Near-optimum decoding and performance bounds for multiple-antenna communication

Ling, Cong

2005

Ling, C. (2005). Near-optimum decoding and performance bounds for multiple-antenna communication. Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/4703>

<https://doi.org/10.32657/10356/4703>

Nanyang Technological University

Downloaded on 26 Apr 2023 21:27:05 SGT

Near-Optimum Decoding and Performance Bounds for Multiple-Antenna Communication

Ling Cong

School of Electrical & Electronic Engineering

A thesis submitted to the Nanyang Technological University
in fulfillment of the requirement for the degree of
Doctor of Philosophy

2005

Acknowledgment

First and foremost, I would like to express my earnest appreciation to my supervisor, Dr. Kwok Hung Li, and my co-supervisor, Dr. Alex C. Kot, for their professional guidance, invaluable assistance and kindly encouragement throughout my PhD study.

I have benefited from the collaboration and discussion with Dr. W. H. Mow, an information theorist who first introduced universal lattice decoding to communications, and Dr. Q. T. Zhang, an expert of performance analysis for fading channels.

It is my great pleasure to thank and acknowledge the fellow students in Communication Lab IV. Appreciation is also extended to all my friends for their care and friendship during my joyful life in Singapore.

Last but not the least, this thesis is dedicated to my family and especially my wife Lu Gan, whose love and support are always the greatest inspiration for me.

Table of Contents

<i>Acknowledgment</i>	<i>i</i>
<i>Table of Contents</i>	<i>ii</i>
<i>Summary</i>	<i>iv</i>
<i>Chapter 1 Introduction</i>	<i>1</i>
1.1 Overview	1
1.2 Objectives and Contributions.....	3
1.3 Organization.....	5
<i>Chapter 2 MIMO Communication and Coding</i>	<i>7</i>
2.1 Information-Theoretic Results	7
2.2 MIMO Codes	11
2.3 MIMO Decoding.....	18
<i>Chapter 3 MIMO Differential Lattice Decoding</i>	<i>27</i>
3.1 Introduction.....	27
3.2 Lattice Formulation.....	28
3.3 Differential Lattice Decoding	31
3.4 Numerical Results	38
3.5 Conclusions.....	42
<i>Chapter 4 Receiver Design for Nondiagonal Constellations</i>	<i>44</i>
4.1 Introduction.....	44
4.2 System Model	45
4.3 Limitation of Linear Prediction Receiver	46
4.4 Noncoherent Sequence Detection	50
4.5 Error Analysis	54
4.6 Reduced-State Detection.....	61
4.7 Conclusions and Discussion	64
<i>Chapter 5 Performance Bounds for Space-Time Codes</i>	<i>67</i>
5.1 Introduction.....	67
5.2 System Model and Union Bound.....	68
5.3 Gallager Bounds.....	71
5.4 Weight Enumeration.....	84
5.5 Numerical Examples and Discussion.....	89
5.6 Correlated Fading.....	95
5.7 Conclusions.....	96
<i>Chapter 6 Performance Bounds for Noncoherent Systems</i>	<i>97</i>

6.1 Introduction.....	97
6.2 Gallager Bounds for Quadratic Decoders	97
6.3 Case Study.....	102
6.4 Conclusions.....	108
<i>Chapter 7 Conclusions and Future Work.....</i>	<i>110</i>
7.1 Conclusions.....	110
7.2 Future Work	110
<i>Appendix</i>	<i>113</i>
A. Importance Sampling	113
B. Cubical Bound.....	113
C. Gallager Bounds: Historical Notes.....	114
D. Gallager's Second Bound for Space-Time Codes.....	116
<i>Author's Publications.....</i>	<i>119</i>
<i>Bibliography</i>	<i>120</i>

Summary

Multiple-antenna communication promises substantially enhanced data rate and diversity over rich-scattering fading channels. Coding for multiple-input multiple-output (MIMO) channels addresses three related problems: design of MIMO codes, MIMO decoding and performance analysis. In this dissertation, we look at selected topics in the latter two problems. In particular, we present near-optimum MIMO differential decoding algorithms, aiming at high decoding speed or improved performance in continuous fading, and derive tight performance bounds for space-time codes. The analysis of decoding algorithms and performance bounds herein may provide useful insights into code design and facilitate computer search for good codes.

A number of MIMO differential decoders have been given in literature. However, most decoders are natural extensions of the traditional differential phase-shift keying (DPSK) techniques and do not really address some fundamental differences of the MIMO differential scheme. As a result, most decoders are limited to small constellation sizes and follow the linear prediction structure, which does not necessarily match the matrix signaling of MIMO. In this dissertation, we propose differential lattice decoding that is significantly faster for large constellation sizes. Moreover, noncoherent decoders well-matched to the matrix signaling over continuously fading channels are presented. Proper approximations are made to obtain practical decoding complexity, which nonetheless guarantee near-optimum performance.

The standard union bound is quite loose (in fact often divergent) for either coherent or noncoherent space-time codes. This phenomenon is especially pronounced in a quasi-static fading channel. We present tight performance bounds for space-time codes in the general framework of Gallager's bounding techniques. Closed-form upper bounds are proposed that are surprisingly tight in terms of the frame error probability. The proposed bounds only assume the knowledge of the weight spectrum, and can be evaluated very fast. In addition, we present novel methods of weight enumeration, which, in conjunction with the bounding techniques, give a complete treatment of performance bounds for space-time codes. The bounds are then extended to noncoherent MIMO schemes.

Chapter 1

Introduction

1.1 Overview

The mathematical model of a digital communication system is most concisely described by Fig. 1, which traces back to Shannon's seminal work on information theory [1]. The channel generally represents the medium through which signals are transmitted. It may be subject to various types of disturbances such as noise, fading, dispersion, filtering, interference, nonlinearity, band-limitation etc. The encoder represents any signal processing of the source output prior to transmission, while the decoder represents the processing of the channel output with the objective of reproducing the source output with acceptable accuracy. In digital communications, the accuracy is typically measured by the error probability of bits, symbols, blocks or frames.

Fig. 1. Mathematical model of a communication system.

Shannon proved that if the source rate is below the channel capacity, then there exist codes with long codewords that lead to arbitrarily small error probabilities. As such, the channel capacity is considered by information theorists the most fundamental measure of a communication system. It is of great significance to communication engineers as well. For instance, the data bandwidth of an optical fiber can easily surpass by many orders that of a telephone line equipped with the most advanced modem, since the former has much higher capacity. This is not the case in wireless communications, however, as the wireless spectrum is a precious resource shared by all users. A meaningful figure of merit is the spectral efficiency, i.e., the capacity per Hz.

Random coding is assumed in Shannon's proof of the coding theorem. For a given channel, hence the capacity, it remains the task of coding theorists to design explicit coding systems that can realize a significant portion of the promised capacity. Channel coding theory generally addresses three closely related problems:

- **Code construction.** This gives concrete block or convolutional codes with low error probabilities on a given channel. Note that the channel capacity may or may not be the guideline of the construction of a concrete code. The more common criterion for code design is the error performance or distance. Construction of good codes is challenging and is seen as an art of invention. This is essentially bypassed in Shannon's random coding approach.
- **Decoding algorithms.** On the receiver side, information sequences are recovered from the channel output in an optimum or near-optimum manner. These include algebraic decoding, maximum-likelihood (ML) decoding, and sequential decoding etc. [2]. Once a code is available, decoding constitutes the major implementation complexity of a coded system. The decoding complexity largely determines the feasibility of a given code. Hence, the efficiency of decoding algorithms is important in real applications. Shannon's random coding approach simply assumes an ML decoder, which is not always practical.
- **Performance bounds.** The performance of a coded system is analyzed by using proper bounding techniques. Since the exact expression of the error probability rarely exists for a coded system, performance bounds are almost the only way out. The union bound is most popular due to its simplicity, but it may be inadequate in many cases. Historically, tight performance bounds of Gallager play a crucial role in proving the coding theorem [3], [4].

Over the past half century, the information theoretical capacity of the traditional single-antenna system has been gradually achieved by powerful codes such as concatenated codes, algebraic-geometry codes, trellis coded modulation (TCM), low-density parity check (LDPC) codes [5] and turbo codes [6]. There is not much room for further improvement in the data bandwidth of a single-antenna channel. To support broadband wireless communication, channels with much higher capacity need to be created. Multiple-antenna technology is an effective way to boost the spectral efficiency, where a multiple-input multiple-output (MIMO) channel is created by employing multiple antennas at the transmitter and the receiver. It is well known that the capacity of a rich-scattering MIMO channel increases linearly with the smaller of the number of transmitter and receiver antennas [7], [8]. This has motivated the spatial multiplexing technology [9], where multiple data streams are transmitted simultaneously from all antennas. More traditionally, multi-antenna technology is used in communications for the purpose of diversity. Now it is recognized that diversity and multiplexing can be obtained simultaneously in MIMO systems [10]. The promised capacity/diversity, of course, can

only be achieved by proper MIMO coding techniques. This dissertation addresses selected topics of the second and third problems of MIMO coding theory, namely MIMO decoding and performance bounds. Although code design is not the concern of this dissertation, the proposed decoding algorithms can influence the design of codes, while the derived performance bounds can provide considerable insights and facilitate computer search for good codes.

1.2 Objectives and Contributions

Communication can be broadly classified as coherent and noncoherent in accordance with the availability of the channel information on the receiver side (the transmitter never knows the channel in the framework of this dissertation.) Noncoherent communication, broadly interpreted as communication under channel uncertainty, is especially needed when it is difficult or uneconomical to estimate a time-varying channel. At present, a popular trend in cellular communication systems is to use pilots or training symbols for channel estimation and then perform coherent decoding with the obtained channel estimates. This pilot-aided approach must be thought of as an ad-hoc, suboptimum method of noncoherent communication rather than the optimum means. Apparently, the capacity of a noncoherent channel is necessarily an upper bound on that of the pilot-based system.

The capacity of noncoherent MIMO systems is investigated in [11], [12]. The information-theoretic results suggest a signaling scheme of unitary space-time modulation for block fading channels [13], i.e., the random fading coefficients are constant during a block. A mobile channel often exhibits continuous fading because of the movement of the mobile user. The results of space-time modulation for continuous fading are rather limited. Hence, the traditional differential scheme, i.e., differential phase-shift keying (DPSK), has been extended to multiple-antenna systems. The resultant multiple-antenna differential scheme is popularly called differential space-time modulation (DSTM) [14], [15], [16].

- **MIMO differential decoding.** A number of noncoherent decoders for the MIMO differential scheme have been given in literature. Most decoders are natural extensions of the existing DPSK results and do not really address some fundamental differences of the MIMO differential scheme. For example, most decoders are limited to small constellation sizes, or otherwise the decoding will be too lengthy. This is not consistent with the goal of high rate transmission in MIMO systems. Moreover, many advanced receivers adopt linear prediction and the decision-feedback mechanism, which again follows that of DPSK. Although this technique is successful for the scalar

signaling of DPSK, it does not necessarily match the matrix signaling of DSTM. In Part I of this dissertation, we look at these two fundamental issues. More specifically, we shall propose fast MIMO differential decoders from a lattice viewpoint, and noncoherent decoders that are well-matched to the matrix signaling. Truly optimum decoding in the two cases is difficult to achieve, if not impossible. Consequently, proper approximations are made, which nonetheless guarantee near-optimum performance. In this sense, we are addressing “near-optimum decoding” for (noncoherent) multiple-antenna communication.

We note that rigorously optimum decoding is often hard to implement in practice because of the associated complexity. Thus the near-optimum decoding philosophy is relatively common to see. Some examples are iterative decoding for turbo-like codes and sequential decoding for long-memory trellis codes. In MIMO differential decoding, the difficulty of optimum decoding is caused by the huge constellation sizes or long memory of the continuously fading channel.

When analyzing the performance of the noncoherent (sequence) decoder, we observe that the traditional union bound is usually quite loose (sometimes even diverges), especially at low signal-to-noise ratios (SNR). We recognize that this is a reflection of the weakness of the union bound. To derive a tight performance bound, some nonstandard bounding techniques are necessary. In the process of literature survey, we notice that, surprisingly, no efficient bounding techniques exist even for coherent MIMO coding! This has motivated us to look at performance bounds of space-time codes in Part II of this dissertation.

- **MIMO Performance bounds.** Probably the most illustrative MIMO channel model is like this: each link between a pair of transmit and receive antennas is subject to flat Rayleigh fading, and the fading coefficient remains constant during a frame, but changes independently from frame to frame; the fading coefficients are statistically independent in space. Communication over this type of quasi-static channels was considered challenging. Since there is no time diversity, conventional coding is not effective. Space-time codes are specially designed to capture the spatial diversity and integrate this with error correction. Thus much better performance is obtained. Standard performance analysis for space-time codes assumes the union bound. Because the error probability only decreases polynomially with the weight in quasi-static fading channels, while the number of codewords is typically exponential, the union bound must diverge. To obtain a meaningful bound, nonstandard bounding techniques are to be adopted in this dissertation. We develop tight and efficient

bounds in the general framework of the Gallager bound [5], which was originally used to analyze the performance of LDPC codes. In the past, Gallager's bound has proven to be very general and be extremely successful in the context of turbo-like codes [17]. The derived bounds for coherent space-time codes are then extended to noncoherent systems, thereby enabling us to return to the problem of noncoherent sequence detection of DSTM.

1.3 Organization

The dissertation is organized as follows. Chapter 2 is the survey of information-theoretic results and explicit designs of MIMO communication and coding. The body of the dissertation is organized into two parts. Part I (Chapters 3 and 4) is devoted to MIMO differential decoding. Specifically, Chapter 3 addresses differential lattice decoding for diagonal DSTM constellations, which is significant faster than brute-force ML decoding. In Chapter 4, we look at near-optimum decoding of non-diagonal DSTM over continuously fading channels, which improves the performance of linear predictive receivers considerably. Part II (Chapters 5 and 6) addresses performance bounds for MIMO coding. In Chapter 5, we derive tight performance bounds for space-time codes with coherent decoding. Chapter 6 is devoted to bounds for noncoherent decoding. This organization is plotted schematically in Fig. 2. In the end we conclude the dissertation and comment on future work.

Fig. 2. Organization of the body of the dissertation.

The two parts of the thesis are linked by the performance analysis of noncoherent sequence detection for DSTM (see Chapter 4), which motivated the performance bound study of Part II. In turn, Part II is closed by a revisit to the performance of noncoherent sequence detection.

In the PhD period, we also proposed an improved linear predictive receiver for DSTM by using the multi-sampling technique [J5]. For the integrity of the dissertation, this sideline work is not included.

Notation: Throughout this dissertation, matrices (vectors) are represented in bold upper (lower) case, $\mathbf{I}_n$ ($\mathbf{0}_n$) denotes the n -by- n identity (null) matrix (the subscript n is omitted if the dimension is clear,) $(\cdot)^*$ complex conjugate, $(\cdot)^T$ transpose, $(\cdot)^H$ Hermitian transpose, $\text{tr}(\cdot)$ trace, $\otimes$ stands for the Kronecker product, $\circ$ stands for the Hadamard product, $\text{diag}(\cdot)$ denotes forming a diagonal matrix from a vector, $|\mathbf{A}|$ denotes the determinant, $\|\mathbf{A}\|^2 = \text{tr}(\mathbf{A}\mathbf{A}^H)$ denotes the Frobenius norm, and $\llbracket \cdot \rrbracket$ denotes rounding towards the nearest integer.

Chapter 2

MIMO Communication and Coding

The use of multiple antennas in communications is not new. Antenna arrays and diversity reception through multiple antennas have existed for decades. In array signal processing, antennas are closely spaced, and the high correlation among antennas is exploited to realize beamforming, nulling, inter-user interference rejection etc. Transmit or receive diversity is a means to combat fading, where fading is considered a nuisance. In contrast, fading is a beneficial factor in MIMO communication. In MIMO or space-time communication, antennas are usually far spaced so as to achieve independent fading (though this is not a must). It in fact takes advantage of the independence naturally provided by fading to create multiple spatial parallel channels, since the channel matrix is well conditioned with high probability. Hence, higher channel capacity is obtained over the same bandwidth, thereby higher spectral efficiency. Diversity reception (or transmission) can also increase the channel capacity, but this is not the most effective way. It is only in MIMO where multiple antennas are placed both at the transmitter side and at the receiver side that the channel capacity increases linearly. Moreover, the theory of MIMO is substantially different from that of array signal processing. It is known that there is significant gain to be realized by viewing the problem from a coding perspective rather than purely from a signal processing point of view.

2.1 Information-Theoretic Results

Fig. 3. Block diagram of a MIMO channel.

Consider an n_R -by- n_T MIMO channel depicted in Fig. 3. The n_R -D received signal $\mathbf{y}$ can be written as

$$\mathbf{y} = \sqrt{\rho/n_T} \mathbf{H} \mathbf{x} + \mathbf{n} \quad (2.1)$$

where $\mathbf{x}$ is the n_T -D complex transmitted vector, $\mathbf{H}$ is the n_R -by- n_T complex Gaussian channel matrix with i.i.d. entries, and $\mathbf{n}$ is the n_R -D i.i.d. complex Gaussian noise vector. Each entry of $\mathbf{H}$ or $\mathbf{n}$ has unit variance. The transmitted signal $\mathbf{x}$ satisfies the unit power constraint so that ρ represents the average SNR at each receive antenna. Precise knowledge about $\mathbf{H}$ is assumed at the receiver.

2.1.1 Capacity of MIMO channels

Conditioned on the channel realization $\mathbf{H}$, the MIMO capacity is given by

$$C_{\mathbf{H}} = \log_2 \left| \mathbf{I} + \frac{\rho}{n_T} \mathbf{H} \mathbf{H}^H \right| \quad (\text{bps/Hz}) \quad (2.2)$$

From this point on, researchers usually proceed in two directions. In one scenario, the channel coefficients remain constant for a long time, from the beginning till the end of the transmission. This scenario is typical of indoor communication, fixed wireless access or a low-speed mobile user. The Shannon capacity of this type of quasi-static fading channels will be zero. This is because there always exists a positive probability that a particular realization of $\mathbf{H}$ (deep fading) is incapable of supporting arbitrarily low error rates, no matter how long the codeword is. For such non-ergodic channels, the complementary cumulative distribution function (ccdf) P_c of the capacity is to be found. A capacity outage is defined as the event that the instantaneous capacity $C_{\mathbf{H}}$ is less than a given rate R . The outage probability is thus

$$P_{\text{out}}(R) = P(C_{\mathbf{H}} < R) = 1 - P_c(R). \quad (2.3)$$

The so-called *outage capacity* is defined as the rate R for which a target outage probability p can be satisfied:

$$C_{\text{outage}} = P_{\text{out}}^{-1}(p). \quad (2.4)$$

Foschini and Gans [7], and even earlier, Winters [18], did not pursue an analytic approach and instead estimated the ccdf and outage capacity by the Monte Carlo method.

In another scenario, the channel exhibits sufficiently fast fading so that it is possible to do coding over many channel realizations. For such ergodic channels, the Shannon capacity can be defined and is equal to

$$C_{\text{ergodic}} = E_{\mathbf{H}}[C_{\mathbf{H}}]. \quad (2.5)$$

Telatar [8] derived the analytic expression for the ergodic capacity

$$C_{\text{ergodic}} = \int_0^\infty \log_2 \left(1 + \frac{\rho}{n_T} \lambda \right) \sum_{k=0}^{m-1} \frac{k!}{(k+n+m)!} [L_k^{n-m}(\lambda)]^2 \lambda^{n-m} e^{-\lambda} d\lambda \quad (2.6)$$

where $m = \min\{n_R, n_T\}$, $n = \max\{n_R, n_T\}$, and $L_k^{n-m}(x)$ is the Laguerre polynomial of order k . He won the 2001 Information Theory Paper Award for this analytic work.

Recently a unified approach was undertaken by Chiani et al. [19]. They derived the characteristic function of the MIMO capacity (2.2), which is a random variable. This enables analytic evaluation of the MIMO capacity in terms of ccdf, the outage capacity and the mean capacity (and all moments as well).

The asymptotic behavior at high SNR provides considerable insights. At high SNR, the ergodic MIMO capacity admits the form [10]

$$C_{\text{ergodic}} = \min\{n_R, n_T\} \log_2 \frac{\rho}{n_T} + \sum_{i=|\min\{n_R, n_T\}+1}^{\max\{n_R, n_T\}} E[\log \chi_{2i}^2] + o(1) \quad (2.7)$$

where χ_{2i}^2 is chi-square distributed with $2i$ degrees of freedom. It is clear that the capacity increases linearly with $\min\{n_R, n_T\}$. Similar behavior exists for the outage capacity.

Fig. 4. Ergodic MIMO capacity.

Fig. 4 contrasts the MIMO capacity (2.6) to the capacity $\log_2(1 + \rho)$ of a single-antenna complex AWGN channel. It is clear that the MIMO capacity can easily surpass that of a Gaussian channel. The MIMO capacity increases linearly with the antenna number. On the contrary, it is easy to check that multiple-antenna does not change the capacity of a

Gaussian channel (this can be done by letting $\mathbf{H}$ in (2.2) be an all-one matrix times $\sqrt{2}/2$.) Hence fading is indeed a beneficial factor. This is in contrast to the common intuition that fading impairs the performance. In fact, the fading penalty is insignificant even for the single antenna case, as shown by the bottom two curves in Fig. 4.

2.1.2 Diversity vs. Multiplexing: Two Aspects of MIMO

Knowing the channel capacity is not always enough. For example, it is often of great interest to know how hard it is to get close to the capacity. The classic error exponent theory of Gallager [4] tells us that there exists a tradeoff between the error probability and the data rate. Specifically, the error probability of random coding in Gaussian channels decreases exponentially with block length L :

$$P_e \leq e^{-LE_r(R)} \quad (2.8)$$

where $E_r(R)$ is the random coding exponent for given rate R . In a quasi-static MIMO fading channel, (2.8) is no longer a good expression because the error probability decreases polynomially at most. Then another parameter comes into play, namely, the diversity order. Consequently, the error probability is a function of the rate and the diversity order and there should be an analogous tradeoff between rate and diversity.

From an engineering perspective, rate and diversity are two resources of a MIMO channel. On one hand, space-time trellis and block codes exploit the diversity to improve performance over a quasi-static fading channel. On the other hand, independent information streams can be transmitted in parallel through multiple antennas so that the data rate is increased, as in BLAST. This boosting effect is called spatial multiplexing. Most research (as well as this dissertation) only looks at one aspect of the MIMO channel. Zheng and Tse [10] put forth a unified point of view that both diversity gain and spatial multiplexing gain can be simultaneously obtained in a MIMO channel and there exists a fundamental tradeoff between how much of each type of gain can be achieved. To be more specific, they consider a family of codes with a spatial multiplexing gain r and a diversity gain d , where the rate of the codes scales like $r \log \rho$, and the average error probability decays like ρ^{-d} . The optimal tradeoff means for each given r the maximum diversity gain $d^*(r)$ achievable by any family of codes. Under mild conditions, they show that the tradeoff is very simple:

$$d^*(r) = (n_T - r)(n_R - r) \quad (2.9)$$

for $r = 0, 1, \dots, \min\{n_T, n_R\}$. Thus, it is as if r transmit and r receive antennas were used

for multiplexing while the remaining $(n_T - r)$ transmit and $(n_R - r)$ receive antennas provided the diversity (and $P_e(r) \leq \rho^{-d^*(r)}$ up to some coefficient). It is shown in [10] that the diversity-multiplexing tradeoff is indeed a parallel theory of error exponents.

In this framework, the tradeoff curve of the Alamouti scheme is given by

$$d(r) = n_T n_R (1 - r)^+ \quad (2.10)$$

where x^+ denotes $\max\{0, x\}$. In the case of $n_R = 1$, the Alamouti scheme is optimal; but in the case of $n_R = 2$, it is suboptimal except at $r = 0$. They further concluded that the orthogonal space-time block codes, fixed-rate space-time trellis codes, V-BLAST, and D-BLAST with zero-forcing (ZF) detection are all suboptimal, while D-BLAST with minimum mean square error (MMSE) detection achieves the optimal tradeoff, if the processing overhead is ignored. The suboptimality of the Alamouti scheme, orthogonal space-time block codes and fixed-rate space-time trellis codes is not difficult to understand, because spatial multiplexing is not a purpose of these codes. The optimality of MMSE D-BLAST suggests that the code design for optimal diversity/multiplexing tradeoff may not be challenging, if the tradeoff is borne in mind. For example, lattice space-time codes given in [20] reach the optimal tradeoff.

2.2 MIMO Codes

We only look at the communication aspects of MIMO in this dissertation. In different lines of thought, researchers proposed different MIMO schemes.

2.2.1 Spatial Multiplexing Schemes

This line of thought is motivated by the promising capacity gain of a MIMO channel. It exploits the degrees of freedom provided by a MIMO channel to transmit independent information streams in parallel through antennas. Therefore, high data rates are sustained. Since it is believed that a maximum-likelihood (ML) decoder will be computationally prohibitive for high data rates¹, an idea of layering is widely followed. Accordingly, spatial multiplexing schemes are often called “layered space-time codes”. Foschini proposed the famous Bell Labs space-time architecture (BLAST) [9], which reportedly achieves spectral efficiency of 20-40 bps/Hz. Layering is in essence a divide-and-conquer strategy. The receiver uses a successive nulling and canceling process so that each layer can be decoded individually. In this way the receiver complexity is greatly reduced, and it

¹ This belief is not always true. As will be shown in Sect. 2.3, a lattice perspective can significantly lower the complexity of ML decoding.

allows the use of 1D channel coding over each layer.

Fig. 5. Block diagram of BLAST.

Fig. 5 shows the BLAST transmitter. In diagonal BLAST (D-BLAST), the input data stream is divided into layers, and each layer is transmitted across antennas/time slots in a diagonal fashion. The modulo- n_T shift block in Fig. 5 corresponds to the diagonal assignment in space/time. This makes it possible for all layers to share the diversity of each transmit antenna. Put in matrix form, the transmitted signal of a 4×4 D-BLAST system is

$$\begin{bmatrix} \mathbf{x}_1^{(1)} & \mathbf{x}_1^{(2)} & \mathbf{x}_1^{(3)} & \mathbf{x}_1^{(4)} & \dots \\ \mathbf{0} & \mathbf{x}_2^{(1)} & \mathbf{x}_2^{(2)} & \mathbf{x}_2^{(3)} & \mathbf{x}_2^{(4)} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{x}_3^{(1)} & \mathbf{x}_3^{(2)} & \mathbf{x}_3^{(3)} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{x}_4^{(1)} & \mathbf{x}_4^{(2)} & \dots \end{bmatrix}$$

where $\mathbf{x}_i^{(k)}$ denotes the symbols transmitted on the i th antenna for layer k . The receiver first estimates $\mathbf{x}_2^{(1)}$, $\mathbf{x}_3^{(1)}$ and $\mathbf{x}_4^{(1)}$ by nulling out the interference from other layers (entries over them in the above matrix), and then these are fed along with $\mathbf{x}_1^{(1)}$ to a joint decoder to decode the first layer. After decoding the first layer, the contribution from this layer is canceled from the received signals and then decoding of the second layer begins. Here, an overhead is required to start the decoding process so that every layer is on equal footing. This corresponds to the $\mathbf{0}$ entries in the above example. To obtain better performance, the nulling (ZF) stage can be replaced by a MMSE receiver. Foschini demonstrated that D-BLAST achieves a fairly tight lower bound on the MIMO capacity:

$$C \geq \sum_{i=1}^{n_T} \log_2 \left[1 + \frac{\rho}{n_T} \chi_{2i}^2 \right] \text{ (bps/Hz)}. \quad (2.11)$$

Vertical BLAST (V-BLAST) is a simpler version. Its block diagram is given by Fig. 5 without the modulo- n_T shift block. In matrix form, the signal is simply given by

$$\begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \\ \mathbf{x}^{(3)} \\ \mathbf{x}^{(4)} \end{bmatrix}$$

where symbols are arranged in a vertical fashion. No overhead is needed here. The receiver uses a similar successive nulling/cancellation process. For a coded system, the above matrix is actually very wide, and a more appropriate name is Horizontal BLAST (H-BLAST). Note that $\mathbf{x}^{(i)}$'s are independent codewords. Since each layer is pegged to a fixed antenna, the layers are not on equal footing. Obviously the first layer has the worst performance, the second layer has the second worst performance, and so on. Thus, V-BLAST is inferior to D-BLAST in performance.

The overhead of D-BLAST is imposed by the nulling/cancellation processing. To avoid this overhead, which results a loss in spectral efficiency, some improvements have been proposed in literature. The basic idea is to replace the modulo- n_T shift block in Fig. 5 by a more general one. For example, a threaded space-time system is given in [21], which unfortunately loses the elegant nulling/cancellation structure and must resort to iterative decoding.

2.2.2 Space-Time Trellis Codes

The design of channel codes is not a specific goal of BLAST. A more coding-theoretic line of work for MIMO communication was initiated by Tarokh et al. [71]. Their space-time codes integrate channel coding and transmit diversity. Data are encoded by a trellis code and are then split into n_T streams that are simultaneously transmitted through n_T antennas. By assuming ML decoding, they derived the error performance and presented several design criteria of space-time codes. For the case of quasi-static Rayleigh fading, the pairwise error probability (PEP) is given by

$$P(\mathbf{c} \rightarrow \mathbf{e}) \leq \left(\prod_{i=1}^r \lambda_i \right)^{-n_R} \left(\frac{E_s}{4N_0} \right)^{-m_R} \quad (2.12)$$

where $\lambda_1, \dots, \lambda_r$ are nonzero eigenvalues of $(\mathbf{c} - \mathbf{e})(\mathbf{c} - \mathbf{e})^H$, the correlation matrix of codeword difference, and $E_s / N_0 = \rho / n_T$ is the symbol SNR per transmit antenna. Therefore, code design should meet the rank and determinant criteria: maximize the minimum rank of $\mathbf{c} - \mathbf{e}$, as well as the minimum determinant of $(\mathbf{c} - \mathbf{e})(\mathbf{c} - \mathbf{e})^H$ (if full-rank) over all pairs of distinct codewords. Tarokh et al. further presented the distance and product criteria for ideal fast fading. Bearing such criteria in mind, they constructed a

bunch of space-time trellis codes for 2 to 4 antennas, whose performance is within several dB of the outage probability.

Tarokh et al.'s codes are handcrafted and no specific efforts are made to optimize the coding gain. Namely, the codes only achieve the diversity gain. Computer search methods and results for codes with improved coding gains are presented in [22], [23], [24]. The systematic design of PSK space-time codes meeting the rank criterion is given in [25]. A rank criterion for QAM space-time codes is given in [26]. As the number of antennas gets large, the MIMO channel behaves more like a Gaussian channel. So Biglieri et al. [27] and Yuan et al. [28] argued that the conventional Euclidean distance rather than the determinant should be maximized.

If implemented by the Viterbi algorithm, the complexity of ML decoding for space-time trellis codes (measured by the number of trellis states) grows exponentially with the data rate. This is often criticized as the disadvantage of space-time trellis codes².

2.2.3 Space-Time Block Codes

To lower the decoding complexity, Tarokh et al. propose a class of space-time block codes (STBC) by applying the theory of orthogonal designs [29]. A key feature of this class of codes is the orthogonality of code matrices that enables separate decoding rather than joint decoding of symbols transmitted from different antennas. They can be used as inner codes in a channel-coded system, which capture the diversity gain of a MIMO fading channel. The Alamouti scheme [30] is the first discovery in this class of codes. Its code matrix is given by

$$\mathbf{X} = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}. \quad (2.13)$$

It can be seen that two rows of $\mathbf{X}$ are orthogonal. STBC is basically an extension of this simple idea to an arbitrary number of transmit antennas. Generally, for symbols $s_1, \dots, s_{n_T}$ to be transmitted, the code matrix of linear STBC takes the form

$$\mathbf{X} = \sum_{i=1}^{n_T} (\bar{s}_i \mathbf{A}_i + \tilde{s}_i \mathbf{B}_i) \quad (2.14)$$

where $\{\mathbf{A}_i, \mathbf{B}_i\}$ are fixed matrices, and $\bar{s}_i = (s_i + s_i^*)/2$, $\tilde{s}_i = (s_i - s_i^*)/2$. Orthogonal STBC has the property

² Again, this criticism is not always justified since there exist low-complexity decoding algorithms for high-rate codes.

$$\mathbf{X}\mathbf{X}^H = \sum_{i=1}^{n_T} |s_i|^2 \cdot \mathbf{I}. \quad (2.15)$$

Suppose that the signal model is $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{N}$, pending proper subsuming of the coefficient $\sqrt{\rho/n_T}$. The ML decoding metric is given by

$$\begin{aligned} \|\mathbf{Y} - \mathbf{H}\mathbf{X}\|^2 &= \|\mathbf{Y}\|^2 - 2 \operatorname{Re} \left[\operatorname{tr}(\mathbf{Y}^H \mathbf{H}\mathbf{X}) \right] + \|\mathbf{H}\mathbf{X}\|^2 \\ &= \|\mathbf{Y}\|^2 - 2 \sum_{i=1}^{n_T} \operatorname{Re} \left[\operatorname{tr}(\mathbf{Y}^H \mathbf{H}\mathbf{A}_i) \right] \bar{s}_i - \operatorname{Im} \left[\operatorname{tr}(\mathbf{Y}^H \mathbf{H}\mathbf{B}_i) \right] \tilde{s}_i + \|\mathbf{H}\|^2 \cdot |s_i|^2 \\ &= \|\mathbf{H}\|^2 \sum_{i=1}^{n_T} \left| s_i - \frac{\operatorname{Re} \left[\operatorname{tr}(\mathbf{Y}^H \mathbf{H}\mathbf{A}_i) \right] - \operatorname{Im} \left[\operatorname{tr}(\mathbf{Y}^H \mathbf{H}\mathbf{B}_i) \right]}{\|\mathbf{H}\|^2} \right|^2 + \text{const}. \end{aligned} \quad (2.16)$$

Equation (2.16) demonstrates that the ML metric decouples into a sum of n_T terms, each of which depends on just one symbol. Accordingly, the detection of s_i is decoupled from the detection of other symbols, thereby significantly reducing the decoding complexity. At the same time, OSTBC guarantees full diversity order.

2.2.4 Noncoherent and Differential Schemes

The decoding of the afore-mentioned MIMO systems generally requires accurate knowledge of channel state information. However, accurate channel estimation is difficult when signals experience fast fading in a mobile environment and especially when many transmit and receive antennas are deployed. Consequently, it is of great interest to develop MIMO systems for unknown channels, which are customarily called noncoherent.

A. Capacity of Noncoherent MIMO Channels

Marzetta and Hochwald studied the capacity of noncoherent multi-antenna channels in piecewise constant fading [11], where the (random) fading coefficients remain constant during T time epochs. Under this model, the received signal is changed from (2.1) to

$$\mathbf{Y} = \sqrt{\rho/n_T} \mathbf{H}\mathbf{X} + \mathbf{N} \quad (2.17)$$

where $\mathbf{Y}$, $\mathbf{X}$ and $\mathbf{N}$ have length T . This piecewise-constant fading model mimics, in a tractable way, the behavior of a continuously fading channel. To derive the noncoherent capacity, the conditional pdf of $\mathbf{Y}$ given $\mathbf{X}$ is to be determined. Here, $\mathbf{H}$ is random with a known pdf, but its value is unknown. They were only able to derive explicitly the capacity of the one transmit antenna and one receive antenna case. The general multiple-antenna case was solved by Zheng and Tse later in [10]. They proved that at high SNR the noncoherent capacity (bps/Hz) takes the form

$$C_{\text{noncoherent}} = K(1 - K/T) \log_2 \rho + c + o(1) \quad (2.18)$$

where $K = \min\{n_T, n_R\}$, $T \geq K + n_R$, and c is a constant. Comparing this with the coherent capacity (2.7), one can see that the coefficient $(1 - K/T)$ represents the noncoherent penalty. It can be shown that the noncoherent capacity approaches the coherent capacity as $T \rightarrow \infty$ (and indeed $(1 - K/T) \rightarrow 1$). The capacity expression has a geometric interpretation as communication on the so-called Grassmann manifold, which, roughly speaking, is the space of noncoherently distinguishable unitary matrices. The capacity-achieving signals only use the directions that are not affected by channel coefficients to communicate.

B. Space-Time Modulation

Motivated by the capacity analysis, Hochwald and Marzetta proposed unitary space-time modulation for noncoherent MIMO communication [13], where signals are orthonormal with respect to time among transmitter antennas. Since the issue of channel coding is not addressed, it works more like conventional orthogonal modulation—so follows its name. Unitary space-time modulation is defined as

$$\mathbf{S} = \sqrt{T} \mathbf{\Phi}, \quad \mathbf{\Phi} \mathbf{\Phi}^H = \mathbf{I} \quad (2.19)$$

where $\mathbf{\Phi}$ is an $n_T \times T$ oblong matrix. All message information is transmitted on the directions of the orthonormal rows of $\mathbf{\Phi}$. Unitary space-time modulation is able to attain the capacity in piecewise-constant fading channels when $T \gg n_T$. The ML noncoherent detector is given by

$$\mathbf{\Phi}_{\text{ML}} = \arg \max_{\mathbf{\Phi}_l} \text{tr} \left\{ \mathbf{Y} \mathbf{\Phi}_l^H \mathbf{\Phi}_l \mathbf{Y}^H \right\}. \quad (2.20)$$

Obviously, if $\mathbf{\Phi}_1 = \mathbf{U} \mathbf{\Phi}_2$, where $\mathbf{U}$ is an $n_T \times n_T$ unitary matrix, then the noncoherent detector is unable to distinguish the two signals. Therefore, signal design for space-time modulation must bear this constraint in mind, e.g., codes on the Grassmann manifold. It is worth mentioning that although OSTBC bears a similarity to the orthogonality (2.19), it has not the noncoherently distinguishable property.

Hochwald and Marzetta briefly addressed the case of continuous fading, but the result is quite limited.

