Electro-elastic analysis of cracks in piezoelectric solids

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Electro-elastic Analysis of Cracks in Piezoelectric Solids

by

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Abstract

In this thesis, several important classes of electro-elastic crack problems involving an arbitrary number of arbitrarily orientated planar cracks are solved. In all the problems considered here, the relevant boundary conditions on the cracks are formulated in terms of a system of hypersingular integral equations which are solved by using an accurate collocation technique. The problems under consideration are as follows.

Firstly, the electro-elastostatic interaction of multiple planar cracks in an infinitely long piezoelectric strip is considered. The cracks are acted upon by suitably prescribed internal stresses and are electrically either permeable or impermeable. To solve the problem, a Green's function which satisfies the conditions of vanishing traction and normal electric displacement on the edges of the strip is first derived using a Fourier transform technique. It is then used to derive an explicit semi-analytic solution for the electro-elastostatic fields around the cracks in the strip. The solution is expressed in terms of integrals over lines representing the cracks. The jumps in the displacement and electric potential across opposite crack faces are the unknown functions in the integrals. They are to be determined by solving a system of hypersingular integral equations which are derived from the boundary conditions on the cracks.

Secondly, special electro-elastostatic Green's functions satisfying stress-free and electrically either permeable or impermeable conditions on multiple planar cracks in a piezoelectric domain of infinite extent are constructed numerically. From the bound-

ary conditions on the cracks, the task of constructing the Green's functions requires solving a system of hypersingular integral equations. Once the hypersingular integral equations are solved, explicit formulae can be obtained for computing numerically the Green's functions. The numerical Green's functions are used to develop a simple but accurate boundary element method for analyzing numerically multiple planar cracks in a piezoelectric solid of finite extent. As the singular behaviors of the stress and electric displacement are analytically built into the Green's functions, the boundary element procedure does not require the crack faces to be discretized into boundary elements. Furthermore, the relevant crack tip stress and electric displacement intensity factors can be extracted very accurately.

Thirdly, the numerical Green's function boundary element approach for electrically impermeable cracks is adapted to deal with electro-elastostatic plane problem involving electrically semi-permeable cracks in a piezoelectric solid of finite extent. An iterative procedure for treating the nonlinear boundary conditions on the semi-permeable cracks is proposed.

Lastly, a semi-analytic method of solution is successfully developed for analyzing the multiple cracks in an infinite piezoelectric domain under dynamic loading. The Laplace transform technique is employed to suppress the second order time derivatives of the displacement in the governing partial differential equations. The displacement and electric potential in the Laplace transform domain are then expressed using suitably constructed exponential Fourier transform representations. The unknown functions in the Fourier transform representations are directly related to the jumps in the Laplace transforms of the displacement and electric potential across opposite crack

faces and are determined by solving a system of hypersingular integral equations. Once the displacement and electric potential are determined in the Laplace transform domain, they can be recovered in the physical time domain by using a numerical formula for inverting Laplace transforms.

For some specific cases of the problems under consideration, the computed crack tip intensity factors are compared with those published in the literature. Some new results are also obtained for certain configurations of the planar cracks.

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List of Symbols

c_{ijkl}	-	elastic moduli of the material
e_{ijk}	-	piezoelectric coefficients of the material
i	-	represents $\sqrt{-1}$
m_i	-	x_i component of the unit vector normal to the crack
u_i	-	displacement in x_i direction
Δu_i	-	x_i component of crack opening displacement
x_1, x_2, x_3	-	Cartesian coordinates
$\mathcal{C}$	-	denotes Cauchy principal value integral
E_i	-	electric field vector
D_i	-	x_i component of the electric displacement vector
$\mathcal{H}$	-	denotes Hadamard finite part integral
$H(x)$	-	Heaviside unit step function
K_I	-	mode I stress intensity factor
K_{II}	-	mode II stress intensity factor
K_{III}	-	mode III stress intensity factor
K_{IV}	-	electric displacement intensity factor
$U^{(j)}(x)$	-	j^{th} order Chebyshev polynomial of second kind
$\gamma^{(k)}$	-	k^{th} crack
γ_{ij}	-	strain tensor
δ_{ik}	-	Kronecker delta function
$\delta(x_1, x_2)$	-	Dirac delta function
ϵ_c	-	permittivity of the medium filling the cracks
$\theta^{(n)}$	-	inclination of the crack from the vertical axis
κ_{jp}	-	dielectric coefficients of the materials
ρ	-	density of the material
σ_{ij}	-	stress tensor
ϕ	-	electric potential
$\Delta\phi$	-	jump in electric potential across the crack faces

Chapter 1

General Introduction

1.1 Motivation

Piezoelectricity was discovered by Pierre and Paul-Jacques Curie in 1880 (Katzir [42] and Schwartz [74]). Electric charges can be generated within a piezoelectric material by applying mechanical pressure on it. Conversely, the material can be mechanically deformed by setting up an electric field in it. Many natural materials such as quartz, wood and human bone are piezoelectric due to the polarization of their crystals. Nevertheless, as the crystals are orientated in a rather arbitrary manner, the alignments of the electric dipoles in such materials result in only a weak coupling between the electric and strain fields. In general, such naturally occurring materials show only weak piezoelectric effects and are not suitable for use in engineering applications.

A synthetic piezoelectric material is manufactured by heating it to a high temperature, exposing it to a strong electric field to unify the polarization direction of the crystals and then cooling it to room temperature. The curing process greatly enhances the piezoelectric property of the material, making it technologically useful.

Paul Langevin's famous invention of a sonar device in 1920 for detecting submarines is among the early applications of piezoelectric effects. Since then, piezoelectric materials have found numerous applications in modern engineering. They are used to generate electric charges in applications such as acoustic pressure sensors, gas igniters, vibration sensors, accelerometers, and hydrophones. Other examples of ap-

plications include piezoelectric motors, piezoelectrically driven relays, piezoelectric transformers, ink-jet heads for printers, noise cancellation systems, video cassette recording head trackers, precise positioners, and deformable mirrors for correcting of optical images.

Though synthetic piezoelectric materials are useful in engineering applications, they are very brittle, hence highly vulnerable to fracturing. Their performance in engineering applications can be adversely affected by the presence of cracks. An understanding of the behaviors of cracks under the influence of electromechanical loads is essential for safe and reliable applications of piezoelectric materials in modern technology. Because of this, the electro-elastic analysis of cracks in piezoelectric solids is a research topic of considerable interest among many researchers.

1.2 Some Prior Works on Piezoelectric Cracks

A brief survey of some prior works on the solutions of piezoelectric crack problems is given here.

One of the earliest works on piezoelectric crack problems was carried out by Parton [69] in the 1970s. In this work, the equations of plane electro-elastostatics were solved using a Fourier transform technique for a single straight crack in a piezoelectric material of infinite extent and the critical stress for crack propagation was given.

Research activities on the analyses of piezoelectric crack problems started to gain significant momentum only in the 1990s, especially during the second half of the decade.

In the early 1990s, Sosa and Pak [81] used the method of eigenfunction expansions to investigate the behaviors of the stress and electrical displacement fields near the tip of an electrically impermeable stress-free semi-infinite planar crack in an infinite homogeneous piezoelectric space. The stress and the electric displacement were found to possess the $1/\sqrt{r}$ singularity behavior (where r is the distance of a point from the crack tip). As in the case of elastic materials, the idea of crack tip stress and electric displacement intensity factors may be introduced for piezoelectric media.

Gao and Fan [29] revisited the plane problem of an elliptical hole in an infinite piezoelectric space, which was solved analytically by Sosa [80] using the Lekhnitskii approach [48] together with the method of complex functions. Letting the minor axis of the elliptical hole tend to zero, they used the solution in [80] to analyze the electro-elastic fields around a planar crack. Dascalu and Homentcovschi [23], Xu and Rajapakse [93] and Zhang and Gao [97] had also used a similar complex variable function approach for the analysis of a piezoelectric crack problem.

For mathematical simplicity, many researchers had assumed that the piezoelectric material under consideration is deformed by antiplane shear stress and inplane electrical static loads (see, for example, Yang and Kao [95]). For such a deformation, if the electro-elastic fields are static (independent of time), the governing partial differential equations may be reduced to a pair of equations which essentially take the form of the two-dimensional Laplace's equation. For certain cases involving cracks of specific shapes and orientations in a piezoelectric material with an idealized geometry (such as an infinitely long strip), the resulting boundary value problems are amenable to mathematical treatments. For example, Li [50] and Zhong

and Li [99] had obtained closed form solutions for a pair of electrically permeable collinear cracks in an infinitely long strip, and Zhong and Meguid [100] had solved the problem of a circular arc-crack in an infinite piezoelectric space using the complex variable method. Lee, Lee and Park [47] derived a solution for investigating the interaction between a semi-infinite crack and a screw dislocation in a piezoelectric materials. They calculated the force on the screw dislocation due to the existence of the semi-infinite crack subject to external electromechanical loads.

Problems involving multiple cracks even in a piezoelectric space having an idealized geometry (for example, a full space or a strip) are in general more complicated to solve, if the piezoelectric space is deformed by inplane mechanical and electrical loads. Following the approach of Horii and Nemat-Nasser [39], Han and Chen [36] had used the so called pseudo-traction electric displacement method to model multiple parallel cracks in a transversely isotropic piezoelectric ceramics of infinite extent under inplane mechanical loads. With the aid of Fourier transforms, Zhou and Wu [102] had formulated the problem of two parallel mode-I or four parallel mode-I electrically permeable cracks in a full piezoelectric space in terms of a system of dual integral equations. The dual integral equations can be solved numerically by expanding the unknown functions in terms of series involving the Jacobi polynomials. Using a continuous distribution of dislocations, Han and Wang [37] had modeled cracks in a full piezoelectric space. The unknown weight functions of the dislocations on each crack were determined by solving numerically a system of Cauchy singular integral equations. The dislocational Green's function approach is also used by Chen, Liew and Xiao [14] and Lu, Tan and Liew [61] for modeling piezoelectric cracks.

The papers cited in the preceding paragraph are concerned with electro-elastostatic deformations of cracks. According to Kuna [46], there are comparatively fewer works on piezoelectric cracks that are acted upon by time dependent loads. An example of work on dynamic piezoelectric crack problem is Shindo [75]. In [75], the problem of a single planar crack in an infinite piezoelectric ceramic under normal impact is formulated in terms of a pair of dual integral equations by representing the displacement and electric potential in the Laplace transform domain by suitable Fourier sine and cosine transform representations. The dual integral equations are solved as explained in Sneddon and Lowengrub [78], by reducing them to Fredholm integral equations of the second kind. The dynamic piezoelectric crack problem can also be formulated in terms of hypersingular integral equations using the approach in a recent paper by García-Sánchez, Zhang, Sládek and Sládek [32]. In [32], the kernels of the hypersingular integral formulation contain second order spatial derivatives of a suitable dynamic Green's function for piezoelectric solids. The Green's function is derived using Radon transform. Its evaluation is a rather involved exercise, requiring the computation of a line integral over a unit circle with integrand that is expressed in terms of exponential integrals (Wang and Zhang [91]).

The piezoelectric crack problems in all the works cited above are solved within the framework of the local theory of linear electro-elasticity. In recent years, there are some attempts to solve piezoelectric crack problems in the context of non-local theory of electro-elasticity (see, for example, Zhou, Sun and Wang [101] and Liang [58]). The non-local theory for elasticity is a macroscopic approach for taking into consideration longer range molecular interactions in the elastic deformation of solids

(Eringen and Edelen [27]). The stress and the electric displacement from the non-local theory of electro-elasticity are bounded and do not possess the physically undesirable feature of $1/\sqrt{r}$ singularity at crack tips.

Boundary value problems involving general geometries and boundary conditions have to be solved using numerical methods. The boundary element method is a well established numerical technique for linear fracture analysis of bodies. During the last ten years or so, there has been considerable interest in the development of the boundary element method for electro-elastic crack problems. Examples of papers giving boundary element solutions for cracks in piezoelectric solids include Garcia-Sanchez, Saez and Dominguez [31], Groh and Kuna [33], Pan [66] and Rajapakse and Xu [70].

More references on solutions of relevant electro-elastic crack problems are given at the beginning of Chapters 3, 4, 5 and 6 of this thesis.

1.3 Boundary Conditions on Cracks

In solving piezoelectric crack problems, the choice of the electrical boundary conditions to impose on a crack is an important issue. On one extreme, the crack may be regarded as electrically permeable, that is, the electrical field is assumed to permeate across opposite crack faces with no jump in the electric potential. On the other extreme, the crack may be viewed as electrically insulated so that the normal component of the electric displacement is zero on the crack faces. For an electrically impermeable crack, the electric potential may be discontinuous across opposite crack faces.

The choice of the two extreme conditions has been discussed by various researchers. Dunn [26] questioned the validity of the impermeable assumption, pointing out that it could lead to significant errors in the effects of the electric fields on crack propagation based on an energy release rate criterion. In a more recent paper, Wang and Mai [88] concluded that the suitability of each of the two extreme electrical conditions on the crack faces would depend on the fracture parameters to be extracted. According to them, the permeable condition may be a suitable choice for the calculation of the crack-opening displacements, while the impermeable condition may be used for extracting fracture parameters like electric displacement intensity factor and energy release rate. Other researchers like Ou and Chen [65] had argued that it may be more reasonable to impose a semi-permeable electrical condition on the crack. The semi-permeable condition, however, gives rise to a non-linear problem which is mathematically more difficult to solve.

As for the mechanical conditions, it is usual to assume that the crack is stress-free. Recently, Li and Chen [49], however, have studied a problem in which a piezoelectric crack is acted upon by an unknown Coulomb traction. They concluded that the Coulombic traction may be negligible and previous investigations under the stress-free crack condition may be still be acceptable, if there is a relatively large mechanical loading and a relatively small electrical field. Otherwise, they claimed that neglecting the Coulombic traction may lead to some erroneous results with over 10% relative errors.

1.4 Fracture Criterion

For an elastic crack under a mixed mode loading, the conventional criterion for determining crack growth is based on the strain energy release rate. The crack will grow if its strain energy release rate exceeds a certain critical value.

An obvious extension of the criterion to piezoelectric materials is to include both mechanical and electrical energies in the calculation of the energy release rate for determining crack growth. Park and Sun [67] had shown that such an extension is, however, not consistent with experimental observations. They argued that fracture is a mechanical process and hence it may be more suitable to use only the mechanical energy release rate as the fracture criterion.

In recent years, researchers are more inclined towards using the strain energy density factor as an alternative fracture criterion. For piezoelectric materials, the strain energy density factor is a function of the crack tip stress and electric displacement intensity factors. For example, for a single planar crack in an infinite piezoelectric space, Zuo and Sih [103] had derived an explicit formula for the strain energy density factor in terms of the crack tip stress and electric displacement intensity factors. A more general formula which takes into consideration the polarization orientation of the electric field may be found in Chue and Weng [19]. Thus, a knowledge of the crack tip stress and electric displacement intensity factors is useful for establishing criterion for crack extension in piezoelectric materials.

1.5 The Present Thesis

In this thesis, several important classes of electro-elastic crack problems involving an arbitrary number of arbitrarily orientated planar cracks are solved. For the problems under consideration, a certain emphasis is placed on the calculation of the crack tip stress and electric displacement intensity factors.

The remaining part of this thesis comprises six chapters as follows.

Chapter 2 lays down the mathematical preliminaries needed in subsequent chapters.

In Chapter 3, the electro-elastostatic interaction of multiple planar cracks in an infinitely long piezoelectric strip is considered. The cracks are acted upon by suitable prescribed internal stresses and are electrically either permeable or impermeable. To solve the problem, a Green's function which satisfies the conditions of vanishing traction and normal electric displacement on the edges of the strip is first derived using a Fourier transform technique. It is then used to derive an explicit solution for the electro-elastostatic fields around the cracks in the strip. The solution is expressed in terms of integrals over lines representing the cracks. The jumps in the displacement and electric potential across opposite crack faces are unknown functions in the integrals, to be determined by solving a system of hypersingular integral equations which are derived by using the boundary conditions on the cracks.

Special electro-elastostatic Green's functions satisfying stress-free and electrically either permeable or impermeable conditions on multiple planar cracks in a piezoelectric domain of infinite extent are constructed numerically in Chapter 4. From the boundary conditions on the cracks, the task of constructing the Green's

functions requires solving a system of hypersingular integral equations. Once the hypersingular integral equations are solved, explicit formulae can be obtained for computing numerically the Green's functions and the corresponding stress components. The numerical Green's functions can be used to develop a simple but accurate boundary element method for analyzing multiple planar cracks in a piezoelectric solid of finite extent. As the singular behaviors of the stress and electric displacement analytically built into the Green's functions, the boundary element procedure does not require the crack faces to be discretized into boundary elements. Furthermore, the relevant crack tip stress and electric displacement intensity factors can be extracted with great accuracy.

In Chapter 5, the above mentioned numerical Green's function boundary element approach for electrically impermeable cracks is adapted to deal with elastostatic plane problem involving electrically semi-permeable cracks in a piezoelectric solid of finite extent. An iterative procedure for treating the nonlinear boundary conditions on the semi-permeable cracks is proposed.

A semi-analytic method of solution is successfully developed in Chapter 6 for the analysis of multiple cracks in an infinite piezoelectric domain under dynamic loading. The Laplace transform technique is employed to suppress the second order time derivatives of the displacement in the governing partial differential equations. The displacement and electric potential in the Laplace transform domain are then expressed using suitably constructed exponential Fourier transform representations. The unknown functions in the Fourier transform representations are directly related to the jumps in the Laplace transforms of the displacement and electric potential

across opposite crack faces and are determined from a system of hypersingular integral equations. Once the displacement and electric potential are determined in the Laplace transform domain, they can be recovered in the physical time domain by using a numerical formula for inverting Laplace transforms.

Lastly, Chapter 7 summarizes the contributions of the thesis and suggests some research extensions.

1.6 Publications

This thesis is written based on the following published papers:

Athanasius L, Ang WT, Sridhar I. Numerical Green's functions for some electroelastic crack problems. *Engineering Analysis with Boundary Elements*. 2009; 33: 778-788.

Athanasius L, Ang WT, Sridhar I. Electro-elastostatic analysis of multiple cracks in an infinitely long piezoelectric strip: a hypersingular integral approach. *European Journal of Mechanics-A/Solids*. 2010; 29: 410-419.

Ang WT and Athanasius L. A boundary integral approach for plane analysis of electrically semi-permeable planar cracks in a piezoelectric solid. *Engineering Analysis with Boundary Elements*. 2011; 35: 647-656.

Ang WT and Athanasius L. Dynamic interaction of multiple arbitrarily oriented planar cracks in a piezoelectric space: A semi-analytic solution. *European Journal of Mechanics - A/Solids*. 2011; 30: 608-618.

Chapter 2

Mathematical Preliminaries

2.1 Basic Equations of Electro-elasticity

The governing partial differential equations of electro-elasticity are derived from the law of conservation of momentum and Gauss law of electric flux. In the Cartesian coordinate system $Ox_1x_2x_3$, they are given by [94]:

$$\begin{aligned}\frac{\partial \sigma_{ij}}{\partial x_j} &= \rho \frac{\partial^2 u_i}{\partial t^2}, \\ \frac{\partial D_j}{\partial x_j} &= 0,\end{aligned}\tag{2.1}$$

where u_i , σ_{ij} and D_i are respectively displacement, stress and electric displacement, ρ is the density of the piezoelectric material and t denotes time. The Einsteinian convention of summing over a repeated index is assumed for lowercase Latin subscripts which take the values of 1, 2 and 3.

The linear constitutive equations relating u_i , σ_{ij} , D_i and the electric potential ϕ are given by

$$\begin{aligned}\sigma_{ij} &= c_{ijkl} \frac{\partial u_k}{\partial x_\ell} + e_{lij} \frac{\partial \phi}{\partial x_\ell}, \\ D_j &= e_{jkl} \frac{\partial u_k}{\partial x_\ell} - \kappa_{jl} \frac{\partial \phi}{\partial x_\ell},\end{aligned}\tag{2.2}$$

where c_{ijkl} , e_{kij} , κ_{jk} are the constant elastic moduli, piezoelectric coefficients and dielectric coefficients respectively.

Substitution of (2.2) into (2.1) yields

$$\begin{aligned} c_{ijk\ell} \frac{\partial^2 u_k}{\partial x_j \partial x_\ell} + e_{lij} \frac{\partial^2 \phi}{\partial x_j \partial x_\ell} &= \rho \frac{\partial^2 u_i}{\partial t^2}, \\ e_{jk\ell} \frac{\partial^2 u_k}{\partial x_j \partial x_\ell} - \kappa_{j\ell} \frac{\partial^2 \phi}{\partial x_j \partial x_\ell} &= 0. \end{aligned} \quad (2.3)$$

Following Barnett and Lothe [11], we can write (2.2) and (2.3) more compactly by defining

$$\begin{aligned} U_J &= \begin{cases} u_j & \text{for } J = j = 1, 2, 3, \\ \phi & \text{for } J = 4, \end{cases} \\ S_{Ij} &= \begin{cases} \sigma_{ij} & \text{for } I = i = 1, 2, 3, \\ D_j & \text{for } I = 4, \end{cases} \\ C_{IjK\ell} &= \begin{cases} c_{ijk\ell} & \text{for } I = i = 1, 2, 3 \text{ and } K = k = 1, 2, 3, \\ e_{lij} & \text{for } I = i = 1, 2, 3 \text{ and } K = 4, \\ e_{jk\ell} & \text{for } I = 4 \text{ and } K = k = 1, 2, 3, \\ -\kappa_{j\ell} & \text{for } I = 4 \text{ and } K = 4. \end{cases} \end{aligned} \quad (2.4)$$

With (2.4), (2.3) and (2.2) become

$$C_{IjK\ell} \frac{\partial^2 U_K}{\partial x_j \partial x_\ell} = B_{IK} \frac{\partial^2 U_K}{\partial t^2}, \quad (2.5)$$

and

$$S_{Ij} = C_{IjK\ell} \frac{\partial U_K}{\partial x_\ell}, \quad (2.6)$$

where

$$B_{IK} = \begin{cases} \rho & \text{if } I = K \text{ and } I \neq 4, \\ 0 & \text{otherwise.} \end{cases} \quad (2.7)$$

The uppercase Latin subscripts are assigned values from 1 to 4, while lowercase ones from 1 to 3.

We may think of U_K and S_{Ij} as the generalized (extended) displacement and stress respectively.

For time independent electro-elastic fields, (2.5) reduces to

$$C_{IjK\ell} \frac{\partial^2 U_K}{\partial x_j \partial x_\ell} = 0. \quad (2.8)$$

2.2 Plane Electro-elastostatics

2.2.1 General Solution

As shown in Clements [20], for the case of plane electro-elastostatic deformations in which the electroelastic fields are functions of only x_1 and x_2 , it is possible to construct a general solution for (2.8).

Specifically, for the plane electro-elastostatic deformations, (2.8) admits solutions of the general form

$$U_K = \text{Re} \left\{ \sum_{\alpha=1}^4 A_{K\alpha} f_\alpha(z_\alpha) \right\}, \quad (2.9)$$

where $z_\alpha = x_1 + \tau_\alpha x_2$, f_α are functions of z_α such that $f_\alpha(z_\alpha)$ is holomorphic (analytic) in the solution domain, τ_α are roots obtained by solving

$$\det [C_{I1K1} + (C_{I1K2} + C_{I2K1})\tau + C_{I2K2}\tau^2] = 0, \quad (2.10)$$

and $A_{K\alpha}$ are non-trivial solutions of the homogeneous system

$$[C_{I1K1} + (C_{I1K2} + C_{I2K1})\tau_\alpha + C_{I2K2}\tau_\alpha^2] A_{K\alpha} = 0. \quad (2.11)$$

Equation (2.10) is an 8-th order polynomial equation in τ . It can be shown through physical consideration that the solutions of (2.10) are complex numbers with non-zero imaginary parts (Barnett and Lothe [11]). Furthermore, as the coefficients of the polynomial equation in τ are real numbers, the solutions occur in complex

conjugate pairs. Now (2.10) can have up to 4 distinct complex conjugate pairs as solutions. For our discussion here, it is assumed that (2.10) has 4 distinct complex conjugate pairs, denoted by τ_1, τ_2, τ_3 and τ_4 , as solutions. In general, the roots τ_1, τ_2, τ_3 and τ_4 can be extracted by using numerical methods such as the Bairstrow method [13].

The extended (generalized) stress corresponding to (2.9) is

$$S_{Ij} = \operatorname{Re} \left\{ \sum_{\alpha=1}^4 L_{Ij\alpha} f'_\alpha(z_\alpha) \right\}, \quad (2.12)$$

where

$$L_{Ij\alpha} = [C_{IjK1} + \tau_\alpha C_{IjK2}] A_{K\alpha}. \quad (2.13)$$

With (2.9) and (2.12), the task of solving a plane electro-elastostatics problem may be reduced to finding complex functions $f_\alpha(z_\alpha)$ which are holomorphic in the solution domain and such that the prescribed boundary conditions are satisfied. For certain solution domains, such as an infinitely long strip, and for relatively simple boundary conditions, it may be possible to use the Fourier transform technique as an aid for constructing $f_\alpha(z_\alpha)$ (see, for example, the derivation of a Green's function for an infinitely long strip in Chapter 3).

2.2.2 Computation of τ_α , $A_{K\alpha}$ and Related Constants

This section explains the computation of the constants τ_α , $A_{K\alpha}$ and $L_{Kj\alpha}$ for certain electrical poling directions. The piezoelectric material deforms in such a way that it becomes elastically transversely isotropic under the action of the electric field, with the transverse plane being perpendicular to the electrical poling direction. The

electro-elastic properties of such a material are characterized by 10 independent constants denoted here by $A, N, F, C, L, e_1, e_2, e_3, \epsilon_1$ and ϵ_2 .

For the electrical poling direction, two cases are considered: inplane poling along the direction of the x_2 axis and antiplane (out of plane) poling along the x_3 axis.

