Micromechanics Investigation on defects in piezoelectric and elastic solids

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Micromechanics Investigation on Defects in Piezoelectric and Elastic Solids

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A thesis submitted to the Nanyang Technological University
in fulfilment of the requirement for the degree of
Doctor of Philosophy

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SUMMARY

The emerging technologies of smart materials and structures have found increasing applications in diverse engineering branches such as aerospace, mechanical, manufacturing and civil engineering, etc. Piezoelectric materials are ideal candidates for functional (smart) materials, and have been a primary focus of attention in the realization of smart (adaptive) structure systems. Due to the fact that piezoelectric materials are very brittle, to find better employment of piezoelectric materials depends on the better understanding of mechanical failure mechanism of piezoelectric materials. Micro-crack is one of the most commonly encountered defects leading to mechanical failure, and it plays an important part in fracture processes. During the entire fracture procedure of a crack, Zener-Stroh crack mechanism controls the initial stage of crack growth and propagation. When the micro-crack is stressed after its propagation, Griffith crack mechanism controls the crack to final fracture. Investigation on Zener-Stroh crack and Griffith crack can help to understand how a crack initiates and gradually grows and find out which cracks constitute an obvious risk for fracture and which do not.

In the current research, the physical problems on micro-cracks near a coated inclusion in piezoelectric materials are investigated; both Zener-Stroh crack and Griffith crack mechanism are studied for the existing micro-cracks. Accordingly, three specific

problems, viz, a screw dislocation interacting with a coated inclusion; a Zener-Stroh crack interacting with a coated inclusion and a Griffith crack interacting with a coated inclusion in piezoelectric solids, are analyzed in detail. In the study, three dissimilar material phases are involved: the matrix, the inclusion and the coating layer. All the three materials are piezoelectric and with different material constants. The solutions for the two crack problems begin with the derivation of stress and electric displacement fields of a screw piezoelectric dislocation in relevant three phase piezoelectric structures. Explicit closed-form analytical solutions are obtained by using the complex variable method. The image force acting on the screw dislocation is calculated by using the generalized Peach-Koehler formula. Numerical examples for different material constant combinations are performed. Following this work, the obtained single screw dislocation solution is used as Green's function to simulate the crack with the application of distributed dislocation method. The crack problems are formulated into a set of singular integral equations and solved with comprehensive numerical methods. The stress and electric displacement intensity factors are derived and expressed in terms of dislocation density functions. The influences of material properties of the inclusion and the coating layer and defects location are studied in numerical examples in detail.

Besides the above solved physical problems, the interaction between an edge dislocation with an interfacial void (the centre of void is at the interface of a bi-material structure) is also studied. The existing void is simulated by distributing continuous dislocations in mathematical sense and the resulting singular integral equations are solved by Gauss-Jacobi quadrature. The image forces on the dislocation are discussed in numerical examples with the influence of material properties and dislocation positions. It is found

that an oscillatory characteristic exists with the existence of interfacial void when the dislocation is very close to the void.

TABLE OF CONTENTS

Acknowledgements..... i

Summary ii

Table of Contents v

List of Figures ix

List of Tables xiv

List of Symbols xv

Chapter 1 Introduction..... 1

 1.1 Introduction..... 1

 1.2 Objectives 4

 1.3 Outline of the Thesis 5

Chapter 2 Literature Review and Basic Theory 8

 2.1 Linear Theory of Piezoelectricity..... 8

 2.1.1 Linear Theory of Piezoelectricity..... 9

 2.1.2 Basic Solution of the Stroh Formalism 13

 2.1.3 Piezoelectric Composites and Related Crack Problems..... 17

2.2 Distributed Dislocation Technique	20
2.2.1 Concept of a Dislocation	20
2.2.2 Introduction to the Technique	24
2.3 Crack Models	27
2.3.1 Griffith Crack	28
2.3.2 Zener-Stroh Crack	29
2.3.3 Energy Consideration of Griffith and Zener-Stroh Crack	32
2.4 Numerical Solution of Singular Integral Equations	34
2.4.1 Gauss-Chebyshev Quadrature	34
2.4.2 Gauss-Jacobi Quadrature	37

Chapter 3 Electro-elastic Stress Analysis for a Screw Dislocation

Interacting with a Coated Inclusion	39
3.1 Introduction	39
3.2 Formulation	40
3.3 Stress and Electric Displacement Fields	44
3.4 Image Force on the Dislocation	45
3.5 Examples and Discussions	46
3.6 Conclusions	59

Chapter 4 Electro-elastic Stress Analysis for a Zener-Stroh Crack

Interacting with a Coated Inclusion	60
4.1 Introduction	60

4.2 Formulation..... 61

4.3 Numerical Procedure..... 65

4.4 Numerical Results and Discussions 66

4.5 Conclusions..... 79

Chapter 5 Electro-elastic Stress Analysis for a Griffith Crack

Interacting with a Coated Inclusion 80

5.1 Introduction 80

5.2 Formulation..... 81

5.3 Numerical Procedure..... 85

5.4 Numerical Results and Discussions 86

 5.4.1 Influence of the Elastic Modulus of the Coating Layer 88

 5.4.2 Influence of the Piezoelectric Coefficient of the Coating Layer. 88

 5.4.3 Influence of the Coating Thickness..... 93

 5.4.4 Influence of the Crack Location..... 98

5.5 Conclusions..... 98

Chapter 6 An Edge Dislocation Interacting with an Interfacial Void100

6.1 Introduction..... 100

6.2 Formulation..... 102

6.3 Singularity and Numerical Solution..... 105

6.4 Image Force on Dislocation 107

6.5 Numerical Examples and Discussions 108

6.6 Conclusions..... 114

Chapter 7 Conclusions and Recommendations..... 115

7.1 Conclusions..... 115

7.2 Recommendations..... 117

References 121

Appendix A 134

Appendix B..... 136

Appendix C 140

Publications..... 146

LIST OF FIGURES

Figure 1.1	Illustration of piezoelectric effect in a cylinder of PZT material (APC International Ltd.: Piezoelectric Ceramics and Application Notes)	2
Figure 2.1	An edge dislocation	21
Figure 2.2	A screw dislocation.....	22
Figure 2.3	The creation of an edge dislocation (a) by an opening displacement (climbing) (b) by a tangential displacement (glide)	23
Figure 2.4	Application of Bueckner's principle (a) overall problem (b) stress in perfect body (c) separated material.....	24
Figure 2.5	Modeling a crack using dislocations (a) an array of dislocations(b) an array over a small element (c) the dislocation density(d) crack opening displacement	26
Figure 2.6	Griffith crack situated between two elastic half planes (a) Mode I crack (b) Mode II crack (c)Mode III crack.....	28
Figure 2.7	Zener's mechanism of crack initiation	30
Figure 2.8	Cottrel's model of crack initiation	30
Figure 2.9	Anti-Zener-Stroh crack model	31
Figure 2.10	Critical crack length.....	33
Figure 3.1	A screw dislocation near a coated inclusion in piezoelectric solid	40

Figure 3.2	$F_{xb_z}^*$ versus $c_{44}^{II} / c_{44}^{III}$ with $e_{15}^I : e_{15}^{II} : e_{15}^{III} = 25 : 1 : 5$, $\kappa_{11}^I : \kappa_{11}^{II} : \kappa_{11}^{III} = 1 : 1 : 1$, $c_{44}^I / c_{44}^{III} = m$	47
Figure 3.3	$F_{xb_\phi}^*$ versus $c_{44}^{II} / c_{44}^{III}$ with $e_{15}^I : e_{15}^{II} : e_{15}^{III} = 25 : 1 : 5$, $\kappa_{11}^I : \kappa_{11}^{II} : \kappa_{11}^{III} = 1 : 1 : 1$, $c_{44}^I / c_{44}^{III} = m$	48
Figure 3.4	$F_{xb_z}^*$ versus $e_{15}^{II} / e_{15}^{III}$ with $c_{44}^I : c_{44}^{II} : c_{44}^{III} = 25 : 1 : 5$, $\kappa_{11}^I : \kappa_{11}^{II} : \kappa_{11}^{III} = 1 : 1 : 1$	50
Figure 3.5	$F_{xb_\phi}^*$ versus $e_{15}^{II} / e_{15}^{III}$ with $c_{44}^I : c_{44}^{II} : c_{44}^{III} = 25 : 1 : 5$, $\kappa_{11}^I : \kappa_{11}^{II} : \kappa_{11}^{III} = 1 : 1 : 1$	51
Figure 3.6	$F_{xb_z}^*$ versus b/a with $c_{44}^I / c_{44}^{III} = 2$, $c_{44}^{II} / c_{44}^{III} = 0.7$, $e_{15}^I / e_{15}^{III} = 2.5$, $e_{15}^{II} / e_{15}^{III} = 0.5$, $\kappa_{11}^I / \kappa_{11}^{III} = 3$, $\kappa_{11}^{II} / \kappa_{11}^{III} = 0.3$	52
Figure 3.7	$F_{xb_\phi}^*$ versus b/a with $c_{44}^I / c_{44}^{III} = 2$, $c_{44}^{II} / c_{44}^{III} = 0.7$, $e_{15}^I / e_{15}^{III} = 2.5$, $e_{15}^{II} / e_{15}^{III} = 0.5$, $\kappa_{11}^I / \kappa_{11}^{III} = 3$, $\kappa_{11}^{II} / \kappa_{11}^{III} = 0.3$	53
Figure 3.8	$F_{xb_z}^*$ versus d/b with $c_{44}^{II} / c_{44}^{III} = 2$, $e_{15}^{II} / e_{15}^{III} = 2.5$, $\kappa_{11}^{II} / \kappa_{11}^{III} = 3$, $e_{15}^I / e_{15}^{III} = 1$, $\kappa_{11}^I / \kappa_{11}^{III} = 1$, $c_{44}^I / c_{44}^{III} = m1$	55
Figure 3.9	$F_{xb_z}^*$ versus d/b with $c_{44}^{II} / c_{44}^{III} = 2$, $e_{15}^{II} / e_{15}^{III} = 2.5$, $\kappa_{11}^{II} / \kappa_{11}^{III} = 3$, $c_{44}^I / c_{44}^{III} = 1$, $\kappa_{11}^I / \kappa_{11}^{III} = 1$, $e_{15}^I / e_{15}^{III} = n1$	56
Figure 3.10	$F_{xb_z}^*$ versus d/b with $c_{44}^I / c_{44}^{III} = 2$, $e_{15}^I / e_{15}^{III} = 2.5$, $\kappa_{11}^I / \kappa_{11}^{III} = 3$, $e_{15}^{II} / e_{15}^{III} = 1$, $\kappa_{11}^{II} / \kappa_{11}^{III} = 1$, $c_{44}^{II} / c_{44}^{III} = m2$	57
Figure 3.11	$F_{xb_z}^*$ versus d/b with $c_{44}^I / c_{44}^{III} = 2$, $e_{15}^I / e_{15}^{III} = 2.5$, $\kappa_{11}^I / \kappa_{11}^{III} = 3$, $\kappa_{11}^{II} / \kappa_{11}^{III} = 1$, $c_{44}^{II} / c_{44}^{III} = 1$, $e_{15}^{II} / e_{15}^{III} = n2$	58
Figure 4.1	A Zener-Stroh crack near a coated inclusion.....	61
Figure 4.2	Normalized K_σ^L verse $c_{44}^{(2)} / c_{44}^{(3)}$ with $e_{15}^{(1)} : e_{15}^{(2)} : e_{15}^{(3)} = 25 : 1 : 5$, $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $c_{44}^{(1)} / c_{44}^{(3)} = 5$	68

Figure 4.3	Normalized K_{σ}^R verse $c_{44}^{(2)}/c_{44}^{(3)}$ with $e_{15}^{(1)}:e_{15}^{(2)}:e_{15}^{(3)} = 25:1:5$, $\kappa_{11}^{(1)}:\kappa_{11}^{(2)}:\kappa_{11}^{(3)} = 1:1:1$, $c_{44}^{(1)}/c_{44}^{(3)} = 5$ 69
Figure 4.4	Normalized K_D^L verse $c_{44}^{(2)}/c_{44}^{(3)}$ with $e_{15}^{(1)}:e_{15}^{(2)}:e_{15}^{(3)} = 25:1:5$, $\kappa_{11}^{(1)}:\kappa_{11}^{(2)}:\kappa_{11}^{(3)} = 1:1:1$, $c_{44}^{(1)}/c_{44}^{(3)} = 5$ 70
Figure 4.5	Normalized K_D^R verse $c_{44}^{(2)}/c_{44}^{(3)}$ with $e_{15}^{(1)}:e_{15}^{(2)}:e_{15}^{(3)} = 25:1:5$, $\kappa_{11}^{(1)}:\kappa_{11}^{(2)}:\kappa_{11}^{(3)} = 1:1:1$, $c_{44}^{(1)}/c_{44}^{(3)} = 5$ 71
Figure 4.6	Normalized K_{σ}^L verse $e_{15}^{(2)}/e_{15}^{(3)}$ with $c_{44}^{(1)}:c_{44}^{(2)}:c_{44}^{(3)} = 25:1:5$, $\kappa_{11}^{(1)}:\kappa_{11}^{(2)}:\kappa_{11}^{(3)} = 1:1:1$, $e_{15}^{(1)}/e_{15}^{(3)} = 5$ 73
Figure 4.7	Normalized K_{σ}^R verse $e_{15}^{(2)}/e_{15}^{(3)}$ with $c_{44}^{(1)}:c_{44}^{(2)}:c_{44}^{(3)} = 25:1:5$, $\kappa_{11}^{(1)}:\kappa_{11}^{(2)}:\kappa_{11}^{(3)} = 1:1:1$, $e_{15}^{(1)}/e_{15}^{(3)} = 5$ 74
Figure 4.8	Normalized K_D^L verse $e_{15}^{(2)}/e_{15}^{(3)}$ with $c_{44}^{(1)}:c_{44}^{(2)}:c_{44}^{(3)} = 25:1:5$, $\kappa_{11}^{(1)}:\kappa_{11}^{(2)}:\kappa_{11}^{(3)} = 1:1:1$, $e_{15}^{(1)}/e_{15}^{(3)} = 5$ 75
Figure 4.9	Normalized K_D^R verse $e_{15}^{(2)}/e_{15}^{(3)}$ with $c_{44}^{(1)}:c_{44}^{(2)}:c_{44}^{(3)} = 25:1:5$, $\kappa_{11}^{(1)}:\kappa_{11}^{(2)}:\kappa_{11}^{(3)} = 1:1:1$, $e_{15}^{(1)}/e_{15}^{(3)} = 5$ 76
Figure 4.10	Normalized K_{σ}^R verse b/a with $\kappa_{11}^{(1)}:\kappa_{11}^{(2)}:\kappa_{11}^{(3)} = 1:1:1$, $c_{44}^{(2)}:c_{44}^{(3)} = 5:1$, $e_{15}^{(2)}:e_{15}^{(3)} = 5:1$ 77
Figure 4.11	Normalized K_D^R verse b/a with $\kappa_{11}^{(1)}:\kappa_{11}^{(2)}:\kappa_{11}^{(3)} = 1:1:1$, $c_{44}^{(2)}:c_{44}^{(3)} = 5:1$, $e_{15}^{(2)}:e_{15}^{(3)} = 5:1$ 78
Figure 5.1	A Griffith crack near a coated piezoelectric inclusion subject to far-field anti-plane shear and in-plane electric field 81
Figure 5.2	Normalized K_{σ}^L verses $c_{44}^{(2)}/c_{44}^{(3)}$ with $e_{15}^{(1)}/e_{15}^{(3)} = 2.5$, $e_{15}^{(2)}/e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)}/\kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$, $c_{44}^{(1)}/c_{44}^{(3)} = 2$ 89
Figure 5.3	Normalized K_D^L verses $c_{44}^{(2)}/c_{44}^{(3)}$ with $e_{15}^{(1)}/e_{15}^{(3)} = 2.5$, $e_{15}^{(2)}/e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)}/\kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$, $c_{44}^{(1)}/c_{44}^{(3)} = 2$ 90

Figure 5.4	Normalized K_{σ}^L verses $e_{15}^{(2)}/e_{15}^{(3)}$ with $c_{44}^{(1)}/c_{44}^{(3)} = 2$, $c_{44}^{(2)}/c_{44}^{(3)} = 0.7$, $\kappa_{11}^{(1)}/\kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$, $e_{15}^{(1)}/e_{15}^{(3)} = 2.5$ 91
Figure 5.5	Normalized K_D^L verses $e_{15}^{(2)}/e_{15}^{(3)}$ with $c_{44}^{(1)}/c_{44}^{(3)} = 2$, $c_{44}^{(2)}/c_{44}^{(3)} = 0.7$, $\kappa_{11}^{(1)}/\kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$, $e_{15}^{(1)}/e_{15}^{(3)} = 2.5$ 92
Figure 5.6	Normalized K_{σ}^L verses b/a with $c_{44}^{(2)}/c_{44}^{(3)} = 0.7$, $e_{15}^{(2)}/e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$, $t_1 = 2.0a$, $t_2 = 2.3a$ 94
Figure 5.7	Normalized K_D^L verses b/a with $c_{44}^{(2)}/c_{44}^{(3)} = 0.7$, $e_{15}^{(2)}/e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$, $t_1 = 2.0a$, $t_2 = 2.3a$ 95
Figure 5.8	Normalized K_{σ}^L verses crack location for $c_{44}^{(1)}/c_{44}^{(3)} = 2$, $c_{44}^{(2)}/c_{44}^{(3)} = 0.7$, $e_{15}^{(1)}/e_{15}^{(3)} = 2.5$, $e_{15}^{(2)}/e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)}/\kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$, $(t_2 - t_1)/b = 0.3$ 96
Figure 5.9	Normalized K_D^L verses crack location for $c_{44}^{(1)}/c_{44}^{(3)} = 2$, $c_{44}^{(2)}/c_{44}^{(3)} = 0.7$, $e_{15}^{(1)}/e_{15}^{(3)} = 2.5$, $e_{15}^{(2)}/e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)}/\kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$, $(t_2 - t_1)/b = 0.3$, 97
Figure 6.1	An interface edge dislocation near a circular interfacial void 101
Figure 6.2	Normalized image force F_x versus a/ρ 110
Figure 6.3	Normalized image force F_y versus a/ρ 111
Figure 6.4	Normalized image force F_x versus μ_2/μ_1 112
Figure 6.5	Normalized image force F_y versus μ_2/μ_1 113
Figure 7.1	An edge dislocation near a coated inclusion in piezoelectriv solids 118
Figure 7.2	An interface crack near a circular interfacial void..... 119
Figure B.1	A three phase piezoelectric solids under remote mechanical and electric fields 137
Figure C.1	An dislocation at position (ξ, η) near a straight interface..... 141

Figure C.2 An edge dislocation on a straight interface..... 144

LIST OF TABLES

Table 6.1 Material Properties..... 109

LIST OF SYMBOLS

a	radius of an inclusion or inside radius of a coating layer of an inclusion
a_{cr}	equilibrium crack length
b	outside radius of a coating layer of an inclusion
b_x, b_y, b_z	components of Burgers vectors for the displacement discontinuity at the dislocation core
b_ϕ	Burgers vector for the electric potential discontinuity across the slip plane at the dislocation core
$b_{-uk}^{III}, b_{-\phi k}^{III}$	coefficient functions associated with the dislocation location and Burgers vector
	Burgers vector for the displacement discontinuity at the dislocation core
$b_z(\xi), b_\phi(\xi)$	charged dislocation density functions along the crack line
c_{ijkl}	elastic moduli
d	distance from dislocation to the center of the inclusion
e_{ikl}	piezoelectric coefficients
f_i	body force
g	Gibbs function per unit volume
$p_\sigma(x)$	mechanical traction function along the plane $\theta = 0$
$p_D(x)$	electric traction function along the plane $\theta = 0$
q_b	body charge
q_s	applied surface charge

s_i	the set of discrete integration points
t_1	distance of the crack left tip to the center of the inclusion
t_2	distance of the crack right tip to the center of the inclusion
t_i	collocation points
u_i	elastic displacement
w	out-of-plane displacement
$x_i (i = 1, 2, 3)$	conventional indicial notation
A_1, A_2	constants which is obtained with the solution of a screw dislocation interacting with a coated inclusion in piezoelectric media
$A_{-1}^{(2)}, A_1^{(2)}$	constants for stress of the intermediate layer which is obtained with the solution of a three-phase piezoelectric cylinder model under a field-field anti-plane mechanical load and a far-field in-plane electrical load
B_1, B_2	material constants which is obtained with the solution of a screw dislocation interacting with a coated inclusion in piezoelectric media
$B_{-1}^{(2)}, B_1^{(2)}$	constants for stress of the intermediate layer which is obtained with the solution of a three-phase piezoelectric cylinder model under a field-field anti-plane mechanical load and a far-field in-plane electrical load
$A^{(1)}, B^{(1)}$	constants for stress of the inclusion which is obtained with the solution of a three-phase piezoelectric cylinder model under a field-field anti-plane mechanical load and a far-field in-plane electrical load
$A_{-1}^{(3)}, B_{-1}^{(3)}$	constants for stress of the matrix which is obtained with the solution of a three-phase piezoelectric cylinder model under a field-field anti-plane mechanical load and a far-field in-plane electrical load
D_∞	far-field in-plane electric loading
D_m	electric displacement
E_m	electric field
$F(t), G(t)$	bounded function
F_x, F_y	image force on dislocation
$G_{z\varphi}(t, s)$	function of the regular part of the kernel

K	stress intensity factor
K_{σ}	stress intensity factor
K_D	electric displacement intensity factor
$N(x)$	normal stress along the crack line
$S(x)$	shear stress along the crack line
$P_N^{(a,b)}$	Jacobi polynomial
T_i	applied surface traction
U	internal energy density
$\omega(s), W(s)$	weight function
α, β	Dundur's parameters
δ	variational symbol
ε_{ij}	strain tensor
ϕ	electric potential
$\phi(s)$	smooth continuous function
φ	generalized stress function
κ_{ij}	dielectric constants
μ	modulus of rigidity
ν	Poisson's ratio
σ_{ij}	stress tensor
τ_{∞}	far-field shear stress
$\Phi(z)$	electrical displacement complex potential function
$\Psi(z), U(z)$	displacement complex potential function
b	Burgers vector
c	elastic moduli matrix
e	piezoelectric coefficients matrix
f	generalized displacement matrix
u	generalized displacement vector
C	material constant matrix for piezoelectric solid
κ	dielectric constants matrix

CHAPTER 1

INTRODUCTION

1.1 Introduction

In nature, some materials exhibit a coupling effect between electric and mechanical fields. This coupling phenomenon is known as *piezoelectricity*. The coupling effect in piezoelectric material comes from two basic phenomena (Figure 1.1). The first phenomenon is known as the direct piezoelectric effect, which implies that, the application of mechanical force or pressure to a piezoelectric material produces an electrical charge or voltage (Figures 1.1b and 1.1c). On the other hand, the application of an electrical charge or voltage to the material induces strain or deformation, which is known as the converse piezoelectric effect (Figures 1.1d-1.1f). Attributing to their intrinsic electromechanical coupling behavior, more recently, the use of piezoelectric materials has gone beyond the traditional application domain of small electric devices due to the emergence of piezoelectric composites. Nowadays, the application of piezoelectric materials has increased rapidly in micro-electro-mechanical-systems (MEMS) devices such as sensors, transducers and actuators.

It was the Curies who first initiated the study of piezoelectricity in 1880. They found that certain crystalline materials generate an electric charge proportional to a mechanical stress. Since then new theories and applications of this field have been constantly advanced. Piezoelectric materials are very brittle and susceptible to fracture during the service. Their reliability as in practical structures depends on the knowledge of applied mechanics and electric and other disturbances. During the fabrication process, various kinds of defects such as dislocations, cracks, cavities, and inclusions will inevitably exist. Since these fracture phenomena on the macro- and micro scale lead to undesirable mechanical and dielectric response for piezoelectric composites, it is of great importance to predict the fracture processes in piezoelectric materials.

Figure 1.1 Illustration of piezoelectric effect in a cylinder of PZT material
(APC International Ltd.: Piezoelectric Ceramics and Application Notes)

In order to obtain a fundamental understanding of material defects and to gain insights into the mechanisms of strengthening and toughening for the piezoelectric material, many researchers have devoted attention to this field and developed some basic solution approaches for crack problems of piezoelectric materials. Generally, the solutions methods commonly used in fracture analysis of piezoelectric materials include the

following approaches: Green's function solution, techniques of integral transformation, the potential function method, the finite element method and the boundary element method. The first three methods are mostly based on analytical approaches, while the last two (FEM and BEM) are the most versatile computational tools to treat fracture problems of piezoelectric material of all the numerical methods. Among the analytical methods, the Green's function approach is the most classical and effective tool in the evaluation on the stress and electric displacement field in crack problem. In fact, Green's function is a solution due to a point load or a dislocation located in a physical solid of infinite domain. Early in 1968, Bilby and Eshelby showed that the displacement jump across the crack face in an isotropic, linear elastic medium could be modeled by a continuous distribution of dislocations. By determining the appropriate distribution of slipping and climbing edge dislocations and screw dislocations, all three modes of fracture can be simulated. Once the dislocation density is obtained through the boundary condition on crack faces and far-field loading conditions, the stresses and displacements for crack problems can be solved by integrating the single dislocation solution according to the dislocation density. Therefore, determination of a single dislocation solution is the key task in Green's function approach to the crack problem. Barnett and Lothe (1975) were the first to show that this method can be extend to the piezoelectric case. They obtained the single dislocation solution by simple generalization of the non-piezoelectric result to the piezoelectric case as

$$\mathbf{U} = \frac{1}{\pi} \text{Im} \left\{ \mathbf{A} \langle \ln z_a \rangle \mathbf{B}^T \right\} \mathbf{b}, \quad (1.1)$$

$$\varphi = \frac{1}{\pi} \text{Im} \left\{ \mathbf{B} \langle \ln z_a \rangle \mathbf{B}^T \right\} \mathbf{b}. \quad (1.2)$$

For a given problem, $\mathbf{b}$ can be determined by satisfying the related loading and boundary conditions. Having obtained the dislocation density $\mathbf{b}(x)$, one can apply it together with

classical Griffith virtual work argument to derive the criterion for stability of the crack under given loads.