C. Differential Space-Time Modulation

To cope with continuous fading, differential space-time modulation (DSTM) was proposed by several authors almost simultaneously. This is an extension of the standard

single-antenna DPSK to multiple-antenna systems.

Fig. 6. Block diagram of DSTM.

Fig. 6 shows the block diagram of DSTM. Transmitted signals are organized into an $n_T \times n_T$ matrix $\mathbf{S}[\tau]$ where row indices represent different antennas and column indices represent time instants $\tau_T, \dots, \tau_T + n_T - 1$. The matrix is properly normalized so that the average power of each column is one. The total transmitted power therefore does not depend on the number of transmit antennas. The constellation $\mathbb{G}$ of a rate- R (bits/channel use) DSTM system comprises of $L = 2^{Rn_T}$ unitary matrices $\mathbf{G}_l$, $l \in \{0, \dots, L-1\}$, of size $n_T \times n_T$ each. The Rn_T bits to be transmitted at time instant τ_T are mapped to an L -ary symbol $a[\tau]$, which selects an element $\mathbf{G}[\tau] = \mathbf{G}_{a[\tau]}$ from $\mathbb{G}$. Signal matrices are then differentially encoded in a fashion similar to DPSK as

$$\mathbf{S}[\tau] = \mathbf{S}[\tau-1]\mathbf{G}_{a[\tau]} \quad (2.21)$$

with $\mathbf{S}[0] = \mathbf{A}$ is a unitary initially transmitted matrix.

As we have commented previously, all (square) unitary matrices of the same size are indistinguishable to a noncoherent detector. That is why they must be differentially encoded before transmission. Moreover, OSTBC can allow noncoherent detection if differentially encoded. For this reason, sometimes DSTM is also called differential STBC.

$\mathbb{G}$ may or may not be a group, depending on the design methodology. Tarokh and Jafarkhani [14] proposed a differential scheme on the basis of Alamouti's orthogonal design for two transmit antennas. It leads to nongroup constellations, except when $R = 1$. Using the powerful tool of the group theory, Hughes [15], [51] and Hochwald and Sweldens [16] presented richer design that can handle an arbitrary number of transmit antennas. An appealing feature of the group design is that matrix multiplication may be replaced by addition and table look-up.

Hughes' and Alamouti's constellations are generally nondiagonal. Hochwald and Sweldens proposed diagonal signals of the form

$$\mathbf{G}_l = (\mathbf{G}_1)^l, \quad \mathbf{G}_1 = \text{diag}[e^{j2\pi u_1/L}, \dots, e^{j2\pi u_{n_T}/L}] \quad (2.22)$$

where $u_m \in \{0, 1, \dots, L-1\}$ for $m = 1, \dots, n_T$. Though diagonal signals result in performance degradation for high rates compared to non-Abelian constellations, they perform quite well for $R \leq 2$. Specifically, it was shown that the diagonal restriction does not affect the error performance provided that the group is Abelian.

Suppose that the fading process is approximately constant over two consecutive block intervals, the optimization criterion of integers u_m for differential detection is the same as that for coherent detection, namely, to maximize the diversity product [16]

$$\zeta = \min_{1 \leq l < L} \prod_{m=1}^{n_T} |\sin(\pi u_m l / L)|^{1/n_T} = \min_{1 \leq l < L} \prod_{m=1}^{n_T} [1 - \gamma_m^2(l)]^{1/2n_T}. \quad (2.23)$$

where $\gamma_m(l) = |\cos(\pi u_m l / L)|$ are the singular values of the matrix $(\mathbf{I} + \mathbf{G}_0^H \mathbf{G}_l)/2$. The diversity product ζ lies between zero and one. The closer to one ζ is, the better the group $\mathbb{G}$ is in terms of error performance. The diagonal structure greatly simplifies constellation design, since performance optimization can be attained through searching over u_m 's. Optimal values of u_m for $R = 1, 2$ and n_T up to 5 are tabulated in [16].

For almost all these diagonal DSTM constellations, it is possible to construct Gray mapping to assign information bits to signal matrix $\mathbf{G}_l$. In order to do so, the nearest neighbor of $\mathbf{G}_0$ is identified as $l_0 = \arg \min_{1 \leq l < L} \prod_{m=0}^{M-1} |\sin(\pi u_m l / L)|^{1/M}$. The indices of the nearest neighbors of $\mathbf{G}_l$ are $l \pm l_0 \pmod{L}$ owing to symmetry. Then information bits are assigned in a way that two nearest neighbors differ in a single bit position. This not only yields a lower BER, but also facilitates the BER calculation.

2.3 MIMO Decoding

A fundamental motivation of MIMO wireless communication is to support enormously high data rates. This however makes a naive ML decoder quickly become impractical, as its complexity grows exponentially with data rate. For example, the ML decoder needs to search $64^8 = 2^{48}$ points for an uncoded 8×8 system with 64QAM signaling, which is plainly hopeless to implement. Consequently, it is crucial to develop efficient decoding algorithms for high-rate MIMO systems. Furthermore, the feasibility and thereupon the design of MIMO systems depend very much on the decoding complexity affordable at the receiver. Layered systems and OSTBC are exactly motivated by the premise that ML decoding is unaffordable for high-rate codes. Recently, people begin to recognize that there is a need of changing this picture, as there exist low-complexity ML or near-ML

decoding algorithms. Lattice decoding and sequential decoding are two such algorithms. This recognition has significant consequences on system design. For example, there will be no need of layering in spatial multiplexing systems. The successive nulling/cancellation processing of BLAST is suboptimum, while ML decoding brings considerable performance gain. The claim that D-BLAST outperforms V-BLAST is not necessarily true under ML decoding. As another example, there will be no need of imposing orthogonality in STBC. This observation leads to the linear dispersion codes [31], which are linear over space and time so that lattice decoding (or of course BLAST-type decoding) is allowable.

2.3.1 Lattice Decoding

Universal lattice decoding has recently received great interests in MIMO due to its asymptotically polynomial average complexity for many communication problems. More generally, it is applicable to any linear systems, such as CDMA. Universal lattice decoding consists of two major stages—basis reduction and enumeration of nearby lattice points, the latter of which is often called sphere decoding in communication literature. The first stage selects a short and fairly orthogonal basis, usually by using the Lenstra-Lenstra-Lovasz (LLL) algorithm [32], while the second finds the closest lattice point from those falling inside a sphere centered at the query point [33], [34], [35]. Mow was the first to introduce the principle of universal lattice decoding to communications [36], where he considered an equalization problem. Viterbo and Boutros presented lattice decoding for fading channels and coined the term “sphere decoding” [37], where their main contribution is the tailoring to finite lattices. See [38], [39], [40] for up-to-date overview of universal lattice decoding.

To see why lattice decoding can be much more efficient than exhaustive ML decoding, consider the problem of finding the nearest point in the integer lattice $\mathbb{Z}^n$ to a query point $\mathbf{q} \in \mathbb{R}^n$. Apparently, there is no need to search $\mathbb{Z}^n$ (and this is impossible), as the nearest point is simply given by rounding off $\hat{\mathbf{x}} = \llbracket \mathbf{q} \rrbracket$. Roughly speaking, lattice decoding is the extension of the trivial rounding off to a general lattice. In fact, the simple rounding off is surprisingly powerful if a proper basis of the lattice is used.

An m -D lattice for basis $\mathbf{B} \in \mathbb{R}^{m \times n}$ is defined as the set of points $L(\mathbf{B}) = \{\mathbf{B}\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^n\}$. The lattice-decoding problem is to find the nearest lattice point

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathbb{Z}^n} \|\mathbf{q} - \mathbf{B}\mathbf{x}\|^2. \quad (2.24)$$

Communication through the MIMO channel can be formulated as a lattice problem. For

convenience, put the signal model as $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$. If the constellation from which the component of $\mathbf{x}$ is drawn is part of a lattice, such as PAM, QAM, BPSK, and QPSK, then ML decoding can be recast into lattice decoding. Here, $\mathbf{H}$ is the lattice basis, and $\mathbf{y}$ is the query point. The dimension will be doubled for complex signaling.

The sphere decoder draws a ball³ centered at the query point $\mathbf{y}$ with radius R (cf. Fig. 7). Obviously it is sufficient to examine the points falling into this ball, as the nearest point inside the ball must be the nearest point in the entire lattice. Thus the search space is greatly reduced. For the case shown in Fig. 7, only four points need be examined. This property alone does not necessarily reduce the overall decoding complexity, because we still have to enumerate the lattice points inside the ball. This is a hard problem for a general lattice. Fortunately, since the transmitted vector in communication problems is perturbed by Gaussian noise, there is high probability that $\mathbf{y}$ is close to a lattice point for medium to high SNR. When this is true, it turns out that the average complexity of sphere decoding is polynomial in the dimension of the lattice [55]. At the extreme of infinite SNR, pseudo inverse will give the exact solution, which has cubic complexity. At another extreme of low SNR, in contrast, the query point $\mathbf{y}$ is quite arbitrary, and the decoding complexity is exponential. Therefore, it is expected that there is a smooth shift from exponential to cubical complexity as the SNR goes high.

Fig. 7. Geometric illustration of sphere decoding.

The speed of sphere decoding is heavily influenced by the basis $\mathbf{B}$. The closer $\mathbf{B}$ is to an orthonormal basis, the faster sphere decoding is. A same lattice can be generated by different bases:

³ More generally, it can be an ellipsoid or box. A sphere is most commonly used.

$$L(\mathbf{B}_1) = L(\mathbf{B}_2) \text{ iff } \mathbf{B}_1 = \mathbf{U}\mathbf{B}_2 \quad (2.25)$$

where $\mathbf{U}$ is a unimodular matrix, i.e., $\mathbf{U}$ has integer entries and $|\mathbf{U}| = 1$. The objective of lattice reduction is to choose a short and nearly orthogonal basis to represent the same lattice. Then sphere decoding is performed on the reduced basis. In this way, lattice reduction can significantly speed up sphere decoding. There are many algorithms of lattice reduction, among them the LLL algorithm guarantees polynomial running time.

While mathematicians assume infinite lattices, lattices are normally finite in communication problems. Sphere decoding can be easily modified to accommodate finite lattices [37]. On the other hand, it is difficult for lattice reduction to accommodate the finite. As such the boundary of a finite lattice is sometimes ignored, and lattice decoding is performed as if it were an infinite lattice. This causes boundary errors, which nevertheless is insignificant for large alphabet sizes [39].

The Pohst-Fincke strategy [33], [34] and the Schnorr-Euchner strategy [35] are two basic search algorithms of sphere decoding. The Schnorr-Euchner strategy can be interpreted as a branch and bound algorithm for tree search. Therefore, its application goes beyond lattice problems. For instance, Lampe et al. applied sphere decoding to MSDD of MDPSK [47], where the MPSK constellation is generally not a lattice. This hints that other tree search algorithms in optimization theory may play a role in fast MIMO decoding.

2.3.2 Sequential Decoding

Sequential decoding is an old decoding algorithm for convolutional codes with long constraint memory. It actually predates the Viterbi algorithm [2], [4]. Unlike the Viterbi algorithm that searches the full trellis, sequential decoding only examines part of a tree. It belongs to a family of tree search algorithms.

To see how a tree is defined for MIMO decoding, consider the metric $\|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2$ for the case $n_R = n_T = n$. Using the QR decomposition $\mathbf{H} = \mathbf{Q}\mathbf{R}$, where $\mathbf{Q}$ is unitary and $\mathbf{R}$ is upper triangular with positive diagonal elements, we have

$$\|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 = \|\mathbf{Q}^H \mathbf{y} - \mathbf{R}\mathbf{x}\|^2 = \|\mathbf{y}' - \mathbf{R}\mathbf{x}\|^2, \quad \mathbf{y}' = \mathbf{Q}^H \mathbf{y}. \quad (2.26)$$

Due to the upper triangular structure of $\mathbf{R}$, it can be put in a more informative form

$$\|\mathbf{y}' - \mathbf{R}\mathbf{x}\|^2 = \sum_{i=1}^n \left| y'_i - \sum_{j=i}^n r_{i,j} x_j \right|^2 = \sum_{i=1}^n w_i(\mathbf{x}_i^n) \quad (2.27)$$

where $\mathbf{x}_i^n = (x_i, x_{i+1}, \dots, x_n)^T$ and $w_i(\mathbf{x}_i^n)$ is implicitly defined. Hence the vector $\mathbf{x}$ can be

represented as a path in a tree, where the symbol x_i labels the branch at depth $n - i + 1$, and the branch metric is $w_i(\mathbf{x}_i^n)$. For example, Fig. 8 shows the tree for 4PAM transmission. Decoding actually starts at the last symbol x_n . Due to this tree representation, any tree search algorithm can be applied to reduced-complexity MIMO decoding. Sphere decoding explores the tree in a branch and bound manner; thereby it avoids computing the metric of every path while guaranteeing the finding of the ML codeword.

Fig. 8. Tree representation of MIMO systems.

Sequential decoding is a heuristic algorithm developed in the coding community. It is based on the intuition that the metric of the ML path will rise on the average, while that of any incorrect path will fall as it extends. Accordingly, a path will be discarded if the metric fall is detected. Since not every path is inspected, the decoding complexity is reduced. The Fano algorithm and the stack algorithm are two basic implementations of sequential decoding. The Fano algorithm bears a similarity to the Schnorr-Euchner strategy of sphere decoding in that they examine only one path at a time and need not store all but one path and its metric (a feature of depth-first search algorithms). The major difference is that the Fano algorithm has a way to compare the likelihood of paths of unequal lengths. This is achieved by the Fano metric, which contains a bias term [2]. Since the comparison is only valid in a probabilistic sense, sequential decoding does not guarantee the finding of the ML codeword. The performance and complexity of sequential decoding are available in [2], [4].

So far, sequential decoding has only been applied to uncoded MIMO systems. We believe the application of sequential decoding (and other tree decoding algorithms) to high-rate space-time trellis codes is an exciting frontier, as lattice decoding is not suited.

This would reverse the common conception on the decoding complexity of space-time trellis codes.

The main disadvantage of all tree-decoding algorithms is the variable computation. The worst case decoding delay can be very long. Furthermore, since they do not have a regular structure, the hardware implementation is not as convenient as the Viterbi decoder.

2.3.3 Iterative Decoding

The achieving of MIMO capacity ultimately involves powerful error control codes such as turbo codes. In quasi-static channels, the use of turbo codes is not worthwhile, as the turbo cliff never shows up. So let us restrict our attention to ergodic channels, where the fading coefficients are changing rapidly. Hochwald and Brink proposed the combination of iterative decoding and spatial multiplexing to approach the ergodic MIMO capacity [56]. Data streams are encoded by an outer code, interleaved, and spatially multiplexed (by a linear mapper) as in Fig. 9. The receiver iterates a soft-in soft-output decoder for spatial multiplexing and a turbo decoder. At the heart of this method is the soft-in soft-output MIMO decoder. Hochwald and Brink used a list sphere decoder [56], while Baro, Hagenauer and Witzke used a list sequential decoder [41]. Iterative decoding is probably the *most pragmatic approach* to achieve the MIMO capacity, as the difficult task of designing MIMO codes is simply bypassed.

Fig. 9. Block diagram of iterative decoding for turbo-coded MIMO systems.

2.4.4 Noncoherent Decoding

Noncoherent decoding is generally referred to any decoding means for noncoherent channels. It is possible to extend all the foregoing decoding algorithms for coherent channels to noncoherent channels, subject to proper modification. In the noncoherent regime, we have to exploit the statistical information of the channel to improve the performance of a decoder. The noncoherent ML decoder for the signal model $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{N}$ can be formulated as a Bayesian detector

$$\hat{\mathbf{X}} = \arg \max_{\mathbf{X}} \int_{\mathbf{H}} p(\mathbf{Y} | \mathbf{H}, \mathbf{X}) p(\mathbf{H}) d\mathbf{H}. \quad (2.28)$$

In essence, this leads to *combined* detection and decoding (Fig. 10) which takes into account both code constraint and channel constraint. This will make a noncoherent decoder more complex than its coherent counterpart, because the decoder must expand the number of states to accommodate the channel memory. As such, we often need to truncate the channel memory or resort to tree decoding.

Fig. 10. Noncoherent communication system.

If even the statistical knowledge about the channel is unavailable, we have to use the generalized-likelihood ratio test (GLRT) that in essence assumes uniform distribution of the channel parameters. It takes the form

$$\hat{\mathbf{X}}_{\text{GLRT}} = \arg \max_{\mathbf{X}} \left\{ \max_{\mathbf{H}} p(\mathbf{Y} | \mathbf{H}, \mathbf{X}) \right\}. \quad (2.29)$$

The GLRT receiver does not work if the variation of channel uncertainty is fast or if the channel deviates significantly from uniform distribution.

There is a large body of literature on noncoherent decoding for single-antenna systems, especially for DPSK. The standard differential detection is an ad hoc method of noncoherent decoding. It only works when the unknown phase, fading coefficient, etc., remain constant over two symbol intervals. If the parameters vary over time, an irreducible error floor occurs. Block detection, or more commonly known as *multiple-symbol differential detection* (MSDD) for DPSK was proposed by Divsalar and Simon

[58]⁴ and Wilson et al. [42]. The computational complexity of brute-force MSDD is exponential in the block length. Mackenthun's work [43] is a remarkable contribution to fast MSDD, which has virtually linear complexity. However, it is only applicable to DPSK over AWGN channels. Recently, Lampe et al. applied sphere decoding to MSDD over fading channels [47], achieving polynomial complexity at high SNR. Another noncoherent decoder with polynomial-complexity at any SNR was proposed in [44]. The above-mentioned strategies are *exact* ML noncoherent decoders. If the channel memory is truncated properly, approximate ML decoding for DPSK can be implemented in the form of *linear predictive trellis decoding*. Further complexity reduction can be achieved by *decision-feedback detection*, because the error propagation effect is insignificant for DPSK. The problem of fast noncoherent decoding is not completely solved for QAM signaling. We believe the noncoherent sequence detection of Colavolpe and Raheli [62], combined with the reduced-state technique, is a very flexible solution in terms of performance/complexity tradeoff. This is essentially a truncated-memory trellis decoder. A comparison of truncated-memory detection and tree search is given in [45], where it is concluded that tree search is more suitable for small channel variation, while truncated-memory detection is more suitable for large channel variation.

When it comes to MIMO, there is another fold of complexity burden in the space dimension, in addition to the time dimension. There have been several extensions of the DPSK receiver structure to DSTM, but some important differences are ignored in our opinion. In Part I of this dissertation, we are going to use lattice decoding and truncated-memory Viterbi decoding in the noncoherent regime. The first work addresses the space dimension, while the second addresses the time dimension. We believe a general framework addressing both the space and time dimension is still open.

⁴ One of the best papers in communications history.

Part I

MIMO Differential Decoding

Chapter 3

MIMO Differential Lattice Decoding

3.1 Introduction

Differential space-time modulation (DSTM) is a noncoherent MIMO scheme specially tailored for continuously fading channels. The differential detection and its improved versions are well-documented in literature. However, most detectors use exhaustive search to find the maximum-likelihood signal point. This approach works well for MDPSK, but its complexity grows quickly for multiple-antenna constellations. Clearly this approach is impractical for high-rate DSTM. As most applications to coherent MIMO communication, the principle of universal lattice decoding is also applicable to noncoherent systems (see, *e.g.*, [46], [47], [48] for the applications of sphere decoding). Notably, Clarkson, Sweldens and Zheng [49] gave an elegant lattice formulation of diagonal DSTM and developed fast, approximate differential decoding (DD)⁵ based on the LLL basis reduction. The achieved complexity is polynomial in the number of transmit antennas and independent of the rate. This technique has been extended to some of the nondiagonal constellations [50], [51] and decision-feedback differential detection (DF-DD) [52]. The concerns of these papers are the applications rather than the investigation of the fast decoding algorithm itself.

In this Chapter, we present improved fast algorithms of conventional DD and DF-DD for *diagonal* DSTM based on the principle of universal lattice decoding. As pointed out in [49], there are three approximations with the decoder of component wise rounding in an LLL-reduced lattice therein. We eliminate two of these by using differential lattice decoding (DLD) which finds the closest lattice point exactly. The remaining approximation of the cosine function is however inherent with the lattice formulation. We therefore develop DLD augmented by local search incorporating the cosine measure, whose performance is indistinguishable from the MLD. It is shown that both the exact and augment DLD can be sped up significantly if the basis is LLL-reduced. Moreover, our decoders always work with an n_T -dimensional (n_T -D) lattice for a system with n_T transmitter antennas and n_R receiver antennas, while the decoder in [49] needs to define

⁵ In line with the literature of lattice decoding, “decoding” and “detection” are used interchangeably in this chapter.

an $n_T n_R$ -D lattice. Not only does this reduce the decoding complexity, the performance also improves since the cosine approximation is less accurate for higher-dimensional lattices.

3.2 Lattice Formulation

A. System Model

Consider an $n_T \times n_R$ DSTM system over a flat, time-selective fading channel, where the time selectivity is described by the Jakes model [53] for instance. The constellation $\mathbb{G}$ of a rate- R_c (bits/channel use) DSTM system is comprised of $L = 2^{R_c n_T}$ unitary matrices $\mathbf{G}_l$, $l \in \{0, \dots, L-1\}$, of size $n_T \times n_T$ each. The $R_c n_T$ bits to be transmitted at time instant m_T are mapped to an L -ary symbol $a[\tau]$, which selects an element $\mathbf{G}_{a[\tau]}$ from $\mathbb{G}$. Signal matrices are then differentially encoded as $\mathbf{S}[\tau] = \mathbf{S}[\tau-1]\mathbf{G}_{a[\tau]}$ with $\mathbf{S}[0] = \mathbf{I}$. The diagonal constellations of Hochwald and Sweldens [16] have the form

$$\mathbf{G}_l = (\mathbf{G}_1)^l, \text{ where } \mathbf{G}_1 = \text{diag}[e^{j2\pi u_1/L}, \dots, e^{j2\pi u_{n_T}/L}], 0 \leq l < L$$

and the integers $u_m \in \{0, 1, \dots, L-1\}$ for $1 \leq m \leq n_T$. One may set $u_1 = 1$ without loss of generality.

Since there is a single transmit antenna active at each time instant for diagonal constellations, we define an $n_R \times n_T$ channel matrix $\mathbf{H}[\tau]$ where the (i, j) th entry $h_{i,j}[\tau]$ denotes the fading coefficient between receive antenna i and transmit antenna j at time instant $m_T + j$. Accordingly, the received signals can be expressed by

$$\mathbf{Y}[\tau] = \sqrt{\rho} \mathbf{H}[\tau] \mathbf{S}[\tau] + \mathbf{W}[\tau] \quad (3.1)$$

where $\mathbf{W}[\tau]$ is the corresponding noise matrix. The entries of both $\mathbf{H}[\tau]$ and $\mathbf{W}[\tau]$ are independent across space and time, and are identically complex normal $\mathcal{CN}(0, 1)$ distributed. Because of the power normalization, ρ is the average signal-to-noise ratio (SNR) at each receive antenna. In the Jakes model, the correlation after k time samples of $h_{i,j}[\tau]$ is given by $J_0(2\pi f_d n_T k)$, where $J_0(\cdot)$ is the zeroth-order Bessel function of the first kind, and f_d is the normalized Doppler shift with respect to the scalar symbol period. Note that with this signal model for diagonal constellations, the effective Doppler shift of the fading process is multiplied by a factor of n_T .

B. Improved Problem Formulation

The maximum-likelihood (ML) DD for diagonal constellations is given by

$$\begin{aligned}
 \hat{a}^{\text{ML}}[\tau] &= \arg \min_l \left\| \mathbf{Y}[\tau] - \mathbf{Y}[\tau-1] \mathbf{G}_l \right\|^2 \\
 &= \arg \max_l \text{Re} \left[\text{tr} \left(\mathbf{G}_l^H \mathbf{Y}^H[\tau] \mathbf{Y}[\tau-1] \right) \right] \\
 &= \arg \max_l \sum_{m=1}^{n_T} \sum_{k=1}^{n_R} \text{Re} \left[e^{j2\pi u_m l / L} y_{k,m}^*[\tau] y_{k,m}[\tau-1] \right].
 \end{aligned} \tag{3.2}$$

Noticing that $e^{j2\pi u_m l / L}$ does not depend on index k , we may rewrite the MLD as

$$\hat{a}^{\text{ML}}[\tau] = \arg \max_l \sum_{m=1}^{n_T} \text{Re} \left[\left(\sum_{k=1}^{n_R} y_{k,m}[\tau] y_{k,m}^*[\tau-1] \right) e^{-j2\pi u_m l / L} \right].$$

Define

$$\begin{aligned}
 A_m &= \left| \sum_{k=1}^{n_R} y_{k,m}[\tau] y_{k,m}^*[\tau-1] \right|^{1/2}, \\
 \phi_m &= \frac{L}{2\pi} \arg \left(\sum_{k=1}^{n_R} y_{k,m}[\tau] y_{k,m}^*[\tau-1] \right),
 \end{aligned} \tag{3.3}$$

where $\arg(\cdot)$ has range $[-\pi, \pi)$ so that $\phi_m \in [-L/2, L/2)$. Then we have

$$\hat{a}^{\text{ML}}[\tau] = \arg \max_l \sum_{m=1}^{n_T} A_m^2 \cos[(u_m l - \phi_m) 2\pi / L] \tag{3.4}$$

Apparently the complexity of MLD is proportional to L . In other words, it is exponential in the number of transmit antennas and the rate. Equation (3.4) is formally the same as [49, (11)], but integrates the case of $n_R > 1$. This reduces the number of summands by a factor of n_R (cf. [49, (24)]).

To derive a fast decoder, a lattice viewpoint is adopted in [49]. Since the cosine function is 2π periodic, the argument can be restricted to $[-\pi, \pi)$ without loss of generality. The argument of the cosine function in (3.4) can thus be rewritten as $[(u_m l - \phi_m) \bmod^* L] 2\pi / L$, where $\bmod^*$ takes values in $[-L/2, L/2)$. The vectors $\mathbf{l} \bmod^* L$ for $l = 0, \dots, L-1$ form a finite integer lattice. Furthermore, an approximation $\cos \alpha \approx 1 - \alpha^2 / 2$ is made in [49]. This brings the lattice decoder into a Euclidean space. The best solution in the Euclidean space

$$\hat{a}^{\text{eucl}}[\tau] = \arg \min_l \sum_{m=1}^{n_T} \left[(A_m u_m l - A_m \phi_m) \bmod^* A_m L \right]^2 \tag{3.5}$$

is supposed to be a good approximation of the ML decision, because the cosine approximation is locally accurate near the maximum at $\alpha = 0$. If we define a basis

$$\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{n_T}] = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ A_2 u_2 & A_2 L & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n_T} u_{n_T} & 0 & \dots & A_{n_T} L \end{bmatrix},$$

the point set $\{\mathbf{B}\mathbf{x}: \mathbf{x} \in \mathbb{Z}^{n_T}\}$ will form an *infinite lattice* in $\mathbb{R}^{n_T}$. Denote by $\mathbf{t}$ the target vector with components $A_m \phi_m$. The decoding problem of (3.5) can be recast into a standard closest (lattice) vector problem (CVP), in which an integer vector $\hat{\mathbf{x}} \in \mathbb{Z}^{n_T}$ is to be found such that $\|\mathbf{B}\mathbf{x} - \mathbf{t}\|^2$ is minimized. It is seen that $\text{mod}^* A_1 L$ is artfully omitted in $\|\mathbf{B}\mathbf{x} - \mathbf{t}\|^2$. Such decoding in an infinite lattice has the convenient feature that no boundary control is needed. Accordingly, the decision is given by

$$\hat{a}^{\text{latt}}[\tau] = \hat{x}_1 \bmod L, \quad (3.6)$$

and the omission of $\text{mod}^* A_1 L$ does not affect the exactness of (3.5).

In essence, the ML Voronoi region is approximated by a polytope in the above lattice formulation. The approximation is quite accurate in low dimensions, but may be less accurate in high dimensions. It is worth mentioning that in our formulation, the lattice *always* has dimension n_T , regardless of the value of n_R . This is in sharp contrast with [49, (24)], where an $n_T n_R$ -D lattice would be used. Thanks to the notation (3.3), we know an n_T -D lattice is sufficient to define the decoding. Therefore, our formulation will improve the performance as well as the speed of DLD for a multi-antenna receiver.

C. DF-DD

Because of the n_T -fold Doppler shift, conventional DD easily suffers from an irreducible error floor in fast fading channels. Linear predictive DF-DD is an appealing technique of improvement for diagonal constellations. To mitigate the error floor, DF-DD makes use of the observations during the past few time epochs [52]. More precisely, it takes the form

$$\hat{a}[\tau] = \arg \max_l \text{Re} \left[\text{tr} \left(\mathbf{G}_l \mathbf{Y}^H[\tau] \sum_{n=1}^{N-1} p_n \mathbf{Y}[\tau - n] \prod_{i=1}^{n-1} \mathbf{G}_{\hat{a}[\tau - i]} \right) \right] \quad (3.7)$$

where $N - 1$ is the prediction order⁶, p_n for $n = 1, \dots, N - 1$ are predictor taps, and $\prod_{i=1}^{n-1} \mathbf{G}_{\hat{a}[\tau - i]} \triangleq \mathbf{I}_{n_T}$ if $n = 1$. Clearly, the lattice formulation is valid for DF-DD verbatim, if we replace $\mathbf{Y}[\tau - 1]$ in (3.2) by the sum in (3.7) that corresponds to linear prediction.

⁶ The reason for the seemingly unnatural notation $N - 1$ is the need of maintaining coherence with later chapters, where N is generally referred to as the observation window length of a noncoherent detector. Accordingly conventional DD has $N = 2$.

Again, an n_T -D lattice suffices and there is no need in going to an $n_T n_R$ -D lattice as in [52].

3.3 Differential Lattice Decoding

In this section, we present several procedures to approximately or exactly find the closest lattice point $\hat{\mathbf{x}} \in \mathbb{Z}^{n_r}$ and to further improve the DLD when the performance loss due to the cosine approximation is not negligible. Clearly, all the procedures are applicable to both conventional DD and DF-DD.

A. Approximate DLD

In addition to functioning as a preprocessing step in universal lattice decoding, basis reduction alone can be used to solve CVP approximately. Ideally, if the basis were orthonormal, simple rounding off would find the closest lattice point. This is the rare case in reality, and basis reduction attempts to select a nearly orthogonal and short basis for a given lattice. It is generally computationally hard to find the most orthogonal and shortest basis. Instead, the celebrated LLL algorithm finds a basis within an exponential factor from the shortest in polynomial time [32].

Basically, LLL reduction is an iteration of size reduction and swap. Let the basis have Gram-Schmidt orthogonalization $\mathbf{B} = \hat{\mathbf{B}}[\mu_{ij}]^T$, where $\hat{\mathbf{B}} = [\hat{\mathbf{b}}_1, \dots, \hat{\mathbf{b}}_{n_r}]$ and $[\mu_{ij}]^T$ is a lower-triangular matrix with unit diagonal elements. Size reduction shortens the lengths of basis vectors as

$$\mathbf{b}_i \leftarrow \mathbf{b}_i - \left\lfloor \frac{\langle \mathbf{b}_i, \hat{\mathbf{b}}_j \rangle}{\|\hat{\mathbf{b}}_j\|^2} \right\rfloor \mathbf{b}_j, \quad \text{for } j = i-1, \dots, 1.$$

After size reduction we have $|\mu_{i+1,i}| \leq 1/2$. Two adjacent basis vectors are swapped if

$$\delta \|\hat{\mathbf{b}}_i\|^2 > \|\hat{\mathbf{b}}_{i+1}\|^2 + \mu_{i+1,i}^2 \|\hat{\mathbf{b}}_i\|^2, \quad 2/3 < \delta < 1.$$

After swap we probably see $|\mu_{i+1,i}| > 1/2$ again. Then size reduction is performed once more. The LLL algorithm in its preliminary form repeats the two steps until no further improvement is achievable. Small values of δ lead to fast convergence, while large values of δ lead to a better-reduced basis. As a tradeoff, $\delta = 3/4$ is the common choice.

Theoretically, the LLL algorithm requires $O\left(n_T^4 \log\left(\max_{1 \leq i \leq n_T} \{\|\mathbf{b}_i\|^2\}\right)\right)$ arithmetic operations if implemented efficiently as in [39]. In this implementation, the orthogonal vectors $\hat{\mathbf{b}}_1, \dots, \hat{\mathbf{b}}_{n_r}$ are not stored, and the Gram-Schmidt orthogonalization is performed only once.

The reduced and unreduced bases are related by $\mathbf{B}_r = \mathbf{B}\mathbf{U}$, where $\mathbf{U} \in \mathbb{Z}^{n_T \times n_T}$ is a unimodular matrix with determinant $|\mathbf{U}| = \pm 1$.

Babai [54] gave two procedures for solving the approximate CVP in an LLL-reduced lattice. Adapted to our DLD problem, Babai's procedures can be rephrased as follows. Let $\mathbf{B}_r$ denote the reduced basis, its Gram-Schmidt orthogonalization be $\mathbf{B}_r = \hat{\mathbf{B}}_r[\mu_{ij}]^T$ and $\boldsymbol{\zeta} = \hat{\mathbf{B}}_r^{-1}\mathbf{t}$. Then the closest vector, up to a factor $1 + 2n_T(9/2)^{n_T/2}$ and $2^{n_T/2}$ (for $\delta = 3/4$), respectively, is given by

$$\text{Rounding Off (LLL-ZF): } \hat{\mathbf{x}} = \mathbf{U} \left\lfloor \mathbf{B}_r^{-1} \mathbf{t} \right\rfloor;$$

$$\text{Nearest Plane (LLL-DFE): } \hat{\mathbf{x}} = \mathbf{U} \hat{\mathbf{z}}, \text{ where } \hat{z}_{n_T} = \left\lfloor \zeta_{n_T} \right\rfloor \text{ and } \hat{z}_j = \left\lfloor \zeta_j - \sum_{i=j+1}^{n_T} \hat{z}_i \mu_{ij} \right\rfloor \text{ for } j = n_T - 1, \dots, 1.$$

The complexity of both procedures is negligible with respect to LLL reduction. While the worst-case factor is exponential in $n_T/2$, the performance is much better in reality. It can be seen that the rounding off procedure is equivalent to the zero-forcing (ZF) detector in communications, while the nearest plane procedure is equivalent to a special type of decision-feedback equalization (DFE), *i.e.*, the nulling and cancellation detector of the vertical Bell Labs layered space-time architecture (V-BLAST) without ordering. It is shown in [39] that V-BLAST ordering cannot improve the performance over the natural ordering of the nearest plane procedure for an LLL-reduced lattice, although it is beneficial for an unreduced lattice. This is because ordering is equivalent to a series of swapping operations, which have already been done in LLL reduction. As such, we will not consider ordering for approximate DLD in a reduced lattice.

Due to the incorporation of decision feedback, LLL-DFE has better performance than LLL-ZF. Thus it is important to make a clear distinction between the two decoders. The “component wise rounding” in [49] is just Babai's rounding off procedure, though this is not explicitly pointed out therein⁷. In the rounding off procedure, the polyhedral Euclidean Voronoi cell is further approximated by a parallelepiped [49, Fig. 3]. In the nearest plane procedure, it is better fitted by a cuboid.

1) *Example:* We reexamine the case considered in [49] where $n_T = 2$, $\mathbf{u} = [1 \ 9]^T$, $L = 32$, and $A_1 = A_2 = 1$. Fig. 11 shows the decision regions of the origin for different decoders. The Euclidean Voronoi cell is the solid-line hexagon. Suppose the shortest basis vectors $[4 \ 4]^T$ and $[3 \ -5]^T$ are identified successfully. Then the dash-dotted diamond is the

⁷ [49, Eq. (22)] is a property of the nearest plane procedure, but the hat over $\mathbf{b}_i$ there might be a typo.

decision region for the ZF decoder. The two orthogonal vectors resulting from Gram-Schmidt orthogonalization are given by $[4 \ 4]^T$ and $[4 \ -4]^T$. Thus the DFE decision region is the dotted rectangle (in fact a square in this special case). Apparently, the DFE decision region has more overlap with the Euclidean Voronoi cell, thereby resulting in improved performance. The two small filled triangles represent the additional overlap due to DFE.

Fig. 11. Decision regions of the origin ($l = 0$) for MLD, ZF and DFE. The two small filled triangles represent the extra overlap due to DFE.

B. Exact DLD

As noted in [49], besides the cosine approximation, there are two other approximations associated with LLL-ZF or LLL-DFE that make it a suboptimum lattice decoder: The Euclidean Voronoi cell is approximated by a parallelepiped or cuboid, and the LLL algorithm does not necessarily find the shortest basis. An exact lattice-decoding algorithm in the Euclidean space is mentioned in [49], which has a running time exponential in n_T . Here we apply sphere decoding to solve the CVP exactly, which has polynomial average complexity [55] for many communication problems at moderately high SNR. In Fig. 11, the Euclidean Voronoi cell corresponds to the decision region of sphere decoding. In the original form of sphere decoding, the Pohst-Fincke strategy [33], [34] enumerates all lattice point satisfying $\|\mathbf{B}\mathbf{x} - \mathbf{t}\|^2 \leq R^2$, where R denotes the chosen radius of the sphere (in fact it is an ellipsoid.) To determine the closest point, each time a closer point is

obtained, R is decreased suitably. Since the Schnorr-Euchner strategy [35] is more efficient in *closest lattice point search*, it is adopted here for DLD.

