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Robust Analysis and Control for Polynomial Nonlinear Systems Using Convex Optimization

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School of Electrical & Electronic Engineering

A thesis submitted to the Nanyang Technological University
in fulfillment of the requirement for the degree of
Doctor of Philosophy

2010

Statement of Originality

I hereby certify that the content of this thesis is the result of work done by me and has not been submitted for a higher degree to any other University or Institution.

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Date

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ZHAO Dan

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Summary

This thesis studies the nonlinear robust control problems for continuous time parameter-dependent nonlinear systems with polynomial vector fields. Several novel approaches have been proposed to overcome the non-convex nature and numerical difficulty in robust nonlinear system design. Several design problems are addressed, including stability analysis and synthesis, estimation of region of attraction, H_∞ synthesis, output feedback design, multi-objective reliable control and H_∞ filter design.

These designs are based on the convex optimization and in particular based on the new sum of squares programming technique. Sufficient conditions guaranteeing the existence of robust nonlinear controllers and filters are formulated as sum of squares based optimization problems. Sum of squares programming technique is applied to obtain computationally tractable solutions. The robust nonlinear controllers and filters obtained by the proposed approaches in this thesis guarantee the closed-loop stability, robustness and optimal performance for parameter-dependent polynomial nonlinear systems.

The parameter-dependent Lyapunov functions and more general assumption and relaxation are provided to reduce the conservativeness involved in the design. By introducing additional matrix variables, the coupling terms involving system matrices, Lyapunov matrices and filter dynamics are eliminated. Then the sufficient conditions have a more suitable structure to deal with parametric uncertainty for the polynomial nonlinear systems. The non-convex nonlinear robust design problems can be transformed into convex optimization problems, and computationally tractable solutions can be obtained via semidefinite programming.

SUMMARY

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The effectiveness of the proposed robust nonlinear controller and filter design approaches are demonstrated by the examples provided in this thesis.

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Acronyms and symbols

Acronyms

HJE	Hamilton-Jacobi Equation
HJI	Hamilton-Jacobi Inequality
LMI	Linear Matrix Inequality
LPV	linear parameter varying
LTI	Linear Time Invariant
MHC	Moving Horizon Control
MPC	Model Predictive Control
NLMI	Nonlinear Matrix Inequality
NLTI	Nonlinear Time Invariant
NMPC	Nonlinear Model Predictive Control
RHC	Receding Horizon Control
SDLMI	State Dependent Linear Matrix Inequality
SDLR	State Dependent Linear Representation
SDMI	State Dependent Matrix Inequality

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SDP	Semidefinite Program
SDRE	State Dependent Riccati Equation
SOS	Sum of Squares
SOSP	Sum of Squares Program

Mathematical Symbols

$\mathbb{R}$	the real numbers.
$\mathbb{R}_+$	nonnegative real numbers.
$\mathbb{R}^n$	n -dimensional real Euclidean space.
$\mathbb{R}^{n \times m}$	$n \times m$ real matrix.
$\mathbb{Z}$	the ring of integers.
$\mathbb{Z}_+$	nonnegative integers.
$\mathcal{R}_n$	the set of polynomials in n variables with real coefficients.
$\mathcal{R}_n^n$	n -dimensional polynomial vector field in n variables.
$\mathcal{R}_{n,d}$	the subset of $\mathcal{R}_n$ that are polynomials with maximum degree d .
$\mathcal{P}_n$	the subset of $\mathcal{R}_n$ that are positive semidefinite polynomials.
$\mathcal{P}_{n,d}$	the subset of $\mathcal{R}_n$ that are positive semidefinite polynomials with maximum degree d .
Σ_n	the subset of $\mathcal{R}_n$ that are sum of squares polynomials.
$\Sigma_{n,d}$	the subset of $\mathcal{R}_n$ that are sum of squares polynomials with maximum degree d .
$\in$	belong to.
$\emptyset$	the empty set.
$P \subseteq Q$	the set P is a subset of Q .
$P \subset Q$	the set P is a strict subset of Q .
$P \cap Q$	set intersection.
$P \cup Q$	set union.

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$\forall$	for all.
$\triangleq$	defined as.
$\otimes$	the Kronecker product.
$\text{diag}\{a_1, \dots, a_n\}$	an $n \times n$ diagonal matrix with a_i as its i_{th} diagonal element.
$\ G\ _2$	H_2 norm of system G .
$\ G\ _\infty$	H_∞ norm of system G .
I	identity matrix.
$I_{n \times n}$	$n \times n$ identity matrix.
A^T	transpose of matrix A .
A^{-1}	inverse of matrix A .
$\text{trace}(A)$	trace of A .
$P \geq 0$	symmetric positive semi-definite matrix.
$P > 0$	symmetric positive definite matrix.
$P \geq Q$	$P - Q \geq 0$ for symmetric positive semi-definite matrices.
$P > Q$	$P - Q > 0$ for symmetric positive semi-definite matrices.
$*$	symmetric entries of a symmetric matrix.

Chapter 1

Introduction

This thesis is concerned with the problems of robust analysis and synthesis for nonlinear systems with polynomial dynamics. Our methodology is based on the primary tool of nonlinear control, the Lyapunov stability theory. The standard method of analysis consists of finding a candidate Lyapunov function such that, when combined with the system, the Lyapunov stability requirements are met. The synthesis method is similar in that it searches for a candidate Lyapunov function as well as a controller that make the system achieve the assumptions of the Lyapunov theorem. This chapter provides an introduction of motivation and related literature.

1.1 Motivation

In recent years, polynomial control systems have received considerable attention in nonlinear control. Polynomial control systems are control systems whose dynamics are described by polynomial functions. The great interest in polynomial control systems stems from its wide application. Many nonlinear control problems can be formulated, transformed to or approximated by polynomial control systems. Moreover, linear control systems can be categorized as a special case of polynomial control systems, and a broad

class of systems can be modelled as polynomial control systems.

Polynomial control systems are in general very difficult to study. However, in combination with semidefinite programming and in particular with the recently developed sum of squares (SOS) based numerical method, many nonlinear analysis and synthesis problems have been solved successfully. This motivates our research on the polynomial nonlinear control systems.

Since unavoidable discrepancies between system models and physical systems can result in the degradation of control system performance even instability, the ability of the control system to guarantee robustness with respect to system uncertainty in the design model has been one of the fundamental problems in control system design. Therefore we are motivated to investigate how the control problems are still solved in spite of parameter uncertainty.

The main focus of this thesis is to develop several methodologies for designing polynomial nonlinear controllers and filters which provide both robust stability and robust performance for parameter-dependent polynomial nonlinear systems. The main advantages of the proposed design procedure are that general polynomial control systems are considered, and no special requirements on the system structure are imposed except for the polynomial description and the affine nonlinear structure.

When control systems are not represented in polynomial vector fields, the SOS based technique cannot be used to analyze the systems directly. However, it has been shown [117] that any system with non-polynomial nonlinearities can be converted to a polynomial system with a larger state dimension using certain algebraic transformation.

1.2 Nonlinear Control System Design

Control theory has been extremely successful in dealing with linear time invariant (LTI) dynamic systems. A blend of state space and frequency domain methods has reached

a level at which feedback control design is systematic, not only with disturbance-free models, but also in the presence of disturbances and modelling errors. There is an abundance of design methodologies for linear models: root locus, Bode plots, LQR-optimal control, eigenstructure assignment, H_∞/H_2 , μ -synthesis, Linear Matrix Inequalities (LMI), etc. Each of these methods can be used to achieve stabilization, tracking, disturbance attenuation and similar design objectives.

The situation is quite different for nonlinear models. Although several nonlinear methodologies have emerged, each method taken alone is very often not sufficient for a satisfactory feedback design, especially for multiobjective problems. A single design method always can not encompass all nonlinear models of practical interest, and the large diversity of nonlinear phenomena suggests that with a single design approach most of the results would end up being unnecessarily conservative. To deal with diverse nonlinear phenomena a diversity of design tools and procedures comparable to linear systems are needed.

In recent years, considerable attention has been paid to the design problems of nonlinear control systems, and some important advances have been presented in numerous publications and papers. A brief review of different design methods relevant to the nonlinear control methods is summarized as follows.

Because of the powerful tools for linear systems, the most practical way to approach the control problems for nonlinear systems is to appeal to the neat results available in the linear case, that is, via linearization. A feedback control law is designed by linearizing the system about a desired equilibrium point and designing a stabilizing linear feedback control for the linearized system. However, linearization alone will not be sufficient for the analysis of nonlinear systems since linearization is an approximation in the neighborhood of an operating point, it can only predict the “local” behavior of the nonlinear system in some neighborhood of that single operating (equilibrium) point, and the dynamics of a nonlinear system are much richer than the dynamics of a linear system and there are “essentially nonlinear phenomena” that can take place only in the presence of

nonlinearity.

Various nonlinear control design methodologies have been paid more and more attention in recent years. Gain scheduling is a technique that can extend the validity of the linearization approach to a range of operating points [127]. Several operating points are selected to cover the whole range of the system's operation. These operating points are indexed by some combination of state variables or reference state trajectories, which are termed as scheduling variables. Then at each of these operating points, a time-invariant linearization of the system is obtained and a linear feedback control law is designed for each linearized system. In between operating points, the control gains are interpolated and scheduled according to the scheduling variables, thus resulting in a global controller, which is called a gain-scheduled controller.

The arbitrariness in gain-scheduling designs makes this method unsatisfactory. First of all, the selection of the scheduling variable is crucial. Secondly, it is based on trial and error, and no systematic approach for devising this scheduling scheme. Besides, a conventional gain-scheduled control law does not even give the correct linearization when linearized about these selected operating points, hence there is no guarantee that a conventional gain-scheduled control law will perform as it is designed.

However, a number of different approaches have been described using nonlinear methods. The exact feedback linearization method was introduced by Hunt et al. [48] in 1983 and this approach has been applied to aircraft control by the same authors. The basic idea of the exact feedback linearization is to use feedback and state transformation to cancel out the nonlinearities in the system. The resulting system is linear and a feedback control can be designed using any linear synthesis method, e.g., LQR method. A serious drawback of the feedback linearization technique is that an exact knowledge of the dynamics is assumed to cancel the nonlinearities of the system, while exact knowledge of the state equation and exact mathematical cancellation of terms are almost impossible [54].

Nonlinear optimal control problem was introduced by Lee and Markus [76]. Consider

the following nonlinear system,

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x) + j(x)u, \quad x \in \mathbb{R}^n, u, y \in \mathbb{R}^m\end{aligned}\tag{1.1}$$

To improve performance, we try to find a feedback control $u(x)$ that stabilizes system (1.1) while minimizing the cost

$$J = \int_0^\infty [l(x) + u^T R(x)u] dt\tag{1.2}$$

with $l(x) \geq 0$ and $R(x) > 0$, for all $x \neq 0$. If there is a function $V(x) \geq 0$ that satisfies the Hamilton-Jacobi-Bellman equation

$$\begin{aligned}L_f V(x) - \frac{1}{4} L_g V(x) R^{-1}(x) (L_g V(x))^T + l(x) &= 0 \\ V(0) &= 0\end{aligned}\tag{1.3}$$

where $L_f V(x) := [\partial V / \partial x](x) f(x)$ and $L_g V(x) := [\partial V / \partial x](x) g(x)$. Then the optimal feedback law is

$$u = -\frac{1}{2} R^{-1}(x) [L_g V(x)]^T\tag{1.4}$$

Under a detectability condition with $l(x)$ as the system output, the optimal control (1.4) is stabilizing. Furthermore, if the value function $V(x)$ is positive definite, it can be used as a Lyapunov function, thus establishing a connection between stability and optimality as discussed in [66, 120].

For the nonlinear system (1.1) and the cost (1.2) with $R(x) = I$, it has been discussed by Moylan and Anderson in [89] that a control law $u = -\mu(x)$ is optimal if and only if the system $\dot{x} = f(x) + g(x)u$ with the input u and the output $y = \mu(x)$ is dissipative with

the rate $w(u, y) \leq u^T y + \frac{1}{2} y^T y$, which establishes the connection between the passivity and optimality.

Dissipativity theory for general nonlinear systems was presented by J. C. Willems [151, 152] in his seminal two-part paper on dissipative dynamical systems. In particular, Willems introduced the definition of dissipativity for general dynamical systems in terms of a dissipation inequality involving a supply rate and a storage function. On the level of analysis, dissipativity can involve conditions on system parameters that render an input, state, output system dissipative. On the synthesis level, dissipativity can be used to design feedback controllers that add dissipation and guarantee stability robustness allowing stabilization to be understood in physical terms, as discussed in [40].

The clear connections between the linear H_∞ control theory and the traditional methods in optimal control stimulates several attempts to generalize the linear H_∞ results in state space to nonlinear systems. It has been shown that the solution of the H_∞ control problem via feedback can be determined from the solution of a Hamilton-Jacobi equation (HJE) or Hamilton-Jacobi inequality (HJI), which is the nonlinear version of the Riccati equation and Riccati inequality for the corresponding linear H_∞ control problem [11, 41, 52, 53, 55, 56, 57, 80, 81, 141, 142, 143, 144]. In particular, in [141, 142, 143] sufficient conditions for the existence of stabilizing solutions of the HJE are given in terms of the existence of a linear (sub)optimal H_∞ controller for the linearized nonlinear controlled system. In parallel, nonlinear dissipativity theory is applied for nonlinear systems with appropriate storage functions and supply rates to obtain gain bounded closed-loop systems in [11, 52, 53, 55, 56, 57, 144].

In the nonlinear control design, HJE or HJI plays a very important role since the existence of the stabilizing feedback controllers are directly related to the solutions of HJEs or HJIs. This problem is also the biggest bottleneck to the applications of nonlinear H_∞ control theory because there is no systematic numerical approach to solve the partial differential equations or inequalities. For the computational method to find the solutions of HJEs, only a few research has been presented on the exact solutions of the

HJE [141, 142]. However, some approximation methods have been reported, including [44, 62, 83] for the method by Taylor series expansion, [13, 14] for the Galerkin approximation, [12, 125, 126] for the viscosity solutions in nonlinear H_∞ control, [116] for investigating the geometric property and structure of HJE by using symplectic geometry, [86] for discretization-interpolation technique and [3] for a transformation approach.

A couple of alternative approaches with promising computational properties have also been developed to handle nonlinear optimal regulation and H_∞ control problems. Motivated by the solutions to the H_2 and H_∞ control problems for linear systems, [23, 24, 88, 90, 122] proposed state dependent linear representation (SDLR) for nonlinear systems and derived synthesis solutions that mimic those of the purely linear ones, including the state dependent Riccati equations (SDRE) nonlinear regulation, SDRE nonlinear H_∞ , SDRE nonlinear H_2 , SDRE nonlinear filtering, existence of solutions as well as stability and optimality properties associated with SDRE controllers. However, this approach provides no guarantee for global asymptotic stability and therefore further stability analysis is usually required.

On the other hand, methods based on nonlinear matrix inequalities (NLMIs) have been developed [47, 82]. In [82] the convexity of the solution to the H_∞ control problem is examined and the solution is then characterized in terms of a set of NLMIs instead of HJIs which are in fact state dependent Linear Matrix Inequalities (SDLMI). Another closely related method is the quasi linear parameter varying (LPV) approach [45, 46] built upon LPV control [155], which represents nonlinear systems as linear forms with varying parameters that are dependent upon state variables.

Last, but not the least, an important nonlinear control technique, backstepping method needs to be mentioned. Backstepping is an effective nonlinear design method which preserves certain useful nonlinearity and gives the controller systematically. Also, compared with exact feedback linearization method, backstepping often has better robustness. Therefore, nonlinear control based on backstepping has become a popular nonlinear design approach in recent years [2, 43, 64, 72, 157, 173]. However, the disadvantage

of the backstepping method lies in that it is limited to a class of pure feedback (lower-triangular) systems and the design procedure is often burdened by recursive construction of stabilizing Lyapunov functions.

1.3 SOS Programming

The aforementioned techniques use mainly Lyapunov and storage functions for analysis and control Lyapunov functions for synthesis. However, these approaches suffer from one drawback: constructing such suitable functions is very difficult and not systematic. In particular, no unified procedure and efficient computational method have been suggested in the previous research work to obtain the Lyapunov function candidates that will stabilize the close-loop systems for general nonlinear systems. This is the main motivation for our research based on a new computationally efficient nonlinear synthesis method using the SOS decomposition and semidefinite programming.

In recent years, there has been a great interest in SOS polynomials and SOS optimization [22, 74, 101, 102, 115, 123]. A new computationally tractable nonlinear analysis method based on the SOS decomposition and semidefinite programming was proposed by Parrilo [101]. From a computational perspective, the SOS based method relaxed the search for positive definite stability certificate functions to a search for SOS certificate functions.

A multivariate polynomial $p(x_1, \dots, x_n) \triangleq p(x)$ is a SOS polynomial if there exist polynomials $f_1(x), \dots, f_m(x)$ such that $p(x) = \sum_{i=1}^m f_i^2(x)$. This naturally implies that $p(x) \geq 0$ for $x \in \mathbb{R}^n$. Indeed, the nonnegativity of a polynomial is NP-hard to test, whereas the SOS decomposition provides a sufficient condition to check if a multivariate polynomial is positive semidefinite. Generally, there is no equivalence between nonnegativeness and SOS except for some special cases [115]. The SOS decomposition has been shown to be equivalent to the existence of a positive semidefinite matrix Q and a vector of monomials $Z(x)$ such that $p(x) = Z^T(x)QZ(x)$ [101]. This equivalence

implies that the SOS decomposition amounts to searching for Q in the intersection of the cone of positive semidefinite matrices and a set defined by some affine constraints, therefore it is a semidefinite programming problem [111].

The nonnegativity of a SOS polynomial is a crucial property in many control applications where polynomial inequalities are replaced with the SOS based conditions. Take as an example the problem of proving stability of a nonlinear system's equilibrium based on the Lyapunov stability results. Consider a nonlinear system with polynomial vector field $\dot{x} = f(x)$ where $f(0) = 0$. If there exists a continuously differentiable function $V(x)$ such that in some neighborhood of the system's equilibrium

$$V(x) > 0 \quad (1.5)$$

$$-\frac{\partial V(x)}{\partial x} f(x) \geq 0 \quad (1.6)$$

then by the Lyapunov's direct method, the nonlinear system's equilibrium point is stable. Instead of asking for the function $V(x)$ to be positive definite and its time derivative negated to be positive semidefinite, the SOS based technique requires the following two expressions are SOS

$$V(x) - \varphi(x) \quad (1.7)$$

$$-\frac{\partial V(x)}{\partial x} f(x) \quad (1.8)$$

where $\varphi(x)$ is some positive definite polynomial. The SOS conditions guarantee that the expressions in (1.7) and (1.8) are nonnegative, and therefore the Lyapunov stability conditions (1.5) and (1.6) are satisfied and $V(x)$ is actually a Lyapunov function to achieve the stability of the system's equilibrium. The SOS conditions provide a computationally tractable relaxation for the original stability problem and can be solved via semidefinite programming.

A sum of squares program (SOSP) is generally a convex optimization problem of the

following form [100, 110]:

$$\text{Minimize } \sum_{j=1}^J w_j c_j \quad (1.9)$$

subjecto to

$$a_{i,0}(x) + \sum_{j=1}^J a_{i,j}(x)c_j \text{ is SOS, for } i = 1, \dots, I, \quad (1.10)$$

where c_j are the scalar real decision variables, w_j are given real numbers and $a_{i,j}(x)$ are given polynomials. Another equivalent form of SOSP has been proposed in [107, 109].

As mentioned above, the SOS decomposition can be efficiently computed via semidefinite programming. Therefore it is desirable to have a computational aid that facilitates the formulation of the semidefinite programs (SDPs) from the SOSPs. SOSTOOLS is exactly one such software and it is also a free, third-party MATLAB toolbox for solving the SOSPs [107, 108, 109, 110]. SOSTOOLS automates the conversion from SOSP to SDP, calls the SDP solver and converts the SDP solution back to the solution of the original SOSP. The current version of SOSTOOLS uses SeDuMi [128] or SDPT3 [135] as the SDP solver. This whole process is depicted in Figure 1.1 [100, 109, 110].

Figure 1.1: The relations between SOSP, SDP, SOSTOOLS and SeDuMi or SDPT3

There are two fundamental features that distinguish the SOS based technique from other nonlinear design methodologies.

- The computational tractability of the SOS decomposition for multivariable polynomials, therefore a relaxed problem can be solved via semidefinite programming.
- The computational complexity involved is polynomial in the problem size, therefore it is computationally attractive.

These two crucial advantages of the SOS based methodology have been applied successfully in several control problems for nonlinear systems with polynomial vector field. The Lyapunov stability analysis and stabilizing synthesis problems have been extensively studied for nonlinear systems [32, 50, 99, 100, 101], time-delay systems [94, 95, 97, 98, 100], parameter-dependent systems [75, 110], fuzzy systems [131, 133], switched and hybrid systems [100, 106], LPV systems [30, 154], as well as applications to aircraft control systems [96]. In parallel, passivity analysis based on the computation of storage functions, and an overview over analysis and design of polynomial control systems using dissipation inequalities have been proposed in [15, 29], resp. In [9, 112] a dual approach based on the so-called density functions [114] was proposed for the joint search of Lyapunov functions and nonlinear controllers using semidefinite programs. In spite of better convexity property, it has been not clear how a performance objective can be incorporated.

Estimation of the region of attraction around an asymptotically stable equilibrium is an important tool in controller verification, which has been studied using the SOS based approach and presented in many publications [21, 59, 71, 118, 130, 134, 137, 138]. In particular, local stability analysis was considered in [130], and the region of attraction inner bound enlargement problem was presented for polynomial systems with uncertain dynamics. In [21], static nonlinear output feedback controllers to enlarge the domain of attraction of equilibrium points were computed.

The problem of synthesizing feedback controllers with guaranteed cost or H_∞ performance objectives by SOS optimization have been addressed in a few papers [38, 84, 111, 132, 136, 154, 170, 171, 172]. In [111], a semidefinite programming approach based on SDLMI was proposed to obtain global stability and performance objectives in case of quadratic Lyapunov functions. In [154], solutions to the LPV control analysis and synthesis were formulated as parameter-dependent LMIs via state feedback or output feedback. A two step hybrid and switched approach was derived to achieve both global stabilization and local performance by combining backstepping and LPV control technique in [170], and by combining SOSP and linear design technique with application for satellites/spacecrafts in [38, 171, 172].

Other control problems in which SOS based technique is applicable are, for instance, observer design [31, 49], model validation [104], safety verification [105], model reduction [113, 124], nonlinear synchronization [65] and contraction analysis [10].

Considering the effective application of the SOS based methodology on several control problems mentioned above, in this thesis, the primary focus is on the nonlinear robust design for polynomial parameter-dependent systems by utilizing the SOS decomposition and semidefinite programming.

For most existing SOS based control design schemes, the nonlinear system is considered with no parametric uncertainty. However, most practical systems are inevitably interfered with uncertainty and external noise. Therefore, the robustness of control systems with respect to system uncertain and noise is quite an challenging control design problem.

When parameters are involved in system matrices, most of the SOS based design methodologies mentioned above can not be utilized further due to the non-convex nature. Therefore, in this thesis, several new design methodologies are proposed to overcome the involved non-convex problems and the numerical difficulties. And these new formulations provide an effective way for the application of the new SOS programming technique. Several control problems including stability analysis, estimation of region of attraction,

H_∞ state feedback design, static output feedback design, multi-objective reliable design and robust H_∞ filtering are explored in an efficient computational manner.

1.4 Objectives

In this thesis, the main objectives are as follows:

- (1) To develop the robust control designs for polynomial nonlinear systems with parameter uncertainty based on Lyapunov stability theories;
- (2) To develop computationally tractable designs for polynomial nonlinear systems based on the new SOS programming technique;
- (3) To reduce the design conservativeness of the performance/stability analysis and synthesis;
- (4) To achieve robust stability and optimized regions of attraction for the closed-loop systems;
- (5) To develop the robust polynomial controller designs for polynomial nonlinear systems based on H_∞ robust control theories;
- (6) To develop a computational scheme to overcome the non-convex nature in nonlinear static output feedback design;
- (7) To guarantee robust stability of the resulting closed-loop systems in the nominal condition as well as when fault occurs;
- (8) To achieve optimized performance in the nominal condition and simultaneously maintain an satisfactory performance in the presence of faults.
- (9) To develop a methodology of solving the robust H_∞ filtering problem for the polynomial nonlinear systems against parametric uncertainty.

1.5 Major Contributions

The semidefinite programming relaxations based on the SOS decomposition are utilized to numerically solve several nonlinear robust control problems, including stability analysis, estimation of region of attraction, H_∞ state feedback design, static output feedback design, multi-objective reliable design and robust H_∞ filtering, for continuous time polynomial nonlinear systems with polytopic uncertainty. In our methodology, more general state dependent linear-like representations are proposed, and the exact variation rate of the state is incorporated as opposed to bounding the variation rate inside a convex set. Major contributions of this thesis are summarized as follows:

- (1) Developed a local stability analysis and synthesis method for polynomial nonlinear systems with parameter uncertainty. Computationally tractable synthesis is proposed to realize the closed-loop stability and to enlarge the provable regions of attraction for the system, using the parameter-dependent Lyapunov functions. The proposed optimization methodology provides efficient numerical method to the nonlinear synthesis and at the same time to the analysis of regions of attraction for closed-loop systems.
- (2) Developed a H_∞ state feedback controller design for parameter-dependent polynomial nonlinear systems. Additional matrix variables are adopted in this approach therefore a sufficient condition for H_∞ controller synthesis is presented, which provides extra freedom for the controller design. The proposed approach employs parameter-dependent Lyapunov functions to reduce the design conservativeness and provides more general assumption and relaxation to deal with the nonlinear terms in the matrix inequalities. No iterative algorithm is involved in the design;
- (3) Developed a new computational scheme of solving the nonlinear static output feedback design problems for polynomial nonlinear systems. The proposed formulation provides an effective way to overcome the numerical difficulty in solving

static output feedback problem, and the output feedback controllers and Lyapunov functions are constructed in an efficient computational manner. Because of the decoupling of the system matrices and Lyapunov matrix, the proposed methodology has a more suitable structure to deal with system uncertainty. More general relaxation and parameter-dependent Lyapunov functions are also utilized to reduce the design conservativeness;

- (4) Developed a new computational synthesis method for solving the multi-objective control problems for parameter-dependent polynomial nonlinear systems based on the concept of passive reliable design. The proposed approach achieves the optimal performance in nominal condition and simultaneously maintains satisfactory performance in the presence of faults. Parameter-dependent Lyapunov functions, one for each and every vertex of the uncertainty polytope, are employed to reduce the conservativeness involved in the design.
- (5) Developed an efficient computational methodology for solving the robust H_∞ filtering problems for the polynomial nonlinear systems against parametric uncertainty. The coupling among system dynamics, filter dynamics and the Lyapunov matrix is eliminated by introducing slack matrix variables. The closed-loop stability with guaranteed H_∞ norm from the norm-bounded disturbance signal to the estimation error are guaranteed by solvable SOS based optimization.

1.6 Organization

The outline of this thesis is as follows:

Chapter 1 introduces the background and latest research work on nonlinear control system design, Lyapunov based design and a new computationally efficient nonlinear analysis methodology using the SOS decomposition and semidefinite programming. Major contributions of this thesis are also summarized.

Chapter 2 provides mathematical background on the properties of multivariate polynomials and in particular some very important concepts from real algebraic geometry. This preliminary material is the basis for problem formulations in the subsequent chapters. A brief overview of polynomial definitions, SOS polynomials, SOS programming, SOS matrices, the Positivstellensatz theorem and its relation to the $\mathcal{S}$ -procedure are presented.

Chapter 3 investigates the stability analysis and state feedback synthesis problem for parameter-dependent nonlinear systems with polynomial vector fields. Based on the classical Lyapunov stability results, computationally tractable design is proposed to test the stability of the closed-loop system and to enlarge the region of attraction for the system. The feedback design methodology is applied to a numerical example and the F-8 aircraft longitudinal model.

Chapter 4 addresses the state feedback design problems for parameter-dependent polynomial nonlinear systems. Less conservative sufficient conditions to guarantee the closed-loop stability with or without H_∞ performance are formulated in terms of nonlinear matrix inequalities, where polynomial state feedback controllers are provided. These inequalities based conditions are formulated as SOS based optimization problems and solved via semidefinite programming. Numerical examples are provided to demonstrate the effectiveness of the proposed method.

Chapter 5 furthers the study to the nonlinear static output feedback design for polynomial nonlinear systems. A simple method is applied to include the measured output in the augmented system, and sufficient conditions to guarantee the closed-loop stability with or without H_∞ performance are derived as nonlinear matrix inequalities. The output feedback controllers and Lyapunov functions are derived in an efficient computational manner via semidefinite programming. Numerical examples are provided to demonstrate the effectiveness of the proposed method.

Chapter 6 discusses the multi-objective reliable control problem for parameter-dependent polynomial nonlinear systems. A fixed controller is designed for the nominal case as

well as fault cases. Not only the optimal performance during normal system operation, but also the satisfactory performance in the presence of faults is guaranteed. A numerical example is demonstrated to show the effectiveness of the proposed method.

Chapter 7 studies the robust H_∞ filtering problem for polynomial nonlinear systems against parametric uncertainty. A stable H_∞ filter is proposed to guarantee the closed-loop stability with guaranteed H_∞ norm from the unknown norm-bounded disturbance signal to the estimation error based on the SOS optimization. A numerical example is demonstrated to show the effectiveness of the proposed method.

Chapter 8 concludes all the approaches proposed in this thesis and gives some recommendations of future research directions.

Chapter 2

Polynomial Background

The properties of multivariate polynomials and in particular some very important concepts from real algebraic geometry form the basis for the formulations in the subsequent chapters. Therefore, the necessary polynomial background is provided in this chapter for the upcoming results.

Let $\mathbb{R}$ denote the set of the real numbers, $\mathbb{R}_+ := [0, \infty) \subset \mathbb{R}$. $\mathbb{R}^n$ is the n -dimensional real space. $\mathbb{Z}_+$ denotes the set of nonnegative integers. These notations will be used in the formal definitions and almost every results in the following chapters.

2.1 Polynomial Definitions

Definition 2.1.1 (Monomials) A *Monomial* m_α in n variables is a function defined as $m_\alpha(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha_i \in \mathbb{Z}_+$. The degree of a monomial is defined as $\deg m_\alpha := \sum_{i=1}^n \alpha_i$.

Definition 2.1.2 (Polynomials) A *Polynomial* p in n variables is a finite linear combi-

nation of monomials

$$p := \sum_{\alpha} c_{\alpha} m_{\alpha} = \sum_{\alpha} c_{\alpha} x^{\alpha} \quad (2.1)$$

with $c_{\alpha} \in \mathbb{R}$. The degree of a polynomial is defined as $\deg p := \max_{\alpha} \deg m_{\alpha}$ (provided the associated c_{α} is non-zero).

The set of polynomials with real coefficients and common independent variables, say, $x_1, \dots, x_n$, is often denoted as $\mathbb{R}[x_1, \dots, x_n]$. To eliminate reference to a particular set of independent variables, we define $\mathcal{R}_n$ to be the set of all polynomials in n variables with real coefficients, with the assumption that if $p \in \mathcal{R}_n$ and $f \in \mathcal{R}_n$, then p and f are functions of the same independent variables.

A polynomial vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f(x) = [f_1(x), \dots, f_n(x)]^T$ is a vector field with $f_i \in \mathcal{R}_n$, i.e., the entries of the vector field are polynomial functions in $x \in \mathbb{R}^n$, and is denoted as $f(x) \in \mathcal{R}_n^n$.

Define a subset of $\mathcal{R}_n$, $\mathcal{R}_{n,d} := \{p \in \mathcal{R}_n | \deg p \leq d\}$. This is the set of all polynomials in n variables that have maximum degree d . If all the monomials of polynomial p are of the same degree d , the polynomial p is a homogenous polynomial of degree d and it satisfies $p(\lambda x) = \lambda^d p(x)$ for any scalar λ .

Another subset of $\mathcal{R}_n$ is the set of positive semidefinite (PSD) polynomials, which are nonnegative on all of $\mathbb{R}^n$. This set is defined as $\mathcal{P}_n := \{p \in \mathcal{R}_n | p(x) \geq 0, \forall x \in \mathbb{R}^n\}$.

We also have $\mathcal{P}_{n,d} := \mathcal{P}_n \cap \mathcal{R}_{n,d}$.

2.2 SOS Polynomials

A basic problem that appears in many control design problems is that of requiring a polynomial to be nonnegative. However, checking the global positivity of a polynomial function is in fact NP-hard when the degree is at least four [101]. A deceptively simple

sufficient condition for a polynomial function to be nonnegative is the existence of a SOS decomposition. In the subsequent chapters, conditions on nonnegativity in the control synthesis formulations are relaxed by the sufficient conditions on the polynomial being a SOS polynomial.

2.2.1 SOS Decomposition

Definition 2.2.1 (SOS Polynomial) A polynomial $p(x)$ in n variables is a SOS polynomial if there exist $f_i(x) \in \mathcal{R}_n$, $i = 1, \dots, m$ such that

$$p(x) = \sum_{i=1}^m f_i^2(x) \quad (2.2)$$

Define Σ_n to be the set of SOS polynomials in n variables, which is a very important subset of the polynomials. Similarly, we have $\Sigma_{n,d} := \Sigma_n \cap \mathcal{R}_{n,d}$.

Note that $p(x)$ being a SOS polynomial implies $p(x) \geq 0 \forall x \in \mathbb{R}^n$, which implies $p(x) \in \mathcal{P}_n$ and $\Sigma_n \subseteq \mathcal{P}_n$. However, it should be noted that there are positive definite polynomials that are not SOS polynomials, ie., $p \in \mathcal{P}_n \not\Rightarrow p \in \Sigma_n$.

For a converse statement, the equivalence between “nonnegativity” and “sum of squares”, proven by Hilbert, are only in the following three special cases [115].

- Univariate polynomials, any (even) degree, i.e., $n = 1$.
- Quadratic polynomials, in any number of variables, i.e., $d = 2$.
- Quartic polynomials in two variables, i.e., $n = 2, d = 4$.

2.2.2 Computation of the SOS Decomposition

The computation of the SOS decomposition has been developed from two main different viewpoints in unrelated fields. One is from a convex optimization perspective [123], the other is from an algebraic perspective and presented as the “Gram matrix” methodology [22]. An implementation of the Gram matrix method is presented in [103] via exact decision methods, but no convexity property is exploited. While, an equivalent characterization of the SOS decomposition has been shown to be a semidefinite program in [101] and given in the following proposition.

Proposition 2.2.1 (Parrilo, [101]) *A polynomial $p(x)$ of degree $2d$ is a SOS if and only if there exists a positive semidefinite matrix Q and a vector of monomials $Z(x)$ containing monomials of degree less than or equal to d given by different products of x such that*

$$p(x) = Z^T(x)QZ(x) \quad (2.3)$$

Proof: See [101]. ■

Since the monomials in $Z(x)$ are not algebraically independent, the representation (2.3) is not unique, and Q may be positive semidefinite for some representations but not for others.

The equivalence between (2.2) and (2.3) makes the SOS decomposition is just a convex LMI feasibility problem and computable via semidefinite programming [101]. Expanding $Z^T(x)QZ(x)$ and equating the coefficients of the resulting monomials to the ones in $p(x)$, a set of affine relations on the elements of Q is obtained. If $Q \geq 0$, a factorization of matrix Q directly provides a SOS decomposition of $p(x)$. Conversely, if $p(x)$ is a SOS polynomial, Q can be always constructed by expanding the terms in monomials. Therefore, finding the SOS decomposition amounts to searching for the matrix Q in the intersection of a linear subspace of matrices Q that satisfy (2.3) with the pos-

itive semidefinite matrix cone. Two simple examples are provided to explain the above mentioned points.

Example 2.2.1 ([101]) Consider the quartic polynomial in two variables $p(x_1, x_2) = 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$. Define $Z(x) = [x_1^2, x_2^2, x_1x_2]^T$, then we have

$$\begin{aligned}
 p(x_1, x_2) &= 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4 \\
 &= Z^T(x) \underbrace{\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}}_Q Z(x) \\
 &= q_{11}x_1^4 + q_{22}x_2^4 + (2q_{12} + q_{33})x_1^2x_2^2 + 2q_{13}x_1^3x_2 + 2q_{23}x_1x_2^3 \quad (2.4)
 \end{aligned}$$

from (2.4) we obtain the following relations:

$$q_{11} = 2, \quad q_{22} = 5, \quad q_{13} = 1, \quad q_{23} = 0, \quad 2q_{12} + q_{33} = -1$$

Finding the SOS decomposition for $p(x)$ amounts to searching for q_{12} and q_{33} satisfying the last equation such that $Q \geq 0$. For $q_{12} = -3$ and $q_{33} = 5$, the matrix Q is positive semidefnite and we have

$$Q = L^T L, \quad \text{where } L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

This gives the following SOS decomposition:

$$p(x) = \frac{1}{2}(2x_1^2 - 3x_2^2 + x_1x_2)^2 + \frac{1}{2}(x_2^2 + 3x_1x_2)^2$$

Example 2.2.2 Consider the quartic polynomial in two variables $p(x) = x_1^4 + x_1^2x_2^2 + x_2^4$. Define $Z(x) = [1, x_1, x_2, x_1^2, x_1x_2, x_2^2]^T$. Two different constant matrices Q_1 and Q_2 can

be obtained such that the SOS decompositions (2.3) are satisfied.

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Note that $Q_1 \geq 0$ while $Q_2 \not\geq 0$. Q_1 's positive semidefiniteness shows that $p(x) \in \Sigma_{2,4}$, while $Q_2 \not\geq 0$ shows nothing.

2.2.3 SOS Programming

Many control problems involving polynomials can be transformed to the problems of searching for the SOS decompositions over convex sets. This type of problems is termed as the SOS problems which have been proved to be the semidefinite programming problems.

Theorem 2.2.1 ([102]) *Given a finite set $\{p_i\}_{i=0}^m \in \mathcal{R}_n$, and convex set $\mathcal{C} \subseteq \mathbb{R}^m$ defined by semidefinite constraints. The existence of $\lambda \in \mathcal{C}$ such that*

$$p_0 + \sum_{i=1}^m \lambda_i p_i \in \Sigma_n \quad (2.5)$$

is a semidefinite programming feasibility problem.

Theorem 2.2.1 provides a useful computation method for the SOS programming problems as follows:

Corollary 2.2.1 *Given $p_0, p_1 \in \mathcal{R}_n$. The existence of $k \in \mathcal{R}_n$ with a given degree such that*

$$p_0 + kp_1 \in \Sigma_n \quad (2.6)$$

is a semidefinite programming problem.

Proof: $k \in \mathcal{R}_n$ means that k can be represented as a linear combination of its monomials $\{m_j\}$, i.e., $k = \sum_{j=1}^s a_j m_j$. Then (2.6) becomes

$$p_0 + kp_1 = p_0 + \sum_{j=1}^s a_j (m_j p_1) \quad (2.7)$$

Since $\{m_j\}$ are monomials and $p_1 \in \mathcal{R}_n$, we know that $(m_j p_1) \in \mathcal{R}_n$. Hence (2.6) is a semidefinite programming problem by the result in Theorem 2.2.1. ■

The basic feasibility problem in SOS programming is formulated as follows:

Feasibility Problem: Given $a_{i,j}(x) \in \mathcal{R}_n$. Find $p_i(x) \in \mathcal{R}_n$ for $i = 1, \dots, \hat{N}$ and $p_i(x) \in \Sigma_n$ for $i = \hat{N} + 1, \dots, N$ such that

$$a_{0,j}(x) + \sum_{i=1}^N p_i(x) a_{i,j}(x) = 0, \text{ for } j = 1, \dots, \hat{J} \quad (2.8)$$

$$a_{0,j}(x) + \sum_{i=1}^N p_i(x) a_{i,j}(x) \in \Sigma_n, \text{ for } j = \hat{J} + 1, \dots, J \quad (2.9)$$

Besides the feasibility problem, many other problems in the convex SOS programming involves the optimization of an objective function. The optimization problem is generally in the form of

Optimization Problem: Given $a_{i,j}(x) \in \mathcal{R}_n$. Find $p_i(x) \in \mathcal{R}_n$ for $i = 1, \dots, \hat{N}$ and $p_i(x) \in \Sigma_n$ for $i = \hat{N} + 1, \dots, N$ such that

$$\begin{aligned} &\text{Minimize} && w^T c \\ &\text{Subject to} && a_{0,j}(x) + \sum_{i=1}^N p_i(x) a_{i,j}(x) = 0, \text{ for } j = 1, \dots, \hat{J} \end{aligned} \quad (2.10)$$

$$a_{0,j}(x) + \sum_{i=1}^N p_i(x) a_{i,j}(x) \in \Sigma_n, \text{ for } j = \hat{J} + 1, \dots, J \quad (2.11)$$

where w is a weighting coefficients vector and c is a vector formed from the unknown coefficients of the polynomial variables $p_i(x)$.

