\mathcal{R}_{e'}^D$-strategy ($e \leq e'$), and η is a substrategy of β. Then, when η has outcome g or d, we will say that β' becomes inactive or injured, and we need to arrange a back-up strategy for β' under this outcome. Moreover, it's not hard to see that, for a fixed e, there are at most finitely many backup $\mathcal{R}_e^C$-strategies ($\mathcal{R}_e^D$-strategy is similar) on any path of the construction tree and the longest $\mathcal{R}_e^C$-strategy is responsible to satisfy the $\mathcal{R}_e^C$ requirement.

Note that, we may have two nested links (β, η) and (β', η') at the same stage in the construction, where β is an $\mathcal{R}_e^C$-strategy and β' is an $\mathcal{R}_{e'}^D$-strategy, and η is an $\mathcal{S}_{e,i}^C$-strategy and η' is an $\mathcal{S}_{e',i'}^D$-strategy with $\beta^\frown\langle\infty\rangle \subset \beta' \subset \beta'^\frown\langle\infty\rangle \subset \eta' \subset \eta'^\frown\langle\mathcal{O}\rangle \subset \eta$, where $\mathcal{O} \in \{g, d\}$. That is, we may have that a gap is open (or reopened) for $\eta'$and creates a link between β' and η' and η reopens a gap and creates a link between β

and η at the same stage, s_0 say. At the next β-expansionary stage s_1, suppose β closes the gap for η unsuccessfully and cancels the link (β, η), so it requires that the $\mathcal{H}$-strategies with ∞ outcome between β and η enumerate the λ-uses into set D to lift λ-uses if needed. (The enumeration of numbers into D at stage s_1 may have effect on the computation involved in the gap for η', but this is allowed since the gap for η' is open at stage s_1.) η will impose a C-restraint below s_1 after stage s_1 till the stage, s_2 say, at which the next gap for η is open. But, before stage s_2, i.e. during this cogap for η, we will travel the link (β', η') created at stage s_0 and β' may close the gap for η' unsuccessfully and so it requires that the $\mathcal{H}$-strategies with ∞ outcome between β' and η' enumerate the λ-uses into set C to lift λ-uses if needed. Note that such λ-uses are already lifted at stage s_1 and hence must be greater than s_1, so we can enumerate them into C and there is no conflict.

3.3 Construction

Before we give the full construction, we first define the priority tree T effectively.

Definition 1 (1) Define *the priority ranking of the requirements* as follows:

$\mathcal{Q}_0^C < \mathcal{Q}_0^D < \mathcal{H}_0 < \mathcal{R}_0^C < \mathcal{S}_{0,0}^C < \mathcal{R}_0^D < \mathcal{S}_{0,0}^D < \mathcal{Q}_1^C < \mathcal{Q}_1^D < \mathcal{H}_1 < \mathcal{R}_1^C < \mathcal{S}_{0,1}^C < \mathcal{S}_{1,0}^C < \mathcal{S}_{1,1}^C < \mathcal{R}_1^D < \mathcal{S}_{0,1}^D < \mathcal{S}_{1,0}^D < \mathcal{S}_{1,1}^D < \cdots < \mathcal{Q}_n^C < \mathcal{Q}_n^D < \mathcal{H}_n < \mathcal{R}_n^C < \mathcal{S}_{0,n}^C < \mathcal{S}_{1,n}^C < \cdots < \mathcal{S}_{n-1,n}^C < \mathcal{S}_{n,0}^C < \mathcal{S}_{n,1}^C < \cdots < \mathcal{S}_{n,n}^C < \mathcal{R}_n^D < \mathcal{S}_{0,n}^D < \mathcal{S}_{1,n}^D < \cdots < \mathcal{S}_{n-1,n}^D < \mathcal{S}_{n,0}^D < \mathcal{S}_{n,1}^D < \cdots < \mathcal{S}_{n,n}^D < \cdots$

where $\mathcal{X} < \mathcal{Y}$ means that $\mathcal{X}$ has higher priority than $\mathcal{Y}$.

(2) An $\mathcal{H}$-, or $\mathcal{R}$-strategy has two possible outcomes $\infty <_L f$.
An $\mathcal{S}$-strategy has four possible outcomes $s <_L g <_L d <_L w$.
The possible outcomes of a $\mathcal{Q}$-strategy are

$$(0, \infty) <_L (0, f) <_L \cdots <_L (j, \infty) <_L (j, f) <_L \cdots .$$

Definition 2 Given $\zeta \in T$.

(1) A requirement $\mathcal{R}_e^C$ is *satisfied at* ζ if there is an $\mathcal{R}_e^C$-strategy β with $\beta^\frown\langle f\rangle \subset \zeta$, or there is an $\mathcal{R}_e^C$-strategy β and an $\mathcal{S}_{e,i}^C$-strategy η for some i with the following properties:

- $\beta^\frown\langle\infty\rangle \subseteq \eta \subset \eta^\frown\mathcal{O} \subset \zeta$, where $\mathcal{O}$ is g or d.

- there is no $\mathcal{S}_{e',i'}^C$-strategy or $\mathcal{S}_{e',i'}^D$-strategy η' such that $\beta^\frown\langle\infty\rangle \subseteq \eta' \subset \eta'^\frown\mathcal{O} \subset \eta$ for any $e' < e$ and any i', where $\mathcal{O}$ is g or d.

In the latter case, β has a Σ_3-outcome at η, and under this outcome, all the non-$\mathcal{H}$-strategies between β and η are said to be injured at η. When a strategy is injured, then all its substrategies are injured.

(2) Similarly, a requirement $\mathcal{R}_e^D$ is *satisfied at* ζ if there is an $\mathcal{R}_e^D$-strategy β with $\beta^\frown\langle f\rangle \subset \zeta$, or there is an $\mathcal{R}_e^D$-strategy β and an $\mathcal{S}_{e,i}^D$-strategy η for some i with the following properties:

- $\beta^\frown\langle\infty\rangle \subseteq \eta \subset \eta^\frown\mathcal{O} \subset \zeta$, where $\mathcal{O}$ is g or d.

- there is no $\mathcal{S}_{e',i'}^C$-strategy or $\mathcal{S}_{e',i'}^D$-strategy η' such that $\beta^\frown\langle\infty\rangle \subseteq \eta' \subset \eta'^\frown\mathcal{O} \subset \eta$ for any $e' \leq e$ and any i', where $\mathcal{O}$ is g or d.

(3) A requirement $\mathcal{R}_e^C$ is *active at* ζ *via* β if $\mathcal{R}_e^C$ is not satisfied at ζ and there is an $\mathcal{R}_e^C$-strategy β such that

- $\beta^\frown\langle\infty\rangle \subset \zeta$,

- there is no $\mathcal{S}_{e',i'}^C$-strategy or $\mathcal{S}_{e',i'}^D$-strategy η' such that $\beta^\frown\langle\infty\rangle \subseteq \eta' \subset \eta'^\frown\mathcal{O} \subset \zeta$ for any $e' < e$ and any i', where $\mathcal{O}$ is g or d.

(4) Similarly, a requirement $\mathcal{R}_e^D$ is *active at* ζ *via* β if $\mathcal{R}_e^D$ is not satisfied at ζ and there is an $\mathcal{R}_e^D$-strategy β such that

- $\beta^\frown\langle\infty\rangle \subset \zeta$,

- there is no $\mathcal{S}_{e',i'}^C$-strategy or $\mathcal{S}_{e',i'}^D$-strategy η' such that $\beta^\frown\langle\infty\rangle \subseteq \eta' \subset \eta'^\frown\mathcal{O} \subset \zeta$ for any $e' \leq e$ and any i', where $\mathcal{O}$ is g or d.

(5) A requirement $\mathcal{S}^C_{e,i}$ is *satisfied at* ζ if either $\mathcal{R}^C_e$ is satisfied at ζ, or $\mathcal{R}^C_e$ is active at ζ via β and there is an $\mathcal{S}^C_{e,i}$-strategy η with $\beta^\frown\langle\infty\rangle \subset \eta \subset \zeta$.

We can define a requirement $\mathcal{S}^D_{e,i}$ to be *satisfied at* ζ in the same manner.

(6) A requirement $\mathcal{Q}^C_e$ (or $\mathcal{Q}^D_e$) is *satisfied at* ζ if there is a $\mathcal{Q}^C_e$-strategy (or $\mathcal{Q}^D_e$-strategy) α with $\alpha^\frown\langle k, -\rangle \subset \zeta$, and α is not injured at ζ, where $- \in \{\infty, f\}$.

(7) A requirement $\mathcal{H}_e$ is *satisfied at* ζ if there is an $\mathcal{H}_e$-strategy τ with $\tau \subset \zeta$.

Now we define the priority tree T as follows.

Definition 3 (1) Define the root node as a $\mathcal{Q}^C_0$-strategy.

(2) The immediate successors of a node are the possible outcomes of the corresponding strategy.

(3) For $\zeta \in T$, ζ works for the highest priority requirement which has neither been satisfied, nor been active at ζ.

(4) Continuing the inductive steps above, the priority tree T is built.

Definition 4 Given an $\mathcal{S}^C_{e,i}$-strategy η, we define the *mother node* of η as the longest $\mathcal{R}^C_e$-strategy β such that $\beta^\frown\langle\infty\rangle \subset \eta$, we use $\mathrm{top}(\eta)$ to denote the mother node of η.

We can define the *mother node* of an $\mathcal{S}^D_{e,i}$-strategy in the same manner.

Definition 5 (Believable computations)

Recall that an $\mathcal{H}$-strategy τ has a parameter $bd(\tau)$, playing a role of the restraint imposed on τ: when τ is visited for the first time, we define $bd(\tau)$ as a fresh number, whenever τ is initialized, we will define $bd(\tau)$ as a fresh number again.

Given a node ζ in T, we say that a computation $\Phi^{A\oplus C}_\zeta(x)$ at stage s is ζ-*believable*, if for each $\mathcal{H}_e$-strategy τ with $\tau^\frown\langle\infty\rangle \subseteq \zeta$, for any existing $\lambda(e,x) > bd(\tau)[s]$, if $\lambda(e,x) \leq \varphi_\zeta(x)[s]$, then $\lambda(e,x) \in C_s \cup D_s$. Here $\varphi_\zeta(x)[s]$ is the use of the computation

$\Phi_\zeta^{A\oplus C}(x)[s]$. (Fix ζ in T, we can define a computation $\Phi_\zeta^{A\oplus D}(x)$ at stage s to be ζ-believable in the same way).

Note that if a computation $\Phi_\zeta^{A\oplus C}(x)[s]$ is ζ-believable, then τ's further enumerations into C with $\tau^\frown\langle\infty\rangle \subseteq \zeta$ will not change this computation. Accordingly, in our construction, when we say that a computation associated with ζ converges (e.g. in a $\mathcal{Q}_e^D$-strategy), then this computation is considered as a ζ-believable computation.

For an $\mathcal{R}_e^C$-strategy β, we modify the definition of the length of agreement function $l(\beta, s)$ at stage s to be $l(\beta, s) = \max\{x < s : (\forall y < x)[W_e(y)[s] = \Phi_e^{A\oplus C}(y)[s]\downarrow$ via β-believable computations$]\}$. For an $\mathcal{R}_e^D$-strategy, we also modify the definition of the length of agreement function in the same way.

In the construction, an $\mathcal{S}$-strategy η has two parameters x_η, z_η, and a p.c. functional Θ_η. When an $\mathcal{S}$-strategy η is initialized, then the parameters x_η, z_η will be cancelled, and the p.c. functional Θ_η will become the empty set.

The full construction is as follows.

Stage 0: Initialize all nodes on T, and let $C_0 = D_0 = \emptyset$.

Stage $s > 0$: This stage has two phases.

Phase I. (finding σ_s)

Substage 0: Let $\sigma_s(0)$ be the root node.

Substage t**:** Given $\zeta = \sigma_s \upharpoonright t$.

If $t = s$ then define $\sigma_s = \zeta$ and initialize all the nodes with lower priority than σ_s. Go to Phase II.

If $t < s$, then take action for ζ and define $\sigma_s(t)$ as follows:

Case 1 $\zeta = \tau$ is an $\mathcal{H}_e$-strategy. There are three subcases.

(τ1) If $bd(\tau)\uparrow$, then define $bd(\tau)$ to be a fresh number.

Let $\sigma_s = \tau$ and initialize all the nodes with priority lower than σ_s. Go to Phase II.

(τ2) If $bd(\tau)\downarrow$, and if s is a τ-expansionary stage, then put all existing $\lambda(e,x)$ with $bd(\tau) < \lambda(e,x) \le s$ into C. Let $\sigma_s(t) = \tau^\frown\langle\infty\rangle$, and go to the next substage.

(τ3) If $bd(\tau)\downarrow$, and if s is not a τ-expansionary stage, let $\sigma_s(t) = \tau^\frown\langle f\rangle$, and go to the next substage.

Case 2 $\zeta = \alpha$ is a $\mathcal{Q}_e^C$-strategy. There are three subcases.

(α1) If α has no cycle started, then start cycle 0 and choose a fresh number $x_{\alpha,0}$ as its attacker. Define $\sigma_s = \alpha^\frown(0,f)$, initialize all nodes with priority lower than σ_s. Go to Phase II.

(α2) If α1 fails, let j be the largest active cycle at the last α-stage.

