Two-point right focal boundary value problems on time scales

Boey, Kok Leong

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Two-point Right Focal Boundary Value Problems on Time Scales

Boey Kok Leong

School of Electrical & Electronic Engineering

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Contents

Acknowledgements	ii
Contents	iii
Summary	iv
1 Introduction	1
2 Properties of Green's Function	8
3 Eigenvalues	18
3.1 Characterization of Eigenvalues	19
3.2 Intervals of Eigenvalues	27
4 One or More Positive Solutions	33
4.1 Existence Criteria	34
4.2 Applications	51
5 Triple Positive Solutions	61
5.1 Existence Criteria	62
5.2 Examples	76
Author's Publications	81
Bibliography	82

SUMMARY

Let $\mathbf{T}$ be a time scale, i.e., $\mathbf{T}$ is a closed subset of $\mathbb{R}$. For any $a, b (> a) \in \mathbf{T}$, define the interval $[a, b]$ as $[a, b] = \{t \in \mathbf{T} \mid a \leq t \leq b\}$. For $t < \sup \mathbf{T}$ and $s > \inf \mathbf{T}$, the *forward jump operator* σ and the *backward jump operator* ρ are respectively defined by $\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\}$ and $\rho(s) = \sup\{\tau \in \mathbf{T} \mid \tau < s\}$. We define $\sigma^n(t) = \sigma(\sigma^{n-1}(t))$ with $\sigma^0(t) = t$.

Let $\lambda > 0$, p, n be fixed integers satisfying $n \geq 2$, $1 \leq p \leq n - 1$, $a, b \in \mathbf{T}$ with $a < \sigma(b)$ and $\rho(\sigma(b)) = b$. We also let $F : [a, \sigma(b)] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Consider the differential equation on $\mathbf{T}$ of the form

$$(-1)^{n-1} y^{\Delta^n}(t) = \lambda (-1)^{p+1} F(t, y(\sigma^{n-1}(t))), \quad t \in [a, b] \quad (1)$$

subject to the two-point right focal boundary conditions

$$\begin{aligned} y^{\Delta^i}(a) &= 0, & 0 \leq i \leq p-1 \\ y^{\Delta^i}(\sigma(b)) &= 0, & p \leq i \leq n-1. \end{aligned} \quad (2)$$

We aim to offer conditions that guarantee the existence of one or more positive solutions of (1), (2).

Chapter 1 begins with an informal introduction to the literature. We also state the necessary definitions and several fixed point theorems that are needed later.

The Green's function related to (1), (2) plays a central role in the development of our results. Chapter 2 contains a systematic analysis of this function. Results in this chapter include an expression for the Green's function. We also establish upper and lower bounds of the Green's function which are very useful in subsequent chapters.

Chapter 3 deals with the existence of positive solutions of (1), (2). We say that y is a *positive solution* of (1), (2) if y satisfies (1), (2) and $y(t) \geq 0$ for $t \in [a, \sigma^n(b)]$. If, for a particular λ , the boundary value problem (1), (2) has a positive solution y ,

then λ is called an eigenvalue of (1), (2). Let

$$E = \{\lambda > 0 \mid (1), (2) \text{ has a positive solution} \}$$

be the set of eigenvalues of the boundary value problem (1),(2). We shall provide conditions under which the set of eigenvalues E either contains an interval or is an interval itself. Further, in the case that E is an interval, criteria are established for E to be a bounded or an unbounded interval.

In Chapter 4, we consider (1) when $\lambda = 1$, i.e.,

$$(-1)^{n-1}y^{\Delta^n}(t) = (-1)^{p+1}F(t, y(\sigma^{n-1}(t))), \quad t \in [a, b] \quad (1)'$$

together with the two-point right focal boundary conditions (2). Our results will guarantee the existence of single, double and multiple positive solutions of (1)', (2). In addition, we shall consider the following special cases of (1)', (2) when $n = 2$ and $p = 1$:

$$\begin{cases} y^{\Delta^2}(t) + h(t)([y(\sigma(t))]^\alpha + [y(\sigma(t))]^\beta) = 0, & t \in [a, b] \\ y(a) = y^{\Delta}(\sigma(b)) = 0. \end{cases} \quad (Q1)$$

and

$$\begin{cases} y^{\Delta^2}(t) + h(t)e^{\zeta[y(\sigma(t))]} = 0, & t \in [a, b] \\ y(a) = y^{\Delta}(\sigma(b)) = 0. \end{cases} \quad (Q2)$$

We shall provide existence results for double positive solutions of (Q1) and (Q2), as well as upper and lower bounds for these solutions. The importance of (Q1) is illustrated in [18, 40] where particular cases in the real and discrete domains are discussed. The boundary value problem (Q2) arises in applications involving the diffusion of heat generated by positive temperature-dependent sources [1]. For example, when $\zeta = I$ the boundary value problem (Q2) occurs in the analysis of Joule losses in electrically conducting solids as well as in frictional heating.

Chapter 5 is concerned with the existence of triple positive solutions of the boundary value problem (1)', (2). We develop existence criteria for (at least) three positive solutions. We remark that knowledge of how many meaningful solutions is

probably most important from a numerical standpoint. If it is known that there are multiple solutions, then naturally one may need to develop methods that produce a specific one of the solutions for efficiency sake.

The motivation of our work comes from the fact that boundary value problems model a wide spectrum of physical phenomena which include gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, just to name a few. In all these problems, only positive solutions are meaningful. Boundary value problems have attracted a lot of attention in the literature, many researchers have discussed the existence of positive solutions of boundary value problems on the real and discrete domains. A recent trend is to consider boundary value problems on time scales, which include the real and the discrete as special cases. Our approach and results in this work unify the analysis for the real and the discrete cases. Furthermore, some of these results, which when reduced to $\mathbb{R}$ and $\mathbb{Z}$, are also *new* in the literature. Throughout, examples are included to illustrate the importance of the results obtained.

Chapter 1

Introduction

We investigate the existence of positive solutions to the differential equation on time scales of the form

$$(-1)^{n-1}y^{\Delta^n}(t) = \lambda(-1)^{p+1}F(t, y(\sigma^{n-1}(t))), \quad t \in [a, b] \quad (1.1)$$

subject to the two-point right focal boundary conditions

$$\begin{aligned} y^{\Delta^i}(a) &= 0, \quad 0 \leq i \leq p-1 \\ y^{\Delta^i}(\sigma(b)) &= 0, \quad p \leq i \leq n-1, \end{aligned} \quad (1.2)$$

where $\lambda > 0$, p, n are fixed integers satisfying $n \geq 2$, $1 \leq p \leq n-1$, $a, b \in \mathbf{T}$ with $a < \sigma(b)$ and $\rho(\sigma(b)) = b$ where $\mathbf{T}$ is a time scale. We also let $F : [a, \sigma(b)] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function.

To understand the notations used in (1.1), (1.2), we recall some standard definitions as follows. The reader may refer to [26] for an introduction to the subject.

Definition 1.1.

- (a) Let $\mathbf{T}$ be a time scale, i.e., $\mathbf{T}$ is a closed subset of $\mathbb{R}$. We assume that $\mathbf{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. Throughout, for any a, b ($> a$), the interval $[a, b]$ is defined as $[a, b] = \{t \in \mathbf{T} \mid a \leq t \leq b\}$. We also use the notation $\mathbb{R}[c, d]$ to denote the real interval $\{t \in \mathbb{R} \mid c \leq t \leq d\}$,

and $\mathbb{Z}[c, d]$ to represent the set $\{t \in \mathbb{Z} \mid c \leq t \leq d\}$. Analogous notations for open and half-open intervals will also be used in the paper.

- (b) For $t < \sup \mathbf{T}$ and $s > \inf \mathbf{T}$, the *forward jump operator* σ and the *backward jump operator* ρ are respectively defined by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T} \quad \text{and} \quad \rho(s) = \sup\{\tau \in \mathbf{T} \mid \tau < s\} \in \mathbf{T}.$$

We define $\sigma^n(t) = \sigma(\sigma^{n-1}(t))$ with $\sigma^0(t) = t$. Similar definition is used for $\rho^n(s)$.

- (c) Fix $t \in \mathbf{T}$. Let $y : \mathbf{T} \rightarrow \mathbb{R}$. We define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|.$$

We call $y^\Delta(t)$ the *delta derivative* of $y(t)$. Define $y^{\Delta^n}(t)$ to be the delta derivative of $y^{\Delta^{n-1}}(t)$, i.e., $y^{\Delta^n}(t) = (y^{\Delta^{n-1}}(t))^\Delta$.

- (d) If $F^\Delta(t) = f(t)$, then we define the integral

$$\int_a^t f(\tau) \Delta\tau = F(t) - F(a).$$

A solution y of (1.1), (1.2) will be sought in $C[a, \sigma^n(b)]$, the space of continuous functions $\{y : [a, \sigma^n(b)] \rightarrow \mathbb{R}\}$.

Definition 1.2. We say that y is a *positive solution* of (1.1), (1.2) if y satisfies (1.1), (1.2) and $y(t) \geq 0$ for $t \in [a, \sigma^n(b)]$. If, for a particular λ , the boundary value problem (1.1), (1.2) has a positive solution y , then λ is called an *eigenvalue* and y a corresponding *eigenfunction* of (1.1), (1.2). We let

$$E = \{\lambda > 0 \mid (1.1), (1.2) \text{ has a positive solution} \}$$

be the set of eigenvalues of the boundary value problem (1.1), (1.2).

This definition relating to positive solutions forms the foundation for much to follow. Criteria for the existence of positive solutions of the boundary value problem (1.1), (1.2) and some of its special cases are developed in subsequent chapters.

Boundary value problems have attracted a lot of attention in the recent literature, due mainly to the fact that they model many physical phenomena which include gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, just to name a few. In all these problems, only *positive* solutions are meaningful. Many papers have discussed the existence of positive solutions of boundary value problems on the real and discrete domains, we refer to [9, 11, 13, 16, 17, 21, 22, 24, 33, 37, 39, 42, 44, 52, 56, 57, 58, 60, 64] and the monographs [8, 10] which give a good documentary of the literature. A recent trend is to consider boundary value problems on time scales, which include the real and the discrete as special cases, see [3, 4, 14, 15, 20, 28, 29, 30, 32, 34, 35, 36, 45, 46, 54, 55, 59]. Our approach in the present work not only unifies the analysis for the real and the discrete cases in [56, 57, 60, 61, 62, 63], but also leads to *new* results which, when reduced to $\mathbb{R}$ and $\mathbb{Z}$, are also *new* in the literature.

We give some examples to illustrate time scales.

Example 1.1. [31] Assume that the life span of a certain species is α units of time. Suppose that just before the species dies, eggs are laid that are then hatched β units of time later. Hence, we are looking at the time scale

$$\mathbf{T} = \bigcup_{k \in \mathbb{N}} \mathbb{R}[k(\alpha + \beta), k(\alpha + \beta) + \alpha].$$

As a specific example, the cicada *magicicada septendecim* lives as a larva for 17 years and as an adult for about 1 week. Here, we would be considering $\alpha = \frac{1}{52}$ year and $\beta = 17$ years in the time scale $\mathbf{T}$. In another instance, the common mayfly *stenonema canadense* remains a larva for 1 year and becomes an adult for about 1

day. For a possible time scale that represents the life cycle of the mayfly, we may let $\alpha = 1$ day and $\beta = 365$ days in T .

Example 1.2. [48] Consider a simple electric circuit with resistance R , capacitance C , and inductance L . Suppose the capacitor is discharged periodically every time unit and assume that the discharging takes $\delta(> 0)$ time units where δ is small. Then this simulation can be modelled using the time scale

$$T = \bigcup_{k \in \mathbf{N}} [k, k + 1 - \delta].$$

If $Q(t)$ is the total charge on the capacitor at time t , then we have

$$LQ^{\Delta^2}(t) + RQ^{\Delta}(t) + \frac{Q}{C} = 0, \quad t \in T.$$

Figure 1.1. An Electric Circuit

For the rest of this chapter, we shall state some definitions and several fixed point theorems that are needed later. Let B be a Banach space equipped with norm $\|\cdot\|$.

Definition 1.3. Let $C (\subset B)$ be a nonempty closed convex set. We say that C is a *cone* provided the following conditions are satisfied:

- (a) if $y \in C$ and $\alpha \geq 0$, then $\alpha y \in C$;
- (b) if $y \in C$ and $-y \in C$, then $y = 0$.

The cone C induces an ordering $\leq$ on B . For $x, y \in B$, we write $x \leq y$ if and only if $y - x \in C$. If $x, y \in B$ with $x \leq y$, we let $\langle x, y \rangle$ denote the closed order interval given by $\langle x, y \rangle = \{z \in B \mid x \leq z \leq y\}$.

Definition 1.4. Let $C (\subset B)$ be a cone. A map ψ is a *nonnegative continuous concave functional* on C if the following conditions are satisfied:

- (a) $\psi : C \rightarrow \mathbb{R}[0, \infty)$ is continuous;
- (b) $\psi(tx + (1 - t)y) \geq t\psi(x) + (1 - t)\psi(y)$ for all $x, y \in C$ and $0 \leq t \leq 1$.

Definition 1.5. Let $C (\subset B)$ be a cone. A map β is a *nonnegative continuous convex functional* on C if the following conditions are satisfied:

- (a) $\beta : C \rightarrow \mathbb{R}[0, \infty)$ is continuous;
- (b) $\beta(tx + (1 - t)y) \leq t\beta(x) + (1 - t)\beta(y)$ for all $x, y \in C$ and $0 \leq t \leq 1$.

Let γ, β, Θ be nonnegative continuous convex functionals on C and α, ψ be nonnegative continuous concave functionals on C . For nonnegative numbers w_i , $1 \leq i \leq 3$, we shall introduce the following notations:

$$C(w_1) = \{y \in C \mid \|y\| < w_1\},$$

$$C(\psi, w_1, w_2) = \{y \in C \mid \psi(y) \geq w_1 \text{ and } \|y\| \leq w_2\},$$

$$P(\gamma, w_1) = \{y \in C \mid \gamma(y) < w_1\},$$

$$P(\gamma, \alpha, w_1, w_2) = \{y \in C \mid \alpha(y) \geq w_1 \text{ and } \gamma(y) \leq w_2\},$$

$$Q(\gamma, \beta, w_1, w_2) = \{y \in C \mid \beta(y) \leq w_1 \text{ and } \gamma(y) \leq w_2\},$$

$$P(\gamma, \Theta, \alpha, w_1, w_2, w_3) = \{y \in C \mid \alpha(y) \geq w_1, \Theta(y) \leq w_2 \text{ and } \gamma(y) \leq w_3\},$$

$$Q(\gamma, \beta, \psi, w_1, w_2, w_3) = \{y \in C \mid \psi(y) \geq w_1, \beta(y) \leq w_2 \text{ and } \gamma(y) \leq w_3\}.$$

The following fixed point theorems are needed later. The first theorem is the Leray-Schauder alternative [53] while the next is due to Krasnosel'skii [49]. The third is usually called *Leggett-Williams' fixed point theorem* [50], and the last is known as the *five-functional fixed point theorem* [19].

Theorem 1.1. [53] Let B be a Banach space with $E \subseteq B$ closed and convex. Assume U is a relatively open subset of E with $0 \in U$ and $S : \overline{U} \rightarrow E$ is a continuous and compact map. Then, either

- (a) S has a fixed point in $\overline{U}$, or
- (b) there exists $y \in \partial U$ and $\lambda \in \mathbb{R}(0, 1)$ such that $y = \lambda Sy$. □

Theorem 1.2. [49] Let $B = (B, \|\cdot\|)$ be a Banach space, and let $C(\subset B)$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of B with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let

$$S : C \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow C$$

be a continuous and completely continuous operator such that, either

- (a) $\|Sy\| \leq \|y\|$, $y \in C \cap \partial\Omega_1$, and $\|Sy\| \geq \|y\|$, $y \in C \cap \partial\Omega_2$, or
- (b) $\|Sy\| \geq \|y\|$, $y \in C \cap \partial\Omega_1$, and $\|Sy\| \leq \|y\|$, $y \in C \cap \partial\Omega_2$

Then, S has a fixed point in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$. □

Theorem 1.3. [50] Let $C(\subset B)$ be a cone, and $w_4 > 0$ be given. Assume that ψ is a nonnegative continuous concave functional on C such that $\psi(y) \leq \|y\|$ for all $y \in \overline{C}(w_4)$, and let $S : \overline{C}(w_4) \rightarrow \overline{C}(w_4)$ be a continuous and completely continuous operator. Suppose that there exist numbers w_1, w_2, w_3 where $0 < w_1 < w_2 < w_3 \leq w_4$ such that

- (a) $\{y \in C(\psi, w_2, w_3) \mid \psi(y) > w_2\} \neq \emptyset$, and $\psi(Sy) > w_2$ for all $y \in C(\psi, w_2, w_3)$;
- (b) $\|Sy\| < w_1$ for all $y \in \overline{C}(w_1)$;
- (c) $\psi(Sy) > w_2$ for all $y \in C(\psi, w_2, w_4)$ with $\|Sy\| > w_3$.

Then, S has (at least) three fixed points y_1 , y_2 and y_3 in $\overline{C}(w_4)$. Furthermore, we have

$$y_1 \in C(w_1), \quad y_2 \in \left\{ y \in C(\psi, w_2, w_4) \mid \psi(y) > w_2 \right\} \quad \text{and} \\ y_3 \in \overline{C}(w_4) \setminus \left(C(\psi, w_2, w_4) \cup \overline{C}(w_1) \right). \quad \square$$

Theorem 1.4. [19] Let $C (\subset B)$ be a cone. Assume that there exist positive numbers w_5, M , nonnegative continuous convex functionals γ, β, Θ on C , and nonnegative continuous concave functionals α, ψ on C , with

$$\alpha(y) \leq \beta(y) \quad \text{and} \quad \|y\| \leq M\gamma(y)$$

for all $y \in \overline{P}(\gamma, w_5)$. Let $S : \overline{P}(\gamma, w_5) \rightarrow \overline{P}(\gamma, w_5)$ be a continuous and completely continuous operator. Suppose that there exist nonnegative numbers w_i , $1 \leq i \leq 4$ with $0 < w_2 < w_3$ such that

- (a) $\{y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(y) > w_3\} \neq \emptyset$, and $\alpha(Sy) > w_3$ for all $y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$;
- (b) $\{y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(y) < w_2\} \neq \emptyset$, and $\beta(Sy) < w_2$ for all $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$;
- (c) $\alpha(Sy) > w_3$ for all $y \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Sy) > w_4$;
- (d) $\beta(Sy) < w_2$ for all $y \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Sy) < w_1$.

Then, S has (at least) three fixed points y_1 , y_2 and y_3 in $\overline{P}(\gamma, w_5)$. Furthermore, we have

$$\beta(y_1) < w_2, \quad \alpha(y_2) > w_3, \quad \text{and} \quad \beta(y_3) > w_2 \quad \text{with} \quad \alpha(y_3) < w_3. \quad \square$$

We end this chapter by introducing the notations

$$\underline{f}_0 = \liminf_{x \rightarrow 0} \frac{f(x)}{x}, \quad \overline{f}_0 = \limsup_{x \rightarrow 0} \frac{f(x)}{x}, \\ \underline{f}_\infty = \liminf_{x \rightarrow \infty} \frac{f(x)}{x}, \quad \overline{f}_\infty = \limsup_{x \rightarrow \infty} \frac{f(x)}{x}.$$

Chapter 2

Properties of Green's Function

The Green's function related to (1.1), (1.2) plays a central role in the development of our results. In this chapter, by using some results of Anderson [12], we provide the explicit expression of the Green's function in Theorem 2.5 and establish upper and lower bounds of the Green's function respectively in Theorems 2.7 and 2.8. To begin, we state some definitions and notations.

Definition 2.1.

