Distributed consensus of second-order multi-agent systems

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Distributed Consensus of Second-Order Multi-Agent Systems

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Summary

In recent years, with the development of robotics, artificial intelligence and communication and network technologies, cooperative control of systems consisting of some "agents" is arousing more and more attention. Consensus is an interesting and attractive topic in cooperative control, and we focus on both continuous- and discrete-time second-order consensus in the thesis.

Firstly, the fastest convergence rate for continuous second-order multi-agent consensus is derived for a specific category of topologies. The fastest convergence rate and its corresponding control gain are explicitly given. Furthermore, a separation principle is established for observer-based output feedback.

Extending the continuous convergence rate to discrete case, a new parameter is involved and the problem becomes more complex. A condition on the parameter to guarantee second-order consensus is provided, and the fastest convergence rate is also given as the solution to an algebraic equation.

As the last part of the thesis, quantization is introduced for discrete second-order consensus, where only quantized innovation of positions of its neighboring agents are available for each agent to construct its control law. An observer-based encoder-decoder pair is proposed and an interaction protocol is designed to achieve exact consensus.

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Chapter 1

Introduction

1.1 Cooperative Control and Consensus

With the development of robotics, artificial intelligence and communication and network technologies, cooperative control of systems consisting of some "agents" is arousing more and more attention. Cooperative control for multi-agent systems can be categorized as either formation control problems with applications to mobile robots, unmanned air vehicles (UAVs), autonomous underwater vehicles (AUVs), satellites, aircraft, spacecraft, and automated highway systems, or nonformation cooperative control problems such as task assignment, payload transport, role assignment, air traffic control, time synchronization, and search [1]. In the cooperative control problems, agents share their information through networks and finally come to a common view.

For cooperative control strategies to be effective, a team of agents must be able to respond to unanticipated situations or changes in the environment that are sensed as a cooperative task is carried out. As the environment changes, the agents on the team must be in agreement as to what changes took place. A direct consequence of the assumption that shared information is a necessary condition for coordination is that cooperation requires that the group of agents reach consensus on the coordination data. In other words, the instantiation of the coordination data on each agent must asymptotically approach a sufficiently common value. In networks of agents (or dynamic systems),

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"consensus" means to reach an agreement regarding a certain quantity of interest that depends on the state of all agents. A "consensus algorithm" (or protocol) is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network[2].

Consensus protocols have potential applications in formation control and cooperative timing and search problems for multiple robots, spacecrafts and UAVs. For example, information consensus for dynamically evolving information was applied to formation flying of multiple space-based interferometers[3].

Almost all the studies on consensus focused on first-order kinematics until the beginning of this century. However, second-order dynamic also has a broad range of applications in reality[3, 4]. In the case of first-order consensus protocols, the final consensus value is a constant. In contrast to the constant final consensus value, it might be better to derive second-order consensus protocols such that some information states converge to a consistent value (e.g. position of the formation centre) while others converge to another consistent value (e.g. velocity of the formation centre). The extension of consensus protocols from first-order to second-order is non-trivial, however, and many researchers are concentrating on this topic. [5] proposes and analyzes consensus problems for second-order dynamics and shows that unlike first-order case, both the topology graph and the coupling strength of relative velocities between neighboring vehicles affect the convergence result in the general case of directed interaction.

Another hot topic is quantized consensus due to the bandwidth limitation of communication channels. [6] has established a series of procedures for first-order quantized consensus.

After reading the papers on consensus, we can find that the issues of convergence rate and quantized second-order consensus are barely addressed. We shall investigate these problems and give some results.

1.2 Scope and Contributions of the Thesis

In this thesis, we focus on second-order consensus problems. Both continuous and discrete cases are discussed, and the following contributions are made:

1. The fastest convergence rate for continuous second-order multi-agent systems with real-eigenvalued Laplacian matrix is explicitly obtained. In addition, an observer-based output feedback is designed and verified to achieve the fastest convergence rate with a proper choice of observer and control gain.
2. The condition for fastest convergence rate of discrete second-order multi-agent consensus is also obtained which is the solution to an algebraic equation.
3. We propose an observer-based encoder-decoder pair and an interaction protocol and show that with proper choices of parameters, exact consensus can be achieved under quantized position innovation feedback.

1.3 Thesis Organization

The rest of the thesis will be divided into five chapters:

Chapter 2 is the review of the results for both first- and second-order consensus study over the several decades. In addition, some simulations are provided.

Chapter 3 studies the convergence rate of second-order consensus problems and proposes an observer-based protocol to achieve fastest convergence using only position information.

Chapter 4 studies the fastest convergence rate for discrete second-order consensus with both position and velocity information.

Chapter 5 extends the problem in Chapter 4 to quantized distributed consensus where only quantized position information is used to design a second-order consensus protocol.

Chapter 6 gives the conclusion and recommendations for future research.

Chapter 2

Literature Review

2.1 Historical Review

The problem of consensus has a long history in computer science. The formal consensus problem was first studied in groups of experts originated in *management science and statistics* in 1960's (See [2] and references therein). As a topic closely related to multi-agent networked systems, consensus problems are popular among researchers world wide currently. Some related subjects include consensus [7, 8], collective behavior of flocks and swarms [9], sensor fusion [10], random networks [11], synchronization of coupled oscillators [12], algebraic connectivity of complex networks [13, 14], asynchronous distributed algorithms [15], formation control for multirobot systems [16], optimization-based cooperative control [17], dynamic graphs [18], complexity of coordinated tasks [19], and consensus-based belief propagation in Bayesian networks [20].

Two approaches are considered in the study of consensus problems. The first one relies on algebraic graph theory, in which graph topologies are connected with the algebraic properties of the corresponding graph matrices. Some other researchers use nonlinear mathematical tools to study consensus problems [21]. However, the research using theories other than graph theory is not unified into a consistent framework, and occupies limited part in the area. Therefore only the main trend is introduced here.

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In [22], the discrete consensus problem was proposed and studied with Vicsek model[23], which proposed some basic principles for discrete consensus and arose all researchers' interest in consensus in recent years. Olfati-Saber and his colleagues investigated the problem for both continuous- and discrete-time cases and extend it to switching topology and time-delay cases[24]. This paper provides the most widely accepted protocol for continuous first-order consensus problem. Some other protocols have been proposed, for example, [25] takes the input of neighbors into account to achieve faster convergence. In 2005, Ren provided a comprehensive survey on the consensus problem[1].

As an extension of first-order consensus, the second-order consensus problem was addressed in [5] and criterion of convergence was provided. Comparing to the first-order case, the second-order consensus problem contains one more parameter and the analysis for system behavior is more complicated. The second-order consensus is widely used in flocking and swarm problems [26].

Recently, the problem of quantized consensus attracted a lot of interest due to the application of digital networks and their limited communication resources. With quantization, the problem becomes much more challenging due to the introduced nonlinearities. [6] proposed a quantized consensus protocol for first order systems and proved that the exact consensus can be achieved by merely one bit of information exchange between agents. In Chapter 5 of this thesis, we shall extend the result to the second-order case.

2.2 Introduction to Graph Theory

Graph theory is the foundation of most of the methods on studying consensus problems, this part is a short introduction to some useful definitions and terminology.[27]

Suppose that the system consists of N agents, denoted by N nodes $\nu_i, i = 1, 2, \dots, N$, and the set of the nodes is denoted as $\mathcal{V}_N$. An ordered pair $(\nu_i, \nu_j), \nu_i, \nu_j \in \mathcal{V}_N$ is an edge. $\mathcal{E}_N \subseteq \mathcal{V}_N^2$ is the set of all the ordered edges between nodes. If node ν_i sends its

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information to ν_j , then the pair $(\nu_i, \nu_j) \in \mathcal{E}_N$. The pair $(\mathcal{V}_N, \mathcal{E}_N)$ models the information exchange between agents, and is called a graph.

If the information exchange between any two nodes of a graph is mutual, which means that $(\nu_i, \nu_j) \in \mathcal{E}_N \Leftrightarrow (\nu_j, \nu_i) \in \mathcal{E}_N$, then the graph is called an undirected graph. On the other hand, if the information exchange is not mutual, the graph is a directed graph and called digraph for short. If there is a directed edge from node ν_i to node ν_j , then ν_i is defined as the parent node and ν_j is defined as the child node. A directed path is a sequence of ordered edges of the form $(\nu_{i_1}, \nu_{i_2}), (\nu_{i_2}, \nu_{i_3}), \dots$, where $\nu_{i_j} \in \mathcal{V}_N$, in a digraph. An undirected path in an undirected graph is defined analogously. A digraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node has exactly one parent except for one node, called the root, which has no parent, and the root has a directed path to every other node. A (directed) spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. We say that a graph has (or contains) a (directed) spanning tree if there exists a (directed) spanning tree that is a subset of the graph. Note that the condition that a digraph has a (directed) spanning tree is equivalent to the case that there exists at least one node having a directed path to all the other nodes. The union of a group of digraphs is a digraph with nodes given by the union of the node sets and edges given by the union of the edge sets of those digraphs. Figure 2.1 is an example of digraph, which has more than one possible (directed) spanning trees, but is not strongly connected.

The adjacency matrix $\mathcal{A} = [a_{ij}]$ of a weighted digraph is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}_N$ where $i \neq j$. The adjacency matrix of a weighted undirected graph is defined analogously except that $a_{ij} = a_{ji}, \forall i \neq j$, since $(j, i) \in \mathcal{E}_N$ implies $(i, j) \in \mathcal{E}_N$. Let matrix $\mathcal{L} = [l_{ij}]$ be defined as $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$, where $i \neq j$. Matrix $\mathcal{L}$ satisfies the

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Figure 2.1: A digraph

following conditions:

$$l_{ij} < 0, i \neq j$$

$$\sum_{j=1}^N l_{ij} = 0, \quad i = 1, 2, \dots, N$$

For an undirected graph, $\mathcal{L}$ is called the Laplacian matrix, which has the property that it is symmetric positive semi-definite. However, matrix $\mathcal{L}$ for a digraph does not have this property. As an example of an adjacency matrix for a weighted digraph, the matrix

$$\mathcal{A} = \begin{bmatrix} 0 & 1.5 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.1 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.2 & 0 \end{bmatrix}$$

can be a valid adjacency matrix corresponding to the digraph in Figure 1.1. Correspondingly, matrix $\mathcal{L}$ is defined as

$$\mathcal{L} = \begin{bmatrix} 1.5 & -1.5 & 0 & 0 & 0 & 0 \\ -0.7 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & -1.1 & 1.1 & 0 & 0 & 0 \\ -0.8 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & -0.2 & 0 & -0.3 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -1.2 & 1.2 \end{bmatrix}$$

Let $\mathcal{I} = 1, 2, \dots, N$. Let $\mathbf{1}$ and $\mathbf{0}$ denote the $N \times 1$ column vector of all ones and all zeros, respectively. Let I_N denote the $N \times N$ identity matrix and $0_{M \times N}$ denote the

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$M \times N$ matrix with all zero entries. Let $M_N(\mathbb{R})$ represent the set of all $N \times N$ real matrices. Given a matrix $\mathcal{A} = [a_{ij}] \in M_N(\mathbb{R})$ the digraph of $\mathcal{A}$, denoted by $\Gamma(\mathcal{A})$ is the digraph on N nodes $\nu_i, i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(\mathcal{A})$ from ν_j to ν_i if and only if $a_{ij} \neq 0$ (cf. [28]).

2.3 First-order Consensus

2.3.1 Problem Formulation

Suppose that there are N vehicles in a team. We use directed graph $\mathcal{G}_N \triangleq (\mathcal{V}_N, \mathcal{E}_N)$, where $\mathcal{V}_N = \{1, \dots, N\}$ is the node set and $\mathcal{E}_N \subseteq \mathcal{V}_N \times \mathcal{V}_N$ is the edge set, to model the interaction topology among the n vehicles. Let $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ and $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ be the adjacency matrix and the nonsymmetrical Laplacian matrix associated with $\mathcal{G}_N$ respectively.

For the convenience that it can be extended to second-order case, the information state of the agent i will be denoted as $\xi_i \in \mathbb{R}^M$. With the agent dynamics

$$\dot{\xi}_i = u_i, \quad i = 1, \dots, N, \quad (\text{Eq. 2.1})$$

where $u_i \in \mathbb{R}^M$ is the information control input for the i th agent. A most commonly used consensus protocol for the above system is

$$u_i(t) = - \sum_{j=1}^N a_{ij}(t)(\xi_i(t) - \xi_j(t)), \quad i = 1, \dots, N \quad (\text{Eq. 2.2})$$

where $a_{ij}(t)$ is the (i, j) entry of adjacency matrix $\mathcal{A}(t) \in \mathbb{R}^{N \times N}$. It is obvious that the states of agents are driven toward the states of their neighbors and will come together under some condition. With (Eq. 2.1) and (Eq. 2.2), the system dynamics is given by

$$\dot{\xi}(t) = -[\mathcal{L}(t) \otimes I_M]\xi, \quad (\text{Eq. 2.3})$$

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where $\xi(t) = [\xi_1^T(t), \dots, \xi_N^T(t)]^T$, $\mathcal{L}(t) \in \mathbb{R}^{N \times N}$ is the nonsymmetric Laplacian matrix at time t and $\otimes$ denotes the Kronecker product. For any initial condition $\xi_i(0)$ and any $i, j = 1, \dots, N$, if $\|\xi_i(t) - \xi_j(t)\| \rightarrow 0$ as $t \rightarrow \infty$, the group of agents reaches or achieves consensus by definition.

2.3.2 First-order Consensus Under Fixed Topology

In this part, the information exchange topology of the system is assumed to be constant and the condition for consensus will be provided. To begin with, we introduces two lemmas from [29].

Lemma 2.1 *Given a matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ii} \leq 0, a_{ij} \geq 0, \forall i \neq j$, and $\sum_{j=1}^N a_{ij} = 0$ for each i , then $\mathcal{A}$ has at least one zero eigenvalue with an associated eigenvector $\mathbf{1}$, and all nonzero eigenvalues are in the open left half plane. Furthermore, $\mathcal{A}$ has exactly one zero eigenvalue if and only if the directed graph of $\mathcal{A}$, denoted by $\Gamma(\mathcal{A})$, has a directed spanning tree.*

The proof for the first part of Lemma 2.1 is straightforward from the definition of matrix $\mathcal{A}$ and Gershgorin's theorem. [29] gives the proof for the second part based on some results in [22].

The next lemma shows that the system states reach consensus asymptotically if and only if $-\mathcal{L}$ has exactly one zero eigenvalue and all other eigenvalues are in the open left half plane. Besides, the equilibrium state is calculated.

Lemma 2.2 *If $\mathcal{L}$ is the $N \times N$ nonsymmetrical Laplacian matrix, then $e^{-\mathcal{L}t}, \forall t \geq 0$, is a row-stochastic matrix(matrix whose entries are all positive and all the row sums are 1) with positive diagonal entries. In addition, $\text{Rank}(\mathcal{L}) = N - 1$ if and only if $\mathcal{L}$ has a simple zero eigenvalue. Furthermore, if $\mathcal{L}$ has a simple zero eigenvalue and $\nu = [\nu_1, \dots, \nu_N]^T \geq 0$ satisfies that $\mathbf{1}^T \nu = 1$ and $\mathcal{L}^T \nu = 0$, then $e^{-\mathcal{L}t} \rightarrow \mathbf{1} \nu^T$, as $t \rightarrow \infty$.*

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Lemma 2.2 is easily proved with the results in [22].

With Lemmas 2.1 and 2.2, the condition for protocol (Eq. 2.2) to reach consensus is found.

Theorem 2.1 [30] *Suppose that $\mathcal{A}$ is a constant matrix. Protocol (Eq. 2.2) achieves consensus asymptotically if and only if directed graph $\mathcal{G}_N$ has a directed spanning tree. In particular, $\xi_i(t) \rightarrow \sum_{i=1}^N \nu_i \xi_i(0)$, as $t \rightarrow \infty$, where $\nu = [\nu_1, \dots, \nu_N]^T \geq 0$, $\mathbf{1}^T \nu = 1$, and $\mathcal{L}^T \nu = 0$.*

Proof: (Sufficiency) Obviously $-\mathcal{L}$ associated with $\mathcal{G}_N$ satisfies the condition of Lemma 2.1 and thus has exactly one zero eigenvalue and all other eigenvalues are in the open left half plane. Then the protocol (Eq. 2.2) will achieve consensus asymptotically according to Lemma 2.2. Furthermore, the equilibrium is $\xi(t) = (e^{-\mathcal{L}(t)} \otimes I_M) \xi(0)$ and thus $\xi_i(t) \rightarrow \sum_{i=1}^N \nu_i \xi_i(0)$, $i = 1, \dots, N$ as $t \rightarrow \infty$.

(Necessity) Suppose that protocol (Eq. 2.2) reaches consensus asymptotically while $\mathcal{G}_N$ does not contain a directed spanning tree. Then there exist at least two vehicles, say i and j , who are not connected. Therefore there is no information exchange between the two vehicles and consensus will never be achieved. This is a contradiction to the condition provided. $\square$

When the state of each agent reaches consensus and the equilibrium state is the average of their initial condition, $\xi_i(t) \rightarrow \sum_{i=1}^N \xi_i(0)/N$, the consensus is called average consensus. It is proved that average consensus is reached asymptotically if and only if the directed graph $\mathcal{G}_N$ is strongly connected and balanced or the undirected graph $\mathcal{G}_N$ is connected.

