Arbitrary-length filter banks and robust frames: theory and factorization

Xu, Zhiming

2009

Xu, Z. (2009). Arbitrary-length filter banks and robust frames: theory and factorization.
Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/46835>

<https://doi.org/10.32657/10356/46835>

Nanyang Technological University

Downloaded on 26 Apr 2023 17:07:44 SGT

ARBITRARY-LENGTH FILTER BANKS AND ROBUST FRAMES: THEORY AND FACTORIZATION

XU ZHIMING

School of Electrical and Electronic Engineering

A thesis submitted to the Nanyang Technological University
in fulfillment of the requirement for the degree of
Doctor of Philosophy

2009

Acknowledgment

It is my pleasure to acknowledge and thank the people who have influenced me through the course of this research. First, I express my gratitude to my supervisor, Prof. Anamitra Makur, for his continuing support, guidance and encouragement. The discussions and meetings with him were always inspiring. It is his encouragement and timely help that keep me working towards the completion of this dissertation. Aside from being an excellent supervisor, he is also a sincere and hospitable friend whom I often consult for various issues.

I also appreciate those people who ever helped me in various ways during these years of my PhD study and research. Thank Prof. Lin Zhiping, from Nanyang Technological University, for the fruitful discussions, illumination and kindly helps during our collaboration. Thank Prof. See-May Phoong, from National Taiwan University, for his kindly help. Thank Prof. Markus Püschel, from Carnegie Mellon University, for his comments on filter bank frames at Hawaii, USA during ICASSP 2007 meeting. Thanks to Dr. Gagan Rath, from INRIA-IRISA of France, for his guidance and interesting discussions during his stay at Nanyang Technological University. Thank Dr. Gan Lu, from University of Liverpool, for her invaluable help at the beginning of my PhD study. Thanks to Dr. Zhang Lei for fruitful discussions and collaborations with him. In addition, great appreciation to the following people who have successfully broaden my horizons on applied and computational mathematics for signal and image processing. Thank Prof. Thomas Yizhao Hou, from California Institute of Technology, for his teaching on MAS734 Multiscale Analysis and Computations, and illuminating discussions as well as his kindly helps and encouragements. Thank Prof. Xue-Cheng Tai, from Bergen University, for his teaching on MAS721 Numerical Partial Differential Equations and fruitful discussions, and kindly guidance on numerical analysis and computations. Thank Prof. Stanley Osher, from University of California at Los Angeles, for his teaching on MAS731 Variational and Partial Differential Equation Techniques for Image Processing and the intriguing discussions. Thank Prof. Alfred Bruckstein, from Israel Institute of Technology Technion, for his teaching on MAS730 Differential Geometry and Variational Methods for Image Analysis and kindly share of his perspectives on image analysis. In addition, thank Prof. Alex Kot and Tan Yap Peng, from Nanyang Technological University, for their help and encouragement for my attendance of various academic conferences.

I am also grateful to all colleagues and friends who have made my stay at Nanyang Technological University and Singapore stimulating and fun, from whom I also have learned a lot. The generous financial support from School of Electrical and Electronic Engineering, Nanyang Technological University is gratefully acknowledged as well, which made this research possible. Finally, I would like give my heartiest appreciation to my parents and older brother for their constant love and unconditional support. Without them, absolutely nothing is possible.

Abstract

Over the past two decades, there have been persistent interests in the study of filter banks (FBs), due to their successes in wide applications such as compression, filtering and communication, and their unifying role for block, lapped and wavelet transforms. The core issue in the study of FBs is the theory and design approaches. Among all the existing methods, one of the most efficient ones is the lattice factorization based design method which employs the classic divide-and-conquer strategy, i.e., constructing a higher-order matrix polynomial by cascading several lower-order modular building blocks. Lattice factorization can offer modular and robust structures which enable the unconstrained optimization based FB design and are friendly to hardware realization in very large scale integrated circuits. Taking account of these advantages, this dissertation is devoted to a systematic investigation of the theory, lattice factorizations and designs for both critically sampled and oversampled FBs, as studied in Part I (Chapters 2-5) and Part II (Chapters 6-8), respectively.

In Part I for critically sampled systems, we firstly review the fundamentals of FBs. Then, we begin to study a large class of M -channel finite impulse response (FIR) FBs with *arbitrary* equal lengths $L = KM + \beta$ ($0 \leq \beta < M$, $K \in \mathbb{N}$), in order to relax the *unnecessary* constraint of $L = KM$ imposed in most existing works. The necessary existence conditions on the filter length L are derived, and various novel lattice factorizations are established for such class of FBs and with various desired properties, including perfect reconstruction (PR), paraunitariness (PU), linear phase (LP) and regularity. The arbitrary filter length L can be easily achieved by appropriately choosing design parameters β and K . More importantly, it can be shown that these lattice factorizations can *completely span* the whole class of both M -channel FIR PUFBs and LPPUFBs with any valid length, and the class of PRFBs and LPPRFBs with lengths $L \leq 2M$. These proposed structures are also shown to be minimal for LPPRFBs in the sense of the number of delay elements used in the implementation. In addition to these outcomes, more theoretical issues for the lattice factorizations of LPPRFBs are further investigated. Firstly, through pointing out the limitation of the existing works, we show the *nonexistence* of any complete order-one lattice factorization. Secondly, we establish the *suboptimality* of the existing lattice factorization based design method for LPPRFBs, through studying the spaces $\mathbb{L}_\beta$ for LPPRFBs spanned by the lattice structures and their closeness property under polynomial transformations. In addition, Several design examples are presented to validate the proposed lattice structures and the LPPRFBs are further evaluated in a transform-based image coding framework to illustrate the length effect on image compression.

Inspired from the successful application of unequal-length FBs in low bit rate image coding, we further investigate a class of unequal-length FBs. The existence conditions and various lattice structures are developed and the structures for LPPRFBs are shown to be minimal. Our derived

existence conditions and proposed lattice structures reveal that there are much more degrees of design freedom and our design approaches can considerably expand the designer's choices. Several design examples are shown to validate the proposed design approaches. The comparison of different FBs in terms of generalized transform coding gain and real image compression at low bit rate region demonstrates the better performance of unequal-length LPPRFBs.

In Part II for oversampled systems, we begin with the review of fundamentals of frame theory and oversampled filter banks (OFBs) as well as their relationship. Then, we systematically investigate the robustness of FB frames to erasures and quantization in infinite dimensional Hilbert space $l^2(\mathbb{Z})$ from the perspectives of both frame and FB theory. We firstly analyze the robustness of FB frames to erasures for robust information transmission over erasure networks. A sufficient condition is given for general FB frames without the unnecessary equal-norm property usually imposed in previous works. Furthermore, we show that our derived robust condition is also *necessary* for tight FB frames. In addition, a more general analysis on maximal robust FB frames is explored based on an explicit constructive approach which can also serve as a guideline for designing maximal robust FB frames. Then, we further study the reconstruction by the pseudoinverse of the subframe which is usually a hybrid FIR and (infinite impulse response) IIR FB. Thus, we specifically examine the *stability* and even possible *FIR* reconstruction of such pseudoinverse. We derive the *necessary and sufficient* condition for the FIR pseudoinverse of the subframe after one erasure of tight FB frames. Because quantization is an inevitable process of digital transmission of information, we continue to analyze the quantization effect on the reconstruction. In particular, we introduce a novel notion, named *frame energy*, which actually dictates the quantization performance of FB frames. Furthermore, we show that tight FB frames are optimal for quantization with bounded frame energy, which reveals *universal optimality* of tight FB frames in contrast to restricted optimality within unit-norm tight frames reported before. Finally, the effect of erasures on the reconstruction mean squared error (MSE) for general FB frames is examined by incorporating prior information on possible erasure distribution into our analysis. We show that tight FB frames are optimal and the optimal tight FB frames are usually not of equal norm. We then have put our analysis into the framework of multiple description coding to show its potential application due to the desirable reduced reconstruction MSE by the correctly received additional subbands.

Finally, we study the theory and factorizations for oversampled FBs. Through the extension of our work on critically sampled FBs, we propose lattice factorizations for oversampled PRFBs and LPPRFBs, respectively, with arbitrary lengths. The difficulty in more rigorous derivation and the limitation of our lattice factorization are discussed. The difference between the forward and backward factorizations for oversampled FBs is presented from both perspectives of design and implementation. Compared to the previous works, our work is more general and can cover a larger class of such FBs than before. Furthermore, the proposed lattice structures are proven to be complete for oversampled LPPUFBs.

Contents

Acknowledgment	i
Abstract	ii
List of Figures	ix
List of Tables	xii
List of Acronyms	xiv
1 Introduction	1
1.1 Signal Compression	1
1.1.1 Classical Transform Coding	1
1.1.2 Multiple Description Coding	2
1.2 Signal Decompositions	3
1.2.1 Bases and Critically Sampled Filter Banks	4
1.2.2 Frames and Oversampled Filter Banks	5
1.2.3 Signal Compression Based on Filter Banks	5
1.3 Outline and Contributions	6
1.3.1 Critically Sampled Filter Banks	6
1.3.2 Oversampled Filter Bank Frames	8
1.4 Notations	9
Part I Critically Sampled Filter Banks	11
2 Fundamentals of Filter Bank	12
2.1 Structures and Representations for Filter Banks	12
2.1.1 Critically Sampled Filter Banks	12
2.1.2 Polyphase Representations	13

2.2	Properties of Filter Banks	16
2.2.1	Desired Properties	16
2.2.2	Performance Criteria for Design	19
2.3	Lattice Factorization	21
2.3.1	The Basic Idea of Lattice Factorization	21
2.3.2	Properties of Lattice Factorization	23
2.4	Connections among Transforms, Filter Banks and Wavelets	23
2.4.1	Block and Lapped Transforms	23
2.4.2	Wavelet Transforms	25
2.4.3	Interconnection among Transforms, Filter Banks and Wavelets	27
2.5	Brief Review of Previous Related Works	30
3	Arbitrary Equal-Length Multichannel Filter Banks	32
3.1	Introduction	32
3.2	Existence Conditions and General Lattice Factorization	34
3.2.1	Restriction on Filter Length	34
3.2.2	General Lattice Factorization	35
3.3	Lattice Factorizations of General Perfect Reconstruction Filter Banks	36
3.3.1	Paraunitary Filter Banks	37
3.3.2	Perfect Reconstruction Filter Banks	41
3.4	Lattice Factorization of Regular Filter Banks	44
3.5	Design Examples	45
3.5.1	Degrees of Freedom for Filter Bank Design	45
3.5.2	Filter Bank Design Examples	46
3.6	Summary	48
4	Arbitrary Equal-Length Linear Phase Filter Banks	49
4.1	Introduction	49
4.2	Refined Existence Conditions and the General Lattice Factorization	52
4.3	Lattice Factorizations of Linear Phase Perfect Reconstruction Filter Banks	54
4.3.1	LPPRFBs with Even β	54
4.3.2	LPPRFBs with Odd β	62

4.3.3	Relationship with Previous Works	64
4.4	Theoretical Properties of the Lattice Factorizations	66
4.4.1	Minimality	66
4.4.2	Completeness	68
4.5	Non-Existence of Complete Order-One Lattice Factorization	70
4.5.1	Review of Existing Proof and Correction	71
4.5.2	Nonexistence of Order-One LPPR Propagating Block with CAFACAFI	72
4.5.3	Revisiting the GenLOT and LBT	76
4.6	On the Closeness of the Space Spanned by the Lattice Structures	76
4.6.1	The Space $\mathbb{L}_\beta$ Spanned by the Lattice Structure	77
4.6.2	Closeness of the Space $\mathbb{L}_\beta$ for General LPPRFBs	79
4.6.3	Closeness of the Space $\mathbb{L}_\beta$ for Regular LPPRFBs	83
4.6.4	Discussion	86
4.7	Design and Application Examples	87
4.7.1	Degrees of Freedom for Filter Bank Design	87
4.7.2	Filter Bank Design Examples	88
4.7.3	Effect of Filter Length on Image Compression	91
4.8	Summary	95
4.9	Appendix: Proof and Remarks for Lemma 4.6	96
5	Unequal-Length Multichannel Filter Banks	99
5.1	Introduction	99
5.2	Filter Length Profile and General Lattice Factorization	101
5.3	Lattice Factorization of Unequal-Length Perfect Reconstruction Filter Banks	102
5.3.1	General Unequal-Length Perfect Reconstruction Filter Banks	102
5.3.2	Regular Unequal-Length Perfect Reconstruction Filter Banks	103
5.4	Lattice Factorization of Unequal-Length LPPRFBs	104
5.4.1	Existence Conditions	104
5.4.2	Even Channel Systems	108
5.4.3	Odd Channel Systems	111
5.4.4	Minimality of the Proposed Structure	112

5.5	Design and Application Examples	112
5.5.1	Degrees of Freedom for Filter Bank Design	113
5.5.2	Filter Bank Design Examples	113
5.5.3	Effect of Unequal Filter Lengths on Image Compression	115
5.6	Summary	119
Part II	Frames and Oversampled Filter Banks	121
6	Fundamentals of Frame Theory and Oversampled Filter Banks	122
6.1	Overcomplete Signal Representations and Frames	122
6.1.1	Overcomplete Signal Representations	123
6.1.2	Frames	124
6.1.3	Nonunique Dual Frames and Various Frame Operators	125
6.2	Filter Bank Frames	127
6.2.1	Oversampled Filter Banks	127
6.2.2	Relationship with Frame Expansions	129
6.2.3	Uniform Filter Bank Frames	130
6.3	Quantized Frame Expansions	132
6.4	Brief Review of Previous Related Works	134
7	Robustness and Quantization of Filter Bank Frames	136
7.1	Introduction	136
7.2	Characterization and Eigen-Structure of Frame Operator	139
7.3	Erasure Effects on the Structures of FB Frames and Reconstruction after Erasures	140
7.3.1	Erasure Effect on Structure of Filter Bank Frames	141
7.3.2	Reconstruction after Erasures	147
7.4	Optimal Quantized Frames without and with Erasures	151
7.4.1	Optimal Quantized Frames without Erasures	151
7.4.2	Optimal Quantized Frames with Erasures	153
7.5	An Example of Multiple Description Coding Based on Quantized Filter Bank Frames: MSE Analysis	159
7.5.1	The Case of $1 \leq K \leq M$	160
7.5.2	The Case of $M < K \leq P - 1$	161
7.6	Summary	163

8	Arbitrary Equal-Length Oversampled Filter Banks	165
8.1	Introduction	165
8.2	Oversampled Perfect Reconstruction Filter Banks and Lattice Factorizations . .	166
8.3	Oversampled Linear Phase Perfect Reconstruction Filter Banks and Lattice Fac- torizations	168
8.3.1	The Existence Conditions	169
8.3.2	General Lattice Factorization and Classification of OLPPRFBs	171
8.3.3	OLPPUFBs with Arbitrary Equal Lengths	172
8.3.4	OLPPRFBs with Arbitrary Equal Lengths	179
8.4	Design Examples	180
8.4.1	Degrees of Freedom for Filter Bank Design	180
8.4.2	Filter Bank Design Examples	181
8.5	Summary	183
9	Conclusions and Future Works	184
9.1	Conclusions	184
9.1.1	Critically Sampled Filter Banks	184
9.1.2	Oversampled Filter Banks and Frames	186
9.2	Future Research Directions	188
9.2.1	Critically Sampled Filter Banks	188
9.2.2	Oversampled Filter Banks and Frames	189
	Author's Publications	191
	References	193

List of Figures

1.1	Examples of three possible sets of vectors which can completely span the two dimensional space $\mathbb{R}^2$	4
1.2	A general framework for signal compression based on filter banks.	6
2.1	The general form of M -channel FBs.	13
2.2	The regular form of M -channel uniform FBs.	13
2.3	The noble identities in multirate signal processing.	14
2.4	Equivalent polyphase representations of M -channel critically sampled uniform FBs shown in Fig. 2.2.	15
2.5	Two equivalent lattice factorizations of the analysis bank of an M -channel critically sampled uniform FB.	22
2.6	Equivalent representations of conventional two-band discrete time wavelet with three levels of decomposition.	26
2.7	A block diagram for illustration of lapped transforms in 2D setting.	28
2.8	Different structures for grouping the transformed coefficients.	29
3.1	Forward lattice factorization for $\mathbf{E}(z)$ of an FIR equal-length FB.	39
3.2	Design example of 4-channel PUFBs with consecutive lengths $L = 12, 14, 15$ and 16.	47
3.3	Design example of a regular 5-channel PRFB with length $L = 13$	48
4.1	The forward lattice structure of the polyphase matrix $\mathbf{E}(z)$ of an M -channel LPPRFB for both odd ($M = 2m + 1$) and even ($M = 2m$) channel systems with length $L = KM + \beta$ ($\beta = 2l$). The dashed line only exists for the odd channel case.	58

4.2	The structure of the initial block $\mathbf{E}_0^f(z)$ of M -channel LPPRFBs ($M = 2m + 1$) with length $L = 2M + \beta$ ($\beta = 2l + 1$).	63
4.3	Interconnections of lattice structures of FIR LPPRFBs with arbitrary equal lengths and the whole space.	78
4.4	Design example I: $M = 7$, $L = 11$ optimized for coding gain and stopband attenuation.	89
4.5	Design example II: $M = 7$, $L = 19$ optimized for stopband attenuation and passband ripple.	89
4.6	Design example III: $M = 8$, $L = 10$ optimized for coding gain and stopband attenuation.	90
4.7	Design example IV: $M = 8$, $L = 14$ optimized for coding gain and stopband attenuation.	91
4.8	Enlarged portions of <i>Lena</i> at 100:1 compression ratio using various transforms and the original image.	93
4.9	Enlarged portions of <i>Barbara</i> at 64:1 compression ratio using various transforms and the original image.	94
5.1	The hybrid lattice structure of the polyphase matrix $\mathbf{E}(z)$ of an M -channel unequal-length PRFB with length profile (5.1).	103
5.2	Block diagram for $\hat{\mathbf{G}}_i^{\text{LP}}(z)$ with odd number of long filters, i.e., $N = 2n + 1$	110
5.3	A design example of 8-channel unequal-length LPPRFB with 3 long and 5 short filters of $L_{av} = 14$	114
5.4	A design example of 8-channel unequal-length LPPRFB with 2 long and 6 short filters of $L_{av} = 14$	116
5.5	Improvement in generalized transform coding gain over DCT as a function of bit rate for several FBs.	117
5.6	Comparison of reconstructed images by different transforms at bit rate of 0.1 bpp with the same averaged length of 14.	118
5.7	Comparison of reconstructed images by different transforms at bit rate 0.1 bpp with the same averaged length of 12.	118
5.8	Comparison of reconstructed images by different transforms at bit rate 0.125 bpp with the same averaged length of 14.	119

6.1 A P -channel FB with subsampling factor M 128

6.2 A block diagram of quantized frame expansion and reconstruction. 132

7.1 Error resilient transmission by a frame expansion implemented by a P channel
OFB with subsampling factor M 140

8.1 Design examples of an 8-channel critically sampled LPPUFB and three OLP-
PUFBs with same subsampling factor M but different filter lengths L 182

List of Tables

2.1	The existing works on FBs with equal lengths $L = KM + \beta$ ($0 \leq \beta < M$).	30
2.2	The remaining problems in the existing literature for FBs with equal lengths $L = KM + \beta$ ($0 \leq \beta < M$), and some of them to be solved in Part I of this dissertation.	31
2.3	The previous works and remaining problems in the existing literature for FBs with unequal lengths, and some of the problems to be solved in Part I of this dissertation.	31
3.1	Degrees of freedom for the lattice factorizations of M -channel PRFBs and PUFBs with lengths $L = KM + \beta$	45
4.1	Existence conditions for M -channel LPPRFBs with lengths $L = KM + \beta$	52
4.2	Degrees of freedom for the lattice factorizations of M -channel LPPRFBs and LPPUFBs with lengths $L = KM + \beta$	88
4.3	Comparisons of three properties of eight-channel transforms.	91
4.4	Comparisons of computational complexity of transforms.	91
4.5	Compression performance (PSNR) using different transforms on the test image. .	92
4.6	Summary of the closeness of $\mathbb{L}_\beta$ for even-channel LPPRFBs with arbitrary lengths.	96
4.7	Summary of the closeness of $\mathbb{L}_\beta$ for even-channel regular LPPRFBs with arbitrary lengths.	96
5.1	Existence conditions for M -channel LPPRFBs with length profile (5.1).	107
5.2	Degrees of freedom for the lattice factorizations of M -channel unequal-length PRFBs and PUFBs with length profile (5.1).	113
5.3	Degrees of freedom for the lattice factorizations of M -channel unequal-length LPPRFBs and LPPUFBs with length profile (5.1).	113

6.1 The existing works on the robustness of frames to quantization and erasures. . . 134

6.2 The existing works on oversampled filter banks with equal lengths $L = KM$. . . 135

8.1 Degrees of freedom for the forward lattice factorization of P -channel oversampled PRFBs and PUFBs with lengths $L = KM + \beta$ 180

8.2 Degrees of freedom for the backward lattice factorization of P -channel oversampled PRFBs and PUFBs with lengths $L = KM + \beta$ 181

8.3 Degrees of freedom for the forward lattice factorization of P -channel type-I oversampled LPPRFBs and PUFBs with lengths $L = KM + \beta$ 181

8.4 Degrees of freedom for the backward lattice factorization of P -channel type-I oversampled LPPRFBs and PUFBs with lengths $L = KM + \beta$ 181

List of Acronyms

Abbreviations	Full Expressions
CG:	Coding gain
DCT:	Discrete cosine transform
DOF:	Degrees of freedom
EL:	Equal-length
ENF:	Equal-norm frame
ENTF:	Equal-norm tight frame
EVD:	Eigen-value decomposition
FB:	Filter bank
FIR:	Finite impulse response
GLBT:	Generalized lapped biorthogonal transform
GenLOT:	Generalized lapped orthogonal transform
GLT:	Generalized lapped transform
IIR:	Infinite impulse response
LBT:	Lapped biorthogonal transform
LOT:	Lapped orthogonal transform
LP:	Linear phase
LPPRFB:	Linear phase perfect reconstruction filter bank
LPPUFB:	Linear phase paraunitary filter bank
LT:	Lapped transform
MDC:	Multiple description coding
MIMO:	Multiple input multiple output
MR:	Maximally robust
OFB:	Oversampled filter bank
OLPPRFB:	Oversampled linear phase perfect reconstruction filter bank
OLPPUFB:	Oversampled linear phase paraunitary filter bank
OPRFB:	Oversampled perfect reconstruction filter bank
OPUFB:	Oversampled perfect paraunitary filter bank
PR:	Perfect reconstruction
PU:	Paraunitary
SVD:	Singular value decomposition
TF:	Tight frame
UF:	Uniform frame
UNF:	Unit-norm frame
UNTF:	Unit-norm tight frame
UL:	Unequal-length
WT:	Wavelet transform

Chapter 1

Introduction

1.1 Signal Compression

The objective of signal compression is to achieve a low bit rate digital representation of an input signal with minimum perceived loss of signal quality [1]. Signal compression is also often called as source coding in the context of information theory [2]. Lossy source coding [3] for image, audio and video signals has been of great interests due to the proliferation of various multimedia contents over Internet and the bandwidth bottleneck of Internet. Most modern source coding schemes are based on signal decomposition [1–8]. The aim of signal decomposition for source coding is to perform the decorrelation, which means the coefficients after decorrelation are uncorrelated in principle, or nearly uncorrelated in practice. In this dissertation, we study various signal decompositions by filter banks for signal compression.

1.1.1 Classical Transform Coding

Transforms are pervasive in signal processing as analytical tools and computational devices for signal decomposition. One of the most visible successes of transforms and their cousins such as filter banks is in signal compression. In the current popular compression methods and standards for audio, image and video signals, various transforms and filter banks are at the heart. The most popular and well-established transform techniques are the celebrated discrete cosine transform (DCT) [9] used in the lossy JPEG, MPEG, and ITU video coder standards, and the discrete wavelet transform (DWT) [6,7] standardized in MPEG-4 and JPEG 2000 [3,10]. The DCT is applied in these standards strictly as a block-based approach usually on blocks with size of 8×8 . The DWT in contrast is usually implemented in JPEG 2000 and MPEG-4 as a frame-based approach applied to entire images, and block partitions are also possible.

The DCT is a so-called compact transform, because most signal energy is compacted into the lower frequency coefficients. Most higher frequency coefficients are small or zero after quantization, and small or zero-valued coefficients tend to be clustered together. Thus, on average

only a small number of quantized DCT coefficients need to be transmitted to the receiver to obtain a good approximated reconstruction. Similar to the DCT approach, most signal energy is compacted by the DWT into the lower-frequency subbands, and small or zero-valued coefficients tend to be clustered together and *more importantly* in a zero-tree structure approximately, whose nonzero coefficients also tend to cluster in the relative *same* spatial positions in different scales [11]. There is good evidence in the past works that DWT provides improved coding gains compared to DCT strategies for image coding due to the superior performance of DWT.

1.1.2 Multiple Description Coding

Traditional source coding techniques compress the source signals only for high compression efficiency. They do not consider the transmission channel characteristics during the compression. In fact, they implicitly assume that the compressed bit stream can be received correctly. This is Shannon's classical separation model of communication system. However, error-free transmission generally cannot be attained in practice. The uncorrected errors can deteriorate the reconstructed signal quality seriously because source coding generally increases the bit sensitivity to errors greatly. Thus, it seems more appropriate to leave the source signal uncompressed if transmission cannot be made error-free even with channel coding. However, to not compress is a rather extreme reaction to the possibility of transmission errors or erasures. A more compromised approach is to account for the possibility of transmission errors or erasures in the source coding design [12]. Such approach is called error-resilient source coding, of which multiple description coding (MDC) is one class.

In MDC, several descriptions of the same source signal are produced such that various reconstruction qualities can be obtained from different subsets of the descriptions. Unlike traditional layered scalable coding, there is no hierarchy of different descriptions. Thus, MDC is more suitable for packet erasure networks such as Internet without priority provisions. Different descriptions are transmitted perhaps over different channels. It is assumed that the decoder knows which channel is lost, so that it can use different methods for signal reconstruction. The decoder can reconstruct the source signal with *any* nonempty subset of descriptions and various reconstruction qualities can be obtained from different subsets. There are two conditions that make MDC useful in real applications [12]. Firstly, the channels are not error free all the time which means one or more descriptions are to be lost during transmission. If all descriptions are always received correctly, the traditional source coding works well and there is no need to use the MDC. Secondly, different reconstruction qualities are all acceptable and differentiable.

1.2 Signal Decompositions

The uneven distribution of signal energy in the frequency domain makes the signal decomposition as the basis for the signal compression, as shown in rate-distortion theory [1,2]. The signal decomposition in this dissertation is confined to be **linear** decomposition. Given any signal $x[n]$ from the infinite-dimensional Hilbert signal space $\ell^2(\mathbb{Z})$ [6,13], we want to find a set of elementary signals $\{\phi_i[n], i \in \mathbb{Z}\}$ for this space so that any signal $x[n]$ can be decomposed as a linear combination of these elementary signals, i.e., $x[n] = \sum_{i \in \mathbb{Z}} c_i \phi_i[n]$, $\forall x[n] \in \ell^2(\mathbb{Z})$. The set $\{\phi_i[n], i \in \mathbb{Z}\}$ is called *complete* for the space if any signal belonging to the space can be decomposed in such form. A complete set of signals $\{\phi_i[n], i \in \mathbb{Z}\}$ has a *dual* set of signals $\{\tilde{\phi}_i[n], i \in \mathbb{Z}\}$ which can be used to compute the expansion coefficients c_i by $c_i = \sum_n x[n] \tilde{\phi}_i[n]$. Such computation for c_i is actually the *inner product* of the signal $x[n]$ and the dual signal $\tilde{\phi}_i[n]$, denoted by $c_i = \langle x, \tilde{\phi}_i \rangle$.

An important particular case is when the set $\{\phi_i[n], i \in \mathbb{Z}\}$ is orthonormal and complete. This means the set is an *orthonormal basis* for space $\ell^2(\mathbb{Z})$ and the dual set is same as itself, $\tilde{\phi}_i = \phi_i$ for all $i \in \mathbb{Z}$. Then, it can be easily shown,

$$\langle \phi_i, \phi_j \rangle = \delta[i - j], \forall i, j \in \mathbb{Z} \quad (1.1)$$

where $\delta[\cdot]$ is the unit pulse function [5]. However, if the set $\{\phi_i[n], i \in \mathbb{Z}\}$ is complete and the vectors ϕ_i are linearly independent, but not necessarily orthogonal, it is called a *biorthogonal basis*. Its basis vectors and the dual vectors satisfy the following biorthogonal condition.

$$\langle \phi_i, \tilde{\phi}_j \rangle = \delta[i - j], \forall i, j \in \mathbb{Z} \quad (1.2)$$

Finally, if the set $\{\phi_i[n], i \in \mathbb{Z}\}$ is complete but *redundant*, which means the number of vectors is more than that needed for the exact reconstruction, then we do not have a basis, but an overcomplete signal representation. One important subclass of this representation is called *frame*. It can be shown that the vectors ϕ_i in the set generally are not linearly independent. We take a simple example [6] to illustrate these basic but very important concepts for this dissertation.

Example: Three Sets of Vectors for the Two Dimensional Euclidean Space $\mathbb{R}^2$

We show three classes of complete set of vectors for the Euclidean space $\mathbb{R}^2$ in Fig. 1.1. The standard Euclidean basis is denoted by e_0 and e_1 . In Fig. 1.1.a, an orthonormal basis is given by $\phi_0 = [1, 1]^T / \sqrt{2}$ and $\phi_1 = [1, -1]^T / \sqrt{2}$. The dual basis is same as itself. In Fig. 1.1.b, a biorthogonal basis is given by $\phi_0 = e_0$ and $\phi_1 = [1, 1]^T$. Now, the dual basis is $\tilde{\phi}_0 = [1, -1]^T$ and $\tilde{\phi}_1 = [0, 1]^T$. Finally, a frame is given by $\phi_0 = e_0$, $\phi_1 = [-1/2, \sqrt{3}/2]^T$ and

Figure 1.1: Examples of three possible sets of vectors which can completely span the two dimensional space $\mathbb{R}^2$.

$\phi_2 = [-1/2, -\sqrt{3}/2]^T$ shown in Fig. 1.1.c. Then it can be easily verified that *one* possible dual frame is identical, within a scale factor, to itself, namely, $\tilde{\phi}_i = 2/3\phi_i$. In contrast to the orthonormal or biorthogonal basis, the dual set of vectors for a frame is not *unique*.

We will confine our expansions to have some **structures**, which means various basis or frame vectors are related to each other by some elementary operations such as shifting in time, scaling and modulation [6]. Because we are concerned with expansions for very high-dimensional space, possibly infinite, the bases or frames without such structures are almost useless in real applications due to implementation complexity. In this dissertation, we select the filter bank as the basic structure for signal expansions because it has a fine structure for implementation and it is also generalized enough to cover various popular transforms as special cases.

1.2.1 Bases and Critically Sampled Filter Banks

It is well known that critically sampled filter banks (FBs) correspond to a class of discrete-time signal expansions by bases, known as shift-invariant expansions [6]. Such relationship has been well studied in [5–7]. The basis functions of FBs can be orthogonal or biorthogonal according to the property held by the underlying FBs. The key idea in FBs is to decompose the signal into different frequency bands by different analysis filters and then to treat those subbands individually for the application at hand. Within the general framework of FBs, various popular transforms can be analyzed and unified [8]. In contrast to block transforms like DCT whose length of the basis functions are constrained to be same as the size of blocks, more freedom for tuning the basis functions is provided by FBs. The lengths of the basis functions of FBs are extended and thus overlapping from the perspective of the block transform. Such overlapping can eliminate the annoying blocking artifacts inherent in low bit rate image compression by

block transform coding. FBs have found applications in almost every area of modern digital signal processing, including audio, image and video processing, adaptive and statistical signal processing, and communications. Practically, of extreme importance is the capability to design FBs for desired applications. We will study the theory, factorizations and designs for critically sampled FBs and their application to image compression in Part I of this dissertation.

1.2.2 Frames and Oversampled Filter Banks

Traditionally, bases, especially orthonormal basis, play an important role in signal processing because of their compactness and uniqueness for signal representation and of the availability of various fast algorithms for computation. However, unique signal expansion imposes strong *constraints* on bases which may be in conflict with other desirable specifications [14]. For example, nontrivial two-channel orthogonal filter banks exclude the possibility of linear phase filters, which is very desirable for image processing. In addition, the conditions for being a basis, referred in (1.1) and (1.2), are so *strong* that even a slight modification of the basis vectors might destroy the basis property. Thus, the extension of basis expansions, i.e., complete and unique signal expansions, to more general and flexible signal decomposition methods is expected, which motivates us to study the overcomplete signal expansions such as frames [14–16] and oversampled FBs. Overcomplete means that the set of vectors or functions used for expansion is larger than that necessarily needed. In other words, although the vectors or functions behave like basis vectors or functions, they are actually *linearly dependent* in contrast to independent basis vectors or functions. Recently, it has been recognized that oversampled FBs, as a class of frames, correspond to overcomplete signal expansions. In this dissertation, we will study the theory and designs of frames and oversampled FBs and their potential uses for robust transmission of information over noisy or erasure channels in Part II. The frame is a powerful competent in such an application due to its more flexible representations than traditional orthonormal or biorthogonal basis expansions [14].

1.2.3 Signal Compression Based on Filter Banks

According to the above introduction, we show a general framework in Fig. 1.2 for signal compression based on FBs as the signal decomposition tool. The input signal $x[n]$ is first segmented by the delay chain denoted by z^{-1} and the downsampling block $\downarrow M$, then passed through a filter bank to obtain the subband signals $u_i[n]$ with $i = 0, 1, \dots, P-1$. Finally, the subband signals are quantized either by independent scalar quantization or by vector quantization of the vector $\mathbf{u}[n] = (u_0[n], u_1[n], \dots, u_{P-1}[n])$. The entropy coding converts the quantized subband signals $u_{q,i}[n]$ into binary digital stream. Finally, depending on the application, one bit

Figure 1.2: A general framework for signal compression based on filter banks.

stream is obtained for storage or transmission through a single channel, or multiple streams are obtained for transmission over packet networks like MDC.

Such model is generalized enough so that most previous works on source coding, especially image and video coding, can be casted into this general framework as special cases. For traditional transform coding, the FB is a critically sampled FB or related transform [6, 17]. However, for MDC, the FB can be a critically sampled FB as well as an oversampled FB or a frame [18–21]. From this framework, we can recognize the essential importance of design of critically sampled and oversampled FBs.

1.3 Outline and Contributions

The main body of this dissertation is divided into two parts: Part I on critically sampled FBs (Chapters 2-5) and Part II on oversampled filter bank frames (Chapters 6-8). The conclusions and some future research directions are summarized in Chapter 9. The outline and contributions of this dissertation are described below.

1.3.1 Critically Sampled Filter Banks

In **Chapter 2**, we review the basic notions, various structures, some important concepts and lattice structures which are employed extensively throughout this dissertation. In addition, we also present the unifying role of FBs for various transforms and discuss interconnections of block and lapped transforms, FBs and wavelets and their applications in image compression.

Chapter 3 studies a large class of M -channel finite impulse response (FIR) FBs with *arbitrary* equal lengths $L = KM + \beta$ ($0 \leq \beta < M, K \in \mathbb{N}$), in contrast to most previous works with the *unnecessary* length constraint of $L = KM$. Our main contribution here is the establishment of the existence conditions, lattice factorizations and designs for such general class of FBs. An arbitrary filter length L can be easily achieved by appropriately choosing design

parameters β and K . The novel lattice factorizations are proposed for both paraunitary (PU) FBs and perfect reconstruction (PR) FBs, respectively, for any number of channels $M \geq 2$. It is shown that these lattice factorizations can completely span the whole class of both M -channel FIR PU FBs with *any* valid length profile, and the class of PR FBs with length $L \leq 2M$. Several design examples are presented to validate the proposed lattice structures.

Chapter 4 further specializes the above work to linear phase perfect reconstruction FBs (LPPRFBs) with arbitrary equal lengths. The existence conditions are reviewed and the lattice factorizations are developed for both even and odd channel systems. It is shown that these proposed lattice factorizations are complete for the whole class of LPPUFBs with *any* valid length and the class of LPPRFBs with length $L \leq 2M$. In addition, we show that these proposed structures are minimal in the sense of the number of delay elements used in the implementation. When considering regular FBs, we present how to structurally impose one-degree of regularity into the proposed lattice structures. In addition to the development of these more general lattice structures, more theoretical issues for the lattice factorizations of LPPRFBs are further investigated. Firstly, we study the problem of the existence of complete order-one lattice factorization. Through pointing out the limitation of the existing order-one LPPR propagating blocks, we show the *nonexistence* of any complete order-one lattice factorization. Secondly, we investigate the optimality of the existing lattice based design method for LPPRFBs, through studying the spaces $\mathbb{L}_\beta$ for general and regular LPPRFBs spanned by the lattice structures and their closeness property under polynomial transformations. We show the closeness of the spaces $\mathbb{L}_\beta$, which establishes the suboptimality of the lattice based design for LPPRFBs. Finally, several design examples for LPPRFBs are provided and evaluated in a transform-based image coding framework to illustrate the length effect, i.e., a flexible tradeoff between the performance and filter length, on image compression.

Inspired from the success in the application of unequal-length FBs in low bit rate image coding, we further investigate a class of unequal-length FBs with much more flexible length profile in **Chapter 5**. Various lattice structures are developed and the structures for LPPRFBs are shown to be minimal. Our derived existence conditions reveal that it is possible to design unequal-length LPPRFBs with odd number of long filters, which corrects a misleading statement reported before. Consequently, our design approaches are more general and can considerably expand the designer's choices in contrast to very limited possible designs by traditional methods. In addition, the limitation of the hybrid lattice structures for some class of LPPRFBs is pointed out together with some other possible solutions for those FBs. Several design examples are shown to validate the proposed design approaches. The comparison of different FBs in terms of generalized coding gain and real image compression at medium and low bit rate region demonstrates the better performance of unequal-length LPPRFBs.

1.3.2 Oversampled Filter Bank Frames

In **Chapter 6**, we review the fundamentals of frame theory and oversampled filter banks (OFBs) as well as their relationship. In addition, some earlier works on the quantized FB frames is presented.

Chapter 7 investigates the robustness of FB frames in infinite dimensional Hilbert space $l^2(\mathbb{Z})$ from the perspectives of both frame and FB theory. Since the erasure of packets is a common phenomenon in information transmission over Internet, we analyze the robustness of FB frames to erasures. A characterization of the basic constraint on the FB frames is given by studying the maximum spectral singular values of the subframes. As a corollary, we show an existence condition for tight FB frames. For being robust to erasures, a sufficient condition is given for general FB frames robust to erasures without the unnecessary equal-norm property in contrast to previous works. Furthermore, we show that our derived robust condition is also *necessary* for tight FB frames. Our works generalize the previous reported results and indicate that the equal-norm property is unnecessary for the robustness to erasures. In addition, a more general analysis on maximal robust (MR) FB frames is explored based on an explicit constructive approach which can also serve as a guideline for designing MR FB frames. Then, we further study the reconstruction by the pseudoinverse of the subframe which is usually a hybrid FIR and infinite impulse response (IIR) FB. Thus, we specifically examine the *stability* and even possible *FIR* reconstruction of such a pseudoinverse. It is shown that the uniformity of the squared norm $P_i(\omega)$ of the analysis filters over any frequency is the *necessary and sufficient* condition for the FIR pseudoinverse of the subframe after one erasure of tight FB frames.

Because quantization is an inevitable process of digital transmission of information, we further analyze the quantization effect on the reconstruction by the pseudoinverse and the optimal FB frames based on the popular additive, pair-wise uncorrelated quantization noise model. In particular, it is shown that the equal-norm property, imposed frequently in previous works, is *not* necessary for optimization of frames. Moreover, this property is not desirable from the viewpoint of designing FB frames. In fact, we introduce a novel notion, named *frame energy*, which actually dictates the quantization performance of FB frames. Furthermore, we show that tight FB frames are optimal for quantization with bounded frame energy over all FB frames. This result reveals the important *universal optimality* of tight FB frames, in contrast to the restricted optimality within unit-norm tight frames reported before. In addition, the design freedom obtained from removing the equal-norm property is explained and illustrated with examples. Finally, the effect of erasures on the reconstruction MSE for general FB frames is examined. Different from before, we incorporate a prior information on possible erasure

distribution into our analysis, which makes our result more general and can cover the previous work as a special case. In particular, we show that the tight FB frames are optimal over very general FB frames. In contrast to the previous works, we show that the optimal tight FB frames are usually not equal-norm tight frames. An illustrative example of two-state erasure network model is shown to further explain our result. We then have put our analysis into the framework of MDC to show its potential application due to the desirable reduced reconstruction MSE by the correctly received additional subbands.

The designs for OFBs are further presented in **Chapter 8**. Through the extension of our work on critically sampled PRFBs, we propose lattice factorizations for oversampled PRFBs with arbitrary lengths. The difficulty for more rigorous derivation and the limitation of our lattice factorization are discussed. The difference between the forward and backward factorizations for critically sampled and oversampled FBs is presented from the perspective of implementation. We further investigate OLPPRFBs with equal lengths and the same symmetry center among all filters. The existence conditions on the symmetry polarity for such FBs is established first. Lattice structures for type-I OLPPRFBs are developed. Compared with the previous works, our work is more general and can cover a larger class of OLPPRFBs than before. Furthermore, the proposed lattice structures are proven to be complete for type-I OLPPRFBs. Finally, two design examples are shown to demonstrate the validity of the proposed lattice structures and to show the design flexibility for the filter length L .

1.4 Notations

This section introduces some notations and abbreviations widely used in this dissertation. Most of our notation parallels those used in [5].

- Bold-faced quantities denote matrices and vectors such as $\mathbf{A}$, $\mathbf{E}(z)$ and $\mathbf{u}$ etc. The subscripts denoting the dimension of matrices or vectors will generally be omitted unless it is not clear from the context. $\mathbf{I}_M$ denotes the identity matrix with size M , and $\mathbf{0}$ is a null matrix of appropriate size. $\mathbf{U}^H$, $\mathbf{U}^T$ and $\mathbf{U}^{-1}$ denote the conjugated transpose, the transpose and the inverse of the matrix $\mathbf{U}$, respectively. The rank of a matrix $\mathbf{U}$ is denoted by $\rho(\mathbf{U})$. The $\text{diag}(\cdot)$ denotes a diagonal (or block diagonal) matrix with specified diagonal elements (or blocks) in bracket. The tilde operation on a polyphase matrix is defined by $\tilde{\mathbf{E}}(z) = \mathbf{E}^H(1/z^*)$, where $*$ denotes complex conjugation for scalars.
- For FIR FBs, the polyphase matrix can be written as $\mathbf{E}(z) = \sum_{i=0}^K \mathbf{E}_i z^{-i}$, where $\mathbf{E}_K \neq \mathbf{0}$. K is defined as the order of $\mathbf{E}(z)$, i.e., the FIR FB. It is related to the maximum possible filter length L of the FB by $L \leq (K + 1)M$, where M is the subsampling factor.

CHAPTER 1. INTRODUCTION

- The delay chain vector $\mathbf{d}_M(z) = [1, z^{-1}, \dots, z^{-(M-1)}]^T$, and $\delta[n]$ is the unit impulse function, defined by $\delta[n] = 1$ if $n = 0$, otherwise $\delta[n] = 0$.
- The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denotes the sets of complex, real, integer and positive integer numbers, respectively. For $x \in \mathbb{R}$, $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the ceiling and floor operation of x , respectively.
- A special matrix useful for linear phase property of FBs is $\mathbf{I}_M^J = \text{diag}(\mathbf{I}_{\lceil M/2 \rceil}, \mathbf{J}_{\lfloor M/2 \rfloor})$. In addition, $\mathbf{W}_{2m}$ and $\mathbf{W}_{2m+1}$ are $2m \times 2m$ and $(2m+1) \times (2m+1)$ butterfly-like orthogonal matrices, respectively, as follows.

$$\mathbf{W}_{2m} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_m & \mathbf{I}_m \\ \mathbf{I}_m & -\mathbf{I}_m \end{bmatrix}, \quad \mathbf{W}_{2m+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m \\ \mathbf{0} & \sqrt{2} & \mathbf{0} \\ \mathbf{I}_m & \mathbf{0} & -\mathbf{I}_m \end{bmatrix}$$

Part I

Critically Sampled Filter Banks

Chapter 2

Fundamentals of Filter Bank

This chapter presents the basic notions and some important concepts which are employed extensively throughout this dissertation. In addition, we also review some previous works on filter banks and their application to image compression. The coverage here is very brief and fundamental. For further details and applications, the interested reader is suggested to refer to several popular textbooks [5–7]. The proofs for the Lemmas presented in this chapter are omitted and can be found in the above references.

2.1 Structures and Representations for Filter Banks

In this section, we review the basics of multirate signal processing and filter bank systems, especially the various structures and representations for multi-channel FBs.

2.1.1 Critically Sampled Filter Banks

The digital FB system is composed by a set of digital filters with downsamplers and upsamplers in between them, as shown in Fig. 2.1. The original idea of using FB systems is to decompose the input signal $x[n]$ into multiple narrow frequency bands by different analysis filters $H_i(z)$ with $i = 0, 1, \dots, M - 1$ and then using the downsampler $\downarrow m_i$ to reduce the sample rate of each subband signal. After appropriate processing of subband signals $u_i[n]$, the synthesis FB increases the sample rate of each subband to the original sample rate of the input signal and then combines them to obtain the reconstructed signal $\hat{x}[n]$.

According to the relationship between the downsampling factors m_i and the number of channels M , we can classify FBs as follows. If all subsampling factors are identical, i.e., $m_i = M$ for $i = 0, \dots, M - 1$, the FB is named as uniform FB as shown in Fig. 2.2, and otherwise as nonuniform FB. Most research works deal with uniform FBs due to its simplicity and wide practical applications in multimedia signal processing. Moreover, a class of nonuniform FBs

Figure 2.1: The general form of M -channel FBs.

can also be achieved by the tree-cascaded uniform FBs. Thus, in this dissertation, we will concentrate our work on uniform FBs. In addition, for uniform FBs, if the downsampling factor is equal to the number of channels M , such kind of FBs is called critically sampled uniform FB with which the part I of this dissertation mainly deals. However, if the downsampling factor is less than the number of channels, the FB is called oversampled uniform FB which the part II of this dissertation mainly deals with. The final class of uniform FBs with the downsampling factor larger than the number of channels is named as undersampled FB which is of little interest as explained in Sec. 2.2.

Figure 2.2: The regular form of M -channel uniform FBs.

2.1.2 Polyphase Representations

The regular form of uniform FBs shown in Fig. 2.2 is not efficient for real implementation since all the analysis filters $H_i(z)$ and synthesis filters $F_i(z)$ operate at the high sampling rate of input signal $x[n]$. A very useful tool in the theory and implementations of multirate signal processing and FBs is the so called noble identities shown in Fig. 2.3. Notice that the filtering

CHAPTER 2. FUNDAMENTALS OF FILTER BANK

operation after applying noble identities on downsampler or upsampler is on the low sample rate, i.e., subband rate, and thus is more efficient from the perspective of real implementation.

Figure 2.3: The noble identities in multirate signal processing.

Besides the advantage for real implementation, the noble identities facilitate a very important advancement in FB theory, i.e., the invention of the polyphase representation [5]. Consider a filter $H_i(z) = \sum_{n=-\infty}^{\infty} h_i[n]z^{-n}$ of an M -channel uniform FB. Such filter can be *uniquely* decomposed into M groups $E_{ij}(z)$ $j = 0, 1, \dots, M-1$, called polyphase components, as follows.

$$H_i(z) = \sum_{j=0}^{M-1} z^{-j} E_{i,j}(z^M), \text{ with } E_{i,j}(z) = \sum_{m=-\infty}^{\infty} h_i[mM+j]z^{-m}, \quad j = 0, 1, \dots, M-1 \quad (2.1)$$

This is the so called Type-I polyphase representation of a digital filter $H_i(z)$. Extending the Type-I polyphase decomposition to the whole analysis bank, we can obtain the important $M \times M$ *analysis polyphase matrix* $\mathbf{E}(z)$ and its relation to the analysis filters as follows,

$$\mathbf{E}(z) = \begin{pmatrix} E_{0,0}(z) & E_{0,1}(z) & \cdots & E_{0,M-1}(z) \\ E_{1,0}(z) & E_{1,1}(z) & \cdots & E_{1,M-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ E_{M-1,0}(z) & E_{M-1,1}(z) & \cdots & E_{M-1,M-1}(z) \end{pmatrix} \quad (2.2)$$

$$\mathbf{H}(z) \triangleq [H_0(z), H_1(z), \dots, H_{M-1}(z)]^T = \mathbf{E}(z^M) \mathbf{d}_M(z) \quad (2.3)$$

where $\mathbf{d}_M(z)$ is the delay chain vector of size M defined in Sec. 1.4.

Similarly, for each synthesis filter $F_i(z) = \sum_{n=-\infty}^{\infty} f_i[n]z^{-n}$ $i = 0, \dots, M-1$, we can decompose it by another polyphase form, i.e., Type-II polyphase representation, as follows,

$$F_i(z) = \sum_{j=0}^{M-1} z^{-j} R_{i,j}(z^M), \text{ with } R_{i,j}(z) = \sum_{m=-\infty}^{\infty} f_i[mM-j]z^{-m}, \quad j = 0, 1, \dots, M-1 \quad (2.4)$$

Then, we can also obtain the corresponding $M \times M$ *synthesis polyphase matrix* $\mathbf{R}(z)$ and its

CHAPTER 2. FUNDAMENTALS OF FILTER BANK

relation to the synthesis filters as follows.

$$\mathbf{R}(z) = \begin{pmatrix} R_{0,0}(z) & R_{1,0}(z) & \cdots & R_{P-1,0}(z) \\ R_{0,1}(z) & R_{1,1}(z) & \cdots & R_{P-1,1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ R_{0,M-1}(z) & R_{1,M-1}(z) & \cdots & R_{P-1,M-1}(z) \end{pmatrix} \quad (2.5)$$

$$\mathbf{F}^T(z) \triangleq [F_0(z), F_1(z) \cdots F_{M-1}(z)] = \tilde{\mathbf{d}}_M(z) \mathbf{R}(z^M) \quad (2.6)$$

Thus, the M -channel FB depicted in Fig. 2.2 can be redrawn as in Fig. 2.4.a by using (2.2) and (2.5). In addition, it can be further simplified to a more efficient representation as shown in Fig. 2.4.b by employing the noble identities shown in Fig. 2.3. The structure shown in Fig. 2.4.b is more efficient since the filtering operation is done only at $1/M$ of the input sample rate, while the filtering operation in the structures shown in Fig. 2.2 and Fig. 2.4.a is still at the input sample rate.

2.4.a: Polyphase representation without the noble identities applied

2.4.b: More efficient polyphase representation with the noble identities applied

Figure 2.4: Equivalent polyphase representations of M -channel critically sampled uniform FBs shown in Fig. 2.2.

Notice that the i th row of the analysis polyphase matrix $\mathbf{E}(z)$ contains the polyphase components of the i th analysis filter $H_i(z)$, while the i th column of the synthesis polyphase matrix

$\mathbf{R}(z)$ consists of the polyphase components of the i th synthesis filter $F_i(z)$. In the polyphase representation shown in Fig. 2.4.b, the combination of the delay chain and the downsampler at the analysis side works as a blocking device, i.e., serial-to-parallel converter, whereas the combination of the upsampler and the delay chain at the synthesis side can be thought as the de-blocking device, i.e., parallel-to-serial converter.

The invention of polyphase representation shown in Fig. 2.4.b not only provided more efficient implementation, but also advanced the FB theory dramatically. It can simplify the previous results greatly and can show the relationship among the block transform like DCT, lapped transforms, FBs and wavelets clearly. More importantly, it enables the study of FBs from the perspective of multi-input multi-output (MIMO) linear systems [22] represented by the polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$. For example, the classic lattice structure in the electric network and speech processing has been widely employed in factorization based FB design, which is also one of the emphasis of this dissertation.

2.2 Properties of Filter Banks

In this section, we present various desired properties of FBs and some criteria for evaluating the performance of FBs. In discussion of properties of FBs, no processing stage is assumed as shown in Fig. 2.2 in between the analysis bank and the synthesis bank.

2.2.1 Desired Properties

Firstly, FBs can be classified with respect to the nature of filters used in a FB. If all analysis and synthesis filters are **finite impulse response (FIR)** filters, the FB is called FIR FB. Furthermore, if the filter lengths of all the analysis and synthesis filters are identical, it is called *equal-length* FIR FB. Otherwise, it is named as *unequal-length* FIR FB. However, if some filters are infinite impulse response (IIR) filters, the FB is named as IIR FB. As digital IIR filters, the stability of IIR FBs is a tough issue for the design and applications. Thus, if not stated clearly, this dissertation is confined to the FIR FBs due to the inherent stability.

After the analysis filtering and downsampling, the low rate subband signals $u_i[n]$ $i = 0, \dots, M - 1$ usually can provide a more efficient and compact representation of input signal than the input signal $x[n]$ itself since the input signals of interest have an uneven distribution of energy in frequency bands, which is often employed for signal compression and denoising, etc. However, it is usually required that the subband decomposition be invertible since we want to reconstruct or synthesize the original input signal from its subband representation. This leads to the concept of **perfect reconstruction (PR)**. With the help of PR, the error analysis of

CHAPTER 2. FUNDAMENTALS OF FILTER BANK

a FB system can be greatly simplified since all the distortion comes from the processing stage only.

Definition 2.1 *A filter bank is said to have perfect reconstruction if the reconstructed signal $\hat{x}[n]$ is a scaled version of time-delayed input signal $x[n]$, i.e., $\hat{x}[n] = cx[n-d]$ for some nonzero constant c and some integer d .*

Without loss of generality, one can assume $c = 1$ and $d = 0$ by using appropriate scaling factor and time-advance element so that $\hat{x}[n] = x[n]$. In this dissertation, we will always make this assumption. It can be easily shown [6] that the PR property cannot be satisfied for undersampled FB in $\ell^2(\mathbb{Z})$ signal space. This can explain why almost all research work on FBs is focused on critically sampled FBs and OFBs.

The above definition states what PR condition leads to. However, it is not very clear about what the relationship between analysis bank $\mathbf{E}(z)$ and $\mathbf{R}(z)$ should be to achieve PR. Here, we can see the advancement provided by the polyphase representation. Since the mapping between the filters and the polyphase matrices is bijective, i.e., one-to-one correspondence, we can see that PR condition requires that there should be no information loss introduced by the analysis bank, which is equivalent to the nonsingularity of the MIMO system $\mathbf{E}(z)$ [22]. Therefore, we can obtain the following necessary and sufficient condition for the PR with normalized assumption in terms of polyphase matrices.

Lemma 2.1 *An M -channel uniform filter bank is said to have perfect reconstruction if and only if its polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ satisfy $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$.*

This Lemma for PR is much more useful than Definition 2.1 and it actually holds the key to design and implementation of a PRFB. For critically sampled FBs, it basically stated that $\mathbf{R}(z)$ is the inverse of $\mathbf{E}(z)$. Since they are both square polynomial matrices, $\mathbf{E}(z)$ is also the inverse of $\mathbf{R}(z)$, i.e., $\mathbf{E}(z)\mathbf{R}(z) = \mathbf{I}$. This Lemma transforms the filter design problem into the problem of how to design a nonsingular polynomial matrix $\mathbf{E}(z)$ with some good desired properties. In the past, one of the most popular approaches to achieve the design problem is to factorize the polynomial matrix $\mathbf{E}(z)$ into simpler matrices with some special structures, such as lattice or lifting structures. In this dissertation, we will focus on the lattice factorization based approach to filter bank design.

In practice, FIR PRFBs are preferred over IIR ones since the former can be efficiently implemented without concerning the stability problem inherent in the IIR FBs. It can be shown easily that an FIR analysis bank $\mathbf{E}(z)$ can have an FIR synthesis bank $\mathbf{R}(z)$ to achieve PR if and only if the determinant of $\mathbf{E}(z)$ is a monomial as stated formally below.

Lemma 2.2 *For FIR filter banks with polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ to achieve perfect reconstruction, it must have*

$$|\mathbf{E}(z)| = z^{-m} \quad \text{and} \quad |\mathbf{R}(z)| = z^{-n}, \quad m, n \in \mathbb{Z}.$$

A subclass of PRFBs is called **paraunitary (PU)** or orthogonal FBs which deserve special attention. In fact, this concept extends the traditional unitary constant matrices, i.e., $\mathbf{A}^H \mathbf{A} = \mathbf{I}$, to the polynomial matrices. A polynomial matrix $\mathbf{E}(z)$ is paraunitary if

$$\tilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I}, \quad \forall z \in \mathbb{C} \quad (2.7)$$

where $\tilde{\mathbf{E}}(z) \triangleq \mathbf{E}^H(1/z^*)$ as defined in [5]. Recall that on the unit circle $z = e^{j\omega}$, $\tilde{\mathbf{E}}(z) = \mathbf{E}^H(e^{j\omega})$. Thus, the notion of paraunitary implies the unitariness or orthogonality of the polyphase matrix $\mathbf{E}(z)$ on the unit circle. Combined with Lemma 2.1, we can see that the PR synthesis matrix $\mathbf{R}(z)$ is just $\tilde{\mathbf{E}}(z)$. Since $\tilde{\mathbf{E}}(z)$ is the conjugate transpose of $\mathbf{E}(z)$ with time reversal and noticing that the analysis and synthesis filters are the rows and columns of $\mathbf{E}(z)$ and $\mathbf{R}(z)$, respectively, we can see that the filter impulse responses of the analysis and synthesis filters are essentially identical. PUFBs have been extensively studied in the past and have found wide applications in signal processing. Compared to the general PRFBs, PUFBs further demand that the signal's energy be preserved after FB transformation. This property is very desirable in some applications in communications and signal processing.

When all the filters $H_i(z)$ and $F_i(z)$ have linear phase (symmetry or antisymmetry of filter impulse responses for real-valued filter coefficients), such FB is called **linear phase (LP)** FB that is very desirable in image and video processing. The linear phase property of FBs can effectively eliminate the phase distortion that is quite disastrous in many image and video processing applications. In addition, linear phase filters can handle the boundaries of finite length signals via simple symmetric extension method [7, 23] which is a popular way of manipulating the border of images. This is very important for achieving a non-expansive representation [24] of the signal. Furthermore, the symmetric extension method can eliminate the energy leakage due to the discontinuities at the borders of an image, compared to the traditional periodic extension method based circular convolution. From the perspective of computational complexity, a more efficient implementation method can also be obtained by exploiting the LP property.

Since we deal with FIR LP FBs with real-valued coefficients, the condition for a filter $h[n]$ with length N to have linear phase is $h[n] = h[N-1-n]$ (symmetric filter) or $h[n] = -h[N-1-n]$ (antisymmetric filter). When all the filters of an LPFB have the same symmetry center, it is called concentric FB. Without loss of generality, we can always arrange the M channel

CHAPTER 2. FUNDAMENTALS OF FILTER BANK

linear phase filters in such an order that all n_s symmetric filters are placed before the n_a anti-symmetric filters. For the case of a concentric FB with the same filter length $L = KM + \beta$ for all analysis and synthesis filters, the associated polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ should satisfy the generalized LP condition [25],

$$\mathbf{E}(z) = z^{-(K-1)} \mathbf{D}_M \mathbf{E}(z^{-1}) \mathbf{J}_M(z), \quad \mathbf{R}(z) = z^{K-1} \mathbf{J}_M(z^{-1}) \mathbf{R}(z^{-1}) \mathbf{D}_M \quad (2.8)$$

where $\mathbf{J}_M(z) = \text{diag}(z^{-1} \mathbf{J}_\beta, \mathbf{J}_{M-\beta})$ and $\mathbf{D}_M = \text{diag}(\mathbf{I}_{n_s}, -\mathbf{I}_{n_a})$, in which the possible value of n_s and n_a are reported in [25] for various cases. Notice that the matrix $\mathbf{J}_M(z)$ will degenerate into the traditional reversal matrix $\mathbf{J}_M$ when $\beta = 0$.

Another desired property of FB is the **regularity** which is closely related to the performance of image compression based on LPPRFBs. In this dissertation, the degree of regularity of a FB is referred to as the number of multiple zeros at the aliasing frequencies [26, 27].

Definition 2.2 *An M -channel FB is said to be (K_a, K_s) -regular if the analysis and synthesis lowpass filters $H_0(z)$ and $F_0(z)$ have a zero of multiplicity K_a and K_s , respectively, at the aliasing frequencies $2\pi k/M$ for $k = 1, 2, \dots, M-1$.*

Note that one degree of regularity corresponds to no DC leakage of a FB, which is desired for avoiding the artificial checker board effect in low bit rate image coding. Regularity also determines the number of vanishing moments of the M -band wavelets [26]. Furthermore, the decay of wavelet coefficients [7, 28] and the accuracy of wavelet approximation [7, 29, 30] are both determined by the degree of regularity, which is further related to the smoothness of the scaling function. In many applications such as image processing, signal compression and numerical analysis, smoothness plays an important role and regularity is thus very desirable.

Previous studies have shown [27] that the regularity constraint is equivalent to the following conditions on the analysis and synthesis polyphase matrices:

$$\left. \frac{d^n}{dz^n} [\mathbf{E}(z^M) \mathbf{d}_M(z)] \right|_{z=1} = c_n \mathbf{e}_0, \quad \left. \frac{d^m}{dz^m} [\mathbf{R}^T(z^M) \mathbf{J}_M \mathbf{d}_M(z)] \right|_{z=1} = d_m \mathbf{e}_0 \quad (2.9)$$

where $n = 0, 1, \dots, K_s - 1$; $m = 0, 1, \dots, K_a - 1$, and $\{c_n, d_m\}$ are some nonzero constants. $\mathbf{d}_M(z)$ is the delay chain vector defined in Sec. 1.4 of Chapter 1 and $\mathbf{e}_0 = [1, 0, \dots, 0]^T$ is the standard orthonormal basis vector.