In sphere decoding, a basis is first transformed into its upper-triangular representation by QR decomposition. Alternatively, it can be transformed into the lower-triangular representation by QL decomposition. As the DSTM basis $\mathbf{B}$ is already lower triangular, it looks plausible to decode on $\mathbf{B}$ directly. This, however, turns out to be extremely inefficient. This is because A_1 , the first diagonal element of $\mathbf{B}$, is typically much smaller than the other diagonal elements $A_m L$. It easily leads to early errors that are quite wasteful. Consequently this method will not be considered and we follow the usual QR decomposition. Noting that $\mathbf{B}$ is a square matrix, we have the QR decomposition $\mathbf{B} = \mathbf{Q}\mathbf{R}$, where $\mathbf{Q}$ is an orthogonal matrix with positive diagonal elements, and $\mathbf{R}$ is upper triangular. The constraint $\|\mathbf{B}\mathbf{x} - \mathbf{t}\|^2 \leq R^2$ can be put in an equivalent form

$$\|\mathbf{R}\mathbf{x} - \mathbf{t}'\|^2 \leq R^2, \text{ where } \mathbf{t}' = \mathbf{Q}^T \mathbf{t}.$$

This can be solved successively by back-substitution, starting at the last element of $\mathbf{x}$. Moreover, since the lattice is infinite, no boundary control is necessary and the subroutine of the Schnorr-Euchner search given in [38] can be applied readily. The initial radius of the Schnorr-Euchner strategy can be set to infinity, and the first point found corresponds to the DFE decision. In the Schnorr-Euchner strategy, every time when a closer lattice point is found, it is stored as a potential output point, the radius R is decreased to the distance between this point and the query point, and the algorithm backtracks without restarting. The final output of the decoder gives the closest lattice point.

There are a number of ways of preprocessing or ordering to speed up the sphere decoder [38], [39], [40]. We compare their applications to DLD in the following.

1) *Basis Reduction*: Basis reduction is an essential preprocessing stage of universal lattice decoding, which improves the speed of sphere decoding drastically. For a finite lattice, basis reduction is sometimes considered inconvenient, as the boundary control is complicated in the reduced lattice [40]. Such an issue does not exist in an infinite DSTM lattice. It raises another issue though. Since the basis $\mathbf{B}$ is governed by A_m 's and in turn by received signal matrices (cf. (3)), it *changes every time*. Basis reduction needs to be re-performed for every incoming symbol. Consequently, it has a significant impact on the overall complexity of the DLD, unlike the usual scenario where the channel remains constant for a long time so that the complexity of basis reduction is negligible. In high dimensions, basis reduction will usually constitute a major computational burden of DLD. Nonetheless, it is still the fastest solution to our CVP for large values of n_T , as will be

shown in the next section. The reason is that the subsequent sphere decoding will typically be much faster than basis reduction, thanks to the nice structure of a reduced basis. It only adds a fraction of complexity to LLL-ZF or LLL-DFE. Suppose $\hat{\mathbf{x}}_r$ is the closest vector on the reduced basis $\mathbf{B}_r$, then its representation in the original lattice is given by $\hat{\mathbf{x}} = \mathbf{U}\hat{\mathbf{x}}_r$.

2) *Ordering*: For an *unreduced* lattice, the order in which the components of $\mathbf{x}$ are decoded may strongly influence the decoding speed. Ordering corresponds to choosing a column permutation matrix $\mathbf{\Pi}$ such that the new basis $\mathbf{B}' = \mathbf{B}\mathbf{\Pi}$ has some desirable properties. Suppose $\hat{\mathbf{x}}'$ is the closest vector on the ordered basis $\mathbf{B}'$, then we have the relation $\hat{\mathbf{x}} = \mathbf{\Pi}\hat{\mathbf{x}}'$. Again, since the DSTM basis $\mathbf{B}$ changes every time, ordering needs to be redone for each incoming received signal matrix. Generally, the computational overhead of ordering is not negligible. The V-BLAST ordering is widely applied prior to sphere decoding for MIMO systems. It identifies the permutation matrix $\mathbf{\Pi}$ such that $\min_{1 \leq i \leq n_T} r_{i,i}$ is maximized over all possible column permutations, where $r_{i,i}$ are the diagonal elements of $\mathbf{R}$ in the standard QR decomposition $\mathbf{B}' = \mathbf{Q}\mathbf{R}$ [40]. The complexity of V-BLAST ordering, as implemented in [40], is $O(n_T^4)$, which is on the same order as that of the LLL algorithm. As will be demonstrated in the next section, ordering for an unreduced basis may work in low dimensions ($n_T \leq 4$), but the LLL reduction results in substantially faster overall decoding in high dimensions.

Fincke and Pohst [34] suggest column ordering according to norms between basis reduction and sphere decoding. We observe no further acceleration of sphere decoding, however, if the basis has been *LLL-reduced*. This observation agrees with coherent universal lattice decoding [39], and can also be interpreted as the consequence of the fact that ordering is a swapping-only operation. This ordering method is applied in [40] to an unreduced lattice, which however is less effective than V-BLAST ordering.

C. Augmented DLD

Although the sphere decoder finds the closest lattice point in the Euclidean space, the cosine approximation remains to be a suboptimum factor, especially in high dimensions. Since the cosine approximation is locally accurate, a search through the nearby region of the sphere decoder output, incorporating the cosine measure, would largely compensate the suboptimality. This is tantamount to enumeration of *all lattice points* inside a sphere, which is exactly achieved by the Pohst-Fincke algorithm [33], [34]. When doing this, it will not shrink the radius R even if a closer point is found. The only distinction of our

problem is that we make the enumeration for the lattice point which maximizes the MLD metric

$$\mathcal{M}_l \triangleq \sum_{m=1}^{n_r} A_m^2 \cos[(u_m l - \phi_m) 2\pi / L].$$

It is easy to integrate the MLD metric into the Pohst-Fincke enumeration. Namely, each time a lattice point $\mathbf{x}_r$ is found, it is converted into the original lattice $\mathbf{x} = \mathbf{U}\mathbf{x}_r$ (if the lattice is reduced), $l = x_1 \bmod L$ is obtained, and $\mathcal{M}_l$ is compared to the maximum metric $\hat{\mathcal{M}}$ so far. If $\mathcal{M}_l$ is larger, $\hat{\mathcal{M}}$ is replaced by $\mathcal{M}_l$, and the most likely symbol $\hat{l}$ so far is replaced by l as well. After the enumeration terminates, $\hat{l}$ is the desired output. Note that the unimodular matrix $\mathbf{U}$ does not change with the particular point once the basis has been reduced.

The Schnorr-Euchner strategy can also be modified to enumerate all lattice points inside a sphere for one with the maximum MLD metric. When it finds a point (namely, the first element x_1 is reached) inside the initial radius R , it takes three different actions: 1) R is never changed; 2) it searches for all possible values of x_1 of in-sphere points rather than just the best one before backtracking; 3) the pair $(\hat{l}, \hat{\mathcal{M}})$ is updated as described above. The Schnorr-Euchner strategy and the Pohst-Fincke strategy have similar complexity in enumeration of *all lattice points* inside a sphere. Either one can be applied to augmented DLD.

Augmented DLD differs from the list sphere decoder of [56] in that we do not maintain a list of candidates. We only store the most likely symbol $\hat{l}$ so far. In this way we make full use of the lattice points inside the sphere. Then it is crucial to select the proper initial radius R for augmented DLD. A too large value of R will slow down the decoder because many points need to be enumerated. On the other hand, if R is too small, then probably no other points will be found and the performance cannot be improved. To determine an appropriate value of R , let us examine the scenario in which the exact DLD errs, whereas the MLD makes a correct decision. It is over here that the improvement of augmented DLD should take place if there is any. This scenario happens if the query point $\mathbf{t} \in \mathcal{V}^{\text{ML}} \cap \bar{\mathcal{V}}^{\text{eucl}}$, where $\mathcal{V}^{\text{ML}}$ and $\mathcal{V}^{\text{eucl}}$ denote the ML and Euclidean Voronoi cell, respectively.

1) *Example:* Fig. 12 demonstrates such a scenario in the lattice of Fig. 11. The ML Voronoi cell is the closeup of dashed lines, where two lattice points have equal likelihood (i.e., they are pairwise decision boundaries.) $\mathcal{V}^{\text{ML}}$ and $\mathcal{V}^{\text{eucl}}$ are virtually indistinguishable

for this lattice (but they may differ more for certain channel realizations or in high dimensions.) For visual ease, the vicinity of the query point $\mathbf{t}$ is enlarged in the upper-right corner, which shows that the exact DLD decides in favor of $l = 25$, whereas the most likely point is in fact $l = 0$. Clearly, the distance d_{ML} between $\mathbf{t}$ and $l = 0$ is approximately equal to the distance d_s between $\mathbf{t}$ and $l = 25$. Accordingly, to include the ML decision in the sphere, the radius R does not need to be much larger than d_s .

Fig. 12. The initial radius R of augmented DLD is not much larger than d_s . In the upper-right corner the vicinity of $\mathbf{t}$ is enlarged.

Generally, the scenario only happens if $\mathbf{t}$ is near the boundary of $\mathcal{V}^{\text{eucl}}$, since the cosine approximation is locally accurate. In other words, the most likely point is *not much further* from $\mathbf{t}$ than the sphere decoder output. Thus R only has to be slightly larger than d_s . We may choose $R = \alpha d_s$, where $\alpha > 1$ is a small constant usually less than three. This way we ensure that not too many points are examined by the augmented DLD. This choice requires the sphere decoder to carry out a second round of search using new radius R . If lattice reduction is performed, then we may simply set $R = \alpha d(\mathbf{t}, \hat{\mathbf{x}}) \geq \alpha d_s$, where $\hat{\mathbf{x}}$ is the output of LLL-ZF or LLL-DFE. Only a single round of search is needed by this approach. While this choice is weaker, the complexity is not much increased, as the LLL-based decision coincides with that of sphere decoding with high probability.

The overall complexity of augmented DLD is on the same order of the LLL algorithm. It

is simpler than another improved decoder based on basis reduction presented in [57], whose complexity is a multiple of that of the LLL algorithm (it works for a finite lattice though.)

DSTM is designed to capture space diversity rather than time diversity of a MIMO fading channel. Thus, to achieve time diversity, it is often concatenated with an outer channel code in practice. Augmented DLD provides a natural means of generating soft output that can be passed to the outer code. Suppose we have some bit mapping rule that assigns a value of l for every $R_c n_T$ input bits. Then the log-likelihood ratio for the i th bit b_i can be closely approximated as

$$LLR(b_i) \approx \log \frac{\sum_{\mathbf{x} \in \mathcal{O}_R, b_i(l)=0} \exp\{\mathcal{M}_l\}}{\sum_{\mathbf{x} \in \mathcal{O}_R, b_i(l)=1} \exp\{\mathcal{M}_l\}} \quad (3.8)$$

where $\mathbf{x} \in \mathcal{O}_R$ means that the lattice point $\mathbf{x}$ lies inside the sphere with radius R , $l = x_1 \bmod L$, and $b_i(l)$ denotes the assigned i th bit of l . In this application, if R is too small to include enough points, it can be increased properly. In this way, the complexity of the exact a posteriori probability (APP) decoder is reduced tremendously.

3.4 Numerical Results

In this section, we compare the performance and complexity of different schemes of conventional DD and DF-DD for diagonal constellations. The performance index is the error rate of DSTM symbols, as bit mapping is of secondary importance to our concern. The Jakes fading model is employed in computer simulation. Constellations are drawn from [49, Tab. I] if available. Clearly, the performance of exact DLD is independent of what kind of preprocessing or ordering is used, though the complexity may vary significantly. The Schnorr-Euchner strategy is followed in augmented DLD, and we set the initial radius as $R = 2d(\mathbf{t}, \hat{\mathbf{x}})$, where $\hat{\mathbf{x}}$ is the output of LLL-DFE. Surprisingly, with this simple choice of R we have *never* observed a single discrepancy between MLD and augmented DLD, throughout all comparisons made in simulation! If MLD is deemed too time-consuming, we plot the union bound for MLD derived in [52], which is sufficiently tight at high SNR. For DF-DD, the depicted union bound is of a genie-aided (*i.e.*, correct feedback) detector, which is often a good approximation of the real error rate.

Fig. 13 shows the performance of conventional DD for $R = 2$, $n_T = 4$, $n_R = 1$, and $f_d = 0.0025$. Our results slightly differ from [49, Fig. 5] at high SNR, but the simulated performance for MLD has excellent match with the analytic union bound. It is seen that LLL-DFE has better performance than LLL-ZF. Note also that the performance of LLL-

DFE is close to MLD, which is typical of low-dimension conventional DD. The decisions of augmented DLD and MLD are compared verbatim; no difference is found. Thus their performance is depicted by the same line.

Fig. 13. Symbol error rate of conventional DD for $R = 2$, $n_T = 4$, $n_R = 1$, and $f_d = 0.0025$.

Fig. 14. Symbol error rate of conventional DD for $R = 2$, $n_T = n_R = 4$, and $f_d = 0.0025$ with LLL-ZF, LLL-DFE in 16 and 4-D lattices.

Conventional DD utilizing an n_T -D lattice is compared with $n_T n_R$ -D lattice decoding in Fig. 14, for the above setting but with $n_R = 4$. LLL-ZF is susceptible to the lattice dimension; decoding in a 16-D lattice leads to performance loss of over 1 dB at high SNR, despite its higher complexity. In comparison, LLL-DFE differs by 0.5 dB only. Again, the performance of LLL-DFE in a 4-D lattice is close to MLD.

The performance gain of exact and augmented DLD in conventional DD is apparent in Fig. 15 for $R = 2$, $n_T = 8$, $n_R = 1$, and $f_d = 0.001$. We just pick a good (but not necessarily the optimum) constellation $\mathbf{u} = [1 \ 1551 \ 3693 \ 5951 \ 10593 \ 10643 \ 25213 \ 29893]^T$ found in random search. It is seen that augmented DLD is over 0.5 dB better than LLL-DFE at high SNR, and the gap between LLL-ZF and LLL-DFE is expanded to 1.5 dB. Exact DLD lies between LLL-DFE and augmented DLD. Remarkably, the performance of augmented DLD approaches the union bound for MLD as the SNR grows. It means that the performance loss due to the cosine approximation is practically recovered. Since this constellation size ($L = 65536$) is very large, we only compare 100 trials of MLD with augmented DLD for each SNR. Every trial sees exactly the same decisions.

Fig. 15. Symbol error rate of conventional DD for $R = 2$, $n_T = 8$, $n_R = 1$, and $f_d = 0.001$.

Fig. 16 illustrates the performance of DF-DD for $R = 2$, $n_T = 5$, $n_R = 1$, $f_d = 0.01$, and $N = 4$. The union bound for conventional DD is also included as a benchmark. Once again augmented DLD and MLD are depicted by the same line, because no different decoding

outcomes are observed. It is seen that LLL-DFE improves over LLL-ZF by 1 dB at high SNR. The augmented DLD improves further by 1 dB, and agrees with the MLD bounds excellently.

Fig. 16. Symbol error rate of DF-DD for $R = 2$, $n_T = 5$, $n_R = 1$, $f_d = 0.01$, and $N = 4$.

Generally, the improvement of exact and augmented DLD tends to be more visible in high dimensions or DF-DD. The former is an effect of the respective approximation of the Voronoi cell; the latter might be an effect of the feedback of occasionally erroneous decisions.

To evaluate the computational complexity, we count the average number of flops required to decode each incoming DSTM symbol. This measure is quite meaningful, as it is largely independent of the particular processor and programming environment. Moreover, it reflects the real hardware complexity more objectively than other measures such as CPU running time. TABLE I shows the average numbers of flops for various conventional DD over a channel with $f_d = 0.0025$ and $\text{SNR} = 20$ dB. The similar trend has been observed for DF-DD. The ratio between the data for MLD and LLL-ZF differs somewhat from [49], but the trend is the same. It is clear that LLL-ZF, LLL-DFE and exact DLD with LLL reduction have similar complexity. Since we don't store the orthogonal vectors in LLL reduction, the QR decomposition makes LLL-DFE slightly more complex than LLL-ZF. Augmented DLD with LLL reduction and DFE initial radius

has less than the double complexity of LLL-ZF. All of these four algorithms are faster than MLD for constellation size $L \geq 64$. Surprisingly, ordering brings virtually no complexity reduction over pure sphere decoding for all constellation sizes shown here. This is because the advantage of ordering in complexity reduction will not appear until the dimension is quite large (≥ 30) [40, Fig. 8]. Furthermore, the computation overhead of V-BLAST ordering will especially offset the saving in sphere decoding if any. Sphere decoding with or without ordering is only faster than LLL-based DLD for small constellation sizes. For large constellations sizes, the LLL reduction makes DLD significantly faster, which is the case of more interest in high data rate MIMO communication. It can be seen that the complexity of LLL-based decoding increases much slower than $O(n_T^4)$ as predicted by theory, for practical numbers of transmit antennas. Another advantage of LLL-based decoding is that the complexity is largely independent of SNR, while pure sphere decoding is more dependent.

TABLE I
COMPLEXITY IN KILOFLOPS FOR AN IMPLEMENTATION OF CONVENTIONAL DD
OVER A CHANNEL WITH $f_d = 0.0025$ AND SNR = 20 dB

n_T	R_c	L	MLD	LLL-ZF	LLL-DFE	Pure sphere decoding	Sphere decoding with norm ordering	Sphere decoding with V-BLAST ordering	Exact DLD with LLL	Augmented DLD with LLL
2	1	4	0.09	0.21	0.30	0.14	0.17	0.27	0.28	0.40
3	1	8	0.24	0.60	0.84	0.33	0.43	1.08	0.81	1.07
4	1	16	0.60	1.34	1.87	0.68	0.86	3.39	1.74	2.42
5	1	32	1.46	2.36	3.41	1.24	1.53	8.87	3.14	4.17
6	1	64	3.48	3.91	5.71	2.38	2.56	20.1	5.26	6.96
2	2	16	0.31	0.26	0.34	0.14	0.17	0.27	0.32	0.47
3	2	64	1.75	0.75	1.04	0.42	0.46	1.08	0.97	1.30
4	2	256	9.24	1.62	2.23	1.29	1.07	3.46	2.12	2.67
5	2	1024	46.1	3.27	4.29	6.39	3.51	9.68	4.08	5.17
6	2	4096	221	5.13	7.08	18.1	9.36	22.5	6.29	8.31
2	3	64	1.17	0.28	0.36	0.16	0.20	0.28	0.33	0.49
3	3	512	13.8	0.68	0.93	0.58	0.62	1.44	0.85	1.37
4	3	4096	147	1.71	2.25	3.32	2.30	3.61	2.05	3.01
5	3	32768	1470	3.50	4.54	26.7	32.1	22.7	4.28	5.96
6	3	262144	13000	5.87	6.81	118	182	119	6.89	10.6

3.5 Conclusions

We have presented improved fast decoding algorithms for multi-antenna differential modulation. We showed the two basis reduction-based approximate DD algorithms, *i.e.*, LLL-ZF and LLL-DFE, exhibit quite different performance. In particular, the latter is

much better in high dimensions. Sphere decoding was applied to find the closest point in the Euclidean space exactly. Furthermore, we proposed augmented DLD to compensate the remaining cosine approximation, which practically achieves the MLD performance. The performance gain of DLD and augmented DLD is clear for large values of n_T and high SNR. While the improvement over LLL-DFE is insignificant at low SNR, augmented DLD can provide soft output for an outer code, which is usually necessary to achieve time diversity of a time-selective fading channel. Moreover, our definition of an n_T -D lattice for the MIMO differential scheme is simple, yet effective in performance improvement and complexity reduction.

Various methods of preprocessing and ordering were examined, and we found LLL reduction is most effective in reducing the overall complexity for large constellation sizes. Ordering is not recommended for practical values of n_T . The four LLL-based DLD algorithms have complexity on the same order, and are all much faster than brute-force MLD for large constellation sizes. The small amount of extra complexity of DLD is rewarding in terms of the obtained performance gain or soft output.

Ordinarily, basis reduction is seen as the preprocessing stage with minor complexity for sphere decoding. Here a different viewpoint is more suitable for multi-antenna DLD: sphere decoding is a postprocessing stage of basis reduction. In contrast to coherent MIMO decoding in a quasi-static fading channel [39], [40], the LLL algorithm is often the bottleneck of decoding speed for multi-antenna DLD. The application of some fast LLL-type reduction algorithms may warrant a serious investigation.

Chapter 4

Receiver Design for Nondiagonal Constellations

4.1 Introduction

DSTM constellations can be classified as diagonal versus nondiagonal ones. In diagonal DSTM the design of constellations was simplified considerably, however the diagonal restriction has negative impacts on performance and transmitting amplifiers. It is known that diagonal restriction may result in performance loss. Furthermore, since signal transmission at each antenna is discontinuous in diagonal DSTM, the instantaneous power is n_T times larger compared to the case in which powers are otherwise distributed on n_T amplifiers evenly. This requires an amplifier with a large linear region, which is often unwanted. Accordingly, it is sometimes desirable to use nondiagonal constellations.

Let N denote the window length of a noncoherent detector. The standard detector for DSTM is the differential detector in which decisions are made over two ($N = 2$) consecutive observation intervals. If the fading process keeps constant in the duration of two DSTM symbols, the performance penalty of differential detection versus coherent detection is merely 3 dB. However, the mobile radio channels are characterized by time-selective fading due to Doppler spreading. In such channels, differential detection suffers an irreducible error floor, analogous to differential phase-shift keying (DPSK). To mitigate the flooring effect, a number of receiver structures that outperform the differential detector have been developed. They have $N > 2$ so that the memory of the noncoherent fading channel can be exploited to improve the performance. These structures applied the reception techniques originally devised for single-antenna systems that basically fall into three categories: block detection [58], [59], decision-feedback detection (DFD) and sequence detection. Schober and Lampe [52] presented multiple-symbol differential detection (MSDD) for DSTM, of which the computational complexity is exponential in the observation length. To overcome the computation burden, DFD based on linear prediction (LP) has been proposed [52], [60]. In addition, Chiavaccini and Vitetta [61] derived sequence detection of DSTM signals by means of the Viterbi algorithm, where the branch metric contains a linear predictor similar to that in DFD.

Except the exhaustive MSDD scheme, an essential limitation of the afore-mentioned popular linear prediction receivers is that they lack optimality in continuously fading

channels for nondiagonal constellations. Diagonal constellations were assumed in many works on decision-feedback detection to simplify the receiver design. Specifically, the structure of the linear predictor is the same as that for DPSK, i.e., a time-invariant linear filter subject to an adverse effect that the Doppler frequency shift is multiplied by n_T . Others like [61] considered nondiagonal constellations, mainly Alamouti's two-antenna code, but made an assumption that the fading process is invariant during a DSTM symbol so as to derive the linear predictor. Though the accuracy of this assumption is acceptable in slow fading, the temporal variation is no longer negligible in fast fading.

In this chapter, we scrutinize the behavior of linear prediction receivers for (mainly) nondiagonal DSTM constellations in continuous fading. We shall show that, in contrast to DPSK, linear prediction receivers may be mismatched to nondiagonal DSTM. Then, we propose noncoherent receivers well-matched to the nondiagonal structure. Both DFD and sequence detection will be investigated.

4.2 System Model

Consider an n_T -antenna DSTM system. Every Rn_T bits to be transmitted at time instant nn_T are mapped to an L -ary symbol $a[n]$, which then selects a signal matrix from *generator constellation* $\mathbb{G}$ as $\mathbf{G}[n] = \mathbf{G}_{a[n]}$. Before transmission takes place, the matrices $\mathbf{G}[n]$ are differentially encoded in a fashion similar to DPSK

$$\mathbf{S}[n] = \mathbf{S}[n-1]\mathbf{G}[n], \quad \mathbf{S}[0] = \mathbf{A}, \quad (4.1)$$

where the initially transmitted matrix $\mathbf{A}$ can be any given unitary matrix. Here we make a special distinction between *generator constellation* $\mathbb{G}$ and the *transmission constellation* $\mathbb{S} = \{\mathbf{S}[n]\}$.

For nongroup constellations, such as those based on Alamouti's two-antenna code, in general the matrix multiplications in (4.1) need to be done explicitly. If $\mathbb{G}$ forms a group under matrix multiplication, then an alternative implementation is

$$\begin{aligned} c[n] &= c[n-1] \oplus a[n], \quad c[0] = 1 \\ \mathbf{S}[n] &= \mathbf{A} \mathbf{G}_{c[n]}, \end{aligned} \quad (4.2)$$

where the addition $\oplus$ is defined in accordance with the matrix multiplication. If $\mathbf{A}$ is further a member of $\mathbb{G}$, no explicit matrix multiplication is needed at all. Moreover, if $\mathbb{G}$ is a cyclic group, $\oplus$ becomes the usual addition modulo L .

Let $\mathbf{s}_i[n]$ be the i th column of $\mathbf{S}[n]$, $\mathbf{H}_i[n]$ be the n_R -by- n_T matrix of channel coefficients seen by $\mathbf{s}_i[n]$, and $\mathbf{W}[n]$ be an n_R -by- n_T noise matrix. Then the received data are given by $\mathbf{Y}[n] = \mathcal{H}[n]\mathbf{S}[n] + \mathbf{W}[n]$, where the n_R -by- n_T^2 matrix $\mathcal{H}[n]$ is obtained by stacking $\mathbf{H}_i[n]$:

$$\mathcal{H}[n] = [\mathbf{H}_0[n], \mathbf{H}_1[n], \dots, \mathbf{H}_{n_T-1}[n]],$$

and $\mathcal{S}[n]$ is a stretched version of $\mathbf{S}[n]$, which is no longer square, but has dimension n_T^2 -by- n_T :

$$\mathcal{S}[n] = \begin{bmatrix} \mathbf{s}_0[n] & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_1[n] & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{s}_{n_T-1}[n] \end{bmatrix} \quad (4.3)$$

If the fading process is constant in the DSTM symbol duration, i.e., $\mathbf{H}_i[n] = \mathbf{H}[n]$ for $i = 1, \dots, n_T - 1$, then it is reduced to the widely used piecewise-constant signal model $\mathbf{Y}[n] = \mathbf{H}[n]\mathbf{S}[n] + \mathbf{W}[n]$. If $\mathbf{S}[n]$ is diagonal, in spite of continuously fading, still a formally similar signal model is possible $\mathbf{Y}[n] = \tilde{\mathbf{H}}[n]\mathbf{S}[n] + \mathbf{W}[n]$ where the effective fading matrix $\tilde{\mathbf{H}}[n]$ is obtained by decimating $\mathcal{H}[n]$. $\tilde{\mathbf{H}}[n]$ can be expressed as $\tilde{\mathbf{H}}[n] = [\boldsymbol{\beta}_0[n], \boldsymbol{\beta}_1[n], \dots, \boldsymbol{\beta}_{n_T-1}[n]]$, where $\boldsymbol{\beta}_i[n]$ is the i th column of $\mathbf{H}_i[n]$. In these two cases, the structure of $\mathbf{S}[n]$ is preserved.

4.3 Limitation of Linear Prediction Receiver

We demonstrate the limitation of the linear prediction receiver by using the example of DFD. We shall also show that, interestingly, it incurs no degradation to diagonal-generator constellations. The same is true for sequence detection.

By embedding a linear predictor, LP-DFD makes use of N observations [52]:

$$\hat{\mathbf{G}}[k] = \arg \max_{\mathbf{G}_l \in \mathbb{G}} \text{Re} \left[\text{tr} \left(\mathbf{G}_l \mathbf{Y}^H[k] \sum_{n=1}^{N-1} p_n \mathbf{Y}[k-n] \prod_{\mu=1}^{n-1} \hat{\mathbf{G}}[k-\mu] \right) \right] \quad (4.4)$$

where $N-1$ is the prediction order, p_n for $n = 1, \dots, N-1$ are predictor taps, and

$$\prod_{\mu=1}^{n-1} \hat{\mathbf{G}}[k-\mu] \triangleq \begin{cases} \hat{\mathbf{G}}[k-(n-1)] \cdots \hat{\mathbf{G}}[k-1], & n > 1 \\ \mathbf{I}_{n_T}, & n = 1 \end{cases}$$

reflects the feedback of previously detected symbols. For diagonal transmission constellations, the strategy (4.4) is known to work well provided that the taps are derived from the Wiener-Hopf equations. It is hoped that LP-DFD would not incur much degradation for nondiagonal $\mathbb{S}$.

In essence, LP-DFD removes past modulation on received signals so that a linear predictor can be applied to the recovered fading-plus-noise samples. More precisely, if decisions are perfectly correct, the argument inside $\text{tr}(\cdot)$ in (4.4) can be expressed as

$$\begin{aligned} & \mathbf{G}_l \mathbf{Y}^H[k] \sum_{n=1}^{N-1} p_n \mathbf{Y}[k-n] \prod_{\mu=1}^{n-1} \hat{\mathbf{G}}[k-\mu] \\ &= \mathbf{G}_l (\mathcal{H}[k] \mathcal{S}[k] + \mathbf{W}[k])^H \sum_{n=1}^{N-1} p_n (\mathcal{H}[k-n] \mathcal{S}[k-n] + \mathbf{W}[k-n]) \prod_{\mu=1}^{n-1} \mathbf{G}[k-\mu]. \end{aligned} \quad (4.5)$$

To suppress modulation prior to time index k , we must have the relation

$$\mathcal{S}[k-1] = \mathcal{S}[k-n] \prod_{\mu=1}^{n-1} \mathbf{G}[k-\mu], \quad \forall k, n$$

which basically requires that $\mathcal{S}[k] = \mathcal{S}[k-1] \mathbf{G}[k]$, $\forall k$. Recall the differential rule $\mathbf{S}[k] = \mathbf{S}[k-1] \mathbf{G}[k]$ for *square* $\mathbf{S}[k]$, we cannot always guarantee this because of the stretching effect of $\mathcal{S}[k]$. For example, one can easily check the invalidity of the above relation for Alamouti's code or Hughes' rate-1 group constellation with

$$\mathbf{G}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

It means that we will probably lose the most fundamental property of differential schemes if nondiagonal constellations are transmitted over continuously changing channels! This problem is peculiar to multi-antenna differential modulation, in the sense that it never shows up in DPSK, because the relation $\mathcal{S}[k] = \mathcal{S}[k-1] \mathbf{G}[k]$, $\forall k$ is always valid for one-dimensional signals (no stretch of a scalar exists.)

Interestingly, there do exist exceptions in nondiagonal transmission constellations satisfying this condition. This is based on the following result.

Proposition: The relation $\mathcal{S}[k] = \mathcal{S}[k-1] \mathbf{G}[k]$, $\forall k$, is valid if and only if the generator constellation $\mathbb{G}$ is diagonal.

Proof: If $\mathbf{G}[k]$ is diagonal, put $\mathbf{G}[k] = \text{diag}[g_0[k], g_1[k], \dots, g_{M-1}[k]]$. Then

$$\begin{aligned} \mathcal{S}[k-1] \mathbf{G}[k] &= \text{diag}[\mathbf{s}_0[k-1], \dots, \mathbf{s}_{M-1}[k-1]] \cdot \text{diag}[g_0[k], \dots, g_{M-1}[k]] \\ &= \text{diag}[\mathbf{s}_0[k-1]g_0[k], \dots, \mathbf{s}_{M-1}[k-1]g_{M-1}[k]]. \end{aligned}$$

Meanwhile, by definition $\mathbf{S}[k] = \mathbf{S}[k-1] \mathbf{G}[k]$, we can easily check that $\mathbf{s}_m[k] = \mathbf{s}_m[k-1]g_m[k]$. Hence, $\mathcal{S}[k] = \text{diag}[\mathbf{s}_0[k], \dots, \mathbf{s}_{M-1}[k]]$ is equal to $\mathcal{S}[k-1] \mathbf{G}[k]$.

On the other hand, suppose that the relation is valid. Then

$$\begin{bmatrix} \mathbf{s}_0[k] & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_1[k] & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{s}_{M-1}[k] \end{bmatrix} = \begin{bmatrix} \mathbf{s}_0[k-1] & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_1[k-1] & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{s}_{M-1}[k-1] \end{bmatrix} \bullet \begin{bmatrix} g_0[k] & g_{01}[k] & \cdots & g_{0,M-1}[k] \\ g_{10}[k] & g_1[k] & \cdots & g_{1,M-1}[k] \\ \vdots & \vdots & \ddots & \vdots \\ g_{M-1,0}[k] & g_{M-1,1}[k] & \cdots & g_{M-1}[k] \end{bmatrix}$$

where we expand $\mathbf{G}[k]$ into its complete form. Let $s_{m,n}[k]$ denote the (m,n) th element of $\mathbf{S}[k]$, and let $\bar{\mathbf{g}}_m[k]$ denote the m th row of $\mathbf{G}[k]$ excluding the diagonal element.

Consider the first row of $\mathcal{J}[k]$. Since the $(0, n)$ th element of $\mathcal{J}[k]$ is zero for $n = 1, \dots, M - 1$, we have $s_{0,0}[k-1]g_{0,n}[k] = 0$ for $n = 1, \dots, M - 1$. So either $s_{0,0}[k-1] = 0$ or $\bar{\mathbf{g}}_0[k] = \mathbf{0}$. The same reasoning for the second to the $(M - 1)$ th rows of $\mathcal{J}[k]$ leads to the result that either $s_{m,0}[k-1] = 0$ or $\bar{\mathbf{g}}_0[k] = \mathbf{0}$. Collectively, it implies that if $\bar{\mathbf{g}}_0[k] \neq \mathbf{0}$, then $\mathbf{s}_0[k-1] = \mathbf{0}$. But this is contradictory with the property that $\mathbf{S}[k-1]$ is unitary. Accordingly, we must have $\bar{\mathbf{g}}_0[k] = \mathbf{0}$.

Same considerations on the remaining rows of $\mathcal{J}[k]$ show that $\bar{\mathbf{g}}_m[k] = \mathbf{0}$ for $m = 1, \dots, M - 1$. Collectively, we know all of the off-diagonal elements of $\mathbf{G}[k]$ are zero, which means that $\mathbf{G}[k]$ is diagonal. $\square$

Note that the diagonality of $\mathbb{G}$ does not necessarily impose the diagonality of $\mathbb{S}$. Diagonal $\mathbb{G}$ can be modified into nondiagonal $\mathbb{S}$ by choosing a nondiagonal initial matrix $\mathbf{A}$. For example, we may choose $\mathbf{A}$ in (8) for two-antenna DSTM. Generally, to distribute signal powers evenly to transmit antennas, we may use the discrete Fourier transform (DFT) to construct the initial matrix $\mathbf{A} = \text{DFT}(\mathbf{I}_{n_T}) / \sqrt{n_T}$ ⁸. In this way, diagonal generator constellations can be employed for nondiagonal transmission, yet permitting the use of LP-DFD in continuous fading.

We caution the use of another way to get nondiagonal transmission constellations, which makes a transformation of diagonal signals as $\mathbf{G}_l \mapsto \mathbf{U}^H \mathbf{G}_l \mathbf{U}$ where $\mathbf{U}$ is a unitary matrix [16]. This transformation has no influence on the performance of LP-DFD in piecewise-constant channels, but it does affect its performance in continuously fading channels.

To obtain a clear impression of the suboptimality of the linear prediction receiver, we present the performance of two constellations in the following. The derivation is omitted and can be seen as a special case of the error analysis given in later sections.

Fig. 17 displays the union bounds of DSTM based on the BPSK Alamouti code over a fast fading channel with $f_d T = 0.03$. Clearly LP-DFD suffers from extremely poor performance; its error rate grows at high SNR and eventually flattens at a level even higher than differential detection! This phenomenon, although somewhat counter-intuitive, has been validated by computer simulation and is caused by the ignorance of fading variation during a matrix symbol. Also shown in the figure is the performance of MS-DFD, which is a degenerate version of noncoherent sequence detection (NSD) proposed in the next section.

⁸ This will probably increase the scalar constellation size. A normalized Hadamard matrix may be used instead if M is a power of 2.

Fig. 17. Performance of DSTM based on the BPSK Alamouti code with LP-DFD and MS-DFD for $f_d T = 0.03$, $N = 4$ and $n_R = 1$.

Fig. 18. Performance of three-antenna DSTM with LP-DFD and MS-DFD for $f_d T = 0.03$, $N = 4$ and $n_R = 1$.

Totally different behaviors of diagonal/nondiagonal generator constellations with LP-

DFD are illustrated in Fig. 18 for three-antenna DSTM. The diagonal generator is $[1 \ 1 \ 3]$ while the nondiagonal generator constellation is obtained as $\mathbf{G}_l \mapsto \mathbf{U}^H \mathbf{G}_l \mathbf{U}$, where $\mathbf{U}$ is given by a DFT matrix. It is seen that the BER of LP-DFD for the nondiagonal generator constellation again grows at high SNR.

In summary, when the channel fades continuously, LP-DFD only matches diagonal generator constellations. This result applies to sequence detection as well.

4.4 Noncoherent Sequence Detection

In this section, we propose approximate maximum-likelihood noncoherent sequence detection (NSD) of DSTM. The philosophy of NSD was originally developed by Colavolpe and Raheli [62] for approximate optimum noncoherent reception of coded linear modulation signals in single-antenna AWGN or slowly fading channels. Basically it is a truncated-memory detector so that a finite-state trellis can be defined. This approach “moves beyond the crude block detection found in this area, to more natural, logical and optimal trellis-based detection schemes.” Inspired by Colavolpe and Raheli, we employ the idea of NSD to overcome the limitation of linear-predictive receivers for DSTM, while maintaining practical implementation complexity.