Later, Deeg (1980), Pak (1992), Suo *et al.* (1992) and other researchers presented more detailed discussion on this topic. Various models were built to understand the mechanical and electrical coupling behavior of piezoelectric material. However, such existing micro-mechanics models for piezoelectric (composite) material are all two-phase models of inclusion (cavity)/matrix or bi-materials with defects. For non-piezoelectric composites the three phase models of inclusion/intermediate phase/matrix composite (Christensen and Lo, 1979; Luo and Chen, 1992a; Xiao and Chen, 2000 and 2001, Chen, 2001) are developed as an extension of Eshelby's ellipsoidal inclusion model (Eshelby, 1957 and 1959). In estimating the effective moduli of the composite materials, the three-phase models provide accurate predictions that are well suited for extreme types of inclusions (from voids to rigid inclusions). To accurately predict the fracture behavior of a crack in a piezoelectric composite and provide insight on the mechanisms of strengthening and toughening, the interaction of the dislocation and crack with the surrounding inclusion should be fully investigated.

1.2 Objectives

The objectives of the current project are as follows.

- To obtain a comprehensive understanding on dislocation and the distributed dislocation technique for the solution of crack problem.

- To solve the interaction problem on a screw dislocation near a coated inclusion in a three-phase piezoelectric material, and discover new features for the obtained Green's function solution.
- To investigate the physical problem on a Zener-Stroh crack interacting with a coated fiber in a three-phase piezoelectric material.
- To investigate the physical problem on a Griffith crack interacting with a coated fiber in a three-phase piezoelectric material.
- To establish an efficient model to investigate the interfacial void in bi-material and its interaction with an interfacial dislocation

Although the purely elastic cases for the three-phase model of crack-inclusion interaction have been investigated by Xiao and Chen (2000 and 2001), some important microstructure mechanism on the fracture behaviors of the same problems in piezoelectric materials are still need to be studied, due to the electromechanical coupling properties as well as the anisotropic behavior in such materials. It brings more new features under various complicated boundary conditions in real engineering structures. The current study begins with the derivation of the stress and electric displacement field due to a single dislocation in a three-phase piezoelectric composite model. Thereafter the obtained solution for a single dislocation is used to model and investigation a Zener-Stroh crack and a Griffith crack problem in the same composite model.

1.3 Outline of the Thesis

To trace the recent development on fracture studies in piezoelectric materials and elastic materials, some representative work on the related crack problems in the context of linear

elastic fracture mechanic is reviewed in Chapter 2. Research on crack problems with three-phase model in piezoelectric material is briefly summarized. The distributed dislocation method used to formulate crack problems and numerical procedures for solving integral equations are also introduced in this chapter.

In Chapter 3, the electro-elastic stress investigation on the interaction problem of a piezoelectric screw dislocation near a coated inclusion in a piezoelectric material has been carried out. In the study, three dissimilar material phases are involved: the matrix, the inclusion and the coating layer. All the three materials are piezoelectric having different material constants. Explicit closed-form analytical solutions for the stress and electric displacement fields are obtained by using the complex variable method. The image force acting on the screw dislocation is calculated by using the generalized Peach-Koehler formula. Numerical examples for different material constant combinations are performed. The influences of material properties of the inclusion and the coating layer on the image forces are examined and discussed.

In Chapter 4, with the aid of the distributed dislocation method, the single dislocation solution obtained in the previous chapter is used as a Green's function to investigate the problem of a Zener-Stroh crack at the matrix of a three-phase piezoelectric composite. A detailed formulation of the problem and the numerical calculation procedure are presented. The emphases are on using the complex variable method to obtain the stress and electric displacement field. The stress and electric displacement intensity factors are derived accordingly and discussed.

The problem of a Griffith crack interacting with the coated inclusion in piezoelectric solids is investigated in Chapter 5. The stress and electric displacement intensity factors are derived in terms of the asymptotic values of the dislocation density functions evaluated from the integral equations. Numerical examples are given for various material constants combinations and geometric parameters.

The problem of an edge interface dislocation interacting with an interfacial void in bi-material is studied in Chapter 6. The existing interfacial void is modeled by distributing continuous dislocations. The singularities on the boundary of the interfacial void were analyzed. The image forces on the dislocation are given out. Numerical examples about the influence of material properties and dislocation position are performed and discussed.

Finally, conclusions of the thesis and recommendations related to the current research topic are given in Chapter 7

CHAPTER 2

LITERATURE REVIEW AND BASIC THEORY

To begin the main part of this thesis, an overview of the linear theory of piezoelectricity is presented first, which is the fundamental block of the current project. As an important crack modelling technique, distributed dislocation method, with which a crack can be simulated into a series of dislocations in physical sense and formulated into singular integral equations through mathematical approaches, is introduced in detail. Also the latest crack related research progress in elastic and piezoelectric solids will also be summarized in this section. Following that, two famous crack models: Griffith crack and Zener-Stroh crack; are introduced with a comparison of their crack nucleation mechanisms and characteristics. Various required numerical methods for solving singular integral equations, such as Gaussian-Jacobi quadrature formulae and Gauss-Chebyshev integration formulae are reviewed in the last section.

2.1 Linear Theory of Piezoelectricity

In this section, the three-dimensional formulation of linear piezoelectric as summarized by Qin (2001) is briefly recalled first. Then the basic fracture concept of a piezoelectric

material is reviewed. A three-dimension Cartesian coordinate system is adopted. The position vector is denoted by x_i ($i=1, 2, 3$). In this thesis, both conventional indicial notation x_i and traditional Cartesian notation (x, y, z) are utilized.

2.1.1 Linear Theory of Piezoelectricity

The three-dimensional constitutive equations for linear piezoelectricity can be derived by considering the full Gibbs function (Parton and Kudryavtsev, 1988) per unit volume, g is defined as

$$g = U - E_m D_m - Ts, \quad (2.1)$$

where U is the internal energy density, and it can be expressed in the differential form as

$$dU = \sigma_{ij} d\varepsilon_{ij} + E_m dD_m + Tds, \quad (2.2)$$

E_m is the electric field, D_m is the electric displacement, T is the absolute temperature and s is the entropy density. E_i is defined by

$$E_i = -\phi_{,i}, \quad (2.3)$$

in which ϕ is electric potential, a comma followed by arguments denotes partial differentiation with respect to the argument. Considering constant temperature situation, the effect of thermal conditions is omitted. From the exact differential of equation (2.1)

$$dg = \sigma_{ij} d\varepsilon_{ij} - D_m dE_m, \quad (2.4)$$

where σ_{ij} is the stress tensor, ε_{ij} is the strain tensor., while ε_{ij} is defined by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad (2.5)$$

in which u_i is the elastic displacement. We obtain

$$\sigma_{ij} = \frac{\partial g}{\partial \varepsilon_{ij}}, \quad D_m = -\frac{\partial g}{\partial E_m}. \quad (2.6)$$

When the function g is expanded with respect to ε_{ij} and E_m within the scope of linear interactions, we have

$$g = \frac{1}{2} \left(\varepsilon_{ij} \frac{\partial}{\partial \varepsilon_{ij}} + E_m \frac{\partial}{\partial E_m} \right) \left(\varepsilon_{kl} \frac{\partial}{\partial \varepsilon_{kl}} + E_n \frac{\partial}{\partial E_n} \right) g. \quad (2.7)$$

The following constants can then be defined:

$$c_{ijkl} = \frac{\partial^2 g}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}, \quad \kappa_{nm} = -\frac{\partial^2 g}{\partial E_n \partial E_m}, \quad e_{mij} = -\frac{\partial^2 g}{\partial \varepsilon_{ij} \partial E_m}, \quad (2.8)$$

where c_{ijkl} are the elastic moduli measured in a constant electric field, κ_{ij} are the dielectric constants measured at constant strain, e_{ikl} are the piezoelectric coefficient.

When the function g is differentiated according to equation (2.4), and above constants are used, we find the constitutive equations as

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl} - e_{kij} E_k, \quad (2.9)$$

$$D_i = e_{ikl} \varepsilon_{kl} + \kappa_{ik} E_k. \quad (2.10)$$

Using the notation defined above, the Gibbs function per unit volume can now be expressed as

$$g = \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \frac{1}{2} \kappa_{ij} E_i E_j - e_{ikl} \varepsilon_{kl} E_i. \quad (2.11)$$

For a linear elastic piezoelectric material occupying region V bounded by surface S , Hamilton's principle can be stated as

$$\delta \int_V (g + E_i D_i) dV - \int_V (f_i \delta u_i - q_b \delta \phi) dV - \int_S (T_i \delta u_i - q_s \delta \phi) dS = 0, \quad (2.12)$$

where δ is the variational symbol, f_i is the body force, u_i is the displacement, q_b is the body charge, T_i is the applied surface traction, q_s is the applied surface charge, and ϕ is the electric potential, the negative gradient of which is the electric field.

In this formulation, the displacement and the electric potential are taken to be the independent field variables. The variational formulation for the piezoelectric continuum provides the following result:

Governing field equation,

$$\sigma_{ij,j} + f_i = 0, \quad D_{i,i} = q_b; \quad (2.13)$$

Boundary conditions,

$$\sigma_{ij}n_j = T_i, \quad D_i n_i = -q_s, \quad (2.14)$$

together with constitutive equations (2.9) and (2.10), where n_i is the outer unit normal vector to S .

The material constants can be reduced by the following consideration. According to definition (2.5), we may write $\varepsilon_{ij} = \varepsilon_{ji}$. It follows that

$$C_{ijkl} = C_{ijlk}. \quad (2.15)$$

Further, from $\sigma_{ij} = \sigma_{ji}$, we have

$$C_{ijkl} = C_{jikl}, \quad e_{kij} = e_{kji}. \quad (2.16)$$

In view of these properties, it is useful to introduce the so-called two-index notation or matrix notation. Two-index notation consists of replacing ij or kl by p or q , i.e.

$c_{ijkl} = c_{pq}$, $e_{ikl} = e_{iq}$, $\sigma_{ij} = \sigma_p$, where i, j, k, l take the values 1-3, and p, q assume

the values 1-6 according to the following replacements $11 \rightarrow 1$, $22 \rightarrow 2$, $33 \rightarrow 3$, 23 or $32 \rightarrow 4$, 13 or $31 \rightarrow 5$, 12 or $21 \rightarrow 6$. The constitutive relations (2.9) and (2.10) then become

$$\sigma_p = c_{pq} \varepsilon_q - e_{kp} E_k, \quad (2.17)$$

$$D_i = e_{iq} \varepsilon_q + \kappa_{ik} E_k \quad (2.18)$$

in which

$$\varepsilon_p = \begin{cases} \varepsilon_{ij}, & \text{when } i = j, \\ 2\varepsilon_{ij}, & \text{when } i \neq j. \end{cases} \quad (2.19)$$

In addition, the elastic, piezoelectric and dielectric constants can now be written in matrix form since they all are described by two indices. The arrays for an arbitrarily anisotropic material are

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix}, \quad (2.20)$$

$$\mathbf{e} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{bmatrix}, \quad (2.21)$$

$$\mathbf{\kappa} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & \kappa_{22} & \kappa_{23} \\ \kappa_{13} & \kappa_{23} & \kappa_{33} \end{bmatrix}. \quad (2.22)$$

It can be seen that there are $21+18+6=45$ independent constants for this material type. For a transversely isotropic piezoelectric material, such as PZT-6H, with x_3 in the poling direction, the related material matrices are

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11} - c_{12}) \end{bmatrix}, \quad (2.23)$$

$$\mathbf{e} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix}, \quad (2.24)$$

$$\mathbf{\kappa} = \begin{bmatrix} \kappa_{11} & 0 & 0 \\ 0 & \kappa_{11} & 0 \\ 0 & 0 & \kappa_{33} \end{bmatrix}. \quad (2.25)$$

Thus it is clear that a material with this type of symmetry is described by 10 independent material constants.

2.1.2 Basic Solution of the Stroh Formalism

There are two powerful solution procedures in the literature for two-dimensional deformations in a general anisotropic piezoelectric material, in which elastic displacement u_i and electric potential ϕ depend on x_1 and x_2 only. One is Lekhnitskii's approach (1981), which starts with equilibrated stress functions, then compatibility equations. Another is Stroh's formalism (1958), which starts with the displacement and electric potential, then equilibrium equations. The equivalency of these two formalisms has been discussed by Suo (1990). In this thesis Stroh's formalism is adopted as the approach to solve the micromechanical problem in piezoelectric solids, so the details of this formalism are given below only.

In 1958 and 1962, Stroh elaborated the work of Eshelby *et al.* (1953) on two-dimensional problems of general anisotropic elasticity involving dislocation, line force and steady waves. This powerful and elegant approach was named the Stroh formalism. Later, Barnett and Lothe (1975) extended Stroh's formalism with the three-dimensional space to four dimensions, so that the piezoelectric effect is included.

In the absence of body forces and free charges, the balance laws (2.13) require

$$\sigma_{ij,j} = 0, D_{i,j} = 0. \quad (2.26)$$

For two-dimensional deformations in which elastic displacement u_i and electric potential ϕ depend on x_1 and x_2 only, a general solution can be obtained by considering an arbitrary analytic function of the form

$$u_J = a_J f(z) \quad (J=1,2,3,4), \quad (2.27)$$

in which

$$z = x_1 + px_2, \quad u_4 = \phi, \quad (2.28)$$

and p, a_J are constants to be determined by inserting eqn (2.27) into eqn (2.9) and (2.10),

and later into eqn (2.26). In matrix notation, we have

$$\mathbf{u} = \mathbf{a}f(z). \quad (2.29)$$

Thus $\mathbf{u}$ and $\mathbf{a}$ are vectors with four components and $\mathbf{u}$ is called the generalized displacement. By defining

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^E & \mathbf{e}_{11} \\ \mathbf{e}_{11}^T & -\kappa_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}^E & \mathbf{e}_{21} \\ \mathbf{e}_{12}^T & -\kappa_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}^E & \mathbf{e}_{22} \\ \mathbf{e}_{22}^T & -\kappa_{22} \end{bmatrix}, \quad (2.30)$$

where

$$(\mathbf{Q}^E)_{ik} = c_{i1k1}, \quad (\mathbf{R}^E)_{ik} = c_{i1k2}, \quad (\mathbf{T}^E)_{ik} = c_{i2k2}, \quad (\mathbf{e}_{ij})_m = e_{ijm}. \quad (2.31)$$

We combine (2.8),(2.9),(2.25) and (2.26) into one equation as

$$\left[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2 \mathbf{T} \right] \mathbf{a} = 0. \quad (2.32)$$

The stress and electric displacement obtained by substituting eqn (2.29) into (2.9) and (2.10) can be written in terms of a generalized stress function φ as

$$\varphi = \mathbf{b}f(z), \quad (2.33)$$

$$\mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a} = -p^{-1}(\mathbf{Q} + p\mathbf{R})\mathbf{a}, \quad (2.34)$$

with

$$\sigma_{i1} = -\varphi_{i,2}, \quad \sigma_{i2} = \varphi_{i,1}, \quad D_1 = -\varphi_{4,2}, \quad D_2 = \varphi_{4,1}. \quad (2.35)$$

The second equality in (2.34) follows from equation (2.32).

There are eight eigenvalues p from equation (2.32) which consists of four pairs of complex conjugates. Let

$$p_{\alpha+4} = \overline{p_\alpha}, \quad \text{Im}\{p_\alpha\} > 0, \quad \alpha = 1, 2, 3, 4; \quad (2.36)$$

where “Im” stands for the imaginary part of a complex number and the overbar denotes the complex conjugate. Assuming that p_α are distinct, the general solution obtained by superposing eight solutions of (2.29) and (2.33), associated with the eight eigenvalues p_α are

$$\mathbf{u} = 2 \text{Re} \left\{ \sum_{\alpha=1}^4 \mathbf{a}_\alpha f_\alpha(z_\alpha) \right\}, \quad \varphi = 2 \text{Re} \left\{ \sum_{\alpha=1}^4 \mathbf{b}_\alpha f_\alpha(z_\alpha) \right\} \quad (2.37)$$

in which “Re” stands the real part and $f_{\alpha+4} = \overline{f_\alpha}$ ($\alpha = 1, 2, 3, 4$) is chosen.

In most application f_α assume the same functional form, so that we may write

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha) \quad (\alpha \text{ not summed}). \quad (2.38)$$

Hence, expressions (2.37) can then be written in compact matrix notation

$$\mathbf{u} = 2 \text{Re} \{ \mathbf{A} \mathbf{f}(z) \} = 2 \text{Re} \{ \mathbf{A} \langle f(z_\alpha) \rangle \mathbf{q} \}, \quad (2.39)$$

$$\varphi = 2 \operatorname{Re} \{ \mathbf{B} \mathbf{f}(z) \} = 2 \operatorname{Re} \{ \mathbf{B} \langle f(z_\alpha) \rangle \mathbf{q} \}, \quad (2.40)$$

where $\mathbf{A}$ and $\mathbf{B}$ are 4×4 matrices given by

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4], \quad (2.41)$$

and $\langle f(z_\alpha) \rangle$ is the 4×4 diagonal matrix

$$\langle f(z_\alpha) \rangle = \operatorname{diag} [f(z_1) \quad f(z_2) \quad f(z_3) \quad f(z_4)]. \quad (2.42)$$

For a given problem, all that is required is to determine the unknown function $f(z_\alpha)$ which is analytic in the solution domain and the complex constant vector $\mathbf{q}$ subject to given boundary condition.

Two special cases are most of interest in two-dimensional deformation problems in a general anisotropic piezoelectric material. One is the plane strain problem. The other is the anti-plane deformation problem. For the later case, when the (x, y) -plane coincide with the isotropic $(1, 2)$ -plane, only the out-of-plane displacement u_z couples with the in-plane electric field E_x, E_y , such that the electrical fields and displacement are written as

$$u_x = u_y = 0, \quad u_z = u_z(x, y), \quad (2.43)$$

$$E_z = 0, \quad E_x = E_x(x, y), \quad E_y = E_y(x, y). \quad (2.44)$$

combining with (2.2), in the absence of body force and free charges, the governing equations (2.15) simplify to

$$c_{44} \nabla^2 u_3 + e_{15} \nabla^2 \phi = 0, \quad e_{15} \nabla^2 u_3 - \kappa_{11} \nabla^2 \phi = 0, \quad (2.45)$$

where ∇^2 is the two-dimensional Laplacian operator,

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}. \quad (2.46)$$

This case has been studied by Pak (1990). The solution of complex analytic function is given by

$$u_z = 2 \operatorname{Re}[f_1(z)], \quad \phi = 2 \operatorname{Re}[f_2(z)], \quad z = x + iy. \quad (2.47)$$

The two characteristic roots are identical, $p_1 = p_2 = i$. Consistent with the notation of solution of Stroh formalism, equation (2.40) can be written as (Suo *et al.*, 1992):

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = i \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix}. \quad (2.48)$$

2.1.3 Piezoelectric Composites and Related Crack Problems

As a typical smart material, piezoelectric materials are widely used in sensors, transducers and actuators, some of which are implemented through micro-electro-mechanical-systems (MEMS). Much effort has been made in solving various mechanics problems in piezoelectric materials (Parton, 1976; Deeg, 1980; Pak, 1990a, 1990b, 1992a, 1992b; Sosa, 1992; Suo *et al.*, 1992; Chung and Ting, 1995; Rajapakse and Zhou, 1999; Zhang *et al.*, 2001; Zhang and Gao, 2004; etc.). Various other solution approaches were also developed for crack problem of piezoelectric materials. Yu and Qin (1996a; 1996b) used Fourier transformation techniques to study the crack-tip singularity and damage properties of thermopiezoelectric materials. Wang and Zheng (1995) obtained the solution by potential function method for a general 3D piezoelectric problem. Against these analytical approaches, numerical methods were also developed to obtain a meaningful solution for electroelastic crack problem. Allik and Hughes (1970) were the first apply finite element method (FEM) to piezoelectricity problem. Sze *et al.* (2001) formulated one-dimensional finite element procedure for computing the edge singularities in piezoelectric material. As an alternative numerical method, boundary element method were also extended and applied in fracture mechanics of piezoelectric materials. Lee and Jiang (1994) first derived the boundary integral equation of piezoelectricity by the method of weighted residuals.

Pan (1999) proposed a single-domain BEM formulation for fracture mechanics analysis in cracked 2D anisotropic piezoelectric solids. Qin (2004) applied boundary element method (BEM) to piezoelectric composites in conjunction with homogenization approach for determining their effective material properties. Sanz *et al.* (2005) presented a boundary element approach with quadratic isoparametric elements, quarter-point elements and singular quarter-point elements for three-dimensional crack problems in piezoelectric solids.

In real engineering application, piezoelectric materials were usually combined with other materials to piezoelectric composites. Most encountered structures were bi-material with block-bonded structure or inclusion-matrix structure. Accordingly, there inevitably exist various micro defects in the interface or in the matrix material, such as dislocation problem (Chung and Ting, 1995; Deng and Meguid, 1999; Huang and Kuang, 2001; Wu *et al.*, 2003; Chen *et al.*, 2002a, 2002b, 2002c, 2003, 2004a, 2004b; etc), crack problem (Diao *et al.*, 1996; Narita and Shindo, 1998; Shen *et al.*, 1999; Ou and Wu, 2003; Xiao *et al.*, 2003; Chue and Liu, 2004; Yin, 2005; etc) and the interaction problem between different defects (Liu *et al.* 1997; Meguid and Deng, 1998; Xiao *et al.*, 2001; Chen *et al.*, 2002b; Li *et al.*, 2003; etc).

Different to the bimaterial structure, a three phase piezoelectric composite materials with related crack problem are treated in this thesis. The three-phase (inclusion/coating/matrix) model is of great practical and theoretical interest in composite mechanics research. It can be used to study stress and electric concentrations in piezoelectric composites and devices. Also, the three-phase model provide good predictions in estimating the effective moduli of composite material, and the method also gives the correct asymptotic behavior of

composites as the inclusion volume fraction fully packed. Some work on the three-phase model in non-piezoelectric and piezoelectric material has been done recently. The investigation for a crack near a coated circular inclusion was carried out by Xiao and Chen (2001). Jiang and Cheung (2001) derived exact closed-form solutions for the electromechanical fields for a three-phase piezoelectric cylinder model under a far-field anti-plane mechanical load and a far-field in-plane electrical load. Using these results, a set of simple algebraic equations was obtained for determining the relevant effective electro-elastic moduli of piezoelectric composites. Sudak (2003) investigated the effect of an inter-phase layer on the electro-elastic stresses within a three-phase elliptic inclusion.

In studying the crack problem in piezoelectric materials, an important issue is the crack-face electric boundary conditions. There are two idealized crack-face electric boundary conditions that are extensively used in the literatures. One commonly used boundary condition treats crack as being electrically impermeable, which means the specification that the normal component of electric displacement along the crack face equals to zero (see, e.g., Deeg, 1980; Sosa and Pak, 1990; Sosa, 1991; Sosa, 1992; Suo *et al.*, 1992; Pak, 1992b; Park and Sun, 1993; Gao and Barnett, 1996; Gao *et al.*, 1997., etc). This boundary condition ignores the permittivity in the medium interior to the crack. The other commonly used boundary considers the crack being electrically permeable (Dunn, 1994; Shindo *et al.*, 2000; Gao and Wang, 2000; Chao and Huang, 2000; etc). For permeable crack, the electric field and the electric displacement intensity factors are found to depend only on the applied mechanical loads. The stress intensity factor and the energy release rate become independent of electrical loads. Other approach towards resolving the piezoelectric crack problems considers the deformation of the crack (Hao and Shen, 1994; McMeeking, 2001; Zhang *et al.*, 1998). This approach is based on the assumption that

under applied mechanical and electrical loads, the crack will open and there is an electrical potential difference on the upper surface and the lower surface of the crack. Wang and Mai (2004) made a discussion on these two commonly used electrical boundary condition assumptions on the crack faces by treating the crack as a notch with a finite thickness. According to their results, the application of an electric field and the ratio of notch thickness to length have no influence on the stress intensity factors. Since a flaw in engineering materials is always a notch of finite thickness rather than a slit crack, the electrically impermeable boundary is a reasonable one for engineering problems. Therefore, in this thesis, the electrically impermeable boundary condition will be adapted in studying the related crack problem in piezoelectric materials.

2.2 Distributed Dislocation Technique

Distributed dislocation technique is very popular in solving crack problem and all the work for this thesis are done with this method. This technique makes it possible to determine explicitly the state of the stress induced by a dislocation in a range of different geometries such as an infinite plane, a half plane, bonded strips and bonded half planes. The crack problem can be reduced to the one of determining the distributed dislocations which generates tractions along the crack line, equal and opposite to those produced by the applied loading, so that the crack face still remain traction-free.