2.2.2 Performance Criteria for Design

Several criteria can be employed to evaluate the performance of a FB and to be used as a guideline for designing FBs. We present three popular criteria widely used for designing FBs.

Stopband Energy and Passband Ripple: One objective function for optimization is minimization of the stopband attenuation and/or passband ripple for ideal filter shape, which is the L^2 -norm of the stopband frequency responses and passband ripples. It is a classical design criterion in the design and optimization of FBs [5]. Denote the passband of the i th filter $H_i(z)$ as $[\omega_{i,L}, \omega_{i,H}]$ and the transition bandwidth as $2\epsilon_i$. For M -channel uniform FBs, the parameters for the i th passband are $\omega_{i,L} = i2\pi/M$ and $\omega_{i,H} = (i+1)2\pi/M$. The cost function to be optimized can be C_{SB} , C_{PB} , or weighted sum of both,

$$\begin{aligned} C_{SB} &= \sum_{i=0}^{M-1} \left(\int_0^{\omega_{i,L}-\epsilon_i} |H_i(e^{j\omega})|^2 d\omega + \int_{\omega_{i,H}+\epsilon_i}^{\pi} |H_i(e^{j\omega})|^2 d\omega \right) \\ C_{PB} &= \sum_{i=0}^{M-1} \int_{\omega_{i,L}+\epsilon_i}^{\omega_{i,H}-\epsilon_i} (|H_i(e^{j\omega})| - 1)^2 d\omega \end{aligned}$$

where ϵ_i are all selected to be $0.25\pi/M$ radian (it is approximately 0.1 radian for 8-channel FB.) in this dissertation when this criterion is used. Better frequency selectivity can be achieved if the stopband energy is minimized. The ideal one is the so called brick-wall FB, in which the stopband energy is zero. It has been shown to be the necessary condition for the optimal orthogonal [31] and biorthogonal [32] subband coders.

Coding Gain: Another optimization criterion which is more related to efficient image compression is the generalized coding gain for PRFBs [33, 34], which for one-dimensional signal is taken as,

$$C_{CG} = 10 \log_{10} \frac{\sigma_x^2}{\left(\prod_{i=0}^{M-1} \sigma_i^2 \|f_i\|^2 \right)^{1/M}} \quad (2.10)$$

where σ_x^2 is the input signal variance, σ_i^2 is the variance of the i th subband signal and $\|f_i\|^2$ is the norm of the i th synthesis filter impulse response $f_i[n]$. We consider the popular first-order Gauss-Markov process with correlation coefficient $\rho = 0.95$ as our source signal [4].

DC leakage: DC leakage refers to the DC component of a signal that leaks out of the lowpass band $\{H_0(z)\}$ because the bandpass and highpass filters $\{H_1(z), \dots, H_{M-1}(z)\}$ do not have zero responses at the DC frequency [7]. For M -channel FBs, assuming the filter length is L , this cost function is as follows.

$$C_{DC} = \sum_{i=1}^{M-1} \sum_{n=0}^{L-1} h_i[n] \quad (2.11)$$

In most signal compression and denoising applications, the higher frequency bands are usually heavily quantized or thresholded. Thus, if the DC component is leaked out to the bandpass or highpass band, the input signal will be distorted greatly since most energy of real life signals

is in lowpass band. In image compression, the DC leakage usually leads to the checkerboard artifact on the reconstructed image. However, note that the DC leakage can be easily avoided by imposing the regularity onto the FBs.

In addition, the combination of the above different design criteria is usually employed for designing FBs for some specific applications. The cost function C for the nonlinear optimization of the filter impulse responses is in the following linear form,

$$C = w_1 C_{CG} + w_2 C_{SB,a} + w_3 C_{SB,s} + w_4 C_{DC} \quad (2.12)$$

where $C_{SB,a}$ and $C_{SB,s}$ are the stopband attenuation for the analysis banks and synthesis banks, respectively. For PU FBs, we only need one of them since they are exactly the same due to the PU property. The weighting factors w_i for $1 \leq w \leq 4$ are adopted from the existing works [35] to be consistent with the literature. Note that the term for DC leakage can be dropped if the regularity is structurally imposed on the FBs and thus the cost function can be further simplified to facilitate the nonlinear optimization involved.

2.3 Lattice Factorization

In this section, we review the basic idea of traditional lattice factorization of FBs and talk about the possible generalization and variants. Then, we talk about some useful theoretical properties of the lattice factorization.

2.3.1 The Basic Idea of Lattice Factorization

As shown in Sec. 2.2, the FB design problem can be transformed into constructing the appropriate polynomial matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ satisfying the desired properties. Although this may be achieved easily for constant matrices, it is definitely not a trivial task for polynomial matrices. In the past, several approaches have been developed to address this problem, in which the lattice factorization is most attractive and popular [5]. From the viewpoint of design, it can give rise to unconstrained optimization since those desired properties can be held *structurally*, which means that the properties of a FB are independent of the variation and quantization of the lattice coefficients. From the perspective of real implementation, the lattice factorization can often leads to VLSI-friendly structure.

Lattice factorizations of FBs have been developed over the past two decades. Instead of directly constructing a higher-order polynomial matrix $\mathbf{E}(z)$ with desired properties, the method of lattice factorization constructs $\mathbf{E}(z)$ by cascading some basic lower-order building blocks which have simpler structures, usually expressed by a product of matrices as follows,

$$\mathbf{E}(z) = \mathbf{G}_{K-1}^f(z) \mathbf{G}_{K-2}^f(z) \cdots \mathbf{G}_1^f(z) \mathbf{E}_0^f(z) \quad (2.13)$$

Figure 2.5: Two equivalent lattice factorizations of the analysis bank of an M -channel critically sampled uniform FB.

where the rightmost block $\mathbf{E}_0^f(z)$ is an $M \times M$ matrix polynomial with order N_0 and filter length $L_0 \leq (N_0 + 1)M$. This served as the *initial* polyphase matrix which corresponds to a FB with certain desired properties like PR, LP and regularity. The other blocks $\mathbf{G}_i^f(z)$ for $i = 1, \dots, K - 1$ with same order N_1 are called *propagating* matrices which can preserve those desired properties held in $\mathbf{E}_0^f(z)$ and increase the order of the underlying polyphase matrix by N_1 , i.e., increasing filter length by $N_1 M$. Such a cascade form would finally generate a FB with filter length $L = (K - 1)N_1 M + L_0$. The lattice structure (2.13) has a similar modular structure of linear-predictive lattice filters widely used in speech processing, which means we can easily obtain the given filter length just by varying the number of blocks $\mathbf{G}_i^f(z)$. In addition, many desired properties like PR, LP and the given filter length L can be *structurally enforced* into the lattice structure (2.13), which means the desired properties are always held *independent* of which specific values are chosen for the lattice parameters. Thus, the FBs can be designed with *exact* desired properties and fast convergence via *unconstrained* optimization routine, in contrast to some other methods only capable of approximation [7, 36]. In addition, the lattice factorization is much more preferred for multichannel FBs over the lifting factorization [37–39] which works very well only for two-channel FBs.

Note that in the above factorization, the blocks $\mathbf{G}_i^f(z)$ are factorized out of $\mathbf{E}(z)$ before $\mathbf{E}_0^f(z)$, thus (2.13) is named as the **forward lattice factorization** (denoted by the superscript f). An alternative way of factorization of $\mathbf{E}(z)$ is called the **backward lattice factorization** (denoted by the superscript b) formulated as follows,

$$\mathbf{E}(z) = \mathbf{E}_0^b(z) \mathbf{G}_1^b(z) \mathbf{G}_2^b(z) \cdots \mathbf{G}_{K-1}^b(z) \quad (2.14)$$

where the leftmost block $\mathbf{E}_0^b(z)$ serves as the initial matrix and the blocks $\mathbf{G}_i^b(z)$ are propagating matrices. For a given FB, both lattice factorizations usually exist if the polyphase matrix $\mathbf{E}(z)$

is factorable by the lattice structure. Furthermore, one type of factorization can often be transformed into the other one by appropriate parameter transformation. However, those basic building blocks may have different structural forms and parameters.

2.3.2 Properties of Lattice Factorization

There are three important concepts associated with a given lattice factorization for a class of FBs [5, 40], as itemized below.

- **Completeness:** The completeness of a lattice structure implies that any FB with certain desired properties like PR, LP and given filter length can always be factorized into the *same* lattice structure.
- **McMillan degree and minimality:** The McMillan degree of a FB $\mathbf{E}(z)$ refers to the minimum number of delay elements required for its implementation. In many applications of signal processing, especially for speech and audio processing, excessive delay should be avoided as much as possible. Thus, it is often expected that a lattice factorization for a class of FBs is *minimal*, i.e., the number of delays used is equal to the McMillan degree.
- **Degrees of freedom:** It refers to the number of free parameters in a lattice structure that can be used for optimization of FBs with respect to some design criteria.

2.4 Connections among Transforms, Filter Banks and Wavelets

In this section, we first review the block and lapped transforms from the perspective of filter length, and present their relationship with FBs. Then, we review the wavelet transforms, followed by a discussion of the interconnections of transforms, FBs and wavelets and their applications in image compression.

2.4.1 Block and Lapped Transforms

We review the basics of block and lapped transforms, especially emphasizing the length effect of the basis functions on the performance in image compression.

2.4.1.1 Block Transforms

In block transforms, the analysis or decomposition is done with a blockwise treatment of the input signal. The input signal is firstly segmented into *nonoverlapping* blocks. Then, each block is transformed independently by some matrix $\mathbf{T}$, usually an orthogonal matrix, into a block of

coefficients. One of the block transforms, i.e., the Discrete Cosine Transform (DCT) [9], has been studied extensively and employed in image and video processing because it is a very good approximation to the statistical / optimal Karhunen-Loève transform (KLT) for a wide class of signals. Due to many fast algorithms with efficient hardware and software implementations available, DCT has been adopted by many image and video compression standards such as JPEG and MPEG families.

In traditional block transforms, the length of the basis functions is equal to the size of the data block. This implies that the forward transform and inverse transform matrices are *square*. Such structure has the least possible freedom in tuning its basis functions and can only meet the requirement of orthogonality. The block transform can be viewed as a degenerate case of FBs since the combination of the delay chain vector and the downsampler works as the blocking device and the transform matrix $\mathbf{T}$ serves as the polyphase matrix. However, in contrast to the constant matrix $\mathbf{T}$ of the block transform, the polyphase matrix $\mathbf{E}(z)$ of a FB is usually dependent on the complex variable z .

The performance of image and video compression by using DCT is quite good, i.e., yielding almost visually lossless image reconstruction quality, at high bit rates. However, it is known to degrade significantly at low bit rates. The reconstructed image usually exhibits the annoying blocking artifact, since each block is processed independently by the DCT and the interblock correlation is not exploited. These shortcomings come from the short, nonoverlapping and discontinuous DCT basis functions at the block boundaries.

2.4.1.2 Lapped Transforms and Generalized Lapped Transforms

In order to address the blocking effect inherent in DCT, the lapped orthogonal transform (LOT) has been introduced for DCT by applying a post-processing to the DCT coefficients. In the conventional blockwise setting, this is equivalent to borrowing samples from the adjacent blocks to produce LOT coefficients. Then, the transform becomes *rectangular* or the decomposition has overlapping data blocks. This overlapping can reduce the blocking effect inherent in block transforms. Doubling the length of the basis functions over the size of the data block gives the lapped transform (LT). Although the blocking artifact is reduced significantly by the LOT, they can still be observed in some cases. The performance can be further improved by the lapped biorthogonal transform (LBT). It has been observed in [41] that for image compression, the functions of the forward transform should maximize the energy compaction, whereas the inverse transform provides blocking-free reconstruction. In addition, the basis functions of the inverse transform should be smooth enough to reduce blocking effects and should be short to avoid excessive ringing noise. Since the basis functions of the forward and inverse transforms

of an orthogonal system are essentially same, it is very difficult to achieve these objectives *simultaneously*. On the other hand, biorthogonal transforms do not suffer from this restriction, and they are more appropriate for this kind of application.

The basis functions of LT can decay smoothly to zero at the boundaries and thus can reduce the blocking artifact greatly. However, the LT cannot give a sharp cutoff in frequency domain with enough attenuation. More freedom for tuning the basis functions is possible if we specifically extend the length of these functions, leading to the so called generalized lapped transform (GLT) with filter length $L = KM$. A typical example is the generalized lapped orthogonal transform (GenLOT) developed in [42]. It was well known that various GLTs actually correspond to a class of LPPRFBs with constrained lengths $L = KM$. Moreover, the LOT and GenLOT are corresponding to the LPPUFBs with lengths $L = KM$. However, most GLTs are based on the DCT and thus possess fast implementation algorithms and good performance in real applications. In fact, the structure of these LTs is essentially same as the forward lattice factorization as shown in Fig. 2.5.a, where the initial matrix $\mathbf{E}_0^f(z)$ is usually reserved by the DCT matrix and the propagating matrices $\mathbf{G}_i^f(z)$ function as the transform-domain postprocessing of DCT coefficients. In turn, the theory and developments of FBs facilitate more efficient designs and implementations of GLTs.

Recently, Tran *et al.* have developed a family of LTs with flexible lengths and efficient structures and implementations [43–45]. In this family of LTs, the postprocessing of DCT coefficients has been transferred cleverly in time-domain just before DCT. This family of LTs is named as time-domain lapped transform (TDLT), which actually corresponds to a class of LPPRFBs that we will present in Chapter 4 where we further discuss the connection between them. In fact, the structure of TDLTs is essentially equivalent to the backward factorization of FBs $\mathbf{E}(z)$ as shown in Fig. 2.5.b, where the initial matrix $\mathbf{E}_0^b(z)$ is usually reserved by the DCT matrix and the propagating matrices $\mathbf{G}_i^b(z)$ function as the time-domain preprocessing. Compared with the conventional LTs, the framework of TDLTs is more compliant with the existing DCT-based signal compression systems like JPEG and MPEG. The time-domain preprocessing blocks can be incorporated easily into the existing softwares or hardware implementations with little changes, which makes it backward compatible to the existing JPEG and MPEG image/video coding standards. Due to its backward compatibility, superior performance improvement and moderate increase of computation complexity, it was recently adopted in Microsoft Windows Video format standard [46].

2.4.2 Wavelet Transforms

The discrete wavelet transform (DWT) has found wide applications in communications, signal processing and image analysis, etc. The traditional two-band wavelet system is obtained

CHAPTER 2. FUNDAMENTALS OF FILTER BANK

by successively iterating a regular two-channel FB on its lowpass band. Thus, it generates an octave-band FB with long low frequency basis functions and short high frequency basis functions. As a result, the blocking artifact can be eliminated completely when the wavelet transform is applied to image compression. Together with other attractive properties and fast implementations of the WT, it has been adopted recently by the new generation of image coding standard JPEG2000 in order to replace the old JPEG standard based on DCT. In addition, the idea of dyadic wavelet has been extended to the more general M -band wavelets. They can also be generated by iterating M -channel regular FBs, which could lead to more compact representation and more efficient processing of signals than the two-band counterpart.

Figure 2.6: Equivalent representations of conventional two-band discrete time wavelet with three levels of decomposition.

Wavelet transform has been proposed as a new multiresolution decomposition tool [47], which provides a description of a signal on various levels of resolutions or scales. The concept of multiresolution is shown in Fig. 2.6.a by an example of three levels of decomposition of a two-band wavelet. In fact, it corresponds to the dyadic subband tree structure of nonuniform FBs. Simply speaking, the wavelet transform permits a decomposition of a signal into the sum of a coarse (lower resolution) signal at certain scale plus sum of detail signals at different resolutions. The input signal $x[n]$ is first split into lowpass and highpass components in the first level of decomposition. This first lowpass subband, usually containing most of the energy, is subsampled and again decomposed into lowpass and highpass subbands in the second level of decomposition. This process can be continued into more levels. (In wavelet theory, it can be iterated infinitely as long as the two-channel underlying FB is regular. The interested reader is suggested to refer to references [15, 28, 48] from the perspective of mathematical exposition

CHAPTER 2. FUNDAMENTALS OF FILTER BANK

and [5–8] from the viewpoint of subband decomposition by FBs.) The coarsest approximation of the input signal is labeled by $c_3[n]$ which is an octave-band lowpass approximated $x[n]$. Moving from the right to the left in this figure, we can see a progression from the coarser to finer signal representation with the highpass components, i.e., the refined details $d_i[n]$, added at the each level. The input signal thus can be approximately represented by different resolutions at each level of the dyadic tree.

It is well known that the discrete wavelet transform corresponds to the iteration of FBs. The kernel of the wavelet transform is obtained through the dilations and translations of a prototype function, called the wavelet function. From the FB point of view, dilation is just a simple but not very efficient method to increase filter length. A wavelet system with *finite* levels of decomposition, i.e., an octave-band iterated nonuniform FB, can also be represented by an M -channel uniform FB as shown in Fig. 2.6.b, but with very long filter length for the lowpass channel and short filter lengths for bandpass and highpass channels. For the popular 9/7 biorthogonal wavelet [6, 7, 15] (the lowpass analysis filter has length 9 and the highpass analysis filter has length 7) with three levels of iteration, the resulting eight filters have lengths of 57, 49, 21, 21, 7, 7, 7, 7 from the lowpass to highpass analysis filters, respectively. From the perspective of FBs, the very long filter length of the equivalent lowpass filter explains the smoothness of the reconstructed image where the blocking artifact is effectively eliminated.

2.4.3 Interconnection among Transforms, Filter Banks and Wavelets

In this section, we discuss the interconnection among transforms, FBs and wavelets. We also present the difference among them in the conventional transform-based image compression setting.

2.4.3.1 Filter Length

Firstly, if the filter length L is *constrained* to be the number of channels of a FB, i.e., $L = M$, the FB is degenerated into a block transform, where the polyphase matrix $\mathbf{E}(z)$ becomes a constant matrix $\mathbf{T}$. In particular, if the filter coefficients are the following cosine functions, the FB degenerates into DCT.

$$h_k[n] = \sqrt{\frac{2}{M}} a(k) \cos \left[\frac{\pi}{2M} (2n+1)k \right], \text{ with } a(k) = \begin{cases} 1/\sqrt{2}, & k = 0 \\ 1, & 1 \leq k \leq M-1 \end{cases} \quad (2.15)$$

However, the filter length of a *nontrivial* FB is larger than M in general. In such case, the polyphase matrix $\mathbf{E}(z)$ is indeed a polynomial matrix, i.e., dependent on the complex variable z . The matrix $\mathbf{P}$ of filter impulse responses is of size $L \times M$ with columns as the basis functions.

Figure 2.7: A block diagram for illustration of lapped transforms in 2D setting.

Note that it is no longer a square matrix as in the case of block transform, but a rectangular matrix.

Based on the structure of matrix $\mathbf{P}$, we can further explain the subband decomposition by a FB from the perspective of block transform as shown in Fig. 2.7. Different from traditional block transform, the size of underlying block to be transformed is different from the size of transform matrix. Here the transform matrix $\mathbf{P}$ has size $L \times M$ with columns as the basis functions. The size of image blocks is $L \times L$, but each block of transform coefficients has size $M \times M$. To have perfect reconstruction, we have to make the underlying block $\mathbf{B}_2$ to overlap the block $\mathbf{B}_1$ as shown in Fig. 2.7. However, the blocks $\mathbf{C}_1$ and $\mathbf{C}_2$ of transformed coefficients are not overlapping. Thus, subband decomposition looks very similar to traditional block transform, but with a sliding window with the size of the filter length of the FB. Although the decomposition stages are similar, one essential difference is the reconstruction. The conventional simple inverse transform method is not applicable to FBs or wavelets due to the overlap of basis functions.

From the design perspective, we indeed lose design freedom when setting the filter length $L = M$ a priori, because the length L is usually not constrained by practical applications like image processing. It is free for the designer or user to try to improve the performance if possible at the expense of increasing computational complexity. Setting L equal to M is just for simplicity, especially in case of reconstruction.

In addition, from the perspective of implementation and computational complexity, the filter length L is closely related to the complexity. From Fig. 2.7, we can see that the subband decomposition is equivalent to a transform with a sliding and overlapping window of size $L \times L$. Without further imposed structures on the underlying FB, the computational complexity for each transform coefficient is L times multiplication and $L - 1$ times addition of real or complex numbers. Thus, the filter length L is almost linearly proportional to the computational complexity.

2.4.3.2 Structure of Grouping Coefficients

Next, we show their relationship and subtle differences of the structure for grouping coefficients in 2D setting as shown in Fig. 2.8.

Figure 2.8: Different structures for grouping the transformed coefficients.

First, we show the block transform case by an example of 4×4 transform over an image with sixteen blocks with four columns and rows, respectively. Each block has sixteen coefficients arranged in conventional setting. The square one is the so called DC coefficient, while the other ones are AC coefficients. We can see that each block of coefficients is *locally spatial*, which means it only contains the information of that spatial block in image pixel domain. However, each block of coefficients has full frequency bands, from DC to all AC coefficients.

On the contrary, the subband coefficients can be seen as a reordering of those block transformed coefficients into subband structure which groups the coefficients of same frequency band together. Therefore, each subband is *local in frequency band* but has full spatial information of the original whole image.

Wavelet structure is very similar to subband structure except they further group some subbands to get the pyramid representation, thus achieving the scale concept. (The popular zero-tree structure is also shown here. Note that it just reflects the spatial relationship among the coefficients at different scales.) Thus, it is just the translation or reintroduction of the spatial structure of the block transform structure into the wavelet. It has been shown and demonstrated that such embedded zero-tree coding and its variants can greatly improve the image compression performance at low bit rates [11, 35, 49, 50].

Here, we briefly summarize the relationships of how to group coefficients after decomposition. FBs group coefficients according to the *frequency band*, while wavelets group the coefficients in a similar way as FBs with an additional *pyramidal structure*. However, the block transforms group coefficients according to their *spatial positions*.

2.5 Brief Review of Previous Related Works

This section reviews some previous related works on critically sampled FBs and also serves as a guideline for our works to be presented in Part I of this dissertation. The details of our works are to be presented in subsequent chapters, and the difference and improvement of our works compared with the existing ones are further explained therein.

Table 2.1: The existing works on FBs with equal lengths $L = KM + \beta$ ($0 \leq \beta < M$).

Filter Banks	Paraunitary	Perfect Reconstruction
Nonlinear phase	Only modulated FBs, especially cosine-modulated FBs were studied [51]. Quadratic-constrained least-square based design of the lowpass prototype filter and FBs [52] Efficient implementation of arbitrary-length cosine-modulated FBs [53]	
Linear phase	Existence conditions were reported in [25].	
	Lattice factorization for even-channel LPPUFBs with even β [25]	Lattice factorization for even-channel LPPRFBs with even β [55]
	Lattice factorization for odd-channel LPPUFBs with odd β [54]	
	Lattice factorization for even-channel cosine-modulated LPPUFBs [56]	
	Generalized lapped transforms by time-domain pre/post-filtering [43]	

The class of FBs with equal lengths $K = KM$ has been extensively investigated in the past two decades, including PR/PU FBs and linear phase FBs. The existence conditions, lattice factorizations and their properties, various design methods and efficient implementation were reported in the literature. However, the unnecessary length constraint $L = KM$, which traditionally comes from the generalization of block transforms, is imposed a priori just for simplification of mathematical derivation. It has been recognized that such unnecessary constraint on the filter length can be relaxed to further generalize FB theory and designs which lead to a better trade-off between the performance and the design/computation complexity. Thus, a larger class of FBs with arbitrary equal lengths $L = KM + \beta$ ($0 \leq \beta < M$) has attracted attention since it can cover the previous works as a special case of $\beta = 0$. We summarize the documented existing works on such class of FBs in Table 2.1 with references therein of some classical works.

In addition to the above existing works, there are still some open problems to be solved. We list some of them in Table 2.2, which is to be investigated in Part I of this dissertation.

Similarly, the past works on some open problems for a class of unequal-length (UL) FBs are summarized in Table 2.3.

CHAPTER 2. FUNDAMENTALS OF FILTER BANK

Table 2.2: The remaining problems in the existing literature for FBs with equal lengths $L = KM + \beta$ ($0 \leq \beta < M$), and some of them to be solved in Part I of this dissertation.

Filter Banks	Remaining problems in the existing literature
Nonlinear phase PRFBs	Restriction on filter length L (Sec. 3.2.1)
	Lattice factorizations for both general PUFBs (Sec. 3.3.1) and PRFBs (Sec. 3.3.2), and completeness of the lattice factorizations
	Lattice factorization for regular FBs (Sec. 3.4)
	Further simplification of the lattice factorizations
Linear phase PRFBs	Refined existence conditions (Sec. 4.2)
	Lattice factorization for odd-channel LPPRFBs with even β (Sec. 4.3.1) and odd β (Sec. 4.3.2)
	Minimality and completeness of lattice factorizations (Sec. 4.4)
	Existence of complete order-one lattice factorization (Sec. 4.5)
	Closeness of the space spanned by the lattice structures and the optimality of lattice based design (Sec. 4.6)

Table 2.3: The previous works and remaining problems in the existing literature for FBs with unequal lengths, and some of the problems to be solved in Part I of this dissertation.

	Unequal-length FBs
Past works	UL-LPPUFBS with two sets of filter lengths $L_l = (K + 1)M$ and $L_s = KM$ [57]
	UL-GenLOTs with flexible lengths $L_i = K_i M$ but with additional constraints of pair of symmetric-antisymmetric filters [58]
	UL lapped transforms with flexible lengths $2M$ and M only [59] and some extensions [60]
Remaining problems	The existence condition for the filter lengths of UL-PRFBs
	Lattice factorizations for both general and regular UL-PRFBs with $L_i = K_i M + \beta$ (Sec. 5.3)
	Existence conditions for UL-LPPRFBs with $L_i = K_i M + \beta$ (Sec. 5.4.1)
	Lattice factorizations for such class of UL-LPPRFBs (Sec. 5.4.2) & (Sec. 5.4.3)
	Minimality of the lattice factorizations of UL-LPPRFBs (Sec. 5.4.4)

Chapter 3

Arbitrary Equal-Length Multichannel Filter Banks

3.1 Introduction

The multichannel critically sampled filter bank, as a powerful tool for time-frequency signal analysis and fast implementation for wavelets, has been extensively studied and employed in various signal processing applications [5–8]. In the past two decades, there has been persistent interest in the study of theory, factorizations and designs for FIR FBs due to their inherent stability and wide applications in signal and image processing.

The fundamental theory and factorization results for multichannel FIR FBs were reported in [61, 62] for paraunitary (PU) FBs and in [63–65] for more general perfect reconstruction (PR) FBs. In addition, they were studied in [66] under the framework of extended lapped transforms. One important advance of the theory for multichannel FBs is the introduction of the polyphase representation and its characterization for FBs with various desired properties. Such advancement facilitates the lattice factorization based FB designs. The lattice factorization for M -channel FIR PUFBs was firstly reported in [61] and further studied in [67–69]. Recently, several more efficient lattice structures were reported in [70, 71]. For general PRFBs, an important subclass of PRFBs, i.e., causal FIR FBs with anticausal FIR inverses (CAFA-CAFI), was introduced in [64, 65] for the characterization of all FIR PRFBs. Recent works on such PRFBs include [72] for imposing structural regularity in the dyadic-based form and [73] for simplifying the lattice structure. In addition, the relationship between FBs and wavelets was further investigated in [26, 27, 72, 74, 75].

In this chapter, we will further study a more general class of M -channel FIR FBs with *arbitrary* equal length $L = KM + \beta$ ($0 \leq \beta < M, K \in \mathbb{N}$), in contrast to most previous works with the *unnecessary* length constraint of $L = KM$. The main motivation of such FBs is to further

develop more general theory, factorizations and designs for FIR FBs to complement previous works. Our work can also yield more *flexible* designs on FBs for some desired applications like low bit rate image coding, whereas traditional works [61, 63, 65, 66, 69–73] with unnecessary length constraint of $L = KM$, i.e., $\beta = 0$, greatly limit the possible designs. For example, if the length is constrained to be maximum $2M$ (lapped transforms) due to computational complexity requirement for power consumption of portable devices, there are only two possible choices in conventional designs, in contrast to M possible choices by our method. The restriction becomes more severe for large M . Moreover, our method can offer better trade-off between the filter length and the performance of FBs than traditional methods. From the perspective of designing FBs, such more flexible filter length profile can also yield a better trade-off between the design and optimization complexities and the filter length than before. In addition to these advantages, such FBs with regularity can generate M -band wavelets with more flexible compact supports, since the nonvanishing support lengths of scaling and wavelet functions are linear functions of the filter length [7].

Previous works for the case of $\beta > 0$ have only been reported in few cases for some restricted forms of FBs, especially for modulated filter banks [51]. In [52], the cosine-modulated PRFBs were studied. In contrast to lattice factorization, they employed the quadratic-constrained least squares formulation to design the lowpass prototype filter $H_0(z)$ and then to obtain PRFBs by cosine-modulation structure. In addition, an efficient implementation of arbitrary-length cosine-modulated FBs was reported in [53]. Besides these works, the FBs with arbitrary filter lengths were studied in [76] based on minimization of mean square error in general subband coding setup. However, their numerical optimization based design methods cannot structurally enforce the PR property. Thus, their designed FBs which are source and bit rate dependent, have no PR property in general, and suffer from quantization of coefficients.

Different from before, we systematically investigate the more general class of arbitrary-length CAFACAFI FBs in this chapter *without* any unnecessary constraint. The existence conditions and lattice factorizations are developed for both perfect reconstruction and paraunitary FBs with any number of channels $M \geq 2$. Furthermore, it is shown that the novel derived lattice is complete for any FIR paraunitary FB. The outline of this chapter is as follows. First, some existence conditions on the filter length L for such FBs are given in Sec. 3.2.1, followed by a brief review of general lattice structures in 3.2.2. Next, arbitrary equal-length PUFBs and PRFBs are studied in Sec. 3.3.1 and Sec. 3.3.2, respectively. Regular FBs are to be further studied in Sec. 3.4. Finally, several design examples are shown in Sec. 3.5 to validate the proposed design method based on our derived lattice factorization.

3.2 Existence Conditions and General Lattice Factorization

3.2.1 Restriction on Filter Length

Due to the PU constraint which implies orthogonality of the filter impulse response to its own shifted versions [5] and the PR constraint which implies the biorthogonality condition (orthogonality on the two sets of functions) on the analysis and synthesis filter impulse responses, there are some restrictions on the filter length L for such (nontrivial) FBs as stated in the following theorem.

Theorem 3.1 *For equal-length perfect reconstruction filter banks with CAFACAFIs, the filter length $L \neq KM + 1$ for any $K \in \mathbb{N}$.*

Proof: Denote the causal analysis and anticausal synthesis polyphase matrices by $\mathbf{E}(z) = \sum_{i=0}^K \mathbf{E}_i z^{-i}$ and $\mathbf{R}(z) = \sum_{i=0}^K \mathbf{R}_i z^i$. The proof is based on the rank constraint of the highest-order blocks of the matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$, i.e., $\mathbf{E}_K$ and $\mathbf{R}_K$. Partition the matrix $\mathbf{E}_K$ column-wise and the matrix $\mathbf{R}_0$ row-wise as follows,

$$\mathbf{E}_K = [\mathbf{e}_{K,0}, \mathbf{0}, \dots, \mathbf{0}], \quad \mathbf{R}_0 = [\mathbf{r}_{0,0}^T, \dots, \mathbf{r}_{0,M-1}^T]^T \quad (3.1)$$

where the vectors $\mathbf{e}_{K,0}$ and $\mathbf{r}_{0,i}$ with $i = 0, \dots, M-1$ are column and row vectors, respectively. The PR condition

$$\mathbf{E}(z)\mathbf{R}(z) = \mathbf{E}_K \mathbf{R}_0 z^{-K} + \dots + (\mathbf{E}_0 \mathbf{R}_0 + \dots + \mathbf{E}_K \mathbf{R}_K) z^0 + \dots + \mathbf{E}_0 \mathbf{R}_K z^K = \mathbf{I}_M \quad (3.2)$$

leads to the matrix equation $\mathbf{E}_K \mathbf{R}_0 = \mathbf{0}_M$ for the power term z^{-K} . From the above partition for these two matrices, we can see it can yield $\mathbf{e}_{K,0} \mathbf{r}_{0,0} = \mathbf{0}$. We can conclude that $\mathbf{e}_{K,0} = \mathbf{0}$ due to the dyadic form and the assumption of $\mathbf{r}_{0,0} \neq \mathbf{0}$. Then, we can see the whole matrix $\mathbf{E}_K$ is a null matrix. Since the vector $\mathbf{e}_{K,0}$ is located at the boundaries of all analysis filters, it leads to the effective filter length of the analysis FB $L < KM + 1$, at maximum to be $L = KM$. Similarly, the PR condition results in the matrix equation $\mathbf{E}_0 \mathbf{R}_K = \mathbf{0}$ for the power term z^K , which would lead to $\mathbf{R}_K$ to be the null matrix. Consequently, the effective filter length of the synthesis bank $L < KM + 1$. ■

Remark 3.1 *The above Theorem can also be proved in the time domain directly by the time-domain condition for PR, i.e., biorthogonality condition [6] on the analysis and synthesis filter impulse responses $h_i[n]$ and $f_i[n]$ with $i = 0, 1, \dots, M-1$ and $n = 0, 1, \dots, KM + \beta - 1$.*

$$\sum_n h_i[n] f_j[n - lM] = \delta[i - j] \delta[l], \quad 0 \leq i, j \leq M-1, \quad \forall l \in \mathbb{N}$$

The proof can be obtained by setting $l = K$ and $l = -K$. In fact, it is equivalent to the above proof in polyphase domain by evaluating the coefficient of matrix polynomial (3.2).

CHAPTER 3. ARBITRARY EQUAL-LENGTH MULTICHANNEL FILTER BANKS

Such existence conditions are very useful guidelines for FB designs. They can narrow down the solution space for FB designer, which is helpful for deriving efficient lattice structures and avoiding impossible design specifications. It also helps to explain why only some solutions exist. For example, there does not exist any two-channel PUFB with odd length [74], which is just a special case $M = 2$ of our result. A similar result reported before for LPPUFBs [77] is also just a special case of Theorem 3.1 since it applies for all *general* equal-length PRFBs.

3.2.2 General Lattice Factorization

Here, we first present the general forward lattice factorization for FIR FBs with the given filter length $L = KM + \beta$, followed by the backward factorization and their relationship. The lattice structure (2.13) presented in Sec. 2.3 can be adapted for this class of FBs in the following form,

$$\mathbf{E}(z) = \mathbf{G}_{K-1}^f(z) \cdots \mathbf{G}_1^f(z) \mathbf{E}_0^f(z) \quad (3.3)$$

where $\mathbf{E}_0^f(z)$ is a FB with prescribed properties like PU/PR and length $L_0 = N_0M + \beta$, and each block $\mathbf{G}_i^f(z)$ with order N_1 usually can simultaneously increase filter length by N_1M ¹ and propagate those desired properties held in $\mathbf{E}_0^f(z)$. In this way, we can easily construct FBs with desired properties and prescribed length $L = [(K-1)N_1 + N_0]M + \beta$. If we can make $N_0 = N_1 = 1$, i.e., order-one blocks, then it can be easily seen that L is just the given filter length. Such kind of idea has been studied and extensively used in digital lattice filters for speech signal processing [1].

Next, we present some existing structures for the desired order-one propagating blocks $\mathbf{G}_i^f(z)$ widely used in the literature [65]. For general PR FBs,

$$\mathbf{G}_i^f(z) = \mathbf{I}_M - \mathbf{U}_i \mathbf{V}_i^H + z^{-1} \mathbf{U}_i \mathbf{V}_i^H \quad (3.4)$$

where $\mathbf{U}_i$ and $\mathbf{V}_i$ are $M \times r_i$ matrices ($1 \leq r_i < M$) and $\mathbf{V}_i^H \mathbf{U}_i$ is an upper-triangular matrix. For the equal filter length L between analysis and synthesis filters, we need to further impose an additional condition $\mathbf{V}_i^H \mathbf{U}_i = \mathbf{I}_{r_i}$ [65, 70]. Then, it can be easily seen that the inverse of the propagating block is $[\mathbf{G}_i^f(z)]^{-1} = \mathbf{G}_i^f(z^{-1}) = \mathbf{I}_M - \mathbf{U}_i \mathbf{V}_i^H + z \mathbf{U}_i \mathbf{V}_i^H$, i.e., order-one but anticausal. For the subclass of PUFBs, we just need to further constrain $\mathbf{V}_i = \mathbf{U}_i$. Note that the above dyadic-form based block $\mathbf{G}_i^f(z)$ can also be rewritten as $\mathbf{W}_i^f(z) = \mathbf{W}_i^f \mathbf{\Lambda}_i(z)$, where $\mathbf{W}_i^f$ is an $M \times M$ invertible matrix for PRFBs or a unitary matrix for PUFBs, and $\mathbf{\Lambda}_i(z)$ is a delay matrix in the form of $\text{diag}(\mathbf{I}_{M-r_i}, z^{-1} \mathbf{I}_{r_i})$. Furthermore, such $\mathbf{W}_i^f(z)$ can be further

¹In some exceptional cases, the block $\mathbf{G}_i^f(z)$ may not be able to increase filter length due to the vanishing of the higher-order coefficient matrices of the product of blocks $\mathbf{G}_i^f(z)$. An example is the block composed by the unimodular matrix [5]. However, this can usually be avoided by parameter checking during design process.

simplified [70, 71, 73] by successively removing redundant parameters in the product of $\mathbf{W}_i^f(z)$ with the remaining lattice structure.

By using simple matrix algebra, we can easily obtain the backward factorization from the forward one (3.3) as follows.

$$\mathbf{E}(z) = \mathbf{E}_0^b(z) \mathbf{G}_1^b(z) \mathbf{G}_2^b(z) \cdots \mathbf{G}_{K-1}^b(z) \quad (3.5)$$

For the case of constrained filter length $L = KM$, it has been shown that $\mathbf{E}_0^f(z)$ becomes a constant invertible matrix $\mathbf{E}_0$ [34]. Then, it can be shown that $\mathbf{E}_0^b(z) = \mathbf{E}_0$ and the block $\mathbf{G}_i^b(z)$ in backward factorization (3.5) can be made in the *same structural form* as the $\mathbf{G}_i^f(z)$, but with different parameter matrices $\hat{\mathbf{U}}_i$ and $\hat{\mathbf{V}}_i$ as follows,

$$\mathbf{G}_i^b(z) = \mathbf{I}_M - \hat{\mathbf{U}}_i \hat{\mathbf{V}}_i^H + z^{-1} \hat{\mathbf{U}}_i \hat{\mathbf{V}}_i^H \quad (3.6)$$

where $\hat{\mathbf{U}}_i = \mathbf{E}_0^{-1} \mathbf{U}_i$ and $\hat{\mathbf{V}}_i = \mathbf{E}_0^H \mathbf{V}_i$. It can be easily seen that $\hat{\mathbf{V}}_i^H \hat{\mathbf{U}}_i = \mathbf{V}_i^H \mathbf{U}_i$ to be an upper triangular or identity matrix. Same as the block $\mathbf{G}_i^f(z)$, each block $\mathbf{G}_i^b(z)$ can also increase the filter length by M . Note that the dyadic-form based block $\mathbf{G}_i^b(z)$ can also be written as $\mathbf{W}_i^b(z) = \mathbf{\Lambda}_i(z) \mathbf{W}_i^b$, where $\mathbf{W}_i^b$ is an $M \times M$ invertible or unitary matrix for PRFBs or PUFBs, respectively.

However, the *difference* between our work on the lattice factorization (3.3) and conventional ones lies in the starting block $\mathbf{E}_0^f(z)$. Contrary to the traditional works [61, 63, 65, 66, 69–73, 78], $\mathbf{E}_0^f(z)$ cannot be made order zero, i.e., constant matrix $\mathbf{E}_0$, for the general case of $\beta \neq 0$, which is treated as a trivial case because it would impose at least $(M - \beta)$ zero filter coefficients at fixed positions including the boundary. Thus, $\mathbf{E}_0^f(z)$ has at least order one, i.e., $N_0 \geq 1$. Similarly, the residual block $\mathbf{E}_0^b(z)$ has at least order one for the case of nonzero β . However, if we use some modified backward blocks $\hat{\mathbf{G}}_i(z)$ in the right most of factorization (3.5), it is possible to have zero-order block $\mathbf{E}_0^b$ shown in the next section. This has also been studied for a class of linear phase filter banks under the frameworks of lapped transforms via pre/post filtering [43, 45] for efficient image and video coding, which will be further discussed in Chapter 4 for linear phase perfect reconstruction filter banks.

3.3 Lattice Factorizations of General Perfect Reconstruction Filter Banks

In this section, we develop the lattice factorizations for general paraunitary and perfect reconstruction filter banks in Sec. 3.3.1 and Sec. 3.3.2, respectively. In the following, we will first derive the forward lattice factorization in detail and then present the backward one with a discussion on their relationship.

3.3.1 Paraunitary Filter Banks

Consider an M -channel PUFB with filter length $L = KM + \beta$. Since the general PU propagating block $\mathbf{G}_i^f(z)$ in Sec. 3.2.2 has order one, i.e., $N_1 = 1$, the minimal order of initial block $\mathbf{E}_0^f(z)$ can be made one, i.e., $N_0 = 1$ and length $M + \beta$. Thus, $\mathbf{E}_0^f(z)$ can be written as,

$$\mathbf{E}_0^f(z) = [\mathbf{E}_{00} + z^{-1}\mathbf{E}_{01}, \mathbf{E}_{02}] \quad (3.7)$$

where $\mathbf{E}_{00}$ and $\mathbf{E}_{01}$ are both $M \times \beta$ matrices, and $\mathbf{E}_{02}$ is an $M \times (M - \beta)$ matrix. Then, with the PU condition $\tilde{\mathbf{E}}_0^f(z)\mathbf{E}_0^f(z) = \mathbf{I}_M$, the following matrix equations can be established.

$$\mathbf{E}_{00}^H\mathbf{E}_{00} + \mathbf{E}_{01}^H\mathbf{E}_{01} = \mathbf{I}_\beta \quad (3.8)$$

$$\mathbf{E}_{02}^H\mathbf{E}_{02} = \mathbf{I}_{M-\beta} \quad (3.9)$$

$$\mathbf{E}_{02}^H\mathbf{E}_{00} = \mathbf{0}_{(M-\beta) \times \beta} = \mathbf{E}_{02}^H\mathbf{E}_{01}, \quad \mathbf{E}_{00}^H\mathbf{E}_{02} = \mathbf{0}_\beta \quad (3.10)$$

With these equations, the rank conditions on the matrices $\mathbf{E}_{00}$, $\mathbf{E}_{01}$ and $\mathbf{E}_{02}$ can be found, which is the *key* to the later lattice factorization.

Lemma 3.3 *For the class of FIR PUFBs with equal lengths and the starting block $\mathbf{E}_0^f(z)$ stated above, the matrix $\mathbf{E}_{02}$ has full column rank $M - \beta$ i.e., $\rho(\mathbf{E}_{02}) = M - \beta$, and $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) = \beta$.*

Proof: From (3.9), it can be seen that matrix $\mathbf{E}_{02}$ has full rank, thus $\rho(\mathbf{E}_{02}) = M - \beta$. According to (3.8) and the rank inequality [79], we can obtain that

$$\beta = \rho(\mathbf{E}_{00}^H\mathbf{E}_{00} + \mathbf{E}_{01}^H\mathbf{E}_{01}) \leq \rho(\mathbf{E}_{00}^H\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}^H\mathbf{E}_{01}) = \rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) \quad (3.11)$$

thus we know $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) \geq \beta$. Define matrix $\mathbf{F} = [\mathbf{E}_{00}, \mathbf{E}_{01}, \mathbf{E}_{02}]$, then from (3.10), we can see easily that $\mathbf{F}^H\mathbf{F} = \text{diag}(\mathbf{E}_{00}^H\mathbf{E}_{00}, \mathbf{E}_{01}^H\mathbf{E}_{01}, \mathbf{I}_{M-\beta})$. Due to such diagonal structure, it can be seen

$$\begin{aligned} \rho(\mathbf{F}^H\mathbf{F}) &= \rho(\mathbf{E}_{00}^H\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}^H\mathbf{E}_{01}) + \rho(\mathbf{I}_{M-\beta}) \\ &= \rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) + M - \beta = \rho(\mathbf{F}) \leq \min\{M, M + \beta\} = M \end{aligned} \quad (3.12)$$

which implies $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) \leq \beta$. Finally, we obtain $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) = \beta$ by combining the above two inequalities. $\blacksquare$

These rank constraints derived from the PU conditions establish the *necessary* constraints on those matrices $\mathbf{E}_{0i}$. Without loss of generality, we can assume the rank of matrix $\mathbf{E}_{01}$ to be α with $0 \leq \alpha \leq \beta$ and thus $\rho(\mathbf{E}_{00}) = \beta - \alpha$ due to Lemma 3.3 (Note that the case of $\alpha = 0$ or $\alpha = \beta$ would lead to the trivial case of the starting block $\mathbf{E}_0^f(z)$ with β zero filter

CHAPTER 3. ARBITRARY EQUAL-LENGTH MULTICHANNEL FILTER BANKS

coefficients at either left boundary or right boundary, which would result in a PUFB $\mathbf{E}(z)$ with β zero coefficients at the boundary. Thus, we only consider $1 \leq \alpha \leq \beta - 1$ in the following for nontrivial cases).

For the lattice factorization of $\mathbf{E}_0^f(z)$, we propose the parameterized forms for $\mathbf{E}_{00}$ and $\mathbf{E}_{01}$ according to the above derived necessary constraints as follows,

$$\mathbf{E}_{00} = \mathbf{U}_{00}\mathbf{A}_0, \quad \mathbf{E}_{01} = \mathbf{U}_{01}\mathbf{A}_1, \quad \mathbf{E}_{02} = \mathbf{U}_{02} \quad (3.13)$$

where matrices $\mathbf{U}_{00}$, $\mathbf{U}_{01}$ and $\mathbf{U}_{02}$ have sizes of $M \times (\beta - \alpha)$, $M \times \alpha$ and $M \times (M - \beta)$, respectively, and matrices $\mathbf{A}_0$ and $\mathbf{A}_1$ have sizes of $(\beta - \alpha) \times \beta$ and $\alpha \times \beta$, respectively. According to our assumption on the rank of $\mathbf{E}_{01}$ and the rank conditions stated in Lemma 3.3, the matrices $\mathbf{U}_{0i}$ should be of full column rank and matrices $\mathbf{A}_i$ should be of full row rank. This can be seen from the reduced singular value decomposition (SVD) [79] in which the columns of $\mathbf{U}_{0i}$ are the left singular vectors and the rows of $\mathbf{A}_i$ are the right singular vectors scaled by the singular values. By the parameterized form (3.13) for matrices $\mathbf{E}_{0i}$, we can establish the lattice factorization of $\mathbf{E}_0^f(z)$ as follows,

$$\begin{aligned} \mathbf{E}_0^f(z) &= [\mathbf{E}_{00} + z^{-1}\mathbf{E}_{01}, \mathbf{E}_{02}] = [\mathbf{U}_{00}\mathbf{A}_0 + z^{-1}\mathbf{U}_{01}\mathbf{A}_1, \mathbf{U}_{02}] \\ &= [\mathbf{U}_{00}, z^{-1}\mathbf{U}_{01}, \mathbf{U}_{02}] \begin{bmatrix} \mathbf{A}_0 & \mathbf{0} \\ \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{M-\beta} \end{bmatrix} \\ &= [\mathbf{U}_{00}, \mathbf{U}_{01}, \mathbf{U}_{02}] \text{diag}(\mathbf{I}_{\beta-\alpha}, z^{-1}\mathbf{I}_{\alpha}, \mathbf{I}_{M-\beta})\mathbf{\Gamma} = \mathbf{E}_0^f\mathbf{\Lambda}_0(z)\mathbf{\Gamma} \end{aligned} \quad (3.14)$$

where $\mathbf{E}_0^f = [\mathbf{U}_{00}, \mathbf{U}_{01}, \mathbf{U}_{02}]$ is a square matrix of size M , $\mathbf{\Lambda}_0(z) = \text{diag}(\mathbf{I}_{\beta-\alpha}, z^{-1}\mathbf{I}_{\alpha}, \mathbf{I}_{M-\beta})$ is a delay matrix and $\mathbf{\Gamma} = \text{diag}(\mathbf{T}_{\beta}, \mathbf{I}_{M-\beta})$ with square matrix $\mathbf{T}_{\beta} = [\mathbf{A}_0^T, \mathbf{A}_1^T]^T$ of size β . It can be easily shown that such factorization can guarantee the PU property of FBs as long as matrices $\mathbf{E}_0^f$ and $\mathbf{T}_{\beta}$ are unitary. Thus, the submatrices $\mathbf{U}_{0i}$ of square matrix $\mathbf{E}_0^f$ must be of full column rank as in our parameterized form (3.13). Similar argument is also applicable to the matrices $\mathbf{A}_i$.

Combining (3.3) and (3.14), we can obtain the overall forward lattice factorization for $\mathbf{E}(z)$ of PUFBs as follows.

$$\mathbf{E}(z) = \mathbf{G}_{K-1}^f(z) \cdots \mathbf{G}_1^f(z) \mathbf{E}_0^f \mathbf{\Lambda}_0(z) \mathbf{\Gamma} \quad (3.15)$$

It can be easily seen that the traditional factorization of PUFBs with length constraint $L = KM$, i.e., $\beta = 0$, is only a special case of the proposed lattice factorization (3.15), since the matrices $\mathbf{\Lambda}_0(z)$ and $\mathbf{\Gamma}$ will disappear in the case of $\beta = 0$, which leads the factorization (3.15) to degenerate into the same one as before [67, 69, 70, 80]. In addition, a special case of $\beta = 1$

CHAPTER 3. ARBITRARY EQUAL-LENGTH MULTICHANNEL FILTER BANKS

deserves more attention. In this case, the matrix $\mathbf{T}_\beta$ becomes a scalar and we can only have $0 \leq \alpha \leq 1$. However, either $\alpha = 0$ or $\alpha = 1$ would lead to the zero filter coefficient at the boundary of $\mathbf{E}_0^f(z)$. Moreover, these zeros will be held at the boundary at every stage of lattice structure (3.15). Thus, any PUFB with length $KM + 1$ would always have zero filter coefficient at the boundary, which is actually a PUFB with shorter length KM (with possible shifting) since the proposed lattice structure is complete (as shown in Theorem 3.2 below). This also explains the statement on the existence conditions for filter lengths in Sec. 3.2.1.

Figure 3.1: Forward lattice factorization for $\mathbf{E}(z)$ of an FIR equal-length FB.

For PUFBs with constrained length $L = KM$, it has been shown that the backward factorization is equivalent to the forward one, i.e., $\mathbf{E}(z) = \mathbf{G}_{K-1}^f(z) \cdots \mathbf{G}_1^f(z) \mathbf{E}_0^f = \mathbf{E}_0^b \mathbf{G}_1^b(z) \cdots \mathbf{G}_{K-1}^b(z)$. Then, by combining the above equivalence and the forward factorization (3.15), we can easily obtain the backward factorization as follows.

$$\mathbf{E}(z) = \mathbf{E}_0^b \mathbf{G}_1^b(z) \cdots \mathbf{G}_{K-1}^b(z) \mathbf{\Lambda}_0(z) \mathbf{\Gamma} \quad (3.16)$$

By defining the block $\hat{\mathbf{G}}(z) = \mathbf{\Lambda}_0(z) \mathbf{\Gamma}$, we can see that such *special* order-one block $\hat{\mathbf{G}}(z)$ exists in the right most of *both* forward factorization (3.15) and backward factorization (3.16). It serves to increase the filter length by β as a modified building block. Similar to the case of the forward factorization (3.15), the matrices $\mathbf{\Lambda}_0(z)$ and $\mathbf{\Gamma}$ will disappear when $\beta = 0$, which leads to the factorization (3.16) to degenerate into the previous works such as the extended lapped transforms [66] and the postfiltering structure [71].

It is clear that the proposed lattice factorization (3.15) can generate M -channel PUFBs with arbitrary lengths $L = KM + \beta$. Furthermore, a stronger result is that the converse, i.e., *completeness* of the proposed lattice factorization, is also true, which is stated in the following theorem.

Theorem 3.2 *Any M -channel FIR PUFB with filter length $L = KM + \beta$ can be factorized into the forward lattice structure (3.15) or the backward lattice structure (3.16).*

CHAPTER 3. ARBITRARY EQUAL-LENGTH MULTICHANNEL FILTER BANKS

Proof: This theorem can be rephrased as follows: if there exists an analysis polyphase matrix $\mathbf{E}(z)$ of a PUFB with equal length of $L = KM + \beta$, then $\mathbf{E}(z)$ can always be factorized as in (3.15) or (3.16). We only provide the proof for the forward lattice factorization (3.15) since the backward lattice is equivalent to the forward lattice. Firstly, through the modified order reduction process [70, 81], i.e., *length reduction*, any higher-order polyphase matrix $\mathbf{E}(z)$ can be transformed into an order-one polyphase matrix $\mathbf{E}_0^f(z)$ (or constant matrix $\mathbf{E}_0^f$ in the case of $\beta = 0$) by extracting out the order-one blocks $\mathbf{G}_i^f(z)$. Then, it is just needed to show that $\mathbf{E}_0^f(z)$ can always be factorized as in (3.14).

Note that matrix equations (3.8)-(3.10) contain all the *necessary* conditions to be satisfied for equal-length PUFBs. It can be easily shown that our parameterized forms as shown in (3.13) are a set of solutions to those matrix equations given that matrices $\mathbf{E}_0^f, \mathbf{T}_\beta$ are orthogonal. Thus, the factorization (3.14) derived from them is sufficient to generate an order-one PUFB $\mathbf{E}_0^f(z)$.

However, completeness property of the lattice structure demands that all possible solutions should be able to be converted into the form of (3.14) for the analysis bank. This can be achieved if the parameterized form (3.13) can span the whole solution space of PU conditions (3.8)-(3.10), i.e., there always exist square orthogonal matrices $\mathbf{E}_0^f = [\mathbf{U}_{00}, \mathbf{U}_{01}, \mathbf{U}_{02}]$ of size M and $\mathbf{T}_\beta = [\mathbf{A}_0^T, \mathbf{A}_1^T]^T$ of size β for any solution to those PU conditions. From the Lemma 3.3 and its proof, we know that the parameterized form (3.13) indeed spans the solution space for $\mathbf{E}_{0i}$ based on the reduced SVD, which results in the factorization (3.14). ■

Theorem 3.2 reveals the theoretical significance of the proposed lattice factorization, i.e., it can completely cover the whole space of FIR PUFBs, rather than the subclass of PUFBs with constrained lengths $L = KM$ reported before [5, 69, 70]. To our knowledge, this is the first complete lattice factorization for *any* FIR PUFB reported in the literature.

Degrees of freedom: As shown in Theorem 3.2, any FIR PUFB is characterized by lattice (3.15) or (3.16). For the case of filter length $L = KM + \beta$, there are $K - 1$ PU building blocks $\mathbf{G}_i(z)$ and two free unitary matrices $\mathbf{U}_0$ and $\mathbf{T}_\beta$. Recall that the simplified order-one building blocks derived in [71] and [73] are equivalent to $\mathbf{G}_i^f(z)$ in Sec. 3.2.2. Each simplified block has $2r_i(M - r_i)$ free parameters for complex coefficient case and $r_i(M - r_i)$ for real coefficient case. The two general square unitary matrices $\mathbf{U}_0$ and $\mathbf{T}_\beta$ need $M^2 + \beta^2$ and $M(M - 1)/2 + \beta(\beta - 1)/2$ parameters for complex and real cases, respectively. Thus, the degrees of freedom are given by $D_L = M^2 + \beta^2 + \sum_{i=1}^{K-1} 2r_i(M - r_i)$ for complex case and $D_L = M(M - 1)/2 + \beta(\beta - 1)/2 + \sum_{i=1}^{K-1} r_i(M - r_i)$ for the case of real-valued coefficients. Note that there still exist redundant parameters in between $\mathbf{U}_0$ and $\mathbf{T}_\beta$ in the above characterization, which will be addressed in future.

3.3.2 Perfect Reconstruction Filter Banks

In this section, we further extend the above lattice factorization to more general M -channel PRFBs with arbitrary lengths $L = KM + \beta$. Since the general PR propagating block $\mathbf{G}_i(z)$ has order one, the initial block $\mathbf{E}_0^f(z)$ can still be made order-one and length $M + \beta$. Thus, $\mathbf{E}_0^f(z)$ can also be written in the same form as in the PUFB, i.e., $\mathbf{E}_0^f(z) = [\mathbf{E}_{00} + z^{-1}\mathbf{E}_{01}, \mathbf{E}_{02}]$, where matrices $\mathbf{E}_{0i}$ have the same sizes as those in Sec. 3.3.1. Due to the class of PRFBs with CAFACAFI, we know that the initial block $\mathbf{R}_0^f(z)$ of the synthesis bank can also be made anticausal with order-one and length $M + \beta$ since the PR propagating block $[\mathbf{G}_i^f(z)]^{-1}$ for the synthesis bank is order-one and anticausal. Thus, we can write $\mathbf{R}_0^f(z)$ in the following form,

$$\mathbf{R}_0^f(z) = [\mathbf{R}_{00}^T + z\mathbf{R}_{01}^T, \mathbf{R}_{02}^T]^T \quad (3.17)$$

where $\mathbf{R}_{00}$ and $\mathbf{R}_{01}$ are both $\beta \times M$ matrices, and $\mathbf{R}_{02}$ is an $(M - \beta) \times M$ matrix.

Similar to the case of PUFBs, we can employ the PR condition to obtain a set of matrix equations (shown later in the proof) which consequently lead to the necessary constraints on the ranks of matrices $\mathbf{E}_{0i}$ and $\mathbf{R}_{0i}$, stated formally in the following lemma.

Lemma 3.4 *For the class of FIR PRFBs with equal lengths and the starting blocks $\mathbf{E}_0^f(z)$ and $\mathbf{R}_0^f(z)$ stated above, the matrix $\mathbf{E}_{02}$ has full column rank $M - \beta$ i.e., $\rho(\mathbf{E}_{02}) = M - \beta$, and $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) = \beta$. Moreover, $\rho(\mathbf{R}_{0i}) = \rho(\mathbf{E}_{0i})$ for $0 \leq i \leq 2$.*

Proof: Different from the PU case, we will see that both PR conditions $\mathbf{R}_0^f(z)\mathbf{E}_0^f(z) = \mathbf{I}_M$ and $\mathbf{E}_0^f(z)\mathbf{R}_0^f(z) = \mathbf{I}_M$ are needed for establishing the above rank conditions on the submatrices $\mathbf{E}_{0i}$ and $\mathbf{R}_{0i}$. Firstly, from the PR condition $\mathbf{R}_0^f(z)\mathbf{E}_0^f(z) = \mathbf{I}_M$, we can obtain a set of matrix equations as follows.

$$\mathbf{R}_{00}\mathbf{E}_{00} + \mathbf{R}_{01}\mathbf{E}_{01} = \mathbf{I}_\beta \quad (3.18)$$

$$\mathbf{R}_{02}\mathbf{E}_{02} = \mathbf{I}_{M-\beta} \quad (3.19)$$

$$\mathbf{R}_{01}\mathbf{E}_{02} = \mathbf{0}, \mathbf{R}_{01}\mathbf{E}_{00} = \mathbf{0} \quad (3.20)$$

$$\mathbf{R}_{00}\mathbf{E}_{01} = \mathbf{0}, \mathbf{R}_{02}\mathbf{E}_{01} = \mathbf{0} \quad (3.21)$$

$$\mathbf{R}_{00}\mathbf{E}_{02} = \mathbf{0}, \mathbf{R}_{02}\mathbf{E}_{00} = \mathbf{0} \quad (3.22)$$

From (3.19) and Sylvester rank inequality [79], we can see that

$$M - \beta = \rho(\mathbf{R}_{02}\mathbf{E}_{02}) \leq \rho(\mathbf{E}_{02}) \leq \min\{M, M - \beta\} = M - \beta \quad (3.23)$$

which leads to $\rho(\mathbf{E}_{02}) = M - \beta$. A similar argument can be applied to $\mathbf{R}_{02}$ and we can see $\rho(\mathbf{E}_{02}) = \rho(\mathbf{R}_{02}) = M - \beta$. Then, by using the similar arguments to the proof in Lemma 3.3

CHAPTER 3. ARBITRARY EQUAL-LENGTH MULTICHANNEL FILTER BANKS

for the PU case, we can also establish the rank condition $\rho(\mathbf{R}_{00}\mathbf{E}_{00}) + \rho(\mathbf{R}_{01}\mathbf{E}_{01}) = \beta$ by replacing the matrices $\mathbf{E}_{0i}^H$ by $\mathbf{R}_{0i}$. However, different from the PUFBs in which $\mathbf{R}_{0i}$ are just $\mathbf{E}_{0i}^H$ and thus $\rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01}) = \beta$ can be obtained immediately, we cannot draw the same conclusion for general PRFBs since the important rank equality $\rho(\mathbf{A}^H\mathbf{A}) = \rho(\mathbf{A})$ for any matrix $\mathbf{A}$ is no longer available here.