Notice that when p_i are restricted to be constants and $a_{i,j}$ are in quadratic forms, the standard LMI problem formulation is recovered.

Both the feasibility and optimization problems formulated above are quite general and can be solved by the semidefinite programming with the help of the software SOS-TOOLS [107, 109].

2.3 The Positivstellensatz

It is notable that SOSP can be used to prove the emptiness of the semi-algebraic sets, i.e., sets of the form

$$\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0, g_j(x) \neq 0, h_k(x) = 0 \ \forall i, j, k \}$$

where $f_i(x)$, $g_j(x)$ and $h_k(x)$ are polynomials. The main tool for this purpose is the Positivstellensatz, which is a central theorem from real algebraic geometry characterizing certificates for infeasibility of the above system of polynomial equalities and inequalities.

With the polynomial definitions presented in the before sections, concepts to state the Positivstellensatz can be defined in this section. the Positivstellensatz is a powerful theorem which generalizes many known results, for example, the S -procedure, as shown in Section 2.4.

Definition 2.3.1 Given $\{g_1, \dots, g_t\} \in \mathcal{R}_n$, the **Multiplicative Monoid** generated by g_j 's is the set of all finite products of g_j 's, including the empty product, which is defined to be 1. It is denoted as $\mathcal{M}(g_1, \dots, g_t)$. For completeness define $\mathcal{M}(\phi) := 1$.

An example: $\mathcal{M}(g_1, g_2) = \{g_1^{k_1} g_2^{k_2} | k_1, k_2 \in \mathbb{Z}_+\}$.

Definition 2.3.2 Given $\{f_1, \dots, f_s\} \in \mathcal{R}_n$, the **Cone** generated by f_i 's is

$$\mathcal{P}(f_1, \dots, f_s) := \left\{ s_0 + \sum_{i=1}^l s_i b_i \mid l \in \mathbb{Z}_+, s_i \in \Sigma_n, b_i \in \mathcal{M}(f_1, \dots, f_s) \right\} \quad (2.12)$$

For completeness note that $\mathcal{P}(\phi) := \Sigma_n$.

Note that if $p \in \Sigma_n$ and $f \in \mathcal{R}_n$, then $pf^2 \in \Sigma_n$ as well.

One example of $p(f_1, f_2) \in \mathcal{P}(f_1, f_2)$ is: $p(f_1, f_2) = \{s_0 + s_1 f_1 + s_2 f_2 + s_3 f_1 f_2\}$, where $s_0, \dots, s_3 \in \Sigma_n$.

Definition 2.3.3 Given $\{h_1, \dots, h_u\} \in \mathcal{R}_n$, the **Ideal** generated by h_k 's is

$$\mathcal{I}(h_1, \dots, h_u) := \left\{ \sum h_k p_k \mid p_k \in \mathcal{R}_n \right\} \quad (2.13)$$

For completeness note that $\mathcal{I}(\phi) := 0$

With these definitions, we can state the following theorem.

Theorem 2.3.1 (Positivstellensatz, [18]) Let $(f_i)_{i=1, \dots, s}$, $(g_j)_{j=1, \dots, t}$, and $(h_k)_{k=1, \dots, u}$ be finite families of polynomials in $\mathcal{R}_n$. Denote by $\mathcal{P}(f_1, \dots, f_s)$ the cone generated by

$(f_i)_{i=1,\dots,s}$, $\mathcal{M}(g_1, \dots, g_t)$ the multiplicative monoid generated by $(g_j)_{j=1,\dots,t}$, and $\mathcal{I}(h_1, \dots, h_u)$ the ideal generated by $(h_k)_{k=1,\dots,u}$. Then the following are equivalent:

1. The set

$$\left\{ x \in \mathbb{R}^n \left| \begin{array}{l} f_i(x) \geq 0, i = 1, \dots, s, \\ g_j(x) \neq 0, j = 1, \dots, t, \\ h_k(x) = 0, k = 1, \dots, u \end{array} \right. \right\} \quad (2.14)$$

is empty,

2. There exist polynomials $f \in \mathcal{P}(f_1, \dots, f_s)$, $g \in \mathcal{M}(g_1, \dots, g_t)$, $h \in \mathcal{I}(h_1, \dots, h_u)$ such that

$$f + g^2 + h = 0 \quad (2.15)$$

From Theorem 2.3.1, we know that the set emptiness condition holds by finding specific f , g , and h such that $f + g^2 + h = 0$. These f , g , and h are known as Positivstellensatz certificates since they certify that the equality holds. In [101], it is presented that the search for bounded degree Positivstellensatz certificates can be done using semidefinite programming, which is presented in the following theorem.

Theorem 2.3.2 (Theorem 5.1, [102]) Given polynomials $\{f_1, \dots, f_s\}$, $\{g_1, \dots, g_t\}$ and $\{h_1, \dots, h_u\}$ in $\mathcal{R}_n$, if the set

$$\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0, g_j(x) \neq 0, h_k(x) = 0, i = 1, \dots, s, j = 1, \dots, t, k = 1, \dots, u \} \quad (2.16)$$

is empty, then the search for bounded degree Positivstellensatz refutations can be done using semidefinite programming. If the degree bound is chosen large enough the semidef-

inite programs will be feasible and give the refutation certificates.

Proof: See [102]. ■

A simple example is provided to illustrate the procedure to obtain an semidefinite programming condition for the Positivstellensatz certificates with the application of the Positivstellensatz and simplifications.

Example 2.3.1 Given $a(x), b(x), c(x) \in \mathcal{R}_n$. Consider the problem of verifying if

$$a(x) = 0, b(x) \geq 0 \Rightarrow c(x) \geq 0 \quad (2.17)$$

satisfies.

(2.17) can be written as a set containment condition of the form:

$$\{ x \in \mathbb{R}^n \mid a(x) = 0, b(x) \geq 0 \} \subseteq \{ x \in \mathbb{R}^n \mid c(x) \geq 0 \} \quad (2.18)$$

Before proceeding to the application of the Positivstellensatz, we formulate the condition (2.18) as a emptiness constraint.

$$\{ x \in \mathbb{R}^n \mid a(x) = 0, b(x) \geq 0, -c(x) \geq 0, c(x) \neq 0 \} \text{ is empty} \quad (2.19)$$

By the Positivstellensatz, this holds if and only if there exist $p \in \mathcal{R}_n$, $s_{\{.\}} \in \Sigma_n$ and $k \in \mathbb{Z}_+$ such that

$$s_0(x) + s_1(x)b(x) - s_2(x)c(x) - s_{12}(x)b(x)c(x) + c^{2k}(x) + p(x)a(x) = 0 \quad (2.20)$$

A special case is obtained by setting $k = 1$, $s_0(x) = 0$, $s_1(x) = 0$ and $p(x) = \hat{p}(x)c(x)$, then the expression reduces to

$$-s_{12}(x)b(x) + c(x) + \hat{p}(x)a(x) \in \Sigma_n \quad (2.21)$$

which clearly implies that (2.17) holds. Since (2.21) is affine in $s_{12}(x)$ and $\hat{p}(x)$, the search for such certificates can be posed as a semidefinite program.

2.4 The $\mathcal{S}$ -Procedure

In this section, the $\mathcal{S}$ -procedure and its generalization for the non-quadratic functions and non-scalar weights are provided. The well known $\mathcal{S}$ -procedure for quadratic forms and nonstrict inequalities has been presented in [19].

Given the symmetric matrices $\{A_i\}_{i=0}^m \in \mathbb{R}^{n \times n}$. The $\mathcal{S}$ -procedure states: if there exist nonnegative scalars $\{\lambda_i\}_{i=1}^m$ such that $A_0 - \sum_{i=1}^m \lambda_i A_i \geq 0$, then the following condition on $\{A_i\}_{i=0}^m$ holds.

$$\bigcap_{i=1}^m \{x \in \mathbb{R}^n \mid x^T A_i x \geq 0\} \subset \{x \in \mathbb{R}^n \mid x^T A_0 x \geq 0\} \quad (2.22)$$

Here, the relationship between the $\mathcal{S}$ -procedure and the Positivstellensatz is analyzed to derive the $\mathcal{S}$ -procedure in the Positivstellensatz formalism.

The equation (2.22) can be described as the following empty set, which is similar to the first emptiness condition (2.14) in Theorem 2.3.1.

$$\left\{ x \in \mathbb{R}^n \left| \begin{array}{l} x^T A_1 x \geq 0, \dots, x^T A_m x \geq 0, \\ -x^T A_0 x \geq 0, x^T A_0 x \neq 0 \end{array} \right. \right\} \quad (2.23)$$

If λ_i exists, define $Q := A_0 - \sum_{i=1}^m \lambda_i A_i \geq 0$, so $x^T Q x \in \Sigma_n$. Define $g(x) := x^T A_0 x$ and

$$f(x) := (x^T Q x)(-x^T A_0 x) + \sum_{i=1}^m \lambda_i (-x^T A_0 x)(x^T A_i x) \quad (2.24)$$

From the nonnegativity of λ_i we have $\lambda_i \in \Sigma_n$, and because $x^T Q x \in \Sigma_n$ we know that the function $f(x) \in \mathcal{P}(x^T A_1 x, \dots, x^T A_m x, -x^T A_0 x)$ and $g(x) \in \mathcal{M}(x^T A_0 x)$. Then, we verify $f + g^2 = 0$,

$$\begin{aligned}
 f + g^2 &= (x^T Q x)(-x^T A_0 x) + \sum_{i=1}^m \lambda_i (-x^T A_0 x)(x^T A_i x) + (x^T A_0 x)^2 \\
 &= -(x^T A_0 x)^2 + \sum_{i=1}^m \lambda_i (x^T A_i x)(x^T A_0 x) + \sum_{i=1}^m \lambda_i (-x^T A_0 x)(x^T A_i x) \\
 &\quad + (x^T A_0 x)^2 \\
 &= 0
 \end{aligned} \tag{2.25}$$

The above equation $f + g^2 = 0$ illustrates that f and g are Positivstellensatz certificates that prove (2.23) is empty.

The $\mathcal{S}$ -procedure given above can be generalized to deal with the non-quadratic functions and non-scalar weights [101]. The following lemma is a generalization of the $\mathcal{S}$ -procedure and it is a special case of the Positivstellensatz theorem, where we are searching over the SOS polynomials $\{s_i\}_{i=1}^m$ instead of the nonnegative scalars $\{\lambda_i\}_{i=1}^m$.

Lemma 2.4.1 (Generalized $\mathcal{S}$ -procedure) Given $\{p_i\}_{i=0}^m \in \mathcal{R}_n$. If there exist $\{s_i\}_{i=1}^m \in \Sigma_n$ such that

$$p_0 - \sum_{i=1}^m s_i p_i = q \tag{2.26}$$

with $q \in \Sigma_n$. Then

$$\bigcap_{i=1}^m \{x \in \mathbb{R}^n \mid p_i(x) \geq 0\} \subset \{x \in \mathbb{R}^n \mid p_0(x) \geq 0\} \tag{2.27}$$

Proof: The related empty set for (2.27) is

$$\{ x \in \mathbb{R}^n \mid p_1(x) \geq 0, \dots, p_m(x) \geq 0, -p_0(x) \geq 0, p_0(x) \neq 0 \} \quad (2.28)$$

Similarly, define $g := p_0$ and

$$f := -qp_0 - \sum_{i=1}^m s_i p_0 p_i \quad (2.29)$$

Since q and s_i belong to Σ_n , then $g \in \mathcal{M}(p_0)$ and $f \in \mathcal{P}(p_1, \dots, p_m, -p_0)$. Verifying the condition $f + g^2 = 0$,

$$\begin{aligned} f + g^2 &= -qp_0 - \sum_{i=1}^m s_i p_0 p_i + p_0^2 \\ &= -\left(p_0 - \sum_{i=1}^m s_i p_i \right) p_0 - \sum_{i=1}^m s_i p_0 p_i + p_0^2 \\ &= 0 \end{aligned} \quad (2.30)$$

The above condition illustrates that f and g provide the Positivstellensatz certificates for the empty set (2.28), then the conclusion (2.27) is obtained. ■

2.5 SOS Polynomial Matrices

The SOS polynomial matrices are matrices with polynomial entries and that are positive semidefinite for all values of the indeterminates. The formal definition has been presented as follows:

Definition 2.5.1 ([37]) Let $S \in \mathbb{R}[x]^{m \times m}$ be a symmetric matrix, and $y = [y_1, \dots, y_m]$ be new indeterminates. The matrix S is an SOS matrix if the scalar polynomial $y^T S y$ is an SOS in $\mathbb{R}[x, y]$.

It is noted that the concept of the SOS matrix reduces to the standard SOS polynomial notion when $m = 1$. And when S is independent of x , the definition implies a positive semidefinite matrix. Therefore, SOS matrices are a common generalization of positive semidefinite matrices and SOS polynomials.

Since the SOS matrix condition is equivalent to a standard SOS test in a polynomial with additional variables, when only symmetric polynomial matrices are involved, the SOS decomposition can provide a computational relaxation stated in the following proposition.

Proposition 2.5.1 ([111]) *Let $F(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in \mathbb{R}^n$. Furthermore, let $Z(x)$ be a column vector whose entries are all monomials in x with degree no greater than d , and consider the following conditions.*

- (1) $F(x) \geq 0$ for all $x \in \mathbb{R}^n$.
- (2) $v^T F(x) v$ is a SOS, where $v \in \mathbb{R}^N$.
- (3) There exists a positive semidefinite matrix Q such that $v^T F(x) v = (v \otimes Z(x))^T Q (v \otimes Z(x))$, where $\otimes$ denotes the Kronecker product.

Then (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3).

Proof: See [111] ■

The converse implication (1) $\Rightarrow$ (2) generally does not hold except for a special case when $n = 1$ [37]. Proposition 2.5.1 implies that the problem to find $Q \geq 0$ can be formulated as an semidefinite programming problem. In other words, we can check the SOS polynomial matrix $F(x)$ with respect to some monomial basis by solving an semidefinite programming problem.

In the upcoming chapters, the extensions of the SOS decomposition are proposed for the general continuous control system design, and the corresponding sufficient conditions

are formulated as the NLMIs (or SDMIs). A problem involving the SDMIs is in general an infinite dimensional convex optimization problem of the form

$$\text{minimize } \sum_{i=1}^m a_i c_i \quad (2.31)$$

$$\text{subject to } F_0(x) + \sum_{i=1}^m c_i F_i(x) \geq 0 \quad (2.32)$$

where a_i are some fixed real coefficients, c_i are the decision variables and $F_i(x)$ are some symmetric matrix functions of the indeterminate $x \in \mathbb{R}^n$.

Solving these SDMIs means solving an infinite set of LMIs and therefore is computationally hard. As it has been discussed in [111], when only symmetric polynomial matrices are involved, by Proposition 2.5.1, any solution to the SOS optimization problem

$$\text{minimize } \sum_{i=1}^m a_i c_i \quad (2.33)$$

subject to

$$v^T \left[F_0(x) + \sum_{i=1}^m c_i F_i(x) \right] v \in \Sigma_{n+N} \quad (2.34)$$

is also a solution to the SDMIs (2.31)-(2.32). However, (2.33)-(2.34) provide a computational relaxation for (2.31)-(2.32), and the semidefinite programming can be utilized to solve these SDMIs.

Chapter 3

Stability Analysis and Controller Synthesis for Parameter-Dependent Polynomial Nonlinear Systems

In the analysis of nonlinear systems, the construction of Lyapunov functions and the verification of the non-negativity of the Lyapunov conditions is a complex task. However, a new computationally tractable analysis method provides a new way of searching for SOS decomposition of polynomials to relax the original problem.

In this chapter, the stability analysis and state feedback synthesis problem is proposed for parameter-dependent nonlinear systems with polynomial vector fields. Besides the analysis of the closed-loop stability for the parameter-dependent system, a fixed state feedback controller is designed to enlarge the region of attraction for the system. The proposed stability analysis and state feedback controller synthesis problem is converted to a SOS based optimization problem which can be solved via the semidefinite programming. In order to reduce the conservativeness involved in the controller design, a parameter-dependent Lyapunov function instead of a fixed one is adopted.

3.1 Problem Formulation

Consider the following parameter-dependent nonlinear system

$$\dot{x} = f(x; \theta) + g(x; \theta)u_c \quad (3.1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u_c \in \mathbb{R}^m$ is the control input, and the uncertain parameter $\theta = [\theta_1 \ \theta_2]^T \in \mathbb{R}^2$ are constants that satisfy

$$\theta \in \Theta \triangleq \left\{ \theta \in \mathbb{R}^2 : \theta_1 \geq 0, \theta_2 \geq 0 \text{ and } \theta_1 + \theta_2 = 1 \right\} \quad (3.2)$$

The polynomial functions $f(x; \theta)$ and $g(x; \theta)$ are in the form of

$$\begin{aligned} f(x; \theta) &= f_1(x)\theta_1 + f_2(x)\theta_2 \\ g(x; \theta) &= g_1(x)\theta_1 + g_2(x)\theta_2 \end{aligned} \quad (3.3)$$

where $f_1(x), f_2(x) \in \mathcal{R}_n^n$ and $g_1(x), g_2(x) \in \mathcal{R}_n^{n \times m}$ are known polynomial vector fields of appropriate dimensions satisfying $f_1(0) = 0$ and $f_2(0) = 0$.

We assume that the equilibrium point $\bar{x} = 0$ does not depend on parameter θ , i.e., $f(\bar{x}; \theta) = 0, \forall \theta \in \Theta$. If we allow the control input to be generated by a state feedback controller

$$u_c = k(x) \in \mathcal{R}_n^m \text{ with } k(0) = 0 \quad (3.4)$$

we have the following closed-loop system

$$\dot{x} = f(x; \theta) + g(x; \theta)k(x) \quad (3.5)$$

The objective is to find a state feedback control law (3.4) which stabilizes the closed-

loop system and enlarges the set of the points that are attracted to the origin point.

3.2 Stabilizing State Feedback Design

The stability analysis and state feedback synthesis problem is proposed based on the Lyapunov stability theorem. In common nonlinear feedback design methods for the uncertain systems (3.1)-(3.3), one fixed Lyapunov function for all the uncertain systems are usually adopted. In our methodology, the state and parameter dependent Lyapunov functions (3.6) are utilized so that the conservatism involved in the design can be reduced.

$$V(x; \theta) = V_1(x)\theta_1 + V_2(x)\theta_2 \quad \theta \in \Theta \quad (3.6)$$

Stabilization Problem: Inspired by [58], we define two regions for a given parameter-dependent system of the form (3.5). One region is of a variable size,

$$\mathcal{O}_\beta := \{x \in \mathbb{R}^n | p(x) \leq \beta\} \quad (3.7)$$

where $\beta > 0$ is the “radius” of $\mathcal{O}_\beta$, and $p(x)$ is a known positive definite polynomial, independent of θ . The other region is

$$\Omega := \{x \in \mathbb{R}^n | V(x; \theta) \leq 1, \forall \theta \in \Theta\} \quad (3.8)$$

with $V(x; \theta)$ being an unknown Lyapunov function of the form (3.6).

Our objective is to search for a parameter-dependent Lyapunov function (3.6) and a polynomial state feedback control law (3.4) to maximize β such that,

$$V(x; \theta) > 0, \forall x \in \mathbb{R}^n \setminus \{0\} \text{ and } V(0) = 0, \forall \theta \in \Theta \quad (3.9)$$

$$\mathcal{O}_\beta \subseteq \Omega \quad (3.10)$$

$$\Omega \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | \dot{V}(x; \theta) < 0, \forall \theta \in \Theta\} \quad (3.11)$$

From the Lyapunov argument, for each fixed $\bar{\theta} \in \Theta$, the region $\Omega_{\bar{\theta}} := \{x \in \mathbb{R}^n | V(x; \bar{\theta}) \leq 1\}$ is an invariant set and also a subset of region of attraction for the system (3.5) with that particular $\bar{\theta}$. Hence every point in the region $\mathcal{O}_\beta$ converges asymptotically to the origin point.

We know that the region Ω is the intersection of $\Omega_{\bar{\theta}}$ over all $\bar{\theta} \in \Theta$, i.e., $\Omega := \bigcap_{\bar{\theta} \in \Theta} \Omega_{\bar{\theta}}$. Even though $\Omega_{\bar{\theta}} := \{x \in \mathbb{R}^n | V(x; \bar{\theta}) \leq 1\}$ is an invariant set for the system (3.5) with that particular $\bar{\theta}$, the intersection Ω is not invariant. However, Ω is a subset of region of attraction for all $\theta \in \Theta$ since every trajectory starting in Ω approaches the origin as $t \rightarrow \infty$.

Lemma 3.2.1 *Consider a parameter-dependent system (3.5) and a fixed positive definite function $p(x) \in \mathcal{R}_n$. If there exist positive definite polynomials $V_1(x), V_2(x) \in \mathcal{R}_n$ with $V_1(0) = 0, V_2(0) = 0$, and $k(x) \in \mathcal{R}_n^m$ with $k(0) = 0$ such that*

$$V_1(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\} \text{ and } V_1(0) = 0 \quad (3.12)$$

$$V_2(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\} \text{ and } V_2(0) = 0 \quad (3.13)$$

$$\{x \in \mathbb{R}^n | p(x) \leq \beta\} \subseteq \{x \in \mathbb{R}^n | V_1(x) \leq 1\} \quad (3.14)$$

$$\{x \in \mathbb{R}^n | p(x) \leq \beta\} \subseteq \{x \in \mathbb{R}^n | V_2(x) \leq 1\} \quad (3.15)$$

$$\{x \in \mathbb{R}^n | V_1(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | M_1(x) < 0\} \quad (3.16)$$

$$\{x \in \mathbb{R}^n | V_1(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | M_2(x) < 0\} \quad (3.17)$$

$$\{x \in \mathbb{R}^n | V_1(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | M_3(x) < 0\} \quad (3.18)$$

$$\{x \in \mathbb{R}^n | V_2(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | M_1(x) < 0\} \quad (3.19)$$

$$\{x \in \mathbb{R}^n | V_2(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | M_2(x) < 0\} \quad (3.20)$$

$$\{x \in \mathbb{R}^n | V_2(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | M_3(x) < 0\} \quad (3.21)$$

where

$$M_1(x) = \nabla V_1(x) [f_1(x) + g_1(x)k(x)] \quad (3.22)$$

$$M_2(x) = \nabla V_2(x) [f_2(x) + g_2(x)k(x)] \quad (3.23)$$

$$M_3(x) = \nabla V_1(x) [f_2(x) + g_2(x)k(x)] + \nabla V_2(x) [f_1(x) + g_1(x)k(x)] \quad (3.24)$$

Then, the conditions (3.9)-(3.11) are satisfied. Hence, for all $\theta \in \Theta$, the region Ω is a subset of region of attraction for the system (3.5), and system (3.5) is asymptotically stable about the origin.

Proof: From (3.6), it is easily seen that (3.12) and (3.13) $\Rightarrow$ (3.9).

Next, we prove that (3.14) and (3.15) $\Rightarrow$ (3.10). From the conditions (3.14) and (3.15), we know that

$$\{x \in \mathbb{R}^n | p(x) \leq \beta\} \subseteq \{x \in \mathbb{R}^n | V_1(x) \leq 1, V_2(x) \leq 1\} \quad (3.25)$$

Then, with (3.6), for any given $x \in \mathbb{R}^n$

$$V_1(x) \leq 1, V_2(x) \leq 1 \Rightarrow V(x; \theta) \leq 1 \quad (3.26)$$

So, we have

$$\{x \in \mathbb{R}^n | V_1(x) \leq 1, V_2(x) \leq 1\} \subseteq \{x \in \mathbb{R}^n | V(x; \theta) \leq 1, \forall \theta \in \Theta\} \quad (3.27)$$

From (3.25) and (3.27) we have

$$\{x \in \mathbb{R}^n | p(x) \leq \beta\} \subseteq \{x \in \mathbb{R}^n | V(x; \theta) \leq 1, \forall \theta \in \Theta\} \quad (3.28)$$

and the condition (3.10) holds.

Finally, we prove that (3.16) - (3.21) $\Rightarrow$ (3.11). From the conditions (3.16)-(3.18), we know

$$\{x \in \mathbb{R}^n | V_1(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | M_1(x) < 0, M_2(x) < 0, M_3(x) < 0\} \quad (3.29)$$

Because for any given $x \in \mathbb{R}^n$

$$M_1(x) < 0, M_2(x) < 0, M_3(x) < 0 \Rightarrow \dot{V}(x; \theta) < 0, \forall \theta \in \Theta \quad (3.30)$$

We have

$$\{x \in \mathbb{R}^n | M_1(x) < 0, M_2(x) < 0, M_3(x) < 0\} \subseteq \{x \in \mathbb{R}^n | \dot{V}(x; \theta) < 0, \forall \theta \in \Theta\} \quad (3.31)$$

From (3.29) and (3.31), we have

$$\{x \in \mathbb{R}^n | V_1(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | \dot{V}(x; \theta) < 0, \forall \theta \in \Theta\} \quad (3.32)$$

Similarly from (3.19)-(3.21) we have

$$\{x \in \mathbb{R}^n | V_2(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | \dot{V}(x; \theta) < 0, \forall \theta \in \Theta\} \quad (3.33)$$

Now we will prove (3.32) and (3.33) $\Rightarrow$ (3.11). With (3.32) and (3.33),

$$\{x \in \mathbb{R}^n | V_1(x) \leq 1\} \cup \{x \in \mathbb{R}^n | V_2(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | \dot{V}(x; \theta) < 0, \forall \theta \in \Theta\} \quad (3.34)$$

We also know that

$$\{x \in \mathbb{R}^n | V(x; \theta) \leq 1, \forall \theta \in \Theta\} \subseteq \{x \in \mathbb{R}^n | V_1(x) \leq 1\} \cup \{x \in \mathbb{R}^n | V_2(x) \leq 1\} \quad (3.35)$$

So, from (3.34) and (3.35) we have, for each $\theta \in \Theta$

$$\{x \in \mathbb{R}^n | V(x; \theta) \leq 1, \forall \theta \in \Theta\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | \dot{V}(x; \theta) < 0, \forall \theta \in \Theta\} \quad (3.36)$$

and the condition (3.11) holds. ■

Remark 3.2.1 Lemma 3.2.1 provides sufficient conditions for (3.9)-(3.11) to achieve the closed-loop stability via a state feedback controller for the parameter-dependent system (3.5). The estimation of the regions of attraction for the systems with different $\bar{\theta}$ is realized by maximizing the region $\mathcal{O}_\beta$, i.e., maximizing the “radius” β over the polynomial functions $V_1(x)$, $V_2(x)$ and state feedback controller $k(x)$. So, finally the stabilization problem becomes

Maximize β

subject to

$$\begin{aligned} & k(x) \in \mathcal{R}_n^m \text{ with } k(0) = 0 \\ & V_1(x), V_2(x) \in \mathcal{R}_n > 0 \text{ and } V_1(0) = 0, V_2(0) = 0 \\ & \text{conditions in (3.12) – (3.21)} \end{aligned}$$

Before proceeding to the application of the Positivstellensatz, we formulate the sufficient

conditions (3.12)-(3.21) as emptiness constraints as follows:

$$\{x \in \mathbb{R}^n | V_1(x) \leq 0, x \neq 0\} = \emptyset \quad (3.37)$$

$$\{x \in \mathbb{R}^n | V_2(x) \leq 0, x \neq 0\} = \emptyset \quad (3.38)$$

$$\{x \in \mathbb{R}^n | p(x) \leq \beta, V_1(x) \geq 1, V_1(x) \neq 1\} = \emptyset \quad (3.39)$$

$$\{x \in \mathbb{R}^n | p(x) \leq \beta, V_2(x) \geq 1, V_2(x) \neq 1\} = \emptyset \quad (3.40)$$

$$\{x \in \mathbb{R}^n | V_1(x) \leq 1, M_1(x) \geq 0, x \neq 0\} = \emptyset \quad (3.41)$$

$$\{x \in \mathbb{R}^n | V_1(x) \leq 1, M_2(x) \geq 0, x \neq 0\} = \emptyset \quad (3.42)$$

$$\{x \in \mathbb{R}^n | V_1(x) \leq 1, M_3(x) \geq 0, x \neq 0\} = \emptyset \quad (3.43)$$

$$\{x \in \mathbb{R}^n | V_2(x) \leq 1, M_1(x) \geq 0, x \neq 0\} = \emptyset \quad (3.44)$$

$$\{x \in \mathbb{R}^n | V_2(x) \leq 1, M_2(x) \geq 0, x \neq 0\} = \emptyset \quad (3.45)$$

$$\{x \in \mathbb{R}^n | V_2(x) \leq 1, M_3(x) \geq 0, x \neq 0\} = \emptyset \quad (3.46)$$

Applying the Positivstellensatz in theorem 2.3.1 and following the simplifying procedure in [58], the stabilization problem can be formulated as the following SOS optimization problem which can be solved via the semidefinite programming.

SOS Optimization Problem:

$$\begin{aligned} \max \quad & \beta \quad \text{over} \quad k(x) \in \mathcal{R}_n^m, V_1(x), V_2(x) \in \mathcal{R}_n \\ & \text{and } V_1(0) = 0, V_2(0) = 0, s_i(x) \in \Sigma_n \end{aligned}$$

subject to

$$V_1(x) - l_1(x) \in \Sigma_n \quad (3.47)$$

$$V_2(x) - l_2(x) \in \Sigma_n \quad (3.48)$$

$$-\{s_1(x) [\beta - p(x)] + [V_1(x) - 1]\} \in \Sigma_n \quad (3.49)$$

$$-\{s_2(x) [\beta - p(x)] + [V_2(x) - 1]\} \in \Sigma_n \quad (3.50)$$

$$-\{s_3(x) [1 - V_1(x)] + s_9(x)M_1(x) + l_3(x)\} \in \Sigma_n \quad (3.51)$$

$$-\{s_4(x) [1 - V_1(x)] + s_{10}(x)M_2(x) + l_4(x)\} \in \Sigma_n \quad (3.52)$$

$$-\{s_5(x) [1 - V_1(x)] + s_{11}(x)M_3(x) + l_5(x)\} \in \Sigma_n \quad (3.53)$$

$$-\{s_6(x) [1 - V_2(x)] + s_{12}(x)M_1(x) + l_6(x)\} \in \Sigma_n \quad (3.54)$$

$$-\{s_7(x) [1 - V_2(x)] + s_{13}(x)M_2(x) + l_7(x)\} \in \Sigma_n \quad (3.55)$$

$$-\{s_8(x) [1 - V_2(x)] + s_{14}(x)M_3(x) + l_8(x)\} \in \Sigma_n \quad (3.56)$$

where $M_1(x)$, $M_2(x)$ and $M_3(x)$ are as in (3.22)-(3.24), $l_j(x) \in \Sigma_n > 0$, $j = 1, \dots, 8$, $l_j(0) = 0$ and $s_i(x) \in \Sigma_n$, $i = 1, \dots, 14$.

Remark 3.2.2 (a) It is noted that finding the Lyapunov functions $V_1(x)$, $V_2(x)$ and state feedback controller $k(x)$ that satisfy (3.47)-(3.56) together is a nonconvex problem in general because of the bilinear conditions in $V_1(x)$, $V_2(x)$ and $s_i(x)$, and the trilinear conditions in $V_1(x)$, $V_2(x)$, $s_i(x)$ and $k(x)$. An iterative algorithm is used here to make this non-convex problem convex. The stability constraints (3.47)-(3.56) are checked by SOS programming. (b) We supply feasible initial Lyapunov functions $V_1(x)$, $V_2(x)$ and controller $k(x)$ over which the iterative algorithm improves the value of β to find optimal solutions. A bad choice of the initial point may render unsatisfactory results. (c) Since $s_i(x) \in \Sigma_n$ are SOS polynomials, they must be of even degrees [42]. Additionally the degrees of the polynomials that appear in the SOS problem must satisfy some polynomial degree constraints [58] to ensure that (3.47)-(3.56) are SOS.

Algorithm 3.2.1 Choose the degrees of Lyapunov functions $V_1(x)$, $V_2(x)$, controller $k(x)$, the SOS polynomials $s_i(x)$, and positive definite polynomials $l_j(x)$. Before the iteration procedure, we find a linear initial controller $k^{[0]}(x)$ and quadratic Lyapunov functions $V_1^{[0]}(x)$, $V_2^{[0]}(x)$ by solving the linearized version of the stability problem. For the k^{th} iteration,

- (1) *SOS polynomials iteration:* Set $V_1(x) = V_1^{[k-1]}(x)$, $V_2(x) = V_2^{[k-1]}(x)$ and $k(x) = k^{[k-1]}(x)$. Solve for SOS polynomials $s_i^{[k]}(x)$, $i = 3, \dots, 14$ subject to the constraints in (3.51)-(3.56).
- (2) *Controller iteration:* Set $V_1(x) = V_1^{[k-1]}(x)$, $V_2(x) = V_2^{[k-1]}(x)$ and $s_i(x) = s_i^{[k]}(x)$, $i = 9, \dots, 14$. Solve for controller $k^{[k]}(x)$ subject to the constraints in (3.51)-(3.56).
- (3) *SOS polynomials iteration:* Set $V_1(x) = V_1^{[k-1]}(x)$, $V_2(x) = V_2^{[k-1]}(x)$. Solve the linesearch on β for $s_1^{[k]}(x)$ and $s_2^{[k]}(x)$ subject to the constraints in (3.49) and (3.50).
- (4) *Lyapunov functions iteration:* Set $k(x) = k^{[k]}(x)$ and $s_i(x) = s_i^{[k]}(x)$, $i = 1, \dots, 14$. Solve the linesearch on β for $V_1^{[k]}(x)$, $V_2^{[k]}(x)$ and $\beta^{[k]}$ subject to the constraints in (3.47) - (3.56).
- (5) If $\beta^{[k]} - \beta^{[k-1]}$ is less than a specified tolerance, the iteration concludes, else increase the index and go to step (1).

Remark 3.2.3 The system (3.1)-(3.3) under consideration is a special class of affine parameter-dependent systems. The above results for two parameters $[\theta_1, \theta_2]$ can be generalized easily to l parameters with $\theta = [\theta_1, \theta_2, \dots, \theta_l]^T \in \mathbb{R}^l$ being constants that satisfy

$$\theta \in \Theta \triangleq \left\{ \theta \in \mathbb{R}^l : \theta_1 \geq 0, \dots, \theta_l \geq 0 \text{ and } \sum_{i=1}^l \theta_i = 1 \right\} \quad (3.57)$$

The system functions $f(x; \theta)$ and $g(x; \theta)$ in (3.1) are in the form of

$$\begin{aligned} f(x; \theta) &= \sum_{i=1}^l f_i(x) \theta_i \\ g(x; \theta) &= \sum_{i=1}^l g_i(x) \theta_i \end{aligned} \quad (3.58)$$

where $f_1(x), \dots, f_l(x) \in \mathcal{R}_n^n$ and $g_1(x), \dots, g_l(x) \in \mathcal{R}_n^{n \times m}$ are known polynomial vector fields of appropriate dimensions satisfying $f_1(0) = 0, \dots, f_l(0) = 0$. Similarly, the following parameter-dependent Lyapunov function (instead of a fixed Lyapunov function) is adopted.

$$V(x; \theta) = \sum_{i=1}^l V_i(x) \theta_i, \quad \theta \in \Theta \quad (3.59)$$

Then for the stabilization problem proposed above, the corresponding Lemma 3.2.2 can be derived.

Lemma 3.2.2 Consider a parameter-dependent system (3.57)-(3.58) and a fixed positive definite function $p(x) \in \mathcal{R}_n$. If there exist positive definite polynomials $V_k(x) \in \mathcal{R}_n$ with $V_k(0) = 0, k = 1, \dots, l$, and $k(x) \in \mathcal{R}_n^m$ with $k(0) = 0$ such that for $i, j = 1, \dots, l$ and $i \neq j$

$$V_k(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\} \text{ and } V_k(0) = 0 \quad (3.60)$$

$$\{x \in \mathbb{R}^n | p(x) \leq \beta\} \subseteq \{x \in \mathbb{R}^n | V_k(x) \leq 1\} \quad (3.61)$$

$$\{x \in \mathbb{R}^n | V_k(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | M_i(x) < 0\} \quad (3.62)$$

$$\{x \in \mathbb{R}^n | V_k(x) \leq 1\} \setminus \{0\} \subseteq \{x \in \mathbb{R}^n | N_{i,j}(x) < 0\} \quad (3.63)$$

where

$$M_i(x) = \nabla V_i(x) [f_i(x) + g_i(x)k(x)] \quad (3.64)$$

$$N_{i,j}(x) = \nabla V_i(x) [f_j(x) + g_j(x)k(x)] + \nabla V_j(x) [f_i(x) + g_i(x)k(x)] \quad (3.65)$$

Then, the conditions (3.9)-(3.11) in the stabilization problem are satisfied, and the system (3.57)-(3.58) is asymptotically stable about the origin.

Proof: The proof of Lemma 3.2.2 is similar with that of Lemma 3.2.1, so it is omitted here. ■

3.3 State Feedback Design Examples

In this section state feedback design examples are presented to demonstrate the proposed controller design approach in Section 3.2.

3.3.1 Academic Example

Consider a parameter-dependent nonlinear system of the form (3.1)-(3.3) with

$$f_1(x) = \begin{bmatrix} x_2 + x_1^3 \\ -x_1 + x_2^3 \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (3.66)$$

$$f_2(x) = \begin{bmatrix} x_2 + x_1(x_1^2 + 3x_2^2) \\ -x_1 + x_2(3x_1^2 + x_2^2) \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix} \quad (3.67)$$

Using the state feedback design constraints in (3.47)-(3.56) we find a state feedback controller $u = k(x)$ which stabilizes the closed-loop system and enlarges the regions of attraction of the origin point. The software package SOSTOOLS [107, 109] is used to solve the SOS optimization problem via semidefinite programming.

The regions of attraction are estimated by the variable sized region $\mathcal{O}_\beta := \{x \in \mathbb{R}^2 | p(x) \leq \beta\}$, where $p(x) = x^T P x$, $P = [4, 0; 0, 0.01]$. The fixed positive definite polynomials $\{l_j(x)\}_{j=1}^8$ are chosen in the following form of $\varepsilon \sum_{i=1}^2 x_i^d$ with some small constant $\varepsilon > 0$ and d is the maximum degree of the corresponding polynomial. Here we are interested in deriving a nonlinear state feedback controller, so we increase the degree of controller $k(x)$ to 3. We supply the following feasible initial Lyapunov functions and controller

candidate:

$$V_1(x) = 2.7248x_1^2 + 0.4622x_1x_2 + 0.2253x_2^2 \quad (3.68)$$

$$V_2(x) = 2.7248x_1^2 + 0.4622x_1x_2 + 0.2253x_2^2 \quad (3.69)$$

$$k(x) = -5.5459x_1 - 5.4077x_2 \quad (3.70)$$

Finally we have the maximum size of the region $\mathcal{O}_\beta$ with the value of $\beta = 0.9629$. The state feedback controller is

$$k_{nl}(x) = -194.6591x_1 - 326.6482x_2 - 5.566x_1^3 - 7.2421x_1^2x_2 + 3.8058x_1x_2^2 - 78.7766x_2^3 \quad (3.71)$$

and the corresponding Lyapunov functions are

$$V_{nl1}(x) = 3.6073x_1^2 + 0.016704x_1x_2 + 0.010125x_2^2 \quad (3.72)$$

$$V_{nl2}(x) = 3.0757x_1^2 + 0.011561x_1x_2 + 0.010106x_2^2 \quad (3.73)$$

The simulation results, i.e., the estimated regions of attraction $\{x \in \mathbb{R}^2 | V(x; \theta) \leq 1\}$ for various values of $0 \leq \theta \leq 1$ are shown in Figure 3.1. Note that $\{x \in \mathbb{R}^2 | p(x) \leq \beta\}$ is contained in these regions of attraction. For comparison, we use parameter-independent Lyapunov function $V(x)$ to obtain $\beta = 0.8824$, which is smaller than that from parameter-dependent Lyapunov function $V(x; \theta)$. The corresponding estimated region of attraction is shown in Figure 3.2. Figure 3.3 shows the closed-loop state trajectories of the system with three initial conditions: one point in the stable region in Figure 3.1 $x_0 = [0.2, 0.5]^T$, one point at the fringe of the stable region $x_0 = [0.5474, -0.4]^T$ and one point outside the stable region $x_0 = [1, 2.5]^T$, where the parameter θ is fixed as $[0.5, 0.5]^T$. Simulations show that the origin of the closed-loop system is locally

asymptotically stable, and the stabilizing feedback controller derived by the proposed state feedback design enlarges the the region of the points that are attracted to the origin, as shown in Figure 3.1.