(α2.1) If A has a change below the corresponding use of some (least) cycle $j' \le j$, then define $\sigma_s(t)$ as $\alpha^\frown(j',\infty)$ if cycle j' has not performed diagonalization so far; and define $\sigma_s(t)$ as $\alpha^\frown(j'+1,f)$ if cycle j' performed diagonalization before. In the former case, go to the next substage. In the latter case, redefine $\xi_\alpha(j')$ as $\xi_\alpha(j'-1)[s]$ since j' is in K_s and define $x_{\alpha,j'+1}$ as a fresh number, now define $\sigma_s = \alpha^\frown(j'+1,f)$, initialize all the nodes with priority lower than σ_s and go to Phase II.

(α2.2) If α2.1 fails and K has a change on some (least) number $j' \le j$ between the last α-stage and stage $s+1$, then let cycle j' act at this stage. So j'-cycle will enumerate $x_{\alpha,j'}$ into C. Say that j'-cycle of α performs diagonalization at stage $s+1$. Initialize all the nodes with priority lower than or equal to $\alpha^\frown(j',f)$ and go to Phase II. Define $\sigma_s = \alpha^\frown(j',f)$. We say that α is satisfied via j'-cycle till A changes below the corresponding use.

(α3) If neither subcases α1 nor α2 is true, then take actions as follows:

If α is at j-cycle, not satisfied yet, and $\Phi_{e(\alpha)}^{A\oplus D}(x_{\alpha,j})\downarrow = 0$, then define $\Xi_\alpha^A(j) = K(j)$. The use $\xi_\alpha(j)$ is defined as $\varphi_{e(\alpha)}(x_{\alpha,j})[s]$ if $j \notin K_s$, and defined as $\xi_\alpha(j-1)[s]$ if $j \in K_s$. And then initialize all the nodes with priority lower than $\alpha^\frown(j, f)$. Start $(j+1)$-cycle by choosing a fresh number $x_{\alpha,j+1}$ as its attacker and define $\sigma_s = \alpha^\frown(j+1, f)$. Go to Phase II.

Otherwise, define $\sigma_s(t)$ as $\alpha^\frown(j, f)$ and go to the next substage.

Case 3 ζ is a $\mathcal{Q}_e^D$-strategy. The strategy is similar to Case 2.

Case 4 $\zeta = \beta$ is an $\mathcal{R}_e^C$-strategy.

(β1) If s is not a β-expansionary stage, then let $\sigma_s(t) = \beta^\frown\langle f\rangle$, and go to the next substage.

(β2) If s is a β-expansionary stage, then we check whether β has a substrategy η such that one of the following is true:

(2.1) η is in a gap or in a cogap, and W_e has a change below z_η associated to this gap, and $\Gamma_\beta^A(z_\eta)$ is undefined.

(2.2) After the last β-expansionary stage, A has changes below z_η.

If there is such a substrategy, then let η be the one with the highest priority. We provide priority to these two cases, if more than one applies to the same η, then (2.1) has higher priority than (2.2).

If (2.1) is true, then we enumerate z_η into X_β, and declare that η is satisfied at stage s. Let $\sigma_s = \eta^\frown\langle s\rangle$, and initialize all the nodes with priority lower than σ_s. Go to Phase II.

If (2.2) is true, then β opens a new gap for η, and let $\sigma_s(t) = \eta^\frown\langle g\rangle$, and go to the next substage. We say that the existing gap is closed completely at stage s.

If such a substrategy does not exist, then we check whether some substrategy η has a gap open.

If no, then let $\sigma_s(t) = \beta^\frown\langle\infty\rangle$, and go to the next substage.

If η has a gap (for z_η) open, and A changes below $\varphi_e(z_\eta)$ since the last β-expansionary stage, then we calculate σ_s and check whether η is accessible at stage s. If η is accessible at stage s, then we just let $\sigma_s(t) = \beta^\frown\langle\infty\rangle$, and go to the next substage. (In this case, this gap is reopened for η when η is accessible at this stage, i.e. let η have outcome d at this stage.)

If η has a gap (for z_η) open, and either A does not change below $\varphi_e(z_\eta)$ since the last β-expansionary stage or η is not accessible at stage s, then let β close this gap for η unsuccessfully. For each $\mathcal{H}_{e'}$-strategy τ with $\beta \subset \tau \subset \tau^\frown\langle\infty\rangle \subseteq \eta$, for any existing $\lambda(e', x)$ use with $bd(\tau) < \lambda(e', x) \leq s$ (if any), then enumerate such $\lambda(e', x)$ into D. Cancel the link (β,η). For all $x \leq z_\eta$, if $\Theta^A_\eta(x)[s] \uparrow$, then define $\Theta^A_\eta(x)[s] = W_e(x)[s]$ with $\theta_\eta(x)[s] = \varphi_e(x)[s]$. Let $\sigma_s(t) = \beta^\frown\langle\infty\rangle$, and go to the next substage.

Case 5 $\zeta = \eta$ is an $\mathcal{S}^C_{e,i}$-strategy. Let β= top(η).

We check whether A has changes below the use of a computation involved in a cogap since the last β-expansionary stage (so this gap is not closed completely yet, and the A-change will reopen it).

If yes, then η reopens this gap, creates a link between β and η. Define $\sigma_s(t) = \eta^\frown\langle d\rangle$ and go to the next substage.

Otherwise, there are four cases.

(η1) If $x_\eta \uparrow$, then define x_η as a fresh number. Let $\sigma_s = \eta^\frown\langle w\rangle$ and initialize all the nodes with priority lower than σ_s. Go to Phase II.

(η2) If $x_\eta \downarrow$, and $\varphi_i(x_\eta)$ does not converge to 0, then let $\sigma_s(t) = \eta^\frown\langle w\rangle$, and go to the next substage.

(η3) If $x_\eta \downarrow$, and $\varphi_i(x_\eta)[s] \downarrow= 0$, then, update z_η as this x_η, and let $x_\eta \uparrow$. Say that η requests β to open a gap for z_η. Let $\sigma_s = \eta^\frown\langle w\rangle$ and initialize all the nodes with priority lower than σ_s. Go to Phase II.

(η4) If η is satisfied, then let $\sigma_s(t) = \eta^\frown\langle s\rangle$, and go to the next substage.

Case 6 ζ is an $\mathcal{R}^D_e$-strategy. The strategy is similar to Case 4.

Case 7 ζ is an $\mathcal{S}^D_{e,i}$-strategy. The strategy is similar to Case 5.

Phase II. Having σ_s, for $\zeta \subset \sigma_s$, do as follows, and then go to the next stage.

Case 1 If $\zeta = \tau$ is an $\mathcal{H}_e$-strategy, then

(1) If s is a τ-expansionary stage, then extend the definition of $\Lambda^{A\oplus C\oplus D}$ to all arguments (e, x) with $x < l(\tau, s)$ and $\Lambda^{A\oplus C\oplus D}(e, x) \uparrow$, define $\Lambda^{A\oplus C\oplus D}(e, x) = 1$ with use -1.

(2) If s is not an τ-expansionary stage, then extend the definition of $\Lambda^{A\oplus C\oplus D}$ to all arguments (e, x) with $x < s$ and $\Lambda^{A\oplus C\oplus D}(e, x) \uparrow$, define $\Lambda^{A\oplus C\oplus D}(e, x) = 0$ with use $\lambda(e, x) = s$.

Case 2 If $\zeta = \beta$ is an $\mathcal{R}_e^C$-strategy ($\mathcal{R}_e^D$-strategy is similar).

If s is a β-expansionary stage, and no link with β as top exists, then

• For each $x < l(\beta, s)$ with $\Delta_e^{W_e}(x)[s] \uparrow$, define $\Delta_e^{W_e}(x)[s] = X_e(x)[s]$ with $\delta_e(x)[s] = x$.

• For each $x < l(\beta, s)$ with $\Gamma_e^A(x)[s] \uparrow$, check that whether $\Gamma_e^A(x)$ has been defined so far (after the last stage at which β is initialized):

If no, then define $\Gamma_e^A(x)[s] = X_e(x)[s]$ with $\gamma_e(x)[s] = s$.

If yes, then check whether the computation $\Phi_e^{A\oplus C}(x)$ has changed from the stage when $\Gamma_e^A(x)$ was defined last time. If the computation keeps the same, then define $\Gamma_e^A(x)[s] = X_e(x)[s]$ with $\gamma_e(x)[s]$ the same as before. If the computation changed, then define $\Gamma_e^A(x)[s] = X_e(x)[s]$ with $\gamma_e(x)[s] = s$.

This completes the construction.

3.4 Verification

Define the true path of the construction to be $\sigma = \liminf_s \sigma_s$, i.e. the leftmost path of the construction. The following lemma implies that the true path σ is infinite.

Lemma 8. *Let ζ be any node on the true path σ. Then*

(1) *ζ can be initialized at most finitely often.*

(2) *ζ has an outcome $\mathcal{O}$ such that $\zeta^\frown\mathcal{O}$ is on σ.*

(3) *ζ can initialize the node $\zeta^\frown\mathcal{O}$ at most finitely often.*

Proof. We prove the lemma by induction on the length of ζ.

When ζ is the root node of the priority tree T, i.e. $\mathcal{Q}_0^C$-strategy, then ζ can never be initialized. So (1) is clearly true.

To show (2), for a contradiction, suppose that $\zeta^\frown\mathcal{O} \not\subset \sigma$ for any outcome $\mathcal{O}$ of ζ. This happens only when ζ runs infinitely many cycles and each of them runs finitely often. That is, for every $j \in \omega$, cycle j stops at step 3 or step 6. So $\Xi_\zeta^A(j) = K(j)$ for all $j \in \omega$, i.e. Ξ_ζ^A computes K correctly, which can not be true since A is incomplete.

We have shown that ζ (the root node) has an outcome $\mathcal{O}$ such that $\zeta^\frown\mathcal{O}$ is on σ. So there is a stage s_0 such that the nodes on the left of $\zeta^\frown\mathcal{O}$ can never be visited after stage s_0. That is, after stage s_0, ζ never initializes the node $\zeta^\frown\mathcal{O}$. So (3) is true.

Suppose that the lemma is true for all $\zeta' \subset \zeta$, we now want to show that the lemma is also true for ζ. Let ζ^- be the immediate predecessor of ζ. By the induction hypothesis, ζ^- can be initialized at most finitely often and ζ^- has a true outcome $\mathcal{O}$ on σ, and ζ^- can initialize the node $\zeta = \zeta^{-\frown}\mathcal{O}$ at most finitely often. Thus, we can fix some (least) stage s_1 such that ζ can never be initialized by higher priority strategies after stage s_1. So (1) is true for ζ.

Now we show that (2) and (3) are true for ζ.

$\zeta = \alpha$ is a $\mathcal{Q}_e^C$-strategy or $\mathcal{Q}_e^D$-strategy. Apply the same argument as for $\mathcal{Q}_0^C$-strategy to α. So (2) and (3) are true for ζ.

$\zeta = \beta$ is an $\mathcal{R}_e^C$-strategy or $\mathcal{R}_e^D$-strategy. By our construction, if there are infinitely many β-expansionary stages then we have $\beta^\frown\langle\infty\rangle \subset \sigma$; if there are only finitely many β-expansionary stages then we have $\beta^\frown\langle f\rangle \subset \sigma$. Note that β never initializes the strategies with lower priority in the construction. So (2), (3) are true for ζ.

$\zeta = \eta$ is an $\mathcal{S}^C_{e,i}$-strategy ($\mathcal{S}^D_{e,i}$-strategy is similar). Let the mother node of η be β, i.e. β is the longest $\mathcal{R}^C_e$-strategy on the true path with $\beta^\frown\langle\infty\rangle \subset \eta \subset \sigma$. By the choice of s_1, η can never be initialized after stage s_1 and hence neither can β. Then, after stage s_1, x_η will be defined by η, say at a stage $s_2 > s_1$.

If there is no stage at which $\varphi_i(x_\eta)$ converges to 0, then we have that $\eta^\frown\langle w\rangle$ on the true path. And η will never initialize the strategies with lower priority in the construction after stage s_2. So (2), (3) are true for ζ.

If there is an η-stage $s_3 > s_2$ such that $\varphi_i(x_\eta)[s_3] \downarrow= 0$, then we define $z_\eta = x_\eta$ and undefine x_η at stage s_3. η requests β to open a gap for z_η when A changes below z_η (i.e. $\Gamma^A_\beta(z_\eta)$ is undefined). Note that stage s_3 is also a β-expansionary stage, and at a β-expansionary stage, if no link with top β exists, then we will extend the definitions of Γ^A_β and $\Delta^{W_e}_\beta$.

At the next β-expansionary stage $s_4 > s_3$, if $\Gamma^A_\beta(z_\eta)[s_4] \uparrow$, then β will open a gap for this z_η and create a link between β and η at stage s_4, and let η have outcome g at this stage. If $\Gamma^A_\beta(z_\eta)[s_4] \downarrow$, and η is visited at stage s_4, then η will define x_η as a fresh number again, and let η have outcome w at this stage. At the next η-stage, if $\varphi_i(x_\eta)$ converges to 0, then we will update z_η as this x_η and undefine x_η again.

Suppose that β opens a gap for z_η at some β-expansionary stage v, then we will create a link between β and η at stage v. This gap is open for W_e to change below z_η after stage v.

At the next β-expansionary stage $s > v$, if W_e changes below z_η, i.e. $\Delta^{W_e}_\beta(z_\eta)[s] \uparrow$, then we close this gap successfully, let η perform the diagonalization by putting this z_η into X_β, cancel the link. So we have $X_\beta(z_\eta) = 1 \neq 0 = \varphi_i(z_\eta)$ and hence η is satisfied via this z_η. In this case, $\eta^\frown\langle s\rangle$ will be on the true path, and η will take no further actions after stage s, so (2), (3) are true for ζ.