Theorem 2.1 proposed the condition for convergence, then a question arises: how fast it will converge to the equilibrium states? According to Gershgorin theorem, we know

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that all eigenvalues of $\mathcal{L}$ in the complex plane are located in a closed disk centered at $D + 0j$ with a radius of $D = \max_i d_i$, which means the maximum degree of a graph. If the graph is undirected, $\mathcal{L}$ is symmetric with real eigenvalues and the set of eigenvalues of $\mathcal{L}$ can therefore be ordered sequentially in an ascending order like

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N \leq 2D$$

For a graph that can reach consensus, the zero eigenvalue is known as the trivial eigenvalue of $\mathcal{L}$. Thus $\lambda_2 > 0$. The second smallest eigenvalue of Laplacian λ_2 is called *algebraic connectivity* of a graph. It can be observed that, for a dense graph λ_2 is relatively large and for a sparse graph λ_2 is relatively small. So the conclusion is that algebraic connectivity of the network topology is a measure of performance/speed of consensus algorithms.

Another important result for invariant information exchange topology is about the upper bound of time-delay of the network in the case that the undirected graph is connected and the time-delay on each link of the network is the same. In this case, the protocol of (Eq. 2.2) is of the following form

$$\dot{\xi}_i(t) = - \sum_{j=1}^N a_{ij} (\xi_i(t - \tau) - \xi_j(t - \tau)) \quad (\text{Eq. 2.4})$$

The upper bound of time-delay is provided in [24].

Theorem 2.2 *The protocol (Eq. 2.4) asymptotically solves the average consensus problem with a uniform one-hop time-delay τ for all initial states if and only if $0 \leq \tau \leq \pi/2\lambda_N$, where λ_N is the largest eigenvalue of Laplacian matrix $\mathcal{L}$.*

2.3.3 First-order Consensus Under Dynamic Topology

To make the problem less complicated, the dwell time (the finite time interval being the lower bound of topology switching[22]) is used. It is assumed that there is a finite, lower

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bound on the time between switches of topology, and the topology remains the same during the dwell time. In this sense, both $\mathcal{A}(t)$ and $\mathcal{L}(t)$ are time-variant and piecewise constant and the protocol (Eq. 2.3) will be

$$\dot{\xi}(t) = -[\mathcal{L}(t_i) \otimes I_M]\xi(t), \quad t \in [t_i, t_i + \tau_i) \quad (\text{Eq. 2.5})$$

[31] presents the following theorem for consensus under dynamic topology.

Theorem 2.3 *Let $t_1, t_2, \dots$ be an infinite time sequence at which the interaction graph or weighting factors switch and $\tau_i = t_{i+1} - t_i \in \Upsilon, i = 0, 1, \dots$. Let $\mathcal{G}_N(t_i) \in \bar{\mathcal{G}}_N$ be a switching interaction graph at time $t = t_i$ and $\sigma_{ij}(t_i) \in \bar{\sigma}$, where $\bar{\sigma}$ is a finite set of arbitrary positive numbers. The continuous-time update scheme (Eq. 2.5) achieves consensus asymptotically if there exists an infinite sequence of uniformly bounded, nonoverlapping time intervals $[t_{i_j}, t_{i_{j+1}}), j = 1, 2, \dots$, starting at $t_i = t_0$, with the property that each interval is uniformly bounded and the union of the graphs across each interval interval has a spanning tree. Furthermore, if the union of the graphs after some finite time does not have a spanning tree, then consensus cannot be achieved asymptotically for $\mathcal{A}$.*

A similar theorem is that

Theorem 2.4 *Suppose that $\mathcal{A}(t) = [a_{ij}(t)] \in \mathbb{R}^{N \times N}$ in (Eq. 2.5) is piecewise continuous and its nonzero and hence positive entries are both uniformly lower and upper bounded (i.e., $a_{ij} \in [\underline{a}, \bar{a}]$, where $0 < \underline{a} < \bar{a}$, if $(j, i) \in \mathcal{E}_N$ and $a_{ij} = 0$ otherwise). Let $t_0, t_1, \dots$ be the time sequence corresponding to the times at which $\mathcal{A}(t)$ switches, where it is assumed that $t_i - t_{i-1} \geq t_L, \forall i = 1, 2, \dots$ with t_L a positive constant. The continuous-time algorithm (Eq. 2.5) achieves consensus asymptotically if there exists an infinite sequence of contiguous, nonempty, uniformly bounded time-intervals $[t_{i_j}, t_{i_{j+1}}), j = 1, 2, \dots$, starting at $t_{i_1} = t_0$, with the property that the union of directed graphs $\mathcal{G}_N(t)$ across each such interval has a directed spanning tree.*

The dynamic topology is not addressed in this thesis, and some related content can be found in [29]

2.3.4 Some Applications

The consensus problems are widely used in variant engineering applications, however, only some typical examples are provided here.

Vehicle formation problems are extensively studied due to their commercial and military applications. The idea of consensus is used in formation applications by representing the formations with vectors of relative positions of neighboring vehicles and the use of consensus-based controllers with input bias. Fax and Murray [32] have proposed a theoretical framework for design and analysis of distributed controllers for multivehicle formation problems. Each vehicle needs to minimize its local cost $U_i(x) = \sum_{j \in \mathcal{N}_i} \|x_j - x_i - \mathbf{r}_{ij}\|^2$ where x_i is the position information of agent i , $\mathcal{N}_i$ is the set of agents which communicate to agent i and $\mathbf{r}_{ij}$ is the desired intervehicle relative-position vector. With gradient-descent algorithm and transformation of protocol (Eq. 2.2)

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i - \mathbf{r}_{ij}) = \sum_{j \in \mathcal{N}_i} (x_j - x_i) + b_i \quad (\text{Eq. 2.6})$$

$$b_i = \sum_{j \in \mathcal{N}_i} \mathbf{r}_{ji}$$

applied to the collective cost $\sum_{i=1}^N U_i(x)$, the objective of each agent will be achieved. Although the final state x_i of each agent is not the same for formation problems, the difference between the ultimate states of any two agents is the same. In this sense the problem can be solved using generalized consensus protocols.

Rendezvous problem is another example of consensus. It is the problem that a group of mobile autonomous agents are required to reach a particular position simultaneously. The synchronous case and asynchronous case are both considered in [33].

There are also some other applications for consensus, see [1, 2] for some details.

2.3.5 Simulations

In this section, an example is given to demonstrate the theorems. Consider a system consisting of four agents with the communication topology shown as Fig. 2.2, it is conspicuous that the graph contains a spanning tree and the corresponding adjacency matrix is

Figure 2.2: An example graph containing a spanning tree

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

In this particular scenario, we have $\xi_i(t) = [x_i(t), y_i(t)]^T$. Applying (Eq. 2.2) to each of the agents with the initial state $[2.3, 3.8]$, $[8.1, 1.1]$, $[4.9, 5.4]$, $[7.4, 9.5]$, we have the result shown in Fig 2.3. Fig 2.4 shows the evolution of the evolutions of ξ_x and ξ_y values of the four agents. The position, time, velocity and acceleration units in all simulations are $m, s, m/s$ and m/s^2 , if not mentioned specifically.

By adding some vectors to the control signal in (Eq. 2.2), the problem is transferred into a formation problem. In order to form the four agents into a square with length 1, we need to add different vector to each control of the agent. For example, a vector $b_1 = [-1, 1]^T$ is added to the control of agent 1 since agent 1 receives information from agent 4 and the vector from agent 4 to agent 1 is $[-1, 1]^T$ (see Eq. 2.6). Adding the

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Figure 2.3: The trajectory of the four agents under first-order consensus protocol.

corresponding vector to the control signal, we have agents trajectory shown in Fig. 2.5 and evolutions of ξ_x and ξ_y in Fig. 2.6.

From the simulations we can see that the agents do form a square with the vectors added to the control.

2.4 Second-order Consensus

2.4.1 Problem Formulation

Consider information states with double-integrator dynamics given by

$$\dot{\xi}_i = \zeta_i, \quad \dot{\zeta}_i = u_i, \quad i = 1, \dots, N, \quad (\text{Eq. 2.7})$$

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Figure 2.4: The evolutions of ξ_x and ξ_y values of the four agents under first-order consensus protocol.

where $\xi_i \in \mathbb{R}^M$ is the information state, $\zeta_i \in \mathbb{R}^M$ is the information state derivative, and $u_i \in \mathbb{R}^M$ is the information control input associated with the i th vehicle.

Note that the interaction topologies for ξ_i and ζ_i among n vehicles might be different under some situations. Let $\mathcal{G}_N^A \triangleq (\mathcal{V}_N, \mathcal{E}_N^A)$, where $\mathcal{V}_N = \{1, \dots, N\}$ is the node set and $\mathcal{E}_N^A \subseteq \mathcal{V}_N \times \mathcal{V}_N$ is the edge set, and $\mathcal{G}_N^B \triangleq (\mathcal{V}_N, \mathcal{E}_N^B)$, where $\mathcal{V}_N$ is the node set and $\mathcal{E}_N^B \subseteq \mathcal{V}_N \times \mathcal{V}_N$ is the edge set to model the interaction topologies for ξ_i and ζ_i among the vehicles respectively. Also, $\mathcal{L}^A = [l_{ij}] \in \mathbb{R}^{N \times N}$ (i.e., $l_{ij} = -a_{ij}, i \neq j, l_{ij} = \sum_{j=1, j \neq i}^N a_{ij}$) and $\mathcal{L}^B = [l_{ij}] \in \mathbb{R}^{N \times N}$ (i.e., $l_{ij} = -b_{ij}, i \neq j, l_{ij} = \sum_{j=1, j \neq i}^N b_{ij}$). If the two topologies happen to be the same, then $\mathcal{G}_N \triangleq (\mathcal{V}_N, \mathcal{E}_N)$ is used to model the topology among vehicles. $\mathcal{A}$ and $\mathcal{L}$ are the adjacency matrix and nonsymmetric Laplacian matrix associated with $\mathcal{G}_N$.

With the above definitions, a consensus protocol for (Eq. 2.7) is proposed as

$$u_i(t) = - \sum_{j=1}^N a_{ij}(t) [(\xi_i(t) - \xi_j(t)) + \gamma(t)(\zeta_i(t) - \zeta_j(t))], \quad i = 1, \dots, N, \quad (\text{Eq. 2.8})$$

where $a_{ij}(t)$ is the (i, j) entry of the adjacency matrix $\mathcal{A}(t) \in \mathbb{R}^{N \times N}$ associated with $\mathcal{G}_N$

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Figure 2.5: The trajectory of the four agents under first-order formation protocol.

at time t and $\gamma(t)$ is a positive scalar at time t . Note that the topologies for $\xi_i(t)$ and $\zeta_i(t)$ are considered to be the same for simplicity. When (Eq. 2.8) is applied to the system, consensus is said to be reached by the group of vehicles if, for all $\xi_i(0)$ and $\zeta_i(0)$ and all $i, j = 1, \dots, N$, $\|\xi_i(t) - \xi_j(t)\| \rightarrow 0$ and $\|\zeta_i(t) - \zeta_j(t)\| \rightarrow 0$, as $t \rightarrow \infty$.

Let $\xi \triangleq [\xi_1^T, \dots, \xi_N^T]^T$ and $\zeta \triangleq [\zeta_1^T, \dots, \zeta_N^T]^T$. With protocol (Eq. 2.8), the (Eq. 2.7) can be written in the matrix form as

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} = (\Theta(t) \otimes I_M) \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \quad (\text{Eq. 2.9})$$

where

$$\Theta = \begin{bmatrix} 0_{N \times N} & I_N \\ -\mathcal{L}(t) & -\gamma(t)\mathcal{L}(t) \end{bmatrix}$$

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Figure 2.6: The evolutions of ξ_x and ξ_y values of the four agents under first-order formation protocol.

with $\mathcal{L}(t) \in \mathbb{R}^{N \times N}$ as the nonsymmetric Laplacian matrix associated with $\mathcal{G}_N$ at time t .

2.4.2 Second-order Consensus Over Fixed Topology

For first-order dynamic systems, protocol (Eq. 2.2) converges if and only if the graph has a spanning tree. For the second-order case, however, it is obvious that if the protocol (Eq. 2.8) converges, the graph must contain a spanning tree. But having a spanning tree is only a necessary condition for this case.

For convenience, $\mathcal{A}$ and γ are taken to be constant during the analysis. For a block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

$\det(M) = \det(AD - CB)$ if A and C are commutable. For Θ in the previous section, the A block of matrix $\lambda I_{2N} - \Theta$ is λI_N , which is commutable with any other matrix.

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$$\begin{aligned} \det(\lambda I_{2N} - \Theta) &= \det \left(\begin{bmatrix} \lambda I_N & -I_N \\ \mathcal{L} & \lambda I_N + \gamma \mathcal{L} \end{bmatrix} \right) \\ &= \det[\lambda^2 I_N + (1 + \gamma \lambda) \mathcal{L}]. \end{aligned} \quad (\text{Eq. 2.10})$$

Also,

$$\det(\lambda I_N + \mathcal{L}) = \prod_{i=1}^N (\lambda - \mu_i), \quad (\text{Eq. 2.11})$$

where μ_i is the i th eigenvalue of $-\mathcal{L}$.

Comparing (Eq. 2.10) and (Eq. 2.11),

$$\det[\lambda^2 I_N + (1 + \gamma \lambda) \mathcal{L}] = \prod_{i=1}^N [\lambda^2 - (1 + \gamma \lambda) \mu_i], \quad (\text{Eq. 2.12})$$

thus solving equation $\lambda^2 = (1 + \gamma \lambda) \mu_i$ will obtain the roots of (Eq. 2.10). The eigenvalues of Θ are

$$\lambda_{i\pm} = \frac{\gamma \mu_i \pm \sqrt{\gamma^2 \mu_i^2 + 4 \mu_i}}{2}, \quad (\text{Eq. 2.13})$$

where λ_{i+} and λ_{i-} are eigenvalues of Θ associated with μ_i .

Since $\mathcal{L} \mathbf{1} = 0$, matrix $-\mathcal{L}$ has at least one zero eigenvalue and hence matrix Θ has at least two zero eigenvalues, denoted as $\lambda_{1+} = \lambda_{1-} = 0$.

Here are two lemmas for convergence of the second-order multi-agent system[5].

Lemma 2.3 *Protocol (Eq. 2.8) achieves consensus asymptotically if and only if Θ has exactly two zero eigenvalues and all other eigenvalues have negative real parts. Specifically, $\xi_i(t) \rightarrow \sum_{i=1}^N p_i \xi_i(0) + t \sum_{i=1}^N p_i \zeta_i(0)$ and $\zeta_i(t) \rightarrow \sum_{i=1}^N p_i \zeta_i(0)$ for large t , where $p = [p_1, \dots, p_N]^T \geq 0$, $\mathbf{1}_N^T p = 1$, and $\mathcal{L}^T p = 0$.*

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It is proved that Θ has two zero eigenvalues but only one corresponding eigenvector and thus has one generalized eigenvector. The Jordan block for zero eigenvalue will be $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and the following proof is similar to that of the first-order case.

Lemma 2.4 *Let*

$$\rho_{\pm} = \frac{\gamma\mu - \alpha \pm \sqrt{(\gamma\mu - \alpha)^2 + 4\mu}}{2}$$

where $\rho, \mu \in \mathbb{C}$. If $\alpha \geq 0$, $\text{Re}(\mu) < 0$, $\text{Im}(\mu) > 0$, and

$$\gamma > \sqrt{\frac{2}{|\mu| \cos \left[\tan^{-1} \frac{\text{Im}(\mu)}{\text{Re}(\mu)} \right]}}, \quad (\text{Eq. 2.14})$$

then $\text{Re}(\rho_{\pm}) < 0$, where $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ represent the real and imaginary parts of a number respectively.

Based on the previous two lemmas, the condition for γ to achieve second-order consensus is proposed[5].

Theorem 2.5 *Consensus protocol (Eq. 2.8) achieves consensus asymptotically if the information exchange topology has a (directed) spanning tree and*

$$\gamma > \max_{\mu_i \neq 0} \sqrt{\frac{2}{|\mu_i| \cos \left(\frac{\pi}{2} - \tan^{-1} \frac{-\text{Re}(\mu_i)}{\text{Im}(\mu_i)} \right)}} \quad (\text{Eq. 2.15})$$

where $\mu_i, i = 1, \dots, N$, are the eigenvalues of $-\mathcal{L}$.