Figure 3.1: Regions of attraction of the closed-loop systems with nonlinear feedback controller (3.71), using $V(x; \theta)$

Figure 3.2: Region of attraction of the closed-loop systems using $V(x)$

Figure 3.3: Closed-loop state trajectories x_1 and x_2

Remark 3.3.1 For the feedback controller $u = k(x) \in \mathcal{R}_n^m$, a bound $\underline{r}_i \leq k_i(x) \leq \bar{r}_i$, $i = 1, \dots, m$ can also be incorporated into the controller design. This is accomplished by appending the following conditions to (3.9)-(3.11) such that for $\forall \theta \in \Theta$

$$\{x \in \mathbb{R}^n | V(x; \theta) \leq 1\} \subseteq \{x \in \mathbb{R}^n | k_i(x) \leq \bar{r}_i\} \quad (3.74)$$

$$\{x \in \mathbb{R}^n | V(x; \theta) \leq 1\} \subseteq \{x \in \mathbb{R}^n | k_i(x) \geq \underline{r}_i\} \quad (3.75)$$

Using the Positivstellensatz again, we obtain constraints as follows:

$$[\bar{r}_i - k_i(x)] - [1 - V_1(x)] s_{15}(x) \in \Sigma_n \quad (3.76)$$

$$[\bar{r}_i - k_i(x)] - [1 - V_2(x)] s_{16}(x) \in \Sigma_n \quad (3.77)$$

$$[k_i(x) - \underline{r}_i] - [1 - V_1(x)] s_{17}(x) \in \Sigma_n \quad (3.78)$$

$$[k_i(x) - \underline{r}_i] - [1 - V_2(x)] s_{18}(x) \in \Sigma_n \quad (3.79)$$

Then the controller design algorithm is similar with the Algorithm 3.2.1 with the addi-

tional constraints (3.76)-(3.79).

3.3.2 Aircraft Application Example

A design example for nonlinear F-8 aircraft model is presented to demonstrate the proposed controller design approach in Section 3.2. Here we consider the aircraft models in normal operation and in the operation with 25% loss of control surface impairment at the elevator.

The F-8 Aircraft Dynamical Model

The candidate nonlinear aircraft model for this study is the F-8 aircraft longitudinal flight dynamics which consists of both phugoid and short period modes. Ignoring drag, the basic nonlinear equations describing the longitudinal flight dynamics are used [147]

$$\dot{u} = -uq \tan \alpha - g \sin \vartheta + \frac{L_w}{m} \sin \alpha + \frac{L_t}{m} \sin \alpha_t \quad (3.80)$$

$$\dot{\alpha} = q + \frac{g}{u} \cos \alpha \cos(\alpha - \vartheta) - \frac{L_w}{um} \cos \alpha - \frac{L_t}{um} \cos \alpha \cos(\alpha - \alpha_t) \quad (3.81)$$

$$\dot{\vartheta} = q \quad (3.82)$$

$$\dot{q} = (M_w + lL_w \cos \alpha - l_t L_t \cos \alpha_t - cq)/I_y \quad (3.83)$$

where

$$\alpha_t = (1 - a_e)\alpha + \delta_e$$

$$L_w = C_L(\alpha)\bar{q}S$$

$$L_t = C_{L_t}(\alpha_t, \delta_e)\bar{q}S_t$$

$$\bar{q} = \frac{\rho u^2}{2 \cos^2 \alpha}$$

The nomenclature is as follows:

- u : forward speed (ft/s);
- α : wing angle of attack (rad);
- ϑ : pitch angle (rad);
- q : pitch rate (rad/s);
- α_t : tail angle of attack (rad);
- δ_e : elevator angle (rad);
- m : mass of aircraft ($slugs$);
- I_y : moment of inertia of aircraft about Y axis ($slugs\ ft^2$);
- L_w, L_t : wing and tail lifts (lb);
- M_w : wing moment;
- l : distance between wing a.c. and aircraft c.g. (ft);
- l_t : distance between tail a.c. and aircraft c.g. (ft);
- $c\dot{\vartheta}$: damping moment ($slugs\ ft^2$);
- C_L, C_{L_t} : wing and tail lift coefficients;
- $\bar{q}$: dynamic pressure (lb/ft^2);
- S, S_t : wing and tail area (ft^2);
- ρ : atmospheric density ($slugs/ft^3$).

Cubic approximation is used for the lift coefficient curves

$$C_L(\alpha) = (C_L^1\alpha - C_L^2\alpha^3)$$

$$C_{L_t}(\alpha_t, \delta_e) = (C_{L_t}^1\alpha_t - C_{L_t}^2\alpha_t^3 + a_e\delta_e)$$

3.3 State Feedback Design Examples

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Consider an altitude of $30,000\text{ft}$ (i.e., with $\rho = 0.00089\text{ slug/ft}^3$ and a speed of sound of 994.85 ft/s), and a level unaccelerated flight at $Mach = 0.85$. The system coefficients for the aircraft model in normal operation are taken as follows:

$$\begin{aligned} C_L^1 &= 4.0, & C_L^2 &= 12, & a_e &= 0.1, & a_{\dot{e}} &= 0.75, \\ S &= 375\text{ft}^2, & S_t &= 93.4\text{ft}^2, & m &= 667.7\text{slugs}, \\ I_y &= 96800\text{slug ft}^2, & l &= 0.189\text{ft}, & l_t &= 16.7\text{ft}, \\ M_w &= 0\text{ lb ft}, & c &= 38332.8\text{ lb ft s}, & g &= 32.2\text{ft/s}^2 \end{aligned}$$

With C_L^1 and C_L^2 given as above, the stall angle of attack at the wing and at the tail can be calculated easily to be about 19.1 deg ($1/3\text{ rad}$). An elevator deflection limit of 25 deg and elevator rate limit of 100 deg/s are applied.

Control surface faults are commonly seen in fighter aircraft. The usual control surface fault is the control surface impairment which will change the aerodynamic characteristics of the aircraft. Control surface impairment can be characterized by the percentage loss of the total control surface area.

With the system coefficients, the trim conditions for the aircraft models in normal operation and in the operation with 25% loss of control surface can be calculated as follows respectively:

Table 3.1: Operating points

	Nominal	25% loss
$u_{trim}(\text{ft/s})$	845	845
$q_{trim}(\text{rad/s})$	0	0
$\alpha_{trim}(\text{rad})$	0.04478619	0.044786176
$\vartheta_{trim}(\text{rad})$	0.04431	0.04432
$\delta_{e_{trim}}(\text{rad})$	-0.008952	-0.008295

Substituting the new variables $x_1 = \alpha - \alpha_{trim}$, $x_2 = \vartheta - \vartheta_{trim}$, $x_3 = q$ as the states, $u_c = \delta_e - \delta_{e_{trim}}$ as the control input and airspeed $u = \text{const}$ into (3.80)-(3.83), and using

the system coefficients, we obtain the following short period aircraft models in normal operation and in operation with 25% loss of control surface:

Aircraft model in normal operation:

$$\dot{x} = f_1(x) + g_1(x)u_c \quad (3.84)$$

with

$$f_1(x) = \begin{bmatrix} -0.878x_1 + x_3 - x_1^2x_3 - 0.0896x_1x_3 - 0.019x_2^2 + 0.473x_1^2 + 3.813x_1^3 \\ x_3 \\ -4.209x_1 - 0.396x_3 - 0.408x_1^2 - 2.137x_1^3 \end{bmatrix}$$

$$g_1(x) = \begin{bmatrix} -0.216 \\ 0 \\ -20.991 \end{bmatrix} \quad (3.85)$$

Aircraft model with 25% loss of control surface:

$$\dot{x} = f_2(x) + g_2(x)u_c \quad (3.86)$$

with

$$f_2(x) = \begin{bmatrix} -0.865x_1 + x_3 - x_1^2x_3 - 0.0896x_1x_3 - 0.019x_2^2 + 0.473x_1^2 + 3.81x_1^3 \\ x_3 \\ -2.929x_1 - 0.396x_3 - 0.409x_1^2 - 2.417x_1^3 \end{bmatrix}$$

$$g_2(x) = \begin{bmatrix} -0.162 \\ 0 \\ -15.742 \end{bmatrix} \quad (3.87)$$

For loss of control surface between 0% and 25%, interpolations of the above $f_1(x)$ and $f_2(x)$, and $g_1(x)$ and $g_2(x)$ can be used to approximate the intermediate models as

follows:

$$\begin{aligned} f(x; \theta) &= f_1(x)\theta_1 + f_2(x)\theta_2 \\ g(x; \theta) &= g_1(x)\theta_1 + g_2(x)\theta_2 \end{aligned} \quad (3.88)$$

where $f_1(x), f_2(x) \in \mathcal{R}_3^3$ and $g_1(x), g_2(x) \in \mathcal{R}_3^{3 \times 1}$ are as in (3.85) and (3.87) with $f_1(0) = 0$ and $f_2(0) = 0$, which represent the vertices of possible control surface impairment for aircraft model. The parameter $\vartheta = [\vartheta_1 \ \vartheta_2]^T \in \mathbb{R}^2$, which provides the interpolation between the two vertices, is constant and satisfies (3.2). As a result, the nonlinear aircraft model with control surface impairment faults between 0% and 25% can be described by an affine parameter-dependent system of the form as in (3.1).

State Feedback Design and Simulation Results

We supply feasible initial Lyapunov functions $V_1(x), V_2(x)$ and controller $k(x)$ in (3.89) over which the iterative Algorithm 3.2.1 improves the value of β to find optimal solutions.

$$\begin{aligned} V_1(x) &= 0.14x_1^2 + 0.3286x_2^2 + 0.0016x_3^2 - 0.1538x_1x_2 - 0.0022x_1x_3 + 0.0048x_2x_3 \\ V_2(x) &= 0.14x_1^2 + 0.3286x_2^2 + 0.0016x_3^2 - 0.1538x_1x_2 - 0.0022x_1x_3 + 0.0048x_2x_3 \\ k(x) &= 1.7042x_1 + 7.4162x_2 + 7.4365x_3 \end{aligned} \quad (3.89)$$

The regions of attraction are estimated by the variable sized region $\mathcal{O}_\beta := \{x \in \mathbb{R}^2 | p(x) \leq \beta\}$, where $p(x) = x^T P x$, $P = [4, -0.1, 0.03; -0.1, 0.59, 0; 0.03, 0, 0.05]$. The fixed positive definite polynomials $\{l_j(x)\}_{j=1}^8$ are chosen in the following form of $\varepsilon \sum_{i=1}^3 x_i^d$ with some small constant $\varepsilon > 0$ and d is the maximum degree of the corresponding polynomials $l_j(x)$.

By solving the SOS optimization problem in Section 3.2, the maximum size of the region $\mathcal{O}_\beta$ is determined with 16 iterations for β to obtain the final value of $\beta = 1.7662$. The iterations for β can be seen in Figure 3.4. The state feedback controller that guarantees the local stability with optimized ROA is

$$\begin{aligned} k_{nl}(x) = & -0.60357x_1 + 0.51918x_2 + 2.6414x_3 + 1.1853x_1^2 - 0.0039012x_2^2 \\ & + 0.15242x_3^2 - 0.2542x_1x_2 - 0.58084x_1x_3 + 0.15914x_2x_3 + 8.0966x_1^3 \\ & + 0.07795x_2^3 + 0.25808x_3^3 - 1.9417x_1^2x_2 - 4.2947x_1^2x_3 + 0.5879x_1x_2^2 \\ & + 0.34048x_1x_3^2 + 0.077199x_2^2x_3 - 0.13242x_2x_3^2 + 0.69357x_1x_2x_3 \end{aligned} \quad (3.90)$$

and the corresponding Lyapunov functions are

$$\begin{aligned} V_{nl1}(x) = & 2.2536x_1^2 + 0.26504x_2^2 + 0.01307x_3^2 - 0.058304x_1x_2 \\ & + 0.02543x_1x_3 + 0.01389x_2x_3 \end{aligned} \quad (3.91)$$

$$\begin{aligned} V_{nl2}(x) = & 2.2456x_1^2 + 0.28935x_2^2 + 0.017715x_3^2 - 0.16074x_1x_2 \\ & + 0.059581x_1x_3 + 0.021701x_2x_3 \end{aligned} \quad (3.92)$$

The estimated regions of attraction $\{x \in \mathbb{R}^2 | V(x; \theta) \leq 1\}$ for various θ as computed using SOS optimization are shown in Figure 3.5. Note that $\{x \in \mathbb{R}^2 | p(x) \leq \beta\}$ is contained in these regions of attraction.

A plot of the response of the closed-loop F-8 aircraft model (3.5) with initial condition $x(0) = [0.5, 0.5, 0.5]^T$, which is within the estimated stability region, is shown in Figure 3.6. In the simulation the uncertain parameter vector θ is assumed to be fixed as $[0.5, 0.5]^T$. In order to test the validity of the computed attraction regions, we also initialize the states of the nonlinear closed-loop F-8 aircraft model outside the estimated region of attraction. The simulation result is shown in Figure 3.7. The initial conditions

Figure 3.4: Iterations of β

Figure 3.5: Regions of attraction of the closed-loop systems with nonlinear feedback controller (3.90)

are simultaneously perturbed in all states.

Figure 3.6: Response of closed-loop system with states initialized within the estimated stability region, $\theta = [0.5, 0.5]^T$

From Figures 3.6 and 3.7, we can see that the nonlinear feedback controller (3.90) can stabilize the nonlinear aircraft model, even when the initial states are outside the estimated region of attraction by SOS optimization. Similar simulation results are obtained when the uncertain parameter vector θ is fixed as other values. The apparent conservativeness of the estimated region of attraction is due to the fact that Lemma 3.2.1 is only a sufficient condition, and that we are fitting a predetermined shape $p(x)$ to the ROA.

The closed-loop system responses of the nonlinear controller (3.90) derived from SOS optimization are compared with that of the third order controller in [36]. The responses of the states α , ϑ and q are shown in Figure 3.8 for the third order controller in [36], and in Figures 3.9 and 3.10 for the proposed nonlinear controller (3.90). Here, in computer simulation, the initial conditions for angle of attack are set to the largest value that stabilization can be achieved, while pitch angle and pitch rate are fixed as zeros. The largest deviation in angle of attack that the third order controller in [36] can sustain is about 0.4727 rad (27.1 degree). While, the proposed nonlinear controller (3.90) can sustain larger deviations in angle of attack at about 0.721 rad (41.3 degree) and 0.7131 rad (40.8 degree) with parameter θ fixed as $[1, 0]^T$ and $[0, 1]^T$ respectively. In summary,

Figure 3.7: Response of closed-loop system with states initialized outside the estimated stability region, $\theta = [0.5, 0.5]^T$

it can be clearly seen that the proposed nonlinear controller (3.90) performs better than the third order controller in [36] in bringing the aircraft back to trim conditions.

Figure 3.8: Response of closed-loop system with the third order controller in [36]

To verify the robustness of our nonlinear stabilizing feedback control law derived from SOS optimization, simulations using the original nonlinear aircraft dynamics in (3.80)-(3.83) are performed, and the results are analyzed. Simulations are carried out for both

Figure 3.9: Response of closed-loop system with proposed nonlinear controller (3.90), $\vartheta = [1, 0]^T$

Figure 3.10: Response of closed-loop system with proposed nonlinear controller (3.90), $\vartheta = [0, 1]^T$

small and large initial conditions in the angle of attack, which represent small and large deviations from the trim conditions due to disturbances. For nonlinear F-8 aircraft models in operation with 25% loss of control surface, the initial conditions for $\alpha(0)$ are selected as $7.6deg$ (5 deg above its operation point value) and $25.7deg$ (corresponding to the positive largest deviation that can be stabilized). The responses of the states α , ϑ , q and elevator δ_e are shown in Figures 3.11 and 3.12, where input saturations are also considered.

Figure 3.11: State trajectories and elevator deflection, initial condition $\alpha(0) = 7.6deg$

Figure 3.12: State trajectories and elevator deflection, initial condition $\alpha(0) = 25.7deg$

It is noticed that in the 3rd order nonlinear simulation (Figure 3.10), the designed nonlinear controller (3.90) can sustain the deviation in angle of attack up to 0.7131 rad (40.8 deg). However, in the robustness simulation (Figure 3.12) using the original nonlinear model (3.80)-(3.83), the largest deviation is only arrived at 25.7 deg. The main reason for this discrepancy is the fact that the proposed stabilizing controller is derived based on the simplified (3rd order) aircraft models in (3.84) and (3.86).

3.4 Conclusion

This chapter discusses the stability analysis and state feedback design problem for the parameter-dependent nonlinear systems whose dynamics are described by parameter-dependent polynomials. Sufficient conditions to test the stability of the closed-loop system and to enlarge the region of attraction for the system are presented based on the classical Lyapunov stability results. To reduce the conservativeness involved in the controller design, parameter-dependent Lyapunov functions are adopted. Compared with other nonlinear feedback design methods, this approach allows computationally tractable design with parameter-dependent Lyapunov functions and the stabilizing feedback control law can be efficiently computed via semidefinite programming. Finally, the feedback controller design method and corresponding iterative algorithm are applied to the longitudinal model of an F-8 aircraft. Robust stability can be guaranteed for the nonlinear aircraft model in normal operation and in the event of control surface faults. Simulation results show the effectiveness of the proposed method.

Chapter 4

Nonlinear H_∞ Synthesis for Parameter-Dependent Polynomial Nonlinear Systems

In Chapter 3 we have discussed the stability analysis and state feedback design for parameter-dependent polynomial nonlinear systems. Sufficient conditions to test the stability of the closed-loop system have been presented and the synthesis problem of stabilizing feedback controller to enlarge the regions of attraction have been solved by the SOS based optimization. Such optimization methodology provides efficient numerical method to the nonlinear synthesis and at the same time to the analysis of regions of attraction for closed-loop systems.

Nevertheless, there are two disadvantages regarding this methodology that should not be neglected. The first one is that bilinear terms and trilinear terms of the decision variables are involved in the design conditions, which naturally result in an iterative algorithm. Moreover, feasible initial Lyapunov candidate and feedback controller need to be supplied to the proposed iterative design procedure. A bad choice of the initial variables may fail to render satisfactory solution in some cases, though the system may

possess a stabilized controller.

The other disadvantage is that disturbances have not been considered in Chapter 3. As we know, disturbances are widely encountered and often adversely affect performance and stability. Therefore, guaranteeing the ability of the control system to reject uncertain exogenous disturbances is also a very important issue in control system design. In non-linear H_∞ control, it has been turned out that the existence of H_∞ controllers is directly related to some HJEs or HJIs. However, there is no systematic numerical algorithm currently available for the solutions of these HJEs or HJIs.

In view of these disadvantages mentioned above, in this chapter, less conservative sufficient conditions to achieve the closed-loop stability with or without bounded H_∞ performance are formulated in terms of nonlinear matrix inequalities, where polynomial state feedback controllers (instead of the rational controllers) are derived. By introducing additional matrix variables, we succeed in eliminating the coupling between system matrices and the Lyapunov matrix. Therefore the proposed methodology is successfully extended to the synthesis for the parameter-dependent polynomial systems. Parameter and state dependent Lyapunov functions are used to reduce the conservatism involved in the design. The SDMI conditions are formulated as SOS based constraints and solved via the semidefinite programming directly, with no iterative algorithm involved.

4.1 Stability Synthesis

Notation: define Σ_{sos} to be the set of SOS polynomials, ignoring the particular dimension index.

This section presents the stability synthesis result for polynomial nonlinear system without considering any performance. Then the result is extended to the synthesis for the parameter-dependent polynomial systems in Section 4.3.

4.1.1 Problem Formulation

Consider the following input-affine nonlinear time invariant (NLTI) system

$$\dot{x} = f(x) + g(x)u \quad (4.1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input. $f(x) \in \mathcal{R}_n^n$ and $g(x) \in \mathcal{R}_n^{n \times m}$ are known polynomial vector fields of appropriate dimensions satisfying $f(0) = 0$.

We write the system (4.1) in the following state dependent linear-like representation:

$$\dot{x} = A(x)Z(x) + B(x)u \quad (4.2)$$

where $A(x)$ and $B(x)$ are polynomial matrices in x with appropriate dimensions, and $Z(x)$ is an $N \times 1$ vector of monomials in x satisfying the following assumption.

Assumption 4.1.1 $Z(x) = 0$ iff $x = 0$.

Remark 4.1.1 *Assumption 4.1.1 ensures that $x = 0, u = 0$ is an equilibrium point of (4.2). It should be noted that, given $f(x) \in \mathcal{R}_n^n$, the representation $f(x) = A(x)Z(x)$ is highly non-unique. Notice that for any $L(x)$ with $L(x)Z(x) = 0$, $A(x) + L(x)$ can also be used as a representation for $f(x)$. A special case of the representation corresponds to $Z(x) = x$, while at the other extreme, $Z(x)$ can be selected to contain all the monomials in $f(x)$ (in which case, $A(x)$ becomes a constant matrix).*

Let $M(x)$ be a $N \times n$ polynomial matrix whose $(i, j)^{\text{th}}$ entry is given by

$$M_{ij}(x) = \frac{\partial Z_i}{\partial x_j}(x), \quad i = 1, \dots, N, j = 1, \dots, n \quad (4.3)$$

If we allow the control input to be generated by a state feedback controller

$$u = K(x)Z(x) \quad \text{with} \quad K(0) = 0 \quad (4.4)$$

We have the following closed-loop system

$$\dot{x} = [A(x) + B(x)K(x)]Z(x) = \tilde{A}(x)Z(x) \quad (4.5)$$

with

$$\tilde{A}(x) = A(x) + B(x)K(x) \quad (4.6)$$

The objective is to design a state feedback control law (4.4) which stabilizes the closed-loop system (4.5). Here we have not included any performance objective in the synthesis.

4.1.2 Stabilizing State Feedback Design

The methodology presented in this chapter is based on the Lyapunov stability argument. Consider the following Lyapunov function

$$V(x) = Z^T(x)P(x)Z(x) \quad (4.7)$$

where $P(x) = P^T(x) > 0$ is a nonsingular polynomial matrix in x and $Q(x) = P^{-1}(x)$.

Theorem 4.1.1 *Consider system (4.2). If there exist $Q(x) = Q^T(x) > 0$ and $Y(x)$ such that the following nonlinear matrix inequality is satisfied*

$$\begin{aligned} M(x)A(x)Q(x) + Q(x)A^T(x)M^T(x) + M(x)B(x)Y(x) \\ + Y^T(x)B^T(x)M^T(x) + Q(x)\dot{P}(x)Q(x) < 0 \end{aligned} \quad (4.8)$$

then the control law (4.4) stabilizes the system with

$$K(x) = Y(x)Q^{-1}(x) \quad (4.9)$$

Proof: With the closed-loop system matrix $\tilde{A}(x)$ in (4.6) and controller matrix in (4.9), the nonlinear matrix inequality (4.8) is equivalent to

$$Q(x)\tilde{A}^T(x)M^T(x) + M(x)\tilde{A}(x)Q(x) + Q(x)\dot{P}(x)Q(x) < 0 \quad (4.10)$$

Multiplying (4.10) from both sides by $P(x) = Q^{-1}(x)$,

$$\tilde{A}^T(x)M^T(x)P(x) + P(x)M(x)\tilde{A}(x) + \dot{P}(x) < 0 \quad (4.11)$$

from the Lyapunov function (4.7), it is very easy to obtain $\dot{V}(x) < 0$. Then we know that the state feedback control law (4.4) stabilizes the system with (4.9). ■

Remark 4.1.2 When $Y(x)$ and $Q(x)$ are polynomial matrices, $K(x)$ in (4.9) is rational. Although a possible solution to avoid rational controller gain can be obtained with a constant matrix Q , it is obviously conservative. Moreover, when parameters are involved in system matrices, the set of $Q(x)$ and $Y(x)$ satisfying the stability inequality is not jointly convex due to the product of $Q(x)$ by $\tilde{A}(x)$. A more suitable structure is proposed to deal with the non-convex problem by the following sufficient condition.

Lemma 4.1.1 Consider system (4.2). If the following matrix inequality in $G(x)$, $H(x)$ and $Q(x) = Q^T(x) > 0$ possesses a solution shown in (4.12)

$$\begin{bmatrix} [G^T(x)\tilde{A}^T(x)M^T(x) + M(x)\tilde{A}(x)G(x) & * \\ +Q(x)\dot{P}(x)Q(x)] & \\ -Q(x) + G(x) - H^T(x)\tilde{A}^T(x)M^T(x) & -H(x) - H^T(x) \end{bmatrix} < 0 \quad (4.12)$$

then the system is stabilized with the control law (4.4).

Proof: If (4.12) possesses a solution, multiplying the inequality from the left and the right by $[I, -M(x)\tilde{A}(x)]$ and $[I, -M(x)\tilde{A}(x)]^T$, then from the result in Theorem 4.1.1, we know that the system is stabilized with state feedback control law (4.4). ■

In order to obtain the state feedback gain matrix $K(x)$ from the latter inequality (4.12), $G(x)$ is chosen as a constant nonsingular matrix and $H(x)$ is chosen as βG , where β is a positive tuning scalar. Substituting the closed-loop matrix $\tilde{A}(x)$ in (4.6) into the sufficient stability condition (4.12) and denoting $Y(x) = K(x)G$, the main result is obtained.

Theorem 4.1.2 Consider system (4.2). If there exist $Q(x) = Q^T(x) > 0$, $Y(x)$, and constant nonsingular matrix G such that, for a positive tuning scalar β , the nonlinear matrix inequality (4.13) is satisfied

$$\Gamma(x) = \begin{bmatrix} [M(x)A(x)G + G^T A^T(x)M^T(x) \\ +M(x)B(x)Y(x) + Y^T(x)B^T(x)M^T(x) \\ +Q(x)\dot{P}(x)Q(x)] & * \\ [-Q(x) + G - \beta G^T A^T(x)M^T(x) \\ -\beta Y^T(x)B^T(x)M^T(x)] & -\beta(G + G^T) \end{bmatrix} < 0 \quad (4.13)$$

then the control law (4.4) stabilizes the system with

$$K(x) = Y(x)G^{-1} \quad (4.14)$$

Remark 4.1.3 (a) The advantage of (4.13) is that it separates the system matrix and Lyapunov matrix, and it only involves affine matrix variable terms in $Q(x)$, G and $Y(x)$ except for a nonlinear term $Q(x)\dot{P}(x)Q(x)$. (b) By choosing G as a constant matrix, it is easy to see that $K(x) = Y(x)G^{-1}$ is a polynomial matrix in x . (c) There is still a

nonlinear term $Q(x)\dot{P}(x)Q(x)$ which leads to the set of matrix variables $Q(x)$, $Y(x)$ and G satisfying the sufficient condition (4.13) is not jointly convex. A more general relaxation for this non-convex problem will be provided in the SOS based optimization.

4.1.3 SOS Based Optimization

The stability condition for the state feedback design is formulated as a nonlinear matrix inequality. Solving this inequality means solving an infinite set of LMIs. By Proposition 2.5.1, when only symmetric polynomial matrices are involved, the SOS decomposition can provide a computational relaxation for the sufficient conditions in Theorem 4.1.2.

Proposition 4.1.1 *Consider the nonlinear system (4.2). If there exist $Q(x) = Q^T(x)$, $Y(x)$, and nonsingular matrix G such that, for a tuning scalar β , a constant $s_1 > 0$ and a SOS polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following expressions are SOS with $\Gamma(x)$ as in (4.13)*

$$v_1^T [Q(x) - s_1 I] v_1 \quad (4.15)$$

$$- \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T (\Gamma(x) + s_2(x)I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (4.16)$$

where $v_1 \in \mathbb{R}^N$ and $v_2 \in \mathbb{R}^N$, then the state feedback stabilization problem is solvable, and the control law (4.4) stabilizes the system with (4.14).

With the definition of SOS polynomial and its decomposition introduced in Chapter 2, the result in Proposition 4.1.1 can be obtained directly, so the proof is omitted here.

Remark 4.1.4 *The nonlinear matrix inequality (4.13) is not convex in the set of matrix variables because of the existence of the nonlinear term $Q(x)\dot{P}(x)Q(x)$. The transformation of this non-convex problem into a convex one needs further discussion.*

Remark 4.1.5 *Instead of assuming system's control and disturbance matrices have some zero rows and Lyapunov matrix $P(\tilde{x})$ only depends on states $\tilde{x}$ whose corresponding rows in control and disturbance matrices are zeroes [111], we define the Lyapunov matrix $P(x)$ that depends on the full states x . Following the technique in [84], a more general relaxation is provided to deal with the nonlinear term in the matrix inequality (4.13).*

Note that

$$\begin{aligned}
 & Q(x)\dot{P}(x)Q(x) \\
 &= Q(x) \left[\sum_{j=1}^n \frac{\partial Q^{-1}(x)}{\partial x_j} \dot{x}_j \right] Q(x) \\
 &= \sum_{j=1}^n Q(x) \frac{\partial Q^{-1}(x)}{\partial x_j} Q(x) \dot{x}_j \\
 &= - \sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} [A_j(x) + B_j(x)K(x)] Z(x)
 \end{aligned} \tag{4.17}$$

where $A_j(x)$ and $B_j(x)$ denotes the j^{th} row of $A(x)$ and $B(x)$, resp. Define

$$\phi_1(x, v_1) = \left[v_1^T \frac{\partial Q(x)}{\partial x_1} v_1, \dots, v_1^T \frac{\partial Q(x)}{\partial x_n} v_1 \right] \tag{4.18}$$

then for the nonlinear term in (4.17), we have

$$\begin{aligned}
 & v_1^T \left[\sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} B_j(x)K(x)Z(x) \right] v_1 \\
 &= \sum_{j=1}^n v_1^T \frac{\partial Q(x)}{\partial x_j} v_1 B_j(x)K(x)Z(x) \\
 &= \left[v_1^T \frac{\partial Q(x)}{\partial x_1} v_1, \dots, v_1^T \frac{\partial Q(x)}{\partial x_n} v_1 \right] B(x)K(x)Z(x) \\
 &= \phi_1(x, v_1)B(x)u(x)
 \end{aligned} \tag{4.19}$$

By imposing a bound on the effect of the nonlinear term as in (4.20) below, we know that if γ_1 in (4.20) has zero minimum, then the nonlinear term $Q(x)\dot{P}(x)Q(x)$ in Proposition 4.1.1 can be replaced by its linear part $-\sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x)Z(x)$.

$$\begin{bmatrix} \gamma_1 & \phi_1(x, v_1)B(x) \\ * & I \end{bmatrix} \geq 0 \quad (4.20)$$

Hence, by including (4.20) as a SOS based constraint, Proposition 4.1.2 follows.

Proposition 4.1.2 *Consider the nonlinear system (4.2). If there exist $Q(x) = Q^T(x)$, $Y(x)$, and nonsingular matrix G such that, for a positive tuning scalars β , a constant $s_1 > 0$ and a SOS polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following optimization problem has zero optimum,*

$$\text{Minimize} \quad \gamma_1$$

subject to

$$v_1^T [Q(x) - s_1 I] v_1 \in \Sigma_{sos} \quad (4.21)$$

$$-\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T (\Upsilon(x) + s_2(x)I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \Sigma_{sos} \quad (4.22)$$

$$v_3^T \begin{bmatrix} \gamma_1 & \phi_1(x, v_1)B(x) \\ * & I \end{bmatrix} v_3 \in \Sigma_{sos} \quad (4.23)$$

where $v_1 \in \mathbb{R}^N$, $v_2 \in \mathbb{R}^N$ and $v_3 \in \mathbb{R}^{m+1}$. $\phi_1(x, v_1)$ is as in (4.18) and

$$\Upsilon(x) = \begin{bmatrix} [M(x)A(x)G + G^T A^T(x)M^T(x) \\ + M(x)B(x)Y(x) + Y^T(x)B^T(x)M^T(x) & * \\ - \sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x)Z(x)] \\ [- Q(x) + G - \beta G^T A^T(x)M^T(x) & -\beta(G + G^T) \\ -\beta Y^T(x)B^T(x)M^T(x) &] \end{bmatrix} \quad (4.24)$$

then the state feedback stabilization problem is solvable, and the control law (4.4) stabilizes the system with (4.14).

4.2 H_∞ Performance Synthesis

In this section, H_∞ performance synthesis result for polynomial nonlinear system is presented. This result is extended to the synthesis for the parameter-dependent polynomial systems in Section 4.3.

4.2.1 Problem Formulation

Consider the following input-affine NLTI system

$$\begin{aligned} \dot{x} &= f(x) + g_u(x)u + g_w(x)w \\ z &= h(x) + l(x)u \end{aligned} \quad (4.25)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^p$ is the exogenous disturbance with bounded energy and $z \in \mathbb{R}^{n_z}$ is the objective signal to be regulated. $f(x) \in \mathcal{R}_n^n$, $g_u(x) \in \mathcal{R}_n^{n \times m}$, $g_w(x) \in \mathcal{R}_n^{n \times p}$, $h(x) \in \mathcal{R}_n^{n_z}$ and $l(x) \in \mathcal{R}_n^{n_z \times m}$

are known polynomial vector fields of appropriate dimensions satisfying $f(0) = 0$.

We write the system (4.25) in the following state dependent linear-like representation:

$$\begin{aligned}\dot{x} &= A(x)Z(x) + B_u(x)u + B_w(x)w \\ z &= C_z(x)Z(x) + D_z(x)u\end{aligned}\tag{4.26}$$

where $A(x)$, $B_u(x)$, $B_w(x)$, $C_z(x)$ and $D_z(x)$ are polynomial matrices in x . Define $M(x)$ and $Z(x)$ as in Section 4.1.

With the state feedback controller (4.4), we have the closed-loop system as follows:

$$\begin{aligned}\dot{x} &= \tilde{A}(x)Z(x) + B_w(x)w \\ z &= \tilde{C}(x)Z(x)\end{aligned}\tag{4.27}$$

with

$$\begin{aligned}\tilde{A}(x) &= A(x) + B_u(x)K(x) \\ \tilde{C}(x) &= C_z(x) + D_z(x)K(x)\end{aligned}\tag{4.28}$$

The objective is to design a state feedback control law (4.4) such that

- The closed-loop system (4.27) is stabilized
- The induced L_2 gain from the exogenous disturbance w to the performance output z is attenuated as

$$\int_0^\infty [z(t)^T z(t)] dt < \gamma^2 \int_0^\infty [w(t)^T w(t)] dt\tag{4.29}$$

4.2.2 H_∞ State Feedback Design

Define the Lyapunov function as in Section 4.1.

Theorem 4.2.1 Consider system (4.26). If there exist $Q(x) = Q^T(x) > 0$ and $Y(x)$ such that the following nonlinear matrix inequality is satisfied

$$\begin{bmatrix} [M(x)A(x)Q(x) + Q(x)A^T(x)M^T(x) \\ +M(x)B_u(x)Y(x) + Y^T(x)B_u^T(x)M^T(x) \\ +Q(x)\dot{P}(x)Q(x)] & * & * \\ B_w^T(x)M^T(x) & -\gamma^2 I & * \\ C_z(x)Q(x) + D_z(x)Y(x) & 0 & -I \end{bmatrix} < 0 \quad (4.30)$$

then the control law (4.4) stabilizes the closed-loop system (4.27) and achieves the H_∞ performance $\|z\|_2 < \gamma\|w\|_2$ with

$$K(x) = Y(x)Q^{-1}(x) \quad (4.31)$$

Proof: With the closed-loop system matrix $\tilde{A}(x)$ and $\tilde{C}(x)$ in (4.28) and controller matrix in (4.31), the nonlinear matrix inequality (4.30) is equivalent to

$$\begin{bmatrix} [Q(x)\tilde{A}^T(x)M^T(x) + M(x)\tilde{A}(x)Q(x) \\ +Q(x)\dot{P}(x)Q(x)] & * & * \\ B_w^T(x)M^T(x) & -\gamma^2 I & * \\ \tilde{C}(x)Q(x) & 0 & -I \end{bmatrix} < 0 \quad (4.32)$$

Multiplying (4.32) from both sides by $\text{diag}\{P(x), I, I\}$, from the Lyapunov function, and by the Schur complement we have $\dot{V} + z^T z - \gamma^2 w^T w < 0$. Then with the zero initial condition, the close-loop system is stablized and the H_∞ performance is achieved as $\|z\|_2 < \gamma\|w\|_2$ with state feedback control law (4.4) with (4.31). ■

As what has been discussed in Remark 4.1.2, a sufficient condition is proposed to deal with the non-convexity problem, which is more suitable for the case when parameters are involved in system matrices.

Lemma 4.2.1 Consider system (4.26). If the following matrix inequality in $G(x)$, $H(x)$ and $Q(x) = Q^T(x) > 0$ possess a solution shown in (4.33)

$$\begin{bmatrix} G^T(x)\hat{A}^T(x) + \hat{A}(x)G(x) + X & * & * \\ \hat{B}^T(x) & -\gamma^2 I & * \\ -\hat{Q}(x) + G(x) - H^T(x)\hat{A}^T(x) & 0 & -H(x) - H^T(x) \end{bmatrix} < 0 \quad (4.33)$$

with

$$\begin{aligned} \hat{A}(x) &= \begin{bmatrix} M(x)\tilde{A}(x) & 0 \\ \tilde{C}(x) & -\frac{1}{2}I \end{bmatrix}, \quad \hat{B}(x) = \begin{bmatrix} M(x)B_w(x) \\ 0 \end{bmatrix} \\ \hat{Q}(x) &= \begin{bmatrix} Q(x) & 0 \\ 0 & I \end{bmatrix}, \quad X = \begin{bmatrix} Q(x)\dot{P}(x)Q(x) & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (4.34)$$

then the closed-loop system (4.27) is stabilized and the H_∞ performance is achieved as $\|z\|_2 < \gamma\|w\|_2$ with the control law (4.4).

Proof: The matrix inequality (4.33) is equivalent to

$$\begin{bmatrix} G^T(x)\hat{A}^T(x) + \hat{A}(x)G(x) + X(x) & * & * \\ -\hat{Q}(x) + G(x) - H^T(x)\hat{A}^T(x) & -H(x) - H^T(x) & * \\ \hat{B}^T(x) & 0 & -\gamma^2 I \end{bmatrix} < 0 \quad (4.35)$$

Multiplying the inequality from the left and the right by $\Psi(x)$ and $\Psi^T(x)$, respectively, where

$$\Psi(x) = \begin{bmatrix} I & -\hat{A}(x) & 0 \\ 0 & 0 & I \end{bmatrix}$$

then by the Schur complement, we have

$$\hat{A}(x)\hat{Q}(x) + \hat{Q}(x)\hat{A}^T(x) + X + \gamma^{-2}\hat{B}(x)\hat{B}^T(x) < 0 \quad (4.36)$$

which is equivalent to (4.32), then from the result in Theorem 4.2.1, we know that the control law (4.4) stabilizes the system and achieves the H_∞ performance $\|z\|_2 < \gamma\|w\|_2$.

■

In order to obtain the state feedback gain matrix $K(x)$ from the latter inequality (4.33), $G(x)$ and $H(x)$ are chosen to possess the following structures:

$$G(x) = \begin{bmatrix} G_{11} & 0 \\ G_{21}(x) & G_{22}(x) \end{bmatrix}, \quad H(x) = \begin{bmatrix} \beta G_{11} & 0 \\ H_{21}(x) & H_{22}(x) \end{bmatrix} \quad (4.37)$$

where β is a positive tuning scalar and G_{11} is a constant nonsingular matrix. Substituting the closed-loop matrix $\tilde{A}(x)$ and $\tilde{C}(x)$ in (4.28) into the sufficient stability condition (4.33) and denoting $Y(x) = K(x)G_{11}$, the main result is obtained.