2.2.1 Concept of a Dislocation

Elastic deformations in a crystal may arise not only by the action of external force on it but also because of the internal structure defects present in the crystal. The principal type

of the defect that influences the mechanical properties of crystal is called a dislocation (Landau *et al.*, 1986). There are two basically different dislocations in the crystal materials, named edge dislocation and screw dislocation respectively. The edge dislocation is defined by the edge of the extra half plane of atoms shown in Figure 2.1. As a result, the upper part of the crystal is compressed on either side of the half plane, while the region below the dislocation experience considerable dilatation. By convention, the bottom edge of the half plane is defined as a positive edge dislocation. Had the extra half plane been introduced into the lower half of the crystal, the regions of localized compression and dilatation would be reversed and the dislocation line defined as a negative edge dislocation.

Figure 2.1 An edge dislocation

(<http://www.cmat.uni-halle.de/~hsl/Realstruktur/Defects%20V%20-%20Dislocation%20topology.pdf>)

The screw dislocation may be visualized as the result of “cutting” the lattice along a half-plane and then shifting the parts of the lattice of either side of the cut in opposite direction

to a distance of one lattice period parallel to the edge of the cut. Such a dislocation converts the lattice planes into a helicoidal surface, like a spiral staircase without the steps. Figure 2.2 shows a diagram of a screw dislocation.

Figure 2.2 A screw dislocation
<http://www.cmat.uni-halle.de/~hsl/Realstruktur/Defects%20V%20-%20Dislocation%20topology.pdf>

In two dimensions, a dislocation may be created in two steps. First make a slit in an infinite solid, from the core of the dislocation to any infinitely remote point, and then impose a constant displacement between adjacent points located either side of the slit by inserting or removing material as necessary and then welding the material back together. The constant relative displacement imposed is Burgers vector. For edge dislocation, where the Burgers vector is perpendicular to dislocation line, while for screw dislocations, the Burgers vector is in the direction of the dislocation line. The state of stress induced is independent of the path cut taken, but does depend on the components of the Burgers vectors (b_x, b_y, b_z) .

Figure 2.3 shows two means of producing an edge dislocation with Burgers vector b_x . In the first, Figure 2.3(a), we make a cut along the positive y-axis, pull the material apart, and insert a thin strip of thickness b_x before rejoining. In the second, Figure 2.3(b), we make a cut along the positive x-axis, and slip the material below the cut in the x direction by an amount b_x before rejoining. It is remarkable that the induced stress fields and displacements in y direction are the same for both dislocated bodies. In fact, this will always be the case as long as the relative displacement between material on either side of the cut is a constant, b_x , in the x direction, where the cut may follow any path from the core to infinity. The sign of b_x is defined in a traditional way: an observer, starting from the core and walking along the cut can distinguish between the positive (+) side, on the right, and the negative (-) side, on the left from Figure 2.3. When traversing a path from the negative to the positive side of the cut, we experience a sudden jump, or discontinuity, in the displacements along the x direction, so that $u_x(+)-u_x(-)=b_x$ along the cut.

Figure 2.3 The creation of an edge dislocation
(a) by an opening displacement (climbing) (b) by a tangential displacement (glide)

Climb and glide dislocations are defined in this way: If the Burgers vector is perpendicular to the path cut, so that a strip is inserted, as in Figure 2.3 (a), the dislocation is climb dislocation; If the Burgers vector is parallel to the cut path, so that there are shear displacements, as in Figure 2.3 (b), the dislocation is glide dislocation.

2.2.2 Introduction to the Technique

Figure 2.4 Application of Bueckner's principle
(a) overall problem (b) stress in perfect body (c) separated material

In this section, the technique of distributed dislocation method (J. Weertman, 1996 and Hills. *et. al.*, 1996) will be introduced through a simple example, a straight plane crack of length $2a$, embedded in an infinitely extended sheet of material, where the material is loaded by a tensile stress, $\sigma_{yy}^{\infty}(x)$, perpendicular to the crack, as shown in Figure 2.4(a). The problem can be decomposed into two subsidiary problems as shown in Figure 2.4(b)

and (c). With the superposition method, the stress field can be the addition of the solution to two problems, viz;

$$\sigma_{ij}(x, y) = \tilde{\sigma}_{ij}(x, y) + \bar{\sigma}_{ij}(x, y) \quad (2.49)$$

where $\tilde{\sigma}_{ij}(x, y)$ is the stress field induced in the material in the absence of the crack along the line of the crack ($y = 0, |x| < a$) as shown in Figure 2.4(b) $\bar{\sigma}_{ij}(x, y)$ is the stress field along the line of the crack induced by the dislocation without any external loading in the same material and geometry as shown in Figure 2.4(c).

Denote the normal and shear stresses along the crack-line by $N(x)$ and $S(x)$ respectively.

Considering $\tilde{\sigma}_{yy}(x, 0) \equiv \sigma_{yy}^{\infty}(x, 0)$ and $\tilde{\sigma}_{xy}(x, 0) \equiv 0$ at all values of y , we may state the boundary conditions as

$$N(x) \equiv \sigma_{yy}(x, 0) = \sigma_{yy}^{\infty}(x, 0) + \bar{\sigma}_{yy}(x, 0) = 0, \quad |x| < a \quad (2.50)$$

$$S(x) \equiv \sigma_{xy}(x, 0) = \bar{\sigma}_{xy}(x, 0) = 0, \quad |x| < a. \quad (2.51)$$

$$\bar{\sigma}_{yy}, \bar{\sigma}_{xy}, \bar{\sigma}_{xx} \rightarrow 0 \quad \text{as } x, y \rightarrow \pm\infty \quad (2.52)$$

The corrective stresses $\bar{\sigma}_{ij}(x, y)$ can be generated by distributing dislocations continuously along the crack-line as shown in Figure 2.5(a). To obtain an expression for the stresses, we first consider a distribution of climb dislocations along an infinitesimal crack-element joining the points ξ and $\xi + \delta\xi$ as shown in Figure 2.5(b). If define $B_y(\xi)$ as the density of the dislocations at point ξ as shown in Figure 2.5(c), the normal stresses due to a single dislocation with Burgers vector δb_y at position ξ is

$$\bar{\sigma}_{yy}(x, 0) = \frac{2\mu}{\pi(1+\nu)} \frac{\delta b_y(\xi)}{x - \xi} = \frac{2\mu}{\pi(1+\nu)} \frac{B_y(\xi)}{x - \xi} \delta\xi \quad (2.53)$$

and the normal stress due to a continuous distribution of dislocations along the crack-line will be

$$\bar{\sigma}_{yy}(x, 0) = \frac{2\mu}{\pi(1+\nu)} \int_{-a}^{+a} \frac{B_y(\xi)}{x-\xi} d\xi, \quad (2.54)$$

where

$$B_y(\xi) = \frac{db_y(\xi)}{d\xi}. \quad (2.55)$$

The opening displacement of the crack at position x as shown in Figure 2.5(d), $g(x)$ can be expressed by $B_y(\xi)$ as

$$g(x) = - \int_{-a}^x B_y(\xi) d\xi. \quad (2.56)$$

Figure 2.5 Modeling a crack using dislocations
 (a) an array of dislocations (b) an array over a small element
 (c) the dislocation density (d) crack opening displacement

With the boundary conditions in (2.50), the equation can be expressed as

$$-\frac{1+\nu}{2\mu} \sigma_{yy}^\infty(x) = \frac{1}{\pi} \int_{-a}^{+a} B_y(\xi) \frac{1}{x-\xi} d\xi, \quad |x| < a, \quad (2.57)$$

The integrand contains two terms. The first is the density; and the second is the kernel of the integral, the $(x - \xi)^{-1}$ term, which is called a simple Cauchy Kernel appearing in all plane crack problem. To make (2.57) have a unique solution, an additional side condition should be imposed:

$$\int_{-a}^{+a} B_y(\xi) d\xi = 0. \quad (2.58)$$

The numerical method used to solve the equation will be introduced in the numerical procedure section.

The general steps using the distributed dislocation technique can be seen from the above example. The boundary conditions, the total crack open displacement and the fundamental solution due to a single dislocation are the preconditions to solve the problem by using this technique.

2.3 Crack Models

In fracture mechanics, the crack is the primary objective of study. There are two important crack models, which named Griffith crack and Zener-Stroh crack respectively, to be used to explain fracture behavior. Griffith crack is famous in explaining brittle fracture by considering production and handling flaws as physical stress concentration cracks. Complementary to Griffith crack, Zener-Stroh crack illustrates the micro-crack's propagation mechanism by dislocation coalescing method. Therefore, a review on the research progress of Griffith crack and Zener-Stroh crack is given below.

2.3.1 Griffith Crack

Early in 1913, Inglis solved the problem of an infinitely long, elliptical hole subjected to either a tensile loading or to one or two types of shear loading. A special case of the Inglis solution is for the sharp crack which the minor axis of the ellipse is set equal to zero. Griffith (1921) made use of the Inglis crack solution to obtain his famous fracture criterion, which is derived later in this chapter. The sharp crack is usually known as a Griffith crack in the literature.

The two-dimensional Griffith crack in an elastic solid is shown in Figure 2.6. The crack plane is considered to be on the x -axis using an (x, y) rectangular coordinate system. The crack tips are situated at $x = -a$ and $x = a$. The crack is infinitely long in the z -direction. By definition, if the crack is stressed in tension as shown in Figure 2.6(a), it is called a mode I Griffith crack. If in shear, as shown in Figure 2.6(b) and Figure 2.6(c), it is called a mode II and a mode III Griffith crack, respectively, depending upon the orientation of the shear stress. The direction of the shear stress is perpendicular to the crack front for a mode II crack. It is parallel to the crack front for mode III crack (J. Weertman, 1996).

Figure 2.6 Griffith crack situated between two elastic half planes
 (a) Mode I crack (b) Mode II crack (c) Mode III crack

As stated in the previous section, the stress field due to a Griffith crack can be simulated by continuously distributed dislocations (Weertman, 1996). A large number of discrete dislocations, each with Burgers vector b , exist on a crack plane; this distribution can be represented by a distribution density function $B(x)$ of infinitesimal dislocations (as shown in Figure 2.5). The crack plane dislocations of a mode I crack are climb edge dislocations; for a mode II crack they are glide edge dislocations; and for a mode III crack they are screw dislocations.

2.3.2 Zener-Stroh Crack

It was Zener (1948) who first proposed that a pileup of edge dislocations that were stopped at an obstacle, such as a grain boundary, could coalesce into a crack nucleus. Later, Koehler (1952) also suggested that edge dislocations could nucleate cracks through his independent work. Stroh (1954) made a detailed calculation to find how many dislocations are needed to form a micro crack. According to his research, 10^3 dislocations are needed to form a crack in hardened metal such as copper and a number of the order of 10^2 is required if a stack of piled-up dislocations on parallel slip planes exist. Through energy consideration, Stroh (1955) argued that the energy needed to initiate a Zener-Stroh crack is possibly lower than that due to other mechanism. Figure 2.7 shows Zener's crack initiation model.

Figure 2.7 Zener's mechanism of crack initiation

From then on, many scientists gave out other crack nucleate models that can be considered as variants of the Zener's model. Cottrell (1958) presented that pileup of dislocations on two intersecting slip planes can coalesce into a micro crack, which is showed in Figure 2.8

Figure 2.8 Cottrell's model of crack initiation

Figure 2.9 shows another mechanism, anti-Zener-Stroh crack model, first suggested by Kikuchi *et al.* (1981). In this model, concentrate stress field makes a slip nucleated near the ends of the particle at the grain boundary, thus dislocations of one sign move away

from this region, and then a crack is leaved as a pileup of leaving opposite sign dislocations. These two invariants together with Zener's mechanism are all pure dislocation based model and the crack is generally called Zener-Stroh crack.

Dislocation based Zener-Stroh crack is the crack complementary to the famous Griffith crack. Physical parameters that are symmetric for the Griffith crack are anti-symmetric for the Zener-Stroh crack and vice versa. Antisymmetric dislocations distribution along the Griffith crack causes symmetric traction stress, while dislocations distribution along Zener-Stroh crack is symmetric and the stress field is antisymmetric as a result. Weertman (1986) compared stress, displacement, dislocation density and stress intensity factors of Griffith crack and those of Zener-Stroh crack. Different to that in Griffith crack, the total sum of Burgers vectors of the dislocations b^T is not equal to zero due to displacement loading mechanism. The crack tip where the dislocation enters the Zener-Stroh crack is called blunt tip and the other sharp tip. The crack always propagates from the sharp tip. This kind of micro crack was often observed in metal matrix composite materials.

Figure 2.9 Anti-Zener-Stroh crack model

In recent years, some researchers focus on the interaction between the structure and the crack to reveal the structures influence on the crack feature. Fan and Xiao (1997) investigated a Zener-Stroh crack near an interface and Xiao and Chen (2001) gave the stress analysis for a Zener-Stroh crack interacting with a coated inclusion. Xiao *et al.* (2003) then carry on the investigation about a dislocation pileup model for micro-crack initiation at tip of a rigid line inhomogeneity in piezoelectric materials which is based on Zener-Stroh crack initiation mechanism. Other researchers mainly applied Zener-Stroh mechanism to explain fatigue fracture procedure in various kinds of material, such as silicon carbide (Shih *et al.*, 2000) and metal fatigue (Lawson *et al.*, 1997).

2.3.3 Energy Consideration of Griffith and Zener-Stroh Crack

The energy concept in classic Griffith crack and Zener-Stroh crack was generalized by Cottrell and others (see J. Weertman, 1986). When a crack is loaded by both net dislocations and external constant traction, the energy stored in the system as a function of critical crack length is schematically depicted in Figure 2.10. It is seen that there are two values of critical crack length according to two mechanisms, namely Zener-Stroh and Griffith mechanisms. Thus its stress intensity factor may be expressed as

$$K = K_{zs} + K_G, \quad (2.59)$$

where

$$K_{zs} = \frac{b^T \mu}{2(1-\nu)\sqrt{\pi a}} \quad (2.60)$$

$$K_G = \sigma_a \sqrt{\pi a} \quad (2.61)$$

for a plane crack in a homogeneous material. Critical crack length can be obtained by setting $K = K_c$. Then substituting (2.60) and (2.61) to (2.59), as the solutions of a second order equation, the equilibrium crack length a_{cr} has two solutions as

$$\sqrt{a_{cr}} = \left(K_c / 2\sigma\sqrt{\pi} \right) \left\{ 1 \pm \left[1 - \frac{2\mu\sigma_a b^T}{(1-\nu)K_c^2} \right]^{\frac{1}{2}} \right\} \quad (2.62)$$

The smaller length a_{cr}^- is controlled by Zener-Stroh crack mechanism and the bigger length a_{cr}^+ by Griffith mechanism. Form Figure 2.10, the small one is energy stable and bigger one unstable. Increasing dislocation loading b^T pushes a_{cr}^- to a higher value, and at the same time, K_{SZ} decreases; while increasing the applied stress forces a_{cr}^+ lower and K_G higher. That means Zener-Stroh crack mechanism controlled critical crack length gives the original size of a micro crack which is initiated from dislocation pileups; while the Griffith mechanism controlled critical crack length predicts the moment of propagation.

Figure 2.10 Critical crack length

2.4 Numerical Solution of Singular Integral Equations

As illustrated in previous section, a crack problem can be formulated in terms of singular integral equations by using the distributed dislocation technique. The kernel of the integral equation is singular. To solve the integral equation, a numerical method is normally used. Several quadrature schemes can be selected to solve the corresponding singular integral equations; such as Lobatto-Chebyshev quadrature developed by Theocaris and Ioakimidis (1977), piecewise polynomial method of Gerasoulis and Srivastav (1981) and Kurtz *et al.* (1994). The most widely used and easily implemented quadrature is the Gauss-Chebyshev quadrature developed by Erdogan and Gupta (1972) and Erdogan *et al.* (1973).

2.4.1 Gauss-Chebyshev Quadrature

The standard form of a first kind singular integral equation, which arise in a two-dimensional treatment of cracks where the kernel of the integral is formed from a dislocation solution, can be found in the form (with the crack lying along $[-1, +1]$)

$$F(t) = \int_{-1}^{+1} \frac{B(s)}{t-s} ds, \quad |t| < 1 \quad (2.63)$$

where $(t-s)^{-1}$ is the Cauchy Kernel, and $B(s)$ is the unknown function or in current study referring to the dislocation density, which is given by

$$B(s) = \omega(s)\phi(s). \quad (2.64)$$

Here, $\phi(s)$ is a smooth continuous function and $\omega(s)$ is a weight function of the form

$$\omega(s) = (1+s)^a (1-s)^b \quad (2.65)$$

where a and b may be real or complex and are constrained by $-1 < \operatorname{Re}(a), \operatorname{Re}(b)$. It was proved by Muskhelishvili (1953) that the exponents a and b must be either $+1/2$ or $-1/2$, so that four separate classes of singular integral equation arises, depending on the behavior of the density $B(s)$ at the end points. These are:

(i). $B(s)$ singular at both ends	$a = -1/2, b = -1/2$
(ii). $B(s)$ singular at $s = -1$ and bounded at $s = +1$	$a = +1/2, b = -1/2$
(iii). $B(s)$ bounded at $s = -1$ and singular at $s = +1$	$a = -1/2, b = +1/2$
(iv). $B(s)$ bounded at both ends	$a = +1/2, b = +1/2$

In class (i) problem, e.g. a crack in an infinite homogeneous sheet, in addition to equation (2.63), we must satisfy a side condition to render the solution unique, namely

$$\int_{-1}^{+1} B(s) ds = A, \text{ Side condition for class (i) singular integral equation.} \quad (2.66)$$

In class (ii) and (iii) problem, the solution is unique and, in class (iv) problems, the solution must satisfy the condition

$$\int_{-1}^{+1} \frac{F(s)}{\omega(s)} ds = 0, \text{ Side condition for class (iv) singular integral equation.} \quad (2.67)$$

Chebyshev polynomials are defined as

$$T_n(x) = \cos n\theta, \quad U_n(x) = \sin(n+1)\theta / \sin \theta, \quad \theta = \cos^{-1}(x) \quad (2.68)$$

For class (i), $\omega(s)$ is the weight of the first kind Chebyshev polynomials $T_j(t)$. The unknown function $\phi(s)$ can be approximated to a sufficient degree of accuracy by

$$\phi(s) = \sum_0^p A_j T_j(s). \quad (2.69)$$

According to Erdogan-Gupta method (1972), the discretized form of the Cauchy singular integral in (2.63) is

$$F(t_k) = \int_{-1}^{+1} \frac{B(s)}{t-s} ds = \frac{\pi}{N} \sum_{i=1}^N \frac{\phi(s_i)}{s_i - t_k}, \quad (2.70)$$

where

$$T_N(s_i) = 0, s_i = \cos \frac{(2i-1)\pi}{2N}, \quad i = 1, \dots, N, \quad (2.71)$$

$$U_{N-1}(t_k) = 0, t_k = \cos \frac{k\pi}{N}, \quad k = 1, \dots, N-1. \quad (2.72)$$

Note that (2.70) is identical to the following standard Gauss-Chebyshev integration formula for bounded function $g(x, s)$:

$$\int_{-1}^{+1} \frac{g(t, s)}{(1-s^2)^{\frac{1}{2}}} ds = \frac{\pi}{N} \sum_{i=1}^N g(t, s_i), \quad s_i = \cos \frac{(2i-1)\pi}{2N}, \quad i = 1, \dots, N, \quad (2.73)$$

with the important difference that (2.73) is valid for any x whereas (2.70) is valid only for certain discrete set of t_k given by (2.72).

For class (iv), $\omega(s)$ is the weight of the second kind Chebyshev polynomials $U_j(s)$. $\omega(s)$ can be expressed as

$$\phi(s) \cong \sum_0^p B_j U_j(s), \quad (2.74)$$

$$F(t_k) = \int_{-1}^{+1} \frac{B(s)}{t-s} ds = \frac{\pi}{N+1} \sum_{i=1}^N \frac{(1-s_i^2)\phi(s_i)}{s_i - t_k}, \quad (2.75)$$

$$s_i = \cos \frac{i\pi}{N+1}, \quad i = 1, \dots, N, \quad (2.76)$$

$$t_k = \cos \frac{(2k-1)\pi}{2(N+1)}, \quad k = 1, \dots, N-1, \quad (2.77)$$

For a bounded function $g(x, s)$:

$$\int_{-1}^{+1} \frac{g(t, s)}{(1-s^2)^{\frac{1}{2}}} ds = \frac{\pi}{N+1} \sum_{i=1}^N (1-s_i^2) g(t, s_i), \quad s_i = \cos \frac{(2i-1)\pi}{2N}, i = 1, \dots, N. \quad (2.78)$$

The above equation is valid for any t .

For class (ii) and (iii), $\omega(s)$ is the weight function of Jacobi polynomials $P_j^{(+\frac{1}{2}, -\frac{1}{2})}$ and $P_j^{(-\frac{1}{2}, +\frac{1}{2})}$ respectively.

$$F(t_k) = \int_{-1}^{+1} \frac{B(s)}{t-s} ds = \frac{2\pi}{2N+1} \sum_{i=1}^N \frac{(1-s_i)\phi(s_i)}{s_i-t_k}, \quad (2.79)$$

$$s_i = \cos \frac{2i\pi}{2N+1}, \quad i = 1, \dots, N, \quad (2.80)$$

$$t_k = \cos \frac{(2k-1)\pi}{2N+1}, \quad k = 1, \dots, N, \quad (2.81)$$

for class (ii); and

$$F(t_k) = \int_{-1}^{+1} \frac{B(s)}{t-s} ds = \frac{2\pi}{2N+1} \sum_{i=1}^N \frac{(1+s_i)\phi(s_i)}{s_i-t_k}, \quad (2.82)$$

$$s_i = \cos \frac{(2i-1)\pi}{2N+1}, \quad i = 1, \dots, N, \quad (2.83)$$

$$t_k = \cos \frac{2k\pi}{2N+1}, \quad k = 1, \dots, N, \quad (2.84)$$

for class (iii).

2.4.2 Gauss-Jacobi Quadrature

When the integral equations has a Generalized Cauchy kernel of the following type:

$$F(t) = \int_{-1}^{+1} B(s) \left[\frac{1}{t-s} + G'(t;s) \right] ds, \quad (2.85)$$

$G'(t;s)$ becomes unbounded as both the integration and collocation variables, t and s , tend to the same end-point. The density function may be singular or bounded at each end-point. The Gauss-Jacobi quadrature (Erdogan *et al.*, 1973) can be used to reduce equation (2.79) to a system of algebraic equations; viz,

$$F(t_k) = \sum_{i=1}^N W_i(s_i) \phi(s_i) \left[\frac{1}{t_k - s_i} + G'(t_k; s_i) \right]. \quad (2.86)$$

In this formula, the integration points s_i are the zeros of corresponding Jacobi polynomials; viz,

$$P_N^{(a,b)}(s_i) = 0, \quad (2.87)$$

and the collocation points t_j are determined from

$$P_{N-1}^{(a+1,b+1)}(t_j) = 0, \quad j = 1 \dots N-1, \quad \text{case I} \quad (2.88)$$

$$P_{N-1}^{(a-1,b+1)}(t_j) = 0, \quad j = 1 \dots N, \quad \text{case II} \quad (2.89)$$

$$P_{N-1}^{(a+1,b-1)}(t_j) = 0, \quad j = 1 \dots N, \quad \text{case III} \quad (2.90)$$

$$P_{N-1}^{(a-1,b-1)}(t_j) = 0, \quad j = 1 \dots N+1. \quad \text{case IV} \quad (2.91)$$

The weights are given by

$$W_i(s_i) = -\frac{2N+a+b}{(N+1)!(N+a+b)} \frac{\Gamma(N+a+1)\Gamma(N+b+1)}{\Gamma(N+a+b+1)} \times \frac{2^{a+b}}{P_N^{(a,b)}(s_i)P_{N+1}^{(a,b)}(s_i)} \quad (2.92)$$

In special case when a and b are both either $\pm 1/2$, the formulae embodied in equation (2.86) to (2.92) reduce to Gauss-Chebyshev formulae given in previous section. Further, that the quadrature is not valid for the case of a or b equal to zero.

CHAPTER 3

ELECTRO-ELASTIC STRESS ANALYSIS FOR A SCREW DISLOCATION INTERACTING WITH A COATED INCLUSION

3.1 Introduction

In this chapter, the physical problem of a screw dislocation interacting with a coated circular inclusion embedded in an infinite solid is investigated. For the current problem, three dissimilar material phases are involved: the matrix, the inclusion and the coating layer. All the three materials are piezoelectric and with different material constants. An analytical solution for the stress field due to a screw dislocation near the coated inclusion is derived. The image force acting on the screw dislocation is calculated by using the generalized Peach-Koehler formula. Finally, the influence of material properties on the dislocation is examined and discussed. The study of three-phase (inclusion, coating and matrix) inclusion model is of great practical significance, as the property of the coating material can influence the overall behavior of the three-phase composite.

3.2 Formulation

The physical problem to be considered is shown in Figure 3.1, where a circular piezoelectric inclusion with an intermediate coating layer is embedded in an infinite piezoelectric material. A screw dislocation $\mathbf{b} = [b_z, b_\phi]^T$ is located at point $(d, 0)$, $d > b$. Here b_ϕ is known as the electro-potential dislocation. The three-phase piezoelectric materials occupy the region *I* (the inclusion), *II* (the coating), and *III* (the matrix), respectively, with each phase having different material properties. All the three phases are assumed to have the same material orientation poled along the z -direction with an isotropic Oxy -plane.