However, there is an intrinsic relationship between $\mathbf{E}_{0i}$ and $\mathbf{R}_{0i}$ by which we establish the desired rank results. We consider another form of PR condition $\mathbf{E}_0^f(z)\mathbf{R}_0^f(z) = \mathbf{I}_M$ from which another set of matrix equations can be obtained as follows.

$$\mathbf{E}_{00}\mathbf{R}_{00} + \mathbf{E}_{01}\mathbf{R}_{01} + \mathbf{E}_{02}\mathbf{R}_{02} = \mathbf{I}_M \quad (3.24)$$

$$\mathbf{E}_{00}\mathbf{R}_{01} = \mathbf{0}, \quad \mathbf{E}_{01}\mathbf{R}_{00} = \mathbf{0} \quad (3.25)$$

Next, we will show that $\mathbf{R}_{0i}$ for $i = 0, 1$ are the reflexive generalized inverses of $\mathbf{E}_{0i}$. We firstly multiply $\mathbf{E}_{00}$ from the left at the both sides of (3.18) and obtain $\mathbf{E}_{00}\mathbf{R}_{00}\mathbf{E}_{00} = \mathbf{E}_{00}$ according to (3.25). Similarly, we can obtain $\mathbf{R}_{00}\mathbf{E}_{00}\mathbf{R}_{00} = \mathbf{R}_{00}$ by multiplying $\mathbf{R}_{00}$ from the right at the both sides of (3.18). From these two equations, we know that $\mathbf{R}_{00}$ is the reflexive generalized inverse [82] of $\mathbf{E}_{00}$. A similar argument can also be applied to matrices $\mathbf{R}_{01}$ and $\mathbf{E}_{01}$. From the rank property of the reflexive generalized inverse [82], we know that $\rho(\mathbf{E}_{0i}) = \rho(\mathbf{R}_{0i})$ for $i = 0, 1$. Then, we can establish $\beta = \rho(\mathbf{R}_{00}\mathbf{E}_{00}) + \rho(\mathbf{R}_{01}\mathbf{E}_{01}) = \rho(\mathbf{E}_{00}) + \rho(\mathbf{E}_{01})$. The same argument applies to matrices $\mathbf{R}_{0i}$. ■

Remark 3.2 Although the statement looks similar to the case of PUFBs, its proof cannot be extended straightforwardly from the proof of Lemma 3.3 since we no longer have the important rank equality $\rho(\mathbf{A}^H\mathbf{A}) = \rho(\mathbf{A})$ for any matrix $\mathbf{A}$. Moreover, the proof for Lemma 3.4 brought a novel perspective of the PR conditions of critically sampled PRFBs. Different from the case of PUFBs, the PR condition $\mathbf{R}_0^f(z)\mathbf{E}_0^f(z) = \mathbf{I}_M$ alone is not enough to characterize those rank constraints on $\mathbf{E}_{0i}$ and $\mathbf{R}_{0i}$. Another PR condition $\mathbf{E}_0^f(z)\mathbf{R}_0^f(z) = \mathbf{I}_M$ which is usually overlooked, is actually needed in the proof for Lemma 3.4 in order to show that the matrices $\mathbf{R}_{0i}$ are the reflexive generalized inverses [82] of matrices $\mathbf{E}_{0i}$ for $i = 0, 1$ and consequently establish the rank conditions stated in Lemma 3.4.

These rank constraints derived from the PR conditions establish *all the necessary* constraints on those matrices $\mathbf{E}_{0i}$ and $\mathbf{R}_{0i}$ for $i = 0, 1, 2$. Without loss of generality, we can assume the rank of matrix $\mathbf{E}_{01}$ to be α with $0 \leq \alpha \leq \beta$ and thus $\rho(\mathbf{E}_{00}) = \beta - \alpha$. Similarly, we know that $\rho(\mathbf{R}_{01}) = \alpha$ and $\rho(\mathbf{R}_{00}) = \beta - \alpha$ due to Lemma 3.4. The parameterized forms of $\mathbf{E}_{0i}$ shown in (3.13) and the corresponding lattice factorization of $\mathbf{E}_0^f(z)$ shown in (3.14) are still applicable

CHAPTER 3. ARBITRARY EQUAL-LENGTH MULTICHANNEL FILTER BANKS

to PRFBs here, since (3.13) and (3.14) only exploit the rank structures of matrices $\mathbf{E}_{0i}$ and the PRFBs have the same rank structures as those of PUFBs, as shown in Lemma 3.3 and 3.4. Different from PUFBs, the factorization of (3.14) can ensure the PR property if and only if the lattice parameters $\mathbf{E}_0$ and $\mathbf{T}_\beta$ are just *nonsingular* matrices, in contrast to more restrictive unitary matrices for PUFBs.

In addition, for the lattice factorization of $\mathbf{R}_0^f(z)$, we can employ similar parameterized forms as in (3.13) for $\mathbf{R}_{00}$ and $\mathbf{R}_{01}$ as follows, according to the necessary constraints shown in Lemma 3.4,

$$\mathbf{R}_{00} = \mathbf{B}_0 \mathbf{W}_{00}, \quad \mathbf{R}_{01} = \mathbf{B}_1 \mathbf{W}_{01}, \quad \mathbf{R}_{02} = \mathbf{W}_{02} \quad (3.26)$$

where matrices $\mathbf{W}_{00}$, $\mathbf{W}_{01}$ and $\mathbf{W}_{02}$ have sizes of $(\beta - \alpha) \times M$, $\alpha \times M$ and $(M - \beta) \times M$, respectively, and matrices $\mathbf{B}_0$ and $\mathbf{B}_1$ have sizes of $\beta \times (\beta - \alpha)$ and $\beta \times \alpha$, respectively. According to the rank constraints for $\mathbf{R}_{0i}$ stated in Lemma 3.4, the matrices $\mathbf{W}_{0i}$ should be of full row rank and matrices $\mathbf{B}_i$ should be of full column rank. Based on the parameterized form (3.26), we can establish the lattice factorization of $\mathbf{R}_0^f(z)$ as follows,

$$\begin{aligned} \mathbf{R}_0^f(z) &= \begin{bmatrix} \mathbf{B}_0 \mathbf{W}_{00} + z \mathbf{B}_1 \mathbf{W}_{01} & \mathbf{W}_{02} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{M-\beta} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{00} \\ z \mathbf{W}_{01} \\ \mathbf{W}_{02} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}_\beta & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{M-\beta} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{\beta-\alpha} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & z \mathbf{I}_\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{M-\beta} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{00} \\ \mathbf{W}_{01} \\ \mathbf{W}_{02} \end{bmatrix} = \hat{\Gamma} \Lambda_0(z^{-1}) \mathbf{R}_0^f \end{aligned} \quad (3.27)$$

where $\mathbf{R}_0^f = [\mathbf{W}_{00}^T, \mathbf{W}_{01}^T, \mathbf{W}_{02}^T]^T$ is a square matrix of size M and $\hat{\Gamma} = \text{diag}(\mathbf{Q}_\beta, \mathbf{I}_{M-\beta})$ with square matrix $\mathbf{Q}_\beta = [\mathbf{B}_0, \mathbf{B}_1]$ of size β . It can be easily shown that such factorization can guarantee the PR property of FBs as long as matrices $\mathbf{R}_0^f$ and $\mathbf{Q}_\beta$ are nonsingular. Moreover, from the PR condition $\mathbf{R}_0^f(z) \mathbf{E}_0^f(z) = \mathbf{I}_M$, we know that the inverses of $\mathbf{R}_0^f$ and $\mathbf{Q}_\beta$ are just $\mathbf{E}_0^f$ and $\mathbf{T}_\beta$, respectively, i.e., $[\mathbf{R}_0^f]^{-1} = \mathbf{E}_0^f$ and $\mathbf{Q}_\beta^{-1} = \mathbf{T}_\beta$. Thus, we can see that the set of matrices $\mathbf{W}_{0i}$ is *biorthogonal* to the set of matrices $\mathbf{U}_{0i}$ due to $\mathbf{R}_0^f \mathbf{E}_0^f = [\mathbf{W}_{00}^T, \mathbf{W}_{01}^T, \mathbf{W}_{02}^T]^T [\mathbf{U}_{00}, \mathbf{U}_{01}, \mathbf{U}_{02}] = \mathbf{I}_M$. Similar argument is also applicable to the sets of matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ due to $\mathbf{T}_\beta \mathbf{Q}_\beta = \mathbf{I}_\beta$.

Compared to the conventional factorizations [65, 72, 73] for PRFBs of length $L = KM$, the proposed one is more general since it can degenerate into the traditional one by setting $\beta = 0$. In addition, the PU FBs shown in Sec. 3.3.1 can be casted into a special case of this general class of PRFBs by further constraining general free nonsingular matrices to be unitary, which leads to the biorthogonal condition degenerates into the conventional orthogonal condition. Furthermore, we present the completeness of the proposed lattice factorization for any order-one PRFBs without length constraint, i.e., $L \leq 2M$, in the following theorem, which generalizes the previous factorization results (cf. [83, Table I]).

Theorem 3.3 *For the class of PRFBs stated above, the proposed forward lattice factorization (3.15) or backward factorization (3.16) is complete for any M -channel PRFB with $L \leq 2M$.*

Proof: Firstly, when the filter length $L = M$, such FB degenerates into traditional block transform $\mathbf{E}_0^f$, thus nonsingular. Secondly, when $L = 2M$, the derived lattice factorization degenerates into the one reported in [65] in which it has been shown to be a complete factorization. Thus, we only need to consider the lengths in between which means no PR propagating block $\mathbf{G}_i(z)$ exists, i.e., just order-one polyphase matrix $\mathbf{E}_0^f(z)$. This indicates that we only need to show that $\mathbf{E}_0^f(z)$ can always be factorized in (3.13) with nonzero β . The proof can be continued along the same lines as those in the proof for Theorem 3.2 to show that the parameterized form (3.13) spans the solution space by virtue of Lemma 3.4. ■

In addition, as in the case of PUFBs, the matrix $\mathbf{T}_\beta$ and $\mathbf{Q}_\beta$ become scalars when $\beta = 1$. Furthermore, the only possible choices $\alpha = 0$ or $\alpha = 1$ would both lead to the zero coefficient at the boundary of $\mathbf{E}_0(z)$ and $\mathbf{R}_0(z)$. Thus, there does not exist any nontrivial PRFB with length $M + 1$ according to our complete lattice factorization for $L \leq 2M$ stated in Theorem 3.3.

Degrees of freedom: The lattice structure of (3.15) can generate PRFBs with length $L = KM + \beta$. It needs $K - 1$ PR building blocks $\mathbf{G}_i^f(z)$ and two free invertible matrices $\mathbf{U}_0$ and $\mathbf{T}_\beta$. Recall that the simplified order-one block $\mathbf{G}_i^f(z)$ [73] has $4r_i(M - r_i)$ free parameters for complex coefficient case and $2r_i(M - r_i)$ for real case. The two square invertible matrices $\mathbf{U}_0$ and $\mathbf{T}_\beta$ need $2(M^2 + \beta^2)$ parameters for complex case and $M^2 + \beta^2$ parameters for real case. Thus, the degrees of freedom for the whole lattice are given by $D_L = 2(M^2 + \beta^2) + \sum_{i=1}^{K-1} 4r_i(M - r_i)$ for complex case and $D_L = M^2 + \beta^2 + \sum_{i=1}^{K-1} 2r_i(M - r_i)$ for real case. Similar to the PUFBs, there still exist redundant parameters in between $\mathbf{U}_0$ and $\mathbf{T}_\beta$ in the above characterization, which will be addressed in future.

3.4 Lattice Factorization of Regular Filter Banks

Another desired property of FB is the regularity which is closely related to the performance of image coding based on FBs. By virtue of the above lattice structures, we can structurally impose the regularity by further structural constraints on lattice parameters. In the following, we only present lattice structures for one degree of regularity. Such FBs have no DC leakage, which is desirable for avoiding the artificial checker board effect in image coding. In addition, it also serves as the necessary condition [26, 27, 72] for M -band wavelets generated by iterating M -channel FBs. The necessary conditions for one degree of regularity in terms of polyphase matrices are,

$$\mathbf{E}(z^M)\mathbf{d}_M(z)|_{z=1} = c\mathbf{e}_0, \quad \tilde{\mathbf{R}}(z^M)\mathbf{J}_M\mathbf{d}_M(z)|_{z=1} = d\mathbf{e}_0 \quad (3.28)$$

CHAPTER 3. ARBITRARY EQUAL-LENGTH MULTICHANNEL FILTER BANKS

where delay chain vector $\mathbf{d}_M(z) = [1, z^{-1}, \dots, z^{-(M-1)}]^T$, Euclidean basis vector $\mathbf{e}_0 = [1, 0, \dots, 0]^T$ and c, d are nonzero constants with $cd = M$.

Next, we impose the above constraints onto the proposed lattice structures (3.15). It can be easily seen that $\mathbf{G}_i^f(z)|_{z=1} = \mathbf{G}_i^b(z)|_{z=1} = \mathbf{I}_M$ and $\mathbf{\Lambda}_0(z)|_{z=1} = \mathbf{I}_M$. Thus, for PRFBs factorized in (3.15), the constraints shown in (3.28) simplify to

$$\mathbf{E}_0^f \text{diag}(\mathbf{T}_\beta, \mathbf{I}_{M-\beta}) \mathbf{1}_M = \mathbf{F} \mathbf{1}_M = c \mathbf{e}_0, \quad \left(\mathbf{E}_0^f\right)^{-H} \text{diag}(\mathbf{T}_\beta^{-H}, \mathbf{I}_{M-\beta}) \mathbf{1}_M = \mathbf{F}^{-H} \mathbf{1}_M = d \mathbf{e}_0 \quad (3.29)$$

where $\mathbf{F} \triangleq \mathbf{E}_0^f \text{diag}(\mathbf{T}_\beta, \mathbf{I}_{M-\beta})$. These constraints on the matrix $\mathbf{F}$ have been studied before in which such $\mathbf{F}$ can be characterized by using the LDU factorization [27] or Householder transform [72]. Denote the Householder matrix $\mathbf{H}_{[\mathbf{1}_M]} = \mathbf{I}_M - 2\mathbf{u}\mathbf{u}^T/\mathbf{u}^T\mathbf{u}$ with $\mathbf{u} = \mathbf{1}_M/\sqrt{M} - \mathbf{e}_0$, which can align the vector $\mathbf{1}_M$ to $\mathbf{e}_0$, i.e., $\mathbf{H}_{[\mathbf{1}_M]} \mathbf{1}_M = \sqrt{M} \mathbf{e}_0$. Then, the constraint (3.29) can be accomplished by further constraining the general nonsingular matrix $\mathbf{F}$ to be of the following form [72],

$$\mathbf{F} = \text{diag}\left(c/\sqrt{M}, \hat{\mathbf{F}}\right) \mathbf{H}_{[\mathbf{1}_M]} \quad (3.30)$$

where $\hat{\mathbf{F}}$ is an $(M-1) \times (M-1)$ arbitrary invertible matrix.

Thus, the design and optimization will involve the unconstrained invertible matrix $\mathbf{T}_\beta$ and the matrix $\mathbf{F}$ with the above constraints, and the matrix $\mathbf{E}_0^f$ in our lattice structure (3.15) can be obtained by $\mathbf{E}_0^f = \mathbf{F} \text{diag}(\mathbf{T}_\beta^{-1}, \mathbf{I}_{M-\beta})$. The unconstrained invertible matrix $\mathbf{T}_\beta$ and its inverse can be easily characterized by SVD or LDU factorization. The same procedure can also be applied to the backward factorization (3.16) just by replacing the matrix $\mathbf{E}_0^f$ by $\mathbf{E}_0^b$.

3.5 Design Examples

3.5.1 Degrees of Freedom for Filter Bank Design

According to the lattice factorizations developed in Sec. 3.3.1 and Sec. 3.3.2 for PU and PR FBs, respectively, the degrees of freedom (DOF) for designing such FBs design are summarized in Table 3.1 for various cases. It shows the number of free parameters within the lattice factorizations.

Table 3.1: Degrees of freedom for the lattice factorizations of M -channel PRFBs and PUFBs with lengths $L = KM + \beta$.

DOF	Complex Case	Real Case
PUFBs	$M^2 + \beta^2 + \sum_{i=1}^{K-1} 2r_i(M - r_i)$	$M(M-1)/2 + \beta(\beta-1)/2 + \sum_{i=1}^{K-1} r_i(M - r_i)$
PRFBs	$2(M^2 + \beta^2) + \sum_{i=1}^{K-1} 4r_i(M - r_i)$	$M^2 + \beta^2 + \sum_{i=1}^{K-1} 2r_i(M - r_i)$

Note that once the filter length L and the degree parameters r_i are chosen for an M -channel FB, the DOF is fixed as shown in the above table since K and β are uniquely determined by a given L . The filter length L and the degree parameters r_i are usually dependent on the requirements of the applications. For example, the delay cannot be too large for speech processing, and the filter length L is better not to be too large for image processing due to the ringing artifacts and computational complexity limitation. However, how to choose the appropriate filter length is still an open problem and the guidelines may be quite different for different kinds of applications. Usually, the filter length is chosen by a balance of performance and complexity during experiments.

3.5.2 Filter Bank Design Examples

In this section, we show design examples for PUFB and PRFB with equal length $L = KM + \beta$ designed by using the proposed lattice factorizations. A good FB can be obtained through unconstrained nonlinear optimization due to the lattice factorizations (3.15) or (3.16) where the lattice parameters are the free nonsingular or unitary matrices for PRFBs or PUFBs, respectively. In different applications, various objective functions could be employed for optimization of the filter coefficients. One objective function is minimization of the stopband attenuation and/or passband ripple for ideal filter shape, which is a classical one in FB design [5] and optimization presented in Sec. 2.2.2.

For our design examples, the free invertible matrices within the lattice structures are decomposed by SVD through orthogonal and diagonal matrices, where the orthogonal matrix is further decomposed by Givens rotation. The invertibility of square matrices is ensured as long as the diagonal elements are nonzero. The unconstrained optimization procedure used for our design examples is the simplex search function *fminunc* available in MATLAB Optimization Toolbox since the lattice factorizations can structurally enforce the desired properties and given filter length L . The initial values for the invertible and orthogonal matrices are chosen as follows. The Givens rotation angles of the orthogonal matrices are initialized to be zero, i.e., identity matrix, and the singular values for the diagonal matrix with SVD are chosen to be one.

We first show a design example for arbitrary equal-length even channel PUFB with $M = 4$ and consecutive lengths $L = 12, 14, 15$ and 16 , i.e., $K = 3$ with $\beta = 2, 3, 4$ and $K = 4$, optimized for stopband attenuation. The degree parameters r_i for the propagating blocks $\mathbf{G}_i^f(z)$ of (3.4) or $\mathbf{G}_i^b(z)$ of (3.6) are $r_i = 2$ for $i = 1, \dots, K - 1$. The degree parameter α with the delay matrix $\mathbf{\Lambda}_0(z)$ is chosen as $\alpha = \lceil \beta/2 \rceil$. The magnitude responses of the analysis bank is shown in Fig. 3.2 for various lengths. As seen from the magnitude responses, the filter shape gets better approximation to ideal one as the filter length increases. The stopband attenuations

Figure 3.2: Design example of 4-channel PUFBs with consecutive lengths $L = 12, 14, 15$ and 16 .

for the analysis bank are 17.32dB, 17.86dB, 17.86dB and 18.05dB for lengths $L = 12, 14, 15$ and 16 , respectively. It is improved gradually with the increment of filter lengths, i.e., a trade-off between the FB performance and its computational complexity. Note that the 4-channel nontrivial PUFB with length $L = 13$, i.e., $\beta = 1$, does not exist as seen from Theorem 3.1.

Then, we show another design example for arbitrary equal-length odd channel PRFB with $M = 5$ and $L = 13$, i.e., $K = 2, \beta = 3$, optimized for the weighted sum of stopband attenuation C_{SB} and coding gain C_{CG} , i.e., $C = w_c C_{\text{CG}} + w_s C_{\text{SB}}$. The weighting factors are same as those in [35]. The degree parameter r_i for the building block $\mathbf{G}_i^f(z)$ is $r_i = 2$. Besides the PR property, we further impose no DC leakage of the FB, i.e., one-degree of regularity, thus no need to add

CHAPTER 3. ARBITRARY EQUAL-LENGTH MULTICHANNEL FILTER BANKS

the term of DC leakage C_{DC} into the cost function C . The magnitude responses of the analysis bank and the corresponding synthesis bank are shown in Fig. 3.3.a and Fig. 3.3.b, respectively. At the right of both magnitude responses, we show the optimized filter coefficients in the wavy lines with filled circles for the corresponding analysis and synthesis filters, respectively. Note the very strong attenuation at the zero frequency for all the bandpass and highpass filters in the magnitude responses of both analysis and synthesis banks, due to the structural enforcement of one-degree of regularity. To our knowledge, this is the first design example for such kind of FBs reported in the literature.

Figure 3.3: Design example of a regular 5-channel PRFB with length $L = 13$.

3.6 Summary

We have presented the existence conditions, lattice factorizations and designs for a class of M -channel FBs with arbitrary lengths. The desired properties like PR or PU can be structurally enforced into the proposed lattice structures. The arbitrary filter length L can be easily achieved by appropriately choosing parameters β and K . In contrast to very limited possible designs by most traditional methods, our design approaches are more general and can considerably expand the designer's choices. In addition, it has been shown that the proposed lattice factorizations can completely span the whole space of M -channel FIR PUFBs with any valid length, and PRFBs with length $L \leq 2M$ (order-one lapped transforms with all lengths). To our knowledge, these are the most general lattice factorizations for multichannel equal-length FIR PU/PR FBs in the literature.

Chapter 4

Arbitrary Equal-Length Linear Phase Filter Banks

4.1 Introduction

Multichannel linear phase perfect reconstruction filter banks (LPPRFBs) have been studied extensively and widely employed in image and video processing. The LP property of FBs can effectively eliminate the phase distortion and handle the boundaries of finite length signals via simple symmetric extension method [23,24] which is a popular way of manipulating the border of images in image processing. From the perspective of computational complexity, a more efficient implementation method can also be obtained by exploiting the LP property. Thus, for various practical purposes, only causal FIR LPPRFBs with real-valued filter coefficients are considered in this chapter. LPPRFBs, as the extension of the block transforms and lapped transforms under another terminology called generalized lapped biorthogonal transforms (GLBTs), have been studied extensively and used for image compression at *low* bit rate region. The theory, factorizations, designs and implementations of LPPRFBs with length $L = KM$ ($K \geq 2, K \in \mathbb{N}$) were studied in [27, 34, 42, 63, 66, 72, 77, 78, 84–95]. The most popular method of analyzing and designing such FBs is the lattice factorization of the polyphase matrix $\mathbf{E}(z)$ and numerous works have been done for designing such FBs by lattice factorizations.

However, most previous works imposed an *unnecessary* condition a priori on the filter length $L = KM$ just for simplification of the lattice factorization and design. In this chapter, we extend the previous works to a larger class of LPPRFBs with real-valued filter coefficients in which all filters have the same *arbitrary* equal length $L = KM + \beta$ ($0 \leq \beta < M, K \in \mathbb{N}$) and same symmetry center. One motivation for such class of LPPRFBs is to design more flexible FBs which may give more possible choices for a given desired application. In most pioneer works for LPPRFBs [34, 42, 77, 93], the filter length must be an integer multiple of M , which greatly

limits the possible designs of LPPRFBs. However, this restriction is due to the traditional FB design methods, *not* inherent in LPPRFBs, which can be seen from the existence conditions in Sec. 4.2. Our lattice factorization based design method can provide more possible LPPRFBs satisfying the given constraint length, and the length increment among FBs can be made as small as possible. For example with $M = 8$ and length $L \leq 2M$, the proposed design approach can give five possible solutions in contrast to the only two choices by traditional methods. Such restriction becomes more severe for larger M . In addition, the designed FBs based on the derived lattice structures can have the LP and PR properties regardless of the value of filter coefficients and are source independent, which is desired in signal and image processing applications. From the perspective of designing FBs, such more flexible filter length profile can also yield a better trade-off between the design and optimization complexities and the performance of FBs than before. In addition to these advantages mentioned before, such FBs with regularity can generate M -band wavelets with more flexible compact supports, since the support lengths of scaling and wavelet functions are linear functions of the filter length [7].

In this chapter, we systematically investigate the lattice factorizations of LPPRFBs with arbitrary filter lengths. There have been some works on the theory, factorizations and designs for this more general class of LPPRFBs. The existence conditions of such class of LPPRFBs in terms of filter lengths and filter symmetry polarities were firstly studied in [77, 85] and then explored thoroughly in [25]. However, most existing design methods [27, 34, 42, 72, 77, 86, 92, 93] concentrated only on the special case $\beta = 0$, i.e., filter lengths of integer multiple of M . As far as the case $\beta > 0$ is concerned, some works have been reported before. As mentioned before, the cosine-modulated FBs with arbitrary lengths were studied in [52], but without the LP property. Another design method was discussed in [25] for the class of restrictive even channel LPPUFBs without constrained cosine-modulated structure. Later, it was also extended to even channel LPPRFBs [55]. In addition, a subclass of such LPPRFBs, i.e., LPPUFBs with both odd M and β , was considered in [54]. Besides these works, the FBs with arbitrary filter lengths were studied in [76] based on minimization of mean square error in general subband coding setup. However, as mentioned before, their least-square constraint optimization method based design cannot structurally enforce LP and PR properties, and suffer from quantization of coefficients. Different from these works, Tran *et al.* have contributed a lot for this class of LPPRFBs under the framework of lapped transforms via time-domain pre- and post-filtering [43–45]. Their smart structure of lapped transforms can be employed for more efficient image and video coding, which is backward compatible to the existing standards. Recently, it was adopted in Microsoft Windows Video format standard [46]. Later, we will show that our derived structure is essentially equivalent to their lapped transform structure. However,

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

different from their presentation, we study such FBs based on the systematic investigation of the necessary constraints held on the polyphase matrix $\mathbf{E}(z)$, which can lead us to derive lattice factorizations and show their connections with other works and some theoretical properties.

The outline of this chapter is as follows. In section 4.2, refined existence conditions are established for the LPPRFBs of interest with arbitrary filter lengths based on the pioneer work [25]. Previous works on lattice factorization and design of M -channel LPPRFBs are also reviewed. In Sec. 4.3.1, lattice factorizations for LPPRFBs with even β are explored in detail. Both even and odd channel FBs are discussed and can be viewed as a whole. We also discuss how to structurally impose regularity constraint onto our lattice structure for filter banks with even β to ensure the smoothness of the basis function, which is very desirable in high performance image compression. These structures are more general compared to the conventional design methods, and cover them as special cases. Thus, our design approaches can expand greatly the family of LPPRFBs with equal lengths. In addition, the relationship of our derived lattice structure with the lapped transforms via time-domain pre/postfiltering and oversampled LPPRFBs is to be presented. The LPPRFBs with odd β are further studied in Sec. 4.3.2 based on a cascading structure. Its relationship with previous works are also discussed. In Sec. 4.4, the theoretical properties of the proposed lattice structures such as the completeness and minimality are presented. Specifically, these lattice structures are proven to be *minimal* in terms of the number of delay elements used in implementation, and *completely* span the class of LPPRFBs with length $L \leq 2M$ and FIR LPPUFBs. In addition to the development of these general lattice structures, more theoretical issues for the lattice factorizations of LPPRFBs are further investigated. Firstly, we study the problem of the existence of complete order-one lattice factorization. Through pointing out the limitation of the existing order-one LPPR propagating blocks, we show the *nonexistence* of any complete order-one lattice factorization. Secondly, we investigate the optimality of the existing lattice based design method for LPPRFBs, through studying the spaces $\mathbb{L}_\beta$ for general and regular LPPRFBs spanned by the lattice structures and their closeness property under polynomial transformations. We show the closeness of the spaces $\mathbb{L}_\beta$, which establishes the *suboptimality* of the lattice based design for LPPRFBs. We finally provide several design examples using the proposed methods, and evaluate them further using a transform-based image coding framework in Sec. 4.7 to show the effect of filter length on the reconstructed image. Simulation results show that these well designed filter banks can obtain comparable and even better performance compared to the existing wavelet transform. Moreover, they consistently yield smoother reconstructed images and better perceptual quality as filter length increases. Therefore, they can provide a desired *flexible* tradeoff between performance and filter length (measure of computational complexity). For simplicity, we only consider the class of causal FIR FBs with anticausal FIR inverses (CAFACFI) [64] in this chapter.

4.2 Refined Existence Conditions and the General Lattice Factorization

In [25], some necessary conditions for LPPRFBs with lengths $L_i = K_i M + \beta$ ($0 \leq \beta < M$) were derived. They established the fundamental constraints for the symmetry polarity and sum of lengths. In particular, for the class of LPPRFBs with equal filter length, i.e., $K_i = K$ for $i = 0, 1, \dots, M-1$, some further refined existence conditions can be obtained and listed in Table 4.1, where n_s and n_a denote the number of symmetric and anti-symmetric filters, respectively. Since the sum $\sum K_i$ needs to be odd for even M -channel case with odd β , it can be easily seen that even channel LPPRFBs with the same length cannot have odd length, i.e., only even lengths are possible. In addition, we can see that only odd filter lengths are possible for odd channel systems.

Table 4.1: Existence conditions for M -channel LPPRFBs with lengths $L = KM + \beta$.

	Symmetry Polarity Condition	Order K	Length L
M even, β even	$n_s = \frac{M}{2}$ and $n_a = \frac{M}{2}$	arbitrary	even
M odd, β even	$n_s = \frac{M+1}{2}$ and $n_a = \frac{M-1}{2}$	odd	odd
M odd, β odd	$n_s = \frac{M+1}{2}$ and $n_a = \frac{M-1}{2}$	even	odd
M even, β odd	Does not exist		

Without loss of generality, we can always arrange the M channel linear phase filters in such an order that all n_s symmetric filters are placed before the n_a anti-symmetric filters. Note that we only consider the class of LPPRFBs with equal lengths and the *same* symmetry center, i.e., concentric LPPRFBs [24,96]. In this way, the associated analysis polyphase matrix $\mathbf{E}(z)$ should satisfy the generalized LP condition [25],

$$\mathbf{E}(z) = z^{-(K-1)} \mathbf{D}_M \mathbf{E}(z^{-1}) \mathbf{J}_M(z) \quad (4.1)$$

where $\mathbf{J}_M(z) = \text{diag}(z^{-1} \mathbf{J}_\beta, \mathbf{J}_{M-\beta})$ and $\mathbf{D}_M = \text{diag}(\mathbf{I}_{n_s}, -\mathbf{I}_{n_a})$, in which the possible values of n_s and n_a are listed in Table 4.1 for various possible cases. Notice that the matrix $\mathbf{J}_M(z)$ degenerates into the traditional reversal matrix $\mathbf{J}_M$ when $\beta = 0$.

By adapting the general lattice structure (2.13) presented in Sec. 2.3 for the class of LP-PRFBs, a *forward* lattice factorization for the analysis polyphase matrix $\mathbf{E}(z)$ of an LPPRFB can be formulated in the following form [65, 71, 95],

$$\mathbf{E}(z) = \mathbf{G}_{K-1}^f(z) \mathbf{G}_{K-2}^f(z) \cdots \mathbf{G}_1^f(z) \mathbf{E}_0^f(z) \quad (4.2)$$

where the starting block $\mathbf{E}_0^f(z)$ with order N_0 and length $N_0 M + \beta$ (with order $N_0 - 1$ and length $N_0 M$ if $\beta = 0$) has both LP and PR properties, and each block $\mathbf{G}_i^f(z)$ with order N_1 can

propagate both LP and PR properties while increasing filter length by $N_1 M$. Such a cascade form would finally generate an LPPRFB with filter length $L = (K N_1 - N_1 + N_0)M + \beta$. Most previous works [25, 34, 42, 66, 77, 85, 86, 92, 93] have been focused on the development of such LPPR propagating blocks. For even channel case, i.e., $M = 2m$, $\mathbf{G}_i^f(z)$ can be made order-one, i.e., $N_1 = 1$, in the following form [93],

$$\mathbf{G}_i^f(z) = \Phi_i \mathbf{W}_M \Lambda_i(z) \mathbf{W}_M \quad (4.3)$$

where the delay matrix $\Lambda_i(z) = \text{diag}(\mathbf{I}_m, z^{-1} \mathbf{I}_m)$ and the block diagonal matrix $\Phi_i = \text{diag}(\mathbf{I}_m, \mathbf{V}_i)$ in which $\mathbf{V}_i$ is an arbitrary $m \times m$ invertible matrix. However, for odd channel case, i.e., $M = 2m + 1$, $\mathbf{G}_i^f(z)$ has at least order-two, i.e., $N_1 = 2$, and can be the following form [93],

$$\mathbf{G}_i^f(z) = \Phi_{i,0} \mathbf{W}_M \Lambda_{i,0}(z) \mathbf{W}_M \Phi_{i,1} \mathbf{W}_M \Lambda_{i,1}(z) \mathbf{W}_M \quad (4.4)$$

where the delay matrices $\Lambda_{i,0}(z) = \text{diag}(\mathbf{I}_{m+1}, z^{-1} \mathbf{I}_m)$ and $\Lambda_{i,1}(z) = \text{diag}(\mathbf{I}_m, z^{-1} \mathbf{I}_{m+1})$ the block diagonal matrices $\Phi_{i,0} = \text{diag}(\mathbf{U}_i, \mathbf{I}_m)$ and $\Phi_{i,1} = \text{diag}(\mathbf{I}_m, \mathbf{V}_i)$ in which $\mathbf{U}_i$ and $\mathbf{V}_i$ are arbitrary invertible matrices with sizes $(m+1) \times (m+1)$ and $m \times m$, respectively.

The *difference* between this work and the previous ones with constrained lengths $L = KM$ lies in the starting block $\mathbf{E}_0^f(z)$ of the forward factorization (4.2). Contrary to previous works [34, 42, 66, 77, 85, 86, 92, 93], for the case of $\beta \neq 0$, $\mathbf{E}_0^f(z)$ cannot be made order zero, i.e., a constant matrix, which is treated as a trivial case because it would impose multiple of $(M - \beta)$ zero-valued filter coefficients at fixed positions. Thus, $\mathbf{E}_0^f(z)$ has at least order one for the case of $\beta \neq 0$, i.e., $N_0 \geq 1$ for the forward factorization.

Similar arguments are also applied to the backward factorization formulated as follows.

$$\mathbf{E}(z) = \mathbf{E}_0^b(z) \mathbf{G}_1^b(z) \mathbf{G}_2^b(z) \cdots \mathbf{G}_{K-1}^b(z) \quad (4.5)$$

For even channel case, the LPPR propagating block $\mathbf{G}_i^b(z)$ in backward factorization (4.5) can be made order one, i.e., increasing the filter length by M , with the following form,

$$\mathbf{G}_i^b(z) = \Lambda_i(z) \mathbf{W}_M^J \Phi_i \mathbf{W}_M^J \quad (4.6)$$

where $\mathbf{W}_M^J = \mathbf{I}_M^J \mathbf{W}_M \mathbf{I}_M^J$. However, for odd channel case, $\mathbf{G}_i^b(z)$ in backward factorization (4.5) can be made order two, i.e., increasing filter length by $2M$, with the following form.

$$\mathbf{G}_i^b(z) = \Lambda_{i,0}(z) \mathbf{W}_M^J \Phi_{i,1} \mathbf{W}_M^J \Lambda_{i,1}(z) \mathbf{W}_M^J \Phi_{i,0} \mathbf{W}_M^J \quad (4.7)$$

Thus, after possible length reduction by such backward blocks $\mathbf{G}_i^b(z)$, the residual block $\mathbf{E}_0^b(z)$ in the backward factorization (4.5) has at least order one for the case of nonzero β . However,

if we use some modified backward LPPR propagating blocks $\hat{\mathbf{G}}_i(z)$ in the factorization (4.5), then it is possible to have zero-order block $\mathbf{E}_0^b$. This has been studied in a heuristic way under the framework of lapped transforms via pre/post filtering [43–45]. The relationship between the forward and backward factorizations and the connection to previous works are further to be discussed after the development of our factorizations in the next section.

As seen from Table 4.1, there are three valid cases for arbitrary-length LPPRFBs. However, we can classify them into two cases according to the minimum order of $\mathbf{E}_0^f(z)$. One case is even β in which the minimum order of the block $\mathbf{E}_0^f(z)$ can always be made one, i.e., $N_0 \geq 1$. Moreover, we can see in the next section that both even and odd channel cases with even β are similar in derivations. The other case is odd β in which the minimum order of the block $\mathbf{E}_0^f(z)$ can be made at least two, i.e., $N_0 \geq 2$. These two cases are to be investigated in detail in the next two subsections, respectively.

4.3 Lattice Factorizations of Linear Phase Perfect Reconstruction Filter Banks

4.3.1 LPPRFBs with Even β

Consider an M -channel LPPRFB with filter length $L = KM + \beta$ ($0 \leq \beta < M$). As seen from Table 4.1, both odd channel systems ($M = 2m + 1$) and even channel systems ($M = 2m$) can have even β . In the following, we will derive the factorization in the same framework for both odd and even channel LPPRFBs. We first derive the forward factorization in detail and then show the backward factorization and their relationship. Next, LPPRFBs with structural regularity is studied. Finally, the relationship of our work with the previous ones is explained. In this section, all the derivations for M -channel LPPRFBs will follow the assumption of arranging all n_s symmetric filters first before the remaining n_a anti-symmetric filters.

4.3.1.1 General Lattice Factorization

Forward factorization:

For even $\beta = 2l$ ($0 \leq l < m$), assume that the LPPRFB with length $L = KM + \beta$ can be factorized into (4.2) with the starting block $\mathbf{E}_0^f(z)$ and the LPPR propagating blocks in the form of (4.3) (The class of LPPRFBs factorable by the forward factorization (4.2) is to be investigated in Sec. 4.4.2 later.). Then, from the linear phase condition (4.1) and the symmetry

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

polarity in Table 4.1, the starting block $\mathbf{E}_0^f(z)$ can *always* be written in the following form¹,

$$\mathbf{E}_0^f(z) = \begin{bmatrix} \mathbf{S}_{00} + z^{-1}\mathbf{S}_{00}\mathbf{J}_\beta & \mathbf{S}_{01} & \mathbf{q} & \mathbf{S}_{01}\mathbf{J}_{\lfloor(M-\beta)/2\rfloor} \\ \mathbf{A}_{00} - z^{-1}\mathbf{A}_{00}\mathbf{J}_\beta & \mathbf{A}_{01} & \mathbf{0} & -\mathbf{A}_{01}\mathbf{J}_{\lfloor(M-\beta)/2\rfloor} \end{bmatrix} \quad (4.8)$$

where matrices $\mathbf{S}_{00}$ and $\mathbf{S}_{01}$ have size $\lceil M/2 \rceil \times \beta$ and $\lceil M/2 \rceil \times \lfloor (M-\beta)/2 \rfloor$, respectively, since there should be $n_s = \lceil M/2 \rceil$ symmetric channels. The matrices $\mathbf{A}_{00}$ and $\mathbf{A}_{01}$ have size $\lfloor M/2 \rfloor \times \beta$ and $\lfloor M/2 \rfloor \times \lfloor (M-\beta)/2 \rfloor$, respectively. For the case of odd channel systems, $\mathbf{q}$ and $\mathbf{0}$ are column vectors with size $\lceil M/2 \rceil$ and $\lfloor M/2 \rfloor$, respectively. However, for even channel systems, they do not exist at all. Similar to analysis bank, the corresponding synthesis bank has a starting block,

$$\mathbf{R}_0^f(z) = \begin{bmatrix} \mathbf{T}_{00} + z\mathbf{J}_\beta\mathbf{T}_{00} & \mathbf{B}_{00} - z\mathbf{J}_\beta\mathbf{B}_{00} \\ \mathbf{T}_{01} & \mathbf{B}_{01} \\ \mathbf{r}^T & \mathbf{0}^T \\ \mathbf{J}_{\lfloor(M-\beta)/2\rfloor}\mathbf{T}_{01} & -\mathbf{J}_{\lfloor(M-\beta)/2\rfloor}\mathbf{B}_{01} \end{bmatrix} \quad (4.9)$$

where matrices $\mathbf{T}_{00}$, $\mathbf{T}_{01}$, $\mathbf{B}_{00}$ and $\mathbf{B}_{01}$ have size $\beta \times \lceil M/2 \rceil$, $\lfloor (M-\beta)/2 \rfloor \times \lceil M/2 \rceil$, $\beta \times \lfloor M/2 \rfloor$ and $\lfloor (M-\beta)/2 \rfloor \times \lfloor M/2 \rfloor$, respectively. For odd channel systems, the row vectors $\mathbf{r}^T$ and $\mathbf{0}^T$ have size $\lceil M/2 \rceil$ and $\lfloor M/2 \rfloor$, respectively. They do not exist in even channel systems.

The above block matrix forms of $\mathbf{E}_0^f(z)$ and $\mathbf{R}_0^f(z)$ have already structurally embedded the linear phase property into them. Then, a set of matrix equations can be obtained by the PR condition $\mathbf{R}_0^f(z)\mathbf{E}_0^f(z) = \mathbf{I}_M$, whose solutions establish an LPPRFB with the structural forms (4.8) and (4.9).

$$\mathbf{T}_{00}\mathbf{S}_{00} + \mathbf{J}_\beta(\mathbf{T}_{00}\mathbf{S}_{00})\mathbf{J}_\beta = \mathbf{I}_{\beta/2}, \quad \mathbf{T}_{00}\mathbf{S}_{00} = \mathbf{B}_{00}\mathbf{A}_{00} \quad (4.10)$$

$$\mathbf{T}_{01}\mathbf{S}_{01} = \frac{1}{2}\mathbf{I}_{\lfloor(M-\beta)/2\rfloor} = \mathbf{B}_{01}\mathbf{A}_{01} \quad (4.11)$$

$$\mathbf{T}_{00}\mathbf{S}_{01} = \mathbf{0}_{\beta \times \lfloor(M-\beta)/2\rfloor} = \mathbf{B}_{00}\mathbf{A}_{01}, \quad \mathbf{T}_{01}\mathbf{S}_{00} = \mathbf{0} = \mathbf{B}_{01}\mathbf{A}_{00} \quad (4.12)$$

$$\mathbf{T}_{00}\mathbf{q} = \mathbf{0}, \quad \mathbf{T}_{01}\mathbf{q} = \mathbf{0}, \quad \mathbf{r}^T\mathbf{S}_{00} = \mathbf{0}, \quad \mathbf{r}^T\mathbf{S}_{01} = \mathbf{0}, \quad \mathbf{r}^T\mathbf{q} = 1 \quad (4.13)$$

From these equations, a set of rank conditions on the matrices which consist of $\mathbf{E}_0^f(z)$ and $\mathbf{R}_0^f(z)$ can also be found. These rank properties will be a key to the following lattice factorization and to the proof of the completeness of the lattice structure.

Lemma 4.5 *For the class of LPPRFBs and its starting block $\mathbf{E}_0^f(z)$ and $\mathbf{R}_0^f(z)$ stated above, $\rho(\mathbf{S}_{01}) = \rho(\mathbf{T}_{01}) = \rho(\mathbf{A}_{01}) = \rho(\mathbf{B}_{01}) = \lfloor (M-\beta)/2 \rfloor$, $\rho(\mathbf{S}_{00}) = \rho(\mathbf{T}_{00}) = \rho(\mathbf{A}_{00}) = \rho(\mathbf{B}_{00}) = \beta/2$. Moreover, for odd channel systems, $\mathbf{q}$ and $\mathbf{r}$ are nonzero vectors satisfying (4.13).*

¹We derive the factorization for the odd channel case in detail. The even channel case is similar to the odd channel case except there do not exist the special column vector $[\mathbf{q}^T, \mathbf{0}^T]^T$ and row vector $[\mathbf{r}^T, \mathbf{0}^T]$ in $\mathbf{E}_0(z)$ and $\mathbf{R}_0(z)$, respectively. Thus, there is no such column and row in various matrices in subsequent derivation for the even channel case.

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

Proof: We first show the ranks of $\mathbf{S}_{00}$ and $\mathbf{S}_{01}$ in detail. From (4.11), we know that

$$\lfloor (M - \beta)/2 \rfloor = \rho(\mathbf{T}_{01}\mathbf{S}_{01}) \leq \rho(\mathbf{S}_{01}) \leq \min\{\lceil M/2 \rceil, \lfloor (M - \beta)/2 \rfloor\} = \lfloor (M - \beta)/2 \rfloor \quad (4.14)$$

Thus, the matrix $\mathbf{S}_{01}$ has full rank, i.e., $\rho(\mathbf{S}_{01}) = \lfloor (M - \beta)/2 \rfloor$. The matrix $\mathbf{T}_{01}$ also has full rank by similar derivation. From (4.10) and the rank inequality for matrices [97], we can obtain that $\beta = \rho(\mathbf{T}_{00}\mathbf{S}_{00} + \mathbf{J}_\beta\mathbf{T}_{00}\mathbf{S}_{00}\mathbf{J}_\beta) \leq 2\rho(\mathbf{T}_{00}\mathbf{S}_{00})$, i.e., $\rho(\mathbf{T}_{00}\mathbf{S}_{00}) \geq \beta/2$. Define matrices $\mathbf{T} = [\mathbf{T}_{00}^T, \mathbf{T}_{01}^T, \mathbf{r}]^T$ and $\mathbf{S} = [\mathbf{S}_{00}, \mathbf{S}_{01}, \mathbf{q}]$, then from (4.11)-(4.13), we can see $\mathbf{TS} = \text{diag}(\mathbf{T}_{00}\mathbf{S}_{00}, \frac{1}{2}\mathbf{I}_{\lfloor (M-\beta)/2 \rfloor}, 1)$, which results in the following inequality.

$$\rho(\mathbf{TS}) = \rho(\mathbf{T}_{00}\mathbf{S}_{00}) + \lfloor (M - \beta)/2 \rfloor + 1 \leq \rho(\mathbf{S}) \leq \min\{\lceil (M + \beta)/2 \rceil, \lceil M/2 \rceil\} = \lceil M/2 \rceil \quad (4.15)$$

Thus the inequality $\rho(\mathbf{T}_{00}\mathbf{S}_{00}) \leq \beta/2$ holds. Combined with above two inequalities, we get $\rho(\mathbf{T}_{00}\mathbf{S}_{00}) = \beta/2$.

Similarly, define matrices $\bar{\mathbf{T}}_0 = [\mathbf{T}_{01}^T, \mathbf{r}]^T$ and $\bar{\mathbf{S}}_{01} = [\mathbf{S}_{01}, \mathbf{q}]$, then it can be shown easily that $\bar{\mathbf{T}}_{01}\bar{\mathbf{S}}_{01} = \text{diag}(\frac{1}{2}\mathbf{I}_{\lfloor (M-\beta)/2 \rfloor}, 1)$, i.e., $\rho(\bar{\mathbf{T}}_{01}) = \lfloor (M - \beta)/2 \rfloor + 1$. However, from $\bar{\mathbf{T}}_{01}\mathbf{S}_{00} = \mathbf{0}$ and Sylvester rank inequality [97], we can obtain that $0 = \rho(\bar{\mathbf{T}}_{01}\mathbf{S}_{00}) \geq \rho(\bar{\mathbf{T}}_{01}) + \rho(\mathbf{S}_{00}) - \lceil M/2 \rceil$ from which the inequality $\rho(\mathbf{S}_{00}) \leq \lceil M/2 \rceil - \rho(\bar{\mathbf{T}}_{01}) = \lceil M/2 \rceil - \lfloor (M - \beta)/2 \rfloor = \beta$ can be established since β is even. Finally, from $\rho(\mathbf{T}_{00}\mathbf{S}_{00}) = \beta/2 \leq \rho(\mathbf{S}_{00})$, we know that $\rho(\mathbf{S}_{00}) = \beta/2$. Similar arguments can be conducted for other matrices stated in the above Lemma. ■

From the sizes of matrices and the properties of their rank, we know that $\mathbf{S}_{01}$, $\mathbf{A}_{01}$, $\mathbf{T}_{01}$ and $\mathbf{B}_{01}$ are simply full rank matrices satisfying (4.11). However, matrix $\mathbf{S}_{00}$ has the rank equal to half of its number of columns and is constrained by (4.10), (4.12) and (4.13). The matrix equations in (4.10)-(4.13) for the PR condition remind us of the previous works on LOT [78], the cosine-modulated FBs with PR [5] and the LPPUFBs [25]. They all share the similar structural forms of those matrix equations for achieving PR. Based on the above analysis and the past works, a solution to the set of matrix equations (4.10)-(4.13) has been found, which can help factorize the initial blocks $\mathbf{E}_0^f(z)$ and $\mathbf{R}_0^f(z)$.

$$\mathbf{S}_{00} = \frac{1}{\sqrt{2}} [\mathbf{U}_{00}\mathbf{\Sigma}, \mathbf{U}_{00}\mathbf{\Delta}], \mathbf{A}_{00} = \frac{1}{\sqrt{2}} [\mathbf{V}_{00}\mathbf{\Sigma}, \mathbf{V}_{00}\mathbf{\Delta}] \quad (4.16)$$

where $\mathbf{U}_{00}$ and $\mathbf{V}_{00}$ have size $\lceil M/2 \rceil \times \beta/2$ and $\lfloor M/2 \rfloor \times \beta/2$, respectively. Further, we denote $\mathbf{\Sigma} = (\mathbf{N}_1 + \mathbf{N}_2)/2$ and $\mathbf{\Delta} = (\mathbf{N}_1 - \mathbf{N}_2)\mathbf{J}_l/2$, where $\mathbf{N}_1$ and $\mathbf{N}_2$ are two arbitrary square nonsingular matrices with size $l \times l$. Replace the matrices $\mathbf{S}_{01}$ and $\mathbf{A}_{01}$ by $\mathbf{U}_{01}/\sqrt{2}$ and $\mathbf{V}_{01}/\sqrt{2}$, respectively, of the same size as them. With these replacements, it can be shown that $\mathbf{E}_0^f(z)$ can be written in the first row of the equations shown in the top of the next page and factorized as in (4.17), where the constant matrix $\hat{\mathbf{E}}_0^f$ can be further factorized in the following form,

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

$$\begin{aligned}
 \mathbf{E}_0^f(z) &= \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{U}_{00}\Sigma + z^{-1}\mathbf{U}_{00}\Delta\mathbf{J}_l & \mathbf{U}_{00}\Delta + z^{-1}\mathbf{U}_{00}\Sigma\mathbf{J}_l & \mathbf{U}_{01} & \sqrt{2}\mathbf{q} & \mathbf{U}_{01}\mathbf{J}_{\lfloor(M-\beta)/2\rfloor} \\ \mathbf{V}_{00}\Sigma - z^{-1}\mathbf{V}_{00}\Delta\mathbf{J}_l & \mathbf{V}_{00}\Delta - z^{-1}\mathbf{V}_{00}\Sigma\mathbf{J}_l & \mathbf{V}_{01} & \mathbf{0} & -\mathbf{V}_{01}\mathbf{J}_{\lfloor(M-\beta)/2\rfloor} \end{bmatrix} \\
 &= \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{U}_{00} & \mathbf{U}_{01} & \sqrt{2}\mathbf{q} & \mathbf{U}_{01}\mathbf{J}_{\lfloor(M-\beta)/2\rfloor} & z^{-1}\mathbf{U}_{00}\mathbf{J}_l \\ \mathbf{V}_{00} & \mathbf{V}_{01} & \mathbf{0} & -\mathbf{V}_{01}\mathbf{J}_{\lfloor(M-\beta)/2\rfloor} & -z^{-1}\mathbf{V}_{00}\mathbf{J}_l \end{bmatrix} \underbrace{\begin{bmatrix} \Sigma & \Delta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{M-\beta} \\ \mathbf{J}_l\Delta\mathbf{J}_l & \mathbf{J}_l\Sigma\mathbf{J}_l & \mathbf{0} \end{bmatrix}}_{\hat{\Gamma}} \\
 &= \frac{1}{\sqrt{2}} \underbrace{\begin{bmatrix} \mathbf{U}_{00} & \mathbf{U}_{01} & \sqrt{2}\mathbf{q} & \mathbf{U}_{01}\mathbf{J}_{\lfloor(M-\beta)/2\rfloor} & \mathbf{U}_{00}\mathbf{J}_{\beta/2} \\ \mathbf{V}_{00} & \mathbf{V}_{01} & \mathbf{0} & -\mathbf{V}_{01}\mathbf{J}_{\lfloor(M-\beta)/2\rfloor} & -\mathbf{V}_{00}\mathbf{J}_{\beta/2} \end{bmatrix}}_{\hat{\mathbf{E}}_0^f} \underbrace{\begin{bmatrix} \mathbf{I}_{M-\beta/2} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I}_{\beta/2} \end{bmatrix}}_{\Lambda_0(z)} \hat{\Gamma} \\
 &= \hat{\mathbf{E}}_0^f \Lambda_0(z) \hat{\Gamma} \tag{4.17}
 \end{aligned}$$

$$\begin{aligned}
 \hat{\mathbf{E}}_0^f &= \begin{bmatrix} \mathbf{U}_{00} & \mathbf{U}_{01} & \mathbf{q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{V}_{00} & \mathbf{V}_{01} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} & \mathbf{I}_m \\ \mathbf{0} & \sqrt{2} & \mathbf{0} \\ \mathbf{I}_m & \mathbf{0} & -\mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_m \end{bmatrix} \\
 &= \text{diag}(\hat{\mathbf{U}}_0, \hat{\mathbf{V}}_0) \mathbf{W}_M \mathbf{I}_M^J \tag{4.18}
 \end{aligned}$$

in which the square matrix $\hat{\mathbf{U}}_0 = [\mathbf{U}_{00}, \mathbf{U}_{01}, \mathbf{q}]$ has size $\lceil M/2 \rceil$ and the square matrix $\hat{\mathbf{V}}_0 = [\mathbf{V}_{00}, \mathbf{V}_{01}]$ has size $\lfloor M/2 \rfloor$.

The special matrix $\hat{\Gamma}$ looks complicated. However, as we see, it can be further factorized into product of simple matrices and then reorganized as follows.

$$\hat{\Gamma} = \begin{bmatrix} \mathbf{I}_l & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{M-\beta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_l \end{bmatrix} \begin{bmatrix} \frac{\mathbf{N}_1 + \mathbf{N}_2}{2} & \frac{\mathbf{N}_1 - \mathbf{N}_2}{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{M-\beta} \\ \frac{\mathbf{N}_1 - \mathbf{N}_2}{2} & \frac{\mathbf{N}_1 + \mathbf{N}_2}{2} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_l & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{M-\beta} \end{bmatrix}$$

The second matrix in the righthand side of the above equation can be further simplified by factoring out $\text{diag}(\mathbf{N}_1, \mathbf{I}_{M-\beta}, \mathbf{N}_1)$ from the left. Using a method similar to [93], this factor can be propagated across the matrix $\text{diag}(\mathbf{I}_l, \mathbf{I}_{M-\beta}, \mathbf{J}_l)$. This leads to $\hat{\Gamma} = \text{diag}(\mathbf{N}_1, \mathbf{I}_{M-\beta}, \mathbf{J}_l \mathbf{N}_1 \mathbf{J}_l) \Gamma$, where Γ has the same structural form as $\hat{\Gamma}$ but with $\mathbf{N}_1 = \mathbf{I}_{\beta/2}$ and another nonsingular matrix $\mathbf{N}_2 = \hat{\mathbf{N}}_1^{-1} \hat{\mathbf{N}}_2$ later simply denoted by $\mathbf{N}$. This modified factor $\text{diag}(\mathbf{N}_1, \mathbf{I}_{M-\beta}, \mathbf{J}_l \mathbf{N}_1 \mathbf{J}_l)$ can be further propagated across $\Lambda_0(z)$ and merged into the block diagonal matrix $\text{diag}(\hat{\mathbf{U}}_0, \hat{\mathbf{V}}_0)$ of (4.18).

By these matrix operations, we have obtained a simplified factorization of $\mathbf{E}_0^f(z)$ equivalent to (4.17) as follows.

$$\mathbf{E}_0^f(z) = \mathbf{E}_0^f \Lambda_0(z) \Gamma = \text{diag}(\mathbf{U}_0, \mathbf{V}_0) \mathbf{W}_M \mathbf{I}_M^J \Lambda_0(z) \Gamma \tag{4.19}$$

where $\mathbf{E}_0^f = \text{diag}(\mathbf{U}_0, \mathbf{V}_0) \mathbf{W}_M \mathbf{I}_M^J$ with general square nonsingular matrices $\mathbf{U}_0$ and $\mathbf{V}_0$ of sizes $\lceil M/2 \rceil$ and $\lfloor M/2 \rfloor$, respectively. The relation between the parameters of the simplified

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

factorization (4.19) and the original one (4.17) is $\mathbf{U}_0 = \hat{\mathbf{U}}_0 \text{diag}(\hat{\mathbf{N}}_1, \mathbf{I}_{[(M-\beta)/2]})$ and $\mathbf{V}_0 = \hat{\mathbf{V}}_0 \text{diag}(\hat{\mathbf{N}}_1, \mathbf{I}_{[(M-\beta)/2]})$.

Figure 4.1: The forward lattice structure of the polyphase matrix $\mathbf{E}(z)$ of an M -channel LPPRFB for both odd ($M = 2m + 1$) and even ($M = 2m$) channel systems with length $L = KM + \beta$ ($\beta = 2l$). The dashed line only exists for the odd channel case.

Finally, by combining (4.2), we can obtain the overall forward factorization for the polyphase matrix $\mathbf{E}(z)$ as follows.

$$\mathbf{E}(z) = \mathbf{G}_{K-1}^f(z) \cdots \mathbf{G}_1^f(z) \mathbf{E}_0^f \mathbf{\Lambda}_0(z) \mathbf{\Gamma} \quad (4.20)$$

A block diagram of the factorization (4.20) is shown in Fig. 4.1. It can be easily seen that such factorization can ensure PR property of $\mathbf{E}(z)$ as long as $\mathbf{V}_i$ with $0 \leq i \leq K - 1$, $\mathbf{U}_0$ and $\mathbf{N}$ are square invertible matrices. In addition, it can be seen that if the filter length is constrained to be $L = KM$, i.e., $\beta = 0$, the factorization (4.20) is same as before [34, 42, 66, 77, 85, 86, 92, 93] since the additional matrices $\mathbf{\Lambda}_0(z)$ and $\mathbf{\Gamma}$ will disappear, which leads to $\mathbf{E}_0^f(z) = \mathbf{E}_0^f$. In this sense, the lattice factorization (4.20) is more general than before, since it can generate LPPRFBs with arbitrary lengths and cover the traditional one as a special case.

Note that this factorization is also applicable to PU systems, which is a special case of this factorization if the matrices $\mathbf{V}_i$ with $0 \leq i \leq K - 1$, $\mathbf{U}_0$ and $\mathbf{N}$ are constrained to be square orthogonal matrices, reported in [25]. However, the general LPPRFB is more desirable at least for the case of $\beta = 2$ [25] since it can avoid the inevitable zero filter coefficients of LPPUFBs at fixed positions. This is because the PU systems demand a null matrix $\mathbf{\Delta}$ (actually a zero-valued scalar), which would lead to a zero column in $\mathbf{S}_{00}$ when $\beta = 2$. Thus, all filters would have the zero-valued coefficients at the same positions. (If we permute the columns in the parameterized form (4.16) to reallocate the zero column at the boundaries, the essential filter length L would reduce to traditional case KM .) However, the general PR systems will not have this problem since it only requires the scalars $\mathbf{N}$ to be nonzero when $\beta = 2$. Thus, no zero filter coefficients

will be imposed as long as $N \neq 1$.

Backward factorization:

Besides the forward factorization, the backward lattice factorization has also been studied for general PR/PU FBs without phase constraint in [65, 66, 71] and for LPPRFBs in [44, 95]. For LPPRFBs with constrained length $L = KM$, it has been shown that these two factorizations are equivalent, i.e., $\mathbf{E}(z) = \mathbf{G}_{K-1}^f(z) \cdots \mathbf{G}_1^f(z) \mathbf{E}_0^f = \mathbf{E}_0^b \mathbf{G}_1^b(z) \cdots \mathbf{G}_{K-1}^b(z)$. Then, by combining the above equivalence and the forward factorization (4.20), we can easily obtain the backward factorization of LPPRFBs with length $L = KM + \beta$ as follows.

$$\mathbf{E}(z) = \mathbf{E}_0^b \mathbf{G}_1^b(z) \cdots \mathbf{G}_{K-2}^b(z) \mathbf{G}_{K-1}^b(z) \mathbf{\Lambda}_0(z) \mathbf{\Gamma} \quad (4.21)$$

where $\mathbf{E}_0^b = \text{diag}(\bar{\mathbf{U}}_0, \bar{\mathbf{V}}_0) \mathbf{W}_M \mathbf{I}_M^J$ with square nonsingular matrices $\bar{\mathbf{U}}_0$ and $\bar{\mathbf{V}}_0$ of sizes $\lceil M/2 \rceil$ and $\lfloor M/2 \rfloor$, respectively. The backward LPPR building blocks $\mathbf{G}_i^b(z)$ are defined in (4.6) and (4.7) for even and odd channel FBs, respectively.

By defining the block $\hat{\mathbf{G}}(z) = \mathbf{\Lambda}_0(z) \mathbf{\Gamma}$, we can see such special order-one block like $\hat{\mathbf{G}}(z)$ exists in the right most place of *both* forward and backward factorizations. Different from the conventional order-one blocks $\mathbf{G}(z)$ as in (4.3) or (4.6) for increasing length by M , the special block like $\hat{\mathbf{G}}(z)$ as a modified LPPR building block can only increase the filter length by β ($\beta < M$). Note that for even channel systems, the above $\hat{\mathbf{G}}(z)$ degenerates into the traditional block shown in (4.6) when $\beta = M$, i.e., $\hat{\mathbf{G}}(z) = \mathbf{G}^b(z)$. Thus, the proposed form is more general than before since the traditional one can again be seen as a special case of $\hat{\mathbf{G}}(z)$.

4.3.1.2 Lattice Factorization for Regular LPPRFBs

Another desired property of FB is the regularity which is closely related to the performance of image compression based on LPPRFB. In this paper, regularity of a FB is referred to as the number of multiple zeros at the aliasing frequencies [26]. An M -channel FB is said to be (K_a, K_s) -regular [27] if the analysis and synthesis lowpass filters $H_0(z)$ and $F_0(z)$ have a zero of multiplicity K_a and K_s , respectively, at the aliasing frequencies $\exp(j2\pi k/M)$ for $k = 1, 2, \dots, M-1$. By virtue of the above lattice structure, we can impose the regularity structurally into our even channel LPPRFB with arbitrary length.