- (a) Define the function $h_k : \mathbf{T} \times \mathbf{T} \rightarrow \mathbb{R}$, $k \in \{0, 1, \dots\}$, recursively as

$$h_0(t, s) = 1 \quad \text{for all } s, t \in \mathbf{T}$$

and

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta\tau, \quad \text{for all } s, t \in \mathbf{T} \text{ and } k = 0, 1, \dots.$$

- (b) Let t_i , $1 \leq i \leq n$ be such that

$$a = t_1 = \dots = t_p < t_{p+1} = \dots = t_n = \sigma(b).$$

- (c) Define $T_i : [a, b] \rightarrow \mathbb{R}$, $0 \leq i \leq n-1$ as

$$T_0(t) \equiv 1$$

and

$$T_i(t) = T_i(t : t_1, \dots, t_i) = \int_{t_1}^t \int_{t_2}^{\tau_1} \cdots \int_{t_i}^{\tau_{i-1}} \Delta \tau_i \cdots \Delta \tau_2 \Delta \tau_1, \quad 1 \leq i \leq n-1.$$

(d) For integers $n \geq 2$ and for $i = 1, 2, \dots, n-1$, define

$$u_{n,i}(t, s) = u_{n,i}(t, s : t_1, t_2, \dots, t_n), \quad s, t \in \mathbf{T}$$

by

$$u_{n,i}(t, s) = (-1)^{n+1} \begin{vmatrix} 0 & h_1(t, t_1) & h_2(t, t_1) & \cdots & h_{n-1}(t, t_1) \\ c_2(s, i) & 1 & h_1(t_2, t_1) & \cdots & h_{n-2}(t_2, t_1) \\ c_3(s, i) & 0 & 1 & \cdots & h_{n-3}(t_3, t_1) \\ c_4(s, i) & 0 & 0 & \cdots & h_{n-4}(t_4, t_1) \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1}(s, i) & 0 & 0 & \cdots & h_1(t_{n-1}, t_1) \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix} \quad (2.1)$$

where $c_j(s, i) = H(j-1-i)h_{n-j}(t_j, \sigma(s))$ for $j = 2, 3, \dots, n-1$ and $i = 1, 2, \dots, n-1$. Here

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

In addition, for integers $n \geq 2$ and for $i = 1, 2, \dots, n-1$, define

$$v_{n,i}(t, s) = u_{n,i}(t, s) + (-1)^{n-1}h_{n-1}(t, \sigma(s)), \quad s, t \in \mathbf{T}. \quad (2.2)$$

We also define for all integers $n \geq 2$ and $s, t \in \mathbf{T}$

$$\begin{aligned} u_{n,0}(t, s) &= (-1)^n h_{n-1}(t, \sigma(s)), \\ v_{n,0}(t, s) &= u_{n,0}(t, s) + (-1)^{n-1} h_{n-1}(t, \sigma(s)). \end{aligned}$$

Although working on a different boundary value problem, the Green's function provided in [12] is closely related to the Green's function for (1.1), (1.2). We shall state without proof the following two results.

Lemma 2.1. [12] Consider the n -point right focal boundary value problem

$$(-1)^{n-1}y^{\Delta^n}(t) = 0, \quad t \in [a, b]$$

$$y^{\Delta^{\tau-1}}(z_\tau) = 0, \quad 1 \leq \tau \leq n$$

where $a = z_1 < \cdots < z_n = \sigma(b)$. The Green's function for this boundary value problem is given by

$$g_n(t, s : t_1, \dots, t_n) = \begin{cases} s \in I_1 & : \begin{cases} u_{n,1}(t, s) & : t \leq \sigma(s) \\ v_{n,1}(t, s) & : t \geq \sigma(s) \end{cases} \\ s \in I_2 & : \begin{cases} u_{n,2}(t, s) & : t \leq \sigma(s) \\ v_{n,2}(t, s) & : t \geq \sigma(s) \end{cases} \\ \vdots & : \vdots \\ s \in I_{n-1} & : \begin{cases} u_{n,n-1}(t, s) & : t \leq \sigma(s) \\ v_{n,n-1}(t, s) & : t \geq \sigma(s), \end{cases} \end{cases}$$

where $I_1 = [a, \rho(z_2)]$, $I_i = [\rho(z_i), \rho(z_{i+1})]$ for $i = 2, 3, \dots, n-2$ and $I_{n-1} = [\rho(z_{n-1}), b]$, with $u_{n,i}(t, s)$ as in (2.1) and $v_{n,i}(t, s)$ as in (2.2). $\square$

Remark 2.1.

- (a) Since $h_j(a, a) = 0$ for $1 \leq j \leq n-1$, from (2.1), $u_{n,j}(a, s)$ has a row of zeros and hence $u_{n,j}(a, s) = 0$ for $1 \leq j \leq n-1$. It then follows from (2.2) that $v_{n,j}(a, s) = (-1)^{n-1}h_{n-1}(a, \sigma(s))$.
- (b) Consider the special case when $z_i = t_i$, $1 \leq i \leq n$. Since $a = t_1 = \cdots = t_p < t_{p+1} = \cdots = t_n = \sigma(b)$ and $\rho(\sigma(b)) = b$, we have $I_1 = [a, \rho(a)]$, $I_2 = \cdots = I_{p-1} = \{\rho(a)\}$, $I_{p+1} = \cdots = I_{n-1} = \{b\}$, and the only nondegenerate interval is $I_p = [\rho(a), b]$.

Lemma 2.2. [12] Consider $u_{n,i}(t, s) = u_{n,i}(t, s : t_1, \dots, t_n)$ and $v_{n,i}(t, s) = v_{n,i}(t, s : t_1, \dots, t_n)$. For $i = 2, 3, \dots, n-1$, when we differentiate with respect to t , we have

$$\begin{aligned} u_{n,i}^{\Delta t}(t, s : t_1, t_2, \dots, t_n) &= -u_{n-1,i-1}(t, s : t_2, t_3, \dots, t_n) \\ v_{n,i}^{\Delta t}(t, s : t_1, t_2, \dots, t_n) &= -v_{n-1,i-1}(t, s : t_2, t_3, \dots, t_n), \end{aligned}$$

while

$$\begin{aligned} u_{n,1}^{\Delta t}(t, s : t_1, t_2, \dots, t_n) &= -u_{n-1,0}(t, s : t_2, t_3, \dots, t_n) = (-1)^n h_{n-2}(t, \sigma(s)) \\ v_{n,1}^{\Delta t}(t, s : t_1, t_2, \dots, t_n) &= -v_{n-1,0}(t, s : t_2, t_3, \dots, t_n) = 0. \end{aligned}$$

Furthermore,

$$u_{n,p}^{\Delta t^{n-1}}(t, s) = (-1)^n \quad \text{and} \quad v_{n,p}^{\Delta t^{n-1}}(t, s) = 0. \quad \square$$

The result in Lemma 2.2 provides a basis for expressing $u_{n,p}(t, s)$ and $v_{n,p}(t, s)$ in terms of $T_i(t)$ and $h_i(t, s)$. We shall begin with $v_{n,p}(t, s)$.

Lemma 2.3. $v_{n,p}(t, s) = (-1)^{n-1} \sum_{i=0}^{p-1} T_i(t) h_{n-1-i}(a, \sigma(s))$.

Proof: We proceed by induction on $n \geq 2$. We denote the statement by $P(n)$.

When $n = 2$, we only have $p = 1$. From definition, we have

$$\begin{aligned} v_{2,1}(t, s) &= u_{2,1}(t, s) + (-1)^1 h_1(t, \sigma(s)) \\ &= (-1)^3 \begin{vmatrix} 0 & h_1(t, t_1) \\ 1 & 1 \end{vmatrix} - h_1(t, \sigma(s)) = \sigma(s) - a. \end{aligned}$$

But $-\sum_{i=0}^0 T_i(t) h_{1-i}(a, \sigma(s)) = -T_0(t) h_1(a, \sigma(s)) = \sigma(s) - a$, and so $P(2)$ is true.

We now assume that $P(k)$ is true for some positive integer $k \geq 2$.

Case 1. Consider $p = 1$. From Lemma 2.2,

$$v_{k+1,1}^{\Delta t}(t, s : t_1, t_2, \dots, t_{k+1}) = -v_{k,0}(t, s : t_2, t_3, \dots, t_{k+1}) = 0.$$

Integrating with respect to t ,

$$v_{k+1,1}(t, s : t_1, \dots, t_{k+1}) - v_{k+1,1}(t_1, s) = \int_{t_1}^t -v_{k,0}(\tau, s) \Delta \tau = 0.$$

It then follows from Remark 2.1(a) that

$$v_{k+1,1}(t, s) = (-1)^k h_k(a, \sigma(s))$$

and thus $P(k+1)$ is true for $p = 1$.

Case 2. Suppose $p \geq 2$. Differentiating $v_{k+1,p}(t, s)$ with respect to t and applying Lemma 2.2, we have

$$\begin{aligned} v_{k+1,p}^{\Delta_t}(t, s : t_1, \dots, t_{k+1}) &= -v_{k,p-1}(t, s : t_2, \dots, t_{k+1}) \\ &= (-1)^k \sum_{i=0}^{p-2} T_i(t : t_2, \dots, t_{i+1}) h_{k-1-i}(a, \sigma(s)). \end{aligned}$$

On integration, this yields

$$\begin{aligned} &v_{k+1,p}(t, s) - v_{k+1,p}(a, s) \\ &= \int_{t_1}^t v_{k+1,p}^{\Delta_t}(\tau, s : t_1, \dots, t_{k+1}) \Delta \tau \\ &= \int_{t_1}^t (-1)^k \sum_{i=0}^{p-2} T_i(\tau : t_2, \dots, t_{i+1}) h_{k-1-i}(a, \sigma(s)) \Delta \tau \\ &= (-1)^k \sum_{i=0}^{p-2} \left(\int_{t_1}^t T_i(\tau : t_2, \dots, t_{i+1}) \Delta \tau \right) h_{k-1-i}(a, \sigma(s)) \\ &= (-1)^k \sum_{i=0}^{p-2} [T_{i+1}(t : t_1, \dots, t_{i+1}) - T_{i+1}(t_1 : t_1, \dots, t_{i+1})] h_{k-1-i}(a, \sigma(s)) \\ &= (-1)^k \sum_{i=1}^{p-1} T_i(t : t_1, \dots, t_i) h_{k-i}(a, \sigma(s)) \end{aligned}$$

since $T_{i+1}(t_1) = 0$. By Remark 2.1(a), we note that $v_{k+1,p}(a, s) = (-1)^k h_k(a, \sigma(s))$, thus it follows that

$$\begin{aligned} v_{k+1,p}(t, s) &= (-1)^k \sum_{i=1}^{p-1} T_i(t) h_{k-i}(a, \sigma(s)) + (-1)^k h_k(a, \sigma(s)) \\ &= (-1)^k \sum_{i=0}^{p-1} T_i(t) h_{k-i}(a, \sigma(s)) \end{aligned}$$

and so $P(k+1)$ is true for $p \geq 2$.

In both cases, $P(k+1)$ is true. Thus, $P(n)$ is true by induction. $\square$

Corollary 2.4. $u_{n,p}(t, s) = (-1)^{n-1} \sum_{i=0}^{p-1} T_i(t) h_{n-1-i}(a, \sigma(s)) + (-1)^n h_{n-1}(t, \sigma(s)).$

Proof. The result follows immediately from (2.2) and Lemma 2.3. $\square$

The last few results prove the following theorem, which is of fundamental importance in obtaining solutions for (1.1), (1.2).

Theorem 2.5. The Green's function of the boundary value problem (1.2),

$$(-1)^{n-1} y^{\Delta^n}(t) = 0, \quad t \in [a, b] \quad (2.3)$$

may be expressed as

$$G(t, s) = \begin{cases} (-1)^{n-1} \sum_{i=0}^{p-1} T_i(t) h_{n-1-i}(a, \sigma(s)) + (-1)^n h_{n-1}(t, \sigma(s)), & t \leq \sigma(s) \\ (-1)^{n-1} \sum_{i=0}^{p-1} T_i(t) h_{n-1-i}(a, \sigma(s)), & t \geq \sigma(s) \end{cases} \quad (2.4)$$

where $t \in [a, \sigma^n(b)]$ and $s \in [a, b]$.

Proof. Applying Lemma 2.3 and Corollary 2.4, we shall prove that

$$G(t, s) = \begin{cases} u_{n,p}(t, s), & t \leq \sigma(s) \\ v_{n,p}(t, s), & t \geq \sigma(s) \end{cases}$$

which is the same as (2.4). First, for $t \in [a, \sigma^n(b)]$ and $s \in [a, b]$, from (2.2) we see that

$$v_{n,p}(t, s) - u_{n,p}(t, s) = (-1)^{n-1} h_{n-1}(t, \sigma(s)).$$

So if $t = \sigma(s)$, then $v_{n,p}(\sigma(s), s) = u_{n,p}(\sigma(s), s)$. Thus, $G(t, s)$ is well-defined.

Next, we shall show that $G(t, s)$ satisfies the boundary conditions (1.2). When $1 \leq j \leq p$, by Lemma 2.2 and Remark 2.1(a), we obtain

$$G^{\Delta_t^{j-1}}(a, s) = u_{n,p}^{\Delta_t^{j-1}}(a, s : t_1, t_2, \dots, t_n) = (-1)^{j-1} u_{n-j+1, p-j+1}(a, s : t_j, \dots, t_n) = 0.$$

On the other hand, when $p + 1 \leq j \leq n$ and $t \geq \sigma(s)$, we find

$$\begin{aligned} G^{\Delta_t^{j-1}}(t, s) &= (-1)^{n-1} \sum_{i=0}^{p-1} (T_i(t))^{\Delta_t^{j-1}} h_{n-1-i}(a, \sigma(s)) \\ &= (-1)^{n-1} \sum_{i=0}^{p-1} (0) h_{n-1-i}(a, \sigma(s)), \end{aligned}$$

from which it follows that $G^{\Delta_t^{j-1}}(\sigma(b), s) = 0$. Hence for each $s \in [a, b]$, we have shown that

$$\begin{aligned} G^{\Delta_t^{j-1}}(a, s) &= 0, \quad 1 \leq j \leq p, \\ G^{\Delta_t^{j-1}}(\sigma(b), s) &= 0, \quad p + 1 \leq j \leq n. \end{aligned}$$

Finally, assume that $f(t)$ is a given right-dense continuous function on $[a, b]$, i.e., f is continuous at all right-dense points of T and has a left sided limit at all the left-dense points of T . Define

$$y(t) = \int_a^{\sigma(b)} G(t, s) f(s) \Delta s.$$

Then for $j = 0, 1, \dots, n$,

$$y^{\Delta_t^j}(t) = \int_a^{\sigma(b)} G^{\Delta_t^j}(t, s) f(s) \Delta s.$$

It is easy to check that y satisfies the boundary conditions (1.2). Moreover, it is clear that

$$\begin{aligned} y^{\Delta^{n-1}}(t) &= \int_a^{\sigma(b)} G^{\Delta_t^{n-1}}(t, s) f(s) \Delta s \\ &= \int_a^t v_{n,p}^{\Delta_t^{n-1}}(t, s) f(s) \Delta s + \int_t^{\sigma(b)} u_{n,p}^{\Delta_t^{n-1}}(t, s) f(s) \Delta s \\ &= 0 + (-1)^n \int_t^{\sigma(b)} f(s) \Delta s \end{aligned}$$

where we have used Lemma 2.2. Hence, for $t \in [a, b]$,

$$(-1)^{n-1} y^{\Delta^n}(t) = - \left(\int_t^{\sigma(b)} f(s) \Delta s \right)^{\Delta_t} = f(t).$$

□

On the other hand, when $p + 1 \leq j \leq n$ and $t \geq \sigma(s)$, we find

$$\begin{aligned} G^{\Delta_t^{j-1}}(t, s) &= (-1)^{n-1} \sum_{i=0}^{p-1} (T_i(t))^{\Delta_t^{j-1}} h_{n-1-i}(a, \sigma(s)) \\ &= (-1)^{n-1} \sum_{i=0}^{p-1} (0) h_{n-1-i}(a, \sigma(s)), \end{aligned}$$

from which it follows that $G^{\Delta_t^{j-1}}(\sigma(b), s) = 0$. Hence for each $s \in [a, b]$, we have shown that

$$\begin{aligned} G^{\Delta_t^{j-1}}(a, s) &= 0, \quad 1 \leq j \leq p, \\ G^{\Delta_t^{j-1}}(\sigma(b), s) &= 0, \quad p + 1 \leq j \leq n. \end{aligned}$$

Finally, assume that $f(t)$ is a given right-dense continuous function on $[a, b]$, i.e., f is continuous at all right-dense points of $\mathbf{T}$ and has a left sided limit at all the left-dense points of $\mathbf{T}$. Define

$$y(t) = \int_a^{\sigma(b)} G(t, s) f(s) \Delta s.$$

Then for $j = 0, 1, \dots, n$,

$$y^{\Delta_t^j}(t) = \int_a^{\sigma(b)} G^{\Delta_t^j}(t, s) f(s) \Delta s.$$

It is easy to check that y satisfies the boundary conditions (1.2). Moreover, it is clear that

$$\begin{aligned} y^{\Delta_t^{n-1}}(t) &= \int_a^{\sigma(b)} G^{\Delta_t^{n-1}}(t, s) f(s) \Delta s \\ &= \int_a^t v_{n,p}^{\Delta_t^{n-1}}(t, s) f(s) \Delta s + \int_t^{\sigma(b)} u_{n,p}^{\Delta_t^{n-1}}(t, s) f(s) \Delta s \\ &= 0 + (-1)^n \int_t^{\sigma(b)} f(s) \Delta s \end{aligned}$$

where we have used Lemma 2.2. Hence, for $t \in [a, b]$,

$$(-1)^{n-1} y^{\Delta_t^n}(t) = - \left(\int_t^{\sigma(b)} f(s) \Delta s \right)^{\Delta_t} = f(t). \quad \square$$

$$\begin{aligned}
 &= \begin{cases} (-1)^n h_{n-2}(t, \sigma(s)), & t \leq \sigma(s) \\ 0, & t \geq \sigma(s) \end{cases} \\
 &= \begin{cases} (-1)^n (-1)^{n-2} g_{n-2}(\sigma(s), t), & t \leq \sigma(s) \\ 0, & t \geq \sigma(s) \end{cases} \geq 0.
 \end{aligned}$$

Hence, $G(t, s)$ is nondecreasing in t and it follows that

$$0 = G(a, s) \leq G(t, s) \leq G(\sigma^n(b), s), \quad (t, s) \in [a, \sigma^n(b)] \times [a, b].$$

Case 2: $p \geq 2$

We proceed by induction on $n \geq 3$. We denote the statement by $P(n)$. For $n = 3$ with $p = 2$, we differentiate $G(t, s)$ with respect to t and apply Lemma 2.2 to obtain

$$\begin{aligned}
 G^{\Delta_t}(t, s) &= \begin{cases} -u_{2,1}(t, s : t_2, \dots, t_n), & t \leq \sigma(s) \\ -v_{2,1}(t, s : t_2, \dots, t_n), & t \geq \sigma(s) \end{cases} \\
 &= \begin{cases} -(\sigma(s) - a) - h_2(t, \sigma(s)), & t \leq \sigma(s) \\ -(\sigma(s) - a), & t \geq \sigma(s) \end{cases} \leq 0.
 \end{aligned}$$

Thus, $G(t, s)$ is nonincreasing in t and so

$$0 = G(a, s) \geq G(t, s) \geq G(\sigma^3(b), s), \quad (t, s) \in [a, \sigma^3(b)] \times [a, b].$$

Hence $P(3)$ is true.

We now assume that $P(k)$ is true for some positive integer $k \geq 3$. Let $n = k + 1$.

Differentiating $G(t, s)$ with respect to t gives

$$G^{\Delta_t}(t, s) = \begin{cases} -u_{k,p-1}(t, s : t_2, \dots, t_{k+1}), & t \leq \sigma(s) \\ -v_{k,p-1}(t, s : t_2, \dots, t_{k+1}), & t \geq \sigma(s). \end{cases}$$

By hypothesis $P(k)$, we observe that $(-1)^{p-1}G^{\Delta_t}(t, s) \geq 0$ for $t \geq a$. Thus, $(-1)^{p-1}G(t, s)$ is nondecreasing in t and we have for $(t, s) \in [a, \sigma^{k+1}(b)] \times [a, b]$,

$$0 = (-1)^{p+1}G(a, s) \leq (-1)^{p+1}G(t, s) \leq (-1)^{p+1}G(\sigma^{k+1}(b), s).$$

Hence $P(k+1)$ is true. Therefore, $P(n)$ is true by induction. $\square$

Remark 2.2. From the proof of Theorem 2.7, it is noted that for $n \geq 2$, $(-1)^{p+1}G(t, s)$ is a nondecreasing function in $t \in [a, \sigma^n(b)]$.

We now determine a lower bound for the Green's function. Throughout the thesis, for a fixed real number $\delta \in \mathbb{R}(0, \frac{1}{2})$, we let

$$\begin{cases} c = \min\{t \in \mathbf{T} \mid t \geq a + \delta(\sigma^n(b) - a)\} \\ d = \max\{t \in \mathbf{T} \mid t \leq \sigma^n(b) - \delta(\sigma^n(b) - a)\} \\ \text{and assume the existence of } c \text{ and } d \text{ such that } a < c < \rho^{n-1}(d) < \sigma(b). \end{cases} \quad (2.5)$$

Theorem 2.8. For $(t, s) \in [c, d] \times [a, b]$,

$$(-1)^{p+1}G(t, s) \geq k(-1)^{p+1}G(\sigma^n(b), s)$$

where $0 < k < 1$ is a constant given by

$$k = \inf_{s \in [a, b]} \frac{G(c, s)}{G(\sigma^n(b), s)}.$$

Proof. Using Theorem 2.7 and Remark 2.2, for $(t, s) \in [c, d] \times [a, b]$, we have

$$\begin{aligned} (-1)^{p+1}G(t, s) &\geq (-1)^{p+1}G(c, s) = (-1)^{p+1} \frac{G(c, s)}{G(\sigma^n(b), s)} \times G(\sigma^n(b), s) \\ &\geq \left[\inf_{s \in [a, b]} \frac{G(c, s)}{G(\sigma^n(b), s)} \right] \times (-1)^{p+1}G(\sigma^n(b), s). \end{aligned}$$

It is clear from Theorem 2.7 that $0 < k < 1$. $\square$

Chapter 3

Eigenvalues

In this chapter, we shall concentrate on the existence of positive solutions of (1.1), (1.2). We shall make use of the properties of the Green's function developed in Chapter 2 as well as Krasnosel'skii's fixed point theorem. We shall provide conditions under which the set of eigenvalues E is either an interval or it contains an interval. Further, in the case that E is an interval, criteria are established for E to be a bounded or an unbounded interval.