Figure 3.1 A screw dislocation near a coated inclusion in piezoelectric solid

As the configuration is an anti-plane problem, only the out-of-plane displacement u_z couples with the in-plane electric field E_x, E_y , such that

$$u_x = u_y = 0, \quad u_z = u_z(x, y), \quad (3.1)$$

$$E_z = 0, \quad E_x = E_x(x, y), \quad E_y = E_y(x, y). \quad (3.2)$$

The equilibrium equations for the stress and electric displacements in the absence of body force and electric charge density are

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0, \quad (3.3)$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0. \quad (3.4)$$

For linear piezoelectric material, the constitutive relations are

$$\sigma_{zx} = c_{44}\gamma_{zx} - e_{15}E_x, \quad \sigma_{zy} = c_{44}\gamma_{zy} - e_{15}E_y, \quad (3.5)$$

$$D_x = e_{15}\gamma_{zx} + \kappa_{11}E_x, \quad D_y = e_{15}\gamma_{zy} + \kappa_{11}E_y. \quad (3.6)$$

The shear strains γ_{zx}, γ_{zy} and the electric fields E_x, E_y are related to the displacement u_z and the electric potential ϕ by the following relations

$$\gamma_{zx} = \frac{\partial u_z}{\partial x}, \quad \gamma_{zy} = \frac{\partial u_z}{\partial y}, \quad E_x = -\frac{\partial \phi}{\partial x}, \quad E_y = -\frac{\partial \phi}{\partial y}. \quad (3.7)$$

Substituting (3.5) and (3.6) into (3.3) and (3.4), by using the matrix notation, we get the constitutive relations as:

$$\begin{bmatrix} \sigma_{zx} \\ D_x \end{bmatrix} = \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix} \begin{bmatrix} \frac{\partial u_z}{\partial x} \\ \frac{\partial \phi}{\partial x} \end{bmatrix} = \mathbf{C} \begin{bmatrix} \frac{\partial u_z}{\partial x} \\ \frac{\partial \phi}{\partial x} \end{bmatrix}, \quad (3.8)$$

$$\begin{bmatrix} \sigma_{zy} \\ D_y \end{bmatrix} = \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix} \begin{bmatrix} \frac{\partial u_z}{\partial y} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \mathbf{C} \begin{bmatrix} \frac{\partial u_z}{\partial y} \\ \frac{\partial \phi}{\partial y} \end{bmatrix}, \quad (3.9)$$

in which

$$\mathbf{C} = \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix}. \quad (3.10)$$

Substituting (3.8) and (3.9) into (3.3) and (3.4), the governing equations are simplified as

$$\mathbf{C} \begin{bmatrix} \nabla^2 u_z \\ \nabla^2 \phi \end{bmatrix} = 0. \quad (3.11)$$

The boundary conditions at the interface between the dissimilar phases can be expressed as

$$\sigma_{zr}^I = \sigma_{zr}^{II}, \quad u_z^I = u_z^{II}, \quad D_r^I = D_r^{II}, \quad \phi_I = \phi_{II}, \quad \text{at } L_1 (r = a), \quad (3.12)$$

$$\sigma_{zr}^{II} = \sigma_{zr}^{III}, \quad u_z^{II} = u_z^{III}, \quad D_r^{II} = D_r^{III}, \quad \phi_{II} = \phi_{III}, \quad \text{at } L_2 (r = b). \quad (3.13)$$

where the superscript and subscript *I*, *II* and *III* denote stress and electric fields in region *I*, *II* and *III* respectively.

The equation (3.11) indicates that the u_z and ϕ are harmonic functions which can be taken as the real part of some analytic functions of the complex variable $z = x + iy$ (Muskhelishvili, 1977), such that

$$\mathbf{f} = \begin{bmatrix} u_z \\ \phi \end{bmatrix} = \text{Re} \begin{bmatrix} U(z) \\ \Phi(z) \end{bmatrix}. \quad (3.14)$$

Follow Xiao and Chen (2000), in our problem, the holomorphic complex functions are taken as

$$U^I(z) = \frac{b_z}{2\pi} \sum_{k=0}^{\infty} (a_{uk}^I + ib_{uk}^I) z^k, \quad (3.15)$$

$$\Phi^I(z) = \frac{b_\phi}{2\pi} \sum_{k=0}^{\infty} (a_{\phi k}^I + ib_{\phi k}^I) z^k, \quad (3.16)$$

$$U^{II}(z) = \frac{b_z}{2\pi} \sum_{k=-\infty}^{\infty} (a_{uk}^{II} + ib_{uk}^{II}) z^k, \quad (3.17)$$

$$\Phi^{II}(z) = \frac{b_\phi}{2\pi} \sum_{k=-\infty}^{\infty} (a_{\phi k}^{II} + ib_{\phi k}^{II}) z^k, \quad (3.18)$$

$$U^{III}(z) = \frac{b_z}{2\pi} \sum_{k=-\infty}^{-1} (a_{uk}^{III} + ib_{uk}^{III}) z^k - \frac{ib_z}{2\pi} \log(z-d), \quad (3.19)$$

$$\Phi^{III}(z) = \frac{b_\phi}{2\pi} \sum_{k=-\infty}^{-1} (a_{\phi k}^{III} + ib_{\phi k}^{III}) z^k - \frac{ib_\phi}{2\pi} \log(z-d), \quad (3.20)$$

when $|z| < d$, we have

$$\log(z-d) = \log d + i\pi - \sum_{k=1}^{\infty} \frac{z^k}{kd^k}. \quad (3.21)$$

Rewriting z as $z = r(\cos\theta + i\sin\theta)$, and substituting $U(z)$ and $\Phi(z)$ into the boundary conditions (3.12) and (3.13), by comparing the coefficient of $\sin k\theta$ and $\cos k\theta$, we obtain three groups of linear equations which determine the coefficients $a_{uk}^i, a_{\phi k}^i, b_{uk}^i, b_{\phi k}^i$ ($i = I, II, III$ for the three phases). By using the matrix notion, the equations are written as followings:

$$\mathbf{M}_1 \begin{bmatrix} a_{u0}^I, a_{u0}^{II}, a_{\phi 0}^I, a_{\phi 0}^{II} \end{bmatrix}^T = \begin{bmatrix} 0, 0, \pi, \pi \end{bmatrix}^T, \quad (3.22)$$

$$\mathbf{M}_2 \begin{bmatrix} a_{uk}^I, a_{\phi k}^I, a_{uk}^{II}, a_{\phi k}^{II}, a_{-uk}^{II}, a_{\phi k}^{II}, a_{-uk}^{III}, a_{\phi k}^{III} \end{bmatrix}^T = 0, \quad (3.23)$$

$$\begin{aligned} \mathbf{M}_3 \begin{bmatrix} b_{uk}^I, b_{\phi k}^I, b_{uk}^{II}, b_{\phi k}^{II}, b_{-uk}^{II}, b_{\phi k}^{II}, b_{-uk}^{III}, b_{\phi k}^{III} \end{bmatrix}^T \\ = \frac{1}{kd^k} \begin{bmatrix} 0, 0, 1, 1, 0, 0, c_{44}^{III} b_z + e_{15}^{III} b_\phi, e_{15}^{III} b_z - \kappa_{11}^{III} b_\phi \end{bmatrix}. \end{aligned} \quad (3.24)$$

where the three matrices $\mathbf{M}_1, \mathbf{M}_2$ and $\mathbf{M}_3$ are shown in the Appendix A.

Solving Eqs.(3.22), (3.23) and (3.24) yields

$$a_{u0}^I = a_{u0}^{II} = a_{\phi 0}^I = a_{\phi 0}^{II} = \pi, \quad (3.25)$$

$$a_{uk}^I = a_{\phi k}^I = a_{uk}^{II} = a_{\phi k}^{II} = a_{-uk}^{II} = a_{\phi k}^{II} = a_{-uk}^{III} = a_{\phi k}^{III} = 0, \quad (3.26)$$

and

$$\begin{aligned}
& [b_{uk}^I, b_{\phi k}^I, b_{uk}^{II}, b_{-uk}^{II}, b_{\phi k}^{II}, b_{-\phi k}^{II}, b_{-uk}^{III}, b_{-\phi k}^{III}]^T \\
& = \frac{1}{kd^k} \mathbf{M}_3^{-1} [0, 0, 1, 1, 0, 0, c_{44}^{III} b_z + e_{15}^{III} b_\phi, e_{15}^{III} b_z - \kappa_{11}^{III} b_\phi]^T.
\end{aligned} \tag{3.27}$$

3.3 Stress and Electric Displacement Fields

The stress and electric displacement fields components due to the screw dislocation in the three-phase piezoelectric material can be calculated using (3.5)~(3.7), together with (3.15)~(3.20),

$$\sigma_{zx}^I = -\frac{c_{44}^I b_z}{2\pi} \sum_{k=1}^{\infty} r^{k-1} k \sin(k-1)\theta \cdot b_{uk}^I - \frac{e_{15}^I b_\phi}{2\pi} \sum_{k=1}^{\infty} r^{k-1} k \sin(k-1)\theta \cdot b_{\phi k}^I, \tag{3.28}$$

$$\sigma_{zy}^I = -\frac{c_{44}^I b_z}{2\pi} \sum_{k=1}^{\infty} r^{k-1} k \cos(k-1)\theta \cdot b_{uk}^I - \frac{e_{15}^I b_\phi}{2\pi} \sum_{k=1}^{\infty} r^{k-1} k \cos(k-1)\theta \cdot b_{\phi k}^I, \tag{3.29}$$

$$D_x^I = -\frac{e_{15}^I b_z}{2\pi} \sum_{k=1}^{\infty} r^{k-1} k \sin(k-1)\theta \cdot b_{uk}^I + \frac{\kappa_{11}^I b_\phi}{2\pi} \sum_{k=1}^{\infty} r^{k-1} k \sin(k-1)\theta \cdot b_{\phi k}^I, \tag{3.30}$$

$$D_y^I = -\frac{e_{15}^I b_z}{2\pi} \sum_{k=1}^{\infty} r^{k-1} k \cos(k-1)\theta \cdot b_{uk}^I + \frac{\kappa_{11}^I b_\phi}{2\pi} \sum_{k=1}^{\infty} r^{k-1} k \cos(k-1)\theta \cdot b_{\phi k}^I, \tag{3.31}$$

$$\begin{aligned}
\sigma_{zx}^{II} = & -\frac{c_{44}^{II} b_z}{2\pi} \sum_{k=1}^{\infty} \left[r^{k-1} k \sin(k-1)\theta \cdot b_{uk}^{II} + r^{-(k+1)} k \sin(k+1)\theta \cdot b_{-uk}^{II} \right] \\
& - \frac{e_{15}^{II} b_\phi}{2\pi} \sum_{k=1}^{\infty} \left[r^{k-1} k \sin(k-1)\theta \cdot b_{\phi k}^{II} + r^{-(k+1)} k \sin(k+1)\theta \cdot b_{-\phi k}^{II} \right],
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\sigma_{zy}^{II} = & -\frac{c_{44}^{II} b_z}{2\pi} \sum_{k=1}^{\infty} \left[r^{k-1} k \cos(k-1)\theta \cdot b_{uk}^{II} - r^{-(k+1)} k \cos(k+1)\theta \cdot b_{-uk}^{II} \right] \\
& - \frac{e_{15}^{II} b_\phi}{2\pi} \sum_{k=1}^{\infty} \left[r^{k-1} k \cos(k-1)\theta \cdot b_{\phi k}^{II} - r^{-(k+1)} k \cos(k+1)\theta \cdot b_{-\phi k}^{II} \right],
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
D_x^{II} = & -\frac{e_{15}^{II} b_z}{2\pi} \sum_{k=1}^{\infty} \left[r^{k-1} k \sin(k-1)\theta \cdot b_{uk}^{II} + r^{-(k+1)} k \sin(k+1)\theta \cdot b_{-uk}^{II} \right] \\
& + \frac{\kappa_{11}^{II} b_\phi}{2\pi} \sum_{k=1}^{\infty} \left[r^{k-1} k \sin(k-1)\theta \cdot b_{\phi k}^{II} + r^{-(k+1)} k \sin(k+1)\theta \cdot b_{-\phi k}^{II} \right],
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
D_y^{II} = & -\frac{e_{15}^{II} b_z}{2\pi} \sum_{k=1}^{\infty} \left[r^{k-1} k \cos(k-1)\theta \cdot b_{uk}^{II} + r^{-(k+1)} k \cos(k+1)\theta \cdot b_{-uk}^{II} \right] \\
& + \frac{\kappa_{11}^{II} b_\phi}{2\pi} \sum_{k=1}^{\infty} \left[r^{k-1} k \cos(k-1)\theta \cdot b_{\phi k}^{II} + r^{-(k+1)} k \cos(k+1)\theta \cdot b_{-\phi k}^{II} \right],
\end{aligned} \tag{3.35}$$

$$\sigma_{zx}^{III} = -\frac{c_{44}^{III} b_z}{2\pi} \left[\frac{y}{(x-d)^2 + y^2} + \sum_{k=1}^{\infty} r^{-(k+1)} k \sin(k+1)\theta \cdot b_{-uk}^{III} \right] - \frac{e_{15}^{III} b_\phi}{2\pi} \left[\frac{y}{(x-d)^2 + y^2} + \sum_{k=1}^{\infty} r^{-(k+1)} k \sin(k+1)\theta \cdot b_{-\phi k}^{III} \right], \quad (3.36)$$

$$\sigma_{yx}^{III} = -\frac{c_{44}^{III} b_z}{2\pi} \left[\frac{x-d}{(x-d)^2 + y^2} + \sum_{k=1}^{\infty} r^{-(k+1)} k \cos(k+1)\theta \cdot b_{-uk}^{III} \right] - \frac{e_{15}^{III} b_\phi}{2\pi} \left[\frac{x-d}{(x-d)^2 + y^2} + \sum_{k=1}^{\infty} r^{-(k+1)} k \cos(k+1)\theta \cdot b_{-\phi k}^{III} \right], \quad (3.37)$$

$$D_x^{III} = -\frac{e_{15}^{III} b_z}{2\pi} \left[\frac{y}{(x-d)^2 + y^2} + \sum_{k=1}^{\infty} r^{-(k+1)} k \sin(k+1)\theta \cdot b_{-uk}^{III} \right] + \frac{\kappa_{11}^{III} b_\phi}{2\pi} \left[\frac{y}{(x-d)^2 + y^2} + \sum_{k=1}^{\infty} r^{-(k+1)} k \sin(k+1)\theta \cdot b_{-\phi k}^{III} \right], \quad (3.38)$$

$$D_y^{III} = -\frac{e_{15}^{III} b_z}{2\pi} \left[\frac{x-d}{(x-d)^2 + y^2} + \sum_{k=1}^{\infty} r^{-(k+1)} k \cos(k+1)\theta \cdot b_{-uk}^{III} \right] + \frac{\kappa_{11}^{III} b_\phi}{2\pi} \left[\frac{x-d}{(x-d)^2 + y^2} + \sum_{k=1}^{\infty} r^{-(k+1)} k \cos(k+1)\theta \cdot b_{-\phi k}^{III} \right]. \quad (3.39)$$

In the case that all the three materials are purely elastic, equations (3.28)~(3.39) are fully reduced to the relations of Xiao and Chen (2000).

3.4 Image Force on the Dislocation

One of the major interests in discussing dislocation problems is the image force acting on the dislocation. Following Pak (1990b), the generalized Peach Koehler forces acting on a screw dislocation with a line force and a line charge can be expressed as

$$F_x = b_z \sigma_{zy}^T + b_\phi D_y^T + p \gamma_{zx}^T + q E_x^T, \quad (3.40)$$

$$F_y = -b_z \sigma_{zx}^T - b_\phi D_x^T + p \gamma_{zy}^T + q E_y^T. \quad (3.41)$$

where p is the line force at the dislocation core, q is the line charge at the dislocation core, and the variable $\sigma_{zy}^T, \sigma_{zx}^T, D_y^T, D_x^T, \gamma_{zy}^T, \gamma_{zx}^T, E_y^T$ and E_x^T can be obtained by subtracting the field generated by the dislocation from Eqs (3.36) to (3.39) in the forms

$$\gamma_{zx}^T = 0, \quad (3.42)$$

$$\gamma_{zy}^T = \frac{b_z}{2\pi} \sum_{k=1}^{\infty} d^{-(k+1)} k \cdot b_{-uk}^{III}, \quad (3.43)$$

$$E_x^T = 0, \quad (3.44)$$

$$E_y^T = -\frac{b_\phi}{2\pi} \sum_{k=1}^{\infty} d^{-(k+1)} k \cdot b_{-\phi k}^{III}, \quad (3.45)$$

$$\sigma_{zx}^T = 0, \quad (3.46)$$

$$\sigma_{zy}^T = \frac{c_{44}^{III} b_z}{2\pi} \left[\sum_{k=1}^{\infty} d^{-(k+1)} k \cdot b_{-uk}^{III} \right] + \frac{e_{15}^{III} b_\phi}{2\pi} \left[\sum_{k=1}^{\infty} d^{-(k+1)} k \cdot b_{-\phi k}^{III} \right], \quad (3.47)$$

$$D_x^T = 0, \quad (3.48)$$

$$D_y^T = \frac{e_{15}^{III} b_z}{2\pi} \left[\sum_{k=1}^{\infty} d^{-(k+1)} k \cdot b_{-uk}^{III} \right] - \frac{\kappa_{11}^{III} b_\phi}{2\pi} \left[\sum_{k=1}^{\infty} d^{-(k+1)} k \cdot b_{-\phi k}^{III} \right]. \quad (3.49)$$

3.5 Examples and Discussions

In the previous sections, a closed-form analytical solution has been derived for a screw dislocation near a coated circular inclusion in piezoelectric material. Image forces due to the existence of the inclusion are then obtained. In this section, examples are given to illustrate how the material properties of the inclusion and the coating layer affect the image forces. In the calculation, we only discuss the case when there is no line force and line charge, i.e. $p = q = 0$, such that the expressions of the image forces are simplified as

$$F_x = \frac{1}{2\pi} \sum_{k=1}^{\infty} \left[b_{-uk}^{III} \left(b_z^2 c_{44}^{III} + b_z b_\phi e_{15}^{III} \right) + b_{-\phi k}^{III} \left(b_z b_\phi e_{15}^{III} - b_\phi^2 \kappa_{11}^{III} \right) \right] d^{-(k+1)} k, \quad (3.50)$$

$$F_y = 0. \quad (3.51)$$

And in below numerical examples (also in Chapter 4 and 5), the parameter k was chosen to be 50 for its convergence and lower computer consuming time according to the trial calculation results.

Figure 3.2~Figure 3.7 depict how the ratio of elastic constant of the coating layer to that of the matrix, the ratio of the piezoelectric constant of the coating layer to that of the matrix, as well as the ratio b/a (the normalized coating thickness) affect the image forces on the dislocation. To view the effects more explicitly, we allow the dislocation to have only one non-zero strength characteristic (either b_z or b_ϕ) and the other one is zero in each plotted curve. The normalizing factors in each curve are given separately by

$$F_{b_z}^0 = \frac{c_{44}^{III} b_z^2}{2\pi b}, F_{b_\phi}^0 = \frac{\kappa_{11}^{III} b_\phi^2}{2\pi b}. \quad (3.52)$$

Figure 3.2 $F_{xb_z}^*$ versus $c_{44}^{II} / c_{44}^{III}$ with $e_{15}^I : e_{15}^{II} : e_{15}^{III} = 25 : 1 : 5$, $\kappa_{11}^I : \kappa_{11}^{II} : \kappa_{11}^{III} = 1 : 1 : 1$,
 $c_{44}^I / c_{44}^{III} = m$.

Figure 3.3 $F_{xb_\phi}^*$ versus c_{44}^{II}/c_{44}^{III} with $e_{15}^I : e_{15}^{II} : e_{15}^{III} = 25:1:5$, $\kappa_{11}^I : \kappa_{11}^{II} : \kappa_{11}^{III} = 1:1:1$,
 $c_{44}^I/c_{44}^{III} = m$

For example, $F_{xb_z}^*$ in the figures takes the expression of $F_x / F_{b_z}^0$, and $F_{xb_\phi}^*$ in the figures takes the expression of $F_x / F_{b_\phi}^0$.

Figure 3.2 shows how the material constant ratio $c_{44}^{II} / c_{44}^{III}$ changes the force on the mechanical dislocation b_z for the cases where elastically the inclusion is “harder” ($m = c_{44}^I / c_{44}^{III} = 5$), “softer” ($m = c_{44}^I / c_{44}^{III} = 0.2$) and the same as the matrix ($m = 1$), respectively. Here the word “harder” is used to represent a larger material modulus and “softer” means smaller material modulus. The inclusion attracts the dislocation when $c_{44}^{II} / c_{44}^{III}$ has small values, while repels it when $c_{44}^{II} / c_{44}^{III}$ increases. As for the electric-potential dislocation b_ϕ , the inclusion only repels the dislocation at small ratio $c_{44}^{II} / c_{44}^{III}$ (Figure 3.3).

The influences of the piezoelectric constant ratio $e_{15}^{II} / e_{15}^{III}$ on the image forces for the mechanical dislocation b_z and electric-potential dislocation b_ϕ can be observed from Figure 3.4 and Figure 3.5, respectively. In the figures, $e_{15}^{II} / e_{15}^{III} < 0$ indicates that the poling directions of the two piezoelectric materials are opposite. By increasing the absolute value of the ratio $e_{15}^{II} / e_{15}^{III}$, the inclusion first attracts the mechanical dislocation, and then repels it. As for the electric-potential dislocation, the inclusion almost always attracts it.

Figure 3.4 $F_{xb_z}^*$ versus $e_{15}^{II} / e_{15}^{III}$ with $c_{44}^I : c_{44}^{II} : c_{44}^{III} = 25 : 1 : 5$, $\kappa_{11}^I : \kappa_{11}^{II} : \kappa_{11}^{III} = 1 : 1 : 1$

Figure 3.5 $F_{xb_\theta}^*$ versus e_{15}^{II}/e_{15}^{III} with $c_{44}^I : c_{44}^{II} : c_{44}^{III} = 25 : 1 : 5$, $\kappa_{11}^I : \kappa_{11}^{II} : \kappa_{11}^{III} = 1 : 1 : 1$

The influences of the coating thickness on the image forces on the mechanical dislocation b_z and the electric-potential dislocation b_ϕ are shown in Figure 3.6 and Figure 3.7, respectively, for the case that $c_{44}^I / c_{44}^{III} = 2$, $c_{44}^{II} / c_{44}^{III} = 0.7$, $e_{15}^I / e_{15}^{III} = 2.5$, $e_{15}^{II} / e_{15}^{III} = 0.5$, $\kappa_{11}^I / \kappa_{11}^{III} = 3$, $\kappa_{11}^{II} / \kappa_{11}^{III} = 0.3$. In such case, increasing the coating layer thickness will change the image force from positive to negative values for the mechanical dislocation, while the situation is reversed for the electric-potential dislocation.

Figure 3.6 $F_{xb_z}^*$ versus b/a with $c_{44}^I / c_{44}^{III} = 2$, $c_{44}^{II} / c_{44}^{III} = 0.7$, $e_{15}^I / e_{15}^{III} = 2.5$, $e_{15}^{II} / e_{15}^{III} = 0.5$, $\kappa_{11}^I / \kappa_{11}^{III} = 3$, $\kappa_{11}^{II} / \kappa_{11}^{III} = 0.3$

Figure 3.7 $F_{xb_\phi}^*$ versus b/a with $c_{44}^I/c_{44}^{III} = 2$, $c_{44}^{II}/c_{44}^{III} = 0.7$, $e_{15}^I/e_{15}^{III} = 2.5$, $e_{15}^{II}/e_{15}^{III} = 0.5$,
 $\kappa_{11}^I/\kappa_{11}^{III} = 3$, $\kappa_{11}^{II}/\kappa_{11}^{III} = 0.3$

The above examples emphasized the influences of the inclusion on the separated dislocation strength characteristic (b_z or b_ϕ), which did not take into account the coupling effects between the mechanical dislocation b_z and the electric-potential dislocation b_ϕ . In the following examples, we assume that the material of the matrix is a practical engineering material PZT-5H, whose material properties are taken from Pak (1992)

$$\mathbf{C}_{III} = \begin{bmatrix} 3.53 \times 10^{10} \text{ N/m}^2 & 17.0 \text{ C/m}^2 \\ 17.0 \text{ C/m}^2 & 151 \times 10^{-10} \text{ C/Vm} \end{bmatrix}. \quad (3.53)$$

A piezoelectric screw dislocation with Burger's vector

$$\mathbf{b} = \begin{bmatrix} b_z & b_\phi \end{bmatrix}^T = \begin{bmatrix} 1.0 \times 10^{-9} \text{ m} & 1.0 \text{ V} \end{bmatrix}^T \quad (3.54)$$

is located at the point $(d, 0)$ in the matrix.

The normalized forces on the dislocation $F_{xb_z}^*$ versus d/b with different values of c_{44}^I / c_{44}^{III} and e_{15}^I / e_{15}^{III} are depicted in Figure 3.8 and Figure 3.9 respectively, when $\mathbf{C}_{II} = 5\mathbf{C}_{III}$; While the normalized forces on the dislocation $F_{xb_z}^*$ versus d/b with different values of $c_{44}^{II} / c_{44}^{III}$ and $e_{15}^{II} / e_{15}^{III}$ are depicted in Figure 3.10 and Figure 3.11 respectively when $\mathbf{C}_I = 5\mathbf{C}_{III}$.