However, from the existing works on image coding based on regular M -channel LPPRFBs, one degree regularity is usually enough for efficient image compression. In addition, when the filter length L of the underlying LPPRFB is less than $2M$, *only* one degree of regularity is possible [27, 45, 72]. Furthermore, when the number of channels M is large, more than one

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

degree of regularity is not efficient and useful, at the expense of fewer free parameters for other more desired property like coding gain. Thus, we only provide the solution of one degree of regularity for LPPRFBs with length $L = KM + \beta$. This also corresponds to no DC leakage, which is desired for avoiding the artificial checkerboard effect in low bit rate image coding. However, note that it can also be extended to more than one degree of regularity in a similar way as in [27, 45, 72] if it is desirable for some specific applications.

Previous studies have shown [27, 72] that the regularity constraint is equivalent to the following conditions on the analysis and synthesis polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$, respectively:

$$\left. \frac{d^n}{dz^n} [\mathbf{E}(z^M) \mathbf{d}_M(z)] \right|_{z=1} = c_n \mathbf{e}_0, \quad \left. \frac{d^m}{dz^m} [\mathbf{R}^T(z^M) \mathbf{J}_M \mathbf{d}_M(z)] \right|_{z=1} = d_m \mathbf{e}_0 \quad (4.22)$$

where $n = 0, 1, \dots, K_s - 1$; $m = 0, 1, \dots, K_a - 1$ and $\{c_n, d_m\}$ are some nonzero constants, and standard orthonormal basis vector $\mathbf{e}_0 = [1, 0, \dots, 0]^T$.

Next, we impose the regularity constraints on the proposed lattice structure to obtain the so called structural regularity. In order to get one degree of regularity, i.e., (1,1)-regular, the polyphase matrix $\mathbf{E}(z)$ must satisfy $[\mathbf{E}(z^M) \mathbf{d}_M(z)]|_{z=1} = c\mathbf{e}_0$ and $[\mathbf{R}^T(z^M) \mathbf{J}_M \mathbf{d}_M(z)]|_{z=1} = d\mathbf{e}_0$. Let $\mathbf{1}_M$ be a constant column vector of size M with all elements being one. By noting that $\mathbf{J}_l \mathbf{1}_l = \mathbf{1}_l$ and $\mathbf{\Sigma} + \mathbf{\Delta} = \mathbf{I}_l$, we can see that $\mathbf{\Gamma} \mathbf{1}_M = \mathbf{1}_M$. Then, we can easily obtain $\mathbf{W}_M \mathbf{I}_M^J \mathbf{\Gamma} \mathbf{1}_M = \sqrt{2}(\mathbf{1}_m^T, 1/\sqrt{2}, \mathbf{0}^T)^T$ for $M = 2m + 1$ and $\mathbf{W}_M \mathbf{I}_M^J \mathbf{\Gamma} \mathbf{1}_M = \sqrt{2}(\mathbf{1}_m^T, \mathbf{0}^T)^T$ for $M = 2m$.

Odd channel regular LPPRFBs:

For odd channel systems, first we can see that when evaluating at $z = 1$, the LPPR building block $\mathbf{G}_i^f(z)$ of (4.4) can be simplified to the following,

$$\mathbf{G}_i^f(z)|_{z=1} = \mathbf{\Phi}_{i,0} \mathbf{W}_{2m+1} \mathbf{W}_{2m+1} \mathbf{\Phi}_{i,1} \mathbf{W}_{2m+1} \mathbf{W}_{2m+1} = \mathbf{\Phi}_{i,0} \mathbf{\Phi}_{i,1} = \text{diag}(\mathbf{U}_{i,0}, \mathbf{V}_{i,1}) \quad (4.23)$$

Then, from our factorization (4.20) of $\mathbf{E}(z)$, we can obtain

$$\mathbf{E}(1) \mathbf{1}_M = \prod_{i=K-1}^0 \text{diag}(\mathbf{U}_{i,0}, \mathbf{V}_{i,1}) \mathbf{W}_M \mathbf{I}_M^J \mathbf{\Gamma} \mathbf{1}_M = \sqrt{2} \text{diag}(\prod_{i=K-1}^0 \mathbf{U}_i, \prod_{i=K-1}^0 \mathbf{V}_i) (\mathbf{1}_m^T, 1/\sqrt{2}, \mathbf{0}^T)^T$$

Thus, we can see that $\mathbf{E}(1) \mathbf{1}_M = c\mathbf{e}_0$ is equivalent to requiring an extra constraint only on the product of matrices $\mathbf{U}_{i,0}$ as follows.

$$\sqrt{2} \prod_{i=K-1}^0 \mathbf{U}_i (\mathbf{1}_m^T, 1/\sqrt{2})^T = c\mathbf{e}_0 \quad (4.24)$$

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

By similar derivation, we can obtain the simplified condition of $\mathbf{R}^T(z^M)\mathbf{J}_M\mathbf{d}_M(z)|_{z=1} = \mathbf{R}^T(1)\mathbf{1}_M = \mathbf{E}^{-T}(1)\mathbf{1}_M = d\mathbf{e}_0$. It also gives an extra constraint only on the product of matrices $\mathbf{U}_{i,0}$ as follows.

$$\sqrt{2} \prod_{i=K-1}^0 \mathbf{U}_i^{-T}(\mathbf{1}_m^T, 1/\sqrt{2})^T = d\mathbf{e}_0 \quad (4.25)$$

In particular, for lapped transforms with length $L < 2M$, we just need to require

$$\sqrt{2}\mathbf{U}_0(\mathbf{1}_m^T, 1/\sqrt{2})^T = c\mathbf{e}_0 \text{ and } \sqrt{2}\mathbf{U}_0^{-T}(\mathbf{1}_m^T, 1/\sqrt{2})^T = d\mathbf{e}_0 \quad (4.26)$$

Even channel regular LPPRFBs:

For even channel FBs, first we can see that when evaluating at $z = 1$, the LPPR building block $\mathbf{G}_i^f(z)$ of (4.3) can be simplified to the following,

$$\mathbf{G}_i^f(z)|_{z=1} = \Phi_i \mathbf{W}_{2m} \mathbf{W}_{2m} = \Phi_i = \text{diag}(\mathbf{I}_m, \mathbf{V}_i) \quad (4.27)$$

Then, from our factorization (4.20) of $\mathbf{E}(z)$, we can obtain

$$\mathbf{E}(1)\mathbf{1}_M = \prod_{i=K-1}^0 \Phi_i \mathbf{W}_{2m} \mathbf{I}_{2m}^J \Gamma \mathbf{1}_M = \sqrt{2} \text{diag}(\mathbf{U}_0, \prod_{i=K-1}^0 \mathbf{V}_i)(\mathbf{1}_m^T, \mathbf{0}^T)^T \quad (4.28)$$

Thus, we can see that $\mathbf{E}(1)\mathbf{1}_M = c\mathbf{e}_0$ is equivalent to requiring an extra constraint only on $\mathbf{U}_0$ as follows.

$$\sqrt{2}\mathbf{U}_0\mathbf{1}_m = c\mathbf{e}_0 \quad (4.29)$$

The other simplified condition $\mathbf{R}^T(1)\mathbf{1}_M = \mathbf{E}^{-T}(1)\mathbf{1}_M = d\mathbf{e}_0$ also gives an extra constraint only on $\mathbf{U}_0$ as follows.

$$\sqrt{2}\mathbf{U}_0^{-T}\mathbf{1}_m = d\mathbf{e}_0 \quad (4.30)$$

The above two conditions on an arbitrary invertible matrix $\mathbf{U}_0$ have been studied and named A_{01} and A_{10} conditions in [27]. By LDU matrix factorization of nonsingular matrix $\mathbf{U}_0$, both A_{01} and A_{10} conditions can be structurally imposed by the lifting structure [27] or Householder transform [72]. Denote the Householder matrix $\mathbf{H}_{[1_m]} = \mathbf{I}_m - 2\mathbf{u}\mathbf{u}^T/\mathbf{u}^T\mathbf{u}$ with $\mathbf{u} = \mathbf{1}_m/\sqrt{m} - \mathbf{e}_0$, which can align the vector $\mathbf{1}_m$ to $\mathbf{e}_0$, i.e., $\mathbf{H}_{[1_m]}\mathbf{1}_m = \sqrt{m}\mathbf{e}_0$. Then, the regularity can be enforced by constraining the general nonsingular matrix $\mathbf{U}_0$ to be of the following special form [72],

$$\mathbf{U}_0 = \text{diag}(c/\sqrt{m}, \hat{\mathbf{U}}_0)\mathbf{H}_{[1_m]} \quad (4.31)$$

where $\hat{\mathbf{U}}_0$ is an $(m-1) \times (m-1)$ arbitrary invertible matrix.

4.3.2 LPPRFBs with Odd β

For M -channel LPPRFBs with odd $\beta = 2l + 1$ ($0 \leq l \leq m$), the number of channels must be odd ($M = 2m + 1$) and K must be even as seen from Table 4.1. In the following, we first present the forward factorization in detail and then show the backward factorization. Since the minimum order of the LPPR building block $\mathbf{G}_i^f(z)$ is two for odd channel systems as seen in (4.4), the starting block $\mathbf{E}_0^f(z)$ has at least order two. This is the essential *difference* in the case of odd β , compared to the previous one. From the linear phase condition (4.1) and the symmetry polarity shown in Table 4.1, we can write the structural form of $\mathbf{E}_0^f(z)$ similar to (4.8) but with order-two as follows,

$$\mathbf{E}_0^f(z) = \begin{bmatrix} \mathbf{S}_0 & \mathbf{S}_1 \\ \mathbf{A}_0 & \mathbf{A}_1 \end{bmatrix} + z^{-1} \begin{bmatrix} \mathbf{S}_2 & \mathbf{q} & \mathbf{S}_2 \mathbf{J}_l & \mathbf{S}_1 \mathbf{J}_{2m-2l} \\ \mathbf{A}_2 & \mathbf{0} & -\mathbf{A}_2 \mathbf{J}_l & -\mathbf{A}_1 \mathbf{J}_{2m-2l} \end{bmatrix} + z^{-2} \begin{bmatrix} \mathbf{S}_0 \mathbf{J}_{2l+1} & \mathbf{0} \\ -\mathbf{A}_0 \mathbf{J}_{2l+1} & \mathbf{0} \end{bmatrix} \quad (4.32)$$

where matrices $\mathbf{S}_0$, $\mathbf{S}_1$ and $\mathbf{S}_2$ have sizes of $(m+1) \times (2l+1)$, $(m+1) \times (2m-2l)$ and $(m+1) \times l$, respectively; matrices $\mathbf{A}_0$, $\mathbf{A}_1$ and $\mathbf{A}_2$ have sizes of $m \times (2l+1)$, $m \times (2m-2l)$ and $m \times l$, respectively; and $\mathbf{q}$ and $\mathbf{0}$ are column vectors with sizes of $m+1$ and m , respectively. Same as before, we can employ the L' structural forms of $\mathbf{E}_0^f(z)$ and $\mathbf{R}_0^f(z)$ and the PR condition to establish a set of matrix equations as the necessary conditions for the parameter matrices of $\mathbf{E}_0^f(z)$ and $\mathbf{R}_0^f(z)$. However, different from the even β case, it is very involved, if not impossible, to obtain useful and efficient rank conditions and possible complete solutions to those matrix equations due to the higher order of $\mathbf{E}_0^f(z)$ than before.

4.3.2.1 General Lattice Factorization

Inspired by the previous works on developing lattice factorization for higher-order LPPRFBs, we also employ the cascade of two modified order-one LPPR building blocks to generate the desired order-two LPPRFB $\mathbf{E}_0^f(z)$. First, we can rewrite the filter length L of $\mathbf{E}_0^f(z)$ as $L = 2M + \beta = M + \beta'$. We can see that $\beta' = M + \beta$ is even. Thus, we are thinking to employ the factorization results in the previous section to achieve the order-two $\mathbf{E}_0^f(z)$. Since the previous results are only applied to the case of even β less than M , we can divide β' into two parts as $\beta' = \beta_0 + \beta_1$, where both $\beta_0 = M - 1$ and $\beta_1 = \beta + 1$ are even integers less than M . Then, by virtue of the prefiltering block $\hat{\mathbf{G}}(z)$ developed in Sec. 4.3.1, we can employ two prefiltering blocks $\hat{\mathbf{G}}_0(z)$ and $\hat{\mathbf{G}}_1(z)$ for increasing filter lengths by β_0 and β_1 , respectively. Therefore, the overall factorization of such order-two LPPRFB $\mathbf{E}_0^f(z)$ can be formulated as follows,

$$\mathbf{E}_0^f(z) = \mathbf{E}_0^f \hat{\mathbf{G}}_0(z) \hat{\mathbf{G}}_1(z) = \mathbf{E}_0^f \mathbf{\Lambda}_{0,0}(z) \mathbf{\Gamma}_0 \mathbf{\Lambda}_{0,1}(z) \mathbf{\Gamma}_1 \quad (4.33)$$

where $\mathbf{E}_0^f = \text{diag}(\mathbf{U}_0, \mathbf{V}_0) \mathbf{W}_M \mathbf{I}_M^J$ with general square nonsingular matrices $\mathbf{U}_0$ and $\mathbf{V}_0$ of sizes $\lceil M/2 \rceil$ and $\lfloor M/2 \rfloor$, respectively. The delay matrices are $\mathbf{\Lambda}_{0,0}(z) = \text{diag}(\mathbf{I}_{m+1}, z^{-1} \mathbf{I}_m)$ and

$\Lambda_{0,1}(z) = \text{diag}(\mathbf{I}_m, z^{-1}\mathbf{I}_{l+1}, \mathbf{I}_{m-l})$, respectively. In addition, the prefiltering matrices $\mathbf{\Gamma}_0$ and $\mathbf{\Gamma}_1$ are as follows,

$$\Gamma_0 = \begin{bmatrix} \Sigma_0 & \mathbf{0} & \Delta_0 \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{J}_m \Delta_0 \mathbf{J}_m & \mathbf{0} & \mathbf{J}_m \Sigma_0 \mathbf{J}_m \end{bmatrix} = \mathbf{I}_M^J \mathbf{W}_M \Phi_{0,0} \mathbf{W}_M \mathbf{I}_M^J \quad (4.34)$$

where the block diagonal matrix $\Phi_{0,0} = \text{diag}(\mathbf{I}_{m+1}, \mathbf{N}_0)$ by using the simplification same as before. $\mathbf{N}_0$ is an arbitrary $m \times m$ nonsingular matrix. Now

$$\Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & \mathbf{I}_{m-l} & 0 \\ \Sigma_1 & \mathbf{b}/\sqrt{2} & \Delta_1 & 0 & 0 \\ \mathbf{c}^T/\sqrt{2} & d & \mathbf{c}^T \mathbf{J}_l/\sqrt{2} & 0 & 0 \\ \mathbf{J}_l \Delta_1 \mathbf{J}_l & \mathbf{J}_l \mathbf{b}/\sqrt{2} & \mathbf{J}_l \Sigma_1 \mathbf{J}_l & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_{m-l} \end{bmatrix} \quad (4.35)$$

where both $\mathbf{b}$ and $\mathbf{c}$ are column vectors of size l , and d is a scalar. Similar to (4.34), the submatrix in the left part of $\mathbf{\Gamma}_1$ with size of $\beta \times \beta$ can be further factorized into simple matrices as follows.

$$\begin{bmatrix} \Sigma_1 & \mathbf{b}/\sqrt{2} & \Delta_1 \\ \mathbf{c}^T/\sqrt{2} & d & \mathbf{c}^T \mathbf{J}_l/\sqrt{2} \\ \mathbf{J}_l \Delta_1 \mathbf{J}_l & \mathbf{J}_l \mathbf{b}/\sqrt{2} & \mathbf{J}_l \Sigma_1 \mathbf{J}_l \end{bmatrix} = \mathbf{I}_\beta^J \mathbf{W}_\beta \Phi_{0,1} \mathbf{W}_\beta \mathbf{I}_\beta^J \quad (4.36)$$

The block diagonal matrix $\Phi_{0,1}$ can be shown to be further simplified into the following form,

$$\Phi_{0,1} = \begin{bmatrix} \bar{\mathbf{N}}_1 & \mathbf{b} & \mathbf{0} \\ \mathbf{c}^T & d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_l \end{bmatrix} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_l \end{bmatrix} \quad (4.37)$$

where $\mathbf{N}_1$ is an arbitrary square nonsingular matrix with size $l + 1$. A block diagram of the factorization (4.33) is shown in Fig. 4.2.

Figure 4.2: The structure of the initial block $\mathbf{E}_0^f(z)$ of M -channel LPPRFBs ($M = 2m + 1$) with length $L = 2M + \beta$ ($\beta = 2l + 1$).

Then, by combining (4.2), we can establish the overall forward factorization as follows,

$$\mathbf{E}(z) = \mathbf{G}_{\frac{K-2}{2}}^f(z) \cdots \mathbf{G}_1^f(z) \mathbf{E}_0^f \Lambda_{0,0}(z) \Gamma_0 \Lambda_{0,1}(z) \Gamma_1 \quad (4.38)$$

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

It can be easily seen that the factorization (4.38) can ensure PR property of $\mathbf{E}(z)$ if and only if $\mathbf{U}_i$ and $\mathbf{V}_i$ with $0 \leq i \leq (K-2)/2$, $\mathbf{N}_0$ and $\mathbf{N}_1$ are square invertible matrices. An LPPUFB $\mathbf{E}(z)$ can be obtained if we further constrain these matrices to be orthogonal ones. In addition, it can be seen that if the filter length is constrained to be $L = (K+1)M$ (note that here K is even), i.e., $\beta = M$, the factorization (4.38) is same as before [34]. Similar to the case of even β , we can obtain the backward factorization for the case of odd β in the following form,

$$\mathbf{E}(z) = \mathbf{E}_0^b \mathbf{G}_1^b(z) \cdots \mathbf{G}_{\frac{K-2}{2}}^b(z) \mathbf{\Lambda}_{0,0}(z) \mathbf{\Gamma}_0 \mathbf{\Lambda}_{0,1}(z) \mathbf{\Gamma}_1 \quad (4.39)$$

It can be easily seen that in the special case of $\beta = M$, the product of the modified block is same as the backward LPPR building block $\mathbf{G}^b(z)$ shown in (4.7), i.e., $\hat{\mathbf{G}}_0(z) \hat{\mathbf{G}}_1(z) = \mathbf{G}^b(z)$. Thus, the factorizations (4.38) and (4.39) are more general than before because they can cover the traditional one as a special case.

4.3.3 Relationship with Previous Works

The lattice structure for M -channel LPPRFBs with arbitrary lengths is not totally new. There are some existing works related to it. In this section, we discuss the relationship between our derived lattice factorizations and the previous works.

4.3.3.1 Relationship with lapped transforms via time-domain pre/postfiltering

Tran *et al.* has done some work in this area. In [25], they first showed some necessary length conditions for M -channel LPPRFBs and thus showed the possibility of the filter length L to be noninteger multiple of M . The lattice factorization has been developed for even channel LPPUFBs with arbitrary lengths. Later, it was extended to even channel LPPRFBs in [55]. Then, for the purpose of more efficient image and video coding which is backward compatible to the existing JPEG standard, their group further developed M -channel LPPRFBs with flexible lengths under the framework of lapped transforms via time-domain pre- and post-filtering [43–45].

In fact, these works are equivalent to a hybrid forward and backward factorization by treating the backward LPPR building block $\mathbf{G}^b(z)$ and the forward LPPR building block $\mathbf{G}^f(z)$ as the pre- and post-filtering part, respectively, and putting the matrix $\mathbf{E}_0$ in between them acting as a block transform. This can be seen as follows.

Even β case:

First, we can use some permutation matrix to change $\mathbf{\Gamma}$ to be a block diagonal matrix $\tilde{\mathbf{\Gamma}} =$

$\text{diag}\left(\begin{bmatrix} \Sigma & \Delta \\ \mathbf{J}_l \Delta \mathbf{J}_l & \mathbf{J}_l \Sigma \mathbf{J}_l \end{bmatrix}, \mathbf{I}_{M-\beta}\right)$. Then, note that the upper-left block of $\bar{\Gamma}$ can be further factorized into simple blocks as follows,

$$\frac{1}{2} \begin{bmatrix} \mathbf{I}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_l \end{bmatrix} \begin{bmatrix} \mathbf{I}_l + \mathbf{N} & \mathbf{I}_l - \mathbf{N} \\ \mathbf{I}_l - \mathbf{N} & \mathbf{I}_l + \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{I}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_l \end{bmatrix} = \mathbf{I}_\beta^J \mathbf{W}_\beta \text{diag}(\mathbf{I}_l, \mathbf{N}) \mathbf{W}_\beta \mathbf{I}_\beta^J = \mathbf{W}_\beta^J \text{diag}(\mathbf{I}_l, \mathbf{V}) \mathbf{W}_\beta^J \quad (4.40)$$

where $\mathbf{V} = \mathbf{J}_l \mathbf{N} \mathbf{J}_l$ is still an invertible matrix. Note that the matrix $\bar{\Gamma}$ and its component (4.40) is just the prefilter $\mathbf{P}$ developed in [43–45]. Correspondingly, it can be seen that the lattice structure shown in Fig. 4.1 with $M = 8, K = 1$ and $\beta = 4$ is equivalent to [43, Fig. 5].

However, it should be pointed out that our structure shown in Fig. 4.1 is minimal in contrast to the *nonminimal* structure shown in [43, Fig. 5] since the authors always fixed the special delay matrix $\Lambda_0(z)$ to be the common delay matrix $\Lambda_i(z)$ and thus employed two unnecessary delay elements (Also can be seen in [44, Fig. 5]). Those nonminimal structures can be easily made minimal by reducing and placing the number of delay elements appropriately. In addition, previous works on lapped transforms via pre/postfiltering were developed *heuristically* based on an important observation [43] that the amount of block overlap can be controlled by the size of the prefilter matrix $\mathbf{P}$. Thus, those pioneering works successfully developed the more efficient and flexible M -channel LPPRFBs with general filter lengths. However, due to the heuristic way of development of those structures, the completeness of those structures are left for further investigation. On the contrary, we started with the necessary conditions of M -channel LPPRFBs with arbitrary lengths and then developed the lattice factorization (4.19) based on the ranks of underlying matrices shown in Lemma 4.5. Thus, we can further show the completeness of the lattice factorization (4.19) for LPPRFBs and LPPUFBs over certain length range in Sec 4.4 later.

The regular lattice structure of M -channel LPPRFBs with arbitrary lengths has also been discussed in [45] but only for lapped transforms with one stage of prefiltering $\hat{\mathbf{G}}(z)$ and constraining the nonsingular matrix $\mathbf{E}_0$ to be the DCT matrix for fast implementation. In contrast to their work, we intend to more systematically derive the regular lattice structures to find those *necessary* conditions without those constraints and allowing multiple pre/post filtering stages.

Odd β case:

The regularity-constrained pre/post filtering for block DCT-based lapped transforms have been studied in [45]. They have shown both even and odd channel systems with length $L = M + \beta$ and $\beta = 2l$. For generalized lapped transforms with longer lengths, they proposed to employ multiple stages of prefiltering to increase the length. This is essentially same as our idea of

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

factorization in Sec. 4.3.2. However, the structure with multiple stages of prefiltering [45] was proposed heuristically by simply cascading the *same* prefiltering block at the boundaries of each stage. However, as we have seen in Fig. 4.2, the two prefiltering stages $\hat{\mathbf{G}}_0(z)$ and $\hat{\mathbf{G}}_1(z)$ are *different* according to the sizes of the matrices $\mathbf{N}_0$ and $\mathbf{N}_1$. In fact, [45] is just a special case of (4.33) since it constrains $\mathbf{b}$ and $\mathbf{c}$ to be null vectors and scalar $d = 1$. In such situation, they can achieve the regular lattice structure easily since $\mathbf{\Gamma}_1 \mathbf{1}_M = \mathbf{1}_M$.

However, note that the structure proposed heuristically in [45] is not a general solution for the odd β case, since those unnecessary constraints would lead a filter to have essential filter length less than L due to the zero filter coefficients at both the boundaries of that filter. On the contrary, the factorization (4.38) can be proven to be complete for LPPUFBs presented in Theorem 4.6 of Sec. 4.4.2. However, for the general factorization (4.33) with nonzero vectors $\mathbf{b}$ and $\mathbf{c}$, it is very involved to obtain an explicit condition for the regularity since $\mathbf{\Gamma}_1 \mathbf{1}_M = (\mathbf{1}_{m-l}, \mathbf{1}_l + \mathbf{b}/\sqrt{2}, \sqrt{2}\mathbf{c}^T \mathbf{1}_l + d, \mathbf{1}_l + \mathbf{J}_l \mathbf{b}/\sqrt{2}, \mathbf{1}_{m-l}) \neq \mathbf{1}_M$. The problem of structurally imposing regularity onto (4.33) is to be further addressed in future.

4.3.3.2 Connection with oversampled LPPRFBs

We may partition the analysis polyphase matrix $\mathbf{E}(z)$ according to the order of polyphase components, i.e., $\mathbf{E}(z) = [\mathbf{E}_l(z), \mathbf{E}_r(z)]$, where $\mathbf{E}_l(z)$ and $\mathbf{E}_r(z)$ have sizes of $M \times \beta$ and $M \times (M - \beta)$, respectively. Correspondingly, we partition the synthesis polyphase matrix $\mathbf{R}(z) = [\mathbf{R}_u(z)^T, \mathbf{R}_d(z)^T]^T$. According to PR condition $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}_M$ and generalized LP condition (4.1), it can be seen that $\mathbf{E}_l(z)$ is an M -channel oversampled LPPRFB with subsampling factor β and filter length $L_l = (K+1)\beta$. Similarly, $\mathbf{E}_r(z)$ is associated with an M -channel oversampled LPPRFB with subsampling factor $M - \beta$ and filter length $L_r = K(M - \beta)$. Such kind of FBs has been investigated in [80] by using forward factorizations and a backward factorization for even channel oversampled LPPRFBs has further been developed in [98]. In fact, it can be shown that our derived lattice factorization (4.20) is equivalent to appropriately combining the backward lattice factorizations of both oversampled LPPRFBs $\mathbf{E}_l(z)$ and $\mathbf{E}_r(z)$.

4.4 Theoretical Properties of the Lattice Factorizations

In this section, we further investigate some theoretical properties of the derived lattice factorizations in previous sections, including the minimality and completeness.

4.4.1 Minimality

The forward lattice factorizations of (4.20) and (4.38), and the backward factorizations (4.21) and (4.39) for both odd and even channel LPPRFBs with arbitrary filter lengths are not only

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

more general than the traditional works, but also attractive in terms of the number of delays used in implementation. The following theorem states that the derived lattice factorizations in previous sections employ the fewest delays in implementation for both even and odd channel systems.

Theorem 4.4 *The forward lattice factorizations of (4.20) and (4.38) for the cases of even β and odd β , respectively, are minimal for both even and odd channel LPPRFBs. The same conclusion is true for the backward factorizations (4.21) and (4.39).*

Proof: The degree of a causal rational system is defined as the minimum number of delays required for its implementation [5]. A structure is said to be minimal if the number of delays used is equal to the degree of the transfer function. For the class of CAFACAFI FBs, it has been proven that [64] $\deg(\mathbf{E}(z)) = \deg(|\mathbf{E}(z)|)$. From the LP condition (4.1), thus we can have

$$\deg(\mathbf{E}(z)) = \deg(|\mathbf{E}(z)|) = \deg\left(z^{-M(K-1)}|\mathbf{D}| \times |\mathbf{E}(z^{-1})| \times |\mathbf{J}_M(z)|\right) \quad (4.41)$$

$|\mathbf{D}| = \pm 1$, which is a constant and independent of complex variable z . Since $|\mathbf{J}_M(z)| = (-1)^{(M-1)/2}z^{-\beta}$ for the case of odd M and $|\mathbf{J}_M(z)| = (-1)^{\beta+M/2}z^{-\beta}$ for the case of even M [25], we know that $\deg(|\mathbf{J}_M(z)|) = \beta$ for both odd and even M . Then, from (4.41) and the equality $\deg(\mathbf{E}(z^{-1})) = -\deg(|\mathbf{E}(z)|) = -\deg(\mathbf{E}(z))$, we can obtain,

$$\deg(\mathbf{E}(z)) = M(K-1) + \beta - \deg(\mathbf{E}(z))$$

which leads to $\deg(\mathbf{E}(z)) = M(K-1)/2 + \beta/2$.

In the case of even channel systems, there are $(K-1)$ order-one building blocks in the forward or backward factorization, in which each block uses $M/2$ delays as seen in (4.3) or (4.6). The initial block $\mathbf{E}_0^f(z)$ also employs $\beta/2$ delays as seen in (4.19). Therefore, the total number of delays in use is $M(K-1)/2 + \beta/2$.

In the case of odd channel systems, from the existence conditions in Table 4.1, we know that K should be odd for the case of even β . In the forward or backward factorization, there are $(K-1)/2$ order-two building blocks, where each employs M delays as seen in (4.4) or (4.7) ($\mathbf{\Lambda}_{i,0}(z)$ has $M/2$ and $\mathbf{\Lambda}_{i,1}(z)$ has $(M+1)/2$ delays). The initial block $\mathbf{E}_0^f(z)$ employs $\beta/2$ delays as seen in (4.19). Therefore, the total number of delays in use is also $M(K-1)/2 + \beta/2$. In addition, in the case of odd β , there are $(K-2)/2$ order two LPPR blocks $\mathbf{G}_i(z)$, thus using $M(K-2)/2$ delays. The initial block $\mathbf{E}_0^f(z)$ uses $m+l+1 = M/2 + \beta/2$ delays as seen in (4.33). The total delays used are $M(K-2)/2 + M/2 + \beta/2$ which is just the degree of the transfer function $\mathbf{E}(z)$, hence it is also a minimal factorization. ■

4.4.2 Completeness

A very important concept for the lattice factorization is completeness [7]. Completeness of a lattice factorization implies that any LPPRFB with certain desired properties like LP, PR and the given filter length can always be factorized into the *same* lattice structure. In the following, we establish the completeness of our derived lattice factorizations for LPPRFBs and LPPUFBs over certain length range, respectively. These results reveal that these lattice factorizations can greatly expand the family of LPPRFBs. First, the following theorem shows the completeness of the lattice factorization (4.19) with length $L \leq 2M$ for any M -channel LPPRFB.

Theorem 4.5 *The lattice factorization (4.19) is complete for both even and odd channel LP-PRFBs with length $L \leq 2M$.*

Proof: Firstly, when the filter length $L = M$, such LPPRFBs degenerate into the traditional block transform as $\mathbf{E}_0$ with the structure shown in (4.18). It can be easily shown that it is a complete parameterization due to the LP and PR properties. Secondly, when $L = 2M$ (it can only apply to even channel LPPRFBs), the derived lattice factorization degenerates into the existing one [99] which has been shown to be a complete factorization. Thus, we only need to consider the lengths in between, which means that the polyphase analysis matrix $\mathbf{E}(z)$ has no LPPR propagating blocks $\mathbf{G}_i(z)$, i.e., just order-one $\mathbf{E}_0(z)$ with the form of (4.8). This indicates that we need to show that $\mathbf{E}_0^f(z)$ can always be factorized as in (4.19) with even $\beta \neq 0$.

Note that the block matrix form of $\mathbf{E}_0^f(z)$ in (4.8) and matrix equations (4.10)-(4.13) contain all the *necessary* conditions to be satisfied for such LPPRFBs. It can be easily shown that our parameterized forms (4.16) and $\mathbf{S}_{01}, \mathbf{A}_{01}$ are a set of solutions to those matrix equations given $\mathbf{U}_0, \mathbf{V}_0$ are nonsingular. Thus, the factorizations of (4.19) derived from them are sufficient to generate an LPPRFB with length $L < 2M$.

However, completeness property of the lattice structure demands that all possible solutions should be able to be converted into the form of (4.19) for analysis bank $\mathbf{E}(z)$ (we only show the analysis bank because the synthesis bank $\mathbf{R}(z)$ is uniquely determined by the given analysis bank $\mathbf{E}(z)$). This can be achieved if the parameterized form (4.16) and $\mathbf{U}_{01}, \mathbf{V}_{01}$ and $\mathbf{q}$ can span the whole solution space of PR conditions (4.10)-(4.13), i.e., there always exist square invertible matrices $\mathbf{U}_0 = [\mathbf{U}_{00}, \mathbf{U}_{01}, \mathbf{q}]$ of size $\lceil M/2 \rceil$, $\mathbf{V}_0 = [\mathbf{V}_{00}, \mathbf{V}_{01}]$ of size $\lfloor M/2 \rfloor$ and $\mathbf{N}$ of size $\beta/2$ for any solution to those PR conditions. From the Lemma 4.5, we know that $\mathbf{S}_{01}$ is a full column rank matrix, i.e., $\lfloor (M - \beta)/2 \rfloor$ independent column vectors of size $\lceil M/2 \rceil$. Since $\mathbf{U}_{01}$ contains $\lfloor (M - \beta)/2 \rfloor$ columns of an arbitrary invertible matrix $\mathbf{U}_0$ of size $\lceil M/2 \rceil$ in factorization (4.19), $\mathbf{U}_{01}$ surely spans the space of all possible $\mathbf{S}_{01}$.

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

Next we show that the parameterized form (4.16) can span the solution space of $\mathbf{S}_{00}$. First note that the parameterized form (4.16) is indeed a solution of those PR equations. Since $\rho(\mathbf{S}_{00}) = \beta/2$, $\rho(\bar{\mathbf{T}}_{01}) = \lceil (M - \beta)/2 \rceil$ and $\bar{\mathbf{T}}_{01}\mathbf{S}_{00} = \mathbf{0}$, we know that the matrix $[\mathbf{S}_{00}, \bar{\mathbf{T}}_{01}^T]$ has full rank $\lceil M/2 \rceil$, i.e., $\lceil M/2 \rceil$ independent column vectors out of its total $\lceil (M + \beta)/2 \rceil$ columns. From $\bar{\mathbf{T}}_{01}\mathbf{S}_{00} = \mathbf{0}$, we know that the column space of $\mathbf{S}_{00}$ should be located in the orthogonal complement subspace of the row space of matrix $\bar{\mathbf{T}}_{01}$. This subspace should have dimension less than or equal to $\beta/2$ due to $\rho(\bar{\mathbf{T}}_{01}) = \lceil (M - \beta)/2 \rceil$. However, $\rho(\mathbf{S}_{00}) = \beta/2$ indicates that the column space of $\mathbf{S}_{00}$ should be exactly *same* as the orthogonal complement subspace of the row space of $\bar{\mathbf{T}}_{01}$. This means that there is a *unique* subspace with dimension $\beta/2$ spanned by the columns of $\mathbf{S}_{00}$, and all possible solutions of $\mathbf{S}_{00}$ to the PR conditions are located in this unique subspace. Thus, as long as the parameterized form (4.16) can span this unique subspace, it can cover all solutions of $\mathbf{S}_{00}$, leading to a complete solution by the parameterized form (4.16). This can be easily seen since $\rho(\mathbf{U}_{00}[\Sigma, \Delta]) = \rho(\mathbf{U}_{00}) = \beta/2$ because $[\Sigma, \Delta]$ is full row rank matrix. ■

Remark 4.3 *Note that we did not claim that it is also a unique parameterized solution to $\mathbf{S}_{00}$, i.e., there may exist other forms of solution of $\mathbf{S}_{00}$. However, from previous analysis, we know that the completeness property is ensured although we don't know if it is the unique complete solution.*

Note also that only two special cases with $L = M$ and $L = 2M$ have been shown to be complete before. Furthermore, counter examples [99] have been constructed to demonstrate the incompleteness of the lattice structures of (4.19) for $L \geq 3M$. Thus, the above theorem fills the gap in between the known results and expands the family of complete lattice factorizations of M -channel LPPRFBs.

Next, we present a much stronger result for the completeness of the derived lattice factorizations for FIR LPPUFBs.

Theorem 4.6 *The proposed forward lattice factorizations as shown in (4.20) and (4.38) are complete for any FIR LPPUFB with length satisfying the existence condition shown in Table 4.1. The same conclusion is true for the backward factorizations.*

Proof: This theorem can be rephrased as follows: if there exists an analysis polyphase matrix $\mathbf{E}(z)$ of an LPPUFB with equal filter length $L = KM + \beta$, which must satisfy LP condition (4.1), then $\mathbf{E}(z)$ can always be factorized as in (4.2), whereas $\mathbf{E}_0^f(z)$ can always be factorized as in (4.19) or (4.33). The former part is achieved by performing the modified order reduction process,

i.e., *length reduction* process, as presented in [81]. (Note that the traditional order reduction process [5, 25, 77] is not convincing enough for the proof of the case $\beta \neq 0$ since the reduced-order polyphase matrix $\mathbf{F}(z) = \mathbf{G}^{-1}(z)\mathbf{E}(z)$ may not have the length $L_F = (K-1)M + \beta$ but be something in between $(K-1)M$ and KM . Thus, the initial block may not be in the special structural form (4.8) or (4.32). In summary, for the case of nonzero β , the traditional order reduction process is not sufficient for proving the reduction of filter length by M , in contrast to the conventional case of $\beta = 0$.)

For the case of even β , after $K-1$ steps of length reduction performed by $\mathbf{G}_i(z)$, the remainder is an order-one LPPUFB with length $M + \beta$ which takes the form in (4.8). Thus, we need only to prove that $\mathbf{E}_0^f(z)$ can always be factorized as in (4.19) for even β . Similar arguments to the proof of Theorem 4.5 can be conducted here just by replacing $\mathbf{T}_{00}$ and $\mathbf{T}_{01}$ by $\mathbf{S}_{00}^T$ and $\mathbf{S}_{01}^T$, respectively, since it is a paraunitary FB.

For the case of odd β , the remainder is an order-two LPPUFB $\mathbf{E}_0^f(z)$ with the filter length $2M + \beta$ and the structural form of (4.32) after the length reduction. It has been shown [54] that the second stage $\hat{\mathbf{G}}_1(z)$ of the prefilter presented in Sec. 4.3.2 can decrease the filter length by $\beta + 1$ and the first stage $\hat{\mathbf{G}}_0(z)$ can decrease length by $M - 1$. Thus, the order-two $\mathbf{E}_0^f(z)$ of (4.32) can always be factorized as in (4.33). ■

It should be pointed that a special case of the above theorem, i.e., even-channel LPPUFBs, has been reported and shown in [25, Theorem 3].

4.5 Non-Existence of Complete Order-One Lattice Factorization

In spite of considerable research works in the last decade and the achievement in this chapter, the problem of *complete* lattice factorization for general M -channel ($M \geq 2$) LPPRFBs with equal lengths has remained open so far. The 2-channel LPPRFBs with equal lengths, i.e., type-A systems, and the associated lattice factorization have been investigated in [63, 84]. However, it was shown that such lattice factorization is not complete through a counter example [63]. Later, a lattice structure which is the special case of our factorization (4.20) with $\beta = 0$,

$$\mathbf{E}(z) = \mathbf{G}_{K-1}(z)\mathbf{G}_{K-2}(z) \cdots \mathbf{G}_1(z)\mathbf{E}_0 \quad (4.42)$$

was asserted to be complete for general even M -channel LPPRFBs [34]. However, it was further studied in [99] that such assertion is incorrect. In fact, the latter M -channel lattice structure degenerates into the previous one for 2-channel type-A systems when $M = 2$, thus

its incompleteness is expected. Moreover, there are some counter examples shown in [99] with filter lengths $L = KM$ with $K \geq 3$.

However, a more fundamental question needs to be answered before attempting to find a possibly complete lattice factorization. It is the issue of the *existence* of such lattice factorization that was overlooked in the past. This is partially because there does exist a complete lattice factorization for LPPUFBs. Thus, it was guessed that there may also exist a complete lattice factorization for LPPRFBs. However, no rigorous investigation is attempted. In this section, we revisit the incompleteness of the lattice factorization (4.42) for M -channel LPPRFBs with equal lengths $L = KM$ ($K \in \mathbb{N}$) and further investigate a more fundamental problem, i.e., the existence of complete lattice factorization by using LPPR propagating blocks of order-one CAFACAFI matrices. In the factorization (4.42), it can be easily seen that $\mathbf{R}(z)$, just by inverting the matrices $\mathbf{G}_i(z)$ and $\mathbf{E}_0$ in reverse order, is an *anticausal* matrix due to the anticausal inverse of $\mathbf{G}_i(z)$. (In [34] and [93], the authors employed unit delays elements z^{-1} to make $\mathbf{R}(z)$ causal. However, it does not change the system-theoretical property of $\mathbf{R}(z)$ developed in [64, 65]. Thus, the synthesis polyphase matrices $\mathbf{R}(z)$ are in essence still anticausal matrices.

Through reviewing the previous works, we point out the limitation of the existing LPPR propagating blocks and then show its consequence for incompleteness of the existing lattice factorizations. Furthermore, we show the *nonexistence* of any order-one LPPR propagating block by using CAFACAFI matrices. In addition, the completeness of lattice factorizations has been re-examined for GenLOT and LBT with linear phase based on the analysis developed in this section. For simplicity, we only consider even-channel critically sampled LPPRFBs within the class of CAFACAFIs [65] and show the results for the forward lattice factorizations. Note that similar work has also been extended to backward factorizations.

4.5.1 Review of Existing Proof and Correction

To help understand the incompleteness of lattice factorization (4.42) and gain some insights into the existence issue, we briefly review the previous works [34] and [99] here. The original proof of completeness in [34] is fulfilled by induction. The essential step is the so called order reduction process that can successively reduce the order of the polyphase matrix $\mathbf{E}(z)$ of an LPPRFB by one, while the reduced-order polyphase matrix $\mathbf{F}(z) = \mathbf{G}_i^{-1}(z)\mathbf{E}(z)$ still retains the LP and PR properties. A critical issue in the order reduction is the elimination of noncausality due to the anticausal property of $\mathbf{G}_i^{-1}(z)$, so that $\mathbf{F}(z)$ is a causal polyphase matrix. Let $\mathbf{G}_i^{-1}(z) = \mathbf{H}(z) = \mathbf{H}_0 + z\mathbf{H}_1$ and $\mathbf{E}(z) = \mathbf{E}_0 + \mathbf{E}_1z^{-1} + \dots + \mathbf{E}_{K-1}z^{-(K-1)}$. It is *necessary* to have $\mathbf{H}_1\mathbf{E}_0 = \mathbf{0}$ for $\mathbf{F}(z)$ to be causal. It was assumed in [34] that if $\rho(\mathbf{E}_0) \leq M/2$, which is a

necessary condition for LPPRFBs with length $L = KM$, then it is always possible to establish $\mathbf{H}_1 \mathbf{E}_0 = \mathbf{0}$ by carefully choosing the basis vectors from the null space of $\mathbf{E}_0$. However, it was later shown in [99] that this assumption is generally incorrect due to the *particular* structure of $\mathbf{G}_i(z)$ in (4.3). The condition $\mathbf{H}_1 \mathbf{E}_0 = \mathbf{0}$ for elimination of noncausality is actually equivalent to the following [99],

$$\mathbf{E}_u = \bar{\mathbf{U}}_i \mathbf{E}_d \quad (4.43)$$

where $\mathbf{E}_0 = [\mathbf{E}_u^T, \mathbf{E}_d^T]^T$ with two $M/2 \times M$ submatrices $\mathbf{E}_u$ and $\mathbf{E}_d$, and $\bar{\mathbf{U}}_i$ is a nonsingular square matrix with size $M/2$. Equation (4.43) implies that matrices $\mathbf{E}_u$ and $\mathbf{E}_d$ have the *same* row space, which is a stronger condition than $\rho(\mathbf{E}_0) \leq M/2$ [99]. Thus, the previous assumption on the existence of $\mathbf{H}_1$ based on $\rho(\mathbf{E}_0) \leq M/2$ is incorrect and consequently, the statement of completeness of the lattice factorization (4.42) is not convincing. In addition, it can be seen that equation (4.43) implies a *special* rank condition on $\mathbf{E}_0$, i.e., $\rho(\mathbf{E}_u) = \rho(\mathbf{E}_d) = \rho(\mathbf{E}_0)$ since $\mathbf{E}_0 = \text{diag}(\bar{\mathbf{U}}_i, \mathbf{I}_m) [\mathbf{E}_d^T, \mathbf{E}_d^T]^T$. Thus, the elimination of noncausality by the lattice structure (4.42) can be guaranteed *if and only if* $\rho(\mathbf{E}_u) = \rho(\mathbf{E}_d) = \rho(\mathbf{E}_0)$.

4.5.2 Nonexistence of Order-One LPPR Propagating Block with CAFA-CAFI

From the analysis in section 4.5.1, we have seen that the incompleteness of lattice factorization (4.42) comes from the LPPR propagating block $\mathbf{G}_i(z)$ with the form of (4.3). Thus, it was expected that maybe we can smartly devise a novel order-one LPPR propagating block to replace that in (4.3) so that a complete lattice factorization can be achieved. However, before attempting to do so, a fundamental question needs to be answered about the existence of such order-one LPPR propagating block for complete lattice factorization. This problem has been overlooked in the past, partially because there does exist such a block for LPPUFBs. In fact, the LPPR propagating block (4.3) has the same structural form as that for LPPUFBs [42] except the orthogonality constraints on the parameter matrices $\mathbf{U}_i$ and $\mathbf{V}_i$. Thus, it was expected that such structure can also make the lattice factorization (4.42) complete for LPPRFBs. However, in the following, we first explicitly point out the unnecessary constraint of the existing LPPR propagating block $\mathbf{G}(z)$ and then study the existence of order-one LPPR propagating blocks.

4.5.2.1 Limitation of the Existing LPPR Propagating Block

Following the original concept of lattice structure [5, 34, 77], we attempt to devise a general block which can propagate both LP and PR properties and increase filter length as well. Since we study FIR FBs, we can always make the analysis FB causal with appropriate number of

unit delays z^{-1} . Thus, the LPPR propagating block $\mathbf{G}(z)$ for analysis FBs can be made causal without any loss of generality. The necessary and sufficient conditions and the unique structure for such $\mathbf{G}(z)$ has been reported in [34, Theorem I]. It has to satisfy the following two conditions: (i) $\mathbf{G}(z)$ is FIR invertible. (ii) $\mathbf{G}(z) = z^{-1}\mathbf{D}_M\mathbf{G}(z^{-1})\mathbf{D}_M$. Expressing $\mathbf{G}(z)$ in the sum-form, i.e., $\mathbf{G}(z) = \mathbf{G}_0 + z^{-1}\mathbf{G}_1$, the second condition is equivalent to $\mathbf{G}_1 = \mathbf{D}_M\mathbf{G}_0\mathbf{D}_M$. Now, we are only left with the first condition, i.e., FIR invertibility. In fact, this is a very weak condition as extensively investigated in [65]. A complete characterization of such polynomial matrices has been reported in [65, Theorem 7.1] using CAFACAFI and unimodular matrices. However, such characterization has received little attention due to the lack of practical use, since there does not exist a general factorization form for unimodular matrices except the first order unimodular matrices [83]. In addition, note that the above characterization can only ensure the FIR inverse, *but not necessarily the anticausal inverse*.

In [34], it was presumed that the inverse of $\mathbf{G}(z)$ can be *simultaneously* made order-one and anticausal inverse without any loss of generality. It is indeed true for LPPUFBs. *However, this assumption is incorrect for LPPRFBs as seen from the above necessary condition (i) and the complete characterization of causal FIR systems with FIR inverse [65, 100]*. The FIR inverse can be easily obtained, and we can always make the inverse anticausal by an appropriate number of unit advance elements z . However, different from LPPUFBs in which the anticausal inverse of order-one block $\mathbf{G}(z)$ is necessarily order one, the constraint of anticausal inverse does not always lead to the anticausal inverse of $\mathbf{G}(z)$ to be order one (See equation (21) and the note below the Example 5.1 of [65]). In addition, the above necessary condition (ii) cannot generally enforce order-one for the anticausal inverse of $\mathbf{G}(z)$, either. Thus, the assumption made in [34] indeed lost some generality in putting *additional unnecessary* constraint of order one for the anticausal inverse of LPPR block $\mathbf{G}(z)$. Thus, the incompleteness of the lattice factorization (4.42) is expected.

4.5.2.2 Nonexistence of Order-One LPPR Propagating Blocks

In this section, we systematically investigate the existence of *any* order-one LPPR propagating block $\mathbf{G}(z)$ (may not be necessarily in the form of (4.3)) with the order-one anticausal inverse for lattice factorization (4.42) to be complete.

Like before, we restrict $\mathbf{G}(z)$ be order-one in order to cover all filter lengths of $L = KM$. Similarly, we also want the LPPR propagating block $\mathbf{G}^{-1}(z)$ to be order-one for the synthesis bank. Denote $\mathbf{H}(z) = \mathbf{G}^{-1}(z) = \mathbf{H}_0 + z\mathbf{H}_1$ and $\mathbf{G}(z) = \mathbf{G}_0 + z^{-1}\mathbf{G}_1$. From the necessary condition (ii) for LPPR propagating block $\mathbf{G}(z)$ shown in Sec. 4.5.2.1, it can be shown by inverting both sides of $\mathbf{G}(z) = z^{-1}\mathbf{D}_M\mathbf{G}(z^{-1})\mathbf{D}_M$ that $\mathbf{H}(z)$ has to satisfy a similar condition

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

$\mathbf{H}(z) = z\mathbf{D}_M\mathbf{H}(z^{-1})\mathbf{D}_M$, i.e., $\mathbf{H}_1 = \mathbf{D}_M\mathbf{H}_0\mathbf{D}_M$. Now, we are only left with the PR condition $\mathbf{G}(z)\mathbf{H}(z) = \mathbf{I}_M$ which leads to the following equivalent matrix equations.

$$\mathbf{G}_0\mathbf{H}_0 + \mathbf{G}_1\mathbf{H}_1 = \mathbf{I}_M \quad (4.44)$$

$$\mathbf{G}_0\mathbf{H}_1 = \mathbf{0}_M \quad (4.45)$$

Following the method in [99], we can derive a special rank structure for matrices $\mathbf{G}_i$ and $\mathbf{H}_i$ with $i = 0, 1$. Let the matrices $\mathbf{G}_0$ and $\mathbf{H}_0$ be partitioned as follows,

$$\mathbf{G}_0 = \begin{bmatrix} \mathbf{G}_u \\ \mathbf{G}_d \end{bmatrix}, \quad \mathbf{H}_0 = [\mathbf{H}_l, \mathbf{H}_r] \quad (4.46)$$

where $(M/2) \times M$ submatrices $\mathbf{G}_u$ and $\mathbf{G}_d$ are the upper and lower submatrices of $\mathbf{G}_0$, respectively, and $M \times (M/2)$ submatrices $\mathbf{H}_l$ and $\mathbf{H}_r$ are the left and right submatrices of $\mathbf{H}_0$, respectively. From the necessary relations $\mathbf{G}_1 = \mathbf{D}_M\mathbf{G}_0\mathbf{D}_M$ and $\mathbf{H}_1 = \mathbf{D}_M\mathbf{H}_0\mathbf{D}_M$, we can obtain the similar partitioned forms for $\mathbf{G}_1$ and $\mathbf{H}_1$ as the following,

$$\mathbf{G}_1 = \begin{bmatrix} \mathbf{G}_u\mathbf{D}_M \\ -\mathbf{G}_d\mathbf{D}_M \end{bmatrix}, \quad \mathbf{H}_1 = [\mathbf{D}_M\mathbf{H}_l, -\mathbf{D}_M\mathbf{H}_r] \quad (4.47)$$

From the above partitions for matrices $\mathbf{G}_i$ and $\mathbf{H}_i$ and the PR condition (4.44), it can be easily seen that $\mathbf{G}_u\mathbf{H}_l = \mathbf{G}_d\mathbf{H}_r = \frac{1}{2}\mathbf{I}_{M/2}$. According to the rank inequality $\rho(\mathbf{AB}) \leq \min\{\rho(\mathbf{A}), \rho(\mathbf{B})\}$, we can see that $\rho(\mathbf{G}_u) \geq M/2$. However, $\rho(\mathbf{G}_u) \leq \min\{M/2, M\} = M/2$ would lead to $\rho(\mathbf{G}_u) = M/2$. The same argument and conclusion can be applied to the other submatrices $\mathbf{G}_d$, $\mathbf{H}_l$ and $\mathbf{H}_r$. In addition, the other PR condition (4.45) can be simplified to $\mathbf{G}_0\mathbf{H}_1 = \mathbf{G}_0\mathbf{D}_M\mathbf{H}_0\mathbf{D}_M = \mathbf{0}_M$, which would result in the rank condition $\rho(\mathbf{G}_0) + \rho(\mathbf{H}_0) \leq M$ due to the Sylvester's rank inequality. Thus, we can establish the necessary rank condition $\rho(\mathbf{H}_0) = M/2$ and consequently $\rho(\mathbf{H}_0) = \rho(\mathbf{H}_l) = \rho(\mathbf{H}_r) = M/2$. This means the independent column vectors of $\mathbf{H}_l$ and $\mathbf{H}_r$ span the same subspace of $\mathbb{R}^M$ with dimension $M/2$. Thus, there exists a nonsingular square matrix $\mathbf{X}$ with size $\frac{M}{2} \times \frac{M}{2}$ such that $\mathbf{H}_l\mathbf{X} = \mathbf{H}_r$. Then, the condition $\mathbf{H}_1\mathbf{E}_0 = \mathbf{0}_M$ for elimination of noncausality can be simplified as follows.

$$\mathbf{H}_1\mathbf{E}_0 = \mathbf{D}_M[\mathbf{H}_l, -\mathbf{H}_r] \begin{bmatrix} \mathbf{E}_u \\ \mathbf{E}_d \end{bmatrix} = \mathbf{D}_M(\mathbf{H}_l\mathbf{E}_u - \mathbf{H}_l\mathbf{X}\mathbf{E}_d) = \mathbf{0}_M \quad (4.48)$$

Thus, we can obtain $\mathbf{H}_l\mathbf{E}_u = \mathbf{H}_l\mathbf{X}\mathbf{E}_d$. Since $\mathbf{H}_l$ is a matrix with full column rank, it is left invertible which means that there exists an $(M/2) \times M$ matrix $\mathbf{Y}$ such that $\mathbf{Y}\mathbf{H}_l = \mathbf{I}_{M/2}$. Thus, we can establish $\mathbf{E}_u = \mathbf{X}\mathbf{E}_d$, which is in the same form of condition as (4.43). In addition, we can also obtain $\rho(\mathbf{E}_0) = \rho(\mathbf{E}_u) = \rho(\mathbf{E}_d) = M/2$. According to the analysis in Sec. 4.5.1, we can conclude that the above $\mathbf{G}(z)$ and $\mathbf{H}(z)$ cannot make the lattice factorization (4.42) complete. Therefore, we negatively answered the fundamental question on the existence issue, which is formally stated in the following theorem.

Theorem 4.7 *For general even-channel LPPRFBs with length $L = KM$ ($K > 2$) of both analysis and synthesis filters, there does not exist any LPPR building block $\mathbf{G}(z)$ given by the order-one CAFACAFI matrix that can make the lattice factorization (4.42) to be complete.*

This theorem helps us further understand the limitation of the existing lattice factorization, and possibly hint at some other directions for complete characterization of LPPRFBs. However, there is one exceptional case of the above theorem. For two-channel LPPRFBs with given length $L = 6$, i.e., $K = 3$, the lattice factorization (4.42) by order-one building blocks is complete [63, Appendix B].

In addition, by using the complete characterization of biorthogonal lapped transforms (BOLTs) [65], it can be shown that the structure of the LPPR building block $\mathbf{G}(z)$ in (4.3) is the *unique* form for any LPPR propagating block by the order-one CAFACAFI matrix. This can be seen as follows. Based on [65], the complete characterization for $\mathbf{G}(z)$ can be represented by the dyadic form, i.e., $\mathbf{G}(z) = \mathbf{N}(\mathbf{I}_M - \mathbf{U}\mathbf{V}^T + z^{-1}\mathbf{U}\mathbf{V}^T)$, where $\mathbf{U}$ and $\mathbf{V}$ are full rank matrices with size $M \times \rho$, and $\mathbf{N}$ is a square invertible matrix with size M . For the anticausal inverse $\mathbf{G}^{-1}(z)$ to be order-one, we need to have $\mathbf{V}^T\mathbf{U} = \mathbf{I}_\rho$ [65]. Furthermore, it can be shown that $\rho = M/2$ due to $\rho(\mathbf{G}_1) = \rho(\mathbf{G}_u) = \rho(\mathbf{G}_d) = M/2$. The nonsingular matrix $\mathbf{N}$ can be simplified to $\mathbf{I}_M$ by moving it through the lattice structure (4.42) in a similar way as in [99]. Then, such $\mathbf{G}(z)$ actually degenerates into the existing LPPR propagating block in standard dyadic form [72], which in fact is equivalent to that shown in (4.3). Thus, the statement of completeness of lattice factorization in [72] is also incorrect.

Remark 4.4 *The problem of nonexistence of LPPR propagating block $\mathbf{G}(z)$ with order-one CAFACAFI mainly comes from the fact that both orders of $\mathbf{G}(z)$ and $\mathbf{G}^{-1}(z)$ are determined a priori. Since the FBs are critically sampled FBs, i.e., square polynomial matrices, we know that the inverse $\mathbf{G}^{-1}(z)$ is uniquely determined by $\mathbf{G}(z)$, including the order of $\mathbf{G}^{-1}(z)$. Thus, we cannot freely choose $\mathbf{G}^{-1}(z)$ to be order one without any loss of generality. This was also illustrated in the Example 5.1 of [65]. Thus, we can conclude that the problem of incompleteness of lattice factorization (4.42) comes from the unnecessary constraint of order-one $\mathbf{G}^{-1}(z)$.*

Similarly, given analysis polyphase matrix $\mathbf{E}(z)$ of an LPPRFB with order $K-1$, i.e., length $L = KM$, we cannot generally determine the order of the PR synthesis matrix $\mathbf{R}(z)$ a priori. Thus, we cannot enforce $\mathbf{R}(z)$ to be order- $(K-1)$ without any loss of generality. This partially explains the nonexistence of complete lattice factorization for LPPRFBs with length $L = KM$ for both analysis and synthesis banks.

4.5.3 Revisiting the GenLOT and LBT

For the GenLOTs [42], i.e., LPPUFBs with length $L = KM$, the polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ can be expressed in the sum-form by $\mathbf{E}(z) = \sum_{i=0}^{K-1} \mathbf{E}_i z^{-i}$ and $\mathbf{R}(z) = \tilde{\mathbf{E}}(z) = \sum_{i=0}^{K-1} \mathbf{E}_i^T z^i$. The LP condition (4.1) yields $\mathbf{E}_i = \mathbf{D}_M \mathbf{E}_{K-1-i} \mathbf{J}_M$. From the PU condition, i.e., $\tilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I}_M$, we can obtain $\mathbf{E}_{K-1}^T \mathbf{E}_0 = \mathbf{0}_M$ due to the inter-orthogonality. By virtue of the LP condition, we can transform the above inter-orthogonality to $\mathbf{E}_0^T \mathbf{D}_M \mathbf{E}_0 = \mathbf{0}_M$. Using the partitioning $\mathbf{E}_0 = [\mathbf{E}_u^T, \mathbf{E}_d^T]^T$, we can obtain the following relationship between $\mathbf{E}_u$ and $\mathbf{E}_d$.

$$\mathbf{E}_0^T \mathbf{D}_M \mathbf{E}_0 = [\mathbf{E}_u^T \ \mathbf{E}_d^T] \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{M/2} \end{bmatrix} \begin{bmatrix} \mathbf{E}_u \\ \mathbf{E}_d \end{bmatrix} = \mathbf{E}_u^T \mathbf{E}_u - \mathbf{E}_d^T \mathbf{E}_d = \mathbf{0}_M \quad (4.49)$$

Thus, we know $\mathbf{E}_u^T \mathbf{E}_u = \mathbf{E}_d^T \mathbf{E}_d$, which leads to the rank relationship $\rho(\mathbf{E}_u) = \rho(\mathbf{E}_d)$. Moreover, we can obtain $\mathbf{E}_0^T \mathbf{E}_0 = \mathbf{E}_u^T \mathbf{E}_u + \mathbf{E}_d^T \mathbf{E}_d = 2\mathbf{E}_u^T \mathbf{E}_u$. Therefore, we can establish the rank condition on $\mathbf{E}_0$, i.e., $\rho(\mathbf{E}_0) = \rho(\mathbf{E}_u) = \rho(\mathbf{E}_d)$. As we have seen in section 4.5.1, such rank condition on $\mathbf{E}_0$ could ensure the elimination of noncausality required for order reduction. Thus, the lattice factorization (4.42) can be complete for LPPUFBs, and it has been shown in [42, 70, 77, 100].

For the LBTs, i.e., LPPRFBs with length $L = 2M$, it has been shown in [99] that the rank condition $\rho(\mathbf{E}_0) = \rho(\mathbf{E}_u) = \rho(\mathbf{E}_d)$ also holds and thus the lattice factorization (4.42) can be complete for LPPRFBs with $K = 2$. In addition, it has been shown in this chapter that the arbitrary equal-length LPPRFBs of lengths $L = KM + \beta$ ($0 \leq \beta < M$) with $L \leq 2M$ can be completely characterized by our derived lattice factorizations (4.20) and (4.38) which can account for those extra β coefficients. However, for $K > 2$, the above rank condition on $\mathbf{E}_0$ is not always valid as seen from the counter examples in [63] and [99].