Many papers have discussed the existence of positive solutions of boundary value problems on the real and discrete domains, refer to [17, 33, 37, 42, 52, 56, 57, 58, 60, 64] and the monographs [8, 10] which give a good documentary of the literature. A recent trend is to consider boundary value problems on time scales, which include the real and the discrete as special cases, see [4, 14, 15, 28, 32, 35, 36, 54, 59]. Our approach in the present work, when $T = \mathbb{R}$ or $\mathbb{Z}$, reduced to the real and the discrete cases in [56, 60] respectively. Furthermore, some of our results are also *new* in the literature when reduced to $\mathbb{R}$ or $\mathbb{Z}$.

3.1 Characterization of Eigenvalues

To begin, we shall list the conditions that are needed later as follows. In these conditions, f, u, v are continuous functions such that $f : \mathbb{R}[0, \infty) \rightarrow \mathbb{R}[0, \infty)$ and $u, v : [a, \sigma(b)] \rightarrow \mathbb{R}[0, \infty)$.

- (A1) f is nondecreasing;
- (A2) for $y \geq 0$ and $t \in [a, \sigma(b)]$, $F(t, y)$ is nonnegative;
- (A3) $u(t)f(y(\sigma^{n-1}(t))) \leq F(t, y(\sigma^{n-1}(t))) \leq v(t)f(y(\sigma^{n-1}(t)))$ for $t \in [a, \sigma(b)]$ and $y \geq 0$;
- (A4) $u(t)$ is nonnegative and it is not identically zero on any nondegenerate subinterval of $[a, b]$, and there exists a number $\eta \in \mathbb{R}(0, 1]$ such that $u(t) \geq \eta v(t)$ for $t \in [a, \sigma(b)]$;
- (A5) for $y \geq 0$ and $t \in [a, \sigma(b)]$, $F(t, y)$ is nondecreasing in y ;
- (A6) $\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s < \infty$.

Further, we define the constant

$$\theta = k\eta \tag{3.1}$$

where k is given in Lemma 2.8. It is clear that $0 < \theta < 1$.

We let the Banach space $B = C[a, \sigma^n(b)]$ be equipped with the norm $\|y\| = \sup_{t \in [a, \sigma^n(b)]} |y(t)|$. Let $\delta \in \mathbb{R}\left(0, \frac{1}{2}\right)$ be fixed, and define

$$C = \left\{ y \in B \mid y(t) \geq 0, t \in [a, \sigma^n(b)]; \min_{t \in [c, d]} y(t) \geq \theta \|y\| \right\}.$$

We note that C is a cone in B .

Let the operator $S : C \rightarrow B$ be defined by

$$Sy(t) = \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)].$$

If (A2) and (A3) hold, then we have for $y \geq 0$ and $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} & \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ & \leq Sy(t) \leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s. \end{aligned} \quad (3.2)$$

To obtain a positive solution of (1.1), (1.2), we shall seek a fixed point of the operator S in the cone C .

Lemma 3.1. The operator $S : C \rightarrow B$ is continuous and completely continuous.

Proof. First we shall prove that the operator S is continuous. Let $\{y_m\}, y \in C$ be such that $\lim_{m \rightarrow \infty} \|y_m - y\| = 0$. Then, it is easy to see that

$$\rho_m = \sup_{s \in [a, b]} |F(s, y_m(\sigma^{n-1}(s))) - F(s, y(\sigma^{n-1}(s)))| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Hence, we get for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} & |Sy_m(t) - Sy(t)| \\ &= \left| \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) [F(s, y_m(\sigma^{n-1}(s))) - F(s, y(\sigma^{n-1}(s)))] \Delta s \right| \\ &\leq \lambda \rho_m \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) \Delta s \\ &\leq \lambda \rho_m \sup_{t \in [a, \sigma^n(b)]} \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) \Delta s \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$. This shows that $S : C \rightarrow B$ is continuous.

Next, to show complete continuity, we shall apply Arzela-Ascoli Theorem [53]. Let Ω be a bounded subset of C and let $y \in \Omega$. Now there exists $L > 0$ such that $\|y\| \leq L$ for all $y \in \Omega$. Let

$$M = \sup_{s \in [a, b], u \in \mathbb{R}[-L, L]} |F(s, u)|.$$

Clearly, we have for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} |Sy(t)| &\leq \lambda M \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) \Delta s \\ &\leq \lambda M \sup_{t \in [a, \sigma^n(b)]} \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) \Delta s \end{aligned}$$

and for $t, t' \in [a, \sigma^n(b)]$,

$$|Sy(t) - Sy(t')| \leq \lambda M \int_a^{\sigma(b)} |G(t, s) - G(t', s)| \Delta s.$$

The Arzela-Ascoli Theorem guarantees that $S\Omega$ is relatively compact, so $S : C \rightarrow B$ is completely continuous. $\square$

Lemma 3.2. Let (A2)–(A4) hold. Then, the operator S maps C into C .

Proof. Let $y \in C$. Then for all $t \in [a, \sigma^n(b)]$, we see from (3.2) that

$$Sy(t) \geq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \geq 0.$$

It also follows from (3.2) and Theorem 2.7 that for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s. \end{aligned}$$

Thus,

$$\|Sy\| \leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s.$$

Using this inequality, together with (3.2), Theorem 2.8 and (3.1), we find for $t \in [c, d]$,

$$\begin{aligned} Sy(t) &\geq \lambda \int_a^{\sigma(b)} k(-1)^{p+1} G(\sigma^n(b), s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\geq \lambda \int_a^{\sigma(b)} k(-1)^{p+1} G(\sigma^n(b), s) \eta v(s) f(y(\sigma^{n-1}(s))) \Delta s \geq \theta \|Sy\|. \end{aligned}$$

Hence,

$$\min_{t \in [c, d]} Sy(t) \geq \theta \|Sy\|.$$

We have shown that $Sy \in C$. $\square$

Theorem 3.3. Let (A2)–(A4) and (A6) hold. Then, there exists constant $r > 0$ such that the real interval $\mathbb{R}(0, r] \subseteq E$.

Proof: Let $L > 0$ be given and $C(L) = \{y \in C \mid \|y\| \leq L\}$. Define

$$r = \frac{L}{\sup_{x \in \mathbb{R}[-L, L]} f(x)} \left[\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \right]^{-1}. \quad (3.3)$$

Let $\lambda \in \mathbb{R}(0, r]$. We shall prove that S maps $C(L)$ into $C(L)$. Let $y \in C(L)$. By Lemma 3.2, S maps C into C . Thus, it remains to check that $\|Sy\| \leq L$. Using (3.2), Theorem 2.7 and (3.3), we get for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\leq r \left(\sup_{x \in \mathbb{R}[-L, L]} f(x) \right) \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s = L. \end{aligned}$$

Hence, $\|Sy\| \leq L$. We have thus shown that $S(C(L)) \subseteq C(L)$. From Lemma 3.1, S is continuous and completely continuous. By Schauder fixed point theorem, S has a fixed point in $C(L)$. It is clear that this fixed point is a positive solution of (1.1), (1.2) and hence λ is an eigenvalue of (1.1), (1.2). Since $\lambda \in \mathbb{R}(0, r]$ is arbitrary, it follows that the interval $\mathbb{R}(0, r] \subseteq E$. $\square$

The next result uses the monotonicity and compactness of the operator S on the cone C .

Theorem 3.4. Let (A2) and (A5) hold. Suppose that $\lambda_0 \in E$. Then, for each $0 < \lambda < \lambda_0$, $\lambda \in E$.

Proof: Let y_0 be the eigenfunction corresponding to the eigenvalue λ_0 . Then

$$y_0(t) = \lambda_0 \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y_0(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)].$$

Define

$$K = \{y \in B \mid 0 \leq y(t) \leq y_0(t), \quad t \in [a, \sigma^n(b)]\}.$$

For $y \in K$ and $\lambda \in \mathbb{R}(0, \lambda_0)$,

$$\begin{aligned} Sy(t) &= \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y(\sigma^{n-1}(s))) \Delta s \\ &\leq \lambda_0 \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y_0(\sigma^{n-1}(s))) \Delta s = y_0(t), \quad t \in [a, \sigma^n(b)]. \end{aligned}$$

Hence, S maps K into K . Moreover, S is continuous and completely continuous, Schauder's fixed point theorem guarantees that S has a fixed point in K , which is a positive solution of (1.1), (1.2). Thus λ is an eigenvalue of (1.1), (1.2). $\square$

Corollary 3.5. Let (A2) and (A5) hold. If E is nonempty, then E is an interval.

Proof: Suppose E is not an interval. Then, there exist $\lambda_0, \lambda'_0 \in E$ ($\lambda_0 < \lambda'_0$) and $r \in \mathbb{R}(\lambda_0, \lambda'_0)$ with $r \notin E$. However, this is not possible as Theorem 3.4 guarantees that $r \in E$. Hence, E is an interval. $\square$

We shall now establish conditions under which E is a bounded or an unbounded interval. For this, we require the following result.

Theorem 3.6. Let (A1)–(A3) and (A6) hold. Let λ be an eigenvalue of (1.1), (1.2) and $y \in C$ be a corresponding eigenfunction. Further, let $\|y\| = q$. Then

$$\lambda \geq \frac{q}{f(q)} \left[\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \right]^{-1} \quad (3.4)$$

and

$$\lambda \leq \frac{q}{f(\theta q)} \left[\int_a^{\rho^{n-1}(d)} k(-1)^{p+1} G(\sigma^n(b), s) u(s) \Delta s \right]^{-1}. \quad (3.5)$$

Proof: Let $t_0 \in [a, \sigma^n(b)]$ be such that $q = \|y\| = y(t_0)$. Applying Theorem 2.7, (3.2) and (A1), we find

$$\begin{aligned} q &= y(t_0) = (Sy)(t_0) \\ &\leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\leq \lambda f(q) \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \end{aligned}$$

which is (3.4).

Next, using $y(s) \geq \theta \|y\| = \theta q$ for $s \in [c, d]$, we have

$$y(\sigma^{n-1}(s)) \geq \theta q, \quad s \in [c, \rho^{n-1}(d)].$$

In view of (3.2), Theorem 2.8 and (A1), it follows that

$$\begin{aligned}
 q &\geq y(c) = \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(c, s) F(s, y(\sigma^{n-1}(s))) \Delta s \\
 &\geq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(c, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\
 &\geq \lambda \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(c, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\
 &\geq \lambda \int_c^{\rho^{n-1}(d)} k(-1)^{p+1} G(\sigma^n(b), s) u(s) f(\theta q) \Delta s
 \end{aligned}$$

which is (3.5). □

Theorem 3.7. Let (A1)–(A6) hold. Define

$$\begin{aligned}
 F_B &= \left\{ f \mid \frac{u}{f(u)} \text{ is bounded for } u \in \mathbb{R}[0, \infty) \right\}, \\
 F_0 &= \left\{ f \mid \lim_{u \rightarrow \infty} \frac{u}{f(u)} = 0 \right\}, \\
 F_\infty &= \left\{ f \mid \lim_{u \rightarrow \infty} \frac{u}{f(u)} = \infty \right\}.
 \end{aligned}$$

- (a) If $f \in F_B$, then $E = \mathbb{R}(0, r)$ or $\mathbb{R}(0, r]$ for some $r \in \mathbb{R}(0, \infty)$.
- (b) If $f \in F_0$, then $E = \mathbb{R}(0, r]$ for some $r \in \mathbb{R}(0, \infty)$.
- (c) If $f \in F_\infty$, then $E = \mathbb{R}(0, \infty)$.

Proof.

- (a) This is immediate from (3.5) and Corollary 3.5.
- (b) Here, we have $F_0 \subseteq F_B$. By Theorem 3.7(a), $E = \mathbb{R}(0, r)$ or $\mathbb{R}(0, r]$ for some $r \in \mathbb{R}(0, \infty)$. In particular, $r = \sup E$.

Let $\{\lambda_n\}_{n=1}^\infty$ be monotonically increasing sequence in E , with $\lim_{n \rightarrow \infty} \lambda_n = r$, and let $\{y_n\}_{n=1}^\infty$ in $\widetilde{K} = \{y \in B \mid y(t) \geq 0, t \in [a, \sigma^n(b)]\}$ be a corresponding sequence of eigenfunctions. We write $q_n = \|y_n\|$. Then (3.5) together with $f \in F_0$ implies that there is no subsequence of $\{q_n\}$ can diverge. Thus, there exists $L > 0$ such that $q_n \leq L$ for all n . So $\|y_n\| \leq L$ for every n and $\{y_n\}_{n=1}^\infty$ is uniformly bounded. Hence

there exists a subsequence of $\{y_n\}_{n=1}^\infty$ relabelled as the original sequence such that it converges to y ($\in \widetilde{K}$) uniformly. Thus,

$$y_n(t) = \lambda_n \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y_n(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)].$$

Letting $n \rightarrow \infty$ in the above inequality gives

$$y(t) = r \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)].$$

This shows that r is an eigenvalue with corresponding eigenfunction y . So $r = \sup E \in E$ and hence $E = \mathbb{R}(0, r]$.

(c) Let $\lambda > 0$ be fixed. Choose $\epsilon > 0$ such that

$$\lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \leq \frac{1}{\epsilon}. \quad (3.6)$$

Since $f \in F_\infty$, there exists $M = M(\epsilon) > 0$ such that for $u \geq M$,

$$f(u) < \epsilon u. \quad (3.7)$$

Let $C(M) = \{y \in C \mid \|y\| \leq M\}$. We shall show that $S(C(M)) \subseteq C(M)$. To begin, let $y \in C(M)$. By Lemma 3.2, we have $Sy \in C$. To show that $\|y\| \leq M$, we use (3.2), Theorem 2.7, (A1), (3.7) and (3.6) to get for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(M) \Delta s \\ &\leq \lambda \epsilon M \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \leq M. \end{aligned}$$

Thus, $\|Sy\| \leq M$ and so $S(C(M)) \subseteq C(M)$. Since S is continuous and completely continuous, Schauder's fixed point theorem guarantees that S has a fixed point in $C(M)$. This fixed point is a positive solution of (1.1), (1.2) and hence λ is an eigenvalue of (1.1), (1.2). Since $\lambda > 0$ is arbitrary, therefore $E = \mathbb{R}(0, \infty)$. $\square$

Before we consider an example, we shall state without proof the following result:

Lemma 3.8. [27] Let $m \in \mathbb{R}^+$. For any $t, s \in m\mathbb{Z}$ with $t > s$, we have

$$h_n(t, s) = \frac{1}{n!} \prod_{j=0}^{n-1} (t - s - jm), \quad n \in \mathbb{Z}^+. \quad \square$$

Example 3.9. Let $m \in \mathbb{R}^+$ and $T = m\mathbb{Z}$. Consider the boundary value problem

$$\begin{aligned} y^{\Delta^3}(t) &= \frac{\lambda(9y(\sigma^2(t)) + 2)^q}{(9\sigma^2(t)\{3\sigma(b)[\sigma(b) - t] + t\sigma(t)\} + 2)^q}, \quad t \in [0, b] \\ y(0) &= y^\Delta(\sigma(b)) = y^{\Delta^2}(\sigma(b)) = 0 \end{aligned}$$

where $\lambda > 0$, $q \geq 0$ and $b \in T$.

In this example, $n = 3$ and $p = 1$. Taking $f(y) = (9y + 2)^q$, we may choose

$$u(t) = v(t) = \frac{1}{(9\sigma^2(t)\{3\sigma(b)[\sigma(b) - t] + t\sigma(t)\} + 2)^q}.$$

It is easy to verify that the conditions (A1)–(A5) are satisfied. Further, we have

$$\frac{y}{f(y)} = \frac{y}{(9y + 2)^q}$$

Note that $v(t) \leq \frac{1}{2^q}$ for $t \in [0, \sigma(b)]$. By Theorem 2.5 and Lemma 3.8, we note that

$$G(t, s) = \begin{cases} \frac{t}{2}(2s - t + 3m), & t \leq \sigma(s) \\ \frac{1}{2}(s + m)(s + 2m), & t \geq \sigma(s). \end{cases}$$

A direct computation yields

$$\begin{aligned} \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s &= \int_0^{\sigma(b)} \frac{1}{2} (s + m)(s + 2m) v(s) \Delta s \\ &\leq \frac{1}{2^{1+q}} \int_0^{\sigma(b)} (s + m)(s + 2m) \Delta s < \infty. \end{aligned}$$

Hence, the condition (A6) holds.

Case 1: $0 \leq q < 1$

Since $f \in F_\infty$, by Theorem 3.7(c), the set $E = \mathbb{R}(0, \infty)$. Note that when $\lambda = 6$, the boundary value problem has a positive solution given by

$$y(t) = t\{3\sigma(b)[\sigma^2(b) - \rho(t)] + \rho(t)\rho^2(t)\}.$$

Observe that when we differentiate with respect to t , by Lemma 3.8, we obtain

$$\begin{aligned} y^\Delta(t) &= \left[3\sigma(b)\sigma^2(b)t - 3\sigma(b)t\rho(t) + t\rho(t)\rho^2(t) \right]^\Delta \\ &= \left[3\sigma(b)\sigma^2(b)t - 6\sigma(b)h_2(t, 0) + 6h_3(t, 0) \right]^\Delta \\ &= 3\sigma(b)(\sigma^2(b) - 2t) + 3t\rho(t). \end{aligned}$$

By a similar approach, we have $y^{\Delta^2}(t) = 6(t - \sigma(b))$.

Case 2: $q = 1$

Since $f \in F_B$, by Theorem 3.7(a), the set $E = \mathbb{R}(0, r)$ or $\mathbb{R}(0, r]$ for some real value r . From Case 1 and Theorem 3.4, E contains the interval $\mathbb{R}(0, 6]$.

Case 3: $q > 1$

Since $f \in F_0$, by Theorem 3.7(b), the set $E = \mathbb{R}(0, r]$ for some real r . From Case 1 and Theorem 3.4, $\mathbb{R}(0, 6] \subseteq E$.

3.2 Intervals of Eigenvalues

In this section, we shall employ Krasnosel'skii's fixed point theorem [49] to derive explicit eigenvalue intervals.

Theorem 3.10. Let (A2)–(A4) and (A6) hold. Then, for each λ satisfying

$$L < \lambda < R$$

where

$$L = \left[\theta \underline{f}_\infty \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \Delta s \right]^{-1}$$

and

$$R = \left[\bar{f}_0 \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \right]^{-1},$$

the boundary value problem (1.1), (1.2) has a positive solution.

Proof: Let $L < \lambda < R$ and let $\epsilon > 0$ be such that

$$\begin{aligned} & \left[\theta(\underline{f}_\infty - \epsilon) \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \Delta s \right]^{-1} \\ & \leq \lambda \leq \left[(\overline{f}_0 + \epsilon) \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \right]^{-1}. \end{aligned} \quad (3.8)$$

Next, we choose $w > 0$ so that

$$f(x) \leq (\overline{f}_0 + \epsilon)x, \quad 0 < x \leq w. \quad (3.9)$$

Let $y \in C$ be such that $\|y\| = w$. Then applying (3.2), Theorem 2.7, (3.9) and (3.8), we find that for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) & \leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ & \leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) (\overline{f}_0 + \epsilon) y(\sigma^{n-1}(s)) \Delta s \\ & \leq \lambda (\overline{f}_0 + \epsilon) \|y\| \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \leq \|y\|. \end{aligned}$$

Hence,

$$\|Sy\| \leq \|y\|. \quad (3.10)$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < w\}$, then (3.10) holds for $y \in C \cap \partial\Omega_1$.

Further, we let $T_1 > w > 0$ be such that

$$f(x) \geq (\underline{f}_\infty - \epsilon)x, \quad x \geq T_1. \quad (3.11)$$

Let $y \in C$ be such that $\|y\| = \frac{T_1}{\theta}$. Then, for $t \in [c, d]$,

$$y(t) \geq \theta \|y\| \geq \theta \cdot \frac{T_1}{\theta} = T_1,$$

which, by (3.11), leads to

$$f(y(\sigma^{n-1}(s))) \geq (\underline{f}_\infty - \epsilon)y(\sigma^{n-1}(s)), \quad s \in [c, \rho^{n-1}(d)]. \quad (3.12)$$

Using (3.2), (3.12) and (3.8),

$$\begin{aligned} Sy(\sigma^n(b)) &\geq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\geq \lambda \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) (\underline{f}_\infty - \epsilon) y(\sigma^{n-1}(s)) \Delta s \\ &\geq \lambda (\underline{f}_\infty - \epsilon) \theta \|y\| \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \Delta s \geq \|y\|. \end{aligned}$$

Therefore,

$$\|Sy\| \geq \|y\|. \quad (3.13)$$

If we set $\Omega_2 = \{y \in B \mid \|y\| < \frac{T_1}{\theta}\}$, then (3.13) holds for $y \in C \cap \partial\Omega_2$.