Proof: If the graph associated with the information exchange topology has a spanning tree, then $-\mathcal{L}$ has a simple zero eigenvalue and all the other eigenvalues are in the left half open plane. Let $\mu_1 = 0$ and $\text{Re}(\mu_i) < 0, i = 2, \dots, N$. If (Eq. 2.14) holds, $\text{Re}(\lambda_{\pm}) <$

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$0, i = 2, \dots, N$ according to Lemma 2.4, where $\lambda_{i\pm}$ are eigenvalues of Θ associated with μ_i . Now Θ has exactly two zero eigenvalues and all other eigenvalues have negative real parts, the protocol reaches consensus by Lemma 2.3. $\square$

The consensus of second-order multi-agent system with time-delay is still a challenge for study, only an upper bound for particular condition is given[34]. For undirected communication topology, we propose the following lemma:

Lemma 2.5 *For an undirected connected topology, protocol (Eq. 2.8) achieves consensus for any positive factor γ .*

Proof: As the graph is undirected and connected, the graph has one zero eigenvalue and all the other eigenvalues are real and negative. With (Eq. 2.13), $\lambda_{i\pm} = \frac{\gamma\mu_i \pm \sqrt{\gamma^2\mu_i^2 + 4\mu_i}}{2}$. If the part inside the square root is positive, since μ_i is negative $\sqrt{\gamma^2\mu_i^2 + 4\mu_i} < |\gamma\mu_i| = -\gamma\mu_i$ and hence $\gamma\mu_i \pm \sqrt{\gamma^2\mu_i^2 + 4\mu_i} < 0$. If the part is negative, $\lambda_{i\pm}$ will be complex with real part being $\gamma\mu_i/2 < 0$. So we see that for any $\gamma > 0$, the eigenvalue set of Θ contains two zero eigenvalues and all the other eigenvalues are either negative real or complex with negative real part. From Lemma 2.3, consensus is achieved. $\square$

The upper bound for time-delay second-order consensus is given by[34]:

Theorem 2.6 *Suppose the time-delay between any two agents is the same, denoted as τ , and the graph is undirected and connected. With protocol*

$$u_i(t) = -k_1 \sum_{j=1}^N a_{ij}(\xi_i(t - \tau) - \xi_j(t - \tau)) - k_2 \sum_{j=1}^N a_{ij}(\zeta_i(t - \tau) - \zeta_j(t - \tau)),$$

where $k_1 > 0, k_2 > 0$, the system reaches consensus asymptotically if and only if $\tau < \tau^*$ with τ^* given by

$$\tau^* = \frac{k_2 \tan^{-1}(\sqrt{\omega})(1 - \omega)}{k_1 \sqrt{\omega}},$$

where ω is the positive root of

$$\omega^3 + \left(1 + \frac{4k_1}{k_2^2 \lambda_{max}}\right) \omega^2 + \left(\frac{4k_1}{k_2^2 \lambda_{max}} - 1\right) \omega - 1 = 0 \quad (\text{Eq. 2.16})$$

with λ_{max} as the largest eigenvalue of $\mathcal{L}$.

The convergence speed of second-order consensus is not addressed much in the past few years, [35] used another protocol and proposed some results similar to ours but the proof is quite different.

2.4.3 Second-order Consensus Over Dynamic Topology

For the first-order consensus algorithm (Eq. 2.2), it is shown that consensus can be achieved asymptotically under switching interaction topologies if there exist infinitely many uniformly bounded, consecutive time intervals such that the union of the directed graphs across each such interval has a directed spanning tree. However, the condition is generally not the truth for second-order cases.

Example in [29] used 3 different Laplacian matrices which might be the case in a dynamic topology system,

$$\mathcal{L}_{3(1)} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathcal{L}_{3(2)} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix}, \mathcal{L}_{3(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

and $\gamma_1, \gamma_2, \gamma_3$ as the corresponding weight of the three matrices. Θ_i is defined as

$$\Theta_i = \begin{bmatrix} 0_{3 \times 3} & I_3 \\ -\mathcal{L}_{3(i)} & -\gamma_i \mathcal{L}_{3(i)} \end{bmatrix}, \quad (\text{Eq. 2.17})$$

where $i = 1, 2, 3$.

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The example shows that the higher the weight of mostly connected $\mathcal{L}$ matrix, the longer the topology remains the mostly connected, the more likely that the system reaches consensus. The conclusion coincides with the intuition, but no formula are provided yet.

2.4.4 Simulations

Still we use the topology as the one in the previous section, and take $[2.3, 3.8]$, $[8.1, 1.1]$, $[4.9, 5.4]$, $[7.4, 9.5]$ as the initial positions, $[-0.1, 0.2]$, $[0.9, 0.2]$, $[0.4, -0.6]$, $[0.6, 0.5]$ as the initial velocities, $\gamma = 2$. The results are shown in Fig. 2.7 through Fig. 2.9.

Figure 2.7: The trajectory of the four agents under second-order consensus protocol.

Add the displacement vector to corresponding part, the protocol is changed to

$$u_i(t) = - \sum_{j=1}^N a_{ij}(t) [(\xi_i(t) + b_i - \xi_j(t)) + \gamma(t)(\zeta_i(t) - \zeta_j(t))], \quad i = 1, \dots, N.$$

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Figure 2.8: The evolutions of ξ_x and ξ_y of the four agents under second-order consensus protocol.

Figure 2.9: The evolutions of ζ_x and ζ_y of the four agents under second-order consensus protocol.

In this case, the simulation results are shown in Fig. 2.10 to Fig. 2.12. We can see that the four agents formed the square as designed.

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Figure 2.10: The trajectory of the four agents under second-order formation protocol.

2.5 Conclusions

In this chapter, the graph theory was introduced as the preliminary knowledge, and the results for first- and second-order multi-agent systems were reviewed. As an application of consensus problems, formation control was also investigated. The simulations result at the end of each section verified the validity of the protocols.

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Figure 2.11: The evolutions of ξ_x and ξ_y of the four agents under second-order formation protocol..

Figure 2.12: The evolutions of ζ_x and ζ_y of the four agents under second-order formation protocol..

Chapter 3

Fastest Convergence Rate for Continuous Second-Order Consensus

The first order consensus problem has been studied for over twenty years with many well-established theories. In some cases, however, the first-order consensus is no longer sufficient. For example, if the formation problem requires the agents to maintain the form at a speed instead of staying static, then it is not only the state ξ but the derivative of state $\zeta = \dot{\xi}$ that must reach consensus. Consensus of the second-order dynamic system has attracted a lot of interest in recent years, it is an extension of the first-order consensus but has many more challenges.

3.1 Fastest Convergence Rate

In this section, we study the fastest convergence rate for the continuous second-order consensus problem, where both position and velocity information is available to design a control signal. It can be seen that different consensus systems converge with different speed, decided by the eigenvalue of Θ which is closest to the real axis. The magnitude of the real part of the eigenvalue is called the convergence rate, indicating the speed of convergence. Note that when the system communication topology is fixed, the only factor affecting the rate is the control gain γ . So the objective is to find the value of γ to maximize the convergence rate. Let us introduce a lemma to begin with.

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With Lemma 2.5, we have the following theorem for the rate of convergence of second-order consensus system.

Theorem 3.1 *If the graph of the second-order multi-agent system is undirected and connected, with Protocol (Eq. 2.8) the convergence rate of the system is decided by the eigenvalue other than zero of Θ whose real part is the largest. When $\gamma = \sqrt{\frac{4\mu_N}{\mu_2^2 - 2\mu_2\mu_N}}$, the system converges fastest at the rate $\sqrt{\frac{\mu_2\mu_N}{\mu_2 - 2\mu_N}}$.*

Proof: The first part of the theorem is easy to prove. Consider the difference of system states, and denote $\epsilon_i = \xi_i - \xi_1$ and $\eta_i = \zeta_i - \zeta_1, i = 2, 3, \dots, N$. Furthermore, let $\bar{\xi} = [\epsilon_2, \dots, \epsilon_N]^T$ and $\bar{\zeta} = [\eta_2, \dots, \eta_N]^T$. Then

$$\begin{bmatrix} \dot{\bar{\xi}} \\ \dot{\bar{\zeta}} \end{bmatrix} = \begin{bmatrix} 0_{(N-1) \times (N-1)} & I_{(N-1)} \\ -\bar{\mathcal{L}} & -\gamma\bar{\mathcal{L}} \end{bmatrix} \begin{bmatrix} \bar{\xi} \\ \bar{\zeta} \end{bmatrix} \quad (\text{Eq. 3.1})$$

where

$$\bar{\mathcal{L}} = \begin{bmatrix} l_{22} - l_{12} & \cdots & l_{2N} - l_{1N} \\ \vdots & \ddots & \vdots \\ l_{N2} - l_{12} & \cdots & l_{NN} - l_{1N} \end{bmatrix}$$

with l_{ij} the (i, j) -entry of the Laplacian matrix $\mathcal{L}$.

If the system (Eq. 3.1) is asymptotically stable, the difference between states of any two agents will converge to zero, i.e. the consensus is reached by the multi-agent system.

It is easy to see that the matrix $\bar{\mathcal{L}}$ has exactly the same $N - 1$ non-zero eigenvalues of $\mathcal{L}$. Denote the state matrix in (Eq. 3.1) as $\bar{\Theta}$. The characteristic polynomial of $\bar{\Theta}$ is given by [5]

$$\begin{aligned} \det(\lambda I - \bar{\Theta}) &= \begin{vmatrix} \lambda I & -I \\ \bar{\mathcal{L}} & \lambda I + \gamma\bar{\mathcal{L}} \end{vmatrix} \\ &= \det(\lambda^2 I + (\lambda\gamma + 1)\bar{\mathcal{L}}) \end{aligned}$$

Suppose that matrix $-\bar{\mathcal{L}}$ has eigenvalues $\mu_i, i = 1, 2, \dots, N - 1$, then each μ_i corresponds to two eigenvalues $\lambda_{i\pm}$ of $\bar{\Theta}$ satisfying $\lambda_{i\pm}^2 = (\lambda_i\gamma + 1)\mu_i$. Indeed, by solving the equation we have

$$\lambda_{i\pm} = \frac{\gamma\mu_i \pm \sqrt{\gamma^2\mu_i^2 + 4\mu_i}}{2},$$

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which are also the non-zero eigenvalues of matrix Θ . From [5] we know that with proper γ , all the eigenvalues of $\bar{\Theta}$ lie in the left half plane and the system is stable which means that the original system can reach second-order consensus. Furthermore, the convergence rate of the original system is determined by the eigenvalue of Θ which has the largest negative real part.

To determine the proper γ to make the system converge most quickly, we need to analyze the set of eigenvalues of matrix Θ . It is obvious that $\lambda_{i-} = (\gamma\mu_i - \sqrt{(\gamma\mu_i)^2 + 4\mu_i})/2$ has real part less than its counterpart $\lambda_{i+} = (\gamma\mu_i + \sqrt{(\gamma\mu_i)^2 + 4\mu_i})/2$, so we only need to investigate λ_{i+} .

From Lemma 2.5, we know that with different γ and μ_i , the part inside the square root may be positive or negative, so it is necessary to discuss the following three cases for γ .

Case 1: $0 < \gamma \leq \sqrt{-\frac{4}{\mu_N}}$

In this case, $(\gamma\mu_i)^2 + 4\mu_i < 0, i = 2, 3, \dots, N$ for all μ_i and the real parts of the nonzero eigenvalues will be $\gamma\mu_i/2$. So $\gamma\mu_2/2$ is the largest real part and the greater the γ , the smaller the $\gamma\mu_2/2$ (since $\mu_2 < 0$) and hence the faster the system converges. When $\gamma_1 = \sqrt{-4/\mu_N}$, the convergence rate is the fastest, which is $r_1 = |\gamma_1\mu_2/2| = -\mu_2/\sqrt{-\mu_N}$.

Case 2: $\sqrt{-\frac{4}{\mu_N}} \leq \dots \leq \sqrt{-\frac{4}{\mu_l}} \leq \gamma \leq \sqrt{-\frac{4}{\mu_2}}$

In this case, $(\gamma\mu_i)^2 + 4\mu_i < 0, i = 2, 3, \dots, l-1$ and $(\gamma\mu_i)^2 + 4\mu_i > 0, i = l, \dots, N$. For $i = 2, 3, \dots, l-1$, it is the same as case 1 and the convergence rate will be $\gamma\mu_2/2$. For the other i , $\lambda_i = (\gamma\mu_i + \sqrt{(\gamma\mu_i)^2 + 4\mu_i})/2$. Neglecting the subscript i and taking partial

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derivative of λ with respect to μ , we have

$$\begin{aligned}\frac{\partial \lambda}{\partial \mu} &= \frac{\gamma}{2} + \frac{\gamma^2 \mu + 2}{2\sqrt{(\gamma \mu)^2 + 4\mu}} \\ &= \frac{\gamma}{2} \left(1 + \frac{\gamma \mu + \frac{2}{\gamma}}{\sqrt{\left(\gamma \mu + \frac{2}{\gamma}\right)^2 - \frac{4}{\gamma^2}}} \right)\end{aligned}$$

For $i = l, \dots, n$, $\sqrt{-4/\mu_i} \leq \gamma \Rightarrow \gamma^2 \mu_i \leq -4 \Rightarrow \gamma \mu_i + 2/\gamma \leq -2/\gamma < 0$ so that we have $\sqrt{(\gamma \mu_i + 2/\gamma)^2 - 4/\gamma^2} < -(\gamma \mu_i + 2/\gamma)$. Thus, the part in the bracket in the partial derivative is negative and so does the partial derivative. The negative derivative means that λ_N corresponding to μ_N has the largest real part among the eigenvalues. Now we need to compare the two values: $\gamma \mu_2/2$ and $(\gamma \mu_N + \sqrt{(\gamma \mu_N)^2 + 4\mu_N})/2$.

Let $\delta = \gamma \mu_2/2 - (\gamma \mu_N + \sqrt{(\gamma \mu_N)^2 + 4\mu_N})/2$, take partial derivative of δ with respect to γ to get

$$\begin{aligned}\frac{\partial \delta}{\partial \gamma} &= \frac{1}{2}(\mu_2 - \mu_N) - \frac{\gamma \mu_N^2}{2\sqrt{(\gamma \mu_N)^2 + 4\mu_N}} \\ &= \frac{1}{2} \left(\mu_2 - \mu_N + \frac{\mu_N}{\sqrt{1 + \frac{4}{\gamma^2 \mu_N}}} \right)\end{aligned}$$

Since $\gamma \leq \sqrt{-4/\mu_2}$, the following inequalities are deduced

$$\begin{aligned}\gamma^2 \leq -\frac{4}{\mu_2} &\Rightarrow 1 + \frac{4}{\gamma^2 \mu_N} \leq 1 - \frac{\mu_2}{\mu_N} \\ &\Rightarrow \frac{\partial \delta}{\partial \gamma} < \frac{1}{2} \left(\mu_2 - \mu_N + \frac{\mu_N}{\sqrt{1 - \frac{\mu_2}{\mu_N}}} \right) \\ &= \frac{\mu_N}{2} \left(\frac{\mu_2}{\mu_N} - 1 + \frac{1}{\sqrt{1 - \frac{\mu_2}{\mu_N}}} \right) \\ &= \frac{\mu_N}{2} \left(-\alpha^2 + \frac{1}{\alpha} \right) \\ &< 0\end{aligned}$$

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where $\alpha = \sqrt{1 - \mu_2/\mu_N}$ and the fact that $\mu_N \leq \mu_2 < 0$ and $0 \leq \alpha < 1$ is used.

On the other hand, we need to check if the two values are the same. Let $\delta = 0$ to calculate $\gamma^* = \sqrt{4\mu_N/(\mu_2^2 - 2\mu_2\mu_N)}$ and it is easy to check that this value lies in the interval of this case. Because $\delta = \gamma\mu_2/2 - (\gamma\mu_N + \sqrt{(\gamma\mu_N)^2 + 4\mu_N})/2$ decreases as the partial derivative is negative, when $\gamma \leq \gamma^*$ and $\delta \geq 0$ we need to choose $\gamma\mu_2/2$ as the convergence rate, the value for fastest convergence is $-\gamma^*\mu_2/2 = \sqrt{\mu_2\mu_N/(\mu_2 - 2\mu_N)}$. When $\gamma \geq \gamma^*$ and $\delta \leq 0$ the convergence rate will be $(\gamma\mu_N + \sqrt{(\gamma\mu_N)^2 + 4\mu_N})/2$. Take partial derivative of λ_N with respect to γ ,

$$\begin{aligned}\frac{\partial \lambda_N}{\partial \gamma} &= \frac{\mu_N}{2} + \frac{\gamma\mu_N^2}{2\sqrt{(\gamma\mu_N)^2 + 4\mu_N}} \\ &= \frac{\mu_N}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{4}{\gamma^2\mu_N}}} \right)\end{aligned}$$

and

$$\gamma^2 \geq -\frac{4}{\mu_N} \Rightarrow -1 \leq \frac{4}{\gamma^2\mu_N} < 0 \Rightarrow 1 < \frac{1}{\sqrt{1 + \frac{4}{\gamma^2\mu_N}}} \Rightarrow \frac{\partial \lambda_N}{\partial \gamma} > 0$$

Therefore λ_N is least when γ takes its minimum value γ^* and again the convergence rate is $\sqrt{\mu_2\mu_N/(\mu_2 - 2\mu_N)}$. So the rate is $r_2 = \sqrt{\mu_2\mu_N/(\mu_2 - 2\mu_N)}$ for case 2.

Case 3: $\gamma \geq \sqrt{-\frac{4}{\mu_2}}$

For case 3, $(\gamma\mu_i)^2 + 4\mu_i > 0, i = 2, \dots, N$ and we have the conclusion from case 2 that $\partial\lambda/\partial\mu < 0$ hence the convergence rate is λ_N . Again from case 2, $\partial\lambda_N/\partial\gamma > 0$ and $\gamma = \sqrt{-4/\mu_2}$ will maximize the convergence rate to $r_3 = \sqrt{-\mu_N}(\sqrt{\mu_N/\mu_2} - \sqrt{\mu_N/\mu_2 - 1})$.