From Figure 3.8 and Figure 3.9, we observed that the “harder” coating layer ($\mathbf{C}_{II} = 5\mathbf{C}_{III}$) always repels the piezoelectric screw dislocation no matter whether the inclusion is “harder” or “softer” than the matrix.

Figure 3.8 $F_{xb_z}^*$ versus d/b with $c_{44}^{II}/c_{44}^{III}=2$, $e_{15}^{II}/e_{15}^{III}=2.5$, $\kappa_{11}^{II}/\kappa_{11}^{III}=3$, $e_{15}^I/e_{15}^{III}=1$,
 $\kappa_{11}^I/\kappa_{11}^{III}=1$, $c_{44}^I/c_{44}^{III}=m1$

Figure 3.9 $F_{xb_z}^*$ versus d/b with $c_{44}^{II}/c_{44}^{III} = 2$, $e_{15}^{II}/e_{15}^{III} = 2.5$, $\kappa_{11}^{II}/\kappa_{11}^{III} = 3$, $c_{44}^I/c_{44}^{III} = 1$,
 $\kappa_{11}^I/\kappa_{11}^{III} = 1$, $e_{15}^I/e_{15}^{III} = n1$

Figure 3.10 $F_{xb_z}^*$ versus d/b with $c_{44}^I/c_{44}^{III} = 2$, $e_{15}^I/e_{15}^{III} = 2.5$, $\kappa_{11}^I/\kappa_{11}^{III} = 3$, $e_{15}^{II}/e_{15}^{III} = 1$,
 $\kappa_{11}^{II}/\kappa_{11}^{III} = 1$, $c_{44}^{II}/c_{44}^{III} = m_2$

Figure 3.11 $F_{xb_z}^*$ versus d/b with $c_{44}^I / c_{44}^{III} = 2$, $e_{15}^I / e_{15}^{III} = 2.5$, $\kappa_{11}^I / \kappa_{11}^{III} = 3$, $\kappa_{11}^{II} / \kappa_{11}^{III} = 1$,
 $c_{44}^{II} / c_{44}^{III} = 1$, $e_{15}^{II} / e_{15}^{III} = n_2$

From Figure 3.10, we can find that when the dislocation is moving away from the inclusion, the inclusion first attracts, then repels the dislocation in the case when the coating layer is elastically “softer” than the matrix ($c_{44}^{II} < c_{44}^{III}$) and always repels the dislocation when the coating layer is elastically “harder” ($c_{44}^{II} > c_{44}^{III}$). The above result is based on the condition that the inclusion is harder than matrix ($\mathbf{C}_I = 5\mathbf{C}_{III}$). On the other hand, it is observed that the influence of the polarity of the coating layer on the image forces is not significant, as shown in Figure 3.11.

3.6 Conclusions

A closed-form analytical solution in series form is obtained for a piezoelectric screw dislocation interacting with a three-phase piezoelectric material by using the complex variable method. Some numerical examples are studied and discussed for the image force on the dislocation. The results indicated that the coating layer plays an important role in the inclusion-dislocation interaction problem. The obtained explicit analytical solution can be used as a Green’s function for solving related crack-inclusion interaction problems in piezoelectric composite materials.

CHAPTER 4

ELECTRO-ELASTIC STRESS ANALYSIS FOR A ZENER-STROH CRACK INTERACTING WITH A COATED INCLUSION

4.1 Introduction

In this chapter, the physical problem of a Zener-Stroh crack interacting with a coated inclusion embedded in an infinite solid has been investigated. The single dislocation solution obtained in the previous chapter is used as a Greens function to formulate the current Zener-Stroh crack problem. Three dissimilar material phases are involved: the matrix, the inclusion and the coating layer. All the three materials are piezoelectric and have different material properties. The crack is in the matrix material and near to the coated inclusion. The engineering background is to study the fracture toughness of fiber-reinforced piezoelectric composites, with the fiber being coated and simulated as a coated circular inclusion. It is to study how the crack behavior is influenced by the coated inclusion. The stress and electric displacement intensity factors are derived and some numerical examples are calculated and discussed.

4.2 Formulation

Figure 4.1 A Zener-Stroh crack near a coated inclusion

The physical problem to be investigated is shown in Figure 4.1, where a circular piezoelectric inclusion with an intermediate coating layer is embedded in an infinite piezoelectric material. A Zener-Stroh micro crack is initiated near the coated circular inclusion, and is located in the radial direction within the matrix phase. The three-phase piezoelectric materials occupy the region *I* (the inclusion), *II* (the coating), and *III* (the matrix), respectively. All the three phases are assumed to have the same material orientation poled along the *z*-direction with isotropic *Oxy*-plane. For the current engineering problem, it is assumed there is no external mechanical loading and electrical field. In other words, the crack is loaded purely by the net mechanical and electro-

potential dislocations inside the crack. It is worth to mention that if external mechanical loading and electrical field exist, we can simply superpose those loads to the current dislocation loading, without changing the formulation procedure drastically.

The Zener-Stroh crack is modeled by a continuous distribution of dislocations along the crack line. The stress and electric displacement fields can be obtained by integrating the influence of all the distributed dislocations along the crack line. In order to get the Green's function solution for the current problem, we consider a single charged screw dislocation at the point $(\xi, 0)$ in the matrix phase first. The Burgers vector for the displacement discontinuity at the dislocation core is denoted by b_z , and the electric potential across the slip plane at the dislocation core is denoted by b_φ , respectively. Based on the work given in the previous chapter, due to the interaction between a single screw dislocation and the coated inclusion, the shear stress and electric displacement are given by

$$\sigma_{zy}(x, \xi)|_{y=0} = \frac{1}{2\pi} \left[c_{44}^{(3)} b_z(\xi) \left(\sum_{k=1}^{\infty} kx^{-(k+1)} b_{-uk}^{III} + \frac{1}{x-\xi} \right) + e_{15}^{(3)} b_\varphi(\xi) \left(\sum_{k=1}^{\infty} kx^{-(k+1)} b_{-\varphi k}^{III} + \frac{1}{x-\xi} \right) \right], \quad (4.1)$$

$$D_y(x, \xi)|_{y=0} = \frac{1}{2\pi} \left[e_{15}^{(3)} b_z(\xi) \left(\sum_{k=1}^{\infty} kx^{-(k+1)} b_{-uk}^{III} + \frac{1}{x-\xi} \right) - \kappa_{11}^{(3)} b_\varphi(\xi) \left(\sum_{k=1}^{\infty} kx^{-(k+1)} b_{-\varphi k}^{III} + \frac{1}{x-\xi} \right) \right]. \quad (4.2)$$

From the result of the previous chapter, the coefficients b_{-uk}^{III} , $b_{-\varphi k}^{III}$ can be expressed as

$$b_{-uk}^{III} = \frac{\xi^{-k}}{kb_z(\xi)} [A_1 b_z(\xi) + A_2 b_\varphi(\xi)], \quad (4.3)$$

$$b_{-\varphi k}^{III} = \frac{\xi^{-k}}{kb_{\varphi}(\xi)} [B_1 b_z(\xi) + B_2 b_{\varphi}(\xi)], \quad (4.4)$$

where A_1 , A_2 , B_1 and B_2 are constants which can be decided by the material properties of the three phases once they are known.

Then Eqs (4.1) and (4.2) can be rewritten as

$$\begin{aligned} \sigma_{zy}(x, \xi)|_{y=0} = & -\frac{1}{2\pi} [c_{44}^{(3)} b_z(\xi) + e_{15}^{(3)} b_{\varphi}(\xi)] \frac{1}{x-\xi} \\ & -\frac{1}{2\pi} \left[(c_{44}^{(3)} A_1 + e_{15}^{(3)} B_1) b_z(\xi) + (c_{44}^{(3)} A_2 + e_{15}^{(3)} B_2) b_{\varphi}(\xi) \right] \sum_{k=1}^{\infty} x^{-(k+1)} \xi^{-k}, \end{aligned} \quad (4.5)$$

$$\begin{aligned} D_y(x, \xi)|_{y=0} = & \frac{1}{2\pi} [e_{15}^{(3)} b_z(\xi) - \kappa_{11}^{(3)} b_{\varphi}(\xi)] \frac{1}{x-\xi} \\ & + \frac{1}{2\pi} \left[(e_{15}^{(3)} A_1 - \kappa_{11}^{(3)} B_1) b_z(\xi) + (e_{15}^{(3)} A_2 - \kappa_{11}^{(3)} B_2) b_{\varphi}(\xi) \right] \sum_{k=1}^{\infty} x^{-(k+1)} \xi^{-k}. \end{aligned} \quad (4.6)$$

With the aid of the above equations the traction free conditions on the crack surface are written as

$$\begin{aligned} & \frac{1}{2\pi} \int_{t_1}^{t_2} \frac{c_{44}^{(3)} b_z(\xi) + e_{15}^{(3)} b_{\varphi}(\xi)}{x-\xi} d\xi \\ & + \frac{1}{2\pi} \int_{t_1}^{t_2} \left[(c_{44}^{(3)} A_1 + e_{15}^{(3)} B_1) b_z(\xi) + (c_{44}^{(3)} A_2 + e_{15}^{(3)} B_2) b_{\varphi}(\xi) \right] \sum_{k=1}^{\infty} x^{-(k+1)} \xi^{-k} d\xi = 0 \end{aligned} \quad (4.7)$$

$t_1 < x < t_2,$

$$\begin{aligned} & \frac{1}{2\pi} \int_{t_1}^{t_2} \frac{e_{15}^{(3)} b_z(\xi) - \kappa_{11}^{(3)} b_{\varphi}(\xi)}{x-\xi} d\xi + \\ & \frac{1}{2\pi} \int_{t_1}^{t_2} \left[(e_{15}^{(3)} A_1 - \kappa_{11}^{(3)} B_1) b_z(\xi) + (e_{15}^{(3)} A_2 - \kappa_{11}^{(3)} B_2) b_{\varphi}(\xi) \right] \sum_{k=1}^{\infty} x^{-(k+1)} \xi^{-k} d\xi = 0 \end{aligned} \quad (4.8)$$

$t_1 < x < t_2.$

The $b_z(\xi)$, $b_{\varphi}(\xi)$ are the charged dislocation density functions along the crack line, respectively. They satisfy

$$\int_{t_1}^{t_2} b_z(\xi) d\xi = b_T \quad (4.9)$$

$$\int_{t_1}^{t_2} b_\varphi(\xi) d\xi = b_D, \quad (4.10)$$

where b_T , b_D are the total sums of Burgers vector b_z and electro-potential dislocation vector b_φ of the net charged dislocations inside the Zener-Stroh crack.

By introducing a set of dimensionless variables as follows

$$x = \frac{t_2 - t_1}{2} s + \frac{t_2 + t_1}{2}, \quad \xi = \frac{t_2 - t_1}{2} t + \frac{t_2 + t_1}{2}, \quad (4.11)$$

and letting the $G_{z\varphi}(t, s)$ denote the regular part of the kernel,

$$G_{z\varphi}(t, s) = \frac{t_2 - t_1}{2} \sum_{k=1}^{\infty} \left(\frac{t_2 - t_1}{2} s + \frac{t_2 + t_1}{2} \right)^{-(k+1)} \left(\frac{t_2 - t_1}{2} t + \frac{t_2 + t_1}{2} \right)^{-k}, \quad (4.12)$$

the Eqs.(4.7)~(4.10) are then written in dimensionless forms as

$$\begin{aligned} & \int_{-1}^1 b_z(t) \left(\frac{c_{44}^{(3)}}{s-t} + (c_{44}^{(3)} A_1 + e_{15}^{(3)} B_1) G_{z\varphi}(t, s) \right) dt \\ & + \int_{-1}^1 b_\varphi(t) \left(\frac{e_{15}^{(3)}}{s-t} + (c_{44}^{(3)} A_2 + e_{15}^{(3)} B_2) G_{z\varphi}(t, s) \right) dt = 0 \quad -1 < s < 1, \end{aligned} \quad (4.13)$$

$$\begin{aligned} & \int_{-1}^1 b_z(t) \left(\frac{e_{15}^{(3)}}{s-t} + (e_{15}^{(3)} A_1 - \kappa_{11}^{(3)} B_1) G_{z\varphi}(t, s) \right) dt \\ & - \int_{-1}^1 b_\varphi(t) \left(\frac{\kappa_{11}^{(3)}}{s-t} - (e_{15}^{(3)} A_2 - \kappa_{11}^{(3)} B_2) G_{z\varphi}(t, s) \right) dt = 0 \quad -1 < s < 1, \end{aligned} \quad (4.14)$$

$$\int_{-1}^1 b_z(t) dt = \frac{2b_T}{t_2 - t_1}, \quad (4.15)$$

$$\int_{-1}^1 b_\varphi(t) dt = \frac{2b_D}{t_2 - t_1}. \quad (4.16)$$

4.3 Numerical Procedure

Since the whole crack is located in the matrix material, the singularities on both crack tips should be inverse square root. As a result, the dislocation density functions must have inverse square root singular behaviour on the both crack tips. Follow Hills *et al.*'s technique (1996), let $F(t)$ and $G(t)$ be bounded function in $[-1, 1]$, and

$$b_z(t) = F(t)(1+t)^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}, \quad (4.17)$$

$$b_\varphi(t) = G(t)(1+t)^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}. \quad (4.18)$$

Eqs (4.13)~(4.16) are rewritten in discretized form as

$$\sum_j^n \frac{1}{n} F(t_j) \left(\frac{c_{44}^{(3)}}{s_i - t_j} + (c_{44}^{(3)} A_1 + e_{15}^{(3)} B_1) G_{z\varphi}(t_j, s_i) \right) + \sum_j^n \frac{1}{n} G(t_j) \left(\frac{e_{15}^{(3)}}{s_i - t_j} + (c_{44}^{(3)} A_2 + e_{15}^{(3)} B_2) G_{z\varphi}(t_j, s_i) \right) = 0, \quad (4.19)$$

$$\sum_j^n \frac{1}{n} F(t_j) \left(\frac{e_{15}^{(3)}}{s_i - t_j} + (e_{15}^{(3)} A_1 - \kappa_{11}^{(3)} B_1) G_{z\varphi}(t_j, s_i) \right) - \sum_j^n \frac{1}{n} G(t_j) \left(\frac{\kappa_{11}^{(3)}}{s_i - t_j} - (e_{15}^{(3)} A_2 - \kappa_{11}^{(3)} B_2) G_{z\varphi}(t_j, s_i) \right) = 0, \quad (4.20)$$

$$\sum_j^n \frac{1}{n} F(t_j) = \frac{b_T}{\pi} \frac{2}{t_2 - t_1}, \quad (4.21)$$

$$\sum_j^n \frac{1}{n} G(t_j) = \frac{b_D}{\pi} \frac{2}{t_2 - t_1}, \quad (4.22)$$

in which

$$s_i = \cos\left(\pi \frac{i}{n}\right), \quad t_j = \cos\pi\left(\frac{2j-1}{2n}\right), \quad j = 1, \dots, n; \quad i = 1, \dots, n-1. \quad (4.23)$$

Eqs. (4.19) to (4.22) provide a system of $2n$ linear algebraic equations to determine the values of the $F(s_i), \dots, F(s_n)$ and $G(s_i), \dots, G(s_n)$.

From the definition of stress intensity factors, the stress and electric displacement intensity factors at the blunt (left) tip and the sharp (right) tip of the crack can be derived as

$$\begin{aligned} K_{\sigma}^L &= -\lim_{\xi \rightarrow t_1} \sqrt{\frac{\pi}{2}} \sqrt{\xi - t_1} [c_{44}^{(3)} b_z(\xi) + e_{15}^{(3)} b_{\varphi}(\xi)] \\ &= -\frac{1}{\sqrt{2\pi(t_2 - t_1)}} [c_{44}^{(3)} b_T F(-1) + e_{15}^{(3)} b_D G(-1)] \end{aligned} \quad (4.24)$$

$$\begin{aligned} K_D^L &= -\lim_{\xi \rightarrow t_1} \sqrt{\frac{\pi}{2}} \sqrt{\xi - t_1} [e_{15}^{(3)} b_z(\xi) - \kappa_{11}^{(3)} b_{\varphi}(\xi)] \\ &= -\frac{1}{\sqrt{2\pi(t_2 - t_1)}} [e_{15}^{(3)} b_T F(-1) - \kappa_{11}^{(3)} b_D G(-1)] \end{aligned} \quad (4.25)$$

$$\begin{aligned} K_{\sigma}^R &= \lim_{\xi \rightarrow t_2} \sqrt{\frac{\pi}{2}} \sqrt{t_2 - \xi} [c_{44}^{(3)} b_z(\xi) + e_{15}^{(3)} b_{\varphi}(\xi)] \\ &= \frac{1}{\sqrt{2\pi(t_2 - t_1)}} [c_{44}^{(3)} b_T F(1) + e_{15}^{(3)} b_D G(1)] \end{aligned} \quad (4.26)$$

$$\begin{aligned} K_D^R &= \lim_{\xi \rightarrow t_2} \sqrt{\frac{\pi}{2}} \sqrt{t_2 - \xi} [e_{15}^{(3)} b_z(\xi) - \kappa_{11}^{(3)} b_{\varphi}(\xi)] \\ &= \frac{1}{\sqrt{2\pi(t_2 - t_1)}} [e_{15}^{(3)} b_T F(1) - \kappa_{11}^{(3)} b_D G(1)] \end{aligned} \quad (4.27)$$

where the superscripts L and R represent the left and right crack tips, respectively.

4.4 Numerical Results and Discussions

For the numerical examples given in this section, the stress and electric displacement intensity factors of the crack are normalized by

$$K_{\sigma}^0 = \frac{c_{44}^{(3)} b_T + e_{15}^{(3)} b_D}{\sqrt{2\pi(t_2 - t_1)}} \quad \text{and} \quad K_D^0 = \frac{e_{15}^{(3)} b_T - \kappa_{11}^{(3)} b_D}{\sqrt{2\pi(t_2 - t_1)}} \quad (4.28)$$

respectively. They are the respective stress and electric displacement intensity factors at the sharp tip of the same size Zener-Stroh crack with the same extended Burgers vectors

(b_T, b_D) in a homogeneous piezoelectric material (without the coated inclusion). In the calculations, it is found that the stress and electric displacement intensity factors at the blunt tip (the left crack tip) are always negative. This is similar to case when all the three materials involved are purely elastic. So the propagation of a Zener-Stroh crack is always along the sharp crack tip.

To reveal the physical mechanism of the problem more explicitly, we allow the total dislocation to have only one non-zero strength characteristic (either b_T or b_D) and the other one is zero in each plotted curve. A practical engineering material PZT-5H is chosen as the matrix material whose properties are (Pak, 1992):

$$C_{III} = \begin{bmatrix} 3.53 \times 10^{10} \text{ N/m}^2 & 17.0 \text{ C/m}^2 \\ 17.0 \text{ C/m}^2 & -151 \times 10^{-10} \text{ C/Vm} \end{bmatrix}. \quad (4.29)$$

Figs 4.2~4.5 plot the variations of normalized stress and electric displacement intensity factors (the SIFs and EDIFs) with the material constant ratio $c_{44}^{(2)} / c_{44}^{(3)}$, for both crack tips and different coating layer thickness $b/a=1.02, 1.5$, respectively. The other parameters are taken as $t_1=1.05b$, $t_2=1.35b$ and $e_{15}^{(1)} : e_{15}^{(2)} : e_{15}^{(3)} = 25 : 1 : 5$, $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $c_{44}^{(1)} / c_{44}^{(3)} = 5$. It is shown that the influence of the elastic moduli of the coated layer becomes less sensitive when b/a increases for the SIF and EDIF. In Fig 4.2 and Fig 4.3, the curves are close to each other and intersect at the point where $c_{44}^{(1)} = c_{44}^{(2)}$. At this point, the current problem is reduced to the two-phase case (without the coating layer). It can be concluded that the coating layer thickness has little influence to the SIFs. In the figure, there is no intersection for the normalized EDIFs, this is different to the SIFs. This phenomenon indicates that the EDIFs are more sensitive to the variation of the coating layer thickness than the elastic material constants. For the right crack tip (Fig. 4.3), the

value of normalized SIF is greater than 1. This indicates that a “hard” inclusion (whose elastic modulus is higher than that of the matrix) makes a displacement loaded Zener-Stroh crack easier to propagate, which is totally opposite to a Griffith crack case: a “hard” inclusion retard the propagation of a stress loaded Griffith crack. It is worth to note that the Zener-Stroh mechanism dominates the crack behaviour only when the crack is just initiated at its micro level. As the crack size increases, the Griffith mechanism will dominate the propagation of the crack.

Figure 4.2 Normalized K_{σ}^L verse $c_{44}^{(2)} / c_{44}^{(3)}$ with $e_{15}^{(1)} : e_{15}^{(2)} : e_{15}^{(3)} = 25 : 1 : 5$,
 $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $c_{44}^{(1)} / c_{44}^{(3)} = 5$

Figure 4.3 Normalized K_σ^R verse $c_{44}^{(2)}/c_{44}^{(3)}$ with $e_{15}^{(1)}:e_{15}^{(2)}:e_{15}^{(3)} = 25:1:5$,
 $\kappa_{11}^{(1)}:\kappa_{11}^{(2)}:\kappa_{11}^{(3)} = 1:1:1$, $c_{44}^{(1)}/c_{44}^{(3)} = 5$

Figure 4.4 Normalized K_D^L verse $c_{44}^{(2)} / c_{44}^{(3)}$ with $e_{15}^{(1)} : e_{15}^{(2)} : e_{15}^{(3)} = 25 : 1 : 5$,
 $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $c_{44}^{(1)} / c_{44}^{(3)} = 5$

Figure 4.5 Normalized K_D^R verse $c_{44}^{(2)} / c_{44}^{(3)}$ with $e_{15}^{(1)} : e_{15}^{(2)} : e_{15}^{(3)} = 25 : 1 : 5$,
 $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $c_{44}^{(1)} / c_{44}^{(3)} = 5$

The variations of normalized SIFs and EDIFs with the piezoelectric constant ratio $e_{15}^{(2)} / e_{15}^{(3)}$ are depicted in Figs. 4.6~4.9, with the other parameters are taken as $t_1=1.05b$, $t_2=1.35b$ and $c_{44}^{(1)} : c_{44}^{(2)} : c_{44}^{(3)} = 25 : 1 : 5$, $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $e_{15}^{(1)} / e_{15}^{(3)} = 5$. In the figures, $e_{15}^{(2)} / e_{15}^{(3)} < 0$ indicates that the poling directions of the two piezoelectric materials (coating and the matrix) are opposite. By increasing the absolute value of the ratio $e_{15}^{(2)} / e_{15}^{(3)}$, the values of the SIFs and EDIFs are both increased. And the variation ranges of the electric displacement intensity factors are smaller than those of the stress intensity factors.

Figure 4.10 and 4.11 plot the variations of normalized SIF and EDIF with the coating layer thickness ratio b/a . Since the SIF and EDIF at the blunt tip are always negative, we only discuss the results of the sharp tip (the right tip). The other parameters are taken as, $t_1=1.05b$, $t_2=1.35b$ and $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $c_{44}^{(2)} : c_{44}^{(3)} = 5 : 1$, $e_{15}^{(2)} : e_{15}^{(3)} = 5 : 1$. For a “soft” inclusion (the Young’s modulus of the inclusion is lower than that of the matrix), the elastic and piezoelectric constants are taken as $c_{44}^{(1)} : c_{44}^{(3)} = 1 : 5$, $e_{15}^{(1)} : e_{15}^{(3)} = 1 : 5$, the curve is plotted as the black line in the Fig 4.10 and 4.11. For a hard inclusion (the Young’s modulus of the inclusion is higher than that of the matrix), the elastic and piezoelectric constants are taken as $c_{44}^{(1)} : c_{44}^{(3)} = 25 : 1$, $e_{15}^{(1)} : e_{15}^{(3)} = 25 : 1$, which is plotted as the dash line in the Figs. 4.10 and 4.11. From the graph, we can find that the SIF and EDIF become a constant with the increasing of coating thickness no matter the inclusion is “soft” or “hard”. Obviously this is because that a very thick coating layer shields the inside inclusion and makes it have little influence on the crack.

Figure 4.6 Normalized K_{σ}^L verse $e_{15}^{(2)}/e_{15}^{(3)}$ with $c_{44}^{(1)} : c_{44}^{(2)} : c_{44}^{(3)} = 25 : 1 : 5$,
 $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $e_{15}^{(1)}/e_{15}^{(3)} = 5$

Figure 4.7 Normalized K_{σ}^R verse $e_{15}^{(2)} / e_{15}^{(3)}$ with
 $c_{44}^{(1)} : c_{44}^{(2)} : c_{44}^{(3)} = 25 : 1 : 5$, $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $e_{15}^{(1)} / e_{15}^{(3)} = 5$

Figure 4.8 Normalized K_D^L verse $e_{15}^{(2)} / e_{15}^{(3)}$ with $c_{44}^{(1)} : c_{44}^{(2)} : c_{44}^{(3)} = 25 : 1 : 5$,
 $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $e_{15}^{(1)} / e_{15}^{(3)} = 5$

Figure 4.9 Normalized K_D^R verse $e_{15}^{(2)}/e_{15}^{(3)}$ with
 $c_{44}^{(1)} : c_{44}^{(2)} : c_{44}^{(3)} = 25 : 1 : 5$, $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1 : 1 : 1$, $e_{15}^{(1)}/e_{15}^{(3)} = 5$

Figure 4.10 Normalized K_σ^R verse b/a with $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1:1:1$, $c_{44}^{(2)} : c_{44}^{(3)} = 5:1$,
 $e_{15}^{(2)} : e_{15}^{(3)} = 5:1$

Figure 4.11 Normalized K_D^R verse b/a with $\kappa_{11}^{(1)} : \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 1:1:1$, $c_{44}^{(2)} : c_{44}^{(3)} = 5:1$,
 $e_{15}^{(2)} : e_{15}^{(3)} = 5:1$

4.5 Conclusions

The problem of a Zener-Stroh crack initiated around a coated inclusion in piezoelectric solid is investigated. The stress and electric displacement intensity factors for both crack tips are derived. The effects of various material constants on the crack are studied in details by numerical examples. It is found the SIFs and EDIFs at the blunt crack tip are always negative, which indicates the propagation of a Zener-Stroh crack is always along its sharp tip.