Suppose that the transmission of DSTM symbols begins at $n = 0$ and ends at $n = N_T$. The starting point of the proposed NSD is the optimum block detection for the entire $N_T + 1$ DSTM symbols as a whole. Stacking the variables involved in the detection yields the notations

$$\begin{aligned}\bar{\mathcal{S}}[N_T] &= \text{diag}[\mathcal{S}[0], \mathcal{S}[1], \dots, \mathcal{S}[N_T]], \\ \bar{\mathcal{H}}[N_T] &= [\mathcal{H}[0], \mathcal{H}[1], \dots, \mathcal{H}[N_T]], \\ \bar{\mathbf{Y}}[N_T] &= [\mathbf{Y}[0], \mathbf{Y}[1], \dots, \mathbf{Y}[N_T]], \\ \bar{\mathbf{W}}[N_T] &= [\mathbf{W}[0], \mathbf{W}[1], \dots, \mathbf{W}[N_T]].\end{aligned}$$

With these we have the signal model

$$\bar{\mathbf{Y}}[N_T] = \bar{\mathcal{H}}[N_T] \bar{\mathcal{S}}[N_T] + \bar{\mathbf{W}}[N_T] \quad (4.6)$$

The brute-force MSDD searches $\bar{\mathcal{S}}[N_T]$ that maximizes the conditional probability density function [52]

$$f(\bar{\mathbf{Y}}[N_T] | \bar{\mathcal{S}}[N_T]) = \frac{1}{(\pi^{n_r(N_T+1)} |\mathbf{R}_{\bar{\mathcal{S}}}[N_T]|)^{n_r}} \exp\{-\text{tr}(\bar{\mathbf{Y}}[N_T] \mathbf{R}_{\bar{\mathcal{S}}}^{-1}[N_T] \bar{\mathbf{Y}}^H[N_T])\} \quad (4.7)$$

where $\mathbf{R}_{\bar{\mathcal{S}}}[N_T]$ is the autocorrelation matrix of the received signals conditioned on $\bar{\mathcal{S}}[N_T]$ at a particular antenna. $\mathbf{R}_{\bar{\mathcal{S}}}[N_T]$ can be expressed as

$$\mathbf{R}_{\bar{\mathcal{S}}}[N_T] = \bar{\mathcal{S}}^H[N_T](\Phi_{N_T+1} \otimes \mathbf{I}_{n_T})\bar{\mathcal{S}}[N_T] \quad (4.8)$$

where $\otimes$ denotes the Kronecker product, and Φ_{N_T+1} is the $n_T(N_T+1)$ -by- $n_T(N_T+1)$ autocorrelation matrix of a fading-plus-noise process, whose (i, j) th entry is given by $\phi[j-i] + \sigma^2 \delta_{ij}$, δ_{ij} being the Kronecker delta function.

$\mathbf{R}_{\bar{\mathcal{S}}}[N_T]$ admits a more compact form than (4.8). It's not hard to show that

$$\mathbf{R}_{\bar{\mathcal{S}}}[N_T] = \Phi_{N_T+1} \circ (\bar{\mathbf{S}}^H[N_T] \bar{\mathbf{S}}[N_T])$$

where $\bar{\mathbf{S}}[N_T] \triangleq [\mathbf{I}, \mathbf{S}[1], \dots, \mathbf{S}[N_T]]$, and $\circ$ denotes the Hadamard product. Collect the differentially encoded DSTM symbols in a matrix $\bar{\mathbf{C}}[N_T] \triangleq [\mathbf{I}, \mathbf{C}[1], \dots, \mathbf{C}[N_T]]$, where we define $\mathbf{C}[n] = \prod_{i=1}^n \mathbf{G}[i] \triangleq \mathbf{G}[1]\mathbf{G}[2]\dots\mathbf{G}[n]$ if $n > 0$, and $\mathbf{C}[n] = \mathbf{I}$ if $n = 0$. We purposely use the notation $\mathbf{C}[n]$ to discriminate from $\mathbf{S}[n]$ that depends on the initial matrix $\mathbf{A}$. Using the differential encoding rule, we have

$$\mathbf{S}^H[n_1]\mathbf{S}[n_2] = \left(\mathbf{S}[0] \prod_{i=1}^{n_1} \mathbf{G}[i] \right)^H \mathbf{S}[0] \prod_{j=1}^{n_2} \mathbf{G}[j] = \left(\prod_{i=1}^{n_1} \mathbf{G}[i] \right)^H \prod_{j=1}^{n_2} \mathbf{G}[j] = \mathbf{C}^H[n_1]\mathbf{C}[n_2].$$

Hence $\mathbf{R}_{\bar{\mathcal{S}}}[N_T]$ can be rewritten as

$$\mathbf{R}_{\bar{\mathcal{S}}}[N_T] = \Phi_{N_T+1} \circ (\bar{\mathbf{C}}^H[N_T] \bar{\mathbf{C}}[N_T]). \quad (4.9)$$

An advantage of this new version is considerable computational reduction relative to (4.8) due to reduced dimension of matrices and element-wise product rather than standard matrix product. It is clear that $\mathbf{R}_{\bar{\mathcal{S}}}[N_T]$ is independent of $\mathbf{S}[0]$, i.e., it depends on $\{\mathbf{G}[n]\}$ exclusively. Since there exists a one-to-one correspondence between the DSTM symbol sequence $\{\mathbf{G}[n]\}$ and information symbol sequence $\mathbf{a} = \{a[1], a[2], \dots, a[N_T]\}$, we write $\mathbf{R}_{\bar{\mathcal{S}}}[N_T] = \mathbf{R}_{\mathbf{a}}[N_T]$ to better reflect its sole dependence on $\mathbf{a}$.

By doing this and taking the natural logarithm of (4.7), the MSDD strategy becomes

$$\hat{\mathbf{a}} = \arg \min_{\tilde{\mathbf{a}}} \left\{ \text{tr} \left(\bar{\mathbf{Y}}[N_T] \mathbf{R}_{\tilde{\mathbf{a}}}^{-1}[N_T] \bar{\mathbf{Y}}^H[N_T] \right) + n_R \ln |\mathbf{R}_{\tilde{\mathbf{a}}}[N_T]| \right\} \quad (4.10)$$

where $\hat{\mathbf{a}}$ is the detected sequence, and $\tilde{\mathbf{a}}$ represents a trial sequence. This optimum detection strategy forms the basis of our proposed method in the sequel.

Applying the philosophy of [62], let us define a partial sequence metric up to the n th DSTM symbol interval

$$\Lambda[n] = \text{tr} \left(\bar{\mathbf{Y}}[n] \mathbf{R}_{\tilde{\mathbf{a}}}^{-1}[n] \bar{\mathbf{Y}}^H[n] \right) + n_R \ln |\mathbf{R}_{\tilde{\mathbf{a}}}[n]| \quad (4.11)$$

as well as an incremental metric

$$\begin{aligned}\Delta[n] &\triangleq \Lambda[n] - \Lambda[n-1] \\ &= \left\{ \text{tr} \left(\bar{\mathbf{Y}}[n] \mathbf{R}_{\tilde{\mathbf{a}}}^{-1}[n] \bar{\mathbf{Y}}^H[n] \right) + n_R \ln |\mathbf{R}_{\tilde{\mathbf{a}}}[n]| \right\} \\ &\quad - \left\{ \text{tr} \left(\bar{\mathbf{Y}}[n-1] \mathbf{R}_{\tilde{\mathbf{a}}}^{-1}[n-1] \bar{\mathbf{Y}}^H[n-1] \right) + n_R \ln |\mathbf{R}_{\tilde{\mathbf{a}}}[n-1]| \right\}.\end{aligned}\quad (4.12)$$

Obviously, the minimization of the sequence metric of (4.11) can be computed recursively via the incremental metrics. As in single-antenna NSD, the implementation difficulty inherent in the optimum metric (4.12) is the infinite memory that would render the complexity of the detector to increase exponentially with n .

In order to limit the complexity, a truncation of the memory is introduced. That is, only the most recent N observations are considered in the incremental metric [62]. This can be thought of as a sliding-window detector with window length N . With this modification, the incremental metric is approximated by

$$\begin{aligned}\lambda[n] &\triangleq \left\{ \text{tr} \left(\bar{\mathbf{Y}}_N[n] \mathbf{R}_{\tilde{\mathbf{a}},N}^{-1}[n] \bar{\mathbf{Y}}_N^H[n] \right) + n_R \ln |\mathbf{R}_{\tilde{\mathbf{a}},N}[n]| \right\} \\ &\quad - \left\{ \text{tr} \left(\bar{\mathbf{Y}}_{N-1}[n-1] \mathbf{R}_{\tilde{\mathbf{a}},N-1}^{-1}[n-1] \bar{\mathbf{Y}}_{N-1}^H[n-1] \right) + n_R \ln |\mathbf{R}_{\tilde{\mathbf{a}},N-1}[n-1]| \right\}\end{aligned}\quad (4.13)$$

where $\bar{\mathbf{Y}}_N[n] = [\mathbf{Y}[n-(N-1)], \mathbf{Y}[n-(N-2)], \dots, \mathbf{Y}[n]]$ is an observation of length N , $\mathbf{R}_{\tilde{\mathbf{a}},N}[n]$ is the Nn_T -by- Nn_T correlation matrix of $\bar{\mathbf{Y}}_N[n]$ associated with the hypothetical sequence $\tilde{\mathbf{a}}$, and $\bar{\mathbf{Y}}_{N-1}[n-1]$, $\mathbf{R}_{\tilde{\mathbf{a}},N-1}[n-1]$ are defined accordingly. From (4.13), it is easy to see that $\mathbf{R}_{\tilde{\mathbf{a}},N}[n]$ only depends on $\tilde{a}[n-(N-2)], \tilde{a}[n-(N-3)], \dots, \tilde{a}[n]$. In this way, the truncation allows the minimization to be realized efficiently by the Viterbi algorithm. Since the truncated incremental metric, or the branch metric in terms of the Viterbi algorithm, depends exclusively on the sequence $(\tilde{a}[n-(N-2)], \tilde{a}[n-(N-3)], \dots, \tilde{a}[n])$ as well, a trellis state may be defined by $N-2$ information symbols

$$\chi[n] \triangleq (\tilde{a}[n-(N-2)], \tilde{a}[n-(N-3)], \dots, \tilde{a}[n-1]).$$

In the trellis $\tilde{a}[n]$ labels the current branch. The resulted number of states of the trellis is $S = L^{N-2}$. There are L paths originating from each state, and L paths leading into each state in the trellis, respectively. Therefore, the complexity of the detector becomes only exponential in $N-1$ rather than in N_T . Since the fading process can be quite accurately modeled as a finite-order auto-regressive (AR) process, the noncoherent sequence detector has the potential to perform close to optimum detection by selecting a value of N no less than the order of the AR process divided by n_T .

As an example, the trellis of Alamouti's two-antenna code for BPSK signaling, or equivalently Hughes' $R=1$ constellation is shown in Fig. 19 in which $L=4$, $N=3$, and $S=4$.

Fig. 19. Four-state trellis diagram for two-antenna DSTM with $R = 1$ and $N = 3$.

The noncoherent sequence detector, in general, does not bring a linear-prediction interpretation of its structure. Two exceptional cases are: a) fading is piecewise constant within the duration of a DSTM symbol; or b) $\mathbb{G}$ is diagonal (which does not necessarily imply that the transmitted DSTM symbols $\mathbf{S}[n]$ are diagonal). In this case, the correlation matrix $\mathbf{R}_{\tilde{\mathbf{a}},N}[n]$ can further be simplified to

$$\mathbf{R}_{\tilde{\mathbf{a}},N}[n] = \bar{\mathbf{Z}}_{\tilde{\mathbf{a}},N}^H[n](\Phi_N' \otimes \mathbf{I}_{M_T})\bar{\mathbf{Z}}_{\tilde{\mathbf{a}},N}[n] \quad (4.14)$$

where $\bar{\mathbf{Z}}_{\tilde{\mathbf{a}},N}[n] \triangleq \text{diag}[\tilde{\mathbf{C}}[n-(N-1)], \dots, \tilde{\mathbf{C}}[n]]$, and Φ_N' is a reduced-dimension correlation matrix of size N -by- N , whose (i, j) th entry is given by $\phi[n_T(j-i)] + \delta_{ij}$. By inspection, it is obvious that $|\mathbf{R}_{\tilde{\mathbf{a}},N}[n]|$ is independent of the hypothetical sequence so that it can be dropped from the branch metric. It is also clear that $\mathbf{R}_{\tilde{\mathbf{a}},N}^{-1}[n] = \bar{\mathbf{Z}}_{\tilde{\mathbf{a}},N}^H[n](\Phi_N'^{-1} \otimes \mathbf{I}_{n_T})\bar{\mathbf{Z}}_{\tilde{\mathbf{a}},N}[n]$. Applying the Cholesky decomposition to $\Phi_N'^{-1} \otimes \mathbf{I}_{n_T}$, we have

$$\text{tr}(\bar{\mathbf{Y}}_N[n]\mathbf{R}_{\tilde{\mathbf{a}},N}^{-1}[n]\bar{\mathbf{Y}}_N^H[n]) = \sum_{k=0}^{N-1} \left\| \mathbf{Y}[n-k]\tilde{\mathbf{C}}^H[n-k] - \sum_{i=1}^{N-1-k} p_i^{N-1-k} \mathbf{Y}[n-k-i]\tilde{\mathbf{C}}^H[n-k-i] \right\|^2 / \xi_k \quad (4.15)$$

where p_i^k is the i th coefficient for a k th order linear predictor, and ξ_k is the corresponding variance of the prediction error [63]. The other term $\text{tr}(\bar{\mathbf{Y}}_{N-1}[n-1]\mathbf{R}_{\tilde{\mathbf{a}},N-1}^{-1}[n-1]\bar{\mathbf{Y}}_{N-1}^H[n-1])$ in the branch metric has a similar expression, but with a smaller range of summation from 0 to $N-2$. Then it is readily seen that the branch metric is reduced to one single term corresponding to $k = N-1$:

$$\lambda[n] = \left\| \mathbf{Y}[n]\tilde{\mathbf{C}}^H[n] - \sum_{i=1}^{N-1} p_i^{N-1} \mathbf{Y}[n-i]\tilde{\mathbf{C}}^H[n-i] \right\|^2 / \xi_{N-1}. \quad (4.16)$$

Because ξ_{N-1} is a constant, it may be dropped without affecting the performance. The $(N$

– 1)th order linear predictor is determined from

$$\begin{bmatrix} 1 \\ -(\mathbf{p}^{N-1})^* \end{bmatrix} = (\Phi_N')^{-1} [\xi_{N-1} \quad 0 \quad \cdots \quad 0]^T \quad (4.17)$$

where $\mathbf{p}^{N-1} = [p_1^{N-1}, p_2^{N-1}, \dots, p_{N-1}^{N-1}]^T$. This is exactly the linear predictive Viterbi receiver derived by Chiavaccini and Vitetta [61].

The implementation complexity of noncoherent sequence detection comprises the algorithmic complexity of the Viterbi algorithm and the computational complexity arising from computing the branch metric. At every level of the trellis, the Viterbi algorithm needs to determine the metrics of L^{N-1} transitions. This is common to NSD and the linear prediction receiver. Such algorithmic complexity can be circumvented effectively by the reduced-state sequence detection, as will be demonstrated later. The branch metric involves calculation of the inversion and determinant of $\mathbf{R}_{\tilde{\mathbf{a}},N}[n]$ and $\mathbf{R}_{\tilde{\mathbf{a}},N-1}[n-1]$. The computational complexity is normally on the order of $(Nn_T)^3$, which can be large in many situations⁹. Currently we do not know of any efficient method to invert $\mathbf{R}_{\tilde{\mathbf{a}},N}[n]$ and $\mathbf{R}_{\tilde{\mathbf{a}},N-1}[n-1]$ for nondiagonal $\mathbb{G}$, because they are not Toeplitz (hence the Levinson-Durbin recursion is not applicable). In comparison, the linear predictive Viterbi receiver has a fixed-coefficient linear filter structure, whose complexity is only linear in N . In case of small values of L and N , a possible solution to reduce the computational complexity of noncoherent sequence detection is to predetermine the inversion and determinant of $\mathbf{R}_{\tilde{\mathbf{a}},N}[n]$ and $\mathbf{R}_{\tilde{\mathbf{a}},N-1}[n-1]$ for every possible branch, and store the results in a look-up table. The resulting size of this table is proportional to L^{N-1} .

A comment on the noncoherent sequence detector for DSTM ends this section. The structure of the correlation matrix implies that two sequences $\{\mathbf{S}[n]\}$ and $\{\mathbf{S}'[n]\}$ will have the same metrics, if one can be obtained from the other after multiplying by any constant unitary matrix. That is, they are indistinguishable to the noncoherent sequence detector. Hence differential encoding is indeed necessary for the applicability of noncoherent sequence detection of the constellation $\mathbb{G}$.

4.5 Error Analysis

We proceed to analyze the BER performance of the proposed noncoherent sequence

⁹ Matrix inversion routines faster than $O(n^3)$ exist. The optimal method by far appears to have asymptotic complexity of $O(n^{2.376})$. However, these methods, reducing the exponent, may suffer from a very large constant. An anonymous examiner is acknowledged for pointing out this issue.

detection. The union bound on the BER is given by

$$P_b^{\text{NSD}} \leq \frac{1}{Rn_T} \sum_{\mathbf{a}} P\{\mathbf{a}\} \sum_{\hat{\mathbf{a}} \neq \mathbf{a}} d(\mathbf{a}, \hat{\mathbf{a}}) P\{\mathbf{a} \rightarrow \hat{\mathbf{a}}\} \quad (4.18)$$

where $d(\mathbf{a}, \hat{\mathbf{a}})$ is the Hamming distance between the two bit sequences associated with $\mathbf{a}$ and $\hat{\mathbf{a}}$, $P\{\mathbf{a}\}$ is the a priori probability of $\mathbf{a}$, and $P\{\mathbf{a} \rightarrow \hat{\mathbf{a}}\}$ is the pairwise error probability (PEP). To evaluate the PEP, consider an erroneous path in the trellis diagram of DSTM that diverges from the correct path at a certain level and remerges with it at a later level. An example is illustrated in Fig. 20 for a 16-state trellis, where the correct path is the all-zero path. Because of the shift property of the states, the last $N - 2$ information symbols before the remerging are the same for the correct and erroneous paths. Of course, the two paths must differ at the first position right after the divergence, and do not contain the same $N - 2$ symbols in a row at other positions. Accordingly the length of an error event can be expressed as $D + N - 2$, where $D = 1, 2, \dots$ is the maximum number of information symbols the two paths differ at [64].

Fig. 20. A major error event in a 16-state trellis for two-antenna DSTM with $R = 1$ and $N = 4$.

The PEP $P\{\mathbf{a} \rightarrow \hat{\mathbf{a}}\}$ is the probability that the accumulated metric of an erroneous path is smaller than that of the correct path. Let the error event start at time n and span $D + N - 2$ symbol intervals. Then the PEP can be expressed as

$$P\{\mathbf{a} \rightarrow \hat{\mathbf{a}}\} = P\left\{\gamma \triangleq \sum_{i=n}^{n+D+N-3} \hat{\lambda}[i] - \lambda[i] < 0\right\}.$$

Our approach to evaluate this error probability is to express the metric difference γ in certain quadratic forms and then determine its characteristic function. Note that $K = D + 2N - 3$ successive observations $(\mathbf{Y}[n - (N - 1)], \dots, \mathbf{Y}[n + D + N - 3])$ are involved in γ .

To express γ in quadratic forms, we rewrite the branch metric (4.13) as

$$\begin{aligned} \lambda[n] = & \text{tr}(\bar{\mathbf{Y}}_N[n] \mathbf{Q}_{\hat{\mathbf{a}},N}[n] \bar{\mathbf{Y}}_N^H[n]) \\ & + n_R \ln |\mathbf{R}_{\hat{\mathbf{a}},N}[n]| - n_R \ln |\mathbf{R}_{\hat{\mathbf{a}},N-1}[n-1]| \end{aligned} \quad (4.19)$$

where

$$\mathbf{Q}_{\hat{\mathbf{a}},N}[n] \triangleq \mathbf{R}_{\hat{\mathbf{a}},N}^{-1}[n] - \begin{bmatrix} \mathbf{R}_{\hat{\mathbf{a}},N-1}^{-1}[n-1] & \mathbf{0}_{n_T} \\ \mathbf{0}_{n_T} & \mathbf{0}_{n_T} \end{bmatrix}.$$

In terms of such matrices, the difference of the path metrics has the expression

$$\gamma = \text{tr}(\bar{\mathbf{Y}}_K[n] \mathbf{F} \bar{\mathbf{Y}}_K^H[n]) - c \quad (4.20)$$

with the $Kn_T \times Kn_T$ matrix $\mathbf{F}$ defined by

$$\mathbf{F} \triangleq \sum_{i=0}^{D+N-3} \text{diag} \begin{bmatrix} \mathbf{0}_{n_T} & \mathbf{Q}_{\hat{\mathbf{a}},N}[n+i] - \mathbf{Q}_{\mathbf{a},N}[n+i] & \mathbf{0}_{(K-N-i)n_T} \end{bmatrix},$$

and c is a constant with respect to the K random observations:

$$\begin{aligned} c = n_R \sum_{i=0}^{D+N-3} & \left\{ \left[\ln |\mathbf{R}_{\mathbf{a},N}[n+i]| - \ln |\mathbf{R}_{\mathbf{a},N-1}[n+i-1]| \right] \right. \\ & \left. - \left[\ln |\mathbf{R}_{\hat{\mathbf{a}},N}[n+i]| - \ln |\mathbf{R}_{\hat{\mathbf{a}},N-1}[n+i-1]| \right] \right\}. \end{aligned}$$

Let $\bar{\mathbf{Y}}_K^j[n]$ denote the j th row of $\bar{\mathbf{Y}}_K[n]$. Then $\eta_j \triangleq \bar{\mathbf{Y}}_K^j[n] \mathbf{F} (\bar{\mathbf{Y}}_K^j[n])^H$ is a quadratic form.

The metric difference can be further rewritten as $\gamma = \sum_{j=0}^{n_R-1} \eta_j - c$.

The characteristic function of η_j is determined by [65]:

$$\Phi_{\eta_j}(s) = \left| \mathbf{I} + s \mathbf{R}_{\mathbf{a},K}[n + D + N - 3] \mathbf{F} \right|^{-1} = \left[\prod_{i=0}^{Kn_T-1} (1 + s \psi_i) \right]^{-1} \quad (4.21)$$

where ψ_i for $i = 0, \dots, Kn_T - 1$ are the eigenvalues of the matrix $\mathbf{R}_{\mathbf{a},K}[n + D + N - 3] \mathbf{F}$.

Since the fading processes are mutually independent between receive antennas, the

characteristic function of $\sum_{j=0}^{n_R-1} \eta_j$ is simply given by $\Phi_{\eta_j}(s)$ raised to the n_R th power.

Consequently, the PEP is given by the integral [66]

$$P\{\mathbf{a} \rightarrow \hat{\mathbf{a}}\} = P\left\{\sum_{j=0}^{M_R-1} \eta_j < c\right\} = \frac{1}{2\pi j} \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} \frac{e^{sc} \Phi_{\eta_i}^{M_R}(s)}{s} ds, \quad (4.22)$$

where the integration path lies within the vertical strip $0 < \varepsilon < \varepsilon_0$. ε_0 is the right boundary of the regularity domain of $\Phi_{\eta_i}(s)$. The integration can be calculated efficiently via a number of methods such as the Gauss-Chebyshev quadrature [67]. To obtain fast convergence in the numerical integration, the optimum value of ε may be identified as the saddle point of the function $e^{sc} \Phi_{\eta_i}^{M_R}(s)$ [66]. Nonetheless the integration is found insensitive to the choice of ε .

Once the PEP is known, the union bound can be evaluated in principle by listing all error events with different values of D . A tight approximation is obtained by truncating the union bound, i.e., we only consider the error events with $D \leq D_0$ such that the probability of the error events with $D > D_0$ is negligible [64]. Practically, this can be achieved by increasing D_0 step by step till no significant change of the sum is observed. This method is conceptually reasonable because the main error events are characterized by small Hamming distance in Rayleigh fast fading channels.

When $\mathbb{G}$ is nondiagonal, the uniform error probability criterion is generally not satisfied. That is, the bit error probability depends on which code sequence is sent. Therefore, it is necessary to consider all transmitted sequences possible when evaluating the BER. Fortunately, a good approximation of the BER may be obtained by averaging over a moderate number of randomly selected sequences $\mathbf{a}$. For diagonal, cyclic group constellations, it can be checked that the uniform error probability criterion is satisfied. Then it suffices to consider an arbitrary sequence when evaluating the BER. Therefore, for equally probable signaling the union bound is simplified to

$$P_b^{\text{NSD}} \leq \frac{1}{Rn_T} \sum_{\hat{\mathbf{a}} \neq \mathbf{a}} d(\mathbf{a}, \hat{\mathbf{a}}) P\{\mathbf{a} \rightarrow \hat{\mathbf{a}}\} \quad (\text{diagonal \& cyclic}) \quad (4.23)$$

If Gray mapping is applied, the BER for noncoherent sequence detection of DPSK in a Gaussian channel is dominated by two most probable error events [64]

$$\mathbf{e}_0 = (e[n] = +1, e[n+1] = -1)$$

and

$$\mathbf{e}'_0 = (e[n] = -1, e[n+1] = +1),$$

where $e[n] = \hat{a}[n] - a[n]$, and $e[n+1] = \hat{a}[n+1] - a[n+1]$. Due to the exponential nature of the PEP for Gaussian channels, these two error events bring a BER approximation that asymptotically coincides with the true union bound. This is not the case in fading

channels, especially for uncoded systems, because the BER typically decreases linearly with the SNR. Accordingly, the two major error events are not so dominant. The BER obtained by considering the two error events is no longer an asymptotic upper bound. Instead, it might be viewed as a lower bound, if their overlap is neglected. Nonetheless, it still represents a good approximation of the actual BER of DSTM in many circumstances. This is justified by the following observations: if the fading is slow, then quite often the union bound is loose; if the fading is fast, then short error events dominate the performance at high SNR, while the union bound is loose at low SNR. Since each error event entails two bits in error under Gray mapping, the BER due to these two error events for cyclic constellations is

$$P_b^{\text{NSD}} \approx \frac{2}{Rn_T} \sum_{\mathbf{a}} P\{\mathbf{a}\} (P\{\mathbf{a} \rightarrow \mathbf{a} + \mathbf{e}_0\} + P\{\mathbf{a} \rightarrow \mathbf{a} + \mathbf{e}'_0\}) \quad (4.24)$$

The difference here is that ± 1 are not necessarily the nearest neighbors in DSTM. Hence $\mathbf{e}_0$ and its complement are usually identified by search.

The error event $\mathbf{e}_0$ relative to the all-zero path is plotted in Fig. 20 for $L = 4$ and $N = 4$, which spans four DSTM symbol intervals. Fig. 21 shows the truncated union bound for Alamouti's code with the trellis depicted in Fig. 20 over a fading channel with $f_d T = 0.01$ and compares with the approximation which counts in the two dominant error events only. In this figure, the curves corresponding to $D_0 = 2, 3$, and 4 are plotted. It can be observed that the bounds converge at high SNR for different values of D_0 . Furthermore, the approximation (4.24) is quite close to the bounds at high SNR. The minor difference between the approximation and the bounds at high SNR is due to the miss of another major error event ($e[n] = +2, e[n+1] = -2$). Computer simulation results are also shown in Fig. 21. Since the union bound diverges at low SNR, it actually overestimates the BER in such circumstances. On the contrary, the approximation fits simulation results quite well. Consequently, we evaluate the BER by the two-error-event approximation henceforth.

The performance of coherent detection of a DSTM system with differential encoding and decoding in static fading channels ($f_d = 0$) is included in Fig. 21 as a benchmark for comparison. Its BER is obtained by doubling that of coherent detection of the constellation $\mathbb{G}$ without differential encoding, since differential decoding typically doubles the BER. The PEP of coherent detection without differential encoding was given in [13, (C.3)] as

$$P\{\mathbf{G}_i \rightarrow \mathbf{G}_j\} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 + 1/4} \prod_{m=0}^{M_T-1} \left[\frac{1}{1 + \rho \mu_m^2 (\omega^2 + 1/4)} \right]^{M_R}$$

where $\rho = 1/\sigma^2$, and μ_m for $m = 0, \dots, M_T - 1$ are the singular values of $(\mathbf{I} + \mathbf{G}_j^H \mathbf{G}_i)/2$.

Fig. 21. Performance of NSD with $N = 4$ of two-antenna DSTM with BPSK signaling (Alamouti's code) for $f_d T = 0.01$ and $n_R = 1$.

Fig. 22. Performance of NSD of two-antenna DSTM with BPSK signaling (Alamouti's

code), for various values of N and $n_R = 1$.

Remarkably, performance close to coherent detection is achieved even with a small value of $N = 4$ for $f_d T = 0.01$. As fading gets faster, larger values of N are needed. Fig. 22 demonstrates how the performance of coherent detection is approached by increasing N in fading channels with $f_d T = 0.03$ and $f_d T = 0.1$, respectively. By selecting a proper value for N , the BER can decrease linearly with the SNR, which is characteristic of static fading channels. In particular, the performance of NSD with $N = 8$ is less than 2 dB away from coherent detection for $f_d T = 0.03$. Such performance characteristics might be useful in selection of DSTM constellations for fast fading channels. That is, if a constellation possesses good performance in static fading, then it can also have good performance in time-selective fading when NSD is employed.

Fig. 23. Performance comparison between NSD and the linear precision (LP) receiver for two-antenna DSTM with BPSK signaling (Alamouti's code), for various values of $f_d T$, $N = 6$, and $n_R = 1$.

In Fig. 23, performance of noncoherent sequence detection of Alamouti's code with $N = 6$ is compared with the linear prediction-based Viterbi receiver in [61]. The linear predictive receiver can be viewed as a noncoherent sequence detector employing a mismatched branch metric for nondiagonal $\mathbb{G}$. Specifically, the correlation matrices

$\mathbf{R}_{\tilde{\mathbf{a}},N}[n]$ and $\mathbf{R}_{\tilde{\mathbf{a}},N-1}[n-1]$ are (incorrectly) expressed as (4.14) so that a linear predictor could be derived. The PEP of the linear predictive receiver is simply obtained through the same analytical procedure for noncoherent sequence detection, except that the expression (4.14) is used from (4.19) to (4.22). The final BER is calculated by (4.24). Other methods simpler than this are possible, but this one is easy to manipulate for our purposes. The results shown in Fig. 23 indicate that though the linear predictive receiver has good performance in slow fading with $f_d T = 0.001$, it suffers an error floor greater than 10^{-6} for fading faster than $f_d T = 0.01$. Based on the observations, we conclude that for nondiagonal $\mathbb{G}$ the application of linear prediction-based receivers is limited to slowly fading channels.

4.6 Reduced-State Detection

The algorithmic complexity of NSD is proportional to the number of states $S = 2^{Rn_T(N-2)}$ of the trellis. Even for moderate values of R and n_T , the complexity increases drastically with N . For example, the trellis will contain $S = 4096$ states if $R = 1$, $n_T = 3$, and $N = 6$. It might be difficult to implement real-time sequence detection with such a large-size trellis in practice. The reduced-state sequence detection technique can be incorporated to overcome the implementation complexity [68], [69]. For this, a trellis state is defined in terms of most recent Γ information symbols, where $\Gamma < N - 2$:

$$\chi'[n] \triangleq (\tilde{a}[n - \Gamma], \tilde{a}[n - (\Gamma - 1)], \dots, \tilde{a}[n - 1]).$$

In this way the number of states is reduced to $S = L^\Gamma$. In reduced-state noncoherent sequence detection, the calculation of the branch metric still involves N observations and $N - 1$ trial symbols, among which the remaining $N - 2 - \Gamma$ symbols unavailable in the state are extracted from the survivor history according to the per-survivor processing (PSP) principle [70]. In the extreme case of $\Gamma = 0$, all the $N - 2$ symbols are found from history NSD degenerates into DFD performing symbol-by-symbol detection, which is generally different from the LP-DFD. We call it MS-DFD since it can be obtained by introducing decisions back into the metric of MSDD.

It is also possible to incorporate other reduced-state techniques such as set-partitioning [68] in NSD when the constellation size L is large. Therefore, NSD is very flexible in tradeoffs between performance and complexity. By controlling N and Γ , we can range between the optimum MSDD and DFD.

Performance analysis of reduced-state NSD is a modification of the procedures for full-state detection described in the previous section. The analysis is complicated to some extent by sporadic erroneous symbols in the survivor history, analogous to decision-

feedback detection. Hence we follow a widely accepted approach that assumes completely correct feedback symbols. It is well known that there is little error propagation in DFD of DPSK and DSTM. Besides, it is known that reduced-state sequence detection has even less effect on error propagation than DFD [68], [69]. Therefore, we expect the genie-aided analysis will yield the BER close to the actual performance.

To be specific, the length of an error event is shortened to $D + \Gamma$. The difference in cumulative metrics becomes $\gamma = \sum_{i=n}^{n+D+\Gamma-1} \hat{\lambda}[i] - \lambda[i]$, which involves $K' = D + \Gamma + N - 1$ successive observations $(\mathbf{Y}[n - (N - 1)], \dots, \mathbf{Y}[n + D + \Gamma - 1])$. The matrices $\mathbf{R}_{\hat{\mathbf{a}}, N}[n]$ and $\mathbf{R}_{\hat{\mathbf{a}}, N-1}[n-1]$ are calculated by imposing correct symbols $\hat{a}[n - (N - 1)] = a[n - (N - 1)]$, $\dots, \hat{a}[n - (\Gamma + 1)] = a[n - (\Gamma + 1)]$ for those in the survivor history. The upper limit of the summation in $\mathbf{F}$ and c is reduced to $D + \Gamma - 1$. After the characteristic function

$$\Phi_{\eta_j}(s) = \left| \mathbf{I} + s \mathbf{R}_{\hat{\mathbf{a}}, K'}[n + D + \Gamma - 1] \mathbf{F} \right|^{-1}$$

is determined, the PEP is evaluated by (4.24).

Considerable insights into the performance of NSD can be gained by a comparison with DFD and MSDD. The performance of DFD can be maintained at the same order as MSDD. Because DFD is the least favorable case of reduced-state NSD from the performance viewpoint, we conclude that reduced-state NSD ($\Gamma > 0$) has better performance than MSDD for equal values of N . This property was also observed in [62] for single-antenna full-state NSD over AWGN channels by inspecting the metrics.

Fig. 24 illustrates the performance of reduced-state NSD as well as two extremes: full-state trellis detection and MS-DFD, where $n_T = 3$, $n_R = 2$, $R = 1$, $N = 6$, and $f_d T = 0.1$. The transmitted signal constellation is generated by a transform $\mathbb{G}' = \mathbf{U}^H \mathbb{G} \mathbf{U}$, where $\mathbb{G}$ is the rate-1 diagonal constellation for three antennas [16], and $\mathbf{U}$ is selected as a unitary matrix given by the discrete Fourier transform (DFT). After such a transform the constellation is nondiagonal so that the linear prediction receiver does not work. Fig. 24 shows that decision-feedback detection exhibits an error floor above 10^{-4} in this fast fading channel, while reduced-state NSD yield significant gains. A noticeable feature is that a major amount of the performance gain can be achieved by an L -state trellis ($\Gamma = 1$) in NSD. If $\Gamma = 2$, the reduced-state detection performs close to full-state detection. Therefore, we do not need a large trellis in order to achieve satisfactory performance.

Fig. 24. Performance of reduced-state NSD versus DFD for three-antenna nondiagonal DSTM, for $f_d T = 0.1$, $N = 6$ and $n_R = 2$.

Fig. 25. Performance comparison between the reduced-state NSD and linear prediction (LP) receiver for Alamouti's code with BPSK signaling, for $N = 6$, $\Gamma = 1$, and $n_R = 1$.

Performance comparison between the reduced-state NSD and linear prediction receiver [61] is given in Fig. 25 for Alamouti's code with BPSK signaling. Both receivers select N

$= 6$, $\Gamma = 1$, i.e., the Viterbi decoding is performed on a four-state trellis. Simulation results are extracted from [61], which conform to our theoretical analysis. It is clear that our proposed NSD performs close to coherent detection, while the linear prediction receiver exhibits an error floor (higher than 10^{-4} if $f_d T = 0.03$). In fact, it was observed in [61] that, in comparison with the standard differential detector, this reduced-state linear prediction receiver can only reduce the error floor by one order in magnitude.

4.7 Conclusions and Discussion

We have shown that linear prediction receivers are mismatched to nondiagonal generator constellations in continuously fading channels. To obtain near-optimum performance, we proposed NSD of DSTM. The idea was to truncate the memory of the MSDD metric so that a trellis for Viterbi decoding could be defined. NSD overcomes the limitation of the existing linear prediction-based Viterbi receiver as it can accommodate nondiagonal constellations. For diagonal constellations the two techniques are equivalent to each other. By tuning N and Γ , reduced-state NSD can range between optimum block detection and DFD. Performance analysis for full-state and reduced-state NSD was carried out, although admittedly, it brought few insights into how the performance changes with system parameters like diversity gain. The approximation considering two major error events might be used as a lower bound on the BER, and is fairly accurate in many cases. The analysis showed that reduced-state NSD with $\Gamma > 0$ has better performance than MSDD for equal values of N , and a major amount of performance gain in NSD could be obtained by an L -state trellis.

Since the number of trellis states can be controlled by Γ , the major implementation difficulty in NSD will arise from computing the branch metrics. The table look-up technique requires a table of size L^{N-1} , even if reduced-state sequence detection is employed. So its application is limited to small values of L and N . When L and/or N are large, we need to compute the inverse and determinant of $\mathbf{R}_{\mathbf{a},N}[n]$ and $\mathbf{R}_{\mathbf{a},N-1}[n-1]$ for each branch transition in real-time. In numerical evaluation of the performance, we found that the BER is nearly unchanged if $|\mathbf{R}_{\mathbf{a},N}[n]|$ and $|\mathbf{R}_{\mathbf{a},N-1}[n-1]|$ are ignored in the branch metric. But the matrix inversion still involves considerable computational burden. Moreover, numerical instability was sometimes observed as a side effect of matrix inversion in performance evaluation at high SNR.

The NSD requires the knowledge of the fading autocorrelation function and noise variance. So we need to estimate the second-order statistics of the channel. When such

knowledge is unavailable at the receiver, the linear prediction receiver is ready to be adapted by the classical least mean-square (LMS) or recursive least-square (RLS) algorithm. The NSD, however, lacks a linear filter structure, thereby seemingly not admitting adaptation. Such implementation-related issues warrant future research.