Having obtained (3.10) and (3.13), it follows from Theorem 1.2 that S has a fixed point $y \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $w \leq \|y\| \leq \frac{T_1}{\theta}$. It is obvious that y is a positive solution of (1.1), (1.2). $\square$

The following result is immediate from Theorem 3.10.

Corollary 3.11. Let (A2)–(A4) and (A6) hold. Then

$$\mathbb{R}(L, R) \subseteq E$$

where L and R are defined in Theorem 3.10. $\square$

Theorem 3.12. Let (A2)–(A4) and (A6) hold. Then, for each λ satisfying

$$L' < \lambda < R'$$

where

$$L' = \left[\theta \underline{f}_0 \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \Delta s \right]^{-1}$$

and

$$R' = \left[\overline{f}_\infty \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \right]^{-1},$$

the boundary value problem (1.1), (1.2) has a positive solution.

Proof. Let $L' < \lambda < R'$. We let $\epsilon > 0$ be such that

$$\begin{aligned} & \left[\theta(\underline{f}_0 - \epsilon) \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \Delta s \right]^{-1} \\ & \leq \lambda \leq \left[(\bar{f}_\infty + \epsilon) \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \right]^{-1}. \end{aligned} \quad (3.14)$$

Let $\bar{w} > 0$ be such that

$$f(x) \geq (\underline{f}_0 - \epsilon)x, \quad 0 < x \leq \bar{w}. \quad (3.15)$$

Let $y \in C$ be such that $\|y\| = \bar{w}$. Then applying (3.2), (3.15) and (3.14), we find that

$$\begin{aligned} Sy(\sigma^n(b)) & \geq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ & \geq \lambda \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) (\underline{f}_0 - \epsilon) y(\sigma^{n-1}(s)) \Delta s \\ & \geq \lambda(\underline{f}_0 - \epsilon) \theta \|y\| \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \Delta s \geq \|y\|. \end{aligned}$$

Hence,

$$\|Sy\| \geq \|y\|.$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < \bar{w}\}$, then $\|Sy\| \geq \|y\|$ holds for $y \in C \cap \partial\Omega_1$.

Next, we choose $T_2 > 0$ such that

$$f(x) \leq (\bar{f}_\infty + \epsilon)x, \quad x \geq T_2. \quad (3.16)$$

We shall consider two cases, f is bounded and f is unbounded.

Case 1. Suppose f is bounded, i.e. there exists $M^* > 0$ such that

$$f(x) \leq M^*, \quad x \in \mathbb{R}[0, \infty). \quad (3.17)$$

We define

$$T_3 = \max \left\{ 2\bar{w}, \lambda M^* \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \right\}.$$

Let $y \in C$ be such that $\|y\| = T_3$. From Theorem 2.7, (3.2) and (3.17), for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\leq \lambda M^* \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \leq T_3 = \|y\|. \end{aligned}$$

Therefore,

$$\|Sy\| \leq \|y\|.$$

Case 2. Suppose f is unbounded. Then there exists $T_3 > \max\{2\bar{w}, T_2\}$ such that

$$f(x) \leq f(T_3), \quad 0 < x \leq T_3. \quad (3.18)$$

Let $y \in C$ be such that $\|y\| = T_3$. Applying Theorem 2.7, (3.18), (3.16) and (3.14), we find for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(T_3) \Delta s \\ &\leq \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) (\bar{f}_\infty + \epsilon) T_3 \Delta s \\ &= \lambda (\bar{f}_\infty + \epsilon) \|y\| \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \leq \|y\|. \end{aligned}$$

Once again, we have

$$\|Sy\| \leq \|y\|.$$

In both cases, if we set $\Omega_2 = \{y \in B \mid \|y\| < T_3\}$, then $\|Sy\| \leq \|y\|$ holds for $y \in C \cap \partial\Omega_2$.

It now follows from Theorem 1.2 that S has a fixed point $y \in C \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $\bar{w} \leq \|y\| \leq T_3$. It is clear that y is a positive solution of (1.1), (1.2). $\square$

From the preceding theorem, we obtain the following corollary.

Corollary 3.13. Let (A2)–(A4) and (A6) hold. Then

$$\mathbb{R}(L', R') \subseteq E$$

where L' and R' are defined in Theorem 3.12. $\square$

Remark 3.1. If f is superlinear (i.e. $\overline{f}_0 = 0$ and $\underline{f}_\infty = \infty$) or sublinear (i.e. $\underline{f}_0 = \infty$ and $\overline{f}_\infty = 0$), we conclude from Corollary 3.11 and Corollary 3.13 respectively that $E = \mathbb{R}(0, \infty)$. Thus the boundary value problem (1.1), (1.2) has a positive solution for any positive λ .

Chapter 4

One or More Positive Solutions

In this chapter, by letting $\lambda = 1$ in (1.1), we present results governing the existence of positive solutions to the differential equation

$$(-1)^{n-1}y^{\Delta^n}(t) = (-1)^{p+1}F(t, y(\sigma^{n-1}(t))), \quad t \in [a, b] \quad (1.1)'$$

subject to the two-point right focal boundary conditions (1.2).

Criteria for the existence of single, double and multiple positive solutions of the boundary value problem (1.1)', (1.2) are developed. In addition, we shall consider the following special cases of (1.1)', (1.2) when $n = 2$ and $p = 1$:

$$\begin{cases} y^{\Delta^2}(t) + h(t)([y(\sigma(t))]^\alpha + [y(\sigma(t))]^\beta) = 0, & t \in [a, b] \\ y(a) = y^\Delta(\sigma(b)) = 0 \end{cases} \quad (Q1)$$

and

$$\begin{cases} y^{\Delta^2}(t) + h(t)e^{\zeta[y(\sigma(t))]} = 0, & t \in [a, b] \\ y(a) = y^\Delta(\sigma(b)) = 0. \end{cases} \quad (Q2)$$

It is assumed that $0 \leq \alpha < 1 < \beta$, $\zeta > 0$ and h is nonnegative. We shall provide conditions under which (Q1) and (Q2) have double positive solutions, and also establish upper and lower bounds for these solutions. The importance of (Q1) is illustrated in [18, 40] where particular cases in the real and discrete domains are discussed. The boundary value problem (Q2) arises in applications involving

the diffusion of heat generated by positive temperature-dependent sources [1]. For instance, when $\zeta = 1$ the boundary value problem (Q2) occurs in the analysis of Joule losses in electrically conducting solids as well as in frictional heating.

We refer to [22, 39] for discussions on the existence of double positive solutions of boundary value problems on the real and discrete domains. Recent works on boundary value problems on time scales, which include the real and the discrete as special cases can be found in [3, 4, 14, 29, 34, 35, 45, 55, 59, 54]. Our approach in this chapter presents work unifies the analysis for the real and the discrete cases in [57, 61], it also leads to *new* results which, when reduced to $\mathbb{R}$ and $\mathbb{Z}$, are also *new* in the literature. Throughout, examples are included to illustrate the importance of the results obtained.

4.1 Existence Criteria

Recall from the previous chapter that we let the Banach space $B = C[a, \sigma^n(b)]$ be equipped with the norm

$$\|y\| = \sup_{t \in [a, \sigma^n(b)]} |y(t)|.$$

Let the operator $S : B \rightarrow B$ be defined by

$$Sy(t) = \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)]. \quad (4.1)$$

It is clear that a fixed point of the operator S is a solution of (1.1)', (1.2).

For clarity, we shall now list some of the conditions used later. In these conditions, we let

$$\tilde{K} = \{y \in B \mid y(t) \geq 0, \quad t \in [a, \sigma^n(b)]\}$$

and

$$K = \{y \in \tilde{K} \mid y(t) > 0 \text{ for some } t \in [a, \sigma^n(b)]\} = \tilde{K} \setminus \{0\}.$$

(B1) The function F is continuous on $[a, \sigma(b)] \times \tilde{K}$ with

$$F(t, y) \geq 0, \quad (t, y) \in [a, \sigma(b)] \times \tilde{K}$$

and

$$F(t, y) > 0, \quad (t, y) \in [a, \sigma(b)] \times K.$$

(B2) There exist continuous functions w, q with $w : \mathbb{R}[0, \infty) \rightarrow \mathbb{R}[0, \infty)$ nondecreasing and $q : [a, \sigma(b)] \rightarrow \mathbb{R}[0, \infty)$ such that for $(t, y) \in [a, \sigma(b)] \times \tilde{K}$,

$$F(t, y) \leq q(t)w(y).$$

(B3) There exists $\tau : [c, \rho^{n-1}(d)] \rightarrow \mathbb{R}^+$ such that for $(t, y) \in [c, \rho^{n-1}(d)] \times K$,

$$F(t, y) \geq \tau(t)w(y).$$

(B4) There exist continuous functions f, u, v with $f : \mathbb{R}[0, \infty) \rightarrow \mathbb{R}[0, \infty)$ and $u, v : [a, \sigma(b)] \rightarrow \mathbb{R}[0, \infty)$ such that for $(t, y) \in [a, \sigma(b)] \times \tilde{K}$,

$$u(t)f(y(\sigma^{n-1}(t))) \leq F(t, y(\sigma^{n-1}(t))) \leq v(t)f(y(\sigma^{n-1}(t))).$$

(B5) $u(t)$ is nonzero for some $t \in [c, \rho^{n-1}(d)]$, and there exists a number $\eta \in \mathbb{R}(0, 1]$ such that $u(t) \geq \eta v(t)$ for $t \in [a, \sigma(b)]$.

(B6) $\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s < \infty$.

Lemma 4.1. Let $F : [a, \sigma(b)] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Then, the operator $S : B \rightarrow B$ is continuous and completely continuous.

Proof. The proof of this result is similar to that of Lemma 3.1. □

We shall now provide an existence criteria for a *general* (not necessarily positive) solution of (1.1)', (1.2).

Theorem 4.2. Let $F : [a, \sigma(b)] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Suppose there exists a constant ρ , independent of λ , such that

$$\|y\| \neq \rho$$

for any solution $y \in B$ of the equation

$$y(t) = \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)], \quad (4.2)$$

where $\lambda \in \mathbb{R}(0, 1)$. Then (1.1)', (1.2) has at least one solution $y \in B$ such that $\|y\| \leq \rho$.

Proof. Solving (4.2) is equivalent to obtaining a fixed point of the equation

$$y = \lambda S y$$

where S is defined in (4.1). By Lemma 4.1, S is continuous and completely continuous. Next, in the context of Theorem 1.1, we define

$$U = \{y \in B \mid \|y\| < \rho\}.$$

The condition $\|y\| \neq \rho$ ensures that we cannot have conclusion (b) of Theorem 1.1, hence (a) must hold, i.e., (1.1)', (1.2) has a solution $y \in \bar{U}$ with $\|y\| \leq \rho$. $\square$

The next result employs Theorem 4.2 to provide the existence of a *positive* solution.

Theorem 4.3. Let (B1) and (B2) hold. Suppose

$$\begin{cases} \text{there exists } \alpha > 0 \text{ such that } \alpha > d \cdot w(\alpha) \\ \text{where } d = \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) q(s) \Delta s. \end{cases} \quad (4.3)$$

Then (1.1)', (1.2) has a positive solution $y \in B$ such that $\|y\| < \alpha$, i.e., $0 \leq y(t) < \alpha$ for $t \in [a, \sigma^n(b)]$.

Proof. Consider the equation

$$y(t) = \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) \hat{F}(s, y(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)] \quad (4.4)$$

where $\hat{F} : [a, \sigma(b)] \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\hat{F}(t, y) = F(t, |y|). \quad (4.5)$$

Since $|y| \in \tilde{K}$, by (B1) the function $\hat{F}$ is well-defined and is continuous.

We shall prove that (4.4) has a solution. For this, we consider the equation

$$y(t) = \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) \hat{F}(s, y(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)] \quad (4.6)$$

where $\lambda \in \mathbb{R}(0, 1)$. Let $y \in B$ be any solution of (4.6). We shall show that $\|y\| \neq \alpha$, then it will follow from Theorem 4.2 that (4.4) has a solution.

By using (4.5), Theorem 2.7 and (B1), we get for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} y(t) &= \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) \hat{F}(s, y(\sigma^{n-1}(s))) \Delta s \\ &= \lambda \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, |y(\sigma^{n-1}(s))|) \Delta s \geq 0. \end{aligned}$$

This means that

$$|y(t)| = y(t), \quad t \in [a, \sigma^n(b)]. \quad (4.7)$$

Applying (4.7), (B2) and Theorem 2.7, for $t \in [a, \sigma^n(b)]$ we see that

$$\begin{aligned} |y(t)| = y(t) &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, |y(\sigma^{n-1}(s))|) \Delta s \\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) q(s) w(|y(\sigma^{n-1}(s))|) \Delta s \\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) q(s) w(\|y\|) \Delta s \\ &= d \cdot w(\|y\|), \end{aligned}$$

which immediately leads to

$$\|y\| \leq d \cdot w(\|y\|). \quad (4.8)$$

Comparing (4.8) and (4.3), we conclude that $\|y\| \neq \alpha$.

It now follows from Theorem 4.2 that (4.4) has a solution $y_0 \in B$ with $\|y_0\| \leq \alpha$, and

$$y_0(t) = \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) \hat{F}(s, y_0(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)].$$

Using a similar argument as above, it can be seen that

$$|y_0(t)| = y_0(t), \quad t \in [a, \sigma^n(b)], \quad \text{and} \quad \|y_0\| \neq \alpha. \quad (4.9)$$

Thus, y_0 is positive and $\|y_0\| < \alpha$. Further, noting (4.5) and (4.9), we have for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} y_0(t) &= \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, |y_0(\sigma^{n-1}(s))|) \Delta s \\ &= \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y_0(\sigma^{n-1}(s))) \Delta s. \end{aligned}$$

Hence, y_0 is in fact a solution of (1.1)', (1.2). The proof is complete. $\square$

Remark 4.1. We note that the last inequality in (B1), viz,

$$F(t, y) > 0, \quad (t, y) \in [a, \sigma(b)] \times K$$

is *not* needed in Theorem 4.3.

Theorem 4.3 provides the existence of a positive solution, which may be trivial. Our next result guarantees the existence of a *nontrivial* positive solution in the cone C_k , defined, for a fixed $\delta \in \mathbb{R}(0, \frac{1}{2})$, as

$$C_k = \left\{ y \in B \mid y(t) \geq 0, \quad t \in [a, \sigma^n(b)]; \quad \min_{t \in [c, d]} y(t) \geq k \|y\| \right\} \quad (4.10)$$

where k is given in Theorem 2.8.

Theorem 4.4. Let $\delta \in \mathbb{R}(0, \frac{1}{2})$ be fixed and assume (B1)–(B3) and (4.3) hold. Suppose

$$\begin{cases} \text{there exists } \beta > 0 \text{ such that for } x \in \mathbb{R}[k\beta, \beta], \text{ we have} \\ x \leq w(x) \int_c^{\rho^{n-1}(d)} k(-1)^{p+1} G(\sigma^n(b), s) \tau(s) \Delta s. \end{cases} \quad (4.11)$$

Then (1.1)', (1.2) has a positive solution $y \in B$ such that

(a) $\alpha < \|y\| \leq \beta$ and $\min_{t \in [c, d]} y(t) > k\alpha$ if $\alpha < \beta$;

(b) $\beta \leq \|y\| < \alpha$ and $\min_{t \in [c, d]} y(t) \geq k\beta$ if $\alpha > \beta$.

Proof: We shall employ Theorem 1.2. To begin, the operator $S : B \rightarrow B$ is continuous and completely continuous by Lemma 4.1.

Next, we shall show that S maps the cone C_k into C_k . For this, we let $y \in C_k$. Since $C_k \subseteq \tilde{K}$, it follows from (B1) that

$$F(t, y) \geq 0, \quad (t, y) \in [a, \sigma(b)] \times C_k. \quad (4.12)$$

Noting (4.12), we obtain for $t \in [a, \sigma^n(b)]$,

$$Sy(t) = \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y(\sigma^{n-1}(s))) \Delta s \geq 0. \quad (4.13)$$

In view of (4.13) and Theorem 2.7, we find

$$|Sy(t)| = Sy(t) \leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) F(s, y(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)],$$

which implies

$$\|Sy\| \leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) F(s, y(\sigma^{n-1}(s))) \Delta s. \quad (4.14)$$

Now, using Theorem 2.8 and (4.14), we get for $t \in [c, d]$,

$$Sy(t) \geq \int_a^{\sigma(b)} k(-1)^{p+1} G(\sigma^n(b), s) F(s, y(\sigma^{n-1}(s))) \Delta s \geq k\|Sy\|.$$

Hence,

$$\min_{t \in [c, d]} Sy(t) \geq k\|Sy\|. \quad (4.15)$$

Finally, we combine (4.13) and (4.15) to obtain $S(C_k) \subseteq C_k$.

To proceed to the next part of the proof, let

$$\Omega_\alpha = \{y \in B \mid \|y\| < \alpha\} \quad \text{and} \quad \Omega_\beta = \{y \in B \mid \|y\| < \beta\}.$$

We shall verify that (i) $\|Sy\| \leq \|y\|$ for $y \in C_k \cap \partial\Omega_\alpha$, and (ii) $\|Sy\| \geq \|y\|$ for $y \in C_k \cap \partial\Omega_\beta$.

To show (i), let $y \in C_k \cap \partial\Omega_\alpha$. Then, $\|y\| = \alpha$. Using (4.13), (B2), Theorem 2.7 and (4.3), we get for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} |Sy(t)| &= Sy(t) \\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) q(s) w(y(\sigma^{n-1}(s))) \Delta s \\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) q(s) w(\alpha) \Delta s = d \cdot w(\alpha) < \alpha = \|y\|, \end{aligned}$$

and so

$$\|Sy\| \leq \|y\|.$$

Next, we shall prove (ii). Let $y \in C_k \cap \partial\Omega_\beta$. Then, $\|y\| = \beta$. Moreover, $k\beta \leq y(t) \leq \beta$ for $t \in [c, d]$ and this implies

$$k\beta \leq y(\sigma^{n-1}(t)) \leq \beta, \quad t \in [c, \rho^{n-1}(d)].$$

Applying (B3) and (4.11), we find

$$\begin{aligned} |Sy(\sigma^n(b))| &= Sy(\sigma^n(b)) \\ &= \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) F(s, y(\sigma^{n-1}(s))) \Delta s \\ &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) F(s, y(\sigma^{n-1}(s))) \Delta s \\ &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) \tau(s) w(y(\sigma^{n-1}(s))) \Delta s \\ &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) \tau(s) \left(\frac{y(\sigma^{n-1}(s))}{k \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), x) \tau(x) \Delta x} \right) \Delta s \\ &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) \tau(s) \left(\frac{k\beta}{k \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), x) \tau(x) \Delta x} \right) \Delta s \end{aligned}$$

$$= \beta = \|y\|,$$

which implies

$$\|Sy\| \geq \|y\|.$$

Having established (i) and (ii), it follows from Theorem 1.2 that S has a fixed point $y \in C \cap (\overline{\Omega}_{\max\{\alpha, \beta\}} \setminus \Omega_{\min\{\alpha, \beta\}})$. Thus, $\min\{\alpha, \beta\} \leq \|y\| \leq \max\{\alpha, \beta\}$. Using a similar argument as in the first part of the proof of Theorem 4.3, we see that $\|y\| \neq \alpha$. Hence, we obtain the first part of conclusions (a) and (b). Further, since $y \in C$, we have

$$\min_{t \in [c, d]} y(t) \geq k\|y\| \geq \min\{\alpha, \beta\},$$

which, together with $\|y\| \neq \alpha$, gives the second part of conclusions (a) and (b). $\square$

Our next result gives the existence of *double* positive solutions.

Theorem 4.5. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed and assume (B1)–(B3), (4.3) and (4.11) hold with $\alpha < \beta$. Then (1.1)', (1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that

$$0 \leq \|y_1\| < \alpha < \|y_2\| \leq \beta \quad \text{with} \quad \min_{t \in [c, d]} y_2(t) > k\alpha.$$

Proof. The existence of y_1 and y_2 is guaranteed by Theorems 4.3 and 4.4 respectively. $\square$

In Theorem 4.5, it is possible to have $\|y_1\| = 0$. Our next result guarantees the existence of *two nontrivial* positive solutions.

Theorem 4.6. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed and assume (B1)–(B3), (4.3), (4.11) and (4.11)| $_{\beta=\tilde{\beta}}$ hold, where $0 < \tilde{\beta} < \alpha < \beta$. Then (1.1)', (1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that

$$0 < \tilde{\beta} \leq \|y_1\| < \alpha < \|y_2\| \leq \beta, \quad \min_{t \in [c, d]} y_1(t) \geq k\tilde{\beta}, \quad \min_{t \in [c, d]} y_2(t) > k\alpha.$$

Proof. The existence of y_1 and y_2 is guaranteed from Theorem 4.4(b) and 4.4(a) respectively. $\square$

The next result generalizes Theorems 4.5 and 4.6 and gives the existence of *multiple* positive solutions of (1.1)', (1.2).