CHAPTER 5

ELECTRO-ELASTIC STRESS ANALYSIS FOR A GRIFFITH CRACK INTERACTING WITH A COATED INCLUSION

5.1 Introduction

In most practical engineering applications, structures are applied by remote mechanical or electrical loads, Griffith cracks become the major pattern of failure for piezoelectric composites. Cracks are formed during manufacturing, or initiated during loading, especially during cyclic loading. To accurately predict the fracture behavior of a crack in a piezoelectric composite, the interaction between a Griffith crack and coated inclusion should be fully investigated.

By using the solution of a screw dislocation interacting with a coated inclusion derived as the Green's function, a Mode III Griffith crack near a coated inclusion embedded in an infinitely extended piezoelectric matrix under remote mechanical and electrical loading condition is studied. Dislocation distribution technique is adopted in the solution process.

The stress and electric displacement intensity factors are derived in terms of the asymptotic values of the dislocation density functions evaluated from the integral equations. Numerical examples are given for various material constant combinations and geometric parameters.

5.2 Formulation

Figure 5.1 A Griffith crack near a coated piezoelectric inclusion subject to far-field anti-plane shear and in-plane electric field

The physical problem to be studied is shown in Figure 5.1, where a circular inclusion with an intermediate coating layer is embedded in an infinite matrix material. A line crack near the coated circular inclusion is situated along the radial direction within the matrix phase.

The three dissimilar material phases are all piezoelectric, occupying the region 1 (the inclusion), 2 (the coating), and 3 (the matrix), respectively. These three material phases are assumed to have the same material orientation poled along the z -direction with isotropic Oxy -plane. For the current engineering problem, we assume that the model is subjected to a far-field anti-plane shear and a far-field in-plane electrical field. A uniform electric field can be induced in a homogeneous piezoelectric material by applying a constant potential jump across the model. In the current configuration, only the out-of-plane displacement w coupled with in-plane electrical field E_x and E_y need to be considered.

To solve the problem, the equivalence between a crack and a pile-up of dislocations is employed. The stress and electric displacement fields can be obtained by integrating the influence of all the distributed dislocations along the crack line. In order to obtain the Green's function solution for a single dislocation case, we consider a single charged screw dislocation at the point $(\xi, 0)$ in the matrix phase first. The Burgers vector for the displacement discontinuity at the dislocation core is denoted by b_z , and the electric potential across the slip plane at the dislocation core is denoted by b_ϕ , respectively. Based on previous work in chapter 3, the shear stress and electric displacement due to the interaction between a single screw dislocation and the coated inclusion, are given by

$$\sigma_{zy}(x, \xi)|_{y=0} = -\frac{1}{2\pi} \left[c_{44}^{(3)} b_z(\xi) \left(\sum_{k=1}^{\infty} k x^{-(k+1)} b_{-uk}^{III} + \frac{1}{x-\xi} \right) + e_{15}^{(3)} b_\phi(\xi) \left(\sum_{k=1}^{\infty} k x^{-(k+1)} b_{-\phi k}^{III} + \frac{1}{x-\xi} \right) \right], \quad (5.1)$$

$$D_y(x, \xi)|_{y=0} = -\frac{1}{2\pi} \left[e_{15}^{(3)} b_z(\xi) \left(\sum_{k=1}^{\infty} k x^{-(k+1)} b_{-uk}^{III} + \frac{1}{x-\xi} \right) - \kappa_{11}^{(3)} b_{\phi}(\xi) \left(\sum_{k=1}^{\infty} k x^{-(k+1)} b_{-\phi k}^{III} + \frac{1}{x-\xi} \right) \right]. \quad (5.2)$$

The coefficients b_{-uk}^{III} , $b_{-\phi k}^{III}$ in the above equations are expressed as

$$b_{-uk}^{III} = \frac{\xi^{-k}}{k b_z(\xi)} [A_1 b_z(\xi) + A_2 b_{\phi}(\xi)], \quad (5.3)$$

$$b_{-\phi k}^{III} = \frac{\xi^{-k}}{k b_{\phi}(\xi)} [B_1 b_z(\xi) + B_2 b_{\phi}(\xi)], \quad (5.4)$$

Eqns (5.1) and (5.2) can be rewritten as

$$\sigma_{zy}(x, \xi)|_{y=0} = -\frac{1}{2\pi} [c_{44}^{(3)} b_z(\xi) + e_{15}^{(3)} b_{\phi}(\xi)] \frac{1}{x-\xi} - \frac{1}{2\pi} [(c_{44}^{(3)} A_1 + e_{15}^{(3)} B_1) b_z(\xi) + (c_{44}^{(3)} A_2 + e_{15}^{(3)} B_2) b_{\phi}(\xi)] \sum_{k=1}^{\infty} x^{-(k+1)} \xi^{-k}, \quad (5.5)$$

$$D_y(x, \xi)|_{y=0} = -\frac{1}{2\pi} [e_{15}^{(3)} b_z(\xi) - \kappa_{11}^{(3)} b_{\phi}(\xi)] \frac{1}{x-\xi} - \frac{1}{2\pi} [(e_{15}^{(3)} A_1 - \kappa_{11}^{(3)} B_1) b_z(\xi) + (e_{15}^{(3)} A_2 - \kappa_{11}^{(3)} B_2) b_{\phi}(\xi)] \sum_{k=1}^{\infty} x^{-(k+1)} \xi^{-k}. \quad (5.6)$$

With the aid of above equations the traction free conditions on the crack surface are written as

$$\frac{1}{2\pi} \int_{t_1}^{t_2} \frac{c_{44}^{(3)} b_z(\xi) + e_{15}^{(3)} b_{\phi}(\xi)}{x-\xi} d\xi + \quad (5.7)$$

$$\frac{1}{2\pi} \int_{t_1}^{t_2} [(c_{44}^{(3)} A_1 + e_{15}^{(3)} B_1) b_z(\xi) + (c_{44}^{(3)} A_2 + e_{15}^{(3)} B_2) b_{\phi}(\xi)] \sum_{k=1}^{\infty} x^{-(k+1)} \xi^{-k} d\xi = p_{\sigma}(x) \\ t_1 < x < t_2,$$

$$\frac{1}{2\pi} \int_{t_1}^{t_2} \frac{e_{15}^{(3)} b_z(\xi) - \kappa_{11}^{(3)} b_{\phi}(\xi)}{x-\xi} d\xi + \quad (5.8)$$

$$\frac{1}{2\pi} \int_{t_1}^{t_2} [(e_{15}^{(3)} A_1 - \kappa_{11}^{(3)} B_1) b_z(\xi) + (e_{15}^{(3)} A_2 - \kappa_{11}^{(3)} B_2) b_{\phi}(\xi)] \sum_{k=1}^{\infty} x^{-(k+1)} \xi^{-k} d\xi = p_D(x) \\ t_1 < x < t_2.$$

Here $b_z(\xi)$ and $b_\varphi(\xi)$ are the charged dislocation density functions along the crack line, respectively. They satisfy

$$\int_{-1}^1 b_z(\xi) d\xi = 0 \quad (5.9)$$

$$\int_{-1}^1 b_\varphi(\xi) d\xi = 0 \quad (5.10)$$

In Equations (5.7) and (5.8), $p_\sigma(x)$ and $p_D(x)$ are the imaginative traction long the crack surface due to the far field anti-plane mechanical loading τ_∞ and in-plane electric loading D_∞ , when the crack has been removed in Figure 5.1. The solution of $p_\sigma(x)$ and $p_D(x)$ can be obtained and derived from Jiang and Cheung's results (2001). Detailed derivations for the expressions of $p_\sigma(x)$ and $p_D(x)$ are given in the Appendix B.

By introducing a set of dimensionless variables as follows

$$x = \frac{t_2 - t_1}{2} s + \frac{t_2 + t_1}{2}, \quad \xi = \frac{t_2 - t_1}{2} t + \frac{t_2 + t_1}{2}, \quad (5.11)$$

and letting the $G_{z\varphi}(t, s)$ denote the regular part of the kernel,

$$G_{z\varphi}(t, s) = \frac{t_2 - t_1}{2} \sum_{k=1}^{\infty} \left(\frac{t_2 - t_1}{2} s + \frac{t_2 + t_1}{2} \right)^{-(k+1)} \left(\frac{t_2 - t_1}{2} t + \frac{t_2 + t_1}{2} \right)^{-k}, \quad (5.12)$$

the Eqns. (5.7)~(5.10) are then written in dimensionless forms as

$$\begin{aligned} & \frac{1}{2\pi} \int_{-1}^1 b_z(t) \left(\frac{c_{44}^{(3)}}{s-t} + (c_{44}^{(3)} A_1 + e_{15}^{(3)} B_1) G_{z\varphi}(t, s) \right) dt \\ & + \frac{1}{2\pi} \int_{-1}^1 b_\varphi(t) \left(\frac{e_{15}^{(3)}}{s-t} + (c_{44}^{(3)} A_2 + e_{15}^{(3)} B_2) G_{z\varphi}(t, s) \right) dt = p_\sigma(s), \end{aligned} \quad (5.13)$$

$$\begin{aligned} & \frac{1}{2\pi} \int_{-1}^1 b_z(t) \left(\frac{e_{15}^{(3)}}{s-t} + (e_{15}^{(3)} A_1 - \kappa_{11}^{(3)} B_1) G_{z\varphi}(t, s) \right) dt \\ & - \frac{1}{2\pi} \int_{-1}^1 b_\varphi(t) \left(\frac{\kappa_{11}^{(3)}}{s-t} - (e_{15}^{(3)} A_2 - \kappa_{11}^{(3)} B_2) G_{z\varphi}(t, s) \right) dt = p_D(s), \end{aligned} \quad (5.14)$$

$$\int_{-1}^1 b_z(t) dt = 0, \quad (5.15)$$

$$\int_{-1}^1 b_{\varphi}(t) dt = 0. \quad (5.16)$$

5.3 Numerical Procedure

Since the whole crack is located in the matrix material, the singularities on both crack tips should be inverse square root. As a result, the dislocation density functions must have inverse square root singular behavior on the both crack tips. Follow Erdogan and Gupta's method (1972) which was summarized and extended by Hills *et al.* (1996), let $F(t)$ and $G(t)$ be bounded function in $[-1, 1]$, and

$$b_z(t) = F(t)(1+t)^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}, \quad (5.17)$$

$$b_{\varphi}(t) = G(t)(1+t)^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}. \quad (5.18)$$

Eqns (5.13) ~ (5.16) are rewritten in discrete form as

$$\begin{aligned} & \sum_j^n \frac{1}{n} F(t_j) \left(\frac{c_{44}^{(3)}}{s_i - t_j} + (c_{44}^{(3)} A_1 + e_{15}^{(3)} B_1) G_{z\varphi}(t_j, s_i) \right) \\ & + \sum_j^n \frac{1}{n} G(t_j) \left(\frac{e_{15}^{(3)}}{s_i - t_j} + (c_{44}^{(3)} A_2 + e_{15}^{(3)} B_2) G_{z\varphi}(t_j, s_i) \right) = 2p_{\sigma}(s_i), \end{aligned} \quad (5.19)$$

$$\begin{aligned} & \sum_j^n \frac{1}{n} F(t_j) \left(\frac{e_{15}^{(3)}}{s_i - t_j} + (e_{15}^{(3)} A_1 - \kappa_{11}^{(3)} B_1) G_{z\varphi}(t_j, s_i) \right) \\ & - \sum_j^n \frac{1}{n} G(t_j) \left(\frac{\kappa_{11}^{(3)}}{s_i - t_j} - (e_{15}^{(3)} A_2 - \kappa_{11}^{(3)} B_2) G_{z\varphi}(t_j, s_i) \right) = 2p_D(s_i), \end{aligned} \quad (5.20)$$

$$\sum_j^n \frac{1}{n} F(t_j) = 0, \quad (5.21)$$

$$\sum_j^n \frac{1}{n} G(t_j) = 0, \quad (5.22)$$

in which

$$s_i = \cos\left(\pi \frac{i}{n}\right), \quad t_j = \cos\left(\pi \frac{2j-1}{2n}\right), \quad j = 1, \dots, n; \quad i = 1, \dots, n-1. \quad (5.23)$$

Eqns. (5.19) to (5.22) provide a system of $2n$ linear algebraic equations to determine the values of the $F(t_i), \dots, F(t_n)$ and $G(t_i), \dots, G(t_n)$.

From the definition of stress intensity factors, the stress and electric displacement intensity factors at the left tip and the right tip of the crack can be derived as

$$\begin{aligned} K_{\sigma}^L &= -\lim_{\xi \rightarrow t_1} \sqrt{\frac{\pi}{2}} \sqrt{\xi - t_1} \left[c_{44}^{(3)} b_z(\xi) + e_{15}^{(3)} b_{\varphi}(\xi) \right] \\ &= -\sqrt{\frac{\pi}{2(t_2 - t_1)}} \left[c_{44}^{(3)} F(-1) + e_{15}^{(3)} G(-1) \right], \end{aligned} \quad (5.24)$$

$$\begin{aligned} K_D^L &= -\lim_{\xi \rightarrow t_1} \sqrt{\frac{\pi}{2}} \sqrt{\xi - t_1} \left[e_{15}^{(3)} b_z(\xi) - \kappa_{11}^{(3)} b_{\varphi}(\xi) \right] \\ &= -\sqrt{\frac{\pi}{2(t_2 - t_1)}} \left[e_{15}^{(3)} F(-1) - \kappa_{11}^{(3)} G(-1) \right], \end{aligned} \quad (5.25)$$

$$\begin{aligned} K_{\sigma}^R &= \lim_{\xi \rightarrow t_2} \sqrt{\frac{\pi}{2}} \sqrt{t_2 - \xi} \left[c_{44}^{(3)} b_z(\xi) + e_{15}^{(3)} b_{\varphi}(\xi) \right] \\ &= \sqrt{\frac{\pi}{2(t_2 - t_1)}} \left[c_{44}^{(3)} F(1) + e_{15}^{(3)} G(1) \right], \end{aligned} \quad (5.26)$$

$$\begin{aligned} K_D^R &= \lim_{\xi \rightarrow t_2} \sqrt{\frac{\pi}{2}} \sqrt{t_2 - \xi} \left[e_{15}^{(3)} b_z(\xi) - \kappa_{11}^{(3)} b_{\varphi}(\xi) \right] \\ &= \sqrt{\frac{\pi}{2(t_2 - t_1)}} \left[e_{15}^{(3)} F(1) - \kappa_{11}^{(3)} G(1) \right], \end{aligned} \quad (5.27)$$

where the superscripts L and R represent the left and right crack tips, respectively.

5.4 Numerical Results and Discussions

For the numerical examples given in this section, the stress and electric displacement intensity factors of the crack are normalized by

$$K_{\sigma}^0 = \tau_{\infty} \sqrt{\pi \left(\frac{t_2 - t_1}{2} \right)} \quad \text{and} \quad K_D^0 = D_{\infty} \sqrt{\pi \left(\frac{t_2 - t_1}{2} \right)}, \quad (5.28)$$

respectively. They are the Mode III stress intensity factor and electric displacement intensity factor of the same size Griffith crack in a homogenous piezoelectric material (Pak, 1990), respectively.

For the numerical examples, a practical engineering material PZT-5H is chosen as the matrix material; its material properties are given by (Deeg, 1980):

$$\begin{aligned} c_{44} &= 3.53 \times 10^{10} \text{ N/m}^2, \\ e_{15} &= 17.0 \text{ C/m}^2, \\ \kappa_{11} &= 151 \times 10^{-10} \text{ C/Vm}, \end{aligned} \tag{5.29}$$

where N is the force in Newtons, C is the charge in coulombs, V is the electric potential in volts and m is the length in meters.

In the following numerical examples, the influence of the three different materials properties, the distance between the crack and inclusion, and the thickness of the coating layer, etc., on the fracture behavior of the crack will be discussed. Since there are too many parameters involved in the numerical examples and discussions, the far-field stress and electric field conditions in our calculations are prescribed as

$$\tau_{yz} \Big|_{|z| \rightarrow \infty} = \tau_{\infty} = 4.2 \times 10^6 \text{ Nm}^{-2}, \tag{5.30}$$

$$D_y \Big|_{|z| \rightarrow \infty} = D_{\infty} = 2 \times 10^{-3} \text{ C/m}^2. \tag{5.31}$$

As we are more interested in the fracture behavior of the left crack tip (it is closer to the inclusion), only the SIFs and EDIFs at the left crack tip for various loading conditions are discussed below.

5.4.1 Influence of the Elastic Modulus of the Coating Layer

The variations of normalized stress and electrical displacement intensity factors at the left crack tip with $c_{44}^{(2)} / c_{44}^{(3)}$ for different coating layer thickness $b/a=1.02, 1.5$ under far-field shear load and electrical field were plotted in Figure 5.2 and Figure 5.3, respectively. The crack length and location are taken as $t_1 = 1.05b$ and $t_2 = 1.35b$. The other material parameter ratios are taken as $e_{15}^{(1)} / e_{15}^{(3)} = 2.5$, $e_{15}^{(2)} / e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)} / \kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)} / \kappa_{11}^{(3)} = 0.3$ and $c_{44}^{(1)} / c_{44}^{(3)} = 2$. It is shown that both the stress and electrical displacement intensity factors decrease with the increasing elastic modulus ratio $c_{44}^{(2)} / c_{44}^{(3)}$. For the stress intensity factor (SIF), when the coating layer is very thin ($b/a=1.02$), the variation of SIF is small when the elastic modulus ratio of the coating layer to the matrix increases. While for the electrical displacement factor (EDIF), the variation magnitude of EDIF is about the same for different coating layer thicknesses.

5.4.2 Influence of the Piezoelectric Coefficient of the Coating Layer

The variations of normalized stress and electrical displacement intensity factors at left crack tip with $e_{15}^{(2)} / e_{15}^{(3)}$ for different coating layer thickness $b/a=1.05, 1.5$ under far-field shear load and electrical field are plotted in Figure 5.4 and Figure 5.5, respectively. The crack length and location are also taken as $t_1 = 1.05b$ and $t_2 = 1.35b$. The other material parameters ratios are taken as $e_{15}^{(1)} / e_{15}^{(3)} = 2.5$, $e_{15}^{(2)} / e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)} / \kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)} / \kappa_{11}^{(3)} = 0.3$ and $c_{44}^{(1)} / c_{44}^{(3)} = 2$. In the figures, $e_{15}^{(2)} / e_{15}^{(3)} < 0$ indicates that the poling directions of the two piezoelectric materials (the coating and the matrix) are opposite. When the poling directions of the coating layer and matrix are the same, both SIF and EDIF decrease with

the increasing piezoelectric coefficient of the coating layer. On the other hand, if the poling direction of the coating layer is opposite to that of the matrix, the variations of the SIF and EDIF change dramatically with the increasing absolute value of the ratio $e_{15}^{(2)} / e_{15}^{(3)}$, especially for the thin coating layer case ($b/a=1.05$). It should be also noted that when the coating layer and matrix material have the same poling direction, the electric displacement intensity factor is less sensitive to the increasing value of the ratio $e_{15}^{(2)} / e_{15}^{(3)}$.

Figure 5.2 Normalized K_{σ}^L versus $c_{44}^{(2)} / c_{44}^{(3)}$ with $e_{15}^{(1)} / e_{15}^{(3)} = 2.5$, $e_{15}^{(2)} / e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)} / \kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)} / \kappa_{11}^{(3)} = 0.3$, $c_{44}^{(1)} / c_{44}^{(3)} = 2$

Figure 5.3 Normalized K_D^L versus $c_{44}^{(2)} / c_{44}^{(3)}$ with $e_{15}^{(1)} / e_{15}^{(3)} = 2.5$, $e_{15}^{(2)} / e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)} / \kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)} / \kappa_{11}^{(3)} = 0.3$, $c_{44}^{(1)} / c_{44}^{(3)} = 2$

Figure 5.4 Normalized K_{σ}^L verses $e_{15}^{(2)}/e_{15}^{(3)}$ with $c_{44}^{(1)}/c_{44}^{(3)} = 2$, $c_{44}^{(2)}/c_{44}^{(3)} = 0.7$, $\kappa_{11}^{(1)}/\kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$, $e_{15}^{(1)}/e_{15}^{(3)} = 2.5$

Figure 5.5 Normalized K_D^L verses $e_{15}^{(2)} / e_{15}^{(3)}$ with $c_{44}^{(1)} / c_{44}^{(3)} = 2$, $c_{44}^{(2)} / c_{44}^{(3)} = 0.7$, $\kappa_{11}^{(1)} / \kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)} / \kappa_{11}^{(3)} = 0.3$, $e_{15}^{(1)} / e_{15}^{(3)} = 2.5$.

5.4.3 Influence of the Coating Thickness

The variations of the normalized stress and electrical displacement intensity factors at the left crack tip with the coating thickness b/a are depicted in Figure 5.6 and Figure 5.7, respectively, for different material constants combination as:

$$c_{44}^{(2)} / c_{44}^{(3)} = 0.7, e_{15}^{(2)} / e_{15}^{(3)} = 0.5, \kappa_{11}^{(2)} : \kappa_{11}^{(3)} = 0.3, t_1 = 2.0a, t_2 = 2.3a. \quad (5.32)$$

For comparison, two different inclusion materials are displayed in the figures, one is a “hard” inclusion (the material constants of the inclusion are higher than those of the matrix), with the parameters taken as $c_{44}^{(1)} / c_{44}^{(3)} = 2, e_{15}^{(1)} / e_{15}^{(3)} = 2.5, \kappa_{11}^{(1)} : \kappa_{11}^{(3)} = 3$; another is a “soft” inclusion (the material constants of the inclusion are lower than those of the matrix), with the parameters taken as $c_{44}^{(1)} / c_{44}^{(3)} = 0.4, e_{15}^{(1)} / e_{15}^{(3)} = 0.35, \kappa_{11}^{(1)} : \kappa_{11}^{(3)} = 0.25$. It is shown that the SIFs and EDIFs both increase with the increasing thickness of coating layer. When the coating layer is getting thicker, the influence of the inclusion on the crack is getting weaker. Our calculation results also show that if the coating layer is “harder” than the inclusion, the trend of the results will be reversed. Those results are not shown here for the sake of reducing the number of the figures.

Figure 5.6 Normalized K_{σ}^L versus b/a with $c_{44}^{(2)}/c_{44}^{(3)} = 0.7$, $e_{15}^{(2)}/e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$,
 $t_1 = 2.0a$, $t_2 = 2.3a$

Figure 5.7 Normalized K_D^L verses b/a with $c_{44}^{(2)}/c_{44}^{(3)}=0.7$, $e_{15}^{(2)}/e_{15}^{(3)}=0.5$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)}=0.3$,
 $t_1 = 2.0a$, $t_2 = 2.3a$

Figure 5.8 Normalized K_{σ}^L versus crack location for $c_{44}^{(1)} / c_{44}^{(3)} = 2$, $c_{44}^{(2)} / c_{44}^{(3)} = 0.7$, $e_{15}^{(1)} / e_{15}^{(3)} = 2.5$, $e_{15}^{(2)} / e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)} / \kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)} / \kappa_{11}^{(3)} = 0.3$, $(t_2 - t_1) / b = 0.3$.

Figure 5.9 Normalized K_D^L versus crack location for $c_{44}^{(1)} / c_{44}^{(3)} = 2$, $c_{44}^{(2)} / c_{44}^{(3)} = 0.7$,
 $e_{15}^{(1)} / e_{15}^{(3)} = 2.5$, $e_{15}^{(2)} / e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)} / \kappa_{11}^{(3)} = 3$,
 $\kappa_{11}^{(2)} / \kappa_{11}^{(3)} = 0.3$, $(t_2 - t_1) / b = 0.3$. $\kappa_{11}^{(2)} / \kappa_{11}^{(3)} = 0.3$, $(t_2 - t_1) / b = 0.3$.

5.4.4 Influence of the Crack Location

The variations of the normalized stress and electrical displacement intensity factors at the left crack tip with $(t_1+t_2)/2b$ (the normalized distance between the centers of the crack and inclusion) are depicted in Figure 5.8 and Figure 5.9, respectively. The other material parameters were taken as $c_{44}^{(1)}/c_{44}^{(3)} = 2$, $c_{44}^{(2)}/c_{44}^{(3)} = 0.7$, $e_{15}^{(1)}/e_{15}^{(3)} = 2.5$, $e_{15}^{(2)}/e_{15}^{(3)} = 0.5$, $\kappa_{11}^{(1)}/\kappa_{11}^{(3)} = 3$, $\kappa_{11}^{(2)}/\kappa_{11}^{(3)} = 0.3$ and $(t_2-t_1)/b = 0.3$. It can be seen from the figure that the normalized SIF increases with the increasing distance for the thin coating layer thickness case ($b/a = 1.02$), but it decreases with the distance for the thick coating layer thickness case ($b/a = 1.30$). This is due to the fact that for the current case, the coating layer has lower material constants, especially lower elastic modulus than the matrix and inclusion, the crack is “shielded” by the thick coating layer. As the distance increases, the shield effect is getting weaker. If the coating layer has higher material constants than the matrix, the crack will be shielded no matter the coating layer is thick or thin. While for the electric displacement intensity factor, it always decreases with the increasing distance between the crack and the inclusion.