Theorem 4.2.2 *Consider system (4.26). If there exist $Q(x) = Q^T(x) > 0$, nonsingular matrix G_{11} , $G_{21}(x)$, $G_{22}(x)$, $H_{21}(x)$, $H_{22}(x)$ and $Y(x)$ such that, for some positive tuning scalar β , the nonlinear matrix inequality (4.38) is satisfied, where $G(x)$ and $H(x)$ possess the structure of (4.37),*

$$\Gamma_h(x) = \begin{bmatrix} \Gamma_{h11}(x) & \hat{B}(x) & \Gamma_{h13}(x) \\ * & -\gamma^2 I & 0 \\ * & * & -H(x) - H^T(x) \end{bmatrix} < 0 \quad (4.38)$$

with $\Gamma_{h_{11}}(x)$ and $\Gamma_{h_{13}}(x)$ as in (4.39),

$$\Gamma_{h_{11}}(x) = \begin{bmatrix} \{M(x)A(x)G_{11} + [M(x)A(x)G_{11}]^T \\ + M(x)B_u(x)Y(x) + [M(x)B_u(x)Y(x)]^T \\ + Q(x)\dot{P}(x)Q(x)\} & * \\ C_z(x)G_{11} + D_z(x)Y(x) - \frac{1}{2}G_{21}(x) & -\frac{1}{2}G_{22}(x) - \frac{1}{2}G_{22}^T(x) \end{bmatrix}$$

$$\Gamma_{h_{13}}(x) = - \begin{bmatrix} Q(x) & 0 \\ 0 & I \end{bmatrix} + G^T(x)$$

$$- \begin{bmatrix} \beta M(x)A(x)G_{11} + \beta M(x)B_u(x)Y(x) & 0 \\ \beta C_z(x)G_{11} + \beta D_z(x)Y(x) - \frac{1}{2}H_{21}(x) & -\frac{1}{2}H_{22}(x) \end{bmatrix} \quad (4.39)$$

then the control law (4.4) stabilizes the closed-loop system (4.27) and achieves the H_∞ performance $\|z\|_2 < \gamma\|w\|_2$ with

$$K(x) = Y(x)G_{11}^{-1} \quad (4.40)$$

4.2.3 SOS Based Optimization

Based on the result in Theorem 4.2.2, the SOS relaxation problem is proposed as follows:

Proposition 4.2.1 *Consider the nonlinear system (4.26). If there exist $Q(x) = Q^T(x)$, nonsingular matrix G_{11} , $G_{21}(x)$, $G_{22}(x)$, $H_{21}(x)$, $H_{22}(x)$ and $Y(x)$ such that, for some tuning scalar β , a constant $s_1 > 0$ and a SOS polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following SOS optimization problem is feasible with $\Gamma_h(x)$ as in (4.38)*

Minimize γ

s.t.

$$v_1^T [Q(x) - s_1 I] v_1 \in \Sigma_{sos} \quad (4.41)$$

$$-\begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Gamma_h(x) + s_2(x)I) \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (4.42)$$

where $v_1 \in \mathbb{R}^N$, $v_2 \in \mathbb{R}^{N+n_z}$, $v_3 \in \mathbb{R}^p$ and $v_4 \in \mathbb{R}^{N+n_z}$, then the state feedback control law (4.4) stabilizes the closed-loop system (4.27) and achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$ with (4.40).

Using the similar technique in Section 4.1.3 to deal with the nonlinear term $Q(x)\dot{P}(x)Q(x)$ in (4.42). Define $\phi_1(x, v_{21})$ as

$$\phi_1(x, v_{21}) = \left[v_{21}^T \frac{\partial Q(x)}{\partial x_1} v_{21}, \dots, v_{21}^T \frac{\partial Q(x)}{\partial x_n} v_{21} \right] \quad (4.43)$$

and by imposing a bound on the effect of the nonlinear term as in (4.44) below, we know that if γ_1 in (4.44) has zero minimum, then the nonlinear term $Q(x)\dot{P}(x)Q(x)$ in Proposition 4.2.1 can be replaced by its linear part $-\sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x)Z(x)$.

$$\begin{bmatrix} \gamma_1 & \phi_1(x, v_{21})B_u(x) & \phi_1(x, v_{21})B_w(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} \geq 0 \quad (4.44)$$

Hence, by including (4.44) as a SOS based constraint, we have the following revised SOS relaxation problem.

Proposition 4.2.2 Consider the nonlinear system (4.26). If there exist $Q(x) = Q^T(x)$, nonsingular matrix G_{11} , $G_{21}(x)$, $G_{22}(x)$, $H_{21}(x)$, $H_{22}(x)$ and $Y(x)$ such that, for some sufficiently small value of γ_1 , some tuning scalar β , a constant $s_1 > 0$ and a SOS

polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following optimization problem has feasible solutions,

$$\text{Minimize } \gamma$$

subject to

$$v_1^T [Q(x) - s_1 I] v_1 \in \Sigma_{sos} \quad (4.45)$$

$$- \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Upsilon_h(x) + s_2(x)I) \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (4.46)$$

$$v_5^T \begin{bmatrix} \gamma_1 & \phi_1(x, v_{21})B_u(x) & \phi_1(x, v_{21})B_w(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} v_5 \in \Sigma_{sos} \quad (4.47)$$

where $v_1 \in \mathbb{R}^N$, $v_2 \in \mathbb{R}^{N+n_z}$, $v_3 \in \mathbb{R}^p$, $v_4 \in \mathbb{R}^{N+n_z}$ and $v_5 \in \mathbb{R}^{m+p+1}$. $G(x)$ and $H(x)$ possesses the structure of (4.37), $\phi_1(x, v_{21})$ is as in (4.43) and

$$\Upsilon_h(x) = \begin{bmatrix} \Upsilon_{h_{1,1}}(x) & \hat{B}(x) & \Upsilon_{h_{1,3}}(x) \\ * & -\gamma^2 I & 0 \\ * & * & -H(x) - H^T(x) \end{bmatrix} \quad (4.48)$$

with

$$\Upsilon_{h_{1,1}}(x) = \begin{bmatrix} \{M(x)A(x)G_{11} + [M(x)A(x)G_{11}]^T \\ + M(x)B_u(x)Y(x) + [M(x)B_u(x)Y(x)]^T & * \\ - \sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x)Z(x)\} & \\ C_z(x)G_{11} + D_z(x)Y(x) - \frac{1}{2}G_{21}(x) & -\frac{1}{2}G_{22}(x) - \frac{1}{2}G_{22}^T(x) \end{bmatrix}$$

$$\Upsilon_{h_{1,3}}(x) = \Gamma_{h_{13}}(x) \text{ in (4.39)} \quad (4.49)$$

then the state feedback control law (4.4) stabilizes the closed-loop system (4.27) and

achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$ with (4.40).

Remark 4.2.1 Proposition 4.2.2 provides a sufficient condition involving additional matrix variables $G(x)$ and $H(x)$ for the stabilizing controller design, which potentially leads to extra freedom in the design. It can be seen that when $G(x) = [G_{11}, 0; 0, I]$ and $H(x) = [\beta G_{11}, 0; 0, I]$, it is the result of Theorem 6 in [159].

4.3 Robust State Feedback Control

The nonlinear systems considered in Section 4.1 and 4.2 assume that all parameters of the systems are known. In this section, we consider systems (4.2) and (4.26) whose matrices are not exactly known with some uncertainty.

4.3.1 Robust Stability Synthesis

Consider the NLTI system in polytopic description

$$\dot{x} = A(x; \theta)Z(x) + B(x; \theta)u \quad (4.50)$$

where $A(x; \theta)$ and $B(x; \theta)$ are polynomial matrices of the form

$$A(x; \theta) = \sum_{i=1}^q A_i(x)\theta_i, \quad B(x; \theta) = \sum_{i=1}^q B_i(x)\theta_i \quad (4.51)$$

The uncertain constant parameter vector $\theta = [\theta_1, \dots, \theta_q]^T \in \mathbb{R}^q$ satisfies

$$\theta \in \Theta \triangleq \left\{ \theta \in \mathbb{R}^q : \theta_i \geq 0, i = 1, \dots, q, \sum_{i=1}^q \theta_i = 1 \right\} \quad (4.52)$$

With the state feedback controller (4.4), we have the following closed-loop system:

$$\dot{x} = [A(x; \theta) + B(x; \theta)K(x)]Z(x) = \tilde{A}(x; \theta)Z(x) \quad (4.53)$$

with

$$\tilde{A}(x; \theta) = A(x; \theta) + B(x; \theta)K(x) = \sum_{i=1}^q \theta_i \tilde{A}_i(x) \quad (4.54)$$

where $\tilde{A}_i(x) = A_i(x) + B_i(x)K(x)$ is the closed-loop matrix at the vertex. The parameter θ provides the interpolation between these vertices.

In order to reduce the conservatism involved in the design, we define the following parameter-dependent Lyapunov function

$$V(x) = Z^T(x)P(x; \theta)Z(x) \quad (4.55)$$

with $P(x; \theta)$ is nonsingular, $P(x; \theta) = P^T(x; \theta) > 0$ and $P(x; \theta) = \sum_{i=1}^q \theta_i P_i(x)$.

With the results in Section 4.1, the main result for robust stabilizing synthesis can be obtained directly.

Proposition 4.3.1 *Consider the nonlinear system (4.50). If there exist $Q_i(x) = Q_i^T(x)$, $Y(x)$ and nonsingular matrix G such that, for a positive tuning scalar β , constants $s_i > 0$ and SOS polynomials $s_{il}(x)$ with $s_{il}(x) > 0$ for $x \neq 0$ ($i, l = 1, \dots, q$), the following optimization problem has zero optimum,*

$$\text{Minimize} \quad \gamma_1$$

subject to

$$v_1^T [Q_i(x) - s_i I] v_1 \in \Sigma_{sos} \quad (4.56)$$

$$- \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T (\Upsilon_{il}(x) + s_{il}(x)I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \Sigma_{sos} \quad (4.57)$$

$$v_3^T \begin{bmatrix} \gamma_1 & \phi_{1i}(x, v_1) B_l(x) \\ * & I \end{bmatrix} v_3 \in \Sigma_{sos} \quad (4.58)$$

where $v_1 \in \mathbb{R}^N$, $v_2 \in \mathbb{R}^N$ and $v_3 \in \mathbb{R}^{m+1}$. $\phi_{1i}(x, v_1)$ is defined as

$$\phi_{1i}(x, v_1) = \left[v_1^T \frac{\partial Q_i(x)}{\partial x_1} v_1, \dots, v_1^T \frac{\partial Q_i(x)}{\partial x_n} v_1 \right] \quad (4.59)$$

and

$$\Upsilon_{il}(x) = \begin{bmatrix} [M(x)A_i(x)G + G^T A_i^T(x)M^T(x) \\ + M(x)B_i(x)Y(x) + Y^T(x)B_i^T(x)M^T(x) \\ - \sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} A_{lj}(x)Z(x)] & * \\ [- Q_i(x) + G - \beta G^T A_i^T(x)M^T(x) \\ - \beta Y^T(x)B_i^T(x)M^T(x)] & -\beta(G + G^T) \end{bmatrix} \quad (4.60)$$

then state feedback controller (4.4) stabilizes the closed-loop system (4.53) with $K(x) = Y(x)G^{-1}$.

Proof: Assume that the solutions that satisfy all the conditions in Proposition 4.3.1 exist, then with the definition of sum of squares polynomials and the Proposition 2.5.1

in Chapter 2, we have

$$Q_i(x) > 0 \quad (4.61)$$

$$\Upsilon_{il}(x) < 0 \quad (4.62)$$

$$\phi_{1i}(x, v_1)B_l(x) = 0 \quad (4.63)$$

Do summation of (4.61)-(4.63) for $i, l = 1, \dots, q$, and with (4.51) and (4.55) we know that the Lyapunov function (4.55) is positive definite and

$$\Upsilon(x; \theta) < 0 \quad (4.64)$$

$$\phi_1(x, v_1; \theta)B(x; \theta)u = 0 \quad (4.65)$$

with

$$\Upsilon(x; \theta) = \begin{bmatrix} \begin{bmatrix} M(x)A(x; \theta)G + G^T A^T(x; \theta)M^T(x) \\ + M(x)B(x; \theta)Y(x) + Y^T(x)B^T(x; \theta)M^T(x) \\ - \sum_{j=1}^n \frac{\partial Q(x; \theta)}{\partial x_j} A_j(x; \theta)Z(x) \end{bmatrix} & * \\ \begin{bmatrix} -Q(x; \theta) + G - \beta G^T A^T(x; \theta)M^T(x) \\ -\beta Y^T(x)B^T(x; \theta)M^T(x) \end{bmatrix} & -\beta(G + G^T) \end{bmatrix} \quad (4.66)$$

$$\phi_1(x, v_1; \theta) = \left[v_1^T \frac{\partial Q(x; \theta)}{\partial x_1} v_1, \dots, v_1^T \frac{\partial Q(x; \theta)}{\partial x_n} v_1 \right] \quad (4.67)$$

Note that

$$\phi_1(x, v_1; \theta)B(x; \theta)u = v_1^T \sum_{j=1}^n \frac{\partial Q(x; \theta)}{\partial x_j} B_j(x; \theta)u v_1 = 0, \text{ for } v_1 \in \mathbb{R}^N$$

Hence we have $\sum_{j=1}^n \frac{\partial Q(x; \theta)}{\partial x_j} B_j(x; \theta) u = 0$. From (4.17), we obtain $Q(x; \theta) \dot{P}(x; \theta) Q(x; \theta) = -\sum_{j=1}^n \frac{\partial Q(x; \theta)}{\partial x_j} A_j(x; \theta) Z(x)$, which implies that

$$\Upsilon(x; \theta) = \begin{bmatrix} [M(x)A(x; \theta)G + G^T A^T(x; \theta)M^T(x) \\ + M(x)B(x; \theta)Y(x) + Y^T(x)B^T(x; \theta)M^T(x) \\ + Q(x; \theta)\dot{P}(x; \theta)Q(x; \theta)] & * \\ [- Q(x; \theta) + G - \beta G^T A^T(x; \theta)M^T(x) \\ - \beta Y^T(x)B^T(x; \theta)M^T(x)] & -\beta(G + G^T) \end{bmatrix}$$

The following proof follows that of Theorem 4.1.2. Then we have the conclusion that the solutions in Proposition 4.3.1 satisfy the condition in Theorem 4.1.1, and state feedback controller (4.4) stabilizes the closed-loop system (4.53) with $K(x) = Y(x)G^{-1}$. ■

4.3.2 Robust H_∞ Synthesis

Consider the NLTI system in polytopic description

$$\begin{aligned} \dot{x} &= A(x; \theta)Z(x) + B_u(x; \theta)u + B_w(x; \theta)w \\ z &= C_z(x; \theta)Z(x) + D_z(x; \theta)u \end{aligned} \quad (4.68)$$

where constant parameter θ is defined as in (4.52). $A(x; \theta)$, $B_u(x; \theta)$, $B_w(x; \theta)$, $C_z(x; \theta)$ and $D_z(x; \theta)$ are polynomial matrices of the form

$$\begin{aligned} A(x; \theta) &= \sum_{i=1}^q A_i(x)\theta_i, \quad B_u(x; \theta) = \sum_{i=1}^q B_{ui}(x)\theta_i \\ B_w(x; \theta) &= \sum_{i=1}^q B_{wi}(x)\theta_i, \quad C_z(x; \theta) = \sum_{i=1}^q C_{zi}(x)\theta_i \end{aligned}$$

$$D_z(x; \theta) = \sum_{i=1}^q D_{zi}(x) \theta_i \quad (4.69)$$

With the state feedback controller (4.4), we have the closed-loop system:

$$\begin{aligned} \dot{x} &= \tilde{A}(x; \theta)Z(x) + B_w(x; \theta)w \\ z &= \tilde{C}(x; \theta)Z(x) \end{aligned} \quad (4.70)$$

with

$$\begin{aligned} \tilde{A}(x; \theta) &= A(x; \theta) + B_u(x; \theta)K(x) = \sum_{i=1}^q \theta_i \tilde{A}_i(x) \\ \tilde{C}(x; \theta) &= C_z(x; \theta) + D_z(x; \theta)K(x) = \sum_{i=1}^q \theta_i \tilde{C}_i(x) \end{aligned} \quad (4.71)$$

where $\tilde{A}_i(x)$ and $\tilde{C}_i(x)$ are the closed-loop matrices at the vertex

$$\begin{aligned} \tilde{A}_i(x) &= A_i(x) + B_{ui}(x)K(x) \\ \tilde{C}_i(x) &= C_{zi}(x) + D_{zi}(x)K(x) \end{aligned} \quad (4.72)$$

With the results in Section 4.2, the main result for robust H_∞ synthesis can be obtained directly.

Proposition 4.3.2 *Consider the nonlinear system (4.68). If there exist $Q_i(x) = Q_i^T(x)$, nonsingular matrix G_{11} , $G_{21}(x)$, $G_{22}(x)$, $H_{21}(x)$, $H_{22}(x)$ and $Y(x)$ such that, for some sufficiently small value of γ_1 , some tuning scalar β , constants $s_i > 0$ and SOS polynomials $s_{il}(x)$ with $s_{il}(x) > 0$ for $x \neq 0$ ($i, l = 1, \dots, q$), the following optimization problem has feasible solutions,*

Minimize γ

subject to

$$v_1^T [Q_i(x) - s_i I] v_1 \in \Sigma_{sos} \quad (4.73)$$

$$- \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Upsilon_{hil}(x) + s_{il}(x)I) \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (4.74)$$

$$v_5^T \begin{bmatrix} \gamma_1 & \phi_{1i}(x, v_{21})B_{ul}(x) & \phi_{1i}(x, v_{21})B_{wl}(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} v_5 \in \Sigma_{sos} \quad (4.75)$$

where $v_1 \in \mathbb{R}^N$, $v_2 \in \mathbb{R}^{N+n_z}$, $v_3 \in \mathbb{R}^p$, $v_4 \in \mathbb{R}^{N+n_z}$ and $v_5 \in \mathbb{R}^{m+p+1}$. $G(x)$ and $H(x)$ possesses the structure of (4.37), $\phi_{1i}(x, v_{21})$ is defined as

$$\phi_{1i}(x, v_{21}) = \left[v_{21}^T \frac{\partial Q_i(x)}{\partial x_1} v_{21}, \dots, v_{21}^T \frac{\partial Q_i(x)}{\partial x_n} v_{21} \right] \quad (4.76)$$

and

$$\Upsilon_{hil}(x) = \begin{bmatrix} \Upsilon_{hil,1}(x) & \hat{B}_i(x) & \Upsilon_{hil,3}(x) \\ * & -\gamma^2 I & 0 \\ * & * & -H(x) - H^T(x) \end{bmatrix} \quad (4.77)$$

with $\Upsilon_{hil,1}$ and $\Upsilon_{hil,3}$ as in (4.78),

$$\Upsilon_{hil,1} = \begin{bmatrix} \{M(x)A_i(x)G_{11} + [M(x)A_i(x)G_{11}]^T \\ + M(x)B_{ui}(x)Y(x) + [M(x)B_{ui}(x)Y(x)]^T \\ - \sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} [A_{lj}(x)Z(x)]\} \\ C_{zi}(x)G_{11} + D_{zi}(x)Y(x) - \frac{1}{2}G_{21}(x) & * \\ & -\frac{1}{2}G_{22}(x) - \frac{1}{2}G_{22}^T(x) \end{bmatrix}$$

$$\Upsilon_{hil_{1,3}}(x) = - \begin{bmatrix} Q_i(x) & 0 \\ 0 & I \end{bmatrix} + G^T(x) - \begin{bmatrix} \beta M(x)A_i(x)G_{11} + \beta M(x)B_{ui}(x)Y(x) & 0 \\ \beta C_{zi}(x)G_{11} + \beta D_{zi}(x)Y(x) - \frac{1}{2}H_{21}(x) & -\frac{1}{2}H_{22}(x) \end{bmatrix} \quad (4.78)$$

then the control law (4.4) stabilizes the closed-loop system (4.70) and achieves the H_∞ performance $\|z\|_2 < \gamma\|w\|_2$ with $K(x) = Y(x)G_{11}^{-1}$.

Proof: : The proof is similar with that of Proposition 4.3.1, hence it is omitted here. ■

Remark 4.3.1 When Lyapunov matrices $Q_i(x)$ are constant matrices, Lyapunov functions are radially unbounded. Therefore Proposition 4.3.1 and Proposition 4.3.2 are global design, which imply that the closed-loop stabilities and optimal upper bound hold globally.

4.4 Numerical Examples

In this section, two state feedback design examples are presented to demonstrate the effectiveness of the proposed controller design approach in Section 4.3.

4.4.1 Example 1

Consider a parameter-dependent nonlinear system of the form (4.50)-(4.51) with system matrices given by

$$A_1(x) = \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 - \frac{3}{4}x_2^2 & \frac{1}{4} - x_1^2 - \frac{1}{2}x_2^2 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_2(x) = \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 & \frac{1}{4} - x_1^2 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix} \quad (4.79)$$

and $Z(x) = [x_1 \ x_2]^T$.

Using Proposition 4.3.1, a state feedback control law can be obtained. The values of positive constants s_1, s_2 are fixed as 0.001, then for $\beta = 0.0001$, the SOS based problem returns constant matrices Q_1, Q_2 with the optimal value of $\gamma_1 = 0.1315 \times 10^{-8}$, and a 2^{nd} order nonlinear stabilizing control law is constructed as follows:

$$u(x) = -6.6065x_1 - 5.8518x_2 - 0.8940x_1^2 - 0.5960x_1x_2 - 0.3351 \times 10^{-9}x_2^2 \quad (4.80)$$

Figure 4.1 shows the open-loop state trajectories with initial condition $x(0) = [2, 2]^T$. It can be seen that the origin point of the open-loop system is unstable. Then, for the same initial condition $x(0)$, Figure 4.2 shows the closed-loop state trajectories of 10 interpolated systems at various values of θ between the two vertices in (4.79). This demonstrates that the state feedback controller derived stabilizes the parameter-dependent system and the origin point is asymptotically stable.

Figure 4.1: States trajectories of the open-loop system

Figure 4.2: States and controller trajectories of the closed-loop systems, stabilizing state feedback control law (4.80)

4.4.2 Example 2

Consider a parameter-dependent nonlinear system of the form (4.68)-(4.69) with system matrices given by

$$\begin{aligned}
 A_1(x) &= \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 - \frac{3}{4}x_2^2 & \frac{1}{4} - x_1^2 - \frac{1}{2}x_2^2 \\ 0 & 0 \end{bmatrix}, \quad B_{u1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
 B_{w1} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_{z1} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad D_{z1} = 1 \\
 A_2(x) &= \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 & \frac{1}{4} - x_1^2 \\ 0 & 0 \end{bmatrix}, \quad B_{u2} = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix} \\
 B_{w2} &= \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}, \quad C_{z2} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad D_{z2} = 1
 \end{aligned} \tag{4.81}$$

and $Z(x) = [x_1 \ x_2]^T$.

Using Proposition 4.3.2, a 2nd order state feedback control law (4.82) can be obtained

by minimizing γ . The values of positive constants s_1, s_2 are fixed as 0.00001, the SOS polynomials $s_{il}(x)$ for $i, l = 1, 2$ are chosen as $0.00001(x_1^2 + x_2^2)$. Then for $\beta = 0.01$, the SOS based optimization problem returns 1.7925 as the optimal value of γ and 0.1×10^{-4} as the value of γ_1 , which implies that the L_2 gain from w to z of the closed-loop system is no greater than $\gamma = 1.7925$. Since the Lyapunov matrices $Q_1(x), Q_2(x)$ returned by the SOS optimization are constant matrices as in (4.83), the performance of the controller designed for the parameter-dependent nonlinear system is guaranteed over the entire state space.

$$u(x) = -2.7798x_1 - 3.7415x_2 - 0.0962x_1^2 - 0.0641x_1x_2 - 0.2832 \times 10^{-8}x_2^2 \quad (4.82)$$

$$Q_1 = \begin{bmatrix} 0.50753 & -0.37677 \\ -0.37677 & 0.56591 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.50601 & -0.37859 \\ -0.37859 & 0.57577 \end{bmatrix} \quad (4.83)$$

In Figure 4.3, it can be seen that when the disturbance input of step signal is introduced from 20 sec to 30 sec, the closed-loop systems' stabilities can be achieved by the derived state feedback control law (4.82). Moreover, the largest value of γ for the 10 interpolated systems is 0.5, which is smaller than the designed value $\gamma = 1.7925$.

Figure 4.3: State trajectories and attenuated output signal of the closed-loop systems, state feedback control law (4.82)

For comparison, a H_∞ state feedback control law is designed with the result of Theorem 6 in [159] and the optimal H_∞ upper bound is achieved as the value of $\gamma = 2.4592$, which is larger than our designed value of $\gamma = 1.7925$. Proposition 4.3.2 provides a SOS optimization algorithm involving additional matrix variables $G(x)$ and $H(x)$, which potentially leads to extra freedom in the design. It can be proven that when $G(x) = [G_{11}, 0; 0, I]$ and $H(x) = [\beta G_{11}, 0; 0, I]$, it is the result of Theorem 6 in [159].

Simulations are carried out to compare the achieved performance by the designed control law (4.82) with that by the H_∞ state feedback control law designed by using the result of [159]. It can be clearly noticed in Figure 4.4 that our design methodology enhances system’s performance by providing smaller L_2 gain.

Figure 4.4: System performance comparison

4.5 Conclusion

This chapter discusses the state feedback synthesis problems for a class of nonlinear polynomial systems. Less conservative sufficient conditions to guarantee the closed-loop stability with or without H_∞ performance via state feedback are presented as SD-MIs. We eliminate the coupling terms between system matrices and the Lyapunov matrix by introducing additional matrix variables, hence SDMI based conditions have a more suitable structure to deal with parameter uncertainty for the parameter-dependent polynomial systems. In order to reduce the conservatism involved in the controller design, parameter and state dependent Lyapunov functions (instead of fixed ones) are used, and more general assumption and relaxation are provided to deal with the nonlinear terms in the matrix inequalities. Finally, two state feedback design examples are presented to demonstrate the effectiveness of the proposed design approach.

Chapter 5

Robust Static Output Feedback Design for Polynomial Nonlinear Systems

In Chapter 3 and Chapter 4 we have proposed SOS based optimizations for state feedback design with or without H_∞ performance objective based on an iterative algorithm and SDMI sufficient conditions, respectively. The nonlinear design techniques discussed for state feedback assume that measurements of all state variables are available. However, in practical nonlinear design, this assumption is often unrealistic because the full state variables are not always accessible or some of them are chosen not to be measured due to technical or economic reasons. Therefore, it is important to extend these techniques to the output feedback design.

Since dynamic output feedback results in high order controllers which may not be practical in industry, the static output feedback design has been of much interest to control practitioners. An added advantage of the static output feedback design is that it preserves the controller structure developed based on the physical intuition from the actual systems. Therefore, extending our study to the static output feedback synthesis is meaningful.

Although the static output feedback problem for nonlinear systems has not been studied

with so much interest as for linear systems [129], there are still some advances presented for continuous nonlinear systems in [7, 8, 54, 63, 66, 156, 162, 168] and the references therein. For example, the stabilization problem based on the Hamilton-Jacobi setup [7, 8], LMI formulation [168], Lyapunov analysis and sliding mode techniques [162], and H_∞ or mixed $\mathcal{L}_2/H_\infty$ design based on the Takagi-Sugeno (T-S) fuzzy model [63, 156]. However, no general result in nonlinear synthesis via static output feedback is available now due to its non-convex nature.

In view of all these mentioned above, in this chapter, we will further our study to the nonlinear static output feedback design for polynomial nonlinear systems. In particular, we apply a simple method to include the measured output into the state of an augmented system. Sufficient conditions to guarantee the closed-loop stability with or without bounded H_∞ performance are derived in terms of nonlinear matrix inequalities, then the SOS programming technique is applied to obtain computationally tractable solutions. Finally, the proposed methodology is extended to the synthesis for the parameter-dependent polynomial systems. The parameter and state dependent Lyapunov functions (instead of fixed ones) are used to reduce the conservatism involved in the design, and robust static output feedback controller is derived.

5.1 Stability Synthesis

In this section, a stabilizing static output feedback design for polynomial nonlinear systems is addressed without considering any performance. Then the result is extended to the robust synthesis for the parameter-dependent polynomial systems in Section 5.3.

5.1.1 Problem Formulation

Consider the following input-affine NLTI system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input and $y \in \mathbb{R}^h$ is the measured output defined as $y = [y_1, \dots, y_h]^T$ where $y_1, \dots, y_h$ are polynomial functions. $f(x) \in \mathcal{R}_n^n$, $g(x) \in \mathcal{R}_n^{n \times m}$ and $h(x) \in \mathcal{R}_n^h$ are known polynomial vector fields of appropriate dimensions satisfying $f(0) = 0$.

We write the system (5.1) in the following state dependent linear-like representation:

$$\begin{aligned}\dot{x} &= A(x)Z(x) + B(x)u \\ y &= CZ(x)\end{aligned}\tag{5.2}$$

where $A(x)$ and $B(x)$ are polynomial matrices in x with appropriate dimensions, C is a constant matrix and $Z(x)$ is an $N \times 1$ vector of monomials in x satisfying the following assumption.

Assumption 5.1.1 $Z(x) = 0$ iff $x = 0$.

Remark 5.1.1 Assumption 5.1.1 ensures that $x = 0, u = 0$ is an equilibrium point of (5.2). As has been discussed in Remark 4.1.1, given $f(x) \in \mathcal{R}_n^n$, the representation $f(x) = A(x)Z(x)$ is highly non-unique. Notice that for any $L(x)$ with $L(x)Z(x) = 0$, $A(x) + L(x)$ can also be used as a representation for $f(x)$.

To avoid introducing non-convex optimization, we assume without loss of generality

that C is constant. Given that the nonlinear system has polynomial vector fields, C in (5.2) can always be made a constant matrix by including in $Z(x)$ all monomials of the output equation.

Let $M(x)$ be a $N \times n$ polynomial matrix whose $(i, j)^{\text{th}}$ entry is given by

$$M_{ij}(x) = \frac{\partial Z_i}{\partial x_j}(x), \quad i = 1, \dots, N, j = 1, \dots, n \quad (5.3)$$

Consider the following static output feedback control law

$$u = K(y)y \quad (5.4)$$

where

$$K(y) = K_0 + \sum_{r=1}^k K_r y_1^{\sigma_{1r}} y_2^{\sigma_{2r}} \dots y_h^{\sigma_{hr}} \quad (5.5)$$

with K_0 and K_r for $r = 1, \dots, k$ being constant matrices. The degree of $K(y)$ is defined as $\deg K(y) := \max(\sigma_{1r} + \sigma_{2r} + \dots + \sigma_{hr})$.

The objective is to design a static output feedback control law (5.4) which stabilizes the closed-loop system. Here we have not included any performance objective in the synthesis.

We augment system (5.2) to include the measured output y , and define the augmented state vector $\xi = [Z^T(x), y^T(x)]^T$ to obtain the following representation of the closed-loop system:

$$\dot{\xi} = \begin{bmatrix} \dot{Z}(x) \\ \dot{y}(x) \end{bmatrix} = \tilde{A}(x)\xi \quad (5.6)$$

where

$$\tilde{A}(x) = \begin{bmatrix} M(x)A(x) & M(x)B(x)K(y) \\ CM(x)A(x) & CM(x)B(x)K(y) \end{bmatrix} \quad (5.7)$$

5.1.2 Stabilizing Output Feedback Design

Consider the following Lyapunov function

$$V(x) = \xi^T P(x) \xi \quad (5.8)$$

with

$$P(x) = Q^{-1}(x) > 0, Q(x) = \begin{bmatrix} Q_{11}(x) & C^T \hat{Q} \\ \hat{Q} C & \alpha \hat{Q} \end{bmatrix} \quad (5.9)$$

where

$$Q_{11}(x) = Q_{11}^T(x), \quad Q_{11}(x) > \frac{1}{\alpha} C^T \hat{Q} C, \quad \hat{Q} = \hat{Q}^T > 0 \text{ and } \alpha > 0 \in \mathbb{R} \quad (5.10)$$

Theorem 5.1.1 Consider system (5.2). If there exist $Q_{11}(x)$, $\hat{Q}$ as in (5.10), Y_0 and Y_r for $r = 1, \dots, k$ such that, for some tuning scalar $\alpha > 0$, the following nonlinear matrix inequality is satisfied

$$\Pi(x) + Q(x) \dot{P}(x) Q(x) < 0 \quad (5.11)$$

where $Q(x)$ is as in (5.9) and

$$\Pi(x) = \begin{bmatrix} \Pi_{1,1} & * \\ \Pi_{2,1} & \Pi_{2,2} \end{bmatrix} \quad (5.12)$$

with

$$\begin{aligned}
\Pi_{1,1} &= M(x)A(x)Q_{11}(x) + M(x)B(x)Y(y)C \\
&\quad + [M(x)A(x)Q_{11}(x)]^T + [M(x)B(x)Y(y)C]^T \\
\Pi_{2,1} &= [M(x)A(x)C^T\hat{Q}]^T + \alpha[M(x)B(x)Y(y)]^T \\
&\quad + CM(x)A(x)Q_{11}(x) + CM(x)B(x)Y(y)C \\
\Pi_{2,2} &= CM(x)A(x)C^T\hat{Q} + \alpha CM(x)B(x)Y(y) \\
&\quad + [CM(x)A(x)C^T\hat{Q}]^T + \alpha[CM(x)B(x)Y(y)]^T
\end{aligned} \tag{5.13}$$

and

$$Y(y) = Y_0 + \sum_{r=1}^k y_1^{\sigma_{1r}} y_2^{\sigma_{2r}} \dots y_h^{\sigma_{hr}} Y_r \tag{5.14}$$

then the control law (5.4) stabilizes the system with

$$\begin{aligned}
K_0 &= Y_0\hat{Q}^{-1} \\
K_r &= Y_r\hat{Q}^{-1}, \quad r = 1, \dots, k
\end{aligned} \tag{5.15}$$

Proof: With $Q(x)$ in (5.9) and the closed-loop system matrix $\tilde{A}(x)$ in (5.7), the non-linear matrix inequality (5.11) is equivalent to

$$Q(x)\tilde{A}^T(x) + \tilde{A}(x)Q(x) + Q(x)\dot{P}(x)Q(x) < 0 \tag{5.16}$$

Multiplying (5.16) from both sides by $P(x) = Q^{-1}(x)$,

$$\tilde{A}^T(x)P(x) + P(x)\tilde{A}(x) + \dot{P}(x) < 0 \quad (5.17)$$

From the Lyapunov function (5.8), we have $\dot{V}(x) < 0$. Then we know that the system is stable with static output feedback control law (5.4). ■

Remark 5.1.2 When uncertain parameters are involved in system matrices, the set of $Q(x)$, Y_0 and Y_r for $r = 1, \dots, k$ satisfying the stability condition (5.11) is not jointly convex due to the product of $Q(x)$ by $A(x)$. And a simultaneous search for such matrix variables is hard. Therefore, a more suitable structure is proposed to deal with uncertainty problem by the following sufficient condition.

Lemma 5.1.1 Consider system (5.2). If the following matrix inequality in nonsingular matrices $\tilde{W}(x)$ and $Q(x) = Q^T(x) > 0$ possess a solution with a sufficiently small positive scalar ϵ

$$\begin{bmatrix} Q(x) - \tilde{W}(x) - \tilde{W}^T(x) & * \\ \tilde{W}(x) + \epsilon\tilde{A}(x)\tilde{W}(x) & -Q(x) + \epsilon[Q(x)\dot{P}(x)Q(x)] \end{bmatrix} < 0 \quad (5.18)$$

then the system is stabilized with the control law (5.4).

Proof: Considering $[Q(x) - \tilde{W}(x)]^T Q^{-1}(x) [Q(x) - \tilde{W}(x)] > 0$, we have $Q(x) - \tilde{W}(x) - \tilde{W}^T(x) > -\tilde{W}^T(x)Q^{-1}(x)\tilde{W}(x)$. Then, from (5.18) we have

$$\begin{bmatrix} -\tilde{W}^T(x)Q^{-1}(x)\tilde{W}(x) & * \\ \tilde{W}(x) + \epsilon\tilde{A}(x)\tilde{W}(x) & -Q(x) + \epsilon[Q(x)\dot{P}(x)Q(x)] \end{bmatrix} < 0 \quad (5.19)$$

Multiplying the above matrix inequality from the left side by $\text{diag}\{\tilde{W}^{-T}(x), I\}$ and the right side by $\text{diag}\{\tilde{W}^{-1}(x), I\}$, and by using the Schur complement formula, the inequality (5.16) is satisfied. From the result in Theorem 5.1.1, we know that the system is stabilized with static output feedback control law (5.4). ■

In order to obtain the output feedback gain matrix $K(y)$ from the latter inequality (5.18), $\tilde{W}(x)$ is chosen to possess the following structure:

$$\tilde{W}(x) = \begin{bmatrix} W_{11}(x) & W_{12}(x) \\ WC & \beta W \end{bmatrix} \quad (5.20)$$

where β is a positive tuning scalar and W is a constant nonsingular matrix. Substituting the closed-loop matrix $\tilde{A}(x)$ in (5.7) into the sufficient stability condition (5.18), denoting $Y_0 = K_0 W$ and $Y_r = K_r W$ for $r = 1, \dots, k$, the main result is obtained.

Theorem 5.1.2 *Consider system (5.2). If there exist $P^{-1}(x) = Q(x)$ with $Q(x) = Q^T(x) > 0$, $W_{11}(x)$, $W_{12}(x)$, nonsingular matrices W , Y_0 and Y_r for $r = 1, \dots, k$ such that, for some positive tuning scalars β and ϵ , the nonlinear matrix inequality shown in (5.21) is satisfied, where $\tilde{W}(x)$ possess the structure of (5.20),*

$$\Gamma(x) = \begin{bmatrix} Q(x) - \tilde{W}(x) - \tilde{W}^T(x) & * \\ \Gamma_{21}(x) & -Q(x) + \epsilon[Q(x)\dot{P}(x)Q(x)] \end{bmatrix} < 0 \quad (5.21)$$

with

$$\Gamma_{21}(x) = \tilde{W}(x) + \epsilon \begin{bmatrix} [M(x)A(x)W_{11}(x) & [M(x)A(x)W_{12}(x) \\ +M(x)B(x)Y(y)C] & +\beta M(x)B(x)Y(y)] \\ [CM(x)A(x)W_{11}(x) & [CM(x)A(x)W_{12}(x) \\ +CM(x)B(x)Y(y)C] & +\beta CM(x)B(x)Y(y)] \end{bmatrix} \quad (5.22)$$

$$Y(y) = Y_0 + \sum_{r=1}^k y_1^{\sigma_{1r}} y_2^{\sigma_{2r}} \dots y_h^{\sigma_{hr}} Y_r \quad (5.23)$$

then the control law (5.4) stabilizes the system with

$$\begin{aligned} K_0 &= Y_0 W^{-1} \\ K_r &= Y_r W^{-1}, \quad r = 1, \dots, k \end{aligned} \quad (5.24)$$

Remark 5.1.3 (a) It can be seen that (5.21) only involves affine matrix variable terms in $Q(x)$, $\tilde{W}(x)$, Y_0 and Y_r except for a nonlinear term $Q(x)\dot{P}(x)Q(x)$. Moreover, compared with the result in Theorem 5.1.1, $Q(x)$ is assumed of a more general form with no structure constraint. Hence, (5.21) has a more suitable formulation to deal with uncertainty involved in system matrices, and the result in Theorem 5.1.2 can be utilized directly in the robust output feedback design in Section 5.3. (b) ϵ should be chosen as a sufficiently small positive scalar so that the conservatism in Theorem 5.1.2 can be decreased.

5.1.3 SOS Based Optimization

The stability condition for the output feedback design in Theorem 5.1.2 is based on a nonlinear matrix inequality. Solving this inequality means solving an infinite set of LMIs. When only symmetric polynomial matrices are involved, the SOS decomposition can provide a computational relaxation for the sufficient condition in Theorem 5.1.2.