At the next β-expansionary stage s, if W_e does not change below z_η, i.e. $\Delta^{W_e}_\beta(z_\eta)[s] \downarrow$, then, if A changes below $\varphi_e(z_\eta)$ since the last β-expansionary stage, and we calculate σ_s and find that η is accessible at stage s, then we let $\beta^\frown\langle\infty\rangle$ be accessible at stage s. In this case, this gap is reopened for η when η is accessible at this stage, i.e. let η have outcome d at this stage; if A does not change below $\varphi_e(z_\eta)$ since the last β-expansionary stage or η is not accessible at stage s, then let β close this gap for η

unsuccessfully. Cancel the link (β,η), and let $\beta^\frown\langle\infty\rangle$ be accessible at stage s. Note that this gap can be closed unsuccessfully and reopened infinitely often. And also, it can be closed completely due to a new gap opened by β afterwards.

So now we assume that η can never perform the diagonalization in the construction. If β opens infinitely many different gaps for η (i.e. each gap associated with different value of z_η), then $\eta^\frown\langle g\rangle$ will be on the true path, and η will never initialize the strategies with lower priority in the construction, hence (2), (3) are true for ζ.

If β opens only finitely many different gaps for η and the last gap is closed unsuccessfully and reopened infinitely often, it must be the case that for the last gap, A changes below the corresponding use $\varphi_e(z_\eta)$ infinitely often. Then, $\eta^\frown\langle d\rangle$ will be on the true path, and η will never initialize the strategies with lower priority in the construction, hence (2), (3) are true for ζ.

Otherwise, β opens only finitely many different gaps for η, and for the last gap, after a stage large enough at which this gap is closed unsuccessfully, it can never be reopened again. We claim that it is impossible that η defines x_η infinitely often. Suppose not, then it must be the case that for each number x_η defined by η, $\varphi_i(x_\eta)\downarrow= 0$ and we update z_η infinitely often, and hence η requests β to open gaps infinitely often. As no more gaps can be open after a stage large enough, we will have that A will not change below z_η in the remainder of the construction. This shows that A is computable, which can not be true since we assume that A is incomputable in our theorem. Thus, η defines x_η only finitely often, and for the last number x_η defined by η, we never see $\varphi_i(x_\eta)\downarrow= 0$. So we have that $\eta^\frown\langle w\rangle\subset\sigma$. And η initializes the strategies with lower priority only finitely often in the construction. So (2), (3) are true for ζ.

$\zeta=\tau$ is an $\mathcal{H}_e$-strategy. We have $\tau^\frown\langle\infty\rangle\subset\sigma$ if there are infinitely many τ-expansionary stages; $\tau^\frown\langle f\rangle\subset\sigma$ otherwise. So (2) is true for ζ. Furthermore, ζ never initializes the strategies with lower priority in the construction. So (3) is also true for ζ.

This completes the proof of Lemma 8.

$\square$

Lemma 9. *For any $e \in \omega$, let α be the longest $\mathcal{Q}_e^C$-strategy ($\mathcal{Q}_e^D$-strategy) on the true path σ. Then $\mathcal{Q}_e^C$ ($\mathcal{Q}_e^D$) is satisfied via α.*

Proof. Fix e, let α be the longest $\mathcal{Q}_e^C$-strategy on the true path σ. By lemma 8, α can be initialized at most finitely often and α has a true outcome $\mathcal{O}$ on σ. Let s_0 be the least stage after which α can never be initialized and no strategy to the left of $\alpha^\frown\mathcal{O}$ is visited again.

By our construction, $\mathcal{O} = (j, -)$ for some j, where $- \in \{\infty, f\}$. Let $s_1 \geq s_0$ be the stage at which $x = x_{\alpha,j}$ is defined. Then this x cannot be cancelled later.

Note that, in our construction, when we say that a computation $\Phi_e^{A\oplus D}(x)$ associated with α converges at some stage, this computation is an α-believable computation actually.

If, after stage s_1, there is no stage at which $\Phi_e^{A\oplus D}(x)$ converges to 0, then cycle j can never take action after stage s_1. In this case, α has outcome (j, f) on the true path σ and $C(x) = 0 \neq \Phi_e^{A\oplus D}(x)$. Thus $\mathcal{Q}_e^C$-requirement is satisfied obviously.

As at any α-stage, if $\Phi_e^{A\oplus D}(x)$ converges to 0, then cycle j imposes a restraint to protect the associated computation and waits for j to enter K, and starts $(j+1)$-cycle simultaneously. As we assume that α has outcome $(j, -)$ on σ, we know that either A changes below the corresponding use or j enters K.

If j entering K happens first, then cycle j can act to enumerate x into C and A will not change below the corresponding use afterwards, since otherwise, $(j, -)$, now is (j, f), could not be the final outcome of α on σ. In this case, $C(x) = 1$ and $\Phi_e^{A\oplus D}(x) = 0$. So $C(x) \neq \Phi_e^{A\oplus D}(x)$, and hence $\mathcal{Q}_e^C$-requirement is satisfied via witness x.

Otherwise, cycle j can not act to enumerate x into C forever, then cycle j will return from step 4 to step 2 infinitely often. So $\Phi_e^{A\oplus D}(x)\uparrow$, and hence $\mathcal{Q}_e^C$-requirement is satisfied via witness x.

Note that, we can apply the same argument to show that $\mathcal{Q}_e^D$-requirement is satisfied for any e.

□

Lemma 10. *For any $e \in \omega$, let β be the longest $\mathcal{R}_e^C$-strategy ($\mathcal{R}_e^D$-strategy) on the true* *path σ. Then β, together with its substrategies, satisfies the $\mathcal{R}_e^C$ ($\mathcal{R}_e^D$) requirement.*

Proof. Fix e, let β be the longest $\mathcal{R}_e^C$-strategy on the true path σ. By lemma 8, β can be initialized at most finitely often. Let s_0 be the least stage after which β can never be initialized. If $\beta^\frown\langle f\rangle \subset \sigma$, i.e. there are only finitely many β-expansionary stages, then $W_e \neq \Phi_e^{A\oplus C}$ and hence $\mathcal{R}_e^C$ is satisfied obviously. So we assume that $\beta^\frown\langle\infty\rangle \subset \sigma$. In this case, we construct a c.e. set X_β and two p.c. functionals Γ_β, Δ_β at β-expansionary stages. By our construction, β can have, finitely or infinitely, many substrategies on the true path. There are three cases.

Case 1 If a substrategy η of β has outcome g on the true path.

By lemma 8, η can be initialized at most finitely often, let $s_1 > s_0$ be the least stage after which η can never be initialized. If we assume that $W_e = \Phi_e^{A\oplus C}$, then we can show that Θ_η^A is totally defined and computes W_e correctly.

In this case, z_η is updated infinitely many times and β opens and closes infinitely many different gaps (each gap associates with different value of z_η) for η after stage s_1. Note that no gap can be closed successfully, so W_e never changes below the corresponding z_η before a gap is closed unsuccessfully (otherwise, η will perform the diagonalization by putting z_η into X_β, η is satisfied and has outcome s on the true path).

After a gap is closed unsuccessfully, if W_e changes below the corresponding z_η later, then as when this gap is closed unsuccessfully, we lift the λ-uses of the $\mathcal{H}$-strategies with ∞ outcome between β and η (if needed) to protect C, W_e changes on some $x \leq z_\eta$ must due to the A-change below the corresponding use $\varphi_e(x)$. (*Note that it may happen that, during the cogap for z_η, A-change below the corresponding use $\varphi_e(x)$ happens first which lifts the use $\varphi_e(x)$, and hence some small number is enumerated into C may lead to W_e-change on x. But the A-change here is the essential reason for this W_e-change, and $\Theta_\eta^A(x)$ is undefined by this A-change in the cogap.*) So this A-change can undefine $\Theta_\eta^A(x)$. Thus, when $\Theta_\eta^A(x)$ is defined later, $\Theta_\eta^A(x)$ is defined as 1, equals $W_e(x)$. This implies that Θ_η^A can compute W_e correctly. If we assume that $W_e = \Phi_e^{A\oplus C}$, then $\Phi_e^{A\oplus C}(x)\downarrow$ eventually and has a fixed use $\varphi_e(x)$ for all x and hence Θ_η^A is totally defined (as we always define $\theta_\eta(x) = \varphi_e(x)$). In this case, we have that $\mathcal{R}_e^C$-strategy is satisfied.

Case 2 If a substrategy η of β has outcome d on the true path.

In this case, as g is not the true outcome, we know that β opens only finitely many different gaps for η, and for the last gap, it is reopened by η infinitely often. Suppose that the last gap is associated with a value z of z_η, then A must change below the corresponding use $\varphi_e(z)$ infinitely often and hence $\Phi_e^{A\oplus C}(z) \uparrow$. Thus, η provides a witness showing the $\mathcal{R}_e^C$ requirement is satisfied.

Case 3 Otherwise, no substrategy of β on the true path has outcome g or d.

In this case, there are infinitely many substrategies of β on the true path, and each substrategy has true outcome s or w. We first show that each subrequirement $\mathcal{S}_{e,i}^C$ is satisfied.

Fix η as a substrategy of β on the true path. If η has outcome s on the true path, then in our construction, η must perform the diagonalization by putting z_η into X_β and hence η is satisfied obviously. If η has outcome w on the true path, then as β opens only finitely many gaps for η, we can assume that after a stage large enough, no gap will be open for η. Without loss of generality, suppose that for every number x_η selected by η, $\varphi_i(x_\eta) \downarrow= 0$, and then z_η is updated infinitely often, and hence η requests β to open gaps infinitely often. Since we assume that no more gaps can be open after a stage large enough, we will have that A will not change below z_η in the remainder of the construction. This shows that A is computable, which can not be true. Therefore, there is a number x_η selected by η such that $\varphi_i(x_\eta)$ never converges to 0, and η is satisfied in an obvious way.

Thus, each substrategy of β on the true path is satisfied and hence X_β is incomputable. Assume that $W_e = \Phi_e^{A\oplus C}$, we now show that both Γ_β^A and $\Delta_\beta^{W_e}$ are well-defined and compute X_β correctly.

To see this, note that when a number z is selected as z_η by some substrategy η, η requests β to open a gap for z when A changes below it (i.e. $\Gamma_\beta^A(z)$ is undefined), and this gap is open for W_e to change below z. If this gap is closed successfully, then it must be that both $\Gamma_\beta^A(z)$ and $\Delta_\beta^{W_e}(z)$ are undefined, and z is enumerated into X_β. Thus, when β defines $\Gamma_\beta^A(z)$ and $\Delta_\beta^{W_e}(z)$ later, both of them are defined as 1, equals $X_\beta(z)$. If this gap is closed unsuccessfully, we lift the λ-uses of the $\mathcal{H}$-strategies with ∞ outcome between β and η by putting the λ-uses into D (if needed) to preserve C, and restrain those strategies with priority lower than $\eta^\frown\langle w\rangle$ to enumerate small numbers into C (these strategies are initialized). If this gap is not reopened by η at the same stage (note that, as we assume that no substrategy of β has outcome d on the true path, there is no gap can be closed and reopened infinitely often, and hence it can not happen that there always exists a link at β-expansionary stages, so we will extend the definitions of Γ_β^A and $\Delta_\beta^{W_e}$ at a later β-expansionary stage), then β will define $\Gamma_\beta^A(z) = 0$ with use $\gamma_\beta(z)$ bigger than $\varphi_e(z)$, and $\Delta_\beta^{W_e}(z) = 0$ with use $\delta_\beta(z) = z$ at that stage. Thus, if W_e changes below z (i.e. $\Delta_\beta^{W_e}(z)$ is undefined) in the cogap for z, A must have changes below $\varphi_e(z)$, and hence below $\gamma_\beta(z)$, and so $\Gamma_\beta^A(z)$ is undefined, which allows us to enumerate z into X_β. Since we assume that $W_e = \Phi_e^{A\oplus C}$, a computation $\Phi_e^{A\oplus C}(z)$ will be fixed after a stage large enough, and hence A does not change anymore below the corresponding use $\varphi_e(z)$. This implies that $\gamma_\beta(z)$ is kept the same after this stage, and hence $\Gamma_\beta^A(z)$ is defined.

By our construction, at β-expansionary stages, we will extend the definitions of Γ_β^A and $\Delta_\beta^{W_e}$ if no link with β as top exists. Thus, the definitions of Γ_β^A and $\Delta_\beta^{W_e}$ will be extended infinitely often (as it's impossible that there always exists a link at β-expansionary stages by our assumption). Note that, for any $x \in \omega$, when we define $\Gamma_\beta^A(x)$ at some β-expansionary stage, we define the use $\gamma_\beta(x)$ bigger than $\varphi_e(x)$ at this stage, and hence $\Gamma_\beta^A(x)$ is defined (otherwise, $\Phi_e^{A\oplus C}(x)\uparrow$). Thus, Γ_β^A is well-defined. Moreover, we always define $\delta_\beta(x) = x$ in our construction, thus, $\Delta_\beta^{W_e}$ is also well-defined.

We have seen that, in our construction, when a number z is enumerated into X_β by some substrategy η of β, it must be that the gap for z is closed successfully, i.e. both $\Gamma_\beta^A(z)$ and $\Delta_\beta^{W_e}(z)$ are undefined when z is enumerated into X_β. Thus, when β defines $\Gamma_\beta^A(z)$ and $\Delta_\beta^{W_e}(z)$ later, they are defined as 1, equals $X_\beta(z)$. Therefore, both Γ_β^A and $\Delta_\beta^{W_e}$ compute X_e correctly.