4.6 On the Closeness of the Space Spanned by the Lattice Structures

The incompleteness of the existing lattice structures has been well known for M -channel FIR LPPRFBs with filter length $L > 2M$ in the literature [63, 99], and even the nonexistence of complete order-one lattice has been established in Sec. 4.5. Thus, a question arises naturally as to how large the space $\mathbb{L}_\beta$ spanned by the existing lattice structure is and how large the gap between the space $\mathbb{L}_\beta$ and the whole space of LPPRFBs is. In addition, it seems that a permutation matrix can solve incompleteness of the existing lattice structure for the counter examples reported in [99]. Then, *a question arises whether we can use some transformations over the existing lattice structures to expand the space $\mathbb{L}_\beta$* . Thus, it is desirable to investigate what property the space $\mathbb{L}_\beta$ has, especially about the closeness of such space under the polynomial transforms $\mathbf{P}(z)$ (matrix polynomials) since all the basic building blocks used in various

factorizations of LPPRFBs are some kinds of polynomial transforms. *The study for such issue can reveal the sense of optimality for the lattice structure based design method for LPPRFBs since the optimality is limited by the achievable design space $\mathbb{L}_\beta$.*

Inspired by the above perspective, this section studies closeness of the space $\mathbb{L}_\beta$ under some transformations. We firstly study the closeness of $\mathbb{L}_\beta$ under the polynomial transformations for LPPRFBs with arbitrary equal lengths. We have shown that the space $\mathbb{L}_\beta$ is closed under the popular polynomial transforms widely used in FB designs, which establishes the suboptimality of the lattice based design methods for LPPRFBs. Furthermore, the explicit relationship between the lattice parameters before and after transformations have been shown for describing the closeness of $\mathbb{L}_\beta$. The outline of this section is as follows. We first present the concept of closeness of the space $\mathbb{L}_\beta$ in Sec. 4.6.1. Next, the closeness of $\mathbb{L}_\beta$ is to be studied for general and regular LPPRFBs in Sec. 4.6.2 and Sec. 4.6.3, respectively. In Sec. 4.6.4, we discuss the relationship of our work with some previous works. For simplicity, we only consider even-channel critically sampled uniform LPPRFBs with arbitrary filter lengths $L = KM + \beta$ ($0 \leq \beta < M$). However, similar work can be extended to odd-channel critically sampled systems and oversampled systems as well. In addition, we only show the results for the forward lattice factorizations, while similar results have also been obtained for backward factorizations.

4.6.1 The Space $\mathbb{L}_\beta$ Spanned by the Lattice Structure

Although lots of works have been done on analysis and design of M -channel LPPRFBs based on lattice factorizations [25, 27, 34, 42–45, 72, 77, 86, 88, 90–93, 99], a complete lattice structure for such FBs is still unknown except when filter length $L \leq 2M$ as shown in Theorem 4.5. For two-channel LPPRFBs with equal (even) lengths, i.e., type-A systems, a counter example with $L = 8$ has been shown in [63] (cf. [63, Appendix B]). Later in [99], several counter examples have been reported for general M -channel ($M > 2$) systems of filter lengths $L = KM$ with $K \geq 3$ (cf. [99, Sec. III-B]). Furthermore, the *nonexistence* of complete lattice structure composed by order-one building blocks is shown in Sec. 4.5. Thus, a question arises naturally that how large the space $\mathbb{L}_\beta$ spanned by the existing lattice structure is, since we have already known such lattice structure cannot be complete. Instead of the completeness, we study a weaker concept, named as the closeness of the space $\mathbb{L}_\beta$ spanned by the lattice structure (4.20), in this section. This work can help us to understand the gap between $\mathbb{L}_\beta$ and the whole space of LPPRFBs and may indicate some hints on possible complete characterization of LPPRFBs. A block diagram for the inter-relationship of the above spaces is shown in Fig. 4.3 according to the past works. The space $\mathbb{L}_0$ is the space spanned by the lattice structure (4.20) with $\beta = 0$, i.e., filter length $L = KM$. The symbol $\mathbb{S}$ in Fig. 4.3 denotes those counter examples reported before which can

be thought as “singular” cases in the whole space of LPPRFBs, which cannot be factorized by the lattice (4.20).

Figure 4.3: Interconnections of lattice structures of FIR LPPRFBs with arbitrary equal lengths and the whole space.

In this section, we attempt to *characterize the above space $\mathbb{L}_\beta$, especially about its closeness under some transformations*. Inspired by the existing factorizations, we consider two classes of transforms. The first is polynomial transforms $\mathbf{P}(z)$ *without* order constraint (maybe order of infinity), i.e., matrix polynomials without order constraint (causal polynomials for analysis banks and anticausal polynomials for synthesis banks due to the class of FBs with CAFACAFI). The second is finite linear transforms, i.e., constant matrices, which actually is a subclass of polynomial transforms. These two classes of matrices have been widely employed in the existing lattice factorizations of FBs as seen from the lattice (4.20). The study for such closeness can reveal the sense of optimality for the lattice structure based design method, since the optimality of the designed FB is limited by the design space $\mathbb{L}_\beta$. To study the closeness of the space $\mathbb{L}_\beta$ over the polynomial transformations, we have to firstly define the so called permissible transform through which the space $\mathbb{L}_\beta$ may get expanded by transforming some $\mathbf{E}(z)$ in the space $\mathbb{L}_\beta$ to some LPPRFB $\mathbf{F}(z)$ with length $L = KM + \beta$ which can *not* be factorized in (4.20), i.e., $\mathbf{F}(z) \notin \mathbb{L}_\beta$. A polynomial transform $\mathbf{P}(z)$ is called *permissible* from the left (or from the right) if *all* the desired properties of $\mathbf{E}(z)$, such as the *given filter length L , PR, LP and/or regularity* have been retained on transformed polyphase matrix $\mathbf{F}(z) = \mathbf{P}(z)\mathbf{E}(z)$ (or $\mathbf{F}(z) = \mathbf{E}(z)\mathbf{P}(z)$). However, note that the permissible $\mathbf{P}(z)$ is usually *not* the same form as the basic building block $\mathbf{G}(z)$ in the existing lattice factorization (4.20) since the order of the transformed polyphase matrix $\mathbf{F}(z)$ is *same* as that of $\mathbf{E}(z)$. Thus, the space $\mathbb{L}_\beta$ may be enlarged by such polynomial transformations. The closeness of the space spanned by the existing lattice structure is defined as follows.

Definition 4.3 *The space $\mathbb{L}_\beta$ is said to be closed under polynomial transforms from the left (or from the right) if the same lattice structure can be applied to the new polyphase matrix $\mathbf{F}(z)$ transformed by any permissible transforms.*

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

In the following, we study the closeness of the space $\mathbb{L}_\beta$ over polynomial transforms from the left and right, respectively. Specifically, given any analysis polyphase matrix $\mathbf{E}(z)$ with LP, PR and length $L = KM + \beta$ factorable by (4.20), we study the closeness of the space $\mathbb{L}_\beta$ under polynomial transformations such as $\mathbf{F}(z) = \mathbf{L}(z)\mathbf{E}(z)$ or $\mathbf{F}(z) = \mathbf{E}(z)\mathbf{R}(z)$. Note that this is different from the lattice factorizations of FBs since the polynomial transform $\mathbf{L}(z)$ or $\mathbf{R}(z)$ does *not* intend to increase the order of the polyphase matrix. Instead of increasing order, we need to derive the necessary conditions on $\mathbf{L}(z)$ or $\mathbf{R}(z)$ (different from synthesis polyphase matrix $\mathbf{R}(z)$) to make them permissible, which can ensure $\mathbf{F}(z)$ still belonging to the same class of FBs as $\mathbf{E}(z)$, i.e., LPPRFBs with *same* length $L = KM + \beta$. Then, we have to check if the transformed polyphase matrix $\mathbf{F}(z)$ can still be factorized in the existing lattice structures, i.e., $\mathbf{F}(z) \in \mathbb{L}_\beta$ or not. In the case of $\mathbf{F}(z) \in \mathbb{L}_\beta$, we further establish the *explicit* relationship between the lattice parameters of $\mathbf{F}(z)$ and $\mathbf{E}(z)$.

4.6.2 Closeness of the Space $\mathbb{L}_\beta$ for General LPPRFBs

In this section, we investigate the closeness of the space $\mathbb{L}_\beta$ from the left and right in Sec. 4.6.2.1 and Sec. 4.6.2.2, respectively, for the general lattice structure (4.20).

4.6.2.1 Closeness of $\mathbb{L}_\beta$ from the left

First, we study the *necessary* condition for the polynomial transform $\mathbf{L}(z)$ to be permissible. Given any analysis polyphase matrix $\mathbf{E}(z)$ with LP, PR and filter length $L = KM + \beta$, a permissible $\mathbf{L}(z)$ means that $\mathbf{F}(z) = \mathbf{L}(z)\mathbf{E}(z)$ also holds LP, PR and same length L . Thus, it should also satisfy LP condition (4.1).

$$\mathbf{F}(z) = z^{-(K-1)}\mathbf{D}\mathbf{F}(z^{-1})\mathbf{J}_M(z) = z^{-(K-1)}\mathbf{D}\mathbf{L}(z^{-1})\mathbf{E}(z^{-1})\mathbf{J}_M(z) = \mathbf{D}\mathbf{L}(z^{-1})\mathbf{D}\mathbf{E}(z) \quad (4.50)$$

Since $\mathbf{E}(z)$ is invertible, we can simplify (4.50) to $\mathbf{L}(z)\mathbf{D} = \mathbf{D}\mathbf{L}(z^{-1})$. Because $\mathbf{L}(z)$ is a causal matrix, $\mathbf{L}(z^{-1})$ must be an anticausal matrix. Thus, we know that $\mathbf{L}(z)$ must be a constant matrix, i.e., zero-order causal matrix $\mathbf{L}$. Then, from the condition $\mathbf{L}\mathbf{D} = \mathbf{D}\mathbf{L}$, it can be easily shown that $\mathbf{L}$ should be a block diagonal matrix $\text{diag}(\mathbf{L}_0, \mathbf{L}_1)$ due to the signature matrix $\mathbf{D}$. In addition, for $\mathbf{F}(z)$ to have the PR property, $\mathbf{L}$ must be invertible, which implies that the $m \times m$ matrices $\mathbf{L}_0$ and $\mathbf{L}_1$ are invertible.

Remark 4.5 *Similar idea has been employed for deriving the special orthogonal matrix in the basic building block which can propagate the pairwise time-reversed property for the lattice factorization of LPPUFBs [77]. However, as noted before, our work is different from [77] since the filter lengths of the LPPRFBs $\mathbf{E}(z)$ and $\mathbf{F}(z)$ are always maintained to be $L = KM + \beta$ before*

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

and after polynomial transformations. Furthermore, we do not impose any special structure or constraint on the polynomial transforms a priori, in contrast to the special order-one structure $\mathfrak{L}(z) = \mathbf{K}\mathbf{\Lambda}_i(z)$ in [77] where $\mathbf{K}$ is an orthogonal matrix and $\mathbf{\Lambda}_i(z)$ has the same form as that presented in Sec. 4.3.1. Note that the special structure on the orthogonal matrix $\mathbf{K}$ derived in [77, Equation 3.6] is valid only at the assumption of the above special order-one structure of $\mathfrak{L}(z)$. If no such strong assumption is available, their result cannot be obtained even by the weaker assumption of just order-one $\mathfrak{L}(z)$. On the contrary, we did not assume any constraint on the structure or the order of $\mathfrak{L}(z)$. Instead of assumptions, we have shown that the permissible left transform $\mathfrak{L}(z)$ must be in the constant block diagonal form. The other polynomial transforms would lead to $\mathbf{F}(z) \notin \mathbb{L}_\beta$, which is invalid for our study of the closeness of $\mathbb{L}_\beta$.

Then, we can show that such a diagonal matrix $\mathfrak{L}$ can be absorbed into the lattice structure (4.20) by using the technique presented in [93], and thus show the closeness of the space $\mathbb{L}_\beta$ from the left.

$$\begin{aligned} \mathfrak{L}\mathbf{G}_i(z) &= \frac{1}{2} \begin{bmatrix} \mathfrak{L}_0 & \mathbf{0} \\ \mathbf{0} & \mathfrak{L}_1\mathbf{V}_i \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{I}_m \\ \mathbf{I}_m & -\mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{I}_m \\ \mathbf{I}_m & -\mathbf{I}_m \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_m & \mathbf{I}_m \\ \mathbf{I}_m & -\mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I}_m \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_m & \mathbf{I}_m \\ \mathbf{I}_m & -\mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathfrak{L}_0 & \mathbf{0} \\ \mathbf{0} & \mathfrak{L}_0 \end{bmatrix} \\ &= \bar{\Phi}_i \mathbf{W}_{2m} \mathbf{\Lambda}_i(z) \mathbf{W}_{2m} \text{diag}(\mathfrak{L}_0, \mathfrak{L}_0) = \bar{\mathbf{G}}_i(z) \text{diag}(\mathfrak{L}_0, \mathfrak{L}_0) \end{aligned} \quad (4.51)$$

where $\bar{\mathbf{V}}_i = \mathfrak{L}_1\mathbf{V}_i\mathfrak{L}_0^{-1}$. By the same way, we can move the matrix $\text{diag}(\mathfrak{L}_0, \mathfrak{L}_0)$ across the delay chain matrix $\mathbf{\Lambda}_i(z)$ and butterfly matrix $\mathbf{W}_{2m}$ until the initial block $\mathbf{E}_0$, which results in the same lattice structure as (4.20) but with different lattice parameters $\bar{\mathbf{V}}_i$ and $\bar{\mathbf{U}}_0$ as follows.

$$\bar{\mathbf{V}}_i = \begin{cases} \mathfrak{L}_1\mathbf{V}_i\mathfrak{L}_0^{-1}, & i = K-1 \\ \mathfrak{L}_0\mathbf{V}_i\mathfrak{L}_0^{-1}, & i = K-2, \dots, 1 \end{cases} \quad \begin{cases} \bar{\mathbf{U}}_0 = \mathfrak{L}_0\mathbf{U}_0 \\ \bar{\mathbf{V}}_0 = \mathfrak{L}_0\mathbf{V}_0 \end{cases} \quad (4.52)$$

4.6.2.2 Closeness of $\mathbb{L}_\beta$ from the right

Here, we study the closeness of $\mathbb{L}_\beta$ over polynomial transforms $\mathfrak{R}(z)$ from the right. Similar to the above case, a permissible $\mathfrak{R}(z)$ should also lead to $\mathbf{F}(z) = \mathbf{E}(z)\mathfrak{R}(z)$ satisfying the LP condition (4.1),

$$\begin{aligned} \mathbf{E}(z)\mathfrak{R}(z) &= \mathbf{F}(z) = z^{-(K-1)}\mathbf{D}\mathbf{F}(z^{-1})\mathbf{J}_M(z) \\ &= z^{-(K-1)}\mathbf{D}\mathbf{E}(z^{-1})\mathfrak{R}(z^{-1})\mathbf{J}_M(z) = \mathbf{E}(z)\mathbf{J}_M^{-1}(z)\mathfrak{R}(z^{-1})\mathbf{J}_M(z) \end{aligned} \quad (4.53)$$

Since $\mathbf{E}(z)$ is invertible, we can also simplify the above equation to $\mathbf{J}_M(z)\mathfrak{R}(z) = \mathfrak{R}(z^{-1})\mathbf{J}_M(z)$. However, different from before, we *cannot* draw a conclusion that $\mathfrak{R}(z)$ must be a constant matrix since $\mathbf{J}_M(z)$ is an order-one causal matrix.

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

In fact, it can be shown that $\mathfrak{R}(z)$ must be a lower triangular unimodular matrix. This can be derived as follows. First, we partition the causal matrix $\mathfrak{R}(z)$ into a block form as,

$$\mathfrak{R}(z) = \begin{bmatrix} \mathfrak{R}_{00}(z) & \mathfrak{R}_{01}(z) \\ \mathfrak{R}_{10}(z) & \mathfrak{R}_{11}(z) \end{bmatrix} \quad (4.54)$$

where the causal submatrices $\mathfrak{R}_{00}(z)$, $\mathfrak{R}_{01}(z)$, $\mathfrak{R}_{10}(z)$ and $\mathfrak{R}_{11}(z)$ have the sizes of $\beta \times \beta$, $\beta \times (M - \beta)$, $(M - \beta) \times \beta$ and $(M - \beta) \times (M - \beta)$, respectively. Then the condition of $\mathbf{J}_M(z)\mathfrak{R}(z) = \mathfrak{R}(z^{-1})\mathbf{J}_M(z)$ is equivalent to the following,

$$\mathbf{J}_{M-\beta}\mathfrak{R}_{11}(z) = \mathfrak{R}_{11}(z^{-1})\mathbf{J}_{M-\beta} \quad (4.55)$$

$$z^{-1}\mathbf{J}_\beta\mathfrak{R}_{00}(z) = z^{-1}\mathfrak{R}_{00}(z^{-1})\mathbf{J}_\beta \quad (4.56)$$

$$z^{-1}\mathbf{J}_\beta\mathfrak{R}_{01}(z) = \mathfrak{R}_{01}(z^{-1})\mathbf{J}_{M-\beta} \quad (4.57)$$

$$\mathbf{J}_{M-\beta}\mathfrak{R}_{10}(z) = z^{-1}\mathfrak{R}_{10}(z^{-1})\mathbf{J}_\beta \quad (4.58)$$

Note that each of $\mathfrak{R}_{ij}(z)$ with $i, j = 0, 1$ is also a causal matrix with same order N of causal matrix $\mathfrak{R}(z)$, i.e., $\mathfrak{R}_{ij}(z) = \sum_{k=0}^N \mathfrak{R}_{ij}[k]z^{-k}$. Then, it can be easily seen that $\mathfrak{R}_{11}(z)$ should be a constant matrix $\mathfrak{R}_{11}$ since the left hand side of (4.55) is causal while the right hand side of (4.55) is anticausal. Similarly, the matrix $\mathfrak{R}_{00}(z)$ is also a constant matrix $\mathfrak{R}_{00}$. By a similar argument, we can see that the matrix $\mathfrak{R}_{01}(z)$ should be a null matrix, i.e., $\mathfrak{R}_{01}(z) = \mathbf{0}$, since the left hand side of (4.57) is a shifted causal matrix while the right hand side of (4.57) is still anticausal. However, we can see that $\mathfrak{R}_{10}(z)$ should be a first-order causal matrix with additional constraint $\mathfrak{R}_{10}[1] = \mathbf{J}_{M-\beta}\mathfrak{R}_{10}[0]\mathbf{J}_\beta$ due to (4.58). Thus, the matrix $\mathfrak{R}(z)$ becomes $\begin{bmatrix} \mathfrak{R}_{00} & \mathbf{0} \\ \mathfrak{R}_{10}(z) & \mathfrak{R}_{11} \end{bmatrix}$ whose determinant $\det(\mathfrak{R}(z)) = \det(\mathfrak{R}_{00})\det(\mathfrak{R}_{11})$ is a constant, which implies $\mathfrak{R}(z)$ to be a unimodular matrix.

For the purpose of simplicity, we only study a subclass of polynomial transforms by further constraining $\mathfrak{R}(z)$ to be a constant matrix $\mathfrak{R}$. (However, note that if the length is constrained to be $L = KM$, i.e., $\beta = 0$, we can conclude that $\mathfrak{R}(z)$ must be a constant matrix since $\mathbf{J}_M(z)$ degenerates into the constant reversal matrix $\mathbf{J}_M$ and consequently the constraint for $\mathfrak{R}(z)$ becomes $\mathbf{J}_M\mathfrak{R}(z) = \mathfrak{R}(z^{-1})\mathbf{J}_M$. Thus, the above constraint is appropriate.) Then, for holding the LP property, we have $\mathbf{J}_M(z)\mathfrak{R} = \mathfrak{R}\mathbf{J}_M(z)$. It can be easily shown that the constant matrix $\mathfrak{R}$ should be a diagonal matrix $\text{diag}(\mathfrak{R}_0, \mathfrak{R}_1)$ with additional constraints $\mathfrak{R}_0\mathbf{J}_\beta = \mathbf{J}_\beta\mathfrak{R}_0$ and $\mathfrak{R}_1\mathbf{J}_{M-\beta} = \mathbf{J}_{M-\beta}\mathfrak{R}_1$. These constraints enforce the following structures on matrices $\mathfrak{R}_0$ and $\mathfrak{R}_1$,

$$\mathfrak{R}_0 = \begin{bmatrix} \mathfrak{R}_{0,u} & \mathbf{J}\mathfrak{R}_{0,l}\mathbf{J}_l \\ \mathfrak{R}_{0,l} & \mathbf{J}\mathfrak{R}_{0,u}\mathbf{J}_l \end{bmatrix}, \quad \mathfrak{R}_1 = \begin{bmatrix} \mathfrak{R}_{1,u} & \mathbf{J}\mathfrak{R}_{1,l}\mathbf{J}_{m-l} \\ \mathfrak{R}_{1,l} & \mathbf{J}\mathfrak{R}_{1,u}\mathbf{J}_{m-l} \end{bmatrix} \quad (4.59)$$

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

where matrices $\mathfrak{R}_{0,u}$ and $\mathfrak{R}_{0,l}$ have sizes of $l \times l$, and matrices $\mathfrak{R}_{1,u}$ and $\mathfrak{R}_{1,l}$ have sizes of $(m-l) \times (m-l)$.

Next, we will show that $\mathbf{E}_0 \Lambda_0(z) \Gamma \mathfrak{R}$ can still be factorized as (4.20), i.e., in the form of $\bar{\mathbf{E}}_0(z) = \bar{\mathbf{E}}_0 \Lambda_0(z) \bar{\Gamma}$. First, by defining $\mathbf{Q}_1 \triangleq \mathfrak{R}_{0,u} + \mathbf{J} \mathfrak{R}_{0,l}$ and $\mathbf{Q}_2 \triangleq \mathfrak{R}_{0,u} - \mathbf{J} \mathfrak{R}_{0,l}$, we can see that the matrix $\mathfrak{R}_0$ can be absorbed into the matrix Γ

$$\begin{aligned} \begin{bmatrix} \Sigma & \Delta \\ \mathbf{J} \Delta \mathbf{J} & \mathbf{J} \Sigma \mathbf{J} \end{bmatrix} \mathfrak{R}_0 &= \frac{1}{2} \begin{bmatrix} \mathbf{I} + \mathbf{N} & (\mathbf{I} - \mathbf{N}) \mathbf{J} \\ \mathbf{J}(\mathbf{I} - \mathbf{N}) & \mathbf{J}(\mathbf{I} + \mathbf{N}) \mathbf{J} \end{bmatrix} \begin{bmatrix} \mathfrak{R}_{0,u} & \mathbf{J} \mathfrak{R}_{0,l} \mathbf{J}_l \\ \mathfrak{R}_{0,l} & \mathbf{J} \mathfrak{R}_{0,u} \mathbf{J}_l \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \mathbf{Q}_1 + \mathbf{N} \mathbf{Q}_2 & (\mathbf{Q}_1 - \mathbf{N} \mathbf{Q}_2) \mathbf{J} \\ \mathbf{J}(\mathbf{Q}_1 - \mathbf{N} \mathbf{Q}_2) & \mathbf{J}(\mathbf{Q}_1 + \mathbf{N} \mathbf{Q}_2) \mathbf{J} \end{bmatrix} = \begin{bmatrix} \bar{\Sigma} & \bar{\Delta} \\ \mathbf{J} \bar{\Delta} \mathbf{J} & \mathbf{J} \bar{\Sigma} \mathbf{J} \end{bmatrix} \end{aligned}$$

through the following transforms on lattice parameters

$$\bar{\Sigma} = (\mathbf{Q}_1 + \mathbf{T})/2 \text{ and } \bar{\Delta} = (\mathbf{Q}_1 - \mathbf{T})\mathbf{J}_l/2, \text{ where } \mathbf{T} = \mathbf{N} \mathbf{Q}_2 \quad (4.60)$$

Then, it can be easily seen,

$$\begin{aligned} \Lambda_0(z) \Gamma \mathfrak{R} &= \Lambda_0(z) \begin{bmatrix} \bar{\Sigma} & \bar{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathfrak{R}_1 \\ \mathbf{J} \bar{\Delta} \mathbf{J} & \mathbf{J} \bar{\Sigma} \mathbf{J} & \mathbf{0} \end{bmatrix} = \Lambda_0(z) \text{diag}(\mathbf{Q}_1, \mathfrak{R}_1, \mathbf{J}_l \mathbf{Q}_1 \mathbf{J}_l) \bar{\Gamma} \\ &= \text{diag}(\mathbf{Q}_1, \mathfrak{R}_1, \mathbf{J}_l \mathbf{Q}_1 \mathbf{J}_l) \Lambda_0(z) \bar{\Gamma} \end{aligned} \quad (4.61)$$

where $\bar{\Gamma}$ has the same structural form as that of Γ but with the parameter $\mathbf{N}$ replaced by $\bar{\mathbf{N}} = \mathbf{Q}_1^{-1} \mathbf{N} \mathbf{Q}_2$. Next, we can further move the matrix $\mathfrak{R}_1$ into $\mathbf{E}_0 = \text{diag}(\mathbf{U}_0, \mathbf{V}_0) \mathbf{W}_{2m} \mathbf{I}_{2m}^J$ as follows,

$$\begin{aligned} \mathbf{W}_{2m} \mathbf{I}_{2m}^J \text{diag}(\mathbf{Q}_1, \mathfrak{R}_1, \mathbf{J}_l \mathbf{Q}_1 \mathbf{J}_l) &= \mathbf{W}_{2m} \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathfrak{R}_{1,u} & \mathbf{J} \mathfrak{R}_{1,l} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_1 \mathbf{J}_l \\ \mathbf{0} & \mathbf{J} \mathfrak{R}_{1,l} & \mathfrak{R}_{1,u} \mathbf{J} & \mathbf{0} \end{bmatrix} \\ &= \mathbf{W}_{2m} \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathfrak{R}_{1,u} & \mathbf{0} & \mathbf{J} \mathfrak{R}_{1,l} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \mathfrak{R}_{1,l} & \mathbf{0} & \mathfrak{R}_{1,u} \end{bmatrix} \mathbf{I}_{2m}^J = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 & \mathbf{0} & \mathbf{S}_1 \\ \mathbf{Q}_1 & \mathbf{0} & -\mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 & \mathbf{0} & -\mathbf{S}_2 \end{bmatrix} \mathbf{I}_{2m}^J \\ &= \text{diag}(\mathbf{Q}_1, \mathbf{S}_1, \mathbf{Q}_1, \mathbf{S}_2) \mathbf{W}_{2m} \mathbf{I}_{2m}^J \end{aligned} \quad (4.62)$$

where $\mathbf{S}_1 = \mathfrak{R}_{1,u} + \mathbf{J}_{m-l} \mathfrak{R}_{1,l}$ and $\mathbf{S}_2 = \mathfrak{R}_{1,u} - \mathbf{J}_{m-l} \mathfrak{R}_{1,l}$. Finally, we can see that the whole matrix $\mathfrak{R}$ is absorbed into $\mathbf{E}_0(z)$ of the lattice structure (4.20) by combining (4.60), (4.61) and (4.62).

$$\begin{aligned} \mathbf{E}_0 \Lambda_0(z) \Gamma \mathfrak{R} &= \text{diag}(\mathbf{U}_0, \mathbf{V}_0) \mathbf{W}_{2m} \mathbf{I}_{2m}^J \text{diag}(\mathbf{Q}_1, \mathfrak{R}_1, \mathbf{J}_l \mathbf{Q}_1 \mathbf{J}_l) \Lambda_0(z) \bar{\Gamma} \\ &= \text{diag}(\mathbf{U}_0, \mathbf{V}_0) \text{diag}(\mathbf{Q}_1, \mathbf{S}_1, \mathbf{Q}_1, \mathbf{S}_2) \mathbf{W}_{2m} \mathbf{I}_{2m}^J \Lambda_0(z) \bar{\Gamma} \\ &= \text{diag}(\bar{\mathbf{U}}_0, \bar{\mathbf{V}}_0) \mathbf{W}_{2m} \mathbf{I}_{2m}^J \Lambda_0(z) \bar{\Gamma} = \bar{\mathbf{E}}_0 \Lambda_0(z) \bar{\Gamma} \end{aligned} \quad (4.63)$$

where the new lattice parameters are,

$$\bar{\mathbf{U}}_0 = \mathbf{U}_0 \text{diag}(\mathbf{Q}_1, \mathbf{S}_1), \quad \bar{\mathbf{V}}_0 = \mathbf{V}_0 \text{diag}(\mathbf{Q}_1, \mathbf{S}_2) \quad (4.64)$$

For the PR property, we need $\mathbf{Q}_1$, $\mathbf{T}$ and $\bar{\mathbf{U}}_0$, $\bar{\mathbf{V}}_0$ to be nonsingular, which is equivalent to invertibility of $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{S}_1$, $\mathbf{S}_2$. This can be seen as follows,

$$\begin{aligned} \begin{bmatrix} \mathbf{I}_m & \mathbf{J}_m \\ \mathbf{I}_m & -\mathbf{J}_m \end{bmatrix} \mathfrak{R}_0 &= \begin{bmatrix} \mathfrak{R}_{0,u} + \mathbf{J}\mathfrak{R}_{0,l} & (\mathbf{J}\mathfrak{R}_{0,l} + \mathfrak{R}_{0,u})\mathbf{J} \\ \mathfrak{R}_{0,u} - \mathbf{J}\mathfrak{R}_{0,l} & (-\mathfrak{R}_{0,u} + \mathbf{J}\mathfrak{R}_{0,l})\mathbf{J} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_1 \\ \mathbf{Q}_2 & -\mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_m \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{I}_m \\ \mathbf{I}_m & -\mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_m \end{bmatrix} \end{aligned}$$

Since all matrices except $\text{diag}(\mathbf{Q}_1, \mathbf{Q}_2)$ are invertible, we know that $\text{diag}(\mathbf{Q}_1, \mathbf{Q}_2)$ must be invertible, which leads to nonsingularity of both $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Similar argument can be applied to $\mathbf{S}_1$ and $\mathbf{S}_2$ just by replacing $\mathfrak{R}_0$ by $\mathfrak{R}_1$.

Therefore, we have shown the closeness of the space $\mathbb{L}_\beta$ over the constant permissible transforms $\mathfrak{R}$ from the right, and further established the relationship between the lattice parameters through (4.60) and (4.64).

4.6.3 Closeness of the Space $\mathbb{L}_\beta$ for Regular LPPRFBs

In this section, we study the closeness of the space $\mathbb{L}_\beta$ from the left and right in Sec. 4.6.3.1 and Sec. 4.6.3.2, respectively, for the regular lattice structure presented in Sec. 4.3.1.2.

4.6.3.1 Closeness of $\mathbb{L}_\beta$ from the left

In Sec. 4.6.2.1, we have shown that the permissible left polynomial transforms $\mathfrak{L}(z)$ for the lattice structure (4.20) must be a constant and block diagonal matrix $\mathfrak{L} = \text{diag}(\mathfrak{L}_0, \mathfrak{L}_1)$. Furthermore, the lattice structure (4.20) is maintained just by transforming lattice parameters through (4.52). From Sec. 4.3.1.2, we know that the polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ of regular LPPRFBs have to satisfy (4.22) with nonzero constants c and d . For further preserving the regularity of LPPRFBs with analysis polyphase matrix $\mathbf{F}(z) = \mathfrak{L}\mathbf{E}(z)$, it is necessary to have,

$$\mathbf{F}(1)\mathbf{1}_M = c'\mathbf{e}_0 \text{ and } \mathbf{F}^{-T}(1)\mathbf{1}_M = d'\mathbf{e}_0 \quad (4.65)$$

where c' and d' are a pair of nonzero scalars with products $c'd' = M$. According to (4.22), the leftside of (4.65) can be simplified to $\mathfrak{L}\mathbf{E}(1)\mathbf{1}_M = c\mathfrak{L}\mathbf{e}_0 = c'\mathbf{e}_0$. By the diagonal structure of matrix $\mathfrak{L}$, it can be further simplified to $c\mathfrak{L}_0\mathbf{e}_0 = c'\mathbf{e}_0$, which leads to $\mathbf{L}_0$ to be a block upper-triangular matrix in the form $\begin{bmatrix} \alpha & \mathfrak{L}_{01} \\ \mathbf{0} & \mathfrak{L}_{00} \end{bmatrix}$, where $\alpha = c'/c$ is a nonzero scalar. In addition,

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

the right side of (4.65) is necessary for the regularity of analysis bank, which can be simplified in a similar way as the above,

$$d' \mathbf{e}_0 = d \mathbf{L}_0^{-T} \mathbf{e}_0 = d \begin{bmatrix} \alpha^{-1} & \mathbf{0} \\ \frac{\mathbf{L}_{01} \mathbf{L}_{00}^{-1}}{-\alpha} & \mathbf{L}_{00}^{-1} \end{bmatrix} \mathbf{e}_0$$

This yields $\mathbf{L}_{01} = \mathbf{0}$, which leads to $\mathbf{L}_0$ to be an $m \times m$ block diagonal matrix $\text{diag}(\alpha, \mathbf{L}_{00})$.

Next, according to the relation of (4.52), we can see that the transformed lattice parameter $\bar{\mathbf{U}}_0$ still holds the same structural form (4.31) as follows.

$$\bar{\mathbf{U}}_0 = \mathbf{L}_0 \mathbf{U}_0 = \text{diag}(c'/\sqrt{m}, \mathbf{L}_{00} \hat{\mathbf{U}}_0) \mathbf{H}_{[1_m]} \quad (4.66)$$

This can lead us to conclude that the space $\mathbb{L}_\beta$ for regular lattice structure is closed from the left.

4.6.3.2 Closeness of $\mathbb{L}_\beta$ from the right

In this section, we continue studying the closeness of $\mathbb{L}_\beta$ spanned by the regular lattice structure over constant permissible matrices from the right. For preserving the regularity of transformed polyphase matrix $\mathbf{F}(z) = \mathbf{E}(z) \mathfrak{R}$, it is necessary to have the following conditions,

$$\mathbf{F}(1) \mathbf{1}_M = c' \mathbf{e}_0, \mathbf{F}(1)^{-T} \mathbf{1}_M = d' \mathbf{e}_0 \quad (4.67)$$

where nonzero constants $c'd' = M$. Because $\mathbf{E}(z)$ has one-degree of regularity, i.e., $\mathbf{E}(1) \mathbf{1}_M = c \mathbf{e}_0$ and $\mathbf{E}(1)^{-T} \mathbf{1}_M = d \mathbf{e}_0$ with $cd = M$, we can see that (4.67) is equivalent to,

$$\mathfrak{R} \mathbf{1}_M = (c'/c) \mathbf{1}_M, \mathfrak{R}^T \mathbf{1}_M = (d/d') \mathbf{1}_M \quad (4.68)$$

Without loss of generality, we can always use a scalar to scale $\mathfrak{R}$ in order to let $c' = c$ and $d' = d$. This will lead to the following conditions on $\mathbf{Q}_1$ and $\mathbf{S}_1$,

$$\mathbf{Q}_1 \mathbf{1}_l = \mathbf{1}_l = \mathbf{Q}_1^T \mathbf{1}_l, \mathbf{S}_1 \mathbf{1}_{m-l} = \mathbf{1}_{m-l} = \mathbf{S}_1^T \mathbf{1}_{m-l} \quad (4.69)$$

since $\mathfrak{R}$ has a diagonal form with components in (4.59) as shown in Sec. 4.6.2.2.

These two conditions are inherently same, which makes the sum of each row and column of an invertible matrix to be unity. In the following, we develop a novel matrix decomposition for any $K \times K$ nonsingular matrix $\mathbf{T}$ with such constraints, i.e., $\mathbf{T} \mathbf{1}_K = \mathbf{1}_K$ and $\mathbf{T}^T \mathbf{1}_K = \mathbf{1}_K$ by using Schur unitary triangularization. Such constraints make $\mathbf{T}$ similar to the doubly stochastic matrix [79], except that elements of the matrix $\mathbf{T}$ are not necessarily nonnegative. Thus, the Birkhoff theorem of characterizing a doubly stochastic matrix by a convex combination of

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

permutation matrices [79] cannot be applied here. However, such constraints reveal that the matrix $\mathbf{T}$ has a special eigen-pair, i.e., an eigenvalue of one and both corresponding left and right eigenvectors of $\mathbf{1}_K$. Thus, we can use the Schur unitary triangularization method to characterize such matrix [5, 101] stated in the following lemma, with proof shown in the Appendix.

Lemma 4.6 *Any $K \times K$ nonsingular matrix $\mathbf{T}$ with constraints of $\mathbf{T}\mathbf{1}_K = \mathbf{1}_K$ and $\mathbf{T}^T\mathbf{1}_K = \mathbf{1}_K$ can always be factorized in the following form,*

$$\mathbf{T} = \mathbf{U}\mathbf{\Delta}\mathbf{U}^T \quad (4.70)$$

where $\mathbf{U} = [\mathbf{u}_0, \dots, \mathbf{u}_{K-1}]$ is an orthonormal matrix with $\mathbf{u}_0 = \mathbf{1}_K/\sqrt{K}$ and $\mathbf{\Delta}$ has the form of $\text{diag}(1, \bar{\mathbf{\Delta}})$ with $(K-1) \times (K-1)$ matrix $\bar{\mathbf{\Delta}}$ being nonsingular ($\bar{\mathbf{\Delta}}$ can be of the form of general invertible matrix or the invertible upper-triangular matrix depending on the matrix $\mathbf{U}$, see Remark 4.7 in the Appendix).

In the following, we show how the transformed matrix $\mathbf{F}(z)$ can still be factorized in the regular lattice structure presented in Sec. 4.3.1.2. First, from (4.63), we can see that $\mathfrak{R}$ can be absorbed into the lattice with modified parameters $\bar{\mathbf{U}}_0$, $\bar{\mathbf{V}}_0$ and $\bar{\mathbf{N}}$. Next, we just need to show that $\bar{\mathbf{U}}_0$ in (4.64) can still be written in the form of (4.31). From (4.69), we know that $\mathbf{Q}_1$ and $\mathbf{S}_1$ are the generalized doubly stochastic matrices. In addition, the matrix $\text{diag}(\mathbf{Q}_1, \mathbf{S}_1)$ can be factorized as in (4.70) due to its diagonal form, i.e., $\text{diag}(\mathbf{Q}_1, \mathbf{S}_1) = \mathbf{U}\text{diag}(1, \hat{\mathbf{\Delta}})\mathbf{U}^T$ with an invertible $(m-1) \times (m-1)$ matrix $\hat{\mathbf{\Delta}}$.

From (4.31), we know (we assume $c = \sqrt{m}$ in (4.31) for simplicity)

$$\bar{\mathbf{U}}_0 = \mathbf{U}_0\text{diag}(\mathbf{Q}_1, \mathbf{S}_1) = \text{diag}(1, \hat{\mathbf{U}}_0)\mathbf{H}_{[1_m]}\text{diag}(\mathbf{Q}_1, \mathbf{S}_1) \quad (4.71)$$

We first show the following equality

$$\mathbf{H}_{[1_m]}\text{diag}(\mathbf{Q}_1, \mathbf{S}_1) = \mathbf{H}_{[1_m]}\mathbf{U}\text{diag}(1, \hat{\mathbf{\Delta}})\mathbf{U}^T = \text{diag}(1, \mathbf{X})\mathbf{H}_{[1_m]}$$

where $\mathbf{X}$ is an invertible $(m-1) \times (m-1)$ matrix. Let us see first the product of $\mathbf{H}_{[1_m]}$ and the orthonormal matrix $\mathbf{U} = [\mathbf{u}_0, \dots, \mathbf{u}_{m-1}]$ as in (4.70).

$$\mathbf{H}_{[1_m]}\mathbf{U} = [\mathbf{e}_0 \mid \mathbf{H}_{[1_m]}\mathbf{u}_1 \mid \dots \mid \mathbf{H}_{[1_m]}\mathbf{u}_{m-1}] \quad (4.72)$$

Since both $\mathbf{H}_{[1_m]}$ and $\mathbf{U}$ are orthogonal matrices, the product is also an orthogonal matrix which leads to $\mathbf{H}_{[1_m]}\mathbf{U}$ to be a block diagonal matrix with the form of $\text{diag}(1, \mathbf{A})$, where $\mathbf{A}$ is an orthogonal matrix with size $m-1$. Thus, we can see that the matrix $\text{diag}(\mathbf{Q}_1, \mathbf{S}_1)$ is similar to a block diagonal matrix by the similarity transform $\mathbf{H}_{[1_m]}$, i.e.,

$$\begin{aligned} \mathbf{H}_{[1_m]}\text{diag}(\mathbf{Q}_1, \mathbf{S}_1)\mathbf{H}_{[1_m]}^T &= \mathbf{H}_{[1_m]}\mathbf{U}\text{diag}(1, \hat{\mathbf{\Delta}})\mathbf{U}^T\mathbf{H}_{[1_m]}^T \\ &= \text{diag}(1, \mathbf{A})\text{diag}(1, \hat{\mathbf{\Delta}})\text{diag}(1, \mathbf{A}^T) = \text{diag}(1, \mathbf{X}) \end{aligned}$$

where $\mathbf{X} = \mathbf{A}\hat{\Delta}\mathbf{A}^T$ is an $(m-1) \times (m-1)$ invertible matrix. This allows us to move $\mathbf{H}_{[1_m]}$ across $\text{diag}(\mathbf{Q}_1, \mathbf{S}_1)$,

$$\mathbf{H}_{[1_m]}\text{diag}(\mathbf{Q}_1, \mathbf{S}_1) = \text{diag}(1, \mathbf{X})\mathbf{H}_{[1_m]} \quad (4.73)$$

Finally, we can obtain the following factorization form for $\bar{\mathbf{U}}_0$ by combining (4.71) and (4.73),

$$\bar{\mathbf{U}}_0 = \text{diag}(1, \hat{\mathbf{U}}_0)\text{diag}(1, \mathbf{X})\mathbf{H}_{[1_m]} = \text{diag}(1, \mathbf{Y})\mathbf{H}_{[1_m]} \quad (4.74)$$

where $\mathbf{Y} = \hat{\mathbf{U}}_0\mathbf{X}$ is an invertible matrix. This is the same structural form as (4.31) with constant $c = \sqrt{m}$. It is trivial to show it is also correct for a constant c other than $\sqrt{m}$.

Therefore, we can factorize the transformed polyphase matrix $\mathbf{F}(z)$ in the same way as before, which shows the closeness of the space $\mathbb{L}_\beta$ for the regular lattice structure over constant permissible matrices from the right.

4.6.4 Discussion

As we have shown in Sec. 4.6.2.1, the space $\mathbb{L}_\beta$ spanned by the existing lattice structure (4.20) is closed under *all* permissible polynomial transforms $\mathfrak{L}(z)$ from the left. In addition, we have shown in Sec. 4.6.2.2 that all the permissible polynomial transforms $\mathfrak{R}(z)$ from the right should be lower-triangular unimodular matrices. For an important subclass of all permissible $\mathfrak{R}(z)$, i.e., constant matrices $\mathfrak{R}$, we have shown that the space $\mathbb{L}_\beta$ is closed under all permissible $\mathfrak{R}$ from the right. Furthermore, we have shown that for an important subclass of LPPRFBs with constrained length $L = KM$, i.e., $\beta = 0$, the permissible polynomial transforms from the right must be a constant matrix. *Thus, our results have shown that the space $\mathbb{L}_0$ is closed under all permissible polynomial transformations from both the left and right.*

Remark 4.6 *The lattice structure (4.20) with $\beta = 0$ has been shown to be complete for LP-PUFBs in [42, 77]. Consequently, the space $\mathbb{L}_0$ for LPPUFBs is closed due to the completeness. However, the lattice structure (4.20) has been shown to be incomplete for general LPPRFBs with $L > 2M$ [63, 99]. Moreover, the nonexistence of order-one lattice structure for such LP-PRFBs has been shown in Sec. 4.5. Thus, the closeness of the space $\mathbb{L}_0$ for general LPPRFBs cannot be generalized straightforwardly from [77]. Instead, through our derivation in Sec. 4.6.2, we have established the closeness of $\mathbb{L}_0$ for general LPPRFBs, including the class of LPPUFBs just as a special case.*

However, we cannot draw the same conclusion for the larger space $\mathbb{L}_\beta$ because we indeed lose some generality by restricting $\mathfrak{R}(z)$ to be constant matrices $\mathfrak{R}$ since the *free* first-order causal submatrix $\mathfrak{R}_{10}(z)$ of (4.54) is restricted *unnecessarily* to be null for the general case of $\beta \neq 0$.

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

Since all permissible polynomial transforms $\mathfrak{R}(z)$ should be first-order unimodular matrices as shown in Sec. 4.6.2.2, we may at first glance think about avoiding such a restriction by employing the complete factorization of lapped unimodular transforms (order-one unimodular FBs) [83] to attempt to study the closeness of the space $\mathbb{L}_\beta$ under all permissible polynomial transforms from the right. Unfortunately, such effort is in vain because the factorization in [83] is for general order-one unimodular matrices, not appropriate for the general permissible $\mathfrak{R}(z)$ due to its special triangular structure as shown in Sec. 4.6.2.2.

Such a unimodular matrix from the right reminds us of the complete factorization of causal FIR systems having FIR inverse (but not necessarily having FIR *anticausal* inverses) shown in [65, Theorem 7.1]. In fact, the counter examples reported in [63] and [99] are just such cases. After possibly fulfilling order-reduction on the polyphase matrix $\mathbf{E}(z) = \mathbf{G}(z)\tilde{\mathbf{E}}(z)$, they leave a residual matrix $\tilde{\mathbf{E}}(z)$ (still an LPPRFB but with lower order) that is a shifted unimodular matrix whose determinant is a monomial in the form of $c(z^M)^{-i}$ with $i \in \mathbb{Z}$ and nonzero constant c . The matrix $\tilde{\mathbf{E}}(z)$ can be transformed to a unimodular matrix by shifting such matrix using delay element z^{-1} or advance element z . Thus, the idea of using permutation matrix to attempt to solve the counter examples in [99] as conjectured in the beginning of Sec. 4.6 is in vain since the constant permutation matrix cannot change the nature of the shifted unimodular matrix, it just gives another same order shifted unimodular matrix with possibly different determinants. Note that shifted unimodular matrix $\tilde{\mathbf{E}}(z)$ can be an LPPRFB. It does not contradict the conclusion of nonexistence of linear phase unimodular FBs with equal lengths [83], since the matrix $\tilde{\mathbf{E}}(z)$ is shifted unimodular, not causal unimodular which is the premise of that conclusion in [83].

4.7 Design and Application Examples

In this section, several LPPRFBs are designed by using the proposed lattice structures for different cases. These design examples are further evaluated in transform coding based image compression and compared with some existing FBs.

4.7.1 Degrees of Freedom for Filter Bank Design

According to the lattice factorizations developed in Sec. 4.3.1 and Sec. 4.3.2 for LPPRFBs with even and odd β , respectively, the degrees of freedom (DOF) for designing such FBs design are summarized in Table 4.2 for various cases. It shows the number of free parameters within those lattice factorizations.

Note that once the filter length L is given for an M -channel FB, the DOF is fixed as shown in the above table since K and β are uniquely determined by a given L . The filter length L is

Table 4.2: Degrees of freedom for the lattice factorizations of M -channel LPPRFBs and LP-PUFBs with lengths $L = KM + \beta$.

DOF	LPPUFBs	LPPRFBs
Even β	$\left(\frac{\lceil \frac{M}{2} \rceil (\lceil \frac{M}{2} \rceil - 1)}{2} + \frac{\lfloor \frac{M}{2} \rfloor (\lfloor \frac{M}{2} \rfloor - 1)}{2} \right) \frac{K+1}{2} + \frac{\beta}{2} \left(\frac{\beta}{2} - 1 \right) / 2$	$\left(\lceil \frac{M}{2} \rceil^2 + \lfloor \frac{M}{2} \rfloor^2 \right) \frac{K+1}{2} + \left(\frac{\beta}{2} \right)^2$
Odd β	$\left(\frac{M-1}{2} \right)^2 \frac{K}{2} + \left(\frac{\beta-1}{2} \right)^2$	$\left(\lceil \frac{M}{2} \rceil^2 + \lfloor \frac{M}{2} \rfloor^2 \right) \frac{K}{2} + \left(\left\lceil \frac{\beta}{2} \right\rceil \right)^2 + \left(\left\lfloor \frac{\beta}{2} \right\rfloor \right)^2$

usually dependent on the requirements of the applications. For example, the filter length L is better not to be too large for image processing due to the ringing artifacts and computational complexity limitation. However how to choose the appropriate filter length is still an open problem and the guidelines may be quite different for different kinds of applications. Usually, the filter length is chosen by a balance of performance and complexity during experiments.

4.7.2 Filter Bank Design Examples

Since our factorizations can structurally impose various desirable properties like LP, PR and regularity into the FB, i.e., regardless of the actual values of free parameters in the structures, the optimal LPPRFB can be designed by unconstrained optimization for any design criteria. Below, all invertible matrices are decomposed by singular value decomposition (SVD) through orthogonal and diagonal matrices, where general orthogonal matrix is decomposed by Givens rotation matrices. The invertibility is ensured as long as the diagonal elements are nonzero.

One objective function for optimization is minimization of the stopband attenuation and/or passband ripple for ideal filter shape, which is a classical one in FB design and optimization. Another optimization criterion which is more related to efficient image compression is the generalized coding gain for LPPRFBs [34] as presented in Chapter 2. Note that optimization of the DC leakage and mirror frequency attenuation is actually already guaranteed by imposing regularity of FBs.

4.7.2.1 Odd channel FBs

We first show two design examples for odd-channel FBs. The first one is a 7-channel LPPRFB with filter length $L = 11$, i.e., $K = 1$ and $\beta = 4$, optimized for the coding gain of the AR(1) process with correlation coefficient $\rho = 0.95$ and the stopband attenuation. Such LPPRFB corresponds to the case of even β presented in Sec. 4.3.1. The analysis bank and its filter impulse responses are shown in Fig. 4.4.a and the corresponding synthesis bank is shown in Fig. 4.4.b.

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

4.4.a: 7-channel analysis bank

4.4.b: 7-channel synthesis bank

Figure 4.4: Design example I: $M = 7$, $L = 11$ optimized for coding gain and stopband attenuation.

The second design example is a 7-channel LPPRFB with filter length $L = 19$, i.e., $K = 2$ and $\beta = 5$, optimized for ideal filter shape. Such LPPRFB corresponds to the case of odd β as investigated in Sec. 4.3.2. The analysis and synthesis banks are shown in Fig. 4.5.a and 4.5.b, respectively. The filter impulse responses are too long for display and thus are omitted here.

4.5.a: 7-channel analysis bank

4.5.b: 7-channel synthesis bank

Figure 4.5: Design example II: $M = 7$, $L = 19$ optimized for stopband attenuation and passband ripple.

4.7.2.2 Even channel FBs

We present two design examples for the even-channel LPPRFB with one regularity. Note that *only* one regularity is possible when the filter length $L < 2M$ for M -channel LPPRFBs. The first one is an 8-channel LPPRFB with filter length $L = 10$, i.e., $K = 1$ and $\beta = 2$, optimized for maximization of coding gain for the AR(1) process and the stopband attenuation. The analysis bank and its filter impulse responses are shown in Fig. 4.6.a and the corresponding synthesis bank is shown in Fig. 4.6.b. The coding gain and stopband attenuation are shown in Table 4.3. Compared to the restrictive paraunitary case in [25], the general biorthogonal FB can avoid the inevitable zero filter coefficient at fixed positions [25], thus obtaining better performance. We show an 8-channel LPPRFB with length $L = 14$, i.e., $K = 1$ and $\beta = 6$, optimized for

Figure 4.6: Design example III: $M = 8$, $L = 10$ optimized for coding gain and stopband attenuation.

maximization of coding gain for the same AR(1) process and the stopband attenuation. The analysis bank and its filter impulse responses are shown in Fig. 4.7.a and the corresponding synthesis bank is shown in Fig. 4.7.b. The coding gain and stopband attenuation are also shown in Table 4.3.

Here we also show the comparison of coding gains for different LPPRFBs for AR(1) process with correlation $\rho = 0.95$. Following the traditional transform coding terminology, the LPPUFB and LPPRFB with eight channels and length $L = 16$ are also called LOT [4] and GLBT [34], respectively. It can be seen that the proposed lattice structure can offer more *flexible* tradeoff between coding gain and the filter length of GLBT. This is very desirable since the filter length is directly related to the computational complexity of the underlying transform. Thus, our design methods can provide much more flexible trade off between performance and computational complexity than before. We also show the degrees of freedom (DOF) in this

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

Figure 4.7: Design example IV: $M = 8$, $L = 14$ optimized for coding gain and stopband attenuation.

Table 4.3: Comparisons of three properties of eight-channel transforms.

	8×8 DCT	8×16 LOT	8×10 GLBT	8×12 GLBT	8×14 GLBT	8×16 GLBT
CG (dB)	8.83	9.22	9.09	9.36	9.51	9.62
DOF	0	18	33	36	41	48
Anal. Att. (dB)	8.66	16.32	8.71	10.17	11.25	12.63
Syn. Att. (dB)	8.66	16.32	9.13	10.31	11.34	12.75

table to illustrate the increased design freedom. A comparison of computational complexity among various popular transforms in terms of number of multiplications and additions needed for 8 transform coefficients is listed in Table 4.4.

Table 4.4: Comparisons of computational complexity of transforms.

	8×16 LOT [4]	8×10 GLBT	8×12 GLBT	8×14 GLBT	8×16 GLBT	$9 / 7$ WT
# Mult.	26	40	48	56	64	63
# Add.	66	72	88	104	120	98
Total	92	112	136	160	184	161

4.7.3 Effect of Filter Length on Image Compression

In this section, several LPPRFBs obtained from earlier design examples are evaluated for an image compression application to illustrate the length effect on the image compression. We follow the popular transform-based image coding setting and replace traditional DCT or wavelet transform by the LPPRFBs obtained from the design examples. The set partitioning in hi-

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

erarchical trees (SPIHT) [49] progressive image compression algorithm is chosen for efficient encoding of the transform coefficients due to its effectiveness and efficiency in approximating optimal nonlinear approximation. The eight-channel transform coefficients are rearranged and grouped into the popular quad-tree structure [35, 50] in analogy to the wavelet representation in order to be compatible with standard SPIHT encoding algorithm. Thus, an eight-channel transform is equivalent to a three level dyadic wavelet decomposition. In addition, the DC sub-band of the eight-channel transform coefficients is further decomposed by a three level dyadic wavelet decomposition. This ensures the fairness of comparison by SPIHT encoding algorithm, using the same tree depth in all cases.

To obtain a fair comparison of various transforms, we fix the encoding algorithm and only modify the transform stage of the encoder and the reconstruction stage by inverse transform in the decoder. The testing images in our experiment are the popular 512×512 8-bit gray scale images *Lena* and *Barbara* which represent typical images with smooth regions and plentiful textures, respectively. We select the transforms listed in Table 4.4 for the experiments. Note that the 9/7 wavelet has four degrees of regularity, while the remaining transforms have only (1, 1) degree of regularity. The objective compression results (PSNR in dB) are tabulated in Table 4.5. As stated before, we only consider the low bit rate image coding with compression ratio (CR) equal to or greater than 16 : 1.

Table 4.5: Compression performance (PSNR) using different transforms on the test image.

Lena		PSNR (dB)				
Compression Ratio	8×16 LOT	8×10 GLBT	8×12 GLBT	8×14 GLBT	8×16 GLBT	9 / 7 WT
16 : 1	36.66	36.55	36.75	36.90	37.12	37.06
32 : 1	33.52	33.46	33.63	33.80	33.96	33.91
64 : 1	30.36	30.19	30.54	30.77	30.90	30.96
100 : 1	28.52	28.10	28.54	28.80	29.07	29.18

Barbara		PSNR (dB)				
Compression Ratio	8×16 LOT	8×10 GLBT	8×12 GLBT	8×14 GLBT	8×16 GLBT	9 / 7 WT
16 : 1	32.64	31.04	31.78	32.18	32.95	31.13
32 : 1	28.54	27.08	27.81	28.27	28.83	27.23
64 : 1	25.32	24.44	24.95	25.29	25.68	24.55
100 : 1	23.80	23.13	23.64	23.72	23.99	23.50

For Lena, the 9/7 wavelet gives best performance in PSNR except for two cases with compression ratio 16:1 and 32:1, where eight-channel 16-tap GLBT is better. This is expected since Lena has lots of low frequency smooth regions, the 9/7 wavelet already does a good job

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

of de-correlation due to its long low frequency basis function. However, experiments indicate that the well designed eight-channel 16-tap regular GLBT can also provide comparable or even better compression performance in comparison with 9/7 wavelet. Among these eight-channel GLBTs, it can be easily seen that the performance gets better as filter length increases according to Table 4.5. This is more evident in perceptual visual quality shown in Fig. 4.8, which shows the zoomed-in part of the reconstructed images coded by various GLBTs with different filter length at 100:1 compression ratio. It is clear that the annoying blocking artifact, a typical phenomenon in low bit rate transform-based image compression, is improved as filter length increases, and even gets completely eliminated by the well designed eight-channel 16-tap regular GLBT. This shows the much more desired *flexible* trade off between filter length and compression performance than before (where the filter length was increased by step size of $M = 8$).

Figure 4.8: Enlarged portions of *Lena* at 100:1 compression ratio using various transforms and the original image.

Similarly, the same set of transforms are applied to the image Barbara which has rich texture information. According to Table 4.5, we can see that all eight-channel GLBTs and LOT are

Figure 4.9: Enlarged portions of *Barbara* at 64:1 compression ratio using various transforms and the original image.

better than the 9/7 wavelet except the 8×10 GLBT due to its too short filter length. This is because the traditional two-channel wavelet transform fails to capture fine details at high frequency in rich textures. Thus, FBs can easily outperform wavelets. The zoomed-in part of the reconstructed images shown in Fig. 4.9 are coded at 64:1 compression ratio. The textures in the scarf around the head are smoothed out by the 9/7 wavelet, however they are much better preserved by the well designed GLBTs. In addition, the blocking artifact also gets improved as filter length increases and thus gain shows the desired flexible trade off between filter length and compression performance. We can see that the well designed regular GLBTs with sufficient filter length can provide comparable smooth reconstruction as wavelets, while preserving the texture details better than wavelets.

In summary, the well designed regular GLBTs based on the proposed lattice structures could provide a supplement to the existing lapped transforms or LPPRFBs with desired flexible tradeoff between the filter length (measure of computational complexity) and the performance.

4.8 Summary

In this chapter, we have extended the theory, lattice factorizations and designs for a larger class of M -channel LPPRFBs, where all FIR filters have the same arbitrary length and same symmetry center. The proposed lattice factorizations can greatly expand the existing family of LPPRFBs with equal lengths and cover the conventional ones as special cases. Furthermore, these structures have been proved to completely span such class of LPPRFBs with length $L \leq 2M$ for any number of channels $M \geq 2$ and the whole class of FIR LPPUFBs. We have also shown that these structures are minimal in the sense of the number of delay elements used in the implementation. Considering regular filter banks, we have presented how to structurally impose one-degree of regularity constraint into our lattice structure for LPPRFBs with even β . In addition, we have discussed the difficulty of obtaining the regular lattice structure for the general case of odd β and pointed out the limitation of the previous works.

Several design examples have been presented based on the proposed lattice structures and evaluated in a transform-based image coding framework to illustrate the length effect on image compression. Due to their regular filters with fine frequency resolution, these regular LPPRFBs could obtain comparable, and even better, performance than that achieved by conventional wavelets both objectively (higher PSNR) and subjectively (better perceptual quality) for typical natural images. In addition, they can consistently yield smoother reconstructed images and better perceptual quality on textures as the filter length gradually increases. Therefore, these regular LPPRFBs succeed in providing a desired flexible tradeoff between performance and filter length which is related to computational complexity.

Due to the incompleteness of the lattice factorization for some length range, we have further investigated a more fundamental problem, i.e., the existence of complete lattice factorization by LPPR propagating blocks of order-one CAFACAFIs. We pointed out that the limitation of the existing order-one LPPR propagating blocks comes from the unnecessary constraint of order-one anticausal inverse. We further showed the nonexistence of any complete order-one LPPR propagating block by using CAFACAFI matrices. Although our result negatively answers the fundamental problem of the existence issue, it indicates some new perspectives for attempting to completely characterize LPPRFBs.

Further inspired from the incompleteness of the lattice factorization and to what extent of the optimality for the existing incomplete lattice based design of LPPRFBs, we have investigated the spaces $\mathbb{L}_\beta$ for general and regular LPPRFBs spanned by the existing lattice structures and their closeness property under polynomial transformations, respectively, since they consist of all the basic building blocks for various lattice factorizations of LPPRFBs. We have shown

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

that the spaces $\mathbb{L}_\beta$ are closed under all polynomial transforms $\mathfrak{L}(z)$ from the left and under all constant matrices $\mathfrak{R}$ from the right. Thus, the spaces $\mathbb{L}_\beta$ unfortunately cannot be expanded by transforming polyphase matrix $\mathbf{\Gamma}(z)$ into $\mathbf{F}(z) = \mathfrak{L}(z)\mathbf{E}(z)$ or $\mathbf{F}(z) = \mathbf{E}(z)\mathfrak{R}$, which establishes the suboptimality of the existing lattice based design of LPPRFBs. Furthermore, the space $\mathbb{L}_0$, i.e., an important subclass of LPPRFBs with length $L = KM$, has been shown to be closed under all polynomial transformations $\mathfrak{L}(z)$ and $\mathfrak{R}(z)$ from both left and right. In addition, we have also shown how the transformed polyphase matrix $\mathbf{F}(z)$ can still be factorized into the existing lattice structures and established the explicit relationship of the lattice parameters before and after transformations. Some of the results are summarized below and listed in Tables 4.6 and 4.7 for convenience.

- (i) For general LPPRFBs, the permissible left transform $\mathfrak{L}(z)$ must be a constant matrix and has the block diagonal form. However, the permissible right transform $\mathfrak{R}(z)$ generally should be a lower-triangular unimodular matrix. We further constrained it to be a constant matrix $\mathfrak{R}$ for simplicity and showed that it must also have the block diagonal form with structural components as shown in (4.59).
- (ii) The closeness of the space $\mathbb{L}_\beta$ has been extensively studied for general LPPRFBs in Sec. 4.6.2 and for regular ones in Sec. 4.6.3, respectively. The explicit relationships between the lattice parameters before and after transformations for general and regular FBs have been shown in Table 4.6 and 4.7, respectively.

Table 4.6: Summary of the closeness of $\mathbb{L}_\beta$ for even-channel LPPRFBs with arbitrary lengths.