Electrical Poling along the x_2 Axis

When the electrical poling of the material is along x_2 axis, the constitutive relations are given by

$$\begin{aligned}
 \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} &= \begin{pmatrix} A & F & N & 0 & 0 & 0 \\ F & C & F & 0 & 0 & 0 \\ N & F & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(A-N) & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{pmatrix} \begin{pmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{32} \\ 2\gamma_{31} \\ 2\gamma_{12} \end{pmatrix} \\
 &\quad - \begin{pmatrix} 0 & e_2 & 0 \\ 0 & e_3 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \\
 \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e_1 \\ e_2 & e_3 & e_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{32} \\ 2\gamma_{31} \\ 2\gamma_{12} \end{pmatrix} \\
 &\quad + \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}.
 \end{aligned} \tag{2.14}$$

where

$$\gamma_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \text{ and } E_i = -\frac{\partial \phi}{\partial x_i}. \tag{2.15}$$

According to (2.11), the matrix $[A_{K\alpha}]$ can be obtained by finding non-trivial solutions of the homogeneous systems

$$\begin{aligned}
(A + L\tau_\alpha^2) A_{1\alpha} + (F + L) \tau_\alpha A_{2\alpha} + (e_1 + e_2) \tau_\alpha A_{4\alpha} &= 0, \\
(F + L) \tau_\alpha A_{1\alpha} + (L + C\tau_\alpha^2) A_{2\alpha} + (e_1 + e_3\tau_\alpha^2) A_{4\alpha} &= 0, \\
\left(\frac{1}{2}A - \frac{1}{2}N + L\tau_\alpha^2\right) A_{3\alpha} &= 0, \\
(e_1 + e_2) \tau_\alpha A_{1\alpha} + (e_1 + e_3\tau_\alpha^2) A_{2\alpha} + (-\epsilon_1 - \epsilon_2\tau_\alpha^2) A_{4\alpha} &= 0. \quad (2.16)
\end{aligned}$$

From the above, it is obvious that one of the solutions of (2.10), which is denoted here as τ_3 , is given by

$$\tau_3 = i\sqrt{\frac{A - N}{2L}} \quad (A > N). \quad (2.17)$$

The remaining solutions of (2.10), denoted by τ_1 , τ_2 and τ_4 , can be obtained from

$$\det \begin{pmatrix} A + L\tau^2 & (F + L)\tau & (e_1 + e_2)\tau \\ (F + L)\tau & L + C\tau^2 & e_1 + e_3\tau^2 \\ (e_1 + e_2)\tau & e_1 + e_3\tau^2 & -\epsilon_1 - \epsilon_2\tau^2 \end{pmatrix} = 0. \quad (2.18)$$

For $\alpha = 3$, a non-trivial solution of (2.16) which forms the third column of the matrix $[A_{K\alpha}]$ is given by

$$\begin{pmatrix} A_{13} \\ A_{23} \\ A_{33} \\ A_{43} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (2.19)$$

For $\alpha = 1, 2$ and 4 , if $(A + L\tau_\alpha^2)(L + C\tau_\alpha^2) - (F + L)^2\tau_\alpha^2 \neq 0$, we may take $A_{3\alpha} = 0$ and $A_{4\alpha} = 1$ and find $A_{1\alpha}$ and $A_{2\alpha}$ by solving

$$\begin{aligned}
(A + L\tau_\alpha^2) A_{1\alpha} + (F + L) \tau_\alpha A_{2\alpha} &= -(e_1 + e_2)\tau_\alpha, \\
(F + L) \tau_\alpha A_{1\alpha} + (L + C\tau_\alpha^2) A_{2\alpha} &= -(e_1 + e_3\tau_\alpha^2). \quad (2.20)
\end{aligned}$$

in order to construct the first, second and fourth columns of the matrix $[A_{K\alpha}]$.

Once τ_α and $A_{K\alpha}$ are known, $L_{Kj\alpha}$ can be easily computed using (2.13) and other constants such as $N_{\alpha J}$ (the inverse of $A_{K\alpha}$) and D_{SR} which arise in the boundary integral equations for plane electro-elastostatics (as explained in later sections of this Chapter) can also be calculated.

Electrical Poling along the x_3 Axis

When the electrical poling of the material is along x_3 axis, the constitutive relations are given by

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} A & N & F & 0 & 0 & 0 \\ N & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(A - N) \end{pmatrix} \begin{pmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{32} \\ 2\gamma_{31} \\ 2\gamma_{12} \end{pmatrix} - \begin{pmatrix} 0 & 0 & e_2 \\ 0 & 0 & e_2 \\ 0 & 0 & e_3 \\ 0 & e_1 & 0 \\ e_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix},$$

and

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_1 & 0 & 0 \\ e_2 & e_2 & e_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{32} \\ 2\gamma_{31} \\ 2\gamma_{12} \end{pmatrix} + \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad (2.21)$$

where γ_{ij} and E_i are as defined in (2.15).

The homogeneous system of linear algebraic equations for working out $A_{K\alpha}$ is given by

$$\begin{aligned}
\left(A + \frac{1}{2}(A - N)\tau_\alpha^2\right) A_{1\alpha} + \left(\frac{1}{2}N + \frac{1}{2}A\right) \tau_\alpha A_{2\alpha} &= 0, \\
\left(\frac{1}{2}N + \frac{1}{2}A\right) \tau_\alpha A_{1\alpha} + \left(\frac{1}{2}A - \frac{1}{2}N + A\tau_\alpha^2\right) A_{2\alpha} &= 0, \\
(L + L\tau_\alpha^2) A_{3\alpha} + (e_1 + e_1\tau_\alpha^2) A_{4\alpha} &= 0, \\
(e_1 + e_1\tau_\alpha^2) A_{3\alpha} + (-\epsilon_1 - \epsilon_1\tau_\alpha^2) A_{4\alpha} &= 0. \tag{2.22}
\end{aligned}$$

If we use (2.22), we find that we cannot construct $[A_{K\alpha}]$ that is invertible. To overcome this minor difficulty, a relatively small amount of anisotropy is introduced into the equations governing u_1 and u_2 . Specifically, we replace $C_{1111} = A$ by $C_{1111} = A + \varepsilon$, where ε is a selected real number whose magnitude is very small compared to A . It follows that we modify (2.22) by

$$\begin{aligned}
\left(A + \varepsilon + \frac{1}{2}(A - N)\tau_\alpha^2\right) A_{1\alpha} + \left(\frac{1}{2}N + \frac{1}{2}A\right) \tau_\alpha A_{2\alpha} &= 0, \\
\left(\frac{1}{2}N + \frac{1}{2}A\right) \tau_\alpha A_{1\alpha} + \left(\frac{1}{2}A - \frac{1}{2}N + A\tau_\alpha^2\right) A_{2\alpha} &= 0, \\
(L + L\tau_\alpha^2) A_{3\alpha} + (e_1 + e_1\tau_\alpha^2) A_{4\alpha} &= 0, \\
(e_1 + e_1\tau_\alpha^2) A_{3\alpha} + (-\epsilon_1 - \epsilon_1\tau_\alpha^2) A_{4\alpha} &= 0. \tag{2.23}
\end{aligned}$$

We can take $\tau_3 = \tau_4 = i$ and τ_1 and τ_2 are two distinct solutions with positive imaginary parts of the quartic equation

$$\det \begin{pmatrix} A + \varepsilon + \frac{1}{2}(A - N)\tau^2 & (N + \frac{1}{2}(A - N))\tau \\ (N + \frac{1}{2}(A - N))\tau & \frac{1}{2}(A - N) + A\tau^2 \end{pmatrix} = 0. \tag{2.24}$$

Note that (2.24) cannot yield two distinct solutions with positive imaginary parts if ε is zero.

From (2.23), we find that A_{Ka} may be chosen to be

$$\begin{aligned} A_{1\alpha} &= -\frac{(N + \frac{1}{2}(A - N))\tau_\alpha}{A + \varepsilon + \frac{1}{2}(A - N)\tau_\alpha^2}(\delta_{\alpha 1} + \delta_{\alpha 2}) \\ A_{2\alpha} &= \delta_{\alpha 1} + \delta_{\alpha 2}, \quad A_{3\alpha} = \delta_{\alpha 3}, \quad A_{4\alpha} = \delta_{\alpha 4}. \end{aligned} \quad (2.25)$$

The matrix $[A_{K\alpha}]$ as constructed in (2.25) is invertible if $\tau_1 \neq \tau_2$.

2.3 Boundary Integral Formulation

2.3.1 A Reciprocal Relation

If $U_K(x_1, x_2)$ and $U_K^*(x_1, x_2)$ are solutions of

$$C_{IjKp} \frac{\partial^2 U_K}{\partial x_j \partial x_p} = h_I(x_1, x_2) \quad \text{and} \quad C_{IjKp} \frac{\partial^2 U_K^*}{\partial x_j \partial x_p} = h_I^*(x_1, x_2) \quad (2.26)$$

inside the region R enclosed by a simple closed curve (on the Ox_1x_2 plane) then

$$\int_C [P_I U_I^* - P_I^* U_I] ds = \int_R [h_I U_I^* - h_I^* U_I] dR, \quad (2.27)$$

where

$$\begin{aligned} P_K &= C_{IjKp} \frac{\partial U_K}{\partial x_p} n_j, \\ P_K^* &= C_{IjKp} \frac{\partial U_K^*}{\partial x_p} n_j \end{aligned} \quad (2.28)$$

and n_i are the components of the unit normal to C pointing away from R .

The derivation of the reciprocal relation (2.27) is given in Clements [20] and Clements and Rizzo [21].

2.3.2 Boundary Integral Equations

Consider the systems

$$C_{IjK\ell} \frac{\partial^2 U_K}{\partial x_j \partial x_\ell} = 0 \text{ for } I = 1, 2, 3, 4 \quad (2.29)$$

and

$$C_{IjKp} \frac{\partial^2 \Phi_{KR}}{\partial x_j \partial x_p} = \delta_{IS} \delta(x_1 - \xi_1, x_2 - \xi_2) \text{ for } I, S = 1, 2, 3, 4, \quad (2.30)$$

where δ_{IS} is the Kronecker-delta function and $\delta(x_1, x_2)$ is the Dirac delta function.

The terms $\delta_{iS} \delta(x_1 - \xi_1, x_2 - \xi_2)$ where $i = 1, 2, 3$ are the components of a body force concentrated along the line defined by $x_1 = \xi_1$, $x_2 = \xi_2$ and $x_3 = u$ for $-\infty < u < \infty$, while $\delta_{4S} \delta(x_1 - \xi_1, x_2 - \xi_2)$ is the concentrated charge density along the same straight line. Thus, Φ_{KR} is a singular (generalized line force) solution as given in (2.32). The function Φ_{KR} in (2.32) is the fundamental (generalized line force) solution for plane electro-elastostatics.

Use of the reciprocal relation in (2.27) on the systems (2.29) and (2.30) in the region R enclosed by a closed curve C yields a boundary integral solution for (2.29), that is,

$$\begin{aligned} \lambda(\xi_1, \xi_2) U_R(\xi_1, \xi_2) &= \int_C [U_I(x_1, x_2) \Xi_{IjR}(x_1, x_2; \xi_1, \xi_2) \\ &\quad - \Phi_{IR}(x_1, x_2; \xi_1, \xi_2) S_{Ij}(x_1, x_2)] n_j(x_1, x_2) ds(x_1, x_2), \end{aligned} \quad (2.31)$$

where

$$\Phi_{KR}(x_1, x_2; \xi_1, \xi_2) = \frac{1}{2\pi} \operatorname{Re} \left\{ \sum_{\alpha=1}^4 A_{K\alpha} N_{\alpha s} \ln(z_\alpha - w_\alpha) D_{SR} \right\}, \quad (2.32)$$

$$\Xi_{IjR}(x_1, x_2; \xi_1, \xi_2) = C_{IjK\ell} \frac{\partial}{\partial x_\ell} [\Phi_{KR}(x_1, x_2; \xi_1, \xi_2)], \quad (2.33)$$

$$\lambda(\xi_1, \xi_2) = \begin{cases} 0 & \text{if } (\xi_1, \xi_2) \notin C \cup R, \\ 1 & \text{if } (\xi_1, \xi_2) \text{ lies inside } R, \\ \frac{1}{2} & \text{if } (\xi_1, \xi_2) \text{ lies on a smooth part of } C, \end{cases} \quad (2.34)$$

S_{Ij} is as defined in (2.6), $n_j(x_1, x_2)$ is the outward unit vector normal to boundary C and $z_\alpha = x_1 + \tau_\alpha x_2$, $w_\alpha = \xi_1 + \tau_\alpha \xi_2$, $[N_{\alpha J}]$ is the inverse of $[A_{K\alpha}]$ and $[D_{SR}]$ is defined by

$$\delta_{IR} = \sum_{\alpha=1}^4 \text{Im} \{ L_{I2\alpha} N_{\alpha S} \} D_{SR}. \quad (2.35)$$

For convenience, (2.31) is rewritten as

$$\begin{aligned} \lambda(\xi_1, \xi_2) U_R(\xi_1, \xi_2) &= \int_C (U_I(x_1, x_2) \Gamma_{IR}(x_1, x_2; \xi_1, \xi_2) \\ &\quad - P_I(x_1, x_2) \Phi_{IR}(x_1, x_2; \xi_1, \xi_2)) ds(x_1, x_2) \end{aligned} \quad (2.36)$$

where

$$\begin{aligned} P_I &= C_{IjKp} \frac{\partial U_K}{\partial x_p} n_j \text{ for } I = 1, 2, 3, 4, \\ \Gamma_{IR}(x_1, x_2; \xi_1, \xi_2) &= C_{IjKp} \frac{\partial \Phi_{KR}}{\partial x_p} n_j \text{ for } I = 1, 2, 3, 4. \end{aligned} \quad (2.37)$$

Note that P_1 , P_2 and P_3 are the components of the traction on the boundary C and P_4 gives the electric displacement vector on C .

Details on the derivation of (2.36) can be found in Clements [20] and Clements and Rizzo [21].

From (2.32) and (2.37), $\Gamma_{IR}(x_1, x_2; \xi_1, \xi_2)$ in (2.36) is given by

$$\Gamma_{IR}(x_1, x_2; \xi_1, \xi_2) = \frac{1}{2\pi} \text{Re} \left\{ \sum_{\alpha=1}^4 L_{Ij\alpha} N_{\alpha S} \frac{1}{(z_\alpha - w_\alpha)} \right\} n_j(x_1, x_2) D_{SR}, \quad (2.38)$$

Equation (2.36) provides useful boundary integral equations for solving boundary value problems in plane electro-elastostatics. Note that if U_K and P_K are all

known on the boundary C then (2.36) with $\lambda(\xi_1, \xi_2) = 1$ provides us a formula for calculating $U_K(\xi_1, \xi_2)$ at any point (ξ_1, ξ_2) in the interior of R .

In Chapters 3, 4 and 5, (2.31) or (2.36) is applied to formulate and solve some plane electro-elastostatic crack problems.

2.4 Green's Functions

The boundary integral equations in (2.36) can be shown to be still valid, if Φ_{KR} and Γ_{IR} in (2.32) and (2.38) are replaced by

$$\begin{aligned}\Phi_{KR}(x_1, x_2; \xi_1, \xi_2) &= \frac{1}{2\pi} \operatorname{Re} \left\{ \sum_{\alpha=1}^4 A_{K\alpha} N_{\alpha s} \ln(z_\alpha - w_\alpha) d_{SR} \right\} \\ &\quad + \Phi_{KR}^*(x_1, x_2; \xi_1, \xi_2), \\ \Gamma_{IR}(x_1, x_2; \xi_1, \xi_2) &= \frac{1}{2\pi} \operatorname{Re} \left\{ \sum_{\alpha=1}^4 L_{Ij\alpha} N_{\alpha s} \frac{1}{(z_\alpha - w_\alpha)} \right\} n_j(x_1, x_2) d_{SR} \\ &\quad + \Gamma_{IR}^*(x_1, x_2; \xi_1, \xi_2),\end{aligned}\tag{2.39}$$

where $\Phi_{KR}^*(x_1, x_2; \xi_1, \xi_2)$ is any function which satisfies

$$C_{IjKp} \frac{\partial^2 \Phi_{KR}^*}{\partial x_j \partial x_p} = 0 \text{ for all points } (x_1, x_2) \text{ in } R\tag{2.40}$$

and $\Gamma_{IR}^*(x_1, x_2; \xi_1, \xi_2)$ is defined by

$$\Gamma_{IR}^*(x_1, x_2; \xi_1, \xi_2) = C_{IjKp} \frac{\partial \Phi_{KR}^*}{\partial x_p} n_j.\tag{2.41}$$

In using (2.36) to develop a solution method, one may choose to use $\Phi_{KR}^* = 0$. Nevertheless, for particular problems, it may be advantageous to choose Φ_{KR}^* such that Φ_{KR} satisfies certain conditions on some parts of the boundary C . In Chapter 3, for example, we use a Fourier transform technique to construct $\Phi_{KR}^*(x_1, x_2; \xi_1, \xi_2)$

as given in (2.39), such that $\Gamma_{IR}(x_1, x_2; \xi_1, \xi_2) = 0$ on the edges of a strip. With such a Green's function Φ_{KR} , the boundary integral equations do not have integrals over the edges of the strip if the traction and normal electric displacement vanishes on the edges. In Chapter 4, we construct $\Phi_{KR}^*(x_1, x_2; \xi_1, \xi_2)$ numerically to satisfy certain conditions on multiple planar cracks. If the numerically constructed $\Phi_{KR}(x_1, x_2; \xi_1, \xi_2)$ are used in the boundary integral equations, it is not necessary to integrate over the crack faces.

A specially constructed function $\Phi_{KR}(x_1, x_2; \xi_1, \xi_2)$ with $\Phi_{KR}^*(x_1, x_2; \xi_1, \xi_2)$ chosen to certain prescribed boundary conditions is called a Green's function.

2.5 Hypersingular Integral Equations

In all the problems considered in this thesis, the boundary conditions on the cracks are expressed in terms of a system of hypersingular integral equations of the general form

$$\begin{aligned} & \mathcal{H} \int_{-1}^1 \frac{r_m(t)dt}{(t-x)^2} + \sum_{k=1}^M \int_{-1}^1 T_{mk}(t, x) r_k(t) dt \\ &= P_m(x) \quad \text{for } -1 < x < 1 \quad (m = 1, 2, \dots, M), \end{aligned} \quad (2.42)$$

where $\mathcal{H}$ denotes that the integral over the interval $(-1, 1)$ is to be interpreted in the Hadamard finite-part sense, $T_{mk}(t, x)$ and $P_m(x)$ are known functions and $r_m(t)$ are unknown functions to be determined. The Hadamard finite-part integral in (2.42) arises directly from the differentiation of a Cauchy principal integral.

The formulation of crack problems using hypersingular integral equations was apparently pioneered by Ioakimidis [41] in the early 1980s. The hypersingular inte-

gral formulation can also be found in the works of many other researchers such as Kaya and Erdogan [43], Matsumoto, Tanaka, Okayama *et al* [62], Nied [64], Lin'kov and Mogilevskaya [59] Takakuda, Koizumi and Shibuya [83] and Zhang [96].

The basic idea of the Hadamard finite-part integrals was introduced by Hadamard [35]. Here we will provide a brief account on the Hadamard finite-part integrals which arise in the formulation of two-dimensional (plane) crack problems in linear elasticity.

2.5.1 Hadamard Finite-part Integrals

A good starting point for discussion is to consider the Cauchy principal integral

$$\mathcal{C} \int_{-1}^1 \frac{p(t)dt}{t-x} \text{ (for } -1 < x < 1). \quad (2.43)$$

The symbol $\mathcal{C}$ is used to denote that the integral must be interpreted in the Cauchy principal sense.

The integral in (2.43) can be defined in a limiting sense by [28]:

$$\mathcal{C} \int_{-1}^1 \frac{p(t)dt}{t-x} \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0^+} \left(\int_{-1}^{x-\epsilon} \frac{p(t)dt}{t-x} + \int_{x+\epsilon}^1 \frac{p(t)dt}{t-x} \right). \quad (2.44)$$

It is assumed that $p(t)$ can be expanded as Taylor series about $t = x$ (for $-1 < x < 1$), that is,

$$p(t) = p(x) + \sum_{m=1}^{\infty} \frac{p^{(m)}(x)}{m!} (t-x)^m. \quad (2.45)$$

From (2.44), the improper integral in (2.43) can be differentiated with respect to x to give

$$\begin{aligned} \frac{d}{dx} \left[\mathcal{C} \int_{-1}^1 \frac{p(t)dt}{t-x} \right] &= \frac{d}{dx} \lim_{\epsilon \rightarrow 0^+} \left(\int_{-1}^{x-\epsilon} \frac{p(t)dt}{t-x} + \int_{x+\epsilon}^1 \frac{p(t)dt}{t-x} \right) \\ &= \frac{d}{dx} (p(x) [\ln(1-x) - \ln(1+x)]) \\ &\quad + \frac{d}{dx} \sum_{m=1}^{\infty} \frac{p^{(m)}(x)}{m!m} [(1-x)^m - (-1-x)^m]. \end{aligned} \quad (2.46)$$

Now if we attempt to express the divergent integral given by

$$\int_{-1}^1 \frac{p(t)dt}{(t-x)^2} \quad (\text{for } -1 < x < 1), \quad (2.47)$$

in terms of a limit by excluding an infinitesimal part containing $t = x$ from the interval of integration, we find that

$$\begin{aligned} \int_{-1}^1 \frac{p(t)dt}{(t-x)^2} &= \lim_{\epsilon \rightarrow 0^+} \left(\int_{-1}^{x-\epsilon} \frac{p(t)dt}{(t-x)^2} + \int_{x+\epsilon}^1 \frac{p(t)dt}{(t-x)^2} \right) \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{2p(x)}{\epsilon} + F(x), \end{aligned} \quad (2.48)$$

where $F(x)$ is given by

$$\begin{aligned} F(x) &= p(x) \left(\frac{1}{-1-x} - \frac{1}{1-x} \right) + p'(x) [\ln(1-x) - \ln(1+x)] \\ &\quad + \sum_{m=1}^{\infty} \frac{p^{(m+1)}(x)}{(m+1)!m} [(1-x)^m - (-1-x)^m]. \end{aligned} \quad (2.49)$$

The function $F(x)$ is called the *finite-part* of the divergent integral in (2.47) and is denoted by

$$\mathcal{H} \int_{-1}^1 \frac{p(t)dt}{(t-x)^2}, \quad (2.50)$$

that is, we define

$$\mathcal{H} \int_{-1}^1 \frac{p(t)dt}{(t-x)^2} \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0^+} \left(\int_{-1}^{x-\epsilon} \frac{p(t)dt}{(t-x)^2} + \int_{x+\epsilon}^1 \frac{p(t)dt}{(t-x)^2} - \frac{2p(x)}{\epsilon} \right) \text{ for } -1 < x < 1. \quad (2.51)$$

From (2.46) and (2.49), we obtain

$$\begin{aligned} & \frac{d}{dx} (p(x) [\ln(1-x) - \ln(1+x)]) \\ & + \frac{d}{dx} \sum_{m=1}^{\infty} \frac{p^{(m)}(x)}{m!m} [(1-x)^m - (-1-x)^m] = F(x). \end{aligned}$$

Thus, we conclude that

$$\frac{d}{dx} [\mathcal{C} \int_{-1}^1 \frac{p(t)dt}{t-x}] = \mathcal{H} \int_{-1}^1 \frac{p(t)dt}{(t-x)^2} \text{ for } -1 < x < 1. \quad (2.52)$$

A alternative equivalent definition for the Hadamard finite-part integrals as given in Ang and Clements [7] is:

$$\mathcal{H} \int_{-1}^1 \frac{p(t)dt}{(t-x)^2} \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0^+} \left(\int_{-1}^1 \frac{(t-x)^2 p(t)dt}{[(t-x)^2 + \epsilon^2]^2} - \frac{\pi}{2\epsilon} p(x) \right). \quad (2.53)$$

2.5.2 Numerical Solution of Hypersingular Integral Equations

For elastic crack problems in which the unknown functions $r_m(t)$ in (2.42) may be written in the form

$$r_m(t) = \sqrt{1-t^2} v_m(t), \quad (2.54)$$

an accurate collocation technique for solving numerically the system of hypersingular integral equations is given in Kaya and Erdogan [43].

In [43], the functions $v_m(t)$ are approximated using

$$v_m(t) \simeq \sum_{j=1}^J \psi_m^{(j)} U^{(j-1)}(t), \quad (2.55)$$

where $U^{(j)}(x) = \sin([j+1] \arccos(x)) / \sin(\arccos(x))$ is the j -th order Chebyshev polynomial of the second kind and $\psi_m^{(j)}$ are the constant coefficients to be determined.

Use of (2.54) and (2.55) in (2.42) together with the well established result

$$\mathcal{H} \int_{-1}^1 \frac{\sqrt{1-t^2} U^{(j-1)}(t)}{(t-x)^2} dt = -\pi j U^{(j-1)}(x) \quad (2.56)$$

gives

$$\begin{aligned} & \sum_{j=1}^J \{ -\psi_m^{(j)} \pi j U^{(j-1)}(x) + \sum_{k=1}^M \psi_k^{(j)} \int_{-1}^1 \sqrt{1-t^2} T_{mk}(t, x) dt \} \\ &= P_m(x) \text{ for } -1 < x < 1 \quad (m = 1, 2, \dots, M). \end{aligned} \quad (2.57)$$

In [43], (2.57) is collocated by letting $x = \cos([2i-1]\pi/[2J])$ for $i = 1, 2, \dots, J$ to set up a system of MJ linear algebraic equations for determining the unknown constants $\psi_k^{(j)}$. Note that the integral over $-1 < t < 1$ in (2.57) can be accurately evaluated by using the numerical quadrature formula (25.4.40) in Abramowitz and Stegun [1].

The numerical method outlined above is used for solving the hypersingular integral equations for the electro-elastic crack problems in Chapters 3, 4, 5 and 6.

2.6 Numerical Inversion of Laplace Transforms

A time dependent electro-elastic crack problem is considered in Chapter 6. The problem is first solved in the Laplace transform domain. The solution is recovered in

the physical time domain by using the explicit approximate formula given in Stehfest [82] for inverting Laplace transforms.

If $\widehat{F}(s)$ is the Laplace transform of $F(t)$ (where s is the Laplace transform parameter and t is time), that is, if

$$\widehat{F}(s) = \int_0^{\infty} F(t) \exp(-st) dt, \quad (2.58)$$

then the Stehfest's algorithm for numerical inversion of Laplace transforms is given by

$$F(t) \simeq \frac{\ln(2)}{t} \sum_{n=1}^{2M} V_n \widehat{F}\left(\frac{n \ln(2)}{t}\right), \quad (2.59)$$

where M is a positive integer and

$$V_n = (-1)^{n+M} \sum_{m=\lceil (n+1)/2 \rceil}^{\min(n,M)} \frac{m^M (2m)!}{(M-m)! m! (m-1)! (n-m)! (2m-n)!}, \quad (2.60)$$

with $\lceil r \rceil$ denoting the integer part of the real number r .

The Stehfest's algorithm requires $\widehat{F}(s)$ to be evaluated for only real values of s . It is widely used to invert Laplace transforms in the solution of time dependent problems in engineering science (see, for example Huang, Tang, Yu *et al* [40], Wen, Zhan, Huang *et al* [90], and Zhang, Zhang, Wang *et al* [98]).

Chapter 3

Cracks in a Piezoelectric Strip

3.1 Introduction

The problem of determining the electro-elastostatic fields around cracks in an infinitely long piezoelectric strip has been a subject of considerable interest among many researchers. From a practical standpoint, such a problem is of interest as thin piezoelectric plates can be found in the design of many electronic devices. Most of the works reported in the literature deal with cracks that have specific geometries and orientations, such as a single straight crack oriented in a direction that is either parallel or perpendicular to the edges of the piezoelectric strip.