5.5 Conclusions

By using the solution of a screw dislocation interacting with a coated inclusion in piezoelectric media as the Green function, the interaction between a Mode III Griffith crack and a coated inclusion is investigated. The stress and electric displacement intensity factors for both crack tips are derived and analyzed. The influence of various parameters such as the elastic modulus and piezoelectric coefficient of the coating layer, the coating

thickness and the crack location, on the stress and electric displacement intensity factors of the crack tip has been analyzed and demonstrated by numerical examples in detail.

CHAPTER 6

AN EDGE DISLOCATION INTERACTING WITH AN INTERFACIAL VOID

6.1 Introduction

For bi-material configuration (two bonded dissimilar materials), the interface between the dissimilar materials is an essential feature. Small imperfections existing on the interface or near the interface will lead to the failure of bi-material structures. Kelly *et al.* (1993) studied the problem of a circular hole near a bi-material interface by using the dislocation distribute method. Selvadurai (2000) examines the problem of the axial displacement of a rigid circular disk-shaped inclusion which is embedded at the interface of two bonded dissimilar isotropic elastic solids. Xiao *et al.* (2002) investigated an edge dislocation near a slightly wavy interface by a regular perturbation scheme. More recently, Prasad *et al.* (2005) obtained the Green's function solution for a point force and dislocation interacting with interfacial elliptical rigid inclusion in a bonded bi-material system. Among the various interfacial defects, voids are the most frequently encountered ones at the interface of dissimilar materials. A number of research works on the subject are focused on the void formation mechanism. Shen (1997) proposed a micromechanism of void formation, with

emphasis on the kinetic process of void nucleation in thin-film metal interconnects. Harik and Cairncross (2000) used a new finite element model to simulate the formation of interfacial voids in the case of metal matrix composites (MMCs). To the author's best knowledge, when there are other kinds of micro defects such as dislocations at the interface of dissimilar materials, the interaction between the dislocations and void so far has not been investigated yet.

Figure 6.1 An interface edge dislocation near a circular interfacial void

In this chapter, the solution is obtained for the problem of an interfacial edge dislocation interacting with a void which is also located at the interface of two dissimilar materials. The image forces on the dislocation are examined and discussed by numerical examples.

6.2 Formulation

Consider two half spaces of dissimilar materials bonded along the interface $y = 0$ (Figure 6.1), with a circular void of radius ρ located symmetrically in the two materials and centred at $(0,0)$. An edge dislocation with Burgers vector (b_x, b_y) sits on the interface location $(-a, 0)$. Here both Cartesian coordinate and polar coordinate are used throughout the whole solution process.

The circular void is assumed traction free along the internal surface, and is simulated by distributing dislocations around the void periphery. Here, the dislocations used are distributed continuously over the entire void boundary, so the boundary conditions on the interface are automatically satisfied. In order to formulate the problem, the stress fields induced by a single dislocation embedded in region 1 and region 2 are needed. Consider a dislocation having the Burgers vector (b_x, b_y) located at (ξ, η) or in polar coordinate notation (ρ, φ) in region 1 first. The stress field in the Cartesian coordinate set can be derived from the Airy stress function of an edge dislocation given by Dundurs and Mura (1964). The same derivation procedure can be applied in region 2. Our following solution procedure will focus on region 1, where the stress field is given by:

$$\frac{\pi(\kappa_1 + 1)}{2\mu_1} \begin{Bmatrix} \sigma_{xx}^{(1)}(x, y) \\ \sigma_{yy}^{(1)}(x, y) \\ \sigma_{xy}^{(1)}(x, y) \end{Bmatrix} = b_x(\xi, \eta) \begin{Bmatrix} G_{xxx}^{(1)}(x, y, \xi, \eta; \alpha, \beta) \\ G_{xyy}^{(1)}(x, y, \xi, \eta; \alpha, \beta) \\ G_{xxy}^{(1)}(x, y, \xi, \eta; \alpha, \beta) \end{Bmatrix} + b_y(\xi, \eta) \begin{Bmatrix} G_{yxx}^{(1)}(x, y, \xi, \eta; \alpha, \beta) \\ G_{yyy}^{(1)}(x, y, \xi, \eta; \alpha, \beta) \\ G_{yyx}^{(1)}(x, y, \xi, \eta; \alpha, \beta) \end{Bmatrix}, \quad (6.1)$$

in which the parameters α and β are the Dundurs constants defined by Dundurs (1967)

$$\alpha = \frac{\Gamma(\kappa_1 + 1) - (\kappa_2 + 1)}{\Gamma(\kappa_1 + 1) + (\kappa_2 + 1)}, \beta = \frac{\Gamma(\kappa_1 - 1) - (\kappa_2 - 1)}{\Gamma(\kappa_1 + 1) + (\kappa_2 + 1)}, \quad (6.2)$$

with $\kappa_j = 3 - 4\nu_j$ ($j = 1, 2$) for plane strain case, $\kappa_j = (3 - \nu_j)/(1 + \nu_j)$ ($j = 1, 2$) for plane stress case, and $\Gamma = \mu_2/\mu_1$, where ν_j and μ_j are the Poisson ratio and modulus of rigidity of material j respectively. The influence functions are a little lengthy and listed explicitly in Appendix C.

The stress state, expressed in polar coordinates (r, θ) for a general position (ρ, θ) along the perimeter of the void due to a dislocation at position (ρ, φ) , can be found through the substitutions

$$\begin{aligned} \frac{x}{r} &= \sin \theta, & \frac{y}{r} &= \cos \theta, \\ \frac{\xi}{\rho} &= \sin \varphi, & \frac{\eta}{\rho} &= \cos \varphi, \end{aligned} \quad (6.3)$$

with the aid of the elementary Mohr transformation:

$$\begin{aligned} G_{jrr} &= G_{jxx} \sin^2 \theta + G_{jyy} \cos^2 \theta + G_{jxy} \sin 2\theta, \\ G_{j\theta\theta} &= G_{jxx} \cos^2 \theta + G_{jyy} \sin^2 \theta - G_{jxy} \sin 2\theta, \\ G_{jr\theta} &= (G_{jxx} - G_{jyy}) \sin \theta \cos \theta + G_{jxy} \cos 2\theta, \end{aligned} \quad (6.4)$$

here $j = x, y$. As a result, Equations (6.3) and (6.4) lead to the stress state in region 1 as

$$\begin{aligned} \frac{\pi(\kappa_1 + 1)}{2\mu_1} \begin{Bmatrix} \sigma_{rr}^{(1)}(\rho, \theta) \\ \sigma_{\theta\theta}^{(1)}(\rho, \theta) \\ \sigma_{r\theta}^{(1)}(\rho, \theta) \end{Bmatrix} &= b_x(\rho, \varphi) \begin{Bmatrix} G_{xrr}^{(1)}(\theta, \varphi; \alpha, \beta) \\ G_{x\theta\theta}^{(1)}(\theta, \varphi; \alpha, \beta) \\ G_{xr\theta}^{(1)}(\theta, \varphi; \alpha, \beta) \end{Bmatrix} \\ &\quad + b_y(\rho, \varphi) \begin{Bmatrix} G_{yrr}^{(1)}(\theta, \varphi; \alpha, \beta) \\ G_{y\theta\theta}^{(1)}(\theta, \varphi; \alpha, \beta) \\ G_{yr\theta}^{(1)}(\theta, \varphi; \alpha, \beta) \end{Bmatrix}. \end{aligned} \quad (6.5)$$

As stated, the disc $r < \rho$ will be punched out by distributing dislocations around its perimeter. An alternate way of understanding the process is as follows: 1) the stress state around the periphery of the intended hole due to the existence of the single interface dislocation is solved under the situation where the void itself is absent. 2) Since the boundary of the void must remain free of traction (even the void is filled by materials), the evaluated stress state $\sigma_{rr}^{(i)}(\rho, \theta)$ and $\sigma_{r\theta}^{(i)}(\rho, \theta)$ around the void periphery should be cancelled by the continuous distribution of dislocations around $-\pi < \varphi < \pi$ for each point (i.e., each angle θ). These two steps lead to the following integral equations,

For region 1:

$$\begin{aligned} -\frac{(\kappa_1 + 1)}{2\mu_1} \sigma_{rr}^{(1)}(\theta) &= \frac{1}{\pi\rho} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ B_x^{(1)}(\varphi) G_{xrr}^{(1)}(\theta, \varphi) + B_y^{(1)}(\varphi) G_{yrr}^{(1)}(\theta, \varphi) \right\} \rho d\varphi, \\ -\frac{(\kappa_1 + 1)}{2\mu_1} \sigma_{r\theta}^{(1)}(\theta) &= \frac{1}{\pi\rho} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ B_x^{(1)}(\varphi) G_{xr\theta}^{(1)}(\theta, \varphi) + B_y^{(1)}(\varphi) G_{yr\theta}^{(1)}(\theta, \varphi) \right\} \rho d\varphi, \end{aligned} \quad (6.6)$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

For region 2:

$$\begin{aligned} -\frac{(\kappa_1 + 1)}{2\mu_1} \sigma_{rr}^{(2)}(\theta) &= \frac{1}{\pi\rho} \times \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left\{ B_x^{(2)}(\varphi) G_{xrr}^{(2)}(\theta, \varphi) + B_y^{(2)}(\varphi) G_{yrr}^{(2)}(\theta, \varphi) \right\} \rho d\varphi, \\ -\frac{(\kappa_1 + 1)}{2\mu_1} \sigma_{r\theta}^{(2)}(\theta) &= \frac{1}{\pi\rho} \times \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left\{ B_x^{(2)}(\varphi) G_{xr\theta}^{(2)}(\theta, \varphi) + B_y^{(2)}(\varphi) G_{yr\theta}^{(2)}(\theta, \varphi) \right\} \rho d\varphi, \end{aligned} \quad (6.7)$$

$$\frac{\pi}{2} < \theta < \frac{3\pi}{2},$$

where

$$B_x^{(j)}(\varphi) = -\frac{\partial \Delta u_x^{(j)}(\varphi)}{\rho \partial \varphi}, B_y^{(j)}(\varphi) = -\frac{\partial \Delta u_y^{(j)}(\varphi)}{\rho \partial \varphi}. \quad (6.8)$$

And $\Delta u_i^{(j)}$ is the relative displacement between the two boundaries (ρ^-, θ) , (ρ^+, θ) in the

i th direction, i.e., $\Delta u_i^{(j)}(\varphi) = u_i^{(j)-}(\varphi) - u_i^{(j)+}(\varphi)$, $i = x, y$. The conditions give

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} B_x^{(1)}(\varphi) d\varphi = \int_{\frac{\pi}{2}}^{+\frac{3\pi}{2}} B_y^{(1)}(\varphi) d\varphi = 0, \quad (6.9)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} B_x^{(2)}(\varphi) d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} B_y^{(2)}(\varphi) d\varphi = 0, \quad (6.10)$$

The tractions induced by the single interfacial edge dislocation can also be derived from

Dundurs and Mura (1964) and expressed in polar coordinate set as

$$\begin{aligned} \hat{\sigma}_{rr}^{(i)}(\theta) &= \frac{2\mu_1}{\pi(\kappa_1 + 1)} \left(b_x \hat{G}_{xrr}^{(i)}(\theta) + b_y \hat{G}_{yrr}^{(i)}(\theta) \right), \\ \hat{\sigma}_{r\theta}^{(i)}(\theta) &= \frac{2\mu_1}{\pi(\kappa_1 + 1)} \left(b_x \hat{G}_{xr\theta}^{(i)}(\theta) + b_y \hat{G}_{yr\theta}^{(i)}(\theta) \right), \end{aligned} \quad (6.11)$$

where the influence functions $\hat{G}_{xjk}^{(i)}, \hat{G}_{yjk}^{(i)}$ are listed in the Appendix C.

To cast equations (6.6) in a standard form, they are normalized by letting $\varphi = \frac{1}{2}\pi s$,

$\theta = \frac{1}{2}\pi t$, which leads to

$$-\frac{(\kappa_1 + 1)}{\mu_1} \hat{\sigma}_{rk}^{(i)}(t) = \int_{-1}^{+1} \left\{ B_x^{(i)}(s) G_{xrk}^{(i)}(t, s) + B_y^{(i)}(s) G_{yrk}^{(i)}(t, s) \right\} ds \quad (6.12)$$

with $k = r, \theta$ and $-1 < t < +1$.

Accordingly, equations (6.9) are expressed as

$$\int_{-1}^{+1} B_x^{(i)}(s) ds = \int_{-1}^{+1} B_y^{(i)}(s) ds = 0. \quad (6.13)$$

6.3 Singularity and Numerical Solution

Once the tractions induced by the single interface dislocation are obtained, the only unknowns in the two integral equations for region 1 are the dislocation densities

$B_x^{(i)}$ and $B_y^{(i)}$. An examination of the influence functions $G_{ijk}^{(i)}$ reveals that the equations

(6.12) have a simple Cauchy kernel, which is $G_{ijk}^{(i)}(t, s) \rightarrow (s - t)^{-1}$ as $s \rightarrow t$. Furthermore,

let us examine the singularity at the end points on the bi-material boundary $s = \mp 1$. Following Bogy's (1971) asymptotic analysis of stress field at these two ends, we assume the stress field is of the order r^λ in a region close to the end-point. The eigenvalue λ is given by the smallest root (or real part of the complex root) of

$$\begin{aligned} (\alpha - \beta)^2 (1 + \lambda)^4 + \{2\beta(\alpha - \beta)\cos^2[\lambda\pi/2] - \alpha^2\} \\ + \cos^2[\lambda\pi/2]\{(\beta^2 - 1)\cos^2[\lambda\pi/2] + 1\} \equiv 0. \end{aligned} \quad (6.14)$$

As a result, the dislocation density functions may be written

$$\begin{aligned} B_x^{(1)}(s) &= D_x^{(1)}(s)(1 - s^2)^\lambda, \\ B_y^{(1)}(s) &= D_y^{(1)}(s)(1 - s^2)^\lambda, \end{aligned} \quad (6.15)$$

where the functions $D_x^{(1)}$ and $D_y^{(1)}$ are Hölder functions continuous in $[-1, +1]$. Now we can apply the Gauss-Jacobi quadrature (Erdogan *et al.*, 1973) which was summarized and extended by Hills *et al.* (1996) to discretize equations (6.12) and (6.13) into a system of algebraic equations given by

$$-\frac{(\kappa_1 + 1)}{\mu_1} \hat{\sigma}_{rk}^{(1)}(t_j) = \sum_{i=1}^N W_i(s_i) \{D_x^{(1)}(s_i) G_{xrk}^{(1)}(t_j, s_i) + D_y^{(1)}(s_i) G_{yrk}^{(1)}(t_j, s_i)\}, \quad (6.16)$$

$$\begin{aligned} \sum_{i=1}^N W_i(s_i) D_x^{(1)}(s_i) &= 0, \\ \sum_{i=1}^N W_i(s_i) D_y^{(1)}(s_i) &= 0, \end{aligned} \quad (6.17)$$

where $k = r, \theta$ and the integration points s_i are the zeros of the corresponding Jacobi polynomials

$$P_N^{(\lambda_1, \lambda_2)}(s_i) = 0, \quad (i = 1, \dots, N), \quad (6.18)$$

and the collocation points are given by

$$P_{N-1}^{(\lambda_1+1, \lambda_2+1)}(t_j) = 0, \quad (j = 1, \dots, N-1). \quad (6.19)$$

The weights are given by

$$W_i(s_i) = -\frac{2N + \lambda_1 + \lambda_2}{(N+1)!(N + \lambda_1 + \lambda_2)} \frac{\Gamma(N + \lambda_1 + 1)\Gamma(N + \lambda_2 + 1)}{\Gamma(N + \lambda_1 + \lambda_2 + 1)} \times \frac{2^{\lambda_1 + \lambda_2}}{P_N^{(\lambda_1, \lambda_2)}(s_i) P_{N+1}^{(\lambda_1, \lambda_2)}(s_i)}. \quad (6.20)$$

For the current case, the same value of λ is used in equations (6.18)~(6.20) .

Once the dislocation densities are solved out, the stress state σ_{jk} at any general point (x, y) can be found through

$$\begin{aligned} \sigma_{jk}^{(1)}(x, y) &= \hat{\sigma}_{jk}^{(1)}(x, y) + \frac{\mu_1}{\kappa_1 + 1} \frac{1}{\pi \rho} \times \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left\{ B_x^{(1)}(\varphi) G_{xjk}^{(1)} + B_y^{(1)}(\varphi) G_{yjk}^{(1)} \right\} \rho d\varphi, \\ \sigma_{jk}^{(2)}(x, y) &= \hat{\sigma}_{jk}^{(2)}(x, y) + \frac{\mu_1}{\kappa_1 + 1} \frac{1}{\pi \rho} \times \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left\{ B_x^{(2)}(\varphi) G_{xjk}^{(2)} + B_y^{(2)}(\varphi) G_{yjk}^{(2)} \right\} \rho d\varphi, \end{aligned} \quad (6.21)$$

where G_{xjk}, G_{yjk} are the appropriate kernels obtained from the influence functions of a dislocation at (ρ, φ) in bonded half-planes, which is in a manner similar to the derivation of the influence functions G_{lmn} of equations(6.5).

6.4 Image Force on Dislocation

The image force F exerted on the edge dislocation is given by the Peach-Koehler relation (Weertman and Weertman, 1992)

$$F_i = -\varepsilon_{ijk} b_l \tau_j \sigma_{kl} \quad (6.22)$$

In Equation (6.22), ε_{ijk} is the unit alternating tensor, b_i is the Burgers vector of the dislocation, τ_j is the unit vector along the dislocation line, and σ_{kl} is the total stress at the

point considered on the dislocation. For the current problem, the Burgers vector of the dislocation has two components (b_x, b_y) . Expanding equation (6.22), we have

$$F_x = b_x \sigma_{yx} + b_y \sigma_{yy}, \quad (6.23)$$

$$F_y = -(b_x \sigma_{xx} + b_y \sigma_{xy}). \quad (6.24)$$

The stress components in equations (6.23) and (6.24) can be obtained by subtracting the field generated by the dislocation from (6.21), in the forms

$$\begin{aligned} \sigma_{jk}^{(1)}(x, y) &= \frac{\mu_1}{\kappa_1 + 1} \frac{1}{\pi \rho} \times \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left\{ B_x^{(1)}(\varphi) G_{xjk}^{(1)} + B_y^{(1)}(\varphi) G_{yjk}^{(1)} \right\} \rho d\varphi, \\ \sigma_{jk}^{(2)}(x, y) &= \frac{\mu_1}{\kappa_1 + 1} \frac{1}{\pi \rho} \times \int_{+\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ B_x^{(2)}(\varphi) G_{xjk}^{(2)} + B_y^{(2)}(\varphi) G_{yjk}^{(2)} \right\} \rho d\varphi. \end{aligned} \quad (6.25)$$

6.5 Numerical Examples and Discussions

In order to obtain an intrinsic understanding of the problem, numerical calculations were carried out and discussed here. In the calculation, we assume only one non-zero strength characteristic existing for the interface edge dislocation (either b_x or b_y) so that the effect of Burgers vectors can be observed more explicitly. For the purpose of demonstration, the number N for the calculation process was typically chosen to be 15. The image forces on the dislocation b_x and b_y are normalized by

$$F_x^0 = \frac{\mu_1 b_x^2}{(\kappa_1 + 1) \rho}, F_y^0 = \frac{\mu_1 b_y^2}{(\kappa_1 + 1) \rho}, \quad (6.26)$$

respectively. First, the variation of the image forces F_x and F_y versus the edge dislocation position is demonstrated in Figure 6.2 and Figure 6.3, respectively. Three different typical engineering material combinations were chosen from MEMS handbook (Gad-el-Hak, 2002). The properties of these materials were listed in Table 6.1.

Table 6.1Material Properties

Materials	Young’s Modulus E (GPa)	Poisson’s Ratio ν
Polysilicon	160	0.22
Silicon Carbide	400	0.25
Silicon Oxide	70	0.18
Aluminium	70	0.30
Epoxy	3.115	0.35

In such material combinations, it is found that the image forces exhibit an oscillation characteristic when the ratio a/ρ is less than 1.1. For the case of a glide dislocation with Burgers vector b_x (shown in Figure 6.2) and taking the material combination of Silicon Oxide with Polysilicon, it can be seen that when the dislocation was very close to the void (a/ρ is less than 1.004), the void would attract the dislocation. While when the dislocation was further away from the void, the void will repel it and then attract it. The points where the curves intersect with the horizontal axis are the balance position of the dislocation. For the climb dislocation with Burgers vector b_y , the image force from the void almost always attracts the dislocation.

Figure 6.2 Normalized image force F_x versus a/ρ

Figure 6.3 Normalized image force F_y versus a/ρ

Figure 6.4 Normalized image force F_x versus μ_2 / μ_1

Figure 6.5 Normalized image force F_y versus μ_2/μ_1

The normalized image forces F_x and F_y versus the bi-material modulus of rigidity ratio $\Gamma = \mu_2/\mu_1$ are depicted in Figures 6.4 and 6.5, respectively. Here, the position of edge dislocation is taken as $a/\rho = 1.02$. From these two figures, it is observed that the variation of the image force changes intensively in the interval of rigidity modulus ratio between 0 and 10. From Figure 6.4, it is found that if the material at the lower half plane (region 2 as shown in Figure 6.1) has lower value of Poisson's ratio than the upper region material, the image force will always be a negative value for the material combination. It means the existing void always repel the glide dislocation. While for the climb dislocation (Figure 6.5), the void first attracts then repels the dislocation. When $\Gamma = 1$ and $\nu_1 = \nu_2$, the problem degenerates to an edge dislocation interacting with a circular void in homogeneous solids. In this case, the characteristics equation (6.14) has no zero solution in the range $-1 < \lambda < 0$, which is the same situation discussed by Bogy (1968). It indicated that the singularity in the stress field at the point of discontinuity on the boundary can not be as strong as r^λ for any $\lambda < 0$.

6.6 Conclusions

The problem of an interfacial edge dislocation interacting with an interfacial void (the centre of void is at the interface) has been solved. The interfacial void is simulated by using distributed dislocation method and the resulting singular integral equations are solved numerically by Gauss-Jacobi quadrature. The image forces on dislocation have been obtained and numerical examples are performed and discussed. It is found that the image force on the dislocation exhibits an oscillation characteristic with the existence of the interfacial void when the dislocation is very near to the void.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

Some physical problems on micro defects in piezoelectric and elastic solids have been solved in this thesis. Four specific problems, viz. a screw dislocation interacting with a coated inclusion, a Zener-Stroh crack interacting with a coated inclusion, a Mode III Griffith crack interacting with a coated inclusion in piezoelectric materials, and an interfacial dislocation-void interaction problem in bi-material elastic solids, were studied in detail.

Concerning micro defect investigation in three phase piezoelectric composite, the interaction between a screw dislocation and a coated inclusion is first investigated. An explicit closed-form analytical solution is obtained by using complex variable method. This solution can be used as the Green function in the later related crack problem. The stress field is derived and force on dislocation is calculated. The influence of various thicknesses of the coating layer and material properties and other parameters were studied numerically. It is found that the coating layer plays an important role in the

inhomogeneity-dislocation interaction problem in piezoelectric solids. From the numerical results, the mechanical screw dislocation will be first attracted then repelled by increasing the coating layer thickness, while the situation is reversed for the electric-potential dislocation. Also by increasing the elastic modulus value of coating layer, the image force changes from negative value to positive value for the mechanical screw dislocation; but for the case of electric-potential dislocation, the image force causes the inclusion to repel the dislocation only at small elastic ratio of the coating layer to inclusion.

Using the above solution as the Green's function, the problems on a crack interacting with a coated inclusion in piezoelectric material have been studied subsequently. Both the Griffith crack and Zener-Stroh crack cases have been analyzed. The problems are formulated into a set of linear Cauchy-type singular integral equations, which are solved numerically by using Erdogan and Gupta's method (1972). The stress and electric displacement intensity factors of the crack are derived and expressed in terms of the asymptotic values of the dislocation density functions which are evaluated from the integral equations. Numerical examples are given and discussed. The results show that the elastic modulus and piezoelectric coefficient of the inclusion and coating layer, the coating thickness and the crack location have various influences on the stress and electric displacement intensity factors of the crack. The results indicated that the Griffith crack and the Zener-Stroh crack form a complementary pair in fracture mechanism, which is similar to the case in conventional pure elastic materials.

The fourth problem investigated in the current thesis involves the interaction between an interfacial edge dislocation and an interfacial void (the centre of the void located at the interface) in bonded dissimilar materials. It is found that the image force on the

dislocation exhibits an oscillation characteristic with the existence of the interfacial void when the ratio of dislocation is very near the void ($a/\rho < 1.1$).

The above findings and results may help mechanical and electrical engineers design more reliable composite structures with piezoelectric mechanism in engineering applications such as in MEMS, sensors and actuators, as designers have a better understanding on how to prevent and reduce micro defects nucleation and propagation. For instance, when a piezoelectric actuator is designed, since the electric field and piezoelectricity influence both the actuating performance and the possible fracture damage, a trade-off should be considered to balance both the performance and reliability requirements of the device.