Theorem 4.7. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed and assume (B1)–(B3) hold. Let (4.3) be satisfied with $\alpha = \alpha_l$, $l = 1, 2, \dots, r$ and (4.11) be satisfied with $\beta = \beta_l$, $l = 1, 2, \dots, m$.

- (a) If $m = r + 1$ and $0 < \beta_1 < \alpha_1 < \dots < \beta_r < \alpha_r < \beta_{r+1}$, then (1.1)', (1.2) has (at least) $2r$ positive solutions $y_1, \dots, y_{2r} \in B$ such that

$$0 < \beta_1 \leq \|y_1\| < \alpha_1 < \dots < \beta_r \leq \|y_{2r-1}\| < \alpha_r < \|y_{2r}\| \leq \beta_{r+1}.$$

- (b) If $m = r$ and $0 < \beta_1 < \alpha_1 < \dots < \beta_r < \alpha_r$, then (1.1)', (1.2) has (at least) $2r - 1$ positive solutions $y_1, \dots, y_{2r-1} \in B$ such that

$$0 < \beta_1 \leq \|y_1\| < \alpha_1 < \dots \leq \beta_r \leq \|y_{2r-1}\| < \alpha_r.$$

- (c) If $r = m + 1$ and $0 < \alpha_1 < \beta_1 < \dots < \alpha_m < \beta_m < \alpha_{m+1}$, then (1.1)', (1.2) has (at least) $2m + 1$ positive solutions $y_0, \dots, y_{2m} \in B$ such that

$$0 \leq \|y_0\| < \alpha_1 < \|y_1\| \leq \beta_1 \leq \dots \leq \beta_m \leq \|y_{2m}\| < \alpha_{m+1}.$$

- (d) If $r = m$ and $0 < \alpha_1 < \beta_1 < \dots < \alpha_m < \beta_m$, then (1.1)', (1.2) has (at least) $2m$ positive solutions $y_0, \dots, y_{2m-1} \in B$ such that

$$0 \leq \|y_0\| < \alpha_1 < \|y_1\| \leq \beta_1 \leq \dots < \alpha_m < \|y_{2m-1}\| \leq \beta_m.$$

Proof. Repetitive applications of Theorem 4.4 yield (a) and (b). In (c) and (d), Theorem 4.3 is used to obtain the existence of $y_0 \in B$ with $0 \leq \|y_0\| < \alpha_1$. The results then follow by repeated use of Theorem 4.4. $\square$

Example 4.8. Let $T = \mathbb{R}[0, 1) \cup \mathbb{Z}^+$. Consider the boundary value problem

$$\begin{cases} y^{\Delta^2}(t) + \frac{2e^{y(\sigma(t))}}{e^{33\sigma(t)-[\sigma(t)]^2}} = 0, & t \in [0, 15] \\ y(0) = y^{\Delta}(\sigma(15)) = 0. \end{cases} \quad (4.16)$$

Here, $n = 2$, $p = 1$, $a = 0$ and $b = 15$. We shall verify that the hypotheses (B1)–(B3), (4.3) and (4.11) are satisfied. Note that (B1) obviously holds with $F(t, y) = 2e^y/e^{33\sigma(t)-[\sigma(t)]^2}$. For conditions (B2) and (B3), we may choose

$$q(t) = \tau(t) = \frac{2}{e^{33\sigma(t)-[\sigma(t)]^2}} \quad \text{and} \quad w(y) = e^y.$$

Next, we compute directly to get

$$d = \int_0^{\sigma(15)} G(\sigma^2(15), s)q(s)\Delta s = \int_0^1 \frac{2s}{e^{33s-s^2}}ds + \sum_{s=1}^{15} \frac{2(s+1)}{e^{33(s+1)-(s+1)^2}} = 1.847 \times 10^{-3}.$$

It follows that the inequality $\alpha > d \cdot e^\alpha$ is true provided $1.85 \times 10^{-3} < \alpha < 8.43$. We may choose $\alpha = 8.4$ so that (4.3) is satisfied.

Using Theorem 2.5 with $n = 2$ and $p = 1$, we get

$$G(t, s) = \begin{cases} t - a, & t \leq \sigma(s) \\ \sigma(s) - a, & t \geq \sigma(s). \end{cases} \quad (4.17)$$

Let $\delta = 0.2$. From (2.5), we get $c = 4$ and $d = 13$. So noting (4.17) and Theorem 2.8, we find

$$k = \inf_{s \in [0, 15]} \frac{G(4, s)}{G(\sigma^2(15), s)} = \min \left\{ \inf_{\sigma(s) \in [0, 4]} \frac{\sigma(s)}{\sigma(s)}, \quad \inf_{\sigma(s) \in [4, \sigma(15)]} \frac{4}{\sigma(s)} \right\} = \frac{4}{\sigma(15)} = \frac{1}{4}.$$

Now the inequality in (4.11) reduces to

$$x \leq w(x) \int_c^{\rho(d)} kG(\sigma^2(15), s)\tau(s)\Delta s = \frac{1}{2}e^x \int_4^{12} \frac{\sigma(s)}{e^{33\sigma(s)-[\sigma(s)]^2}}\Delta s = (3.95 \times 10^{-61})e^x,$$

and is true if $x \leq 3.95 \times 10^{-61}$ or $x \geq 144.05$. Hence, (4.11) is fulfilled if we choose $\beta = 580$ or $\beta = 3.9 \times 10^{-61}$.

It now follows from Theorem 4.6 that the boundary value problem (4.16) has (at least) two positive solutions y_1 and y_2 such that

$$\begin{cases} 3.9 \times 10^{-61} \leq \|y_1\| < 8.4 < \|y_2\| \leq 580, \\ \min_{t \in [4,13]} y_1(t) \geq 9.75 \times 10^{-62}, \quad \min_{t \in [4,13]} y_2(t) > 2.1. \end{cases} \quad (4.18)$$

In fact, a known positive solution is $y(t) = t(33-t)$ with $\|y\| = 272$ and $\min_{t \in [4,13]} y(t) = 116$, both are within the ranges given in (4.18).

We have so far established results based on assumptions (B1)–(B3) on the non-linear term F . We shall now consider conditions (B4)–(B6) on F and develop further existence criteria.

Note that if (B4) holds, then for $(t, y) \in [a, \sigma^n(b)] \times \tilde{K}$,

$$\begin{aligned} \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ \leq S y(t) \leq \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s. \end{aligned} \quad (4.19)$$

To obtain a positive solution of (1.1)', (1.2), we shall seek a fixed point of the operator S in the cone C , defined, for a fixed $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$, as

$$C = \left\{ y \in B \mid y(t) \geq 0, t \in [a, \sigma^n(b)]; \min_{t \in [c,d]} y(t) \geq \theta \|y\| \right\} \quad (4.20)$$

where $\theta = k\eta$, k and η are defined in Theorem 2.8 and (B5) respectively. It is clear that $0 < \theta < 1$.

Lemma 4.9. Let (B1), (B4) and (B5) hold. Then, the operator S maps C into C .

Proof. The proof of this lemma is similar to that of Lemma 3.2. $\square$

Theorem 4.10. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed and assume (B4)–(B6) hold. Let $w > 0$ be given and suppose that f satisfies

$$0 < f(x) \leq w \left[\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \right]^{-1}, \quad 0 < x \leq w. \quad (4.21)$$

(a) If $\underline{f}_0 = \infty$, then (1.1)', (1.2) has a positive solution $y_1 \in B$ such that

$$0 < \|y_1\| \leq w.$$

(b) If $\underline{f}_\infty = \infty$, then (1.1)', (1.2) has a positive solution $y_2 \in B$ such that

$$\|y_2\| \geq w \quad \text{with} \quad \min_{t \in [c, d]} y_2(t) \geq \theta w.$$

(c) If $\underline{f}_0 = \underline{f}_\infty = \infty$, then (1.1)', (1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that

$$0 < \|y_1\| \leq w \leq \|y_2\| \quad \text{with} \quad \min_{t \in [c, d]} y_2(t) \geq \theta w.$$

Proof: (a) We let

$$Q = \left[\theta \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \Delta s \right]^{-1}. \quad (4.22)$$

Since $\underline{f}_0 = \infty$, there exists $0 < r < w$ such that

$$f(x) \geq Qx, \quad 0 < x \leq r. \quad (4.23)$$

Let $y \in C$ be such that $\|y\| = r$. Then applying (4.19), (4.23) and (4.22), we find

$$\begin{aligned} Sy(\sigma^n(b)) &\geq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) Qy(\sigma^{n-1}(s)) \Delta s \\ &\geq Q \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \theta \|y\| \Delta s = \|y\|. \end{aligned}$$

This implies that

$$\|Sy\| \geq \|y\|. \quad (4.24)$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < r\}$, then $\|Sy\| \geq \|y\|$ for $y \in C \cap \partial\Omega_1$.

Next, we let $y \in C$ be such that $\|y\| = w$. Applying (4.19), Theorem 2.7 and (4.21), we get for $t \in [a, \sigma^n(b)]$,

$$Sy(t) \leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \leq w = \|y\|.$$

Hence, we have

$$\|Sy\| \leq \|y\|. \quad (4.25)$$

If we set $\Omega_2 = \{y \in B \mid \|y\| < w\}$, then $\|Sy\| \leq \|y\|$ for $y \in C \cap \partial\Omega_2$.

Having obtained (4.24) and (4.25), it now follows from Theorem 1.2 that S has a fixed point $y_1 \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $r \leq \|y_1\| \leq w$. It is clear that y_1 is a positive solution of (1.1)', (1.2).

(b) From the proof of Theorem 4.10(a), we note that the condition (4.21) gives rise to (4.25). Hence by letting $\Omega_1 = \{y \in B \mid \|y\| < w\}$, we have $\|Sy\| \leq \|y\|$ for $y \in C \cap \partial\Omega_1$.

Next, let Q be defined as in (4.22). Since $f_\infty = \infty$, we may choose $T > w$ such that

$$f(x) \geq Qx, \quad x \geq T. \quad (4.26)$$

Let $y \in C$ be such that $\|y\| = \frac{T}{\theta}$. Then for $t \in [c, d]$, we have

$$y(t) \geq \theta \|y\| = \theta \times \frac{T}{\theta} = T.$$

So it follows from (4.26) that

$$f(y(t)) \geq Qy(t), \quad t \in [c, d]$$

which implies

$$f(y(\sigma^{n-1}(t))) \geq Qy(\sigma^{n-1}(t)), \quad t \in [c, \rho^{n-1}(d)]. \quad (4.27)$$

Now, using (4.19), (4.27) and (4.22), we have

$$Sy(\sigma^n(b)) \geq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) u(s) f(y(\sigma^{n-1}(s))) \Delta s$$

$$\begin{aligned} &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) Q y(\sigma^{n-1}(s)) \Delta s \\ &\geq Q \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \theta \|y\| \Delta s = \|y\| \end{aligned}$$

which leads to (4.24). If we set $\Omega_2 = \{y \in B \mid \|y\| < \frac{T}{\theta}\}$, then $\|Sy\| \geq \|y\|$ for $y \in C \cap \partial\Omega_2$.

It follows from Theorem 1.2 that S has a fixed point $y_2 \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $w \leq \|y_2\| \leq \frac{T}{\theta}$. It is clear that y_2 is a positive solution of (1.1)', (1.2).

(c) This follows from (a) and (b). $\square$

Theorem 4.11. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed and assume (B4)–(B6) hold. Suppose that $\underline{f}_0 = \underline{f}_\infty = \infty$ and f satisfies (4.21) with $w = w_l$, $l = 1, \dots, m$, where $w_1 < w_2 < \dots < w_m$. Then (1.1)', (1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that

$$0 < \|y_1\| \leq w_1, \quad \|y_2\| \geq w_m, \quad \min_{t \in [c, d]} y_2(t) \geq \theta w_m.$$

Proof. The result follows by repeated use of Theorem 4.10. $\square$

Theorem 4.11 ensures the existence of (at least) two positive solutions when f is bounded. Next, we turn to f that is bounded below.

Theorem 4.12. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed and assume (B4)–(B6) hold. Let $w > 0$ be given and suppose that f satisfies

$$f(x) \geq w \left[\int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) \Delta s \right]^{-1}, \quad \theta w \leq x \leq w. \quad (4.28)$$

(a) If $\overline{f}_0 = 0$, then (1.1)', (1.2) has a positive solution $y_1 \in B$ such that

$$0 < \|y_1\| \leq w.$$

(b) If $\overline{f}_\infty = 0$ and f is nondecreasing, then (1.1)', (1.2) has a positive solution $y_2 \in B$ such that

$$\|y_2\| \geq w \quad \text{with} \quad \min_{t \in [c, d]} y_2(t) \geq \theta w.$$

- (c) If $\bar{f}_0 = \bar{f}_\infty = 0$ and f is nondecreasing, then (1.1)', (1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that

$$0 < \|y_1\| \leq w \leq \|y_2\| \quad \text{with} \quad \min_{t \in [c, d]} y_2(t) \geq \theta w.$$

Proof: (a) We define

$$\epsilon = \left[\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \right]^{-1}. \quad (4.29)$$

Since $\bar{f}_0 = 0$, there exists $0 < \gamma < w$ such that

$$f(x) \leq \epsilon x, \quad 0 < x \leq \gamma. \quad (4.30)$$

Let $y \in C$ be such that $\|y\| = \gamma$. Then applying (4.19), Theorem 2.7, (4.30) and (4.29), we find for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \epsilon y(\sigma^{n-1}(s)) \Delta s \\ &\leq \epsilon \|y\| \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s = \|y\|. \end{aligned}$$

This immediately implies (4.25). If we set $\Omega_1 = \{y \in B \mid \|y\| < \gamma\}$, then $\|Sy\| \leq \|y\|$ for $y \in C \cap \partial\Omega_1$.

Next, let $y \in C$ be such that $\|y\| = w$. Clearly,

$$y(\sigma^{n-1}(t)) \in [\theta w, w], \quad t \in [c, \rho^{n-1}(d)]$$

which, together with (4.19) and (4.28), gives

$$Sy(\sigma^n(b)) \geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(\sigma^n(b), s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \geq w = \|y\|.$$

Hence, we have (4.24). Set $\Omega_2 = \{y \in B \mid \|y\| < w\}$, then we get $\|Sy\| \geq \|y\|$ for $y \in C \cap \partial\Omega_2$.

It now follows from Theorem 1.2 that S has a fixed point $y_1 \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $\gamma \leq \|y_1\| \leq w$. It is clear that y_1 is a positive solution of (1.1)', (1.2).

(b) From the proof of Theorem 4.12(a), we note that the condition (4.28) gives rise to (4.24). So by letting $\Omega_1 = \{y \in B \mid \|y\| < w\}$, then $\|Sy\| \geq \|y\|$ for $y \in C \cap \partial\Omega_1$.

Next, let ϵ be defined as in (4.29). Since $\bar{f}_\infty = 0$, we may choose $T > w$ such that

$$f(x) \leq \epsilon x, \quad x \geq T. \quad (4.31)$$

Let $y \in C$ be such that $\|y\| = T$. Then, using (4.19), Theorem 2.7, the fact that f is nondecreasing, (4.31) and (4.29), we find for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(\|y\|) \Delta s \\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \epsilon \|y\| \Delta s = \|y\|. \end{aligned}$$

Thus, we have (4.25). If we set $\Omega_2 = \{y \in B \mid \|y\| < T\}$, then $\|Sy\| \leq \|y\|$ for $y \in C \cap \partial\Omega_2$.

We have now obtained (4.24) and (4.25). Once again, it follows from Theorem 1.2 that S has a fixed point $y_2 \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $w \leq \|y_2\| \leq T$. It is clear that y_2 is a positive solution of (1.1)', (1.2).

(c) This follows from (a) and (b). □

Theorem 4.13. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed and assume (B4)–(B6) hold. Suppose that $\bar{f}_0 = \bar{f}_\infty = 0$, f is nondecreasing and f satisfies (4.28) with $w = w_l$, $l = 1, \dots, m$, where $w_1 < w_2 < \dots < w_m$. Then (1.1)', (1.2) has (at least) two positive solutions $y_1, y_2 \in B$ such that

$$0 < \|y_1\| \leq w_1, \quad \|y_2\| \geq w_m, \quad \min_{t \in [c, d]} y_2(t) \geq \theta w_m.$$

Proof: Repeated applications of Theorem 4.12 yield the result. □

Example 4.14. Let $m \in \mathbb{R}^+$ and $T = m\mathbb{Z}$. Consider the boundary value problem

$$\begin{cases} y^{\Delta^2}(t) + \frac{24[y(\sigma(t))^2 + 1872m^4]}{[300m\sigma(t) - 12(\sigma(t))^2]^2 + 1872m^4} = 0, & t \in [0, 11m] \\ y(0) = y^{\Delta}(12m) = 0. \end{cases} \quad (4.32)$$

Here, $n = 2$, $p = 1$, $a = 0$ and $b = 11m$. Taking $f(y) = y^2 + 1872m^4$, we may choose

$$u(t) = v(t) = \frac{24}{[300m\sigma(t) - 12(\sigma(t))^2]^2 + 1872m^4}.$$

It is easy to check that (B4)–(B6) are satisfied with $\eta = 1$.

Clearly, $\underline{f}_0 = \underline{f}_\infty = \infty$. We shall find some $w > 0$ such that condition (4.21) is fulfilled. First, using (4.17) we obtain

$$\begin{aligned} & \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \\ &= \int_0^{12m} (s+m) \frac{24}{[300m\sigma(s) - 12(\sigma(s))^2]^2 + 1872m^4} \Delta s \\ &\geq 24 \int_0^{12m} \frac{s+m}{(1872m^2)^2 + 1872m^4} \Delta s \\ &= \frac{1872m^2}{(1872m^2)^2 + 1872m^4} = \frac{1}{1873m^2} \end{aligned}$$

where we have substituted $s = 11m$ to get the inequality. Next, to ensure that (4.21) is true, we set

$$\begin{aligned} 0 < f(y) &\leq w^2 + 1872m^4 \\ &\leq w \left[\int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s \right]^{-1} \leq 1873m^2 w, \quad 0 < y \leq w, \end{aligned}$$

which gives the inequality $w^2 + 1872m^4 \leq 1873m^2 w$. This holds if and only if

$$m^2 \leq w \leq 1872m^2. \quad (4.33)$$

Hence, (4.21) holds for any $w \in \mathbb{R}[m^2, 1872m^2]$. By Theorem 4.11, there exist two positive solutions y_1 and y_2 of (4.32) such that for a fixed $\delta \in \mathbb{R}(0, \frac{1}{2})$,

$$0 < \|y_1\| \leq m^2, \quad \|y_2\| \geq 1872m^2, \quad \min_{t \in [c, d]} y_2(t) \geq \theta(1872m^2). \quad (4.34)$$

Let $\delta = 0.3$. Then, $c = 4m$, $d = 9m$ and $\theta = k = \frac{1}{3}$. We note that one positive solution of (4.32) is given by $y(t) = 12t(25m - t)$ with $\|y\| = 1872m^2$ and $\min_{t \in [4m, 9m]} y(t) = 1008m^2$, both are within the ranges given in (4.34).

4.2 Applications

In this section, we shall apply the results obtained in Section 4.1 to two special cases of (1.1)', (1.2), namely (Q1) and (Q2). In addition to providing easily verifiable criteria for the existence of double positive solutions, we also establish upper and lower bounds for these solutions.

Before proceeding further, it is noted that (Q1) and (Q2) are particular cases of (1.1)', (1.2) when $n = 2$ and $p = 1$. Hence, by Theorem 2.5 the related Green's function is explicitly given in (4.17). Moreover, for a fixed $\delta \in \mathbb{R}(0, \frac{1}{2})$, using Theorem 2.8 and (4.17) we get

$$\begin{aligned} k &= \inf_{s \in [a, b]} \frac{G(c, s)}{G(\sigma^2(b), s)} = \inf_{s \in [a, b]} \frac{G(c, s)}{\sigma(s) - a} \\ &= \min \left\{ \inf_{\sigma(s) \in [\sigma(a), c]} \frac{\sigma(s) - a}{\sigma(s) - a}, \inf_{\sigma(s) \in [c, \sigma(b)]} \frac{c - a}{\sigma(s) - a} \right\} = \frac{c - a}{\sigma(b) - a}. \end{aligned} \quad (4.35)$$

We shall begin with the boundary value problem

$$\begin{cases} y^{\Delta^2}(t) + h(t)([y(\sigma(t))]^\alpha + [y(\sigma(t))]^\beta) = 0, & t \in [a, b] \\ y(a) = y^\Delta(\sigma(b)) = 0 \end{cases} \quad (Q1)$$

where $0 \leq \alpha < 1 < \beta$. It is assumed that

(B7) $h(t)$ is continuous and nonnegative on $[a, \sigma(b)]$;

(B8) $h(t)$ is nonzero for some $t \in [c, \rho(d)]$;

(B9) $\int_a^{\sigma(b)} G(\sigma^2(b), s)h(s)\Delta s = \int_a^{\sigma(b)} [\sigma(s) - a]h(s)\Delta s < \infty$.