General LPPRFBs	Structures of permissible $\mathbf{P}(z)$	Modified parameters
Left transforms $\mathfrak{L}(z)$	$\mathfrak{L}(z) = \text{diag}(\mathfrak{L}_0, \mathfrak{L}_1)$	(4.52)
Right transforms $\mathfrak{R}(z)$	$\mathfrak{R}(z) = \begin{pmatrix} \mathfrak{R}_{00} & \mathbf{0} \\ \mathfrak{R}_{10}(z) & \mathfrak{R}_{11} \end{pmatrix},$ or $\mathfrak{R} = \text{diag}(\mathfrak{R}_0, \mathfrak{R}_1)$	(4.60) and (4.64)

Table 4.7: Summary of the closeness of $\mathbb{L}_\beta$ for even-channel regular LPPRFBs with arbitrary lengths.

Regular LPPRFBs	Structures of permissible $\mathbf{P}(z)$	Modified parameters
Left transforms $\mathfrak{L}(z)$	$\mathfrak{L}(z) = \text{diag}(\alpha, \mathfrak{L}_{00}, \mathfrak{L}_1)$	(4.52) and (4.66)
Right transforms $\mathfrak{R}(z)$	Same as Table 4.6	(4.60), (4.64) and (4.74)

4.9 Appendix: Proof and Remarks for Lemma 4.6

In this appendix, we provide the proof and remarks for Lemma 4.6 on the novel matrix decomposition for the generalized doubly stochastic matrix.

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

Proof: From Schur's theorem [79], any square matrix $\mathbf{A}$ can be triangularized by an orthogonal transform $\mathbf{U}$, i.e. $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^T$, where $\mathbf{\Sigma}$ is an upper-triangular matrix and $\mathbf{U}$ is an orthonormal matrix. By applying Schur's theorem to $\mathbf{T}$ and noting that $\mathbf{T}$ has a pair of eigenvalue and eigenvector of 1 and $\mathbf{1}_K$, we can obtain the following (by appropriate permutation)

$$\mathbf{T} = \mathbf{U} \begin{bmatrix} 1 & \delta^T \\ \mathbf{0} & \bar{\mathbf{\Sigma}} \end{bmatrix} \mathbf{U}^T \quad (4.75)$$

where $\delta = [\delta_1, \dots, \delta_{K-1}]^T$ is a column vector with length $K-1$, $\bar{\mathbf{\Sigma}}$ is a square upper-triangular matrix with size $K-1$ and the orthonormal matrix $\mathbf{U}$ has the same form as in (4.70), i.e., the first column vector being $\mathbf{u}_0$. By first transposing (4.75) and then multiplying the matrix $\mathbf{U}$ from the right on the both sides, we can obtain

$$[\mathbf{T}^T \mathbf{u}_0 \mid \mathbf{T}^T \mathbf{u}_1 \mid \dots \mid \mathbf{T}^T \mathbf{u}_{K-1}] = \mathbf{T}^T \mathbf{U} = \mathbf{U} \begin{bmatrix} 1 & \mathbf{0}^T \\ \delta & \bar{\mathbf{\Sigma}}^T \end{bmatrix} \quad (4.76)$$

Now, by focusing on the first column of the above matrix equation, we can see that $\mathbf{T}^T \mathbf{u}_0 = \mathbf{U}[1, \delta^T]^T$ which can be rewritten into,

$$\mathbf{T}^T \mathbf{u}_0 = [\mathbf{u}_0, \dots, \mathbf{u}_{K-1}] \begin{bmatrix} 1 \\ \delta \end{bmatrix} = \mathbf{u}_0 + \sum_{n=1}^{K-1} \mathbf{u}_n \delta_n$$

Then, it can be further simplified to $\sum_{n=1}^{K-1} \mathbf{u}_n \delta_n = \mathbf{0}$ due to the special eigen-pair of $\mathbf{T}^T$, i.e., an eigenvalue of 1 and the corresponding eigenvector $\mathbf{u}_0$. Thus, we can conclude that the vector δ must be a null vector since the column vectors $\mathbf{u}_k$ of the orthonormal matrix $\mathbf{U}$ are linearly independent. Therefore, $\mathbf{T}$ has the form of (4.70) with Δ to be a block diagonal matrix in the form of $\text{diag}(1, \bar{\mathbf{\Sigma}})$. ■

Remark 4.7 For a complete characterization of the generalized doubly stochastic matrices, we can always fix the orthonormal matrix $\mathbf{U}$ to trade off the general $(K-1) \times (K-1)$ invertible matrix $\bar{\Delta}$, in contrast to the constrained form of the upper-triangular matrix $\bar{\mathbf{\Sigma}}$, i.e., moving the degrees of freedom in the orthonormal matrix $\mathbf{U}$ (since the column vectors except the first one are not totally determined) into the submatrix $\bar{\mathbf{\Sigma}}$ so as to make it to be a general invertible matrix. This is desired from the viewpoint of designing LPPRFs by unconstrained numerical optimization.

Remark 4.8 Such trade-off can also be seen from the degrees of freedom. If we fix the matrix $\mathbf{U}$ to trade-off the general invertible matrix $\bar{\Delta}$, the total $(K-1)^2$ free parameters all come from matrix $\bar{\Delta}$. On the contrary, if we only fix the first column $\mathbf{u}_0$ of the orthonormal matrix $\mathbf{U}$,

CHAPTER 4. ARBITRARY EQUAL-LENGTH LINEAR PHASE FILTER BANKS

there are $K(K-1)/2 - (K-1)$ free parameters left. In addition, the upper-triangular matrix $\bar{\Delta}$ has degrees of freedom of $\sum_{n=1}^{K-1} n = K(K-1)/2$. Thus, there are total $K(K-1)/2 + K(K-1)/2 - (K-1) = (K-1)^2$ free parameters, which is the same as the previous case. Note also that the parameterization form (4.70) stated in Lemma 4.6 is minimal in terms of degrees of freedom. This can be seen as follows. There are K^2 free parameters for any nonsingular matrix $\mathbf{T}$, but the two constraints in matrix equations of $\mathbf{T}\mathbf{1}_K = \mathbf{1}_K$ and $\mathbf{T}^T\mathbf{1}_K = \mathbf{1}_K$ reduce $2K-1$ free parameters (because there is one scalar equation in common). Thus, the total degrees of freedom is given by $K^2 - (2K-1) = (K-1)^2$.

Chapter 5

Unequal-Length Multichannel Filter Banks

5.1 Introduction

There has been enormous research works in the area of filter banks (FB) [5–8] which find many applications in image/audio/video processing and communication systems. Of extreme importance is the capability to design filter banks for some desired applications. Continuing the study for critically sampled FBs with more flexible lengths, this chapter moves to study a class of FBs with filters of different lengths, i.e., unequal-length (UL) FBs. We systematically investigate the existence conditions, lattice factorizations and designs for such class of M -channel FBs with various desirable properties.

The main motivation for our work on UL FBs is to design more flexible FBs which cater to low bit rate image coding application. In most traditional works [27, 34, 42, 63, 66, 72, 77, 78, 84–95], the filter lengths of different filters are equal and multiple of M , i.e., $L = KM$, which greatly limits the possible designs of FBs. More importantly, it is not optimal and efficient in the sense of filter length distribution for low bit rate image coding in which the annoying blocking and ringing artifacts are most noticeable. In such situation, long basis functions for low frequency subbands and short basis functions for high frequency ones in both analysis and synthesis FB are desirable to *simultaneously* obtain the energy compaction capability of the analysis bank, to reduce both the blocking and ringing artifacts, and to get smooth reconstruction by the synthesis bank. Thus, UL FBs, especially long low pass filters and short bandpass and high pass filters, are more desirable and suitable for low bit rate subband image coding [57, 102]. Such FBs can achieve a better trade-off between efficiency and filter length than that of equal-length (EL) FBs in the sense of filter length distribution for low bit rate image coding, because they can not only simultaneously reduce annoying blocking and ringing artifacts which are most noticeable

in such application but also keep the averaged length $L_{av} = \sum_{i=0}^{M-1} L_i/M$ reasonably short even if the lowpass filter is very long. In addition, from the perspective of implementation and computational complexity, it is also desirable for such kind of UL FBs since the averaged length L_{av} is almost linearly proportional to the computational complexity, as the case of equal-length FBs seen in Sec. 2.4.3.1.

In this chapter, we study a more general class of FBs with unequal filter lengths. A typical example of such FBs is the biorthogonal 9/7 wavelet FB adopted in the recent image compression standard JPEG2000 [10] which intends to replace the older JPEG standard. Since the success of 9/7 wavelet FB, there has been great interests in the analysis and design of UL FBs. In [102], such FBs have been analyzed in the context of image coding which takes into account the statistics of natural images and the effect of quantization errors in the reconstructed images, which has successfully demonstrated the motivation for unequal length structure of FBs. Then, a generalized algorithm for designing UL LPPUFBs has been proposed in [103] by further extending the factorization result reported in [77]. UL LPPUFBs have also been studied under the framework of generalized lapped transforms in [58] for efficient fast implementation based on DCT and the size-limited structures for their work have also been reported for sub-band image coding. Later, a biorthogonal lapped transform with different lengths has been investigated in [59] and the optimal short basis functions in the sense of energy compaction, when given the long basis functions of such variable-length LBT, were derived by virtue of the theory of KLT in a subspace. The authors continued their study of variable-length LBTs by combining multiple synthesis FBs for efficient image coding [60], which defines the synthesis filters freely first and then obtains the analysis filters by numerical methods. Thus, the whole FB is usually not an exact PRFB and may suffer from the noise amplification problem. Different from these works, Malvar proposed the hierarchical structure by appropriately combining two undersampled LBTs to construct a variable-length LBT, named as hierarchical LBT (HLBT). Based on the structure of TDLT, Tran *et al.* have proposed the factorization and designs for unequal-length LPPUFBs and later extended to LPPRFBs [43–45, 57]. In addition, Ikehara *et al.* have proposed an unequal-length lapped regular transform with integer coefficients by using lifting scheme and maximally flat M th-band filter [104].

However, most of previous works [57–60] concentrated on the constrained filter lengths to be multiple of M , and imposed an additional constraint of even number of long filters for constructing LPPRFBs by butterfly structure, which is *not* a necessary constraint in some situations shown later in this chapter. Although an example of FBs with only one long filter was reported in [104], the long synthesis filter would necessitate corresponding short analysis filter and vice versa. This is not good for image coding. In this chapter, we systematically

study more general class of UL FBs, especially on UL LPPRFBs. The outline of this chapter is as follows. First, the general UL PRFBs are studied in Sec. 5.3 and the important hybrid lattice factorization is presented. Then, we investigate UL LPPRFBs in detail. The existence conditions are derived for such FBs in Sec. 5.4.1, which shows that the number of long filters can be arbitrary in some cases unlike earlier works which limit it to even number. Next, the lattice factorizations are developed for both even and odd channel systems in Sec. 5.4.2 and Sec. 5.4.3, respectively, with the structural enforcement of the unequal length property, linear phase and perfect reconstruction. Especially, we show LPPRFBs with odd number of long filters are possible which corrects the misleading statement in [60]. The minimality of the proposed lattice structures are also shown in Sec. 5.4.4. Moreover, we discuss the limitation of the hybrid lattice structure by pointing out its failure for some situations. Finally, several design examples are presented in Sec. 5.5 to confirm the validity of the proposed design approaches and to compare the performances of different FBs.

5.2 Filter Length Profile and General Lattice Factorization

Since the filter lengths are not equal for all channels of filters, we have to specify the filter length profile for an UL-PRFB. For the purpose of simplicity, let us consider FBs with filters of lengths $L_i = K_i M + \beta$, where

$$K_i = \begin{cases} K_l & , \quad 0 \leq i \leq N-1 \leq M-1 \\ K_s & , \quad N \leq i \leq M-1 \end{cases} \quad \delta_K \triangleq K_l - K_s \geq 0 \quad (5.1)$$

and l, s denote long and short filters, respectively, and N denotes the number of long filters. Obviously, this length profile can be extended to more general setting, i.e., more than two sets of lengths like in [57, 58, 103]. The previous chapters on equal-length FBs can also be obtained from this length profile by setting $K_l = K_s$ (thus $N = 0$), i.e., a special class of UL FBs. In this chapter, we always assume $K_l > K_s$ and $1 \leq N \leq M-1$.

Like before, the lattice structure for UL FBs with length profile (5.1) is expressed by a factorization of polyphase matrix $\mathbf{E}(z)$,

$$\mathbf{E}(z) = \mathbf{F}_{K-1}^f(z) \cdots \mathbf{G}_1^f(z) \mathbf{E}_0^f(z) \quad (5.2)$$

where $\mathbf{E}_0^f(z)$ is a FB with prescribed properties like filter length and PR. Each $M \times M$ block $\mathbf{G}_i^f(z)$ (or $\mathbf{F}_i^f(z)$) is to increase the order and propagate those desired properties held in $\mathbf{E}_0^f(z)$. By this way, we can easily construct FBs with desired properties for some oriented applications and customized filter lengths just by varying the number of such blocks.

However, different from the case of equal-length FBs, each propagating block $\mathbf{G}_i^f(z)$ here may have the same or different structures, thus we specifically use a different notation $\mathbf{F}_i^f(z)$ for a possible different propagating block. The block $\mathbf{G}_i^f(z)$ serves to increase the length of all filters whereas the block $\mathbf{F}_i^f(z)$ only increases the length of the desired long filters. In this way, we can easily generate different filter lengths by varying the structures of $\mathbf{G}_i^f(z)$ and $\mathbf{F}_i^f(z)$ and by different ways of cascading them. Moreover, the initial block $\mathbf{E}_0^f(z)$ is usually not a constant matrix, due to the partial block overlapping, i.e, nonzero "extra" β coefficients. To maintain the desired *modular* structure, $\mathbf{E}_0^f(z)$ is to take care of β , which results in $\mathbf{E}_0^f(z)$ to be a polynomial matrix with length $N_0M + \beta$ ($N_0 \geq 1$). Therefore, for the FBs with simple length profile (5.1), we can design such UL FBs by first designing an equal-length FB with length $L_s = K_sM + \beta$ and further cascading some specifically designed blocks to the channels where the long filters are desired.

5.3 Lattice Factorization of Unequal-Length Perfect Reconstruction Filter Banks

In this section, we study the lattice factorization for UL-PRFBs without phase constraint. The general UL-PRFBs are firstly studied in Sec. 5.3.1 and then regular UL-PRFBs are studied in Sec. 5.3.2.

5.3.1 General Unequal-Length Perfect Reconstruction Filter Banks

To our knowledge, the existence conditions for the filter length profile (5.1) are so far still open problems for general UL PRFBs. Thus, we only consider a subclass of such FBs, i.e., the class of UL PRFBs with *same* filter length L_i for each pair of analysis filter $h_i[n]$ and synthesis filter $f_i[n]$ ($0 \leq i \leq M - 1$). This kind of distribution on the filter length is also desirable from the perspective of image processing [35, 41, 102], especially for image coding, which can simultaneously reduce the annoying blocking and ringing artifacts by long filters and short filters, respectively.

From Sec. 3.2.2, we see that each block $\mathbf{G}_i^f(z)$ of the form (3.4) can increase filter length by M for *all* filters. Thus, the lattice factorization of (5.2) with only traditional building block $\mathbf{G}_i^f(z)$ of (3.4) is sufficient for constructing UL FBs. However, we can modify the blocks in such lattice structure to account for different filter lengths. *The key idea is to use a new propagating block $\hat{\mathbf{G}}_i(z)$ to increase filter length only for desired long filters while maintaining length of short filters unchanged* [57]. Without loss of generality, we can permute the first N long filters

5.4 Lattice Factorization of Unequal-Length LPPRFBs

Unequal-length LPPRFBs have been widely employed in image and video processing, such as the popular biorthogonal 9/7 wavelet FB [15] adopted in recent image compression standard JPEG2000 [10]. In this section, we first show some necessary existence conditions for such LPPRFBs in Sec. 5.4.1 and then develop various lattice structures for both even and odd channel systems in Sec. 5.4.2 and Sec. 5.4.3, respectively, according to our derived existence conditions. Finally, the minimality of our proposed lattice structures is to be studied in Sec. 5.4.4.

5.4.1 Existence Conditions

For such class of LPPRFBs, the LP condition (2.8) is not valid since the lengths of different filters are not same. For the FB with length profile (5.1), it has the LP property if the polyphase matrix $\mathbf{E}(z)$ satisfies,

$$\mathbf{E}(z) = z^{-(K_s-1)} \hat{\mathbf{Z}}(z) \mathbf{D}_M \mathbf{E}(z^{-1}) \mathbf{J}_M \quad (5.5)$$

where $\mathbf{D}_M$ is a signature matrix whose entry is +1 when the corresponding filter is symmetric and -1 for antisymmetric filters, and $\hat{\mathbf{Z}}(z) = \text{diag}(z^{-\delta_K} \mathbf{I}_N, \mathbf{I}_{M-N})$. In traditional LPPRFBs with equal lengths, the matrix $\hat{\mathbf{Z}}(z)$ becomes an identity matrix and the LP condition (5.5) degenerates into the previous one (2.8).

Due to the PR and LP constraints, there are some necessary conditions for such class of LPPRFBs. We first present the existence conditions in terms of the filter length distribution stated in the following theorem.

Theorem 5.8 *For the class of LPPRFBs with length profile (5.1), the number of long filters N should satisfies the following necessary conditions:*

- (i) *In the case of both even M and β , N must be even if δ_K is odd. However, N could be arbitrary if δ_K is even.*
- (ii) *In the case of even M and odd β , N must be odd if δ_K is odd. Moreover, δ_K cannot be even.*
- (iii) *In the case of odd M and even β , there are three subcases available. First, if K_s is even, then both N and K_l must be odd. Second, when K_s is odd, N must be even if δ_K is odd. Third, N could be arbitrary if δ_K are even.*

CHAPTER 5. UNEQUAL-LENGTH MULTICHANNEL FILTER BANKS

- (iv) In the case of both odd M and β , there are three subcases available. First, if K_s is odd, then N must be odd and K_l must be even. Second, when K_s is even, N must be even if δ_K is odd. Third, N could be arbitrary if δ_K are even.

Proof: The proof for this theorem can be divided into four cases depending on even and odd values of M and β . Here, only the case (iii) is shown in detail. However, the proofs for the other three cases can be obtained in a similar manner. First, the two cases $N = 0$ and $N = M$ would degenerate into LPPRFBs with equal lengths, which have been well studied in Chapter 4. Thus, we here only consider the range $1 \leq N \leq M - 1$ for UL FBs. The proof is mainly adapted from the length conditions of M -channel LPPRFBs [25, Theorem 2]. For the case of odd M and even β , the sum of K_i , i.e., $\sum_{i=0}^{M-1} K_i = NK_l + (M - N)K_s = N\delta_K + MK_s$ has to be odd. Thus, if K_s is even, then NK_l must be odd, which implies both K_l and N must be odd. However, if K_s is odd, then $N\delta_K$ must be even, which implies the other two possible solutions in case (iii). Note that in case (ii), $\sum_{i=0}^{M-1} K_i$ has to be odd, which implies $N\delta_K$ to be odd. Thus, the case of even δ_K does not exist for the case (ii). ■

Then, the existence conditions on the symmetry polarity are presented in the following theorem. We denote the number of long symmetric filters, long antisymmetric filters, short symmetric filters and short antisymmetric filters, by $n_{l,s}$, $n_{l,a}$, $n_{s,s}$ and $n_{s,a}$, respectively. The conditions are only specified for the symmetric filters since $n_{l,a} = N - n_{l,s}$ and $n_{s,a} = M - N - n_{s,s}$.

Theorem 5.9 *For the class of LPPRFBs with length profile (5.1), the symmetry polarity should satisfy the following conditions:*

- (i) In the case of both even M and β , if δ_K is odd, $n_{l,s} = N/2$ and $n_{s,s} = (M - N)/2$. However, if δ_K is even, the overall LPPRFB should have $M/2$ symmetric filters, i.e., $n_{l,s} + n_{s,s} = M/2$. The specific values of $n_{l,s}$ and $n_{s,s}$ could be chosen freely.
- (ii) In the case of even M and odd β , $n_{l,s} = (N + 1)/2$ and $n_{s,s} = (M - N + 1)/2$.
- (iii) In the case of odd M and even β , there are three subcases available. First, if K_s is even, then $n_{l,s} = (N + 1)/2$ and $n_{s,s} = (M - N)/2$. Second, when K_s is odd, if δ_K is odd, then $n_{l,s} = N/2$ and $n_{s,s} = (M - N + 1)/2$. Third, if δ_K are even, the overall LPPRFB should have $(M + 1)/2$ symmetric filters, i.e., $n_{l,s} + n_{s,s} = (M + 1)/2$. The specific values of $n_{l,s}$ and $n_{s,s}$ could be chosen freely.
- (iv) In the case of both odd M and β , there are three subcases available. First, if K_s is odd, then $n_{l,s} = (N + 1)/2$ and $n_{s,s} = (M - N)/2$. Second, when K_s is even, if δ_K is odd, then

CHAPTER 5. UNEQUAL-LENGTH MULTICHANNEL FILTER BANKS

$n_{l,s} = N/2$ and $n_{s,s} = (M - N + 1)/2$. Third, if δ_K are even, the overall LPPRFB should have $(M + 1)/2$ symmetric filters, i.e., $n_{l,s} + n_{s,s} = (M + 1)/2$. The specific values of $n_{l,s}$ and $n_{s,s}$ could be chosen freely.

Proof: The proof is similar to that in [57, Theorem II] and here we extend it to the more general class of UL LPPRFBs with length profile (5.1). Thus, we just outline the proof here. Here, only the case (iii) is shown in detail. However, the proofs for the other three cases can be obtained in a similar manner. Without loss of generality, we can always arrange the M channel linear phase filters in such an order that the first N long filters are on the top. From the LP condition (5.5), we can decompose $\mathbf{E}(z)$ into two submatrices $\mathbf{E}_l(z)$ and $\mathbf{E}_s(z)$ with size $N \times M$ and $(M - N) \times M$, respectively, i.e., $\mathbf{E}(z) = [\mathbf{E}_l^T(z), \mathbf{E}_s^T(z)]^T$. It can be easily seen that $\mathbf{E}_l(z)$ and $\mathbf{E}_s(z)$ still have LP property, i.e., satisfying LP condition (5.5) with appropriate dimensions,

$$\mathbf{E}_l(z) = z^{-(K_l-1)} \mathbf{D}_l \mathbf{E}_l(z^{-1}) \mathbf{J}_M(z), \quad \mathbf{E}_s(z) = z^{-(K_s-1)} \mathbf{D}_s \mathbf{E}_s(z^{-1}) \mathbf{J}_M(z) \quad (5.6)$$

where matrices $\mathbf{D}_l$ and $\mathbf{D}_s$ with size $N \times N$ and $(M - N) \times (M - N)$, respectively, are the signature matrices for the symmetry polarity. Similar partition is done on $\mathbf{R}(z)$ columnwise to obtain $M \times N$ matrix $\mathbf{R}_l(z)$ and $M \times (M - N)$ matrix $\mathbf{R}_s(z)$, i.e., $\mathbf{R}(z) = [\mathbf{R}_l(z), \mathbf{R}_s(z)]$. Then from the PR condition $\mathbf{E}(z)\mathbf{R}(z) = \mathbf{I}_M$, we can obtain the following matrix equations,

$$\mathbf{E}_l(z)\mathbf{R}_l(z) = z^{-(K_l-1)} \mathbf{D}_l \mathbf{E}_l(z^{-1}) \mathbf{J}_M(z) \mathbf{R}_l(z) = \mathbf{I}_N \quad (5.7)$$

$$\mathbf{E}_s(z)\mathbf{R}_s(z) = z^{-(K_s-1)} \mathbf{D}_s \mathbf{E}_s(z^{-1}) \mathbf{J}_M(z) \mathbf{R}_s(z) = \mathbf{I}_{M-N} \quad (5.8)$$

From them, we can derive,

$$\mathbf{D}_l = z^{-(K_l-1)} \mathbf{E}_l(z^{-1}) \mathbf{J}_M(z) \mathbf{R}_l(z), \quad \mathbf{D}_s = z^{-(K_s-1)} \mathbf{E}_s(z^{-1}) \mathbf{J}_M(z) \mathbf{R}_s(z) \quad (5.9)$$

Then, taking the trace operation on both $\mathbf{D}_l$ and $\mathbf{D}_s$ and using the equality $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$, we can obtain,

$$\text{tr}[\mathbf{D}_l] = z^{-(K_l-1)} \text{tr}[\mathbf{E}_l(z^{-1}) \mathbf{J}_M(z) \mathbf{R}_l(z)] = z^{-(K_l-1)} \text{tr}[\mathbf{J}_M(z) \mathbf{R}_l(z) \mathbf{E}_l(z^{-1})] \quad (5.10)$$

$$\text{tr}[\mathbf{D}_s] = z^{-(K_s-1)} \text{tr}[\mathbf{E}_s(z^{-1}) \mathbf{J}_M(z) \mathbf{R}_s(z)] = z^{-(K_s-1)} \text{tr}[\mathbf{J}_M(z) \mathbf{R}_s(z) \mathbf{E}_s(z^{-1})] \quad (5.11)$$

Then, from another PR condition $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}_M$, we can obtain

$$\mathbf{R}_l(z)\mathbf{E}_l(z) + \mathbf{R}_s(z)\mathbf{E}_s(z) = \mathbf{I}_M \quad (5.12)$$

CHAPTER 5. UNEQUAL-LENGTH MULTICHANNEL FILTER BANKS

By evaluating the complex variable z in (5.10) and (5.11) at $+1$, we can obtain the relationship between $\text{tr}[\mathbf{D}_l]$ and $\text{tr}[\mathbf{D}_s]$ through (5.12) as follows,

$$\text{tr}[\mathbf{D}_s] = \text{tr}[\mathbf{J}_M(1)\mathbf{R}_s(1)\mathbf{E}_s(1)] = \text{tr}[\mathbf{J}_M(1)] - \text{tr}[\mathbf{J}_M(1)\mathbf{R}_l(1)\mathbf{E}_l(1)] = \text{tr}[\mathbf{J}_M(1)] - \text{tr}[\mathbf{D}_l] \quad (5.13)$$

Similarly, when evaluating $z = -1$, we can obtain

$$\text{tr}[\mathbf{D}_s] = (-1)^{K_s-1}\text{tr}[\mathbf{J}_M(-1)] - (-1)^{K_l-K_s}\text{tr}[\mathbf{D}_l] \quad (5.14)$$

Finally, we can obtain the symmetry polarity conditions stated above by evaluating the trace on $\mathbf{D}_l$ and $\mathbf{D}_s$ for various cases, along with the known result on $\text{tr}[\mathbf{J}_M(z)]$ [25] and the result for N given in Theorem 5.8. For example, in the case (iii), i.e., odd M and even β , $\text{tr}[\mathbf{J}_M(z)] = 1$. Thus, we can simplify (5.13) and (5.14) to the following,

$$\text{tr}[\mathbf{D}_s] = 1 - \text{tr}[\mathbf{D}_l], \quad \text{tr}[\mathbf{D}_s] = (-1)^{-(K_s-1)} - (-1)^{-(K_l-K_s)}\text{tr}[\mathbf{D}_l] \quad (5.15)$$

Then, the symmetry polarity conditions can be obtained by considering the different cases of even or odd values for K_s and K_l . ■

Table 5.1: Existence conditions for M -channel LPPRFBs with length profile (5.1).

	K_l and K_s	N	Symmetry Polarity
Even M , even β	odd δ_K	Even	$n_{l,s} = \frac{N}{2}, n_{s,s} = \frac{M-N}{2}$
	even δ_K	Arbitrary	$n_{l,s} + n_{s,s} = \frac{M}{2}$
Even M , odd β	odd δ_K	Odd	$n_{l,s} = \frac{N+1}{2}, n_{s,s} = \frac{M-N+1}{2}$
Odd M , even β	even K_s	Odd (K_l must be odd)	$n_{l,s} = \frac{N+1}{2}, n_{s,s} = \frac{M-N}{2}$
	odd K_s , even K_l	Even	$n_{l,s} = \frac{N}{2}, n_{s,s} = \frac{M-N+1}{2}$
	odd K_s , odd K_l	Arbitrary	$n_{l,s} + n_{s,s} = \frac{M+1}{2}$
Odd M , odd β	odd K_s	Odd (K_l must be even)	$n_{l,s} = \frac{N+1}{2}, n_{s,s} = \frac{M-N}{2}$
	even K_s , odd K_l	Even	$n_{l,s} = \frac{N}{2}, n_{s,s} = \frac{M-N+1}{2}$
	even K_s , even K_l	Arbitrary	$n_{l,s} + n_{s,s} = \frac{M+1}{2}$

We summarize the above existence conditions in Table 5.1 for various cases. Note that results in this section do *not* require each analysis filter $h_i[n]$ to have the same length as that of corresponding synthesis filter $f_i[n]$. These existence conditions are very useful guidelines for FB design. They can narrow down the solution space for FB designers before the design process. This is very helpful for deriving the efficient lattice structures and avoiding impossible design specifications, in contrast to the previous work [103] in which the permissible distribution of filter lengths and symmetry polarity can only be determined during the design process. It also helps to explain why only some solutions exist. For example, there does not exist any even

channel LPPRFB with odd and equal length. Compared with earlier works [57–59], our results are more general due to the general form of the our length profile (5.1). With even β and $\delta_K = 1$, one can check it is consistent with [57]. However, the other cases are new results and interesting since N can be chosen freely for both β and δ_K to be even. In addition, our results correct the misleading statement[†] in [60], since the number of long filters N need not be limited to even. In fact, for the choice of parameters in their cited example [60], N could be odd as illustrated in GLBT35 of section 5.5. From the above theorems, we can see that the case of even δ_K does not impose any extra limitation on FB other than LP and PR conditions.

Moreover, the M -channel LPPRFBs reported in the literature so far fall into the different cases of our theorems. For example, the 2-channel type-A systems with different lengths, i.e., 2-channel FBs with a symmetric and an antisymmetric filter of both even lengths, is an example for the case (i) with even δ_K . The 2-channel type-B systems, i.e., 2-channel FBs with both symmetric filters of odd different lengths, is an example for the case (ii). Another example for this case was reported in [89] for a 4-channel LPPRFB with length (33, 33, 33, 69) and symmetry polarity (S, A, S, S). In addition, a 3-channel LPPRFB with length (53, 44, 44) and symmetry polarity (S, A, S) was reported in [90], which corresponds to the case (iii) with even $K_s = 14$, odd $K_l = 17$ and $\beta = 2$. Another 3-channel LPPRFB with length (56, 53, 56) and symmetry polarity (S, A, S) was documented in [85], which corresponds to the case (iii) with odd $K_s = 17$, even $K_l = 18$ and $\beta = 2$. Besides these examples, a 9-channel LPPUFB with length (32, 32, 32, 32, 32, 32, 23, 23, 23) and symmetry polarity (S, S, S, A, A, A, S, S, A) was designed in [103], which corresponds to the case (iv) with even $K_s = 2$, odd $K_l = 3$ and $\beta = 5$. Another 5-channel LPPRFB with length (76, 76, 76, 76, 61) and symmetry polarity (S, S, A, A, S) was documented in [90], which corresponds to the case (iv) with even $K_s = 12$, odd $K_l = 15$ and $\beta = 1$.

5.4.2 Even Channel Systems

From the proof of Theorem 5.9, we can see that $\mathbf{E}_l(z)$ with size $N \times M$ and order $K_l - 1$, which represents the long analysis filters, forms a similar system to an N -channel LPFB. In fact, $\mathbf{E}_l(z)$ is an N -channel undersampled LPFB with subsampling factor M ($N < M$). Similarly, $\mathbf{E}_s(z)$ is an $(M - N)$ -channel undersampled LPFB with the same subsampling factor as that of $\mathbf{E}_l(z)$. These two systems cannot have the PR property by themselves, respectively. However, an M -channel critically sampled LPPRFB can be achieved by combining appropriately these two undersampled LPFBs through the hierarchical structure reported in [105].

Here, we continue our study of lattice factorization for LPPRFBs with length profile (5.1). In this section, the lattice structures are to be derived for various cases list in Table 5.1 based

on the idea presented in Sec. 5.3.1. For LPPRFBs, the hybrid lattice structure (5.4) should be changed in the following form to adapt for the linear phase constraint,

$$\mathbf{E}(z) = \hat{\mathbf{G}}_{\delta_K}^{\text{LP}}(z) \cdots \hat{\mathbf{G}}_1^{\text{LP}}(z) \mathbf{P}_e \mathbf{E}_{K_s}^{\text{LP}}(z) \quad (5.16)$$

where $\mathbf{E}_{K_s}^{\text{LP}}(z)$ is an EL LPPRFB whose factorization has been well presented in Chapter 4 and $M \times M$ matrix $\mathbf{P}_e$ is a permutation matrix for arranging N long filters to satisfy the symmetry polarity constraints as shown in Theorem 5.9. It should be pointed that that hierarchical structure in [105] corresponds to the hybrid lattice structure (5.16) by reordering the signal flow. In addition, such hybrid lattice structure has also been studied in [57, 58, 103] for LPPUFBs and [43, 44] under the framework of time-domain lapped transforms. According to the existence conditions in Table 5.1, there are three cases for even channel systems based on even and odd values of β and δ_K . We study them in the following subsections, respectively.

5.4.2.1 Even β and odd δ_K

According to the existence conditions presented in Sec. 5.4.1, we know that the number of long filters N must be even, i.e., $N = 2n$, and half of long and short filters are symmetric. Thus, the modified building block $\hat{\mathbf{G}}_i^{\text{LP}}(z)$ can have such form,

$$\hat{\mathbf{G}}_i^{\text{LP}}(z) = \text{diag}(\hat{\Phi}_i \mathbf{W}_N \hat{\Lambda}_i(z) \mathbf{W}_N, \mathbf{I}_{M-N}) \quad (5.17)$$

where $\hat{\Phi}_i = \text{diag}(\mathbf{U}_i, \mathbf{I}_n)$ in which $\mathbf{U}_i$ is an $n \times n$ nonsingular matrix and $\hat{\Lambda}_i(z) = \text{diag}(\mathbf{I}_n, z^{-1} \mathbf{I}_n)$. Here, the top-left corner of the diagonal matrix $\hat{\mathbf{G}}_i^{\text{LP}}(z)$ has the same structural form as that of $\mathbf{G}_i(z)$ in section 4.3, but with reduced dimension N . The invertibility of the block $\hat{\mathbf{G}}_i^{\text{LP}}(z)$ is guaranteed by the $n \times n$ nonsingular matrix $\mathbf{U}_i$. Note that this simple case has also been reported in [57, 58, 103], but only for LPPUFBs.

5.4.2.2 Even β and even δ_K

In this case, we know that N could be chosen freely according to Table 5.1. If N is even, the block in (5.17) is still applicable. However, if N is odd, i.e., $N = 2n + 1$, the block in (5.17) fails. For odd N , we propose the following second-order form,

$$\hat{\mathbf{G}}_i^{\text{LP}}(z) = \text{diag}(\hat{\Phi}_{i,0} \mathbf{W}_0^o(z) \hat{\Phi}_{i,1} \mathbf{W}_1^o(z), \mathbf{I}_{M-N}) \quad (5.18)$$

where $\hat{\Phi}_{i,0} = \text{diag}(\mathbf{U}_{i,0}, \mathbf{I}_n)$, $\hat{\Phi}_{i,1} = \text{diag}(\mathbf{I}_{n+1}, \mathbf{V}_{i,1})$, $\mathbf{W}_0^o(z) \triangleq \mathbf{W}_N \hat{\Lambda}_0^o(z) \mathbf{W}_N$ and $\mathbf{W}_1^o(z) \triangleq \mathbf{W}_N \hat{\Lambda}_1^o(z) \mathbf{W}_N$. The delay matrices $\hat{\Lambda}_0^o(z) = \text{diag}(\mathbf{I}_{n+1}, z^{-1} \mathbf{I}_n)$ and $\hat{\Lambda}_1^o(z) = \text{diag}(\mathbf{I}_n, z^{-1} \mathbf{I}_{n+1})$. Such second-order form is viable and consistent with our existence conditions due to the even

5.4.3 Odd Channel Systems

In this section, we study the odd channel systems still based on the hybrid lattice structure (5.16). However, note that most previous works only concentrated on the even-channel FBs, whereas we show that (5.16) can also be easily applied to odd-channel systems in the same manner. From the existence conditions in Sec. 5.4.1, we can see that the constraint for the number of long filters N is same for the even or odd values of β , only dependent on δ_K or K_s . Thus, we investigate the different cases in the following subsections, respectively.

5.4.3.1 Odd δ_K and even N

According to the existence conditions presented in Sec. 5.4.1, we know that the number of symmetric long filters $n_{l,s} = N/2 = n$ and the number of short symmetric filters $n_{s,s} = (M - N + 1)/2$. Thus, the new building block $\hat{\mathbf{G}}_i^{\text{LP}}(z)$ can have such form,

$$\hat{\mathbf{G}}_i^{\text{LP}}(z) = \text{diag}(\hat{\Phi}_i \mathbf{W}_N \hat{\Lambda}_i(z) \mathbf{W}_N, \mathbf{I}_{M-N}) \quad (5.19)$$

where $\hat{\Phi}_i = \text{diag}(\mathbf{U}_i, \mathbf{I}_n)$ in which $\mathbf{U}_i$ is an $n \times n$ nonsingular matrix and $\hat{\Lambda}_i(z) = \text{diag}(\mathbf{I}_n, z^{-1} \mathbf{I}_n)$. Here, the top-left corner of the diagonal matrix $\hat{\mathbf{G}}_i^{\text{LP}}(z)$ has the same structural form as that of $\mathbf{G}_i(z)$ in section 4.3, but reduced dimension N . The invertibility of the block $\hat{\mathbf{G}}_i^{\text{LP}}(z)$ is guaranteed by the $n \times n$ nonsingular matrix $\mathbf{U}_i$.

5.4.3.2 Even δ_K

In this case, we know that N could be chosen freely according to Table 5.1. If N is even, the block in (5.17) is still applicable. However, if N is odd, i.e., $N = 2n + 1$, the block in (5.17) fails. For odd N , we propose the following form,

$$\hat{\mathbf{G}}_i^{\text{LP}}(z) = \text{diag}(\hat{\Phi}_{i,0} \mathbf{W}_0^o(z) \hat{\Phi}_{i,1} \mathbf{W}_1^o(z), \mathbf{I}_{M-N}) \quad (5.20)$$

where $\hat{\Phi}_{i,0} = \text{diag}(\mathbf{U}_{i,0}, \mathbf{I}_n)$, $\hat{\Phi}_{i,1} = \text{diag}(\mathbf{I}_{n+1}, \mathbf{V}_{i,1})$, $\mathbf{W}_0^o(z) \triangleq \mathbf{W}_N \hat{\Lambda}_0^o(z) \mathbf{W}_N$ and $\mathbf{W}_1^o(z) \triangleq \mathbf{W}_N \hat{\Lambda}_1^o(z) \mathbf{W}_N$. The delay matrices $\hat{\Lambda}_0^o(z) = \text{diag}(\mathbf{I}_{n+1}, z^{-1} \mathbf{I}_n)$ and $\hat{\Lambda}_1^o(z) = \text{diag}(\mathbf{I}_n, z^{-1} \mathbf{I}_{n+1})$. The invertibility is ensured as long as the $(n+1) \times (n+1)$ matrix $\mathbf{U}_{i,0}$ and $n \times n$ matrix $\mathbf{V}_{i,1}$ are nonsingular.

5.4.3.3 Odd δ_K and odd N : failure of the hybrid lattice structure

In this case, the filter length $L_s = K_s M + \beta$ must be even due to odd number of channels. Thus, if we want to use the lattice structure of (5.16) to generate such FBs, an odd channel EL LPPRFB $\mathbf{E}_{K_s}(z)$ is needed with even length $L_s = K_s M + \beta$. However, from Sec. 4.2,

we know that such FBs do not exist due to odd β and even δ_K (here $\delta_K = 0$). Therefore, we can conclude that the lattice structure of (5.16) can never be employed to generate even channel LPPRFBs with odd length. However, note that such FBs *do* exist. One example is the 3-channel LPPRFB reported in [90], corresponding to our case of $\beta = 2$, $K_s = 14$ and $K_l = 17$.

5.4.4 Minimality of the Proposed Structure

Theorem 5.10 *The proposed lattice structure in (5.16) is minimal in terms of the number of delays used in implementation.*

Proof: A lattice structure is said to be minimal if the number of delays used is equal to the McMillan degree of the transfer function [5]. For the class of FBs described above, it can be proven [64] that the McMillan degree of a FB is just the order of the determinant of the polyphase matrix, i.e., $\deg(\mathbf{E}(z)) = \deg(|\mathbf{E}(z)|)$. By the LP condition (5.5), we have

$$\deg(|\mathbf{E}(z)|) = \deg\left(z^{-M(K_s-1)}|\hat{\mathbf{Z}}(z)| \times |\mathbf{E}(z^{-1})| \times |\mathbf{J}_M(z)|\right) \quad (5.21)$$

where the signature matrix $\mathbf{D}$ has no effects on degree due to their constant determinant. Then, from (5.21) and the equality $\deg(\mathbf{E}(z^{-1})) = -\deg(|\mathbf{E}(z)|) = -\deg(\mathbf{E}(z))$, we obtain,

$$\deg(\mathbf{E}(z)) = M(K_s - 1) + N(K_l - K_s) - \deg(\mathbf{E}(z)) + \beta$$

which means $\deg(\mathbf{E}(z)) = M(K_s - 1)/2 + N(K_l - K_s)/2 + \beta/2$.

In our lattice structure, the equal-length LPPRFB $\mathbf{E}_{K_s}(z)$ is factorized in the same lattice structure presented in Sec. 4.3, which has been shown to be minimal in Sec. 4.4.1, i.e., employing $M(K_s - 1)/2 + \beta/2$ delays. The other $K_l - K_s$ blocks $\hat{\mathbf{G}}_i(z)$ employ $N/2$ delays each for the case of even N seen in (5.17) or (5.19). However, the $(K_l - K_s)/2$ blocks $\hat{\mathbf{G}}_i(z)$ employ N delays each for the case of odd N seen in (5.18) or (5.20). Therefore, the total number of delays in use is $M(K_s - 1)/2 + N(K_l - K_s)/2 + \beta/2$ for both even and odd N , which is just the McMillan degree of the transfer function $\mathbf{E}(z)$. ■

5.5 Design and Application Examples

We show several design examples and their applications to low bit rate image compression in this section. First, the degrees of freedom for design are shown in Sec. 5.5.1. Then, two unequal-length LPPRFBs are designed by using the proposed lattice structures for different cases. These design examples are further evaluated in transform coding based image compression and compared with some existing FBs.

CHAPTER 5. UNEQUAL-LENGTH MULTICHANNEL FILTER BANKS

Table 5.2: Degrees of freedom for the lattice factorizations of M -channel unequal-length PRFBs and PUFBs with length profile (5.1).

DOF	Complex Case	Real Case
PUFBs	$D_{L_s}^{\text{PU},c} + \sum_{j=1}^{\delta_K} 2r_j(N - r_j)$	$D_{L_s}^{\text{PU},r} + \sum_{j=1}^{\delta_K} r_j(M - r_j)$
PRFBs	$D_{L_s}^{\text{PR},c} + \sum_{j=1}^{\delta_K} 4r_j(N - r_j)$	$D_{L_s}^{\text{PR},r} + \sum_{j=1}^{\delta_K} 2r_j(N - r_j)$

5.5.1 Degrees of Freedom for Filter Bank Design

Based on the lattice factorization (5.4) developed in Sec. 5.3 for unequal-length PR/PU FBs, the degrees of freedom for designing such FBs are summarized in Table 5.2 for various cases. It shows the number of free parameters within the lattice factorization for the analysis bank $\mathbf{E}(z)$. The symbol $D_{L_s}^{\text{PU},c}$ shows the DOF of the PUFB with equal-length $L_s = K_s M + \beta$ and complex-valued coefficients, which is given in Table 3.1 of Chapter 3. The other symbols in Table 5.2 denote the similar meaning.

Next, the degrees of freedom for designing unequal-length LPPRFBs and LPPUFBs are summarized in Table 5.2 for various cases factorized by our proposed structures presented in both Sec. 5.4.2 and Sec. 5.4.3. The symbol $D_{L_s}^{\text{LPPU}}$ and $D_{L_s}^{\text{LPPR}}$ denote the DOF of the LPPUFB and LPPRFB with equal-length $L_s = K_s M + \beta$, respectively, which are given in Table 4.2 of Chapter 4.

Table 5.3: Degrees of freedom for the lattice factorizations of M -channel unequal-length LPPRFBs and LPPUFBs with length profile (5.1).

	LPPUFBs	LPPRFBs
DOF	$D_{L_s}^{\text{LPPU}} + \frac{\delta_K}{2} \left(\left\lceil \frac{N}{2} \right\rceil \left(\left\lceil \frac{N}{2} \right\rceil - 1 \right) + \left\lfloor \frac{N}{2} \right\rfloor \left(\left\lfloor \frac{N}{2} \right\rfloor - 1 \right) \right) \frac{1}{2}$	$D_{L_s}^{\text{LPPR}} + \frac{\delta_K}{2} \left(\left\lceil \frac{N}{2} \right\rceil^2 + \left\lfloor \frac{N}{2} \right\rfloor^2 \right)$

5.5.2 Filter Bank Design Examples

In this section, two unequal-length LPPRFBs are constructed by using the proposed lattice structures and are compared with other equal-length LPPRFBs. For both cases of unequal-length FBs, the free invertible matrices in lattice structures are decomposed by SVD through orthogonal and diagonal matrices. The invertibility of square matrices is ensured as long as the diagonal elements are nonzero. Following traditional transform coding terminology, equal-length LPPRFB is also called GLBT. In contrast to it, unequal-length LPPRFB is named as UL-GLBT in this dissertation. However, for most natural images, an odd channel FB cannot achieve the nonexpansive symmetric extension [24] transform due to the evenness of image size. Thus, for practical purposes, only even channel LPPRFBs with real-valued coefficients

CHAPTER 5. UNEQUAL-LENGTH MULTICHANNEL FILTER BANKS

are considered in this section. Besides the classic optimization criterion of filter shape, coding gain is also chosen as the optimization criterion in the following design examples, which is an approximate measure of the FB energy compaction capability and correlates most consistently with higher objective performance (PSNR).

5.3.a: Magnitude response of the analysis bank

5.3.b: Magnitude response of the synthesis bank

5.3.c: The impulse responses of synthesis filters

Figure 5.3: A design example of 8-channel unequal-length LPPRFB with 3 long and 5 short filters of $L_{av} = 14$.

A design example of 8-channel LPPRFB with 3 long filters of length 24 and 5 short filters of length of 8, i.e., $N = 3$, $K_l = 3$ and $K_s = 1$, optimized for coding gain and stopband attenuation for the AR(1) process with correlation coefficient $\rho = 0.95$ is shown in Fig. 5.3. The unconstrained optimization procedure used for this design example is the simplex search

function *fminunc* available in MATLAB Optimization Toolbox, since the proposed lattice factorizations can structurally enforce the desired properties and filter length profile L_i . The magnitude responses of the analysis bank and the corresponding synthesis bank are shown in Fig. 5.3.a and Fig. 5.3.b, respectively. The impulse responses of corresponding synthesis filters are shown in Fig. 5.3.c in the wavy lines with filled circles, which clearly demonstrates the unequal lengths of the FB. Note that the averaged filter length L_{av} is just 14 although the long filters are of length 24. It also should be pointed that the computational complexity of such design example is the same as the equal-length LPPRFBs of same length $L = 14$ by the polyphase implementation, which is clearly presented in Sec. 4.7.2 of Chapter 4. This example is also denoted by UL-GLBT35 under the terminology of the lapped transform, with the two numbers denoting the number of long filters and short filters, respectively.

Another example of 8-channel unequal-length LPPRFB with 2 long filters of length 26 and 6 short filters of length 10, i.e., $N = 2$, $K_l = 3$, $K_s = 1$ and $\beta = 2$, optimized for both coding gain and stopband attenuation is shown in Fig. 5.4. The magnitude responses of the analysis bank and the corresponding synthesis bank are shown in Fig. 5.4.a and Fig. 5.4.b, respectively. The impulse responses of synthesis filters are also shown in Fig. 5.4.c to demonstrate the unequal lengths of this FB. Note that the averaged filter length L_{av} is also 14 although the long filters are of length 26. In addition, similar to the previous example, very strong attenuation is guaranteed at the zero frequency for all the bandpass and highpass filters in the magnitude responses of both analysis and synthesis banks, due to the structural enforcement of one-degree of regularity for efficient image compression. This example is also denoted by UL-GLBT26 for the reference in next section.

5.5.3 Effect of Unequal Filter Lengths on Image Compression

Although the coding gain formula in (2.10) is usually employed to design FBs, it is not informative enough for comparing different FBs at *low bit rate* region due to the unconstrained optimal bit allocation in its derivation. In practical encoders, the minimal bits for subband signals can only be zero in contrast to possibly negative bits in *unconstrained* optimal bit allocation. Therefore, a generalized transform coding gain, denoted by G_{TC} introduced by Malvar [105] is employed as a objective function for comparing different FBs at different bit rates in terms of their energy compaction efficiency.

We have computed G_{TC} at different bit rates for 8-channel DCT, LOT [4], two GLBTs, the UL-GLBT of our design example with 3 long filters (cf. UL-GLBT35), and another designed UL-GLBT with same lengths but with only 2 long filters denoted by UL-GLBT26 reported before [57,60]. The results are shown in Fig. 5.5 in terms of the improvement of G_{TC} of various

CHAPTER 5. UNEQUAL-LENGTH MULTICHANNEL FILTER BANKS

5.4.a: Magnitude response of the analysis bank

5.4.b: Magnitude response of the synthesis bank

5.4.c: The impulse responses of synthesis filters

Figure 5.4: A design example of 8-channel unequal-length LPPRFB with 2 long and 6 short filters of $L_{av} = 14$.

transforms over DCT, i.e., $\Delta G_{TC} = G_{TC}^{GLBT} - G_{TC}^{DCT}$, for different bit rates. Thus, the horizontal axis is just for the DCT which is set as a baseline for the improvements of various well-designed transforms. For UL-GLBTs, the averaged length $L_{av} = \sum_{i=0}^{M-1} L_i / M$ is also shown in the legend for the purpose of comparison. It can be seen that at medium and low bit rate region, i.e., rate less than 0.5 bit per sample (bpp), the UL-GLBT is more efficient than equal-length GLBT with the same averaged length. For example, at 0.3 bit per sample, the UL-GLBT35 and the equal-length GLBT with the same averaged length of 14 have G_{TC} of 1.11 dB and 0.83 dB, respectively. Similarly, the UL-GLBT26 and the GLBT with the same averaged length of 12 have G_{TC} of 0.99 dB and 0.76 dB, respectively. Similar phenomenon is shown obviously in

CHAPTER 5. UNEQUAL-LENGTH MULTICHANNEL FILTER BANKS

Figure 5.5: Improvement in generalized transform coding gain over DCT as a function of bit rate for several FBs.

the low bit rate region, i.e., rate less or equal to 0.25 bpp, the UL-GLBTs can achieve about 0.3 ~ 0.4 dB higher G_{TC} than that of the GLBTs with same averaged lengths.

Next, we show the comparison of real image compression at low bit rates based on our designed GLBTs and UL-GLBTs but same averaged length, in order to illustrate the length effect on the image compression. Same as Chapter 4, we follow the popular transform-based image coding setting by fixing the encoding algorithm and only modify the transform stage of both encoder and decoder. The testing images in our experiment are the popular 512×512 8-bit gray scale images *Peppers*, *Baboon* and *Lena* which represent typical images with very smooth regions, plentiful textures and intermediate structures, respectively.

The reconstructed images for *Peppers* are shown in Fig. 5.6.a for GLBT of length 14 with PSNR 26.25 dB and in Fig. 5.6.b for UL-GLBT of same averaged length with improved PSNR 27.02 dB, respectively. It can be clearly seen that UL-GLBT can greatly reduce the blocking artifacts clearly visible in Fig. 5.6.a.

A similar phenomenon is shown in the reconstructed *Lena* images in Fig. 5.7.a for GLBT of length 12 with PSNR 28.30 dB and in Fig. 5.7.b for UL-GLBT of same averaged length

5.6.a: Reconstructed image by GLBT

5.6.b: Reconstructed image by UL-GLBT35

Figure 5.6: Comparison of reconstructed images by different transforms at bit rate of 0.1 bpp with the same averaged length of 14.

5.7.a: Reconstructed image by GLBT

5.7.b: Reconstructed image by UL-GLBT26

Figure 5.7: Comparison of reconstructed images by different transforms at bit rate 0.1 bpp with the same averaged length of 12.

5.8.a: Reconstructed image by GLBT

5.8.b: Reconstructed image by UL-GLBT35

Figure 5.8: Comparison of reconstructed images by different transforms at bit rate 0.125 bpp with the same averaged length of 14.

with approximately same PSNR 28.33 dB. However, the reconstructed image by UL-GLBT is smoother than the other one, especially on the perceptually important face area, which leads to better perceptual performance. In addition, the reconstructed images for Baboon are shown in Fig. 5.8.a for GLBT of length 14 with PSNR 21.06 dB and in Fig. 5.8.b for UL-GLBT of same averaged length with approximately same PSNR 21.03 dB, respectively. The blocking artifacts on the nose area by UL-GLBT is less visible and the beards around the mouth are reconstructed better than those by GLBT since the ringing noise is specifically limited by the short filters.

5.6 Summary

We have presented the existence conditions, various lattice structures and designs for a class of FBs with much more flexible unequal lengths. The desired properties like PR, LP and regularity were structurally imposed into the proposed modular lattice structures, which leads to FB design based on unconstrained optimization. Our derived existence conditions revealed that it is possible to design unequal-length LPPRFBs with odd number of long filters, which corrects a misleading statement reported before. Various lattice structures according to the even and odd values of M and β have been developed and the structures for LPPRFBs have

CHAPTER 5. UNEQUAL-LENGTH MULTICHANNEL FILTER BANKS

been shown to be minimal. In addition, we also discuss the regular lattice structures which are desired for high-performance image compression. Furthermore, we discussed the limitation of the hybrid lattice structures for some class of LPPRFBs and pointed out some other solutions for those FBs. In contrast to very limited possible designs by most traditional methods, our design approaches are more general and can considerably expand the designer's choices. We have also shown several design examples to validate the proposed design approaches. The comparison of different FBs in terms of GTC and real image compression at low bit rates demonstrated the better performance of unequal-length FBs.

Part II

Frames and Oversampled Filter Banks

Chapter 6

Fundamentals of Frame Theory and Oversampled Filter Banks

This chapter presents the fundamentals of frame theory and oversampled filter banks as well as some earlier foundation works. The coverage here is very brief and basic. For further details and applications, the interested reader is suggested to refer to several books and tutorial papers [14, 15, 28, 106–112]. The presentation in this chapter is mainly based on the treatments given in [14, 15, 106, 108, 113]. To be rigorous, we will always use the linear operator theory [13, 114] and its terminology in this chapter because we assume the underlying signals belong to the infinite dimensional Hilbert space $\ell^2(\mathbb{Z})$ [13]. However, in the simpler case of finite dimensional space $\mathbb{R}^M$ or $\mathbb{C}^M$, the linear operator degenerates to a finite dimensional matrix [107]. All the analysis and computations of linear operators become the conventional matrix analysis and computations, which makes it easily understandable for the interested reader without the background of linear operator theory.

6.1 Overcomplete Signal Representations and Frames

As discussed in Chapter 1, linear signal expansion is found in many applications in signal processing area. The basis functions usually constitute a nonredundant set. It is well known that the use of redundancy in communication systems improves robustness and numerical stability. The channel coding over Galois field is such an example to add the structural redundancy. Motivated by this observation, the study of redundant signal expansions and representations over real or complex number field, i.e., overcomplete signal representations, has been studied recently and found widespread uses in many different engineering disciplines. Recent examples include nonuniform sampling theory [115], A/D conversion, joint source channel coding [116], oversampled filter banks [117, 118], multiple description coding [12, 19], error correcting codes

[119], image denoising and segmentation [120–122], detection and estimation, space-time coding for wireless communications [109], etc.

In this dissertation, we will study the frame expansion [15, 16] and its potential uses for robust transmission of information, especially transmission over noisy or erasure channels. The frame is competent in this application due to its more flexible representation than traditional orthonormal or biorthogonal representations [14]. This section aims to present an introduction and brief review of overcomplete signal representations and frames.

6.1.1 Overcomplete Signal Representations

Traditionally, bases, especially the orthonormal basis, play an important role in signal processing area because of their compactness for signal representation and of the availability of various fast algorithms for calculation. Orthogonal transform coding for signal compression is such an example. The basis is complete in the signal space $\ell^2(\mathbb{Z})$ in that every signal belonging to $\ell^2(\mathbb{Z})$ can be decomposed by the basis. Another beautiful property of basis desirable for some applications is the *unique representation*. For every signal that belongs to $\ell^2(\mathbb{Z})$, one and only one linear expansion of the signal exists, which means the expansion coefficients are unique. However, unique signal expansion imposes *strong constraints* on bases which may be in conflict with other desirable specifications [117]. For example, nontrivial two-channel orthogonal filter bank excludes the possibility of linear phase filters, which is very desirable for image coding applications. In addition, the conditions for being a basis, referred in (1.1) and (1.2), are so strong that even a slight modification of the basis vectors might destroy the basis property. Thus, extension of basis expansions, i.e., complete and unique signal expansions, to more general and flexible signal decomposition method is expected. This motivates us to study the overcomplete linear signal expansions. We generally suppress the word “linear” because we confine the expansion form to be linear expansion. *Overcomplete* means that the set of vectors or functions used for expansion is larger than that necessarily needed. In other words, although the vectors or functions behave like basis vectors or functions, they are actually linearly dependent in contrast to independent basis vectors or functions.

There are several types of overcomplete signal representations. Originally, Gabor studied the problem of information transmission by series expansion, namely Gabor expansion [123], which is the first overcomplete signal expansion in the engineering literature. He employed the time shifting and frequency modulation to construct an overcomplete signal expansion to replace the common Fourier transform which is a complete signal expansion, but has an undesirable time-frequency property. In addition, the non-decimated wavelet generated by the non-decimated FB is another form of overcomplete signal representation [117]. This overcomplete system has a

desirable property, namely shift-invariance in time. Mallat and Zhang developed another highly overcomplete signal representation called matching pursuit for signal compression [124]. Their system is designed for sparseness of signal representations, or more efficient signal compression.

6.1.2 Frames

Frames were first introduced by Duffin and Schaeffer [16] in the context of nonharmonic Fourier series. Outside of harmonic analysis area, frames did not seem to initiate much interest until the breakthrough work by Daubechies, Grossman and Meyer [125] for studying wavelets. Since then, the theory of frames, as a subclass of overcomplete signal representations, began to be more widely studied in the areas of applied and computational mathematics such as approximation theory and wavelet analysis, and has found wide applications in signal and image processing. We first give the definition of frames in the context of Hilbert space below.

Definition 6.4 *A family of functions $\{\varphi_j\}_{j \in \mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is called a frame if there exists constants $A > 0, B < \infty$ so that*

$$A\|f\|^2 \leq \sum_{j \in \mathbb{Z}} |\langle f, \varphi_j \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H} \quad (6.1)$$

The constants A and B are called the lower and upper frame bounds, respectively. In fact, they are not unique. If a frame ceases to be a frame when an arbitrary function is removed from the function set $\{\varphi_j\}_{j \in \mathbb{Z}}$, it is called an *exact* frame. If the two frame bounds are equal, $A = B$, then the frame is called *tight* frame. In a tight frame, we have, for all $f \in \mathcal{H}$,

$$\sum_{j \in \mathbb{Z}} |\langle f, \varphi_j \rangle|^2 = A\|f\|^2$$

Thus, by the polarization identity [15], we can get,

$$f = \frac{1}{A} \sum_{j \in \mathbb{Z}} \langle f, \varphi_j \rangle \varphi_j \quad (6.2)$$

It can be seen easily that (6.2) is very similar to the orthonormal expansion of f . But it is very important to note that frames, even tight frames, generally are *not* orthonormal bases, which is illustrated by the example in Chapter 1. Actually, the orthonormal basis is just a special class of frames, explained by the following theorem.

Theorem 6.11 (cf. [106]) *If $\{\varphi_j\}_{j \in \mathbb{Z}}$ is a tight frame with frame bound $A = 1$ and all $\|\varphi_j\|^2 = 1$, then the frame degenerates to an orthonormal basis.*

In the conventional basis, the reconstruction from expansion coefficients is fulfilled by a dual set of functions, i.e. dual basis. The dual basis is related to the original basis by orthogonal or biorthogonal condition. For tight frames, (6.2) gives a way to reconstruct original signal f from the frame expansion coefficients $\langle f, \varphi_j \rangle$. However, for general frames, we need to use the *dual* frame, defined below, for perfect reconstruction from frame expansion coefficients.

Definition 6.5 For a given frame $\{\varphi_j[n]\}_{j \in \mathbb{Z}}$, any frame $\{\gamma_j[n]\}_{j \in \mathbb{Z}}$ satisfying the condition

$$f = \sum_{j \in \mathbb{Z}} \langle f, \gamma_j \rangle \varphi_j[n], \quad \forall f \in \mathcal{H} \quad (6.3)$$

is called a *dual frame*.

The existence of dual frame is shown in [14, Lemma 5.6.1].

6.1.3 Nonunique Dual Frames and Various Frame Operators

Following the classical transform coding setup, we firstly introduce the frame analysis operator which fulfills the linear signal expansion similar to the functionality of the orthonormal or biorthogonal transform.