TABLE I DETECTOR COMPLEXITY COMPARISON	
Detector	Computational Complexity
MSDD	$O(N^2 e^{N-1})$
NSD	$O(N^3 e^{\Gamma+1}), 1 \leq \Gamma \leq N-2$
MS-DFD	$O(N^3)$
LP-DFD	$O(N)$

In Table I, we compare the computational complexity per DSTM symbol, evaluated by the number of arithmetic operations required, of various detectors in terms of N . Firstly, the number of candidates $\mathbf{C}[k]$ in MSDD is exponential in $N-1$, and for each candidate it needs to invert $\mathbf{R}_c[k]$ whose size is proportional to N . Standard matrix inversion has complexity N^3 . Hence the overall complexity is $O(N^2 e^{N-1})$, as a block of $N-1$ symbols are detected simultaneously. Secondly, NSD has complexity $O(N^3 e^{\Gamma+1})$, as it can be viewed as a sliding-window version of MSDD with the complexity controlled by Γ . Thirdly, the number of candidates in MS-DFD does not depend on observation length. Since it is a scheme of symbol-by-symbol detection, the complexity is $O(N^3)$ reflecting computation of the inverse matrix. Lastly, we only need to implement an LP of $N-1$ taps in LP-DFD, so the complexity is linear in N . Obviously LP-DFD has the lowest complexity, while MS-DFD has the complexity in between LP-DFD and MSDD. Though MS-DFD is computationally simpler than MSDD, it is necessarily more complex than LP-DFD because it caters for nondiagonal generator constellations.

As a final remark, we have seen that the union bound for NSD is loose at low to medium SNR's. This phenomenon is not unique to NSD. In fact it is quite general in fading channels where the error probability decreases polynomially rather than exponentially as in Gaussian channels. In Part II of this dissertation, we are going to address this problem in a systematic way.

Part II

MIMO Performance Bounds

Chapter 5

Performance Bounds for Space-Time Codes

5.1 Introduction

As we have seen in Chapter 4, the union bound for a noncoherent sequence detector is not tight at low to moderate SNR. The looseness of the union bound is more often observed in coding problems where a large number of error events exist. In some cases it even may diverge, i.e., the resulted bound may be greater than unity so that it is virtually meaningless. Typical examples are turbo-like codes when operating beyond the cutoff rate [17] and space-time codes when transmitted over quasi-static fading channels [71]. This necessitates nonstandard analytical tools that produce sharp bounds reflecting the actual performance. In the regime of turbo-like codes, Gallager bounds and their variations have been applied with great success (see [17] and references therein.) In this chapter, we use Gallager's techniques to develop tight bounds for space-time codes in quasi-static fading channels. The simplicity of quasi-static fading eases the analysis and provides insights into the behavior of the bounds. It allows the extension to other realistic fading scenarios, such as time-selective fading in the problem of noncoherent sequence detection in Chapter 4. This extension will be presented in the next chapter (Chapter 6).

A quasi-static fading channel is often assumed in space-time coding where no other form of diversity exists. Currently, performance bounds of space-time codes in quasi-static fading [72], [73], [74] rely mainly on the "limit-before-average" (LBA) technique, which was originally developed by Malkamäki and Leib [75] in a quite heuristic manner for convolutional codes over block fading channels. This technique involves a multi-dimensional integral that represents a challenging problem of integration. The integration, typically implemented in the Monte Carlo method, is lengthy. Such a technique is unlikely to find its application in computer search for best codes. In this chapter, we address efficient bounding techniques for space-time codes in the general framework of Gallager bounds. In this framework, Malkamäki and Leib's LBA technique serves as a limit of the optimum bound. We propose *ellipsoidal* and *spherical upper bounds* that come as close as within a few tenths of a decibel to simulated frame error probabilities, while permitting fast computation because they are derived in closed form. A similar upper bound exploiting Gallager's another technique, which was first published in 1965

[3], is also pursued. The bounds as well as the Malkamäki-Leib bound require knowledge about the weight spectrum of a space-time code. To meet this requirement, we present two new approaches of weight enumeration on the basis of state reduction, each of which may be more suited to certain applications. In particular, the scalar transfer-function bound is a *true* upper bound of space-time codes in quasi-static fading, since truncation of the weight spectrum is needed in the other approach. Collectively, the bounding techniques, state reduction, and weight enumeration give a complete set of analytic tools for space-time codes. They yield surprisingly tight performance bounds for all codes considered, in stark contrast to the common intuition that such bounds are not supposed to be tight in the case of quasi-static fading.

Related work on Gallager bounds was given by Bouzekri and Miller [76], but the validity of their derived bounds appears questionable. Byun et al. [77] also proposed an expression, claimed as an upper bound, which has a computational advantage that only a single integral is required. In this chapter we shall show that, unfortunately, it is unnecessarily a valid upper bound. Hence the Gallager bounds of this chapter provide the only efficient bounding techniques for space-time codes. Other relevant works include Gallager's bounding technique applied to the multi-antenna random coding regime [10] and the sphere-packing lower bound for space-time codes [78], which are not oriented to a specific code.

5.2 System Model and Union Bound

Consider a space-time coded system with n_T transmit antennas and n_R receive antennas. At each time slot t , n_T parallel output symbols of the encoder $c_t^1, c_t^2, \dots, c_t^{n_T}$ are simultaneously transmitted from n_T antennas. Following the convention [71], we assume that the elements of the signal constellations are contracted by a factor $\sqrt{E_s}$ so that the average energy of the constellation is unity. Suppose that the frame length is L . The space-time code matrix of size $n_T \times L$ is defined as

$$\mathbf{c} = \begin{bmatrix} c_1^1 & c_2^1 & \cdots & c_L^1 \\ c_1^2 & c_2^2 & \cdots & c_L^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{n_T} & c_2^{n_T} & \cdots & c_L^{n_T} \end{bmatrix}.$$

The received signal at the j th receive antenna is a noisy superposition of n_T transmitted signals corrupted by fading, given by

$$y_t^j = \sqrt{E_s} \sum_{i=1}^{n_T} \alpha_t^{i,j} c_t^i + \eta_t^j \quad (5.1)$$

where the complex additive white Gaussian noise (AWGN) η_t^j has zero mean and variance $N_0/2$ per dimension, and $\alpha_t^{i,j}$ is the fading gain from transmit antenna i to receive antenna j , modeled as complex Gaussian with zero mean and variance 0.5 per dimension. Accordingly $\gamma_s \triangleq E_s/N_0$ is the symbol SNR per transmit antenna (hence the symbol SNR per receive antenna is given by $n_T \gamma_s$.) Here we focus on quasi-static fading, i.e., the gain $\alpha_t^{i,j} = \alpha^{i,j}$ remains constant during a frame and varies independently from one frame to another. It is further assumed that the fading coefficients are spatially white. Later on we shall make extensions to correlated fading.

In matrix form, we have the signal model

$$\mathbf{y} = \sqrt{E_s} \boldsymbol{\alpha} \mathbf{c} + \boldsymbol{\eta} \quad (5.2)$$

where $\mathbf{y}$ is the $n_R \times L$ received signal matrix obtained by stacking r_t^j , $\boldsymbol{\eta}$ is the $n_R \times L$ noise matrix obtained by stacking η_t^j , for $j = 1, 2, \dots, n_R$ and $t = 1, 2, \dots, L$, and $\boldsymbol{\alpha}$ is the fading matrix given by

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha^{1,1} & \alpha^{2,1} & \dots & \alpha^{n_T,1} \\ \alpha^{1,2} & \alpha^{2,2} & \dots & \alpha^{n_T,2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{1,n_R} & \alpha^{2,n_R} & \dots & \alpha^{n_T,n_R} \end{bmatrix}.$$

Given the fading realization $\boldsymbol{\alpha}$, the pairwise error probability that the decoder decides another codeword $\hat{\mathbf{c}}$ while $\mathbf{c}$ being actually sent can be expressed as

$$P_e(\mathbf{c}, \hat{\mathbf{c}} | \boldsymbol{\alpha}) = Q\left(\sqrt{\frac{\gamma_s}{2}} d^2(\mathbf{c}, \hat{\mathbf{c}} | \boldsymbol{\alpha})\right) \quad (5.3)$$

where

$$d^2(\mathbf{c}, \hat{\mathbf{c}} | \boldsymbol{\alpha}) = \sum_{t=1}^L \sum_{j=1}^{n_R} \left| \sum_{i=1}^{n_T} \alpha^{i,j} (c_t^i - \hat{c}_t^i) \right|^2 \quad (5.4)$$

is the squared Euclidean distance for given $\boldsymbol{\alpha}$. It can be rewritten as [71]

$$d^2(\mathbf{c}, \hat{\mathbf{c}} | \boldsymbol{\alpha}) = \text{tr}(\boldsymbol{\alpha} \mathbf{C} \boldsymbol{\alpha}^H) = \sum_{j=1}^{n_R} \sum_{i=1}^{n_T} \lambda_i |\beta^{i,j}|^2 \quad (5.5)$$

where $\beta^{i,j}$'s are independent and identically distributed (iid) complex Gaussian random variables with zero mean and variance 0.5 per dimension, and λ_i for $i = 1, 2, \dots, n_T$ are eigenvalues of the codeword-difference correlation matrix $\mathbf{C}$ defined as

$$\mathbf{C} \triangleq \mathbf{B}(\mathbf{c}, \hat{\mathbf{c}}) \mathbf{B}^H(\mathbf{c}, \hat{\mathbf{c}}), \quad \mathbf{B}(\mathbf{c}, \hat{\mathbf{c}}) \triangleq \mathbf{c} - \hat{\mathbf{c}}.$$

Using the alternative form of the Q function [79]

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} e^{-\frac{x^2}{2\sin^2\theta}} d\theta, \quad x > 0$$

one may express the unconditional pairwise error probability as

$$P_e(\mathbf{c}, \hat{\mathbf{c}}) = \frac{1}{\pi} \int_0^{\pi/2} \prod_{i=1}^{\rho} \left(1 + \frac{\lambda_i \gamma_s}{4\sin^2\theta} \right)^{-n_R} d\theta \quad (5.6)$$

where ρ denotes the rank of $\mathbf{C}$, and λ_i for $i = 1, 2, \dots, \rho$ are the corresponding nonzero eigenvalues of $\mathbf{C}$. If one applies the Chernoff upper bound on $Q(x)$ (or the fact $\sin^2\theta \leq 1$), then

$$P_e(\mathbf{c}, \hat{\mathbf{c}}) \leq \frac{1}{2} \left(\prod_{i=1}^{\rho} \lambda_i \right)^{-n_R} \left(\frac{\gamma_s}{4} \right)^{-\rho n_R}.$$

The rank-determinant criterion [71] follows immediately: $\mathbf{C}$ has to be of full rank for any code matrices $\mathbf{c}$ and $\hat{\mathbf{c}}$ in order to achieve the maximum diversity order $n_T n_R$, and the minimum determinant $|\mathbf{C}|$ over all pairs $(\mathbf{c}, \hat{\mathbf{c}})$ must be maximized. The Chernoff technique doesn't yield an asymptotically tight bound on the pairwise error probability (5.6). There have been several attempts to improve the Chernoff bound on pairwise error probabilities (see, e.g., [80], [81]). Nonetheless, the rank-determinant criterion remains unchanged, as the Chernoff bound only differs from these tightened bounds by a constant factor in the high SNR regime.

A common feature of all these bounds is that the pair-wise error probability is given in terms of $|\mathbf{C}|$. Let D denote the geometric mean of nonzero eigenvalues of $\mathbf{C}$. By sorting the values of D in ascending order and counting the associated frequency of occurrence A_D over all pairs of $\mathbf{c}$ and $\hat{\mathbf{c}}$, the *distance spectrum* $\{D, A_D\}$ of the code is defined. Consequently, by employing the standard union bound, the frame error rate can be expressed in terms of the distance spectrum. This gives a more complete vision of the code performance than just the worst-case pairwise error probability [82]. Similarly, denoting by A_λ the average multiplicity of first error events having eigenvalues $\lambda = (\lambda_1, \dots, \lambda_{n_T})$, we may define the set of pairs $\{\lambda, A_\lambda\}$ as the *eigenvalue spectrum* and give the exact union bound by employing (5.6). Unfortunately, union bounds are substantially loose in quasi-static fading and may diverge. Since the union bound continues rising as more error events are included, there is no indication of dominant error

events. As a consequence, it is sometimes criticized that the rank-determinant criterion is established on “shaky” foundation.

5.3 Gallager Bounds

In this section we develop tight bounding techniques for space-time codes in the general framework of Gallager bounds. We focus on frame error rates, but the extension to bit error rates is straightforward.

Some works (e.g., [72]) viewed space-time codes as block codes in performance analysis. This is necessary for short-frame codes. If the frame size is moderately large so that edge effects are negligible, it is more convenient to treat space-time codes as trellis codes. Using error-event characterization of trellis codes, we can bound the frame error rate as

$$P_f \leq 1 - (1 - P_E)^L \leq LP_E, \quad \text{for } P_E \leq 1$$

where P_E denotes the first error event probability. Conditioned on fading gains α , P_E is given by

$$P_E(\alpha) = \frac{1}{|\mathbb{C}|} \sum_{\mathbf{c}} \sum_{\hat{\mathbf{c}} \neq \mathbf{c}} Q \left(\sqrt{\frac{\gamma_s}{2}} d^2(\mathbf{c}, \hat{\mathbf{c}} | \alpha) \right) = \sum_{\mathbf{c}} A_{\mathbf{c}} Q \left(\sqrt{\frac{\gamma_s}{2}} \text{tr}(\alpha \mathbf{C} \alpha^H) \right) \quad (5.7)$$

where $|\mathbb{C}|$ stands for the cardinality of the subcode $\mathbb{C}$ of sequences starting at a given time epoch, $\hat{\mathbf{c}}$ runs over *first error events*¹⁰ diverging from $\mathbf{c}$ at this time epoch, and $A_{\mathbf{c}}$ denotes the average multiplicity of first error events having codeword-difference correlation matrix $\mathbf{C}$. The set of pairs $\{\mathbf{C}, A_{\mathbf{c}}\}$ is defined as the (multivariate) *weight spectrum* of a space-time code. Average over $\mathbf{c}$ can be omitted for geometrically uniform codes. The remainder of this chapter adopts this error event characterization. Nonetheless, the results are equally applicable to the block-code characterization subject to minor change.

In his 1960 dissertation on low-density parity-check (LDPC) codes, Gallager introduced a fundamental bounding technique [5, p. 24]. In the context of space-time coding, the bound takes the form

$$\begin{aligned} \text{Prob}(\text{error}) &= \text{Prob}(\text{error}, \alpha \notin \mathcal{R}) + \text{Prob}(\text{error}, \alpha \in \mathcal{R}) \\ &\leq \text{Prob}(\text{error}, \alpha \notin \mathcal{R}) + \text{Prob}(\alpha \in \mathcal{R}) \end{aligned}$$

where $\mathcal{R}$ is the Gallager region, selected as a subset in the space of fading gains which includes the events when all fading gains are relatively small, i.e., the channel is in deep fade. It is expected that the union bound will diverge with high probability in this region;

¹⁰ The sum is restricted to so-called “simple error events” for block codes [72].

thereupon it is simply bounded by unity. Then the union bound is applied outside of $\mathcal{R}$, yielding the Gallager bound

$$P_f \leq P_f^G \triangleq L \int_{\bar{\mathcal{R}}} P_E(\alpha) f_\alpha(\alpha) d\alpha + \int_{\mathcal{R}} f_\alpha(\alpha) d\alpha \quad (5.8)$$

where $f_\alpha(\alpha) = \pi^{-n_R n_T} \exp\{-\text{tr}(\alpha \alpha^H)\}$ is the probability density function (pdf) of α , and $\bar{\mathcal{R}}$ denotes the complement of $\mathcal{R}$. $\mathcal{R}$ is to be optimized for a tight upper bound.

When the bit error probability is our concern, the Gallager bound takes the form

$$P_b \leq P_b^G \triangleq \int_{\bar{\mathcal{R}}} \sum_{C,i} \frac{i}{k} A_{C,i} Q\left(\sqrt{\frac{\gamma_s}{2} \text{tr}(\alpha C \alpha^H)}\right) f_\alpha(\alpha) d\alpha + \frac{1}{2} \int_{\mathcal{R}} f_\alpha(\alpha) d\alpha \quad (5.9)$$

where k is the number of input bits per trellis level, $A_{C,i}$ is the number of error events with correlation matrix C and bit errors i , and the factor $1/2$ is an upper bound on the bit error probability.

A. Malkamäki-Leib Bound (Optimum Bound)

If we bound $LP_E(\alpha)$ by unity wherever it exceeds unity, we shall obtain the tightest Gallager bound. That is, the optimum Gallager region is given by $\mathcal{R}^0 = \{\alpha : LP_E(\alpha) \geq 1\}$. This is a region around the origin with a complicated geometric shape. Interestingly, although the optimum bound has no closed form, it turns out to be equivalent to the “limit-before-average” technique of Malkamäki and Leib [75]

$$P_f \leq P_f^{\text{ML}} \triangleq \int \min[1, LP_E(\alpha)] f_\alpha(\alpha) d\alpha \quad (5.10)$$

This technique also follows from the fact that the error probability never exceeds unity for a given fading realization α . Surprisingly, this simple technique indeed gives the optimum Gallager bound, without any need of explicit optimization.

It is noteworthy that there is a minor difference in use of the relation $P_f \leq 1 - (1 - P_E)^L$ between the two techniques. The Malkamäki-Leib bound can be made slightly tighter by replacing $\min[1, LP_E(\alpha)]$ with $1 - (1 - \min[1, P_E(\alpha)])^L$ [73]. This appears inapplicable to the Gallager bound since $1 - (1 - P_E(\alpha))^L$ might be negative for some α . Anyway, this improvement is found quite marginal, so we employ (5.8) and (5.10) throughout this chapter.

The inherent difficulty in evaluating the Malkamäki-Leib bound is the multi-dimensional integral with respect to α . Due to the function $\min(1, x)$, a closed form is impossible and the integral must be performed numerically. Existing analytical tools for

fading channels play no role here. Such an integral involving the Gaussian pdf is considered a challenging problem even in the discipline of mathematics [83]. In the simplest nontrivial case of $n_T = 2$, $n_R = 1$, it is already a four-dimensional real integral. The direct brute-force integration is extremely time-consuming, while multiple quadrature methods are often found unstable for space-time codes (it is widely acknowledged that quadrature is not as effective in high dimensions [83].) Such deterministic methods quickly go beyond computing capability as the number of antennas increases. Consequently the Monte Carlo method is usually the only solution, which is sometimes claimed not suffering from the dimension effect, but does not provide sufficient accuracy [83]. To obtain accurate results, the Monte Carlo method is still lengthy, and the time increases as the error probability decreases (importance sampling [83] may improve the convergence, see Appendix A.) The computational burden is especially felt for low error probabilities or a large number of antennas. Therefore, the value of the Malkamäki-Leib bound is more a theoretic reference than a practical means of performance analysis. Moreover, an integral form is not attractive even from a purely theoretic point of view.

B. Ellipsoidal Bound

In this subsection we propose a Gallager bound to circumvent the difficulty of the Malkamäki-Leib bound. The bound is derived in closed form and near optimum. We notice that the complicated shape of the optimum Gallager region $\mathcal{R}^O$ is the obstacle of deriving a closed-form bound. Hence it is necessary to find a geometric shape to approximate $\mathcal{R}^O$ with reasonable accuracy such that the derivation becomes *tractable*. There are two fundamental requirements in choosing the geometric shape. Firstly, the shape must permit the derivation of a closed-form bound. Secondly, the bound must admit fast optimization.

We decide to choose an ellipsoid to fit $\mathcal{R}^O$, defined by $\mathcal{R}^E = \{\mathbf{a} : \text{tr}(\mathbf{a}\mathbf{F}\mathbf{a}^H) \leq 1\}$ where $\mathbf{F}$ is a Hermitian positive semi-definite matrix. $\mathbf{F}$ is to be optimized for a minimum upper bound. The reason for the choice of an ellipsoid is as follows. First, it is the optimum shape if there were only one matrix $\mathbf{C}$, since then

$$\mathcal{R}^O = \left\{ \mathbf{a} : LA_c Q \left(\sqrt{\frac{\gamma_s}{2} \text{tr}(\mathbf{a}\mathbf{C}\mathbf{a}^H)} \right) \geq 1 \right\} = \left\{ \mathbf{a} : \text{tr}(\mathbf{a}\mathbf{C}\mathbf{a}^H) \leq \frac{2}{\gamma_s} \left[Q^{-1} \left(\frac{1}{LA_c} \right) \right]^2 \right\}$$

where $Q^{-1}(x)$ denotes the inverse function of $Q(x)$. Intuitively, it will also give good fitting for $\mathcal{R}^O$ in the general case. The reason of using $\text{tr}(\cdot)$ in the definition of $\mathcal{R}^E$ is that there

is no point in discriminating between receive antennas because of spatial independence. The relation of $\mathcal{R}^O$ and $\mathcal{R}^E$ is graphically illustrated in Fig. 26, where a sphere is also plotted. Furthermore, the expression $\mathbf{a}\mathbf{F}\mathbf{a}^H$ pertains to classic analytical results on the pdf of a Hermitian quadratic form in complex Gaussian variates [65]; our derivation actually benefits from classical literature of fading channels. Last but not least, the optimization with respect to a Hermitian positive semi-definite matrix is relatively easy. At first glance, there will be no advantage in this approach since the number of parameters in $\mathbf{F}$ is larger than the number of integral variables. Nonetheless, a crucial point is that optimization can be much more efficient than integration.

Fig. 26. Graphic illustration of Gallager regions in the vicinity of the origin $\mathbf{O}$.

Let us look at the first term on the right-hand side of (5.8) under an ellipsoidal Gallager region. Using the alternative form of $Q(x)$, we have

$$\begin{aligned}
 & L \int_{\mathcal{R}^E} P_E(\mathbf{a}) f_a(\mathbf{a}) d\mathbf{a} \\
 &= L \int_{\mathcal{R}^E} \sum_c A_c \frac{1}{\pi} \int_0^{\pi/2} \exp \left\{ -\frac{\gamma_s}{4 \sin^2 \theta} \text{tr}(\mathbf{a} \mathbf{C} \mathbf{a}^H) \right\} d\theta \frac{1}{\pi^{n_R n_T}} \exp \left\{ -\text{tr}(\mathbf{a} \mathbf{a}^H) \right\} d\mathbf{a} \\
 &= L \sum_c A_c \frac{1}{\pi} \int_0^{\pi/2} \left| \mathbf{I}_{n_T} + \frac{\gamma_s \mathbf{C}}{4 \sin^2 \theta} \right|^{-n_R} \underbrace{\frac{1}{\pi^{n_R n_T} \left| \mathbf{I}_{n_T} + \frac{\gamma_s \mathbf{C}}{4 \sin^2 \theta} \right|^{-n_R}} \int_{\mathcal{R}^E} \exp \left\{ -\text{tr} \left(\mathbf{a} \left(\mathbf{I}_{n_T} + \frac{\gamma_s \mathbf{C}}{4 \sin^2 \theta} \right) \mathbf{a}^H \right) \right\} d\mathbf{a} d\theta}_{P_0}
 \end{aligned} \tag{5.11}$$

where $\mathbf{I}_{n_T}$ denotes the n_T -by- n_T identity matrix. Our key observation is that the quantity P_0 defined above may be interpreted as the probability that $\text{tr}(\tilde{\mathbf{a}} \mathbf{F} \tilde{\mathbf{a}}^H) > 1$, where the rows $\tilde{\mathbf{a}}^j$ ($j = 1, \dots, n_R$) in the random matrix $\tilde{\mathbf{a}}$, statistically independent of each other, have a common correlation matrix $E[(\tilde{\mathbf{a}}^j)^H \tilde{\mathbf{a}}^j] = (\mathbf{I}_{n_T} + \gamma_s \mathbf{C} / 4 \sin^2 \theta)^{-1}$. The evaluation of such

probability is a standard analytical problem in correlated fading channels that is easily resolved by means of the characteristic-function method. The Hermitian quadratic form $X_1 = \tilde{\mathbf{a}}^j \mathbf{F} (\tilde{\mathbf{a}}^j)^H$ has a rational characteristic function [65]

$$\Phi_{X_1}(s) \triangleq E[e^{-sX_1}] = \left| \mathbf{I}_{n_T} + s \left(\mathbf{I}_{n_T} + \frac{\gamma_s \mathbf{C}}{4 \sin^2 \theta} \right)^{-1} \mathbf{F} \right|^{-1} = \prod_{i=1}^{n_T} (1 + s\mu_i)^{-1}, \quad (5.12)$$

where μ_i 's are eigenvalues of $(\mathbf{I}_{n_T} + \gamma_s \mathbf{C} / 4 \sin^2 \theta)^{-1} \mathbf{F}$. It is easily seen that μ_i 's are real, nonnegative. Due to statistical independence, the characteristic function of $\text{tr}(\tilde{\mathbf{a}} \mathbf{F} \tilde{\mathbf{a}}^H)$ is simply given by $\Phi_{X_1}(s)$ raised to the n_R -th power. By inverting the characteristic function, we can express P_0 as

$$P_0 = 1 - \int_{-\infty - j\infty + \varepsilon}^1 \int_{-j\infty + \varepsilon}^{j\infty + \varepsilon} e^{sx} \Phi_{X_1}^{n_R}(s) \frac{ds dx}{2\pi j} = 1 - \int_{-j\infty + \varepsilon}^{j\infty + \varepsilon} s^{-1} e^s \Phi_{X_1}^{n_R}(s) \frac{ds}{2\pi j} \quad (5.13)$$

where $\varepsilon > 0$ such that the integration path is chosen in the right half plane. This integral has a closed-form expression

$$P_0 = - \sum_{k=1}^{K_1} \text{Res} \left[s^{-1} e^s \prod_{i=1}^{n_T} (1 + s\mu_i)^{-n_R}, s_k^1 < 0 \right] \quad (5.14)$$

where s_k^1 is the k th of the K_1 ($1 \leq K_1 \leq n_T$) distinct, negative poles of $\Phi_{X_1}(s)$, and $\text{Res}(\cdot)$ denotes the residue. It is noteworthy that P_0 is positive in spite of the minus ahead of the sum.

The second term on the right-hand side of (5.8) is derived in a similar fashion. Let $\mathbf{a}^j$ be the j th row of $\mathbf{a}$. The Hermitian quadratic form $X_2 = \mathbf{a}^j \mathbf{F} (\mathbf{a}^j)^H$ has a rational characteristic function $\Phi_{X_2}(s) = |\mathbf{I}_{n_T} + s\mathbf{F}|^{-1}$. Since the integral $\int_{\mathcal{R}^E} f_{\mathbf{a}}(\mathbf{a}) d\mathbf{a}$ is the probability that $\text{tr}(\mathbf{a} \mathbf{F} \mathbf{a}^H) \leq 1$, we have

$$\int_{\mathcal{R}^E} f_{\mathbf{a}}(\mathbf{a}) d\mathbf{a} = \int_{-\infty - j\infty + \varepsilon}^1 \int_{-j\infty + \varepsilon}^{j\infty + \varepsilon} e^{sx} \Phi_{X_2}^{n_R}(s) \frac{ds dx}{2\pi j} = \int_{-j\infty + \varepsilon}^{j\infty + \varepsilon} s^{-1} e^s \Phi_{X_2}^{n_R}(s) \frac{ds}{2\pi j} \quad (5.15)$$

where $\varepsilon > 0$ as well. Again, this can be expressed in terms of residues on the left plane:

$$\int_{\mathcal{R}^E} f_{\mathbf{a}}(\mathbf{a}) d\mathbf{a} = 1 + \sum_{k=1}^{K_2} \text{Res} \left[s^{-1} e^s \prod_{i=1}^{n_T} (1 + s\nu_i)^{-n_R}, s_k^2 < 0 \right] \quad (5.16)$$

where ν_i 's (real, nonnegative) are eigenvalues of $\mathbf{F}$, and s_k^2 is the k th of the K_2 ($1 \leq K_2 \leq n_T$) distinct, negative poles of $\Phi_{X_2}(s)$.

Combining these results we arrive at the ellipsoidal bound in *Theorem 5.1*.

Theorem 5.1 (Ellipsoidal Bound): Let $\{C, A_C\}$ denote the weight spectrum of a space-time code, then the frame error rate is upper-bounded by

$$P_f^E = \min_{F \geq 0} \left\{ -L \sum_C A_C \frac{1}{\pi} \int_0^{\pi/2} \left| I_{n_T} + \frac{\gamma_s C}{4 \sin^2 \theta} \right|^{-n_R} \sum_{k=1}^{K_1} \text{Res} \left[s^{-1} e^s \left| I_{n_T} + s \left(I_{n_T} + \frac{\gamma_s C}{4 \sin^2 \theta} \right)^{-1} F \right|^{-n_R}, s_k^1 < 0 \right] d\theta \right. \\ \left. + 1 + \sum_{k=1}^{K_2} \text{Res} \left[s^{-1} e^s \left| I_{n_T} + s F \right|^{-n_R}, s_k^2 < 0 \right] \right\} \quad (5.17)$$

where the notation $F \geq 0$ stands for “ F is positive semi-definite”.

The ellipsoidal bound involves an integral of the form $\int_0^{\pi/2} g(\sin^2 \theta) d\theta / \pi$, which nonetheless can be easily evaluated by using the Gauss-Chebyshev quadrature rule [74, (22)]

$$\frac{1}{\pi} \int_0^{\pi/2} g(\sin^2 \theta) d\theta \doteq \frac{1}{N} \sum_{i=0}^{(N-1)/2-1} g \left(\cos^2 \left(\frac{2i+1}{2N} \pi \right) \right), \quad N \text{ odd},$$

or may be bypassed by using the newly proposed bound

$$Q(x) \leq \frac{1}{2N} \sum_{i=1}^N \exp \left(-\frac{x^2}{2 \sin^2 \theta_i} \right), \quad \theta_i = \frac{i\pi}{2N}$$

and approximation of $Q(x)$ of Chiani *et al.* [84]. Extremely good accuracy is obtained by using no more than ten points in both methods. As such it is considered a closed-form bound. The Chernoff bound corresponds to the case $N = 1$, which will render the final performance bound looser.

For a function $f(s)$ in the form as shown in the brackets of (5.14) and (5.16), the residue at a pole $s_k = -a$ of order m is defined as

$$\text{Res}[f(s), s_k = -a] = \frac{1}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \left[(s-a)^m f(s) \right] \Big|_{s=-a}.$$

If $n_R = 1$ and there are no repeated eigenvalues of μ_i ’s, we can express P_0 in a very simple form as

$$P_0 = \sum_{k=1}^{K_1} \frac{e^{-\mu_k^{-1}}}{\prod_{j=1, j \neq k}^{n_T} (1 - \mu_j / \mu_k)} \quad (5.18)$$

A similar expression exists for (5.16). Repeated eigenvalues require high-order derivatives, which may complicate the expression. There are several methods available in literature to circumvent this difficulty. One is the perturbation method of Siwamogsatham *et al.* [80]. They showed that the pdf of Hermitian quadratic form like those considered in this subsection will increase or decrease if any pole of the characteristic function is move toward or away from the origin. Then by perturbing the multiple poles properly so that the

characteristic function has only simple (negative) poles, we are able to use the single-pole formula to upper- and lower-bound (5.14) and (5.16). The two bounds can be made close by reducing the perturbation magnitude. The case of $n_R > 1$ may be treated as repeated poles. This method is quite convenient from a computational point of view. Other methods include the Euler summation technique (for a positive random variable as for coherent space-time codes) [79, App. 9B] and the saddle-point integration [66], in which the Laplace inversions in (5.13) and (5.15) are performed numerically. These methods are quite efficient and the computational costs are not necessarily higher than that of the residue method. In fact, they are often more stable in the numerical aspect.

In short, the ellipsoidal bound given $\mathbf{F}$ can be computed very efficiently. The minimization of the ellipsoidal bound subject to constraint $\mathbf{F} \geq 0$ is not challenging as well. We convert it to an unconstrained optimization problem by applying the Cholesky decomposition $\mathbf{F} = \mathbf{T}^H \mathbf{T}$ where $\mathbf{T}$ is an upper-triangular complex matrix. This guarantees that $\mathbf{F}$ is Hermitian, positive semi-definite. Then the unconstrained optimization is carried out over $\mathbf{T}$. There exist a wide spectrum of algorithms for unconstrained optimization [85], many of which are available in standard scientific software such as MATLAB. Algorithms of unconstrained optimization can be broadly categorized in terms of the derivative information used. Search methods that only use function evaluations are suitable for highly nonlinear problems such as the ellipsoidal bound. They are generally less efficient but more robust than gradient-based and high-order methods. We suggest the function “fminsearch” in the Optimization Toolbox of MATLAB 6.1, which uses the simplex search method. Convergence of the solution is observed for all codes examined, though a formal proof is currently unavailable. The number of real-valued free parameters in optimization is n_T^2 , for the diagonal elements of $\mathbf{T}$ can be set real-valued without loss of generality. Moreover, the computational complexity in optimization can be traded-off against the tightness of the bound. For instance, we may restrict the optimization over the n_T diagonal elements of $\mathbf{F}$. This corresponds to optimizing the axes of the ellipsoid while neglecting the orientation.

C. Spherical Bound and Its Uniqueness

If we make the axes of the ellipsoid of equal length, it reduces to a spherical region $\mathcal{R}^S = \{\boldsymbol{\alpha} : \text{tr}(\boldsymbol{\alpha}\boldsymbol{\alpha}^H) \leq r^2\}$ where r is the radius. This corresponds to the degenerate case $\mathbf{F} = \mathbf{I} / r^2$. It will result in a spherical bound with a single free parameter that permits exceedingly fast optimization. It is necessarily looser than the ellipsoidal bound; however

it may virtually be as tight if the code has good symmetry. The spherical bound is in a sense unique, for it only requires the knowledge of the eigenvalue spectrum rather than the weight spectrum. As such it is treated separately in this subsection.

Theorem 5.2 (Spherical Bound): Let $\{\lambda, A_\lambda\}$ denote the eigenvalue spectrum of a space-time code, then the frame error rate can be upper-bounded by

$$P_f^S = -L \sum_{\lambda} A_{\lambda} \frac{1}{\pi} \int_0^{\pi/2} \sum_{k=1}^K \text{Res} \left[\frac{e^{sr_0^2}}{s} \prod_{i=1}^{n_T} \left(s + 1 + \frac{\lambda_i \gamma_s}{4 \sin^2 \theta} \right)^{-n_R}, s_k < 0 \right] d\theta + 1 - e^{-r_0^2} \sum_{n=0}^{n_R n_T} \frac{r_0^{2n}}{n!} \quad (5.19)$$

where K is the number of distinct nonzero elements in λ , and r_0 is the root of the following equation

$$L \sum_{\lambda} A_{\lambda} \frac{1}{\pi} \int_0^{\pi/2} \sum_{k=1}^K \text{Res} \left[e^{sr_0^2} \prod_{i=1}^{n_T} \left(s + 1 + \frac{\lambda_i \gamma_s}{4 \sin^2 \theta} \right)^{-n_R}, s_k < 0 \right] d\theta = \frac{r_0^{2(n_R n_T - 1)}}{(n_R n_T - 1)!}. \quad (5.20)$$

Proof: For a spherical region, the first term on the right-hand side of (5.8) can be obtained from (5.17) by setting $F = I_{n_T}$ and changing e^s to e^{sr^2} . This gives

$$L \int_{\mathcal{R}^S} P_E(\alpha) f_{\alpha}(\alpha) d\alpha = -L \sum_{\mathbf{C}} A_{\mathbf{C}} \frac{1}{\pi} \int_0^{\pi/2} \sum_{k=1}^K \text{Res} \left[s^{-1} e^{sr^2} \left| s + I_{n_T} + \frac{\gamma_s \mathbf{C}}{4 \sin^2 \theta} \right|^{-n_R}, s_k < 0 \right] d\theta.$$

Obviously it depends on $\mathbf{C}$ only through its eigenvalues, therefore can be expressed in the form as the first term of (5.19). The second integral in (5.8) over $\mathcal{R}^S$ is recognized as the pdf of a chi-square random variable with $2n_R n_T$ degrees of freedom, given by

$$\int_{\mathcal{R}^S} f_{\alpha}(\alpha) d\alpha = 1 - e^{-r^2} \sum_{n=0}^{n_R n_T} \frac{r^{2n}}{n!}.$$

Combining these two integrals we obtain the spherical bound as in (5.19), with the radius r remaining to be optimized.

This is achieved by setting $\partial P_f^G / \partial r^2 = 0$. To solve for the partial derivative, note that the derivative and finite integral commute; the derivative and $\text{Res}(\cdot)$ also commute, since $\text{Res}(\cdot)$ involves at most a partial derivative with respect to s . Hence we are able to move the partial derivative inside $\text{Res}(\cdot)$ to write

$$\frac{\partial P_f^G}{\partial r^2} = -L \sum_{\lambda} A_{\lambda} \frac{1}{\pi} \int_0^{\pi/2} \sum_{k=1}^K \text{Res} \left[e^{sr^2} \prod_{i=1}^{n_T} \left(s + 1 + \frac{\lambda_i \gamma_s}{4 \sin^2 \theta} \right)^{-n_R}, s_k < 0 \right] d\theta + \frac{r^{2(n_R n_T - 1)}}{(n_R n_T - 1)!} = 0.$$

This completes the proof. $\square$

Equation (5.20) is well defined and has a unique root. To see this, note that the left-hand side of (5.20) decreases monotonically with r_0 , while the right-hand side increases monotonically. Furthermore, the left-hand side approaches zero when $r_0 \rightarrow \infty$, while the

right-hand side equals zero when $r_0 = 0$. Hence there must exist one and only one positive root of r_0 . By using the bisection method, we can easily solve for the optimum value r_0 . Convergence is always guaranteed.