For mathematical simplicity, many researchers have studied cases in which the piezoelectric strip is deformed by antiplane shear stress and inplane electrical static loads. For example, Li [51] and Shindo, Narita and Tanaka [76] applied a Fourier transform technique to reduce the problem of a straight crack to solving a Fredholm integral equation, and Li [50], Li and Duan [52] and Zhong and Li [99] derived closed-form formulae for the electroelastic field intensity factors and energy release rates of a pair of collinear cracks. Some other works of related interest include those of Li and Lee [55] and Kwon and Lee [44] on a straight crack in a piezoelectric strip of finite length subject to an antiplane deformation.

If the piezoelectric strip is deformed by inplane mechanical and electrical loads, the problem is more complicated to solve. Particular plane problems involving piezo-

electric strips with relatively simple crack configurations were solved by Shindo, Watanabe and Narita [77] and Wang and Han [85, 86, 87, 89].

This chapter¹ considers the problem of an infinitely long piezoelectric strip containing an arbitrary number of arbitrarily oriented straight cracks under mixed mode electro-elastostatic loads. The cracks are assumed to be either electrically impermeable or permeable. The solution approach here is to construct an appropriate Green's function for the governing equations of linear electro-elasticity and use it to reduce the problem under consideration to solving hypersingular integral equations which describe the conditions on the cracks. The Green's function which satisfies the conditions of vanishing traction and normal electric displacement on the edges of the piezoelectric strip is derived with aid of exponential Fourier transformation.

Of interest here is the calculation of the crack tip stress and electric displacement intensity factors. Such intensity factors can be computed using different methods such as the J -integral [72] and M -integral [10]. In our approach here, the intensity factors are extracted directly from the full solution of the problem under consideration. Once the crack opening displacements and the jump in the electrical potential across opposite crack faces are determined from the hypersingular integral equations, the intensity factors can be easily computed as explained in Section 3.5. Such an approach has been shown to be highly efficient and accurate (see, for example, Kaya and Erdogan [43] and Nied [64]).

¹ The work reported in this chapter is published in the journal *European Journal of Mechanics-A/Solids*, with details as follows: Athanasius L, Ang WT, Sridhar I, Electro-elastostatic analysis of multiple cracks in an infinitely long piezoelectric strip: a hypersingular integral approach. *European Journal of Mechanics-A/Solids*, 2010; 29: 410-419.

3.2 Statement of Problem

With reference to an $Ox_1x_2x_3$ Cartesian coordinate system, consider an infinitely long piezoelectric strip $-\infty < x_1 < \infty$, $0 < x_2 < h$, $-\infty < x_3 < \infty$, where h is a given positive constant. The strip contains N arbitrarily oriented straight cracks whose geometries do not change along the x_3 axis. The cracks are denoted by $\gamma^{(1)}$, $\gamma^{(2)}$, $\dots$, $\gamma^{(N-1)}$ and $\gamma^{(N)}$. On the Ox_1x_2 plane, the tips of the k -th crack $\gamma^{(k)}$ are given by $(a^{(k)}, b^{(k)})$ and $(c^{(k)}, d^{(k)})$. Refer to Figure 3.1. The cracks do not intersect with one another or the edges $x_2 = 0$ and $x_2 = h$. It is also assumed that the electroelastic deformation of the cracked piezoelectric strip does not depend on the spatial coordinate x_3 and time.

Figure 3.1: A sketch of the problem on the Ox_1x_2 plane.

The problem of interest requires solving (2.8) for U_K in the region $0 < x_2 < h$. In the Barnett and Lothe [11] notation introduced in Section 2.1, the boundary

conditions on the edges of the strip are given by

$$\left. \begin{aligned} S_{I2}(x_1, 0) &= 0 \\ S_{I2}(x_1, h) &= 0 \end{aligned} \right\} \text{ for } -\infty < x_1 < \infty, \quad (3.1)$$

while those on the cracks are given by

$$\begin{aligned} S_{Ij}(x_1, x_2)m_j^{(k)} &\rightarrow -S_{Ij}^{(0)}(\xi_1, \xi_2)m_j^{(k)} \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \gamma_+^{(k)} \quad (k = 1, 2, \dots, N) \text{ for } I = 1, 2, 3, \end{aligned} \quad (3.2)$$

and either

$$\begin{aligned} S_{4j}(x_1, x_2)m_j^{(k)} &\rightarrow -S_{4j}^{(0)}(\xi_1, \xi_2)m_j^{(k)} \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \gamma_+^{(k)} \quad (k = 1, 2, \dots, N) \\ &\text{if the cracks are electrically impermeable,} \end{aligned} \quad (3.3)$$

or

$$\begin{aligned} \Delta U_4(x_1, x_2) &\rightarrow 0 \text{ as } (x_1, x_2) \rightarrow (\xi_1, \xi_2) \in \gamma_+^{(k)} \text{ for } k = 1, 2, \dots, N \\ &\text{if the cracks are electrically permeable,} \end{aligned} \quad (3.4)$$

where the superscript (0) (in $S_{Ij}^{(0)}$) denotes the internal electro-elastostatic fields in the piezoelectric strip, $\gamma_+^{(k)}$ denotes the “upper face” of the crack $\gamma^{(k)}$, $m_i^{(k)}$ being the components of a unit magnitude normal vector to $\gamma_+^{(k)}$ are given by

$$\begin{aligned} m_1^{(k)} &= \frac{d^{(k)} - b^{(k)}}{\ell^{(k)}}, \quad m_2^{(k)} = \frac{a^{(k)} - c^{(k)}}{\ell^{(k)}}, \quad m_3^{(k)} = 0, \\ \ell^{(k)} &= \sqrt{(c^{(k)} - a^{(k)})^2 + (d^{(k)} - b^{(k)})^2}, \end{aligned} \quad (3.5)$$

$$\begin{aligned} \Delta U_I(x_1, x_2) &= \lim_{\varepsilon \rightarrow 0} [U_I(x_1 - |\varepsilon|m_1^{(k)}, x_2 - |\varepsilon|m_2^{(k)}) \\ &\quad - U_I(x_1 + |\varepsilon|m_1^{(k)}, x_2 + |\varepsilon|m_2^{(k)})] \\ &\quad \text{for } (x_1, x_2) \in \gamma_+^{(k)}. \end{aligned} \quad (3.6)$$

In addition, it is required that $S_{Ij} \rightarrow 0$ as $|x_1| \rightarrow \infty$.

3.3 Green's Function for a Piezoelectric Strip

For the piezoelectric strip $0 < x_2 < h$, we seek to construct a special Green function

$\Phi_{KR}(x_1, x_2; \xi_1, \xi_2)$ as defined by (2.39) with

$$\left. \begin{aligned} \Xi_{I2R}(x_1, 0; \xi_1, \xi_2) &= 0 \\ \Xi_{I2R}(x_1, h; \xi_1, \xi_2) &= 0 \end{aligned} \right\} \text{ for } -\infty < x_1 < \infty, \quad (3.7)$$

where

$$\Xi_{IjR}(x_1, x_2; \xi_1, \xi_2) = C_{IjK\ell} \frac{\partial}{\partial x_\ell} [\Phi_{KR}(x_1, x_2; \xi_1, \xi_2)]. \quad (3.8)$$

Guided by the analysis in Clements [20], we take Φ_{KR}^* which satisfies (2.40)

to be

$$\begin{aligned} & \Phi_{KR}^*(x_1, x_2; \xi_1, \xi_2) \\ = & -\frac{1}{2\pi} \operatorname{Re} \left\{ \sum_{\alpha=1}^4 A_{K\alpha} M_{\alpha P} \sum_{\beta=1}^4 \bar{L}_{P2\beta} \bar{N}_{\beta S} \ln(z_\alpha - \bar{c}_\beta) \right\} D_{SR} \\ & + \frac{1}{2\pi} \int_0^\infty \operatorname{Re} \left\{ \sum_{\alpha=1}^4 A_{K\alpha} M_{\alpha P} [E_{PR}(u; \xi_1, \xi_2) \exp(iuz_\alpha) \right. \\ & \quad \left. - \bar{E}_{PR}(u; \xi_1, \xi_2) \exp(-iuz_\alpha) + F_{PR}(u; \xi_1, \xi_2)] \right\} du, \quad (3.9) \end{aligned}$$

where the overhead bar denotes the complex conjugate of a complex number, $i = \sqrt{-1}$, $E_{PR}(u; \xi_1, \xi_2)$ and $F_{PR}(u; \xi_1, \xi_2)$ are arbitrary functions to be determined, $c_\alpha = \xi_1 + \tau_\alpha \xi_2$, $[N_{\alpha S}]$ is the inverse of $[A_{K\alpha}]$, $[M_{\alpha P}]$ is the inverse of $[L_{I2\alpha}]$ and $[D_{SR}]$ are real constants defined in (2.35) and (2.13).

From (2.39), (3.8) and (3.9), we write

$$\begin{aligned} & \Xi_{KjR}(x_1, x_2; \xi_1, \xi_2) \\ = & \frac{1}{2\pi} \operatorname{Re} \left\{ \sum_{\alpha=1}^4 L_{Kj\alpha} N_{\alpha S} (z_\alpha - c_\alpha)^{-1} \right\} D_{SR} + \Xi_{KjR}^*(x_1, x_2; \xi_1, \xi_2) \\ & \text{for } 0 < \xi_2 < h, \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} & \Xi_{KjR}^*(x_1, x_2; \xi_1, \xi_2) \\ = & -\frac{1}{2\pi} \operatorname{Re} \left\{ \sum_{\alpha=1}^4 L_{Kj\alpha} M_{\alpha P} \sum_{\beta=1}^4 \bar{L}_{P2\beta} \bar{N}_{\beta S} (z_\alpha - \bar{c}_\beta)^{-1} \right\} D_{SR} \\ & + \frac{1}{2\pi} \int_0^\infty \operatorname{Re} \left\{ \sum_{\alpha=1}^4 i L_{Kj\alpha} M_{\alpha P} u [E_{PR}(u; \xi_1, \xi_2) \exp(iu z_\alpha) \right. \\ & \quad \left. + \bar{E}_{PR}(u; \xi_1, \xi_2) \exp(-iu z_\alpha)] \right\} du. \end{aligned} \quad (3.11)$$

It can be shown that (2.39) and (3.11) satisfies (2.30) and the boundary conditions given on the first line in (3.7). The boundary conditions on the second line in (3.7) are fulfilled if

$$\begin{aligned} & \operatorname{Re} \left\{ \sum_{\alpha=1}^4 L_{K2\alpha} N_{\alpha S} (x_1 + \tau_\alpha h - c_\alpha)^{-1} \right\} D_{SR} \\ & - \operatorname{Re} \left\{ \sum_{\alpha=1}^4 L_{K2\alpha} M_{\alpha P} \sum_{\beta=1}^4 \bar{L}_{P2\beta} \bar{N}_{\beta S} (x_1 + \tau_\alpha h - \bar{c}_\beta)^{-1} \right\} D_{SR} \\ & + \int_0^\infty \operatorname{Re} \left\{ \sum_{\alpha=1}^4 i L_{K2\alpha} M_{\alpha P} u [E_{PR}(u; \xi_1, \xi_2) \exp(iu [x_1 + \tau_\alpha h]) \right. \\ & \quad \left. + \bar{E}_{PR}(u; \xi_1, \xi_2) \exp(-iu [x_1 + \tau_\alpha h])] \right\} du \\ = & 0 \text{ for } -\infty < x_1 < \infty. \end{aligned} \quad (3.12)$$

Taking the exponential Fourier transform of both sides of (3.12) over the interval $-\infty < x_1 < \infty$, we find that

$$\begin{aligned}
& u \sum_{\alpha=1}^4 \{L_{K2\alpha} M_{\alpha P} \exp(iu\tau_{\alpha}h) - \bar{L}_{K2\alpha} \bar{M}_{\alpha P} \exp(iu\bar{\tau}_{\alpha}h)\} E_{PR}(u; \xi_1, \xi_2) \\
&= \sum_{\alpha=1}^4 L_{K2\alpha} N_{\alpha S} \exp(-iu[c_{\alpha} - \tau_{\alpha}h]) D_{SR} \\
&\quad - \sum_{\alpha=1}^4 L_{K2\alpha} M_{\alpha P} \sum_{\beta=1}^4 \bar{L}_{P2\beta} \bar{N}_{\beta S} \exp(-iu[\bar{c}_{\beta} - \tau_{\alpha}h]) D_{SR}. \tag{3.13}
\end{aligned}$$

We can invert (3.13) as a system of linear algebraic equations to obtain the value of $E_{PR}(u; \xi_1, \xi_2)$. The functions $E_{PR}(u; \xi_1, \xi_2)$ are not well defined at $u = 0$. It can be shown that the integrand of the improper integral in (3.9) is bounded over the interval $0 < u < \infty$ if we choose $F_{PR}(u; \xi_1, \xi_2)$ to be given by

$$F_{PR}(u; \xi_1, \xi_2) = \bar{E}_{PR}(u; \xi_1, \xi_2) - E_{PR}(u; \xi_1, \xi_2). \tag{3.14}$$

Note that the functions $E_{PR}(u; \xi_1, \xi_2)$ tend to zero as the width h tends to infinity (that is, for a piezoelectric half-space $x_2 > 0$).

3.4 Hypersingular Integral Equations

If we apply (2.31) together with the Green's function $\Phi_{KR}(x_1, x_2; \xi_1, \xi_2)$ and the corresponding stress function $\Xi_{KR}(x_1, x_2; \xi_1, \xi_2)$ as given by (2.39), (3.9), (3.10), (3.11) and (3.13) to the crack problem under consideration in this Chapter, we obtain

$$\begin{aligned}
U_R(\xi_1, \xi_2) &= \sum_{k=1}^N \int_{\gamma_+^{(k)}} \Delta U_I(x_1, x_2) m_p^{(k)} \Xi_{IPR}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2) \\
&\quad \text{for } 0 < \xi_2 < h, \tag{3.15}
\end{aligned}$$

where $\Delta U_I(x_1, x_2)$ is as defined in (3.6). In (3.15), $\gamma_+^{(k)}$ (the “upper face” of the crack $\gamma^{(k)}$) is taken to be the straight line from $(a^{(k)}, b^{(k)})$ to $(c^{(k)}, d^{(k)})$.

The integration in (3.15) is only over the crack faces. The integrals over $x_2 = 0$ and $x_2 = h$ vanish because of (3.1) and (3.7). Also, note that the far field conditions that $S_{I1} \rightarrow 0$ as $|x_1| \rightarrow 0$ is used in deriving (3.15).

From (2.6) and (3.15), we obtain

$$\begin{aligned}
 S_{Kj}(\xi_1, \xi_2) &= \sum_{k=1}^N \int_{\gamma_+^{(k)}} \Delta U_I(x_1, x_2) C_{KjR\ell} m_p^{(k)} \\
 &\quad \times \frac{\partial}{\partial \xi_\ell} [\Xi_{IpR}(x_1, x_2; \xi_1, \xi_2)] ds(x_1, x_2) \\
 &\quad \text{for } 0 < \xi_2 < h.
 \end{aligned} \tag{3.16}$$

Conditions on the cracks given by (3.2) and either (3.3) (for electrically impermeable cracks) or (3.4) (for electrically permeable cracks) can be used to derive a system of hypersingular integral equations containing the unknown functions $\Delta U_I(x_1, x_2)$ for $(x_1, x_2) \in \gamma_+^{(k)}$ ($k = 1, 2, \dots, N$). The unknown functions can be determined by solving numerically the system of hypersingular integral equations.

3.4.1 Electrically Impermeable Cracks

For electrically impermeable cracks, using (3.16), the system of hypersingular integral equations derived from (3.2) and (3.3) is given by

$$\mathcal{H} \int_{-1}^{+1} \frac{\chi_{IK}^{(q)} \Delta U_I^{(q)}(v)}{(t-v)^2} dv + \sum_{\substack{n=1 \\ n \neq q}}^N \int_{-1}^{+1} \Delta U_I^{(n)}(v) \Lambda_{IK}^{(nq)}(v, t) dv$$

$$\begin{aligned}
& + \sum_{n=1}^N \int_{-1}^{+1} \Delta U_I^{(n)}(v) \Psi_{IK}^{(nq)}(v, t) dv = -S_{Kj}^{(0)}(X_1^{(q)}(t), X_2^{(q)}(t)) m_j^{(q)} \\
& \text{for } -1 < t < 1, K = 1, 2, 3, 4 \text{ and } q = 1, 2, \dots, N \quad , \quad (3.17)
\end{aligned}$$

where $\mathcal{H}$ denotes that the integral is to be interpreted in the Hadamard finite-part sense (see Chapter 2) and

$$\begin{aligned}
& \Delta U_I^{(q)}(v) = \Delta U_I(X_1^{(q)}(v), X_2^{(q)}(v)), \\
& \Lambda_{IK}^{(nq)}(v, t) = \frac{1}{4\pi} \operatorname{Re} \sum_{\alpha=1}^4 \left\{ \frac{Q_{IKrj\alpha} m_j^{(n)} m_r^{(q)} \ell^{(n)}}{([X_1^{(n)}(v) - X_1^{(q)}(t)] + \tau_\alpha [X_2^{(n)}(v) - X_2^{(q)}(t)])^2} \right\}, \\
& \Psi_{IK}^{(nq)}(v, t) = \frac{m_j^{(n)} m_r^{(q)} \ell^{(n)}}{4\pi} \operatorname{Re} \left\{ \int_0^\infty \sum_{\alpha=1}^4 iu L_{Ij\alpha} M_{\alpha P} \right. \\
& \quad \times [C_{KrRs} \frac{\partial}{\partial \xi_s} (E_{PR}(u; \xi_1, \xi_2)) \exp(iu Z_\alpha^{(n)}(v)) \\
& \quad + C_{KrRs} \frac{\partial}{\partial \xi_s} (\bar{E}_{PR}(u; \xi_1, \xi_2)) \exp(-iu Z_\alpha^{(n)}(v))] du \\
& \quad - \sum_{\alpha=1}^4 L_{Ij\alpha} M_{\alpha P} \sum_{\beta=1}^4 B_{PKr\beta} \\
& \quad \times \frac{1}{([X_1^{(n)}(v) - X_1^{(q)}(t)] + \tau_\alpha X_2^{(n)}(v) - \bar{\tau}_\beta X_2^{(q)}(t))^2} \Big\}, \\
& X_1^{(q)}(v) = \frac{(c^{(q)} + a^{(q)})}{2} + \frac{(c^{(q)} - a^{(q)})}{2} v, \\
& X_2^{(q)}(v) = \frac{(d^{(q)} + b^{(q)})}{2} + \frac{(d^{(q)} - b^{(q)})}{2} v, \\
& Z_\alpha^{(q)}(v) = X_1^{(q)}(v) + \tau_\alpha X_2^{(q)}(v), \\
& \chi_{IK}^{(q)} = \frac{1}{\pi} \operatorname{Re} \sum_{\alpha=1}^4 \left\{ \frac{Q_{IKrj\alpha} m_j^{(q)} m_r^{(q)} \ell^{(q)}}{([c^{(q)} - a^{(q)}] + \tau_\alpha [d^{(q)} - b^{(q)}])^2} \right\}, \\
& Q_{IKrj\alpha} = (C_{KrR1} + \tau_\alpha C_{KrR2}) T_{Ij\alpha R}, \quad T_{Ij\alpha S} = L_{Ij\alpha} N_{\alpha R} D_{RS}, \\
& B_{PKr\beta} = (C_{KrR1} + \bar{\tau}_\beta C_{KrR2}) H_{P\beta R}, \quad H_{P\beta R} = \bar{L}_{P2\beta} \bar{N}_{\beta S} D_{SR}. \quad (3.18)
\end{aligned}$$

The numerical method in Kaya and Erdogan [43] as outlined in Subsection 2.5.2 can be applied to solve (3.17) approximately for $\Delta U_I^{(q)}(v)$ as follows.

Let

$$\Delta U_P^{(n)}(v) \simeq \sqrt{1-v^2} \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(v), \quad (3.19)$$

where $U^{(j)}(x) = \sin([j+1] \arccos(x)) / \sin(\arccos(x))$ is the j^{th} order Chebyshev polynomial of the second kind and $\psi_P^{(nj)}$ are the constants to be determined.

Substitution of (3.19) into (3.17) yields

$$\begin{aligned} & - \sum_{j=1}^J j \pi \psi_I^{(qj)} \chi_{IK}^{(q)} U^{(j-1)}(t) \\ & + \sum_{\substack{n=1 \\ n \neq q}}^N \sum_{j=1}^J \psi_I^{(nj)} \int_{-1}^{+1} \sqrt{1-v^2} U^{(j-1)}(v) \Lambda_{IK}^{(nq)}(v, t) dv \\ & + \sum_{n=1}^N \sum_{j=1}^J \psi_I^{(nj)} \int_{-1}^{+1} \sqrt{1-v^2} U^{(j-1)}(v) \Psi_{IK}^{(nq)}(v, t) dv \\ = & -S_{Kj}^{(0)}(X_1^{(q)}(t), X_2^{(q)}(t)) m_j^{(q)} \end{aligned} \quad (3.20)$$

for $-1 < t < 1$, $K = 1, 2, 3, 4$ and $q = 1, 2, \dots, N$.

Note that (3.20) contains $4JN$ unknown constants $\psi_P^{(nj)}$ ($P = 1, 2, 3, 4$; $n = 1, 2, \dots, N$; $j = 1, 2, \dots, J$). By letting $t = \cos([2i-1]\pi/[2J])$ for $i = 1, 2, \dots, J$, we can generate a system of $4JN$ linear algebraic equations which can be solved for the unknown constants.

3.4.2 Electrically Permeable Cracks

From (3.4), $\Delta U_4^{(q)}(v) = 0$ for $-1 < v < 1$ and $q = 1, 2, \dots, N$, if the cracks are electrically permeable. According to (3.2), the unknown functions $\Delta U_1^{(q)}(v)$, $\Delta U_2^{(q)}(v)$ and $\Delta U_3^{(q)}(v)$ are governed by (3.17) (with $\Delta U_4^{(q)}(v) = 0$) for $K = 1, 2$,

3 (instead of $K = 1, 2, 3, 4$). The functions $\Delta U_1^{(q)}(v)$, $\Delta U_2^{(q)}(v)$ and $\Delta U_3^{(q)}(v)$ can be approximated using (3.19) and the unknown constants $\psi_1^{(nj)}$, $\psi_2^{(nj)}$ and $\psi_3^{(nj)}$ are given by from (3.20) with $\psi_4^{(nj)} = 0$ for $K = 1, 2, 3$ (instead of $K = 1, 2, 3, 4$). As before, we can let $t = \cos([2i - 1]\pi/[2J])$ for $i = 1, 2, \dots, J$, to generate a system of $3JN$ linear algebraic equations to solve for the unknowns.

3.5 Stress and Electric Displacement Intensity Factors

At the tips $(a^{(n)}, b^{(n)})$ and $(c^{(n)}, d^{(n)})$ of the n -th crack $\gamma^{(n)}$, the stress and electric displacement intensity factors are defined as follows:

$$\begin{aligned}
 K_I(a^{(n)}, b^{(n)}) &= \lim_{t \rightarrow -1^-} \sqrt{\frac{\ell^{(n)}}{2}} \sqrt{-2(t+1)} (S_{1j}(X_1^{(n)}(t), X_2^{(n)}(t))m_1^{(n)} \\
 &\quad + S_{2j}(X_1^{(n)}(t), X_2^{(n)}(t))m_2^{(n)})m_j^{(n)}, \\
 K_{II}(a^{(n)}, b^{(n)}) &= \lim_{t \rightarrow -1^-} \sqrt{\frac{\ell^{(n)}}{2}} \sqrt{-2(t+1)} (S_{1j}(X_1^{(n)}(t), X_2^{(n)}(t))m_2^{(n)} \\
 &\quad - S_{2j}(X_1^{(n)}(t), X_2^{(n)}(t))m_1^{(n)})m_j^{(n)}, \\
 K_{III}(a^{(n)}, b^{(n)}) &= \lim_{t \rightarrow -1^-} \sqrt{\frac{\ell^{(n)}}{2}} \sqrt{-2(t+1)} S_{3j}(X_1^{(n)}(t), X_2^{(n)}(t))m_j^{(n)}, \\
 K_{IV}(a^{(n)}, b^{(n)}) &= \lim_{t \rightarrow -1^-} \sqrt{\frac{\ell^{(n)}}{2}} \sqrt{-2(t+1)} S_{4j}(X_1^{(n)}(t), X_2^{(n)}(t))m_j^{(n)}, \\
 K_I(c^{(n)}, d^{(n)}) &= \lim_{t \rightarrow 1^+} \sqrt{\frac{\ell^{(n)}}{2}} \sqrt{2(t-1)} (S_{1j}(X_1^{(n)}(t), X_2^{(n)}(t))m_1^{(n)} \\
 &\quad + S_{2j}(X_1^{(n)}(t), X_2^{(n)}(t))m_2^{(n)})m_j^{(n)}, \\
 K_{II}(c^{(n)}, d^{(n)}) &= \lim_{t \rightarrow 1^+} \sqrt{\frac{\ell^{(n)}}{2}} \sqrt{2(t-1)} (S_{1j}(X_1^{(n)}(t), X_2^{(n)}(t))m_2^{(n)} \\
 &\quad - S_{2j}(X_1^{(n)}(t), X_2^{(n)}(t))m_1^{(n)})m_j^{(n)},
 \end{aligned}$$

$$\begin{aligned}
K_{III}(c^{(n)}, d^{(n)}) &= \lim_{t \rightarrow 1^+} \sqrt{\frac{\ell^{(n)}}{2}} \sqrt{2(t-1)} S_{3j}(X_1^{(n)}(t), X_2^{(n)}(t)) m_j^{(n)}, \\
K_{IV}(c^{(n)}, d^{(n)}) &= \lim_{t \rightarrow 1^+} \sqrt{\frac{\ell^{(n)}}{2}} \sqrt{2(t-1)} S_{4j}(X_1^{(n)}(t), X_2^{(n)}(t)) m_j^{(n)}. \quad (3.21)
\end{aligned}$$

Using (3.16) and (3.19), we find that (3.21) gives

$$\begin{aligned}
K_I(a^{(n)}, b^{(n)}) &\simeq \sqrt{\frac{\ell^{(n)}}{2}} \pi (\chi_{P1}^{(n)} m_1^{(n)} + \chi_{P2}^{(n)} m_2^{(n)}) \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(-1), \\
K_{II}(a^{(n)}, b^{(n)}) &\simeq \sqrt{\frac{\ell^{(n)}}{2}} \pi (\chi_{P1}^{(n)} m_2^{(n)} - \chi_{P2}^{(n)} m_1^{(n)}) \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(-1), \\
K_{III}(a^{(n)}, b^{(n)}) &\simeq -\sqrt{\frac{\ell^{(n)}}{2}} \pi \chi_{P3}^{(n)} \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(-1), \\
K_{IV}(a^{(n)}, b^{(n)}) &\simeq -\sqrt{\frac{\ell^{(n)}}{2}} \pi \chi_{P4}^{(n)} \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(-1), \\
K_I(c^{(n)}, d^{(n)}) &\simeq \sqrt{\frac{\ell^{(n)}}{2}} \pi (\chi_{P1}^{(n)} m_1^{(n)} + \chi_{P2}^{(n)} m_2^{(n)}) \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(+1), \\
K_{II}(c^{(n)}, d^{(n)}) &\simeq \sqrt{\frac{\ell^{(n)}}{2}} \pi (\chi_{P1}^{(n)} m_2^{(n)} - \chi_{P2}^{(n)} m_1^{(n)}) \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(+1), \\
K_{III}(c^{(n)}, d^{(n)}) &\simeq -\sqrt{\frac{\ell^{(n)}}{2}} \pi \chi_{P3}^{(n)} \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(+1), \\
K_{IV}(c^{(n)}, d^{(n)}) &\simeq -\sqrt{\frac{\ell^{(n)}}{2}} \pi \chi_{P4}^{(n)} \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(+1). \quad (3.22)
\end{aligned}$$

3.6 Specific Problems

Some specific cases of the electroelastic crack problem stated in Section 3.2 are solved here using the analysis presented in this Chapter.