Proposition 5.1.1 Consider the nonlinear system (5.2). If there exist $Q(x) = Q^T(x)$, $W_{11}(x)$, $W_{12}(x)$, nonsingular matrices W , Y_0 and Y_r for $r = 1, \dots, k$ such that, for some tuning scalars β and ϵ , a scalar constant $s_1 > 0$ and a SOS polynomial $s_2(x)$ with

$s_2(x) > 0$ for $x \neq 0$, the following expressions are SOS with $\Gamma(x)$ as in (5.21)

$$v_1^T [Q(x) - s_1 I] v_1 \quad (5.25)$$

$$- \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T (\Gamma(x) + s_2(x)I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (5.26)$$

where $v_1 \in \mathbb{R}^{N+h}$ and $v_2 \in \mathbb{R}^{N+h}$, then the output feedback stabilization problem is solvable, and the control law (5.4) with (5.24) stabilizes the system.

With the definition of SOS polynomial and its decomposition, the result in Proposition 5.1.1 can be obtained directly, so the proof is omitted here.

Remark 5.1.4 For the existence of the nonlinear term $Q(x)\dot{P}(x)Q(x)$, the set of matrix variables $Q(x)$, $\tilde{W}(x)$, Y_0 and Y_r for $r = 1, \dots, k$ satisfying (5.21) is not jointly convex. Hence the search for matrix variables is hard. By following the technique discussed in Section 4.1.3, a more general relaxation is provided to transform the static output feedback design problem into a convex semidefinite programming problem.

Define

$$\phi_1(x, v_2) = \left[v_2^T \frac{\partial Q(x)}{\partial x_1} v_2, \dots, v_2^T \frac{\partial Q(x)}{\partial x_n} v_2 \right] \quad (5.27)$$

By imposing a bound on the effect of the nonlinear term as in (5.28) below, we know that if γ_1 in (5.28) has zero minimum, then the nonlinear term $Q(x)\dot{P}(x)Q(x)$ in Proposition 5.1.1 can be replaced by its linear part $-\sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x)Z(x)$.

$$\begin{bmatrix} \gamma_1 & \phi_1(x, v_2)B(x) \\ * & I \end{bmatrix} \geq 0 \quad (5.28)$$

By including (5.28) as a SOS based constraint, Proposition 5.1.2 follows.

Proposition 5.1.2 Consider the nonlinear system (5.2). If there exist $Q(x) = Q^T(x)$, $W_{11}(x)$, $W_{12}(x)$, nonsingular matrices W , Y_0 and Y_r for $r = 1, \dots, k$ such that, for some positive tuning scalars β and ϵ , a scalar constant $s_1 > 0$ and a SOS polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following optimization problem has zero optimum,

$$\text{Minimize} \quad \gamma_1$$

subject to

$$v_1^T [Q(x) - s_1 I] v_1 \in \Sigma_{sos} \quad (5.29)$$

$$- \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T (\Upsilon(x) + s_2(x)I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \Sigma_{sos} \quad (5.30)$$

$$v_3^T \begin{bmatrix} \gamma_1 & \phi_1(x, v_2)B(x) \\ * & I \end{bmatrix} v_3 \in \Sigma_{sos} \quad (5.31)$$

where $v_1 \in \mathbb{R}^{N+h}$, $v_2 \in \mathbb{R}^{N+h}$ and $v_3 \in \mathbb{R}^{m+1}$. $\phi_1(x, v_2)$ is as in (5.27) and

$$\Upsilon(x) = \begin{bmatrix} \Upsilon_{1,1} & * \\ \Upsilon_{2,1} & \Upsilon_{2,2} \end{bmatrix} \quad (5.32)$$

with

$$\begin{aligned} \Upsilon_{1,1} &= Q(x) - \tilde{W}(x) - \tilde{W}^T(x) \\ \Upsilon_{2,1} &= \Gamma_{21}(x) \text{ in (5.22)} \\ \Upsilon_{2,2} &= -Q(x) - \epsilon \sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x) Z(x) \end{aligned} \quad (5.33)$$

and $\tilde{W}(x)$ possesses the structure of (5.20), then the output feedback stabilization problem is solvable, and the control law (5.4) with (5.24) stabilizes the system.

5.2 H_∞ Performance Synthesis

In this section, H_∞ synthesis result for polynomial nonlinear systems is proposed. This result is extended to the synthesis for the parameter-dependent polynomial systems in Section 5.3.

5.2.1 Problem Formulation

Consider the following input-affine NLTI system

$$\begin{aligned}\dot{x} &= f(x) + g_u(x)u + g_w(x)w \\ y &= h(x)\end{aligned}\tag{5.34}$$

with the objective vector

$$z = h_z(x) + l_z(x)u\tag{5.35}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^p$ is the exogenous disturbance with bounded energy, $y \in \mathbb{R}^h$ is the measured output defined as $y = [y_1, \dots, y_h]^T$ where $y_1, \dots, y_h$ are polynomial functions and $z \in \mathbb{R}^{n_z}$ is the objective signal to be regulated. $f(x) \in \mathcal{R}_n^n$, $g_u(x) \in \mathcal{R}_n^{n \times m}$, $g_w(x) \in \mathcal{R}_n^{n \times p}$, $h(x) \in \mathcal{R}_n^h$, $h_z(x) \in \mathcal{R}_n^{n_z}$ and $l_z(x) \in \mathcal{R}_n^{n_z \times m}$ are known polynomial vector fields of appropriate dimensions.

Write the system (5.34)-(5.35) in the following state dependent linear-like representa-

tion:

$$\begin{aligned}\dot{x} &= A(x)Z(x) + B_u(x)u + B_w(x)w \\ y &= CZ(x)\end{aligned}\tag{5.36}$$

with the objective vector

$$z = C_z(x)Z(x) + D_z(x)u\tag{5.37}$$

where $A(x)$, $B_u(x)$, $B_w(x)$, $C_z(x)$ and $D_z(x)$ are polynomial matrices in x . C is a constant matrix. Define $M(x)$ and $Z(x)$ as in Section 5.1.1.

The objective is to design a static output feedback control law (5.4) such that

- The closed-loop system is stabilized
- The induced ℓ_2 gain from the exogenous disturbance w to the performance output z is attenuated as

$$\int_0^\infty [z(t)^T z(t)] dt < \gamma^2 \int_0^\infty [w(t)^T w(t)] dt\tag{5.38}$$

Augmenting the system (5.36)-(5.37) to include the measured output y , we define the augmented state vector $\xi = [Z^T(x), y^T(x)]^T$ and obtain the following representation of the closed-loop system:

$$\begin{aligned}\dot{\xi} &= \begin{bmatrix} \dot{Z}(x) \\ \dot{y}(x) \end{bmatrix} = \tilde{A}(x)\xi + \tilde{B}(x)w \\ z &= \tilde{C}(x)\xi\end{aligned}\tag{5.39}$$

where

$$\begin{aligned}\tilde{A}(x) &= \begin{bmatrix} M(x)A(x) & M(x)B_u(x)K(y) \\ CM(x)A(x) & CM(x)B_u(x)K(y) \end{bmatrix} \\ \tilde{B}(x) &= \begin{bmatrix} M(x)B_w(x) \\ CM(x)B_w(x) \end{bmatrix} \\ \tilde{C}(x) &= \begin{bmatrix} C_z(x) & D_z(x)K(y) \end{bmatrix}\end{aligned}\quad (5.40)$$

5.2.2 H_∞ Output Feedback Design

Define the Lyapunov function $V(x) = \xi^T P(x) \xi$ as in Section 5.1.2 with (5.9)-(5.10)

Theorem 5.2.1 Consider system (5.36)-(5.37). If there exist $Q_{11}(x)$, $\hat{Q}$ as in (5.10), Y_0 and Y_r for $r = 1, \dots, k$ such that, for some tuning scalar $\alpha > 0$, the following nonlinear matrix inequality is satisfied

$$\begin{bmatrix} \Pi_h(x) + Q(x)\dot{P}(x)Q(x) & * & * \\ \tilde{B}^T(x) & -\gamma^2 I & * \\ \tilde{C}(x)Q(x) & 0 & -I \end{bmatrix} < 0 \quad (5.41)$$

where $Q(x)$ is as in (5.9) and

$$\begin{aligned}\Pi_h(x) &= \begin{bmatrix} \Pi_{h1,1} & * \\ \Pi_{h2,1} & \Pi_{h2,2} \end{bmatrix}, \tilde{B}(x) = \begin{bmatrix} M(x)B_w(x) \\ CM(x)B_w(x) \end{bmatrix} \\ \tilde{C}(x)Q(x) &= \begin{bmatrix} [C_z(x)Q_{11}(x) + D_z(x)Y(y)C]^T \\ [C_z(x)C^T\hat{Q} + \alpha D_z(x)Y(y)]^T \end{bmatrix}^T\end{aligned}\quad (5.42)$$

with

$$\begin{aligned}
\Pi_{h_{1,1}} &= M(x)A(x)Q_{11}(x) + M(x)B_u(x)Y(y)C \\
&\quad + [M(x)A(x)Q_{11}(x)]^T + [M(x)B_u(x)Y(y)C]^T \\
\Pi_{h_{2,1}} &= [M(x)A(x)C^T\hat{Q}]^T + \alpha[M(x)B_u(x)Y(y)]^T \\
&\quad + CM(x)A(x)Q_{11}(x) + CM(x)B_u(x)Y(y)C \\
\Pi_{h_{2,2}} &= CM(x)A(x)C^T\hat{Q} + \alpha CM(x)B_u(x)Y(y) \\
&\quad + [CM(x)A(x)C^T\hat{Q}]^T + \alpha[CM(x)B_u(x)Y(y)]^T
\end{aligned} \tag{5.43}$$

and

$$Y(y) = Y_0 + \sum_{r=1}^k y_1^{\sigma_{1r}} y_2^{\sigma_{2r}} \dots y_h^{\sigma_{hr}} Y_r \tag{5.44}$$

then the control law (5.4) stabilizes the system and achieves the H_∞ performance $\|z\|_2 < \gamma\|w\|_2$ with

$$\begin{aligned}
K_0 &= Y_0\hat{Q}^{-1} \\
K_r &= Y_r\hat{Q}^{-1}, \quad r = 1, \dots, k
\end{aligned} \tag{5.45}$$

Proof: With the $Q(x)$ in (5.9) and the closed-loop system matrices $\tilde{A}(x)$, $\tilde{B}(x)$ and $\tilde{C}(x)$ in (5.40), the nonlinear matrix inequality (5.41) is equivalent to

$$\begin{bmatrix} Q(x)\tilde{A}^T(x) + \tilde{A}(x)Q(x) + Q(x)\dot{P}(x)Q(x) & * & * \\ \tilde{B}^T(x) & -\gamma^2 I & * \\ \tilde{C}(x)Q(x) & 0 & -I \end{bmatrix} < 0 \tag{5.46}$$

Multiplying (5.46) from both sides by $\text{diag}\{P(x), I, I\}$, from the Lyapunov function, and by the Schur complement we have $\dot{V} + z^T z - \gamma^2 w^T w < 0$. Then with the zero initial condition, the system is stable and the H_∞ performance is achieved as $\|z\|_2 < \gamma\|w\|_2$ with static output feedback control law (5.4). ■

Similar to the discussion in Remark 5.1.2, a sufficient condition is proposed to deal with the non-convex problem, which is more suitable for the case when parameters are involved in system matrices.

Lemma 5.2.1 Consider system (5.36)-(5.37). If the following matrix inequality in non-singular matrices $\tilde{W}(x)$ and $Q(x) = Q^T(x) > 0$ possess a solution with a sufficiently small positive scalar ϵ

$$\begin{bmatrix} Q(x) - \tilde{W}(x) - \tilde{W}^T(x) & * & * & * \\ \tilde{W}(x) + \epsilon \tilde{A}(x) \tilde{W}(x) & -Q(x) + \epsilon [Q \dot{P} Q] & * & * \\ 0 & \tilde{B}^T(x) & -\epsilon^{-1} \gamma^2 I & * \\ \tilde{C}(x) \tilde{W}(x) & 0 & 0 & -\epsilon^{-1} I \end{bmatrix} < 0 \quad (5.47)$$

then the system is stabilized and the H_∞ performance is achieved as $\|z\|_2 < \gamma\|w\|_2$ with the control law (5.4).

Proof: Similar to the proof of Lemma 5.1.1, we have

$$\begin{bmatrix} -\tilde{W}^T(x) Q^{-1}(x) \tilde{W}(x) & * & * & * \\ \tilde{W}(x) + \epsilon \tilde{A}(x) \tilde{W}(x) & -Q(x) + \epsilon [Q \dot{P} Q] & * & * \\ 0 & \tilde{B}^T(x) & -\epsilon^{-1} \gamma^2 I & * \\ \tilde{C}(x) \tilde{W}(x) & 0 & 0 & -\epsilon^{-1} I \end{bmatrix} < 0 \quad (5.48)$$

Multiplying the above from the left side by $\text{diag}\{\tilde{W}^{-T}(x), I, I, I\}$ and the right side by $\text{diag}\{\tilde{W}^{-1}(x), I, I, I\}$, then by the Schur complement formula, and finally multiplying

from both sides by $\text{diag} \left\{ \epsilon^{-\frac{1}{2}} I, \epsilon^{\frac{1}{2}} I, \epsilon^{\frac{1}{2}} I \right\}$,

$$\begin{bmatrix} Q(x)\tilde{A}^T(x) + \tilde{A}(x)Q(x) & * & * \\ +Q(x)\dot{P}(x)Q(x) & & \\ \tilde{B}^T(x) & -\gamma^2 I & * \\ \tilde{C}(x)Q(x) & 0 & -I \end{bmatrix} + \epsilon \begin{bmatrix} \tilde{A}(x) \\ 0 \\ \tilde{C}(x) \end{bmatrix} Q(x) \begin{bmatrix} \tilde{A}(x) \\ 0 \\ \tilde{C}(x) \end{bmatrix}^T < 0 \quad (5.49)$$

Then the inequality (5.46) is satisfied. From the result in Theorem 5.2.1, we know that the control law (5.4) stabilizes the system and achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$. ■

Substituting the closed-loop matrices in (5.40) into the sufficient condition (5.47), denoting $Y_0 = K_0 W$ and $Y_r = K_r W$ for $r = 1, \dots, k$, the main result is obtained.

Theorem 5.2.2 Consider system (5.36)-(5.37). If there exist $P^{-1}(x) = Q(x)$ with $Q(x) = Q^T(x) > 0$, $W_{11}(x)$, $W_{12}(x)$, constant nonsingular matrices W , Y_0 and Y_r for $r = 1, \dots, k$ such that, for some positive tuning scalars β and ϵ , the nonlinear matrix inequality shown in (5.50) is satisfied,

$$\Gamma_h(x) = \begin{bmatrix} Q(x) - \tilde{W}(x) - \tilde{W}^T(x) & * & * & * \\ \Gamma_{h_{21}}(x) & -Q(x) + \epsilon[Q\dot{P}Q] & * & * \\ 0 & \tilde{B}^T(x) & -\epsilon^{-1}\gamma^2 I & * \\ \Gamma_{h_{41}}(x) & 0 & 0 & -\epsilon^{-1}I \end{bmatrix} < 0 \quad (5.50)$$

with

$$\Gamma_{h21}(x) = \tilde{W}(x) + \epsilon \begin{bmatrix} [M(x)A(x)W_{11}(x) & [M(x)A(x)W_{12}(x) \\ +M(x)B_u(x)Y(y)C] & +\beta M(x)B_u(x)Y(y)] \\ [CM(x)A(x)W_{11}(x) & [CM(x)A(x)W_{12}(x) \\ +CM(x)B_u(x)Y(y)C] & +\beta CM(x)B_u(x)Y(y)] \end{bmatrix}$$

$$\Gamma_{h41}(x) = \begin{bmatrix} C_z(x)W_{11}(x) + D_z(x)Y(y)C & C_z(x)W_{12}(x) + \beta D_z(x)Y(y) \end{bmatrix} \quad (5.51)$$

$$Y(y) = Y_0 + \sum_{r=1}^k y_1^{\sigma_{1r}} y_2^{\sigma_{2r}} \dots y_h^{\sigma_{hr}} Y_r \quad (5.52)$$

where $\tilde{W}(x)$ possess the structure of (5.20), then the control law (5.4) stabilizes the system and achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$ with

$$\begin{aligned} K_0 &= Y_0 W^{-1} \\ K_r &= Y_r W^{-1}, \quad r = 1, \dots, k \end{aligned} \quad (5.53)$$

5.2.3 SOS Based Optimization

Based on the result in Theorem 5.2.2, the SOS relaxation problem is proposed as follows:

Proposition 5.2.1 Consider the nonlinear system (5.36)-(5.37). If there exist $P^{-1}(x) = Q(x)$ with $Q(x) = Q^T(x)$, $W_{11}(x)$, $W_{12}(x)$, nonsingular matrices W , Y_0 and Y_r for $r = 1, \dots, k$ such that, for some tuning scalars β and ϵ , a scalar constant $s_1 > 0$ and a SOS polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following SOS optimization problem is feasible with $\Gamma_h(x)$ in (5.50)

$$\text{Minimize} \quad \gamma$$

s.t.

$$v_1^T [Q(x) - s_1 I] v_1 \in \Sigma_{sos} \quad (5.54)$$

$$- \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Gamma_h(x) + s_2(x)I) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (5.55)$$

where $v_1 \in \mathbb{R}^{N+h}$, $v_2 \in \mathbb{R}^{N+h}$, $v_3 \in \mathbb{R}^p$ and $v_4 \in \mathbb{R}^{n_z}$, then output feedback controller (5.4) with (5.53) achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$.

A similar technique as in Section 5.1.3 can be used to deal with the nonlinear term $Q(x)\dot{P}(x)Q(x)$ in (5.55). Define $\phi_1(x, v_2)$ as in (5.27) and bound the effect of the nonlinear terms as in (5.56), then the linear part $-\sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x) Z(x)$ can be used to replace the nonlinear term $Q(x)\dot{P}(x)Q(x)$.

$$\begin{bmatrix} \gamma_1 & \phi_1(x, v_2)B_u(x) & \phi_1(x, v_2)B_w(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} \geq 0 \quad (5.56)$$

Proposition 5.2.2 Consider the nonlinear system (5.36)-(5.37). If there exist $Q(x) = Q^T(x)$, $W_{11}(x)$, $W_{12}(x)$, nonsingular matrices W , Y_0 and Y_r for $r = 1, \dots, k$ such that, for some sufficiently small value of γ_1 , some tuning scalars β and ϵ , a scalar constant $s_1 > 0$ and a SOS polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following optimization problem has feasible solutions,

$$\text{Minimize } \gamma$$

subject to

$$v_1^T [Q(x) - s_1 I] v_1 \in \Sigma_{sos} \quad (5.57)$$

$$- \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Upsilon_h(x) + s_2(x)I) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (5.58)$$

$$v_5^T \begin{bmatrix} \gamma_1 & \phi_1(x, v_2)B_u(x) & \phi_1(x, v_2)B_w(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} v_5 \in \Sigma_{sos} \quad (5.59)$$

where $v_1 \in \mathbb{R}^{N+h}$, $v_2 \in \mathbb{R}^{N+h}$, $v_3 \in \mathbb{R}^p$, $v_4 \in \mathbb{R}^{n_z}$ and $v_5 \in \mathbb{R}^{m+p+1}$. $\phi_1(x, v_2)$ is as in (5.27),

$$\Upsilon_h(x) = \begin{bmatrix} \Upsilon_{h1,1} & * & * & * \\ \Upsilon_{h2,1} & \Upsilon_{h2,2} & * & * \\ 0 & \tilde{B}^T(x) & -\epsilon^{-1}\gamma^2 I & * \\ \Upsilon_{h4,1} & 0 & 0 & -\epsilon^{-1}I \end{bmatrix} \quad (5.60)$$

with $\tilde{B}(x)$ as in (5.40),

$$\begin{aligned} \Upsilon_{h1,1} &= Q(x) - \tilde{W}(x) - \tilde{W}^T(x) \\ \Upsilon_{h2,1} &= \Gamma_{h21}(x), \quad \Upsilon_{h4,1} = \Gamma_{h41}(x) \text{ in (5.51)} \\ \Upsilon_{h2,2} &= -Q(x) - \epsilon \sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x) Z(x) \end{aligned} \quad (5.61)$$

and $\tilde{W}(x)$ as in (5.20), then output feedback controller (5.4) with (5.53) achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$.

5.3 Robust Static Output Feedback Control

The nonlinear systems considered in Section 5.1 and Section 5.2 assume that all parameters of the systems are known. In this section, we consider systems (5.2) and (5.36)-(5.37) whose matrices are not exactly known.

5.3.1 Robust Stability Synthesis

Consider the NLTI system in polytopic description

$$\begin{aligned}\dot{x} &= A(x; \theta)Z(x) + B(x; \theta)u \\ y &= CZ(x)\end{aligned}\tag{5.62}$$

where $A(x; \theta)$ and $B(x; \theta)$ are polynomial matrices of the form

$$A(x; \theta) = \sum_{i=1}^q A_i(x)\theta_i, \quad B(x; \theta) = \sum_{i=1}^q B_i(x)\theta_i\tag{5.63}$$

The uncertain constant parameter vector $\theta = [\theta_1, \dots, \theta_q]^T \in \mathbb{R}^q$ satisfies

$$\theta \in \Theta \triangleq \left\{ \theta \in \mathbb{R}^q : \theta_i \geq 0, i = 1, \dots, q, \sum_{i=1}^q \theta_i = 1 \right\}\tag{5.64}$$

Using the same technique as in Section 5.1, we can obtain the augmented closed-loop system with the augmented state vector $\xi = [Z^T(x), y^T(x)]^T$ as follows:

$$\dot{\xi} = \begin{bmatrix} \dot{Z}(x) \\ \dot{y}(x) \end{bmatrix} = \tilde{A}(x; \theta)\xi\tag{5.65}$$

with

$$\tilde{A}(x; \theta) = \begin{bmatrix} M(x)A(x; \theta) & M(x)B(x; \theta)K(y) \\ CM(x)A(x; \theta) & CM(x)B(x; \theta)K(y) \end{bmatrix} = \sum_{i=1}^q \theta_i \tilde{A}_i(x) \quad (5.66)$$

where $\tilde{A}_i(x)$ is closed-loop matrix at the vertex. The parameter θ provides the interpolation between these vertices.

$$\tilde{A}_i(x) = \begin{bmatrix} M(x)A_i(x) & M(x)B_i(x)K(y) \\ CM(x)A_i(x) & CM(x)B_i(x)K(y) \end{bmatrix} \quad (5.67)$$

In order to reduce the conservatism involved in the design, we define the following parameter-dependent Lyapunov function

$$V(x) = \xi^T P(x; \theta) \xi \quad (5.68)$$

with $P(x; \theta) = Q^{-1}(x; \theta) > 0$ and $Q(x; \theta) = \sum_{i=1}^q \theta_i Q_i(x)$.

With the results in Section 5.1.2 and $\tilde{W}(x)$ in (5.20), the main result for robust stabilizing synthesis can be obtained directly.

Theorem 5.3.1 Consider system (5.62). If there exist $Q_i(x) = Q_i^T(x) > 0$, $W_{11}(x)$, $W_{12}(x)$, nonsingular matrices W , Y_0 and Y_r for $r = 1, \dots, k$ such that, for some positive tuning scalars β and ϵ , the following nonlinear matrix inequalities shown in (5.69) for $i, l = 1, \dots, q$ are satisfied,

$$\Gamma_{il}(x) = \begin{bmatrix} Q_i(x) - \tilde{W}(x) - \tilde{W}^T(x) & * \\ \Gamma_{il21}(x) & -Q_i(x) - \epsilon \sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} [A_{lj}(x)Z(x) + B_{lj}(x)K(y)y] \end{bmatrix} < 0 \quad (5.69)$$

with $\tilde{W}(x)$ as in (5.20) and

$$\Gamma_{il21} = \tilde{W}(x) + \epsilon \begin{bmatrix} [M(x)A_i(x)W_{11}(x) & [M(x)A_i(x)W_{12}(x) \\ +M(x)B_i(x)Y(y)C] & +\beta M(x)B_i(x)Y(y)] \\ [CM(x)A_i(x)W_{11}(x) & [CM(x)A_i(x)W_{12}(x) \\ +CM(x)B_i(x)Y(y)C] & +\beta CM(x)B_i(x)Y(y)] \end{bmatrix} \quad (5.70)$$

$$Y(y) = Y_0 + \sum_{r=1}^k y_1^{\sigma_{1r}} y_2^{\sigma_{2r}} \dots y_h^{\sigma_{hr}} Y_r \quad (5.71)$$

then the static output feedback control law (5.4) stabilizes the system with

$$\begin{aligned} K_0 &= Y_0 W^{-1} \\ K_r &= Y_r W^{-1}, \quad r = 1, \dots, k \end{aligned} \quad (5.72)$$

The proof of Theorem 5.3.1 only needs summation of $\Gamma_{il}(x)$ for $i, l = 1, \dots, q$, hence the details are omitted here. $A_{lj}(x)$ and $B_{lj}(x)$ in (5.69) denote the j^{th} row of system matrices $A_l(x)$ and $B_l(x)$ at the vertex l .

Define $\phi_{1i}(x, v_2)$ as

$$\phi_{1i}(x, v_2) = \left[v_2^T \frac{\partial Q_i(x)}{\partial x_1} v_2, \dots, v_2^T \frac{\partial Q_i(x)}{\partial x_n} v_2 \right] \quad (5.73)$$

and γ_1 as the upper bound of $[\phi_{1i}(x, v_2) B_l(x)] [\phi_{1i}(x, v_2) B_l(x)]^T$. Then we have the corresponding revised SOS optimization problem.

Proposition 5.3.1 Consider the nonlinear system (5.62). If there exist $Q_i(x) = Q_i^T(x)$, $W_{11}(x)$, $W_{12}(x)$, nonsingular matrices W , Y_0 and Y_r for $r = 1, \dots, k$ such that, for some tuning scalars β and ϵ , constants $s_i > 0$ and SOS polynomials $s_{il}(x)$ with $s_{il}(x) > 0$ for $x \neq 0$ ($i, l = 1, \dots, q$), the following optimization problem has zero optimum,

Minimize γ_1

subject to

$$v_1^T [Q_i(x) - s_i I] v_1 \in \Sigma_{sos} \quad (5.74)$$

$$- \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T (\Upsilon_{il}(x) + s_{il}(x)I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \Sigma_{sos} \quad (5.75)$$

$$v_3^T \begin{bmatrix} \gamma_1 & \phi_{1i}(x, v_2)B_l(x) \\ * & I \end{bmatrix} v_3 \in \Sigma_{sos} \quad (5.76)$$

where $v_1 \in \mathbb{R}^{N+h}$, $v_2 \in \mathbb{R}^{N+h}$ and $v_3 \in \mathbb{R}^{m+1}$. $\tilde{W}(x)$ possesses the structure of (5.20), $\phi_{1i}(x, v_2)$ as in (5.73) and

$$\Upsilon_{il}(x) = \begin{bmatrix} \Upsilon_{il1,1} & * \\ \Upsilon_{il2,1} & \Upsilon_{il2,2} \end{bmatrix} \quad (5.77)$$

with

$$\begin{aligned} \Upsilon_{il1,1} &= Q_i(x) - \tilde{W}(x) - \tilde{W}^T(x) \\ \Upsilon_{il2,1} &= \Gamma_{il21}(x) \text{ in (5.70)} \\ \Upsilon_{il2,2} &= -Q_i(x) - \epsilon \sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} A_{lj}(x) Z(x) \end{aligned} \quad (5.78)$$

then output feedback controller (5.4) with (5.72) stabilizes the system.

Proof: The proof is similar with that of Proposition 4.3.1, hence it is omitted here. ■

5.3.2 Robust H_∞ Synthesis

Consider the NLTI system in polytopic description

$$\begin{aligned}\dot{x} &= A(x; \theta)Z(x) + B_u(x; \theta)u + B_w(x; \theta)w \\ y &= CZ(x)\end{aligned}\tag{5.79}$$

with the objective vector

$$z = C_z(x; \theta)Z(x) + D_z(x; \theta)u\tag{5.80}$$

where constant parameter θ is defined as in (5.64). $A(x; \theta)$, $B_u(x; \theta)$, $B_w(x; \theta)$, $C_z(x; \theta)$ and $D_z(x; \theta)$ are polynomial matrices of the form

$$\begin{aligned}A(x; \theta) &= \sum_{i=1}^q A_i(x)\theta_i, \quad B_u(x; \theta) = \sum_{i=1}^q B_{ui}(x)\theta_i \\ B_w(x; \theta) &= \sum_{i=1}^q B_{wi}(x)\theta_i, \quad C_z(x; \theta) = \sum_{i=1}^q C_{zi}(x)\theta_i \\ D_z(x; \theta) &= \sum_{i=1}^q D_{zi}(x)\theta_i\end{aligned}\tag{5.81}$$

Augmenting the system (5.79)-(5.80) to have

$$\begin{aligned}\dot{\xi} &= \begin{bmatrix} \dot{Z}(x) \\ \dot{y}(x) \end{bmatrix} = \tilde{A}(x; \theta)\xi + \tilde{B}(x; \theta)w \\ z &= \tilde{C}(x; \theta)\xi\end{aligned}\tag{5.82}$$

with

$$\begin{aligned}
 \tilde{A}(x; \theta) &= \sum_{i=1}^q \theta_i \tilde{A}_i(x) = \begin{bmatrix} M(x)A(x; \theta) & M(x)B_u(x; \theta)K(y) \\ CM(x)A(x; \theta) & CM(x)B_u(x; \theta)K(y) \end{bmatrix} \\
 \tilde{B}(x; \theta) &= \sum_{i=1}^q \theta_i \tilde{B}_i(x) = \begin{bmatrix} M(x)B_w(x; \theta) \\ CM(x)B_w(x; \theta) \end{bmatrix} \\
 \tilde{C}(x; \theta) &= \sum_{i=1}^q \theta_i \tilde{C}_i(x) = \begin{bmatrix} C_z(x; \theta) & D_z(x; \theta)K(y) \end{bmatrix}
 \end{aligned} \tag{5.83}$$

where $\tilde{A}_i(x)$, $\tilde{B}_i(x)$ and $\tilde{C}_i(x)$ are closed-loop matrices at the vertex

$$\begin{aligned}
 \tilde{A}_i(x) &= \begin{bmatrix} M(x)A_i(x) & M(x)B_{ui}(x)K(y) \\ CM(x)A_i(x) & CM(x)B_{ui}(x)K(y) \end{bmatrix} \\
 \tilde{B}_i(x) &= \begin{bmatrix} M(x)B_{wi}(x) \\ CM(x)B_{wi}(x) \end{bmatrix} \\
 \tilde{C}_i(x) &= \begin{bmatrix} C_{zi}(x) & D_{zi}(x)K(y) \end{bmatrix}
 \end{aligned} \tag{5.84}$$

With the results in Section 5.2.2 and $\tilde{W}(x)$ in (5.20), the main result for robust H_∞ synthesis can be obtained directly.

Theorem 5.3.2 *Consider system (5.79)-(5.80). If there exist $Q_i(x) = Q_i^T(x) > 0$, $W_{11}(x)$, $W_{12}(x)$, nonsingular matrices W , Y_0 and Y_r for $r = 1, \dots, k$ such that, for some positive tuning scalars β and ϵ , the following nonlinear matrix inequalities shown*

in (5.85) for $i, l = 1, \dots, q$ are satisfied,

$$\Gamma_{hil}(x) = \begin{bmatrix} Q_i(x) - \tilde{W}(x) - \tilde{W}^T(x) & * \\ \Gamma_{hil_{21}}(x) & \left\{ -Q_i(x) - \epsilon \sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} [A_{lj}(x)Z(x) + B_{ulj}(x)K(y)y + B_{wlj}(x)w] \right\} \\ 0 & \tilde{B}_i^T(x) \\ \Gamma_{hil_{41}}(x) & 0 \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ -\epsilon^{-1}\gamma^2 I & * \\ 0 & -\epsilon^{-1}I \end{bmatrix} < 0 \quad (5.85)$$

with

$$\Gamma_{hil_{21}}(x) = \tilde{W}(x) + \epsilon \begin{bmatrix} [M(x)A_i(x)W_{11}(x) & [M(x)A_i(x)W_{12}(x) \\ +M(x)B_{ui}(x)Y(y)C] & +\beta M(x)B_{ui}(x)Y(y)] \\ [CM(x)A_i(x)W_{11}(x) & [CM(x)A_i(x)W_{12}(x) \\ +CM(x)B_{ui}(x)Y(y)C] & +\beta CM(x)B_{ui}(x)Y(y)] \end{bmatrix}$$

$$\Gamma_{hil_{41}}(x) = \begin{bmatrix} C_{zi}(x)W_{11}(x) + D_{zi}(x)Y(y)C & C_{zi}(x)W_{12}(x) + \beta D_{zi}(x)Y(y) \end{bmatrix} \quad (5.86)$$

$$Y(y) = Y_0 + \sum_{r=1}^k y_1^{\sigma_{1r}} y_2^{\sigma_{2r}} \dots y_h^{\sigma_{hr}} Y_r \quad (5.87)$$

then the output feedback controller (5.4) with (5.72) achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$.

Define $\phi_{1i}(x, v_2)$ as in (5.73) and γ_1 as the upper bound of $[\phi_{1i}(x, v_2)B_{ul}(x)][\phi_{1i}(x, v_2)B_{ul}(x)]^T$ and $[\phi_{1i}(x, v_2)B_{wl}(x)][\phi_{1i}(x, v_2)B_{wl}(x)]^T$. Then we have the following SOS

optimization result.

Proposition 5.3.2 *Consider the nonlinear system (5.79)-(5.80). If there exist $Q_i(x) = Q_i^T(x)$, $W_{11}(x)$, $W_{12}(x)$, nonsingular matrices W , Y_0 and Y_r for $r = 1, \dots, k$ such that, for some sufficiently small value of γ_1 , some tuning scalars β and ϵ , constants $s_i > 0$ and SOS polynomials $s_{il}(x)$ with $s_{il}(x) > 0$ for $x \neq 0$ ($i, l = 1, \dots, q$), the following optimization problem has feasible solutions,*

$$\text{Minimize} \quad \gamma$$

subject to

$$v_1^T [Q_i(x) - s_i I] v_1 \in \Sigma_{\text{sos}} \quad (5.88)$$

$$- \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Upsilon_{hil}(x) + s_{il}(x)I) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{\text{sos}} \quad (5.89)$$

$$v_5^T \begin{bmatrix} \gamma_1 & \phi_{1i}(x, v_2)B_{ul}(x) & \phi_{1i}(x, v_2)B_{wl}(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} v_5 \in \Sigma_{\text{sos}} \quad (5.90)$$

where $v_1 \in \mathbb{R}^{N+h}$, $v_2 \in \mathbb{R}^{N+h}$, $v_3 \in \mathbb{R}^p$, $v_4 \in \mathbb{R}^{n_z}$ and $v_5 \in \mathbb{R}^{m+p+1}$. $\tilde{W}(x)$ possesses the structure of (5.20), $\phi_{1i}(x, v_2)$ is as in (5.73) and

$$\Upsilon_{hil}(x) = \begin{bmatrix} \Upsilon_{hil1,1} & * & * & * \\ \Upsilon_{hil2,1} & \Upsilon_{hil2,2} & * & * \\ 0 & \tilde{B}_i^T(x) & -\epsilon^{-1}\gamma^2 I & * \\ \Upsilon_{hil4,1} & 0 & 0 & -\epsilon^{-1}I \end{bmatrix} \quad (5.91)$$

with

$$\begin{aligned}\Upsilon_{hil_{1,1}} &= Q_i(x) - \tilde{W}(x) - \tilde{W}^T(x) \\ \Upsilon_{hil_{2,1}} &= \Gamma_{hil_{21}}(x), \Upsilon_{hil_{4,1}} = \Gamma_{hil_{41}}(x) \text{ in (5.86)} \\ \Upsilon_{hil_{2,2}} &= -Q_i(x) - \epsilon \sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} [A_{lj}(x)Z(x)]\end{aligned}\quad (5.92)$$

then the output feedback controller (5.4) with (5.72) achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$.

Proof: The proof is similar with that of Proposition 4.3.1, hence it is omitted here. ■

Remark 5.3.1 In our methodology, the nonlinear output feedback controller to be derived are defined as in the form of $u = K(y)y$, where $K(y)$ is a polynomial matrix variable in y as in (5.5), hence the output feedback controller is a polynomial controller in y . When the controller are chosen as $u = \tilde{K}\phi(y)$ where $\phi(y)$ is a monomial vector in y without constant terms, the sufficient conditions to guarantee the closed-loop stability and H_∞ performance upper bound proposed in Section 5.1, Section 5.2 and Section 5.3 are also satisfied since $u = \tilde{K}\phi(y)$ can be transformed to the form of $u = K(y)y$.

Remark 5.3.2 The proposed design methodology can also be extended to a polynomial nonlinear system whose performance output z is influenced by disturbance w as follows:

$$\begin{aligned}\dot{x} &= A(x; \theta)Z(x) + B_u(x; \theta)u + B_w(x; \theta)w \\ z &= C_z(x; \theta)Z(x) + D_z(x; \theta)u + D_w(x; \theta)w \\ y &= CZ(x)\end{aligned}\quad (5.93)$$

where constant parameter θ is defined as in (5.64). $A(x; \theta)$, $B_u(x; \theta)$, $B_w(x; \theta)$, $C_z(x; \theta)$

and $D_z(x; \theta)$ are as in (5.81). $D_w(x; \theta)$ is a polynomial matrix in the similar form as

$$D_w(x; \theta) = \sum_{i=1}^q D_{wi}(x) \theta_i \quad (5.94)$$

The SOS based optimization result for the system (5.93) can be derived as that in Proposition 5.3.2 where $\tilde{W}(x)$ possesses the structure of (5.20), $\phi_{1i}(x, v_2)$ is as in (5.73) and $\Upsilon_{hil}(x)$ is modified as

$$\Upsilon_{hil}(x) = \begin{bmatrix} \Upsilon_{hil1,1} & * & * & * \\ \Upsilon_{hil2,1} & \Upsilon_{hil2,2} & * & * \\ 0 & \tilde{B}_i^T(x) & -\epsilon^{-1}\gamma^2 I & * \\ \Upsilon_{hil4,1} & 0 & \epsilon^{-1}D_{wi}(x) & -\epsilon^{-1}I \end{bmatrix} \quad (5.95)$$

with

$$\begin{aligned} \Upsilon_{hil1,1} &= Q_i(x) - \tilde{W}(x) - \tilde{W}^T(x) \\ \Upsilon_{hil2,1} &= \Gamma_{hil21}(x), \Upsilon_{hil4,1} = \Gamma_{hil41}(x) \text{ in (5.86)} \\ \Upsilon_{hil2,2} &= -Q_i(x) - \epsilon \sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} [A_{lj}(x)Z(x)] \end{aligned} \quad (5.96)$$

and $\tilde{B}_i(x)$ as in (5.84).

Remark 5.3.3 When Lyapunov matrices $Q_i(x)$ are constant matrices, Lyapunov functions are radially unbounded, and the closed-loop stabilities and optimal H_∞ upper bounds proposed in Proposition 5.3.1 and Proposition 5.3.2 hold globally.

5.4 Numerical Examples

In this section, two static output feedback design examples are presented to demonstrate the effectiveness of the proposed controller design approach in Section 5.3. The SOS relaxations and computation algorithms have been provided in Proposition 5.3.1 and Proposition 5.3.2 for the robust output feedback design problems. The SOS programming is utilized to solve these convex optimization problems to obtain computationally tractable solutions.