Theorem 4.15. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed and let $w > 0$ be given. Suppose that

$$\int_a^{\sigma(b)} [\sigma(s) - a] h(s) \Delta s \leq \frac{w}{w^\alpha + w^\beta}. \quad (4.36)$$

Then, the boundary value problem (Q1) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that

$$0 < \|y_1\| \leq w \leq \|y_2\| \quad \text{with} \quad \min_{t \in [c, d]} y_2(t) \geq \frac{(c-a)w}{\sigma(b) - a}.$$

Proof: The boundary value problem (Q1) is a particular case of (1.1)', (1.2) when $F(t, y) = h(t)(y^\alpha + y^\beta)$. Pick $f(x) = x^\alpha + x^\beta$ and $u(t) = v(t) = h(t)$. Then, in view of (B7)–(B9), conditions (B4)–(B6) are now satisfied with $\eta = 1$. Moreover, $\underline{f}_0 = \underline{f}_\infty = \infty$. We shall apply Theorem 4.10. Since we have

$$0 < f(x) \leq w^\alpha + w^\beta, \quad 0 < x \leq w,$$

to ensure that (4.21) is satisfied, we shall impose (noting (4.17))

$$w^\alpha + w^\beta \leq w \left[\int_a^{\sigma(b)} G(\sigma^2(b), s) v(s) \Delta s \right]^{-1} = w \left\{ \int_a^{\sigma(b)} [\sigma(s) - a] h(s) \Delta s \right\}^{-1}$$

which leads to (4.36). The conclusion now follows immediately from Theorem 4.10(c) and (4.35). $\square$

Theorem 4.16. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed. Suppose (4.36) is satisfied with $w = w_l$, $l = 1, \dots, m$, where $w_1 < w_2 < \dots < w_m$. Then (Q1) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that

$$0 < \|y_1\| \leq w_1, \quad \|y_2\| \geq w_m, \quad \min_{t \in [c, d]} y_2(t) \geq \frac{(c-a)w_m}{\sigma(b) - a}.$$

Proof: The result follows by repeated use of Theorem 4.15. $\square$

We shall now establish upper and lower bounds for the two positive solutions of (Q1).

Theorem 4.17. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed. We define

$$\phi(x) = \left(\frac{c-a}{\sigma(b)-a} \right)^x \int_c^{\rho(d)} [\sigma(s) - a] h(s) \Delta s.$$

Let

$$w_1 = [\phi(\alpha)]^{\frac{1}{1-\alpha}} \quad \text{and} \quad w_2 = [\phi(\beta)]^{\frac{1}{1-\beta}}.$$

Suppose that $w > 0$ is given and (4.36) holds. Then, the boundary value problem

(Q1) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that

(a) if $w < \min\{w_1, w_2\}$, then

$$0 < \|y_1\| \leq w \leq \|y_2\| \leq \min\{w_1, w_2\} \quad \text{with} \quad \min_{t \in [c, d]} y_2(t) \geq \frac{(c-a)w}{\sigma(b)-a};$$

(b) if $\min\{w_1, w_2\} < w < \max\{w_1, w_2\}$, then

$$\begin{aligned} \min\{w_1, w_2\} &\leq \|y_1\| \leq w \leq \|y_2\| \leq \max\{w_1, w_2\}, \\ \min_{t \in [c, d]} y_1(t) &\geq \frac{(c-a) \min\{w_1, w_2\}}{\sigma(b)-a}, \quad \min_{t \in [c, d]} y_2(t) \geq \frac{(c-a)w}{\sigma(b)-a}; \end{aligned}$$

(c) if $w > \max\{w_1, w_2\}$, then

$$\begin{aligned} \max\{w_1, w_2\} &\leq \|y_1\| \leq w \leq \|y_2\|, \\ \min_{t \in [c, d]} y_1(t) &\geq \frac{(c-a) \max\{w_1, w_2\}}{\sigma(b)-a}, \quad \min_{t \in [c, d]} y_2(t) \geq \frac{(c-a)w}{\sigma(b)-a}. \end{aligned}$$

Proof: Since (4.36) is satisfied, it follows from Theorem 4.15 that (Q1) has double positive solutions y_3 and y_4 such that

$$0 < \|y_3\| \leq w \leq \|y_4\|. \quad (4.37)$$

Next, the operator S defined in (4.1) becomes

$$Sy(t) = \int_a^{\sigma(b)} G(t, s) h(s) \left([y(\sigma(s))]^\alpha + [y(\sigma(s))]^\beta \right) \Delta s, \quad t \in [a, \sigma^2(b)] \quad (4.38)$$

where $G(t, s)$ is given in (4.17). Moreover, the cone $C_k \subset B = C[a, \sigma^2(b)]$ defined in (4.10) becomes (noting (4.35))

$$C_k = \left\{ y \in B \mid y(t) \geq 0, \quad t \in [a, \sigma^2(b)]; \quad \min_{t \in [c, d]} y(t) \geq \frac{c-a}{\sigma(b)-a} \|y\| \right\}. \quad (4.39)$$

We shall employ Theorem 1.2. Let $y \in C_k$ be such that $\|y\| = w$. Then, in view of Theorem 2.7, (4.17) and (4.36), we have

$$\begin{aligned} Sy(t) &\leq \int_a^{\sigma(b)} G(\sigma^2(b), s) h(s) \left([y(\sigma(s))]^\alpha + [y(\sigma(s))]^\beta \right) \Delta s \\ &\leq \int_a^{\sigma(b)} [\sigma(s) - a] h(s) \left(w^\alpha + w^\beta \right) \Delta s \leq w = \|y\|, \quad t \in [a, \sigma^2(b)]. \end{aligned}$$

Thus,

$$\|Sy\| \leq \|y\|. \quad (4.40)$$

By setting $\Omega = \{y \in B \mid \|y\| < w\}$, we see that (4.40) holds for $y \in C_k \cap \partial\Omega$.

Next, it is clear that for $y \in C_k$,

$$\begin{aligned} \|Sy\| &= \sup_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t, s) h(s) \left([y(\sigma(s))]^\alpha + [y(\sigma(s))]^\beta \right) \Delta s \\ &= \int_a^{\sigma(b)} G(\sigma^2(b), s) h(s) \left([y(\sigma(s))]^\alpha + [y(\sigma(s))]^\beta \right) \Delta s \\ &\geq \int_c^{\rho(d)} [\sigma(s) - a] h(s) \left([y(\sigma(s))]^\alpha + [y(\sigma(s))]^\beta \right) \Delta s \\ &\geq \int_c^{\rho(d)} [\sigma(s) - a] h(s) \left[\left(\frac{c-a}{\sigma(b)-a} \right)^\alpha \|y\|^\alpha + \left(\frac{c-a}{\sigma(b)-a} \right)^\beta \|y\|^\beta \right] \Delta s. \end{aligned}$$

It follows that

$$\|Sy\| \geq \phi(\alpha) \|y\|^\alpha + \phi(\beta) \|y\|^\beta, \quad y \in C_k. \quad (4.41)$$

Let $y \in C_k$ be such that $\|y\| = w_1$. Then, (4.41) provides

$$\|Sy\| \geq \phi(\alpha) \|y\|^\alpha \geq \phi(\alpha) \|y\|^{\alpha-1} \|y\| = \|y\|. \quad (4.42)$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < w_1\}$, then (4.42) holds for $y \in C_k \cap \partial\Omega_1$. Having obtained (4.40) and (4.42), it follows from Theorem 1.2 that S has a fixed point $y_5 \in C_k$ such that

$$\min\{w_1, w\} \leq \|y_5\| \leq \max\{w_1, w\}. \quad (4.43)$$

Similarly, if we let $y \in C_k$ be such that $\|y\| = w_2$, then from (4.41) we get

$$\|Sy\| \geq \phi(\beta)\|y\|^\beta \geq \phi(\beta)\|y\|^{\beta-1}\|y\| = \|y\|. \quad (4.44)$$

By setting $\Omega_2 = \{y \in B \mid \|y\| < w_2\}$, we see that (4.44) holds for $y \in C_k \cap \partial\Omega_2$. With (4.40) and (4.44), once again by Theorem 1.2, we conclude that S has a fixed point $y_6 \in C_k$ such that

$$\min\{w_2, w\} \leq \|y_6\| \leq \max\{w_2, w\}. \quad (4.45)$$

Our result follows by combining (4.37), (4.43) and (4.45). For case (a), we may pick

$$y_1 = y_3 \quad \text{and} \quad y_2 = \begin{cases} y_5, & w_1 \leq w_2 \\ y_6, & w_1 \geq w_2. \end{cases}$$

In case (b), we choose

$$(y_1, y_2) = \begin{cases} (y_5, y_6), & w_1 \leq w_2 \\ (y_6, y_5), & w_1 \geq w_2. \end{cases}$$

Finally in case (c), we take

$$y_1 = \begin{cases} y_6, & w_1 \leq w_2 \\ y_5, & w_1 \geq w_2 \end{cases} \quad \text{and} \quad y_2 = y_4. \quad \square$$

Example 4.18. Consider the boundary value problem (Q1) with

$$\mathbf{T} = \{2^k \mid k \in \mathbf{Z}\} \cup \{0\}, \quad a = 1, \quad b = 8.$$

Let $w = 1$. Then, condition (4.36) reduces to

$$\int_1^{\sigma(8)} (2s - 1)h(s)\Delta s \leq \frac{1}{2}. \quad (4.46)$$

By Theorem 4.15, for any $h(t)$ that satisfies (4.46), the boundary value problem has (at least) double positive solutions y_1 and y_2 such that for a fixed $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$,

$$0 < \|y_1\| \leq 1 \leq \|y_2\| \quad \text{with} \quad \min_{t \in [c, d]} y_2(t) \geq \frac{c-1}{15}.$$

Some examples of such $h(t)$ are $\frac{1}{180}$ and $\frac{1}{160} \sin^2 t$.

Example 4.19. Let $T = \{2^k | k \in \mathbb{Z}\} \cup \{0\}$. Consider the boundary value problem

$$\begin{cases} y^{\Delta^2}(t) + \frac{[y(\sigma(t))]^{0.1} + [y(\sigma(t))]^{1.1}}{(\frac{1}{2}\sigma(t) + 5)^2} = 0, & t \in \left[\frac{1}{2}, 4\right] \\ y\left(\frac{1}{2}\right) = y^{\Delta}(4) = 0. \end{cases} \quad (4.47)$$

Here, $\alpha = 0.1$, $\beta = 1.1$ and $h(t) = \left(\frac{1}{2}\sigma(t) + 5\right)^{-2}$. Condition (4.36) yields

$$\frac{w}{w^{0.1} + w^{1.1}} \geq \int_{\frac{1}{2}}^{\sigma(4)} [\sigma(s) - a] h(s) \Delta s = \int_{0.5}^8 \frac{2s - 0.5}{(\frac{1}{2}\sigma(s) + 5)^2} \Delta s = 0.563,$$

and this is satisfied for any $1.39 \leq w \leq 302$.

Let $\delta = 0.05$. Then, $c = 2$ and $d = 8$. By direct computation, we get

$$w_1 = [\phi(0.1)]^{\frac{10}{9}} = 0.0962 \quad \text{and} \quad w_2 = [\phi(1.1)]^{-10} = 1.38 \times 10^{16}.$$

Since $w_1 < w < w_2$, by Theorem 4.17(b), the boundary value problem (4.47) has (at least) two positive solutions y_1 and y_2 such that

$$0.0962 \leq \|y_1\| \leq w \leq \|y_2\| \leq 1.38 \times 10^{16}, \quad \min_{t \in [2, 8]} y_1(t) \geq 0.01924, \quad \min_{t \in [2, 8]} y_2(t) \geq 0.2w.$$

Since $1.39 \leq w \leq 302$, we further conclude that

$$\begin{cases} 0.0962 \leq \|y_1\| \leq 1.39, & 302 \leq \|y_2\| \leq 1.38 \times 10^{16}, \\ \min_{t \in [2, 8]} y_1(t) \geq 0.01924, & \min_{t \in [2, 8]} y_2(t) \geq 0.2(302) = 60.4. \end{cases}$$

For the rest of this section, we shall consider the boundary value problem

$$\begin{cases} y^{\Delta^2}(t) + h(t)e^{\zeta[y(\sigma(t))]} = 0, & t \in [a, b] \\ y(a) = y^{\Delta}(\sigma(b)) = 0 \end{cases} \quad (Q2)$$

where $\zeta > 0$. It is assumed that $h(t)$ satisfies conditions (B7)–(B9).

Theorem 4.20. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed and let $w > 0$ be given. Suppose that

$$\int_a^{\sigma(b)} [\sigma(s) - a]h(s)\Delta s \leq we^{-\zeta w}. \quad (4.48)$$

Then, the boundary value problem (Q2) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that

$$0 < \|y_1\| \leq w \leq \|y_2\| \quad \text{with} \quad \min_{t \in [c, d]} y_2(t) \geq \frac{(c-a)w}{\sigma(b)-a}.$$

Proof. The boundary value problem (Q2) is a special case of (1.1)', (1.2) when $F(t, y) = h(t)e^{\zeta y}$. Choose $f(x) = e^{\zeta x}$ and $u(t) = v(t) = h(t)$. Then, in view of (B7)–(B9), conditions (B4)–(B6) are now satisfied with $\eta = 1$. Further, $\underline{f}_0 = \underline{f}_\infty = \infty$. We shall employ Theorem 4.10. Since

$$f(x) \leq e^{\zeta w}, \quad 0 < x \leq w,$$

condition (4.21) will be satisfied if we set

$$e^{\zeta w} \leq w \left[\int_a^{\sigma(b)} G(\sigma^2(b), s)v(s)\Delta s \right]^{-1} = w \left\{ \int_a^{\sigma(b)} [\sigma(s) - a]h(s)\Delta s \right\}^{-1},$$

which is inequality (4.48). The conclusion is now immediate from Theorem 4.10(c) and (4.35). $\square$

Theorem 4.21. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed. Suppose (4.48) is satisfied with $w = w_l$, $l = 1, \dots, m$, where $w_1 < w_2 < \dots < w_m$. Then (Q2) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that

$$0 < \|y_1\| \leq w_1, \quad \|y_2\| \geq w_m, \quad \min_{t \in [c, d]} y_2(t) \geq \frac{(c-a)w_m}{\sigma(b)-a}.$$

Proof. The result is obtained by repeated use of Theorem 4.20. $\square$

The next result offers upper and lower bounds for the two positive solutions of (Q2).

Theorem 4.22. Let $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$ be fixed and $i, j \in \mathbb{Z}[2, \infty) \cup \{0\}$ be fixed distinct integers. We define

$$\psi(x) = \frac{1}{x!} \left(\frac{\zeta(c-a)}{\sigma(b)-a} \right)^x \int_c^{\rho(d)} (\sigma(s)-a)h(s)\Delta s.$$

Let

$$w_1 = [\psi(j)]^{\frac{1}{1-j}} \quad \text{and} \quad w_2 = [\psi(i)]^{\frac{1}{1-i}}.$$

Suppose that $w > 0$ is given and (4.48) holds. Then, the boundary value problem (Q2) has (at least) two positive solutions $y_1, y_2 \in C[a, \sigma^2(b)]$ such that conclusions (a)–(c) of Theorem 4.17 hold.

Proof: Since (4.48) is satisfied, by Theorem 4.20 the boundary value problem (Q2) has double positive solutions y_3 and y_4 such that (4.37) holds.

Next, the operator S defined in (4.1) becomes

$$Sy(t) = \int_a^{\sigma(b)} G(t, s)h(s)e^{\zeta y(\sigma(s))}\Delta s, \quad t \in [a, \sigma^2(b)] \quad (4.49)$$

where $G(t, s)$ is given in (4.17). Moreover, the cone C_k defined in (4.10) reduces to that in (4.39). We shall once again employ Theorem 1.2. Let $y \in C_k$ be such that $\|y\| = w$. Then, in view of Theorem 2.7, (4.17) and (4.48), for $t \in [a, \sigma^2(b)]$ we have

$$Sy(t) \leq \int_a^{\sigma(b)} G(\sigma^2(b), s)h(s)e^{\zeta y(\sigma(s))}\Delta s \leq \int_a^{\sigma(b)} [\sigma(s)-a]h(s)e^{\zeta w}\Delta s \leq w = \|y\|.$$

Thus, (4.40) holds. By setting $\Omega = \{y \in B \mid \|y\| < w\}$, we see that $\|Sy\| \leq \|y\|$ for $y \in C \cap \partial\Omega$.

Now, using the inequality

$$e^x \geq \frac{x^i}{i!} + \frac{x^j}{j!}, \quad x > 0, \quad (4.50)$$

we find for $y \in C_k$,

$$\|Sy\| = \sup_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t, s)h(s)e^{\zeta y(\sigma(s))}\Delta s$$

$$\begin{aligned}
&= \int_a^{\sigma(b)} G(\sigma^2(b), s) h(s) e^{\zeta y(\sigma(s))} \Delta s \\
&\geq \int_c^{\rho(d)} [\sigma(s) - a] h(s) e^{\zeta y(\sigma(s))} \Delta s \\
&\geq \int_c^{\rho(d)} [\sigma(s) - a] h(s) \exp \left(\frac{\zeta(c-a)}{\sigma(b)-a} \|y\| \right) \Delta s \\
&\geq \int_c^{\rho(d)} [\sigma(s) - a] h(s) \left[\left(\frac{\zeta(c-a)}{\sigma(b)-a} \right)^i \frac{\|y\|^i}{i!} + \left(\frac{\zeta(c-a)}{\sigma(b)-a} \right)^j \frac{\|y\|^j}{j!} \right] \Delta s.
\end{aligned}$$

Hence, we have

$$\|Sy\| \geq \psi(i)\|y\|^i + \psi(j)\|y\|^j, \quad y \in C_k. \quad (4.51)$$

Using a similar technique as in the proof of Theorem 4.17, from (4.51) we get

$$\|Sy\| \geq \|y\| \quad (4.52)$$

for $y \in C \cap \partial\Omega_1$ and for $y \in C \cap \partial\Omega_2$, where $\Omega_1 = \{y \in B \mid \|y\| < w_1\}$ and $\Omega_2 = \{y \in B \mid \|y\| < w_2\}$.

Having obtained (4.40) and (4.52), by Theorem 1.2 the operator S has fixed points y_5 and y_6 satisfying (4.43) and (4.45). Finally, just as in the proof of Theorem 4.17, the conclusions (a)–(c) follow from a combination of (4.37), (4.43) and (4.45). $\square$

Example 4.23. Let $T = m\mathbb{Z}$ where $m \in \mathbb{R}^+$. Consider the boundary value problem

$$\begin{cases} y^{\Delta^2}(t) + h(t)e^{\frac{1}{2}y(\sigma(t))} = 0, & t \in [0, 11m] \\ y(0) = y^{\Delta}(12m) = 0. \end{cases} \quad (4.53)$$

Here $\zeta = \frac{1}{2}$. Let $w = 1$ be given. Then, condition (4.48) reduces to

$$\int_0^{12m} (s+m)h(s)\Delta s \leq e^{-\frac{1}{2}}. \quad (4.54)$$

By Theorem 4.20, if $h(t)$ satisfies (4.54), then the boundary value problem (4.53) has double positive solutions y_1 and y_2 such that for a fixed $\delta \in \mathbb{R} \left(0, \frac{1}{2}\right)$,

$$0 < \|y_1\| \leq 1 \leq \|y_2\| \quad \text{with} \quad \min_{t \in [c, d]} y_2(t) \geq \frac{c}{12m}.$$

Some examples of such $h(t)$ are $\frac{1}{150m^2}$ and $\frac{1}{130} \cos^2 t$.

Example 4.24. Let $T = \mathbb{R}[0, 1) \cup \mathbb{Z}^+$. Consider the boundary value problem

$$\begin{cases} y^{\Delta^2}(t) + \frac{2e^{y(\sigma(t))/2}}{e^{[19(\sigma(t)) - (\sigma(t))^2]/2}} = 0, & t \in [0, 8] \\ y(0) = y^{\Delta}(\sigma(8)) = 0. \end{cases} \quad (4.55)$$

Here, $\zeta = \frac{1}{2}$ and $h(t) = 2e^{-[19(\sigma(t)) - (\sigma(t))^2]/2}$. We check that condition (4.48) is true provided

$$we^{-\frac{1}{2}w} \geq \int_0^{\sigma(8)} [\sigma(s)]h(s)\Delta s = \int_0^1 \frac{2s}{e^{[19s - s^2]/2}} ds + \sum_{s=1}^8 \frac{2(s+1)}{e^{[19(s+1) - (s+1)^2]/2}} = 0.0229,$$

and this inequality is satisfied for any $0.0232 \leq w \leq 12.6$.