Problem 1

Consider the case of a single horizontal straight crack $-a < x_1 < a$, $x_2 = b$, $-\infty < x_3 < \infty$, in the infinitely-long piezoelectric strip with electrical poling along the x_2 direction. Note that $a > 0$ and $0 < b < h$. We take the tips of the crack to be $(a^{(1)}, b^{(1)}) = (a, b)$ and $(c^{(1)}, d^{(1)}) = (-a, b)$. The crack is assumed to be electrically impermeable. A geometrical sketch of the problem is given in Figure 3.2. In Wang and Noda [89], the same problem is formulated in terms of singular integral equations (that is, by using an approach equivalent to modeling the crack as a continuous distribution of dislocations).

Figure 3.2: A horizontal electrically impermeable crack in the strip.

We take the electrical poling direction along the x_2 direction. The relevant material constants and calculation of constants such as τ_α and $A_{K\alpha}$ are calculated as explained in Section 2.2.2.

To compare our results with those in Wang and Noda [89], we use the following material constants:

$$\begin{aligned}
 A &= 12.6 \times 10^{10}, \quad N = 5.5 \times 10^{10}, \quad F = 8.41 \times 10^{10}, \\
 C &= 11.7 \times 10^{10}, \quad L = 2.3 \times 10^{10}, \\
 e_1 &= 17.44, \quad e_2 = -6.5, \quad e_3 = 23.3, \\
 \epsilon_1 &= 150.3 \times 10^{-10}, \quad \epsilon_2 = 130.0 \times 10^{-10}.
 \end{aligned} \tag{3.23}$$

The values of A , N , F , C and L above are in N/m^2 , e_1 , e_2 and e_3 are in C/m^2 , and ϵ_1 and ϵ_2 are in C/(Vm) .

In Figure 3.3, for internal loads on the crack given by $S_{12}^{(0)} = 0$, $S_{22}^{(0)} = \sigma_0$, $S_{32}^{(0)} = 0$ and $S_{42}^{(0)} = 0$ (σ_0 is a positive constant) and for the height ratio $h/a = 4.0$, the normalized crack tip stress and electrical displacement intensity factors $K_I(a, b)/(\sigma_0\sqrt{a})$, $CK_{II}(a, b)/(F\sigma_0\sqrt{a})$ and $CK_{IV}(a, b)/(e_3\sigma_0\sqrt{a})$ are plotted against the normalized distance b/h for $0.10 \leq b/h \leq 0.50$ and compared with the numerical values given by Wang and Noda in [89]. (Note that $K_{III}(a, b) = 0$ for the problem under consideration.) For $0.20 \leq b/h \leq 0.50$, the plots are almost visually indistinguishable from those obtained using the numerical values of Wang and Noda. For b/h nearer to 0, there is, however, a small but noticeable difference between the two sets of values for the intensity factors. Except for b/h smaller than 0.20, the numerical values of the intensity factors are observed to converge to at least 4 significant figures when J in (3.19) is increased from 5 to 10. For b/h which is smaller than 0.20, that is, when there is a stronger interaction between the crack and the edge

$x_2 = 0$, convergence to 2 or more significant figures is observed when we increase J from 10 to 20.

In Figure 3.4, for the crack under uniform internal loads $S_{12}^{(0)} = 0$, $S_{22}^{(0)} = \sigma_0$, $S_{32}^{(0)} = 0$ and $S_{42}^{(0)} = D_0$ (σ_0 is a positive constant and D_0 a non-negative constant) and for $h/a = 4.0$ and $b/h = 0.50$, we plot $K_I(a, b)/(\sigma_0\sqrt{a})$ and $CK_{IV}(a, b)/(e_3\sigma_0\sqrt{a})$ against the normalized electrical load $CD_0/(e_3\sigma_0)$ for $0 \leq CD_0/(e_3\sigma_0) \leq 7.0$. The plots (obtained using $J = 5$ in (3.19)) agree well with the numerical values from Wang and Noda [89].

Figure 3.3: Plots of $K_I(a, b)/(\sigma_0\sqrt{a})$, $CK_{II}(a, b)/(F\sigma_0\sqrt{a})$ and $CK_{IV}(a, b)/(e_3\sigma_0\sqrt{a})$ against b/h .

Figure 3.4: Plots of $K_I(a, b)/(\sigma_0 \sqrt{a})$ and $CK_{IV}(a, b)/(e_3 \sigma_0 \sqrt{a})$ against $CD_0/(e_3 \sigma_0)$.

Problem 2

Here the electrical poling is taken to be along the x_3 direction. Refer to Section 2.2.2 in Chapter 2 for the calculation of the relevant constants required in the solution.

We consider the case of two electrically permeable collinear cracks which are centrally located in the piezoelectric strip, as studied by Li [50]. Specifically, the cracks lie on the plane $x_1 = 0$ and their crack tips are given by $(a^{(1)}, b^{(1)}) = (0, h/2 - d - 2a)$, $(c^{(1)}, d^{(1)}) = (0, h/2 - d)$, $(a^{(2)}, b^{(2)}) = (0, h/2 + d)$ and $(c^{(2)}, d^{(2)}) = (0, h/2 + d + 2a)$, that is, $2a$ is the length of each of the crack and $2d$ is the distance separating the inner crack tips. A geometrical sketch of the problem is given in Figure 3.5. The electrically permeable cracks are acted upon by uniform internal loads given by $S_{11}^{(0)} = 0$, $S_{21}^{(0)} = 0$ and $S_{31}^{(0)} = \tau_0$ (τ_0 is a positive constant).

Figure 3.5: Two electrically permeable collinear crack centrally located in the strip.

Figure 3.6: Plots of $K_{III}(0, h/2 - d - 2a)/(\tau_0 \sqrt{a})$ and $K_{III}(0, h/2 - d)/(\tau_0 \sqrt{a})$ against d/a .

To obtain some numerical results, the relevant material constants are taken to be

$$\begin{aligned} A &= 12.6 \times 10^{10}, \quad N = 5.5 \times 10^{10}, \quad F = 5.3 \times 10^{10}, \\ L &= 3.53 \times 10^{10}, \quad e_1 = 17.0, \quad \epsilon_1 = 151 \times 10^{-10}. \end{aligned} \quad (3.24)$$

The values of C , e_2 , e_3 and ϵ_2 are not given above as they do not play a role in the computation of K_{III} here.

Using the constants in (3.24) and $J = 5$ in (3.19), we compute the normalized stress intensity factors $K_{III}(0, h/2 - d - 2a)/(\tau_0\sqrt{a})$ (at an outer tip) and $K_{III}(0, h/2 - d)/(\tau_0\sqrt{a})$ (at an inner tip) for $h/a = 9$. In Figure 3.6, the non-dimensional stress intensity factors are plotted against d/a for $0.10 \leq d/a \leq 2.40$. The values of the stress intensity factors are in good agreement with those calculated using the analytical formulae given in Li [50].

Problem 3

Consider now three parallel cracks in the infinitely-long piezoelectric strip as sketched in Figure 3.7. Specifically, the middle crack is of length $2a$ and has tips given by $(a, h/2)$ and $(-a, h/2)$. The tips of the crack above the middle crack are $(b, d + h/2)$ and $(-b, d + h/2)$ and those of the crack below are $(b, -d + h/2)$ and $(-b, -d + h/2)$. The top and bottom cracks have equal length $2b$. The uniform internal loads on the electrically impermeable cracks are given by $S_{12}^{(0)} = 0$, $S_{22}^{(0)} = \sigma_0$, $S_{32}^{(0)} = 0$ and $S_{42}^{(0)} = D_0$ with $D_0/\sigma_0 = 10^{-10} \text{ CN}^{-1}$ (σ_0 and D_0 are positive constants). As in Problem 1 above, the electrical poling is in the x_2 direction and the material constants of the strip are given by (3.23). For $h/a = 4$ and $b/a = 1$, plots of the normalized

Figure 3.7: Three parallel cracks in the strip.

Figure 3.8: Plots of $K_I(-a, h/2)/(\sigma_0\sqrt{a})$ and $K_{IV}(-a, h/2)/(D_0\sqrt{a})$ against d/a .

stress intensity factor $K_I(-a, h/2)/(\sigma_0\sqrt{a})$ and the electrical displacement intensity factor $K_{IV}(-a, h/2)/(D_0\sqrt{a})$ at the tip $(-a, h/2)$ of the middle crack against d/a for $0.50 \leq d/a \leq 1.60$ are given in Figure 3.8. From these plots, it may be observed that the intensity factors increase as the other two cracks move away from the middle crack. A plausible explanation for this observation may be given as follows. When the cracks are very near to one another, the stress flow lines are diverted from the tips of the middle crack, giving rise to intensity factors of lower magnitudes. As d/a increases, the stress flow lines diverted by the top and bottom cracks realign themselves perpendicularly to the planes bounding the strip, thereby interacting more strongly with the tips of the middle crack. It is clear that the top and bottom cracks have a shielding effect on the middle crack. The shielding effect can also be observed

Figure 3.9: Plots of $K_I(-a, h/2)/(\sigma_0\sqrt{a})$ and $K_{IV}(-a, h/2)/(D_0\sqrt{a})$ against b/a .

by altering the half crack length b of the top and bottom cracks. For $h/a = 4$ and $d/a = 1$, the normalized stress intensity factor $K_I(-a, h/2)/(\sigma_0\sqrt{a})$ and the electrical displacement intensity factor $K_{IV}(-a, h/2)/(D_0\sqrt{a})$ at the tip $(-a, h/2)$ of the middle crack are plotted against b/a for $0 \leq b/a \leq 1$ in Figure 3.9. It is obvious that the intensity factors decrease with increasing b/a . Their variations are quite slow and gradual as b/a increases from 0 to 0.50 and only start to become more pronounced for $b/a > 0.50$.

Problem 4

Here we study the interaction between two inclined cracks and a horizontal crack. A geometrical sketch of the problem is given in Figure 3.10. Specifically, the horizontal crack lies in the region $-a < x_1 < a$, $x_2 = h/2$, $-\infty < x_3 < \infty$. The tips of the inclined crack on the left are given by $(-d + a \cos \theta, h/2 + a \sin \theta)$ and $(-d - a \cos \theta, h/2 - a \sin \theta)$ and those of the other inclined crack by $(d - a \cos \theta, h/2 + a \sin \theta)$ and $(d + a \cos \theta, h/2 - a \sin \theta)$. The uniform internal loads on the electrically impermeable cracks are given by $S_{12}^{(0)} = 0$, $S_{22}^{(0)} = \sigma_0$, $S_{32}^{(0)} = 0$ and $S_{42}^{(0)} = D_0$ with $D_0/\sigma_0 = 10^{-10} \text{ CN}^{-1}$ (σ_0 and D_0 are positive constants). The electrical poling is in the x_2 direction and the material constants of the strip are given by (3.23).

We examine the effect of the inclined cracks on the stress and electrical displacement intensity factors of the horizontal crack as the distance d changes. For $h/a = 4.0$, Figures 3.11 and 3.12 give plots of intensity factors $K_I(-a, h/2)/(\sigma_0\sqrt{a})$ and $K_{IV}(-a, h/2)/(D_0\sqrt{a})$ respectively against $(d - a)/a$ for $0.50 \leq (d - a)/a \leq 3.5$ for three different values of the angle θ . As expected, we observe that each of

Figure 3.10: Two inclined cracks and a horizontal crack.

Figure 3.11: Plots of $K_I(-a, h/2)/(\sigma_0\sqrt{a})$ against $(d-a)/a$.

the intensity factors tends to a fixed value for all the three values of the angle θ , as the distance $(d - a)/a$ increases. For $0 \leq \theta \leq \pi/2$, the inclined cracks appear to have a greater influence on the intensity factors of the horizontal crack if the angle θ is smaller. For $\theta = \pi/6$ and $\theta = \pi/3$, each of the intensity factors has a peak value at a particular value of $(d - a)/a$. It may be of some interest to note that the variations of $K_I(-a, h/2)/(\sigma_0\sqrt{a})$ with $(d - a)/a$ are qualitatively the same with those of $K_{IV}(-a, h/2)/(D_0\sqrt{a})$.

Figure 3.12: Plots of $K_{IV}(-a, h/2)/(D_0\sqrt{a})$ against $(d - a)/a$.

For $\theta = \pi/6$, $K_I(-a, h/2)/(\sigma_0\sqrt{a})$ is found to be negative when the normalized distance $(d - a)/a$ is smaller than 0.50. This suggests that the inclined cracks may possibly generate a compressive load on the horizontal crack near its tips. Thus, depending on the angle θ and the distance $(d - a)/a$, opposite faces of the cracks

in Figure 3.10 may possibly come into contact with each other near the crack tips. The solution in Section 3.4 assumes that the cracks open up completely under the action of suitably prescribed internal tractions and hence may not be physically valid if crack closure occurs.

Figure 3.13: Plots of $K_I(-a, h/2)/(\sigma_0\sqrt{a})$, $K_{II}(-a, h/2)/(\sigma_0\sqrt{a})$ and $K_{IV}(-a, h/2)/(D_0\sqrt{a})$ against $(d-a)/a$ for $\theta = \pi/4$.

For either $(d-a)/a \rightarrow \infty$ or $\theta = \pi/2$, the mode II crack tip stress intensity factor of the horizontal crack is zero (since $S_{12}^{(0)} = 0$). In general, the presence of the inclined cracks may, however, cause a mode II deformation at the tips of the horizontal cracks. In Figure 3.13, for $h/a = 4.0$ and $\theta = \pi/4$, the normalized intensity factors $K_I(-a, h/2)/(\sigma_0\sqrt{a})$, $K_{II}(-a, h/2)/(\sigma_0\sqrt{a})$ and $K_{IV}(-a, h/2)/(D_0\sqrt{a})$ (at the tip $(-a, h/2)$ of the horizontal crack) are plotted against $(d-a)/a$ for $0.30 \leq (d-a)/a \leq 3.5$. Note that, as before, the variation of $K_I(-a, h/2)/(\sigma_0\sqrt{a})$ with

$(d - a)/a$ shows the same qualitative feature as that of $K_{IV}(-a, h/2)/(D_0\sqrt{a})$. For $(d - a)/a \geq 1$, the mode II stress intensity factor $K_{II}(-a, h/2)/(\sigma_0\sqrt{a})$ is relatively small in magnitude. The effect of the inclined cracks on $K_{II}(-a, h/2)/(\sigma_0\sqrt{a})$ becomes more pronounced as the distance $(d - a)/a$ decreases. From Figure 3.13, it appears that the value of the intensity factors $K_I(-a, h/2)/(\sigma_0\sqrt{a})$, $K_{II}(-a, h/2)/(\sigma_0\sqrt{a})$ and $K_{IV}(-a, h/2)/(D_0\sqrt{a})$ increase rapidly as the crack tip $(-a, h/2)$ of the horizontal crack approaches the inclined cracks.

3.7 Summary

Hypersingular integral equations are derived for an arbitrary number of arbitrarily oriented straight cracks in an infinitely long piezoelectric strip. The unknown functions in the integral equations are given by the displacement and electric potential jumps across opposite crack faces. Once the unknown functions are determined, the stress and electric displacement intensity factors at the tips of each crack can be easily computed using explicit formulae.

The hypersingular integral equations are solved for specific cases of the problem under consideration. For two of the cases, the computed values of crack tip stress and electric displacement intensity factors agree well with those published in the literature, thus verifying the validity of the solution presented here. The crack tip intensity factors for the other cases which have not been previously solved exhibit qualitative features which are physically interesting as well as intuitively acceptable.

Chapter 4

Numerical Green's Functions and Boundary Elements

4.1 Introduction

A well established boundary element approach for solving crack problems is to use special Green's functions (modified fundamental solutions) chosen to satisfy the boundary conditions on the cracks. With an appropriate Green's function, the boundary integral formulation of the crack problem under consideration does not require integration over the crack faces. Consequently, difficulties associated with modeling the crack faces, such as singular stress at the crack tips and degenerate systems of linear algebraic equations, may be neatly avoided.

Such an approach for solving crack problems numerically was pioneered by Snyder and Cruse [79] when they derived an analytical Green's function for a single planar crack in an orthotropic elastic space of infinite extent. Subsequently, Clements and Haselgrove [22] extended the work in [79] to a general anisotropic elastic space, and Ang and Clements [6] further modified the Green's function to include the case of a fully closed planar crack. Special Green's functions for a planar crack and an arc crack in an isotropic elastic space were derived by Ang in [2] and [3] respectively.

In general, it is difficult to derive Green's functions analytically for cracks with arbitrary geometries, configurations and boundary conditions. To solve a wider range of crack problems, Telles, Castor and Guimarães [84] proposed to derive the required

Green's function numerically based on the hypersingular integral formulation of a suitable crack problem (see also Guimarães and Telles [34]). More recently, Ang and Telles [9] extended the numerical Green's function approach in [84] to solve an elastostatic problem involving multiple interacting planar cracks in an anisotropic body.

During the last ten years or so, there has been considerable interest in the development of the boundary element method for fracture analysis of piezoelectric materials. Using the Lekhnitskii's formalism and dislocation modeling, Rajapakse and Xu [70] obtained an analytical Green's function for a single traction free and electrically impermeable crack in a piezoelectric space. More recently, Garcia-Sanchez, Saez and Dominguez [31] and Groh and Kuna [33] presented boundary element procedures based on boundary integral equations derived by using fundamental solution which does not satisfy the boundary conditions on the crack faces. In [33], opposite crack faces were modeled by using the so called subdomain technique and quarter-point elements were employed to deal with the singular behaviors of the stress and electric displacement at the crack tips, while a dual (mixed) boundary integral formulation was used in [31] with the conditions on the cracks treated by a differentiated form of the usual boundary integral equations. Earlier works on boundary element methods for electroelastic crack problems include Pan [66], Xu and Rajapakse [92], Ding, Wang and Chen [25] and Gao and Fan [30].

In this chapter², using the hypersingular integral approach, we derive numerical Green's functions for an arbitrary number of arbitrarily located planar cracks in an infinite piezoelectric space. The Green's functions are chosen to satisfy certain electro-elastic boundary conditions on the cracks. Specifically, the boundary conditions are such that the cracks are stress-free and electrically either permeable or impermeable. With the use of the Green's functions, a boundary integral solution which does not require integration over the crack faces is obtained for a plane electro-elastic problem involving planar cracks in a piezoelectric body. A simple boundary element procedure is outlined for the numerical solution of the crack problem. The displacement and electric potential jumps across opposite crack faces as well as the crack tip stress and electric displacement intensity factors may be readily and accurately computed once the elastic displacements, tractions, electric potential and electric displacement are all known on the boundary. To check the validity of the numerical Green's functions, the boundary element procedure is applied to solve some specific problems.

There are other numerical approaches, such as the finite element method, for fracture analysis. In the finite element method [12], the entire solution domain has to be discretized into elements. For problems involving domains of infinite extent, the remote boundary has to be artificially introduced. Thus, finite element formulations result in relatively larger systems of linear algebraic equations to be solved. In the

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boundary integral approach, the method of superposition may be used ensure that the far-field boundary conditions are exactly satisfied without having to introduce an artificial remote boundary.

The modeling of the cracks as cuts or discontinuity in the solution domain and the treatment of the singular behaviors of the crack tip stress and electric displacement are challenging issues for numerical methods. Special elements are used for cracks in finite element analysis of fracture problems (see, for example, Piltner [71]). In the Green's function boundary element approach here, the two-dimensional cracks can be geometrically modeled as lines and the singular behaviors of the crack tip stress and electric displacement are analytically built into the boundary integral equations. This allows for easy extraction and accurate computation of the crack tip stress and electric displacement intensity factors. The evaluation of the special Green's functions may, however, require some computational efforts.

4.2 Statement of Problem

With reference to a Cartesian coordinate frame denoted by $0x_1x_2x_3$, consider a homogeneous piezoelectric solid which contains M arbitrarily orientated planar cracks as shown in Figure 4.1. The geometries of the solid and the cracks do not change along the x_3 direction. The interior of the solid is denoted by R , the exterior boundary by B and the k -th crack by $\gamma^{(k)}$. It is assumed that the cracks do not intersect with one another or the exterior boundary B . On the plane $x_3 = 0$, the boundary B appears as a simple closed curve and the crack $\gamma^{(k)}$ as a straight cut with tips $(a^{(k)}, b^{(k)})$

Figure 4.1: A geometrical sketch of the problem.

and $(c^{(k)}, d^{(k)})$. For the purpose of the present analysis, $\gamma^{(k)}$ is taken to be the directed straight line segment from $(a^{(k)}, b^{(k)})$ to $(c^{(k)}, d^{(k)})$.

At each and every point on the boundary B , either the displacements or the tractions and either the electric potential or the electric displacement vector are prescribed. The prescribed conditions on B are independent of the spatial coordinate x_3 and time t and are such that the cracks become stress-free. Also, as before, we consider separately two extreme cases: (a) electrically impermeable cracks and (b) electrically permeable cracks. The conditions on the cracks can be written as

$$S_{Ij}(x_1, x_2)m_j^{(k)} \rightarrow 0 \quad (4.1)$$

as $(x_1, x_2) \rightarrow (\xi_1, \xi_2) \in \gamma^{(k)}$ ($k = 1, 2, \dots, N$) for $I = 1, 2, 3$,

and either

$$\begin{aligned}
 S_{4j}(x_1, x_2)m_j^{(k)} &\rightarrow 0 \\
 \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \gamma^{(k)} \ (k = 1, 2, \dots, N) \\
 &\text{if the cracks are electrically impermeable,} \quad (4.2)
 \end{aligned}$$

or

$$\begin{aligned}
 \Delta U_4(x_1, x_2) &\rightarrow 0 \text{ as } (x_1, x_2) \rightarrow (\xi_1, \xi_2) \in \gamma^{(k)} \text{ for } k = 1, 2, \dots, N \\
 &\text{if the cracks are electrically permeable.} \quad (4.3)
 \end{aligned}$$

where $[m_1^{(k)}, m_2^{(k)}, m_3^{(k)}] = [(d^{(k)} - b^{(k)})/\ell^{(k)}, (a^{(k)} - c^{(k)})/\ell^{(k)}, 0]$ is a unit normal vector to the crack $\gamma^{(k)}$, $\ell^{(k)} = \sqrt{(d^{(k)} - b^{(k)})^2 + (a^{(k)} - c^{(k)})^2}$ is the length of $\gamma^{(k)}$ and ΔU_4 is the jump in the electric potential ϕ across opposite crack faces.

The problem is to determine the displacements U_K throughout the cracked piezoelectric solid.

4.3 Numerical Green's Functions

For the multiple planar cracks, we seek to construct a special Green function Φ_{KR} as defined by (2.39) with

$$\begin{aligned}
 \Xi_{IjR}(x_1, x_2; \xi_1, \xi_2)m_j^{(k)} &\rightarrow 0 \\
 \text{as } (x_1, x_2) &\rightarrow (y_1, y_2) \in \gamma^{(k)} \text{ for } I = 1, 2, 3, 4 \text{ and } k = 1, 2, \dots, M \\
 &\text{if the cracks are electrically impermeable,} \quad (4.4)
 \end{aligned}$$

or

$$\begin{aligned} \Xi_{IjR}(x_1, x_2; \xi_1, \xi_2)m_j^{(k)} &\rightarrow 0, \Delta\Phi_{4R}(x_1, x_2; \xi_1, \xi_2) \rightarrow 0 \\ \text{and } \Delta\Xi_R(x_1, x_2; \xi_1, \xi_2) &\rightarrow 0 \text{ as } (x_1, x_2) \rightarrow (y_1, y_2) \in \gamma^{(k)} \\ \text{for } I = i = 1, 2, 3 \text{ and } k &= 1, 2, \dots, M \end{aligned}$$

$$\text{if the cracks are electrically permeable,} \quad (4.5)$$

$\Delta\Xi_R$ is defined by

$$\begin{aligned} \Delta\Xi_R(x_1, x_2; \xi_1, \xi_2) &= \lim_{\varepsilon \rightarrow 0} [\Xi_{4jR}(x_1 - |\varepsilon|m_1^{(k)}, x_2 - |\varepsilon|m_2^{(k)}; \xi_1, \xi_2) \\ &\quad - \Xi_{4jR}(x_1 + |\varepsilon|m_1^{(k)}, x_2 + |\varepsilon|m_2^{(k)}; \xi_1, \xi_2)]m_j^{(k)} \\ &\quad \text{for } (x_1, x_2) \in \gamma^{(k)}, \end{aligned} \quad (4.6)$$

and $\Delta\Phi_{4R}$ denotes the jump of Φ_{4R} across opposite crack faces as defined in (4.10) below. Note that

$$\Xi_{IjR}(x_1, x_2; \xi_1, \xi_2) = C_{IjK\ell} \frac{\partial}{\partial x_\ell} [\Phi_{KR}(x_1, x_2; \xi_1, \xi_2)]. \quad (4.7)$$

Guided by the analysis in Ang and Park [8] and Ang and Telles [9], we take $\Phi_{RS}^*(x_1, x_2; \xi_1, \xi_2)$ in (2.39) to be given by

$$\Phi_{RS}^*(x_1, x_2; \xi_1, \xi_2) = \sum_{k=1}^M \int_{\gamma^{(k)}} \Delta\Phi_{PS}(y_1, y_2; \xi_1, \xi_2) \Lambda_{PR}^{(k)}(x_1, x_2; y_1, y_2) ds(y_1, y_2), \quad (4.8)$$

where

$$\begin{aligned} \Lambda_{IS}^{(k)}(x_1, x_2; y_1, y_2) &= -\frac{1}{2\pi} \operatorname{Re} \sum_{\alpha=1}^4 \left\{ \frac{T_{Ij\alpha S} m_j^{(k)}}{[x_1 - y_1] + \tau_\alpha [x_2 - y_2]} \right\}, \\ T_{Ij\alpha S} &= L_{Ij\alpha} N_{\alpha R} d_{RS}, \end{aligned} \quad (4.9)$$

and

$$\begin{aligned}\Delta\Phi_{PS}(x_1, x_2; \xi_1, \xi_2) &= \lim_{\varepsilon \rightarrow 0} [\Phi_{PS}(x_1 - |\varepsilon|m_1^{(k)}, x_2 - |\varepsilon|m_2^{(k)}; \xi_1, \xi_2) \\ &\quad - \Phi_{PS}(x_1 + |\varepsilon|m_1^{(k)}, x_2 + |\varepsilon|m_2^{(k)}; \xi_1, \xi_2)] \\ &\quad \text{for } (x_1, x_2) \in \gamma^{(k)}.\end{aligned}\quad (4.10)$$

Note that the integration over $\gamma^{(k)}$ in (4.8) is one over a directed straight line segment from $(a^{(k)}, b^{(k)})$ to $(c^{(k)}, d^{(k)})$. It is assumed that (ξ_1, ξ_2) does not lie on any of the cracks.