Therefore, in this case, β, together with its substrategies on the true path, satisfies the $\mathcal{R}_e^C$ requirement.

Note that, we can apply the same argument to show that the $\mathcal{R}_e^D$ requirement is satisfied for any e.

$\square$

Lemma 11. *For any $e \in \omega$, let τ be the $\mathcal{H}_e$-strategy on the true path σ. Then $\mathcal{H}_e$ is satisfied via τ.*

Proof. Let τ be the $\mathcal{H}_e$-strategy on the true path σ. Note that Λ is a global p.c. functional built by us through the whole construction. By our construction, if $\tau^\frown\langle f\rangle \subset \sigma$, i.e. there are only finitely many τ-expansionary stages, then $\Lambda^{A\oplus C\oplus D}(e,x)$ is defined as 0 for almost all x and hence $\lim_x \Lambda^{A\oplus C\oplus D}(e,x) = 0$; if $\tau^\frown\langle\infty\rangle \subset \sigma$, i.e. there are infinitely many τ-expansionary stages, then $\Lambda^{A\oplus C\oplus D}(e,x)$ is defined as 1 for almost all x and hence $\lim_x \Lambda^{A\oplus C\oplus D}(e,x) = 1$. Therefore, we have that $Tot(e) = \lim_x \Lambda^{A\oplus C\oplus D}(e,x)$. $\mathcal{H}_e$ is satisfied via τ.

$\square$

This completes the proof of Theorem 2. $\diamond\diamond$

Chapter 4

INFIMA OF N-C.E. DEGREES

4.1 Introduction

Lachlan observed that the infimum of two c.e. degrees considered in the c.e. degrees coincides with the one considered in the Δ_2^0 degrees. It is not true anymore for the n-c.e. degrees ($n > 1$). Kaddah [31] proved that, for all $n > 1$, there are n-c.e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and an (n+1)-c.e. degree $\mathbf{x}$ such that $\mathbf{a}$ is the infimum of $\mathbf{b}, \mathbf{c}$ in the n-c.e. degrees, but not in the (n+1)-c.e. degrees, as $\mathbf{a} < \mathbf{x} < \mathbf{b}, \mathbf{c}$. In this chapter, we extend Kaddah's result by showing that such a structural difference occurs densely in the c.e. degrees.

Theorem 3. *For $n > 1$ and c.e. degrees $\mathbf{u} < \mathbf{v}$, there are n-c.e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and an $(n+1)$-c.e. degree $\mathbf{x}$ between $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{a} < \mathbf{x} < \mathbf{b}, \mathbf{c}$ and $\mathbf{b}$ and $\mathbf{c}$ have infimum $\mathbf{a}$ in the n-c.e. degrees.*

Theorem 3 implies that the isolated $(n+1)$-*c.e.* degrees are dense in the *c.e.* degrees since there is no *n-c.e.* degree between $\mathbf{a}$ and $\mathbf{x}$, which was first proved by LaForte [39].

4.2 Requirements and strategies

To prove Theorem 3, let U be a c.e. set in $\mathbf{u}$, V be a c.e. set in $\mathbf{v}$, So $U <_T V$. We will construct three n-$c.e.$ sets A, B, C, an $(n+1)$-$c.e.$ set X and p.c. functionals Γ_1, Γ_2 and Δ_e for $e \in \omega$ satisfying the following requirements:

$\mathcal{G}$**:** $A, B, C, X \leq_T V$;

$\mathcal{R}$**:** $X = \Gamma_1^{B,A,U} = \Gamma_2^{C,A,U}$;

$\mathcal{P}_e$**:** $X \neq \Phi_e^{A,U}$;

$\mathcal{N}_e$**:** $\Phi_i^{B,A,U} = \Phi_i^{C,A,U} = E_j \Rightarrow E_j = \Delta_e^{A,U}$,

where $e = \langle i, j \rangle$ and $\{E_j : j \in \omega\}$ is some standard listing of all n-$c.e.$ sets.

Let $\mathbf{a}, \mathbf{x}, \mathbf{b}, \mathbf{c}$ be the degrees of $A \oplus U, X \oplus A \oplus U, B \oplus A \oplus U, C \oplus A \oplus U$ respectively. By the requirements $\mathcal{G}$ and $\mathcal{R}$, $\mathbf{u} \leq \mathbf{a}, \mathbf{x}, \mathbf{b}, \mathbf{c} \leq \mathbf{v}$. By the $\mathcal{P}$-requirements, $\mathbf{a} < \mathbf{x}$. By the $\mathcal{N}$-requirements, $\mathbf{a} = \mathbf{b} \wedge \mathbf{c}$ in the n-$c.e.$ degrees. Thus, the above requirements are sufficient to prove Theorem 3. Note that we only require that these degrees are between $\mathbf{u}$ and $\mathbf{v}$ here. Actually, Sacks' density theorem can ensure that these degrees are strictly between $\mathbf{u}$ and $\mathbf{v}$.

4.2.1 The $\mathcal{G}$ and $\mathcal{R}$-strategies

We satisfy $\mathcal{G}$ by applying the delayed permission argument as that in the Sacks' density theorem.

For the $\mathcal{R}$-requirement, we will build the p.c. functionals Γ_1, Γ_2 by the following rules:

(1) When $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ are first defined at a stage s, we define $\gamma_1(x)$ and $\gamma_2(x)$ as fresh numbers.

(2) If x enters or exits X at a stage t after $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ are defined, then $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ must be undefined to ensure that they are defined agreeing with $X(x)$ eventually. In the construction, we put or, extract a number less than or equal to $\gamma_i(x)[t]$ from $B \cup A$ or $C \cup A$, $i = 1, 2$.

(3) If some $y \leq \gamma_1(x)$ enters $B \cup A$ after $\gamma_1(x)$ is defined, then we will lift $\gamma_1(x)$ to a bigger number; but if later y leaves $B \cup A$, then we will redefine $\gamma_1(x)$ to be the previous value. The definition of $\gamma_2(x)$ is similar.

4.2.2 A $\mathcal{P}$ strategy

A $\mathcal{P}_e$-strategy α is a Sacks coding strategy. That is, even though U is not in our control, we can still satisfy the $\mathcal{P}_e$-requirement by the assumption that $U <_T V$. α will run cycles j for $j \in \omega$, all cycles of α define a functional Ψ_α jointly. α starts cycle 0 first. Each cycle j can start only cycle $j+1$, but can stop or cancel any cycle j' with $j' > j$. Each cycle j tries to find a number x_j such that $X(x_j) \neq \Phi_e^{A \oplus U}(x_j)$, and if cycle j fails to make it, then this cycle will define $\Psi_\alpha^U(j) = V(j)$ successfully. If we fail to satisfy a $\mathcal{P}_e$-strategy, then we will threaten $V = \Psi_\alpha^U$ to get a contradiction.

Cycle j proceeds as follows:

(1) Choose x_j as a fresh number.

(2) Wait for a stage s_0 such that $\Phi_e^{A,U}(x_j)[s_0] \downarrow= 0$.

(3) Preserve $A \upharpoonright \varphi_e(x_j)[s_0]$, and define $\Psi_\alpha^U(j)[s_0] = V(j)[s_0]$ with use $\psi_\alpha(j) = \varphi_e(x_j)[s_0]$. Start cycle $j+1$ simultaneously.

(4) Wait for $U \upharpoonright \varphi_e(x_j)[s_0]$ or $V(j)$ to change.

 (a) If $U \upharpoonright \varphi_e(x_j)[s_0]$ changes first, then cancel all cycles $j' > j$ and drop the A-restraint of cycle j to 0. Go back to step 2.

 (b) If $V(j)$ changes first, then stop cycles $j' > j$, and go to step 5.

(5) Put x_j into X and wait for $U \upharpoonright \varphi_e(x_j)[s_0]$ to change.

(6) Define $\Psi_\alpha^U(j) = V(j) = 1$ with use 0, and start cycle $j+1$.

α has two sorts of outcomes:

(j, f) : Some cycle j waits forever at step 2 or 5.

> (*If* α *waits forever at step 2 then* $\Phi_e^{A,U}(x_j) \downarrow= 0$ *is not true. If* α *waits forever at step 5 then* $\Phi_e^{A,U}(x_j) \downarrow= 0$ *and* $x_j \in X$*. In any case, the requirement* $\mathcal{P}_e$ *is satisfied via witness* x_j*.*)

(j, ∞) : Some (least) cycle j runs infinitely often.

> (*It must be true that cycle* j *returns from step 4 to 2 infinitely often. Thus* $\Phi_e^{A,U}(x_j)$ *diverges. So the requirement* $\mathcal{P}_e$ *is satisfied via* x_j*.*)

By the $\mathcal{P}$-strategy, whenever a cycle j is started, any previous version of it has already been cancelled by a U-change. So Ψ_α^U is well-defined. Note that if there are stages s_j for all $j \in \omega$ such that no cycle j runs after stage s_j but there are infinitely many stages at which some cycle runs, then for each j, cycle j will stop at step 4 or step 6 eventually, and hence all these cycles will define Ψ_α^U totally and $\Psi_\alpha^U = V$, i.e. $V \leq_T U$, a contradiction.

Now, we consider the interaction between strategies $\mathcal{P}$ and $\mathcal{G}$ and $\mathcal{R}$. When α wants to put a number x_j into X at step 5, if $\Gamma_1^{B,A,U}(x_j)$ and $\Gamma_2^{C,A,U}(x_j)$ are defined then we must make them undefined. For this purpose, we will put $\gamma_1(x_j)$ into B and $\gamma_2(x_j)$ into C. But for $\mathcal{G}$ requirement, it requires $B, C, X \leq_T V$, so we have to get a V-permission to do this. In the construction the V-permission is realized via a $V(j)$-change (at step 4(b)). So it's enough to modify step 5 to be step $5'$:

($5'$) Put x_j into X, $\gamma_1(x_j)$ into B and $\gamma_2(x_j)$ into C, and wait for $U \upharpoonright \varphi_e(x_j)[s_0]$ to change.

For the convenience of description, we introduce some notions here. Say that *cycle* j *acts* if it chooses a fresh number x_j as its attacker at step 1 or, it changes the value of $X(x_j)$ by enumerating x_j into X at step $5'$. Say that *cycle* j *is active* at stage s if at this stage, when α is visited, α is running cycle j, except the situation that cycle j is just started at stage s.

4.2.3 An $\mathcal{N}$ strategy

Assume that a strategy β works on $\mathcal{N}_e$ with $e = \langle i, j\rangle$, we define the length of agreement function $l(\beta, s)$ at stage s to be

$$l(\beta, s) = \max\{x < s : \forall y < x(\Phi_i^{B,A,U}(y)[s] \downarrow= \Phi_i^{C,A,U}(y)[s] \downarrow= E_j(y)[s])\},$$

and the maximum length of agreement function at stage s is defined to be

$$m(\beta, s) = \max\{l(\beta, t) : t < s \ \&\ t \text{ is a } \beta\text{-stage}\}.$$

Say a stage s is β-expansionary if $s = 0$ or s is a β-stage with $l(\beta, s) > m(\beta, s)$.

β is to build a p.c. functional Δ_β such that if $\Phi_i^{B,A,U} = \Phi_i^{C,A,U} = E_j$ then $\Delta_\beta^{A,U}$ is totally defined and computes E_j correctly. For simplicity, we will omit the subscripts below. If we do not consider the U-change nor the V-permission here, then the basic strategy for this requirement is exactly the same as the one introduced by Kaddah [31]. That is, for some z, after $\Delta^{A,U}(z)$ was defined, the computations $\Phi^{B,A,U}(z)$ and $\Phi^{C,A,U}(z)$ are allowed to be destroyed simultaneously (by $\mathcal{P}$-strategies) instead of preserving one of them as in Lachlan's minimal pair construction. Since B, C are constructed to be n-c.e. by us, we can remove numbers from one of them to recover a computation to a previous one, and force a disagreement between E and $\Phi^{B,A,U}$ or E and $\Phi^{C,A,U}$ at argument z. The basic strategy is as follows:

(1) At some β-expansionary stage $s(0)$, $\Phi^{B,A,U}(z)[s(0)] = \Phi^{C,A,U}(z)[s(0)] = E_{s(0)}(z) = 0$. Define $\Delta^{A,U}(z)[s(0)] = E_{s(0)}(z)$ with $\delta(z) = s(0)$. (We assume that $E_{s(0)}(z) = 0$ here, mainly because it is the most complicated situation.)

(2) Some $\mathcal{P}_e$-strategy with lower priority enumerates a small number x into X_{s_0} at a stage $s_0 \geq s(0)$, for the global requirement $\mathcal{R}$, we put $\gamma_1(x)[s_0]$ into B_{s_0} and $\gamma_2(x)[s_0]$ into C_{s_0} simultaneously. The computations $\Phi^{B,A,U}(z)[s(0)]$ and $\Phi^{C,A,U}(z)[s(0)]$ could be destroyed by these enumerations.

(3) At stage $s_1 > s_0$, z enters E_{s_1}.

(4) At some β-expansionary stage $s(1) > s_1$, $\Phi^{B,A,U}(z)[s(1)] = \Phi^{C,A,U}(z)[s(1)] = E_{s(1)}(z) = 1$. We force that $E(z) \neq \Phi^{B,A,U}(z)$ by extracting $\gamma_1(x)[s_0]$ from $B_{s(1)}$ to recover the computation $\Phi^{B,A,U}(z)[s(0)] = 0$. To ensure that $\Gamma_1^{B,A,U}$ and $\Gamma_2^{C,A,U}$ are well-defined and compute X correctly, we extract x from X, and put s_0 into A. We put s_0, instead of the current $\gamma_2(x)[s(1)]$, into A to undefine the new $\Gamma_2^{C,A,U}(x)$, mainly because we need to consider the consistency between β's action and those higher $\mathcal{N}$-strategies. (*Step 1 of the disagreement strategy*)

(5) At stage $s_2 > s(1)$, z goes out of E_{s_2}.