Finally, we need to prove $r_2 \geq r_1$ and $r_2 \geq r_3$. Define $\beta = \mu_N/\mu_2$, we can write

$$\begin{aligned} r_1 &= \frac{\sqrt{-\mu_2}}{\sqrt{\beta}} \\ r_2 &= \frac{\sqrt{-\mu_N}}{\sqrt{2\beta-1}} \\ r_3 &= \frac{\sqrt{-\mu_N}}{\sqrt{\beta} + \sqrt{\beta-1}} \end{aligned}$$

$(r_2/r_1)^2 = \beta^2/(2\beta-1) \geq 1$ and $\sqrt{2\beta-1} \leq \sqrt{\beta} + \sqrt{\beta-1}$ hence $r_2 \geq r_3$. So we conclude that r_2 is the fastest convergence rate. $\square$

3.2 Second-order Consensus without Velocity Information

In previous sections, the control vector contains both the agents' positions and velocities: $u = -\mathcal{L}\xi - \gamma\mathcal{L}\zeta$, where ξ and ζ are position and velocity vector. However, it is sometimes not applicable to get the velocities and only positions are available. In such scenarios we need to estimate the velocities and feedback the estimates to the system to achieve consensus. The objective is to design an observer to estimate the velocities of the agents and the control is changed to $u = -\mathcal{L}\xi - \gamma\mathcal{L}\hat{\zeta}$, where $\hat{\zeta}$ is the vector of velocity estimations based on the positions.

For simplicity, each state is considered to be one dimensional and it is easy to extend to high dimension with Kronecker product. For the original system, only state is available and the output matrix $C = [I_N \ 0]$, which is an $N \times 2N$ matrix. Suppose the gain matrix K has the form $K = [K_1^T \ K_2^T]^T$, where K_1 and K_2 are $N \times N$ matrices, the observer can be expressed as the following

$$\begin{bmatrix} \dot{\hat{\xi}} \\ \dot{\hat{\zeta}} \end{bmatrix} = (\Theta - KC) \begin{bmatrix} \hat{\xi} \\ \hat{\zeta} \end{bmatrix} + K\xi$$

Expand the transition matrix and substitute corresponding block in to get

$$\begin{bmatrix} \dot{\hat{\xi}} \\ \dot{\hat{\zeta}} \end{bmatrix} = \begin{bmatrix} -K_1 & I \\ -\mathcal{L} - K_2 & -\gamma\mathcal{L} \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ \hat{\zeta} \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \xi \quad (\text{Eq. 3.2})$$

The problem is to select K_1 and K_2 . Note that the observer must be distributed so that each agent generates its estimate only based on its own position and positions of its neighbors. This requirement means that the structures of K_1 and K_2 are restricted, any entry of K_1 and K_2 must be zero if the corresponding entry of $\mathcal{L}$ is zero since zero implies no communication. We have the following lemma

Lemma 3.1 *The observer (Eq. 3.2) can be distributed and its output can converge to the actual value at any desired rate with proper choice of matrices K_1 and K_2 .*

Proof: Using estimates, the closed-loop system has the form

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{L} & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} - \begin{bmatrix} 0 \\ \gamma\mathcal{L}\hat{\zeta} \end{bmatrix} \quad (\text{Eq. 3.3})$$

Subtract (Eq. 3.2) from (Eq. 3.3) to obtain the observer error dynamics:

$$\begin{bmatrix} \dot{\xi} - \dot{\hat{\xi}} \\ \dot{\zeta} - \dot{\hat{\zeta}} \end{bmatrix} = \begin{bmatrix} -K_1 & I \\ -\mathcal{L} - K_2 & 0 \end{bmatrix} \begin{bmatrix} \xi - \hat{\xi} \\ \zeta - \hat{\zeta} \end{bmatrix} \quad (\text{Eq. 3.4})$$

Consider the characteristic polynomial

$$\begin{aligned} \det \left(\lambda I - \begin{bmatrix} -K_1 & I \\ -\mathcal{L} - K_2 & 0 \end{bmatrix} \right) &= \begin{vmatrix} \lambda I + K_1 & -I \\ \mathcal{L} + K_2 & \lambda I \end{vmatrix} \\ &= \det(\lambda^2 I + \lambda K_1 + \mathcal{L} + K_2) \end{aligned}$$

Select $K_1 = \mathcal{K}_1$ and $K_2 = \mathcal{K}_2 - \mathcal{L}$, where $\mathcal{K}_1 = \text{diag}(k_{1i})$ and $\mathcal{K}_2 = \text{diag}(k_{2i})$ are

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diagonal matrices whose entries are to be determined. Such choices satisfy the distributed requirement we mentioned before, and the characteristic polynomial will be $\det(\lambda^2 I + \lambda K_1 + \mathcal{L} + K_2) = \prod_{i=1}^N (\lambda^2 + \lambda k_{1i} + k_{2i})$. With different values of k_{1i} and k_{2i} , the roots of the characteristic polynomial can be any desired values. Thus the convergence rate can be designed by setting different K_1 and K_2 .

Now we have the theorem for multi-agent systems via output feedback.

Theorem 3.2 *For the system (Eq. 3.3) and observer (Eq. 3.2), if the network topology has a spanning tree and all the eigenvalues of matrix $\mathcal{L}$ are real, the system can reach second-order consensus at the same fastest rate as the full state feedback case in Theorem 3.1.*

Proof: To prove the theorem, we write the system dynamics in a generalized form as

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \\ \dot{\hat{\xi}} \\ \dot{\hat{\zeta}} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 \\ -\mathcal{L} & 0 & 0 & -\gamma\mathcal{L} \\ K_1 & 0 & -K_1 & I \\ K_2 & 0 & -\mathcal{L} - K_2 & -\gamma\mathcal{L} \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \\ \hat{\xi} \\ \hat{\zeta} \end{bmatrix} \quad (\text{Eq. 3.5})$$

Then the characteristic polynomial is

$$\begin{aligned} \rho(\lambda) &= \begin{vmatrix} \lambda I & -I & 0 & 0 \\ \mathcal{L} & \lambda I & 0 & \gamma\mathcal{L} \\ -K_1 & 0 & \lambda I + K_1 & -I \\ -K_2 & 0 & \mathcal{L} + K_2 & \lambda I + \gamma\mathcal{L} \end{vmatrix} \\ &= \begin{vmatrix} \lambda I & -I & 0 & 0 \\ \lambda^2 I + \mathcal{L} & 0 & 0 & \gamma\mathcal{L} \\ -K_1 & 0 & \lambda I + K_1 & -I \\ -K_2 & 0 & \mathcal{L} + K_2 & \lambda I + \gamma\mathcal{L} \end{vmatrix} \\ &= \begin{vmatrix} \lambda^2 I + \mathcal{L} & 0 & \gamma\mathcal{L} \\ -K_1 & \lambda I + K_1 & -I \\ -K_2 & \mathcal{L} + K_2 & \lambda I + \gamma\mathcal{L} \end{vmatrix} \\ &= \det(\lambda^2 I + \lambda K_1 + \mathcal{L} + K_2) \det(\lambda^2 I + (\lambda\gamma + 1)\mathcal{L}) \end{aligned}$$

From the equation, we can see that the polynomial contains two parts, which are exactly

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the characteristic polynomial of the observer and that of the second-order consensus system, as provided in previous sections.

As proved before, the observer can converge at any desired rate. So we can always let the observer converge faster than the consensus system. On the other hand, we can select $\gamma = \gamma^*$ to have the system reach second-order consensus at fastest rate, since all eigenvalues of $\mathcal{L}$ are real. Thus the theorem is proved.

Remark 3.1 *Ren has proposed a consensus algorithm based on a passivity approach without the information of derivative[36]. However, the algorithm requires matrix calculation to determine the parameters and the complexity increases enormously when the dimension is high. Besides the convergence rate cannot be explicitly determined for the approach of [36].*

Remark 3.2 *We can see that the estimation converges at speed $\min_i \operatorname{Re}(\frac{-d_i \pm \sqrt{d_i^2 - 4d_i}}{2})$, but still we need to consider the convergence rate of the system itself, which is mainly determined by γ . The whole convergence rate will be the smaller one of the two rates.*

3.3 Simulations

In simulation, we testify the result we got in Theorem 3.1. For convenience, the topology is assumed to be undirected so that all eigenvalues are real and it is easy to compare the performance with Ren's approach. Fig. 3.1 is the communication topology.

We have the corresponding Laplacian matrix

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

and it is easy to calculate the four eigenvalues of $-\mathcal{L}$: $\lambda_i = 0, -1, -3, -4, i = 1 - 4$.

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Figure 3.1: Undirected communication topology of the system

From Theorem 3.1, the largest convergence rate is $r = 0.756$ corresponding to $\gamma^* = 1.512$. Figures 3.2 to 3.4 show the evolutions of the velocity in x and y direction corresponding to $\gamma=1, 1.512$ and 5 . It is conspicuous that the system converges fastest and performs best for $\gamma = 1.512$ and it is consistent with Theorem 3.1.

Figure 3.2: Evolutions of the $\dot{\zeta}_x$ and $\dot{\zeta}_y$ under second-order consensus protocol with $\gamma=1$.

Still use the symmetric $\mathcal{L}$ to testify Theorem 3.2, let $K_1 = 2I$ and $K_2 = I - \mathcal{L}$ and the observer will converge with rate 1, which is faster than the second-order consensus convergence rate and guarantees that the whole system converges at the consensus convergence rate of the state feedback case. Figures 3.5 to 3.7 show the evolutions of ξ and ζ under conditions $\gamma = 1$, $\gamma = 1.512$ and $\gamma = 2$.

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Figure 3.3: Evolutions of the ζ_x and ζ_y under second-order consensus protocol with $\gamma=1.512$.

Figure 3.4: Evolutions of the ζ_x and ζ_y under second-order consensus protocol with $\gamma=5$.

Fig. 3.8 is the evolution of position and velocity using passivity protocol proposed in [36] with parameters $p = 2$ and $\gamma = -2$ (different from the γ in the thesis). It can be seen that the passivity approach reaches second-order consensus slower than the case when $\gamma = 1.512$.

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Figure 3.5: Evolutions of the ξ and ζ under second-order consensus protocol with $\gamma=1$.

Figure 3.6: Evolutions of the ξ and ζ under second-order consensus protocol with $\gamma=1.512$.

3.4 Conclusions

In this chapter we discussed the fastest convergence rate for continuous second-order multi-agent systems with all eigenvalues of the Laplacian matrix being real. Furthermore, when the agents' velocities are not available, we can still have the system achieve second-order consensus at the fastest convergence rate. The simulations showed the effectiveness of the approaches.

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Figure 3.7: Evolutions of the ξ and ζ under second-order consensus protocol with $\gamma=2$.

Figure 3.8: Evolutions of the ξ and ζ for passivity approach

Chapter 4

Fastest Convergence Rate for Discrete Second-Order Consensus

4.1 Introduction

In the last chapter we focused on the fastest convergence rate for continuous systems under the condition that all eigenvalues of the Laplacian matrix are real. In practice, however, computer control systems are mostly discrete with sampling period h to apply the control protocol step by step. When the original continuous system is sampled every h interval, the discrete system dynamics will be related to both the control gain in protocol and the sampling interval characteristic. In addition, the requirement for consensus is changed to that all eigenvalues of the system transition matrix lie in the unit circle. In such scenarios, is it possible to determine a proper control gain such that the convergence rate is fastest, as what was derived for the continuous case? This problem will be investigated in this chapter.

4.2 Problem Formulation

Assume that we have an undirected communication topology, meaning that any agent sends its information to one of its neighbors if it receives the information from that neighbor and vice versa. With the undirected network topology, the Laplacian matrix is

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symmetric and thus has real eigenvalues. Furthermore, we only discuss the one dimensional case for simplicity, which is easy to extend to higher dimensions with Kronecker product.

For a discrete second-order system with sampling period h , the dynamics of agent i is given by

$$\begin{cases} \xi_i(t+1) = \xi_i(t) + h\zeta_i(t) \\ \zeta_i(t+1) = \zeta_i(t) + hu_i(t) \end{cases} \quad (\text{Eq. 4.1})$$

where $i = 1, 2, \dots, N$ and t is the time instance and the terms with second and higher order of h are omitted. Similar to the continuous case, with full information of the agent itself and its neighbors, we consider a control protocol of the form

$$u_i(t) = \sum_{j=1}^N a_{ij}(\xi_j(t) - \xi_i(t)) + \gamma \sum_{j=1}^N a_{ij}(\zeta_j(t) - \zeta_i(t)) \quad (\text{Eq. 4.2})$$

Denote $\xi(t) = [\xi_1(t), \dots, \xi_N(t)]^T$, $\zeta(t) = [\zeta_1(t), \dots, \zeta_N(t)]^T$, and the closed-loop multi-agent system can be written in matrix form as

$$\begin{bmatrix} \xi(t+1) \\ \zeta(t+1) \end{bmatrix} = \begin{bmatrix} I & hI \\ -h\mathcal{L} & I - h\gamma\mathcal{L} \end{bmatrix} \begin{bmatrix} \xi(t) \\ \zeta(t) \end{bmatrix} \quad (\text{Eq. 4.3})$$

The second-order consensus is said to be achieved if, as $t \rightarrow \infty$ both $\xi(t) \rightarrow p_0(t)\mathbf{1}$ and $\zeta(t) \rightarrow v_0(t)\mathbf{1}$ hold, where $\mathbf{1}$ is the column vector with all entries of 1 and $p_0(t)$ and $v_0(t)$ are the consensus equilibrium of $\xi_i(t)$ and $\zeta_i(t)$ respectively.

Denote the transition matrix as Θ_d and recall from Chapter 3 that we have its counterpart in the continuous case

$$\Theta = \begin{bmatrix} 0 & I \\ -\mathcal{L} & -\gamma\mathcal{L} \end{bmatrix}$$

It can be easily seen that $\Theta_d = I + h\Theta$.

Since the eigenvalues of matrix Θ are $\lambda_{i\pm} = \frac{1}{2}(\gamma\mu_i \pm \sqrt{\gamma^2\mu_i^2 + 4\mu_i})$, the eigenvalues of Θ_d have the form $\eta_{i\pm} = 1 + h\lambda_{i\pm}$, where μ_i are the eigenvalues of matrix $-\mathcal{L}$. The

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two zero eigenvalues $\lambda_{1\pm}$ of Θ corresponding to the zero eigenvalue μ_1 result in $\eta_{1\pm} = 1$. To guarantee consensus, all other eigenvalues $\eta_{i\pm}, i = 2, \dots, N$ must lie in the unit circle. It is easy to see that the convergence rate is decided by the eigenvalue with the largest magnitude other than 1, and the smaller the magnitude is, the faster the system converges. The objective of this chapter is to find a proper γ such that the largest magnitude of the eigenvalues $\eta_{i\pm}, i = 2, 3, \dots, N$ is the smallest among all γ values.

4.3 Main Results

For the problem we proposed in the last section, we have the following theorem.

Theorem 4.1 *Suppose the communication topology is undirected and connected, arrange the eigenvalues of matrix $-\mathcal{L}$ in a descending order $\mu_N \leq \dots \leq \mu_2 < \mu_1 = 0$. Then we have:*

- i) *The multi-agent system reaches consensus only if $h < \frac{2}{\sqrt{-\mu_N}}$*
- ii) *The discrete system (Eq. 4.3) will reach second-order consensus at fastest rate when γ solves the following equations:*

$$\begin{aligned} \sqrt{1 + \mu_2 h(\gamma - h)} &= 1 + h \frac{\gamma \mu_N + \sqrt{\gamma^2 \mu_N^2 + 4\mu_N}}{2}, \\ \text{for } 0 < h &\leq \sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}} \end{aligned} \quad (\text{Eq. 4.4})$$

or

$$\begin{aligned} \sqrt{1 + \mu_2 h(\gamma - h)} &= -1 - h \frac{\gamma \mu_N - \sqrt{\gamma^2 \mu_N^2 + 4\mu_N}}{2}, \\ \text{for } \sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}} &\leq h < \frac{2}{\sqrt{-\mu_N}} \end{aligned} \quad (\text{Eq. 4.5})$$

Proof: i) First of all, let us determine the upper bound for h . To begin with, we consider only the real eigenvalues and the restriction is $\eta_{i-} = 1 + h\lambda_{i-} > -1$ (note that

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$\eta_{i-} < 1$ since $\lambda_{i\pm} < 0$). Since

$$\frac{\partial \lambda_{i-}}{\partial \mu_i} = \frac{1}{2} \left(\gamma - \frac{\gamma^2 \mu_i + 2}{\sqrt{(\gamma \mu_i)^2 + 4\mu_i}} \right)$$

and for real eigenvalues, $(\gamma \mu_i)^2 + 4\mu_i > 0 \Rightarrow \gamma^2 \mu_i + 4 < 0 \Rightarrow \gamma^2 \mu_i + 2 < -2 < 0$ together with the positive γ , we have $\frac{\partial \lambda_{i-}}{\partial \mu_i} > 0$. Hence $1 + h\lambda_{N-} > -1$ will guarantee that the specified condition holds for all real eigenvalues. Hence $h < -\frac{2}{\lambda_{N-}}$, and when $\lambda_{N-} < 0$ takes its maximum at a γ value, the corresponding h is the upper bound for h . On the other hand we have

$$\frac{\partial \lambda_{i-}}{\partial \gamma} = \frac{1}{2} \left(\mu_i - \frac{\gamma \mu_i^2}{\sqrt{(\gamma \mu_i)^2 + 4\mu_i}} \right) < 0$$

and λ_{N-} reaches its maximum $-\sqrt{-\mu_N}$ with $\gamma = \frac{2}{\sqrt{-\mu_N}}$ which is the minimum of γ to guarantee λ_{N-} to be real. So $h < -\frac{2}{\lambda_{N-}} < \frac{2}{\sqrt{-\mu_N}}$ for real eigenvalues. In the following proof we shall see that $0 < h < \frac{2}{\sqrt{-\mu_N}}$ is also a valid range of h for complex eigenvalues to ensure that $\eta_{i\pm}$ lie in the unit circle with certain γ values. Thus the range of h is $0 < h < \frac{2}{\sqrt{-\mu_N}}$.

ii) We shall discuss the different range of γ , like what we did for the continuous case.