5.4.1 Example 1

Consider a parameter-dependent nonlinear system of the form (5.62)-(5.63) with system matrices given by

$$\begin{aligned} A_1(x) &= \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 - \frac{3}{4}x_2^2 & \frac{1}{4} - x_1^2 - \frac{1}{2}x_2^2 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ A_2(x) &= \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 & \frac{1}{4} - x_1^2 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & -1 \end{bmatrix} \end{aligned} \quad (5.97)$$

and $Z(x) = [x_1 \ x_2]^T$. The polynomial controller matrix is defined by

$$K(y) = K_0 + K_1y + K_2y^2 \quad (5.98)$$

Based on the SOS relaxations in Proposition 5.3.1, the SOS programming is utilized to obtain a static output feedback control law by minimizing γ_1 . The values of positive constants s_1, s_2 are fixed as 0.0001, then for $\beta = 1$, $\epsilon = 0.001$, the SOS based optimization problem returns 0.10998×10^{-5} as the optimal value of γ_1 . With the order of

Q_1, Q_2 fixed as 2, we obtain $K_0 = 1.7849$, $K_1 = 0.003919$ and $K_2 = -0.08859$. Then a corresponding 3rd order nonlinear stabilizing control law is given by

$$u = 1.7849y + 0.003919y^2 - 0.08859y^3 \quad (5.99)$$

Figure 5.1 shows the open-loop state trajectories with initial condition $x(0) = [2, 2]^T$. It can be seen that the origin point of the open-loop system is not asymptotically stable. Then, for the same initial condition $x(0)$, Figure 5.2 shows the closed-loop state trajectories of 10 interpolated systems at various values of θ between the two vertices in (5.97). This demonstrates that the output feedback controller derived stabilizes the parameter-dependent system and the origin point is asymptotically stable.

Figure 5.1: State trajectories of the open-loop system, $x(0) = [2, 2]^T$

Figure 5.2: State and control trajectories of the closed-loop systems, stabilizing output feedback control law (5.99)

5.4.2 Example 2

Consider a parameter-dependent nonlinear system of the form (5.79)-(5.80) with system matrices given by

$$\begin{aligned}
 A_1(x) &= \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 - \frac{3}{4}x_2^2 & \frac{1}{4} - x_1^2 - \frac{1}{2}x_2^2 \\ 0 & 0 \end{bmatrix}, \quad B_{u1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
 B_{w1} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_{z1} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad D_{z1} = 1 \\
 A_2(x) &= \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 & \frac{1}{4} - x_1^2 \\ 0 & 0 \end{bmatrix}, \quad B_{u2} = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix} \\
 B_{w2} &= \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}, \quad C_{z2} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad D_{z2} = 1, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix} \quad (5.100)
 \end{aligned}$$

and $Z(x) = [x_1 \ x_2]^T$. The polynomial controller matrix is defined by

$$K(y) = K_0 + K_1 y \quad (5.101)$$

Based on the SOS optimization in Proposition 5.3.2, a nonlinear polynomial output feedback control law can be obtained with $K_0 = 0.5014$ and $K_1 = -0.0061$. The SOS polynomials $s_{il}(x)$ for $i, l = 1, 2$ are chosen as $0.001(x_1^2 + x_2^2)$. Then for $\beta = 1$, $\epsilon = 0.001$, the SOS based optimization problem returns 1.8071 as the optimal value of γ and 10^{-4} as the value of γ_1 , which implies that the L_2 gain from w to z of the closed-loop system is no greater than $\gamma = 1.8071$. Since the Lyapunov matrices $Q_1(x), Q_2(x)$ in (5.103) returned by SOS optimization are constant matrices, the performance of the corresponding control input in (5.102) designed for the parameter-dependent nonlinear system is guaranteed over the entire state space.

$$u = 0.5014y - 0.0061y^2 \quad (5.102)$$

$$Q_1 = \begin{bmatrix} 4.0843 & -4.9391 & 1.726 \\ -4.9391 & 7.9682 & -1.7586 \\ 1.726 & -1.7586 & 1.7669 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 4.0905 & -4.9534 & 1.7268 \\ -4.9534 & 7.9938 & -1.7662 \\ 1.7268 & -1.7662 & 1.7659 \end{bmatrix} \quad (5.103)$$

Figure 5.3 shows the state trajectories and the measured output signal of 10 interpolated closed-loop systems at various values of θ between the two vertices in (5.100), where constant disturbance $w = 1$ is applied. Simulations show that the closed-loop system is stabilized by the derived static output feedback control law. Since the γ designed in our methodology is an upper bound on the guaranteed level of attenuation in the worst situation, the H_∞ norms of the closed-loop systems are lower than the designed value of $\gamma = 1.8071$. In Figure 5.4, it can be seen that when the disturbance input of step signal is introduced from 30sec to 35sec, the largest value of γ for the interpolated systems is 0.3102, which is smaller than the designed value 1.8071.

Figure 5.3: State trajectories and attenuated output signal of the closed-loop systems, output feedback control law (5.102)

Figure 5.4: State trajectories and attenuated output signal of the closed-loop systems, $\gamma = 1.8071$

Static output feedback design can be viewed as a structurally constrained state feedback control [139]. For comparison, a H_∞ state feedback controller is designed by using the relaxed SOS optimization in Proposition 5.3.2 with the output matrix assumed as $C = I$. In this case, an optimal L_2 gain of 0.7247 is achieved for the closed-loop systems. In general, system state provides the most explicit and detailed information about the system, therefore state feedback control should be more effective in improving system performance and robustness properties. Although the L_2 gain obtained from the static output feedback design is higher than that of the state feedback design, in most practical systems where the entire set of system states is not directly measurable, the static output feedback design is more easily implemented.

5.5 Conclusion

This chapter discusses the nonlinear static output feedback design problems for polynomial nonlinear systems. A simple method is applied to include the measured output in the augmented system, and sufficient conditions to guarantee the closed-loop stability with or without H_∞ performance via static output feedback are presented as SDMIs. Because of the elimination of the coupling terms between system matrices and the Lyapunov matrix, the proposed methodology is extended to the synthesis for the parameter-dependent polynomial systems. The parameter and state dependent Lyapunov functions and more general assumption and relaxation are provided to reduce the conservatism involved in the design. Finally, two static output feedback design examples are provided to demonstrate the effectiveness of the proposed approach.

Chapter 6

Multi-objective Robust and Reliable Control for Polynomial Nonlinear Systems

In many control system design, it is often assumed that all the system components are in good conditions. Therefore, the resulting controller usually works well in nominal conditions but may not maintain an acceptable performance in the presence of faults, even in some cases the stability cannot be guaranteed. Based on this, the fault tolerant ability is a realistic and important issue in the control design.

Reliable control system is a control system designed to optimize the performance in normal condition and maintain certain closed-loop stability and performance requirements in the presence of faults simultaneously. Reliable control system is particular suitable for critical practical systems, such as, aircraft, chemical plants and nuclear power plants, etc.

In this chapter, we extend the concept of reliable control design to solve the multi-objective control problem for polynomial nonlinear parameter-dependent systems. In particular, one of the main reliable design methods, the passive approach is utilized to

solve the multi-objective design problem, where a fixed controller is designed for the nominal case as well as fault cases. Sufficient conditions to guarantee the closed-loop stability are proposed in terms of nonlinear matrix inequalities via state feedback. At the same time, not only the optimal performance during normal system operation, but also the satisfactory performance in the presence of faults is guaranteed. Compared with some existing reliable control designs, the main advantage of the proposed design is that the proposed SDMI formulation provides an effective way for the application of the new SOS programming technique. Then the numerical difficulty in solving multi-objective design problem is overcome, and the reliable controllers and Lyapunov functions are constructed in an efficient computational manner. In the synthesis for parameter-dependent systems, parameter and state dependent Lyapunov functions, one for each and every vertex of the uncertainty polytope, are employed to reduce the conservatism involved in the controller design.

As no real-time fault detection, isolation and system reconfiguration procedures are needed, this approach is more suitable for the case that the available reaction time for the system is very little after the occurrence of a severe fault. This crucial property has motivated great interest in the passive reliable control design as in [77, 78, 163, 164, 165, 169] and the references therein.

6.1 Multi-objective Control

In this section, a multi-objective reliable control design for polynomial nonlinear systems is addressed with considering H_∞ performance and bounded energy of the performance output. The result will be extended to the robust synthesis for the parameter-dependent polynomial systems in Section 6.2.

6.1.1 Problem Formulation

Consider the following input-affine NLTI system

$$\begin{aligned}\dot{x} &= f(x) + g_u(x)u + g_w(x)w \\ z &= h(x) + l(x)u \\ x(0) &= x_0\end{aligned}\tag{6.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^p$ is the exogenous disturbance with bounded energy and $z \in \mathbb{R}^{n_z}$ is the objective signal to be regulated. $f(x) \in \mathcal{R}_n^n$, $g_u(x) \in \mathcal{R}_n^{n \times m}$, $g_w(x) \in \mathcal{R}_n^{n \times p}$, $h(x) \in \mathcal{R}_n^{n_z}$ and $l(x) \in \mathcal{R}_n^{n_z \times m}$ are known polynomial vector fields of appropriate dimensions satisfying $f(0) = 0$.

We write the system (6.1) in the following state dependent linear-like representation:

$$\begin{aligned}\dot{x} &= A(x)Z(x) + B_u(x)u + B_w(x)w \\ z &= C_z(x)Z(x) + D_z(x)u \\ x(0) &= x_0\end{aligned}\tag{6.2}$$

where $A(x)$, $B_u(x)$, $B_w(x)$, $C_z(x)$ and $D_z(x)$ are polynomial matrices in x , and $Z(x)$ is an $N \times 1$ vector of monomials in x satisfying the following assumption.

Assumption 6.1.1 $Z(x) = 0$ iff $x = 0$.

Remark 6.1.1 Assumption 6.1.1 ensures that $x = 0, u = 0$ is an equilibrium point of (6.2). It should be noted that, given $f(x) \in \mathcal{R}_n^n$, the representation $f(x) = A(x)Z(x)$ is highly non-unique. And for any $L(x)$ with $L(x)Z(x) = 0$, $A(x) + L(x)$ can also be used as a representation for $f(x)$.

Let $M(x)$ to be a $N \times n$ polynomial matrix whose $(i, j)^{\text{th}}$ entry is given by

$$M_{ij}(x) = \frac{\partial Z_i}{\partial x_j}(x), \quad i = 1, \dots, N, j = 1, \dots, n \quad (6.3)$$

If we allow the control input to be generated by a state feedback controller

$$u = K(x)Z(x) \quad \text{with} \quad K(0) = 0 \quad (6.4)$$

we have the following closed-loop system

$$\begin{aligned} \dot{x} &= \tilde{A}(x)Z(x) + B_w(x)w \\ z &= \tilde{C}(x)Z(x) \end{aligned} \quad (6.5)$$

with

$$\begin{aligned} \tilde{A}(x) &= A(x) + B_u(x)K(x) \\ \tilde{C}(x) &= C_z(x) + D_z(x)K(x) \end{aligned} \quad (6.6)$$

The objective is to design a state feedback control law (6.4) which stabilizes the closed-loop system (6.5) and achieves the following multi-objective performance (6.7)-(6.9).

$$\min_{u(x)} \max_{w(t)} \mathcal{J}_\infty(t) \quad (6.7)$$

subject to

$$\int_0^\infty [z(t)^T z(t)] dt < \gamma \int_0^\infty [w(t)^T w(t)] dt, \quad (6.8)$$

with

$$\mathcal{J}_\infty(t) = \int_0^\infty [z(t)^T z(t)] dt \quad (6.9)$$

where $\sqrt{\gamma}$ with $\gamma > 0$ corresponds to the H_∞ norm of the mapping from the input $w(t)$ to the performance output $z(t)$.

6.1.2 State Feedback Design

The methodology presented in this paper is based on the Lyapunov stability argument. Define the Lyapunov function as

$$V(x) = Z^T(x)P(x)Z(x) \quad (6.10)$$

where $P(x) = P^T(x) > 0$ is a nonsingular polynomial matrix in x and $Q(x) = P^{-1}(x)$.

Theorem 6.1.1 *Consider the nonlinear system (6.2) and the performance (6.7)-(6.9). For given upper bound $\gamma > 0$, if there exist polynomial matrices $Q(x)$ with $Q(x) = Q^T(x) > 0$ and $Y(x)$ such that the following nonlinear matrix inequality is satisfied*

$$\begin{bmatrix} M(x)A(x)Q(x) + M(x)B_u(x)Y(x) + [M(x)A(x)Q(x)]^T & * & * \\ +[M(x)B_u(x)Y(x)]^T + Q(x)\dot{P}(x)Q(x) & & \\ B_w^T(x)M^T(x) & -\gamma I & * \\ C_z(x)Q(x) + D_z(x)Y(x) & 0 & -I \end{bmatrix} < 0 \quad (6.11)$$

then the closed-loop system is asymptotically stabilized by

$$K(x) = Y(x)Q^{-1}(x) \quad (6.12)$$

Furthermore, for any initial condition $x(0) = x_0$ in the region of attraction of the origin

equilibrium, we have the upper bound of performance index

$$\mathcal{J}_\infty(t) < Z^T(x_0)Q^{-1}(x_0)Z(x_0) + \gamma \int_0^\infty w^T(t)w(t)dt \quad (6.13)$$

Proof: Assume that there exist solutions $Q(x)$ and $Y(x)$ to (6.11). The condition $Q(x) > 0$ and Assumption 6.1.1 imply that the Lyapunov function $V(x)$ is positive definite for $x \neq 0$ and $V(0) = 0$. Then the time derivative of $V(x)$ along the closed-loop trajectories is given by

$$\begin{aligned} \dot{V}(x) = Z^T(x) \left\{ \tilde{A}^T(x)M^T(x)P(x) + P(x)M(x)\tilde{A}(x) + \dot{P}(x) \right\} Z(x) \\ + w^T B_w^T(x)M^T(x)P(x)Z(x) + Z^T(x)P(x)M(x)B_w(x)w \end{aligned} \quad (6.14)$$

Applying the Schur complement formula and multiplying from both sides by $P(x) = Q^{-1}(x)$, (6.11) is equivalent to

$$\begin{aligned} P(x)M(x)\tilde{A}(x) + \tilde{A}^T(x)M^T(x)P(x) + \dot{P}(x) + \tilde{C}^T(x)\tilde{C}(x) \\ + \frac{1}{\gamma}P(x)M(x)B_w(x)B_w^T(x)M^T(x)P(x) < 0 \end{aligned} \quad (6.15)$$

Since $\gamma > 0$, from (6.15) we have that

$$P(x)M(x)\tilde{A}(x) + \tilde{A}^T(x)M^T(x)P(x) + \dot{P}(x) < 0 \quad (6.16)$$

From the time derivative of $V(x)$ in (6.14) we have that $\dot{V}(x) < 0$ for $x \neq 0$, and therefore the closed-loop system is asymptotically stable about the origin equilibrium. Furthermore,

$$\left[\sqrt{\gamma}w - \frac{1}{\sqrt{\gamma}}B_w^T(x)M^T(x)P(x)Z(x) \right]^T \left[\sqrt{\gamma}w - \frac{1}{\sqrt{\gamma}}B_w^T(x)M^T(x)P(x)Z(x) \right] \geq 0 \quad (6.17)$$

With (6.12), (6.14), (6.15) and (6.17), we have

$$\begin{aligned}
 \mathcal{J}_\infty(t) &= \int_0^\infty Z^T(x) \tilde{C}^T(x) \tilde{C}(x) Z(x) dt \\
 &< - \int_0^\infty \left[\dot{V}(x) + \frac{1}{\gamma} Z^T(x) P(x) M(x) B_w(x) B_w^T(x) M^T(x) P(x) Z(x) \right. \\
 &\quad \left. - Z^T(x) P(x) M(x) B_w(x) w - w^T B_w^T(x) M^T(x) P(x) Z(x) \right] dt \\
 &\leq - \int_0^\infty \dot{V}(x) dt + \gamma \int_0^\infty w^T(t) w(t) dt \\
 &< Z^T(x_0) Q^{-1}(x_0) Z(x_0) + \gamma \int_0^\infty w^T(t) w(t) dt
 \end{aligned} \tag{6.18}$$

This completes the proof. ■

Remark 6.1.2 When $Y(x)$ and $Q(x)$ are polynomial matrices, $K(x)$ in (6.12) is rational. Although a possible solution to avoid rational controller gain can be obtained with a constant matrix Q , it is obviously conservative. Moreover, when parameters are involved in system matrices, the set of matrix variables satisfying the stability inequality is not jointly convex due to the product of $Q(x)$ by $\tilde{A}(x)$. A more suitable structure is proposed to deal with the non-convexity problem by the following sufficient condition.

Theorem 6.1.2 Consider the nonlinear system (6.2) and the performance (6.7)-(6.9). For given upper bound $\gamma > 0$, if there exist polynomial matrix $Q(x)$ with $Q(x) = Q^T(x) > 0$, $Y(x)$ and constant nonsingular matrix W such that, for some positive tuning scalar ϵ , the nonlinear matrix inequality (6.19) is satisfied

$$\Gamma(x) = \begin{bmatrix} Q(x) - W - W^T & * \\ W + \epsilon M(x) [A(x)W + B_u(x)Y(x)] & -Q(x) + \epsilon [Q(x)\dot{P}(x)Q(x)] \\ 0 & B_w^T(x)M^T(x) \\ C_z(x)W + D_z(x)Y(x) & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & * \\ * & * \\ -\epsilon^{-1}\gamma I & * \\ 0 & -\epsilon^{-1}I \end{bmatrix} < 0 \quad (6.19)$$

then the closed-loop system is asymptotically stabilized by

$$K(x) = Y(x)W^{-1} \quad (6.20)$$

Furthermore, for any initial condition $x(0) = x_0$ in the region of attraction of the origin equilibrium, we have the upper bound of performance index as in (6.13).

Proof: Assume that there exist solutions $Q(x)$, $Y(x)$ and W to (6.19). Considering $[Q(x) - W]^T Q^{-1}(x)[Q(x) - W] > 0$, we have $Q(x) - W - W^T > -W^T Q^{-1}(x)W$. Then, from (6.19) we have

$$\begin{bmatrix} -W^T Q^{-1}(x)W & * \\ W + \epsilon M(x)[A(x)W + B_u(x)Y(x)] & -Q(x) + \epsilon[Q(x)\dot{P}(x)Q(x)] \\ 0 & B_w^T(x)M^T(x) \\ C_z(x)W + D_z(x)Y(x) & 0 \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ -\epsilon^{-1}\gamma I & * \\ 0 & -\epsilon^{-1}I \end{bmatrix} < 0 \quad (6.21)$$

Multiplying the above from the left side by $\text{diag}\{W^{-T}, I, I, I\}$ and the right side by $\text{diag}\{W^{-1}, I, I, I\}$, then by the Schur complement formula, and finally multiplying

from both sides by $\text{diag} \left\{ \epsilon^{-\frac{1}{2}} I, \epsilon^{\frac{1}{2}} I, \epsilon^{\frac{1}{2}} I \right\}$,

$$\begin{aligned}
 & \begin{bmatrix} [Q(x)\dot{P}(x)Q(x) & * & * \\ +M(x)\tilde{A}(x)Q(x) + Q(x)\tilde{A}^T(x)M^T(x) & & \\ B_w^T(x)M^T(x) & -\gamma I & * \\ \tilde{C}(x)Q(x) & 0 & -I \end{bmatrix} \\
 & + \epsilon \begin{bmatrix} M(x)\tilde{A}(x) \\ 0 \\ \tilde{C}(x) \end{bmatrix} Q(x) \begin{bmatrix} M(x)\tilde{A}(x) \\ 0 \\ \tilde{C}(x) \end{bmatrix}^T < 0 \quad (6.22)
 \end{aligned}$$

The rest follows the proof of Theorem 6.1.1. This completes the proof. ■

Remark 6.1.3 (a) Similarly as the result in Chapter 5, the separation of the system matrix and Lyapunov matrix formulates a suitable structure of (6.19) to deal with system uncertainty. ϵ should be chosen as a sufficiently small positive scalar. By choosing W as a constant matrix, the controller matrix $K(x) = Y(x)W^{-1}$ is a polynomial matrix in x . (b) As has been discussed in Remark 6.1.1, the state dependent linear-like representation (6.2) is highly non-unique. Therefore, the sufficient condition (6.19) can be affected by this non-uniqueness, which results in the success of synthesis being dependent on the chosen representation.

Remark 6.1.4 For given initial condition $x(0)$, (6.13) gives an upper bound of performance index $\mathcal{J}_\infty(t)$. Since $w(t)$ is assumed to be energy bounded, i.e., $\int_0^\infty w^T(t)w(t)dt \leq \hat{w}^2 < \infty$, let $Z^T(x_0)Q^{-1}(x_0)Z(x_0) \leq \delta$, therefore

$$\mathcal{J}_\infty(t) < \delta + \gamma \hat{w}^2 \quad (6.23)$$

By Schur complement, $Z^T(x_0)Q^{-1}(x_0)Z(x_0) \leq \delta$ is equivalent to (6.26) below. Hence

the multi-objective control problem described in (6.7)-(6.9) becomes

$$\text{Minimize } \delta \quad (6.24)$$

subject to

$$\Gamma(x) < 0 \quad (6.25)$$

$$\begin{bmatrix} \delta & Z^T(x_0) \\ Z(x_0) & Q(x_0) \end{bmatrix} \geq 0 \quad (6.26)$$

where $\Gamma(x)$ is as in (6.19).

6.1.3 SOS Based Optimization

The sufficient condition for the multi-objective state feedback design is based on a non-linear matrix inequality. Solving this inequality means solving an infinite set of LMIs. When only symmetric polynomial matrices are involved, the SOS decomposition can provide a computational relaxation for the sufficient condition in Theorem 6.1.2 [111].

Proposition 6.1.1 *Consider the nonlinear system (6.2) and the performance (6.7)-(6.9). For given upper bound $\gamma > 0$, if there exist polynomial matrix $Q(x)$ with $Q(x) = Q^T(x)$, $Y(x)$ and constant nonsingular matrix W such that, for some positive tuning scalar ϵ , a constant $s_1 > 0$ and a SOS polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following SOS optimization problem is feasible with $\Gamma(x)$ as in (6.19)*

$$\text{Minimize } \delta$$

s.t.

$$v_1^T [Q(x) - s_1 I] v_1 \in \Sigma_{sos} \quad (6.27)$$

$$-\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Gamma(x) + s_2(x)I) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (6.28)$$

$$v_5^T \begin{bmatrix} \delta & Z^T(x_0) \\ Z(x_0) & Q(x_0) \end{bmatrix} v_5 \in \Sigma_{sos} \quad (6.29)$$

where $v_1 \in \mathbb{R}^N$, $v_2 \in \mathbb{R}^N$, $v_3 \in \mathbb{R}^p$, $v_4 \in \mathbb{R}^{n_z}$ and $v_5 \in \mathbb{R}^{N+1}$, then the closed-loop system is asymptotically stabilized by (6.20). Furthermore, for any initial condition $x(0) = x_0$ in the region of attraction of the origin equilibrium, we have the upper bound of performance index as in (6.23).

With the definition of SOS polynomial and its decomposition, the result in Proposition 6.1.1 can be obtained directly, so the proof is omitted here.

Remark 6.1.5 The nonlinear matrix inequality (6.19) is not jointly convex in the set of matrix variables because of the existence of the nonlinear term $Q(x)\dot{P}(x)Q(x)$. A more general relaxation is provided to transform this nonconvex problem into a convex semidefinite programming problem by following the technique discussed in Section 4.1.3.

Define

$$\phi_1(x, v_2) = \left[v_2^T \frac{\partial Q(x)}{\partial x_1} v_2, \dots, v_2^T \frac{\partial Q(x)}{\partial x_n} v_2 \right] \quad (6.30)$$

By imposing a bound on the effect of the nonlinear term as in (6.31) below, we know that if γ_1 in (6.31) has zero minimum, then the nonlinear terms $Q(x)\dot{P}(x)Q(x)$ can be

replaced by its linear part $-\sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x) Z(x)$.

$$\begin{bmatrix} \gamma_1 & \phi_1(x, v_2)B_u(x) & \phi_1(x, v_2)B_w(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} \geq 0 \quad (6.31)$$

Hence, by including (6.31) as a SOS based constraint, Proposition 6.1.2 follows.

Proposition 6.1.2 *Consider the nonlinear system (6.2) and the performance (6.7)-(6.9). For given upper bound $\gamma > 0$, if there exist polynomial matrix $Q(x)$ with $Q(x) = Q^T(x)$, $Y(x)$ and constant nonsingular matrix W such that, for some sufficiently small value of γ_1 , some positive tuning scalar ϵ , a constant $s_1 > 0$ and a SOS polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following SOS optimization problem has feasible solutions*

$$\text{Minimize} \quad \delta$$

subject to

$$v_1^T [Q(x) - s_1 I] v_1 \in \Sigma_{\text{SOS}} \quad (6.32)$$

$$-\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Upsilon(x) + s_2(x)I) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{\text{SOS}} \quad (6.33)$$

$$v_5^T \begin{bmatrix} \delta & Z^T(x_0) \\ Z(x_0) & Q(x_0) \end{bmatrix} v_5 \in \Sigma_{\text{SOS}} \quad (6.34)$$

$$v_6^T \begin{bmatrix} \gamma_1 & \phi_1(x, v_2)B_u(x) & \phi_1(x, v_2)B_w(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} v_6 \in \Sigma_{\text{SOS}} \quad (6.35)$$

where $v_1 \in \mathbb{R}^N$, $v_2 \in \mathbb{R}^N$, $v_3 \in \mathbb{R}^p$, $v_4 \in \mathbb{R}^{n_z}$, $v_5 \in \mathbb{R}^{N+1}$ and $v_6 \in \mathbb{R}^{m+p+1}$. $\phi_1(x, v_2)$

is as in (6.30) and

$$\Upsilon(x) = \begin{bmatrix} Q(x) - W - W^T & * \\ W + \epsilon M(x)[A(x)W + B_u(x)Y(x)] & -Q(x) - \epsilon \sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x)Z(x) \\ 0 & B_w^T(x)M^T(x) \\ C_z(x)W + D_z(x)Y(x) & 0 \\ * & * \\ * & * \\ -\epsilon^{-1}\gamma I & * \\ 0 & -\epsilon^{-1}I \end{bmatrix} \quad (6.36)$$

then the closed-loop system is asymptotically stabilized by (6.20). Furthermore, for any initial condition $x(0) = x_0$ in the region of attraction of the origin equilibrium, we have the upper bound of performance index as in (6.23).

6.2 Multi-objective Robust and Reliable Control

The nonlinear systems considered in Section 6.1 assume that all parameters of the systems are known. In this section, we consider systems (6.2) whose matrices are not exactly known.

Consider the NLTI system in polytopic description

$$\begin{aligned} \dot{x} &= A(x; \theta)Z(x) + B_u(x; \theta)u + B_w(x; \theta)w \\ z &= C_z(x; \theta)Z(x) + D_z(x; \theta)u \\ x(0) &= x_0 \end{aligned} \quad (6.37)$$

where $A(x; \theta)$, $B_w(x; \theta)$, $B_u(x; \theta)$, $C_z(x; \theta)$ and $D_z(x; \theta)$ are polynomial matrices in x

of the form

$$\begin{aligned}
 A(x; \theta) &= \sum_{i=0}^q A_i(x) \theta_i, \quad B_u(x; \theta) = \sum_{i=0}^q B_{ui}(x) \theta_i \\
 B_w(x; \theta) &= \sum_{i=0}^q B_{wi}(x) \theta_i, \quad C_z(x; \theta) = \sum_{i=0}^q C_{zi}(x) \theta_i \\
 D_z(x; \theta) &= \sum_{i=0}^q D_{zi}(x) \theta_i
 \end{aligned} \tag{6.38}$$

The uncertain constant parameter vector $\theta = [\theta_0, \theta_1, \dots, \theta_q]^T \in \mathbb{R}^{q+1}$ satisfies

$$\theta \in \Theta \triangleq \left\{ \theta \in \mathbb{R}^{q+1} : \theta_i \geq 0, i = 0, 1, \dots, q, \text{ and } \sum_{i=0}^q \theta_i = 1 \right\} \tag{6.39}$$

and $Z(x)$ is an $N \times 1$ vector of monomials in x satisfying Assumption 6.1.1.

There are $q + 1$ vertices in the polytopic system (6.37). One vertex corresponds to the nominal case (no fault), and the remaining q vertices correspond to the fault cases. The parameter θ provides the interpolation between these vertices. Without loss of generality, suppose that $i = 0$ corresponds to the nominal case.

With the state feedback controller (6.4), we have the following closed-loop system

$$\begin{aligned}
 \dot{x} &= \tilde{A}(x; \theta) Z(x) + B_w(x; \theta) w \\
 z &= \tilde{C}(x; \theta) Z(x)
 \end{aligned} \tag{6.40}$$

with

$$\begin{aligned}
 \tilde{A}(x; \theta) &= A(x; \theta) + B_u(x; \theta) K(x) = \sum_{i=0}^q \theta_i \tilde{A}_i(x) \\
 \tilde{C}(x; \theta) &= C_z(x; \theta) + D_z(x; \theta) K(x) = \sum_{i=0}^q \theta_i \tilde{C}_i(x)
 \end{aligned} \tag{6.41}$$

where $\tilde{A}_i(x)$ and $\tilde{C}_i(x)$ are closed-loop matrices at the vertex

$$\begin{aligned}\tilde{A}_i(x) &= A_i(x) + B_{ui}(x)K(x) \\ \tilde{C}_i(x) &= C_{zi}(x) + D_{zi}(x)K(x)\end{aligned}\tag{6.42}$$

The objective is to design a state feedback control law (6.4) which stabilizes the closed-loop system (6.40) and achieves the multi-objective performance (6.7)-(6.9). In the design of the reliable robust controller, stability and performance are guaranteed not only for the vertices of the possible faults, but also for the intermediate faults between the vertices. The conventional method based on quadratic stability is very conservative, here the design method of the reliable robust control via parameter-dependent Lyapunov function is proposed.

Define the parameter-dependent Lyapunov function as

$$V(x) = Z^T(x)P(x; \theta)Z(x)\tag{6.43}$$

with $P(x; \theta)$ is nonsingular, $P(x; \theta) = P^T(x; \theta) > 0$ and $P(x; \theta) = Q^{-1}(x; \theta)$.

With the results in Section 6.1, the main result for robust state feedback synthesis can be obtained directly.

Theorem 6.2.1 *Consider the parameter-dependent nonlinear system (6.37) and the performance (6.7)-(6.9). For every given performance bound $\gamma_i > 0$, $i = 0, 1, \dots, q$, if there exist polynomial matrix $Q_i(x)$ with $Q_i(x) = Q_i^T(x) > 0$, $Y(x)$ and constant non-singular matrix W such that, for some positive tuning scalar ϵ , the nonlinear matrix*

inequalities (6.44) are satisfied for $i, l = 0, 1, \dots, q$

$$\Gamma_{il}(x) = \begin{bmatrix} Q_i(x) - W - W^T & * \\ W + \epsilon M(x)[A_i(x)W + B_{ui}(x)Y(x)] & \left\{ -Q_i(x) - \epsilon \sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} [A_{lj}(x)Z(x) + B_{ulj}(x)K(x)Z(x) + B_{wlj}(x)w] \right\} \\ 0 & B_{wi}^T(x)M^T(x) \\ C_{zi}(x)W + D_{zi}(x)Y(x) & 0 \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ -\epsilon^{-1}\gamma_i I & * \\ 0 & -\epsilon^{-1}I \end{bmatrix} < 0 \quad (6.44)$$

then the closed-loop system is asymptotically stabilized by (6.20). Furthermore, for any initial condition $x(0) = x_0$ in the region of attraction of the origin equilibrium, we have the upper bound of performance index as follows

$$\mathcal{J}_\infty(t) < Z^T(x_0)Q^{-1}(x_0; \theta)Z(x_0) + \gamma(\theta) \int_0^\infty w^T(t)w(t)dt \quad (6.45)$$

where θ is as in (6.39) and

$$Q(x; \theta) = \sum_{i=0}^q \theta_i Q_i(x) \quad \gamma(\theta) = \sum_{i=0}^q \gamma_i \theta_i \quad (6.46)$$

The proof of Theorem 6.2.1 only needs summation of $\Gamma_{il}(x)$ for $i, l = 0, 1, \dots, q$, hence the details are omitted here. $A_{lj}(x)$, $B_{ulj}(x)$ and $B_{wlj}(x)$ in (6.44) denote the j^{th} row of $A_l(x)$, $B_{ul}(x)$ and $B_{wl}(x)$ at the vertex l , resp.

Remark 6.2.1 (a) Since the suitable structure of (6.19) we can extend the multi-objective synthesis methodology of Section 6.1 to the parameter-dependent polynomial systems (6.37). (b) We adopt a parameter-dependent Lyapunov function with Lyapunov matrix $Q(x; \theta)$. When $Q_i(x) = Q(x)$ for $i = 0, 1, \dots, q$, $Q(x; \theta) = Q(x)$ and $V(x) =$

$Z^T(x)Q^{-1}(x)Z(x)$. Therefore the sufficient conditions in Theorem 6.2.1 are less conservative than that with a fixed Lyapunov function. (c) It is noted that there are $(q+1) \times (q+1)$ matrix inequalities in (6.44), and $(q+1) \times (q+2)$ matrix inequalities in Theorem 6.2.1.

Remark 6.2.2 Similarly, from (6.45) we know that with bounded energy disturbance, i.e., $\int_0^\infty w^T(t)w(t)dt \leq \hat{w}^2 < \infty$, let $Z^T(x_0)Q^{-1}(x_0; \theta)Z(x_0) \leq \delta(\theta)$ with $\delta(\theta) = \sum_{i=0}^q \delta_i \theta_i$, therefore

$$\mathcal{J}_\infty(t) < \delta(\theta) + \gamma(\theta)\hat{w}^2 \quad (6.47)$$

By Schur complement, $Z^T(x_0)Q^{-1}(x_0; \theta)Z(x_0) \leq \delta(\theta)$ is equivalent to (6.51). Hence the multi-objective control problem described in (6.7)-(6.9) becomes

$$\text{Minimize } \delta_0 \quad (6.48)$$

subject to

$$\delta_k < \bar{\delta}_k, \quad k = 1, \dots, q \quad (6.49)$$

$$\Gamma_{il}(x) < 0 \quad (6.50)$$

$$\begin{bmatrix} \delta(\theta) & Z^T(x_0) \\ Z(x_0) & Q(x_0; \theta) \end{bmatrix} \geq 0 \quad (6.51)$$

where $\Gamma_{il}(x)$ are as in (6.44).

Using the similar technique as in Section 4.1.3 to deal with the nonlinear term in (6.44).

Define $\phi_{1i}(x, v_2)$ as

$$\phi_{1i}(x, v_2) = \left[v_2^T \frac{\partial Q_i(x)}{\partial x_1} v_2, \dots, v_2^T \frac{\partial Q_i(x)}{\partial x_n} v_2 \right] \quad (6.52)$$

and r_1 as the upper bound of $[\phi_{1i}(x, v_2)B_{ul}(x)][\phi_{1i}(x, v_2)B_{ul}(x)]^T$ and $[\phi_{1i}(x, v_2)B_{wl}(x)][\phi_{1i}(x, v_2)B_{wl}(x)]^T$ then we have the corresponding SOS optimization problem.

Proposition 6.2.1 *Consider the parameter-dependent nonlinear system (6.37) and the performance (6.7)-(6.9). For every given performance bounds γ_i , $i = 0, 1, \dots, q$ and δ_k , $k = 1, \dots, q$, if there exist polynomial matrix $Q_i(x)$ with $Q_i(x) = Q_i^T(x)$, $Y(x)$ and constant nonsingular matrix W such that, for some sufficiently small value of r_1 , some positive tuning scalar ϵ , constants $s_i > 0$ and SOS polynomials $s_{il}(x)$ with $s_{il}(x) > 0$ for $x \neq 0$ ($i, l = 0, 1, \dots, q$), the following SOS optimization problem has feasible solutions*

$$\text{Minimize} \quad \delta_0$$

subject to

$$v_1^T [Q_i(x) - s_i I] v_1 \in \Sigma_{sos} \quad (6.53)$$

$$- \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Upsilon_{il}(x) + s_{il}(x)I) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (6.54)$$

$$v_5^T \begin{bmatrix} \delta_0 & Z^T(x_0) \\ Z(x_0) & Q_0(x_0) \end{bmatrix} v_5 \in \Sigma_{sos} \quad (6.55)$$

$$v_5^T \begin{bmatrix} \delta_k & Z^T(x_0) \\ Z(x_0) & Q_k(x_0) \end{bmatrix} v_5 \in \Sigma_{sos} \quad (6.56)$$

$$v_6^T \begin{bmatrix} r_1 & \phi_{1i}(x, v_2)B_{ul}(x) & \phi_{1i}(x, v_2)B_{wl}(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} v_6 \in \Sigma_{sos} \quad (6.57)$$

where $v_1 \in \mathbb{R}^N$, $v_2 \in \mathbb{R}^N$, $v_3 \in \mathbb{R}^p$, $v_4 \in \mathbb{R}^{n_z}$, $v_5 \in \mathbb{R}^{N+1}$ and $v_6 \in \mathbb{R}^{m+p+1}$. $\phi_{1i}(x, v_2)$ is as in (6.52) and

$$\Upsilon_{il}(x) = \begin{bmatrix} Q_i(x) - W - W^T & * \\ W + \epsilon M(x)[A_i(x)W + B_{ui}(x)Y(x)] & -Q_i(x) - \epsilon \sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} A_{lj} Z \\ 0 & B_{wi}^T(x) M^T(x) \\ C_{zi}(x)W + D_{zi}(x)Y(x) & 0 \\ * & * \\ * & * \\ -\epsilon^{-1} \gamma_i I & * \\ 0 & -\epsilon^{-1} I \end{bmatrix} \quad (6.58)$$

then the closed-loop system is asymptotically stabilized by (6.20). Furthermore, for any initial condition $x(0) = x_0$ in the region of attraction of the origin equilibrium, we have the upper bound of performance index as follows

$$\mathcal{J}_\infty(t) < \sum_i \theta_i \delta_i + \gamma(\theta) \hat{w}^2 \quad (6.59)$$

Proof: The proof is similar with that of Proposition 4.3.1, hence it is omitted here. ■

Remark 6.2.3 Proposition 6.2.1 provides a multi-objective reliable controller design method for polynomial nonlinear systems. The reliable state feedback controller $u = Y(x)W^{-1} Z(x)$ is derived to minimize the performance upper bound δ_0 of the nominal system, while simultaneously guarantee that the performance indices meet certain upper bounds $\bar{\delta}_k$, $k = 1, \dots, q$ for all probable fault systems. It should be noted that different performance indices are chosen for different faults.

Remark 6.2.4 The result in Proposition 6.2.1 is a global design when Lyapunov matrices $Q_i(x)$ are constant matrices, then the closed-loop stability and optimal upper bound hold globally.

Remark 6.2.5 The proposed methodology in this chapter can be extended to output feedback controllers satisfying the multi-objective performance (6.7)-(6.9). The SOS optimization algorithm can be obtained as in Proposition 6.2.1 where $v_1 \in \mathbb{R}^{N+h}$, $v_2 \in \mathbb{R}^{N+h}$, $v_3 \in \mathbb{R}^p$, $v_4 \in \mathbb{R}^{n_z}$, $v_5 \in \mathbb{R}^{N+h+1}$ and $v_6 \in \mathbb{R}^{m+p+1}$, $\phi_{1i}(x, v_2)$ is as in (6.52). $\Upsilon_{il}(x)$ in (6.54), SOS constraints (6.55) and (6.56) are modified as follows:

$$\Upsilon_{il}(x) = \begin{bmatrix} \Upsilon_{il1,1} & * & * & * \\ \Upsilon_{il2,1} & \Upsilon_{il2,2} & * & * \\ 0 & \tilde{B}_i^T(x) & -\epsilon^{-1}\gamma_i I & * \\ \Upsilon_{il4,1} & 0 & 0 & -\epsilon^{-1}I \end{bmatrix} \quad (6.60)$$

$$v_5^T \begin{bmatrix} \delta_0 & \xi^T(x_0) \\ \xi(x_0) & Q_0(x_0) \end{bmatrix} v_5 \in \Sigma_{sos} \quad (6.61)$$

$$v_5^T \begin{bmatrix} \delta_k & \xi^T(x_0) \\ \xi(x_0) & Q_k(x_0) \end{bmatrix} v_5 \in \Sigma_{sos} \quad (6.62)$$

with $\xi = [Z^T(x), y^T(x)]^T$. $\tilde{B}_i(x)$, $\tilde{W}(x)$ are as in (5.84) and (5.20), resp., and

$$\begin{aligned} \Upsilon_{il1,1} &= Q_i(x) - \tilde{W}(x) - \tilde{W}^T(x) \\ \Upsilon_{il2,1} &= \Gamma_{hil21}(x), \Upsilon_{il4,1} = \Gamma_{hil41}(x) \text{ in (5.86)} \\ \Upsilon_{il2,2} &= -Q_i(x) - \epsilon \sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} [A_{lj}(x)Z(x)] \end{aligned} \quad (6.63)$$

then similarly, the closed-loop system is asymptotically stabilized by $u = K(y)y$. And the upper bound of performance index is achieved as

$$\mathcal{J}_\infty(t) < \sum_i \theta_i \delta_i + \gamma(\theta) \hat{w}^2 \quad (6.64)$$

6.3 Numerical Example

In this section, a reliable control design example is presented to demonstrate the effectiveness of the proposed controller design approach in Section 6.2. The SOS relaxation and computation algorithm have been provided in Proposition 6.2.1 for the reliable control design problem. The SOS programming is utilized to solve the convex optimization problem to obtain computationally tractable solutions.