(6) At some β-expansionary stage $s(2) > s_2$, $\Phi^{B,A,U}(z)[s(2)] = \Phi^{C,A,U}(z)[s(2)] = E_{s(2)}(z) = 0$ again. Now, we force that $\Phi^{C,A,U}(z) \neq E(z)$ by extracting s_0 from A to recover the computation $\Phi^{C,A,U}(z)[s(1)] = 1$. At the same time, we enumerate x into X, and put $\gamma_1(x)[s_0]$ into $B_{s(2)}$ again to correct $\Gamma_1^{B,A,U}(x)$. (*Step 2 of the disagreement strategy*)

(7) At stage $s_3 > s(2)$, z enters E_{s_3} again.

(8) At some β-expansionary stage $s(3) > s_3$, $\Phi^{B,A,U}(z)[s(3)] = \Phi^{C,A,U}(z)[s(3)]$ $= E_{s(3)}(z) = 1$. We force that $E(z) \neq \Phi^{C,A,U}(z)$ by putting s_0 into A (to recover the computation $\Phi^{C,A,U}(z)[s(2)] = 0$) and extracting x from X. Note that the action of putting s_0 into A can undefine both $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$, so this action can make sure that $\Gamma_1^{B,A,U}$ and $\Gamma_2^{C,A,U}$ are well-defined and compute X correctly. (*Step 3 of the disagreement strategy*)

(9) At stage $s_4 > s(3)$, z goes out of E_{s_4}.

(10) At some β-expansionary stage $s(4) > s_4$, $\Phi^{B,A,U}(z)[s(4)] = \Phi^{C,A,U}(z)[s(4)]$ $= E_{s(4)}(z) = 0$. We force that $E(z) \neq \Phi^{C,A,U}(z)$ by extracting s_0 from A (to recover the computation $\Phi^{C,A,U}(z)[s(1)] = 1$) and putting x into X. Note that the action of extracting s_0 from A can make sure that $\Gamma_1^{B,A,U}$ and $\Gamma_2^{C,A,U}$ are well-defined and compute X correctly. (*Step 4 of the disagreement strategy*)

Since E is n-c.e., the membership of z in E can change at most n many times. Thus we assume that $E_{s(0)}(z) \neq E_{s(1)}(z), \cdots, E_{s(k)}(z) \neq E_{s(k+1)}(z)$ for $0 \leq k \leq$

$n-1$ (the most complicated situation) and $s(0) < s(1) < \cdots < s(k) < \cdots$ are the consecutive β-expansionary stages. β will continue to proceed as follows:

$k > 4$ odd: At some β-expansionary stage $s(k) > s_k$, $\Phi^{B,A,U}(z)[s(k)] = \Phi^{C,A,U}(z)[s(k)] = E_{s(k)}(z) = 1$. We force that $E(z) \neq \Phi^{C,A,U}(z)$ by putting s_0 into A (to recover the computation $\Phi^{C,A,U}(z)[s(2)] = 0$) and extracting x from X. Note that the action of putting s_0 into A can undefine both $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$, so this action can make sure that $\Gamma_1^{B,A,U}$ and $\Gamma_2^{C,A,U}$ are well-defined and compute X correctly. (*Step $k > 4$ odd of the disagreement strategy*)

$k > 4$ even: At some β-expansionary stage $s(k) > s_k$, $\Phi^{B,A,U}(z)[s(k)] = \Phi^{C,A,U}(z)[s(k)] = E_{s(k)}(z) = 0$. We force that $E(z) \neq \Phi^{C,A,U}(z)$ by extracting s_0 from A (to recover the computation $\Phi^{C,A,U}(z)[s(1)] = 1$) and putting x into X. Note that the action of extracting s_0 from A can make sure that $\Gamma_1^{B,A,U}$ and $\Gamma_2^{C,A,U}$ are well-defined and compute X correctly. (*Step $k > 4$ even of the disagreement strategy*)

If the membership of z in E changes n many times and so β executes all the n steps of the disagreement strategy described above, then β has forced a disagreement since E is n-c.e. and thus there will be no more β-expansionary stages (U-change is not considered now, here we assume that there is no U-change during the above process). Note that the set B as built here is 3-c.e. and C is *c.e.*, but A in general is n-c.e..

Now we consider the U-change and V-permission involved in such a process. Note that, when β executes the disagreement strategy as above, we always assume that U does not change below the corresponding use otherwise we may not be able to recover the desired computations. For example, the step 1 of the disagreement strategy in (4) stated above assumes that $U \upharpoonright \varphi(B, A, U; z)[s(0)]$ does not change after stage s_0, otherwise the computation $\Phi^{B,A,U}(z)[s(0)]$ can not be recovered. If $U \upharpoonright \varphi(B, A, U; z)[s(0)]$ changes, then it can undefine $\Delta^{A,U}(z)[s(0)]$, and hence this U-change allows us to correct $\Delta^{A,U}(z)[s(0)]$. In particular, if this U-change happens after we performing the step 1 of the disagreement strategy in (4), then we will deal with this by threatening $V \leq_T U$ via a functional. That is, we will make infinitely many attempts to satisfy this $\mathcal{N}$-strategy as above by an infinite sequence of cycles. Note that, for $\mathcal{G}$-requirement, when we perform the disagreement strategy, we must obtain the V-permission. The formal description of our idea is given as follows.

To combine all the n steps of the disagreement strategy, we need n many V-permissions, and as in [42], we arrange the basic $\mathcal{N}$-strategy in $\overbrace{\omega \times \cdots \times \omega}^{n}$ many cycles $(k_1, \cdots, k_n)$ where $k_i \in \omega, 1 \leq i \leq n$. Their priority is arranged by the lexicographical ordering. $(0, \cdots, 0)$-cycle starts first, and each $(k_1, \cdots, k_n)$-cycle can start cycles $(k_1, \cdots, k_i+1, 0, \cdots, 0)$ $(1 \leq i \leq n)$ and stop, or cancel cycles $(\hat{k_1}, \cdots, \hat{k_n})$ for $(k_1, \cdots, k_n) < (\hat{k_1}, \cdots, \hat{k_n})$. Each $(k_1, \cdots, k_n)$-cycle defines it's version of $\Delta^{A,U}$. For any $z \in \omega$, once $\Delta^{A,U}(z)$ is defined with use $\delta(z)$, if $U \upharpoonright \delta(z)$ changes later, and if we do not undefine $\Delta^{A,U}(z)$ explicitly in the construction, then it is redefined with the same use as it was last defined. That is, if $U \upharpoonright \delta(z)$ changes, $\Delta^{A,U}(z)$ becomes undefined due to this U-change if and only if we undefine $\Delta^{A,U}(z)$ explicitly in the construction.

$(k_1, \cdots, k_n)$-cycle runs as follows:

(1) Wait for a β-expansionary stage, s say.

(2) If $\Delta^{A,U}(z)[s] = E_s(z)$ for each z with $\Delta^{A,U}(z)[s] \downarrow$, then go to (3). Otherwise, let z be the least one such that $\Delta^{A,U}(z)[s] \downarrow \neq E_s(z)$, and then go to (4).

(3) For all $z < l(\beta, s)$, if $\Delta^{A,U}(z)[s] \uparrow$, then define $\Delta^{A,U}(z)[s] = E_s(z)$ with $\delta(z) = s$, then go back to (1).

(4) Assume that $0 = \Delta^{A,U}(z)[s] \downarrow \neq E_s(z) = 1$ (without loss of generality, we assume that z enters E_s here), then set $\Theta^U_{(k_1,k_2,\cdots,k_{n-1})}(k_n) = V_s(k_n)$ with $\theta_{(k_1,k_2,\cdots,k_{n-1})}(k_n)[s] = \delta(z)[s]$ $(\delta(z)[s] < s)$. Start cycle $(k_1, k_2, \cdots, k_{n-1}, k_n + 1)$ simultaneously. Go to (5).

(5) Wait for $U \upharpoonright \theta_{(k_1,k_2,\cdots,k_{n-1})}(k_n)[s]$ or $V(k_n)$ to change.

   (a) If $U \upharpoonright \theta_{(k_1,k_2,\cdots,k_{n-1})}(k_n)[s]$ changes first then cancel cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$, drop the A, B, C-restraints of $(k_1, \cdots, k_n)$-cycle to 0, undefine $\Delta^{A,U}(z)[s]$ and go back to (1).

(b) If $V(k_n)$ changes first at stage $t > s$ then we *perform the step 1 of the disagreement strategy*, i.e. remove $\gamma_1(x)[s_0]$ from B and x from X_t, put s_0 into A, where $s_0 < s$ is the stage at which x is enumerated into X. Now stop cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$ and go back to (1). If there is a new β-expansionary stage and $U \upharpoonright \delta(z)[s]$ does not change by this new stage, then it must be that z goes out of E again. Go to (6). If there is a new β-expansionary stage but $U \upharpoonright \delta(z)[s]$ changes by this new stage, then this U-change can cancel cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$ and undefine $\Delta^{A,U}(z)[s]$, so drop the A, B, C-restraints of $(k_1, \cdots, k_n)$-cycle to 0 and go to (2). *Note that here we select to run cycle* $(k_1, \cdots, k_n)$ *again, since* $\Theta^U_{(k_1,k_2,\cdots,k_{n-1})}(k_n)$ *is also undefined by such a* U*-change, we can define it later when needed. Furthermore, such a* U*-change gives a chance to correct* $\Delta^{A,U}(z)[s]$.

(6) Set $\Theta^U_{(k_1,k_2,\cdots,k_{n-2})}(k_{n-1}) = V_s(k_{n-1})$ with $\theta_{(k_1,k_2,\cdots,k_{n-2})}(k_{n-1})[s] = \delta(z)[s]$ $(\delta(z)[s] < s)$. Start cycle $(k_1, k_2, \cdots, k_{n-1}+1, 0)$ simultaneously. Go to (7).

(7) Wait for $U \upharpoonright \theta_{(k_1,k_2,\cdots,k_{n-2})}(k_{n-1})[s]$ or $V(k_{n-1})$ to change.

(a) If $U \upharpoonright \theta_{(k_1,k_2,\cdots,k_{n-2})}(k_{n-1})[s]$ changes first then cancel cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$, drop the A, B, C-restraints of $(k_1, \cdots, k_n)$-cycle to 0, undefine $\Delta^{A,U}(z)[s]$ and go back to (1).

(b) If $V(k_{n-1})$ changes first at stage $t > s$, then we *perform the step 2 of the disagreement strategy*, i.e. remove s_0 from A and, put x into X and $\gamma_1(x)[s_0]$ into B, where $s_0 < s$ is the stage at which x is enumerated into X. Now stop cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$ and go back to (1). If there is a new β-expansionary stage and $U \upharpoonright \delta(z)[s]$ does not change by this new stage, then it must be that z enters E again. Go to (8), taking $l = 3$ in (8). If there is a new β-expansionary stage but $U \upharpoonright \delta(z)[s]$ changes by this new stage, then this U-change can cancel cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$ and undefine $\Delta^{A,U}(z)[s]$, so drop the A, B, C-restraints of $(k_1, \cdots, k_n)$-cycle to 0 and go to (2). *Note that here we select to run cycle* $(k_1, \cdots, k_n)$*, since*

$\Theta^U_{(k_1,k_2,\cdots,k_{n-2})}(k_{n-1})$ is now undefined, we can define it later when needed. Furthermore, such a U-change gives a chance to correct $\Delta^{A,U}(z)[s]$.

(8) Set $\Theta^U_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1}) = V_s(k_{n-l+1})$ with $\theta_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1})[s] = \delta(z)[s]$ ($\delta(z)[s] < s$). Start cycle $(k_1, k_2, \cdots, k_{n-l}, k_{n-l+1}+1, 0, \cdots, 0)$ simultaneously. Go to (9). (Here, l is an *odd* number with $1 < l \leq n$.)

Note that when $l = n$, $\Theta^U_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1})$ is denoted as $\Theta^U(k_1)$.

(9) Wait for $U \upharpoonright \theta_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1})[s]$ or $V(k_{n-l+1})$ to change.

(a) If $U \upharpoonright \theta_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1})[s]$ changes first then cancel cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$, drop the A, B, C-restraints of $(k_1, \cdots, k_n)$-cycle to 0, undefine $\Delta^{A,U}(z)[s]$ and go back to (1).

(b) If $V(k_{n-l+1})$ changes first at stage $t > s$ then we *perform the step $l > 1$ odd of the disagreement strategy*, i.e. put s_0 into A and remove x from X_t, where $s_0 < s$ is the stage at which x is enumerated into X. Now stop cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$ and go back to (1) (*note that if $l = n$, and if $U \upharpoonright \delta(z)[s]$ does not change, then there will never be any new expansionary stages appear, i.e. if a new expansionary stage appears then it must be that $U \upharpoonright \delta(z)[s]$ changes*). If there is a new β-expansionary stage and $U \upharpoonright \delta(z)[s]$ does not change by this new stage, then it must be that z goes out of E again. Go to (10), taking $l = 4$ in (10). If there is a new β-expansionary stage but $U \upharpoonright \delta(z)[s]$ changes by this new stage, then this U-change can cancel cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$ and undefine $\Delta^{A,U}(z)[s]$, so drop the A, B, C-restraints of $(k_1, \cdots, k_n)$-cycle to 0 and go to (2). *Note that we select to run cycle $(k_1, \cdots, k_n)$ here, as $\Theta^U_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1})$ is now undefined, so we can define it later when needed. Furthermore, such a U-change gives a chance to correct $\Delta^{A,U}(z)[s]$.*

(10) Set $\Theta^U_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1}) = V_s(k_{n-l+1})$ with $\theta_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1})[s] = \delta(z)[s]$ ($\delta(z)[s] < s$). Start cycle $(k_1, k_2, \cdots, k_{n-l}, k_{n-l+1}+1, 0, \cdots, 0)$ simultaneously. Go to (11). (Here, l is an *even* number with $1 < l \leq n$.)