Fix $\delta = 0.1$, $i = 3$ and $j = 0$. Then, $c = 1$, $d = 9$, and by direct computation we have

$$w_1 = \zeta(0) = 1.6582 \times 10^{-7} \quad \text{and} \quad w_2 = [\zeta(3)]^{-\frac{1}{2}} = 459370.$$

Since $w_1 < w < w_2$, by Theorem 4.22(b), the boundary value problem (4.55) has two positive solutions y_1 and y_2 such that

$$\begin{aligned} 1.6582 \times 10^{-7} &\leq \|y_1\| \leq w \leq \|y_2\| \leq 459370, \\ \min_{t \in [1, 9]} y_1(t) &\geq 1.8424 \times 10^{-8}, \quad \min_{t \in [1, 9]} y_2(t) \geq \frac{w}{9}. \end{aligned}$$

Since $0.0232 \leq w \leq 12.6$, we can further conclude that

$$\begin{cases} 1.6582 \times 10^{-7} \leq \|y_1\| \leq 0.0232, & 12.6 \leq \|y_2\| \leq 459370, \\ \min_{t \in [1, 9]} y_1(t) \geq 1.8424 \times 10^{-8}, & \min_{t \in [1, 9]} y_2(t) \geq \frac{12.6}{9} = 1.4. \end{cases} \quad (4.56)$$

In fact, a positive solution is given by $y(t) = t(19 - t)$ and we notice that $\|y\| = 90$ and $\min_{t \in [1, 9]} y(t) = 18$ are well within the ranges obtained in (4.56).

Chapter 5

Triple Positive Solutions

We now turn to the existence of triple positive solutions of the boundary value problem $(1.1)'$, (1.2) . While papers on the existence of (at least) three solutions on boundary value problems for the real and discrete domains can be found in [9, 11, 13, 16, 21, 24, 44], recent works on time scales can be found in [20, 30, 46, 54].

In this chapter, we shall establish criteria so that the boundary value problem $(1.1)'$, (1.2) has at least three positive solutions in $C[a, \sigma^n(b)]$. In addition, estimates on the norms of these solutions will also be provided. Not only that *new* results have been achieved (to date in the literature), we also discuss the relation between the results in terms of generality, and illustrate the importance of the results through some examples. Moreover, we are considering $(1.1)'$, (1.2) with very *general* nonlinear term F – this yields a much more appropriate and robust model for many nonlinear phenomena [38, 51]. The work in this chapter unifies the analysis for the real and the discrete cases in [62, 63]. The results, when reduced to $\mathbb{R}$ and $\mathbb{Z}$, are also *new* in the literature.

5.1 Existence Criteria

As in the previous chapter, we let the Banach space $B = C[a, \sigma^n(b)]$ be equipped with the usual norm

$$\|y\| = \sup_{t \in [a, \sigma^n(b)]} |y(t)|.$$

For clarity, we shall now list the conditions used in this chapter. In these conditions, $\xi_1, \xi_2, \xi_3, \xi_4$ are some numbers in $[a, \sigma^n(b)]$, and the sets $\tilde{K}, K$ are given by

$$\tilde{K} = \{y \in B \mid y(t) \geq 0, t \in [a, \sigma^n(b)]\}$$

and

$$K = \{y \in \tilde{K} \mid y(t) > 0 \text{ for some } t \in [a, \sigma^n(b)]\} = \tilde{K} \setminus \{0\}.$$

(C1) The function $F : [a, \sigma(b)] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous with

$$F(t, y) \geq 0, \quad (t, y) \in [a, \sigma(b)] \times \tilde{K}$$

and

$$F(t, y) > 0, \quad (t, y) \in [a, \sigma(b)] \times K.$$

(C2) There exist continuous functions f, u, v with $f : \mathbb{R}[0, \infty) \rightarrow \mathbb{R}[0, \infty)$ and $u, v : [a, \sigma(b)] \rightarrow \mathbb{R}[0, \infty)$ such that for $(t, y) \in [a, \sigma(b)] \times \tilde{K}$,

$$u(t)f(y(\sigma^{n-1}(t))) \leq F(t, y(\sigma^{n-1}(t))) \leq v(t)f(y(\sigma^{n-1}(t))).$$

(C3) There exists a number $\eta \in \mathbb{R}(0, 1]$ such that $u(t) \geq \eta v(t)$ for $t \in [a, \sigma(b)]$.

(C4) $G(\sigma^n(b), s)v(s)$ is nonzero for some $s \in [a, \sigma(b)]$.

(C5) $G(c, s)u(s)$ is nonzero for some $s \in [c, \rho^{n-1}(d)]$.

(C6) $G(\xi_2, s)u(s)$ is nonzero for some $s \in [\xi_2, \rho^{n-1}(\xi_3)]$.

(C7) $G(\xi_4, s)v(s)$ is nonzero for some $s \in [\xi_1, \rho^{n-1}(\xi_4)]$.

To proceed, we define the operator $S : B \rightarrow B$ by

$$Sy(t) = \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) F(s, y(\sigma^{n-1}(s))) \Delta s, \quad t \in [a, \sigma^n(b)]. \quad (5.1)$$

It is clear that a fixed point of the operator S is a solution of (1.1)', (1.2). If (C1) and (C2) holds, then we have the following for $(t, y) \in [a, \sigma^n(b)] \times \tilde{K}$,

$$\begin{aligned} & \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ & \leq Sy(t) \leq \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s. \end{aligned} \quad (5.2)$$

Next, we define a cone in B as

$$C = \left\{ y \in B \mid y(t) \geq 0, \quad t \in [a, \sigma^n(b)]; \quad \min_{t \in [c, d]} y(t) \geq \theta \|y\| \right\},$$

where $\theta = k\eta \in \mathbb{R}(0, 1)$, k and η are given in Theorem 2.8 and (C3) respectively.

Clearly, $C \subseteq \tilde{K}$.

We also introduce the following constants which will be used in our results:

$$\begin{aligned} q &= \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \Delta s, \\ \tau &= \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(c, s) u(s) \Delta s, \\ d_1 &= \int_{\xi_2}^{\rho^{n-1}(\xi_3)} (-1)^{p+1} G(\xi_2, s) u(s) \Delta s, \\ d_2 &= \int_{\xi_1}^{\rho^{n-1}(\xi_4)} (-1)^{p+1} G(\xi_4, s) v(s) \Delta s, \\ d_3 &= \int_a^{\xi_1} (-1)^{p+1} G(\xi_4, s) v(s) \Delta s + \int_{\rho^{n-1}(\xi_4)}^{\sigma(b)} (-1)^{p+1} G(\xi_4, s) v(s) \Delta s. \end{aligned} \quad (5.3)$$

We begin with Lemmas 5.1–5.3.

Lemma 5.1. Let $F : [a, \sigma(b)] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Then, the operator $S : B \rightarrow B$ is continuous and completely continuous.

Proof: The proof of Lemma 5.1 is similar to that of Lemma 3.1. □

Lemma 5.2. Let (C1)–(C3) hold. Then, the operator S maps C into C .

Proof: The proof of this lemma is similar to that of Lemma 3.2. $\square$

Lemma 5.3. Let (C1)–(C4) hold. Suppose there exists a number $\omega > 0$ such that $f(x) < \frac{\omega}{q}$ for $x \in \mathbb{R}[0, \omega]$. Then,

$$S(\overline{C}(\omega) \subseteq C(\omega) \subset \overline{C}(\omega).$$

Proof: Let $y \in \overline{C}(\omega)$ and so $y \in \mathbb{R}[0, w]$. An application of (5.2), Theorem 2.7, (C4) and (5.3) gives for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} Sy(t) &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &< \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \frac{\omega}{q} \Delta s = q \frac{\omega}{q} = \omega. \end{aligned}$$

Thus, $\|Sy\| < \omega$. Coupling with the fact that $Sy \in C$ (Lemma 5.2), we have $Sy \in C(\omega)$. Therefore, we have shown that $S(\overline{C}(\omega) \subseteq C(\omega) \subset \overline{C}(\omega)$. $\square$

We are now ready to derive our first theorem using Leggett-Williams' fixed point theorem (Theorem 1.3).

Theorem 5.4. Let (C1)–(C5) hold. Suppose there exist $w_i \in \mathbb{R}^+$, $1 \leq i \leq 3$ where

$$0 < w_1 < w_2 < \frac{w_2}{\theta} \leq w_3$$

such that the following conditions are satisfied:

(H1) $f(x) < \frac{w_1}{q}$ for $x \in \mathbb{R}[0, w_1]$;

(H2) one of the following holds:

(i) $\limsup_{x \rightarrow \infty} \frac{f(x)}{x} < \frac{1}{q}$;

(ii) there exists a number $m (\geq w_3)$ such that if $x \in \mathbb{R}[0, m]$, then $f(x) < \frac{m}{q}$;

(H3) $f(x) > \frac{w_2}{\tau}$ for $x \in \mathbb{R}[w_2, w_3]$.

Then, the boundary value problem (1.1)', (1.2) has (at least) three positive solutions $y_1, y_2, y_3 \in C$ such that

$$\|y_1\| < w_1; \quad y_2(t) > w_2, \quad t \in [c, d]; \quad \|y_3\| > w_1 \quad \text{and} \quad \min_{t \in [c, d]} y_3(t) < w_2. \quad (5.4)$$

Proof: We shall employ Theorem 1.3. First, we shall prove that condition (H2) implies the existence of a number w_4 where $w_4 \geq w_3$ such that

$$S(\overline{C}(w_4)) \subset \overline{C}(w_4). \quad (5.5)$$

Suppose that (H2)(ii) holds. Then, by Lemma 5.3 we have (5.5) immediately when we pick $w_4 = m$. Suppose now that (H2)(i) is satisfied. Then, there exist $T > 0$ and $\epsilon < \frac{1}{q}$ such that

$$\frac{f(x)}{x} < \epsilon, \quad x > T. \quad (5.6)$$

Define $M = \max_{x \in \mathbb{R}[0, T]} f(x)$. In view of (5.6), it is clear that

$$f(x) \leq M + \epsilon x, \quad x \geq 0. \quad (5.7)$$

Now, let w_4 be such that

$$w_4 > \max \left\{ w_3, M \left(\frac{1}{q} - \epsilon \right)^{-1} \right\}. \quad (5.8)$$

For $y \in \overline{C}(w_4)$ and $t \in [a, \sigma^n(b)]$, we use (5.2), Theorem 2.7, (5.7), (5.3) and (5.8) to get

$$\begin{aligned} Sy(t) &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) (M + \epsilon w_4) \Delta s \\ &= q(M + \epsilon w_4) < q \left[w_4 \left(\frac{1}{q} - \epsilon \right) + \epsilon w_4 \right] = w_4. \end{aligned}$$

Hence, $\|Sy\| < w_4$ and so $Sy \in C(w_4) \subset \overline{C}(w_4)$. Thus, (5.5) follows immediately.

Let $\psi : C \rightarrow \mathbb{R}[0, \infty)$ be defined by

$$\psi(y) = \min_{t \in [c, d]} y(t). \quad (5.9)$$

Clearly, ψ is a nonnegative continuous concave functional on C and $\psi(y) \leq \|y\|$ for all $y \in C$.

We shall verify that condition (a) of Theorem 1.3 is satisfied. For this, note that

$$\{y \in C(\psi, w_2, w_3) \mid \psi(y) > w_2\} \neq \emptyset$$

since $y(t) \equiv \frac{1}{2}(w_2 + w_3)$ is a member of the set. Next, let $y \in C(\psi, w_2, w_3)$. Then, $w_2 \leq \psi(y) \leq \|y\| \leq w_3$ and so we have

$$y(s) \in \mathbb{R}[w_2, w_3], \quad s \in [c, d]$$

which implies

$$y(\sigma^{n-1}(s)) \in \mathbb{R}[w_2, w_3], \quad s \in [c, \rho^{n-1}(d)].$$

Using this together with (5.2), Remark 2.2, (H3), (C5) and (5.3), we get

$$\begin{aligned} \psi(Sy) &= \min_{t \in [c, d]} Sy(t) \geq \min_{t \in [c, d]} \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &= \int_a^{\sigma(b)} (-1)^{p+1} G(c, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\geq \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(c, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &> \int_c^{\rho^{n-1}(d)} (-1)^{p+1} G(c, s) u(s) \frac{w_2}{\tau} \Delta s = w_2. \end{aligned}$$

Thus, we have shown that $\psi(Sy) > w_2$ for all $y \in C(\psi, w_2, w_3)$ and hence condition (a) of Theorem 1.3 holds.

Next, we note from Lemma 5.3 and condition (H1) that

$$S(\overline{C}(w_1)) \subseteq C(w_1) \subset \overline{C}(w_1). \quad (5.10)$$

Hence, condition (b) of Theorem 1.3 is met.

Finally, to show that condition (c) of Theorem 1.3 holds, let $y \in C(\psi, w_2, w_4)$ with $\|Sy\| > w_3$. Applying (5.2), Theorem 2.8, (C3) and (5.3), we find

$$\begin{aligned} \psi(Sy) &= \min_{t \in [c, d]} Sy(t) \geq \min_{t \in [c, d]} \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\geq \int_a^{\sigma(b)} k(-1)^{p+1} G(\sigma^n(b), s) \eta v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\geq \theta \|Sy\| > \theta w_3 \geq \theta \frac{w_2}{\theta} = w_2. \end{aligned}$$

Hence, $\psi(Sy) > w_2$ for all $y \in C(\psi, w_2, w_4)$ with $\|Sy\| > w_3$.

Consequently, it follows from Theorem 1.3 that the boundary value problem (1.1)', (1.2) has (at least) three positive solutions $y_1, y_2, y_3 \in \overline{C}(w_4)$ that satisfy (5.4). $\square$

We shall now employ the five-functional fixed point theorem (Theorem 1.4) to give other existence criteria. In applying Theorem 1.4 it is possible to choose the functionals and constants in many different ways. We shall present two results to show the arguments involved. In particular the first result is a generalization of Theorem 5.4.

Theorem 5.5. Assume there exist $\xi_1, \xi_2, \xi_3, \xi_4 \in T$ with

$$a \leq \xi_1 \leq c \leq \xi_2 < \xi_3 \leq d \leq \xi_4 \leq \sigma^n(b),$$

$$\xi_1 < \rho^{n-1}(\xi_4), \quad \xi_2 < \rho^{n-1}(\xi_3) \quad \text{and} \quad \rho^{n-1}(\xi_4) \leq \sigma(b).$$

Let (C1)–(C3), (C6) and (C7) hold. Suppose there exist $w_i \in \mathbb{R}^+$, $2 \leq i \leq 5$ where

$$0 < w_2 < w_3 < \frac{w_3}{\theta} \leq w_4 \leq w_5 \quad \text{and} \quad w_2 > \frac{w_5 d_3}{q}$$

such that the following conditions are satisfied:

$$(H4) \quad f(x) < \frac{1}{d_2} \left(w_2 - \frac{w_5 d_3}{q} \right) \text{ for } x \in \mathbb{R}[0, w_2];$$

$$(H5) \quad f(x) \leq \frac{w_5}{q} \text{ for } x \in \mathbb{R}[0, w_5];$$

$$(H6) \quad f(x) > \frac{w_3}{d_1} \text{ for } x \in \mathbb{R}[w_3, w_4].$$

Then, the boundary value problem (1.1)', (1.2) has (at least) three positive solutions $y_1, y_2, y_3 \in \overline{C}(w_5)$ such that

$$\begin{aligned} y_1(t) < w_2, \quad t \in [\xi_1, \xi_4]; \quad y_2(t) > w_3, \quad t \in [\xi_2, \xi_3]; \\ \max_{t \in [\xi_1, \xi_4]} y_3(t) > w_2 \quad \text{and} \quad \min_{t \in [\xi_2, \xi_3]} y_3(t) < w_3. \end{aligned} \quad (5.11)$$

Proof: Theorem 1.4 will be applied. We begin by defining the following functionals on C :

$$\begin{aligned} \psi(y) &= \min_{t \in [c, d]} y(t), \\ \beta(y) &= \Theta(y) = \max_{t \in [\xi_1, \xi_4]} y(t), \\ \alpha(y) &= \min_{t \in [\xi_2, \xi_3]} y(t), \\ \gamma(y) &= \|y\|. \end{aligned} \quad (5.12)$$

First, we shall show that the operator S maps $\overline{P}(\gamma, w_5)$ into $\overline{P}(\gamma, w_5)$. Let $y \in \overline{P}(\gamma, w_5)$. Thus, we have $y \in \mathbb{R}[0, w_5]$. Using (5.2), Theorem 2.7, (H5) and (5.3) gives for $t \in [a, \sigma^n(b)]$,

$$\begin{aligned} |Sy(t)| = Sy(t) &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) \frac{w_5}{q} \Delta s = q \frac{w_5}{q} = w_5. \end{aligned}$$

Hence,

$$\gamma(Sy) = \|Sy\| \leq w_5.$$

From Lemma 5.2, we already have $Sy \in C$, thus it follows that $Sy \in \overline{P}(\gamma, w_5)$. Hence, we have shown that $S : \overline{P}(\gamma, w_5) \rightarrow \overline{P}(\gamma, w_5)$.

We next prove that condition (a) of Theorem 1.4 holds. We observe that

$$\{y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(y) > w_3\} \neq \emptyset$$

since $y(t) \equiv \frac{1}{2}(w_3 + w_4)$ is an element of the set. Let $y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$. Then, $\alpha(y) \geq w_3$ and $\Theta(y) \leq w_4$ imply that

$$y(s) \in \mathbb{R}[w_3, w_4], \quad s \in [\xi_2, \xi_3]$$

and so

$$y(\sigma^{n-1}(s)) \in \mathbb{R}[w_3, w_4], \quad s \in [\xi_2, \rho^{n-1}(\xi_3)].$$

Using this together with (5.2), Remark 2.2, (C6), (H6) and (5.3), we get

$$\begin{aligned} \alpha(Sy) &= \min_{t \in [\xi_2, \xi_3]} Sy(t) \geq \min_{t \in [\xi_2, \xi_3]} \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &= \int_a^{\sigma(b)} (-1)^{p+1} G(\xi_2, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\geq \int_{\xi_2}^{\rho^{n-1}(\xi_3)} (-1)^{p+1} G(\xi_2, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &> \int_{\xi_2}^{\rho^{n-1}(\xi_3)} (-1)^{p+1} G(\xi_2, s) u(s) \frac{w_3}{d_1} \Delta s = d_1 \frac{w_3}{d_1} = w_3. \end{aligned}$$

Hence, $\alpha(Sy) > w_3$ for all $y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$ and so condition (a) of Theorem 1.4 holds.

Next, we verify that condition (b) of Theorem 1.4 is satisfied. Let w_1 be such that $0 < w_1 < w_2$. Note that

$$\{y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(y) < w_2\} \neq \emptyset$$

since $y(t) \equiv \frac{1}{2}(w_1 + w_2)$ is an element. Let $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$. Then, we have $\beta(y) \leq w_2$ and $\gamma(y) \leq w_5$ which lead to

$$y(s) \in \mathbb{R}[0, w_2], \quad s \in [\xi_1, \xi_4] \quad \text{and} \quad y(s) \in \mathbb{R}[0, w_5], \quad s \in [a, \sigma^n(b)]$$

and subsequently

$$y(\sigma^{n-1}(s)) \in \mathbb{R}[0, w_2], \quad s \in [\xi_1, \rho^{n-1}(\xi_4)] \quad \text{and} \quad y(\sigma^{n-1}(s)) \in \mathbb{R}[0, w_5], \quad s \in [a, \sigma(b)]. \quad (5.13)$$

Then, an application of (5.2), Remark 2.2, (C7), (H4), (H5) and (5.3) yield

$$\begin{aligned} \beta(Sy) &= \max_{t \in [\xi_1, \xi_4]} Sy(t) \leq \max_{t \in [\xi_1, \xi_4]} \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &= \int_a^{\sigma(b)} (-1)^{p+1} G(\xi_4, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &= \int_{\xi_1}^{\rho^{n-1}(\xi_4)} (-1)^{p+1} G(\xi_4, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\quad + \int_a^{\xi_1} (-1)^{p+1} G(\xi_4, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &\quad + \int_{\rho^{n-1}(\xi_4)}^{\sigma(b)} (-1)^{p+1} G(\xi_4, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\ &< \int_{\xi_1}^{\rho^{n-1}(\xi_4)} (-1)^{p+1} G(\xi_4, s) v(s) \frac{1}{d_2} \left(w_2 - \frac{w_5 d_3}{q} \right) \Delta s \\ &\quad + \left[\int_a^{\xi_1} (-1)^{p+1} G(\xi_4, s) v(s) \Delta s + \int_{\rho^{n-1}(\xi_4)}^{\sigma(b)} (-1)^{p+1} G(\xi_4, s) v(s) \Delta s \right] \frac{w_5}{q} \\ &= d_2 \frac{1}{d_2} \left(w_2 - \frac{w_5 d_3}{q} \right) + d_3 \frac{w_5}{q} = w_2. \end{aligned}$$

Hence, $\beta(Sy) < w_2$ for all $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$. Thus, condition (b) of Theorem 1.4 is fulfilled.