Consider a parameter-dependent nonlinear system of the form (6.37) with system matrices given by

Nominal system:

$$\begin{aligned} A_0(x) &= \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 & \frac{1}{4} - x_1^2 \\ 0 & 0 \end{bmatrix}, B_{u0} = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix} \\ B_{w0} &= \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}, C_{z0} = \begin{bmatrix} 0 & 1 \end{bmatrix}, D_{z0} = 0.1 \end{aligned} \quad (6.65)$$

Fault system:

$$\begin{aligned} A_1(x) &= \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 - \frac{3}{4}x_2^2 & \frac{1}{4} - x_1^2 - \frac{1}{2}x_2^2 \\ 0 & 0 \end{bmatrix}, B_{u1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ B_{w1} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_{z1} = \begin{bmatrix} 0 & 1.5 \end{bmatrix}, D_{z1} = 1 \end{aligned} \quad (6.66)$$

and $Z(x) = [x_1 \ x_2]^T$.

Based on the SOS relaxations in Proposition 6.2.1, the SOS programming is utilized to obtain a reliable nonlinear state feedback control law (6.67) with the initial condition

given as $x_0 = [1, 1]^T$. For $\epsilon = 0.001$, $\gamma_0 = \gamma_1 = 4$ and $\delta_1 = 3$, the SOS based optimization problem returns 2.7077 as the optimal value of δ_0 and 10^{-6} as the value of r_1 .

$$u = -0.9366x_1 - 2.6533x_2 - 0.0129x_1^2 - 0.008907x_1x_2 + 0.4958 \times 10^{-11}x_2^2 \tag{6.67}$$

Table 6.1 lists the minimum vales of the performance upper bound δ_0 and δ_1 obtained from SOS optimization, and the corresponding values of $Z^T(x_0)Q_0^{-1}(x_0)Z(x_0)$ and $Z^T(x_0)Q_1^{-1}(x_0)Z(x_0)$. The actual achieved cost is also given in Table 6.1, which is calculated by solving (6.15) for the nominal system and fault system, respectively.

Table 6.1: Performance of reliable control

	Nominal system (i=0)	Fault system (i=1)
Upper bound (δ_i)	2.7077	3
$Z^T(x_0)Q_i^{-1}(x_0)Z(x_0)$	2.456	2.5796
Achieved cost	0.5981	1.0755

Simulations are also carried out to verify the effectiveness of the designed controller. Figure 6.1 and Figure 6.2 show the state trajectories and the measured output signal of the closed-loop systems, where a disturbance of step signal is introduced from 15 sec to 20 sec. It can be seen that the closed-loop systems are stabilized by the designed reliable control law (6.67), and the L_2 gains from w to z of the closed-loop systems are no greater than $\sqrt{\gamma_0} = \sqrt{\gamma_1} = 2$.

Figure 6.3 shows the state trajectories and the measured output signal of 10 interpolated closed-loop systems at various values of θ between the two vertices in (6.65) and (6.66), where a disturbance of step signal is introduced from 15 sec to 20 sec. Simulations show that not only the performance of the vertices of the possible faults, but also that of the intermediate faults between the vertices are guaranteed.

Figure 6.1: State and controller trajectories of the closed-loop systems, nonlinear reliable control law (6.67)

Figure 6.2: Attenuated output signal of the closed-loop nominal and fault systems

Figure 6.3: States and attenuated output signal of the interpolated closed-loop systems

6.4 Conclusion

This chapter discusses the multi-objective reliable control problem for parameter-dependent polynomial nonlinear systems. In particular, the concept of the passive reliable design is utilized for the multi-objective design problem, where a fixed controller is designed for the nominal case as well as fault cases. Sufficient conditions to guarantee the closed-loop stability are proposed in terms of nonlinear matrix inequalities via state feedback. At the same time not only the optimal performance during normal system operation, but also the satisfactory performance in the presence of faults is guaranteed. We eliminate the coupling terms between system matrices and the Lyapunov matrix by introducing additional matrix variables, hence SDMIs based conditions have a more suitable structure to deal with parameter uncertainty. Then these SDMIs based conditions are formulated to SOS based multi-objective optimization problems, which can be solved in an efficient computational manner. Finally, a reliable control design example is provided to demonstrate the effectiveness of the proposed approach.

Chapter 7

Robust H_∞ Filter Design for Polynomial Nonlinear Systems

All state variables are not always accessible for direct measurement in practical feedback control systems. Therefore, it is an essential need for the state variables estimation, in particular in the case of model-based controller synthesis or process monitoring.

The Kalman filtering approach has been a popular approach to deal with the state estimation problem with the assumptions of exact known system dynamics and a prior information on the external noise [161]. However, modelling error and system uncertainties are always unavoidable in the plant model, and the statistic information of the noise may not be exactly known. In such cases, H_∞ filtering was introduced [91, 121]. The problem of H_∞ filtering consist of designing an H_∞ filter that ensures the worst case H_∞ norm from the unknown norm-bounded exogenous disturbance signal to the estimation error minimized.

The robust H_∞ filter design for linear systems has been paid considerable attention over the past decades and a lot of results have been reported [27, 34, 35, 61, 73, 91, 93, 149, 150, 166, 167]. When the system is nonlinear and has uncertainties the H_∞ filter design turns out to be much more difficult. Although the robust H_∞ filter design for nonlinear

systems has not been studied so widely as for linear cases, there are still some advances presented in [1, 4, 25, 28, 119, 140, 148, 158, 160] and the references therein.

In this chapter, we propose a computational scheme of solving the robust H_∞ filtering problems for the polynomial nonlinear systems against parametric uncertainty and admissible norm-bounded disturbance input. Sufficient conditions to guarantee the augmented system stability with guaranteed H_∞ norm from the unknown norm-bounded disturbance signal to the estimation error are derived in terms of SDMIs via a stable H_∞ filter. The main contribution of the proposed design lies in that the coupling terms among the system dynamics, filter dynamics and the Lyapunov matrix are eliminated, then the SDMIs based conditions are more suitable for the nonlinear robust filter design for the polynomial systems with polytopic uncertainty, and the robust H_∞ filter and Lyapunov functions can be constructed in an efficient computational manner.

7.1 H_∞ Filter Design

In this section, H_∞ Filter Design result for polynomial nonlinear systems is proposed. This result is extended to the synthesis for the parameter-dependent polynomial systems in Section 7.2.

7.1.1 Problem Formulation

Consider the following NLTI system

$$\begin{aligned}\dot{x} &= f(x) + g(x)w \\ y &= h_y(x) + l_y(x)w \\ z &= h_z(x) + l_z(x)w\end{aligned}\tag{7.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^p$ is the exogenous disturbance with bounded energy, $y \in \mathbb{R}^h$ is the measured output and $z \in \mathbb{R}^{n_z}$ is the signal to be estimated. $f(x) \in \mathcal{R}_n^n$, $g(x) \in \mathcal{R}_n^{n \times p}$, $h_y(x) \in \mathcal{R}_n^h$, $l_y(x) \in \mathcal{R}_n^{h \times p}$, $h_z(x) \in \mathcal{R}_n^{n_z}$ and $l_z(x) \in \mathcal{R}_n^{n_z \times p}$ are known polynomial vector fields of appropriate dimensions satisfying $f(x) = 0$.

We write the system (7.1) in the following state dependent linear-like representation:

$$\begin{aligned}\Sigma: \quad \dot{x} &= A(x)x + B(x)w \\ y &= C(x)x + D(x)w \\ z &= C_z(x)x + D_z(x)w\end{aligned}\tag{7.2}$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$, $C_z(x)$ and $D_z(x)$ are polynomial matrices in x .

We propose a full-order filter for the system Σ in (7.2) described by

$$\begin{aligned}\mathcal{F}: \quad \dot{\hat{x}} &= \hat{A}(\hat{x})\hat{x} + \hat{B}(\hat{x})y \\ \hat{z} &= \hat{C}_z(\hat{x})\hat{x} + \hat{D}_z(\hat{x})y\end{aligned}\tag{7.3}$$

where $\hat{x} \in \mathbb{R}^n$ is the filter state vector, and $\hat{z} \in \mathbb{R}^{n_z}$ is the estimation of signal z . $\hat{A}(\hat{x})$, $\hat{B}(\hat{x})$, $\hat{C}_z(\hat{x})$ and $\hat{D}_z(\hat{x})$ are the filter coefficient matrices to be determined. It is assumed that the initial condition of system Σ , as well as the initial condition of filter $\mathcal{F}$ are both zero.

Connecting the filter $\mathcal{F}$ and the system Σ together, and denoting $x_{cl} = [x^T \ \hat{x}^T]^T$ and $z_{cl} := z - \hat{z}$. It follows from (7.2) and (7.3) that

$$\begin{aligned}\dot{x}_{cl} &= A_{cl}(x, \hat{x})x_{cl} + B_{cl}(x, \hat{x})w \\ z_{cl} &= C_{cl}(x, \hat{x})x_{cl} + D_{cl}(x, \hat{x})w\end{aligned}\tag{7.4}$$

where

$$\begin{aligned}
 A_{cl}(x, \hat{x}) &= \begin{bmatrix} A(x) & 0 \\ \hat{B}(\hat{x})C(x) & \hat{A}(\hat{x}) \end{bmatrix} \\
 B_{cl}(x, \hat{x}) &= \begin{bmatrix} B(x) \\ \hat{B}(\hat{x})D(x) \end{bmatrix} \\
 C_{cl}(x, \hat{x}) &= \begin{bmatrix} C_z(x) - \hat{D}_z(\hat{x})C(x) & -\hat{C}_z(\hat{x}) \end{bmatrix} \\
 D_{cl}(x, \hat{x}) &= D_z(x) - \hat{D}_z(\hat{x})D(x)
 \end{aligned} \tag{7.5}$$

The objective considered in this section is to investigate the following H_∞ filtering problem.

H_∞ filtering problem: Given the nonlinear system Σ in (7.2) and assume that $w \in \ell_2[0, \infty)$, find an H_∞ filter $\mathcal{F}$ of the form (7.3), if exists, such that

- The augmented system (7.4)-(7.5) is asymptotically stable.
- Under the zero-initial condition, the induced ℓ_2 gain from the exogenous disturbance w to the estimation error $z_{cl} = z(t) - \hat{z}(t)$ satisfies

$$\int_0^\infty [z(t) - \hat{z}(t)]^T [z(t) - \hat{z}(t)] dt < \gamma^2 \int_0^\infty [w(t)^T w(t)] dt \tag{7.6}$$

for all non-zero $w \in \ell_2[0, \infty)$, where $\gamma > 0$ is a prescribed scalar.

7.1.2 H_∞ Filter

In this section, the result to derive an H_∞ filter $\mathcal{F}$ for the nonlinear system Σ is formulated in terms of nonlinear matrix inequalities which involve affine matrix variables. Therefore SOS programming technique can be utilized to obtain computationally tractable solutions.

The methodology presented in this paper is based on the Lyapunov stability argument. Define the Lyapunov function as

$$V(x, \hat{x}) = x_{cl}^T P(x) x_{cl} \quad (7.7)$$

Theorem 7.1.1 Consider nonlinear system Σ in (7.2). If there exist $P_1(x)$, $P_2(x)$, $P_3(x)$ with $P_1(x) = P_1^T(x)$, $P_3(x) = P_3^T(x)$, $H_{11}(x, \hat{x})$, $H_{13}(x, \hat{x})$, $H_{21}(x, \hat{x})$, $H_{23}(x, \hat{x})$, $H_{31}(x, \hat{x})$, $H_{33}(x, \hat{x})$, $G_{11}(x, \hat{x})$, $G_{13}(x, \hat{x})$, $G_{21}(x, \hat{x})$, $G_{23}(x, \hat{x})$, $G_{31}(x, \hat{x})$, $G_{33}(x, \hat{x})$, $L_A(\hat{x})$, $L_B(\hat{x})$, $\hat{C}_z(\hat{x})$, $\hat{D}_z(\hat{x})$ and nonsingular constant matrix G such that, for given constant matrices Y_1 , Y_2 , and some positive tuning scalars λ_1 , λ_2 , α_1 , α_2 and α_3 , the nonlinear matrix inequalities in (7.8)-(7.9) are satisfied,

$$P(x) = \begin{bmatrix} P_1(x) & P_2(x) \\ P_2^T(x) & P_3(x) \end{bmatrix} > 0 \quad (7.8)$$

$$\begin{bmatrix} \Pi_{1,1}(x, \hat{x}) & * & * \\ \Pi_{2,1}(x, \hat{x}) & \Pi_{2,2}(x, \hat{x}) & * \\ \Pi_{3,1}(x, \hat{x}) & 0 & -\gamma^2 I \end{bmatrix} < 0 \quad (7.9)$$

where

$$\begin{aligned} \Pi_{1,1}(x, \hat{x}) &= \begin{bmatrix} \Psi_1 + \dot{P}(x) & * \\ \Psi_2 & \Psi_3 \end{bmatrix} \\ \Pi_{2,1}(x, \hat{x}) &= \begin{bmatrix} \Phi_7 & \Phi_8 & \Phi_9 \\ \Phi_{10} & \Phi_{11} & \Phi_{12} \\ \Phi_{13} & \Phi_{14} & \Phi_{15} \end{bmatrix} \\ \Pi_{2,2}(x, \hat{x}) &= \begin{bmatrix} -H_{11}(x, \hat{x}) - H_{11}^T(x, \hat{x}) & * \\ -H_{21}(x, \hat{x}) - \alpha_1 G^T & -\alpha_2 G - \alpha_2 G^T \\ -H_{31}(x, \hat{x}) - H_{13}^T(x, \hat{x}) & -\alpha_3 Y_2 G - H_{23}^T(x, \hat{x}) \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} * \\ * \\ -H_{33}(x, \hat{x}) - H_{33}^T(x, \hat{x}) \end{bmatrix}$$

$$\Pi_{3,1}(x, \hat{x}) = \begin{bmatrix} C_z(x) - \hat{D}_z(\hat{x})C(x) & -\hat{C}_z(\hat{x}) & D_z(x) - \hat{D}_z(\hat{x})D(x) \end{bmatrix} \quad (7.10)$$

with

$$\Psi_1 = \begin{bmatrix} \Phi_1 & * \\ \Phi_2 & \Phi_3 \end{bmatrix}, \Psi_2 = \begin{bmatrix} \Phi_4 & \Phi_5 \end{bmatrix}, \Psi_3 = \Phi_6 \quad (7.11)$$

and

$$\begin{aligned} \Phi_1 &= G_{11}(x, \hat{x})A(x) + L_B(\hat{x})C(x) + [G_{11}(x, \hat{x})A(x) + L_B(\hat{x})C(x)]^T \\ \Phi_2 &= G_{21}(x, \hat{x})A(x) + \lambda_1 L_B(\hat{x})C(x) + L_A^T(\hat{x}) \\ \Phi_3 &= \lambda_1 [L_A(\hat{x}) + L_A^T(\hat{x})] \\ \Phi_4 &= G_{31}(x, \hat{x})A(x) + \lambda_2 Y_1 L_B(\hat{x})C(x) + [G_{11}(x, \hat{x})B(x) \\ &\quad + L_B(\hat{x})D(x) - \frac{1}{2}G_{13}(x, \hat{x})]^T \\ \Phi_5 &= \lambda_2 Y_1 L_A(\hat{x}) + [G_{21}(x, \hat{x})B(x) + \lambda_1 L_B(\hat{x})D(x) - \frac{1}{2}G_{23}(x, \hat{x})]^T \\ \Phi_6 &= G_{31}(x, \hat{x})B(x) + \lambda_2 Y_1 L_B(\hat{x})D(x) - \frac{1}{2}G_{33}(x, \hat{x}) \\ &\quad + [G_{31}(x, \hat{x})B(x) + \lambda_2 Y_1 L_B(\hat{x})D(x) - \frac{1}{2}G_{33}(x, \hat{x})]^T \\ \Phi_7 &= P_1(x) - G_{11}^T(x, \hat{x}) + H_{11}(x, \hat{x})A(x) + \alpha_1 L_B(\hat{x})C(x) \\ \Phi_8 &= P_2(x) - G_{21}^T(x, \hat{x}) + \alpha_1 L_A(\hat{x}) \\ \Phi_9 &= -G_{31}^T(x, \hat{x}) + H_{11}(x, \hat{x})B(x) + \alpha_1 L_B(\hat{x})D(x) - \frac{1}{2}H_{13}(x, \hat{x}) \\ \Phi_{10} &= P_2^T(x) - G^T + H_{21}(x, \hat{x})A(x) + \alpha_2 L_B(\hat{x})C(x) \\ \Phi_{11} &= P_3(x) - \lambda_1 G^T + \alpha_2 L_A(\hat{x}) \\ \Phi_{12} &= -\lambda_2 [Y_1 G]^T + H_{21}(x, \hat{x})B(x) + \alpha_2 L_B(\hat{x})D(x) - \frac{1}{2}H_{23}(x, \hat{x}) \\ \Phi_{13} &= -G_{13}^T(x, \hat{x}) + H_{31}(x, \hat{x})A(x) + \alpha_3 Y_2 L_B(\hat{x})C(x) \\ \Phi_{14} &= -G_{23}^T(x, \hat{x}) + \alpha_3 Y_2 L_A(\hat{x}) \end{aligned}$$

$$\Phi_{15} = I - G_{33}^T(x, \hat{x}) + H_{31}(x, \hat{x})B(x) + \alpha_3 Y_2 L_B(\hat{x})D(x) - \frac{1}{2}H_{33}(x, \hat{x}) \quad (7.12)$$

then an H_∞ filter in the form of (7.3) solves the H_∞ filtering problem with

$$\hat{A}(\hat{x}) = G^{-1}L_A(\hat{x}), \hat{B}(\hat{x}) = G^{-1}L_B(\hat{x}) \quad (7.13)$$

Proof: Denote

$$\begin{aligned} \tilde{A}(x, \hat{x}) &= \begin{bmatrix} A_{cl}(x, \hat{x}) & B_{cl}(x, \hat{x}) \\ 0 & -\frac{1}{2}I \end{bmatrix} \\ \tilde{C}(x, \hat{x}) &= \begin{bmatrix} C_{cl}(x, \hat{x}) & D_{cl}(x, \hat{x}) \end{bmatrix} \\ \tilde{P}(x) &= \begin{bmatrix} P(x) & 0 \\ 0 & I \end{bmatrix} \\ X(x) &= \begin{bmatrix} \dot{P}(x) & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (7.14)$$

where $A_{cl}(x, \hat{x})$, $B_{cl}(x, \hat{x})$, $C_{cl}(x, \hat{x})$ and $D_{cl}(x, \hat{x})$ are as in (7.5), and $P(x)$ is as in (7.8). The condition (7.8) implies that the Lyapunov function (7.7) is positive definite for $x_{cl} \neq 0$ and $V(0) = 0$. Firstly, we prove that if the inequality (7.15) is satisfied, then with the zero initial condition, the system is stable and H_∞ performance is achieved as in (7.6).

$$\begin{bmatrix} \tilde{G}(x, \hat{x})\tilde{A}(x, \hat{x}) + \tilde{A}^T(x, \hat{x})\tilde{G}^T(x, \hat{x}) + X(x) & * & * \\ \tilde{P}(x) - \tilde{G}^T(x, \hat{x}) + \tilde{H}(x, \hat{x})\tilde{A}(x, \hat{x}) & -\tilde{H}(x, \hat{x}) - \tilde{H}^T(x, \hat{x}) & * \\ \tilde{C}(x, \hat{x}) & 0 & -\gamma^2 I \end{bmatrix} < 0 \quad (7.15)$$

Multiplying the inequality (7.15) from the left and the right by $\Psi(x, \hat{x})$ and $\Psi^T(x, \hat{x})$,

respectively, where

$$\Psi(x, \hat{x}) = \begin{bmatrix} I & \tilde{A}^T(x, \hat{x}) & 0 \\ 0 & 0 & I \end{bmatrix}$$

then by the Schur complement, we have

$$\tilde{A}^T(x, \hat{x})\tilde{P}(x) + \tilde{P}(x)\tilde{A}(x, \hat{x}) + X(x) + \gamma^{-2}\tilde{C}^T(x, \hat{x})\tilde{C}(x, \hat{x}) < 0 \quad (7.16)$$

With $\tilde{A}(x, \hat{x})$, $\tilde{C}(x, \hat{x})$, $\tilde{P}(x)$ and $X(x)$ in (7.14), the inequality (7.16) is equivalent to

$$\begin{bmatrix} P(x)A_{cl}(x, \hat{x}) + A_{cl}^T(x, \hat{x})P(x) + \dot{P}(x) & * & * \\ B_{cl}^T(x, \hat{x})P(x) & -I & * \\ C_{cl}(x, \hat{x}) & D_{cl}(x, \hat{x}) & -\gamma^2 I \end{bmatrix} < 0 \quad (7.17)$$

The time derivative of the Lyapunov function $V(x, \hat{x})$ in (7.7) along the augmented trajectories is given by

$$\begin{aligned} \dot{V}(x, \hat{x}) &= x_{cl}^T [P(x)A_{cl}(x, \hat{x}) + A_{cl}^T(x, \hat{x})P(x) + \dot{P}(x)]x_{cl} \\ &\quad + x_{cl}^T P(x)B_{cl}(x, \hat{x})w + w^T B_{cl}^T(x, \hat{x})P(x)x_{cl} \end{aligned} \quad (7.18)$$

From (7.17) we have that $\dot{V}(x, \hat{x}) < 0$ for $x_{cl} \neq 0$ and $w = 0$. Therefore the augmented system is asymptotically stable about the origin equilibrium. Furthermore,

$$\begin{aligned} &\dot{V}(x, \hat{x}) + \gamma^{-2}z_{cl}^T z_{cl} - w^T w \\ &= \begin{bmatrix} x_{cl} \\ w \end{bmatrix}^T \left(\begin{bmatrix} A_{cl}^T(x, \hat{x})P(x) + P(x)A_{cl}(x, \hat{x}) + \dot{P}(x) & * \\ B_{cl}^T(x, \hat{x})P(x) & -I \end{bmatrix} \right. \\ &\quad \left. + \gamma^{-2} \begin{bmatrix} C_{cl}^T(x, \hat{x})C_{cl}(x, \hat{x}) & * \\ D_{cl}^T(x, \hat{x})C_{cl}(x, \hat{x}) & D_{cl}^T(x, \hat{x})D_{cl}(x, \hat{x}) \end{bmatrix} \right) \begin{bmatrix} x_{cl} \\ w \end{bmatrix} \end{aligned} \quad (7.19)$$

From (7.17) and by the Schur complement we have $\dot{V}(x, \hat{x}) + \gamma^{-2} z_{cl}^T z_{cl} - w^T w < 0$. Then with the zero initial condition, the system is stable and the H_∞ performance is achieved as in (7.6).

We partition $\tilde{G}(x, \hat{x})$ and $\tilde{H}(x, \hat{x})$ in the following blocked matrices:

$$\begin{aligned} \tilde{G}(x, \hat{x}) &= \begin{bmatrix} G_{11}(x, \hat{x}) & G_{12}(x, \hat{x}) & G_{13}(x, \hat{x}) \\ G_{21}(x, \hat{x}) & G_{22}(x, \hat{x}) & G_{23}(x, \hat{x}) \\ G_{31}(x, \hat{x}) & G_{32}(x, \hat{x}) & G_{33}(x, \hat{x}) \end{bmatrix} \\ \tilde{H}(x, \hat{x}) &= \begin{bmatrix} H_{11}(x, \hat{x}) & H_{12}(x, \hat{x}) & H_{13}(x, \hat{x}) \\ H_{21}(x, \hat{x}) & H_{22}(x, \hat{x}) & H_{23}(x, \hat{x}) \\ H_{31}(x, \hat{x}) & H_{32}(x, \hat{x}) & H_{33}(x, \hat{x}) \end{bmatrix} \end{aligned} \quad (7.20)$$

For the full order filter design, we assume that $G_{12}(x, \hat{x})$, $G_{22}(x, \hat{x})$, $H_{12}(x, \hat{x})$ and $H_{22}(x, \hat{x})$ are nonsingular. It is obvious that with $T = \text{diag}\{I, G_{12}(x, \hat{x})G_{22}^{-1}(x, \hat{x}), I\}$,

$$T\tilde{G}(x, \hat{x})T^T = \begin{bmatrix} G_{11} & G_{12}G_{22}^{-T}G_{12}^T & G_{13} \\ G_{12}G_{22}^{-1}G_{21} & G_{12}G_{22}^{-T}G_{12}^T & G_{12}G_{22}^{-1}G_{23} \\ G_{31} & G_{32}G_{22}^{-T}G_{12}^T & G_{33} \end{bmatrix} \quad (7.21)$$

and the first and second block entries in the second column of the above $T\tilde{G}(x, \hat{x})T^T$ are identical. According to the congruence transformation in (7.21), we assume that $\tilde{G}(x, \hat{x})$ in (7.15) possess the following form:

$$\tilde{G}(x, \hat{x}) = \begin{bmatrix} G_{11}(x, \hat{x}) & G & G_{13}(x, \hat{x}) \\ G_{21}(x, \hat{x}) & \lambda_1 G & G_{23}(x, \hat{x}) \\ G_{31}(x, \hat{x}) & \lambda_2 Y_1 G & G_{33}(x, \hat{x}) \end{bmatrix} \quad (7.22)$$

where G is a nonsingular and constant matrix, Y_1 is a given constant matrix, and λ_1 and

λ_2 are positive tuning scalars. We further consider $\tilde{H}(x, \hat{x})$ in (7.15) has the similar form

$$\tilde{H}(x, \hat{x}) = \begin{bmatrix} H_{11}(x, \hat{x}) & \alpha_1 G & H_{13}(x, \hat{x}) \\ H_{21}(x, \hat{x}) & \alpha_2 G & H_{23}(x, \hat{x}) \\ H_{31}(x, \hat{x}) & \alpha_3 Y_2 G & H_{33}(x, \hat{x}) \end{bmatrix} \quad (7.23)$$

Let $G\hat{A}(\hat{x}) = L_A(\hat{x})$ and $G\hat{B}(\hat{x}) = L_B(\hat{x})$. The inequality (7.9) can be obtained by substituting the above matrices $\tilde{G}(x, \hat{x})$ and $\tilde{H}(x, \hat{x})$ in (7.22) and (7.23) into (7.15). Furthermore, the filter dynamics are given by (7.13). ■

Remark 7.1.1 *The advantage of the result in Theorem 7.1.1 is that by introducing slack variables $\tilde{G}(x, \hat{x})$ and $\tilde{H}(x, \hat{x})$, (7.9) separates the system matrix and Lyapunov matrix. Therefore, the sufficient conditions in Theorem 7.1.1 have more suitable structure in the filter design for the polynomial nonlinear systems with polytopic uncertainty.*

Remark 7.1.2 *It should be noted that there is still a nonlinear term $\dot{P}(x)$ in (7.9) which leads to the set of matrix variables satisfying the sufficient condition (7.9) is not jointly convex. A more general relaxation for this nonlinear term will be provided in the next section so as the SOS programming technique can be utilized to solve the H_∞ filtering problem via semidefinite programming.*

Remark 7.1.3 *By choosing G as a constant matrix, it is easy to see that the filter dynamics $\hat{A}(\hat{x}) = G^{-1}L_A(\hat{x})$ and $\hat{B}(\hat{x}) = G^{-1}L_B(\hat{x})$ are polynomial matrices in $\hat{x}$. A possible solution to reduce the involved conservativeness can be obtained with a polynomial matrix $G(\hat{x})$, in this case, the filter parameters $\hat{A}(\hat{x})$ and $\hat{B}(\hat{x})$ are rational.*

7.1.3 SOS Based Optimization

The sufficient conditions for the H_∞ filter design are formulated as the nonlinear matrix inequalities. Solving these inequalities means solving an infinite set of LMIs. By

Proposition 2.5.1, when only symmetric polynomial matrices are involved, the SOS decomposition can provide a computational relaxation for the sufficient conditions in Theorem 7.1.1.

Because of the existence of the nonlinear term $\dot{P}(x)$, the set of matrix variables satisfying the nonlinear matrix inequality (7.9) is not jointly convex. Hence the search for matrix variables can not be solved by the semidefinite programming directly. The transformation of this non-convex problem into a convex semidefinite programming problem is discussed as follows:

Since the Lyapunov matrix $P(x)$ depends on the full states x , and by following the technique discussed in Section 4.1.3, we define

$$\phi_1(x, v_{21}) = \left[v_{21}^T \frac{\partial P(x)}{\partial x_1} v_{21}, \dots, v_{21}^T \frac{\partial P(x)}{\partial x_n} v_{21} \right] \quad (7.24)$$

By imposing a bound on the effect of the nonlinear term as in (7.25) below, we know that if γ_1 in (7.25) has zero minimum, then the nonlinear terms $\dot{P}(x)$ can be replaced by its linear part $\sum_{j=1}^n \frac{\partial P(x)}{\partial x_j} A_j(x) \bar{x}$.

$$\begin{bmatrix} \gamma_1 & \phi_1(x, v_{21})B(x) \\ * & I \end{bmatrix} \geq 0 \quad (7.25)$$

Hence, by including (7.25) as a SOS based constraint, the SOS based optimization result is proposed, where $A_j(x)$ denotes the j^{th} row of $A(x)$.

Proposition 7.1.1 *Consider nonlinear system Σ in (7.2). If there exist polynomial matrices $P_1(x)$, $P_2(x)$, $P_3(x)$ with $P_1(x) = P_1^T(x)$, $P_3(x) = P_3^T(x)$, $H_{11}(x, \hat{x})$, $H_{13}(x, \hat{x})$, $H_{21}(x, \hat{x})$, $H_{23}(x, \hat{x})$, $H_{31}(x, \hat{x})$, $H_{33}(x, \hat{x})$, $G_{11}(x, \hat{x})$, $G_{13}(x, \hat{x})$, $G_{21}(x, \hat{x})$, $G_{23}(x, \hat{x})$, $G_{31}(x, \hat{x})$, $G_{33}(x, \hat{x})$, $L_A(\hat{x})$, $L_B(\hat{x})$, $\hat{C}_z(\hat{x})$, $\hat{D}_z(\hat{x})$ and nonsingular constant matrix G such that, for given constant matrices Y_1 , Y_2 , some sufficiently small value of γ_1 , some positive tuning scalars λ_1 , λ_2 , α_1 , α_2 and α_3 , a constant $s_1 > 0$ and a SOS polynomial $s_2(x, \hat{x})$ with $s_2(x, \hat{x}) > 0$ for $x \neq 0$, $\hat{x} \neq 0$, the following SOS optimization problem*

has feasible solutions

$$\text{Minimize } \gamma$$

subject to

$$v_1^T [P(x) - s_1 I] v_1 \in \Sigma_{sos} \quad (7.26)$$

$$- \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Upsilon(x, \hat{x}) + s_2(x, \hat{x}) I) \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (7.27)$$

$$v_5^T \begin{bmatrix} \gamma_1 & \phi_1(x, v_{21})B(x) \\ * & I \end{bmatrix} v_5 \in \Sigma_{sos} \quad (7.28)$$

where $v_1 \in \mathbb{R}^{2n}$, $v_2 \in \mathbb{R}^{2n+p}$, $v_3 \in \mathbb{R}^{2n+p}$, $v_4 \in \mathbb{R}^{n_z}$ and $v_5 \in \mathbb{R}^{p+1}$. $P(x)$ is as in (7.8), $\phi_1(x, v_{21})$ as in (7.24) and

$$\Upsilon(x, \hat{x}) = \begin{bmatrix} \Upsilon_{1,1}(x, \hat{x}) & * & * \\ \Pi_{2,1}(x, \hat{x}) & \Pi_{2,2}(x, \hat{x}) & * \\ \Pi_{3,1}(x, \hat{x}) & 0 & -\gamma^2 I \end{bmatrix} \quad (7.29)$$

with $\Pi_{2,1}(x, \hat{x})$, $\Pi_{2,2}(x, \hat{x})$, $\Pi_{3,1}(x, \hat{x})$ are as in (7.10) and

$$\Upsilon_{1,1}(x, \hat{x}) = \begin{bmatrix} \Psi_1 + \sum_{j=1}^n \frac{\partial P(x)}{\partial x_j} A_j(x) x & * \\ \Psi_2 & \Psi_3 \end{bmatrix} \quad (7.30)$$

where Ψ_1 , Ψ_2 and Ψ_3 are as in (7.11), then an H_∞ filter in the form of (7.3) solves the H_∞ filtering problem with (7.13).

Proof: With the definition of SOS polynomial and its decomposition, the result in Proposition 7.1.1 can be obtained directly, so the proof is omitted here. ■

Remark 7.1.4 When positive scalars $\lambda_1, \lambda_2, \alpha_1, \alpha_2$ and α_3 of $\tilde{G}(x, \hat{x})$ and $\tilde{H}(x, \hat{x})$ in (7.14) are set to be fixed parameters, with the transformation of $\dot{P}(x)$ into a linear term $\sum_{j=1}^n \frac{\partial P(x)}{\partial x_j} A_j(x)x$, (7.26)-(7.28) are linear in the variables for the solutions of Proposition 7.1.1. This provides convex optimization and the SOS programming can be utilized to obtain computationally tractable solutions.

7.2 Robust H_∞ Filter Design

The nonlinear systems (7.2) considered in Section 7.1 assume that all parameters of the systems are known. In this section, we extend the result in Theorem 7.1.1 to the robust H_∞ filter design for the polynomial nonlinear systems with polytopic uncertainty.

Consider the system

$$\begin{aligned} \Sigma_\theta : \quad \dot{x} &= A(x; \theta)x + B(x; \theta)w \\ y &= C(x; \theta)x + D(x; \theta)w \\ z &= C_z(x; \theta)x + D_z(x; \theta)w \end{aligned} \quad (7.31)$$

where $A(x; \theta), B(x; \theta), C(x; \theta), D(x; \theta), C_z(x; \theta)$ and $D_z(x; \theta)$ are polynomial matrices in x of the form

$$\begin{aligned} A(x; \theta) &= \sum_{i=1}^q A_i(x)\theta_i & B(x; \theta) &= \sum_{i=1}^q B_i(x)\theta_i \\ C(x; \theta) &= \sum_{i=1}^q C_i(x)\theta_i & D(x; \theta) &= \sum_{i=1}^q D_i(x)\theta_i \\ C_z(x; \theta) &= \sum_{i=1}^q C_{zi}(x)\theta_i & D_z(x; \theta) &= \sum_{i=1}^q D_{zi}(x)\theta_i \end{aligned} \quad (7.32)$$

The uncertain constant parameter vector $\theta = [\theta_1, \theta_2, \dots, \theta_q]^T \in \mathbb{R}^q$ satisfies

$$\theta \in \Theta \triangleq \left\{ \theta \in \mathbb{R}^q : \theta_i \geq 0, i = 1, 2, \dots, q, \text{ and } \sum_{i=1}^q \theta_i = 1 \right\} \quad (7.33)$$

Similarly, connecting the filter $\mathcal{F}$ and the system Σ_θ together, and denoting $x_{cl} = [x^T \ \hat{x}^T]^T$ and $z_{cl} := z - \hat{z}$. It follows from (7.31) and (7.3) that

$$\begin{aligned} \dot{x}_{cl} &= A_{cl}(x, \hat{x}; \theta)x_{cl} + B_{cl}(x, \hat{x}; \theta)w \\ z_{cl} &= C_{cl}(x, \hat{x}; \theta)x_{cl} + D_{cl}(x, \hat{x}; \theta)w \end{aligned} \quad (7.34)$$

where

$$\begin{aligned} A_{cl}(x, \hat{x}; \theta) &= \begin{bmatrix} A(x; \theta) & 0 \\ \hat{B}(\hat{x})C(x; \theta) & \hat{A}(\hat{x}) \end{bmatrix} \\ B_{cl}(x, \hat{x}; \theta) &= \begin{bmatrix} B(x; \theta) \\ \hat{B}(\hat{x})D(x; \theta) \end{bmatrix} \\ C_{cl}(x, \hat{x}; \theta) &= \begin{bmatrix} C_z(x; \theta) - \hat{D}_z(\hat{x})C(x; \theta) & -\hat{C}_z(\hat{x}) \end{bmatrix} \\ D_{cl}(x, \hat{x}; \theta) &= D_z(x; \theta) - \hat{D}_z(\hat{x})D(x; \theta) \end{aligned} \quad (7.35)$$

The objective is to investigate the following robust H_∞ filtering problem.

Robust H_∞ filtering problem: Given the nonlinear system Σ_θ in (7.31)-(7.32) and assume that $w \in \ell_2[0, \infty)$, find a robust H_∞ filter $\mathcal{F}$ of the form (7.3), if exists, such that

- The augmented system (7.34)-(7.35) is asymptotically stable.
- Under the zero-initial condition, the induced ℓ_2 gain from the exogenous disturbance w to the estimation error $z_{cl} = z(t) - \hat{z}(t)$ satisfies (7.6) for all non-zero

$w \in \ell_2[0, \infty)$, where $\gamma > 0$ is a prescribed scalar.

The conventional method based on quadratic stability is quite conservative, here the robust filter design methodology via polynomial parameter-dependent Lyapunov function is proposed. Define the parameter-dependent Lyapunov function as

$$V(x; \theta) = x_{cl}^T P(x; \theta) x_{cl} \quad (7.36)$$

where $P(x; \theta)$ is nonsingular in the form of

$$P(x; \theta) = \begin{bmatrix} P_1(x; \theta) & P_2(x; \theta) \\ P_2^T(x; \theta) & P_3(x; \theta) \end{bmatrix} \quad (7.37)$$

with $P_1(x; \theta) = P_1^T(x; \theta)$, $P_3(x; \theta) = P_3^T(x; \theta)$ and $P(x; \theta) = \sum_{i=1}^q P_i(x) \theta_i$, where $P_i(x)$ is the Lyapunov matrix at the vertex

$$P_i(x) = \begin{bmatrix} P_{1i}(x) & P_{2i}(x) \\ P_{2i}^T(x) & P_{3i}(x) \end{bmatrix} \quad (7.38)$$

With the results in Theorem 7.1.1, the sufficient condition for the robust H_∞ filter design can be obtained directly.