Note that when $l = n$, $\Theta^U_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1})$ is denoted as $\Theta^U(k_1)$.

(11) Wait for $U \upharpoonright \theta_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1})[s]$ or $V(k_{n-l+1})$ to change.

(a) If $U \upharpoonright \theta_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1})[s]$ changes first then cancel cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$, drop the A, B, C-restraints of $(k_1, \cdots, k_n)$-cycle to 0, undefine $\Delta^{A,U}(z)[s]$ and go back to (1).

(b) If $V(k_{n-l+1})$ changes first at stage $t > s$ then we *perform the step* $l > 1$ *even of the disagreement strategy*, i.e. remove s_0 from A and put x into X_t, where $s_0 < s$ is the stage at which x is enumerated into X. Now stop cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$ and go back to (1) (*note that if* $l = n$, *and if* $U \upharpoonright \delta(z)[s]$ *does not change, then there will never be any new expansionary stages, i.e. if a new expansionary stage appears then it must be that* $U \upharpoonright \delta(z)[s]$ *changes*). If there is a new β-expansionary stage and $U \upharpoonright \delta(z)[s]$ does not change by this new stage, then it must be that z enters E again. Go to (8), taking $l = 5$ in (8). (*Note that, we may perform the step* $l > 1$ *of the disagreement strategy for* l *is odd and even, alternatively as the case that* $l = 3$ *and* $l = 4$, *until* $l = n$.) If there is a new β-expansionary stage but $U \upharpoonright \delta(z)[s]$ changes by this new stage, then this U-change can cancel cycles $(\hat{k_1}, \cdots, \hat{k_n}) > (k_1, \cdots, k_n)$ and undefine $\Delta^{A,U}(z)[s]$, so drop the A, B, C-restraints of $(k_1, \cdots, k_n)$-cycle to 0 and go to (2). *Note that we select to run cycle* $(k_1, \cdots, k_n)$ *here, as* $\Theta^U_{(k_1,k_2,\cdots,k_{n-l})}(k_{n-l+1})$ *is now undefined, so we can define it later when needed. Furthermore, such a* U*-change gives a chance to correct* $\Delta^{A,U}(z)[s]$.

β has two outcomes:

$(k_1, \cdots, k_n)$: Some (least) cycle $(k_1, \cdots, k_n)$ runs infinitely often, then the corresponding $\Delta^{A,U}$ is totally defined and computes E correctly.

(f): Some cycle waits at step (1) forever, i.e. there are only finitely many β-expansionary stages, then a disagreement appears between $\Phi^{B,A,U}$ and E, or between $\Phi^{C,A,U}$ and E.

β also has two pseudo-outcomes:

- There is an $1 \leq i \leq n-1$ such that, for fixed $k_1, \cdots, k_i$, and for every $x \in \omega$ there is a stage s_x such that no cycle of the form $(k_1, \cdots, k_i, x, k_{i+2}, \cdots, k_n)$ acts after stage s_x for some $k_{i+2}, \cdots, k_n$. Then for every $x \in \omega$, such cycle $(k_1, \cdots, k_i, x, k_{i+2}, \cdots, k_n)$ gets stuck for waiting for U or V to change (i.e. each such cycle runs only finitely often). So $\Theta^U_{(k_1,k_2,\cdots,k_i)}(x) = V(x)$ for all $x \in \omega$, i.e. $\Theta^U_{(k_1,k_2,\cdots,k_i)}$ computes V correctly, a contradiction.

- For every x, there is a stage s_x such that no cycle of the form $(x, k_2, \cdots, k_n)$ acts after stage s_x for some $k_2, \cdots, k_n$. Then for every $x \in \omega$, such cycle $(x, k_2, \cdots, k_n)$ gets stuck for waiting for U or V to change (i.e. each such cycle runs only finitely often). So $\Theta^U(x) = V(x)$ for all $x \in \omega$, i.e. Θ^U computes V correctly, a contradiction.

For the convenience of description, we introduce some notion here. Say that *cycle* $(k_1, \cdots, k_n)$ *is active* at stage s if at this stage, cycle $(k_1, \cdots, k_n)$ is started or running and it is not cancelled by smaller cycles (by U-change).

4.3 Construction

We first arrange the priority ranking of the requirements as follows:

$$\mathcal{G} < \mathcal{R} < \mathcal{N}_0 < \mathcal{P}_0 < \mathcal{N}_1 < \mathcal{P}_1 < \cdots < \mathcal{N}_n < \mathcal{P}_n < \cdots,$$

where $\mathcal{X} < \mathcal{Y}$ means that $\mathcal{X}$ has higher priority than $\mathcal{Y}$.

Each $\mathcal{N}$-strategy has outcomes $(k_1, \cdots, k_n)$, $k_1, \cdots, k_n \in \omega$, and f in order type $\omega^n + 1$. Their priority is arranged as follows: the priority of cycles $(k_1, \cdots, k_n)$, $k_1, \cdots, k_n \in \omega$, is ordered by the lexicographical ordering, and f has the lowest priority.

Each $\mathcal{P}$-strategy has outcomes (j, f), (j, ∞) for $j \in \omega$ in order type ω. Their priority is arranged as follows:

$$(0,\infty) <_L (0,f) <_L \cdots <_L (n,\infty) <_L (n,f) < \cdots .$$

The requirements $\mathcal{G}$ and $\mathcal{R}$ are both global, and hence we will not put them on the priority tree. The priority tree, T, is constructed recursively by the outcomes of the strategies corresponding to the requirements.

The full construction is given as follows. The construction will proceed by stages. At odd stages, we define Γ_i's; at even stages, we approximate the true path.

Full Construction

Stage 0: Initialize all nodes on T, and let $A_0 = B_0 = C_0 = X_0 = \emptyset$.

Stage $s = 2n+1 > 0$: Define $\Gamma_1^{B,A,U}(x) = \Gamma_2^{C,A,U}(x) = X_s(x)$ for the least $x < s$ for which Γ's are not defined, with uses $\gamma_1(x)[s] = s$ and $\gamma_2(x)[s] = s$.

Stage $s = 2n > 0$: (finding σ_s)

Substage 0: Let $\sigma_s(0) = \lambda$ the root node.

Substage t**:** Given $\xi = \sigma_s \upharpoonright t$. First initialize all the nodes $>_L \xi$. If $t = s$ then define $\sigma_s = \xi$ and initialize all the nodes with lower priority than σ_s.

If $t < s$, take action for ξ and define $\sigma_s(t)$ depending on which requirement ξ works for.

Case 1 $\xi = \alpha$ is a $\mathcal{P}$-strategy. There are three subcases.

Subcase ($\alpha 1$**)** If α has no cycle started, then start cycle 0 and choose a fresh number $x_{\xi,0}$ as its attacker. Define $\sigma_s = \xi^\frown(0,f)$, initialize all nodes with priority lower than σ_s.

Subcase ($\alpha 2$**)** If $\alpha 1$ fails, let j be the largest active cycle at the last ξ-stage.

(α2.1) If U has a change below the restraint of some (least) cycle $j' \le j$, then define $\sigma_s(t)$ as $\alpha^\frown(j',\infty)$ if cycle j' has not received the j'-permission so far; and define $\sigma_s(t)$ as $\alpha^\frown(j'+1, f)$ if cycle j' received the j'-permission before. In the former case, go to the next substage. In the latter case, redefine $\psi_\alpha(j')$ as $\psi_\alpha(j'-1)[s]$ since j' is in V_s and define $x_{\alpha,j'+1}$ as a fresh number, now define $\sigma_s = \alpha^\frown(j'+1, f)$, initialize all the nodes with priority lower than σ_s and go to the next stage.

(α2.2) If α2.1 fails and V has a change on some (least) number $j' \leq j$ between the last α-stage and stage $s+1$, then let cycle j' act at this stage. So j'-cycle will enumerate $x_{\alpha,j'}$ into X and $\gamma_1(x_{\alpha,j'})$ into B and $\gamma_2(x_{\alpha,j'})$ into C. Say that j'-cycle of α receives j'-permission at stage $s+1$. Initialize all the nodes with priority lower than or equal to $\alpha^\frown(j', f)$ and go to the next stage. Define $\sigma_s = \alpha^\frown(j', f)$. We say that α is satisfied via j'-cycle till U changes below the corresponding use.

Subcase (α3) If neither subcases α1 nor α2 is true, then take actions as follows:

If α is at j-cycle, not satisfied yet, and $\Phi^{A,U}_{e(\alpha)}(x_{\alpha,j}) \downarrow = 0$, then define $\Psi^U_\alpha(j) = V(j)$. The use $\psi_\alpha(j)$ is defined as $\varphi_{e(\alpha)}(x_{\alpha,j})[s]$ if $j \notin V_s$, and defined as $\psi_\alpha(j-1)[s]$ if $j \in V_s$. And then initialize all the nodes with priority lower than $\alpha^\frown(j, f)$. Start $(j+1)$-cycle by choosing a fresh number $x_{\alpha,j+1}$ as its attacker and define $\sigma_s = \alpha^\frown(j+1, f)$.

Otherwise, define $\sigma_s(t)$ as $\alpha^\frown(j, f)$ and go to the next substage.

Case 2 $\xi = \beta$ is an $\mathcal{N}$-strategy. Suppose cycle $(k_1, \cdots, k_n)$ is the largest cycle (in lexicographical ordering) active at stage s.

Subcase (β1) If there is a cycle $(k'_1, \cdots, k'_n) \leq (k_1, \cdots, k_n)$ such that either U changes below the corresponding use, or V changes on the corresponding number since the last β-stage, then find the least such $(k'_1, \cdots, k'_n)$.

(β1.1) If U changes below the corresponding use of cycle $(k'_1, \cdots, k'_n)$, then define $\sigma_s(t)$ as $\beta^\frown(k'_1, \cdots, k'_n)$, and go to the next substage. Note that all cycles bigger than $(k'_1, \cdots, k'_n)$ become inactive because of this U-change.

(β1.2) If V changes on the corresponding number of cycle $(k'_1, \cdots, k'_n)$, then, say V changes on k'_i $(1 \leq i \leq n)$.

If $i = n$ for $(k'_1, \cdots, k'_i, \cdots, k'_n)$, then we remove the corresponding $\gamma_1(x)[s_0]$ from B and x from X, put s_0 into A, where $s_0 < s$ is the stage at which x is first enumerated into X. Now stop cycles $> (k'_1, \cdots, k'_i, \cdots, k'_n)$, initialize all the nodes with priority lower than $\beta^\frown(k'_1, \cdots, k'_i, \cdots, k'_n)$. Define σ_s as $\beta^\frown(k'_1, \cdots, k'_i, \cdots, k'_n)$ and go to the next stage.

If $i = n - 1$ for $(k'_1, \cdots, k'_i, \cdots, k'_n)$, then we remove s_0 from A and, put x into X and $\gamma_1(x)[s_0]$ into B, where $s_0 < s$ is the stage at which x is first enumerated into X. Now stop cycles $> (k'_1, \cdots, k'_i, \cdots, k'_n)$, initialize all the nodes with priority lower than $\beta^\frown(k'_1, \cdots, k'_i, \cdots, k'_n)$. Define σ_s as $\beta^\frown(k'_1, \cdots, k'_i, \cdots, k'_n)$ and go to the next stage.

If $i = n - l + 1$ for $(k'_1, \cdots, k'_i, \cdots, k'_n)$ and $l > 1$ is odd $(l \leq n)$, then we just put s_0 into A and remove x from X, where $s_0 < s$ is the stage at which x is first enumerated into X. Now stop cycles $> (k'_1, \cdots, k'_i, \cdots, k'_n)$, initialize all the nodes with priority lower than $\beta^\frown(k'_1, \cdots, k'_i, \cdots, k'_n)$. Define σ_s as $\beta^\frown(k'_1, \cdots, k'_i, \cdots, k'_n)$ and go to the next stage.

If $i = n - l + 1$ for $(k'_1, \cdots, k'_i, \cdots, k'_n)$ and $l > 2$ is even $(l \leq n)$, then we just remove s_0 from A and put x into X, where $s_0 < s$ is the stage at which x is first enumerated into X. Now stop cycles $> (k'_1, \cdots, k'_i, \cdots, k'_n)$, initialize all the nodes with priority lower than $\beta^\frown(k'_1, \cdots, k'_i, \cdots, k'_n)$. Define σ_s as $\beta^\frown(k'_1, \cdots, k'_i, \cdots, k'_n)$ and go to the next stage.

Subcase (β2) If β1 does not hold, then we work on cycle $(k_1, \cdots, k_n)$. Check whether s is β-expansionary.

If s is not β-expansionary, then we define $\sigma_s(t) = \beta^\frown f$ and go to the next substage.