Definition 6.6 The linear operator T is defined as the frame analysis operator that maps the Hilbert space $\mathcal{H}$ into the space $\ell^2(\mathbb{Z})$, $T : \mathcal{H} \mapsto \ell^2(\mathbb{Z})$. It assigns each signal $x[n] \in \ell^2(\mathbb{Z})$ to the sequence of inner product $\langle x, \varphi_j \rangle$, i.e.

$$T : x \mapsto \langle x, \varphi_j \rangle, \quad \text{or equivalently, } (Tx)_j = \langle x, \varphi_j \rangle$$

For numerically stable reconstruction from frame expansion coefficients, the frame analysis operator T should satisfy some constraints [113]. First, T should be bounded. Secondly, T should have an inverse, say, *left inverse* T^{-1} . Finally, the left inverse should also be bounded. It can be shown [15] that the frame condition (6.1) ensures a numerically stable reconstruction from the frame expansion coefficients, i.e., it imposes the above desirable constraints on the frame analysis operator T by (6.1). *This reveals the importance of the frame condition.* Furthermore, it can be easily seen that the frame condition is less restrictive than that of traditional orthogonal or biorthogonal basis. Accordingly, a frame can offer some advantages over traditional bases, such as more degrees of freedom for design.

Next, we present the synthesis operator T^* which is a mapping from the expansion coefficients to a signal in original Hilbert signal space.

Definition 6.7 The linear operator T^* is defined as the frame synthesis operator that maps the expansion coefficient space $\ell^2(\mathbb{Z})$ into the original Hilbert space $\mathcal{H}$, $T^* : \ell^2(\mathbb{Z}) \mapsto \mathcal{H}$. It assigns each sequence $\{c_j\} \in \ell^2(\mathbb{Z})$ to a signal $\hat{x}[n]$, i.e.

$$T^* : \{c_j\} \mapsto \hat{x}[n], \text{ or equivalently, } (T^*c_j)[n] = \sum_j c_j \varphi_j[n]$$

It can be shown [113] that the synthesis operator T^* is actually the adjoint operator of frame analysis operator T . It should be noted that the synthesis operator is *not* the conventional inverse transform which can perform perfect reconstruction with exact transform coefficients. After the introduction of frame operator, we will discuss how to implement perfect reconstruction using another operator.

Note that for orthogonal or biorthogonal basis, the dual basis which can provide perfect reconstruction is unique. However, for general frames, the dual frame is *not unique*. There are potentially infinite dual frames satisfying the dual frame condition (6.3). In addition, we have seen how the reconstruction is fulfilled by the dual frame as shown in (6.3), which is very similar to the conventional signal decomposition and synthesis by basis. One particular dual frame, named canonical dual frame, is very important in characterizing all dual frames [14]. The canonical dual frame is related to the original frame by the *frame operator* introduced below.

Definition 6.8 For a given frame $\{\varphi_j[n]\}_{j \in \mathbb{Z}}$, the frame operator $S : \mathcal{H} \mapsto \mathcal{H}$ is defined as

$$S : x[n] \mapsto \sum_{j \in \mathbb{Z}} \langle x, \varphi_j \rangle \varphi_j[n], \text{ or equivalently, } S = T^*T$$

Note that the frame condition (6.1) can be equivalently stated as $A\|f\|^2 \leq \langle Sf, f \rangle \leq B\|f\|^2$ for all $f \in \mathcal{H}$ or equivalently in the form of operator inequalities [15], $A Id \leq S \leq B Id$, where Id is the identity operator on $\mathcal{H}$. It can be easily seen, $S = A Id$ for tight frames. The following theorem states the important properties of the frame operator.

Theorem 6.12 (cf. [14, 15, 106]) For a given frame $\{\varphi_j[n]\}_{j \in \mathbb{Z}}$, the frame operator S is linear, positive-definite, invertible and $A Id \leq S \leq B Id$. Furthermore, $\{(S^{-1}\varphi_j)[n]\}_{j \in \mathbb{Z}}$ is also a frame with bounds B^{-1} and A^{-1} , i.e., $B^{-1}Id \leq S^{-1} \leq A^{-1}Id$.

The new frame generated by the inverse of frame operator S is the *canonical dual frame* denoted as $\{\tilde{\varphi}_j[n]\}_{j \in \mathbb{Z}}$. The associated frame analysis operator $\tilde{T}$ for the canonical dual frame is the mapping,

$$\tilde{T} : x \mapsto \langle x, \tilde{\varphi}_j \rangle, \text{ or equivalently, } (\tilde{T}x)_j = \langle x, \tilde{\varphi}_j \rangle$$

It can be shown [15], the adjoint operator $\tilde{T}^*$ for the canonical dual frame can implement the perfect reconstruction, i.e., $\tilde{T}^*T = Id$. Specifically, for tight frames, $S^{-1} = A^{-1}Id$. Thus, the canonical dual frame becomes $\tilde{\varphi}_j = S^{-1}\varphi_j = \frac{1}{A}\varphi_j$, for all $j \in \mathbb{Z}$. This is very similar to the orthonormal basis except the scale factor A . However, note that the dual frame is not unique in contrast to orthogonal basis. There are potentially infinite dual frames. The canonical dual frame is important in characterizing all dual frames [14] as shown in the next theorem.

Theorem 6.13 (cf. [14]) *For a given frame $\{\varphi_j[n]\}_{j \in \mathbb{Z}}$ in $\mathcal{H}$, all the dual frames can be characterized by*

$$\{\gamma_j[n]\}_{j \in \mathbb{Z}} = \{(S^{-1}\varphi_j)[n] + h_j[n] - \sum_{k \in \mathbb{Z}} \langle S^{-1}\varphi_j, \varphi_k \rangle h_k[n]\}_{j \in \mathbb{Z}}$$

where $\{h_k\}_{k \in \mathbb{Z}}$ is a free Bessel sequence in $\mathcal{H}$.

The additional free Bessel sequence increases the degree of freedom for designing dual frames in contrast to the unique dual basis.

6.2 Filter Bank Frames

Filter banks actually implement a class of discrete-time signal expansions [6], known as shift-invariant expansions. The relationship between critically sampled FBs and signal expansions has been investigated in [5–7]. Recently, it has been recognized that oversampled filter banks (OFBs) correspond to a class of frame expansions in infinite dimensional space $\ell^2(\mathbb{Z})$. The interests for filter bank frames are due to their wide applications to signal processing and their simple structures and efficient implementations. Furthermore, they can offer some desired advantages over traditional critically sampled FBs. First, more degrees of design freedom are provided because the synthesis FB yielding PR is not unique. Second, more residual redundancy is left in the subband signals, which can be exploited for error-resilient coding such as multiple description coding. Third, it results in an improved noise resistance and noise shaping capability [126, 127]. All these advantages come at the expense of an increased number of subband samples than that of critically sampled FBs. In this section, we briefly review the foundation works on OFBs reported in [117, 118] by using the theory of frames.

6.2.1 Oversampled Filter Banks

We consider a P -channel oversampled filter bank with downsampling factor M in each channel shown in Fig. 6.1.a, where the transfer functions of analysis and synthesis filters are denoted by $H_k(z)$ and $F_k(z)$ ($k = 0, 1, \dots, P-1$), respectively, and the corresponding impulse responses are

Figure 6.1: A P -channel FB with subsampling factor M .

denoted by $h_k[n]$ and $f_k[n]$, respectively. In a critically sampled FB, the number of channels P is equal to the subsampling factor M . Thus, the subband signals $u_k[n]$ with $k = 0, 1, \dots, P-1$ contain exactly the same number of samples (per unit time) as the input signal $x[n]$. However, in the case of OFBs, we have $P > M$. This means the subband signals $u_k[n]$ are redundant because they have more samples (per unit time) than the input signal $x[n]$. When $P = M$, OFBs degenerate to traditional critically sampled FBs, i.e., the critically sampled FBs are a special class of OFBs. Thus, the theory developed for OFBs can also be applied to critically sampled FBs, which shows the theoretical significance of OFBs.

Similar to the polyphase representation of the critically sampled FBs as presented in Sec. 2.1.2 of Chapter 2, the analysis filters $H_k[z]$ can be decomposed in the same way as before, i.e., $H_k(z) = \sum_{j=0}^{M-1} z^{-j} E_{k,j}(z^M)$, where $E_{k,j}(z^M) = \sum_{m=-\infty}^{\infty} h_k[mM+j]z^{-m}$ is the j th polyphase component of the k th analysis filter $H_k(z)$. Then, the analysis polyphase matrix $\mathbf{E}(z) = [E_{i,j}(z)]$ with $i = 0, 1, \dots, P-1$ and $j = 0, 1, \dots, M-1$, is a $P \times M$ rectangular polynomial matrix. Similarly, the synthesis polyphase matrix $\mathbf{R}(z) = [R_{i,j}(z)]$ with $i = 0, 1, \dots, P-1$ and $j = 0, 1, \dots, M-1$, is an $M \times P$ rectangular polynomial matrix, where $R_{i,j}(z) = \sum_{m=-\infty}^{\infty} f_i[mM-j]z^{-m}$. The corresponding polyphase representation of the OFB is shown in Fig. 6.1.b.

Besides the polyphase-domain analysis, the time-domain analysis of FBs is insightful in

the context of signal decompositions or signal expansions [6]. In addition, we can find a close relationship between OFBs and frames. Important links have been established in [117, 118] between the analysis filters $h_k[n]$ and the frame vector set $\{h_{k,m}\}$ as discussed in the next section. To be explicit, the subband signals $u_k[m]$ (cf. Fig. 6.1.a) are given by

$$u_k[m] = \sum_{n=-\infty}^{\infty} x[n]h_k[mM - n], \quad k = 0, 1, \dots, P-1 \quad (6.4)$$

The reconstructed signal is

$$\hat{x}[n] = \sum_{k=0}^{P-1} \sum_{m=-\infty}^{\infty} v_k[m]f_k[n - mM] \quad (6.5)$$

Generally, the subband signals $u_k[m]$ will be processed given some desired applications, such as quantization for signal compression and thresholding for signal denoising. Thus, the processed subband signals $v_k[m]$ are not equal to $u_k[m]$. However, the processing is dependent on specific applications. As far as the theoretical study for OFBs and their relation to frame expansions are concerned, we assume $v_k[m] = u_k[m]$ for all P channels, which is extensively used for studying FBs. Therefore, we only need to use $v_k[m]$ to denote the subband signals in future.

6.2.2 Relationship with Frame Expansions

From the perspective of frame theory, we can view the subband signal as the inner product of the input signal $x[n]$ and the vector set $\{h_{k,m}\}$, if we put all discrete-time signals into the vector space framework such as $\ell^2(\mathbb{Z})$ space [117, 118]. Thus, the subband signals $v_k[m]$ can be written as

$$v_k[m] = \langle x, h_{k,m} \rangle, \quad \text{with } h_{k,m}[n] = h_k^*[mM - n] \quad (6.6)$$

where $*$ denotes complex conjugate operator. The symbols x and $h_{k,m}$ denote the corresponding signal vector and filter vectors in the $\ell^2(\mathbb{Z})$ space, respectively, thus suppressing the time variable n . Then, according to the PR property and (6.5), we can obtain

$$x[n] = \hat{x}[n] = \sum_{k=0}^{P-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle f_{k,m}[n], \quad \text{with } f_{k,m}[n] = f_k[n - mM] \quad (6.7)$$

This reveals the signal decompositions in $\ell^2(\mathbb{Z})$ inherent from the FBs. These are expansions underlying signal analysis through a sliding window using a selected set of elementary waveforms [117]. Generally, the set $\{f_{k,m}[n]\}$ is not orthogonal. Thus the subband signals $v_k[m] =$

CHAPTER 6. FUNDAMENTALS OF FRAME THEORY AND OVERSAMPLED FILTER BANKS

$\langle x, h_{k,m} \rangle$, i.e., the expansion coefficients, are obtained by projecting the signal $x[n]$ onto a “dual” set of functions $\{h_{k,m}[n]\}$.

Following the presentation of frames in Sec. 6.1.2, the relationship between OFBs and frames in $\ell^2(\mathbb{Z})$ can be established as follows [113]. The frame vector set $\{\gamma_j[n], j \in \mathcal{J}\}$ defined in Sec. 6.1.2 is now replaced by the set of analysis filter impulse responses and their translations $\{h_{k,m}[n]\}$, where the single index of $j \in \mathcal{J}$ is replaced by the double index $\{k, m \mid k = 0, 1, \dots, P-1, -\infty < m < \infty\}$. In an OFB, the frame analysis operator T defined in Sec. 6.1.2 corresponds to the mapping from input signal $x[n] \in \ell^2(\mathbb{Z})$ to the subband signals $v_k[m]$, $k = 0, 1, \dots, P-1$ which are located in the subband signal space $[\ell^2(\mathbb{Z})]^P$, i.e., $T : x \rightarrow v_k[m] = \langle x, h_{k,m} \rangle$.

From the FB point of view, we would like to impose some constraints on $h_{k,m}[n]$, or equivalently, the frame analysis operator T [113]. First, the FB should satisfy the PR property. This means the synthesis FB exists given an analysis FB. This requirement actually is equivalent to the left invertibility of the frame analysis operator T . Thus, the input signal $x[n]$ can be reconstructed from the expansion coefficients $\langle x, h_{k,m} \rangle$. Second, the BIBO stability is desirable for the FB, which means the subband signals should have finite energy if the input signal has finite energy, i.e., belonging to $\ell^2(\mathbb{Z})$ space. This requirement constrains the frame analysis operator T to be bounded, hence continuous [114]. The third constraint is the numerical stability in the sense that small perturbations of the subband signals, such as quantization, only lead to small perturbations of the reconstructed signal. Thus, the left-inverse operator T^{-1} should also be bounded [15], hence continuous. Combining all three requirements above, we can obtain the frame condition (6.1) in terms of the filter impulse responses $h_k[n]$ as follows,

$$A\|x\|^2 \leq \sum_{k=0}^{P-1} \sum_{m=-\infty}^{\infty} |\langle x, h_{k,m} \rangle|^2 \leq B\|x\|^2, \quad \forall x[n] \in \ell^2(\mathbb{Z}) \quad (6.8)$$

with frame bounds $A > 0$ and $B < \infty$.

6.2.3 Uniform Filter Bank Frames

So far, we can define a large class of FBs corresponding to a large class of frames [113].

Definition 6.9 A function set $\{h_{k,m}[n]\}$ defined as in (6.6) satisfying (6.8) will be named as a uniform filter bank frame (UFBF).

From the above definition, we can see that the UFBF has a special structure, “shift-invariant” structure since $h_{k,m}[n]$ and $f_{k,m}[n]$ are generated by uniformly time shifting the filter impulse response $h_k^*[n]$ and $f_k[n]$ of the underlying FB.

CHAPTER 6. FUNDAMENTALS OF FRAME THEORY AND OVERSAMPLED FILTER BANKS

It has been shown [117,118] that the analysis polyphase matrix of the underlying FB provides a matrix representation of the frame analysis operator for the UFBF, i.e., $\mathbf{E}(z)$ corresponding to T . In addition, the product of $\mathbf{E}(z)$ and its Hermitian transpose corresponds to the frame operator S . These useful links can translate the frame condition (6.1) onto the polyphase matrix in the following theorem.

Theorem 6.14 (cf. [117,118]) *An oversampled filter bank implements a frame expansion if and only if its analysis polyphase matrix $\mathbf{E}(z)$ is of full column rank on the unit circle. Specifically, a paraunitary oversampled filter bank, i.e., $\tilde{\mathbf{E}}(z)\mathbf{E}(z) = c\mathbf{I}$, corresponds to a tight frame expansion.*

It should be emphasized that this theorem is valid for *both* FIR FBs and stable IIR FBs. Thus, this theorem is so far the most fundamental theoretical result on UFBF. As we are often interested in FIR FBs, we want to know what condition should be satisfied for FIR UFBF. The following theorem explains this issue.

Theorem 6.15 (cf. [117]) *An FIR analysis UFBF can have an FIR synthesis bank yielding perfect reconstruction if and only if the polynomials on the diagonal of the Smith-McMillan form of the analysis polyphase matrix $\mathbf{E}(z)$ are monomials. Specifically, the canonical dual frame $\{S^{-1}h_{k,m}\}$ consists of finite length vectors if and only if the matrix representation of frame operator $\mathbf{S}(z)$ is unimodular.*

As we indicated before, the dual frame for a give frame is not unique. There are potentially infinite dual frames. A characterization of all dual frames by the frame operator S has been given in Theorem 6.13. Now, we give an equivalent form of that theorem in the context of UFBFs.

Theorem 6.16 (cf. [118]) *Given a UFBF whose analysis polyphase matrix $\mathbf{E}(z)$ is of full column rank, the corresponding PR synthesis filter bank $\mathbf{R}(z)$, i.e., $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}_M$, is not unique and can be completely characterized as*

$$\mathbf{R}(z) = \mathbf{R}^\dagger(z) + \mathbf{U}(z) \left[\mathbf{I}_P - \mathbf{E}(z)\mathbf{R}^\dagger(z) \right] \quad (6.9)$$

where $\mathbf{R}^\dagger(z)$ is the Moore-Penrose inverse, or pseudo-inverse, of $\mathbf{E}(z)$, given by

$$\mathbf{R}^\dagger(z) = \left[\tilde{\mathbf{E}}(z)\mathbf{E}(z) \right]^{-1} \tilde{\mathbf{E}}(z) \quad (6.10)$$

and $\mathbf{U}(z)$ is a $M \times P$ matrix with arbitrary elements $[\mathbf{U}(z)]_{i,j}$ satisfying $|\mathbf{U}(e^{j2\pi\theta})|_{i,j} < \infty$.

This theorem clearly shows the non-unique solutions of PR synthesis FBs. The non-uniqueness of PR synthesis FBs for a given analysis FB entails more degrees of freedom for FB design, which does not exist in the case of critically sampled FBs [113]. In addition, this parameterization allows us to optimize the PR synthesis FBs through unconstrained optimization, in contrast to the unique PR synthesis FBs in the case of critically sampled FBs.

To prevent confusion of various terms used in the literature, the standard notations established by [111] are adopted and expanded in this dissertation by introducing a new class of frames named as uniform frames.

- λ -tight frame (λ -TF): Tight frame with frame bound λ . When normalizing the frame bound $\lambda = 1$, it is also called Parseval tight frame (PTF).
- Equal-norm frame (ENF): Frame where all the elements have the same norm, $\|\phi_m\| = \|\phi_n\|$ for all $m, n \in \mathcal{I}$.
- Unit-norm frame (UNF): Frame where all the elements have norm 1, $\|\phi_m\| = 1$ for all $m \in \mathcal{I}$.
- Uniform frame (UF): FB Frame where the squared norm of the polyphase vector of each filter is independent of frequency, i.e., $P_i(\omega) = \sum_{j=1}^M |E_{ij}(\omega)|^2 = \|h_i[n]\|^2$ [110].

6.3 Quantized Frame Expansions

According to information theory, the distortion after source coding of a continuous signal is due to quantization. The finer the quantization is done, the less is the distortion. For traditional transform coding, the independent scalar quantization is generally employed because the transform coefficients are nearly uncorrelated. After the reconstruction by the inverse transform, the mean-square error (MSE) of reconstruction is same as the MSE of quantization due to the unitariness of traditional transforms.

Now consider a simple finite-dimensional frame expansion-quantization-reconstruction scheme depicted in Fig. 6.2. A vector $\mathbf{x} \in \mathbb{C}^N$ is pre-multiplied by a matrix $\mathbf{T} \in \mathbb{C}^{M \times N}$ ($N \leq M$) with

Figure 6.2: A block diagram of quantized frame expansion and reconstruction.

full rank N to get the frame expansion coefficients $\mathbf{y} \in \mathbb{C}^M$. The components of the vector $\mathbf{y}$ are

scalar quantized independently. Then, reconstructed vector $\hat{\mathbf{x}}$ is obtained by the dual frame operated on quantized frame expansion vector $\hat{\mathbf{y}}$. From Sec. 6.1.2, we know perfect reconstruction can be obtained by the dual frame with no quantization on expansion coefficients.

However, for practical applications, the expansion coefficients are generally to be degraded in some way. The most likely situation is additive noise due to quantization. We are interested in this issue for its importance in source coding. Suppose the additive quantization noise is white with zero mean, i.e.,

$$\mathbf{y}_q = \mathbf{Q}[\mathbf{y}] = \mathbf{y} + \mathbf{n} = \mathbf{T}\mathbf{x} + \mathbf{n}$$

where $\mathbf{n}$ is zero mean white noise with variance σ^2 , uncorrelated with signal $\mathbf{x}$ and hence $\mathbf{y}$. We want to find an optimal linear estimate of $\mathbf{x}$ in the sense of minimum mean-square error. Although this problem is very similar to that of orthogonal transform coding, the *difference* is that the expansion vector is located in a larger dimensional space $\mathbb{C}^M$ than the original signal vector space $\mathbb{C}^N$. Furthermore, the expansion vector actually is located in some subspace of $\mathbb{C}^M$ with dimension equal to or possibly less than N . Thus, the component of noise $\mathbf{n}$ orthogonal to the subspace of expansion vector can be suppressed by projection of the expansion vector $\mathbf{y}_q$ onto the subspace. This principle is similar to the interference rejection by spread spectrum communications [128].

To be specific, for finite dimensional cases corresponding to oversampled block transforms (the infinite dimensional cases and corresponding oversampled filter banks are discussed in the next chapter in detail), the frame analysis operator T is a finite dimensional matrix. By Hermitian transpose of canonical dual frame analysis operator denoted by $\tilde{T}^H$ which is also a finite dimensional matrix and indeed the pseudo-inverse of T , the linear estimate of $\mathbf{x}$ is given by [15],

$$\hat{\mathbf{x}} = \tilde{T}^H \mathbf{y}_q = \tilde{T}^H (T\mathbf{x} + \mathbf{n}) \quad (6.11)$$

From Sec. 6.1.3, we know that the linear estimate or reconstruction $\hat{\mathbf{x}}$ is perfect reconstruction of $\mathbf{x}$ without quantization, i.e. $\mathbf{x} = \tilde{T}^H T\mathbf{x}$. Moreover, the component of $\mathbf{n}$ orthogonal to the subspace of the expansion vector does not contribute to the linear estimate of $\hat{\mathbf{x}}$. It is expected that $\|\mathbf{x} - \hat{\mathbf{x}}\| = \|\tilde{T}^H \mathbf{n}\|$ is to be less than $\|\mathbf{n}\|$. The following theorem states this phenomenon.

Theorem 6.17 (cf. [129]) *With zero mean white noise model and the linear reconstruction method (6.11), for any frame, the normalized MSE defined by $N^{-1}E\|\mathbf{x} - \hat{\mathbf{x}}\|^2$ is bounded by*

$$\frac{\sigma^2}{B} \leq \text{MSE} \leq \frac{\sigma^2}{A} \quad (6.12)$$

especially for tight frame, the MSE is exactly equal to $A^{-1}\sigma^2$.

From the theory of orthonormal transform coding, we know that the linear reconstruction can only yield normalized MSE equal to σ^2 , known as the by-product of Parseval's equality. However, a tight frame can yield MSE only $1/A$ of that yielded by general orthonormal transform coding. This is one of many advantages offered by frames over traditional bases. The frame has a powerful capability to suppress noise. Certainly, this advantage is obtained at the expense of more expansion coefficients and increased computational complexity.

6.4 Brief Review of Previous Related Works

Frames, as a class of overcomplete signal representation, have been studied extensively in the literature. Furthermore, frames that are robust to erasures and quantization including finite and filter bank frames, have been investigated recently with desired applications to robust transmission of information over erasure channels and networks. We summarize some existing works on such frames in Table 6.2 with references therein of some classical works. In addition,

Table 6.1: The existing works on the robustness of frames to quantization and erasures.

Robustness	Finite frames	Filter Bank Frames
Quantization	Analysis and implementation [127, 131, 132]	Analysis [110, 126, 130]
Erasures	Equal-norm frames [111, 129, 133–135]	Equal-norm FB frames [110, 137]
	Maximal robust frames [136]	
	Causal compensation [138]	

the structures and computational procedures for the finite equal-norm tight frames are reported in [139, 140].

Besides the works on frames studied by the mathematicians in the area of harmonic and wavelet analysis, frames have also been studied under the framework of OFBs from the perspective of signal processing. Recently, the class of OFBs with equal lengths $K = KM$ has attracted a lot of attention in signal processing community, including PR/PU FBs and linear phase FBs. However, the unnecessary length constraint $L = KM$, which traditionally comes from the generalization of critically sampled FBs, is imposed a prior just for simplification of mathematical derivation. We summarize some documented existing works on such class of FBs in Table 6.2 with references therein of some classical works.

Although some advancement have been achieved by the above mentioned works, there are still some open problems left and we list some of them to be solved in Part II of this dissertation below.

Table 6.2: The existing works on oversampled filter banks with equal lengths $L = KM$.

OFBs	Paraunitary	Perfect Reconstruction
Nonlinear phase	Forward lattice factorization [5]	cosine-modulated FBs [141, 142]
		Direct design for constrained length $L = 2M$ [143]
		State-space parameterization [144]
Linear phase	Forward lattice factorization for rational oversampling ratio [145]	Forward lattice factorization for rational oversampling ratio [80, 146]
	Ad hoc design for lapped tight FB frames with $L = 2M$ [147]	Backward lattice factorization for the case of both even P and M [98]

- The necessary condition for general nonequal-norm FB frames to be robust to erasures. (Sec. 7.3.1)
- The stability of reconstruction of FB frames after erasures and the possibility of FIR reconstruction. (Sec. 7.3.2)
- The optimal condition for general nonequal-norm FB frames without and with erasures. (Sec. 7.4)
- Forward and backward lattice factorizations for general oversampled perfect reconstruction FBs with arbitrary lengths $L = KM + \beta$ ($0 \leq \beta < M$). (Sec. 8.2)
- The existence conditions for oversampled linear phase perfect reconstruction FBs with arbitrary lengths $L = KM + \beta$ ($0 \leq \beta < M$). (Sec. 8.3.1)
- Forward and backward lattice factorizations for oversampled linear phase paraunitary and perfect reconstruction FBs with arbitrary lengths $L = KM + \beta$ ($0 \leq \beta < M$). (Sec. 8.3.3 and Sec. 8.3.4)

Chapter 7

Robustness and Quantization of Filter Bank Frames

7.1 Introduction

Recently, frames have gained great interests due to their structural redundancy for robustness to quantization noises and transmission erasures [108, 111] and more degrees of design freedom over traditional bases induced by critically sampled FBs [14]. This chapter investigates the robustness of infinite dimensional frames implemented by oversampled filter banks, named as filter bank frames, in Hilbert space $l^2(\mathbb{Z})$ to quantization and erasures from the perspectives of both frame and FB theory.

The motivation for our research on FB frames is due to their structural redundancy which can enhance resilience to erasures for information transmission over erasure networks like internet. Considering a P -channel OFB with downsampling factor M ($M < P$), after the analysis FB which transforms the input signal into subbands, the P subband signals are scalar quantized separately by uniform scalar quantizers and sent over P independent channels, like the source coding model shown in Fig. 1.2 presented in Chapter 1. Each channel is lossless or totally lost during transmission. The decoder receives $P - e$ channel information, where e is the number of erasures, and tries to reconstruct the source by a linear method, i.e., synthesis FB. In this chapter, we do *not* assume any prior information on the input signal, and only use the popular uncorrelated quantization noise model. Thus, the performance of reconstruction after quantization or erasures only depends on the characteristics of the underlying FB frame. We investigate properties of the FB frame induced by being optimally robust to quantization and erasures.

Most previous works on robust information transmission by frame expansions with quantization and erasures were focused on finite frames in finite dimensional space $\mathbb{C}^M$ or $\mathbb{R}^M$.

CHAPTER 7. ROBUSTNESS AND QUANTIZATION OF FILTER BANK FRAMES

In [131], the quantization effects on overcomplete expansions for $\mathbb{R}^M$ were firstly investigated and more works were reported later in [111, 127, 129, 132]. Later, it was extended to quantized FB frames [110, 126]. The study on the robustness of frames to erasures was pioneered by Goyal, Kovačević and Kelner in [129] for the case of finite frames. A class of finite frames with special structures, named as discrete Fourier transform codes, and its robustness to erasures, was further studied in [133]. In addition, Holmes and Paulsen [134] studied finite frames from the perspective of coding theory and classified the optimal finite frames under a numerical setting. It was followed by [135] for a further study on the bounds of reconstruction error based on graph theory. Vershynin further studied finite frame expansions with erasures by using an approach through the non-commutative operator theory in [148].

Recently, it has been extended to FB frame in $l^2(\mathbb{Z})$ firstly reported in [110, 149]. Then, a class of tree-structured FB frames was studied in [116] as joint source-channel codes for robust transmission of images over erasure channels. In addition, Labeau *et al.* [119] studied FB frames from the perspective of error correcting codes and impulse noise correction. In [130], the error amplification resulting from FB frame based coding scheme was analyzed. In addition, Püschel and Kovačević studied the maximal robust frames [136] through the perspective of polynomial transforms. Recently, Gan and Ling further studied the computation of the dual frames [137] and independently reported a result on the condition for being robust to one erasure which is the same as ours. Different from the previous works, the causal compensation for erasures in frame representations was reported in [138], which uses orthogonal projections at the transmitter to compensate for an erasure.

However, an unnecessary condition of equal-norm for FB frames is frequently imposed in these existing works which leads to the *restriction* of the optimality of their results only over the class of equal-norm FB frames. In addition, the equal-norm property is not desirable from the perspective of designing FB frames. Even for finite frames, it is difficult to design an equal-norm tight frame, i.e., a tall rectangular orthogonal matrix with orthogonal columns and equal-norm of rows, while leaving some degrees of freedom for given design specifications P and M and other properties for some desired applications. Casazza and Leonhard have listed the very few known equal-norm frames with some structures known so far [139]. In addition, a computational procedure for harmonic frames in finite dimensional spaces $\mathbb{C}^M$ was reported in [140]. Recently, a new design of equal-norm lapped tight frames, i.e., tight FB frames with constrained length $L = 2M$, was reported in [147]. More general design methods for equal-norm FB frames, especially tight FB frames, are still open.

However, different from before, we show in this chapter that the equal-norm property is *not* always necessary and tight FB frames are actually optimal over more general FB frames.

CHAPTER 7. ROBUSTNESS AND QUANTIZATION OF FILTER BANK FRAMES

The outline of this chapter is as follows. A characterization of the FB frame operator is given by a form of eigenvalue decomposition for matrix polynomials in Sec. 7.2. Then, the erasure effect on the FB frames is investigated in Sec. 7.3. A characterization of the spectral singular values for general FB frames is first shown and then we specialize it to establish the existence conditions for tight FB frames. Based on this characterization, the robustness of FB frames to erasures is analyzed and a *necessary and sufficient* condition is given for the case of tight FB frames. Our result generalizes the previous works and indicates that the equal-norm property is unnecessary for the robustness to erasures. In addition, maximal robust FB frames are further explored based on an explicit constructive approach which can serve as a guideline for designing MR FB frames. Then, the stability issue and even possible FIR reconstruction by the pseudoinverse of the subframe is studied since the pseudoinverse is usually a hybrid FIR and IIR FB. It is shown that our introduced new concept, i.e., the uniformity of the squared norm $P_i(\omega)$ of the analysis filters over frequency, is the *necessary and sufficient* condition for the FIR reconstruction by the pseudoinverse of the subframe after one erasure of tight FB frames. As an example, we study the order-one tight FB frames in detail and show that it is impossible to obtain an FIR pseudoinverse of the subframe after one erasure for linear phase order-one tight FB frames with odd subsampling factor. Because quantization is an inevitable process of digital transmission of information, we continue to analyze the quantization effect on the reconstruction by the pseudoinverse in Sec. 7.4 and the optimal FB frames based on the popular additive, pair-wise uncorrelated quantization noise model which has been justified for overcomplete signal representations in [150]. In particular, it is shown that the equal-norm property, imposed frequently in previous works, is *not* necessary for optimization of frames. Moreover, this property is not desirable from the viewpoint of designing FB frames. In fact, we introduce a novel notion, named as *frame energy*, which actually dictates the quantization performance of FB frames in Sec. 7.4.1. In addition, we show that tight FB frames are optimal for quantization with a constraint on frame energy over *all* FB frames. Furthermore, the design freedom obtained from removing the equal-norm property is explained and illustrated with examples. In addition, the effect of erasures on the reconstruction MSE for general quantized FB frames is examined in Sec. 7.4.2. Different from before, we incorporate a prior information on possible probability distribution of erasures into our analysis, which makes our analysis more general and can cover the previous works as special cases. In particular, we show that the tight FB frames are optimal over general FB frames and the optimal tight FB frames are usually not equal-norm tight frames. The distribution of norms of the optimal frames is shown to be *dependent* on the probability distribution of erasures and to follow the reverse water-filling principle. The MSE analysis for more than one erasure is continued and various bounds are

obtained for successive reconstruction. An illustrative example of two-state erasure network model is shown to further explain our result. We then put our analysis into the framework of MDC to show its potential application due to the desirable reduced reconstruction MSE by the additional correctly received subbands. Finally, Sec. 7.6 summarizes this chapter.

7.2 Characterization and Eigen-Structure of Frame Operator

It is well known that the frame operator $\mathbf{S}(\omega)$ is a self-adjoint matrix which is an analytic function of the radial parameter $\omega \in [0, 2\pi)$. According to the perturbation theorem for self-adjoint matrices [151], it turns out that the eigenvalues and eigenvectors are also analytic and such matrix function $\mathbf{S}(\omega)$ can be diagonalized, *simultaneously*, for all ω by an analytic matrix function $\mathbf{U}(\omega)$ that satisfies $\mathbf{U}(\omega)\mathbf{U}^*(\omega) = \mathbf{I}$, which is similar to the eigenvalue decomposition (EVD) of a Hermitian matrix.

Lemma 7.7 (Gohberg et al. [151]) *Let $\mathbf{S}(\omega)$ be an analytic self-adjoint $n \times n$ matrix function of a real variable $\omega \in [0, 2\pi)$. Then, there exist n scalar functions $\lambda_1(\omega), \dots, \lambda_n(\omega)$ and an analytic matrix-valued function $\mathbf{U}(\omega)$ for all $\omega \in [0, 2\pi)$ such that,*

$$\mathbf{S}(\omega) = \mathbf{U}^*(\omega)\text{diag}[\lambda_1(\omega), \dots, \lambda_n(\omega)]\mathbf{U}(\omega). \quad (7.1)$$

Those analytic scalar functions $\lambda_i(\omega)$, $i = 1, \dots, n$ are just the spectral eigenvalues of the frame operator $\mathbf{S}(\omega)$. From the above Lemma, we can see that the tightest upper and lower frame bounds are just the essential supremum and infimum of those n spectral eigenvalues $\lambda_i(\omega)$, respectively.

$$B = \text{ess sup}_{\omega \in [0, 2\pi), i=0,1,\dots,n-1} \lambda_i(\omega), \quad A = \text{ess inf}_{\omega \in [0, 2\pi), i=0,1,\dots,n-1} \lambda_i(\omega) \quad (7.2)$$

In the following sections, we will see that the eigenstructure of the frame operator $\mathbf{S}(\omega)$ plays an important role in the analysis of the robustness of FB frames. There is a fundamental equality on the integral sum of the spectral eigenvalues [110, 111], as shown below.

$$\int_{-\pi}^{\pi} \sum_{i=1}^M \lambda_i(\omega) \frac{d\omega}{2\pi} = \int_{-\pi}^{\pi} \sum_{i=1}^P P_i(\omega) \frac{d\omega}{2\pi} = \sum_{i=1}^P \|h_i[n]\|^2 \quad (7.3)$$

For tight FB frames, i.e., paraunitary OFBs, spectral eigenvalues are all equal to the frame bound A and constant over the unit circle. Thus $MA = \sum_{i=1}^P \|h_i[n]\|^2$. In addition, tight FB frames have a very nice property that the pseudo-inverse of $\mathbf{E}(z)$ can be easily obtained by $\mathbf{E}^\dagger(z) = A^{-1}\tilde{\mathbf{E}}(z)$.

7.3 Erasure Effects on the Structures of FB Frames and Reconstruction after Erasures

In this section, we study the robustness of FB frame to erasures, which is a typical phenomenon in information transmission over packet based networks like Internet. Assuming a FB frame

Figure 7.1: Error resilient transmission by a frame expansion implemented by a P channel OFB with subsampling factor M .

with $P \times M$ polyphase matrix $\mathbf{E}(\omega)$ and an erasure pattern indexed by a set E , we denote the polyphase matrix after e erasures by $\mathbf{E}_S(\omega)$, where e is the cardinality of the set E , i.e., $e = |E|$ and $S = \mathcal{I}_P - E$ with $s = |S| \leq P$. The matrix $\mathbf{E}_S(\omega)$ is a $s \times M$ matrix obtained by deleting the rows indexed by the set E from the original $\mathbf{E}(\omega)$. Mathematically, the erasure pattern can be denoted by a matrix $\mathbf{L}_E$ with size $s \times P$ which is dependent on the set E . It can be easily shown that matrix $\mathbf{L}_E$ can be written in this form, $\mathbf{L}_E = [\mathbf{I}_s, \mathbf{0}_{s \times e}] \mathbf{P}$, where $\mathbf{P}$ is a permutation matrix. Thus, the overall equivalent FB frame expansion is $\mathbf{E}_S(\omega) = \mathbf{L}_E \mathbf{E}(\omega)$.

When $\mathbf{E}_S(\omega)$ is still of full rank on the unit circle, i.e., left invertible, it is called a subframe. For such erasure pattern, the perfect reconstruction can still be achieved by the left inverse of $\mathbf{E}_S(\omega)$ without quantization in subbands. It can be easily seen that one of the necessary conditions for such perfect reconstruction is $e \leq P - M$.

Definition 7.10 A FB frame is called robust to specific erasure pattern E if the matrix $\mathbf{E}_S(\omega)$ is of full rank, i.e., the subframe $\mathbf{E}_S(\omega)$ exists.

However, the subframe $\mathbf{E}_S(\omega)$ is usually dependent on the erasure pattern E , which means that a subframe may not exist for some erasure pattern. From the viewpoint of robust transmission over erasure network, it is desirable that the subframe after e erasures always exists, i.e., independent of specific erasure pattern E . An example of such frame is the geometrically uniform frame [152].

Definition 7.11 A FB frame is said to be robust to e erasures if a subframe exists for any erasure pattern E with $e = |E|$.

7.3.1 Erasure Effect on Structure of Filter Bank Frames

Frames have found many applications in signal processing and communications [108, 109] since they have many desirable properties. Here, we first derive an important property of FB frames unnoticed before.

Lemma 7.8 *For any FB frame with $P \times M$ analysis polyphase matrix $\mathbf{E}(\omega)$ and frame bounds A and B , given any subset $S \subseteq \mathcal{I}_P$ with $s = |S|$, the maximum spectral singular value of the $|S| \times M$ submatrix $\mathbf{E}_S(\omega)$ is upper bounded by $\sqrt{B}$ on the unit circle, i.e., $\max_i \{\sigma_i(\omega), i = 0, 1, \dots, \min(M, s) - 1\} \leq \sqrt{B}$ for all $\omega \in [0, 2\pi]$. When the FB frame is tight, it is upper bounded by the frame bound $\sqrt{A}$.*

Proof: First, let us consider an extreme case with $S = \mathcal{I}_P$. In such case, the frame operator matrix is $\mathbf{S}(\omega) = \mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)$ due to $\mathbf{E}(\omega) = \mathbf{E}_S(\omega)$. Denote by $\lambda_i(\omega)$ the spectral eigenvalues of $\mathbf{S}(\omega)$, and by $\sigma_i(\omega)$ the spectral singular eigenvalues of $\mathbf{E}_S(\omega)$. We know that $\sigma_i(\omega) = \sqrt{\lambda_i(\omega)}$ for all $i = 0, 1, \dots, M - 1$. Since the eigenvalues $\lambda_i(\omega)$ of the positive definite $\mathbf{S}(\omega)$ has upper bound B , we can obtain $\sigma_i(\omega) \leq \sqrt{B}$ for all $i = 0, 1, \dots, M - 1$, i.e., $\max_i \{\sigma_i(\omega)\} \leq \sqrt{B}$.

Next, consider the case of $S \subset \mathcal{I}_P$ with $s = |S|$. Denote the set E as the complement set of S with respect to $\mathcal{I}_P$, i.e., $E = \mathcal{I}_P - S$. Then, the following equality holds,

$$\begin{aligned} \mathbf{S}(\omega) &= \mathbf{E}^*(\omega)\mathbf{E}(\omega) = \sum_{i=0}^{P-1} \mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega) = \sum_{k \in S} \mathbf{h}_k^*(\omega)\mathbf{h}_k(\omega) + \sum_{k \in E} \mathbf{h}_k^*(\omega)\mathbf{h}_k(\omega) \\ &= \mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega) + \mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega) \end{aligned} \quad (7.4)$$

For any complex vector z with length M ,

$$z^*\mathbf{S}(\omega)z = z^*[\mathbf{E}^*(\omega)\mathbf{E}(\omega)]z = z^*[\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)]z + z^*[\mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega)]z$$

From the inequality on frame operator $A\mathbf{I}_M \leq \mathbf{S}(\omega) \leq B\mathbf{I}_M$, we know that for any complex vector z , the quadratic form $z^*\mathbf{S}(\omega)z$ is upper bounded by B , i.e., $z^*\mathbf{S}(\omega)z \leq B$ for all $\omega \in [0, 2\pi]$. From Rayleigh-Ritz theorem [79], there exists a vector $x(\omega)$ with unit norm such that $x(\omega)^*[\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)]x(\omega) = \max_i \{\sigma_i^2(\omega), i = 0, \dots, \min(s, M) - 1\}$ for any given $\omega \in [0, 2\pi]$. Since the term $z^*[\mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega)]z = \|\mathbf{E}_E(\omega)z\|^2 \geq 0$ for any vector z , we can obtain

$$\max_i \{\sigma_i^2(\omega)\} = x(\omega)^*[\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)]x(\omega) \leq B\|x(\omega)\|^2 - x(\omega)^*[\mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega)]x(\omega) \leq B$$

For tight FB frames, the upper bound on the spectral singular values is obtained since the frame bounds are equal, i.e., $A = B$. ■

A special case $|S| = 1$ of the above lemma deserves more attention since it is directly related to the analysis filters of FB frame. Furthermore, it is an existence condition about the range of the squared norm of the analysis filter $h_i[n]$ for tight FB frames. We state it formally as a corollary of Lemma 7.8.

Corollary 7.1 *For any FB frame with frame bounds A and B , the squared norms $P_i(\omega)$ of the polyphase vectors are upper bounded by B on the unit circle, i.e., $P_i(\omega) \leq B$ for all $\omega \in [0, 2\pi]$ and all $i = 0, 1, \dots, P-1$. In addition, the squared norms $\|h_i[n]\|^2$ of the analysis filters $h_i[n]$ are also upper bounded by B . When the underlying FB frame is tight, the frame bound A serves as the upper bound for both $P_i(\omega)$ and the squared norm $\|h_i[n]\|^2$.*

Proof: It can be easily seen that $\mathbf{E}_S(\omega)$ simplifies to a polyphase row vector $\mathbf{h}_i(\omega)$ for some i ($0 \leq i \leq P-1$) for the case $|S| = 1$. Then, there is only one spectral singular value for $\mathbf{E}_S(\omega)$ with $\sigma(\omega) = \sqrt{P_i(\omega)}$. Thus, we can obtain the upper bound on the squared norm $P_i(\omega)$. From the equality $\int_{-\pi}^{\pi} P_i(\omega) d\omega / 2\pi = \|h_i[n]\|^2$, we know that the squared norm $\|h_i[n]\|^2$ is upper bounded by B for all $i = 0, 1, \dots, P-1$. For tight frames, it follows from the equal frame bounds $A = B$. ■

This corollary reveals a *novel* perspective on the frame bounds of FB frames. The bound B establishes an upper bound on the squared norm $P_i(\omega)$ of each filter. In addition, the frame bound A sets an upper bound on the squared norm of the impulse responses of analysis filters $h_i[n]$ whose time-reversal, i.e., $h_i[-n]$, are the elements $\phi_i[n]$ of the FB frame. Furthermore, it can be shown that the above necessary condition is also sufficient [153], i.e., is the existence condition for tight FB frames. Note that the above corollary is also applicable to finite frames. Especially for finite tight frames, the frame bound A serves as the upper bound of squared Euclidean norms of frame vectors in finite dimensional space $\mathbb{H}^M$, which is called the fundamental inequality of finite tight frames [154].

Since we are interested in applying FB frames for robust information transmission over erasure networks, we firstly attempt to find that being robust to erasures would impose what further conditions on $\mathbf{E}(\omega)$ except the full rank property. We state it formally in the following theorem.

Theorem 7.18 *For any FB frame associated with a FIR or BIBO stable IIR OFB with $P \times M$ polyphase matrix $\mathbf{E}(\omega)$ and given erasure pattern E with $e = |E|$ ($1 \leq e \leq P-M$), denote the $e \times M$ matrix $\mathbf{E}_E(\omega)$ obtained by extracting the rows indexed by the erasure pattern E from $\mathbf{E}(\omega)$, and its spectral singular values $\sigma_i(\omega)$ ($0 \leq i \leq e-1$). The FB frame is robust to the erasure pattern E if the maximum squared spectral singular value is strictly less than the lower frame bound A over the unit circle, i.e., $\max_i \{\sigma_i^2(\omega)\} < A$ for all $\omega \in [0, 2\pi]$. In addition, the FB frame is robust to e erasures if it is robust to any erasure pattern E with $e = |E|$. Furthermore, if the underlying FB frame is tight, the above conditions are also necessary, i.e., are the existence conditions for robustness to erasures.*

Proof: Denote the set $S = \mathcal{I}_P - E$ and $s = |S| = P - e \geq M$. Call the remaining $s \times M$ polyphase matrix $\mathbf{E}_S(\omega)$ after erasure pattern E . Then from (7.4) we have,

$$\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega) = \mathbf{S}(\omega) - \mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega) = \mathbf{S}(\omega)[\mathbf{I}_M - \mathbf{S}^{-1}(\omega)\mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega)] \quad (7.5)$$

The last step comes from the positive definiteness of the frame operator $\mathbf{S}(\omega)$. For being robust to erasures, the full rank of $\mathbf{E}_S(\omega)$ is required, i.e., the determinant of $\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)$ is nonzero. By the equality [79] for determinant $\det(\mathbf{I} + \mathbf{A}\mathbf{B}) = \det(\mathbf{I} + \mathbf{B}\mathbf{A})$, we can obtain the following equality.

$$\begin{aligned} \det[\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)] &= \det[\mathbf{S}(\omega)]\det[\mathbf{I}_M - \mathbf{S}^{-1}(\omega)\mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega)] \\ &= \det[\mathbf{S}(\omega)]\det[\mathbf{I}_e - \mathbf{E}_E(\omega)\mathbf{S}^{-1}(\omega)\mathbf{E}_E^*(\omega)] \end{aligned} \quad (7.6)$$

Thus, it is only needed to force $\det[\mathbf{I}_e - \mathbf{E}_E(\omega)\mathbf{S}^{-1}(\omega)\mathbf{E}_E^*(\omega)]$ to be nonzero, which is equivalent to constraining its eigenvalues to be nonzero. Denote the spectral eigenvalues of the $e \times e$ matrix $\mathbf{E}_E(\omega)\mathbf{S}^{-1}(\omega)\mathbf{E}_E^*(\omega)$ by $\lambda_i(\omega)$ ($0 \leq i \leq e-1$). It can be easily shown that the spectral eigenvalues of $\mathbf{I}_e - \mathbf{E}_E(\omega)\mathbf{S}^{-1}(\omega)\mathbf{E}_E^*(\omega)$ are equal to $1 - \lambda_i(\omega)$ ($0 \leq i \leq e-1$). Thus it is equivalent to constraining $\lambda_i(\omega) \neq 1$ for all $\omega \in [0, 2\pi]$ and all $i = 0, 1, \dots, e-1$. From the inequality of the frame operator $\mathbf{S}(\omega)$, we can obtain

$$B^{-1}\mathbf{E}_E(\omega)\mathbf{E}_E^*(\omega) \leq \mathbf{E}_E(\omega)\mathbf{S}^{-1}(\omega)\mathbf{E}_E^*(\omega) \leq A^{-1}\mathbf{E}_E(\omega)\mathbf{E}_E^*(\omega) \quad (7.7)$$

Since the spectral singular values of $\mathbf{E}_E(\omega)$ are $\sigma_i(\omega)$, the spectral eigenvalues of the $e \times e$ matrix $\mathbf{E}_E(\omega)\mathbf{E}_E^*(\omega)$ are equal to $\sigma_i^2(\omega)$. We can get $\max_i\{\lambda_i(\omega)\} \leq A^{-1}\max_i\{\sigma_i^2(\omega)\}$ for all $\omega \in [0, 2\pi]$ since there exists a unit norm column vector $\mathbf{u}(\omega)$ to let the quadratic form

$$\max_i\{\lambda_i(\omega)\} = \mathbf{u}^*(\omega)\mathbf{E}_E(\omega)\mathbf{S}^{-1}(\omega)\mathbf{E}_E^*(\omega)\mathbf{u}(\omega) \leq A^{-1}\mathbf{u}^*(\omega)\mathbf{E}_E(\omega)\mathbf{E}_E^*(\omega)\mathbf{u}(\omega) \leq A^{-1}\max_i\{\sigma_i^2(\omega)\} \quad (7.8)$$

Similarly, we can get $\min_i\{\lambda_i(\omega)\} \geq B^{-1}\min_i\{\sigma_i^2(\omega)\}$ for all $\omega \in [0, 2\pi]$.

Since the underlying FB is FIR or stable IIR, the frequency responses of polyphase components of analysis filters are continuous. The spectral eigenvalues $\sigma_i^2(\omega)$ are also continuous function because the eigenvalues are continuous functions of the entries of the underlying matrix [79]. Then from (7.8), we can obtain two sufficient conditions $\max_i\{\sigma_i^2(\omega)\} < A$ or $\min_i\{\sigma_i^2(\omega)\} > B$ which induce the inequalities $\lambda_i(\omega) \neq 1$ for all $\omega \in [0, 2\pi]$ and all $i = 0, 1, \dots, e-1$. From Lemma 7.8, we know that $\sigma_i^2(\omega) \leq B$ for all $i = 0, 1, \dots, e-1$. Thus, we establish a sufficient condition for being robust to erasure pattern E , i.e., $\max_i\{\sigma_i^2(\omega)\} < A$ for all $\omega \in [0, 2\pi]$. Based on Definition 3, the FB frame is robust to e erasures if and only it is robust to any erasure pattern E with $e = |E|$.

CHAPTER 7. ROBUSTNESS AND QUANTIZATION OF FILTER BANK FRAMES

Furthermore, if the original FB frame is a tight frame, i.e., $\mathbf{S}(\omega) = A\mathbf{I}_M$, we can see that the eigenvalues of the matrix $\mathbf{E}_E(\omega)\mathbf{S}^{-1}(\omega)\mathbf{E}_E^*(\omega)$ are $\lambda_i(\omega) = A^{-1}\sigma_i^2(\omega)$. Thus $\lambda_i(\omega) \neq 1$ implies the inequalities $\sigma_i^2(\omega) \neq A$ for all $\omega \in [0, 2\pi]$ and all $i = 0, \dots, P-1$. From Lemma 7.8, the above inequalities are held if and only if $\max_i \{\sigma_i^2(\omega)\} < A$ for all $\omega \in [0, 2\pi]$. ■

As before, we consider a special case of one erasure, i.e., $e = 1$, since it is more related to the analysis filters $h_i[n]$ or the corresponding polyphase vector $\mathbf{h}_i(\omega)$ of the underlying FB frame. Furthermore, the condition shown in below corollary of the above theorem, is both sufficient and necessary for the robustness of FB frames to one erasure.

Corollary 7.2 *Any FB frame implemented by an FIR or BIBO stable IIR OFB is robust to erasure of the i th channel filter $h_i[n]$ if and only if the modified squared norm $\hat{P}_i(\omega) \triangleq \mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)$ of the polyphase vector is strictly less than unity over the entire unit circle, i.e., $\hat{P}_i(\omega) < 1$ for all $\omega \in [0, 2\pi]$. In addition, the FB frame is robust to any one erasure if and only if $\hat{P}_i(\omega) < 1$ over the unit circle and all channels. For the class of tight FB frames, they are robust to one erasure if and only if $P_i(\omega) < A$ for all $\omega \in [0, 2\pi)$ and $i = 0, 1, \dots, P-1$. However, the subframe of any nontrivial tight FB frame after one erasure cannot be a tight frame again.*

Proof: It can be easily seen that $\mathbf{E}_E(\omega)$ simplifies to the erased polyphase row vector $\mathbf{h}_i(\omega)$ for some i ($0 \leq i \leq P-1$). From (7.6) in the proof for Theorem 7.18, we just need to guarantee the nonnegative quadratic form $\hat{P}_i(\omega) \neq 1$. Next, we show that $\hat{P}_i(\omega) \leq 1$, which leads to our necessary and sufficient condition for general FB frames to be robust to one erasure. From Theorem 6.12, we can see that $\hat{P}_i(\omega) \leq A^{-1}\mathbf{h}_i(\omega)\mathbf{h}_i^*(\omega) = A^{-1}P_i(\omega)$. According to Corollary 7.1, we can establish the necessary upper bound, $\hat{P}_i(\omega) \leq B/A \leq 1$.

In addition, for the case of tight FB frames, we can see that $\hat{P}_i(\omega) = A^{-1}P_i(\omega)$ since $\mathbf{S}(\omega) = A\mathbf{I}_M$. Thus, the necessary and sufficient condition for robustness just simplifies to $P_i(\omega) < A$ for all $\omega \in [0, 2\pi)$ and $i = 0, 1, \dots, P-1$. For the subframe to be tight, the frame operator $\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)$ for the subframe must be in the form of $c\mathbf{I}_M$ for some nonzero constant c . From (7.5), we can deduce that the dyadic matrix

$$\mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega) = (A - c)\mathbf{I}_M \quad (7.9)$$

since $\mathbf{S}(\omega) = A\mathbf{I}_M$ for tight FB frames. However, due to the rank constraint for the dyadic matrix, we know that (7.9) is generally invalid, which dictates the nonexistence of tight subframe $\mathbf{E}_S(\omega)$. (Note that (7.9) is valid if and only if $c = A$ which corresponds to the filter $h_i[n] = 0$, which is considered as a trivial filter in this dissertation.) ■

Remark 7.9 *Note that this corollary is also applicable to finite frames since they are a class of degenerate FB frames, i.e., FBs with order zero. A similar result has been reported in [129] and [110] only for finite and FB tight frames, respectively. However, Corollary 7.2 is more general than before and can apply to general FB frames, including tight FB frames as a special case. In addition, the previous works always imposed an unnecessary a priori property - “uniformness of $P_i(\omega)$ with equal norm” - on the underlying frames, which is only helpful for their proof. On the contrary, our analysis reveals that such a priori condition, which is imposed in [129] and [110], is not necessary for robustness to one erasure. It shows more degrees of design freedom on FB frames with robustness to one erasure, since the unnecessary condition of equal norm greatly limits the design freedom, especially for tight FB frames. Note that a similar result is independently reported recently in [137] using the Greville formulas for computation of the dual frames.*

The previous analysis only considers theoretical parts of FB frames with robustness to erasures without explicit design methods. Next we show some structures and design methods for FB frames being robust to erasures. Apparently, if the number of erasures $e > P - M$, the subframe cannot exist. Moreover, the subframe may not exist for some erasure pattern E with $|E| = e = P - M$. Thus, a class of FB frames with polyphase matrix $\mathbf{E}(\omega)$ is called maximally robust (MR) to erasures, i.e., $e = P - M$, if every $M \times M$ submatrix of $\mathbf{E}(\omega)$ is invertible on the unit circle. MR frames are very desirable from the perspective of robust information transmission over erasure networks like multiple description coding [12] and thus received much attention recently. Finite MR frames have been found and studied recently in [111, 129, 133, 136]. They are derived from the so called Harmonic Frames which are mainly based on the beautiful structure of DFT matrix. A computational procedure for all finite harmonic frames and their properties have been documented in [140]. (A good overview of equal norm Parseval frames and their structures can be found in [139].) However, they basically have no design freedom since the finite MR frame is fixed when the design specifications P and M are given. In the following Lemma, we give a parameterized structure which has the MR property, but still has some degrees of design freedom.

Lemma 7.9 *For any given $P \times P$ nonsingular Vandermonde matrix $\mathbf{V}$, we can obtain a maximally robust frame $\mathbf{F}$ represented by a $P \times M$ matrix after deleting the right $P - M$ columns of $\mathbf{V}$.*

Since every $M \times M$ submatrix of $\mathbf{F}$ is still a nonsingular Vandermonde matrix (due to the nonsingular matrix $\mathbf{V}$), $\mathbf{F}$ is a MR frame. This class of MR frames has also been studied

in [136] by using polynomial transforms. The lowpass DFT codes [133] and complex harmonic frames [111, 129] are special cases of this type of MR frames. In addition to the above finite structurally MR frames, we can also design structurally MR FB frames $\mathbf{E}(\omega)$ by backward factorization of the analysis polyphase matrix.

Theorem 7.19 *A MR FB frame can be constructed by the pre-filtering structure, i.e., the analysis polyphase matrix $\mathbf{E}(\omega) = \mathbf{F}\mathbf{U}(\omega)$, where $\mathbf{F}$ is any MR finite frame in $\mathbb{H}^M$ and $\mathbf{U}(\omega)$ is an $M \times M$ polyphase matrix nonsingular over the unit circle.*

Proof: It can be easily shown that every $M \times M$ submatrix of $\mathbf{E}(\omega)$ is nonsingular over the unit circle since the erasures only affect $\mathbf{F}$. ■

The above analysis gives some design procedures for constructing structurally MR finite or FB frames.

Remark 7.10 *It can be checked that the proposed finite MR frames in [111, 129, 133] all satisfy the above theorem due to the special structural form of frames. Our proposed MR finite frames in Lemma 7.9 also satisfy the above theorem. However, they still have some degrees of design freedom in contrast to the previous MR frames which have no design freedom when the design specifications P and M are given.*

Tight frames are very desirable for many applications in signal processing and communications [109]. Thus, we want to investigate the property of the subframe after erasures on tight FB frames. Unfortunately, the subframes are usually no longer tight frames for nontrivial tight FB frames with oversampling ratio $r < 2$. We state it formally by the following theorem.

Theorem 7.20 *For any nontrivial tight FB frame with $P \times M$ polyphase matrix $\mathbf{E}(\omega)$, the subframe (if exists) after any erasure pattern with $e \geq 1$ erasures cannot be a tight frame again if e is strictly less than M , i.e., $1 \leq e < M$. However, when $e \geq M$, the subframe (if exists) after some erasure pattern can be a tight frame if and only if the oversampling ratio $r = P/M \geq 2$, i.e., $P \geq 2M$, and the erasures themselves form a tight frame. This demands that the original tight FB frame is a union of multiple tight FB frames.*

Proof: Assume the same setup as that of the proof for the Lemma 7.8. After e erasures, we get $\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega) = \mathbf{S}(\omega) - \mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega) = \mathbf{A}\mathbf{I}_M - \mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega)$. If the subframe exists, i.e., $\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)$ is of full rank, it can be diagonalized by some unitary matrix $\mathbf{U}(\omega)$.

$$\Sigma_S(\omega) = \mathbf{U}^*(\omega)\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)\mathbf{U}(\omega) = \mathbf{A}\mathbf{I}_M - \mathbf{U}^*(\omega)\mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega)\mathbf{U}(\omega) = \mathbf{A}\mathbf{I}_M - \mathbf{V}_E^*(\omega)\mathbf{V}_E(\omega) \quad (7.10)$$

where $\Sigma_S(\omega)$ is a positive definite, diagonal matrix and $\mathbf{V}_E(\omega) = \mathbf{E}_E(\omega)\mathbf{U}(\omega)$ is an $e \times M$ matrix. The subframe $\mathbf{E}_S(\omega)$ is a tight frame if and only if $\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega) = A_S\mathbf{I}_M$ for some positive A_S . This demands $\mathbf{V}_E^*(\omega)\mathbf{V}_E(\omega) = c\mathbf{I}_M$.

If $1 \leq e < M$, the matrix $\mathbf{V}_E^*(\omega)\mathbf{V}_E(\omega)$ is an $M \times M$ matrix with rank at most e . Thus, it cannot be of the form of $c\mathbf{I}_M$ unless $c = 0$. The constant c is zero if and only if matrix $\mathbf{V}_E(\omega)$ is a null matrix, which means $\mathbf{E}_E(\omega) = \mathbf{0}$, i.e., the erased channels correspond to zero filters. Thus, the subframe cannot be a tight frame for the nontrivial FB frames.

However, if the number of erasures $e \geq M$, the subframe could exist only if $P - M \geq e \geq M$. This demands that the oversampling ratio $r \geq 2$, i.e., $P \geq 2M$. In such case, it is possible to make $\mathbf{V}_E^*(\omega)\mathbf{V}_E(\omega) = c\mathbf{I}_M$ for some positive c . This is true if and only if $\mathbf{E}_E^*(\omega)\mathbf{E}_E(\omega) = c\mathbf{I}_M$, i.e., the erased filters themselves form a tight frame. Then, the subframe is a tight frame with frame bound $A - c$ if $c < A$. From the Lemma 7.8, we know $c \leq A$. It can be easily seen that $c \neq A$ for nontrivial FB frames since the case $c = A$ would constrain $\mathbf{E}_S(\omega)$ to be zero filters. Thus, we can see that the subframe $\mathbf{E}_S(\omega)$ is a subframe if and only if the original tight FB frame is a union of two tight subframes $\mathbf{E}_E(\omega)$ and $\mathbf{E}_S(\omega)$. ■

7.3.2 Reconstruction after Erasures

From the previous subsection, we know that the subframe $\mathbf{E}_S(z)$ exists for some FB frames. Thus, the pseudo-inverse of the subframe exists, $\mathbf{E}_S^\dagger(z) = [\tilde{\mathbf{E}}_S(z)\mathbf{E}_S(z)]^{-1}\tilde{\mathbf{E}}_S(z)$, and can be used for reconstruction since they are optimally resistant to quantization noise (seen from Theorem 7.22). However, even for FB frame $\mathbf{E}(\omega)$ implemented by an FIR OFB, the pseudo-inverse $\mathbf{E}_S^\dagger(z)$ usually becomes a hybrid FB with both FIR and IIR filters because $[\tilde{\mathbf{E}}_S(z)\mathbf{E}_S(z)]^{-1}$ is usually not FIR invertible. Therefore, it is necessary and interesting to investigate the condition for stability of IIR filters and even FIR solution of the pseudo-inverse $\mathbf{E}_S^\dagger(z)$.