7.2 Recommendations

Although numerous research works have been done in the field of micromechanics in piezoelectric solids and composite materials in recent decades, there is a high demanding on the investigation of interacting micro defects in such materials. Recommendations related to the aforesaid research field are made for possible future studies.

Firstly, the physical problem of an edge dislocation interacting with a coated inclusion in piezoelectric media shall be investigated. This is the foundation to investigate the Mode I and II crack problems in piezoelectric solids. The physical model is shown in Fig. 7.1. In this problem, the general solution procedure of Stroh formalism shall be applied.. Actually, the author had attempted to use different mathematical methods such as conformal mapping, alternating technique, etc to solve this problem. But they can not reach the analytic solution. After finished the work in Chapter 6, the author considered the approach

used in it may be extended to this case, that is, the interfaces of the inclusion may be modeled using a distribution of dislocations.

Figure 7.1 An edge dislocation near a coated inclusion in piezoelectric solids

Secondly, the physical problem on the interaction between an interface crack and an interfacial void shall be investigated, as show in Figure 7.2. Since the solution for an edge dislocation interacting with an interfacial void has been obtained in Chapter 6, it can be used as Green's function to model the corresponding crack problem. Both the Zener-Stroh crack and Griffith crack can be studied, as they represent different fracture mechanisms in predicting crack propagation. The above problems shall also be extended to piezoelectric materials case, as the piezoelectric materials exhibit a coupling effect between the mechanical and electric fields. Due to the introduction of the electric field, new physical features on the stress and electric fields could be found and investigated.

Figure 7.2 An interface crack near a circular interfacial void

Thirdly, in this thesis the solutions of two-dimensional micro defects (mainly in cracks) in piezoelectric and elastic solids have been examined in detail from the start of investigating single dislocation problems. Although the two-dimensional cracks problems give a good approximation to crack geometries encountered, most defects existing in engineering materials and structure components are three-dimensional in character. To investigate three-dimensional deformation problems, basically the same strategies in two dimensional problems can be followed by distributing the infinitesimal dislocation loop over the crack faces (Hill *et. al.* 1996). To derive the associated singular integral equations we must first find the stress field due to an infinitesimal dislocation loop with arbitrary Burgers vector

in piezoelectric solid, also the piezoelectric coupling effect should be considered in the derivation.

Another recommendation about the current research is to validate the proposed fracture criteria such as the stress and electric displacement intensity factors. Coordinated experimental studies are required to understand the real electric and mechanical boundary conditions.

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APPENDIX A

EXPLICIT EXPRESSION FOR MATRIX PARAMETERS

The explicit express for the matrix $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ in chapter 3 are as follows:

$$\mathbf{M}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (\text{A1})$$

$$\mathbf{M}_2 = \begin{bmatrix} 1 & 0 & -1 & -\frac{1}{a^{2k}} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -\frac{1}{a^{2k}} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{b^{2k}} & 0 & 0 & -\frac{1}{b^{2k}} & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{b^{2k}} & 0 & -\frac{1}{b^{2k}} \\ c_{44}^I b_z & e_{15}^I b_\phi & -c_{44}^{II} b_z & \frac{c_{44}^{II} b_z}{a^{2k}} & -e_{15}^{II} b_\phi & \frac{e_{15}^{II} b_\phi}{a^{2k}} & 0 & 0 \\ e_{15}^I b_z & -\kappa_{11}^I b_\phi & -e_{15}^{II} b_z & \frac{e_{15}^{II} b_z}{a^{2k}} & \kappa_{11}^{II} b_\phi & -\frac{\kappa_{11}^{II} b_\phi}{a^{2k}} & 0 & 0 \\ 0 & 0 & c_{44}^{II} b_z & -\frac{c_{44}^{II} b_z}{b^{2k}} & e_{15}^{II} b_\phi & -\frac{e_{15}^{II} b_\phi}{b^{2k}} & \frac{c_{44}^{III} b_z}{b^{2k}} & \frac{e_{15}^{III} b_\phi}{b^{2k}} \\ 0 & 0 & e_{15}^{II} b_z & -\frac{e_{15}^{II} b_z}{b^{2k}} & -\kappa_{11}^{II} b_\phi & \frac{\kappa_{11}^{II} b_\phi}{b^{2k}} & \frac{e_{15}^{III} b_z}{b^{2k}} & -\frac{\kappa_{11}^{III} b_\phi}{b^{2k}} \end{bmatrix}, \quad (\text{A2})$$

Appendix A

$$\mathbf{M}_3 = \begin{bmatrix} 1 & 0 & -1 & \frac{1}{a^{2k}} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & \frac{1}{a^{2k}} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{b^{2k}} & 0 & 0 & \frac{1}{b^{2k}} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{b^{2k}} & 0 & \frac{1}{b^{2k}} \\ c_{44}^I b_z & e_{15}^I b_\phi & -c_{44}^{II} b_z & -\frac{c_{44}^{II} b_z}{a^{2k}} & -e_{15}^{II} b_\phi & -\frac{e_{15}^{II} b_\phi}{a^{2k}} & 0 & 0 \\ e_{15}^I b_z & -\kappa_{11}^I b_\phi & -e_{15}^{II} b_z & -\frac{e_{15}^{II} b_z}{a^{2k}} & \kappa_{11}^{II} b_\phi & \frac{\kappa_{11}^{II} b_\phi}{a^{2k}} & 0 & 0 \\ 0 & 0 & c_{44}^{II} b_z & \frac{c_{44}^{II} b_z}{b^{2k}} & e_{15}^{II} b_\phi & \frac{e_{15}^{II} b_\phi}{b^{2k}} & -\frac{c_{44}^{III} b_z}{b^{2k}} & -\frac{e_{15}^{III} b_\phi}{b^{2k}} \\ 0 & 0 & e_{15}^{II} b_z & \frac{e_{15}^{II} b_z}{b^{2k}} & -\kappa_{11}^{II} b_\phi & -\frac{\kappa_{11}^{II} b_\phi}{b^{2k}} & -\frac{e_{15}^{III} b_z}{b^{2k}} & \frac{\kappa_{11}^{III} b_\phi}{b^{2k}} \end{bmatrix}. \quad (\text{A3})$$

APPENDIX B

STRESS AND ELECTRIC DISPLACEMENT FIELD OUTSIDE A COATED INCLUSION IN A PIEZOELECTRIC SOLID

Consider the physical problem in chapter 5 without the crack as show in Figure B.1, from Jiang and Cheung's results (2001), the general solutions of the displacement w and electrical potential ϕ can be expressed by analytical functions or complex potential $\Psi(z)$ and $\Phi(z)$ respectively,

$$w = \text{Re} \Psi(z), \quad \phi = \text{Re} \Phi(z), \quad (\text{B1})$$

where the complex variable $z = x + iy$. Without loss of generality, the complex potential are taken as the following finite terms of the series which can be used to derive the exact solution.

In phase I (inclusion):

$$\Psi^{(I)} = iA^{(I)}z, \quad \Phi^{(I)} = iB^{(I)}z. \quad (\text{B2})$$

In phase II (coating layer):

Appendix B

$$\Psi^{(2)} = i \left(A_{-1}^{(2)} \frac{1}{z} + A_1^{(2)} z \right), \quad \Phi^{(2)} = i \left(B_{-1}^{(2)} \frac{1}{z} + B_1^{(2)} z \right). \quad (\text{B3})$$

In phase III (matrix):

$$\Psi^{(3)} = i \left(A_{-1}^{(3)} \frac{1}{z} + A_1^{(3)} z \right), \quad \Phi^{(3)} = i \left(B_{-1}^{(3)} \frac{1}{z} + B_1^{(3)} z \right). \quad (\text{B4})$$

Figure B.1 A three phase piezoelectric solids under remote mechanical and electric fields

The stress and electric displacement field can be calculated by using the mechanical and electrical coupled constitutive equations (Tiersten, 1969).

In phase I (inclusion):

$$\tau_{yz}^{(1)} = -c_{44}^{(1)} A^{(1)} - e_{15}^{(1)} B^{(1)}, \quad D_y^{(1)} = -e_{15}^{(1)} A^{(1)} + \kappa_{11}^{(1)} B^{(1)}. \quad (\text{B5})$$

In phase II (coating layer):

Appendix B

$$\tau_{yz}^{(2)} = c_{44}^{(2)} \left(\frac{\cos 2\theta}{r^2} A_{-1}^{(2)} - A_1^{(2)} \right) + e_{15}^{(2)} \left(\frac{\cos 2\theta}{r^2} B_{-1}^{(2)} - B_1^{(2)} \right), \quad (B6)$$

$$D_y^{(2)} = e_{15}^{(2)} \left(\frac{\cos 2\theta}{r^2} A_{-1}^{(2)} - A_1^{(2)} \right) - \kappa_{11}^{(2)} \left(\frac{\cos 2\theta}{r^2} B_{-1}^{(2)} - B_1^{(2)} \right). \quad (B7)$$

In phase III (matrix):

$$\tau_{yz}^{(3)} = c_{44}^{(3)} \left(\frac{\cos 2\theta}{r^2} A_{-1}^{(3)} - A_1^{(3)} \right) + e_{15}^{(3)} \left(\frac{\cos 2\theta}{r^2} B_{-1}^{(3)} - B_1^{(3)} \right), \quad (B8)$$

$$D_y^{(3)} = e_{15}^{(3)} \left(\frac{\cos 2\theta}{r^2} A_{-1}^{(3)} - A_1^{(3)} \right) - \kappa_{11}^{(3)} \left(\frac{\cos 2\theta}{r^2} B_{-1}^{(3)} - B_1^{(3)} \right). \quad (B9)$$

The far-field stress and electric field loading (boundary condition) are taken as:

$$\tau_{yz} \Big|_{|z| \rightarrow \infty} = \tau_{\infty}, \quad D_y \Big|_{|z| \rightarrow \infty} = D_{\infty}. \quad (B10)$$

Then from equations (B8) and (B9), one obtain

$$-c_{44}^{(3)} A_1^{(3)} - e_{15}^{(3)} B_1^{(3)} = \tau_{\infty}, \quad (B11)$$

$$-e_{15}^{(3)} A_1^{(3)} + \kappa_{11}^{(3)} B_1^{(3)} = D_{\infty}. \quad (B12)$$

The unknown constants $A_1^{(3)}$ and $B_1^{(3)}$ are determined by solving Eqs.(B11) and (B12).

The other unknown constants $A^{(1)}$, $B^{(1)}$, $A_{-1}^{(2)}$, $A_1^{(2)}$, $B_{-1}^{(2)}$, $B_1^{(2)}$, $A_{-1}^{(3)}$, $B_{-1}^{(3)}$ are determined by the continuity of displacement and traction at the interface $r=a$ and $r=b$. The detail expression for these constants can refer to Jiang and Cheung's paper (2001)

Equations (B8) and (B9) express the stress and electrical displacement components in phase 3, at $\theta = 0$:

$$p_{\sigma}(x) = c_{44}^{(3)} \left(\frac{1}{x^2} A_{-1}^{(3)} - A_1^{(3)} \right) + e_{15}^{(3)} \left(\frac{\cos 2\theta}{x^2} B_{-1}^{(3)} - B_1^{(3)} \right), \quad (B13)$$

$$p_D(x) = e_{15}^{(3)} \left(\frac{1}{x^2} A_{-1}^{(3)} - A_1^{(3)} \right) - \kappa_{11}^{(3)} \left(\frac{1}{x^2} B_{-1}^{(3)} - B_1^{(3)} \right). \quad (B14)$$

where $p_\sigma(x)$, $p_D(x)$ are the mechanical and electric tractions along the plane $\theta = 0$ respectively, for the problem shown in Figure 5.1 without the crack.

APPENDIX C

STRESS FIELD FOR AN EDGE DISLOCATION IN A BI-MATERIAL

Figure C.1 shows two bonded half-plane with a dislocation lying in Region 1 at the position (ξ, η) . The problem was solved by Dundurs and Mura (1964). The stresses at a point (x, y) are expressed as

$$\frac{\pi(\kappa_1 + 1)}{2\mu_1} \begin{Bmatrix} \sigma_{xx}^{(i)}(x, y) \\ \sigma_{yy}^{(i)}(x, y) \\ \sigma_{xy}^{(i)}(x, y) \end{Bmatrix} = b_x(\xi, \eta) \begin{Bmatrix} G_{xxx}^{(i)}(x, y, \xi, \eta; \alpha, \beta) \\ G_{xyy}^{(i)}(x, y, \xi, \eta; \alpha, \beta) \\ G_{xxy}^{(i)}(x, y, \xi, \eta; \alpha, \beta) \end{Bmatrix} + b_y(\xi, \eta) \begin{Bmatrix} G_{yxx}^{(i)}(x, y, \xi, \eta; \alpha, \beta) \\ G_{yyy}^{(i)}(x, y, \xi, \eta; \alpha, \beta) \\ G_{yyx}^{(i)}(x, y, \xi, \eta; \alpha, \beta) \end{Bmatrix}, \quad (C1)$$

where the stresses in region 1 are given by taking $i = 1$, the stress in region 2 by taking $i = 2$. The influence functions then are given by the following equations.

Appendix C

Figure C.1 An dislocation at position (ξ, η) near a straight interface

$$\begin{aligned}
 G_{xxx}^{(I)} = & \frac{(x-\xi)}{r_1^4} \left[-(x-\xi)^2 + (y-\eta)^2 \right] + \\
 & \frac{(\alpha - \beta)(x-\xi)}{(1+\beta)r_2^2} \left\{ C_1 \frac{[-(x-\xi)^2 + (y+\eta)^2]}{r_2^2} + C_2 \frac{2(x-\xi)^2}{r_2^2} \right. \\
 & \left. - \frac{12\eta(y+\eta)}{r_2^2} + \frac{16\eta(y+\eta)^3}{r_2^4} - \frac{16\eta^2(y+\eta)^2}{r_2^4} + \frac{4\eta^2}{r_2^2} \right\},
 \end{aligned} \tag{C2}$$

$$G_{xyy}^{(1)} = -\frac{(x-\xi)}{r_1^4} \left[(x-\xi)^2 + 3(y-\eta)^2 \right] + \frac{(\alpha-\beta)(x-\xi)}{(1+\beta)r_2^2} \left\{ -C_1 \frac{[(x-\xi)^2 + 3(y+\eta)^2]}{r_2^2} + C_2 \frac{2(y+\eta)^2}{r_2^2} - \frac{12\eta(y+\eta)}{r_2^2} + \frac{16\eta(y+\eta)(x-\xi)^2}{r_2^4} - \frac{16\eta^2(x-\xi)^2}{r_2^4} + \frac{12\eta^2}{r_2^2} \right\}, \quad (C3)$$

$$G_{xyx}^{(1)} = \frac{(y-\eta)}{r_1^4} \left[-(x-\xi)^2 + (y-\eta)^2 \right] + \frac{(\alpha-\beta)(y+\eta)}{(1+\beta)r_2^2} \left\{ C_1 \frac{[-(x-\xi)^2 + (y+\eta)^2]}{r_2^2} + C_2 \frac{2(x-\xi)^2}{r_2^2} - \frac{16\eta(y+\eta)(x-\xi)^2}{r_2^4} + \frac{16\eta^2(x-\xi)^2}{r_2^4} - \frac{4\eta^2}{r_2^2} \right\} + \frac{(\alpha-\beta)2\eta}{(1+\beta)r_2^4}, \quad (C4)$$

$$G_{yxx}^{(1)} = \frac{(y-\eta)}{r_1^4} \left[3(x-\xi)^2 + (y-\eta)^2 \right] + \frac{(\alpha-\beta)(y+\eta)}{(1+\beta)r_2^2} \left\{ C_1 \frac{[3(x-\xi)^2 + (y+\eta)^2]}{r_2^2} + C_2 \frac{2(x-\xi)^2}{r_2^2} - \frac{16\eta(y+\eta)}{r_2^2} + \frac{16\eta(y+\eta)^3}{r_2^4} - \frac{16\eta^2(y+\eta)^2}{r_2^4} + \frac{12\eta^2}{r_2^2} \right\} + \frac{(\alpha-\beta)2\eta}{(1+\beta)r_2^4}, \quad (C5)$$

$$G_{yyy}^{(1)} = \frac{(y-\eta)}{r_1^4} \left[-(x-\xi)^2 + (y-\eta)^2 \right] + \frac{(\alpha-\beta)(y+\eta)}{(1+\beta)r_2^2} \left\{ C_1 \frac{[-(x-\xi)^2 + (y+\eta)^2]}{r_2^2} - C_2 \frac{2(y+\eta)^2}{r_2^2} - \frac{6\eta(y+\eta)}{r_2^2} + \frac{16\eta(y+\eta)(x-\xi)^2}{r_2^4} - \frac{16\eta^2(x-\xi)^2}{r_2^4} + \frac{4\eta^2}{r_2^2} \right\} + \frac{(\alpha-\beta)2\eta(x-\xi)^2}{(1+\beta)r_2^4}, \quad (C6)$$

Appendix C

$$\begin{aligned}
G_{xy}^{(1)} = & \frac{(x-\xi)}{r_1^4} \left[(x-\xi)^2 + 3(y-\eta)^2 \right] + \\
& \frac{(\alpha-\beta)(x-\xi)}{(1+\beta)r_2^2} \left\{ C_1 \frac{[(x-\xi)^2 + 3(y+\eta)^2]}{r_2^2} - C_2 \frac{2(y+\eta)^2}{r_2^2} \right. \\
& \left. + \frac{7\eta(y+\eta)}{r_2^2} - \frac{16\eta(y+\eta)^3}{r_2^4} + \frac{16\eta^2(y+\eta)^2}{r_2^4} - \frac{4\eta^2}{r_2^2} \right\},
\end{aligned} \tag{C7}$$

where

$$r_1 = (x-\xi)^2 + (y-\eta)^2, \quad r_2 = (x-\xi)^2 + (y+\eta)^2, \tag{C8}$$

and

$$C_1 = \frac{\alpha + \beta^2}{(1-\beta)(\alpha-\beta)}, \quad C_2 = \frac{(1+\alpha)\beta}{(1-\beta)(\alpha-\beta)}. \tag{C9}$$

Figure C.2 shows an edge dislocation lying on the interface between two dissimilar half-plane. For the dislocation located at $(-a, 0)$, the stresses around the circumference of $r = \rho$ can also be derived from Dundurs and Mura's solution (1964). They are expressed in polar coordinate as:

$$\begin{aligned}
\hat{\sigma}_{rr}^{(i)}(\theta) &= \frac{2\mu_1}{\pi(\kappa_1+1)} \left(b_x \hat{G}_{xrr}^{(i)}(\theta) + b_y \hat{G}_{yrr}^{(i)}(\theta) \right), \\
\hat{\sigma}_{r\theta}^{(i)}(\theta) &= \frac{2\mu_1}{\pi(\kappa_1+1)} \left(b_x \hat{G}_{xr\theta}^{(i)}(\theta) + b_y \hat{G}_{yr\theta}^{(i)}(\theta) \right), \\
\hat{\sigma}_{\theta\theta}^{(i)}(\theta) &= \frac{2\mu_1}{\pi(\kappa_1+1)} \left(b_x \hat{G}_{x\theta\theta}^{(i)}(\theta) + b_y \hat{G}_{y\theta\theta}^{(i)}(\theta) \right),
\end{aligned} \tag{C10}$$

where i also is taken as 1 or 2, the influence functions $\hat{G}_{jkl}^{(i)}(\theta)$ are given as

Figure C.2 An edge dislocation on a straight interface

$$\hat{G}_{xrr}^{(1)}(\theta) = -\frac{1+\alpha}{1-\beta^2} \frac{\cos \theta}{(a^2 + \rho^2 + 2a\rho \sin \theta)^2} \left[(1+2\beta)\rho^3 + 4\beta\rho^2 a \sin \theta \right. \\ \left. + (2-2\beta)\rho a^2 \cos^2 \theta + (2\beta-3)\rho a^2 - 2a^3 \sin \theta \right], \quad (\text{C11})$$

$$\hat{G}_{yrr}^{(1)}(\theta) = \frac{1+\alpha}{1-\beta^2} \frac{1}{(a^2 + \rho^2 + 2a\rho \sin \theta)^2} \left[(1+2\beta)\rho^3 \sin \theta + (3+6\beta)\rho^2 a \right. \\ \left. - 4\beta\rho^2 a \cos^2 \theta - (2\beta-2)\rho a^2 \sin \theta \cos^2 \theta \right. \\ \left. + (3+6\beta)\rho a^2 \sin \theta + (2\beta \sin^2 \theta + 1)a^3 \right], \quad (\text{C12})$$

$$\hat{G}_{xr\theta}^{(1)}(\theta) = -\frac{1+\alpha}{1-\beta^2} \frac{1}{(a^2 + \rho^2 + 2a\rho \sin \theta)^2} \left[\rho^3 \sin \theta + 3\rho^2 a + (2\beta-2)\rho^2 a \cos^2 \theta \right. \\ \left. + (2\beta-2)\rho a^2 \sin \theta \cos^2 \theta + 3\rho a^2 \sin \theta + (1-2\cos^2 \theta)a^3 \right], \quad (\text{C13})$$

Appendix C

$$\hat{G}_{yr\theta}^{(1)}(\theta) = -\frac{1+\alpha}{1-\beta^2} \frac{\cos\theta}{(a^2 + \rho^2 + 2a\rho\sin\theta)} \left[\rho^3 + (2-2\beta)\rho^2 a \sin\theta \right. \\ \left. + (2\beta-2)\rho a^2 \cos^2\theta + (1-4\beta)\rho a^2 - 2\beta a^3 \sin\theta \right], \quad (C14)$$

$$\hat{G}_{x\theta\theta}^{(1)}(\theta) = -\frac{1+\alpha}{1-\beta^2} \frac{\cos\theta}{(a^2 + \rho^2 + 2a\rho\sin\theta)^2} \left[\rho^3 + 4\rho^2 a \sin\theta \right. \\ \left. + 5\rho a^2 + (2\beta-2)\rho a^2 \cos^2\theta + 2a^3 \sin\theta \right], \quad (C15)$$

$$\hat{G}_{y\theta\theta}^{(1)}(\theta) = \frac{1+\alpha}{1-\beta^2} \frac{1}{(a^2 + \rho^2 + 2a\rho\sin\theta)^2} \left[\rho^3 \sin\theta - 4\rho^2 a \cos^2\theta \right. \\ \left. + 3\rho^2 a + (2\beta-2)\rho a^2 \sin\theta \cos^2\theta + 3\rho a^2 \sin\theta + (1+2\beta \cos^2\theta)a^3 \right], \quad (C16)$$

$$\hat{G}_{xrr}^{(2)}(\theta) = -\frac{1+\alpha}{1-\beta^2} \frac{\cos\theta}{(a^2 + \rho^2 + 2a\rho\sin\theta)^2} \left[(1-2\beta)\rho^3 - 4\beta\rho^2 a \sin\theta \right. \\ \left. + (2+2\beta)\rho a^2 \cos^2\theta - (2\beta+3)\rho a^2 - 2a^3 \sin\theta \right], \quad (C17)$$

$$\hat{G}_{yrr}^{(2)}(\theta) = \frac{1+\alpha}{1-\beta^2} \frac{1}{(a^2 + \rho^2 + 2a\rho\sin\theta)} \left[(1-2\beta)\rho^3 \sin\theta + (3-6\beta)\rho^2 a \right. \\ \left. + 4\beta\rho^2 a \cos^2\theta + (2\beta+2)\rho a^2 \sin\theta \cos^2\theta \right. \\ \left. + (3-6\beta)\rho a^2 \sin\theta - (2\beta \sin^2\theta - 1)a^3 \right], \quad (C18)$$

$$\hat{G}_{xr\theta}^{(2)}(\theta) = -\frac{1+\alpha}{1-\beta^2} \frac{1}{(a^2 + \rho^2 + 2a\rho\sin\theta)^2} \left[\rho^3 \sin\theta + 3\rho^2 a - (2\beta+2)\rho^2 a \cos^2\theta \right. \\ \left. - (2\beta+2)\rho a^2 \sin\theta \cos^2\theta + 3\rho a^2 \sin\theta + (1-2\cos^2\theta)a^3 \right], \quad (C19)$$

$$\hat{G}_{yr\theta}^{(2)}(\theta) = -\frac{1+\alpha}{1-\beta^2} \frac{\cos\theta}{(a^2 + \rho^2 + 2a\rho\sin\theta)} \left[\rho^3 + (2+2\beta)\rho^2 a \sin\theta \right. \\ \left. - (2\beta+2)\rho a^2 \cos^2\theta + (1+4\beta)\rho a^2 + 2\beta a^3 \sin\theta \right], \quad (C20)$$

$$\hat{G}_{x\theta\theta}^{(2)}(\theta) = -\frac{1+\alpha}{1-\beta^2} \frac{\cos\theta}{(a^2 + \rho^2 + 2a\rho\sin\theta)^2} \left[\rho^3 + 4\rho^2 a \sin\theta \right. \\ \left. + 5\rho a^2 - (2\beta+2)\rho a^2 \cos^2\theta + 2a^3 \sin\theta \right], \quad (C21)$$

$$\hat{G}_{y\theta\theta}^{(2)}(\theta) = \frac{1+\alpha}{1-\beta^2} \frac{1}{(a^2 + \rho^2 + 2a\rho\sin\theta)^2} \left[\rho^3 \sin\theta - 4\rho^2 a \cos^2\theta \right. \\ \left. + 3\rho^2 a - (2\beta+2)\rho a^2 \sin\theta \cos^2\theta + 3\rho a^2 \sin\theta + (1-2\beta \cos^2\theta)a^3 \right]. \quad (C22)$$

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