If s is β-expansionary, then we check whether there is some argument z with

$\Delta^{A,U}(z)[s] \downarrow$ (defined in cycle $(k_1, \cdots, k_n)$) such that cycle $(k_1, \cdots, k_n)$ performed the disagreement strategy for argument z upon the $V(k_i)$-change for some $1 < i \le n$ at the last β-stage. (Note that $\Delta^{A,U}(z)[s] \downarrow$ implies that U does not change below the corresponding use since the last β-stage.)

If yes, then define functional $\Theta^U_{(k_1,\cdots,k_{i-2})}(k_{i-1}) = V_s(k_{i-1})$ with

$\theta_{(k_1,\cdots,k_{i-2})}(k_{i-1})[s] = \delta(z)[s]$ ($\delta(z)[s] < s$). Let cycle $(k_1, \cdots, k_{i-2}, k_{i-1} + 1, 0, \cdots, 0)$ be active. Define σ_s as $\beta^\frown(k_1, \cdots, k_{i-2}, k_{i-1} + 1, 0, \cdots, 0)$, and initialize all the nodes with priority lower than σ_s. Go to the next stage.

If no, then we check whether $\Delta^{A,U}$ is defined correctly.

If $\Delta^{A,U}$ is defined correctly, then for all $z < l(\beta, s)$ with $\Delta^{A,U}(z)[s] \uparrow$, we define $\Delta^{A,U}(z)[s] = E_{j(\beta)}(z)[s]$ with $\delta(z) = s$. Define $\sigma_s(t) = \beta^\frown(k_1, \cdots, k_n)$ and go to the next substage.

If for some z, $\Delta^{A,U}(z)[s] \downarrow\neq E_{j(\beta)}(z)[s]$, let z be the least such number. Then, z is caught as an attacker by cycle $(k_1, \cdots, k_n)$, so define $\Theta^U_{(k_1,\cdots,k_{n-1})}(k_n) = V_s(k_n)$ with $\theta_{(k_1,\cdots,k_{n-1})}(k_n)[s] = \delta(z)[s]$ ($\delta(z)[s] < s$). Let cycle $(k_1, \cdots, k_{n-1}, k_n+1)$ be active. Define $\sigma_s = \beta^\frown(k_1, \cdots, k_{n-1}, k_n+1)$, initialize all the nodes with priority lower than σ_s. Go to the next stage.

4.4 Verification

Define the true path of the construction to be $\sigma = \liminf_s \sigma_{2s}$, i.e. the leftmost path of the construction. The following lemma implies that the true path σ is infinite.

Lemma 12. *Let ξ be any node on the true path σ. Then*

(1) *ξ can be initialized at most finitely often.*

(2) *ξ has an outcome $\mathcal{O}$ such that $\xi^\frown\mathcal{O}$ is on σ.*

(3) *ξ can initialize $\xi^\frown\mathcal{O}$ at most finitely often.*

Proof. We prove this lemma by induction.

When $\xi = \lambda$, the root of the priority tree, it is an $\mathcal{N}_0$-strategy. It is obvious that (1) is true.

To show (2), for a contradiction, suppose that $\lambda^\frown\mathcal{O} \not\subset \sigma$ for any outcome $\mathcal{O}$ of λ. This happens only when there are infinitely many λ-expansionary stages, λ runs infinitely many cycles and each of them runs finitely often. So either there is an $1 \leq i \leq n-1$ such that, for fixed $k_1, \cdots, k_i$, and for every $x \in \omega$ there is a stage s_x such that no cycle $(k_1, \cdots, k_i, x, k_{i+2}, \cdots, k_n)$ acts after stage s_x for some $k_{i+2}, \cdots, k_n$. Then for every $x \in \omega$, cycle $(k_1, \cdots, k_i, x, k_{i+2}, \cdots, k_n)$ gets stuck for waiting for U or V to change (i.e. each such cycle runs only finitely often). So $\Theta^U_{(k_1,k_2,\cdots,k_i)}(x) = V(x)$ for all $x \in \omega$, i.e. $\Theta^U_{(k_1,k_2,\cdots,k_i)}$ computes V correctly, a contradiction; or for every k_1, there is a stage s_{k_1} such that no cycle $(k_1, k_2, \cdots, k_n)$ acts after stage s_{k_1} for some $k_2, \cdots, k_n$. Then for every $k_1 \in \omega$, cycle $(k_1, k_2, \cdots, k_n)$ gets stuck for waiting for U or V to change (i.e. each such cycle runs only finitely often). So $\Theta^U(x) = V(x)$ for all $x \in \omega$, i.e. Θ^U computes V correctly, a contradiction.

If there are finitely many λ-expansionary stages, then $\lambda^\frown(f) \subset \sigma$. So (3) is clearly true for λ. If there are infinitely many λ-expansionary stages, then by (2), we have that $\lambda^\frown(k_1, k_2, \cdots, k_n) \subset \sigma$ for some $k_1, k_2, \cdots, k_n \in \omega$. So there is a stage large enough after which no nodes to the left of $\lambda^\frown(k_1, k_2, \cdots, k_n)$ is visited again. Thus, $\lambda^\frown(k_1, k_2, \cdots, k_n)$ can never be initialized after this stage. So (3) is true for λ.

Now, for any ξ on σ (ξ is not λ), let ξ^- be the immediate predecessor of ξ. We assume that the lemma holds for ξ^-. Then there is a stage s_0 after which ξ^- cannot be initialized. There are two cases.

Case 1. $\xi^- = \beta$ is an $\mathcal{N}$-strategy.

By our assumption, ξ^- has a true outcome on the true path. Let $\xi = \xi^{-\frown}(k_1, \cdots, k_n)$ for some $k_1, \cdots, k_n \in \omega$ (the case of $\xi = \xi^{-\frown}(f)$ is similar), then only ξ^-'s cycles on the left of $(k_1, \cdots, k_n)$ can initialize ξ after stage s_0. Since ξ is on σ, ξ^- can have outcome on the left of $(k_1, \cdots, k_n)$ at most finitely often, and thus the actions done by ξ^- via these outcomes are at most finitely often, thus ξ can be initialized by ξ^- at most finitely often after stage s_0. So there is a stage $s_1 > s_0$ after which ξ cannot be initialized by higher priority strategies. Thus, (1) is true for ξ.

To prove (2), for a contradiction, we assume that $\xi^\frown\mathcal{O} \not\subset \sigma$ for any outcome $\mathcal{O}$ of ξ (ξ is a $\mathcal{P}$-strategy in this case). Then, ξ runs infinitely many cycles and each cycle runs only finitely often. Thus, for each cycle j, there is a stage s_j such that after which cycle j never runs again, by our construction, $\Psi^U(j)[s_j]$ computes $V(j)$ correctly, and hence $\Psi^U = V$, i.e., $V \leq_T U$, a contradiction. So (2) is true for ξ.

We have shown that ξ has an outcome $\mathcal{O}$ such that $\xi^\frown\mathcal{O}$ is on σ. So there is a stage $s_2 > s_1$ after which the nodes on the left of $\xi^\frown\mathcal{O}$ can never be visited. Note that the $\mathcal{P}$-strategy ξ acts to put a number into X at most once after stage s_2, and ξ will take no further action afterwards. So (3) is true for ξ.

Case 2. $\xi^- = \alpha$ is a $\mathcal{P}$-strategy.

In this case, ξ is an $\mathcal{N}$-strategy. By our assumption, ξ^- has a true outcome on the true path. Let $\xi = \xi^{-\frown}(j,-)$ for some $j \in \omega$, where $- \in \{\infty, f\}$, then only ξ^-'s cycles on the left of cycle j can initialize ξ after stage s_0. Since ξ is on σ, ξ^- can have outcome on the left of $\xi^{-\frown}(j,-)$ at most finitely often, and thus the actions done by ξ^- via these outcomes are at most finitely often, thus ξ can be initialized by ξ^- at most finitely often after stage s_0. So there is a stage $t > s_0$ after which ξ cannot be initialized by higher priority strategies. Thus, (1) is true for ξ.

Now, we apply the same argument with that for λ, i.e. the $\mathcal{N}_0$-strategy. It is easy to see that (2) and (3) are true for ξ in this case.

□

Lemma 13. *For any $e \in \omega$, let ξ be the $\mathcal{P}_e$-strategy on the true path σ. Then $\mathcal{P}_e$ requirement is satisfied via ξ.*

Proof. Fix e, let ξ be the $\mathcal{P}_e$-strategy on the true path σ. By lemma 12, ξ can be initialized at most finitely often and ξ has a true outcome $\mathcal{O}$ on σ. Let s_0 be the least stage after which ξ can never be initialized and no nodes to the left of $\xi^\frown\mathcal{O}$ are visited again.

By our construction, $\mathcal{O} = (j,-)$ for some $j \in \omega$, where $- \in \{\infty, f\}$. Let $s_1 \geq s_0$ be the stage at which $x = x_{\xi,j}$ is defined. Then this x cannot be cancelled later.

If, after stage s_1, there is no stage at which $\Phi^{A,U}_{e(\xi)}(x)$ converges to 0, then cycle j can never take action after stage s_1. In this case, ξ has outcome (j, f) on the true path σ, and $X(x) = 0$ and $\Phi^{A,U}_{e(\xi)}(x)$ does not converge to 0. Thus, $X \neq \Phi^{A,U}_{e(\xi)}$. So $\mathcal{P}_{e(\xi)}$-requirement is satisfied obviously.

As at any ξ-stage, if $\Phi^{A,U}_{e(\xi)}(x)$ converges to 0, then cycle j imposes a restraint to protect the associated computation and waits for j to enter V, and starts $(j+1)$-cycle simultaneously. As we assume that ξ has outcome $(j, -)$ on σ, we know that either U changes below the corresponding use or j enters V.

If j entering V happens first, then cycle j can act to enumerate x into X and U will not change below the corresponding use afterwards, since otherwise, $(j, -)$, now is (j, f), could not be the final outcome of ξ on σ. In this case, $X(x) = 1$ and $\Phi^{A,U}_{e(\xi)}(x) = 0$. So $X \neq \Phi^{A,U}_{e(\xi)}$, and hence $\mathcal{P}_{e(\xi)}$-requirement is satisfied.

Otherwise, cycle j can not act to enumerate x into X forever, then cycle j will return from step 5 to step 2 infinitely often. So $\Phi^{A,U}_{e(\xi)}(x) \uparrow$, and hence $\mathcal{P}$ is satisfied via witness x.

$\square$

Lemma 14. *For any $e \in \omega$, let ξ be the $\mathcal{N}_e$-strategy on the true path σ. Then $\mathcal{N}_e$ requirement is satisfied via ξ.*

Proof. Fix e, let ξ be the $\mathcal{N}_e$-strategy on the true path σ. By lemma 12, ξ can be initialized at most finitely often and ξ has a true outcome $\mathcal{O}$ on σ. Let s_0 be the least stage after which ξ can never be initialized and no nodes to the left of $\xi^\frown\mathcal{O}$ are visited again.

If $\xi^\frown f \subset \sigma$, i.e. there are only finitely many ξ-expansionary stages, then $\mathcal{N}_e$-requirement is satisfied obviously. So we assume that there are infinitely many ξ-expansionary stages, then ξ must have a true outcome $\mathcal{O}$ of the form $(k_1, \cdots, k_n)$ on σ, for some $k_1, \cdots, k_n \in \omega$. Assume that $\Phi^{B,A,U}_i = \Phi^{C,A,U}_i = E_j$, where $e = \langle i, j\rangle$. We now show that $\Delta^{A,U}$ defined by the cycle $(k_1, \cdots, k_n)$ is totally defined and computes E_j correctly.

By the choice of s_0, no strategy to the left of $\xi^\frown(k_1, \cdots, k_n)$ is visited after stage s_0. Thus, the functional Δ defined by cycle $(k_1, \cdots, k_n)$ can not be cancelled by higher priority strategies after stage s_0. To show that $\Delta^{A,U}$ is totally defined, it is enough to show that for every x, $\Delta^{A,U}(x)$ can be undefined at most finitely often. For a contradiction, let z be the least one such that $\Delta^{A,U}(z)$ is undefined infinitely often. By our construction, this z must be caught as an attacker by cycle $(k_1, \cdots, k_n)$ at some (least) stage $s_1 > s_0$, say. Note that, when z is caught as an attacker by cycle $(k_1, \cdots, k_n)$, we wait for either U changes below the corresponding use or k_n to enter V. At the same time, cycle $(k_1, \cdots, k_n + 1)$ is started.

Firstly, assume that $k_n \notin V$. At stage s_1, it must be that $l(\xi, s_1) > z$ and $\Delta^{A,U}(z)[s_1] \downarrow \neq E_{j,s_1}(z)$. As we assume that $\xi^\frown(k_1, \cdots, k_n) \subset \sigma$, and k_n never enters V, U must change below the corresponding use $\theta_{(k_1,\cdots,k_{n-1})}(k_n)[s_1] = \delta(z)[s_1]$ after stage s_1 (otherwise, as ξ starts a new cycle $(k_1, \cdots, k_n + 1)$ when it waits for k_n to enter V, then cycle $(k_1, \cdots, k_n)$ can not be run again, so the true outcome of ξ lies to the right of $(k_1, \cdots, k_n)$, a contradiction). So $\Delta^{A,U}(z)[s_1]$ is undefined by this U-change, at the next ξ-expansionary stage, $\Delta^{A,U}(z)$ will be redefined correctly. Now, if $E_j(z)$ does not change later, then the definition of $\Delta^{A,U}(z)$ is correct forever and we will never undefine it by our construction (since this z can never be caught as an attacker again). In this case, $\Delta^{A,U}(z)$ is undefined finitely often. But E_j is n-c.e., $E_j(z)$ may change afterwards. If so, this z could be caught as an attacker by cycle $(k_1, \cdots, k_n)$ for the second time. Then, by applying the same argument as above, $\Delta^{A,U}(z)$ must be undefined by a U-change and ξ can correct $\Delta^{A,U}(z)$ at the next ξ-expansionary stage again. Since E_j is n-c.e. set, $E_j(z)$ changes at most n many times. For each such change, if z is caught as an attacker by cycle $(k_1, \cdots, k_n)$ then $\Delta^{A,U}(z)$ must be undefined via a U-change and then we have a chance to correct $\Delta^{A,U}(z)$. Therefore, $\Delta^{A,U}(z)$ is undefined finitely often and redefined correctly eventually.