Consider a general FB frame implemented by an FIR OFB with analysis polyphase matrix $\mathbf{E}(z)$. The frame operator $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\mathbf{E}(z)$ is a Laurent matrix polynomial, but its inverse is usually not a Laurent matrix polynomial. Assume the subframe $\mathbf{E}_S(z)$ exists after the erasure pattern E with $e = |E|$, then the matrix $\tilde{\mathbf{E}}_S(z)\mathbf{E}_S(z)$ is nonsingular on the unit circle. From the equality $\tilde{\mathbf{E}}_S(z)\mathbf{E}_S(z) = \tilde{\mathbf{E}}(z)\mathbf{E}(z) - \tilde{\mathbf{E}}_E(z)\mathbf{E}_E(z) = \mathbf{S}(z) - \tilde{\mathbf{E}}_E(z)\mathbf{E}_E(z)$ and the matrix inversion lemma $(\mathbf{A} - \mathbf{BCD})^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1} - \mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}$, with $\mathbf{A} = \mathbf{S}(z)$, $\mathbf{B} = \tilde{\mathbf{E}}_E(z)$, $\mathbf{C} = \mathbf{I}_e$ and $\mathbf{D} = \mathbf{E}_E(z)$, we can obtain

$$[\tilde{\mathbf{E}}_S(z)\mathbf{E}_S(z)]^{-1} = \mathbf{S}^{-1}(z) + \mathbf{S}^{-1}(z)\tilde{\mathbf{E}}_E(z)[\mathbf{I}_e - \mathbf{E}_E(z)\mathbf{S}^{-1}(z)\tilde{\mathbf{E}}_E(z)]^{-1}\mathbf{E}_E(z)\mathbf{S}^{-1}(z)$$

Thus, the pseudo-inverse of the subframe is,

$$\mathbf{E}_S^\dagger(z) = \mathbf{S}^{-1}(z)\tilde{\mathbf{E}}_S(z) + \mathbf{S}^{-1}(z)\tilde{\mathbf{E}}_E(z)[\mathbf{I}_e - \mathbf{E}_E(z)\mathbf{S}^{-1}(z)\tilde{\mathbf{E}}_E(z)]^{-1}\mathbf{E}_E(z)\mathbf{S}^{-1}(z)\tilde{\mathbf{E}}_S(z) \quad (7.11)$$

CHAPTER 7. ROBUSTNESS AND QUANTIZATION OF FILTER BANK FRAMES

However, it is very difficult to analyze the stability of the pseudo-inverse for such general setting because we don't know much about the behavior of $\mathbf{S}^{-1}(z)$ on the z -plane except that it is positive definite on the unit circle with bounded eigenvalues. Since TF is optimal for quantization and desirable for many applications [109], here we only consider the tight FB frames implemented by FIR OFBs, i.e., $\mathbf{S}(z) = \mathbf{A}\mathbf{I}_M$. In such case, the pseudo-inverse simplifies to

$$\mathbf{E}_S^\dagger(z) = \mathbf{A}^{-1}\tilde{\mathbf{E}}_S(z) + \mathbf{A}^{-2}\tilde{\mathbf{E}}_E(z)[\mathbf{I}_e - \mathbf{A}^{-1}\mathbf{E}_E(z)\tilde{\mathbf{E}}_E(z)]^{-1}\mathbf{E}_E(z)\tilde{\mathbf{E}}_S(z) \quad (7.12)$$

We can see that there is only one IIR FB embedded in the $\mathbf{E}_S^\dagger(z)$ denoted by $\mathbf{D}_E^{-1}(z) = [\mathbf{I}_e - \mathbf{A}^{-1}\mathbf{E}_E(z)\tilde{\mathbf{E}}_E(z)]^{-1}$. Since the original OFB is FIR, The pseudo-inverse $\mathbf{E}_S^\dagger(z)$ is stable if and only if the embedded IIR FB $\mathbf{D}_E^{-1}(z)$ is stable. In addition, we can see that $\mathbf{E}_S^\dagger(z)$ is an FIR FB if and only if $\det[\mathbf{D}_E(z)] = c \cdot z^{-n_0}$ for some positive constant c and integer n_0 . Actually the matrix $\mathbf{D}_E(z)$ simplifies to a unimodular matrix, i.e., $n_0 = 0$ because it is para-Hermitian. Denote the spectral singular values of $\mathbf{E}_E(\omega)$ by $\sigma_i(\omega)$ for $i = 0, 1, \dots, e-1$. The necessary and sufficient condition for FIR solution of $\mathbf{E}_S^\dagger(\omega)$ is,

$$\det[\mathbf{D}_E(\omega)] = \det[\mathbf{I}_e - \mathbf{A}^{-1}\mathbf{E}_E(\omega)\mathbf{E}_E^*(\omega)] = \prod_{i=0}^{e-1} [1 - \mathbf{A}^{-1}\sigma_i^2(\omega)] = c \quad (7.13)$$

Although the above analysis holds for any erasure pattern E with $e = |E| \leq P-M$, it is very difficult to characterize the original FB frame $\mathbf{E}(z)$ according to (7.13) when $e > 1$. However, we can obtain a simple characterization for the case $e = 1$, i.e., erasure of one channel. We state it formally by the following theorem on the necessary and sufficient condition for stable or FIR solution of the pseudo-inverse of the subframe $\mathbf{E}_S(z)$.

Theorem 7.21 *For any tight FB frame $\mathbf{E}(\omega)$ implemented by an FIR OFB, the pseudo-inverse of the subframe $\mathbf{E}_S(\omega)$ after the erasure of i th channel $h_i[n]$ ($0 \leq i \leq P-1$) is BIBO stable if and only if the Laurent polynomial $D_i(z) = 1 - \mathbf{A}^{-1}\mathbf{h}_i(z)\tilde{\mathbf{h}}_i(z)$ has no zeros except the possible zero at the origin. Furthermore, FIR solution for the pseudo-inverse $\mathbf{E}_S^\dagger(\omega)$ is possible if and only if the original tight FB frame $\mathbf{E}(\omega)$ is uniform, i.e., $P_i(\omega)$ is independent of frequency.*

Proof: According to the above analysis, $\tilde{\mathbf{E}}_S(z)\mathbf{E}_S(z) = \mathbf{A}\mathbf{I}_M - \tilde{\mathbf{h}}_i(z)\mathbf{h}_i(z)$, where $\mathbf{h}_i(z)$ is the polyphase row vector of the i th analysis filter $h_i[n]$ and the set $S = \mathcal{I}_P - \{i\}$. From (7.12), we can get the pseudo-inverse of subframe

$$\begin{aligned} \mathbf{E}_S^\dagger(z) &= \mathbf{A}^{-1}\tilde{\mathbf{E}}_S(z) + \mathbf{A}^{-1}\mathbf{I}_M\tilde{\mathbf{h}}_i(z)[1 - \mathbf{A}^{-1}\mathbf{h}_i(z)\tilde{\mathbf{h}}_i(z)]^{-1}\mathbf{h}_i(z)\mathbf{A}^{-1}\mathbf{I}_M\tilde{\mathbf{E}}_S(z) \\ &= \mathbf{A}^{-1}\tilde{\mathbf{E}}_S(z) + \mathbf{A}^{-2}[D_i(z)]^{-1}\tilde{\mathbf{h}}_i(z)\mathbf{h}_i(z)\tilde{\mathbf{E}}_S(z) \end{aligned} \quad (7.14)$$

There is an IIR filter $1/D_i(z)$ embedded in $\mathbf{E}_S^\dagger(z)$. Thus, the stability of the pseudo-inverse only depends on the poles of such IIR filters because the other parts of $\mathbf{E}_S^\dagger(z)$ are FIR. Due to its special form, the zeros of Laurent polynomial $D_i(z)$ are the poles of the IIR filter. Since the underlying FB is FIR with real-valued coefficients, $D_i(z) = 1 - A^{-1}\mathbf{h}_i(z)\mathbf{h}_i^T(z^{-1})$ corresponds to a type-I linear phase filter [155], i.e., a real-valued symmetric filter with odd length. Thus, the zeros of $D_i(z)$ have a structural distribution on the z -plane, i.e., if z_0 is a zero point, then $z_0^*, 1/z_0$ and $1/z_0^*$ are all zeros of $D_i(z)$. For a stable IIR filter, the poles must be located in the unit circle. However, if any zero of $D_i(z)$, i.e., the pole of IIR filter $1/D_i(z)$, is in the unit circle, there must be at least one pole $1/z_0$ outside of the unit circle. In addition, the pole on the unit circle $|z_0| = 1$ is not allowed either. Therefore, $D_i(z)$ cannot have any zeros except the possible zero at the infinity.

Furthermore, the pseudo-inverse $\mathbf{E}_S^\dagger(z)$ is FIR if and only if $D_i(z)$ is a nonzero constant over the unit circle, i.e., $D_i(\omega)$ is independent of ω . From the condition on robustness to one erasure, i.e., $P_i(\omega) < A$, we know that $D_i(\omega) = 1 - A^{-1}\mathbf{h}_i(\omega)\mathbf{h}_i^*(\omega) = 1 - A^{-1}P_i(\omega) > 0$. Thus, $D_i(\omega)$ becomes a nonzero constant if and only if $P_i(\omega)$ is independent of frequency. This results in the uniform property of original FB frame $\mathbf{E}(\omega)$. ■

As an example of tight FB frames, we study real-valued oversampled paraunitary FBs with filter length $L \leq 2M$ in detail. Assume the analysis filter $h_i[n]$ with polyphase components $\mathbf{h}_i(z) = \mathbf{a}_i + z^{-1}\mathbf{b}_i$ for $i = 0, 1, \dots, P-1$, where $\mathbf{a}_i$ and $\mathbf{b}_i$ are real-valued row vectors with length M . Then, the Laurent polynomial $D_i(z)$ simplifies to,

$$\begin{aligned} D_i(z) &= 1 - A^{-1}(\mathbf{a}_i + z^{-1}\mathbf{b}_i)(\mathbf{a}_i^T + z\mathbf{b}_i^T) = 1 - A^{-1}(\mathbf{a}_i\mathbf{a}_i^T + \mathbf{b}_i\mathbf{b}_i^T) - A^{-1}\mathbf{a}_i\mathbf{b}_i^T(z + z^{-1}) \\ &= 1 - A^{-1}\|\mathbf{h}_i\|^2 - A^{-1}\mathbf{a}_i\mathbf{b}_i^T(z + z^{-1}) \end{aligned} \quad (7.15)$$

- If $\mathbf{a}_i\mathbf{b}_i^T$ is nonzero

If $D_i(z)$ has complex-valued zero point z_0 , then it must also have zero point z_0^* since real-valued $D_i(z)$ corresponds to a type-I linear phase FIR filter. From $D_i(z_0) = D_i(z_0^*)$, we can see that $z_0 + 1/z_0 = z_0^* + 1/z_0^*$, which is equivalent to the constraint $|z_0| = 1$, zero point on the unit circle. However, the existence of the subframe requires that $D_i(z)$ is invertible on the unit circle. Thus, $D_i(z)$ cannot have complex zeros when $\mathbf{a}_i\mathbf{b}_i^T \neq 0$.

If $D_i(z)$ has real-valued zero point x , then $D_i(x) = 0$ leads to $x + x^{-1} = (A - \|\mathbf{h}_i\|^2)/\mathbf{a}_i\mathbf{b}_i^T$. The equation $x + x^{-1} = C$ has real-valued solutions if and only if $|C| \leq 2$. In addition, the two real-valued solutions of x have mirror distribution over the unit circle, i.e., if one solution $|x_1| \geq 1$ then the other solution $|x_2| \leq 1$. Thus, we must constrain $|(A - \|\mathbf{h}_i\|^2)/\mathbf{a}_i\mathbf{b}_i^T| < 2$ for the stability of the pseudo-inverse of the subframe. In addition,

$|(A - \|h_i\|^2)/\mathbf{a}_i \mathbf{b}_i^T| \neq 0$ since the condition for being robust to one erasure is that $P_i(\omega) < A$ for all $\omega \in [0, 2\pi]$. We can easily obtain $P_i(\omega)|_{\omega=0} = \|h_i[n]\|^2 < A$.

- If $\mathbf{a}_i \mathbf{b}_i^T$ is zero

In such case, the Laurent polynomial $D_i(z) = 1 - A^{-1}\|h_i[n]\|^2$ becomes a positive constant on the z -plane. Thus, the stability of the pseudo-inverse of the subframe is guaranteed. Furthermore, it is obvious that the pseudo-inverse is also an FIR FB. It can be checked that,

$$\begin{aligned} P_i(\omega) &= \mathbf{h}_i(\omega) \mathbf{h}_i^*(\omega) = (\mathbf{a}_i + \mathbf{b}_i e^{-j\omega})(\mathbf{a}_i^T + \mathbf{b}_i^T e^{j\omega}) \\ &= \mathbf{a}_i \mathbf{a}_i^T + \mathbf{b}_i \mathbf{b}_i^T + \mathbf{a}_i \mathbf{b}_i^T (e^{-j\omega} + e^{j\omega}) = \|h_i[n]\|^2 \end{aligned} \quad (7.16)$$

It is consistent with the theorem 7.21, i.e., uniform property of original tight FB frames.

For general oversampled paraunitary FBs with length $L \leq 2M$, the condition for FIR solution of the pseudo-inverse of the subframe $\mathbf{a}_i \mathbf{b}_i^T = 0$ for all $i = 0, 1, \dots, P-1$ does not contradict with the PU condition. However, it is incompatible with an important subclass of OPUFB, i.e., oversampled linear phase paraunitary FBs.

Corollary 7.3 *For any oversampled linear phase paraunitary FBs with odd M and length $L \leq 2M$, it is impossible for being robust to one erasure and having FIR solution of the pseudo-inverse of the subframe.*

Proof: For linear phase FBs, there is a relation held on $\mathbf{b}_i$, i.e., $\mathbf{b}_i = \mathbf{a}_i \mathbf{J}_M$ for $i = 0, 1, \dots, n_s - 1$ and $\mathbf{b}_i = -\mathbf{a}_i \mathbf{J}_M$ for $i = n_s, \dots, P-1$, where n_s denotes the number of symmetric filters. Then, the FIR condition for the pseudo-inverse becomes $\mathbf{a}_i \mathbf{J}_M \mathbf{a}_i^T = 0$ for all $i = 0, 1, \dots, P-1$. Take the trace and summation for the above equalities,

$$\sum_{i=0}^{P-1} \text{tr}[\mathbf{a}_i \mathbf{J}_M \mathbf{a}_i^T] = \sum_{i=0}^{P-1} \text{tr}[\mathbf{a}_i^T \mathbf{a}_i \mathbf{J}_M] = \sum_{i=0}^{P-1} \text{tr}[\mathbf{J}_M \mathbf{a}_i^T \mathbf{a}_i] \quad (7.17)$$

For such LP FBs, the analysis polyphase matrix $\mathbf{E}(z) = [\mathbf{h}_0^T(z), \mathbf{h}_1^T(z), \dots, \mathbf{h}_{P-1}^T(z)]^T$. The PU condition $\tilde{\mathbf{E}}(z) \mathbf{E}(z) = \mathbf{E}^T(z^{-1}) \mathbf{E}(z) \mathbf{A} \mathbf{I}_M$ leads to

$$\sum_{i=0}^{P-1} \mathbf{a}_i^T \mathbf{a}_i + \mathbf{b}_i^T \mathbf{b}_i = \mathbf{A} \mathbf{I}_M = \sum_{i=0}^{P-1} \mathbf{a}_i^T \mathbf{a}_i + \mathbf{J}_M \mathbf{a}_i^T \mathbf{a}_i \mathbf{J}_M \quad (7.18)$$

When left multiplying by $\mathbf{J}_M$ on both sides, it becomes

$$\sum_{i=0}^{P-1} \mathbf{J}_M \mathbf{a}_i^T \mathbf{a}_i + \mathbf{a}_i^T \mathbf{a}_i \mathbf{J}_M = \mathbf{A} \mathbf{J}_M \quad (7.19)$$

Take the trace on both sides. It can be seen that the right hand side becomes zero. However, the left hand side is equal to $A \neq 0$ when M is odd. ■

7.4 Optimal Quantized Frames without and with Erasures

In this section, we investigate the sensitivity of FB frame expansion to quantization noise added to the subband signals in absence of erasures. The reconstruction method from subbands is constrained to be linear, which is equivalent to using a synthesis FB. For mathematical tractability, we assume the quantization noise vector to be white and pairwise uncorrelated with identical variances σ_q^2 . It is well known that the optimal PR synthesis FB is the pseudo-inverse $\mathbf{E}^\dagger(z)$ and the minimum mean square error (MSE) is [126],

$$\sigma_e^2 = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \text{tr}[\mathbf{E}^\dagger(\omega) \mathbf{E}^{\dagger*}(\omega)] \frac{d\omega}{2\pi} = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \sum_{i=1}^M \frac{1}{\lambda_i(\omega)} \frac{d\omega}{2\pi} \quad (7.20)$$

7.4.1 Optimal Quantized Frames without Erasures

The above result reveals the optimal synthesis bank given a FB frame. In the rest of this chapter, we will assume using the pseudo-inverse for reconstruction when quantization is involved. Here, we want to further investigate the optimal FB frame under the same quantization model and find what properties of FB frame are induced by such optimality. Although we don't impose any constraint on the underlying frames, there is a constraint as in (7.3) on the integral sum of the spectral eigenvalues $\lambda_i(\omega)$ of FB frames. This leads naturally to defining an important quantity of FB frames, namely frame energy.

Definition 7.12 *The frame energy T of a FB frame is defined as $T \triangleq \sum_{i=1}^P \|h_i[n]\|^2$. For finite frame, it is just the sum of square of norms of the frame vectors.*

Without loss of generality, we can always scale matrix $\mathbf{E}(\omega)$ to let all frame elements with norm $\|\phi_i\| \leq 1$. Thus, the frame energy has an upper bound P . Now, our optimization problem becomes trying to minimize σ_e^2 given the constraint of upper bounded frame energy. This optimal FB frame in the sense of MSE, is the tight frame as stated in the following theorem.

Theorem 7.22 *When encoding with a FB frame in $l^2(\mathbb{Z})$ with frame energy T upper bounded by C , i.e., $T \leq C$ ($0 < C \leq P$), and decoding with the pseudo-inverse under the additive white and pairwise uncorrelated noise model, the reconstruction MSE is minimum if and only if the FB frame is tight, and the minimum MSE is $\frac{M}{C} \sigma_q^2$. Furthermore, the optimal frame is generally not unique although the equal-norm tight frame (ENTF) is always an optimal one.*

Proof: From the equality (7.3), we can easily see that the optimal frame occurs when frame energy T attains its upper bound C . In addition, from the form of MSE in (7.20) and the Euler-Lagrange condition of the calculus of variations [156], we know that the minimization of

MSE can be treated as a succession of ordinary minimization problems indexed by ω between 0 and 2π . Then we can see that the MSE is minimum when spectral eigenvalues $\lambda_i(\omega)$ are identical and constant over the unit circle, which is true if and only if the FB frame is tight. Thus, the sum of eigenvalues $\sum_{i=1}^M \lambda_i = M\lambda = C$, i.e., $\lambda_i(\omega) = C/M$. The minimum MSE then follows from (7.20). ■

This theorem reveals the optimality of tight FB frames to quantization. Note that this result is also applicable to finite frames. Compared to the previous results in [129] and [110] for finite and FB frames, respectively, which *only* stated the optimality of TFs among unit-norm frames, our result is more general since it states the optimality of tight frames over *all* frames and it can cover their results as a special case of ours. Moreover, theorem 7.22 reveals that it is the frame energy, not the unit-norm property, that dictates the optimal quantization performance of FB frames. In addition, the equal-norm condition is not desirable from the perspective of designing tight frames since it is very difficult, if not impossible, to impose both the column-wise condition (tight) and row-wise condition (equal-norm) *simultaneously* on frame $\mathbf{F}$, but still leave some degree of *design freedom*. So far, the well known Harmonic frames are the only known equal-norm, tight finite frames [111, 153] and the newly reported equal-norm FB frames with constrained length $L = 2M$ [147]. However, they are fixed when design specifications P and M are given. On the contrary, we can obtain some degrees of design freedom by removing equal-norm condition without destroying optimality on MSE. We give two examples of optimal tight frames with the same minimum MSE to validate the *nonuniqueness* of the optimal frames with respect to quantization. For the purpose of simplicity, we use finite frames with frame energy $T \leq P$ for illustration.

Example 1: *Optimal tight frame is an ENTF.*

For such frame, the frame vector norm $\|\phi_i\| = T/P$ for $i = 0, 1, \dots, P-1$. The eigenvalues of the frame operator are $\lambda_i = T/M$ for $i = 0, 1, \dots, M-1$. The minimum MSE is $\frac{M}{T}\sigma_q^2$. The existence of ENTF is shown by explicit construction in [111]. When $T = P$, it is just the unit-norm tight frame (UNTF) studied in [110, 129].

Example 2: *Optimal tight frame is an unequal norm TF.*

We show this by an explicit construction. Starting with any $P \times P$ orthogonal matrix $\mathbf{A}$ (for example $P \times P$ DCT matrix), we scale $\mathbf{A}$ with factor $\sqrt{T/M}$ and then delete any $P-M$ columns, which results in a matrix $\mathbf{F}$. It can be easily shown that $\mathbf{F}^T \mathbf{F} = \frac{T}{M} \mathbf{I}_M$, i.e., $\mathbf{F}$ represents a TF with frame bound T/M . Its eigenvalues are same as those of Example 1, thus it has the same MSE performance. However, the norms of frame vectors usually may not be equal to T/P although their sum is still equal to the frame energy T . However, in the extreme case of $T = P$, the optimal frame must be an UNTF.

7.4.2 Optimal Quantized Frames with Erasures

In the previous section, it has been shown that it is possible to design FB frames which are robust to e ($0 < e \leq P - M$) erasures. We assume such FB frames for the rest of this paper.

Here, we want to examine the effect of erasures on the performance of reconstruction by the pseudo-inverse of the subframe since it is optimally resistant to quantization noise shown in Theorem 7.22. For such source-independent reconstruction method, the reconstruction performance measured by MSE is determined only by the given FB frame and quantization variance. Thus, we attempt to find the optimal FB frame given quantization variance, and its properties induced by this optimality in the sense of MSE.

7.4.2.1 MSE with one erasure

Different from the previous studies [110, 129, 149] with assumption of equal probability distribution of channel failure, we incorporate a prior information (if impossible to obtain, just assume equal probability distribution) on the channel failures into our analysis. We state it formally by the following theorem.

Theorem 7.23 *When encoding with any FB frame implemented by an FIR or stable IIR OFB $\mathbf{E}(\omega)$ with given frame energy T upper bounded by some constant C , i.e., $T \leq C$ ($0 < C \leq P$), and decoding with the pseudo-inverse $\mathbf{E}_S^\dagger(\omega)$ of the subframe under the additive, pairwise uncorrelated noise model with equal variance σ_q^2 , assuming the probability of i th channel failure is q_i ($0 \leq q_i \leq 1$) with a constraint $\sum_{i=0}^{P-1} q_i = 1$, the reconstruction MSE averaged statistically over all possible one channel erasure is minimum if and only if the original FB frame is tight and uniform with the squared norm $P_i(\omega)$ given by,*

$$P_i(\omega) = \frac{C}{M} [1 - (P - M)\alpha_i], \text{ where } \alpha_i \triangleq \frac{\sqrt{q_i}}{\sum_{k=0}^{P-1} \sqrt{q_k}} \quad (7.21)$$

The minimum averaged MSE is

$$\overline{\text{MSE}}_1 = \left[1 + \frac{\sum_{i=0}^{P-1} \sum_{k=0}^{P-1} \sqrt{q_i q_k}}{M(P - M)} - \frac{1}{M} \right] \text{MSE}_0 \quad (7.22)$$

where $\text{MSE}_0 = \frac{M}{C} \sigma_q^2$ is the optimal reconstruction MSE without any erasure.

Proof: Assume the i th channel associated with the i th analysis filter $h_i[n]$ is lost. Denote the set $S = \mathcal{I}_P - \{i\}$ and corresponding subframe by $\mathbf{E}_S(\omega)$. From Theorem 7.22, we know that,

$$\overline{\text{MSE}}_1 = \sum_{i=1}^P q_i \frac{\sigma_q^2}{M} \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr}\{[\mathbf{E}_S^*(\omega) \mathbf{E}_S(\omega)]^{-1}\} d\omega \quad (7.23)$$

CHAPTER 7. ROBUSTNESS AND QUANTIZATION OF FILTER BANK FRAMES

By matrix inversion lemma $(\mathbf{A} - \mathbf{BCD})^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1} - \mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}$, we can obtain

$$[\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)]^{-1} = \mathbf{S}^{-1}(\omega) + \mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)[1 - \mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)]^{-1}\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega) \quad (7.24)$$

where $\mathbf{h}_i(\omega)$ is the polyphase row vector of the i th analysis filter and $\mathbf{S}(\omega)$ is the frame operator matrix with frame bounds A and B , i.e., $A\mathbf{I}_M \leq \mathbf{S}(\omega) \leq B\mathbf{I}_M$. Thus, we obtain

$$\text{tr}[\mathbf{E}_S^*(\omega)\mathbf{E}_S(\omega)]^{-1} = \text{tr}[\mathbf{S}^{-1}(\omega)] + [1 - \mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)]^{-1}\text{tr}[\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)] \quad (7.25)$$

since scalars can be extracted out of trace operation. By the equality $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$, the term $\text{tr}[\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)]$ can be simplified to $\text{tr}[\mathbf{h}_i(\omega)\mathbf{S}^{-2}(\omega)\mathbf{h}_i^*(\omega)]$ which is a scalar. The average MSE becomes

$$\overline{\text{MSE}}_1 = q_i \frac{\sigma_q^2}{M} \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr}[\mathbf{S}^{-1}(\omega)] d\omega + \sum_{i=1}^P q_i \frac{\sigma_q^2}{M} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathbf{h}_i(\omega)\mathbf{S}^{-2}(\omega)\mathbf{h}_i^*(\omega)}{1 - \mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)} d\omega \quad (7.26)$$

The first term is minimized if and only if the frame is tight and the frame energy attains its upper bound C , as seen from Theorem 7.22. In the following, we try to minimize the second term. From Corollary 7.1, we know $P_i(\omega) = C_i \leq B$. From the positive definiteness of $\mathbf{S}(\omega)$, we know that $B^{-1}\mathbf{I}_M \leq \mathbf{S}^{-1}(\omega) \leq A^{-1}\mathbf{I}_M$ which leads to the inequality $1 - \mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega) \leq 1 - B^{-1}\mathbf{h}_i(\omega)\mathbf{h}_i^*(\omega) = 1 - B^{-1}P_i(\omega)$. Similarly, we get $\mathbf{h}_i(\omega)\mathbf{S}^{-2}(\omega)\mathbf{h}_i^*(\omega) \geq \frac{P_i(\omega)}{B^2}$. Thus the critical part of the second term in (7.26) has a lower bound,

$$\sum_{i=0}^{P-1} \frac{q_i}{2\pi} \int_{-\pi}^{\pi} \frac{\mathbf{h}_i(\omega)\mathbf{S}^{-2}(\omega)\mathbf{h}_i^*(\omega)}{1 - \mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)} d\omega \geq \sum_{i=0}^{P-1} \frac{q_i}{2\pi} \int_{-\pi}^{\pi} \frac{B^{-2}P_i(\omega)}{1 - B^{-1}P_i(\omega)} d\omega \quad (7.27)$$

Equality can be achieved if and only if the frame is tight, i.e., $\mathbf{S}(\omega) = A\mathbf{I}_M = B\mathbf{I}_M$. Thus, we know that the tight frame is a necessary condition for being optimal in the sense of MSE.

Define the nonnegative variables $z_i(\omega) = P_i(\omega)/A \leq 1$ (seen from Corollary 7.1), then the lower bound becomes $A^{-1} \sum_{i=0}^{P-1} \frac{q_i}{2\pi} \int_{-\pi}^{\pi} \frac{z_i(\omega)}{1 - z_i(\omega)} d\omega$. Next, it becomes a standard optimization problem with objective function $J = \sum_{i=0}^{P-1} \frac{q_i}{2\pi} \int_{-\pi}^{\pi} \frac{z_i(\omega)}{1 - z_i(\omega)} d\omega$ subject to the constraint on frame energy $\int_{-\pi}^{\pi} \sum_{i=0}^{P-1} P_i(\omega) d\omega / 2\pi = \sum_{i=0}^{P-1} \|h_i[n]\|^2 = T = C$ which is equivalent to $\int_{-\pi}^{\pi} \sum_{i=0}^{P-1} z_i(\omega) d\omega / 2\pi = C/A$. Since J is a convex function over $z_i(\omega)$, by Euler-Lagrange condition in the calculus of variations [156], we can show that J attains minimum if and only if the partial derivatives of J vanish. Thus, we can obtain,

$$z_i(\omega) = 1 - (P - \frac{C}{A}) \frac{\sqrt{q_i}}{\sum_{k=0}^{P-1} \sqrt{q_k}} = 1 - (P - \frac{C}{A}) \alpha_i \quad (7.28)$$

CHAPTER 7. ROBUSTNESS AND QUANTIZATION OF FILTER BANK FRAMES

First, it can be easily seen that the optimal FB frame should be a uniform frame since $P_i(\omega) = Az_i(\omega)$ is independent of frequency. Next, the optimal FB frame usually demands the unequal-norm property of the original frame $\mathbf{E}(\omega)$ except when all α_i are equal, i.e., equal probability of channel failure. Since the FB frame is tight, the spectral eigenvalues $\lambda_i(\omega) = A = C/M$ for all $\omega \in [0, 2\pi]$ and $i = 0, \dots, M-1$. Thus, the squared norm of $P_i(\omega)$ is a constant, but dependent on the ratio α_i , i.e., $P_i(\omega) = C_i = \frac{C}{M}[1 - (P-M)\alpha_i]$. Furthermore, it can be easily seen that such $P_i(\omega)$ satisfies the existence condition stated in Corollary 7.1. In addition, it can be shown that it is also sufficient for the existence of a tight FB frame [153].

From (7.26) and the inequality (7.27), we can derive a useful formula for averaged MSE with one erasure given tight FB frames with specific set of squared norm $P_i(\omega)$.

$$\begin{aligned} \overline{\text{MSE}}_1 &= \frac{M}{C}\sigma_q^2 + \frac{\sigma_q^2}{M} \sum_{i=1}^P q_i \frac{A^{-2}C_i}{1 - A^{-1}C_i} = \frac{M}{C}\sigma_q^2 + \frac{\sigma_q^2}{C} \sum_{i=1}^P q_i \frac{C_i}{A - C_i} \\ &= \left[1 + \sum_{i=1}^P q_i \frac{C_i}{C - MC_i} \right] \frac{M}{C}\sigma_q^2 \end{aligned} \quad (7.29)$$

Finally, the minimum averaged MSE comes from the optimal distribution of squared norm $P_i(\omega)$ according to a prior information on channel conditions q_i . Thus, the minimum averaged MSE of (7.22) can be obtained by replacing optimal $C_i = P_i(\omega)$ of (7.21) in the formula (7.29). ■

From the above theorem, we can see that the ratio α_i related to probability distribution of channel failures determines the squared norm $P_i(\omega)$ for being optimal in the sense of MSE performance. For equal probability distribution, i.e., $q_i = 1/P$ for all $i = 0, 1, \dots, P-1$, it can be easily seen that $P_i(\omega) = C/P$ for all channels, i.e., ENTF. This has been studied in [129] and [110, 149] for finite frames and FB frames, respectively. In their derivation, the equal norm property is imposed *a priori*. Thus, their results only showed the optimality of UNTF over the class of equal norm frames. However, compared with them, this theorem reveals the equal-norm property is actually *necessary* for being optimal in the sense of MSE. This shows the optimality of ENTF over a larger class of frames than theirs and covers their result as a special case.

However, for unequal probability distribution of channel failures, the optimal FB frame is an unequal norm tight frame. According to the formula on squared norm $P_i(\omega)$, we can see that the optimal solution is similar to the water-filling problem in information theory [2].

From our assumption and physical limitations, the squared norm $P_i(\omega) = C_i$ must be constrained in the interval $[0, 1]$. By the formula of $P_i(\omega)$, we can see that $\max_i\{\alpha_i\}$ corresponds to $\min_i\{C_i\}$. Thus, the condition $C_i \geq 0$ derives an upper bound on the ratio $\alpha_i \leq \frac{1}{P-M}$. It can be seen that the equal probability distribution of channel failure, i.e., $\alpha_i = 1/P$ for all i , satisfies the above condition.

7.4.2.2 MSE with more than one erasures

In this section, we analyze the FB frames robust to more than one erasures and further show a nice property of such frames that the reconstruction gets better with *any* additional subbands after receiving at least the necessary $P - e$ subbands.

Theorem 7.24 *For any FB frame being robust to e ($1 \leq e \leq P - M$) erasures, after receiving any $P - e$ subbands, the reconstruction MSE is reduced by any additional subband.*

Proof: Denote $\mathbf{E}_S(\omega)$ as the subframe with $P - e$ subbands received correctly. If one more subband is received correctly, the subframe is denoted by $\mathbf{E}_{\hat{S}}(\omega)$. Assuming such additional subband corresponding to the analysis filter $h_i[n]$, it can be easily seen that $\mathbf{E}_{\hat{S}}(\omega)$ corresponds to the partitioned matrix form $\mathbf{P} [\mathbf{E}_S^T(\omega), \mathbf{h}_i^T(\omega)]^T$, where $\mathbf{P}$ is a square permutation matrix with size $P - e + 1$ and $\mathbf{h}_i(\omega)$ is the row vector of polyphase components of the analysis filter $h_i[n]$. Since the matrix $\mathbf{P}$ has no impact on our later eigenvalue analysis and following results, we can omit it in the following analysis, which is equivalent to $\mathbf{P} = \mathbf{I}$. Now, we can obtain the relationship of the frame operator matrix and Gram matrix between the two subframes $\mathbf{E}_S(\omega)$ and $\mathbf{E}_{\hat{S}}(\omega)$, as following,

$$\hat{\mathbf{S}}(\omega) = \mathbf{E}_{\hat{S}}^*(\omega) \mathbf{E}_{\hat{S}}(\omega) = \mathbf{E}_S^*(\omega) \mathbf{E}_S(\omega) + \mathbf{h}_i^*(\omega) \mathbf{h}_i(\omega) = \mathbf{S}(\omega) + \mathbf{h}_i^*(\omega) \mathbf{h}_i(\omega) \quad (7.30)$$

$$\hat{\mathbf{G}}(\omega) = \mathbf{E}_{\hat{S}}(\omega) \mathbf{E}_{\hat{S}}^*(\omega) = \begin{bmatrix} \mathbf{G}(\omega) & \mathbf{E}_S(\omega) \mathbf{h}_i^*(\omega) \\ \mathbf{h}_i(\omega) \mathbf{E}_S^*(\omega) & P_i(\omega) \end{bmatrix} \quad (7.31)$$

where $\hat{\mathbf{S}}(\omega)$ and $\mathbf{S}(\omega)$ denote the frame operator matrices for the subframes $\mathbf{E}_{\hat{S}}(\omega)$ and $\mathbf{E}_S(\omega)$, respectively. Similarly, the matrices $\hat{\mathbf{G}}(\omega)$ and $\mathbf{G}(\omega)$ represent the Gram matrices for $\mathbf{E}_{\hat{S}}(\omega)$ and $\mathbf{E}_S(\omega)$, respectively.

The reconstruction MSE achieved by the pseudo-inverse of the subframe $\mathbf{E}_{\hat{S}}(\omega)$ is,

$$\hat{\sigma}_e^2 = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \text{tr} [\mathbf{E}_{\hat{S}}^*(\omega) \mathbf{E}_{\hat{S}}(\omega)]^{-1} \frac{d\omega}{2\pi} = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \sum_{i=0}^{M-1} \frac{1}{\hat{\lambda}_i(\omega)} \frac{d\omega}{2\pi} \quad (7.32)$$

where $\hat{\lambda}_i(\omega)$ are the spectral eigenvalues of the frame operator $\hat{\mathbf{S}}(\omega)$. Similarly, the MSE by subframe $\mathbf{E}_S(\omega)$ can be obtained as follows,

$$\sigma_e^2 = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \text{tr} [\mathbf{E}_S^*(\omega) \mathbf{E}_S(\omega)]^{-1} \frac{d\omega}{2\pi} = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \sum_{i=0}^{M-1} \frac{1}{\lambda_i(\omega)} \frac{d\omega}{2\pi} \quad (7.33)$$

where $\lambda_i(\omega)$ are the spectral eigenvalues of the frame operator $\mathbf{S}(\omega)$.

In the following, we show an explicit relation between the two sets of positive eigenvalues $\{\hat{\lambda}_i(\omega)\}$ and $\{\lambda_i(\omega)\}$, which leads to the inequality for reduction of MSE and furthermore gives

CHAPTER 7. ROBUSTNESS AND QUANTIZATION OF FILTER BANK FRAMES

more insights over the impact of the additional subband on the eigenvalue perturbation. First, it can be easily seen that $\mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega)$ is a positive semidefinite, Hermitian matrix. Assume that both sets of eigenvalues $\hat{\lambda}_i(\omega)$ and $\lambda_i(\omega)$ are arranged in increasing order, i.e., $\lambda_0(\omega) \leq \lambda_1(\omega) \leq \dots \leq \lambda_{M-1}(\omega)$. Then we can obtain the following set of inequalities from Weyl theorem [79],

$$\lambda_i(\omega) \leq \hat{\lambda}_i(\omega) \quad \text{for all } i = 0, 1, \dots, M-1 \quad (7.34)$$

Moreover, if we take the trace operation on both sides of (7.30), we can get

$$\sum_{i=0}^{M-1} \hat{\lambda}_i(\omega) = \text{tr}[\hat{\mathbf{S}}(\omega)] = \text{tr}[\mathbf{S}(\omega)] + \text{tr}[\mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega)] = \sum_{i=0}^{M-1} \lambda_i(\omega) + P_i(\omega) \quad (7.35)$$

We can see that all the M equalities in the set of inequalities (7.34) may be attained if and only if $P_i(\omega) = 0$, i.e., the i th analysis filter $h_i[n]$ is a zero filter. Thus, for nontrivial FB frames, there is at least one strict inequality that holds in (7.34). The reduction on reconstruction MSE, by such one additional subband corresponding to nonzero analysis filter $h_i[n]$, follows from the MSE formula shown in (7.32) that gives a strict inequality in (7.34). ■

In addition, we know that the eigenvalues $\hat{\lambda}_i(\omega)$ are also the M positive eigenvalues of the Gram matrix $\hat{\mathbf{G}}(\omega)$ with size $P - e + 1$. All the other eigenvalues of $\hat{\mathbf{G}}(\omega)$ are zero. Similarly, $\lambda_i(\omega)$ are the M positive eigenvalues of the Gram matrix $\mathbf{G}(\omega)$. We still assume the increasing order on both sets of positive eigenvalues $\hat{\lambda}_i(\omega)$ and $\lambda_i(\omega)$. Then, from the special form of $\hat{\mathbf{G}}(\omega)$ in (7.31) and the interlacing eigenvalues theorem for bordered matrices [79], we can establish the following inequality,

$$\lambda_0(\omega) \leq \hat{\lambda}_0(\omega) \leq \lambda_1(\omega) \leq \hat{\lambda}_1(\omega) \leq \dots \leq \lambda_{M-1}(\omega) \leq \hat{\lambda}_{M-1}(\omega) \quad (7.36)$$

Next, we show a bound on the reduction of MSE by one additional subband $h_i[n]$. The reduction of MSE is,

$$\Delta\text{MSE} \triangleq \sigma_e^2 - \hat{\sigma}_e^2 = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \sum_{i=0}^{M-1} \left[\frac{1}{\lambda_i(\omega)} - \frac{1}{\hat{\lambda}_i(\omega)} \right] \frac{d\omega}{2\pi} = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \sum_{i=0}^{M-1} \left[\frac{\hat{\lambda}_i(\omega) - \lambda_i(\omega)}{\lambda_i(\omega)\hat{\lambda}_i(\omega)} \right] \frac{d\omega}{2\pi} \quad (7.37)$$

Then, we can obtain a simple lower bound on the ΔMSE . Since the explicit interlacing eigenvalues are related by one more subband, we know that all the eigenvalues in the inequality (7.36) is upper bounded by the frame bound B of the original frame. Thus, we can get,

$$\Delta\text{MSE} \geq \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \frac{1}{B^2} \sum_{i=0}^{M-1} [\hat{\lambda}_i(\omega) - \lambda_i(\omega)] \frac{d\omega}{2\pi} = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \frac{P_i(\omega)}{B^2} \frac{d\omega}{2\pi} = \frac{\sigma_q^2}{M} \frac{\|h_i[n]\|^2}{B^2} \quad (7.38)$$

CHAPTER 7. ROBUSTNESS AND QUANTIZATION OF FILTER BANK FRAMES

The last step in the above derivation comes from the orthogonality of the polyphase components of the analysis filter $h_i[n]$, i.e., the equality $\int_{-\pi}^{\pi} P_i(\omega) d\omega / 2\pi = \|h_i[n]\|^2$ holds. Since the frame bound $0 < B < \infty$, we know the ΔMSE is always positive for nonzero filter $h_i[n]$, i.e., *the MSE is always reduced by nonzero subbands*. However, without any other property of original FB frame, we cannot further simplify the above lower bound.

However, if the original FB frame $\mathbf{E}(\omega)$ is tight, $A = B = T/M$, then the lower bound of reduction of MSE simplifies to,

$$\Delta\text{MSE} \geq \frac{\sigma_q^2 \|h_i[n]\|^2}{M T^2/M^2} = \frac{\|h_i[n]\|^2}{T} \left[\frac{M}{T} \sigma_q^2 \right] = \frac{\|h_i[n]\|^2}{T} \text{MSE}_0 \quad (7.39)$$

We can see that the squared norm of the analysis filter corresponding to additional received subband contributes to the lower bound of ΔMSE . As a special case, we consider the UNTF with $e = 1$, then our lower bound becomes $\frac{1}{P} \text{MSE}_0$. However, we already know that the exact reduction of MSE is $\overline{\text{MSE}}_1 - \text{MSE}_0 = \frac{1}{P-M} \text{MSE}_0 > \frac{1}{P} \text{MSE}_0$. The lower bound is loose when the oversampling ratio r is near to 1 and the number of subbands P is large. In addition, the lower bound is independent of the number of erasures e .

Next, we consider the upper bound on the reduction of MSE by one more additional subband. It can be easily seen from (7.37) and the inequality (7.36),

$$\Delta\text{MSE} \leq \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \sum_{i=0}^{M-1} \left[\frac{\hat{\lambda}_i(\omega) - \lambda_i(\omega)}{\lambda_i^2(\omega)} \right] \frac{d\omega}{2\pi} \quad (7.40)$$

However, without further information on the subframe $\mathbf{E}_S(\omega)$, we cannot obtain further result other than the integral form in (7.40). Due to the desirable properties of tight frames, we further assume the subframe $\mathbf{E}_S(\omega)$ is tight, i.e., constrain a subclass of FB frames which have at least one tight subframe. Since $\mathbf{S}(\omega) = A_S \mathbf{I}_M$ where A_S is the frame bound for $\mathbf{E}_S(\omega)$, we can get the exact eigenvalue distribution of $\hat{\mathbf{S}}(\omega)$ from (7.30), assuming the increasing order among the eigenvalues $\hat{\lambda}_i(\omega)$,

$$\hat{\lambda}_1(\omega) = \hat{\lambda}_2(\omega) = \dots = \hat{\lambda}_{M-2}(\omega) = A_S, \quad \hat{\lambda}_{M-1}(\omega) = A_S + P_i(\omega) \quad (7.41)$$

Thus, the reduction of MSE can be computed as follows,

$$\Delta\text{MSE} = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \left[\frac{1}{A_S} - \frac{1}{A_S + P_i(\omega)} \right] \frac{d\omega}{2\pi} = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \frac{P_i(\omega)}{A_S(A_S + P_i(\omega))} \frac{d\omega}{2\pi} \quad (7.42)$$

From Corollary 7.2, we know that $P_i(\omega)$ is bounded, i.e., $0 \leq P_i(\omega) \leq B$. Therefore, we can bound ΔMSE as follows,

$$\frac{\sigma_q^2}{M} \frac{\|h_i[n]\|^2}{A_S(A_S + B)} = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \frac{P_i(\omega)}{A_S(A_S + B)} \frac{d\omega}{2\pi} \leq \Delta\text{MSE} \leq \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \frac{P_i(\omega)}{A_S^2} \frac{d\omega}{2\pi} = \frac{\sigma_q^2}{M} \frac{\|h_i[n]\|^2}{A_S^2} \quad (7.43)$$

CHAPTER 7. ROBUSTNESS AND QUANTIZATION OF FILTER BANK FRAMES

The frame bound is $A_S = T_S/M$, where T_S is the frame energy of the tight subframe $\mathbf{E}_S(\omega)$. Since $T_S = T - \sum_{k \in E} \|h_k[n]\|^2$, where E is the index set for erasure pattern, then the frame bound is $A_S = (T - \sum_{k \in E} \|h_k[n]\|^2)/M$. Thus, the reduction of MSE when $\mathbf{E}_S(\omega)$ is a tight subframe is upper bounded by,

$$\Delta \text{MSE} \leq \frac{M\sigma_q^2}{T - \sum_{k \in E} \|h_k[n]\|^2} \frac{\|h_i[n]\|^2}{T - \sum_{k \in E} \|h_k[n]\|^2} = \left[\frac{M}{T} \sigma_q^2 \right] \frac{\|h_i[n]\|^2}{(T - \sum_{k \in E} \|h_k[n]\|^2)^2/T} \quad (7.44)$$

When the original FB frame $\mathbf{E}(\omega)$ is tight (note that in such case, $P \geq 2M$ as seen from Theorem 7.20, combined with the lower bound in (7.39), we can obtain the bounds for the reduction of MSE,

$$\frac{\|h_i[n]\|^2}{T} \text{MSE}_0 \leq \Delta \text{MSE} \leq \frac{\|h_i[n]\|^2}{(1 - \sum_{k \in E} \frac{\|h_k[n]\|^2}{T})(T - \sum_{k \in E} \|h_k[n]\|^2)} \text{MSE}_0 \quad (7.45)$$

Moreover, if the original FB frame has the property of uniformness and equal norm, then the above bounds can be simplified to,

$$\frac{1}{P} \text{MSE}_0 \leq \Delta \text{MSE} \leq \frac{1}{(1 - e/P)(P - e)} \text{MSE}_0 \quad (7.46)$$

where e is the number of erasures, $e = |E|$.

The above theorem and analysis reveal a very good property of FB frames and shows their potential application to multiple description coding for erasure networks like Internet [12] to be presented in the next section. However, the generality of these results, especially the tightness of various bounds on the reduction of MSE ΔMSE , is still needed to be further investigated in future.

7.5 An Example of Multiple Description Coding Based on Quantized Filter Bank Frames: MSE Analysis

As an illustration, we use a two-state channel model to show the optimal unequal norm tight FB frames for transmission of information over erasure networks.

One group of K ($1 \leq K \leq P - 1$) channels with equal probability q_g of channel failure represents the good channels; the other group of $P - K$ channels with same q_b represents the bad channels. Denote the ratio α_g and α_b as in (7.21) for the good channels and bad channels with an assumption $\alpha_g = \frac{\epsilon}{K}$ for $i = 0, 1, \dots, K - 1$ and $\alpha_b = \frac{\epsilon}{P - K}$ for $i = K, \dots, P - 1$, respectively. Due to the symmetric form, we only need to consider ϵ to be very small. In our setting, $\epsilon \leq K/P$ is enough since this corresponds to $\alpha_g \leq \alpha_b$. For some $\epsilon > K/P$, there exist an $\epsilon' \leq K/P$ and different K with same MSE performance as ϵ . This can be achieved just

by interchanging the labels “good” and “bad” for the two groups of channels. For example, if there is a pair $\{\epsilon_1, K_1\}$ with $\epsilon_1 > K/P$, then there exists another pair $\{\epsilon_2, K_2\}$ with $\epsilon_2 = 1 - \epsilon_1$ and $K_2 = P - K_1$. It can be shown that $\epsilon_2 < K_2/P$.

Thus, we only consider the case of $\epsilon \leq K/P$ (corresponding to $\alpha_g \leq \alpha_b$) with frame energy $T \leq P$. If $\epsilon = K/P$, then $\alpha_g = \alpha_b = 1/P$, which corresponds to the equal probability distribution with $q_i = 1/P$ for all $i = 0, 1, \dots, P-1$. After some algebra, it can be shown that the probabilities of channel failure are,

$$q_g = \frac{\epsilon^2(P-K)/K}{(P-K)\epsilon^2 + K(1-\epsilon)^2}, \quad q_b = \frac{(1-\epsilon)^2K/(P-K)}{(P-K)\epsilon^2 + K(1-\epsilon)^2} \quad (7.47)$$

Since we need $\alpha_i \leq \frac{1}{P-M}$ for all $i = 0, 1, \dots, P-1$, we have $\min_i\{\alpha_i\} = \alpha_b = \frac{1-\epsilon}{P-K} \leq \frac{1}{P-M}$. This gives a lower bound as $\epsilon \geq \frac{K-M}{P-M}$. Thus, we need to consider two cases according to different ranges of K .

7.5.1 The Case of $1 \leq K \leq M$

In this case, the lower bound on ϵ becomes nonpositive. Thus, we have $\alpha_i \leq \frac{1}{P-M}$ for all $i = 0, 1, \dots, P-1$, which means the squared norm $P_i(\omega)$ is nonnegative, $P_i(\omega) \geq 0$. According to the upper bound C of frame energy, we need to consider two cases described below.

[a] when the upper bound of frame energy $C \leq \frac{M}{1-(P-M)\alpha_g}$.

In this case, $\max_i\{C_i\} = C_g = \frac{T}{M}[1 - (P-M)\frac{\epsilon}{K}] \leq 1$. Thus, it is a realizable optimal solution. Since $\sum_{i=0}^{P-1} \sum_{k=0}^{P-1} \sqrt{q_i q_k} = K^2 q_g + (P-K)^2 q_b + 2K(P-K)\sqrt{q_g q_b}$ and,

$$K^2 q_g = \frac{K(P-K)\epsilon^2}{(P-K)^2\epsilon^2 + K(1-\epsilon)^2} \quad (7.48)$$

$$(P-K)^2 q_b = \frac{K(P-K)(1-\epsilon)^2}{(P-K)^2\epsilon^2 + K(1-\epsilon)^2} \quad (7.49)$$

$$2K(P-K)\sqrt{q_g q_b} = \frac{2K(P-K)\epsilon(1-\epsilon)}{(P-K)\epsilon^2 + K(1-\epsilon)^2} \quad (7.50)$$

we can obtain,

$$\sum_{i=0}^{P-1} \sum_{k=0}^{P-1} \sqrt{q_i q_k} = \frac{K(P-K)}{(P-K)\epsilon^2 + K(1-\epsilon)^2} \quad (7.51)$$

Thus, the optimal MSE can be obtained from (7.22)

$$\overline{\text{MSE}}_1 = \left[1 + \frac{K(P-K)}{M(P-M)} \times \frac{1}{(P-K)\epsilon^2 + K(1-\epsilon)^2} - \frac{1}{M} \right] \frac{M}{C} \sigma_q^2 \quad (7.52)$$

Similar to the case of no erasure (as seen from Theorem 7.22), it can be seen that the optimal MSE averaged over all possible erasures of one channel is also inversely proportional to the frame energy.

[b] For $\epsilon < k/P$, the frame energy T can be increased to a maximum up to P .

When $\epsilon = K/P$, the upper bound frame energy $C = \frac{M}{1-(P-M)/P} = P$, i.e., is already the maximum. However, when $\epsilon < K/P$, i.e., $C = \frac{M}{1-(P-M)\epsilon/K} < P$, and the frame energy is still increased from C up to the maximum P , the optimal solution in Theorem 7.23 is not realizable because the squared norm $P_i(\omega) = C_g = \frac{C}{M}[1 - (P - M)\epsilon/K] > 1$, which violates our assumption on $P_i(\omega)$. In such a case, we have to modify the optimal solution to be realizable. It is a modified water-filling problem, i.e., water-filling with bound on power increase, which is similar to the optimal bit allocation problem in optimal subband coding [1].

In the current setting, it can be easily seen that the optimal solution can be achieved by constraining $C_g = 1$ for K good channels and distributing the energy left equally to the other $P - K$ channels, i.e., $C_b = \frac{C-KC_g}{P-K} = \frac{C-K}{P-K} \leq 1$. Then, the MSE can be computed by (7.29). Since $\sum_{i=0}^{P-1} q_i \frac{C_i}{C-MC_i} = Kq_g \frac{C_g}{C-MC_g} + (P-K)q_b \frac{C_b}{C-MC_b}$,

$$Kq_g \frac{C_g}{C-MC_g} = \frac{(P-K)\epsilon^2}{(P-K)\epsilon^2 + K(1-\epsilon)^2} \frac{1}{C-M} \quad (7.53)$$

$$(P-K)q_b \frac{C_b}{C-MC_b} = \frac{K(1-\epsilon)^2}{(P-K)\epsilon^2 + K(1-\epsilon)^2} \frac{C-K}{C(P-K)-M(C-K)} \quad (7.54)$$

Finally, the minimum averaged MSE computed from (7.29) is,

$$\overline{\text{MSE}}_1 = \left[1 + \frac{\epsilon^2 \frac{P-K}{C-M} + (1-\epsilon)^2 \frac{K(C-K)}{C(P-K)-M(C-K)}}{(P-K)\epsilon^2 + K(1-\epsilon)^2} \right] \frac{M}{C} \sigma_q^2 \quad (7.55)$$

It can be easily checked that if $C = P$, the above formula is same as that of UNTF. In addition, if $C = \frac{M}{1-(P-M)\epsilon/K}$, it is same as (7.52).

7.5.2 The Case of $M < K \leq P - 1$

First, note that $P > M + 1$ is required for the existence of K in this case. The lower bound on ϵ becomes positive. We must partition the range of ϵ into two intervals $I_1 = [0, \frac{K-M}{P-M})$ and $I_2 = [\frac{K-M}{P-M}, \frac{K}{P}]$, and consider them separately.

[a] For the interval I_1 , we have $\alpha_b > \frac{1}{P-M}$, which leads to negative $P_i(\omega)$ from Theorem 7.23. Thus, we have to change the unrealizable optimal solution to a realizable “optimal solution” discussed later. First, let us check about the unrealizable optimal solution.

- In this interval, $C_b(\omega) = \frac{C}{M}[1 - (P - M)\alpha_b] < 0$ due to $\alpha_b > \frac{1}{P-M}$. The unrealizable minimum averaged MSE is same as (7.52). Let us assume $\epsilon = 0$, then the MSE

CHAPTER 7. ROBUSTNESS AND QUANTIZATION OF FILTER BANK FRAMES

simplifies to $\overline{\text{MSE}}_1 = [1 + \frac{M-K}{(P-M)M}] \frac{M}{C} \sigma_q^2$. Since $K > M$, we can see that $\overline{\text{MSE}}_1 < \text{MSE}_0 = \frac{M}{C} \sigma_q^2$, which means that the MSE after one erasure is even less than that without erasures! This is due to the negative squared norm of the erased channel $h_i[n]$. However, it is obviously a contradiction in a physically realizable system.

- In such case, we need to modify our solution to be realizable and try to find the optimal one. First, we consider the case of the upper bound C of frame energy to be less than or equal to K . It can be seen that we need to constrain $C_b = 0$ for $j = K, \dots, P-1$ and let $C_g = C/K \leq 1$ for $i = 0, \dots, K-1$. Then, it can be easily computed,

$$\sum_{i=0}^{P-1} q_i \frac{C_i}{C - MC_i} = K q_g \frac{C_g}{C - MC_g} = \frac{(P-K)\epsilon^2}{(P-K)\epsilon^2 + K(1-\epsilon)^2} \frac{1}{K-M} \quad (7.56)$$

Thus, the averaged MSE can be calculated from (7.29),

$$\overline{\text{MSE}}_1 = \left[1 + \frac{P-K}{K-M} \frac{\epsilon^2}{(P-K)\epsilon^2 + K(1-\epsilon)^2} \right] \frac{M}{C} \sigma_q^2 \quad (7.57)$$

It can be seen that the modified averaged MSE is still inversely proportional to the upper bound C of the frame energy.

- Let the upper bound C be increased from K up to the maximum P , i.e., $K \leq C \leq P$. From the previous case, we know that we can only increase C_b from zero to a positive number. Thus, the norm distribution is $C_g = 1$ for $i = 0, \dots, K-1$ and let $C_b = \frac{T-K}{P-K} \leq 1$ for $j = K, \dots, P-1$. Since the norm distribution is the same as case 1)-[b], we know that the MSE formula is same as (7.55).

[b] For the interval $I_2 = [\frac{K-M}{P-M}, \frac{K}{P}]$, it can be seen that the nonnegativeness of $P_i(\omega)$ is ensured.

- When the upper bound of frame energy $C \leq \frac{M}{1-(P-M)\alpha_g}$, The optimal water-filling solution is realizable. Thus, the MSE formula is the same as (7.52). Same as before, the optimal averaged MSE is inversely proportional to the upper bound of the frame energy. In addition, it can be easily checked that if $\epsilon = K/P$, i.e., equal probability distribution, the upper bound C is just the maximum frame energy P , and the averaged MSE is the same as that of UNTF.
- However, when $\epsilon < K/P$, i.e., $C = \frac{M}{1-(P-M)\epsilon/K} < P$, if the upper bound C of the frame energy is still increased up to the maximum P , the optimal solution in Theorem 7.23 is not realizable because the squared norm $P_i(\omega) = C_g = \frac{T}{M}[1 - (P -$

$M)\epsilon/K] > 1$, which violates our assumption on $P_i(\omega)$. In such a case, we have to modify the optimal solution to be realizable. In the current setting, it can be easily seen that the optimal solution can be achieved by a constraint $C_g = 1$ for K good channels and distributing the left energy equally to the other $P - K$ channels, i.e., $C_b = \frac{T-KC_g}{P-K} = \frac{T-K}{P-K} \leq 1$. Then, the MSE formula is the same as (7.55).

7.6 Summary

This chapter studies FB frames in infinite dimensional Hilbert space $l^2(\mathbb{Z})$ from the perspective of both frame and FB theory. Borrowing the concepts from these two areas and the fundamental link between them, we can conduct general theoretical analysis as well as consider real design of FB frames, simultaneously. It has been shown that the eigen-structures of the polynomial matrix representation of frame operator play an important role in the analysis of FB frames. We gave a classification of various FB frames according to the existing notation and expanded it by introducing a new class of FB frames, named uniform frames, which is very desirable for the possible FIR pseudoinverse of the subframes.

Since quantization is an inevitable part of digital transmission of information, we have analyzed the quantization effect on the reconstruction by the pseudoinverse and the optimal FB frames based on the popular additive, pair-wise uncorrelated quantization noise model. In particular, it has been shown that the equal-norm property, imposed frequently in the literature, is not necessary for optimization of frames under the quantization model. Moreover, this property is not desirable from the viewpoint of designing FB frames. In this chapter, we have introduced a novel and important notion, named frame energy, which actually dictates the quantization performance of FB frames. In addition, we have shown that tight FB frames are optimal for quantization with a constraint on frame energy over all FB frames. This result reveals the important universal optimality of tight FB frames, in contrast to the restricted optimality only among UNTFs reported before. Furthermore, the design freedom obtained from removing the equal-norm property has been explained and illustrated with examples.

We have then analyzed the robustness of FB frames to erasures. Firstly, we have given a characterization of the maximum spectral singular values of the possible subframes and then specialized it to characterize the basic constraint on the analysis filters of FB frames. Especially, an existence condition is given for tight FB frames. In case of erasures, a sufficient condition is obtained for general FB frames robust to e ($0 < e \leq P - M$) erasures without the unnecessary equal-norm property. Furthermore, we have shown that our robust condition is also necessary for tight FB frames. A special case of our result generalized the previous reported result and

indicated the equal-norm property is useless for being robust to erasures. In addition, a more general analysis on MR FB frames has been explored based on explicit constructive approach which can serve as a guideline for designing MR FB frames. Based on these analyses, we have investigated the properties of the possible subframes of tight FB frames. A classification has been given according to the tightness of the subframes, while the necessary and sufficient conditions has been shown for tight subframes. The pseudoinverse of the subframe is usually a hybrid FIR and IIR FB. Thus, we have examined the stability and even possible FIR solution of such pseudoinverse. It has been shown that the uniformity of the squared norm $P_i(\omega)$ of the analysis filter over all frequency is the necessary and sufficient condition for the FIR pseudoinverse of the subframe after one erasure of tight FB frames. As an example, we have studied the order-one tight FB frames in detail and shown that it is impossible to obtain an FIR pseudoinverse of the subframe after one erasure of order-one tight FB frames with linear phase filters and odd M .

Finally, the effect of erasures on the reconstruction MSE for general FB frames has been examined. Different from before, we have incorporated a priori information on possible erasure distribution into our analysis, which makes our result more general and meaningful. In particular, we have shown that the tight FB frames are optimal over very general FB frames. In contrast to the previous results, we have shown that the optimal tight FB frames are usually not ENTF. The norm distribution is dependent on the a priori information of the channel failure and follows the reverse water-filling principle. An illustrative example of two-state erasure network model has been shown to further explain our result. We then put our analysis in the framework of MDC to show its potential application by the desirable reduced reconstruction MSE by the additional subbands received correctly. The various bounds on the reduction of MSE have been further analyzed.