Case 1: $\gamma \leq \frac{2}{\sqrt{-\mu_N}}$

In this case, all eigenvalues $\lambda_{i\pm} = \frac{\gamma \mu_i \pm \sqrt{\gamma^2 \mu_i^2 + 4\mu_i}}{2}$ are complex numbers and so are $\eta_{i\pm} = 1 + h\lambda_{i\pm}$. Consider the magnitudes of eigenvalues, $|\eta_{i\pm}|^2 = (1 + \frac{\gamma \mu_i h}{2})^2 - \frac{1}{4}h^2(\gamma^2 \mu_i^2 + 4\mu_i) = 1 + \mu_i h(\gamma - h)$. For convergence, it requires that the magnitudes lie in the range $[0, 1)$, which is ensured by the fact that $\mu_i < 0$ and $\gamma > h$. On the other hand, $|\eta_{n\pm}|^2 = 1 + \mu_N h(\gamma - h) \geq 1 + \mu_N h(\frac{2}{\sqrt{-\mu_N}} - h) = (1 - h\sqrt{-\mu_N})^2 \geq 0$ is the smallest value among the magnitudes.

From the above discussion we know that the only restriction for convergence under such condition is that $\gamma > h$, and hence $h < \frac{2}{\sqrt{-\mu_N}}$ must be satisfied, which is consistent with the condition when we determine the upper bound of h . With γ larger than h , the

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magnitude reaches its maximum when γ takes its maximum. Thus $|\eta_{2\pm}|^2$ is the largest among all eigenvalues $\eta_{i\pm}$ and we come to the conclusion that for this range of γ : with $0 < h < \frac{2}{\sqrt{-\mu_N}}$, the value $|\eta_{2\pm}| = \sqrt{1 + \mu_2 h (\frac{2}{\sqrt{-\mu_N}} - h)}$ decides the fastest convergence rate.

Case 2: $\gamma \geq \frac{2}{\sqrt{-\mu_2}}$

In this case, all eigenvalues will be real, and it is easy to see that $\eta_{i\pm} = 1 + h\lambda_{i\pm} < 1$. So we only need to guarantee that $-1 < \eta_{i-} \leq \eta_{i+}$. Besides, $\frac{\partial \eta_{i-}}{\partial \mu_i} > 0$ and hence $1 + h\lambda_{N-} > -1$ will guarantee the convergence. For $1 + h\lambda_{N-}$ we have

$$\eta_{N-} = 1 + h \frac{\gamma \mu_N - \sqrt{\gamma^2 \mu_N^2 + 4\mu_N}}{2} > -1 \Rightarrow 0 \leq h \sqrt{\gamma^2 \mu_N^2 + 4\mu_N} < 4 + \gamma \mu_N h \Rightarrow \gamma < \frac{h}{2} - \frac{2}{\mu_N h}$$

and by condition $0 \leq 4 + \gamma \mu_N h$ we have $\gamma \leq -\frac{4}{\mu_N h}$. Since $0 < h < \frac{2}{\sqrt{-\mu_N}}$, γ satisfies $\frac{2}{\sqrt{-\mu_2}} \leq \gamma < \frac{h}{2} - \frac{2}{\mu_N h} < -\frac{4}{\mu_N h}$ and hence $h < \frac{2}{\sqrt{-\mu_2}} \left(1 - \sqrt{1 - \frac{\mu_2}{\mu_N}}\right) \leq \frac{2}{\sqrt{-\mu_N}}$. With such h value, $|\eta_{N-}| < 1$ guarantees the system convergence. In addition,

$$\frac{\partial \eta_{i-}}{\partial \gamma} = \frac{h}{2} \left(\mu_i - \frac{\gamma \mu_i^2}{\sqrt{(\gamma \mu_i)^2 + 4\mu_i}} \right) < 0$$

and γ should be as small as possible to make η_{N-} largest and hence $|\eta_{N-}|$ smallest which results in the fastest convergence rate. So $\gamma = \frac{2}{\sqrt{-\mu_2}}$ gives the fastest convergence rate for η_{N-} .

On the other hand,

$$\begin{aligned} \frac{\partial \eta_{i+}}{\partial \mu_i} &= \frac{h}{2} \left(\gamma + \frac{\gamma^2 \mu_i + 2}{\sqrt{(\gamma \mu_i)^2 + 4\mu_i}} \right) \\ &= \frac{h\gamma}{2} \left(1 + \frac{\gamma \mu_i + \frac{2}{\gamma}}{\sqrt{(\gamma \mu_i + \frac{2}{\gamma})^2 - \frac{4}{\gamma^2}}} \right) < 0 \end{aligned}$$

and η_{N+} is the largest one among η_{i+} . Besides,

$$\frac{\partial \eta_{i+}}{\partial \gamma} = \frac{h\mu_i}{2} \left(1 + \frac{\gamma \mu_i}{\sqrt{(\gamma \mu_i)^2 + 4\mu_i}} \right) > 0$$

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hence η_{N+} also takes minimum when $\gamma = \frac{2}{\sqrt{-\mu_2}}$, like η_{N-} .

To compare the values of η_{N-} and η_{N+} , we add the two values to get

$$\eta_{N-} + \eta_{N+} = 2 + h \frac{2}{\sqrt{-\mu_2}} \mu_N$$

If $h < -\frac{\sqrt{-\mu_2}}{\mu_N}$, the sum of the two values is positive which means that η_{N+} is larger than its counterpart η_{N-} in magnitude, and the convergence rate is decided by η_{N+} . Otherwise if $-\frac{\sqrt{-\mu_2}}{\mu_N} < h < \frac{2}{\sqrt{-\mu_2}} \left(1 - \sqrt{1 - \frac{\mu_2}{\mu_N}}\right)$, $|\eta_{n-}|$ will determine the convergence rate.

Case 3: $\frac{2}{\sqrt{-\mu_N}} \leq \gamma \leq \frac{2}{\sqrt{-\mu_2}}$

From the previous two parts, we have the following facts:

Firstly, for case 1 when γ is sufficiently large, $\eta_{i\pm}$ will become real values in a descending order with respect to i , and the largest magnitude of complex eigenvalues is still $|\eta_{2\pm}|$ which decreases as γ increases. In fact $|\eta_{2\pm}| = \sqrt{1 + \mu_2 h(\gamma - h)}$ decides the fastest convergence rate among all complex eigenvalues whenever there exist complex eigenvalues. In addition, the value has the largest magnitude in all eigenvalues when $\gamma = \frac{2}{\sqrt{-\mu_N}}$.

Secondly, for case 2, η_{N+} or $|\eta_{N-}|$ decides the fastest convergence rate for all real eigenvalues, and the magnitude is the largest among all eigenvalues when $\gamma = \frac{2}{\sqrt{-\mu_2}}$. Also, when γ decreases from the right end of the range, the magnitude decreases, causing the convergence rate determined by real eigenvalues to increase although some eigenvalues become complex.

Based on the two facts, we increase γ from $\frac{2}{\sqrt{-\mu_N}}$ and decrease the value from $\frac{2}{\sqrt{-\mu_2}}$ such that the largest magnitude of eigenvalues decreases for both cases to get faster convergence rate. Furthermore, since the two expressions are uniformly ascending and descending with respect to γ , the magnitude reaches its minimum when the two values meet, which means that γ takes the solution to (Eq. 4.4) or (Eq. 4.5).

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Let $\eta_{N+} + \eta_{N-} = 2 + \gamma\mu_N h = 0$ and solve the equation to get $\gamma = -\frac{2}{\mu_N h}$. If $h \geq \frac{1}{\sqrt{-\mu_N}} \Rightarrow -\frac{2}{\mu_N h} \leq \frac{2}{\sqrt{-\mu_N}}$, then all γ values in this case will result in $\eta_{N+} + \eta_{N-} \leq 0$ and (Eq. 4.5) is valid. If $h \leq -\frac{\sqrt{-\mu_2}}{\mu_N}$, then $-\frac{2}{\mu_N h} \geq \frac{2}{\sqrt{-\mu_2}}$, (Eq. 4.4) is valid. Otherwise $-\frac{\sqrt{-\mu_2}}{\mu_N} \leq \gamma \leq \frac{1}{\sqrt{-\mu_N}}$ and $\frac{2}{\sqrt{-\mu_N}} \leq -\frac{2}{\mu_N h} \leq \frac{2}{\sqrt{-\mu_2}}$, we need to compare the magnitudes for the complex eigenvalue case and real eigenvalue case.

For complex eigenvalues, $|\eta_{2\pm}| = \sqrt{1 + \mu_2 h(-\frac{2}{\mu_N h} - h)} = \sqrt{-\mu_2 h^2 + 1 - \frac{2\mu_2}{\mu_N}}$. For real eigenvalues, $\eta_{N+} = |\eta_{N-}| = 1 + \frac{h}{2} \left(-\frac{2}{h} + \sqrt{\frac{4}{h^2} + 4\mu_N} \right) = \sqrt{1 + \mu_N h^2}$. Denote $\eta_2 = |\eta_{2\pm}|$ and $\eta_N = \max(\eta_{N+}, |\eta_{N-}|)$ for simplicity and $\eta_2^2 - \eta_N^2 = -(\mu_2 + \mu_N)h^2 - \frac{2\mu_2}{\mu_N}$. When $-\frac{\sqrt{-\mu_2}}{\mu_N} \leq h \leq \sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}}$, $\eta_2 \leq \eta_N$ and γ should decrease to drive the two values to reach a same magnitude. For a smaller γ , $\eta_{N+} + \eta_{N-} = 2 + \gamma\mu_N h > 0$ and (Eq. 4.4) is valid. Otherwise for $\sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}} \leq h \leq \frac{1}{\sqrt{-\mu_N}}$, (Eq. 4.5) works.

Combining the results, we have the conclusion that for $0 < h \leq \sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}}$, the γ value solving (Eq. 4.4) gives the fastest convergence rate. For other $\sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}} \leq h < \frac{2}{\sqrt{-\mu_N}}$, γ should be the solution to (Eq. 4.5).

In the end, we prove that there must exist such a γ value which falls in the range $\left[\frac{2}{\sqrt{-\mu_N}}, \frac{2}{\sqrt{-\mu_2}} \right]$ solving the corresponding equation. For $0 < h \leq \sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}}$, when $\gamma = \frac{2}{\sqrt{-\mu_N}}$, we have $\eta_2^2 = 1 + \mu_2 h \left(\frac{2}{\sqrt{-\mu_N}} - h \right) = 1 - \mu_2 h \left(h - \frac{2}{\sqrt{-\mu_N}} \right)$ and $\eta_N^2 = (1 - h\sqrt{-\mu_N})^2 = 1 - \mu_N h \left(h - \frac{2}{\sqrt{-\mu_N}} \right)$. Since $h < \frac{2}{\sqrt{-\mu_N}}$, $\eta_2^2 > \eta_N^2$ and consequently $\eta_2 - \eta_N > 0$. Then let $\gamma = \frac{2}{\sqrt{-\mu_2}}$, $\eta_2 = \sqrt{1 + \mu_2 h \left(\frac{2}{\sqrt{-\mu_2}} - h \right)} = |1 - h\sqrt{-\mu_2}| = 1 - h\sqrt{-\mu_2}$, and

$$\begin{aligned} \eta_n &= 1 + \frac{h}{2} \left(\mu_n \frac{2}{\sqrt{-\mu_2}} + \sqrt{\frac{4\mu_N^2}{-\mu_2} + 4\mu_N} \right) \\ &= 1 - h\sqrt{-\mu_N} \left(\sqrt{\frac{\mu_N}{\mu_2}} - \sqrt{\frac{\mu_N}{\mu_2} - 1} \right) \\ &= 1 - \frac{h\sqrt{-\mu_N}}{\sqrt{\frac{\mu_N}{\mu_2}} + \sqrt{\frac{\mu_N}{\mu_2} - 1}} \geq 1 - h\sqrt{-\mu_2} = \eta_2 \end{aligned}$$

Thus $\eta_2 - \eta_N > 0$ when $\gamma = \frac{2}{\sqrt{-\mu_N}}$ and $\eta_2 - \eta_N \leq 0$ when $\gamma = \frac{2}{\sqrt{-\mu_2}}$, along with the condition that function $\eta_2 - \eta_N$ is continuous for all γ in the range, there must exist at least

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one zero for $\eta_2 - \eta_N$ which is a solution to (Eq. 4.4). Then for $\sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}} \leq h < \frac{2}{\sqrt{-\mu_N}}$, when $\gamma = \frac{2}{\sqrt{-\mu_N}}$, still we have $\eta_2 - \eta_N > 0$. Let $\gamma = \frac{2}{\sqrt{-\mu_2}}$ and $\eta_2 = |1 - h\sqrt{-\mu_2}|$,

$$\begin{aligned}\eta_N &= -1 - \frac{h}{2} \left(\mu_N \frac{2}{\sqrt{-\mu_2}} - \sqrt{\frac{4\mu_N^2}{-\mu_2} + 4\mu_N} \right) \\ &= h \frac{-\mu_N}{\sqrt{-\mu_2}} \left(1 + \sqrt{1 - \frac{\mu_2}{\mu_N}} \right) - 1 \\ &= \frac{\mu_N}{\mu_2} \left(1 + \sqrt{1 - \frac{\mu_2}{\mu_N}} \right) h \sqrt{-\mu_2} - 1 \\ &= \frac{1}{1 - \sqrt{1 - \frac{\mu_2}{\mu_N}}} h \sqrt{-\mu_2} - 1\end{aligned}$$

Let $\alpha = \frac{\mu_2}{\mu_N} \leq 1$ and $\beta = \frac{1}{1 - \sqrt{1 - \alpha}} \geq 1$. We calculate $\eta_2^2 - \eta_N^2$ to compare the magnitude of the two values.

$$\begin{aligned}\eta_2^2 - \eta_N^2 &= (1 - 2h\sqrt{-\mu_2} - \mu_2 h^2) - (1 - 2\beta h\sqrt{-\mu_2} - \mu_2 \beta^2 h^2) \\ &= 2(\beta - 1)h\sqrt{-\mu_2} + (\beta^2 - 1)\mu_2 h^2 \\ &= (\beta - 1)h\sqrt{-\mu_2}(2 - (\beta + 1)h\sqrt{-\mu_2})\end{aligned}$$

and we have

$$\begin{aligned}(\beta + 1)h\sqrt{-\mu_2} &\geq \left[\frac{\mu_N}{\mu_2} \left(1 + \sqrt{1 - \frac{\mu_2}{\mu_N}} \right) + 1 \right] \sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}} \sqrt{-\mu_2} \\ &= \left(1 + \frac{\mu_2}{\mu_N} + \sqrt{1 - \frac{\mu_2}{\mu_N}} \right) \sqrt{\frac{2\mu_N}{\mu_2 + \mu_N}} \\ &= \sqrt{2} \left(\sqrt{1 + \frac{\mu_2}{\mu_N}} + \sqrt{\frac{\mu_N - \mu_2}{\mu_N + \mu_2}} \right) \\ &= \sqrt{2} \left(\sqrt{1 + \alpha} + \sqrt{\frac{1 - \alpha}{1 + \alpha}} \right)\end{aligned}$$

Furthermore

$$\begin{aligned}\frac{\partial \left(\sqrt{1 + \alpha} + \sqrt{\frac{1 - \alpha}{1 + \alpha}} \right)}{\partial \alpha} &= \frac{1}{2\sqrt{1 + \alpha}} + \frac{-1}{2\sqrt{1 - \alpha}} \frac{1}{\sqrt{1 + \alpha}} - \frac{1}{2(1 + \alpha)^{3/2}} \sqrt{1 - \alpha} \\ &= \frac{1}{2\sqrt{1 + \alpha}} \left(1 - \frac{1}{\sqrt{1 - \alpha}} - \frac{\sqrt{1 - \alpha}}{1 + \alpha} \right) \leq 0\end{aligned}$$

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which means that the value reaches its minimum $\sqrt{2}$ when α takes its maximum 1. Hence $(\beta + 1)h\sqrt{-\mu_2} \geq 2$ and $\eta_2^2 - \eta_N^2 \leq 0$. In this case, there exists a solution to (Eq. 4.5) between $\left[\frac{2}{\sqrt{-\mu_N}}, \frac{2}{\sqrt{-\mu_2}}\right]$.

Now we come to the conclusion:

i) For $0 < h \leq \sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}}$, the solution to (Eq. 4.4) gives the fastest convergence rate.

ii) For $\sqrt{\frac{2\mu_2}{-(\mu_2 + \mu_N)\mu_N}} \leq h < \frac{2}{\sqrt{-\mu_N}}$, γ should take the solution to (Eq. 4.5) so that the system converges fastest. $\square$

Remark 4.1 *The result we obtained above is the fastest convergence rate of the discrete second-order consensus system, under the condition that the full position and velocity information of each agent and its neighbors are available. Further studies on the case of position feedback only is possible. We note that unlike the continuous case, the fastest convergence rate cannot be explicitly given. Furthermore, we can see that μ_2 and μ_N play critical roles again, like the continuous case.*

Remark 4.2 *If we decrease the sampling time h to approach zero, meaning that the system approaches a continuous one, we can calculate the optimal γ value by solving (Eq. 4.4)*

$$\sqrt{1 + \mu_2 h(\gamma - h)} = 1 + h \frac{\gamma \mu_N + \sqrt{\gamma^2 \mu_N^2 + 4\mu_N}}{2}$$

Square both sides and omit the terms with h of order higher than 1 since h is very small, we have

$$\gamma \mu_2 = \gamma \mu_N + \sqrt{(\gamma \mu_N)^2 + 4\mu_N}$$

Solving the equation and we have $\gamma = \sqrt{\frac{4\mu_N}{\mu_2^2 - 2\mu_2\mu_N}}$, which is exactly the optimal γ we found for the continuous case.