4.3.1 Electrically Impermeable Cracks

The conditions (4.4) on the electrically impermeable cracks require that

$$\begin{aligned}C_{IjRp} \frac{\partial \Phi_{RS}^*}{\partial x_p} m_j^{(k)} &\rightarrow \Lambda_{IS}^{(k)}(x_1, x_2, \xi_1, \xi_2) \\ \text{as } (x_1, x_2) &\rightarrow (y_1, y_2) \in \gamma^{(k)} \text{ for } k = 1, 2, \dots, M.\end{aligned}\quad (4.11)$$

From (4.8), the conditions (4.11) for electrically impermeable cracks give rise to the system of hypersingular integral equations

$$\begin{aligned}&\mathcal{H} \int_{-1}^1 \frac{\chi_{PK}^{(q)} \Delta\Phi_{PS}^{(q)}(v, \xi_1, \xi_2) dv}{(t-v)^2} + \sum_{\substack{n=1 \\ n \neq q}}^M \int_{-1}^1 \Delta\Phi_{PS}^{(n)}(v, \xi_1, \xi_2) Y_{PK}^{(nq)}(v, t) dv \\ &= \Lambda_{KS}^{(q)}(X_1^{(q)}(t), X_2^{(q)}(t), \xi_1, \xi_2) \\ &\quad \text{for } -1 < t < 1, K = 1, 2, 3, 4, S = 1, 2, 3, 4 \text{ and } q = 1, 2, \dots, M,\end{aligned}\quad (4.12)$$

where $\mathcal{H}$ indicates that the integral is to be interpreted in the Hadamard finite-part sense and

$$\begin{aligned}\Delta\Phi_{PS}^{(n)}(v, \xi_1, \xi_2) &= \Delta\Phi_{PS}(X_1^{(n)}(v), X_2^{(n)}(v); \xi_1, \xi_2), \\ \chi_{PK}^{(q)} &= \frac{1}{\pi} \operatorname{Re} \sum_{\alpha=1}^4 \left\{ \frac{\ell^{(q)} Q_{PKrj\alpha} m_r^{(q)} m_j^{(q)}}{[(c^{(q)} - a^{(q)}) + \tau_\alpha (d^{(q)} - b^{(q)})]^2} \right\}, \\ Y_{PK}^{(nq)}(v, t) &= \frac{1}{4\pi} \operatorname{Re} \sum_{\alpha=1}^4 \left\{ \frac{\ell^{(n)} Q_{PKrj\alpha} m_r^{(q)} m_j^{(n)}}{[\Xi^{(nq)}(v, t) + \tau_\alpha \Theta^{(nq)}(v, t)]^2} \right\},\end{aligned}\quad (4.13)$$

where $Q_{PKrj\alpha} = (c_{KrI1} + \tau_\alpha c_{KrI2}) T_{Pj\alpha I}$, $\Xi^{(nq)}(v, t) = X_1^{(n)}(v) - X_1^{(q)}(t)$, $\Theta^{(nq)}(v, t) = X_2^{(n)}(v) - X_2^{(q)}(t)$, $2X_1^{(n)}(t) = [c^{(n)} + a^{(n)}] + [c^{(n)} - a^{(n)}]t$ and $2X_2^{(n)}(t) = [d^{(n)} + b^{(n)}] + [d^{(n)} - b^{(n)}]t$.

The method in Kaya and Erdogan [43] as explained in Subsection 2.5.2 is chosen to solve (4.12) numerically for $\Delta\Phi_{PS}^{(n)}(v, \xi_1, \xi_2)$. Let $\Delta\Phi_{PS}^{(n)}(v, \xi_1, \xi_2)$ be given approximately by

$$\Delta\Phi_{PS}^{(n)}(v, \xi_1, \xi_2) \simeq \sqrt{1-v^2} \sum_{j=1}^J \phi_{PS}^{(nj)}(\xi_1, \xi_2) U^{(j-1)}(v), \quad (4.14)$$

where $U^{(j)}(x)$ is the j -th order Chebyshev polynomial of the second kind and the parameters $\phi_{PS}^{(nj)}(\xi_1, \xi_2)$ are to be determined.

Through substituting (4.14) into (4.12) and collocating (4.12) by letting $t = t^{(i)} \equiv \cos([2i-1]\pi/[2J])$ for $i = 1, 2, \dots, J$, a system of linear algebraic equations

containing the unknowns $\phi_{PS}^{(ni)}(\xi_1, \xi_2)$ can be obtained as follows:

$$\begin{aligned}
& - \sum_{j=1}^J j\pi \phi_{PS}^{(qj)}(\xi_1, \xi_2) \chi_{PK}^{(q)} U^{(j-1)}(t^{(i)}) \\
& + \sum_{j=1}^J \sum_{\substack{n=1 \\ n \neq q}}^M \phi_{PS}^{(nj)}(\xi_1, \xi_2) \int_{-1}^1 \sqrt{1-v^2} U^{(j-1)}(v) Y_{PK}^{(nq)}(v, t^{(i)}) dv \\
& = \Lambda_{KS}^{(q)}(X_1^{(q)}(t^{(i)}), X_2^{(q)}(t^{(i)}), \xi_1, \xi_2)
\end{aligned}$$

for $i = 1, 2, \dots, J$, $K = 1, 2, 3, 4$, $S = 1, 2, 3, 4$ and $q = 1, 2, \dots, M$.

(4.15)

Once $\phi_{PS}^{(ni)}(\xi_1, \xi_2)$ are determined from (4.15), $\Phi_{RS}^*(x_1, x_2, \xi_1, \xi_2)$ can be calculated approximately using

$$\begin{aligned}
\Phi_{RS}^*(x_1, x_2, \xi_1, \xi_2) & \simeq \frac{1}{2} \sum_{n=1}^M \ell^{(n)} \sum_{j=1}^J \phi_{PS}^{(nj)}(\xi_1, \xi_2) \int_{-1}^1 \sqrt{1-t^2} \\
& \times U^{(j-1)}(t) \Lambda_{PR}^{(n)}(x_1, x_2, X_1^{(n)}(t), X_2^{(n)}(t)) dt. \quad (4.16)
\end{aligned}$$

If the points (x_1, x_2) and (ξ_1, ξ_2) do not lie on any of the cracks, the numerical evaluation of $\Phi_{RS}^*(x_1, x_2, \xi_1, \xi_2)$ as given by (4.16) does not pose any mathematical difficulties. The definite integrals over the interval $[-1, 1]$ in (4.15) and (4.16) can be easily and accurately computed by using the numerical quadrature formula (25.4.40) listed in Abramowitz and Stegun [1].

Note that in (4.15) the coefficient of the unknown $\phi_{PS}^{(qj)}(\xi_1, \xi_2)$ is independent of the uppercase Latin subscript S and the point (ξ_1, ξ_2) . Thus, in solving (4.15) to determine $\phi_{PS}^{(qj)}(\xi_1, \xi_2)$ for different values of the subscript S and for different points (ξ_1, ξ_2) , the square matrix containing the coefficients of the unknowns has to be computed and processed only once. For example, if the LU decomposition

technique together with backward substitutions is used to solve (4.15), we have to decompose the square matrix only once.

4.3.2 Electrically Permeable Cracks

Conditions (4.5) for electrically permeable cracks require the hypersingular integral equations (4.12) to be modified by taking $\Delta\Phi_{4S}^{(n)}(v, \xi_1, \xi_2) = 0$ and replacing $K = 1, 2, 3, 4$ with $K = k = 1, 2, 3$. Consequently, if the cracks are electrically permeable, $\Phi_{RS}^*(x_1, x_2, \xi_1, \xi_2)$ can still be computed by using (4.16) but with $\phi_{4S}^{(nj)}(\xi_1, \xi_2) = 0$. The remaining functions $\phi_{1S}^{(nj)}(\xi_1, \xi_2)$, $\phi_{2S}^{(nj)}(\xi_1, \xi_2)$ and $\phi_{3S}^{(nj)}(\xi_1, \xi_2)$ required by (4.16) for computing $\Phi_{RS}^*(x_1, x_2, \xi_1, \xi_2)$ are to be determined by solving the system (4.15) (with $\phi_{4S}^{(nj)}(\xi_1, \xi_2) = 0$) for $K = k = 1, 2, 3$ (instead of $K = 1, 2, 3, 4$).

4.4 Boundary Element Method

If the Green's function $\Phi_{IK}(x_1, x_2, \xi_1, \xi_2)$ satisfying either (4.4) or (4.5) (depending on the electrical boundary conditions on the cracks), as given in Section 4.3 is used, a direct boundary integral formulation for the crack problem considered in this Chapter is given by:

$$\begin{aligned} \lambda(\xi_1, \xi_2)U_K(\xi_1, \xi_2) &= \int_B [U_I(x_1, x_2)\Gamma_{IK}(x_1, x_2; \xi_1, \xi_2) \\ &\quad - P_I(x_1, x_2)\Phi_{IK}(x_1, x_2, \xi_1; \xi_2)]ds(x_1, x_2). \end{aligned} \quad (4.17)$$

Note that the path of integration in (4.17) is over only the exterior boundary B of the piezoelectric solid.

From the boundary conditions on the exterior boundary B , either $U_I = u_i$ or $P_I = p_i$ for $I = i = 1, 2, 3$, and either $U_4 = \phi$ or P_4 are known at each and every point on B . The boundary B and the integral equations (4.17) can be discretized to determine approximately the unknown generalized displacements U_I and/or tractions P_I on B . To do this, the boundary B is approximated using N straight line segments denoted by $B^{(1)}, B^{(2)}, \dots, B^{(N-1)}$ and $B^{(N)}$. Across the segment $B^{(m)}$, the displacements U_I and the tractions P_I are approximated by constants $U_I^{(m)}$ and $P_I^{(m)}$ respectively, where $U_I^{(m)}$ and $P_I^{(m)}$ are the displacements and tractions at the midpoint of the line segment $B^{(m)}$. Through approximating (4.17), the unknown constants on the boundary elements $U_I^{(m)}$ and/or tractions $P_I^{(m)}$ can be determined from the system of linear algebraic equations:

$$\begin{aligned} \frac{1}{2}U_K^{(m)} &= \sum_{n=1}^N U_I^{(n)} \int_{B^{(n)}} \Gamma_{IK}(x_1, x_2; \xi_1^{(m)}, \xi_2^{(m)}) ds(x_1, x_2) \\ &\quad - \sum_{n=1}^N P_I^{(n)} \int_{B^{(n)}} \Phi_{IK}(x_1, x_2; \xi_1^{(m)}, \xi_2^{(m)}) ds(x_1, x_2) \\ &\quad \text{for } m = 1, 2, \dots, N, \end{aligned} \quad (4.18)$$

where $(\xi_1^{(m)}, \xi_2^{(m)})$ is the midpoint of $B^{(m)}$. The system (4.18) generates $4N$ equations each with $4N$ unknowns. Since either $U_I^{(m)}$ or $P_I^{(m)}$ is known as the boundary condition of the problem, the unknown value can be found by solving (4.18).

Once $U_I^{(m)}$ and $P_I^{(m)}$ are all determined, the generalized displacements U_K (and hence the stresses S_{Ij}) at any interior point (ξ_1, ξ_2) in R can be computed approxi-

mately using

$$\begin{aligned}
 U_K(\xi_1, \xi_2) &= \sum_{n=1}^N U_I^{(n)} \int_{B^{(n)}} \Gamma_{IK}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2) \\
 &\quad - \sum_{n=1}^N P_I^{(n)} \int_{B^{(n)}} \Phi_{IK}(x_1, x_2; \xi_1, \xi_2) ds(x_1, x_2). \quad (4.19)
 \end{aligned}$$

4.4.1 Electrically Impermeable Cracks

If the cracks are electrically impermeable then $\Delta U_K(x_1, x_2)$ can be determined by solving the hypersingular integral equations (see Ang and Telles [9]):

$$\begin{aligned}
 \mathcal{H} \int_{-1}^1 \frac{\chi_{PK}^{(q)} \Delta U_P^{(q)}(v) dv}{(t-v)^2} + \sum_{\substack{n=1 \\ n \neq q}}^M \int_{-1}^1 \Delta U_P^{(n)}(v) Y_{PK}^{(nq)}(v, t) dv &= S_K^{(q)}(t) \\
 \text{for } -1 < t < 1, K = 1, 2, 3, 4 \text{ and } q = 1, 2, \dots, M, & \\
 & \quad (4.20)
 \end{aligned}$$

where $\Delta U_P^{(q)}(v)$ ($-1 < v < 1$) is a function that gives $\Delta U_P(x_1, x_2)$ at the point $(X_1^{(q)}(v), X_2^{(q)}(v))$ of the crack $\gamma^{(q)}$, and

$$\begin{aligned}
 S_K^{(q)}(t) &= \sum_{n=1}^N C_{KjRs} m_j^{(q)} \int_{B^{(n)}} \left\{ -U_I^{(n)} \frac{\partial}{\partial \xi_s} [\Gamma_{IR}^0(x_1, x_2, \xi_1, \xi_2)] \right\} \Big|_{(\xi_1, \xi_2) = (X_1^{(q)}(t), X_2^{(q)}(t))} \\
 &\quad + P_I^{(n)} \frac{\partial}{\partial \xi_s} [\Phi_{IR}^0(x_1, x_2, \xi_1, \xi_2)] \Big|_{(\xi_1, \xi_2) = (X_1^{(q)}(t), X_2^{(q)}(t))} \} ds(x_1, x_2), \quad (4.21)
 \end{aligned}$$

where

$$\begin{aligned}
 \Phi_{RS}^0(x_1, x; \xi_1, \xi_2) &= \frac{1}{2\pi} \operatorname{Re} \sum_{\alpha=1}^4 \{ A_{R\alpha} N_{\alpha J} \ln([x_1 - \xi_1] + \tau_\alpha [x_2 - \xi_2]) \} d_{JS}, \\
 \Gamma_{IK}^0(x_1, x_2; \xi_1, \xi_2) &= C_{IjRs} n_j(x_1, x_2) \frac{\partial}{\partial x_s} [\Phi_{RK}^0(x_1, x_2, \xi_1, \xi_2)]. \quad (4.22)
 \end{aligned}$$

Note that the system (4.20) is derived using the boundary conditions in (4.1) and (4.2) and $S_K^{(q)}(t)$ is regarded as known after (4.18) is solved.

The system (4.20) can be solved numerically using the same method for (4.12). The unknown functions $\Delta U_P^{(n)}(v)$ are approximated using

$$\Delta U_P^{(n)}(v) \simeq \sqrt{1-v^2} \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(v), \quad (4.23)$$

where $\psi_P^{(nj)}$ are constants determined by the system of linear algebraic equations

$$\begin{aligned} & - \sum_{j=1}^J j\pi \psi_P^{(qj)} \chi_{PK}^{(q)} U^{(j-1)}(t^{(i)}) \\ & + \sum_{j=1}^J \sum_{\substack{n=1 \\ n \neq q}}^M \psi_P^{(nj)} \int_{-1}^1 \sqrt{1-v^2} U^{(j-1)}(v) Y_{PK}^{(nq)}(v, t^{(i)}) dv \\ & = S_K^{(q)}(t^{(i)}) \text{ for } i = 1, 2, \dots, J, K = 1, 2, 3, 4 \text{ and } q = 1, 2, \dots, M, \end{aligned} \quad (4.24)$$

where $t^{(i)} = \cos([2i-1]\pi/[2J])$ as in (4.15).

Note that the unknown $\psi_P^{(qj)}$ in (4.24) has the same coefficient as $\phi_{PS}^{(qj)}(\xi_1, \xi_2)$ in (4.15). Thus, in solving (4.24) for the unknowns $\psi_P^{(qj)}$, it is not necessary to set up and process again the matrix containing the coefficients of the unknowns.

Once the unknowns $\psi_P^{(qj)}$ are determined, $\Delta U_K(x_1, x_2)$ can be approximately computed using (4.23) and the stress and electric displacement intensity factors as defined in (3.21) can then be computed using (3.22).

4.4.2 Electrically Permeable Cracks

If the cracks are electrically permeable then (4.20) has to be modified by setting $\Delta U_4^{(n)}(v) = 0$ and replacing $K = 1, 2, 3, 4$ by $K = k = 1, 2, 3$. It follows that we

can solve (4.24), with $\psi_4^{(qj)} = 0$ and $K = k = 1, 2, 3$, for $\psi_1^{(qj)}$, $\psi_2^{(qj)}$ and $\psi_3^{(qj)}$ in order to determine $\Delta U_1^{(n)}(v)$, $\Delta U_2^{(n)}(v)$ and $\Delta U_3^{(n)}(v)$. As before, the stress and electric displacement intensity factors can be computed using (3.22).

4.5 Specific Problems

In this Section, the boundary element procedure together with the numerical Green's functions is applied to solve some specific problems involving certain piezoelectric materials.

Problem 1

The exterior boundary of the solution domain R (on the plane $x_3 = 0$) is taken to be the sides of a square with vertices (h, h) , $(-h, h)$, $(-h, -h)$ and $(h, -h)$. The interior of R contains a single electrically impermeable crack which occupies the region $-a < x_1 < a$, $x_2 = 0$, where h and a are positive constants such that $a < h$. Here we take the crack tips to be $(a^{(1)}, b^{(1)}) = (-a, 0)$ and $(c^{(1)}, d^{(1)}) = (a, 0)$. The configuration of the crack is as shown in Figure 4.2. The electrical poling direction is taken to be along the x_2 direction. Refer Section 2.2.2 for the derivation of the material matrices.

Figure 4.2: A sketch of Problem 1. For the boundary conditions, the electroelastic fields in (4.25) are used to generate generalized displacements and tractions respectively on the horizontal and vertical sides of the square domain.

A particular solution U_K satisfying (2.8) in the whole of the $0x_1x_2$ plane with a cut in the region $-a < x_1 < a$, $x_2 = 0$ and the corresponding S_{Kj} are given by

$$\begin{aligned} U_K &= \operatorname{Re}\left\{\sum_{\alpha=1}^4 A_{K\alpha}(M_{\alpha 2} + M_{\alpha 4})(z_{\alpha}^2 - a^2)^{1/2}\right\}, \\ S_{Kj} &= \operatorname{Re}\left\{\sum_{\alpha=1}^4 L_{Kj\alpha}(M_{\alpha 2} + M_{\alpha 4})\frac{z_{\alpha}}{(z_{\alpha}^2 - a^2)^{1/2}}\right\}, \end{aligned} \quad (4.25)$$

where $z_{\alpha} = x_1 + \tau_{\alpha}x_2$ and $[M_{\alpha S}]$ is the inverse matrix of $[L_{K2\alpha}]$. It may be verified that with (4.25) the conditions that the crack $-a < x_1 < a$, $x_2 = 0$ is traction-free and electrically impermeable are satisfied, that is, $S_{K2} = 0$ (for $K = 1, 2, 3$ and 4) on the crack. Note that the branch for $(z_{\alpha}^2 - a^2)^{1/2}$ in (4.25) is chosen such that

$$\lim_{|z_{\alpha}| \rightarrow \infty} \frac{(z_{\alpha}^2 - a^2)^{1/2}}{z_{\alpha}} = 1. \quad (4.26)$$

For a particular test problem involving the electrically impermeable crack, (4.25) is used to generate boundary values of U_K and P_K on the horizontal and vertical sides of the square domain R respectively. The material constants of a class of PZT4 piezoceramics is used in the calculation, that is,

$$\begin{aligned} A &= 13.9 \times 10^{10}, N = 7.78 \times 10^{10}, F = 7.43 \times 10^{10}, \\ C &= 11.3 \times 10^{10}, L = 2.56 \times 10^{10}, \\ e_1 &= 13.44, e_2 = -6.98, e_3 = 13.84, \\ \epsilon_1 &= 60 \times 10^{-10}, \epsilon_2 = 54.7 \times 10^{-10}. \end{aligned} \quad (4.27)$$

The values of A , N , F , C and L above are in N/m^2 , e_1 , e_2 and e_3 are in C/m^2 , and ϵ_1 and ϵ_2 are in C/(Vm) . These values are taken from Park and Sun [68].

Each side of the square region $-h < x_1 < h$, $-h < x_2 < h$ is discretized into N_0 equal length elements, so that the total number of elements is $4N_0$. For $a = 1$ and $h = 2$, two sets of numerical calculations are carried out using the boundary element method. Set A is obtained by using $N_0 = 10$ (40 elements), while Set B by $N_0 = 40$ (160 elements). The numerical Green's function for the impermeable crack is calculated using $J = 10$ in (4.16).

Numerical values of the elastic displacement ($U_1 \times 10^{12}$, $U_2 \times 10^{12}$) and the electric potential $U_4 \times 10^3$ at selected points in the interior of the square domain are compared with the exact values computed using (4.25) in Tables 4.1 and 4.2 respectively. (Note that $U_3 = 0$ for the particular problem here.) Both sets of numerical values for U_1 , U_2 and U_4 are reasonably close to the exact ones. On the whole, the

Table 4.1: Numerical and exact values of $(U_1 \times 10^{12}, U_2 \times 10^{12})$ at selected interior points.

Point (x_1, x_2)	Set A (U_1, U_2) $\times 10^{12}$	Set B (U_1, U_2) $\times 10^{12}$	Exact (U_1, U_2) $\times 10^{12}$
(1.10, 0.00)	(2.5808, 0.0000)	(2.6165, 0.0000)	(2.6297, 0.0000)
(0.50, 0.80)	(4.9294, 15.936)	(4.9304, 15.840)	(4.9312, 15.807)
(0.10, 0.70)	(1.0309, 17.915)	(1.0311, 17.815)	(1.0312, 17.780)
(1.90, 0.10)	(9.2099, 0.82173)	(9.3035, 0.74540)	(9.3331, 0.74065)
(0.90, 0.20)	(5.8260, 8.8090)	(5.8412, 8.7584)	(5.8655, 8.7773)
(1.05, 1.05)	(8.4852, 12.839)	(8.4864, 12.746)	(8.4887, 12.716)

Table 4.2: Numerical and exact values of $U_4 \times 10^3$ at selected interior points.

Point (x_1, x_2)	Set A $U_4 \times 10^3$	Set B $U_4 \times 10^3$	Exact $U_4 \times 10^3$
(1.10, 0.00)	0.0000	0.0000	0.0000
(0.50, 0.80)	2.6621	2.7870	2.8228
(0.10, 0.70)	6.3796	6.4938	6.5263
(1.90, 0.10)	-1.5416	-1.1523	-1.1403
(0.90, 0.20)	3.2009	3.2672	3.3500
(1.05, 1.05)	-5.2272	-5.0658	-5.0197

numerical values in Set B are more accurate than those in Set A and show significant convergence towards the exact values.

A graphical comparison between the numerical and the exact crack-opening displacement $\Delta U_2^{(1)}(v)$ for the crack is given in Figure 4.3 for $0 \leq v \leq 1$, where v is a non-dimensional parameter giving the position of points on a crack as defined in (4.23). Similarly, plots of the numerical and the exact electric potential jump $\Delta U_4^{(1)}(v)$ across opposite faces of the crack are given in Figure 4.4. The graphs of the numerical $\Delta U_2^{(1)}(v)$ and $\Delta U_4^{(1)}(v)$ are close to the exact ones. For the particular problem here, note that $\Delta U_2^{(1)}(v) = \Delta U_2^{(1)}(-v)$ and $\Delta U_4^{(1)}(v) = \Delta U_4^{(1)}(-v)$ as well as $\Delta U_1^{(1)}(v) = 0$ and $\Delta U_3^{(1)}(v) = 0$ for $-1 \leq v \leq 1$.

Figure 4.3: Plots of $\Delta U_2^{(1)}(v) \times 10^{10}$ over $0 \leq v \leq 1$.

Figure 4.4: Plots of $\Delta U_4^{(1)}(v)$ over $0 \leq v \leq 1$.

Table 4.3: Numerical and exact values of the stress and electric displacement intensity factors.

Intensity factor	Set A	Set B	Exact
$K_I(a, 0)$	1.00400	1.00102	1.00000
$K_I(-a, 0)$	1.00400	1.00102	1.00000
$K_{IV}(a, 0) \times 10^{10}$	1.02638	1.00627	1.00000
$K_{IV}(-a, 0) \times 10^{10}$	1.02638	1.00627	1.00000

Note that $K_{II}(a, 0) = K_{II}(-a, 0) = 0$ and $K_{III}(a, 0) = K_{III}(-a, 0) = 0$ for the particular problem here. The numerically obtained values of K_{II} and K_{III} are not exactly zero but extremely small in magnitude of the order 10^{-15} . A comparison of the numerical and exact values of only K_I and K_{IV} are given in Table 4.3. The numerical values are in good agreement with the exact ones, even for Set A in which the discretization of the exterior boundary of the solution domain is relatively crude.

Problem 2

The geometry of the solution domain and the direction of the electrical poling are as in Problem 1 above. Here the crack is, however, electrically permeable.

Take

$$\begin{aligned}
 U_K &= \operatorname{Re}\left\{\sum_{\alpha=1}^4 A_{K\alpha} M_{\alpha 1} (1 + (z_{\alpha}^2 - a^2)^{1/2})\right\}, \\
 S_{Kj} &= \operatorname{Re}\left\{\sum_{\alpha=1}^4 L_{Kj\alpha} M_{\alpha 1} \frac{z_{\alpha}}{(z_{\alpha}^2 - a^2)^{1/2}}\right\},
 \end{aligned} \tag{4.28}$$

as a particular electro-elastic solution of (2.8) in the whole of the $0x_1x_2$ plane with a cut in the region $-a < x_1 < a, x_2 = 0$.

Table 4.4: Numerical and exact values of $(U_1 \times 10^{12}, U_2 \times 10^{12})$ at selected interior points.