The computational cost of a Gallager bound is roughly given by the number of steps in optimization times the cost in calculating (5.8) for once. The latter cost is similar for ellipsoidal and spherical bounds. Since it is easier to determine r_0 , the computation of the spherical bound is significantly faster.

When there are no repeated eigenvalues and $n_R = 1$, a similar bound (without a derivation of r_0) has been attempted in [76] by using a Chernoff-like bound on $Q(x)$, but the expression derived therein appears problematic. Moreover, they were unable to cope with repeated eigenvalues or $n_R > 1$ (in essence a case of repeated eigenvalues), which are commonplace in space-time coding. In contrast, these cases are easily handled in our bound by solving for the residue or by using the perturbation method. In the latter method, substituting the perturbed eigenvalues $\tilde{\lambda}_i, i = 1, \dots, n_R n_T$ into (5.19), we have

$$P_f^S = L \sum_{\lambda} A_{\lambda} \frac{1}{\pi} \int_0^{\pi/2} \sum_{i=1}^{n_R n_T} \rho_i \left(1 + \frac{\tilde{\lambda}_i \gamma_s}{4 \sin^2 \theta} \right)^{-1} \exp \left[- \left(1 + \frac{\tilde{\lambda}_i \gamma_s}{4 \sin^2 \theta} \right) r_0^2 \right] d\theta + 1 - e^{-\gamma_s^2} \sum_{n=0}^{n_R n_T} \frac{r_0^{2n}}{n!} \quad (5.21)$$

where

$$\rho_i = \prod_{\substack{j \neq i \\ j=1}}^{n_R n_T} \frac{4 \sin^2 \theta}{(\tilde{\lambda}_j - \tilde{\lambda}_i) \gamma_s}.$$

A nice feature of this form is that the solution of r_0 is independent of the SNR, since it is given by the root of the following equation

$$L \sum_{\lambda} A_{\lambda} \frac{1}{\pi} \int_0^{\pi/2} \sum_{i=1}^{n_R n_T} \left(\prod_{\substack{j \neq i \\ j=1}}^{n_R n_T} \frac{4 \sin^2 \theta}{(\tilde{\lambda}_j - \tilde{\lambda}_i)} \right) \exp \left(- \frac{\tilde{\lambda}_i \gamma_s r_0^2}{4 \sin^2 \theta} \right) d\theta = \frac{(\gamma_s r_0^2)^{n_R n_T - 1}}{(n_R n_T - 1)!} \quad (5.22)$$

Apparently we don't need to solve for $\gamma_s r_0^2$ at each new SNR. It is easily checked that the expression of [76] disagrees with (5.19) if the alternative form of $Q(x)$ is replaced with a Chernoff-like bound.

Next we illustrate that the dependence on the eigenvalue spectrum is in fact *unique* to a spherical Gallager region. Consider a generalized Gallager region of arbitrary shape. Using the decomposition (5.5), we can rewrite (5.8) as

$$P_f^G = L \sum_C A_C \int_{\mathcal{R}} \mathcal{Q} \left(\sqrt{\frac{\gamma_s}{2} \sum_{j=1}^{n_R} \sum_{i=1}^{n_T} \lambda_i |\beta^{i,j}(C)|^2} \right) f_a(\mathbf{a}) d\mathbf{a} + \int_{\mathcal{R}} f_a(\mathbf{a}) d\mathbf{a} \quad (5.23)$$

where the notation $\beta^{i,j}(C)$ reflects its dependency on C . To see the dependency, define

$\beta(C) = [(\beta^1(C))^T, \dots, (\beta^{n_R}(C))^T]^T$ where $\beta^j(C) = [\beta^{1,j}(C), \dots, \beta^{n_T,j}(C)]$, and note the relation $\beta(C) = \alpha U(C)$ where $U(C)$ is a unitary matrix from eigen-decomposition $C = U(C) \Lambda U^H(C)$. Then one tries to change variables of this integral from α to $\beta(C)$. However, the region over which the integral is evaluated will generally be rotated, and such rotation $U(C)$ varies with C (this appears to be overlooked in [76].) The only exception is a sphere. Since $\beta(C) \beta^H(C) = \alpha U(C) U^H(C) \alpha^H = \alpha \alpha^H$, a region given as

$$\mathcal{R} = \left\{ \alpha : \sum_{i=1}^{n_T} |\alpha^{i,j}|^2 \leq r_j^2, j = 1, \dots, n_R \right\}$$

is rotationally invariant. Then $\mathcal{R}^S = \left\{ \alpha : \text{tr}(\alpha \alpha^H) \leq r^2 \right\}$ is unique, again as there is no point in discriminating between receive antennas. This property agrees with the traditional union bound given in terms of the eigenvalue spectrum, since it might be seen as a degenerate spherical bound with $r_0 = 0$.

On the other hand, it is *inadequate* to determine Gallager bounds with other shapes of $\mathcal{R}$ by using the eigenvalue spectrum alone. For instance, the Malkamäki-Leib bound needs full information of the weight spectrum. Equations like [77, (10),(13),(14)], [86, (25)] given in terms of the eigenvalue spectrum are not enough. This is a fundamental difference between Gallager bounds and the standard union bound, as the pairwise error probability in the standard union bound is irrelevant to $U(c, \hat{c})$. An immediate implication is that the “cubical bound” ignoring $U(c, \hat{c})$ as given in [76] is *not a true Gallager bound*. It is unclear at this stage whether it is a valid upper bound or not, though empirically it appears to be [76]. Although a cubical Gallager region still yields an upper bound, its derivation appears intractable due to rotation. At last, it is worth mentioning that the cubical bound is tractable for convolutional codes over block-fading channel, because there is no issue of rotation. Its derivation is given in Appendix B.

D. Remarks

Using the transform (5.5) again, one can rewrite (5.8) as

$$P_f^{\text{ML}} = \int \min \left[1, L \sum_C A_C \mathcal{Q} \left(\sqrt{\frac{\gamma_s}{2} \sum_{i=1}^{\rho} \lambda_i \zeta^i(C)} \right) \right] f_{\alpha}(\alpha) d\alpha \quad (5.24)$$

where $\zeta^i(C) \triangleq \sum_{j=1}^{n_R} |\beta^{i,j}(C)|^2$ for $i = 1, 2, \dots, \rho$ are iid chi-square random variables of $2n_R$ degrees of freedom. Byun, Park and Lee (BPL) [77] derived an expression requiring

only a single integral that permits fast evaluation¹¹

$$P_f^{\text{BPL}} = \int_0^\infty \min \left[1, L \sum_D A_D Q \left(\sqrt{\frac{\gamma_s}{2}} D \xi \right) \right] f_\xi(\xi) d\xi \quad (5.25)$$

where $\xi \triangleq \sum_{i=1}^{n_T} \zeta^i(\mathbf{c}, \hat{\mathbf{c}})$ is a chi-square random variable of $2n_T n_R$ degrees of freedom. The right-hand side of (5.25) is a well-defined integral. Change of variables is justified because $\xi = \text{tr} \{ \boldsymbol{\beta}(\mathbf{C}) \boldsymbol{\beta}^H(\mathbf{C}) \} = \text{tr} \{ \mathbf{a} \mathbf{a}^H \}$ is independent of the codeword pair $\mathbf{C}$ as long as $\mathbf{C}$ has full rank. P_f^{BPL} is quite insightful, as it clearly indicates the role of the distance spectrum.

Byun et al. claimed that P_f^{BPL} is an upper bound on P_f^{ML} , by proving the cumulative distribution function (cdf) of $\sum_{i=1}^\rho \lambda_i \zeta^i(\mathbf{C})$ is less than or equal to that of $D \sum_{i=1}^\rho \zeta^i(\mathbf{C})$. Here we shall show that, however, P_f^{BPL} is an approximation to rather than an upper bound on P_f^{ML} .

To see why P_f^{BPL} is unnecessarily an upper bound, let us reexamine Byun et al's proof. They stated (5.25) is an upper bound for one set of eigenvalues $\boldsymbol{\lambda}$; then this was extended to the entire code as “it equally applies to all different” $\boldsymbol{\lambda}$'s [77]. These two steps are problematic. Firstly, since the eigenvalue spectrum is inadequate to describe the Malkamäki-Leib bound, P_f^{BPL} is unnecessarily an upper bound even for one set of eigenvalues; secondly, the statement “it equally applies to all different” $\boldsymbol{\lambda}$'s corresponds to the case that the random variables $\sum_{i=1}^\rho \lambda_i \zeta^i(\mathbf{C})$ are statistically independent for different codeword pairs, but this is generally not satisfied. In fact, P_f^{BPL} is an upper bound only if there is a single matrix $\mathbf{C}$ (P_f^{BPL} will be asymptotically tight.) It will be more clear in the following counter-example, the purpose of which is to show P_f^{BPL} is unnecessarily an upper bound for one set of eigenvalues.

Counter-example 1: For clarity, consider a binary rate-1/2 convolutional code transmitted over block Rayleigh fading channels. The system model was given in [75]. Suppose we have one set of Hamming distances $\{d_1, d_2\}$, but in different order as (d_1, d_2) and (d_2, d_1) , and let $d = \sqrt{d_1 d_2}$. This system is equivalent to a space-time code with two

¹¹ Byun et al. assumed full rank $\rho = n_T$. This condition is indeed necessary, while it is not required by Gallager bounds.

sets of eigenvalues $(4d_1, 4d_2)$ and $(4d_2, 4d_1)$, $D = 4\sqrt{d_1 d_2}$, $n_T = 2$, and $n_R = 1$, except for 3 dB difference in SNR. Then the Malkamäki-Leib bound is given by

$$P_f^{\text{ML}} = \int_0^\infty \int_0^\infty \min \left\{ 1, L \left[Q \left(\sqrt{2\gamma_s (d_1 x_1 + d_2 x_2)} \right) + Q \left(\sqrt{2\gamma_s (d_2 x_1 + d_1 x_2)} \right) \right] \right\} e^{-(x_1 + x_2)} dx_1 dx_2. \quad (5.26)$$

Correspondingly,

$$P_f^{\text{BPL}} = \int_0^\infty \min \left\{ 1, 2L \left[Q \left(\sqrt{2\gamma_s dx} \right) \right] \right\} e^{-x} dx. \quad (5.27)$$

Setting $d_1 = 1$, $d_2 = 4$, $d = 2$, $L = 1000$, we plot the two curves in Fig. 27 by using numerical integration. The ellipsoidal bound is also included, which degenerates into the spherical bound due to the symmetry of this counter-example. It is clear that P_f^{BPL} is lower than P_f^{ML} , while the ellipsoidal bound is valid.

Fig. 27. Comparison of P_f^{ML} , P_f^{BPL} , and $P_f^{\text{E}} (P_f^{\text{S}})$ for the counterexample (viewed as a space-time code) for $n_T = 2$, $n_R = 1$ and $L = 1000$.

To see how Byun et al.'s proof breaks down, let $Y_1 = d_1 X_1 + d_2 X_2$ and $Y_2 = d_2 X_1 + d_1 X_2$. Via change of variables, P_f^{ML} can be expressed as

$$P_f^{\text{ML}} = \iint_{\Omega} \min \left\{ 1, L \left[Q \left(\sqrt{2\gamma_s y_1} \right) + Q \left(\sqrt{2\gamma_s y_2} \right) \right] \right\} f_{Y_1 Y_2} (y_1, y_2) dy_1 dy_2 \quad (5.28)$$

where Ω is the integration region. If Y_1 and Y_2 were statistically independent, we would be able to break up the double integral so that

$$P_f^{\text{ML}} = \int_0^\infty \int_0^\infty \min \left\{ 1, L \left[Q \left(\sqrt{2\gamma_s y_1} \right) + Q \left(\sqrt{2\gamma_s y_2} \right) \right] \right\} f_{Y_1}(y_1) dy_1 f_{Y_2}(y_2) dy_2 \quad (5.29)$$

and make use of Byun et al.'s inequality for cdf's to arrive at (5.27). But it is obvious that Y_1 and Y_2 are not statistically independent. $\square$

In general, P_f^{BPL} and P_f^{ML} differ by a non-diminishing factor with increasing SNR. At this point, we are unable to determine the factor. This appears to be surprisingly difficult due to the nonlinearity introduced by $\min(1, x)$.

E. Summary

Gallager bounds proposed in this chapter provide a heretofore single solution of efficient, tight upper bounds of space-time codes. Moreover, their applications are not restricted to full-rank codes.

TABLE II
BOUNDING TECHNIQUES

	Tightness	Required information	Computation cost
Malkamäki-Leib bound	Tightest	$\{C, A_C\}$	High
Ellipsoidal (& other Gallager) bound	Tighter	$\{C, A_C\}$	Low
Spherical bound	Tight	$\{\lambda, A_\lambda\}$	Lower
Union bound	Divergent	$\{\lambda, A_\lambda\}$	Lowest
Union-Chernoff (& improved) bound	Divergent	$\{D, A_D\}$	Lowest

The required amount of information, tightness and computational cost of various bounding techniques are summarized in TABLE II. When looking at TABLE II, one may ask a natural question: does a convergent bounding technique requiring only the information of $\{D, A_D\}$ (as P_f^{BPL}) exist? Some researchers calculate the Malkamäki-Leib bound by using several leading terms in $\{D, A_D\}$ [73]. Indeed, the distance spectrum is a sensible measure, if we are to describe performance of space-time codes by using a counterpart of distance spectrum for convolutional codes. The existence of this missing bound remains as an interesting open question. If the answer is positive, it is necessarily looser than the spherical bound, and the truncation in [73] will be more justified.

The Gallager region is defined for fading gains in this section. An alternative is to apply the Gallager bound to Gaussian noise. This is motivated by the fact that this method

promises a bound always less than $\min(1, \text{union bound})$ in P_f^{ML} . In fact, the resultant bound turns out to be slightly tighter than P_f^{ML} over all range of SNR. However, it is observed that improvement is quite marginal. Compared to the Q function, the integral in this alternative is more complex computationally. As such this technique is not further pursued.

The bounding technique used in this section is Gallager's first bounding technique [17]. In Appendix D, we make a further effort to derive Gallager's second bound for space-time codes, which appears to be looser. An overview of Gallager bounds is also given in Appendix C.

5.4 Weight Enumeration

Performance bounds of a space-time code require knowledge of the weight, eigenvalue, or distance spectrum. Accordingly, an analytic treatment is incomplete without including a procedure to get such spectra. Since the eigenvalue or determinant of $\mathbf{C}$ is not an additive measure in time but $\mathbf{C}$ itself is, weight enumeration will be invariably involved. Basically, we need to enumerate multiplicities of matrices $\mathbf{C}$. This is based on the recursive relation

$$\mathbf{C}_\tau = \mathbf{C}_{\tau-1} + \boldsymbol{\chi}_\tau, \quad \tau = 1, 2, \dots \quad (5.30)$$

where $\boldsymbol{\chi}_\tau = [c_\tau^1 - \hat{c}_\tau^1, \dots, c_\tau^{n_r} - \hat{c}_\tau^{n_r}]^H [c_\tau^1 - \hat{c}_\tau^1, \dots, c_\tau^{n_r} - \hat{c}_\tau^{n_r}]$. Weights can be enumerated by using the trellis-search algorithm of Aktas and Fitz [82], even though the objective there was to compute the distance spectrum alone. In this section, we follow a more traditional coding-theory approach to give two new methods of weight enumeration. Each of the three methods has its advantage and disadvantage and may be suited to specific applications.

A. State Reduction

Due to the lack of geometric uniformity of space-time codes in general, weight enumeration requires a product-state diagram of N_s^2 states for an N_s -state code. Since N_s^2 can be large even for code with a moderate trellis size, state-reduction techniques are usually necessary. The reduction can be significant for codes with high symmetry. For geometrically uniform codes such as those constructed in [71], state reduction is unnecessary; since the error probability is irrelevant to the reference codeword, the N_s -state encoder state-diagram suffices.

A product state is defined as the pair encoder-decoder states (i_1, i_2) for $1 \leq i_1, i_2 \leq N_s$. It is labeled as *good* if $i_1 = i_2$ and *bad* otherwise. A first error event is a path through the product-state diagram, stemming from a good state and terminating at a good state once and only once. For convenience suppose that there are no parallel transitions in the code trellis and a trellis stage spans only one symbol period, but the results easily extend to parallel transitions or multiple symbol periods. An edge in the product-state diagram connecting the (j_1) th state and the (j_2) th state is labeled by $1/2^k \cdot W^{\chi(j_1 \rightarrow j_2)}$, where k is the number of source bit per trellis stage, W is a dummy variable, and the time index τ is omitted because of time invariance of the trellis. It is understood that the exponent is taken entry-wise in $W^{\chi(j_1 \rightarrow j_2)}$. The factor $1/2^k$ accounts for the average over reference codewords. An N_s^2 -by- N_s^2 state-transition matrix $\mathbf{M}$ is defined in which the (j_1, j_2) th entry is either given by the corresponding label or set null if no transition exists. $\mathbf{M}$ has the partitioned form

$$\mathbf{M} = \begin{bmatrix} \times & \mathbf{d} \\ \mathbf{b} & \mathbf{P} \end{bmatrix}$$

where $\mathbf{d}$, $\mathbf{b}$, and $\mathbf{P}$ is a diverging, merging and parallel section, respectively, and $\times$ denotes an irrelevant matrix for our purpose. The *transfer function* is, in essence, a complete list of correlation matrices $\mathbf{C}$ and their multiplicities, given by

$$T(W) \triangleq \sum_{\mathbf{C}} A_{\mathbf{C}} W^{\mathbf{C}} = \mathbf{p} \mathbf{d} \sum_{\tau=0}^{\infty} \mathbf{P}^{\tau} \mathbf{b} \mathbf{1} = \mathbf{p} \mathbf{d} \left(\mathbf{I}_{N_s^2 - N_s} - \mathbf{P} \right)^{-1} \mathbf{b} \mathbf{1} \quad (5.31)$$

where $\mathbf{1}$ is the N_s -by-1 vector of all ones, and $\mathbf{p} = \mathbf{1}^T / N_s$. We assume here the infinite series converges. For the bit error probability, we simply add another exponent i , which stands for the number of bit errors of the branch.

Aktas and Fitz [82] have applied the finite-state machine (FSM) theory-based state-reduction technique of Schlegel [87] to space-time codes. Compared to other methods, this technique is more general. In this technique, states are grouped into equivalent classes, where the equivalence is defined in the forward direction. This is the case in classic FSM theory. Nonetheless, Shi and Wesel recently showed that it can be extended to the backward direction when it comes to weight enumeration in coding theory [88]. The reason is that the definition of equivalence is relaxed, as there is no need for two equivalent FSM's to produce identical responses for each input here. In coding theory, two FSM's are seen as equivalent if they have identical weight enumerators.

In the context of trellis codes, Shi and Wesel further proved that all good states can be combined into one state by defining $\hat{\mathbf{d}} = \mathbf{p} \mathbf{d}$ and $\hat{\mathbf{b}} = \mathbf{b} \mathbf{1}$; the number of bad states can be

reduced by half for linear codes [88]. We discover that the reduction by half is always possible for all trellis codes. This is based on the following observation. Consider a transition $(j_1, j_2) \rightarrow (i_1, i_2)$ for $j_1 \neq j_2$ and $i_1 \neq i_2$. Due to the structure of a product-state diagram, there must also exist a mirror-image transition $(j_2, j_1) \rightarrow (i_2, i_1)$. Since the correlation matrix χ is apparently the same for the two transitions, product states (j_1, j_2) and (j_2, j_1) must be forward equivalent. This observation is meaningful, as many space-time codes are nonlinear. Hence, the reduced product-state diagram contains no more than $(N_s^2 - N_s)/2 + 1$ states for any space-time trellis code.

Then forward and backward-equivalence are checked iteratively to arrive at a minimal product-state diagram with N_R states. In this fashion it is possible to obtain a smaller product-state diagram than [82]. In each forward (backward)-equivalent class, we strike all but one row (column) of $\mathbf{M}$, and sum the corresponding columns (rows). Denote by $\mathbf{d}_R$, $\mathbf{b}_R$, and $\mathbf{P}_R$ the diverging, merging, and parallel sections in the reduced transition matrix $\mathbf{M}_R$, respectively. The transfer function may be rewritten as

$$T(W) = \mathbf{d}_R \sum_{\tau=0}^{\infty} \mathbf{P}_R^{\tau} \mathbf{b}_R = \mathbf{d}_R \left(\mathbf{I}_{N_R-1} - \mathbf{P}_R \right)^{-1} \mathbf{b}_R \quad (5.32)$$

B. Transfer-Function Bound

The transfer function can be used to derive a closed-form expression of the conditional first-event error probability $P_E(\alpha)$. In a more convenient way, we modify the exponent of W as $\text{tr}(\alpha \chi \alpha^H)$. This enables us to take advantage of the recursion

$$\text{tr}(\alpha \mathbf{C}_{\tau} \alpha^H) = \text{tr}(\alpha \mathbf{C}_{\tau-1} \alpha^H) + \text{tr}(\alpha \chi_{\tau} \alpha^H) \quad (5.33)$$

analogous to (5.30). Then we have

$$\begin{aligned} P_E(\alpha) &= \frac{1}{\pi} \int_0^{\pi/2} \sum_{\mathbf{C}} A_{\mathbf{C}} \exp \left[-\frac{\gamma_s}{4 \sin^2 \theta} \text{tr}(\alpha \mathbf{C} \alpha^H) \right] d\theta \\ &= \frac{1}{\pi} \int_0^{\pi/2} T(W) \Big|_{W=\exp\left(-\frac{\gamma_s}{4 \sin^2 \theta}\right)} d\theta \\ &= \frac{1}{\pi} \int_0^{\pi/2} \mathbf{d}_R \left(\mathbf{I}_{N_R-1} - \mathbf{P}_R \right)^{-1} \mathbf{b}_R \Big|_{W=\exp\left(-\frac{\gamma_s}{4 \sin^2 \theta}\right)} d\theta. \end{aligned} \quad (5.34)$$

This can be combined with the Malkamäki-Leib technique to give a bound. It is a *true* upper bound in a strict mathematic sense, for the entire weight spectrum is covered. Interestingly, $T(W)$ is a univariate function in our approach. Other methods in literature, e.g., Stefanov and Duman [72] or Caire and Colavolpe [74], require multivariate weight

enumerators. The series of $T(W)$ will converge if the largest eigenvalue of $\mathbf{P}_R$ is less than one. This brings no difficulty, since we may bound $LP_E(\alpha)$ by unity anyway, once an eigenvalue greater than or equal to one is detected.

There is no need to perform the above matrix inversion symbolically. Numerical inversion at each SNR is much more convenient. The inversion of this N_R -by- N_R matrix is the main computational burden ($O(N_R^3)$, in the order of N_R^3) of this approach. Unfortunately, it appears that this approach is inapplicable to Gallager bounds or the standard union bound.

C. Truncated Bound

When N_R is large, the matrix inversion is a sizable task. When Gallager bounds are desired, we need symbolical matrix inversion to obtain the eigenvalue spectrum. This can be hopelessly complicated even for codes with moderate-size trellis. Then we usually need to truncate the weight spectrum at a level I as $T_I(W) = \mathbf{d}_R \sum_{\tau=0}^I \mathbf{P}_R^\tau \mathbf{b}_R$. Since polynomial matrix computation of $\mathbf{P}_R^\tau$ for large values of τ is overwhelming, we propose a new recursive method to accomplish the computation:

$$\hat{T}_0 = \mathbf{d}_R, \quad \hat{T}_\tau = \mathbf{d}_R + \hat{T}_{\tau-1} \mathbf{P}_R. \quad (5.35)$$

The final weight spectrum is given by $T_I(W) = \hat{T}_I \mathbf{b}_R$. In actual implementation, we update multiplicities of the exponents of W step by step, which will give multiplicities of $\mathbf{C}$ when the recursion converges. Since $\mathbf{C}$ is Hermitian, we need multi-dimensional arrays $A_{W_1, W_2, \dots, W_m}^j$ ($j = 1, \dots, N_R - 1$), where $m = (n_T^2 + n_T)/2 + 1$ for real signaling and $m = n_T^2 + n_T + 1$ for complex signaling, respectively. The implementation will be demonstrated in later numerical examples. This method is analogous to Divsalar's calculation for convolutional codes [89].

To keep storage memory from growing infinitely, we impose another truncation $|W_i| \leq H$, where H is a parameter to trade off accuracy against memory complexity. Due to the truncation, the weight enumerator converges fast, and a value I of five to ten times of the code memory is enough. Accordingly, only this type of truncation has effects on accuracy, though we have introduced two types of truncation. The overall memory complexity of the algorithm is roughly $N_R H^{(n_T^2 + n_T)/2 + 1}$ for real signaling and $N_R H^{n_T^2 + n_T + 1}$ for complex signaling, respectively. Further reduction is possible by employing certain properties of signals. For instance, the values of W_i 's must be multiples of four for BPSK codes.

Since no matrix inversion is involved, this method works for large-trellis codes. However, it has a limitation that the memory complexity increases exponentially with $(n_T^2 + n_T)/2 + 1$ or $n_T^2 + n_T + 1$. Therefore, its application is restricted to small values of n_T .

D. Trellis Search

A third method of weight enumeration has been proposed by Aktas and Fitz [82], which searches the reduced trellis for N_D leading terms of the distance spectrum. In the search process, three lists need to be stored: a list of N_D terminated first error events with full-rank correlation matrices, a list of unterminated first error events with full-rank correlation matrices, and a list of unterminated first error events with rank-deficient correlation matrices. It is hard to estimate the memory complexity of this algorithm, for the latter two lists grow dynamically and may be unbounded. But one thing is clear: the memory complexity is $O(n_T^2)$ with respect to n_T . The computational complexity is hard to be estimated, too. It appears to grow exponentially at the initial stages of the algorithm, because every first error event has to be examined. It grows as $O(n_T^3)$ with respect to n_T , which is the computational complexity of matrix rank and determinants.

Unlike the regular realization structure in the second method, the lists need to be managed delicately in the searching process. At each step, the rank and determinant of the signal-difference correlation matrix need to be checked. Sometimes the algorithm does not converge [82]. On the other hand, this algorithm has an advantage that it is amicable to large values of n_T .

In TABLE III, we summarize the applicability, memory and computational complexity of weight enumeration techniques.

TABLE III
WEIGHT ENUMERATION TECHNIQUES

	Bounds	Memory	Computation
Transfer function	P_f^{ML}	$O(N_R^2)$	$O(N_R^3)$
Truncated transfer function	All	$O(N_R H^{(n_T^2 + n_T)/2 + 1})$ or $O(N_R H^{n_T^2 + n_T + 1})$	$O(IN_R H^{(n_T^2 + n_T)/2 + 1})$ or $O(IN_R H^{n_T^2 + n_T + 1})$
Trellis search	All	$O(N_R n_T^2) \times$ Unknown	$O(N_R n_T^3) \times$ Unknown exponential

E. Implication of Gallager Bounds on Uniformity

Geometric uniformity requires the existence of an isometry for any two signal points that leaves the signal set invariant [71]. It is a sufficient, but not necessary condition for the property that performance bounds are independent of the transmitted codeword. In other words, there exist codes that are not geometrically uniform, but are uniform with respect to performance bounds. In fact, a code is uniform with respect to a certain performance bound as long as every codeword has the same set of weights, eigenvalues, or distances. The condition for code uniformity obviously varies with the bounding technique adopted. In such cases, an N_S -state diagram is sufficient to evaluate the bound under consideration. An in-depth treatment of uniformity may be beyond the scope of this chapter. In example 1 of the next section, we shall see an example of non-geometrically uniform codes that are uniform with respect to the spherical bound.

5.5 Numerical Examples and Discussion

Equipped with a set of bounding techniques and a set of weight enumeration methods, we are ready to compute the performance bounds of space-time codes. Following the convention, we plot frame error rates against $n_T \gamma_s$, the symbol SNR per receive antenna, for $L = 130$ in the following figures.

Example 1 ((1, 3) Code): The purpose of this example is to demonstrate the convergence of the truncated bound and uniformity. Consider the two-antenna, rate-1, 2-state (1, 3) BPSK code with a natural spatial parser directing coded bits to antennas, which is the same code used in [72, Figs. 2-4]. The code trellis is shown in Fig. 28(a) along with its product-state trellis in Fig. 28(b). Since the codeword-difference correlation matrix is Hermitian, only the (1, 1)th, (1, 2)th, and (2, 2)th entries are displayed as the exponents of W .

Fig. 28. (a) Trellis of the (1, 3) code; (b) Product trellis of the 2-state BPSK space-time code.

The state-transition matrix of this code is given by

$$\mathbf{M} = \frac{1}{2} \begin{bmatrix} 1 & 1 & W^{(0,0,4)} & W^{(0,0,4)} \\ 1 & 1 & W^{(0,0,4)} & W^{(0,0,4)} \\ W^{(4,4,4)} & W^{(4,-4,4)} & W^{(4,0,0)} & W^{(4,0,0)} \\ W^{(4,4,4)} & W^{(4,-4,4)} & W^{(4,0,0)} & W^{(4,0,0)} \end{bmatrix}.$$

The first step of state reduction already indicates that the reduced trellis contains no more than $(N_s^2 - N_s)/2 + 1 = 2$ states. Accordingly, the reduced state-transition matrix is given by

$$\mathbf{M}_R = \begin{bmatrix} 1 & \frac{1}{2}W^{(0,0,4)} \\ W^{(4,4,4)} + W^{(4,-4,4)} & W^{(4,0,0)} \end{bmatrix}.$$

With this matrix the transfer function bound on the conditional error probability is easily seen as

$$P_E(\alpha) = \frac{1}{2\pi} \int_0^{\pi/2} W^{\text{tr}(\alpha C_1 \alpha^H)} \left(1 - W^{\text{tr}(\alpha C_2 \alpha^H)}\right)^{-1} \left[W^{\text{tr}(\alpha C_3 \alpha^H)} + W^{\text{tr}(\alpha C_4 \alpha^H)} \right] \Big|_{W=\exp\left(-\frac{\gamma_s}{4\sin^2 \theta}\right)} d\theta \quad (5.36)$$

where

$$\mathbf{C}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, \mathbf{C}_2 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{C}_3 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}, \mathbf{C}_4 = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}.$$

This transfer function bound is easy to compute, and it always converges.

The computation of the truncated weight spectrum is implemented in an efficient numerical method. Noting that $T(W)$ for this code is in fact a trivariate function, we define a three-dimensional array A_{W_1, W_2, W_3}^1 to store multiplicities of given tuples (W_1, W_2, W_3) . The initial condition is given by

$$A_{W_1, W_2, W_3}^1(0) = \begin{cases} 1/2, & \text{if } W_1 = 0, W_2 = 0, W_3 = 4; \\ 0, & \text{otherwise.} \end{cases}$$

Then the iteration is given by

$$A_{W_1, W_2, W_3}^1(t) = A_{W_1-4, W_2, W_3}^1(t-1) + A_{W_1, W_2, W_3}^1(0) \quad (5.37)$$

for $1 \leq t < I$. The final result

$$A_{W_1, W_2, W_3}^1(I) = A_{W_1-4, W_2-4, W_3-4}^1(I-1) + A_{W_1-4, W_2+4, W_3-4}^1(I-1) \quad (5.38)$$

is a list of weights and their multiplicities.

Fig. 29 depicts simulated frame error rates and the Malkamäki-Leib bound with different weight enumeration techniques. It clearly exhibits two distinguished features. One is that

all the bounds are surprisingly tight compared to simulated results. The other is that truncated bounds converge to the transfer-function bound very rapidly. It is seen that truncated bounds with increasing values of H almost coincide, and are extremely close to the transfer function bound.

Fig. 29. Malkamäki-Leib bounds versus simulation results for the 2-state BPSK code for $n_T = 2$, $n_R = 2$ and $L = 130$.

There exist two noticeable differences between the bounds herein and [72]. The analytic bounds in [72] are about 3 dB away from simulation results for all codes examined therein. We believe this discrepancy is because the number 4 in Fig. 28(b) is mistaken for 2 in [72]. Under the signal model of this chapter as well as [72], the squared Euclidean distance between symbols 1 and -1 is 4 rather than 2. The tightness of our bounds agrees with des Rosiers and Siegel [73]. Moreover, our bounds appear to converge faster than [72]. It might be attributed to the inclusion of the factor $1/2^k$ in state-transition equations, or to the recursive implementation of weight enumeration.

The first error events of this code are in the form of 11...10. It is easy to see from Fig. 28(b) that the codeword-difference correlation matrix for a first error event of i ($i = 1, 2, \dots$) ones followed by a zero is given by

$$\mathbf{C} = \begin{bmatrix} 4i & 4 \\ 4 & 8 \end{bmatrix} \text{ or } \begin{bmatrix} 4i & -4 \\ -4 & 8 \end{bmatrix}, \quad i = 1, 2, \dots,$$

each of which has multiplicity 0.5. Obviously, the code is not uniform with respect to the

Malkamäki-Leib bound or the ellipsoidal bound. However, the two matrices have the same eigenvalues (and determinants). Thus the code is uniform with respect to the spherical bound or the standard union bound, and a 2-state code trellis suffices to compute the two bounds.

The naive union bound for this code is given by

$$P_f^U = \frac{1}{16} \sum_{i=1}^{\infty} \frac{1}{2i-1},$$

which clearly diverges to infinity. $\square$

Example 2 ((5,7) Code): The purpose of this example is to demonstrate the advantage of our state reduction and weight enumeration. Consider the two-antenna, rate-1, BPSK space-time code using the (5, 7) code in [82, Tables II, V]. The original diagram has 16 states. The sought-after reduced transition matrix is given by

$$\mathbf{M}_R = \begin{bmatrix} 1 & W^{(4,4,4)} + W^{(4,-4,4)} & 0 & 0 \\ 0 & 0 & W^{(0,0,4)} / 2 & W^{(4,0,0)} \\ W^{(4,4,4)} + W^{(4,-4,4)} & 2 & 0 & 0 \\ 0 & 0 & W^{(4,0,0)} / 2 & W^{(0,0,4)} \end{bmatrix}. \quad (5.39)$$

Again, the (1, 1)th, (1, 2)th, and (2, 2)th entries of the codeword-difference correlation matrix are shown. The reduced product-state diagram has 4 states, which are less than 6 states in the reduced diagram of [82].

Define three-dimensional arrays A_{W_1, W_2, W_3}^j ($j = 1, \dots, N_R - 1$) to store multiplicities of given tuples (W_1, W_2, W_3) . The initial condition is given by

$$A_{W_1, W_2, W_3}^j(0) = \begin{cases} 1, & \text{if } j = 1 \text{ and } W_1 = 4, W_2 = \pm 4, W_3 = 4; \\ 0, & \text{otherwise.} \end{cases}$$

Then state-transition equations are given by

$$\begin{aligned} A_{W_1, W_2, W_3}^1(t) &= 2A_{W_1, W_2, W_3}^2(t-1) + A_{W_1, W_2, W_3}^1(0) \\ A_{W_1, W_2, W_3}^2(t) &= \frac{1}{2}A_{W_1, W_2, W_3-4}^1(t-1) + \frac{1}{2}A_{W_1-4, W_2, W_3}^3(t-1) \\ A_{W_1, W_2, W_3}^3(t) &= A_{W_1-4, W_2, W_3}^1(t-1) + A_{W_1, W_2, W_3-4}^3(t-1) \end{aligned} \quad (5.40)$$

for $1 \leq t < I$. The final result is

$$A_{W_1, W_2, W_3}(I) = A_{W_1-4, W_2-4, W_3-4}^2(I-1) + A_{W_1-4, W_2+4, W_3-4}^2(I-1) \quad (5.41)$$

Thereafter, the eigenvalue and distance spectra are readily computable.

Fig. 30 shows performance bounds and simulation results for this code and the 2-state code. Once again, all bounds are rather tight. For these two codes, it appears that the ellipsoidal bound brings little advantage over the spherical bound. However, there exist

codes for which the advantage may be noticeable, as will be seen next. □

Fig. 30. Performance bounds versus simulation results for the 2 and 4-state BPSK codes for $n_T = 2$, $n_R = 1$ and $L = 130$.

Fig. 31. Performance bounds versus simulation results for the TSC 4 and 16-state QPSK codes for $n_T = 2$, $n_R = 1$ and $L = 130$.

In Fig. 31, we provide performance bounds versus simulation results for the 4 and 16-state rate-2 QPSK codes of Tarokh, Seshadri and Calderbank (TSC) [71]. These codes are geometrically uniform. Trellis search is a simple matter. The set $\{\mathbf{C}, A_C\}$ corresponding to three leading terms of the distance spectrum is used to calculate the bounds. We note that the ellipsoidal bound is 0.5-dB tighter than the spherical bound for the 16-state code and is near optimum. For the 4-state code, the three bounds cannot be distinguished. The reason is as follows. The first three terms in the distance spectrum of this code are $\{A_D = 2, 4, 1\}$ and $\{D^2 = 4, 12, 16\}$. Its performance is dominated by the correlation matrix $\mathbf{C} = \text{diag}(2, 2)$ with multiplicity $A_C = 2$. Obviously, the ellipsoidal bound is near optimum (for one correlation matrix), and spherical bound is also near optimum (for one diagonal correlation matrix with equal entries). Meanwhile, the first three terms of the 16-state code are $\{A_D = 1, 2, 5\}$ and $\{D^2 = 12, 20, 28\}$. It is found that $\mathbf{C} = \text{diag}(2, 6)$ for the error event with $D^2 = 12$. Thus the spherical bound is not supposed to be optimum.

We also compute P_f^{BPL} when plotting the performance bounds. We observe $P_f^{\text{BPL}} < P_f^{\text{ML}}$ for most of the codes and the contrary for others. For the TSC 4-state code, P_f^{BPL} and P_f^{ML} are very close to each other. Again, this is because $\mathbf{C} = \text{diag}(2, 2)$ dominates the performance, and Byun et al's proof applies to a single correlation matrix. This helps to explain why P_f^{BPL} appears to be a valid upper bound in [77], where only this code was examined. However, P_f^{BPL} is found lower than simulation points for some other codes.