Remark 4.3 For further investigation, we can determine the valid range for γ . First of all, the lower bound of γ is h from case 1. When γ increases and grows larger than the value for fastest convergence rate, the real eigenvalues will dominate. If $0 < h \leq \sqrt{\frac{2\mu_2}{-(\mu_2+\mu_N)\mu_N}}$, $|\eta_{i-}| < \eta_{i+} = 1 + h\lambda_i < 1$ since $\lambda_i < 0$. Hence γ can be infinitely large. But when $\sqrt{\frac{2\mu_2}{-(\mu_2+\mu_N)\mu_N}} \leq h < \frac{2}{\sqrt{-\mu_N}}$, $\eta_{i+} < |\eta_{i-}| = -1 - h\frac{\gamma\mu_i - \sqrt{(\gamma\mu_i)^2 + 4\mu_i}}{2} < 1$ will result in $\gamma < \frac{h}{2} - \frac{2}{h\mu_i} \leq \frac{h}{2} - \frac{2}{h\mu_N}$. Note that $h < \frac{2}{\sqrt{-\mu_N}}$, which guarantees that $h < \frac{h}{2} - \frac{2}{h\mu_i}$ and hence the valid range is $h < \gamma < \frac{h}{2} - \frac{2}{h\mu_N}$.

4.4 Numerical Examples

In this part, we use MATLAB to show the evolution of the maximum magnitude of all eigenvalues of matrix Θ_d as γ varies with different sampling time h . Still we use the Laplacian matrix in previous chapters

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

whose eigenvalues are 0,1,3,4. To begin with, set $h = 0.01$ and solve (Eq. 4.4) to get $\gamma = 1.514$ and the corresponding largest eigenvalue magnitude $\eta = 0.99245$, and the evolution of η is shown in Fig. 4.1. Note that the optimal value for continuous case is $\gamma = 1.512$ and the value we get here is close to that of continuous case.

Increasing h to 0.1 and the evolution of η is in Fig. 4.2. Solving (Eq. 4.4) to get $\gamma = 1.531$.

In the above two cases, h is relatively small and (Eq. 4.4) works, we can derive the switch value as $h = \sqrt{\frac{2\mu_2}{-(\mu_2+\mu_N)\mu_N}} = 0.316$. So we take $h = 0.4$ as the last example and solve (Eq. 4.5) to get $\gamma = 1.342$. In addition, from the Remark 4.3 we can determine the valid range of γ as $0.4 = h < \gamma < \frac{h}{2} - \frac{2}{h\mu_N} = 1.45$. The evolution of η is shown in Fig. 4.3.

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Figure 4.1: Evolution of η as γ varies and $h = 0.01$

4.5 Conclusions

The consensus problem of discrete second-order multi-agent system was discussed in this chapter. Unlike its continuous counterpart, the fastest convergence rate cannot be explicitly given but can be given by solving an algebraic equation. Finally the simulations verified our conclusions.

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Figure 4.2: Evolution of η as γ varies and $h = 0.1$

Figure 4.3: Evolution of η as γ varies and $h = 0.4$

Chapter 5

Quantized Discrete Second-order Consensus

The previous chapter addressed the discrete second-order consensus problem under ideal communication channels. Like the scenario we discussed in the continuous case, however, the velocity information may not be available and we shall design the control with only agent position information. Furthermore, considering the network transmission bandwidth limitation, quantization is applied in the protocol to achieve consensus under low bandwidth cost. In this chapter, the quantized information is transmitted between agents and used for position and velocity estimation of agents.

5.1 Problem Formulation

For convenience, the discrete system sampling interval is omitted and integrated with the corresponding input terms, and the system dynamics have the form

$$\begin{cases} \xi_i(t+1) = \xi_i(t) + \zeta_i(t), \\ \zeta_i(t+1) = \zeta_i(t) + u_i(t), \\ \psi_i(t) = \xi_i(t), \quad i = 1, 2, \dots, N, \quad t = 0, 1, \dots \end{cases} \quad (\text{Eq. 5.1})$$

where $\xi_i(t)$, $\zeta_i(t)$ and $u_i(t)$ are the position, velocity and control input of agent i and $\psi_i(t)$ the output of the i th agent which is in fact the position. Also the communication

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topology is modeled as an undirected connected graph $\mathcal{G}_N = \{\mathcal{V}_N, \mathcal{E}_N\}$, where the agents and the communication channels between agents are represented by the nodes set $\mathcal{V}_N$ and edges set $\mathcal{E}_N$ respectively. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix of $\mathcal{G}_N$. A communication channel from agent j to i is denoted as a pair (j, i) and we can see that $(j, i) \in \mathcal{E}_N$ if and only if $(i, j) \in \mathcal{E}_N$ since the topology is undirected. Define $\mathcal{N}_i = \{j \in \mathcal{V}_N | (i, j) \in \mathcal{E}_N\}$ as the neighborhood of the i th agent, and for any $i, j \in \mathcal{V}_N$, $a_{ij} > 0$ if and only if $j \in \mathcal{N}_i$. The undirected nature of communication topology also results $a_{ij} = a_{ji}$ and thus define $\deg_i = \sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji}$ as the degree of agent i , and $d^* = \max_i \deg_i$ is called the degree of $\mathcal{G}_N$. If $a_{ij} = 1 \forall (i, j) \in \mathcal{E}_N$, and $|\mathcal{N}_i| = d, i = 1, 2, \dots, N$ where $d > 0$ is a constant, we say that the graph $\mathcal{G}_N$ is d -regular. Let $\mathcal{D} = \text{diag}(\deg_1, \dots, \deg_N)$ and define the Laplacian matrix of $\mathcal{G}_N$ as $\mathcal{L} = \mathcal{D} - \mathcal{A}$.

Suppose that all communication channels between agents are digital channels with finite channel capacity, which limit each agent to transmit a coded version of its current and past position in one time step, denoted as $s_i(t)$. Inspired by [6], the finite memory encoder of agent i is proposed as

$$\begin{cases} \hat{\xi}_i(0) = 0, \\ \hat{\zeta}_i(0) = 0, \\ \hat{\xi}_i(t) = \hat{\xi}_i(t-1) + \hat{\zeta}_i(t-1) + g(t-1)s_i(t), \\ \hat{\zeta}_i(t) = \hat{\zeta}_i(t-1) + g(t-1)s_i(t), \\ s_i(t) = q_t\left(\frac{\psi_i(t) - \hat{\xi}_i(t-1) - \hat{\zeta}_i(t-1)}{g(t-1)}\right), t = 1, 2, \dots \end{cases} \quad (\text{Eq. 5.2})$$

where $g(t)$ is a decaying scaling function and $q_t(\cdot)$ the quantizer. We can see that the encoder can be viewed as a state observer based on the output $\psi_i(t)$ and the quantized innovation. Furthermore, the velocity will also be estimated based on the innovation and thus reduce the communication load. For the encoder, $\hat{\xi}_i(t)$ and $\hat{\zeta}_i(t)$ are the internal

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states, and $q_t(\cdot)$ is defined as a finite-level uniform quantizer as

$$q_t(y) = \begin{cases} 0, & -\frac{1}{2} < y < \frac{1}{2}, \\ i, & i - \frac{1}{2} \leq y < i + \frac{1}{2}, \\ & i = 1, 2, \dots, K(t) - 1, \\ K(t), & y \geq K(t) - \frac{1}{2}, \\ -q(-y), & y \leq -\frac{1}{2}, \end{cases} \quad (\text{Eq. 5.3})$$

where $K(t) \in \{1, 2, \dots\}$ is the number of quantization levels of $q_t(\cdot)$.

On the other hand, when the quantized information is transmitted to the agent's neighbor j , it is decoded as

$$\begin{cases} \hat{\xi}_{ij}(0) = 0, \\ \hat{\zeta}_{ij}(0) = 0, \\ \hat{\xi}_{ij}(t) = \hat{\xi}_{ij}(t-1) + \hat{\zeta}_{ij}(t-1) + g(t-1)s_i(t), \\ \hat{\zeta}_{ij}(t) = \hat{\zeta}_{ij}(t-1) + g(t-1)s_i(t), t = 1, 2, \dots \end{cases} \quad (\text{Eq. 5.4})$$

where $\hat{\xi}_{ij}(t)$ and $\hat{\zeta}_{ij}(t)$ are the output of decoder and also the estimates of agent i by agent j .

It is obvious from their definition that $\hat{\xi}_{ij}(t) \equiv \hat{\xi}_i(t)$ and $\hat{\zeta}_{ij}(t) \equiv \hat{\zeta}_i(t)$. Furthermore, $\hat{\xi}_i(t)$ and $\hat{\zeta}_i(t)$ can be viewed as the estimates of $\xi_i(t)$ and $\zeta_i(t-1)$ respectively. Denote $\Delta_i(t) = s_i(t) - \frac{\psi_i(t) - \hat{\xi}_i(t-1) - \hat{\zeta}_i(t-1)}{g(t-1)}$ as the quantization error, we have

$$\hat{\xi}_i(t) = \xi_i(t) + g(t-1)\Delta_i(t), \quad t = 1, 2, \dots$$

Let $e_{\xi_i}(t) = \hat{\xi}_i(t) - \xi_i(t)$, which is the estimation error of $\xi_i(t)$, and the error can be expressed as

$$\begin{cases} e_{\xi_i}(0) = -\xi_i(0) \\ e_{\xi_i}(t) = g(t-1)\Delta_i(t), \quad t = 1, 2, \dots \end{cases} \quad (\text{Eq. 5.5})$$

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Similarly define $e_{\zeta_i}(t-1) = \hat{\zeta}_i(t) - \zeta_i(t-1)$ as the estimation error of $\zeta_i(t-1)$, it is derived that

$$\begin{cases} e_{\zeta_i}(0) = \xi_i(0) + g(0)\Delta_i(1), \\ e_{\zeta_i}(t-1) = g(t-1)\Delta_i(t) - g(t-2)\Delta_i(t-1), t = 2, 3, \dots \end{cases} \quad (\text{Eq. 5.6})$$

With its definition, we can see that $s_i(t)$ is the quantized innovation with scaling. If we have access to the control input of each agent and consider no quantization, the position and velocity of each agent could be estimated using standard state observer based on output $\psi_i(t)$. In applications, however, the control input is to be designed using the estimation of position and velocity, and the decoder (Eq. 5.4) is used, which induces one-step delay for the velocity estimation, namely $\hat{\zeta}_i(t)$ is in fact the estimate of $\zeta_i(t-1)$.

Both the encoder (Eq. 5.2) and the decoder (Eq. 5.4) can be viewed as state observers based on output $\psi_i(t)$ and the quantized innovation. We call the encoder (Eq. 5.2) a joint encoder-observer and the decoder (Eq. 5.4) a joint decoder-observer. Though the velocity $\zeta_i(t)$ is not measurable, the neighbors of agent i can make an estimate for the overall state $[\xi_i(t), \zeta_i(t)]^T$ based on a joint decoder-observer. This gives us new inspiration of a tremendous reduction of the communication load of the whole network by distributed joint encoding-decoding-observation. At each time step, each agent only needs to send a quantized innovation to its neighbors based on its partial state components, then the neighbors can use joint decoder-observers to get estimates for its overall state components. However, generally speaking, there is no separation principle for the encoder-decoder-observer design and the control design. Compared with classic non-quantized and centralized state observers, due to the nonlinearity of the quantization and the coupling of all agents' states, the convergence of a given joint encoding-decoding-observation strategy depends on the control inputs of all agents and the close-loop dynamics of the whole network.

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A distributed protocol based on quantized communication is a group of control input $\{u_i, i = 1, 2, \dots, N\}$, where the control input of agent i can only depend on its output, the states and the output of its encoder and the outputs of the decoders associated with the channel from its neighbors to itself, that is, u_i can only depend on the information set $\{\psi_i(t), s_i(t), \hat{\xi}_i(t), \hat{\zeta}_i(t), \hat{\xi}_{ji}(t), \hat{\zeta}_{ji}(t), j \in \mathcal{N}_i, t = 0, 1, \dots$. Here we propose a distributed protocol as

$$\begin{cases} u_i(0) = u_{i_0} \\ u_i(t) = k_1 \sum_{j \in \mathcal{N}_i} a_{ij} [\hat{\xi}_{ji}(t-1) - \hat{\xi}_i(t-1)] + k_2 \sum_{j \in \mathcal{N}_i} a_{ij} [\hat{\zeta}_{ji}(t-1) - \hat{\zeta}_i(t-1)], \quad t = 1, 2, \dots \end{cases} \quad (\text{Eq. 5.7})$$

where $u_{i_0} \in \mathbb{R}$ is the initial control input, $k_1 > 0$ and $k_2 > 0$ are the control gains to be determined.

5.2 Main Results

To begin with, we have the following lemma

Lemma 5.1 *Let λ be a positive constant. The roots of polynomial $s^3 - 2s^2 + (k_2\lambda + 1)s + (k_1 - k_2)\lambda$ are all inside the unit circle if and only if*

$$\begin{aligned} (i) & k_2 > 2k_1, \\ (ii) & \lambda < \frac{k_2 - 2k_1}{(k_2 - k_1)^2} \end{aligned}$$

Furthermore, if the condition is met, the polynomial has a real nonzero root and two conjugate complex roots.

Proof: By Jury stability test[37], the roots of the polynomial are all inside the unit circle if and only if

$$\begin{cases} 1 - 2 + (k_2\lambda + 1) + (k_1 - k_2)\lambda > 0, \\ -1 - 2 - (k_2\lambda + 1) + (k_1 - k_2)\lambda < 0, \\ |k_2 - k_1|\lambda < 1, \\ 1 - (k_2 - k_1)^2\lambda^2 > |2\lambda(k_2 - k_1) - k_2\lambda - 1| \end{cases}$$

which is equivalent to

$$\begin{cases} k_1\lambda < 4 + 2k_2\lambda, \\ (k_2 - k_1)\lambda < 1, \\ 1 - (k_2 - k_1)^2\lambda^2 > |2\lambda(k_2 - k_1) - k_2\lambda - 1| \end{cases} \quad (\text{Eq. 5.8})$$

If $k_1 \geq k_2$ in (Eq. 5.8), the last inequality will be

$$1 - (k_2 - k_1)^2\lambda^2 > 2\lambda(k_1 - k_2) + k_2\lambda + 1$$

which is impossible. Thus (Eq. 5.8) is in fact equivalent to

$$\begin{cases} k_1 < k_2, \\ (k_2 - k_1)\lambda < 1, \\ 1 - (k_2 - k_1)^2\lambda^2 > |2\lambda(k_2 - k_1) - k_2\lambda - 1| \end{cases} \quad (\text{Eq. 5.9})$$

Since k_1, k_2 and λ are all positive, $2\lambda(k_2 - k_1) - k_2\lambda - 1 = (k_2 - 2k_1)\lambda - 1 < (k_2 - k_1)\lambda - 1 < 0$, according to (Eq. 5.9). Hence (Eq. 5.9) is in fact

$$\begin{cases} k_1 < k_2, \\ (k_2 - k_1)\lambda < 1, \\ 1 - (k_2 - k_1)^2\lambda^2 > 1 - (k_2 - 2k_1)\lambda \end{cases} \quad (\text{Eq. 5.10})$$

which results in $\lambda < \frac{k_2 - 2k_1}{(k_2 - k_1)^2}$ and together with $\lambda > 0$ gives $k_2 > 2k_1$.

Let $p = k_2\lambda + 1 - \frac{(-2)^2}{3}$, $q = \frac{2}{27}(-2)^3 - \frac{1}{3}(-2)(k_2\lambda + 1) + (k_1 - k_2)\lambda$, the roots of polynomial $s^3 - 2s^2 + (k_2\lambda + 1)s + (k_1 - k_2)\lambda$ are given by

$$\begin{cases} s_1 = \frac{2}{3} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}, \\ s_2 = \frac{2}{3} + \omega \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \omega^2 \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}, \\ s_3 = \frac{2}{3} + \omega^2 \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \omega \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \end{cases} \quad (\text{Eq. 5.11})$$

where $\omega = e^{i\frac{2\pi}{3}}$.