Point (x_1, x_2)	Set A (U_1, U_2) $\times 10^{12}$	Set B (U_1, U_2) $\times 10^{12}$	Exact (U_1, U_2) $\times 10^{12}$
(1.10, 0.00)	(0.00000, -10.926)	(0.0000, -11.068)	(0.0000, -11.111)
(0.50, 0.80)	(32.931, -8.0315)	(32.883, -8.0920)	(32.893, -8.1062)
(0.10, 0.70)	(29.721, -7.5773)	(29.680, -7.5899)	(29.691, -7.5928)
(1.90, 0.10)	(5.7181, -19.721)	(4.3235, -19.796)	(4.3217, -19.899)
(0.90, 0.20)	(13.742, -7.1452)	(13.712, -7.2682)	(13.688, -7.2885)
(1.05, 1.05)	(41.822, -11.510)	(41.739, -11.623)	(41.746, -11.653)

For the particular values of the constants $A, N, F, C, L, e_1, e_2, e_3, \epsilon_1$ and ϵ_2 used in Problem 1, as given in (4.27), the matrices $[A_{K\alpha}]$ and $[M_{\alpha S}]$ are such that

$$\text{Im}\left\{\sum_{\alpha=1}^4 A_{4\alpha} M_{\alpha 1}\right\} = 0. \quad (4.29)$$

Because of (4.29), the electric potential U_4 given by (4.28) satisfies

$$\lim_{\varepsilon \rightarrow 0^+} [U_4(x_1, \varepsilon) - U_4(x_1, -\varepsilon)] = 0 \text{ for } -a < x_1 < a, \quad (4.30)$$

With the material constants in (4.27), the functions U_K and S_{Kj} in (4.28) satisfy the traction-free and electrically permeable conditions ($S_{12} = S_{22} = S_{32} = 0$ and $\Delta U_4^{(1)} = 0$) on the crack $-a < x_1 < a, x_2 = 0$. For a particular test problem to check the boundary element procedure and the numerical Green's function for electrically permeable cracks, we use (4.28) together with (4.27) to generate boundary values of U_K and P_K on the horizontal and vertical sides of the square domain R respectively.

To obtain some numerical results, we take $a = 1$ and $h = 2$, divide each side of the square domain into N_0 of equal length and carry out two sets of numerical calculations (Sets A and B as in Problem 1) using the boundary element method. The numerical Green's function for the permeable crack is computed using $J = 10$ in (4.16). In Tables 4.4 and 4.5, numerical values of $U_1 \times 10^{12}, U_2 \times 10^{12}$ and U_4 at se-

Table 4.5: Numerical and exact values of $U_4 \times 10^2$ at selected interior points.

Point (x_1, x_2)	Set A $U_4 \times 10^2$	Set B $U_4 \times 10^2$	Exact $U_4 \times 10^2$
(1.10, 0.00)	2.7061	2.7330	2.7425
(0.50, 0.80)	2.8303	2.8399	2.8435
(0.10, 0.70)	2.0639	2.0659	2.0666
(1.90, 0.10)	4.8273	4.9076	4.9262
(0.90, 0.20)	2.8701	2.8906	2.9028
(1.05, 1.05)	3.8568	3.8776	3.8845

Table 4.6: Numerical and exact values of the stress intensity factors.

Intensity factor	Set A	Set B	Exact
$K_{II}(a, 0)$	1.00155	0.99962	1.00000
$K_{II}(-a, 0)$	1.00155	0.99962	1.00000

lected points in the interior of the solution domain are compared with the exact values computed using (4.28). For the particular problem here, $K_{II}(a, 0)$ and $K_{II}(-a, 0)$ are the only intensity factors which have non-zero values. Table 4.6 compares the numerical and the exact values of $K_{II}(a, 0)$ and $K_{II}(-a, 0)$. The numerical values of $K_{II}(a, 0)$ and $K_{II}(-a, 0)$ are in good agreement with the exact ones for both Sets A and B.

Problem 3

Let us take the solution domain R to be $-h < x_1 < h$, $-h < x_2 < h$, with three parallel electrically impermeable cracks $\gamma^{(1)}$, $\gamma^{(2)}$ and $\gamma^{(3)}$, where h are given positive constants. The tips of crack $\gamma^{(1)}$ are given by $(-a, 0)$ and $(a, 0)$, tips of $\gamma^{(2)}$ given by $(-a, d)$ and (a, d) , and tips of $\gamma^{(3)}$ given by $(-a, -d)$ and $(a, -d)$, where a and d are given positive constants (with $a < h$). The configuration of the cracks are as shown in Figure 4.5.

Figure 4.5: Three parallel cracks in a square domain.

The boundary conditions on the exterior boundary of R are given by

$$\left. \begin{array}{l} P_1 = \pm S_0 \\ P_2 = \pm T_0 \\ P_3 = 0 \\ P_4 = \pm D_0 \end{array} \right\} \text{ for } -h < x_1 < h \text{ on } x_2 = \pm h,$$

$$\left. \begin{array}{l} P_1 = 0 \\ P_2 = \pm S_0 \\ P_3 = 0 \\ P_4 = 0 \end{array} \right\} \text{ for } -h < x_2 < h \text{ on } x_1 = \pm h, \quad (4.31)$$

where S_0 , T_0 and D_0 are given positive constants.

The normalized mode I and mode II stress intensity factors and the normalized electric displacement intensity factor at the tip $(a, 0)$ of the crack $\gamma^{(1)}$ are given by $K_I(a, 0)/(T_0\sqrt{a})$, $K_{II}(a, 0)/(S_0\sqrt{a})$ and $K_{IV}(a, 0)/(D_0\sqrt{a})$ respectively. Note that the mode III stress intensity factor is zero here.

Plots of $K_I(a, 0)/(T_0\sqrt{a})$, $K_{II}(a, 0)/(S_0\sqrt{a})$ and $K_{IV}(a, 0)/(D_0\sqrt{a})$ against d/a are given in Han and Wang [37] for $h/a \rightarrow \infty$ using the material constants

$$\begin{aligned} A &= 12.6 \times 10^{10}, N = 5.5 \times 10^{10}, F = 5.3 \times 10^{10}, \\ C &= 11.7 \times 10^{10}, L = 3.53 \times 10^{10}, \\ e_1 &= 17.0, e_2 = -6.5, e_3 = 23.3, \\ \epsilon_1 &= 151 \times 10^{-10}, \epsilon_2 = 130 \times 10^{-10}, \end{aligned} \quad (4.32)$$

where the values of A , N , F , C and L are in N/m^2 , e_1 , e_2 and e_3 are in C/m^2 , and ϵ_1 and ϵ_2 are in C/(Vm) . In Han and Wang [37], planar cracks are modeled as continuous distributions of dislocations with density functions to be determined using a numerical procedure.

We employ the boundary element method described in this Chapter to compute the stress and electrical intensity factors $K_I(a, 0)/(T_0\sqrt{a})$, $K_{II}(a, 0)/(S_0\sqrt{a})$ and $K_{IV}(a, 0)/(D_0\sqrt{a})$. The exterior boundary of the region R is discretized into 80 boundary elements. To compute the numerical Green's function for the impermeable crack, we use at least $J = 10$ in (4.16). A larger value of J is needed if the normalized distance d/a separating the cracks is smaller. For the purpose of comparing the normalized intensity factors here with those in Han and Wang [37], we use the material constants in (4.32) and take $h/a = 30$, $S_0/T_0 = 1$ and $D_0/T_0 = 10^{-10} \text{ C/N}$. In Figure 4.6, we compare plots of the normalized intensity factors against d/a with those extracted from Han and Wang [37] for the corresponding case in which $h/a \rightarrow \infty$. The two sets of values appear to agree reasonably well with each other.

Figure 4.6: Plots of $K_I(a,0)/(T_0\sqrt{a})$, $K_{II}(a,0)/(S_0\sqrt{a})$ and $K_{IV}(a,0)/(D_0\sqrt{a})$ against d/a .

Problem 4

Here the electrical poling is taken to be along the x_3 direction. For a particular problem in which the electrical poling is along the x_3 direction, let us take the solution domain R to be $-h < x_1 < h$, $-h < x_2 < h$, with two collinear permeable crack lying in the regions $-b < x_1 < -a$, $x_2 = 0$, and $a < x_1 < b$, $x_2 = 0$, where a , b and h are positive constants such that $a < b < h$. The boundary conditions on the exterior boundary of R are given by

$$\left. \begin{array}{l} P_1 = 0 \\ P_2 = 0 \\ P_3 = \pm S_0 \\ P_4 = \pm D_0 \end{array} \right\} \text{ for } -h < x_1 < h \text{ on } x_2 = \pm h,$$

$$\left. \begin{array}{l} P_1 = 0 \\ P_2 = 0 \\ P_3 = 0 \\ P_4 = 0 \end{array} \right\} \text{ for } -h < x_2 < h \text{ on } x_1 = \pm h, \quad (4.33)$$

where S_0 and D_0 are non-negative constants.

Let K_{III}^{inner} and K_{III}^{outer} respectively denote the mode III stress intensity factor at the inner and outer tips of the collinear cracks. The crack energy release rates at the inner and outer tips are then respectively given by

$$G^{\text{inner}} = \frac{\pi}{2L} (K_{III}^{\text{inner}})^2 \text{ and } G^{\text{outer}} = \frac{\pi}{2L} (K_{III}^{\text{outer}})^2. \quad (4.34)$$

In Li [50], it is analytically given that

$$\left(\frac{4LG^{\text{inner}}}{\pi(b-a)S_0^2}, \frac{4LG^{\text{outer}}}{\pi(b-a)S_0^2} \right) \rightarrow \left(\frac{2[b^2\lambda - a^2]^2}{a(b-a)(b^2 - a^2)}, \frac{2b^3[1 - \lambda]^2}{(b-a)(b^2 - a^2)} \right) \\ \text{as } 2h/(b-a) \rightarrow \infty, \quad (4.35)$$

where

$$\lambda = \frac{\int_0^{\pi/2} [1 - (1 - (a/b)^2) \sin^2 t]^{1/2} dt}{\int_0^{\pi/2} [1 - (1 - (a/b)^2) \sin^2 t]^{-1/2} dt}. \quad (4.36)$$

Using the material constants in (4.32) and taking $2h/(b-a) = 20$, we use the boundary element method with the Green's function for the permeable cracks to compute the crack energy release rates G^{inner} and G^{outer} according to (4.34) (after calculating numerically the mode III stress intensity factors). As explained in Section 2.7.2, in perturbing the elastic modulus C_{1111} to construct an invertible matrix $[A_{K\alpha}]$, we choose $\varepsilon = 10^2$, that is, we take C_{1111} to be given by $(12.6 + 10^{-8}) \times 10^{10}$ N/m² instead of 12.6×10^{10} N/m². The outer boundary of the solution domain is discretized

into 160 elements. The numerical Green's function is calculated using at least $J = 10$ in (4.16). If the inner tips of the cracks are close to each other then $J = 30$ is used. If the outer tips are near the vertical sides, we use $J = 20$ and add another 40 elements on each of the vertical sides.

Figure 4.7: Plots of $4LG_T^{\text{inner}}/(\pi(b-a)S_0^2)$, against $2a/(b-a)$.

Plots of the normalized crack energy release rates $4LG^{\text{inner}}/(\pi(b-a)S_0^2)$ and $4LG^{\text{outer}}/(\pi(b-a)S_0^2)$ against $2a/(b-a)$ (for $0.50 \leq 2a/(b-a) \leq 17.50$) are given in Figures 4.7 and 4.8 respectively. In the figures, we also compare the numerical crack energy release rates with the values calculated from (4.35) (given by Li [50] for $2h/(b-a) \rightarrow \infty$). The numerical crack energy release rates are found to agree very well with (4.35) for small values of $2a/(b-a)$. This is expected as (4.35) is valid only for $2h/(b-a) \rightarrow \infty$ (that is, for an infinite piezoelectric material).

Note that the crack tip energy release rate G for the corresponding problem involving only a single crack of length $b - a$ in an infinite piezoelectric material is given by $4LG/(\pi(b - a)S_0^2) = 1$. Thus, it is not surprising that $4LG^{\text{inner}}/(\pi(b - a)S_0^2)$ and $4LG^{\text{outer}}/(\pi(b - a)S_0^2)$ computed using the boundary element method are quite close to 1 when the inner crack tips are several crack lengths apart and the outer tips are not yet so close to the vertical sides of the solution domain. As $2a/(b - a)$ approaches 18 (that is, as the outer crack tips approach the vertical sides of the solution domain), the crack energy release rates calculated using the boundary element method begin to deviate more significantly from (4.35). As expected, as is obvious in Figure 4.8, $4LG^{\text{outer}}/(\pi(b - a)S_0^2)$ shows a significant increase in magnitude when the outer crack tips interact strongly with the vertical sides of the solution domain.

Figure 4.8: Plots of $4LG_I^{\text{outer}}/(\pi(b - a)S_0^2)$, against $2a/(b - a)$.

For the particular problem under consideration here, it is known theoretically that $4LG^{\text{inner}}/(\pi(b-a)S_0^2)$ for $2a/(b-a) = \xi$ (where ξ is a positive real number such that $0 < \xi < 18$) is equal to $4LG^{\text{outer}}/(\pi(b-a)S_0^2)$ for $2a/(b-a) = 18 - \xi$. Also, $4LG^{\text{outer}}/(\pi(b-a)S_0^2)$ for $2a/(b-a) = \xi$ is equal to $4LG^{\text{inner}}/(\pi(b-a)S_0^2)$ for $2a/(b-a) = 18 - \xi$. In Figures 4.7 and 4.8, the graphs for the numerical values of $4LG^{\text{inner}}/(\pi(b-a)S_0^2)$ and $4LG^{\text{outer}}/(\pi(b-a)S_0^2)$ as obtained from the boundary element method reflect this theoretical observation. For example, we find that the values of $4LG^{\text{inner}}/(\pi(b-a)S_0^2)$ for $2a/(b-a) = 0.50$ and $4LG^{\text{outer}}/(\pi(b-a)S_0^2)$ for $2a/(b-a) = 17.50$ are respectively given by 1.2425 and 1.2407 (which differ from each other by less than 0.2%), and $4LG^{\text{outer}}/(\pi(b-a)S_0^2)$ for $2a/(b-a) = 0.50$ and $4LG^{\text{inner}}/(\pi(b-a)S_0^2)$ for $2a/(b-a) = 17.50$ are respectively given by 1.1024 and 1.1021 (less than 0.03% difference).

4.6 Summary

Green's functions are constructed numerically for multiple arbitrarily located planar cracks in an infinite electroelastic space. The cracks are stress-free and electrically either permeable or impermeable. We apply the Green's functions to derive a simple boundary element method for the numerical solution of some plane electroelastic crack problems involving solution domains of finite extent. As the Green's functions satisfy the boundary conditions on the cracks, the boundary element procedure requires only the exterior boundary of the solution domain to be discretized into boundary elements, that is, no discretization of the crack faces is needed.

To check the validity of the numerical Green's functions and the boundary element method, some specific electroelastic crack problems are solved. Numerical values obtained for the relevant intensity factors and the crack energy release rate at the crack tips are in good agreement with the values computed from known solutions in the literature.

A boundary element approach based on the standard fundamental solution with $\Phi_{RS}^* = 0$ may possibly be developed to solve the problem considered in the present chapter. In such an approach, the unknown jumps in the displacements and electrical potential across opposite crack faces are to be determined together with the unknown electroelastic quantities on the boundary. Consequently, the resulting system of linear algebraic equations is larger than the one in the Green's function approach here, although the computation of the standard fundamental solution is computationally less intensive than that of special Green's functions.

Chapter 5

Boundary Element Method for Electrically Semi-permeable Cracks

5.1 Introduction

In this chapter³, the numerical Green's function for the impermeable cracks is used to obtain boundary integral equations for multiple stress-free electrically semi-permeable cracks. Because of the electrically semi-permeable conditions on the cracks, the boundary integral equations contain integrals whose integrands are given by a nonlinear function of the crack opening displacement and the electric potential jump across opposite crack faces. The boundary integral equations can be solved by using a simple numerical procedure if the crack opening displacement and the electric potential jump are known. Those physical quantities on the cracks are, however, not known a priori. A predictor-corrector approach which iterates to and fro estimating the crack opening displacement and the electrical potential jump and solving the boundary integral equations is presented here for the numerical solution of the semi-permeable crack problem.

A brief review of existing boundary integral approaches for solving elastostatic crack problems is given in Chapter 4. The boundary element solutions

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are mostly for impermeable and conducting cracks. It appears that the only boundary element treatment of the semi-permeable crack problem is given in Denda [24]. In [24], the whole crack singular element is employed together with an iterative scheme to treat a single semi-permeable crack in a piezoelectric solid. The iterative procedure here appears to be more direct and efficient than the one in [24].

5.2 Statement of Problem

A sketch of the problem is as given in Figure 4.1 in Section 4.2. The cracks are assumed to open up and become stress free under the prescribed boundary conditions, but they are assumed to be electrically semi-permeable. Mathematically, the stress-free conditions on the cracks are given by

$$\sigma_{ij}(x_1, x_2)m_j^{(k)} \rightarrow 0 \text{ as } (x_1, x_2) \rightarrow (y_1, y_2) \in \gamma^{(k)} \text{ for } k = 1, 2, \dots, M, \quad (5.1)$$

and the electrical conditions for semi-permeable cracks as proposed in Hao and Shen [38] (based on the analogy of electrical capacitors) are given by

$$\begin{aligned} D_j(x_1, x_2)m_j^{(k)} \Delta u_p(x_1, x_2)m_p^{(k)} &= -\epsilon_c \Delta \phi(x_1, x_2) \\ \text{for } (x_1, x_2) &\in \gamma^{(k)} \text{ for } k = 1, 2, \dots, M, \end{aligned} \quad (5.2)$$

where ϵ_c is the permittivity of the medium filling the cracks, σ_{ij} and D_i are respectively the stresses and the electric displacements. The problem is to determine the displacements u_k and the electric potential ϕ throughout the cracked piezoelectric solid.

The notation used in Chapter 4 is closely followed here.

5.3 Boundary Integral Equations and Numerical Green's Function

As in Chapter 4, the boundary consists of the outer boundary B and the faces of the cracks $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(M-1)}$ and $\gamma^{(M)}$. From (2.36), the boundary integral equations can be written as

$$\begin{aligned} \lambda(\xi_1, \xi_2)U_K(\xi_1, \xi_2) = & \int_B [U_I(x_1, x_2)\Gamma_{IK}(x_1, x_2; \xi_1, \xi_2) \\ & - P_I(x_1, x_2)\Phi_{IK}(x_1, x_2; \xi_1, \xi_2)]ds(x_1, x_2) \\ & + \sum_{k=1}^M \int_{\gamma_+^{(k)}} [\Delta U_I(x_1, x_2)\Gamma_{IK}(x_1, x_2; \xi_1, \xi_2) \\ & - P_I(x_1, x_2)\Delta\Phi_{IK}(x_1, x_2; \xi_1, \xi_2)]ds(x_1, x_2), \end{aligned} \quad (5.3)$$

where $\Delta U_I(x_1, x_2)$ is the jump in the generalized displacements across opposite crack faces, $\gamma_+^{(k)}$ (the “upper face” of the crack $\gamma^{(k)}$) is taken to be the straight line from $(a^{(k)}, b^{(k)})$ to $(c^{(k)}, d^{(k)})$ and

$$\begin{aligned} \Delta\Phi_{IK}(x_1, x_2; \xi_1, \xi_2) = & \lim_{\epsilon \rightarrow 0} [\Phi_{IK}(x_1 - |\epsilon|m_1^{(k)}, x_2 - |\epsilon|m_2^{(k)}; \xi_1, \xi_2) \\ & - \Phi_{IK}(x_1 + |\epsilon|m_1^{(k)}, x_2 + |\epsilon|m_2^{(k)}; \xi_1, \xi_2)] \\ & \text{for } (x_1, x_2) \in \gamma^{(k)}. \end{aligned} \quad (5.4)$$

The numerical Green's function $\Phi_{IK}(x_1, x_2; \xi_1, \xi_2)$ for the electrically impermeable cracks, as derived in Chapter 4, is used here in the boundary integral equations in (5.3), that is, we take $\Phi_{IK}(x_1, x_2; \xi_1, \xi_2)$ to be given by (2.39), (4.15) and (4.16). Thus,

$$\Gamma_{IK}(x_1, x_2; \xi_1, \xi_2) = 0 \text{ for } (x_1, x_2) \in \gamma^{(k)}. \quad (5.5)$$

From (5.1), (5.2) and (5.5), we find that (5.3) is reduced to

$$\begin{aligned} \lambda(\xi_1, \xi_2)U_K(\xi_1, \xi_2) &= \int_B [U_I(x_1, x_2)\Gamma_{IK}(x_1, x_2; \xi_1, \xi_2) \\ &\quad - P_I(x_1, x_2)\Phi_{IK}(x_1, x_2; \xi_1, \xi_2)]ds(x_1, x_2) \\ &\quad - \frac{1}{2} \sum_{n=1}^M \ell^{(n)} \int_{-1}^1 D^{(n)}(t) \Delta\Phi_{4K}^{(n)}(t; \xi_1, \xi_2) dt, \end{aligned} \quad (5.6)$$

where $\Delta\Phi_{4K}^{(n)}(t; \xi_1, \xi_2) = \Delta\Phi_{4K}(X_1^{(n)}(t), X_2^{(n)}(t); \xi_1, \xi_2)$, $2X_1^{(n)}(t) = [c^{(n)} + a^{(n)}] + [c^{(n)} - a^{(n)}]t$, $2X_2^{(n)}(t) = [d^{(n)} + b^{(n)}] + [d^{(n)} - b^{(n)}]t$ and

$$D^{(n)}(t) = - \frac{\epsilon_c \Delta U_4(X_1^{(n)}(t), X_2^{(n)}(t))}{\Delta U_1(X_1^{(n)}(t), X_2^{(n)}(t))m_1^{(n)} + \Delta U_2(X_1^{(n)}(t), X_2^{(n)}(t))m_2^{(n)}}. \quad (5.7)$$

5.4 Numerical Procedure

A numerical method based on the boundary integral equations in (5.6) is described below for solving the semi-permeable crack problem stated in Section 5.2.

5.4.1 Boundary Elements

From the given boundary conditions on the exterior boundary B , either $U_I = u_i$ or $P_I = p_i$ for $I = i = 1, 2, 3$, and either $U_4 = \phi$ or P_4 are known at each and every point on B . If $D^{(n)}(t)$ is assumed known, the boundary B and the integral equations (5.6) can be discretized to determine approximately the unknown generalized displacements U_I and/or tractions P_I on B . To do this, the boundary B is approximated using N straight line segments denoted by $B^{(1)}, B^{(2)}, \dots, B^{(N-1)}$ and $B^{(N)}$. Across the segment $B^{(m)}$, the displacements U_I and the tractions P_I are approximated by constants $U_I^{(m)}$ and $P_I^{(m)}$ respectively. Through approximating (5.6), the unknown

constants on the boundary elements $U_I^{(m)}$ and/or tractions $P_I^{(m)}$ can be determined from the system of linear algebraic equations

$$\begin{aligned} \frac{1}{2}U_K^{(m)} &= \sum_{n=1}^N U_I^{(n)} \int_{B^{(n)}} \Gamma_{IK}(x_1, x_2; \xi_1^{(m)}, \xi_2^{(m)}) ds(x_1, x_2) \\ &\quad - \sum_{n=1}^N P_I^{(n)} \int_{B^{(n)}} \Phi_{IK}(x_1, x_2; \xi_1^{(m)}, \xi_2^{(m)}) ds(x_1, x_2) \\ &\quad - \frac{1}{2} \sum_{n=1}^M \ell^{(n)} \int_{-1}^1 D^{(n)}(t) \Delta \Phi_{4K}^{(n)}(t; \xi_1^{(m)}, \xi_2^{(m)}) dt \\ &\quad \text{for } m = 1, 2, \dots, N, \end{aligned} \quad (5.8)$$

where $(\xi_1^{(m)}, \xi_2^{(m)})$ is the midpoint of $B^{(m)}$.

From (4.14), the last integral in (5.8) can be approximated as

$$\begin{aligned} &\int_{-1}^1 D^{(n)}(t) \Delta \Phi_{4K}^{(n)}(t; \xi_1^{(m)}, \xi_2^{(m)}) dt \\ &\simeq \sum_{j=1}^J \phi_{4S}^{(nj)}(\xi_1^{(m)}, \xi_2^{(m)}) \int_{-1}^1 \sqrt{1-t^2} U^{(j-1)}(t) D^{(n)}(t) dt. \end{aligned} \quad (5.9)$$

As explained before, the integral on the right hand side of (5.9) can be accurately computed by using the numerical quadrature formula (25.4.40) listed in Abramowitz and Stegun [1], if the function $D^{(n)}(t)$ is known.

5.4.2 Generalized Crack Opening Displacements

Once $U_I^{(m)}$ and $P_I^{(m)}$ are all known, the generalized crack opening displacements on the crack $\gamma^{(n)}$, that is, $\Delta U_P^{(n)}(t) = \Delta U_P(X_1^{(n)}(t), X_2^{(n)}(t))$ for $-1 < t < 1$, can be computed numerically. Specifically, $\Delta U_P^{(n)}(t)$ is given approximately by

$$\Delta U_P^{(n)}(t) \simeq \sqrt{1-t^2} \sum_{j=1}^J \psi_P^{(nj)} U^{(j-1)}(t), \quad (5.10)$$

with the constants $\psi_P^{(nj)}$ to be determined by solving the system of linear algebraic equations

$$\begin{aligned}
& - \sum_{j=1}^J j\pi\psi_P^{(qj)}\chi_{PK}^{(q)}U^{(j-1)}(t^{(i)}) \\
& + \sum_{j=1}^J \sum_{\substack{n=1 \\ n \neq q}}^M \psi_P^{(nj)} \int_{-1}^1 \sqrt{1-v^2}U^{(j-1)}(v)Y_{PK}^{(nq)}(v,t^{(i)})dv \\
& = S_K^{(q)}(t^{(i)}) \text{ for } i = 1, 2, \dots, J, K = 1, 2, 3, 4 \text{ and } q = 1, 2, \dots, M,
\end{aligned} \tag{5.11}$$

where $t^{(i)} = \cos([2i - 1]\pi/[2J])$ and $S_K^{(q)}(t)$ is given by

$$\begin{aligned}
S_K^{(q)}(t) &= \sum_{n=1}^N C_{KjRs}m_j^{(q)} \int_{B^{(n)}} \left\{ P_I^{(n)} \frac{\partial}{\partial \xi_s} [\Phi_{IR}^0(x_1, x_2, \xi_1, \xi_2)] \right\} \Big|_{(\xi_1, \xi_2)=(X_1^{(q)}(t), X_2^{(q)}(t))} \\
&\quad - U_I^{(n)} \frac{\partial}{\partial \xi_s} [\Gamma_{IR}^0(x_1, x_2, \xi_1, \xi_2)] \Big|_{(\xi_1, \xi_2)=(X_1^{(q)}(t), X_2^{(q)}(t))} \} ds(x_1, x_2) \\
&\quad + \delta_{K4} D^{(q)}(t).
\end{aligned} \tag{5.12}$$

5.4.3 Iterative Solution

From (5.7), the function $D^{(n)}(t)$ is given by an expression which is a nonlinear function of the generalized crack opening displacements $\Delta U_1^{(n)}(t)$, $\Delta U_2^{(n)}(t)$ and $\Delta U_4^{(n)}(t)$. Thus, it is an unknown function to be determined in the process of solving the crack problem under consideration. An iterative procedure for solving the problem is given in the steps below.

1. Make a guess of $D^{(n)}(t)$. If a solution of the problem for some value of ϵ_c which is close to the desired permittivity of the medium filling the cracks is known, it can be used to provide an initial estimate of $D^{(n)}(t)$ through the formula (5.7).