The rapid convergence of the truncated bounds motivates us to reexamine the role of so-called “dominant error events”. A somewhat accepted conception in literature drawn from the conventional union bound is that there are *no* dominant error events in quasi-static fading. As a result, the rank-determinant criterion is considered to be established on “shaky” foundation. However, the behavior of the truncated bound clearly indicates *the existence of dominant error events in quasi-static fading*. It is the weakness of the standard union bound that masks their dominance. We give a qualitative explanation of the behavior by referring to the spherical bound (5.21). Roughly speaking, the spherical bound decreases exponentially, rather than polynomially as in the union bound, with the eigenvalues of $\mathbf{C}$ at high SNR. Since the eigenvalues are proportional to H , rapid convergence with respect to H is not surprising. Because the minimum-determinant term belongs to dominant error events, the foundation of the rank-determinant criterion is not that “shaky”, although still suboptimum. This property may help to reduce the stack size

N_D in trellis search, for the leading terms are enough to capture the code performance. Similar convergent behavior of the Malkamäki-Leib bound and dominant error events were also observed by des Rosiers and Siegel [73]¹².

5.6 Correlated Fading

In this section, we extend the performance bounds to spatially correlated fading. The fading processes are still assumed to be quasi-static in time. Note that the Malkamäki-Leib bound already admits correlated fading.

Let $\underline{\alpha}^H \triangleq \text{vec}(\alpha^H)$, where the operator $\text{vec}(\alpha^H)$ denotes stacking columns of α^H , left to right. The pdf of $\underline{\alpha}$ is given by

$$f_{\underline{\alpha}}(\underline{\alpha}) = \frac{1}{\pi |\underline{\Sigma}|} \exp(-\underline{\alpha} \underline{\Sigma}^{-1} \underline{\alpha}^H)$$

where $\underline{\Sigma} = E[\underline{\alpha}^H \cdot \underline{\alpha}]$ is the spatial correlation matrix. Likewise, define the ellipsoidal Gallager region as $\mathcal{R}^E = \{\underline{\alpha} : \text{tr}(\underline{\alpha} \underline{F} \underline{\alpha}^H) \leq 1\}$ where $\underline{F}$ is an $n_R n_T$ -by- $n_R n_T$ positive semi-definite matrix. The squared Euclidean distance (4) can be written into a quadratic form

$$d^2(c, \hat{c} | \alpha) = \underline{\alpha} [\underline{I}_{n_R} \otimes \underline{C}] \underline{\alpha}^H$$

where $\otimes$ stands for the Kronecker product. It is easy to show that the ellipsoidal bound is given by

$$P_f^E = \min_{\underline{F} \geq 0} \left\{ -L \sum_c A_c \frac{1}{\pi} \int_0^{\pi/2} \left| \underline{I}_{n_R n_T} + \frac{\gamma_s \underline{\Sigma} (\underline{I}_{n_R} \otimes \underline{C})}{4 \sin^2 \theta} \right|^{-1} \sum_{k=1}^{K_1} \text{Res} \left[s^{-1} e^s \left| \underline{I}_{n_R n_T} + s \left(\underline{\Sigma}^{-1} + \frac{\gamma_s \underline{I}_{n_R} \otimes \underline{C}}{4 \sin^2 \theta} \right)^{-1} \underline{F} \right|^{-1}, s_k^1 < 0 \right] d\theta \right. \\ \left. + 1 + \sum_{k=1}^{K_2} \text{Res} \left[s^{-1} e^s \left| \underline{I}_{n_R n_T} + s \underline{\Sigma} \underline{F} \right|^{-1}, s_k^2 < 0 \right] \right\} \quad (5.42)$$

where $K_1 \leq n_R n_T$ and $K_2 \leq n_R n_T$ are the number of distinct negative poles.

In correlated fading channels, the standard union bound requires knowledge of the eigenvalues of $\underline{\Sigma} (\underline{I}_{n_R} \otimes \underline{C})$ [81]. We would like to derive a spherical bound that keeps this property. To achieve this goal, we choose a Gallager region $\mathcal{R}^S = \{\underline{\alpha} : \text{tr}(\underline{\alpha} \underline{\Sigma}^{-1} \underline{\alpha}^H) \leq r^2\}$, which corresponds to $\underline{F} = \underline{\Sigma}^{-1} r^{-2}$. It can be verified that the property is fulfilled by substituting $\underline{F} = \underline{\Sigma}^{-1} r^{-2}$ into (5.42). We give the expression of the spherical bound, omitting the details, as

¹² It is interesting that the results of [73] are fully correct, but many of the other papers contain more or less misconceptions or mistakes.

$$P_f^S = -L \sum_{\lambda} A_{\lambda} \frac{1}{\pi} \int_0^{\pi/2} \sum_{k=1}^K \text{Res} \left[\frac{e^{sr^2}}{s} \prod_{i=1}^{n_R n_T} \left(s + 1 + \frac{\bar{\lambda}_i \gamma_s}{4 \sin^2 \theta} \right)^{-1}, s_k < 0 \right] d\theta + 1 - e^{-r^2} \sum_{n=0}^{n_R n_T} \frac{r^{2n}}{n!} \quad (5.43)$$

where $\bar{\lambda}_i$'s are eigenvalues of $\Sigma(\mathbf{I}_{n_R} \otimes \mathbf{C})$ and $K \leq n_R n_T$ is the number of distinct negative poles. Note that, strictly speaking, $\mathcal{R}^S = \{\underline{\alpha} : \text{tr}(\underline{\alpha} \Sigma^{-1} \underline{\alpha}^H) \leq r^2\}$ is not a sphere geometrically.

5.7 Conclusions

We have proposed expeditious bounding techniques for space-time codes in the general framework of Gallager bounds, which reduce the computational burden of the “limit-before-average” technique dramatically. The proposed bounds are flexible in terms of tightness-complexity tradeoffs. They are simple to use if the weight spectrum or eigenvalue spectrum is known. In our experience of plotting performance curves with MATLAB on a PC with 2.4 GHz CPU, the ellipsoidal bound can be computed in minutes and the spherical bound in seconds, but one need at least hours to compute the Malkamäki-Leib bound. It is noteworthy that the computation cost of the proposed bounds is independent of the SNR, while the Malkamäki-Leib bound is not if implemented in Monte Carlo. Along with a bundle of weight enumeration methods, this chapter provides a thorough analytic treatment of the performance of space-time codes. For the first time, systematic code search on the basis of practical, meaningful performance bounds becomes possible.

In the Malkamäki-Leib bound, basically what one needs to do is to derive error probabilities for an equivalent AWGN channel. Classical analytic tools for fading channels play no role. When deriving Gallager bounds, we have been able to make use of the classical characteristic-function approach. This bridge between Gallager bounds and an abundance of classical results on correlated fading [79], [65] has far-reaching implications. We believe it opens up the door to tight performance bounds in other scenarios such as other types of fading and noncoherent schemes. Indeed, this is pursued in the next chapter.

Chapter 6

Performance Bounds for Noncoherent Systems

6.1 Introduction

In Chapter 4, we have seen that the conventional union bound for noncoherent sequence detection is quite loose at low to medium SNR's. In fact, the looseness of the union bound is a ubiquitous phenomenon in sequence and (long-)block decoders if fading is not fast enough to give sufficient diversity. The weakness of the union bound for noncoherent systems is most obvious for the example of MSDD over slow fading channels. Here, the number of error events is exponential in the observation length, while the diversity order is nearly constant. As a result, it is often observed that the union bound increases with the observation length, which is contradictory to the fact that increasing the observation length can only improve the performance!

In this chapter, we develop performance bounds for noncoherent MIMO systems. Since a noncoherent detector usually appears in the form of a quadratic receiver, the performance bounds will be developed generally for quadratic decoders. Then it will be specialized for noncoherent detectors. Obviously, the performance bound for coherent space-time codes in Chapter 5 can be put into the framework here. The numerical computation of the performance bounds for quadratic decoders is however more challenging, since two-dimensional characteristic functions will be involved. Consequently, we often need resorting to the Chernoff-bound technique, which will inevitably weaken the bounds.

6.2 Gallager Bounds for Quadratic Decoders

A. Union Bound

Let $\mathbf{y}$ and $\boldsymbol{\alpha}$ denote the (column) vectors of received signals and fading coefficients, respectively. A quadratic receiver is usually defined as

$$\hat{i} = \arg \min_i \mathbf{x}^H \mathbf{K}_i \mathbf{x} + c_i \quad (6.1)$$

where $\mathbf{K}_i$ is a Hermitian matrix corresponding to the i th transmitted signal, c_i is a real constant, and $\mathbf{x}$ is a complex Gaussian (column) vector depending on $\mathbf{y}$. The pairwise error probability (PEP) is given by

$$P(i \rightarrow j) = P(\mathbf{x}^H \mathbf{K}_{ij} \mathbf{x} < c_{ij}) \quad (6.2)$$

where $\mathbf{K}_{ij} = \mathbf{K}_j - \mathbf{K}_i$ and $c_{ij} = c_i - c_j$. For convenience, we first consider geometrically uniform signals, for which the zeroth signal can be assumed the transmitted signal without loss of generality. The standard union bound on the block error probability of a block decoder is

$$P_B \leq \sum_{j \neq 0} P(0 \rightarrow j) = \sum_{j \neq 0} P(\mathbf{x}^H \mathbf{K}_{0j} \mathbf{x} < c_{0j}). \quad (6.3)$$

The calculation of PEP is well documented in literature [65, 66]. There are many available ways of the calculation, including exact evaluation, Chernoff bounds and asymptotically tight bounds [80, 81]. Unfortunately, the union bound is rather loose in the context of coding.

B. Limit-Before-Average (LBA) Bound

A tight upper bound can be obtained by using Gallager's technique. The idea is the same as that for coherent space-time codes: we simply overbound the error probability by unity if the channel experiences deep fading; otherwise we use the union bound. Similarly, the optimum Gallager region is given by

$$R^O = \left\{ \alpha \left| \sum_{j \neq 0} P(\mathbf{x}^H \mathbf{K}_{0j} \mathbf{x} < c_{0j} | \alpha) = 1 \right. \right\}, \quad (6.4)$$

and the LBA technique gives an upper bound on the block error probability

$$P_B \leq \int_{\alpha} \min \left(1, \sum_{j \neq 0} P(\mathbf{x}^H \mathbf{K}_{0j} \mathbf{x} < c_{0j} | \alpha) \right) f_{\alpha}(\alpha) d\alpha. \quad (6.5)$$

Other than the multi-dimensional numerical integral involved, there is another fold of computational burden in (6.5). Unlike the Q function for coherent systems, the conditional error probability in (6.5) does not admit a closed-form expression. Therefore, one must perform numerical integration. It may be easy to perform an integration for a given value of α , but it is difficult to do integrations automatically for many values of α . This is because the parameters of the numerical algorithm need be adjusted with α , and the results easily become inaccurate.

One way out of this difficulty is to use a Chernoff bound on the conditional PEP. This, of course, will sacrifice some tightness. Let $\bar{\mathbf{x}} = E[\mathbf{x} | \alpha]$ be the mean of $\mathbf{x}$ conditioned on the fading coefficients. Since we assume white Gaussian noise superimposed on the signal, the correlation matrix given α is simply $\mathbf{R}_{\mathbf{x}|\alpha} = E[(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})^H] = \sigma^2 \mathbf{I}$. Invoking Stein's

classic results [65], we can put the characteristic function of $\mathbf{x}^H \mathbf{K}_{0j} \mathbf{x}$ for a given $\boldsymbol{\alpha}$ as

$$\Phi_{0j}(s) = \frac{\exp\{-s\bar{\mathbf{x}}^H (\mathbf{K}_{0j}^{-1} + s\sigma^2 \mathbf{I})^{-1} \bar{\mathbf{x}}\}}{|\mathbf{I} + \sigma^2 s \mathbf{K}_{0j}|}. \quad (6.6)$$

According to the Chernoff bound, the conditional PEP is upper-bounded as

$$\begin{aligned} P(\mathbf{x}^H \mathbf{K}_{0j} \mathbf{x} < c_{0j} | \boldsymbol{\alpha}) &\leq \min_{s>0} \left\{ e^{sc_{0j}} \Phi_{0j}(s) \right\} \\ &= \min_{s>0} \frac{\exp\{sc_{0j} - s\bar{\mathbf{x}}^H (\mathbf{K}_{0j}^{-1} + s\sigma^2 \mathbf{I})^{-1} \bar{\mathbf{x}}\}}{|\mathbf{I} + \sigma^2 s \mathbf{K}_{0j}|} \end{aligned} \quad (6.7)$$

where s is real such that $\mathbf{I} + \sigma^2 s \mathbf{K}_{0j}$ is positive definite. Then we may substitute this into (6.5) and employ the Monte Carlo method to evaluate the multi-integral.

C. Ellipsoidal Bound

An expeditious method is the ellipsoidal bound, which is given by

$$\begin{aligned} P_B &\leq P(\text{error}, \boldsymbol{\alpha}^H \mathbf{F} \boldsymbol{\alpha} \geq 1) + P(\boldsymbol{\alpha}^H \mathbf{F} \boldsymbol{\alpha} < 1) \\ &\leq \sum_{j \neq 0} P(\mathbf{x}^H \mathbf{K}_{0j} \mathbf{x} < c_{0j}, \boldsymbol{\alpha}^H \mathbf{F} \boldsymbol{\alpha} \geq 1) + P(\boldsymbol{\alpha}^H \mathbf{F} \boldsymbol{\alpha} < 1). \end{aligned} \quad (6.8)$$

The determination of the conditional PEP hinges on the bivariate characteristic function.

Let $z_1 = \mathbf{x}^H \mathbf{K}_{0j} \mathbf{x}$, and $z_2 = \boldsymbol{\alpha}^H \mathbf{F} \boldsymbol{\alpha}$. Their joint characteristic function is defined as

$$\Phi(s_1, s_2) = E \left[e^{-(s_1 z_1 + s_2 z_2)} \right]. \quad (6.9)$$

To derive the characteristic function, we write $s_1 z_1 + s_2 z_2$ into quadratic form

$$s_1 z_1 + s_2 z_2 = \begin{pmatrix} \mathbf{x}^H & \boldsymbol{\alpha}^H \end{pmatrix} \begin{pmatrix} s_1 \mathbf{K}_{0j} & \mathbf{0} \\ \mathbf{0} & s_2 \mathbf{F} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\alpha} \end{pmatrix}. \quad (6.10)$$

Note that generally this quadratic form is not Hermitian, since s_1 and s_2 are complex. Thus the results in [65, Appendix B] do not apply, and we need to exploit the properties of circular Gaussian random processes described by Helstrom [66, Appendix B]. Suppose that the vector $(\mathbf{x}, \boldsymbol{\alpha})$ has the correlation matrix

$$\mathbf{Q} = E \left[\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\alpha} \end{pmatrix} \begin{pmatrix} \mathbf{x}^H & \boldsymbol{\alpha}^H \end{pmatrix} \right]. \quad (6.11)$$

Then, by definition the joint characteristic function is given by

$$\Phi(s_1, s_2) = \frac{1}{\pi^n |\mathbf{Q}|} \iint \exp \left\{ - \begin{pmatrix} \mathbf{x}^H & \boldsymbol{\alpha}^H \end{pmatrix} \left[\begin{pmatrix} s_1 \mathbf{K}_{0j} & \mathbf{0} \\ \mathbf{0} & s_2 \mathbf{F} \end{pmatrix} + \mathbf{Q}^{-1} \right] \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\alpha} \end{pmatrix} \right\} d\mathbf{x} d\boldsymbol{\alpha} \quad (6.12)$$

Had s_1 and s_2 been real, the integrand would be recognized as the pdf of a Gaussian random vector. When this is not the case, Helstrom showed that still a formally similar

result exists:

$$\begin{aligned}\Phi(s_1, s_2) &= |\mathbf{Q}|^{-1} \left| \begin{pmatrix} s_1 \mathbf{K}_{0j} & \mathbf{0} \\ \mathbf{0} & s_2 \mathbf{F} \end{pmatrix} + \mathbf{Q}^{-1} \right|^{-1} \\ &= \left| \mathbf{I} + \begin{pmatrix} s_1 \mathbf{K}_{0j} & \mathbf{0} \\ \mathbf{0} & s_2 \mathbf{F} \end{pmatrix} \mathbf{Q} \right|^{-1}.\end{aligned}\quad (6.13)$$

It must be stressed that the multi-integral in (6.12) only converges under the condition that

$$\begin{pmatrix} s_1 \mathbf{K}_{0j} & \mathbf{0} \\ \mathbf{0} & s_2 \mathbf{F} \end{pmatrix} + \mathbf{Q}^{-1} \geq 0, \quad (6.14)$$

i.e., the above matrix is positive definite.

Aided by the characteristic function, we are able to express the conditional PEP as

$$\begin{aligned}P(z_1 < c_{0j}, z_2 \geq 1) &= \int_{-\infty}^{c_{0j}} dz_1 \int_1^{\infty} dz_2 f_{z_1, z_2}(z_1, z_2) \\ &= \int_{-\infty}^{c_{0j}} dz_1 \int_1^{\infty} dz_2 \frac{1}{(2\pi j)^2} \iint \Phi(s_1, s_2) e^{s_1 z_1 + s_2 z_2} ds_1 ds_2.\end{aligned}\quad (6.15)$$

By exchanging the order of integration, we have

$$P(z_1 < c_{0j}, z_2 \geq 1) = \frac{1}{4\pi^2} \iint \frac{e^{s_1 c_{0j} + s_2}}{s_1 s_2} \Phi(s_1, s_2) ds_1 ds_2, \quad \text{Re}(s_1) > 0, \text{Re}(s_2) < 0 \quad (6.16)$$

where the integration contour must be carefully chosen so as to assure convergence. In principle, this double integral can be numerically computed by using a bivariate version of Helstrom's saddle point integration, recently given in [90]. However, we are once again facing the formidable task of automatically computing the integral for every possible value of α . The adaptation of the involved parameters is difficult, if not impossible.

Accordingly, we resort to the Chernoff upper bound at the expense of some looseness. In fact, this is exactly the same approach pursued by Gallager in the very beginning of his bounding technique [5]. Gallager considered a memoryless channel and derived a simple expression. However, literature survey indicates that there are surprisingly few results on the bivariate Chernoff bound for a general setting. For our problem, we overbound (6.15) by

$$\begin{aligned}P(z_1 < c_{0j}, z_2 \geq 1) &\leq \int_{-\infty}^{c_{0j}} dz_1 \int_1^{\infty} dz_2 e^{-[s_1(z_1 - c_{0j}) + s_2(z_2 - 1)]} f_{z_1, z_2}(z_1, z_2) \\ &\leq \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 e^{-[s_1(z_1 - c_{0j}) + s_2(z_2 - 1)]} f_{z_1, z_2}(z_1, z_2) \\ &= e^{s_1 c_{0j} + s_2} \Phi(s_1, s_2)\end{aligned}\quad (6.17)$$

where $s_1 > 0$ and $s_2 < 0$ are real numbers such that the double integral converges. This amounts to the condition that

$$\begin{pmatrix} s_1 \mathbf{K}_{0j} & \mathbf{0} \\ \mathbf{0} & s_2 \mathbf{F} \end{pmatrix} + \mathbf{Q}^{-1} \geq 0 \quad (6.18)$$

for $s_1 > 0$ and $s_2 < 0$ real. Hence, the bivariate Chernoff bound is given by

$$P(z_1 < c_{0j}, z_2 \geq 1) \leq \min_{s_1 > 0, s_2 < 0} e^{s_1 c_{0j} + s_2} \Phi(s_1, s_2) \quad (6.19)$$

subject to the constraint (6.18).

The overall ellipsoidal bound can be expressed as

$$\begin{aligned} P_B &\leq \min_{\mathbf{F} \geq 0} \{P(\text{error}, \mathbf{a}^H \mathbf{F} \mathbf{a} \geq 1) + P(\mathbf{a}^H \mathbf{F} \mathbf{a} < 1)\} \\ &\leq \min_{\mathbf{F} \geq 0} \left\{ \sum_{j \neq 0} \min_{s_1 > 0, s_2 < 0} e^{s_1 c_{0j} + s_2} \Phi_j(s_1, s_2) + P(\mathbf{a}^H \mathbf{F} \mathbf{a} < 1) \right\} \end{aligned} \quad (6.20)$$

where the subscript of the characteristic function shows its dependence on j . The Chernoff bound is individually minimized for each error event. We emphasize that the constrained minimization in (6.20) is however not a trivial task. As a result, it is usually performed numerically.

When BER is our concern, the ellipsoidal bound has the form

$$P_b \leq \min_{\mathbf{F} \geq 0} \left\{ \sum_{j \neq 0} \frac{k_j}{N_b} \min_{s_1 > 0, s_2 < 0} e^{s_1 c_{0j} + s_2} \Phi_j(s_1, s_2) + \frac{1}{2} P(\mathbf{a}^H \mathbf{F} \mathbf{a} < 1) \right\} \quad (6.21)$$

where N_b denotes the length in bits of the information sequence, and k_j is the number of bit differences.

D. Gallager's Second Bound

Using Gallager's second bound (see Appendix D), we can overbound (6.5) as

$$P_B \leq \left\{ \int \psi^{1-1/\rho}(\mathbf{a}) \cdot \sum_{j \neq 0} P(\mathbf{x}^H \mathbf{K}_{0j} \mathbf{x} < c_{0j} | \mathbf{a}) \cdot f_a^{1/\rho}(\mathbf{a}) d\mathbf{a} \right\}^\rho \quad (6.22)$$

where $\psi(\mathbf{a})$ is an arbitrary pdf, and $0 \leq \rho \leq 1$. Suppose that the vector $\mathbf{a}$ has length l and pdf

$$f_a(\mathbf{a}) = \frac{1}{\pi^l |\mathbf{R}_a|} e^{-\mathbf{a}^H \mathbf{R}_a^{-1} \mathbf{a}}.$$

For simplicity, we just choose a complex Gaussian pdf for $\psi(\mathbf{a})$:

$$\psi(\mathbf{a}) = \frac{1}{\pi^l |\mathbf{F}|} e^{-\mathbf{a}^H \mathbf{F}^{-1} \mathbf{a}}, \quad \mathbf{F} \geq \mathbf{0}.$$

In addition, suppose that $\bar{\mathbf{x}}$ admits the form $\bar{\mathbf{x}} = S_j \mathbf{a}$, where S_j is a matrix determined by the j th transmitted signal. By further invoking the Chernoff bound (6.7), we have

$$\begin{aligned}
 P_B &\leq \left\{ \sum_{j \neq 0} \min_{s>0} \int \frac{\exp\{s c_{0j} - s \mathbf{a}^H S_j^H (\mathbf{I} + s \sigma^2 \mathbf{K}_{0j})^{-1} \mathbf{K}_{0j} S_j \mathbf{a}\}}{|\mathbf{I} + \sigma^2 s \mathbf{K}_{0j}|} \psi^{1-1/\rho}(\mathbf{a}) f_a^{1/\rho}(\mathbf{a}) d\mathbf{a} \right\}^\rho \\
 &= \left\{ \sum_{j \neq 0} \min_{s>0} \frac{e^{s c_{0j}}}{|\mathbf{I} + \sigma^2 s \mathbf{K}_{0j}|} \int \frac{e^{-\mathbf{a}^H \left[s S_j^H (\mathbf{I} + s \sigma^2 \mathbf{K}_{0j})^{-1} \mathbf{K}_{0j} S_j + \left(1 - \frac{1}{\rho}\right) \mathbf{F}^{-1} + \frac{1}{\rho} \mathbf{R}_a^{-1}\right] \mathbf{a}}}}{(\pi^l |\mathbf{F}|)^{1-1/\rho} (\pi^l |\mathbf{R}_a|)^{1/\rho}} d\mathbf{a} \right\}^\rho \quad (6.23) \\
 &= \left\{ \sum_{j \neq 0} \min_{s>0} \frac{e^{s c_{0j}} |\mathbf{I} + \sigma^2 s \mathbf{K}_{0j}|^{-1} |\mathbf{F}|^{-1+1/\rho} |\mathbf{R}_a|^{-1/\rho}}{\left| s S_j^H (\mathbf{I} + s \sigma^2 \mathbf{K}_{0j})^{-1} \mathbf{K}_{0j} S_j + \left(1 - \frac{1}{\rho}\right) \mathbf{F}^{-1} + \frac{1}{\rho} \mathbf{R}_a^{-1} \right|} \right\}^\rho
 \end{aligned}$$

where the integral converges if the matrix $s S_j^H (\mathbf{I} + s \sigma^2 \mathbf{K}_{0j})^{-1} \mathbf{K}_{0j} S_j + (1 - 1/\rho) \mathbf{F}^{-1} + \mathbf{R}_a^{-1}/\rho$ is positive definite. The upper bound is to be minimized over the above constraints.

Similarly, the bit error probability is upper-bounded by

$$P_b \leq \left(\frac{1}{2} \right)^{1-\rho} \left\{ \sum_{j \neq 0} \frac{k_j}{N_b} \min_{s>0} \frac{e^{s c_{0j}} |\mathbf{I} + \sigma^2 s \mathbf{K}_{0j}|^{-1} |\mathbf{F}|^{-1+1/\rho} |\mathbf{R}_a|^{-1/\rho}}{\left| s S_j^H (\mathbf{I} + s \sigma^2 \mathbf{K}_{0j})^{-1} \mathbf{K}_{0j} S_j + \left(1 - \frac{1}{\rho}\right) \mathbf{F}^{-1} + \frac{1}{\rho} \mathbf{R}_a^{-1} \right|} \right\}^\rho \quad (6.24)$$

subject to the same constraints.

6.3 Case Study

A. MSDD of DPSK

To illustrate the tightness of the bound, let us consider the performance of MSDD of MDPSK over time-selective fading channels. If the block length is N in symbols, $M^N - 1$ error events have to be taken into account in the union bound. As mentioned earlier, this gives rise to a very loose bound, even if some error events can be expurgated according to the concept of “simple error events”. In the extreme case of quasi-static fading¹³, the union bound is completely misleading, as it diverges whereas the actual performance improves with increasing N . To overcome this difficulty, a common theme in literature is to restrict the union bound over the so-called “dominant error events”. There are $2N$ such

¹³ Of course, there is no need of using MSDD in quasi-static fading. Our aim here is to take advantage of its simplicity to illustrate the power of the Gallager bound.

error events for $N > 2$. This method was first proposed by Divsalar and Simon [58] for AWGN channels, and was applied by Ho and Fung [59] to fading channels later on. We also followed this method in the analysis of MSDD for DSTM in Chapter 4. Clearly this method only yields an approximation. In many cases it is not necessarily an upper bound. The accuracy of the approximation is hard to know a priori in fading channels, even if it is quite acceptable in AWGN channels since it is asymptotically tight. Therefore, it is theoretically interesting to see the behavior of the Gallager bound for MSDD in fading channels.

The received signal of DPSK is given by

$$x_n = \alpha_n b_n + w_n \quad (6.25)$$

where α_n is the fading coefficient, b_n is the DPSK symbol, and w_n is the noise. Put in vector form, the signal model is $\mathbf{x} = \boldsymbol{\alpha} \circ \mathbf{b} + \mathbf{w}$. In MSDD, a block of symbols for $n = 1, \dots, N$ are simultaneously detected. MSDD minimizes the following metric [59]

$$\mathcal{M}(\hat{\mathbf{b}}) = \mathbf{x}^H \mathbf{K}_{\hat{\mathbf{b}}} \mathbf{x} = \mathbf{x}^H \left(\text{diag}(\hat{\mathbf{b}})(\mathbf{R} + \sigma^2 \mathbf{I}) \text{diag}(\hat{\mathbf{b}}^H) \right)^{-1} \mathbf{x} \quad (6.26)$$

where both vectors have length N , the Hermitian matrix $\mathbf{K}_{\hat{\mathbf{b}}}$ is implicitly defined, $\mathbf{R}$ is the N -by- N autocorrelation matrix of the fading process, and the constant in (6.1) is equal to zero for DPSK signaling.

Since MDPSK is geometrically uniform under MSDD, $\mathbf{b}$ can be assumed to be the all-one vector. Consequently, the conditional mean of $\mathbf{x}$ is $\bar{\mathbf{x}} = \boldsymbol{\alpha} \circ \mathbf{1} = \boldsymbol{\alpha}$, and

$$\mathbf{K}_{1,\hat{\mathbf{b}}} = \text{diag}(\hat{\mathbf{b}})(\mathbf{R} + \sigma^2 \mathbf{I})^{-1} \text{diag}(\hat{\mathbf{b}}^H) - (\mathbf{R} + \sigma^2 \mathbf{I})^{-1}. \quad (6.27)$$

The Chernoff bound on the conditional PEP is thus

$$P(\mathbf{x}^H \mathbf{K}_{1,\hat{\mathbf{b}}} \mathbf{x} < 0 \mid \boldsymbol{\alpha}) \leq \min_{s>0} \frac{\exp \left\{ -s \boldsymbol{\alpha}^H (\mathbf{I} + s \sigma^2 \mathbf{K}_{1,\hat{\mathbf{b}}})^{-1} \mathbf{K}_{1,\hat{\mathbf{b}}} \boldsymbol{\alpha} \right\}}{\left| \mathbf{I} + \sigma^2 s \mathbf{K}_{1,\hat{\mathbf{b}}} \right|}, \quad (6.28)$$

for s real and $\mathbf{I} + \sigma^2 s \mathbf{K}_{1,\hat{\mathbf{b}}}$ positive definite. The final form of the LBA bound on BER is given by

$$P_b \leq \int_{\boldsymbol{\alpha}} \min \left(\frac{1}{2}, \sum_{\mathbf{b} \neq \mathbf{1}} \frac{k_{1,\hat{\mathbf{b}}}}{(N-1) \log_2 M} P(\mathbf{x}^H \mathbf{K}_{0j} \mathbf{x} < c_{0j} \mid \boldsymbol{\alpha}) \right) f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) d\boldsymbol{\alpha}. \quad (6.29)$$

Some numerical results are shown in the following figures. Fig. 32 shows the performance bounds for DQPSK with $N = 3, 4$, and 5 in a quasi-static fading channel. The union-Chernoff bound clearly increases with N , which is misleading. It in fact diverges to infinity at every SNR for increasing N . In stark contrast, the limit-before-average bound converges and reflects the truth of better performance for larger N (although it is hard to

tell the improvement from the curves.) It is seen that the limit-before-average bound is about 5 dB above the simulation results. This 5 dB gap is partially due to the Chernoff technique we employ in (6.28). In this special case of quasi-static fading, we are able to see how much the Chernoff technique weakens the bound, for numerical instability is not severe here in the computation of the exact conditional PEP. Fig. 33 compares the two performance bounds with and without Chernoff bounding for DQPSK with $N = 4$. We can see that the LBA bound with exact conditional PEP is 3 dB above simulation results, and the Chernoff technique roughly weakens by 2 dB. The “dominant error events” approximation with exact calculation of the PEP of $2L_s$ error events is included in the figure as well for comparison. It happens to be above the simulation results.

Fig. 32. Performance bounds for MSDD of DQPSK with $N = 3, 4$, and 5 in a quasi-static fading channel.

We show in Fig. 34 the performance bounds in a fast fading channel with $f_d T = 0.01$. The trend is the same as that in Fig. 32. It is worth mentioning that if one has a numerically robust method to compute the exact conditional PEP, the LBA bound will be tighter. The case of $f_d T = 0.03$ is shown in Fig. 35. Here things are a little bit different. Since the fading is relatively fast, the union bound appears to converge at high SNR.

Fig. 33. Performance bounds for MSDD of DQPSK with $N = 4$ in a quasi-static fading channel.

Fig. 34. Performance bounds for MSDD of DQPSK with $N = 3, 4$, and 5 in a fading channel with $f_d T = 0.01$.

Fig. 35. Performance bounds for MSDD of DQPSK with $N = 3, 4$, and 5 in a fading channel with $f_d T = 0.03$.

The Monte Carlo method is lengthy especially for low error probabilities when one generate the performance curves. To derive the ellipsoidal bound, first note that the correlation matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{R} + \sigma^2 \mathbf{I} & \mathbf{R} \\ \mathbf{R} & \mathbf{R} \end{bmatrix}. \quad (6.30)$$

Second, it is reasonable to assume that the fading coefficients do not change much during the block for typical fading rate, e.g., $f_d T = 0.01$. Thereupon, a spherical bound is expected to be sufficient, which corresponds to $\mathbf{F} = \mathbf{I}/r^2$. Then the characteristic function is given by

$$\Phi(s_1, s_2) = \left| \mathbf{I} + \begin{pmatrix} s_1 \mathbf{K}_{1,\hat{b}} & \mathbf{0} \\ \mathbf{0} & s_2 \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{R} + \sigma^2 \mathbf{I} & \mathbf{R} \\ \mathbf{R} & \mathbf{R} \end{pmatrix} \right|^{-1}. \quad (6.31)$$

and the spherical bound is

$$P_b \leq \min_r \left\{ \sum_{\hat{b} \neq 1} \frac{k_{1,\hat{b}}}{(N-1) \log_2 M} \min_{s_1 > 0, s_2 < 0} e^{s_2 r^2} \Phi_{\hat{b}}(s_1, s_2) + \frac{1}{2} P(\mathbf{a}^H \mathbf{a} < r^2) \right\}. \quad (6.32)$$

Fig. 36 illustrates the spherical bound in a quasi-static fading channel. The spherical bound is further looser than the LBA-Chernoff bound by 2.5 dB, due to the bivariate Chernoff technique we employ. Nevertheless, the curves are on top of each other, which

clearly shows that the performance improves with increasing L_S .

Fig. 36. Performance bounds for MSDD of DQPSK with $N = 3, 4$, and 5 in a quasi-static fading channel.

B. NSD of DSTM

Now we are ready to refine the bounds for NSD of DSTM over time-selective fading channels (see Chapter 4). The derivation is formally the same as that for MSDD of DPSK, except for two small differences: 1) the constant c_{ij} in (6.2) is nonzero, and 2) the signals are not geometrically uniform. Let $|\mathcal{N}|$ denote the number of codewords i , and w_{ij} denote the Hamming distance (in bits) between codewords i and j . The LBA bound on BER is given by

$$P_b \leq \int_{\mathbf{a}} \min \left(\frac{1}{2}, \frac{1}{|\mathcal{N}|} \sum_{i=1}^{|\mathcal{N}|} \sum_{j \neq i} \frac{w_{ij}}{Rn_T} P(i \rightarrow j | \mathbf{a}) \right) f_{\mathbf{a}}(\mathbf{a}) d\mathbf{a} \quad (6.33)$$

where the inner sum is carried over first error events at a time epoch. To maintain the notational coherence in this chapter, we rewrite the metric difference $\gamma = \bar{\mathbf{Y}}_K[n] \mathbf{F} \bar{\mathbf{Y}}_K^H[n] - c < 0$ in (4.20) for $n_R = 1$ as $\mathbf{x}^H \mathbf{K}_{ij} \mathbf{x} < c_{ij}$. Then the conditional PEP for $n_R > 1$ is given by

$$P(i \rightarrow j < c_{ij} | \mathbf{a}) \leq \min_{s>0} \frac{\exp \{ s c_{ij} - s n_R \bar{\mathbf{x}}^H (\mathbf{I} + s \sigma^2 \mathbf{K}_{ij})^{-1} \mathbf{K}_{ij} \bar{\mathbf{x}} \}}{|\mathbf{I} + \sigma^2 s \mathbf{K}_{ij}|^{n_R}} \quad (6.34)$$

for s real and $\mathbf{I} + \sigma^2 s \mathbf{K}_{ij}$ positive definite. Here we have the following correspondence with the notations of Chapter 4:

$$\begin{aligned} c_{ij} &\Leftrightarrow c \\ \mathbf{a} &\Leftrightarrow \bar{\mathcal{H}}_K[n] \triangleq [\mathcal{H}[n-(K-1)], \dots, \mathcal{H}[n]], \quad K = D + 2N - 3 \\ \mathbf{K}_{ij} &\Leftrightarrow \mathbf{F} = \sum_{i=0}^{D+N-3} \text{diag} \begin{bmatrix} \mathbf{0}_{n_T} & \mathbf{Q}_{\hat{\mathbf{a}},N}[n+i] - \mathbf{Q}_{\mathbf{a},N}[n+i] & \mathbf{0}_{(K-N-i)n_T} \end{bmatrix} \\ \bar{\mathbf{x}} &= \bar{\mathcal{H}}_K[n] \bar{\mathcal{S}}_K[n], \quad \bar{\mathcal{S}}_K[n] \triangleq \text{diag}[\mathcal{S}[n-(K-1)], \dots, \mathcal{S}[n]]. \end{aligned}$$

Fig. 37. LBA-Chernoff bounds for NSD with $N = 4$ of two-antenna DSTM with BPSK signaling (Alamouti's code) for $f_d T = 0.01$ and $n_R = 1$.

We examine the behavior of the LBA-Chernoff bound for NSD of two-antenna DSTM for $f_d T = 0.01$, for which the union bound was previously shown in Fig. 21. For fair comparison, the Chernoff version of the union bound is depicted. It is seen that the LBA bound converges, which contrasts to the divergence of the union bound at low SNR. The LBA bound almost saturates as D goes beyond 2.

6.4 Conclusions

We have extended Gallager bounds to noncoherent MIMO systems. The derived bounds are in fact applicable to any quadratic decoders over fading channels. For example, we can apply the ellipsoidal bound to coherent space-time codes in time-selective fading.

Due to the numerical issues, we usually make use of the Chernoff version of the LBA

and ellipsoidal bounds. This weakens the bounds, but nonetheless keep the convergence property. Therefore, the bounds of this chapter are more meaningful than the traditional union bound.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

In this dissertation, we have addressed the decoding and performance aspects of MIMO coding theory. Although these are more on the receiver side, the results are also useful to signal design on the transmitter side. Both coherent and noncoherent systems were investigated. In particular, we have selected the two topics of differential decoding (Part I) and performance bounds (Part II) for MIMO.