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When $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0$, s_1 will be real and s_2 and s_3 are conjugate complex roots. In the rest of the proof, we prove that when (i) and (ii) hold, $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0$. Let $\gamma = \frac{k_2}{k_1}$, $k_\lambda = k_1 \lambda$ and

$$D_\gamma(k_\lambda) = \frac{1}{27} \left(\gamma k_\lambda - \frac{1}{3} \right)^3 + \frac{1}{4} \left[\frac{16}{27} - \frac{2}{3}(\gamma k_\lambda + 1) + (\gamma - 1)k_\lambda \right]^2$$

In fact $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = D_\gamma(k_\lambda)$ and take partial derivative to have

$$\begin{aligned} \frac{dD_\gamma(k_\lambda)}{dk_\lambda} &= \frac{1}{9} \gamma \left(\gamma k_\lambda - \frac{1}{3} \right)^2 + \frac{1}{2} \left[\left(\frac{1}{3} \gamma - 1 \right) k_\lambda - \frac{2}{27} \right] \left(\frac{1}{3} \gamma - 1 \right) \\ &= \frac{1}{9} \gamma^3 k_\lambda^2 + \left[\frac{1}{2} \left(\frac{1}{3} \gamma - 1 \right)^2 - \frac{2}{27} \gamma^2 \right] k_\lambda + \frac{1}{27} \\ &= \frac{1}{9} \gamma^3 k_\lambda^2 + \left(\frac{1}{2} - \frac{1}{3} \gamma - \frac{1}{54} \gamma^2 \right) k_\lambda + \frac{1}{27} \end{aligned} \quad (\text{Eq. 5.12})$$

Define the discriminant of $\frac{dD_\gamma(k_\lambda)}{dk_\lambda}$ as $\overline{D}(\gamma) = \left(\frac{1}{2} - \frac{1}{3} \gamma - \frac{1}{54} \gamma^2 \right)^2 - \frac{4}{243} \gamma^3$, and $\gamma^* = \min\{\gamma \in [2, \infty), \overline{D}(\gamma) = 0\}$. As $\overline{D}(2) < 0$ and $\lim_{\gamma \rightarrow \infty} \overline{D}(\gamma) = +\infty$, the existence of γ^* is guaranteed, and $\overline{D}(\gamma) < 0, \gamma \in (2, \gamma^*)$. So

$$\frac{dD_\gamma(k_\lambda)}{dk_\lambda} > 0, \forall k_\lambda \geq 0, \gamma \in (2, \gamma^*) \quad (\text{Eq. 5.13})$$

Also it can be verified that

$$\left[\left(\frac{1}{54} \gamma^2 + \frac{1}{3} \gamma - \frac{1}{2} \right) (\gamma - 1)^2 - \frac{2}{9} \gamma^3 (\gamma - 2) \right]^2 > \overline{D}(\gamma) (\gamma - 1)^4, \forall \gamma > 2,$$

which implies

$$\frac{\frac{1}{54} \gamma^2 + \frac{1}{3} \gamma - \frac{1}{2} - \sqrt{\overline{D}(\gamma)}}{\frac{2}{9} \gamma^3} \geq \frac{\gamma - 2}{(\gamma - 1)^2}, \forall \gamma \in [\gamma^*, \infty)$$

So

$$\frac{dD_\gamma(k_\lambda)}{dk_\lambda} > 0, \forall k_\lambda \in \left[0, \frac{\gamma - 2}{(\gamma - 1)^2} \right), \gamma \in [\gamma^*, \infty) \quad (\text{Eq. 5.14})$$

Combining (Eq. 5.13) and (Eq. 5.14) to get

$$\frac{dD_\gamma(k_\lambda)}{dk_\lambda} > 0, \forall k_\lambda \in \left[0, \frac{\gamma - 2}{(\gamma - 1)^2} \right), \gamma \in (2, \infty) \quad (\text{Eq. 5.15})$$

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When (i) and (ii) hold, $\gamma \in (2, \infty)$ and $k_\lambda \in \left(0, \frac{\gamma-1}{(\gamma-1)^2}\right)$, therefore $\frac{dD_\gamma(k_\lambda)}{dk_\lambda} > 0$. Noting that $D_\gamma(0) = 0$, together with the positive derivative leads to $D_\gamma(k_\lambda) = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0$.

With positive $D_\gamma(k_\lambda)$, s_1 must be real and s_2 and s_3 conjugate complex in (Eq. 5.11). And (i) guarantees that $s_1 \neq 0$. $\square$

For a communication topology $\mathcal{G}$, denote its Laplacian matrix as $\mathcal{L}$, and the corresponding eigenvalues in ascending order as $\lambda_i(\mathcal{L}), i = 1, 2, \dots, N$. We have the following assumptions for the rest of this chapter.

A1) The communication graph $\mathcal{G}$ is undirected and connected.

A2) $k_2 > 2k_1$.

A3) $\lambda_i(\mathcal{L}) < \frac{k_2 - 2k_1}{(k_2 - k_1)^2}, i = 2, 3, \dots, N$.

Lemma 5.2 Suppose Assumptions A1)-A3) hold. Let

$$G_i(k_1, k_2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -k_1\lambda_i(\mathcal{L}) & -k_2\lambda_i(\mathcal{L}) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then the eigenvalues of $G_i(k_1, k_2)$ are given by $\lambda_1(G_i(k_1, k_2)) = 0, \lambda_2(G_i(k_1, k_2)) = s_1, \lambda_3(G_i(k_1, k_2)) = s_2, \lambda_4(G_i(k_1, k_2)) = s_3$, where $s_i, i = 1, 2, 3$ are the roots in (Eq. 5.11) with λ replaced with $\lambda_i(\mathcal{L})$. And for $i = 2, 3, \dots, N$, there exists a non-singular matrix $R_i(k_1, k_2)$, such that $G_i(k_1, k_2) = R_i^{-1}(k_1, k_2) \text{diag}\{0, s_1, s_2, s_3\} R_i(k_1, k_2)$.

Proof: The characteristic equation of $G_i(k_1, k_2)$ is

$$s(s^3 - 2s^2 + (k_2\lambda_i + 1)s + (k_1 - k_2))\lambda_i = 0.$$

By Assumption A1), we can see $\lambda_i(\mathcal{L}) > 0$. Then by noting Assumptions A2)-A3) and Lemma 5.1 the proof is completed. $\square$

Now we denote $\xi(t) = [\xi_1(t), \dots, \xi_N(t)]^T, \zeta(t) = [\zeta_1(t), \dots, \zeta_N(t)]^T, u(t) = [u_1(t), \dots, u_N(t)]^T, e_\xi(t) = [e_{\xi_1}(t), \dots, e_{\xi_N}(t)]^T, e_\zeta(t) = [e_{\zeta_1}(t), \dots, e_{\zeta_N}(t)]^T, \delta_\xi(t) = (I - J_N)\xi(t), \delta_\zeta(t) =$

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$(I - J_N)\zeta(t)$, $\Delta(t) = [\Delta_1(t), \dots, \Delta_N(t)]^T$, where $J_N = \frac{1}{N}\mathbf{1}\mathbf{1}^T$. Define the unitary matrix $T = [\frac{1}{\sqrt{N}}, \eta_2, \dots, \eta_N]$, where $\eta_i^T \mathcal{L} = \lambda_i(\mathcal{L})\eta_i^T$, $i = 2, 3, \dots, N$. Let $R_i(k_1, k_2)$, $i = 2, 3, \dots, N$ be non-singular such that

$$G_i(k_1, k_2) = R_i^{-1}(k_1, k_2) \text{diag}\{0, s_1, s_2, s_3\} R_i(k_1, k_2).$$

Denote $\rho_i[(k_1, k_2)] = \rho(G_i(k_1, k_2))$, $r_i[(k_1, k_2)] = \|R_i^{-1}(k_1, k_2)\|_\infty \|R_i(k_1, k_2)\|_\infty$, $\rho(k_1, k_2) = \max_{2 \leq i \leq N} \rho_i$, $r(k_1, k_2) = \max_{2 \leq i \leq N} r_i$.

Theorem 5.1 Suppose Assumptions A1)-A3) hold. Let the scaling function $g(t) = g_0\gamma^t$, where $g_0 > 0$ and $\gamma \in (\rho(k_1, k_2), 1)$. If the number of quantization levels of the quantizer $q_t(\cdot)$, $t = 1, 2, \dots$ satisfy

$$\begin{cases} K(1) \geq \frac{1}{g_0} \|\xi(0) + \zeta(0)\|_\infty - \frac{1}{2}, \\ K(2) \geq \frac{1}{g_0\gamma} \|\xi(0) - u(0)\|_\infty + \frac{1}{\gamma} - \frac{1}{2}, \\ K(3) \geq \frac{k_2 d^* \|\xi(0) + \zeta(0)\|_\infty}{g_0\gamma^2} + \frac{k_2 d^* + 1}{2\gamma^2} + \frac{1}{\gamma} - \frac{1}{2}, \\ K(4) \geq \frac{k_1 d^* \|\xi(0) + \zeta(0)\|_\infty + k_2 d^* \|u(0) + \zeta(0)\|_\infty}{g_0\gamma^3} + \frac{|k_1 - k_2| d^*}{2\gamma^3} + \frac{k_2 d^* + 1}{2\gamma^2} + \frac{1}{\gamma} - \frac{1}{2} \end{cases} \quad (\text{Eq. 5.16})$$

and

$$\begin{aligned} K(t+1) \geq & r(k_1 + k_2) d^* \|T\|_\infty \left[\frac{\rho^{t-2}}{g_0\gamma^t} \|T^{-1}\|_\infty (\max\{\|\delta_\xi(0)\|_\infty + \|\delta_\zeta(0)\|_\infty, \|\delta_\zeta(0)\|_\infty + \|u(0)\|_\infty\}) \right. \\ & + \frac{\rho^{t-3}}{g_0\gamma^t} \left(|k_1 - k_2| \lambda_N \|\xi(0)\|_\infty + \frac{g_0 k_2 \lambda_N}{2} \right) + \frac{(k_1 + k_2) \lambda_N}{2\gamma^3(\gamma - \rho)} + \frac{k_2 \lambda_N}{2\gamma^2(\gamma - \rho)} \\ & \left. + \frac{|k_1 - k_2| d^*}{2\gamma^3} + \frac{k_2 d^* + 1}{2\gamma^2} + \frac{1}{\gamma} - \frac{1}{2}, t = 4, 5, \dots \right] \end{aligned}$$

Then under the protocol proposed in (Eq. 5.2), (Eq. 5.4) and (Eq. 5.7), the closed-loop system satisfies

$$\lim_{t \rightarrow \infty} [\xi_i(t) - \xi_j(t)] = 0, \lim_{t \rightarrow \infty} [\zeta_i(t) - \zeta_j(t)] = 0, \quad i, j = 1, 2, \dots, N. \quad (\text{Eq. 5.17})$$

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Proof: The control input in (Eq. 5.7) can be written as

$$u(t) = -k_1 \mathcal{L}\xi(t-1) - k_2 \mathcal{L}\zeta(t-1) - k_1 \mathcal{L}e_\xi(t-1) - k_2 \mathcal{L}e_\zeta(t-1), t = 1, 2, \dots$$

Substitute the control into system (Eq. 5.1) to get

$$\begin{cases} \delta_\xi(t+1) = \delta_\xi(t) + \delta_\zeta(t), \\ \delta_\zeta(t+1) = \delta_\zeta(t) - k_1 \mathcal{L}\delta_\xi(t-1) - k_2 \mathcal{L}\delta_\zeta(t-1) - k_1 \mathcal{L}e_\xi(t-1) - k_2 \mathcal{L}e_\zeta(t-1). \end{cases} \quad (\text{Eq. 5.18})$$

Let $\delta_\xi(t) = T\tilde{\delta}_\xi(t)$, $\delta_\zeta(t) = T\tilde{\delta}_\zeta(t)$ and denote the i th component of $\tilde{\delta}_\xi(t)$ and $\tilde{\delta}_\zeta(t)$ by $\tilde{\delta}_{\xi_i}(t)$ and $\tilde{\delta}_{\zeta_i}(t)$ respectively. Then $\tilde{\delta}_{\xi_1}(t) = \tilde{\delta}_{\zeta_1}(t) = 0$ and

$$\begin{cases} \tilde{\delta}_{\xi_i}(t+1) = \tilde{\delta}_{\xi_i}(t) + \tilde{\delta}_{\zeta_i}(t), \\ \tilde{\delta}_{\zeta_i}(t+1) = \tilde{\delta}_{\zeta_i}(t) - k_1 \lambda_i \tilde{\delta}_\xi(t-1) - k_2 \lambda_i \tilde{\delta}_\zeta(t-1) \\ \quad - k_1 \lambda_i \eta_i^T e_\xi(t-1) - k_2 \lambda_i \eta_i^T e_\zeta(t-1), i = 2, 3, \dots, N \end{cases} \quad (\text{Eq. 5.19})$$

Denote $y_i(t) = [\tilde{\delta}_{\xi_i}(t), \tilde{\delta}_{\zeta_i}(t), \tilde{\delta}_{\xi_i}(t-1), \tilde{\delta}_{\zeta_i}(t-1)]^T$, $E_i(k_1, k_2, t-1) = [0, e_i(k_1, k_2, t-1), 0, 0]^T$, where $e_i(k_1, k_2, t-1) = -k_1 \lambda_i \eta_i^T e_\xi(t-1) - k_2 \lambda_i \eta_i^T e_\zeta(t-1)$, $t = 1, 2, \dots, i = 2, 3, \dots, N$.

Then (Eq. 5.19) can be rewritten as

$$y_i(t+1) = G_i(k_1, k_2)y_i(t) + E_i(k_1, k_2, t-1), t = 1, 2, \dots \quad (\text{Eq. 5.20})$$

By Lemma 5.2, we have

$$\|y_i(t+1)\|_\infty \leq r_i \left[\rho_i^t \|y_i(1)\|_\infty + \rho_i^{t-1} \|E_i(k_1, k_2, 0)\|_\infty + \sum_{j=2}^t \rho_i^{t-j} \|E_i(k_1, k_2, j-1)\|_\infty \right], t = 2, 3, \dots$$

With the definition in (Eq. 5.5) and (Eq. 5.6), we have

$$\begin{aligned} \|E_i(k_1, k_2, j-1)\|_\infty &= |k_1 \lambda_i \eta_i^T e_\xi(j-1) + k_2 \lambda_i \eta_i^T e_\zeta(j-1)| \\ &\leq k_1 \lambda_i \|\eta_i^T\|_\infty \|e_\xi(j-1)\|_\infty + k_2 \lambda_i \|\eta_i^T\|_\infty \|e_\zeta(j-1)\|_\infty \\ &\leq k_1 \lambda_i \|\eta_i^T\|_\infty \|e_\xi(j-1)\|_\infty + k_2 \lambda_i \|\eta_i^T\|_\infty \|e_\zeta(j-1)\|_\infty \\ &= k_1 \lambda_i \|e_\xi(j-1)\|_\infty + k_2 \lambda_i \|e_\zeta(j-1)\|_\infty \\ &= k_1 \lambda_i g(j-2) \|\Delta_i(j-1)\|_\infty + k_2 \lambda_i (g(j-1) \|\Delta_i(j)\|_\infty \\ &\quad + g(j-2) \|\Delta_i(j-1)\|_\infty) \\ &= (k_1 + k_2) \lambda_i j g(j-2) \|\Delta_i(j-1)\|_\infty + k_2 \lambda_i g(j-1) \|\Delta_i(j)\|_\infty, j = 2, 3, \dots \end{aligned}$$

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Hence

$$\begin{aligned}
\|y_i(t+1)\|_\infty &\leq r_i \left[\rho_i^t \|y_i(1)\|_\infty + \rho_i^{t-1} \|E_i(k_1, k_2, 0)\|_\infty \right. \\
&\quad + \sup_{1 \leq j \leq t-1} \|\Delta_i(j)\|_\infty \frac{\rho_i^t (k_1 + k_2) \lambda_i g_0}{\gamma^2} \sum_{j=2}^t \left(\frac{\gamma}{\rho_i} \right)^j \\
&\quad \left. + \sup_{1 \leq j \leq t} \|\Delta_i(j)\|_\infty \frac{\rho_i^t k_2 \lambda_i g_0}{\gamma} \sum_{j=2}^t \left(\frac{\gamma}{\rho_i} \right)^j \right] \\
&\leq r_i \left[\rho_i^t \|y_i(1)\|_\infty + \rho_i^{t-1} \|E_i(k_1, k_2, 0)\|_\infty \right. \\
&\quad + \sup_{1 \leq j \leq t-1} \|\Delta_i(j)\|_\infty \frac{\rho_i^t (k_1 + k_2) \lambda_i g_0}{\gamma^2} \frac{\left(\frac{\gamma}{\rho_i} \right)^2 \left[\left(\frac{\gamma}{\rho_i} \right)^{t-1} - 1 \right]}{\frac{\gamma}{\rho_i} - 1} \\
&\quad + \sup_{1 \leq j \leq t} \|\Delta_i(j)\|_\infty \frac{\rho_i^t k_2 \lambda_i g_0}{\gamma} \frac{\left(\frac{\gamma}{\rho_i} \right)^2 \left[\left(\frac{\gamma}{\rho_i} \right)^{t-1} - 1 \right]}{\frac{\gamma}{\rho_i} - 1} \left. \right] \\
&\leq r_i \left[\rho_i^t \|y_i(1)\|_\infty + \rho_i^{t-1} \|E_i(k_1, k_2, 0)\|_\infty \right. \\
&\quad + \sup_{1 \leq j \leq t-1} \|\Delta_i(j)\|_\infty \frac{(\gamma^{t-1} - \rho_i^{t-1})(k_1 + k_2) \lambda_i g_0}{\gamma - \rho_i} \\
&\quad \left. + \sup_{1 \leq j \leq t} \|\Delta_i(j)\|_\infty \frac{\gamma(\gamma^{t-1} - \rho_i^{t-1}) k_2 \lambda_i g_0}{\gamma - \rho_i} \right], t = 2, 3, \dots
\end{aligned}$$

By the definition of $\tilde{\delta}_\xi(t)$ and $\tilde{\delta}_\zeta(t)$, we have

$$\begin{aligned}
\|[\delta_\xi^T(t), \delta_\zeta^T(t)]^T\|_\infty &\leq \max\{\|T\|_\infty \|\tilde{\delta}_\xi(t)\|_\infty, \|T\|_\infty \|\tilde{\delta}_\zeta(t)\|_\infty\} \\
&= \|T\|_\infty \|[\tilde{\delta}_\xi^T(t), \tilde{\delta}_\zeta^T(t)]\|_\infty \\
&\leq \|T\|_\infty \max_{i=2,3,\dots,N} \|y_i(t+1)\|_\infty
\end{aligned}$$