For a cold start, $D^{(n)}(t) = 0$ which corresponds to the case of impermeable cracks may be used. Go to Step 2.

2. Solve (5.8) for the unknown generalized displacements U_I and/or tractions P_I on the exterior boundary B using the latest estimate of $D^{(n)}(t)$. Go to Step 3.
3. Solve (5.11) for $\psi_P^{(qj)}$ using the latest values of $U_I^{(n)}$ and $P_I^{(n)}$ and use (5.7) and (5.10) to obtain a new estimate of $D^{(n)}(t)$, that is,

$$D^{(n)}(t) = -\epsilon_c \left\{ \sum_{j=1}^J \psi_4^{(nj)} U^{(j-1)}(t) \right\} \left[\sum_{p=1}^2 \sum_{j=1}^J \psi_p^{(nj)} U^{(j-1)}(t) m_p^{(n)} \right]^{-1}. \quad (5.13)$$

Check whether the newly obtained values $D^{(n)}(t^{(i)})$ ($t^{(i)} = \cos([2i - 1]\pi/[2J])$ for $i = 1, 2, \dots, J$) agree with the previous values to within a specified number of significant figures. If the required convergence is not achieved, go back to Step 2. If convergence is achieved, use the latest values of $\psi_P^{(nj)}$ to compute the stress and electric displacement intensity factors according to the formulae (3.22).

In Step 2, (5.8) gives a system of $4N$ linear algebraic equations in $4N$ unknowns, that is, it can be written in the matrix form $\mathbf{A}\mathbf{X} = \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are respectively known $4N \times 4N$ and $4N \times 1$ matrices and $\mathbf{X}$ is an unknown $4N \times 1$ matrix. The square matrix $\mathbf{A}$ does not change during the iterations between Steps 2 and 3. Similarly, the square matrix in the linear system of algebraic equations in (5.11) for determining $\psi_P^{(qj)}$ in Step 3 remains the same throughout the iterative procedure.

Thus, those square matrices have to be set up and processed only once for solving the systems of linear algebraic equations.

Note that the iterative procedure proposed above differs from the one given in Denda [24] for a single semi-permeable crack. In [24], the jump in the electric potential over the impermeable crack is first obtained. This jump in electric potential is then progressively decreased to zero over the entire crack by gradually changing the value of a control parameter p . For each value of p , the solution for the corresponding permeable crack is used to compute the normal electric displacement and the generalized displacement jumps over the crack in order to estimate the permittivity ϵ_c which corresponds to the control parameter p . Physical quantities of interest such as the extended stress intensity factors are then plotted against ϵ_c or p . The desired solution for any other value of ϵ_c is finally obtained through interpolation. For each value of p , the approximation of ϵ_c from the solution of the permeable crack may, however, contain some errors, particularly for the case in which the crack lies in a body of finite extent, and further investigation is needed to improve the algorithm in [24].

The iterative approach here is a more direct one. The numerical Green's function for the impermeable cracks is used in each iterative step and the iteration is between the calculation of the generalized displacements and tractions on the exterior boundary and the normal electric displacements on the cracks. As explained above, for any desired value of ϵ_c , we start with an initial guess of the normal electric displacement $D^{(n)}(t)$ and iterate until the change in $D^{(n)}(t)$ is sufficiently small. As shown in the numerical examples, convergence to the final solution may be slow

for ϵ_c which is relatively large. The convergence may, however, be improved significantly by replacing (5.13) with (5.23), that is, by introducing a relaxation parameter ω .

5.5 Analytic Solution for a Semi-permeable Crack

An analytic solution to the problem of a semi-permeable crack in an infinite piezoelectric medium is derived here. Later on, it will be used as a check for the numerical procedure in Section 5.4 above.

Let the generalized stress be given by

$$S_{Kj} = S_{Kj}^{(0)} + S_{Kj}^{(1)}, \quad (5.14)$$

where $S_{Kj}^{(0)}$ are the constants giving the uniform state of the generalized stress in the absence of the crack and $S_{Kj}^{(1)}$ are induced by the crack such that $S_{K2}^{(0)} + S_{K2}^{(1)} = 0$ on the crack for $K = 1, 2$ and 3 .

Taking the crack to be in the region $-a < x_1 < a$, $x_2 = 0$, we find that the induced generalized stress $S_{Kj}^{(1)}$ and its corresponding generalized displacement $U_K^{(1)}$ are given by

$$\begin{aligned} U_K^{(1)} &= \operatorname{Re}\left\{\sum_{\alpha=1}^4 A_{K\alpha} M_{\alpha S} P_S[-z_\alpha + (z_\alpha^2 - a^2)^{1/2}]\right\}, \\ S_{Kj}^{(1)} &= \operatorname{Re}\left\{\sum_{\alpha=1}^4 L_{Kj\alpha} M_{\alpha S} P_S\left[-1 + \frac{z_\alpha}{(z_\alpha^2 - a^2)^{1/2}}\right]\right\}, \end{aligned} \quad (5.15)$$

where $P_1 = S_{12}^{(0)}$, $P_2 = S_{22}^{(0)}$, $P_3 = S_{32}^{(0)}$ and, as in Hao and Shen [38], P_4 is assumed to be a constant yet to be determined. Note that $(z_\alpha^2 - a^2)^{1/2}$ is defined in such a way

that

$$\lim_{|z_\alpha| \rightarrow \infty} \frac{z_\alpha}{(z_\alpha^2 - a^2)^{1/2}} = 1. \quad (5.16)$$

It is easy to check that $S_{K2}^{(1)} = -P_K$ on the crack. Thus, $S_{Kj}^{(1)}$ satisfies the stress-free conditions on the crack, that is, $S_{K2}^{(0)} + S_{K2}^{(1)} = 0$ on the crack for $K = 1, 2$ and 3 , as required. Furthermore, it can be shown that $S_{Kj}^{(1)} \rightarrow 0$ as $|z_\alpha| \rightarrow \infty$, that is, the generalized stress is given by $S_{Kj}^{(0)}$ at infinity.

The crack is electrically semi-permeable, that is,

$$S_{42}\Delta U_2 = -\epsilon_c \Delta U_4 \text{ on the crack.} \quad (5.17)$$

From (5.14) and (5.15), $S_{42} = S_{42}^{(0)} - P_4$ on the crack, and if P_4 is to be a real, then

$$\Delta U_2 = \operatorname{Re}\left\{\sum_{\alpha=1}^4 2iA_{2\alpha}M_{\alpha S}\right\}P_S(a^2 - x_1^2)^{1/2} \text{ for } -a < x_1 < a. \quad (5.18)$$

It follows that (5.17) is satisfied if

$$(S_{42}^{(0)} - P_4)V_{2S}P_S = -\epsilon_c V_{4S}P_S, \quad (5.19)$$

where

$$V_{KS} = \operatorname{Re}\left\{\sum_{\alpha=1}^4 2iA_{K\alpha}M_{\alpha S}\right\}. \quad (5.20)$$

Note that (5.19) is a quadratic equation in the unknown parameter P_4 . If a unique constant P_4 satisfying (5.19) and the inequality $V_{2S}P_S > 0$ (so that $\Delta U_2 > 0$) can be found, we have obtained an analytic solution to the problem of the single semi-permeable crack. The value of P_4 for the special case of an electrically impermeable crack ($\epsilon_c = 0$) or a permeable crack ($\epsilon_c \rightarrow \infty$) can be easily obtained from (5.19).

Specifically,

$$P_4 = \begin{cases} S_{42}^{(0)} & \text{for an impermeable crack,} \\ -V_{44}^{-1} \sum_{k=1}^3 V_{4k}P_k & \text{for a permeable crack.} \end{cases} \quad (5.21)$$

Once P_4 is determined, the crack tip stress and electric displacement intensity factors can be easily extracted from (5.15).

5.6 Specific Problems

The numerical procedure in Section 5.4 is applied here to some specific problems involving semi-permeable cracks in piezoelectric solids.

Problem 1

Let us first consider a single crack $-a < x_1 < a$, $x_2 = 0$, in the square region $-h < x_1 < h$, $-h < x_2 < h$, where a and h are given positive constants. The boundary conditions on the exterior boundary B are given by

$$\begin{aligned} P_1 &= 0 \text{ and } P_3 = 0 \text{ on } B, \\ P_2 &= \begin{cases} T_0 & \text{for } -h < x_1 < h, x_2 = h, \\ -T_0 & \text{for } -h < x_1 < h, x_2 = -h, \\ 0 & \text{for } -h < x_2 < h, x_1 = \pm h, \end{cases} \\ P_4 &= \begin{cases} D_0 & \text{for } -h < x_1 < h, x_2 = h, \\ -D_0 & \text{for } -h < x_1 < h, x_2 = -h, \\ 0 & \text{for } -h < x_2 < h, x_1 = \pm h, \end{cases} \end{aligned}$$

where T_0 and D_0 are given positive constants.

The electrical poling direction is taken to be along the x_2 direction. For illustrative purpose, the material constants of a class of PZT4 piezoceramics is used in the calculation, that is,

$$\begin{aligned} A &= 13.9 \times 10^{10}, N = 7.78 \times 10^{10}, F = 7.43 \times 10^{10}, \\ C &= 11.3 \times 10^{10}, L = 2.56 \times 10^{10}, \end{aligned}$$

$$\begin{aligned}
e_1 &= 13.44, \quad e_2 = -6.98, \quad e_3 = 13.84, \\
\epsilon_1 &= 60 \times 10^{-10}, \quad \epsilon_2 = 54.7 \times 10^{-10}.
\end{aligned} \tag{5.22}$$

The values of A , N , F , C and L above are in N/m^2 , e_1 , e_2 and e_3 are in C/m^2 , and ϵ_1 and ϵ_2 are in C/(Vm) . The load ratio is taken to be given by $D_0/T_0 = 10^{-10}$ C/N.

Table 5.1: Numerical and analytical values of $K_{IV}(a, 0)/(D_0\sqrt{a})$ for selected values of normalized permittivity.

$\epsilon_c T_0/D_0^2$	Numerical	Analytical
0	1.00006	1.00000
0.001	1.00013	1.00007
0.01	1.00074	1.00068
0.05	1.00346	1.00339
0.1	1.00683	1.00677
0.2	1.01353	1.01346
0.5	1.03323	1.03314
1	1.06478	1.06467
5	1.27215	1.27151
10	1.45431	1.45496
20	1.69190	1.69216
50	2.02325	2.02027
∞	2.53459	2.53300

For the limiting case in which $h/a \rightarrow \infty$ (the case of a piezoelectric solid of an infinite extent), an analytical solution of the problem can be derived as shown in the Section 5.5, if the electric displacement D_2 can be assumed to be a constant on the crack. To check the validity of the iterative scheme proposed in Section 5.4 for electrically semi-permeable cracks, the numerical electric displacement intensity factor $K_{IV}(a, 0)/(D_0\sqrt{a})$ (at the crack tip $(a, 0)$) obtained from the iterative scheme using $h/a = 30, 40$ equal length elements and $J = 10$ (10 collocation points on

the crack) are compared in Table 1 with the exact one extracted from the analytic solution in Section 5.5 for various values of the normalized permittivity $\epsilon_c T_0 / D_0^2$.

It is clear there is a good agreement between the numerical and analytical values of $K_{IV}(a, 0) / (D_0 \sqrt{a})$ in Table 1. The numerical values are obtained by gradually increasing $\epsilon_c T_0 / D_0^2$. For example, to solve the problem $\epsilon_c T_0 / D_0^2 = 10.0$, the solution for $\epsilon_c T_0 / D_0^2 = 5.0$ may be used as an initial solution to find the numerical solution for $\epsilon_c T_0 / D_0^2 = 5.50$. Subsequently, the solution for $\epsilon_c T_0 / D_0^2 = 5.50$ is used to obtain the solution for $\epsilon_c T_0 / D_0^2 = 6.0$. The value of $\epsilon_c T_0 / D_0^2$ is gradually increased by 0.5 until the final solution for $\epsilon_c T_0 / D_0^2 = 10.0$ is obtained. For smaller $\epsilon_c T_0 / D_0^2$, the required numerical solution may be obtained in three or four iterations. (In the iterative scheme, the convergence criterion used is that the values of $D^{(n)}(t)$ (at the collocation points on the crack) as calculated in (5.13) do not change by more than 0.5% in two consecutive iterations.) For $\epsilon_c T_0 / D_0^2 > 10$, $D^{(n)}(t)$ as calculated using (5.13) may converge very slowly or in an oscillatory manner. For larger $\epsilon_c T_0 / D_0^2$, the convergence of the solution may be improved significantly by modifying (5.13) for updating $D^{(n)}(t)$ as

$$D^{(n)}(t) = -\omega \epsilon_c \left\{ \sum_{j=1}^J \psi_4^{(nj)} U^{(j-1)}(t) \right\} \left[\sum_{p=1}^2 \sum_{j=1}^J \psi_p^{(nj)} U^{(j-1)}(t) m_p^{(n)} \right]^{-1} + (1 - \omega) D_{\text{last}}^{(n)}(t) \quad (5.23)$$

where $D_{\text{last}}^{(n)}(t)$ is the approximation of $D^{(n)}(t)$ in the last iteration and ω is an appropriately chosen relaxation parameter. Using $\omega = 1/2$, we manage to obtain convergence for $D^{(n)}(t)$ for up to $\epsilon_c T_0 / D_0^2 = 50$. (The values of $K_{IV}(a, 0) / (D_0 \sqrt{a})$ in Table 1 for $\epsilon_c T_0 / D_0^2 = 20$ and $\epsilon_c T_0 / D_0^2 = 50$ are computed by using $\omega = 1/2$.) The

numerical value of $K_{IV}(a, 0)/(D_0\sqrt{a})$ in Table 1 for $\epsilon_c T_0/D_0^2 \rightarrow \infty$ is obtained directly by using the numerical Green's function for permeable cracks as explained in Section 4.3.2.

Figure 5.1: Plots of $K_{IV}(a, 0)/(D_0\sqrt{a})$ against $D_0^2/(\epsilon_c T_0)$ for $h/a = 5, 10$ and 30 .

In Figure 5.1, $K_{IV}(a, 0)/(D_0\sqrt{a})$ is plotted against $D_0^2/(\epsilon_c T_0)$ for $h/a = 5, 10$ and 30 . For a fixed h/a , $K_{IV}(a, 0)/(D_0\sqrt{a})$ decreases in magnitude and tends to a limiting value as $D_0^2/(\epsilon_c T_0)$ increases. Note the effects of the exterior boundary of the piezoelectric solid on $K_{IV}(a, 0)/(D_0\sqrt{a})$. For a particular $D_0^2/(\epsilon_c T_0)$, when the boundary is closer to the crack, it appears that $K_{IV}(a, 0)/(D_0\sqrt{a})$ has a higher magnitude.

In Figure 5.2, plots of $D_2(x_1, 0)/D_0$ against $-1 < x_1/a < 1$ are given for $\epsilon_c T/D_0^2 = 0.01$ and selected values of h/a . For smaller values of h/a , it may be

Figure 5.2: Plots of $D_2(x_1, 0)/D_0$ against $-1 < x_1/a < 1$ for selected values of h/a with $\epsilon_c T_0/D_0^2 = 0.01$.

necessary to employ a larger number of boundary elements. For the numerical calculation to obtain the plots in Figure 5.2, up to 80 elements are employed on the exterior boundary. For larger values of h/a , such as $h/a = 5$ and $h/a = 10$, $D_2(x_1, 0)/D_0$ is almost a constant over $-1 < x_1/a < 1$. This observation is consistent with the assumption of constant D_2 over the crack used in the derivation of the analytic solution for $h/a \rightarrow \infty$ in Section 5.5. For smaller values of h/a , the variation of $D_2(x_1, 0)/D_0$ over $-1 < x_1/a < 1$ is more prominent. The value of $D_2(x_1, 0)/D_0$ is minimum at the center of the crack and increases towards the tips, as the field lines of D_2 perpendicular to the crack always tend to deviate towards the tips.

Figure 5.3 shows the variation of D_2/D_0 along the crack for $h/a = 30$ and selected values of $\epsilon_c T/D_0^2$. It can be seen that D_2/D_0 has a bigger magnitude for a

higher value of $\epsilon_c T_0 / D_0^2$. This is as expected because there is a lower resistance to the electrical conductance if the permittivity in the crack is higher.

Figure 5.4 gives plots of $D_0 \Delta \phi(x_1, 0) / (2hT_0)$ against $0 \leq x_1/a \leq 1$ for $h/a = 30$ and some selected values of $\epsilon_c T_0 / D_0^2$. As may be expected, the value of $D_0 \Delta \phi(x_1, 0) / (2hT_0)$ at each point on the crack increases with decreasing $\epsilon_c T_0 / D_0^2$.

Figure 5.3: Plots of $D_2(x_1, 0)/D_0$ against $-1 < x_1/a < 1$ for selected values of $\epsilon_c T_0 / D_0^2$ with $h/a = 30$.

Problem 2

For another problem, let us consider three parallel cracks $\gamma^{(1)}$, $\gamma^{(2)}$ and $\gamma^{(3)}$ in the domain $-h < x_1 < h$, $-h < x_2 < h$, where h is a given positive constant. The crack $\gamma^{(1)}$ lies in the region $-a < x_1 < a$, $x_2 = 0$, $\gamma^{(2)}$ in $-b < x_1 < b$, $x_2 = d$, and $\gamma^{(3)}$ in $-b < x_1 < b$, $x_2 = -d$, where a , b and d are given positive constants. Refer to Figure 5.5.

Figure 5.4: Plots of $D_0 \Delta \phi(x_1, 0) / (2hT_0)$ against $0 \leq x_1/a \leq 1$ for some selected values of $\epsilon_c T_0 / D_0^2$ with $h/a = 30$.

Figure 5.5: Three parallel cracks in a square domain.

The boundary conditions on the sides of the square domain are given by

$$\left. \begin{array}{l} P_1 = \pm S_0 \\ P_2 = \pm T_0 \\ P_3 = 0 \\ P_4 = \pm D_0 \end{array} \right\} \text{ for } -h < x_1 < h \text{ on } x_2 = \pm h,$$

$$\left. \begin{array}{l} P_1 = 0 \\ P_2 = \pm S_0 \\ P_3 = 0 \\ P_4 = 0 \end{array} \right\} \text{ for } -h < x_2 < h \text{ on } x_1 = \pm h, \quad (5.24)$$

where S_0 , T_0 and D_0 are given positive constants.

For the purpose of computation, the loads S_0 , T_0 and D_0 are given the ratios $D_0/T_0 = 10^{-10}$ C/N and $D_0/S_0 = 10^{-10}$ C/N and the material constants in (5.22) are used.

For fixed $h/a = 30$ and $b/a = 1$, in Figures 5.6, 5.7 and 5.8, $K_I(a, 0)/(T_0\sqrt{a})$, $K_{II}(a, 0)/(S_0\sqrt{a})$ and $K_{IV}(a, 0)/(D_0\sqrt{a})$ (at the tip $(a, 0)$ of the center crack) are plotted against d/a for some values of $\epsilon_c T_0/D_0^2$ including $\epsilon_c T_0/D_0^2 = 0$ (impermeable cracks) and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ (permeable cracks). The calculation for $\epsilon_c T_0/D_0^2 \rightarrow \infty$ is carried out using the numerical Green's function for permeable cracks as explained in Section 4.3.

In Figure 5.6, the graphs of $K_I(a, 0)/(T_0\sqrt{a})$ for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 1$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ are visually indistinguishable. Similarly, there is no distinction between the graphs of $K_{II}(a, 0)/(S_0\sqrt{a})$ for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 1$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ in Figure 5.7. It appears that the permittivity of the medium filling the cracks has no significant effect on $K_I(a, 0)/(T_0\sqrt{a})$ and $K_{II}(a, 0)/(S_0\sqrt{a})$. In Figure 5.8, however, the magnitude of $K_{IV}(a, 0)/(D_0\sqrt{a})$ appears to increase with $\epsilon_c T_0/D_0^2$. Note that the normalized stress intensity factors $K_I(a, 0)/(T_0\sqrt{a})$ and $K_{II}(a, 0)/(S_0\sqrt{a})$ in Figures 5.6 and 5.7 are close to unity for large d/a . This is as

Figure 5.6: Plots of $K_I(a, 0)/(T_0\sqrt{a})$ against d/a for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 1$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ with $h/a = 30$ and $b/a = 1$.

Figure 5.7: Plots of $K_{II}(a, 0)/(S_0\sqrt{a})$ against d/a for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 1$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ with $h/a = 30$ and $b/a = 1$.

Figure 5.8: Plots of $K_{IV}(a, 0)/(D_0\sqrt{a})$ against d/a for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 1$, $\epsilon_c T_0/D_0^2 = 5$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ with $h/a = 30$ and $b/a = 1$.

Figure 5.9: Plots of G/G_0 against d/a for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 5$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ with $h/a = 30$ and $b/a = 1$.

expected since $K_I(a, 0)/(T_0\sqrt{a})$ and $K_{II}(a, 0)/(S_0\sqrt{a})$ should both be unity for the corresponding case of a single crack in an infinite piezoelectric space.

The energy release rate at a crack tip can be computed in terms of the crack tip intensity factors by (see [73])

$$G = k_p H_{pj} k_j, \quad (5.25)$$

where $k_1 = K_{II}$, $k_2 = K_I$, $k_3 = K_{III}$, $k_4 = K_{IV}$ and

$$H_{pj} = \frac{1}{2} \operatorname{Re} \left\{ i \sum_{\alpha=1}^4 A_{p\alpha} \psi_{\alpha j} \right\}, \quad (5.26)$$

where

$$[\psi_{\alpha j}] = \begin{pmatrix} -L_{111}/\tau_1 & -L_{112}/\tau_2 & -L_{113}/\tau_3 & -L_{114}/\tau_4 \\ -L_{211}/\tau_1 & -L_{212}/\tau_2 & -L_{213}/\tau_3 & -L_{214}/\tau_4 \\ -L_{311}/\tau_1 & -L_{312}/\tau_2 & -L_{313}/\tau_3 & -L_{314}/\tau_4 \\ -L_{411}/\tau_1 & -L_{412}/\tau_2 & -L_{413}/\tau_3 & -L_{414}/\tau_4 \end{pmatrix}. \quad (5.27)$$

Figure (5.9) gives plots of the normalized energy release rate G/G_0 against d/a for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 5$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$. The plots of G/G_0 follows the same trends as $K_I(a, 0)/(T_0\sqrt{a})$. Note that G_0 is calculated using (5.25) with $k_i = \delta_{i2} T_0\sqrt{a}$.

For fixed $h/a = 30$ and $d/a = 1$, Figures 5.10, 5.11 and 5.12 show the plots of $K_I(a, 0)/(T_0\sqrt{a})$, $K_{II}(a, 0)/(S_0\sqrt{a})$ and $K_{IV}(a, 0)/(D_0\sqrt{a})$ against b/a ($0 \leq b/a \leq 1$) for various values of $\epsilon_c T_0/D_0^2$. As in Figures 5.6 and 5.7, $K_I(a, 0)/(T_0\sqrt{a})$ and $K_{II}(a, 0)/(S_0\sqrt{a})$ for $\epsilon_c T_0/D_0^2 \rightarrow \infty$ in Figures 5.10 and 5.11 are not distinguishable from the corresponding normalized intensity factors for $\epsilon_c T_0/D_0^2 = 0$ and $\epsilon_c T_0/D_0^2 = 1$. For very small b/a , $K_I(a, 0)/(T_0\sqrt{a})$ and $K_{II}(a, 0)/(S_0\sqrt{a})$ are close to one. This is consistent with the observation in Figures 5.6 and 5.7 that the normalized intensity factors are close to one for larger d/a . Figure (5.13) gives plots of the

Figure 5.10: Plots of $K_I(a, 0)/(T_0\sqrt{a})$ against b/a for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 1$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ with $h/a = 30$ and $d/a = 1$.

Figure 5.11: Plots of $K_{II}(a, 0)/(S_0\sqrt{a})$ against b/a for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 1$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ with $h/a = 30$ and $d/a = 1$.

Figure 5.12: Plots of $K_{IV}(a, 0)/(D_0\sqrt{a})$ against b/a for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 1$, $\epsilon_c T_0/D_0^2 = 5$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ with $h/a = 30$ and $d/a = 1$.

Figure 5.13: Plots of G/G_0 against b/a for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 5$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$ with $h/a = 30$ and $d/a = 1$.

normalized energy release rate G/G_0 against b/a for $\epsilon_c T_0/D_0^2 = 0$, $\epsilon_c T_0/D_0^2 = 5$ and $\epsilon_c T_0/D_0^2 \rightarrow \infty$.

For fixed $\epsilon_c T_0/D_0^2 = 1$, $h/a = 30$ and $d/a = 1$, Figure 5.14 shows the variation of D_2/D_0 along the center crack and various values of the crack length ratio b/a . It appears that as b/a increases the normalized normal electrical displacement D_2/D_0 becomes larger in magnitude and exhibits a greater variation over the center crack.

Figure 5.14: Plots of $D_2(x_1, 0)/D_0$ against x_1/a for various values of b/a with $\epsilon_c T_0/D_0^2 = 1$, $h/a = 30$ and $d/a = 1$.

5.7 Summary

An iterative method based on the electro-elastostatic boundary integral equations together with the numerical Green's function for impermeable cracks has been success-

fully implemented for the analysis of electrically semi-permeable cracks in a piezoelectric solid. As the conditions on the semi-permeable cracks are not fully satisfied by the Green's function (for impermeable cracks), the resulting boundary integral formulation involves integrals whose integrands are given by a nonlinear function of the crack opening displacement and the electrical potential jump on the cracks. Nevertheless, if the crack opening displacement and the electric potential jump are assumed known, a simple boundary element procedure which involves only unknowns on the exterior boundary of the piezoelectric solid can be devised. The approach proposed here for the numerical solution of the electroelastic crack problem is to iterate to and fro estimating the crack opening displacement and the electrical potential jump and using the simple boundary element procedure to determine the unknowns on the exterior boundary.

For a particular problem involving a single planar crack which is centrally located in a very large piezoelectric plate under uniform loads, the numerical values obtained for the crack tip stress and electric displacement intensity factors are found to be in good agreement with those computed using analytical formulae. Qualitatively acceptable results are also obtained for some specific problems including one which involves the interaction of three parallel planar crack.

Chapter 6

A Dynamic Crack Problem

6.1 Introduction

Most works giving semi-analytic solutions for dynamic piezoelectric crack problems assume that the cracks undergo antiplane deformations, so that the governing partial differential equations can be reduced to a pair of relatively simple equations which comprises the two-dimensional Helmholtz and Laplace's equations. For example, the dynamic response of a single electrically impermeable planar crack in an infinite transversely isotropic piezoelectric material under pure electric load and undergoing an antiplane deformation is investigated by Chen [15]. Chen and Karihaloo [16], Chen and Meguid [17], Li and Fan [53] and Li and Tang [57] have solved problems involving a single planar crack in an infinitely long piezoelectric strip under antiplane deformations. Coplanar cracks undergoing antiplane deformations in piezoelectric materials are examined in Chen and Worswick [18] and Meguid and Chen [63]. Kwon and Lee [45] and Li and Lee [54, 56] have investigated the antiplane deformation of edge cracks in piezoelectric materials.