Theorem 7.2.1 Consider nonlinear system Σ_θ in (7.31)-(7.32). If there exist $P_{1i}(x)$, $P_{2i}(x)$, $P_{3i}(x)$ with $P_{1i}(x) = P_{1i}^T(x)$, $P_{3i}(x) = P_{3i}^T(x)$, $H_{11}(x, \hat{x})$, $H_{13}(x, \hat{x})$, $H_{21}(x, \hat{x})$, $H_{23}(x, \hat{x})$, $H_{31}(x, \hat{x})$, $H_{33}(x, \hat{x})$, $G_{11}(x, \hat{x})$, $G_{13}(x, \hat{x})$, $G_{21}(x, \hat{x})$, $G_{23}(x, \hat{x})$, $G_{31}(x, \hat{x})$, $G_{33}(x, \hat{x})$, $L_A(\hat{x})$, $L_B(\hat{x})$, $\hat{C}_z(\hat{x})$, $\hat{D}_z(\hat{x})$ and nonsingular constant matrix G such that, for given constant matrices Y_1 , Y_2 , and some positive tuning scalars λ_1 , λ_2 , α_1 , α_2 and

α_3 , the nonlinear matrix inequalities in (7.39)-(7.40) are satisfied for $i, l = 1, \dots, q$

$$P_i(x) = \begin{bmatrix} P_{1i}(x) & P_{2i}(x) \\ P_{2i}^T(x) & P_{3i}(x) \end{bmatrix} > 0 \quad (7.39)$$

$$\Pi_{il}(x, \hat{x}) = \begin{bmatrix} \Pi_{il,1}(x, \hat{x}) & * & * \\ \Pi_{il,2,1}(x, \hat{x}) & \Pi_{il,2,2}(x, \hat{x}) & * \\ \Pi_{il,3,1}(x, \hat{x}) & 0 & -\gamma^2 I \end{bmatrix} < 0 \quad (7.40)$$

where

$$\begin{aligned} \Pi_{il,1}(x, \hat{x}) &= \begin{bmatrix} \Psi_{1i} + \sum_{j=1}^n \frac{\partial P_i(x)}{\partial x_j} [A_{lj}(x)x + B_{lj}(x)w] & * \\ & \Psi_{2i} & \Psi_{3i} \end{bmatrix} \\ \Pi_{il,2,1}(x, \hat{x}) &= \begin{bmatrix} \Phi_{7i} & \Phi_{8i} & \Phi_{9i} \\ \Phi_{10i} & \Phi_{11i} & \Phi_{12i} \\ \Phi_{13i} & \Phi_{14i} & \Phi_{15i} \end{bmatrix} \\ \Pi_{il,2,2}(x, \hat{x}) &= \begin{bmatrix} -H_{11}(x, \hat{x}) - H_{11}^T(x, \hat{x}) & * \\ -H_{21}(x, \hat{x}) - \alpha_1 G^T & -\alpha_2 G - \alpha_2 G^T \\ -H_{31}(x, \hat{x}) - H_{13}^T(x, \hat{x}) & -\alpha_3 Y_2 G - H_{23}^T(x, \hat{x}) \\ & * \\ & * \\ & -H_{33}(x, \hat{x}) - H_{33}^T(x, \hat{x}) \end{bmatrix} \end{aligned}$$

$$\Pi_{il,3,1}(x, \hat{x}) = \begin{bmatrix} C_{zi}(x) - \hat{D}_z(\hat{x})C_i(x) & -\hat{C}_z(\hat{x}) & D_{zi}(x) - \hat{D}_z(\hat{x})D_i(x) \end{bmatrix} \quad (7.41)$$

with

$$\Psi_{1i} = \begin{bmatrix} \Phi_{1i} & * \\ \Phi_{2i} & \Phi_{3i} \end{bmatrix}, \Psi_{2i} = \begin{bmatrix} \Phi_{4i} & \Phi_{5i} \end{bmatrix}, \Psi_{3i} = \Phi_{6i} \quad (7.42)$$

and

$$\begin{aligned}
\Phi_{1i} &= G_{11}(x, \hat{x})A_i(x) + L_B(\hat{x})C_i(x) + [G_{11}(x, \hat{x})A_i(x) + L_B(\hat{x})C_i(x)]^T \\
\Phi_{2i} &= G_{21}(x, \hat{x})A_i(x) + \lambda_1 L_B(\hat{x})C_i(x) + L_A^T(\hat{x}) \\
\Phi_{3i} &= \lambda_1 [L_A(\hat{x}) + L_A^T(\hat{x})] \\
\Phi_{4i} &= G_{31}(x, \hat{x})A_i(x) + \lambda_2 Y_1 L_B(\hat{x})C_i(x) + [G_{11}(x, \hat{x})B_i(x) \\
&\quad + L_B(\hat{x})D_i(x) - \frac{1}{2}G_{13}(x, \hat{x})]^T \\
\Phi_{5i} &= \lambda_2 Y_1 L_A(\hat{x}) + [G_{21}(x, \hat{x})B_i(x) + \lambda_1 L_B(\hat{x})D_i(x) - \frac{1}{2}G_{23}(x, \hat{x})]^T \\
\Phi_{6i} &= G_{31}(x, \hat{x})B_i(x) + \lambda_2 Y_1 L_B(\hat{x})D_i(x) - \frac{1}{2}G_{33}(x, \hat{x}) \\
&\quad + [G_{31}(x, \hat{x})B_i(x) + \lambda_2 Y_1 L_B(\hat{x})D_i(x) - \frac{1}{2}G_{33}(x, \hat{x})]^T \\
\Phi_{7i} &= P_{1i}(x) - G_{11}^T(x, \hat{x}) + H_{11}(x, \hat{x})A_i(x) + \alpha_1 L_B(\hat{x})C_i(x) \\
\Phi_{8i} &= P_{2i}(x) - G_{21}^T(x, \hat{x}) + \alpha_1 L_A(\hat{x}) \\
\Phi_{9i} &= -G_{31}^T(x, \hat{x}) + H_{11}(x, \hat{x})B_i(x) + \alpha_1 L_B(\hat{x})D_i(x) - \frac{1}{2}H_{13}(x, \hat{x}) \\
\Phi_{10i} &= P_{2i}^T(x) - G^T + H_{21}(x, \hat{x})A_i(x) + \alpha_2 L_B(\hat{x})C_i(x) \\
\Phi_{11i} &= P_{3i}(x) - \lambda_1 G^T + \alpha_2 L_A(\hat{x}) \\
\Phi_{12i} &= -\lambda_2 [Y_1 G]^T + H_{21}(x, \hat{x})B_i(x) + \alpha_2 L_B(\hat{x})D_i(x) - \frac{1}{2}H_{23}(x, \hat{x}) \\
\Phi_{13i} &= -G_{13}^T(x, \hat{x}) + H_{31}(x, \hat{x})A_i(x) + \alpha_3 Y_2 L_B(\hat{x})C_i(x) \\
\Phi_{14i} &= -G_{23}^T(x, \hat{x}) + \alpha_3 Y_2 L_A(\hat{x}) \\
\Phi_{15i} &= I - G_{33}^T(x, \hat{x}) + H_{31}(x, \hat{x})B_i(x) + \alpha_3 Y_2 L_B(\hat{x})D_i(x) - \frac{1}{2}H_{33}(x, \hat{x})
\end{aligned}$$

then an H_∞ filter in the form of (7.3) solves the robust H_∞ filtering problem with

$$\hat{A}(\hat{x}) = G^{-1}L_A(\hat{x}), \hat{B}(\hat{x}) = G^{-1}L_B(\hat{x}) \quad (7.44)$$

Proof: With the result in Theorem 7.1.1, the proof of Theorem 7.2.1 only needs summation of $\Pi_{il}(x, \hat{x})$ for $i, l = 1, \dots, q$, hence the details are omitted here. ■

Remark 7.2.1 (a) Since the suitable structure of (7.9) we can extend the H_∞ filter design methodology of Section 7.1 to the parameter-dependent polynomial systems (7.31)-(7.32). $A_{lj}(x)$ and $B_{lj}(x)$ in (7.41) denote the j^{th} row of $A_l(x)$ and $B_l(x)$ at the vertex l , resp. (b) We adopt a parameter-dependent Lyapunov function (7.36) with Lyapunov matrix $P(x; \theta)$. When $P_i(x) = P(x)$ for $i = 1, \dots, q$, $P(x; \theta) = P(x)$ and $V(x) = x_{cl}^T(x)P(x)x_{cl}$. Therefore the sufficient conditions in Theorem 7.2.1 are less conservative than that with a fixed Lyapunov function for all vertices.

Using the similar technique as in Section 7.1 to deal with the nonlinear term $\sum_{j=1}^n \frac{\partial P_i(x)}{\partial x_j} [A_{lj}(x)x + B_{lj}(x)w]$ in (7.41). Define $\phi_{1i}(x, v_{21})$ as

$$\phi_{1i}(x, v_{21}) = \left[v_{21}^T \frac{\partial P_i(x)}{\partial x_1} v_{21}, \dots, v_{21}^T \frac{\partial P_i(x)}{\partial x_n} v_{21} \right] \quad (7.45)$$

and γ_1 as the upper bound of $[\phi_{1i}(x, v_{21})B_l(x)][\phi_{1i}(x, v_{21})B_l(x)]^T$, then we have the corresponding SOS optimization problem.

Proposition 7.2.1 Consider nonlinear system Σ_θ in (7.31)-(7.32). If there exist polynomial matrices $P_{1i}(x)$, $P_{2i}(x)$, $P_{3i}(x)$ with $P_{1i}(x) = P_{1i}^T(x)$, $P_{3i}(x) = P_{3i}^T(x)$, $H_{11}(x, \hat{x})$, $H_{13}(x, \hat{x})$, $H_{21}(x, \hat{x})$, $H_{23}(x, \hat{x})$, $H_{31}(x, \hat{x})$, $H_{33}(x, \hat{x})$, $G_{11}(x, \hat{x})$, $G_{13}(x, \hat{x})$, $G_{21}(x, \hat{x})$, $G_{23}(x, \hat{x})$, $G_{31}(x, \hat{x})$, $G_{33}(x, \hat{x})$, $L_A(\hat{x})$, $L_B(\hat{x})$, $\hat{C}_z(\hat{x})$, $\hat{D}_z(\hat{x})$ and nonsingular constant matrix G such that, for given constant matrices Y_1 , Y_2 , some sufficiently small value of γ_1 , some positive tuning scalars λ_1 , λ_2 , α_1 , α_2 and α_3 , constants $s_i > 0$ and SOS polynomials $s_{il}(x, \hat{x})$ with $s_{il}(x, \hat{x}) > 0$ for $x \neq 0$, $\hat{x} \neq 0$ ($i, l = 1, \dots, q$), the following SOS optimization problem has feasible solutions

$$\text{Minimize} \quad \gamma$$

subject to

$$v_1^T [P_i(x) - s_i I] v_1 \in \Sigma_{sos} \quad (7.46)$$

$$- \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Upsilon_{il}(x, \hat{x}) + s_{il}(x, \hat{x}) I) \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (7.47)$$

$$v_5^T \begin{bmatrix} \gamma_1 & \phi_{1i}(x, v_{21}) B_l(x) \\ * & I \end{bmatrix} v_5 \in \Sigma_{sos} \quad (7.48)$$

where $v_1 \in \mathbb{R}^{2n}$, $v_2 \in \mathbb{R}^{2n+p}$, $v_3 \in \mathbb{R}^{2n+p}$, $v_4 \in \mathbb{R}^{n_z}$ and $v_5 \in \mathbb{R}^{p+1}$. $P_i(x)$ is as in (7.38), $\phi_{1i}(x, v_{21})$ as in (7.45) and

$$\Upsilon_{il}(x, \hat{x}) = \begin{bmatrix} \Upsilon_{il,1}(x, \hat{x}) & * & * \\ \Pi_{il,2,1}(x, \hat{x}) & \Pi_{il,2,2}(x, \hat{x}) & * \\ \Pi_{il,3,1}(x, \hat{x}) & 0 & -\gamma^2 I \end{bmatrix} \quad (7.49)$$

with $\Pi_{il,2,1}(x, \hat{x})$, $\Pi_{il,2,2}(x, \hat{x})$, $\Pi_{il,3,1}(x, \hat{x})$ are as in (7.41) and

$$\Upsilon_{il,1}(x, \hat{x}) = \begin{bmatrix} \Psi_{1i} + \sum_{j=1}^n \frac{\partial P_i(x)}{\partial x_j} A_{lj}(x) x & * \\ \Psi_{2i} & \Psi_{3i} \end{bmatrix} \quad (7.50)$$

where Ψ_{1i} , Ψ_{2i} and Ψ_{3i} are as in (7.42), then an H_∞ filter in the form of (7.3) solves the robust H_∞ filtering problem with (7.44).

Proof: The proof is similar with that of Proposition 4.3.1, hence it is omitted here. ■

Remark 7.2.2 In practise, sensors failures may result in a large degree of filter performance degradation and, more importantly, possible hazard. In such cases, reliable filters need to be designed considering both the normal and sensor faulty cases. The polytopic model (7.31) can be used to describe the nonlinear systems with faults, then

the robust H_∞ filtering result in Proposition 7.2.1 can be extended to the reliable filter synthesis.

Remark 7.2.3 Proposition 7.2.1 gives a global filtering design in general case, since the Lyapunov stabilizing conditions are guaranteed in the entire state-space and Lyapunov functions are radially unbounded.

7.3 Numerical Example

In this section, an robust H_∞ filtering design example is presented to demonstrate the proposed filter design approach in Section 7.2.

Consider a parameter-dependent nonlinear system of the form (7.31)-(7.32) with system matrices given by

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 0 & 1 \\ -0.8 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} \\
 C_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_1 = 0.2 \\
 C_{z1} &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D_{z1} = 0 \\
 A_2 &= \begin{bmatrix} 0.01 & 1.005 \\ -1 - x_1^2 & -0.99 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
 C_2 &= \begin{bmatrix} 0.98 & 0.08 \end{bmatrix}, \quad D_2 = 0.2 \\
 C_{z2} &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D_{z2} = 0
 \end{aligned} \tag{7.51}$$

Using Proposition 7.2.1, an robust H_∞ filter can be obtained as in (7.52). The values of positive constants s_1, s_2 are fixed as 0.001, then for $\lambda_1 = \lambda_2 = 1, \alpha_1 = \alpha_2 = \alpha_3 = 1$, the SOS based optimization problem returns 3.5117 as the optimal value of γ and 10^{-5}

as the value of γ_1 . The filter dynamics are as follows:

$$\begin{aligned}\hat{A} &= \begin{bmatrix} (-8.5796 - 0.176 \times 10^{-2} \hat{x}_1^2 & (0.6903 - 0.1169 \times 10^{-2} \hat{x}_1^2 \\ -0.154 \times 10^{-2} \hat{x}_2^2) & -0.1368 \times 10^{-2} \hat{x}_2^2) \\ (-25.2944 - 0.117 \times 10^{-1} \hat{x}_1^2 & (-1.0853 - 0.3348 \times 10^{-1} \hat{x}_1^2 \\ +0.3938 \times 10^{-2} \hat{x}_1 \hat{x}_2 & +0.3773 \times 10^{-2} \hat{x}_1 \hat{x}_2 \\ -0.6268 \times 10^{-2} \hat{x}_2^2) & -0.3497 \times 10^{-1} \hat{x}_2^2) \end{bmatrix} \\ \hat{B} &= \begin{bmatrix} -8.6019 - 0.1704 \times 10^{-2} \hat{x}_1^2 - 0.1725 \times 10^{-2} \hat{x}_2^2 \\ (-23.189 - 0.1135 \times 10^{-1} \hat{x}_1^2 - 0.1982 \times 10^{-2} \hat{x}_1 \hat{x}_2 \\ -0.1185 \times 10^{-1} \hat{x}_2^2) \end{bmatrix} \\ \hat{C}_z &= \begin{bmatrix} 2.7421 & -0.8311 \end{bmatrix} \\ \hat{D}_z &= 3.8153 \end{aligned} \quad (7.52)$$

Simulations are carried out to verify the effectiveness of the proposed filter design methodology for the polynomial system (7.31)-(7.32) with system matrices given by (7.51). Figure 7.1 shows the trajectories of the estimation errors of 10 interpolated augmented systems at various values of θ between the two vertices in (7.51). The initial conditions for the polynomial systems and the filter are $x(0) = [2, 2]^T$ and $\hat{x}(0) = [1, 1]^T$, resp., and a disturbance of step signal is introduced from 30 sec to 40 sec as follows:

$$w(t) = \begin{cases} 1, & 30 \leq t \leq 40, \\ 0, & \text{otherwise.} \end{cases} \quad (7.53)$$

It can be seen that the augmented systems are stable with the designed robust filter (7.52), and the L_2 gains from disturbance w to the estimation errors $z_{cl} = z - \hat{z}$ of the augmented systems are no greater than $\gamma = 3.5117$.

Correspondingly, the state trajectories of the polynomial systems and the filter are shown

Figure 7.1: Trajectories of the estimation errors of the interpolated systems

in Figure 7.2 and Figure 7.3, resp. It can be seen that the state responses $x(t)$ and $\hat{x}(t)$ asymptotically converge to the equilibrium point, which implies the effectiveness of the proposed design methodology.

Figure 7.2: State trajectories of the interpolated polynomial systems

In order to provide some insight into the strength of the proposed algorithm, a comparative study involving our methodology and some existing work in [49] is implemented. In [49], the Example 2 of Section VI can be transformed into the state dependent linear-like

Figure 7.3: State trajectories of the filter when $\theta \in \Theta$ are fixed as different values

system (7.2) with

$$\begin{aligned}
 A(x) &= \begin{bmatrix} -1 + x_1 - 1.5x_1^2 - 0.75x_2^2 & 0.25 - x_1^2 - 0.5x_2^2 \\ 0.3430 & -0.6775 \end{bmatrix} \\
 B(x) &= \begin{bmatrix} \sqrt{10} & 0 & 0 \\ 0 & \sqrt{0.1} & 0 \end{bmatrix} \\
 C(x) &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D(x) = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \\
 C_z(x) &= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D_z(x) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}
 \end{aligned} \tag{7.54}$$

Since there is no uncertainty involved in the system dynamics (7.54), Proposition 7.1.1 is utilized to derive a stabilizing filter. For $s_1 = 0.001$, $\lambda_1 = \lambda_2 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the SOS based optimization problem returns 0.5089 as the optimal value of γ and 10^{-5} as the value of γ_1 . The filter dynamics are as follows:

$$\begin{aligned}
 \hat{A} &= \begin{bmatrix} -40.7963 & -2.6590 \\ 0.3059 & -1.8964 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} -0.1804 \\ -0.8957 \end{bmatrix} \\
 \hat{C}_z &= \begin{bmatrix} 0.72746 \times 10^{-3} & -0.14448 \end{bmatrix}, \quad \hat{D}_z = 0.57335
 \end{aligned} \tag{7.55}$$

The initial conditions for the polynomial system and the filter are $x(0) = [1, 1]^T$ and $\hat{x}(0) = [1, 1]^T$, resp., and a disturbance of step signal is introduced from 30 sec to 40 sec as in (7.53). In Figure 7.4, it can be seen that the augmented system is stable with the designed H_∞ filter (7.55). Moreover, by using our methodology, the designed L_2 gain from disturbance w to the estimation error $z_{cl} = z - \hat{z}$ of the augmented system is $\gamma = 0.5089$, which is smaller than the designed value of $\gamma = 1$ in [49]. Similarly as in [49], the initial condition of filter is given as $\hat{x}(0) = 0$. Figure 7.5 shows the trajectories of the states error $x - \hat{x}$ which converges to zero with the time approaching infinity. Therefore, from these simulation results, it can be seen that our methodology is effective in the stabilizing H_∞ filter design.

Figure 7.4: Trajectories of system states, filter states and estimation error

Simulations are also carried out to compare the achieved performance by the designed filter (7.55) with that by the filter designed in [49]. It can be clearly seen in Figure 7.6 that our design methodology enhances system's performance by providing smaller L_2 gain.

Figure 7.5: Trajectories of states error

Figure 7.6: Trajectories of estimation errors, $|w| = 1$

7.4 Conclusions

This chapter discusses the robust H_∞ filtering problems for the polynomial nonlinear systems against parametric uncertainty. Sufficient conditions to guarantee the filter stability with guaranteed H_∞ norm from the unknown norm-bounded disturbance signal to the estimation error are proposed in terms of SDMIs. We eliminate the coupling terms among system dynamics, filter dynamics and the Lyapunov matrix by introducing additional matrix variables, hence SDMIs based conditions have a more suitable structure to deal with robust filter design for the parameter-dependent polynomial systems. These SDMIs based conditions are formulated to sum-of-squares based constraints, which can be solved via the semidefinite programming relaxations based on the SOS decomposition. In order to reduce the conservatism involved in the controller design, parameter and state dependent Lyapunov functions are utilized and more general assumption and relaxation are provided to deal with the nonlinear terms in the matrix inequalities. Finally, an robust H_∞ filter design example is provided to demonstrate the effectiveness of the proposed approach.

Chapter 8

Conclusion and Future Work

8.1 Conclusion

This thesis studies the nonlinear robust control problems for continuous time parameter-dependent nonlinear systems whose dynamics are described by polynomial functions. Since a broad class of systems including linear control systems can be modelled as polynomial control systems, and the ability of the control system to guarantee robustness with respect to uncertainty has been a fundamental problem, the proposed design in this thesis for parameter-dependent polynomial nonlinear systems is quite promising.

The designs proposed in this thesis are based on semidefinite programming and in particular based on the recent developed sum of squares programming technique. Sufficient conditions guaranteeing the existence of robust nonlinear controllers and filters are formulated as sum of squares based optimization problems, then solved via semidefinite programming to obtain computationally tractable solutions.

In this thesis, general polynomial control systems are considered and no special requirements on the system structure are imposed except for the polynomial description and the affine nonlinear structure. Several new approaches have been proposed to address

several design problems, including stability analysis and synthesis, estimation of region of attraction, H_∞ synthesis, output feedback design, multi-objective reliable control and H_∞ filter design.

First of all, in Chapter 3 a local stability analysis and synthesis method for polynomial nonlinear systems with parameter uncertainty is presented. Sufficient conditions to achieve the closed-loop stability have been presented and the synthesis problem of stabilizing feedback controller to enlarge the regions of attraction have been solved by the sum of squares based optimization. Such optimization methodology provides efficient numerical method to the nonlinear synthesis and at the same time to the analysis of regions of attraction for closed-loop systems.

However, the bilinear terms and trilinear terms in decision variables result in an iterative algorithm for the design. Feasible initial conditions are needed to obtain the optimal solutions. Furthermore, no disturbance has been considered in the synthesis. These disadvantages motivates the results in Chapter 4. In Chapter 4 H_∞ controller synthesis problem has been fully addressed and sufficient conditions have been formulated as nonlinear matrix inequalities by introducing additional matrix variables. More general relaxation and parameter-dependent Lyapunov functions are utilized to reduce the design conservativeness.

The proposed state feedback design approaches in Chapter 3 and Chapter 4 assume that measurements of all state variables are available. Nevertheless, such assumption is often unrealistic since the full state variables are not always accessible or some of them are chosen not to be measured due to technical or economic reasons. Therefore we further our study to the nonlinear static output feedback design in Chapter 5. The proposed formulation provides an effective way to overcome the non-convex nature of the static output feedback problem. Output feedback controllers and Lyapunov functions are constructed in an efficient computational manner via semidefinite programming.

Then in Chapter 6, we extend our state feedback design methodology to the multi-objective reliable control problem. Passive reliable control concept is utilized such

that a fixed controller is designed for the nominal case as well as fault cases. Optimal performance in nominal condition and satisfactory performance in fault conditions are achieved simultaneously. Parameter-dependent Lyapunov functions, one for each and every vertex of the uncertainty polytope, are employed to reduce the design conservativeness.

Finally, in Chapter 7 the state estimation, which is one of the essential design problems in control area, has been investigated. In this chapter, a numerical methodology for solving the robust H_∞ filtering problem has been proposed for the polynomial nonlinear system against the parameter uncertainty. A stable H_∞ filter is designed to guarantee the closed-loop stability with guaranteed H_∞ norm from the unknown norm-bounded disturbance signal to the estimation error based on the sum of squares optimization.

In summary, Lyapunov based robust control design approaches have been studied in this thesis for continuous time polynomial nonlinear systems with uncertainty. Based on the recent sum of squares programming and semidefinite programming techniques, new computationally tractable nonlinear analysis and synthesis methods are proposed to overcome the numerical difficulty in nonlinear system design. The effectiveness of the proposed robust nonlinear controller and filter design approaches has been demonstrated by the examples provided in this thesis.

Finally, some inherent weaknesses of the proposed algorithms in this thesis are discussed. Firstly, the drawback of representing a nonlinear system in a state dependent linear-like form is that such representation is not unique. The proposed optimization constraints are affected by the non-uniqueness, and the success of synthesis depends on the chosen representation. Secondly, as SOS decomposition is a sufficient condition, the proposed SOS implementations can only provide sufficient conditions for the existence of stabilizing nonlinear polynomial controllers, no necessary conditions exist. Moreover, a certain conservativeness is introduced for deriving the SOS based relaxations for the Lyapunov condition $dV(x)/dt < 0$. Lastly, since the SOS programming problem's size increases exponentially with the increase of the degrees of the polynomials and

the number of variables, the computational complexity will limit the dimensions of the problems considered.

8.2 Future Work

8.2.1 Model Predictive Control for Constrained Nonlinear Systems

Most practical control systems must deal with process constraints. Generally the constraints on manipulated variables result from physical limitations of the actuators, and the constraint on the states may be imposed for the safety or productivity. However, methods for handling constraints have not been abundant.

Model predictive control (MPC), also known as moving horizon control (MHC) or receding horizon control (RHC), is an efficient technique for the control design of slow dynamical systems in the presence of constraints. At each sampling time, MPC uses an explicit process model and information about input and output constraints to compute process inputs so as to optimize future plant behavior over the prediction horizon. Although more than one input move is generally calculated, only the first element is implemented. At the next sampling time, the whole procedure is repeated with new measurement obtained from the system. This receding horizon idea is illustrated in Figure 8.1 [16].

The essential features of predictive control include: an explicit internal model, the receding horizon idea, and computation of the control signal by optimizing predicted plant behavior. These fundamental attributes differentiate MPC from other control methods.

MPC is usually formulated in the state space. The system to be controlled can be described by the difference equation,

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0, \quad (8.1)$$

Figure 8.1: Receding horizon strategy: only the first one of the computed moves $u(t)$ is implemented

where $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$ denote the state and control input at time k , respectively. The control objective is usually to steer the state to the origin under state and input constraints:

$$x(k) \in \mathbb{X}, \quad u(k) \in \mathbb{U}, \quad (8.2)$$

where $\mathbb{X}$ is a convex closed subset of $\mathbb{R}^{n_x}$ and $\mathbb{U}$ is a convex compact subset of $\mathbb{R}^{n_u}$, both containing the origin in the interior. Let N denote the prediction horizon. A terminal constraint is sometimes imposed for stability reasons,

$$x(k+N) \in X_f \subset \mathbb{X}. \quad (8.3)$$

A typical formulation of the open loop optimization problem is given as follows.

$$J(x_k) = \min_{u(k)} \left\{ E(x(k+N)) + \sum_{i=1}^{N-1} l(x(k+i|k), u(k+i|k)) \right\} \quad (8.4)$$

subject to

$$x(k+i+1|k) = f(x(k+i|k), u(k+i|k)), \quad i = 1, 2, \dots, N-1, \quad (8.5)$$

$$x(k+i|k) \in \mathbb{X}, \quad i = 1, 2, \dots, N-1, \quad (8.6)$$

$$x(k+N|k) \in X_f, \quad (8.7)$$

$$x(k|k) = x_k, \quad (8.8)$$

$$u(k+i|k) \in \mathbb{U}, \quad i = 1, 2, \dots, N-1, \quad (8.9)$$

where $l(x(i), u(i))$ is the stage cost, and $E(x(k+N))$ is the terminal cost.

For linear MPC a linear model $f(x(k), u(k)) = Ax(k) + Bu(k)$ is used to predict the system states on the horizon. The cost function is quadratic in the states and inputs, and the constraints are linear. The resulting optimization problem becomes a convex quadratic program. In nonlinear MPC the idea is the same as in linear MPC, but with a nonlinear model describing the process dynamics. The resulting optimization problem becomes a nonlinear program.

Although practical processes are inherently nonlinear, the vast majority of MPC applications to date are based on linear dynamic models (see [26, 51, 67, 70, 87, 145, 153] and references therein). Nevertheless, there are cases where nonlinear effects are significant enough to justify the use of nonlinear model predictive control (NMPC) technology, which is defined as MPC using a nonlinear model. Much progress has been made on NMPC design for nonlinear systems (see [5, 6, 17, 20, 68, 69, 85, 146] and reference therein).

The on-line optimization procedure typically leads to a huge MPC computational effort that has to be finished in each and every sampling period. This is the foremost factor that keeps preventing MPC from being populated on fast response systems and large-scale processes. Moreover, the huge computational burden of a SOS program advises against the implementation of a related optimization procedure at each sampling instant.

Nonetheless, as has been discussed in [33], if the SOS relaxation polynomial problem is off-line moved, the on-line computational demand can be reduced theoretically and the on-line controller could be relatively easier to set up.

In the future, an off-line MPC formulation for polynomial nonlinear systems could be proposed based on the computation of a nested sequence of asymptotically stable invariant sets (see [145]). The main promising aspects to be studied in the off-line MPC design could be,

- Constraints (input saturations/state limits/objective requirements);
- Disturbances rejection;
- uncertainty;
- Efficient SOS based MPC algorithms.

8.2.2 Nonlinear Time-Delay Systems Design

Time-delay is frequently encountered in various practical systems, such as chemical engineering systems, pneumatic and hydraulic systems, network communication systems and population dynamic model. The presence of delays can have an effect on the system performance and even cause instabilities. Ignoring them may lead to design flaws and incorrect analysis conclusions. Therefore, time delays have been taken into account in many system analysis and design problems, and the stability and stabilization of time-delay systems have attracted the attention of many researchers in control area.

To address the nonlinear time delay system design, [94, 95] proposes some analysis and design results based on the SOS programming technique. In [95], the problem of memory controller synthesis for delay-independent and delay-dependent stabilization of uncertain nonlinear time-delay systems has been discussed by constructing Lyapunov-Krasovskii functions algorithmically. The time-delay system is represented in a linear-

like fashion as follows:

$$\dot{x}(t) = A_0(z_1, z_2, p)Z(z_1) + A_1(z_1, z_2, p)Z(z_2) + B(z_1, z_2, p)u \quad (8.10)$$

where $p \in \Delta$ is a parameter set which takes the form

$$\Delta = \{p \in \mathbb{R}^k | g_i(p) \leq 0, i = 1, \dots, N\} \quad (8.11)$$

The control law is assumed to have the form

$$u = K_0(z_1, z_2)Z(z_1) + K_1(z_1, z_2)Z(z_2) \quad (8.12)$$

The stability analysis for nonlinear time delay system can be proposed using the time-domain (Lyapunov-based) method. Stability can be classified as delay-independent if it is retained irrespective of the delay size, and delay-dependent if the stability is lost at a certain delay value. The delay-independent stabilization aims in constructing a controller (8.12) and Lyapunov function (8.13) such that the resulting system is delay-independent stable.

$$V(x_t) = a_0(x(t)) + \int_{-\tau}^0 a_1(x(t+\theta))d\theta \quad (8.13)$$

For system (8.10), we know that if there exist symmetric matrices P and Q , polynomial matrices S_0 and S_1 , a constant $\epsilon_1 > 0$ and a SOS $\epsilon_2(z_1)$ such that (8.14)-(8.15) are satisfied and (8.16) is a SOS matrix for $p \in \Delta$,

$$P - \epsilon_1 I \geq 0, \quad (8.14)$$

$$Q \geq 0, \quad (8.15)$$

$$- \begin{bmatrix} \begin{pmatrix} M(A_0P + BS_0) + \epsilon_2(z_1) \\ +(A_0P + BS_0)^T M^T + Q \end{pmatrix} & M(A_1P + BS_1) \\ (A_1P + BS_1)^T M^T & -Q \end{bmatrix} \quad (8.16)$$

then the stabilizing state feedback controller is given by

$$u(x) = S_0(x)P^{-1}Z(x(t)) + S_1(x)P^{-1}Z(x(t - \tau)) \quad (8.17)$$

For the delay-dependent stabilization result, the details can be referred to [95]. It should be noted that, for the time-delay system designs based on the SOS programming technique, only stability and stabilization problems have been considered without any performance and disturbance attenuation. In the future we plan to extend the robust synthesis results of this thesis for the nonlinear time-delay systems.

In general there are two types of Lyapunov methods by using Lyapunov-Krasovskii functionals and Lyapunov-Razumikhin functions. In the linear case, the Lyapunov-Razumikhin LMI criteria are in general more conservative than the Lyapunov-Krasovskii ones [39]. Recently, the free weighting matrix method has received wide application because of its attractive nature of the simple principle. By utilizing Newton-Leibniz formula, the free weighting matrix method may reduce the conservatism induced by the fixed weighting matrix in the model transformation method. However, many slack variables need to be introduced in the free weighting matrix method, which always lead to the increase of computational complexity. The Jensen integral inequality approach provides a promising way for obtaining less conservativeness and reducing the computational complexity in the time delay system design [175]. On the other hand, a delay decomposition method can be proposed to deal with time delay problems. Since more information of states can be utilized, the obtained results are less conservative. These novel stability analysis methods for delay systems have been successfully used to study neural networks and networked control systems in [174].

8.2.3 Robust Tracking Control Problem

A problem of major importance in control theory is that of controlling a plant to have its output asymptotically tracking any reference trajectory in a prescribed family. In many

control system design, robust tracking performance is required, namely, the outputs track reference signals well and the closed-loop stability is guaranteed in the presence of uncertainty and/or exogenous disturbance.

In the future, the tracking control design under study is to find a controller such that:

- The closed-loop system is robustly stable for all uncertainties in the predefined set;
- The output $Sy(t)$ tracks the reference signal $r(t)$ without steady-state error, that is

$$\lim_{t \rightarrow \infty} e(t) = 0, \quad e(t) = r(t) - Sy(t) \quad (8.18)$$

and with optimized closed-loop performance. Here S is a known constant matrix of appropriate dimension used to form the output required to track the reference signals.

Based on the above design objective, we formulate the robust tracking problem for the following system:

$$\begin{aligned} \dot{x} &= A(x)Z(x) + B_u(x)u + B_w(x)w \\ y &= C_y(x)Z(x) \\ z &= C_z(x)Z(x) + D_z(x)u \end{aligned} \quad (8.19)$$

In order to obtain a robust tracking controller with zero steady state tracking error for step input, an integral action is introduced on the tracking error and the following con-

tinuous time augmented state-space description is constructed.

$$\begin{aligned}\dot{x}_\zeta &= A_\zeta(x)Z_\zeta(x) + B_{\zeta u}(x)u + B_{\zeta w}(x)w_\zeta(t) \\ y_\zeta &= C_{\zeta y}(x)Z_\zeta(x) \\ z &= C_{\zeta z}(x)Z_\zeta(x) + D_z(x)u\end{aligned}\tag{8.20}$$

where

$$\begin{aligned}x_\zeta &= \begin{bmatrix} \int_0^t e(t)dt \\ x(t) \end{bmatrix}, Z_\zeta = \begin{bmatrix} \int_0^t e(t)dt \\ Z(x) \end{bmatrix}, y_\zeta = \begin{bmatrix} \int_0^t e(t)dt \\ y(t) \end{bmatrix}, w_\zeta = \begin{bmatrix} r(t) \\ w(t) \end{bmatrix} \\ A_\zeta(x) &= \begin{bmatrix} 0 & -SC_y(x) \\ 0 & A(x) \end{bmatrix}, B_{\zeta u}(x) = \begin{bmatrix} 0 \\ B_u(x) \end{bmatrix}, B_{\zeta w}(x) = \begin{bmatrix} I & 0 \\ 0 & B_w(x) \end{bmatrix} \\ C_{\zeta y}(x) &= \begin{bmatrix} I & 0 \\ 0 & C_y(x) \end{bmatrix}, C_{\zeta z}(x) = \begin{bmatrix} 0 & C_z(x) \end{bmatrix}\end{aligned}\tag{8.21}$$

The relationship between x_ζ and Z_ζ can be derived as:

$$\dot{Z}_\zeta(x) = M_\zeta(x)\dot{x}_\zeta\tag{8.22}$$

with

$$M_\zeta(x) = \begin{bmatrix} I & 0 \\ 0 & M(x) \end{bmatrix}\tag{8.23}$$

where $M(x)$ is defined as in (4.3). The proposed methodology in this thesis can then be utilized for the tracking controller design. For example, if a state feedback controller

$u = K(x)Z_\zeta(x)$ is considered, then the closed-loop system is:

$$\begin{aligned}\dot{x}_\zeta &= [A_\zeta(x) + B_{\zeta u}(x)K(x)] Z_\zeta(x) + B_{\zeta w}(x)w_\zeta(t) \\ y_\zeta &= C_{\zeta y}(x)Z_\zeta(x) \\ z &= [C_{\zeta z}(x) + D_z(x)K(x)] Z_\zeta(x)\end{aligned}\tag{8.24}$$

Following the technique in Chapter 4, the corresponding SOS based optimization algorithm for robust tracking controller can be obtained. If we obtain a controller to stabilize the augmented system (8.20), then it also stabilizes the original system (8.19) and guarantees a zero steady state error.

8.2.4 Aircraft Control Application

Despite having the software tools for SOS programming, we still run into dimensionality problems. The computational complexity for the SOS decomposition of a polynomial depends on two factors: the degree of the polynomial and the number of independent variables. In particular, the number of decision variables increases exponentially with the number of independent variables and the degree of the polynomial. Therefore, the number of constraints finally in the SOS programming can be large, especially when using many variables and high degree polynomials.

For this reasons, more research work is needed in the implementation of the proposed methodologies on practical systems like the aircraft control systems. The proposed design algorithms in this thesis can provide the preliminary results for the real applications in the future. There are several research directions that can be considered for this area.

- Because the SOS programming problem's size increases exponentially with the increase of the degrees of the polynomials and the number of variables, the computational complexity will limit the size of the problems considered. One way to

solve this problem is to utilize a two step hybrid/switched approach by combining the SOS method with other control techniques. Some recent interesting results have been published, such as the combination of backstepping and LPV control technique in [170], the combination of SOSP and linear design technique in [38].

- The second direction to tackle the SOS programming complexity is to exploit the structure and sparsity of the problem so that the SOS programs can be formulated into SDPs with smaller size. Some promising comments have been provided in [37, 60, 92, 102].
- Thirdly, the degrees of the polynomials should be kept as low as possible, since the use of high-degree polynomial functions can be problematic. First, high-degree polynomials are sensitive to numerical errors. Second, many additional monomial terms are introduced in the monomial bases in the high-dimensional polynomials. This always leads to extremely rapid increase in the number of optimization decision variables and the size of the corresponding SDP. Therefore, it is suggested that only necessary terms should be introduced to reduce the computational complexity.
- In the decision polynomials returned by the SOS solver, there might be some terms with sufficiently small coefficients compared with other entries. These redundant entries might lead to the numerical errors in the optimization. Therefore, one way to solve this numerical problem is to re-examine the magnitude of each coefficient of the decision polynomials returned by the SOS solver, then eliminate the redundant terms and perform the optimization again to obtain more feasible solutions.
- Finally, it is hoped that there will be significant advances in the development of efficient solvers, and there will be future advances in computing power, which will help make SOS more suitable to handle higher dimensional problems. Besides SOSTOOLS, a recent parser YALMIP [79], which is a modeling language for advanced modeling and solution of convex and nonconvex optimization problems,

can be utilized to solve the SOSP.

Publication list

Journal Papers:

1. **Dan Zhao**, Jianliang Wang, “Robust H_∞ filtering for Polynomial Nonlinear Systems Using Sum of Squares Optimization”, Submitted to Journal of Optimization Theory and Applications.
2. **Dan Zhao**, Jianliang Wang, “Multi-objective Robust and Reliable Control for Polynomial Nonlinear Systems Using Sum of Squares Optimization”, Submitted to International Journal of Control , April 2009.
3. **Dan Zhao**, Jianliang Wang, “Robust Static Output Feedback Design for Polynomial Nonlinear Systems”, Accepted by International Journal of Robust and Nonlinear Control, 2009.

Conference Papers:

1. **Dan Zhao**, Jianliang Wang, “An Improved Nonlinear H_∞ Synthesis for Parameter-Dependent Polynomial Nonlinear Systems Using SOS Programming”, American Control Conference 2009, pp. 796 - 801, St. Louis, MO, USA, June 10-12, 2009.
2. **Dan Zhao**, Jianliang Wang, “Stability Analysis and Design for Polynomial Nonlinear Systems Using SOS with Application to Aircraft Flight Control”,

- Proceedings of the 17th World Congress, The International Federation of Automatic Control, pp. 8684-8689, Seoul, Korea, July 6-11, 2008.
3. **Dan Zhao**, Jianliang Wang, Engkee Poh, Fang Liao, "Stability Analysis and Controller Synthesis for Parameter-Dependent Polynomial Nonlinear Systems", American Control Conference 2007, pp. 6015-6020, New York, NY, USA, July 9-13, 2007.
 4. **Dan Zhao**, Jianliang Wang, Fang Liao, Engkee Poh, "Nonlinear Optimal Control for Parameter-Dependent Polynomial Nonlinear Systems", IEEE International Conference on Control and Automation 2007, pp. 1710-1715, Guangzhou, CHINA, May 30 to June 1, 2007.
 5. Lan Li, Zhu Kuanyi, **Dan Zhao**, "Stabilizing Function of the λ -Model with the Tiny Moment of Inertia in a Single Joint Limb System", IEEE International Conference on Control and Automation 2007, pp. 2085-2090, Guangzhou, CHINA, May 30 to June 1, 2007.

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