Secondly, suppose $k_n \in V$. By the above paragraph, it is sufficient to consider the case that there is a stage $t > s_1$ at which cycle $(k_1, \cdots, k_n)$ gets the $V(k_n)$-permission to perform the first step of the disagreement strategy and stop all cycles $> (k_1, \cdots, k_n)$. Assume $E_j(z)$ never changes again. If U does not change below the corresponding use of cycle $(k_1, \cdots, k_n)$ later, then the final outcome of ξ will be f, a contradiction. So U must change below the corresponding use of cycle $(k_1, \cdots, k_n)$ after stage t, and hence $\Delta^{A,U}(z)$ will be redefined correctly and it is correct forever.

Thus, $\Delta^{A,U}(z)$ is undefined finitely often in this case. Suppose $E_j(z)$ changes after stage t at which cycle $(k_1, \cdots, k_n)$ performs the first step of the disagreement strategy. Then this z could be caught as an attacker by cycle $(k_1, \cdots, k_n)$ to do the second step of the disagreement strategy. Since E_j is n-c.e., it may happen that cycle $(k_1, \cdots, k_n)$ performs all the n steps of the disagreement strategy for this z, but eventually U must change below the corresponding use since otherwise the disagreement strategy succeeds and hence a disagreement point appears, then the final outcome of ξ will be f. Thus, $\Delta^{A,U}(z)$ must be undefined via this U-change and then we have a chance to correct $\Delta^{A,U}(z)$. After $E_j(z)$ changes for the last time, this z can never be caught as an attacker by cycle $(k_1, \cdots, k_n)$ and hence $\Delta^{A,U}(z)$ can never be undefined. Therefore, $\Delta^{A,U}(z)$ is undefined finitely often and redefined correctly eventually.

$\square$

Lemma 15. *$A, B, C \leq_T V \oplus U$.*

Proof. We will apply the delayed permission argument here. To show that $A \leq_T V \oplus U$, fix a number $m \in \omega$, it's sufficient to show that we can find a stage s using $V \oplus U$ as the oracle such that $m \in A$ if and only if $m \in A_s$. By our construction, only $\mathcal{N}$-strategies can put or extract numbers into or from A and if m can be enumerated into A, then it must be the case that some $\mathcal{P}$-strategy, α say, enumerates a number x into X, and $\gamma_1(x)[m]$ into B and $\gamma_2(x)[m]$ into C at stage m, and later some $\mathcal{N}$-strategy β with $\beta \subset \alpha$ performs the disagreement strategy by removing $\gamma_1(x)[m]$ from B and extracting x from X and enumerating m into A simultaneously. So if at stage m, no number is enumerated into X, then $m \notin A$; otherwise, assume some number x is enumerated into X at stage m by some $\mathcal{P}$-strategy α, and $\gamma_1(x)[m]$ is enumerated into B and $\gamma_2(x)[m]$ is enumerated into C at stage m simultaneously. Check whether there is some $\mathcal{N}$-strategy $\beta \subset \alpha$ with $\beta^\frown\mathcal{O} \subseteq \alpha$ such that for some z with $\Delta^{A,U}_{\beta^\frown\mathcal{O}}(z)$ defined, the computations $\Phi^{B,A,U}_{e(\beta)}(z)$ and $\Phi^{C,A,U}_{e(\beta)}(z)$ are injured by the enumerations of $\gamma_i(x)$-uses ($i = 1, 2$) and hence z is caught as an attacker. If there is no such β, then m will not be enumerated into A. Now we assume that such $\mathcal{N}$-strategy (with the highest priority) $\beta \subset \alpha$ exists.

For the convenience of description, we first introduce the following definition.

Given $\xi \in T$, the *permission-bound* at stage s, $b(\xi_s)$, is defined as

$$\max(\{k_1+1, \cdots, k_n+1, j+1 : \exists \xi'(\xi'^\frown(k_1, \cdots, k_n) \leq \xi$$

$$\text{or } \xi'^\frown(j, -) \leq \xi, - \in \{\infty, f\})\}).$$

Obviously, $b(\xi_s)$ is computable. Let $S(\xi_s)$ be the least stage such that

$$V_{S(\xi_s)} \upharpoonright b(\xi_s) = V \upharpoonright b(\xi_s).$$

Then $S(\xi_s)$ is V-computable.

Before we continue the proof of $A \leq_T V \oplus U$, we introduce the following claim:

Claim: *Assume that a $\mathcal{P}$-strategy ξ runs cycle j at stage s_0 and $s_1 \geq \max\{s_0, S(\xi^\frown(j, f))\}$ is a U-true stage. If ξ does not run cycle j at stage s_1, then after stage s_1, if ξ runs cycle j again, at stage s_2 say, cycle j must have been cancelled or initialized between stage s_0 and s_2.*

A similar result is true for an $\mathcal{N}$-strategy ξ instead of j cycle with $(k_1, \cdots, k_n)$ cycle.

This claim can be proved by the same argument with that in [42]. One can also refer to Lemma 5.2 in [12].

We continue the proof of $A \leq_T V \oplus U$ now. It is obvious that for such $\mathcal{N}$-strategy β, the outcome $\mathcal{O}$ must be $(k_1, \cdots, k_n)$ for some $k_1, \cdots, k_n \in \omega$ such that $\beta^\frown(k_1, \cdots, k_n) \subseteq \alpha$. Let $\xi = \beta^\frown(k_1, \cdots, k_n)$.

Now if $k_n \in V_m$ or $k_n \notin V$, then $m \notin A$. Assume k_n enters V at a stage $t > m$, let $t' > t$ be the least U-true stage greater than both t and $S(\xi_t)$. t' is $U \oplus V$-computable. At stage t', if β is not visited or β is visited but does not run cycle $(k_1, \cdots, k_n)$, then by the claim, we have $m \in A$ iff $m \in A_{t'}$. If β runs cycle $(k_1, \cdots, k_n)$ at stage t', then it is easy to see that if $m \notin A_{t'}$ then $m \notin A$ since by the choice of $t' > S(\xi_t)$, m cannot get V-permission to enter A after stage t'; if $m \in A_{t'}$ then, by the choice of t', m can not get the V-permission to exit A after stage t'. Thus, $m \in A$ iff $m \in A_{t'}$, so $A \leq_T U \oplus V$.

Because the arguments for $B \leq_T V \oplus U$ and $C \leq_T V \oplus U$ are similar, without loss of generality, we only show the more complicated one $B \leq_T V \oplus U$ since C is in fact a *c.e.* set. Fix a number $m \in \omega$, we will show how to $V \oplus U$-computably determine whether $m \in B$ or not. First note that if m is not a $\gamma_1(x)[s]$ use for any x, s, then $m \notin B$. Find the first y, s such that $\gamma_1(y)[s] > m$. If so far m is not assigned as a γ_1-use for any number $z < y$ at any stage $t < s$, then $m \notin B$.

Assume $m = \gamma_1(x)[s]$ for some x, s. By our construction, $\gamma_1(x)[s]$ is enumerated into B only when x is enumerated into X at a stage $t > s$. We can effectively check whether x is chosen as an attacker by some $\mathcal{P}$-strategy or not. If x is not chosen as an attacker by any $\mathcal{P}$-strategy, then $m \notin B$. Otherwise, assume that x is chosen as an attacker by some $\mathcal{P}$-strategy, α say, for some cycle j at some stage s_0. Let $\xi = \alpha^\frown(j, f)$. If $j \in V_{s_0}$ or $j \notin V$, then it is obvious that $m \notin B$ since m cannot get V-permission forever. If j enters V at a stage $s_1 > s_0$, let $s_2 > s_1$ be the least U-true stage greater than both s_1 and $S(\xi_{s_1})$. s_2 is $U \oplus V$-computable. At stage s_2, if α is not visited, or if α is visited but does not run cycle j, then $m \in B$ iff $m \in B_{s_2}$ (by the claim). If α runs cycle j at stage s_2, it is easy to see that if $m \notin B_{s_2}$ then $m \notin B$, since after stage s_2, m cannot get V-permission to enter B. We now assume that $m \in B_{s_2}$ in the following.

If the enumerations of $m = \gamma_1(x)[s]$ into B and $\gamma_2(x)[s]$ into C do not injure any computations $\Phi^{B,A,U}_{e(\beta)}(z)$ and $\Phi^{C,A,U}_{e(\beta)}(z)$ simultaneously for any z with $\Delta^{A,U}_{\beta^\frown\mathcal{O}}(z)$ defined, where β is an $\mathcal{N}$-strategy with $\beta^\frown\mathcal{O} \subseteq \alpha$, then m will not be removed from B after stage s_2 and hence $m \in B$. Otherwise, suppose that β is an $\mathcal{N}$-strategy of the highest priority such that $\beta^\frown(k_1, \cdots, k_n) \subseteq \alpha$ for some $k_1, \cdots, k_n \in \omega$, and the enumerations of $m = \gamma_1(x)[s]$ into B and $\gamma_2(x)[s]$ into C injured the computations $\Phi^{B,A,U}_{e(\beta)}(z)$ and $\Phi^{C,A,U}_{e(\beta)}(z)$ simultaneously for some z with $\Delta^{A,U}_{\beta^\frown\mathcal{O}}(z)$ defined and hence z is caught as an attacker by β's cycle $(k_1, \cdots, k_n)$. Let $\xi = \beta^\frown(k_1, \cdots, k_n)$. Now let $s_3 > s_2$ be the least U-true stage greater than both s_2 and $S(\xi_{s_2})$. s_3 is $U \oplus V$-computable. At stage s_3, if β is not visited, or if β is visited but does not run cycle $(k_1, \cdots, k_n)$, then $m \in B$ iff $m \in B_{s_3}$ (by the claim). If β runs cycle $(k_1, \cdots, k_n)$ at stage s_3, it is easy to see that $m \in B$ iff $m \in B_{s_3}$ since after stage s_3, m cannot get V-permission to enter or exist B. Thus, $m \in B$ iff $m \in B_{s_3}$. Therefore $B \leq_T U \oplus V$.

$\square$

Lemma 16. $\Gamma_1^{B,A,U}$ *and* $\Gamma_2^{C,A,U}$ *are totally defined and compute* X *correctly. Furthermore,* $X \leq_T V$.

Proof. We will show that $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ are undefined at most finitely often and $\Gamma_1^{B,A,U}(x)$=$\Gamma_2^{C,A,U}(x)$=$X(x)$ eventually for all $x \in \omega$. We prove this by induction on $x \in \omega$.

Fix an $x \in \omega$, suppose that this is true for all $y < x$, i.e. there is some (least) stage s_0 after which $\Gamma_1^{B,A,U}(y)$ and $\Gamma_2^{C,A,U}(y)$ for all $y < x$ cannot be undefined again. After stage s_0, suppose that $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ are defined for the first time at stage s_1. If x is not enumerated into X at any stage $s > s_1$, then $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ will never be undefined. Otherwise, suppose that x is enumerated into X at some stage $s > s_1$ by some $\mathcal{P}$-strategy, α say, then, by our construction, we will undefine $\Gamma_1^{B,A,U}(x)[s]$ by enumerating $\gamma_1(x)[s]$ into B and undefine $\Gamma_2^{C,A,U}(x)[s]$ by enumerating $\gamma_2(x)[s]$ into C. So $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ will be redefined correctly later. If there is some $\mathcal{N}$-strategy, β say, performing the disagreement strategy by extracting $\gamma_1(x)[s]$ from B at some stage $s(1) > s$, it will remove x from X and put s into A, then we will redefine $\gamma_1(x)$ as the same as $\gamma_1(x)[s]$ and lift $\gamma_2(x)$ use, redefine $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ agreeing with $X_{s(1)}(x)$. If β performs the disagreement strategy for the second time by extracting s from A at a stage $s(2) > s(1)$, β will enumerate x into X and put $\gamma_1(x)[s]$ into B again, then we will lift $\gamma_1(x)$ use and restore the $\gamma_2(x)$ to be that defined before enumerating s into A, and redefine $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ agreeing with $X_{s(2)}(x)$. After this, by our construction, β may perform the disagreement strategy at most $(n-2)$ many times more later, at each time of such β-actions, $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ can be undefined or restored by enumerating s into A or removing s from A, and redefined correctly. Thus, $\Gamma_1^{B,A,U}(x)$ and $\Gamma_2^{C,A,U}(x)$ are undefined at most finitely often and $\Gamma_1^{B,A,U}(x)$=$\Gamma_2^{C,A,U}(x)$=$X(x)$ eventually. So, $\Gamma_1^{B,A,U}$ and $\Gamma_2^{C,A,U}$ are totally defined and compute X correctly.

Note that, for any $x \in \omega$, we change the value of $X(x)$ only when it gets the corresponding V-permission. So $X \leq_T V$.

$\square$

This completes the proof of Theorem 3. $\diamond\diamond$

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