Differential lattice decoding (DLD) presented in Chapter 3 has low complexity and near-optimum performance. This is an improvement over Clarkson et al.'s work [49]. Note that we only gave DLD for diagonal constellations, but the extension to some nondiagonal constellations is possible (cf. [50], [51]). This is because nondiagonal group constellations usually contain large diagonal subgroups. We may simply apply DLD with each subgroup and use a naive approach across subgroups.

Noncoherent sequence detection (NSD) presented in Chapter 4 has overcome the limitation of the linear predictive receivers [52], [61]. It takes into account the matrix signaling structure of DSTM so that it can achieve near-optimum performance for nondiagonal constellations over continuously fading channels.

In Part II we developed tight performance bounds for MIMO systems. The starting point is the LBA bound of Stefanov and Duman [72], which was originally proposed by Malkamäki and Leib [75]. In Chapter 5 we employed Gallager's bounding techniques to derive efficient bounds for coherent space-time codes. Our bounds are in closed form so that they can be evaluated fast. An extension to noncoherent systems was also attempted in Chapter 6, and the problem was partially solved at least. A distinguished feature of the new bounds is that they converge as more error events are added in. Accordingly, the new bounds are more meaningful than the conventional union bound.

7.2 Future Work

The research of this dissertation points out several directions for future work.

Optimality and other applications of DLD. The empirical evidence of the identical outcomes of MLD and augmented DLD makes us believe that there exists a lower bound

on the value of α for the initial radius $R = \alpha d_s$, above which augmented DLD is strictly optimum. However, we are currently unable to provide a proof. Clearly, such a lower bound, if existent, would depend on the geometric relation of $\mathcal{V}^{\text{ML}}$ and $\mathcal{V}^{\text{euc}}$. Moreover, an extension of DLD to MSDD for DSTM is of great interest; the major difficulty will be the ordering of (matrix) constellation points, since the lattice formulation is not exact.

Coherent lattice decoding for finite constellations. We only considered differential decoding in Chapter 3, where an infinite lattice is formulated. In contrast, we always have finite lattices in coherent MIMO decoding. There will be boundary errors in lattice decoding if the finiteness is ignored. So far there is no efficient way of boundary control in a reduced lattice. It remains an open problem as to how boundary errors should be controlled in universal lattice decoding.

Low-complexity NSD. The reduced-state NSD only overcomes the complexity growth with respect to the observation length N . Its complexity is still exponential in n_T and R . It would be very interesting to put NSD in perspective of lattice decoding or sequential decoding. That is, the tree-search method is likely to play a role.

Fig. 38. Behaviors of bounds as D_{th} increases for TSC QPSK codes, $n_T = 2$, $n_R = 1$, $L = 130$ at receive symbol SNR of 23 dB.

Code search under the criterion of Gallager bounds. The rank/determinant criterion is employed in early search for space-time codes [22]. Clearly, an upper bound counting in

the distance (weight) spectrum will provide more complete information about the performance. The traditional union bound does not necessarily lead to the truly optimum code. This is illustrated as follows. Truncation of the bound is usually needed when one calculates it. Fig. 38 shows the bounds as functions of the threshold D_{th} where the distance spectrum is truncated. The union bound appears to increase linearly. Ultimately it predicts that the 4-state code has better performance than the 16-state code! This is obviously a paradox. The average distance is suggested in [23], [24] as a performance measure, which is essentially equivalent to the union bound.

In contrast, since the Gallager bounds are efficient and tight, they can predict the performance of codes more accurately. As shown in Fig. 38, the ellipsoidal bound converges as more components of the weight spectrum are accounted. It gives the correct comparison of the two codes. Therefore, if we use Gallager bounds as the search criterion, probably better codes can be found.

Gallager bounds for ergodic MIMO channels. In this dissertation, we derived performance bounds in quasi-static or slowly fading channels (fast fading with, *e.g.*, $f_d T = 0.03$ considered in this dissertation, is still slow fading in comparison with ideally interleaved fading.) It is an exciting topic to derive Gallager bounds for turbo-coded spatial multiplexing systems over fully interleaved channels [56], where the obvious “turbo cliff” hints that the union bound is not supposed to be tight.

Tighter bounds for noncoherent decoders. Since the noncoherent bounds are not as tight, this appears to be a promising line of future work. Addressing the computational difficulty associated with the bounds could lead to tighter bounds, particularly if the use of the Chernoff bound could somehow be eliminated¹⁴.

Last but not the least, the similar geometric regions appearing in lattice decoding and Gallager bounds may warrant a little thinking. The region in lattice decoding can be a box, ellipsoid or sphere; so is the Gallager region. Is this interrelated or just coincidental?

¹⁴ Following the comments of an anonymous examiner.

Appendix

A. Importance Sampling

The idea of importance sampling is to select a biased pdf $f_a^*(\alpha)$ so that quicker convergence of the estimator is obtained. Note that

$$\begin{aligned} P_f^{\text{ML}} &= E[\min\{1, LP_E(\alpha)\}] \\ &= E_*[\min\{1, LP_E(\alpha)\} \cdot f_a(\alpha) / f_a^*(\alpha)] \end{aligned} \quad (8.1)$$

where E_* denotes the expectation with respect to the biased pdf. Since $P_E(\alpha)$ is very small for most realizations of α at high SNR, much time is wasted in the Monte Carlo method. Intuitively, a simple choice of the biased pdf is just a Rayleigh one with smaller variance. The variance is properly determined to minimize the estimator variance.

B. Cubical Bound

The cubical bound applies to 1D convolutional codes over block-fading channels. Note that since the integration is performed individually for each component, repeated eigenvalues are as just easy to handle as single eigenvalues. We omit the much easier derivation and present the final form.

Theorem 3 (Cubical bound for convolutional codes): Let M be the number of blocks, each of equal length L , and A_d for $\mathbf{d} = (d_1, d_2, \dots, d_M)$ denote the multi-component distance spectrum, then the frame error rate of a n_R -fold diversity decoder can be upper-bounded by

$$P_f^C = L \sum_d A_d \frac{1}{\pi} \int_0^{\pi/2} \left[\prod_{i=1}^M \left(1 + \frac{d_i \gamma_s}{4 \sin^2 \theta} \right)^{-n_R} \right] \exp \left[-n_R \sum_{i=1}^M \left(1 + \frac{d_i \gamma_s}{4 \sin^2 \theta} \right) \xi_0 \right] d\theta + (1 - e^{-\xi_0})^{n_R} \quad (8.2)$$

where ξ_0 is the root of the following equation

$$L \sum_d A_d \frac{1}{\pi} \int_0^{\pi/2} \frac{\sum_{i=1}^M \left(1 + \frac{d_i \gamma_s}{4 \sin^2 \theta} \right)}{\prod_{i=1}^M \left(1 + \frac{d_i \gamma_s}{4 \sin^2 \theta} \right)^{n_R}} \exp \left[-n_R \sum_{i=1}^M \left(1 + \frac{d_i \gamma_s}{4 \sin^2 \theta} \right) \xi_0 \right] d\theta = \frac{M (1 - e^{-\xi_0})^{n_R M - 1}}{e^{\xi_0}}. \quad (8.3)$$

It appears that the optimization for ξ_0 depends on the SNR.

We brief the application of the cubical bound to coded $n \times n$ BLAST with ZF/DFE processing. To analyze the *frame error probability*, it is sufficient to consider the

equivalent channel model [10]

$$\mathbf{y}_i = \sqrt{\frac{\gamma_s}{n}} g_i \mathbf{x}_i + \mathbf{w}_i, \quad \text{for } i = 1, 2, \dots, n \quad (8.4)$$

for V-BLAST, and

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} = \sqrt{\frac{\gamma_s}{n}} \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} + \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_n \end{bmatrix} \quad (8.5)$$

for D-BLAST, where $\mathbf{x}_i$, $\mathbf{y}_i$, and $\mathbf{w}_i$ are the transmitted, received signals and the noise for the i th stream, and $g_i^2 \sim \chi_{2i}^2$ is the SNR at the output of the i th decorrelator. The difference between (8.4) and (8.5) is that the transmitted signals $\mathbf{x}_i$'s of D-BLAST belong to the same substream and we can apply an outer code over them; in contrast, the $\mathbf{x}_i$'s of V-BLAST belong to independent substreams. Obviously, coded BLAST can be put into the framework of convolutional codes over block-fading channels, thereby admitting the cubical bound analysis.

C. Gallager Bounds: Historical Notes

Historically, Gallager's two bounding techniques play a central role in the field of tight performance bounds. The Gallager bounds are instances of techniques of overcoming the weakness of the union bound. Performance analysis is tantamount to "performance bound" in the coding area, because there is no efficient method to compute $P\left(\bigcup_{j=1}^M A_j\right)$ exactly for large M . It is easiest to use the union upper bound

$$P\left(\bigcup_{j=1}^M A_j\right) \leq \sum_{j=1}^M P(A_j), \quad (8.6)$$

but this does not give a sharp bound for many problems.

In essence, every new bound must improve the union bound in some way. There are such results in probability theory, which are generally called Bonferroni-type bounds [91]. Bonferroni showed that the series

$$P\left(\bigcup_{j=1}^M A_j\right) = \sum_{j=1}^M (-1)^{j+1} S_j, \quad S_k = \sum P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) \quad (8.7)$$

is an alternating sum. It means that we will obtain an upper bound if we truncate at S_1 ; this is the conventional union bound. We will have an improved upper bound

$$P\left(\bigcup_{j=1}^M A_j\right) \leq S_1 - S_2 + S_3 \quad (8.8)$$

if we truncate at S_3 . A somewhat more elaborated bound that involves only second-order intersections is given by

$$P\left(\bigcup_{j=1}^M A_j\right) \leq S_1 - \sum_{(i,j) \in T_0} P(A_i \cap A_j) \quad (8.9)$$

where T_0 denotes a minimum spanning tree in which A_i 's are represented as nodes. Simple truncation at S_2 gives a lower bound. Lower bounds are however of less interest to the performance of a code.

Unfortunately, Bonferroni-type inequalities are not effective in tightening the bound despite its complicity. Furthermore, it is not convenient to compute the high-order quantities in (8.7)-(8.9), since this requires more knowledge of the code than the pairwise error probability and weight spectrum.

The pairwise error probability and weight spectrum are central concepts of coding theory. In essence, the two concepts are associated with the union bound. Good trellis codes are often found by computer search under the criterion of the weight spectrum. Other measures like the high-order quantities in Bonferroni inequalities will very much complicate performance analysis and code search. A salient feature of the Gallager bounds is that only the knowledge of the weight spectrum is assumed. This is because he did not really invent a new bound on the probability of a union, but applied the union bound in a clever way. Therefore, a lesson learned from Gallager is that we are supposed to *use the union bound wisely*, rather than to look for a different probabilistic bound.

Gallager's 1960 PhD dissertation contains at least two fundamental contributions [5]. One is the LDPC code as the title reads, which is now widely recognized as the earliest Shannon-limit code. The other, which is less well-known, is a new bounding technique. Many bounding techniques appearing later can be thought of as its special cases. In a subsequent 1965 paper on the proof of the coding theorem, he proposed another bounding technique for random codes. It is until 1998 that Duman and Salehi extended this bound to explicit codes [92]. The underlying ideas of Gallager's two bounds are surprisingly simple, although the final bounds may take complicated form. Gallager's first bounding technique can be sketched as

$$\begin{aligned} P(\text{error}) &= P(\text{error, bad state}) + P(\text{error, good state}) \\ &\leq P(\text{bad state}) + P(\text{error, good state}). \end{aligned} \quad (8.10)$$

The conventional union bound then is applied at the good state. If the channel is at the bad state, the union bound is not supposed to be tight; therefore, it is simply upper-bounded by unity. So essentially it uses nothing other than the trivial fact that the probability is never greater than unity! Gallager's second bounding technique takes

advantage of the inequality

$$P(\cup A_i) \leq \left[\sum P(A_i) \right]^\rho, \quad 0 \leq \rho \leq 1 \quad (8.11)$$

which is sometimes called the *generalized union bound*. Something more surprising is that these two are essentially the only improved bounding techniques available in literature up to now! Shamaï and Sason gave a comprehensive overview of the variations of Gallager bounds and connections [17]. They won the 2003 Joint IT/ComSoc Paper Award for this paper. An earlier survey of Gallager bounds was presented by Divsalar [89].

D. Gallager's Second Bound for Space-Time Codes

The starting point of our derivation is again the limit-before-average (LBA) bound $P_f \leq \int \min[1, LP_E(\alpha)] f_\alpha(\alpha) d\alpha$. Using the inequality (or the generalized union bound (8.11))

$$\min[1, LP_E(\alpha)] \leq [LP_E(\alpha)]^\rho, \quad 0 \leq \rho \leq 1 \quad (8.12)$$

we have

$$P_f \leq \int [LP_E(\alpha)]^\rho f_\alpha(\alpha) d\alpha. \quad (8.13)$$

Let $\psi(\alpha)$ be an arbitrary pdf. We rewrite the above bound as

$$\begin{aligned} P_f &\leq \int \psi(\alpha) \psi^{-1}(\alpha) [LP_E(\alpha)]^\rho f_\alpha(\alpha) d\alpha \\ &= \int \psi(\alpha) \left\{ \psi^{-1/\rho}(\alpha) [LP_E(\alpha)] f_\alpha^{1/\rho}(\alpha) \right\}^\rho d\alpha. \end{aligned} \quad (8.14)$$

Invoking the Jensen inequality, we have

$$P_f \leq \left\{ \int \psi^{1-1/\rho}(\alpha) [LP_E(\alpha)] f_\alpha^{1/\rho}(\alpha) d\alpha \right\}^\rho. \quad (8.15)$$

For simplicity, we choose a complex Gaussian pdf with zero mean

$$\psi(\alpha) = \frac{1}{\pi^{n_T} |\mathbf{F}|} e^{-\text{tr}(\alpha \mathbf{F}^{-1} \alpha^H)}, \quad \mathbf{F} \geq \mathbf{0}. \quad (8.16)$$

Substituting this pdf into (8.15), we have

$$\begin{aligned}
 P_f &\leq \left\{ \int L \sum_c A_c \frac{1}{\pi} \int_0^{\pi/2} e^{-\frac{\gamma_s}{4\sin^2\theta} \text{tr}(aCa^H)} d\theta \frac{1}{\pi^{n_T} |\mathbf{F}|^{1-1/\rho}} e^{-\text{tr}(a[(1-\frac{1}{\rho})\mathbf{F}^{-1} + \frac{1}{\rho}\mathbf{I}]a^H)} d\mathbf{a} \right\}^\rho \\
 &= \left\{ \int L \sum_c A_c \frac{1}{\pi} \int_0^{\pi/2} d\theta \frac{1}{\pi^{n_T} |\mathbf{F}|^{1-1/\rho}} e^{-\text{tr}\left(-a\left[\frac{\gamma_s \mathbf{C}}{4\sin^2\theta} + (1-\frac{1}{\rho})\mathbf{F}^{-1} + \frac{1}{\rho}\mathbf{I}\right]a^H\right)} d\mathbf{a} \right\}^\rho \\
 &= \left\{ L \sum_c A_c \frac{1}{\pi} \int_0^{\pi/2} |\mathbf{F}|^{1/\rho} \left| \frac{\gamma_s \mathbf{C}\mathbf{F}}{4\sin^2\theta} + (1-\frac{1}{\rho})\mathbf{I} + \frac{1}{\rho}\mathbf{F} \right|^{-1} d\theta \right\}^\rho.
 \end{aligned} \tag{8.17}$$

The bound has to be minimized over the following constraints

$$\begin{aligned}
 0 &\leq \rho \leq 1, \quad \mathbf{F} \geq \mathbf{0}, \\
 \frac{\gamma_s \mathbf{C}}{4\sin^2\theta} + (1-\frac{1}{\rho})\mathbf{F}^{-1} + \frac{1}{\rho}\mathbf{I} &\geq \mathbf{0}.
 \end{aligned} \tag{8.18}$$

A numerical example of the TSC 16-state code is shown in Fig. 39, where only the first three terms of the distance spectrum are considered. This bound is 2 dB looser than the ellipsoidal bound, but has the advantage that it need not compute the residues.

Fig. 39. Performance bounds for the TSC 16-state QPSK code for $n_T = 2$, $n_R = 1$ and $L = 130$.

To derive Gallager's second bound for the bit error probability, we also starts from the LBA bound

$$P_b \leq \int \min \left[\frac{1}{2}, \sum_{c,i} \frac{i}{k} A_{c,i} Q \left(\sqrt{\frac{\gamma_s}{2} \text{tr}(\mathbf{a} \mathbf{C} \mathbf{a}^H)} \right) \right] f_a(\mathbf{a}) d\mathbf{a}. \quad (8.19)$$

Aided by the inequality

$$\min\{a, b\} \leq a^\rho b^{1-\rho}, \text{ for } 0 \leq \rho \leq 1, a, b \geq 0, \quad (8.20)$$

we obtain

$$P_b \leq \int_{\mathcal{R}} \left(\frac{1}{2} \right)^{1-\rho} \left[\sum_{c,i} \frac{i}{k} A_{c,i} Q \left(\sqrt{\frac{\gamma_s}{2} \text{tr}(\mathbf{a} \mathbf{C} \mathbf{a}^H)} \right) \right]^\rho f_a(\mathbf{a}) d\mathbf{a}. \quad (8.21)$$

The factor $(1/2)^{1-\rho}$ is an improvement over Gallager's generalized union bound (8.11).

Proceeding as for the frame error probability, we arrive at the final form

$$P_b \leq \left(\frac{1}{2} \right)^{1-\rho} \left\{ \sum_{c,i} \frac{i}{k} A_{c,i} \frac{1}{\pi} \int_0^{\pi/2} |\mathbf{F}|^{1/\rho} \left| \frac{\gamma_s \mathbf{C} \mathbf{F}}{4 \sin^2 \theta} + \left(1 - \frac{1}{\rho}\right) \mathbf{I} + \frac{1}{\rho} \mathbf{F} \right|^{-1} d\theta \right\}^\rho, \quad (8.22)$$

which needs to be minimized under the constraints (8.18).

It is interesting that the inequality (8.20) is also used to derive the Chernoff bound in information-theoretic literature, which reduces to the Bhattacharyya bound if $\rho = 1/2$. Hence both Gallager's second bound and the Chernoff bound can be derived from exactly the same inequality (8.20)!!

Author's Publications

Journal Papers

- [J1] Cong Ling, W. H. Mow, K. H. Li and A. C. Kot, "Multiple-Antenna differential lattice decoding," *IEEE J-SAC Special Issue on Differential and Noncoherent Wireless Communications*, vol. 23, Sept. 2005, in press.
- [J2] Cong Ling, K. H. Li and A. C. Kot, "Performance of space-time codes: Gallager bounds and weight enumeration," submitted to *IEEE Trans. Inform. Theory*, Nov. 2003.
- [J3] Cong Ling, K. H. Li and A. C. Kot, "On decision-feedback detection of differential space-time modulation in continuous fading," *IEEE Trans. Commun.*, vol. 52, pp. 1613-1617, Oct. 2004.
- [J4] Cong Ling, K. H. Li and A. C. Kot, "Noncoherent sequence detection of differential space-time modulation," *IEEE Trans. Inform. Theory*, vol. 49, pp. 2727-2734, Oct. 2003.
- [J5] Cong Ling, K. H. Li, A. C. Kot, and Q. T. Zhang, "Multisampling decision-feedback linear prediction receivers for differential space-time modulation over Rayleigh fast fading channels," *IEEE Trans. Commun.*, vol. 51, pp. 1214-1223, July 2003.

Conference Papers

- [C1] Cong Ling, W. H. Mow, K. H. Li and A. C. Kot, "Differential lattice decoding in noncoherent MIMO communication," *ICC'05*, Seoul, Korea, May 2005.
- [C2] Cong Ling, K. H. Li and A. C. Kot, "Gallager bounds for space-time codes," *IEEE Communication Theory Workshop 2004*, Capri, Italy, May 2004.
- [C3] Cong Ling, K. H. Li and A. C. Kot, "On Decision-Feedback Detection of Nondiagonal Differential Space-Time Modulation in Temporally Correlated Fading Channels," in *Proc. ICC'03*, Anchorage, AL, May 2003, pp. 2648-2652.
- [C4] Cong Ling, K. H. Li and A. C. Kot, "Decision-feedback multiple symbol differential detection of differential space-time modulation in continuously fading channels," in *Proc. ICASSP'03*, Hong Kong, China, May 2003, pp. IV_45-IV_48.

Bibliography

- [1] C. E. Shannon, "A mathematical theory of communication," *Bell Syst. Tech. J.*, vol. 27, pp. 379-423, July 1948, and vol. 27, pp. 623-656, Oct. 1948.
- [2] A. J. Viterbi and J. K. Omura, *Principles of Digital Communication and Coding*. New York, NY: McGraw-Hill, 1979.
- [3] R. G. Gallager, "A simple derivation of the coding theorem and some applications," *IEEE Trans. Inform. Theory*, vol. 11, pp. 3-18, Jan. 1965.
- [4] R. G. Gallager, *Information Theory and Reliable Communication*. New York, NY: John Wiley & Sons, 1968.
- [5] R. G. Gallager, *Low-Density Parity-Check Codes*. Cambridge, MA: MIT Press, 1963.
- [6] C. Berrou and A. Glavieux, "Near optimum error correction coding and decoding: Turbo-codes," *IEEE Trans. Commun.*, vol. 44, pp. 1261-1271, Oct. 1996.
- [7] G. J. Foschini, Jr. and M. J. Gans, "On limits of wireless communication in a fading environment when using multiple antennas," *Wireless Personal Commun.*, vol. 6, no. 3, pp. 311-335, Mar. 1998.
- [8] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *Eur. Trans. Telecom.*, vol. 10, pp. 585-595, Nov. 1999.
- [9] G. J. Foschini, "Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas," *Bell Labs. Tech. J.*, vol. 1, pp. 41-59, 1996.
- [10] L. Zheng and D. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple antenna channels," *IEEE Trans. Inform. Theory*, vol. 49, pp. 1073-1096, May 2003.
- [11] T. L. Marzetta and B. M. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading," *IEEE Trans. Inform. Theory*, vol. 45, pp. 139-157, Jan. 1999.
- [12] L. Zheng and D. Tse, "Communication on the Grassmann manifold: A geometric approach to the noncoherent multiple-antenna channel," *IEEE Trans. Inform. Theory*, vol. 48, pp. 359-383, Feb. 2002.
- [13] B. M. Hochwald and T. L. Marzetta, "Unitary space-time modulation for multiple-antenna communication in Rayleigh flat-fading," *IEEE Trans. Inform. Theory*, vol. 46, pp. 543-564, Mar. 2000.
- [14] V. Tarokh and H. Jafarkhani, "A differential detection scheme for transmit diversity," *IEEE J. Select. Areas Commun.*, vol. 18, pp. 1169-1174, July 2000.
- [15] B. L. Hughes, "Differential space-time modulation," *IEEE Trans. Inform. Theory*, vol. 46, pp. 2567-2578, Nov. 2000.
- [16] B. M. Hochwald and W. Sweldens, "Differential unitary space-time modulation," *IEEE Trans.*

- Commun.*, vol. 48, pp. 2041-2052, Dec. 2000.
- [17] S. Shamai (Shitz) and I. Sason, "Variations on the Gallager bounds, connections, and applications," *IEEE Trans. Inform. Theory*, vol. 48, pp. 3029-3051, Dec. 2002.
 - [18] J. H. Winters, "On the capacity of radio communication systems with diversity in Rayleigh fading environment," *IEEE J. Select. Areas Commun.*, vol. 5, pp. 871-878, June 1987.
 - [19] M. Chiani, M. Z. Win and A. Zanella, "On the capacity of spatially correlated MIMO Rayleigh-fading channels," *IEEE Trans. Inform. Theory*, vol. 49, pp. 2363-2371, Oct. 2003.
 - [20] H. El Gammal, G. Caire, and M. O. Damen, "Lattice coding and decoding achieve the optimal diversity-multiplexing tradeoff of MIMO channels," *IEEE Trans. Inform. Theory*, vol. 50, pp. 968-985, June 2004.
 - [21] H. El Gamal and A. R. J. Hammons, "A new approach to layered space-time coding and signal processing," *IEEE Trans. Inform. Theory*, vol. 47, no. 6, pp. 2321-2334, Sept. 2001.
 - [22] S. Baro, G. Bauch and A. Hansmann, "Improved codes for space-time trellis-coded modulation," *IEEE Commun. Lett.*, vol. 4, pp. 20-22, Jan. 2000.
 - [23] R. Blum, "Some analytic tools for the design of space-time convolutional codes," *IEEE Trans. Commun.*, vol. 50, pp. 1593-1599, Oct. 2002.
 - [24] Q. Yan and R. Blum, "Improved space-time convolutional codes for quasi-static slow fading channels," *IEEE Trans. Wireless Commun.*, vol. 1, pp. 563-570, Oct. 2002.
 - [25] H. El Gamal and A. R. Hammons, "On the design and performance of algebraic space-time codes for PSK modulation," *IEEE Trans. Inform. Theory*, vol. 48, pp. 907-913, June 2002.
 - [26] Y. Liu, M. P. Fitz and O. Y. Takeshita, "A rank criterion for QAM space-time codes," *IEEE Trans. Inform. Theory*, vol. 48, pp. 3062-3079, Dec. 2002.
 - [27] E. Biglieri, G. Taricco, and A. Tulino, "Performance of space-time codes for a large number of antennas," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1794-1803, July 2002.
 - [28] J. Yuan, Z. Chen, B. Vucetic and W. Firmanto, "Performance and design of space-time coding in fading channels," *IEEE Trans. Commun.*, vol. 51, pp. 1991-1996, Dec. 2003.
 - [29] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456-1467, July 1999.
 - [30] S. M. Alamouti, "A simple transmit diversity techniques for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1451-1458, Oct. 1998.
 - [31] B. Hassibi and B. M. Hochwald, "High rate codes that are linear in space and time," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1804-1824, July 2002.
 - [32] A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász, "Factoring polynomials with rational coefficients," *Math. Ann.*, vol. 261, pp. 515-534, 1982.
 - [33] M. Pohst, "On the computation of lattice vectors of minimal length, successive minima and reduced bases with applications," *ACM SIGSAM Bull.*, vol. 15, pp. 37-44, Feb. 1981.
 - [34] U. Fincke and M. Pohst, "Improved methods for calculating vectors of short length in a

- lattice, including a complexity analysis,” *Math. Comput.*, vol. 44, pp. 463–471, Apr. 1985.
- [35] C. P. Schnorr and M. Euchner, “Lattice basis reduction: Improved practical algorithms and solving subset sum problems,” *Math. Program.*, vol. 66, pp. 181–191, 1994.
- [36] W. H. Mow, “Maximum likelihood sequence estimation from the lattice viewpoint,” *IEEE Trans. Inform. Theory*, vol. 40, pp. 1591–1600, Sept. 1994.
- [37] E. Viterbo and J. Boutros, “A universal lattice code decoder for fading channels,” *IEEE Trans. Inform. Theory*, vol. 45, pp. 1639–1642, July 1999.
- [38] E. Agrell, T. Eriksson, A. Vardy, and K. Zeger, “Closest point search in lattices,” *IEEE Trans. Inform. Theory*, vol. 48, pp. 2201–2214, Aug. 2002.
- [39] W. H. Mow, “Universal lattice decoding: Principle and recent advances,” *Wireless Communications and Mobile Computing*, vol. 3, pp. 553–569, Aug. 2003.
- [40] O. Damen, H. El Gamal, and G. Caire, “On maximum likelihood detection and the search for the closest lattice point,” *IEEE Trans. Inform. Theory*, vol. 49, pp. 2389–2402, Oct. 2003.
- [41] S. Baro, J. Hagenauer and M. Witzke, “Iterative detection of MIMO transmission using a list-sequential (LISS) detector,” in *Proc. ICC’2003*, pp. 2653–2657, May 2003.
- [42] S. G. Wilson, J. Freebersyser and C. Marshall, “Multi-symbol detection of M-DPSK,” in *Proc. Globecom’89*, pp. 1692–1697, Nov. 1989.
- [43] K. M. Mackenthun, Jr., “A fast algorithm for multiple-symbol differential detection of MPSK,” *IEEE Trans. Commun.*, vol. 42, pp. 1471–1474, Feb.-Apr. 1994.
- [44] I. Motedayen-Aval, and A. Anastasopoulos, “Polynomial-complexity noncoherent symbol-by-symbol detection with application to adaptive iterative decoding of turbo-like codes,” *IEEE Trans. Commun.*, vol. 51, pp. 197–207, Feb. 2003.
- [45] G. Ferrari, A. Anastasopoulos, G. Colavolpe, and R. Raheli, “Adaptive iterative detection for the phase-uncertain channel: Limited-tree-search versus truncated-memory detection,” *IEEE Trans. Veh. Tech.*, vol. 53, pp. 433–442, Mar. 2004.
- [46] Y. Jing and B. Hassibi, “Unitary space-time modulation via Cayley transform,” *IEEE Trans. Signal Process.*, vol. 51, pp. 2891–2904, Nov. 2003.
- [47] L. Lampe, R. Schober, V. Pauli, C. Windpassinger, “Application of sphere decoding to MSDD,” In *5th International ITG Conference on Source and Channel Coding (SCC)*, pp. 251–258, Erlangen, Germany, Jan. 2004.
- [48] B. Hassibi and B. M. Hochwald, “Cayley differential unitary space-time codes,” *IEEE Trans. Inform. Theory*, vol. 48, pp. 1458–1503, June 2002.
- [49] K. L. Clarkson, W. Sweldens, and A. Zheng, “Fast multiple-antenna differential decoding,” *IEEE Trans. Commun.*, vol. 49, pp. 253–261, Feb. 2001.
- [50] A. Shokrollahi, B. Hassibi, B. M. Hochwald, and W. Sweldens, “Representation theory for high-rate multiple-antenna code design,” *IEEE Trans. Inform. Theory*, vol. 47, pp. 2335–2367, Sept. 2001.

- [51] B. L. Hughes, "Optimal space-time constellations from groups," *IEEE Trans. Inform. Theory*, vol. 49, pp. 401-410, Feb. 2003.
- [52] R. Schober and L. H.-J. Lampe, "Noncoherent receivers for differential space-time modulation," *IEEE Trans. Commun.*, vol. 50, pp. 768-777, May 2002.
- [53] W. C. Jakes, *Microwave Mobile Communications*. Piscataway, NJ: IEEE Press, 1993.
- [54] L. Babai, "On Lovász' lattice reduction and the nearest lattice point problem," *Combinatorica*, vol. 6, no. 1, pp. 1-13, 1986.
- [55] H. Vikalo and B. Hassibi, "On sphere decoding algorithm—I: Expected complexity; II: Generalizations, second order statistics, and applications to communications," *IEEE Trans. Signal Proc.*, in press.
- [56] B. M. Hochwald and S. ten Brink, "Achieving near-capacity on a multiple-antenna channel," *IEEE Trans. Commun.*, vol. 51, pp. 389-399, Mar. 2003.
- [57] C. Windpassinger, L. H.-J. Lampe, R. F. H. Fischer, "From lattice-reduction-aided detection towards maximum-likelihood detection in MIMO systems," In *Proceedings of the International Conference on Wireless and Optical Communications*, Banff, Canada, July 2003.
- [58] D. Divsalar and M. K. Simon, "Multiple-symbol differential detection of MPSK," *IEEE Trans. Commun.*, vol. 38, pp. 300-308, Mar. 1990.
- [59] P. Ho and D. Fung, "Error performance of multiple-symbol differential detection of PSK signals transmitted over correlated Rayleigh fading channels," *IEEE Trans. Commun.*, vol. 40, pp. 1566-1569, Oct. 1992.
- [60] G. Wang, A. Song and X.-G. Xia, "Linear equalization combined with multiple symbol decision feedback detection for differential space-time modulation," *EURASIP J. Appl. Signal Process.*, vol. 5, pp. 532-537, May 2002.
- [61] E. Chiavaccini and G. Vitetta, "Further results on Tarokh's space-time differential technique," *Proc. ICC'02*, New York, NY, Apr.-May 2002, pp. 1778-1782.
- [62] G. Colavolpe and R. Raheli, "Noncoherent sequence detection," *IEEE Trans. Commun.*, vol. 47, pp. 1376-1385, Sept. 1999.
- [63] J. H. Lodge and M. L. Moher, "Maximum likelihood sequence estimation of CPM signals transmitted over Rayleigh flat-fading channels," *IEEE Trans. Commun.*, vol. 38, pp. 787-794, June 1990.
- [64] G. Colavolpe and R. Raheli, "Theoretical analysis and performance limits of noncoherent sequence detection of coded PSK," *IEEE Trans. Inform. Theory*, vol. 46, pp. 1483-1494, July 2000.
- [65] M. Schwartz, W. R. Bennett, and S. Stein, *Communication Systems and Techniques*. New York: McGraw-Hill, 1966.
- [66] C. W. Helstrom, *Elements of Signal Detection and Estimation*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

- [67] E. Biglieri, G. Caire, G. Taricco and J. Ventura-Traveset, "Simple method for evaluating error probabilities," *Electron. Lett.*, vol. 32, pp. 191-192, Feb. 1996.
- [68] M. V. Eyuboglu and S. U. H. Qureshi, "Reduced-state sequence estimation with set partitioning and decision feedback," *IEEE Trans. Commun.*, vol. 36, pp. 13-20, Jan. 1988.
- [69] A. Duel-Hallen and C. Heegard, "Delayed decision-feedback sequence estimation," *IEEE Trans. Commun.*, vol. 37, pp. 428-436, May 1989.
- [70] R. Raheli, A. Polydoros, and C. K. Tzou, "Per-survivor processing: A general approach to MLSE in uncertain environment," *IEEE Trans. Commun.*, vol. 43, pp. 354-364, Feb./Mar./Apr. 1995.
- [71] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744-765, Mar. 1998.
- [72] A. Stefanov and T. M. Duman, "Performance bounds for space-time trellis codes," *IEEE Trans. Inform. Theory*, vol. 49, pp. 2134-2140, Sept. 2003.
- [73] A. P. des Rosiers and P. H. Siegel, "Space-time code performance bounds on quasi-static fading channels," in *Proc. ICC'2003*, Anchorage, Alaska, May 2003, pp. 3160-3164.
- [74] G. Caire and G. Colavolpe, "On low-complexity space-time coding for quasi-static fading channels," *IEEE Trans. Inform. Theory*, vol. 49, pp. 1400-1416, June 2003.
- [75] E. Malkamäki and H. Leib, "Evaluating the performance of convolutional codes over block fading channels," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1643-1646, July 1999.
- [76] H. Bouzekri and S. L. Miller, "Performance analysis of space-time trellis codes over quasi-static fading channels," in *Proc. Allerton Conf. Communication, Control and Computing*, Oct. 2001.
- [77] M.-K. Byun, D. Park and B. G. Lee, "Performance analysis of space-time trellis coded modulations in quasi-static Rayleigh fading channels," in *Proc. ICC'2002*, New York, Apr.-May, 2002, pp. 1596-1600.
- [78] M. Fozunbal, S. W. McLaughlin, and R. W. Schafer, "On performance limits of space-time codes: A sphere-packing bound approach," *IEEE Trans. Inform. Theory*, vol. 49, pp. 2681-2687, Oct. 2003.
- [79] M. K. Simon and M.-S. Alouini, *Digital Communication Over Fading Channels*. New York: Wiley, 2000.
- [80] S. Siwamogsatham, M. P. Fitz , and J. Grimm, "A new view of performance analysis of transmit diversity schemes in correlated Rayleigh fading," *IEEE Trans. Inform. Theory*, vol. 48, pp. 950-956, Apr. 2002.
- [81] M. Brehler and M. K. Varanasi, "Asymptotic error probability analysis of quadratic receivers in Rayleigh-fading channels with applications to a unified analysis of coherent and noncoherent space-time receivers," *IEEE Trans. Inform. Theory*, vol. 47, pp. 2383-2399,

Sept. 2001.

- [82] D. Aktas and M. P. Fitz, "Distance spectrum analysis of space-time trellis-coded modulations in quasi-static Rayleigh fading channels," *IEEE Trans. Inform. Theory*, vol. 49, pp. 1335-1344, Dec. 2003.
- [83] M. Evans and T. Swartz, *Approximating Integrals via Monte Carlo and Deterministic Methods*. Oxford, U.K.: Oxford University Press, 2000.
- [84] M. Chiani, D. Dardari and M. K. Simon, "New exponential bounds and approximations for the computation of error probability in fading channels," *IEEE Trans. Wireless Commun.*, vol. 2, pp. 840-845, July 2003.
- [85] B. S. Gottfried and J. Weisman, *Introduction to Optimization Theory*, Ch. 3. Englewood Cliffs, NJ: Prentice-Hall, 1973.
- [86] U. Vilaipornsawai and H. Leib, "On the performance evaluation of space-time trellis codes over quasi-static Rayleigh fading channels," in *Proc. WCNC'04*.
- [87] C. Schlegel, *Trellis Coding*. New York: IEEE Press, 1997.
- [88] J. Shi and R. Wesel, "Efficient computation of trellis code generating functions," *IEEE Trans. Commun.*, vol. 52, pp. 219-227, Feb. 2004.
- [89] D. Divsalar, "A simple tight bound on error probability of block codes with application to Turbo codes," TMO Progress Report 42-139, NASA, JPL, Pasadena, CA, 1999.
- [90] Y. Ma and Q. T. Zhang, "Accurate evaluation for MDPSK with noncoherent diversity," *IEEE Trans. Commun.*, vol. 50, pp. 1189-1200, July 2002.
- [91] S. Yousefi, "Bounds on the performance of maximum-likelihood decoded binary block codes in AWGN interference," Ph.D. dissertation, University of Waterloo, Canada, 2003.
- [92] T. M. Duman and M. Salehi, "New performance bounds for turbo codes," *IEEE Trans. Commun.*, vol. 46, pp. 717-723, June 1998.