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With $K(1)$ satisfying (Eq. 5.16), it is guaranteed that $\|\Delta(1)\|_\infty \leq \frac{1}{2}$. And (Eq. 5.2) results in that

$$\begin{aligned}
 s_i(2) &= q \left(\frac{\xi_i(2) - \hat{\xi}_i(1) - \hat{\zeta}_i(1)}{g(1)} \right) \\
 &= q \left(\frac{\xi_i(1) - \hat{\xi}_i(1) + \zeta_i(0) - \hat{\zeta}_i(1) + u_i(0)}{g(1)} \right) \\
 &= q \left(\frac{\xi_i(0) + 2\zeta_i(0) + u_i(0) - \hat{\zeta}_i(1) - \hat{\xi}_i(1)}{g(1)} \right) \\
 &= q \left(\frac{\xi_i(0) + 2\zeta_i(0) + u_i(0) - 2g(0)s_i(1)}{g(1)} \right) \\
 &= q \left(\frac{u_i(0) - \xi_i(0) + 2g(0)\Delta_i(1)}{g(1)} \right), i = 1, 2, \dots, N
 \end{aligned}$$

With $K(2)$ satisfying (Eq. 5.16), $\|\Delta(1)\|_\infty \leq \frac{1}{2}$. Since the quantization error for time step $t + 1$ is

$$\begin{aligned}
 &\frac{\xi(t+1) - \hat{\xi}(t) - \hat{\zeta}(t)}{g(t)} \\
 &= \frac{\xi(t) - \hat{\xi}(t) + \zeta(t-1) - \hat{\zeta}(t) + u(t-1)}{g(t)} \\
 &= \frac{-2g(t-1)\Delta(t) + g(t-2)\Delta(t-1) + u(t-1)}{g(t)} \quad (\text{Eq. 5.21}) \\
 &= \frac{-2\Delta(t)}{\gamma} + \frac{\Delta(t-1)}{\gamma^2} \\
 &\quad + \frac{-k_1\mathcal{L}\xi(t-2) - k_2\mathcal{L}\zeta(t-2) - k_1\mathcal{L}e_\xi(t-2) - k_2\mathcal{L}e_\zeta(t-2)}{g(t)}, t = 2, 3, \dots,
 \end{aligned}$$

substitute $t = 2, 3$ to get

$$\begin{aligned}
 &\frac{\xi(3) - \hat{\xi}(2) - \hat{\zeta}(2)}{g(2)} \\
 &= \frac{-2\Delta(2)}{\gamma} + \frac{\Delta(1)}{\gamma^2} + \frac{-k_1\mathcal{L}\xi(0) - k_2\mathcal{L}\zeta(0) - k_1\mathcal{L}e_\xi(0) - k_2\mathcal{L}e_\zeta(0)}{g(2)} \\
 &= \frac{-2\Delta(2)}{\gamma} + \frac{\Delta(1)}{\gamma^2} + \frac{-k_2\mathcal{L}(\zeta(0) + \xi(0)) - g(0)k_2\mathcal{L}\Delta(t)}{g(2)}
 \end{aligned}$$

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and

$$\begin{aligned}
& \frac{\xi(4) - \hat{\xi}(3) - \hat{\zeta}(3)}{g(3)} \\
&= \frac{-2\Delta(3)}{\gamma} + \frac{\Delta(2)}{\gamma^2} + \frac{-k_1\mathcal{L}\xi(1) - k_2\mathcal{L}\zeta(1) - k_1\mathcal{L}e_\xi(1) - k_2\mathcal{L}e_\zeta(1)}{g(3)} \\
&= \frac{-2\Delta(3)}{\gamma} + \frac{\Delta(2)}{\gamma^2} + \frac{-k_1\mathcal{L}(\xi(0) + \zeta(0)) - k_2\mathcal{L}(\zeta(0) + u(0))}{g(3)} \\
&\quad - \frac{(k_1 - k_2)g(0)\mathcal{L}\Delta(1) + k_2\mathcal{L}g(1)\Delta(2)}{g(3)}
\end{aligned}$$

As $\|\Delta(1)\|_\infty \leq \frac{1}{2}$ and $\|\Delta(2)\|_\infty \leq \frac{1}{2}$, with $K(3)$ and $K(4)$ satisfying (Eq. 5.16), we have $\|\Delta(3)\|_\infty \leq \frac{1}{2}$ and $\|\Delta(4)\|_\infty \leq \frac{1}{2}$. Consider the magnitude of quantization error,

$$\begin{aligned}
& \left\| \frac{\xi(t+1) - \hat{\xi}(t) - \hat{\zeta}(t)}{g(t)} \right\|_\infty \\
& \leq \frac{2\|\Delta(t)\|_\infty}{\gamma} + \frac{\|\Delta(t-1)\|_\infty}{\gamma^2} \\
& \quad + \frac{k_1\|\mathcal{L}\|_\infty\|\delta_\xi(t-1)\|_\infty + k_2\|\mathcal{L}\|_\infty\|\delta_\zeta(t-2)\|_\infty}{g(t)} \\
& \quad + \frac{|k_1 - k_2|\|\mathcal{L}\|_\infty\|\Delta(t-2)\|_\infty}{\gamma^3} + \frac{k_2\|\mathcal{L}\|_\infty\|\Delta(t-1)\|_\infty}{\gamma^2} \\
& \leq \frac{2\|\Delta(t)\|_\infty}{\gamma} + \frac{(k_2d^* + 1)\|\Delta(t-1)\|_\infty}{\gamma^2} + \frac{|k_1 - k_2|d^*\|\Delta(t-2)\|_\infty}{\gamma^3} \\
& \quad + \frac{(k_1 + k_2)d^*\|T\|_\infty\|y(t-1)\|_\infty}{g(t)} \\
& \leq \frac{2\|\Delta(t)\|_\infty}{\gamma} + \frac{(k_2d^* + 1)\|\Delta(t-1)\|_\infty}{\gamma^2} + \frac{|k_1 - k_2|d^*\|\Delta(t-2)\|_\infty}{\gamma^3} \\
& \quad + r(k_1 + k_2)d^*\|T\|_\infty \left[\frac{\rho^{t-2}}{g_0\gamma^t} \max_{2 \leq i \leq N} \|y_i(1)\|_\infty + \frac{\rho^{t-3}}{g_0\gamma^t} \max_{2 \leq i \leq N} \|E_i(k_1, k_2, 0)\|_\infty \right. \\
& \quad \left. + \sup_{1 \leq j \leq t-3} \|\Delta(j)\|_\infty \frac{(k_1 + k_2)\lambda_N}{\gamma^3(\gamma - \rho)} + \sup_{1 \leq j \leq t-2} \|\Delta(j)\|_\infty \frac{k_2\lambda_N}{\gamma^2(\gamma - \rho)} \right], t = 4, 5, \dots
\end{aligned}$$

For the residual of velocity estimation

$$\begin{aligned}
|e_i(k_1, k_2, 0)| &= |(k_1 - k_2)\lambda_i\eta_i^T\xi(0) + g_0k_2\lambda_i\eta_i^T\Delta(1)| \\
&\leq |k_1 - k_2|\lambda_N\|\eta_i\|_\infty\|\xi(0)\|_\infty + \frac{g_0k_2\lambda_N}{2}\|\eta_i\|_\infty \\
&\leq |k_1 - k_2|\lambda_N\|\eta_i\|_2\|\xi(0)\|_\infty + \frac{g_0k_2\lambda_N}{2}\|\eta_i\|_2 \\
&= |k_1 - k_2|\lambda_N\|\xi(0)\|_\infty + \frac{g_0k_2\lambda_N}{2}.
\end{aligned}$$

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So we have $\max_{2 \leq i \leq N} \|E_i(k_1, k_2, 0)\|_\infty \leq |k_1 - k_2| \lambda_N \|\xi(0)\|_\infty + \frac{g_0 k_2 \lambda_N}{2}$. Note that

$$\begin{aligned} \max_{2 \leq i \leq N} \|y_i(1)\|_\infty &= \max_{2 \leq i \leq N} \max\{\tilde{\delta}_{\xi_i}(1), \tilde{\delta}_{\zeta_i}(1), \tilde{\delta}_{\xi_i}(0), \tilde{\delta}_{\zeta_i}(0)\} \\ &= \max\{\|\tilde{\delta}_{\xi}(1)\|_\infty, \|\tilde{\delta}_{\zeta}(1)\|_\infty, \|\tilde{\delta}_{\xi}(0)\|_\infty, \|\tilde{\delta}_{\zeta}(0)\|_\infty\} \\ &\leq \|T^{-1}\|_\infty \max\{\|\delta_{\xi}(1)\|_\infty, \|\delta_{\zeta}(1)\|_\infty, \|\delta_{\xi}(0)\|_\infty, \|\delta_{\zeta}(0)\|_\infty\} \\ &\leq \|T^{-1}\|_\infty \max\{\|\delta_{\xi}(0)\|_\infty + \|\delta_{\zeta}(0)\|_\infty, \|\delta_{\xi}(0)\|_\infty + \|u(0)\|_\infty\} \end{aligned}$$

Since $\|\Delta(t)\|_\infty \leq \frac{1}{2}$, $t = 1 - 4$ and with the quantization error, we have $\sup_{t \geq 1} \|\Delta(t)\|_\infty \leq \frac{1}{2}$. So $y_i(t) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$, meaning $\|\tilde{\delta}_{\xi}(t)\|_\infty \rightarrow 0$ and $\|\tilde{\delta}_{\zeta}(t)\|_\infty \rightarrow 0$. With the definition of $\tilde{\delta}_{\xi}(t)$ and $\tilde{\delta}_{\zeta}(t)$, the theorem is proved. $\square$

Remark 5.1 *It is easy to find that the required quantization level in (Eq. 5.16) increases at the beginning and decreases from $K(5)$, meaning that $K(5)$ is the largest number of quantization level. For simplicity we can choose $K = K(5)$ to avoid a time-varying quantization level. In addition, as the following simulation shows, the number of quantization level proposed in (Eq. 5.16) is conservative in most applications since it corresponds to the worst case.*

5.3 Simulations

For the simulation, we use the same topology matrix like previous chapters:

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

with eigenvalues $\lambda_i = 0, 1, 3, 4$, $i = 1 - 4$. For the discrete system with sampling time $h = 0.1$, choose $k_1 = 0.01$ and $k_2 = 0.05$ so that $4 = \lambda_4 < \frac{k_2 - 2k_1}{(k_2 - k_1)^2} = 18.75$. We also have $\rho_2 = 0.9845$, $\rho_3 = 0.9495$, $\rho_4 = 0.9295$ and $r_2 = 27.0573$, $r_3 = 21.1459$, $r_4 = 21.3532$, thus $\rho = \rho_2 = 0.9845$ and $r = r_2 = 27.0573$. The degree of the topology graph is $d^* = 3$, and with the definition of matrix $T\mathcal{L} = \text{diag}\{0, 1, 3, 4\}T$, $\|T\|_\infty = 2$ and $\|T^{-1}\|_\infty = 1.904$. Suppose

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that the four agents start from their initial positions: $[2.3, 3.8], [8.1, 1.1], [4.9, 5.4], [7.4, 9.5]$ with initial velocities: $[-0.1, 0.3], [0.9, 0.2], [0.4, -0.6], [0.6, 0.5]$, and all the initial control inputs and estimates take zero value. Let the decreasing factor of scaling function be $\gamma = 0.99 \in (\rho, 1)$ and we can determine the initial value of the scaling function and the number of quantization level required.

As mentioned before, if a number of quantization level K meets the requirement for $K(5)$, it will also satisfy all the other conditions so that we can take $K(t) = K(5), t = 1, 2, \dots$ for simplicity. Substitute all the values into the equation, we have

$$\begin{aligned}
 K &\geq r(k_1 + k_2)d^* \|T\|_\infty \left[\frac{\rho^{t-2}}{g_0 \gamma^t} \|T^{-1}\|_\infty (\max\{\|\delta_\xi(0)\|_\infty + \|\delta_\zeta(0)\|_\infty, \|\delta_\zeta(0)\|_\infty + \|u(0)\|_\infty\}) \right. \\
 &\quad \left. + \frac{\rho^{t-3}}{g_0 \gamma^t} \left(|k_1 - k_2| \lambda_N \|\xi(0)\|_\infty + \frac{g_0 k_2 \lambda_N}{2} \right) + \frac{(k_1 + k_2) \lambda_N}{2\gamma^3(\gamma - \rho)} + \frac{k_2 \lambda_N}{2\gamma^2(\gamma - \rho)} \right] \\
 &\quad + \frac{|k_1 - k_2|d^*}{2\gamma^3} + \frac{k_2 d^* + 1}{2\gamma^2} + \frac{1}{\gamma} - \frac{1}{2} \\
 &= 27.0573 \times 0.06 \times 3 \times 2 \left[1.904 \times \frac{0.9845^2}{0.99^4 g_0} \times 9.55 + 0.04 \times 4 \times 9.5 \times \frac{0.9845}{0.99^4 g_0} \right. \\
 &\quad \left. + \frac{0.9845 \times 0.05 \times 4}{2 \times 0.99^4} + \frac{0.06 \times 4}{2(0.99 - 0.9845)0.99^3} + \frac{0.05 \times 4}{2(0.99 - 0.9845)0.99^2} \right] \\
 &\quad + \frac{0.04 \times 3}{2 \times 0.99^3} + \frac{0.05 \times 3 + 1}{2 \times 0.99^2} + \frac{1}{0.99} - \frac{1}{2} \\
 &= 401.86 + \frac{193.87}{g_0}
 \end{aligned}$$

Let $g_0 = 4$, and $K(t) = K = 512, t = 1, 2, \dots$ so that all quantization errors are restricted to $\frac{1}{2}$. The trajectory of all four agents and evolutions of their states are shown below. It is seen that the trajectory is not so smooth as that in continuous case due to the quantization.

If we investigate the quantization level of all agents, Fig. 5.6 and 5.7 show the corresponding quantization level versus time step. We can see that almost all quantization levels fall in $[-1, 1]$.

The main factor of the large K comes from the terms in the expression of $K(t+1)$ containing $\gamma - \rho$, where ρ is close to 1 in most cases and $\gamma \in (\rho, 1)$ makes $\gamma - \rho$ close

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Figure 5.1: Trajectory of the four agents under quantized second-order consensus protocol.

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Figure 5.2: Evolution of the ξ_x of the four agents under quantized second-order consensus protocol.

to zero. In addition, the two terms with the multiplicand of $\frac{1}{\gamma-\rho}$ and r , which is also large since matrices $R_i(k_1, k_2)$ are nearly singular. Hence the product is very large in most scenarios. Fortunately, however, usually we can choose parameters satisfying the previous four conditions and have the system reach second-order consensus. Let $g_0 = 2$, $K = 2$ and remain the other parameters the same as the previous simulation, we have Fig. 5.8 through 5.12. We can see that the the system still achieves second-order consensus finally.

5.4 Conclusions

Quantization is a solution to the bandwidth limitation of communication channel, and we discussed the problem in this chapter. Due to the nonlinearity introduced by the quantization, it is quite a challenge to analyze the performance of the system. In the chapter, we have proposed an observer based encoder-decoder pair and an interaction

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Figure 5.3: Evolution of the ξ_y of the four agents under quantized second-order consensus protocol.

protocol and shown that the exact consensus can be achieved with proper choices of parameters.

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Figure 5.4: Evolution of the ζ_x of the four agents under quantized second-order consensus protocol.

Figure 5.5: Evolution of the ζ_y of the four agents under quantized second-order consensus protocol.

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Figure 5.6: Quantization level of ξ_x for the four agents.

Figure 5.7: Quantization level of ξ_y for the four agents.

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Figure 5.8: Trajectory of the four agents under quantized second-order consensus protocol.

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Figure 5.9: Evolution of the ξ_x of the four agents under quantized second-order consensus protocol.

Figure 5.10: Evolution of the ξ_y of the four agents under quantized second-order consensus protocol.

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Figure 5.11: Evolution of the ζ_x of the four agents under quantized second-order consensus protocol.

Figure 5.12: Evolution of the ζ_y of the four agents under quantized second-order consensus protocol.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

It can be seen that the first-order consensus problem has been studied for a long period of time and a lot of well established theorems and tools have been developed. The second-order consensus, however, is still open though challenging. The contribution of the thesis lies in the following aspects:

1. The fastest convergence rate for continuous second-order multi-agent systems with real-eigenvalued Laplacian matrix has been explicitly obtained. In addition, an observer-based output feedback has been designed and verified to achieve the fastest convergence rate.
2. The condition for fastest convergence rate of discrete second-order multi-agent systems has also been given which is the solution to an algebraic equation.
3. We have proposed an observer-based encoder-decoder pair and an interaction protocol and showed that with proper choices of parameters, exact consensus can be achieved under quantized position innovation feedback.

6.2 Future Work

The future work will focus on the following parts:

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1. Theorem 4.2 gives the best γ for system convergence rate. However the assumptions for the theorem are somewhat strict as there might be some complex eigenvalues of Laplacian matrix $\mathcal{L}$. We will investigate the convergence rate for more general topology.
2. Theorem 4.4 investigates the possibility of reaching consensus without the knowledge of derivative of the states, and a solution is proposed. The result is under the condition that no noise is considered. In applications, however, noise is unavoidable. Future work will include the study on the effect of noise.
3. Quantized consensus remains a challenging problem, and we have discussed the problem under ideal condition. Channel noise, packet drop-out and transmitting errors can be considered in the future.

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