We shall next show that condition (c) of Theorem 1.4 is met. Observe that, by (5.2) and Theorem 2.7, we have for $y \in C$,

$$\Theta(Sy) = \max_{t \in [\xi_1, \xi_4]} Sy(t) \leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s. \quad (5.14)$$

Moreover, using (5.2), Theorem 2.8 and (C3), we get for $y \in C$,

$$\begin{aligned}
 \alpha(Sy) &= \min_{t \in [\xi_2, \xi_3]} Sy(t) \geq \min_{t \in [\xi_2, \xi_3]} \int_a^{\sigma(b)} (-1)^{p+1} G(t, s) u(s) f(y(\sigma^{n-1}(s))) \Delta s \\
 &\geq \int_a^{\sigma(b)} k(-1)^{p+1} G(\sigma^n(b), s) \eta v(s) f(y(\sigma^{n-1}(s))) \Delta s \\
 &= \theta \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s.
 \end{aligned} \tag{5.15}$$

A combination of (5.14) and (5.15) gives

$$\alpha(Sy) \geq \theta \Theta(Sy), \quad y \in C. \tag{5.16}$$

Let $y \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Sy) > w_4$. Then, it follows from (5.16) that

$$\alpha(Sy) \geq \theta \Theta(Sy) > \theta w_4 \geq \theta \frac{w_3}{\theta} = w_3.$$

Thus, $\alpha(Sy) > w_3$ for all $y \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Sy) > w_4$. Hence, condition (c) of Theorem 1.4 holds.

Finally, we shall prove that condition (d) of Theorem 1.4 is fulfilled. Let $y \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Sy) < w_1$. Then, we have $\beta(y) \leq w_2$ and $\gamma(y) \leq w_5$ which give (5.13). Using a similar argument as in proving condition (b) of Theorem 1.4, we obtain $\beta(Sy) < w_2$. Thus, condition (d) of Theorem 1.4 is satisfied.

Now, an application of Theorem 1.4 ensures that the boundary value problem (1.1)', (1.2) has (at least) three positive solutions $y_1, y_2, y_3 \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$ such that

$$\beta(y_1) < w_2, \quad \alpha(y_2) > w_3, \quad \text{and} \quad \beta(y_3) > w_2 \quad \text{with} \quad \alpha(y_3) < w_3,$$

which in turn reduce to (5.11) immediately. $\square$

Now, let us consider the special case when

$$\xi_1 = a, \quad \xi_2 = c, \quad \xi_3 = d \quad \text{and} \quad \xi_4 = \sigma^n(b).$$

Then,

$$d_1 = \tau, \quad d_2 = q \quad \text{and} \quad d_3 = 0.$$

In this case Theorem 5.5 yields the following corollary.

Corollary 5.6. Let (C1)–(C3) hold, and also assume the following:

$$(C6)' \quad G(c, s)u(s) \text{ is nonzero for some } s \in [c, \rho^{n-1}(d)];$$

$$(C7)' \quad G(\sigma^n(b), s)v(s) \text{ is nonzero for some } s \in [a, \sigma(b)].$$

Suppose there exist $w_i \in \mathbb{R}^+$, $2 \leq i \leq 5$ where

$$0 < w_2 < w_3 < \frac{w_3}{\theta} \leq w_4 \leq w_5$$

such that the following conditions are satisfied:

$$(H4)' \quad f(x) < \frac{w_2}{q} \text{ for } x \in \mathbb{R}[0, w_2];$$

$$(H5)' \quad f(x) \leq \frac{w_5}{q} \text{ for } x \in \mathbb{R}[0, w_5];$$

$$(H6)' \quad f(x) > \frac{w_3}{\tau} \text{ for } x \in \mathbb{R}[w_3, w_4].$$

Then, the boundary value problem (1.1)', (1.2) has (at least) three positive solutions $y_1, y_2, y_3 \in \overline{C}(w_5)$ such that

$$\|y_1\| < w_2; \quad y_2(t) > w_3, \quad t \in [c, d]; \quad \|y_3\| > w_2 \quad \text{and} \quad \min_{t \in [c, d]} y_3(t) < w_3. \quad \square$$

Remark 5.1.

- (a) Corollary 5.6 is actually Theorem 5.4. Hence, Theorem 5.5 is more general than Theorem 5.4. This also shows that the five-functional fixed point theorem (Theorem 1.4), which is used to obtain Theorem 5.5, generalizes Leggett-Williams' fixed point theorem (Theorem 1.3), which is the main tool for Theorem 5.4.

- (b) The proof of Theorem 5.4 is definitely simpler than that of Theorem 5.5. Indeed, in the proof of Theorem 5.4 there is only one functional to be determined as compared to five functionals in Theorem 5.5, thus the arguments used in Theorem 5.5 are different from those in Theorem 5.4 and are much more complicated.
- (c) The existence criteria obtained in Theorem 5.4 are relatively simple to check as compared to those in Theorem 5.5.
- (d) Leggett-Williams' fixed point theorem is well known in the literature, possibly because of the ease to apply and also it produces easily verifiable criteria, in fact till today many authors are still finding new applications of this theorem, the reader is referred to the very recent work of [5, 6, 7, 41, 43, 47, 65]. Five-functional fixed point theorem, though more general, needs greater skill to apply and the results obtained are more difficult to check. Consequently, it is not as popular as Leggett-Williams' fixed point theorem. Still, a number of work has made good use of this theorem, see [13, 20, 23, 25].

The next result illustrates another application of Theorem 1.4.

Theorem 5.7. Assume there exist $\xi_1, \xi_2, \xi_3, \xi_4 \in T$ with

$$c \leq \xi_1 \leq \xi_2 < \xi_3 \leq \xi_4 \leq d,$$

$$\xi_1 < \rho^{n-1}(\xi_4), \quad \xi_2 < \rho^{n-1}(\xi_3), \quad \text{and} \quad \rho^{n-1}(\xi_4) \leq \sigma(b).$$

Let (C1)–(C4), (C6) and (C7) hold. Suppose there exist $w_i \in \mathbb{R}^+$, $1 \leq i \leq 5$ where

$$0 < w_1 \leq w_2\theta < w_2 < w_3 < \frac{w_3}{\theta} \leq w_4 \leq w_5 \quad \text{and} \quad w_2 > \frac{w_5 d_3}{q}$$

such that (H5), (H6) hold and also

$$(H7) \quad f(x) < \frac{1}{d_2} \left(w_2 - \frac{w_5 d_3}{q} \right) \text{ for } x \in \mathbb{R}[w_1, w_2].$$

Then, the boundary value problem (1.1)', (1.2) has (at least) three positive solutions $y_1, y_2, y_3 \in \overline{C}(w_5)$ such that

$$\begin{aligned} y_1(t) < w_2, \quad t \in [\xi_1, \xi_4]; \quad y_2(t) > w_3, \quad t \in [\xi_2, \xi_3]; \\ \max_{t \in [\xi_1, \xi_4]} y_3(t) > w_2 \quad \text{and} \quad \min_{t \in [\xi_2, \xi_3]} y_3(t) < w_3. \end{aligned} \quad (5.17)$$

Proof: In the context of Theorem 1.4, we define the following functionals on C :

$$\begin{aligned} \psi(y) &= \min_{t \in [\xi_1, \xi_4]} y(t), \\ \beta(y) &= \max_{t \in [\xi_1, \xi_4]} y(t), \\ \Theta(y) &= \max_{t \in [\xi_2, \xi_3]} y(t), \\ \alpha(y) &= \min_{t \in [\xi_2, \xi_3]} y(t), \\ \gamma(y) &= \|y\|. \end{aligned} \quad (5.18)$$

Using a similar argument as in the proof of Theorem 5.5, we can show that the operator S maps $\overline{P}(\gamma, w_5)$ into $\overline{P}(\gamma, w_5)$, and also condition (a) of Theorem 1.4 is satisfied.

We now verify that condition (b) of Theorem 1.4 holds. Just as before, we note that

$$\{y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(y) < w_2\} \neq \emptyset$$

since $y(t) \equiv \frac{1}{2}(w_1 + w_2)$ is an element. Let $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$, then $\psi(y) \geq w_1$, $\beta(y) \leq w_2$ and $\gamma(y) \leq w_5$ lead to

$$y(s) \in \mathbb{R}[w_1, w_2], \quad s \in [\xi_1, \xi_4] \quad \text{and} \quad y(s) \in \mathbb{R}[0, w_5], \quad s \in [a, \sigma^n(b)]$$

which imply

$$y(\sigma^{n-1}(s)) \in \mathbb{R}[w_1, w_2], \quad s \in [\xi_1, \rho^{n-1}(\xi_4)] \quad \text{and} \quad y(\sigma^{n-1}(s)) \in \mathbb{R}[0, w_5], \quad s \in [a, \sigma(b)].$$

Then, using (5.2), Remark 2.2, (C7), (H7), (H5) and (5.3), we find

$$\begin{aligned}
 \beta(Sy) &\leq \int_a^{\sigma(b)} (-1)^{p+1} G(\xi_4, s) v(s) f(y(\sigma^{n-1}(s))) \Delta s \\
 &< \int_{\xi_1}^{\rho^{n-1}(\xi_4)} (-1)^{p+1} G(\xi_4, s) v(s) \frac{1}{d_2} \left(w_2 - \frac{w_5 d_3}{q} \right) \Delta s \\
 &\quad + \left[\int_a^{\xi_1} (-1)^{p+1} G(\xi_4, s) v(s) \Delta s + \int_{\rho^{n-1}(\xi_4)}^{\sigma(b)} (-1)^{p+1} G(\xi_4, s) v(s) \Delta s \right] \frac{w_5}{q} \\
 &= d_2 \frac{1}{d_2} \left(w_2 - \frac{w_5 d_3}{q} \right) + d_3 \frac{w_5}{q} = w_2.
 \end{aligned}$$

Thus, $\beta(Sy) < w_2$ for all $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$ and condition (b) of Theorem 1.4 is satisfied.

Next, we shall show that condition (c) of Theorem 1.4 is met. By (5.2) and Theorem 2.7, we obtain for $y \in C$,

$$\Theta(Sy) = \max_{t \in [\xi_2, \xi_3]} Sy(t) \leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s. \quad (5.19)$$

Using a previous argument we also have (5.15) which, when combines with (5.19), gives (5.16). The rest of the proof is similar to that of Theorem 5.5.

Finally, we shall check that condition (d) of Theorem 1.4 is satisfied. By (5.2) and Theorem 2.7, we see that for $y \in C$,

$$\beta(Sy) = \max_{t \in [\xi_1, \xi_4]} Sy(t) \leq \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s. \quad (5.20)$$

On the other hand, it follows from (5.2), Theorem 2.8 and (C3) that for $y \in C$,

$$\psi(Sy) = \min_{t \in [\xi_1, \xi_4]} Sy(t) \geq \theta \int_a^{\sigma(b)} (-1)^{p+1} G(\sigma^n(b), s) v(s) f(y(\sigma^{n-1}(s))) \Delta s. \quad (5.21)$$

A combination of (5.20) and (5.21) gives

$$\psi(Sy) \geq \theta \beta(Sy), \quad y \in C. \quad (5.22)$$

Let $y \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Sy) < w_1$. Then, (5.22) leads to

$$\beta(Sy) \leq \frac{1}{\theta} \psi(Sy) < \frac{1}{\theta} w_1 \leq \frac{1}{\theta} w_2 \theta = w_2.$$

Thus, $\beta(Sy) < w_2$ for all $y \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Sy) < w_1$. Hence, condition (d) of Theorem 1.4 is fulfilled.

It now follows from Theorem 1.4 that the boundary value problem (1.1)', (1.2) has (at least) three positive solutions $y_1, y_2, y_3 \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$ such that

$$\beta(y_1) < w_2, \quad \alpha(y_2) > w_3, \quad \text{and} \quad \beta(y_3) > w_2 \quad \text{with} \quad \alpha(y_3) < w_3,$$

which in turn reduce to (5.17) immediately. $\square$

5.2 Examples

In this section we shall provide examples to illustrate the usefulness of the results obtained in the previous section.

Example 5.8. Let $T = \mathbb{R}[0, 1) \cup \mathbb{Z}^+$. Consider the boundary value problem

$$\begin{cases} y^{\Delta^3}(t) = \frac{-6f(y(\sigma^2(t)))}{1 + |\sin[(\sigma^2(t))^2(30 - \sigma^2(t))]|}, & t \in [0, 8] \\ y(0) = y^{\Delta}(0) = y^{\Delta^2}(\sigma(8)) = 0 \end{cases} \quad (5.23)$$

where

$$f(y) = \begin{cases} 1 + |\sin y|, & y \leq 3000 \\ e^{y-3000} + |\sin y|, & 3000 \leq y \leq 25252 \\ e^{22252} + |\sin 25252|, & y \geq 25252. \end{cases}$$

Here, $n = 3$ and $p = 2$. We shall check the conditions of Theorem 5.4. Pick

$$u(t) = v(t) = \frac{6}{1 + |\sin[(\sigma^2(t))^2(30 - \sigma^2(t))]|}.$$

By Theorem 2.5, we note that

$$G(t, s) = \begin{cases} -\int_0^t x \Delta x, & t \leq \sigma(s) \\ \sigma(s)[\sigma(s) - t] - \int_0^{\sigma(s)} x \Delta x, & t \geq \sigma(s). \end{cases}$$

A direct computation gives

$$\begin{aligned} q &= \int_0^{\sigma(8)} -G(\sigma^3(8), s)v(s)\Delta s \\ &= -3 \int_0^1 \frac{s^2 - 22s}{1 + |\sin[s^2(30 - s)]|} ds - 3 \sum_{s=1}^8 \frac{s^2 - 19s - 21}{1 + |\sin[(s + 2)^2(28 - s)]|} = 1249. \end{aligned}$$

Next, we fix $\delta = 0.3$. It follows from (2.2) that $c = 4$, $d = 7$ and Theorem 2.8 yields

$$\begin{aligned} k &= \inf_{s \in [0, 8]} \frac{G(4, s)}{G(11, s)} \\ &= \min \left\{ \inf_{\sigma(s) \in [0, 1]} \frac{s^2 - 8s}{s^2 - 22s}, \inf_{\sigma(s) \in [1, 4]} \frac{s^2 - 5s - 7}{s^2 - 19s - 21}, \inf_{\sigma(s) \in [4, 9]} \frac{-13}{s^2 - 19s - 21} \right\} \\ &= 0.119. \end{aligned}$$

Also, from (5.3) we get

$$\tau = \int_4^5 -G(4, s)u(s)\Delta s = \frac{13}{2} u(4) = 36.7.$$

It is clear that the conditions (C1)–(C5) are satisfied with $\eta = 1$. Now, choose $w_1 = 2995$, then we see that for $x \in [0, w_1]$,

$$f(x) \leq 2 < \frac{w_1}{q} = 2.40.$$

Thus, (H1) holds. Next, since

$$\limsup_{x \rightarrow \infty} \frac{f(x)}{x} = \limsup_{x \rightarrow \infty} \frac{e^{22252} + |\sin 25252|}{x} = 0,$$

condition (H2)(i) is satisfied. Finally, we shall look for some w_2, w_3 ($w_1 < w_2 < \frac{w_2}{\theta} \leq w_3$) such that condition (H3) holds. Let $w_2 = 3005$ and $w_3 = \frac{w_2}{k} = 25252$.

Then, for $x \in \mathbb{R}[w_2, w_3]$,

$$f(x) \geq e^{3005-3000} > \frac{w_2}{\tau} = 81.9$$

and thus (H3) holds. It now follows from Theorem 5.4 that the boundary value problem (5.23) has (at least) three positive solutions y_1, y_2 and y_3 such that

$$\begin{aligned} \|y_1\| &< 2995; & y_2(t) &> 3005, \quad t \in [4, 7]; \\ \|y_3\| &> 2995 \quad \text{and} \quad \min_{t \in [4, 7]} y_3(t) &< 3005. \end{aligned}$$

Indeed, a known positive solution is $y_1(t) = t^2(30 - t)$ and $\|y_1\| = 1408$ is within the ranges given above.

Example 5.9. Let $T = \mathbb{R}[0, 1) \cup \mathbb{Z}^+$. Consider the boundary value problem (1.1)', (1.2) with $n = 2$, $p = 1$, $a = 0$, $b = 9$ and the term

$$F(t, y(\sigma^2(t))) = f(y(\sigma^2(t))) = \begin{cases} \frac{1}{2d_2} \left(w_2 - \frac{w_5 d_3}{q} \right), & 0 \leq y \leq w_2 \\ l(y), & w_2 \leq y \leq w_3 \\ \frac{w_3}{2} \left(\frac{1}{\tau} + \frac{1}{d_1} \right), & y \geq w_3 \end{cases} \quad (5.24)$$

where $l(y)$ satisfies

$$\frac{d^2}{dy^2} l(y) = 0, \quad l(w_2) = \frac{1}{2d_2} \left(w_2 - \frac{w_5 d_3}{q} \right), \quad l(w_3) = \frac{w_3}{2} \left(\frac{1}{\tau} + \frac{1}{d_1} \right) \quad (5.25)$$

and w_i , $i = 2, 3, 5$ are as in the context of Theorem 5.5.

We shall try to apply Theorem 5.5. Let $\delta = 0.15$, then $c = 2$ and $d = 9$. Further, let $\xi_1 = 1$, $\xi_2 = 3$, $\xi_3 = 9$, $\xi_4 = 10$, and the functions $u = v \equiv 1$ (hence $\eta = 1$). By Theorem 2.5, we get $G(t, s)$ as in Example 4.8 and hence a direct computation yields

$$k = \theta = 0.2, \quad q = 54.5, \quad \tau = 12, \quad d_1 = 15, \quad d_2 = 44, \quad d_3 = 10.5.$$

Thus, the w_i 's are assumed to satisfy the following relation:

$$0 < w_2 < w_3 < \frac{w_3}{\theta} = 5w_3 \leq w_4 \leq w_5 < \frac{q}{d_3} w_2 = 5.19w_2. \quad (5.26)$$

It is clear that the conditions (C1)–(C3), (C6), (C7) and (H4) are satisfied. Next, since

$$\tau < d_1 < d_2 \quad \text{and} \quad \frac{w_3}{\tau} < \frac{w_5}{q} \quad \left(\text{i.e., } w_5 > \frac{q}{r} w_3 = 4.54 w_3 \right),$$

we find that for $0 \leq y \leq w_5$,

$$f(y) \leq \max\{l(w_2), l(w_3)\} = l(w_3) = \frac{1}{2} \left(\frac{w_3}{\tau} + \frac{w_3}{d_1} \right) < \frac{1}{2} \left(\frac{w_3}{\tau} + \frac{w_3}{\tau} \right) = \frac{w_3}{\tau} < \frac{w_5}{q}.$$

Thus, condition (H5) holds. Finally, we have for $w_3 \leq y \leq w_4$,

$$f(y) = \frac{1}{2} \left(\frac{w_3}{\tau} + \frac{w_3}{d_1} \right) > \frac{1}{2} \left(\frac{w_3}{d_1} + \frac{w_3}{d_1} \right) = \frac{w_3}{d_1},$$

and hence condition (H6) is fulfilled.

It now follows from Theorem 5.5 that the boundary value problem (1.1)', (1.2) with (5.24)–(5.26) has (at least) three positive solutions $y_1, y_2, y_3 \in \overline{C}(w_5)$ such that

$$y_1(t) < w_2, \quad t \in [1, 10]; \quad y_2(t) > w_3, \quad t \in [3, 9]; \quad (5.27)$$

$$\max_{t \in [1, 10]} y_3(t) > w_2 \quad \text{and} \quad \min_{t \in [3, 9]} y_3(t) < w_3.$$

In fact, we note that two positive solutions y_1, y_2 are given by

$$y_i(t) = c_i (10t - h_2(t, 0)), \quad i = 1, 2$$

where $c_1 = \frac{1}{2d_2} \left(w_2 - \frac{w_5 d_3}{q} \right)$ and $c_2 = \frac{w_3}{2} \left(\frac{1}{\tau} + \frac{1}{d_1} \right)$. As an example, we fix

$$w_2 = 1, \quad w_3 = 1.03 \quad \text{and} \quad w_5 = 5.$$

so that (5.26) is satisfied. Then, it can be easily checked that

$$\|y_1\| < 5, \quad 3.96 \times 10^{-3} \leq y_1(t) \leq 2.27 \times 10^{-2} < w_2 \quad \text{for } t \in [1, 10],$$

and

$$\|y_2\| < 5, \quad w_3 < 2.05 \leq y_2(t) \leq 4.13 \quad \text{for } t \in [3, 9].$$

Hence, y_1 and y_2 do exhibit what (5.27) has predicted.

Remark 5.2. In Example 5.9, we note that for $w_3 \leq y \leq w_4$,

$$f(y) = \frac{w_3}{2} \left(\frac{1}{\tau} + \frac{1}{d_1} \right) < \frac{w_3}{2} \left(\frac{1}{\tau} + \frac{1}{\tau} \right) = \frac{w_3}{\tau}.$$

Hence, condition (H6)' of Corollary 5.6 does not hold. Recall from Remark 5.1(a) that Corollary 5.6 is actually Theorem 5.4, Example 5.9 illustrates a case when Theorem 5.5 is applicable but not Theorem 5.4. This shows that Theorem 5.5 is indeed a more general result than Theorem 5.4.

Author's Publication

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