There are apparently few articles in which semi-analytic solutions for cracks undergoing dynamic inplane deformations in piezoelectric materials may be found (Kuna [46]). One of them is Shindo *et al* [75]. In [75], the problem of a single planar crack in a piezoelectric ceramic under normal impact is formulated in terms of a pair of dual integral equations by representing the displacement and electric poten-

tial in the Laplace transform domain by suitable Fourier sine and cosine transform representations. The dual integral equations are solved as explained in Sneddon and Lowengrub [78], by reducing them to Fredholm integral equations of the second kind. The dynamic piezoelectric crack problem can also be formulated in terms of hypersingular integral equations using the approach in a recent article by García-Sánchez, Zhang, Sládek *et al* [32]. In [32], the kernels of the hypersingular integral formulation contain second order spatial derivatives of a suitable dynamic Green's function for piezoelectric solids. The Green's function is derived using Radon transform. Its evaluation is a rather involved exercise, requiring the computation of a line integral over a unit circle with integrand that is expressed in terms of exponential integrals (Wang and Zhang [91]).

In this chapter⁴, a semi-analytic solution for an electro-elastic problem involving an arbitrary number of arbitrarily oriented planar cracks in an infinite piezoelectric space is derived. The cracks are acted upon by internal stresses that are time dependent and are either electrically impermeable or permeable. The displacement and electric potential in the Laplace transform domain are expressed as a linear combination of suitably constructed exponential Fourier transform representations. The integrands of the Fourier integrals contain unknown functions that are directly related to the jumps in the Laplace transforms of the displacement and electrical potential across opposite crack faces. The unknown functions are to be determined by solving

⁴ The work reported in this chapter is published in the journal *European Journal of Mechanics - A/Solids*, with details as follows: Ang WT and Athanasius L, Dynamic interaction of multiple arbitrarily oriented planar cracks in a piezoelectric space: A semi-analytic solution, *European Journal of Mechanics - A/Solids* 2011; 30: 608-618.

numerically a system of hypersingular integral equations. Once they are determined, the displacements and electric potential and other physical quantities of interest, such as the crack tip stress and electric displacement intensity factors, can be computed with the help of a suitable algorithm for inverting Laplace transforms. The analysis presented here is general in that it covers both inplane and antiplane deformations. The crack tip stress and electric displacement intensity factors are computed for some specific cases of the problem. For the case involving a single planar crack, the values of the stress and electric displacement intensity factors computed are compared with those published in the literature.

6.2 Statement of Problem

Referring to an $Ox_1x_2x_3$ Cartesian coordinate system, consider an infinite piezoelectric space that contains N arbitrarily oriented non-intersecting planar cracks whose geometries do not change along the x_3 axis. The cracks are denoted by $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(N-1)}$ and $\gamma^{(N)}$. The n -th planar crack $\gamma^{(n)}$ lies in the region

$$-\ell^{(n)} < a_{j1}^{(n)}(x_j - c_j^{(n)}) < \ell^{(n)}, \quad a_{j2}^{(n)}(x_j - c_j^{(n)}) = 0, \quad -\infty < x_3 < \infty, \quad (6.1)$$

where

$$[a_{ij}^{(n)}] = \begin{pmatrix} \sin(\theta^{(n)}) & \cos(\theta^{(n)}) & 0 \\ -\cos(\theta^{(n)}) & \sin(\theta^{(n)}) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.2)$$

On the Ox_1x_2 plane, the crack $\gamma^{(n)}$ is a straight line cut, $2\ell^{(n)}$ is the length of the crack, $(c_1^{(n)}, c_2^{(n)})$ is the midpoint of the crack and $\theta^{(n)}$ is the angle between the crack and the vertical line passing through $(c_1^{(n)}, c_2^{(n)})$ (such that $0 \leq \theta^{(n)} \leq \pi$), as shown

in Figure 6.1. Note that the usual convention of summing over a repeated subscript is adopted in (6.1) for lower case Latin subscripts that run from 1 to 3.

It will be assumed that here the electroelastic deformation of the cracked piezoelectric space does not vary along the x_3 direction. The problem is to solve (2.5) subject to boundary conditions and to determine the displacements $u_k(x_1, x_2, t)$ and electric potential $\phi(x_1, x_2, t)$ in the piezoelectric space for time $t > 0$ such that suitably prescribed boundary conditions on the cracks are satisfied. Furthermore, it is assumed that the displacements u_k and its partial derivative with respect to time (that is, $\partial u_k / \partial t$) are both zero at time $t = 0$.

Figure 6.1: A geometrical sketch of the problem.

The conditions on cracks are given by ,

$$\begin{aligned} S_{Ij}(x_1, x_2, t)m_j^{(n)} &\rightarrow -P_I^{(n)}(\xi_1, \xi_2, t) \quad (I = 1, 2, 3) \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \gamma^{(n)} \quad (n = 1, 2, \dots, N), \end{aligned} \quad (6.3)$$

and either

$$\begin{aligned} S_{4j}(x_1, x_2, t)m_j^{(n)} &\rightarrow -P_4^{(n)}(\xi_1, \xi_2, t) \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \gamma^{(n)} \quad (n = 1, 2, \dots, N) \end{aligned}$$

if the cracks are electrically impermeable, (6.4)

or

$$\begin{aligned} \Delta U_4(x_1, x_2, t) &\rightarrow 0 \text{ as } (x_1, x_2) \rightarrow (\xi_1, \xi_2) \in \gamma^{(n)} \quad (n = 1, 2, \dots, N) \\ &\text{if the cracks are electrically permeable,} \end{aligned} \quad (6.5)$$

where

$$\begin{aligned} \Delta U_I(x_1, x_2, t) &= \lim_{\varepsilon \rightarrow 0} [U_I(x_1 - |\varepsilon|m_1^{(n)}, x_2 - |\varepsilon|m_2^{(n)}) \\ &\quad - U_I(x_1 + |\varepsilon|m_1^{(n)}, x_2 + |\varepsilon|m_2^{(n)})] \\ &\text{for } (x_1, x_2) \in \gamma^{(n)}. \end{aligned} \quad (6.6)$$

In addition, it is required that $S_{Ij}(x_1, x_2, t) \rightarrow 0$ as $x_1^2 + x_2^2 \rightarrow \infty$, $P_1^{(n)}(\xi_1, \xi_2, t)$, $P_2^{(n)}(\xi_1, \xi_2, t)$, $P_3^{(n)}(\xi_1, \xi_2, t)$ and $P_4^{(n)}(\xi_1, \xi_2, t)$ are suitably prescribed functions for $(\xi_1, \xi_2) \in \gamma^{(n)}$, $m_i^{(n)} = -a_{i2}^{(n)}$ are the components of a unit magnitude normal vector to the crack $\gamma^{(n)}$. The initial-boundary conditions are given by

$$U_K = 0 \text{ and } \frac{\partial U_K}{\partial t} = 0 \text{ at } t = 0 \quad (K = 1, 2, 3). \quad (6.7)$$

6.3 Formulation in Laplace Transform Domain

We denote the Laplace transformation of $F(x_1, x_2, t)$ over time $t \geq 0$ by $\widehat{F}(x_1, x_2, s)$, that is, we define

$$\widehat{F}(x_1, x_2, s) = \int_0^{\infty} F(x_1, x_2, t) \exp(-st) dt, \quad (6.8)$$

where s is the Laplace transformation parameter.

Application of the Laplace transformation on both sides of (2.5) together with the initial conditions (6.7) yields

$$C_{IjK\ell} \frac{\partial^2 \widehat{U}_K}{\partial x_j \partial x_\ell} - s^2 B_{IK} \widehat{U}_K = 0 \quad (I = 1, 2, 3, 4). \quad (6.9)$$

In the Laplace transform domain, the problem is to solve (6.9) subject to

$$\begin{aligned} \widehat{S}_{Ij}(x_1, x_2, s) m_j^{(n)} &\rightarrow -\widehat{P}_I^{(n)}(\xi_1, \xi_2, s) \quad (I = 1, 2, 3) \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \gamma^{(n)} \quad (n = 1, 2, \dots, N), \end{aligned} \quad (6.10)$$

and either

$$\begin{aligned} \widehat{S}_{4j}(x_1, x_2, s) m_j^{(n)} &\rightarrow -\widehat{P}_4^{(n)}(\xi_1, \xi_2, s) \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \gamma^{(n)} \quad (n = 1, 2, \dots, N) \end{aligned}$$

if the cracks are electrically impermeable, (6.11)

or

$$\Delta \widehat{U}_4(x_1, x_2, s) \rightarrow 0 \text{ as } (x_1, x_2) \rightarrow (\xi_1, \xi_2) \in \gamma^{(n)} \quad (n = 1, 2, \dots, N)$$

if the cracks are electrically permeable. (6.12)

It is also required that $\widehat{S}_{Ij}(x_1, x_2, s) \rightarrow 0$ as $x_1^2 + x_2^2 \rightarrow \infty$.

6.4 Method of Solution

In this Section, a method based on the theory of exponential Fourier transformation is proposed for solving (6.9) subject to (6.10)-(6.12).

6.4.1 Solution in Fourier Integral Form

For the solution of the piezoelectric crack problem, let

$$\begin{aligned}\widehat{U}_K(x_1, x_2, s) &= \sum_{n=1}^N \operatorname{Re} \left\{ \sum_{\alpha=1}^4 \int_0^{\infty} A_{K\alpha}^{(n)}(\xi, s) [H(a_{r2}^{(n)}(x_r - c_r^{(n)})) F_{1\alpha}^{(n)}(\xi, s) \right. \\ &\quad \times \exp(i\xi(a_{j1}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{j2}^{(n)}))(x_j - c_j^{(n)}) \\ &\quad + H(-a_{r2}^{(n)}(x_r - c_r^{(n)})) F_{2\alpha}^{(n)}(\xi, s) \\ &\quad \left. \times \exp(-i\xi(a_{j1}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{j2}^{(n)}))(x_j - c_j^{(n)})] d\xi \right\}, \quad (6.13)\end{aligned}$$

where $H(x)$ is the unit-step Heaviside function, $F_{1\alpha}^{(n)}(\xi, s)$ and $F_{2\alpha}^{(n)}(\xi, s)$ are functions yet to be determined, $\tau_{\alpha}^{(n)}(\xi, s)$ ($n = 1, 2, \dots, N$) are roots, with positive imaginary parts, of the 8-th order polynomial equation (in τ) given by

$$\begin{aligned}\det \left[\frac{s^2}{\xi^2} B_{IK} + (a_{11}^{(n)} + \tau a_{12}^{(n)})^2 C_{I1K1} \right. \\ \left. + (a_{21}^{(n)} + \tau a_{22}^{(n)})(a_{11}^{(n)} + \tau a_{12}^{(n)})(C_{I1K2} + C_{I2K1}) \right. \\ \left. + (a_{21}^{(n)} + \tau a_{22}^{(n)})^2 C_{I2K2} \right] = 0, \quad (6.14)\end{aligned}$$

$A_{K\alpha}^{(n)}(\xi, s)$ ($n = 1, 2, \dots, N$) are non-trivial solutions of the system

$$\begin{aligned}\left[\frac{s^2}{\xi^2} B_{IK} + (a_{11}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{12}^{(n)})^2 C_{I1K1} \right. \\ \left. + (a_{21}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{22}^{(n)})(a_{11}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{12}^{(n)})(C_{I1K2} + C_{I2K1}) \right. \\ \left. + (a_{21}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{22}^{(n)})^2 C_{I2K2} \right] A_{K\alpha}^{(n)} = 0. \quad (6.15)\end{aligned}$$

From (2.6) and (6.13), we obtain

$$\begin{aligned}
\widehat{S}_{Ij}(x_1, x_2, s) = & \sum_{n=1}^N \operatorname{Re} \left\{ \sum_{\alpha=1}^4 \int_0^{\infty} i\xi L_{Ij\alpha}^{(n)}(\xi, s) [H(a_{r2}^{(n)}(x_r - c_r^{(n)})) F_{1\alpha}^{(n)}(\xi, s) \right. \\
& \times \exp(i\xi(a_{j1}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{j2}^{(n)})(x_j - c_j^{(n)})) \\
& - H(-a_{r2}^{(n)}(x_r - c_r^{(n)})) F_{2\alpha}^{(n)}(\xi, s) \\
& \left. \times \exp(-i\xi(a_{j1}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{j2}^{(n)})(x_j - c_j^{(n)}))] d\xi \right\}, \quad (6.16)
\end{aligned}$$

$L_{Ij\alpha}^{(n)}(\xi, s)$ ($n = 1, 2, \dots, N$) are given by

$$\begin{aligned}
L_{Ij\alpha}^{(n)}(\xi, s) = & [(a_{11}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{12}^{(n)})C_{IjK1} \\
& + (a_{21}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{22}^{(n)})C_{IjK2}]A_{K\alpha}^{(n)}. \quad (6.17)
\end{aligned}$$

The integral representation for $\widehat{U}_K(x_1, x_2, s)$ in (6.13) is obtained by generalizing the analyses in Ang [5, 4] and Clements [20].

Note that $\widehat{U}_K(x_1, x_2, s)$ and $\widehat{S}_{Ij}(x_1, x_2, s)$ in (6.13) and (6.16) respectively are represented by different integral expressions in different parts of the piezoelectric space. To ensure that $\widehat{S}_{Ij}(x_1, x_2, s)m_j^{(n)}$ are continuous on $a_{r2}^{(n)}(x_r - c_r^{(n)}) = 0$, the functions $F_{1\alpha}^{(n)}(\xi, s)$ and $F_{2\alpha}^{(n)}(\xi, s)$ are chosen to be given by

$$F_{1\alpha}^{(n)}(\xi, s) = M_{\alpha P}^{(n)}(\xi, s)\psi_P^{(n)}(\xi, s) \text{ and } F_{2\alpha}^{(n)}(\xi, s) = M_{\alpha P}^{(n)}(\xi, s)\overline{\psi_P^{(n)}}(\xi, s), \quad (6.18)$$

where $\psi_P^{(n)}(\xi, s)$ are functions to be determined and the overhead bar denotes the complex conjugate of a complex number and $M_{\alpha P}^{(n)}(\xi, s)$ are defined by

$$\sum_{\alpha=1}^4 m_j^{(n)} L_{Ij\alpha}^{(n)}(\xi, s) M_{\alpha P}^{(n)}(\xi, s) = \delta_{IP} \quad (n = 1, 2, \dots, N). \quad (6.19)$$

The functions $\widehat{U}_K(x_1, x_2, s)$ are continuous on the plane $a_{r2}^{(n)}(x_r - c_r^{(n)}) = 0$ at points not on any of the cracks if $\psi_P^{(n)}(\xi, s)$ are chosen to be

$$\psi_P^{(n)}(\xi, s) = iT_{PJ}^{(n)}(\xi, s) \int_{-\ell^{(n)}}^{\ell^{(n)}} r_J^{(n)}(u, s) \exp(-i\xi u) du, \quad (6.20)$$

where $i = \sqrt{-1}$, $r_J^{(n)}(u, s)$ are real functions yet to be determined and $T_{PJ}^{(n)}(\xi, s)$ are real functions defined by

$$i \sum_{\alpha=1}^4 [A_{K\alpha}^{(n)}(\xi, s) M_{\alpha P}^{(n)}(\xi, s) - \overline{A}_{K\alpha}^{(n)}(\xi, s) \overline{M}_{\alpha P}^{(n)}(\xi, s)] T_{PJ}^{(n)}(\xi, s) = \delta_{KJ}. \quad (6.21)$$

Use of (6.21) in (6.13) together with

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \int_{-\ell}^{\ell} \frac{\psi(u)}{\epsilon^2 + (v - u)^2} du = \pi \psi(v) \text{ for } -\ell < v < \ell, \quad (6.22)$$

gives

$$\begin{aligned} r_K^{(n)}(a_{j1}^{(n)}(x_j - c_j^{(n)}), s) &= \frac{1}{\pi} \Delta \widehat{U}_K(x_1, x_2, s) \\ \text{for } -\ell^{(n)} &< a_{j1}^{(n)}(x_j - c_j^{(n)}) < \ell^{(n)}, \quad a_{j2}^{(n)}(x_j - c_j^{(n)}) = 0, \end{aligned} \quad (6.23)$$

where $\Delta \widehat{U}_K(x_1, x_2, s)$ are the Laplace transform of the generalized crack opening displacements as defined in (6.6).

The functions $\widehat{S}_{Ij}(x_1, x_2, s)$ in (6.16) can now be written as

$$\begin{aligned} \widehat{S}_{Ij}(x_1, x_2, s) &= - \sum_{n=1}^N \int_{-\ell^{(n)}}^{\ell^{(n)}} r_K^{(n)}(u, s) \operatorname{Re} \left\{ \sum_{\alpha=1}^4 \int_0^{\infty} \xi [H(a_{r2}^{(n)}(x_r - c_r^{(n)})) \right. \\ &\quad \times L_{Ij\alpha}^{(n)}(\xi, s) M_{\alpha P}^{(n)}(\xi, s) \exp(i\xi \tau_{\alpha}^{(n)}(\xi, s) a_{k2}^{(n)}(x_k - c_k^{(n)})) \\ &\quad + H(-a_{r2}^{(n)}(x_r - c_r^{(n)})) \overline{L}_{Ij\alpha}^{(n)}(\xi, s) \overline{M}_{\alpha P}^{(n)}(\xi, s) \\ &\quad \times \exp(i\xi \overline{\tau}_{\alpha}^{(n)}(\xi, s) a_{k2}^{(n)}(x_k - c_k^{(n)}))] \\ &\quad \times T_{PK}^{(n)}(\xi, s) \exp(i\xi [a_{k1}^{(n)}(x_k - c_k^{(n)}) - u]) d\xi \Big\} du. \end{aligned} \quad (6.24)$$

6.4.2 Electrically Impermeable Cracks

From (6.24), conditions (6.10) and (6.11) for electrically impermeable cracks give the hypersingular integral equations

$$\begin{aligned}
& \frac{1}{\ell^{(q)}} \mathcal{H} \int_{-1}^1 \frac{D_{IK}^{(q)} \Delta \hat{U}_K^{(q)}(u, s)}{(v-u)^2} du + \ell^{(q)} \int_{-1}^1 \Delta \hat{U}_K^{(q)}(u, s) \Omega_{IK}^{(q)}(u, v, s) du \\
& + \ell^{(q)} \mathcal{C} \int_{-1}^1 s^2 G_{IK}^{(q)} \Delta \hat{U}_K^{(q)}(u, s) \cosh(\ell^{(q)} \eta |v-u|) \ln(\ell^{(q)} \eta |v-u|) du \\
& + \sum_{\substack{n=1 \\ n \neq q}}^N \ell^{(n)} \int_{-1}^1 \Delta \hat{U}_K^{(n)}(u, s) \Theta_{IK}^{(nq)}(u, v, s) du \\
& = -\pi \hat{P}_I^{(q)}(X_1^{(q)}(v), X_2^{(q)}(v), s) \quad (I = 1, 2, 3, 4) \\
& \quad \text{for } -1 < v < 1 \quad (q = 1, 2, \dots, N),
\end{aligned} \tag{6.25}$$

where $\Delta \hat{U}_K^{(q)}(u, s) = \pi r_K^{(q)}(\ell^{(q)} u, s)$, $X_1^{(q)}(v) = c_1^{(q)} + \ell^{(q)} v \sin(\theta^{(q)})$, $X_2^{(q)}(v) = c_2^{(q)} - \ell^{(q)} v \cos(\theta^{(q)})$, $\mathcal{C}$ denotes that the integral is to be interpreted in the Cauchy principal sense and $\mathcal{H}$ denotes that the integral is to be interpreted in the Hadamard finite-part sense, $D_{IK}^{(q)}$, $G_{IK}^{(q)}$ and $W_{IK}^{(q)}(\xi, s)$ are given by

$$\begin{aligned}
D_{IK}^{(q)} &= \lim_{(\xi/s) \rightarrow \infty} T_{IK}^{(q)}(\xi, s), \\
G_{IK}^{(q)} &= \lim_{(\xi/s) \rightarrow \infty} \left(\frac{\xi}{s}\right)^2 [T_{IK}^{(q)}(\xi, s) - D_{IK}^{(q)}], \\
W_{IK}^{(q)}(\xi, s) &= T_{IK}^{(q)}(\xi, s) - D_{IK}^{(q)} - \frac{s^2 G_{IK}^{(q)}}{\xi^2 + \eta^2} \quad (\eta > 0),
\end{aligned} \tag{6.26}$$

and $\Omega_{IK}^{(q)}(u, v, s)$ and $\Theta_{IK}^{(nq)}(u, v, s)$ are respectively defined by

$$\begin{aligned}\Omega_{IK}^{(q)}(u, v, s) = & - \int_0^\infty \xi W_{IK}^{(q)}(\xi, s) \cos(\ell^{(q)} \xi [v - u]) d\xi \\ & - s^2 G_{IK}^{(q)} [\text{Shi}(\ell^{(q)} \eta |v - u|) \sinh(\ell^{(q)} \eta |v - u|) \\ & - \frac{1}{2} \cosh(\ell^{(q)} \eta |v - u|) (\text{Ei}(\ell^{(q)} \eta |v - u|) - E_1(\ell^{(q)} \eta |v - u|)) \\ & + \cosh(\ell^{(q)} \eta |v - u|) \ln(\ell^{(q)} \eta |v - u|)],\end{aligned}\quad (6.27)$$

and

$$\begin{aligned}\Theta_{IK}^{(nq)}(u, v, s) = & - \text{Re} \left\{ \sum_{\alpha=1}^4 \int_0^\infty \xi m_j^{(q)} [H(Y_2^{(nq)}(u, v)) \right. \\ & \times L_{Ij\alpha}^{(n)}(\xi, s) M_{\alpha P}^{(n)}(\xi, s) \exp(i\xi \tau_\alpha^{(n)}(\xi, s) Y_2^{(nq)}(u, v)) \\ & + H(-Y_2^{(nq)}(u, v)) \overline{L}_{Ij\alpha}^{(n)}(\xi, s) \overline{M}_{\alpha P}^{(n)}(\xi, s) \\ & \times \exp(i\xi \overline{\tau}_\alpha^{(n)}(\xi, s) Y_2^{(nq)}(u, v))] \\ & \left. \times T_{PK}^{(n)}(\xi, s) \exp(i\xi Y_1^{(nq)}(u, v)) d\xi \right\} \\ & \text{if it is assumed that } Y_2^{(nq)}(u, v) \neq 0,\end{aligned}\quad (6.28)$$

with $Y_p^{(nq)}(u, v) = a_{kp}^{(n)}(X_k^{(q)}(v) - c_k^{(n)}) - \ell^{(n)} \delta_{p1} u$ and

$$\begin{aligned}\text{Shi}(u) &= \int_0^u \frac{\sinh(x)}{x} dx, \\ \text{Ei}(u) &= -\mathcal{C} \int_{-u}^\infty \frac{\exp(-x)}{x} dx, \\ E_1(u) &= \int_u^\infty \frac{\exp(-x)}{x} dx.\end{aligned}\quad (6.29)$$

The functions $W_{IK}^{(q)}(\xi, s)$ behave as $O(s^4/\xi^4)$ for very large ξ . Thus, the improper integral over $[0, \infty)$ that appears in the definition of $\Omega_{IK}^{(q)}(u, v, s)$ in (6.27) is well defined.

Note that $\Theta_{IK}^{(nq)}(u, v, s)$ as given in (6.28) is valid for $Y_2^{(nq)}(u, v) \neq 0$. If in any case $Y_2^{(nq)}(u, v) = 0$ then (6.28) has to be modified accordingly. The modification gives

$$\begin{aligned}
\Theta_{IK}^{(nq)}(u, v, s) = & \operatorname{Re}\left\{ \frac{\tilde{D}_{IK}^{(nq)}}{[Y_1^{(nq)}(u, v)]^2} - \int_0^\infty \xi \tilde{W}_{IK}^{(nq)}(\xi, s) \exp(i\xi Y_1^{(nq)}(u, v)) d\xi \right. \\
& - s^2 \tilde{G}_{IK}^{(nq)} [\operatorname{Shi}(\eta |Y_1^{(nq)}(u, v)|) \sinh(\eta |Y_1^{(nq)}(u, v)|) \\
& - \frac{1}{2} \cosh(\eta |Y_1^{(nq)}(u, v)|) \\
& \times (\operatorname{Ei}(\eta |Y_1^{(nq)}(u, v)|) - E_1(\eta |Y_1^{(nq)}(u, v)|)) \\
& + i \frac{\pi}{2} \operatorname{sgn}(Y_1^{(nq)}(u, v)) \\
& \left. \times (\cosh(\eta |Y_1^{(nq)}(u, v)|) - \sinh(\eta |Y_1^{(nq)}(u, v)|)) \right\} \\
& \text{if } Y_2^{(nq)}(u, v) = 0,
\end{aligned} \tag{6.30}$$

where $\operatorname{sgn}(x)$ denotes the sign of x and

$$\begin{aligned}
\tilde{D}_{IK}^{(nq)} &= \lim_{(\xi/s) \rightarrow \infty} \tilde{T}_{IK}^{(nq)}(\xi, s), \\
\tilde{T}_{IK}^{(nq)}(\xi, s) &= \sum_{\alpha=1}^4 m_j^{(q)} L_{Ij\alpha}^{(n)}(\xi, s) M_{\alpha P}^{(n)}(\xi, s) T_{IK}^{(n)}(\xi, s), \\
\tilde{G}_{IK}^{(nq)} &= \lim_{(\xi/s) \rightarrow \infty} \left(\frac{\xi}{s} \right)^2 [\tilde{T}_{IK}^{(nq)}(\xi, s) - \tilde{D}_{IK}^{(nq)}] \\
\tilde{W}_{IK}^{(nq)}(\xi, s) &= \tilde{T}_{IK}^{(nq)}(\xi, s) - \tilde{D}_{IK}^{(nq)} - \frac{s^2 \tilde{G}_{IK}^{(nq)}}{\xi^2 + \eta^2} (\eta > 0).
\end{aligned} \tag{6.31}$$

If the cracks $\gamma^{(n)}$ and $\gamma^{(q)}$ are coplanar then $m_j^{(q)} = m_j^{(n)}$, hence $\tilde{T}_{IK}^{(nq)}(\xi, s) = T_{IK}^{(n)}(\xi, s)$, $\tilde{D}_{IK}^{(nq)} = D_{IK}^{(n)}$, $\tilde{G}_{IK}^{(nq)} = G_{IK}^{(n)}$ and $\tilde{W}_{IK}^{(nq)}(\xi, s) = W_{IK}^{(n)}(\xi, s)$. Note that $Y_2^{(nq)}(u, v) = 0$ for all u and v if $\gamma^{(n)}$ and $\gamma^{(q)}$ are coplanar.

The derivations of (6.25), (6.27) and (6.30) make use of the following results:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \int_{-1}^1 \frac{(\epsilon^2 - (v-u)^2)\psi(u)}{(\epsilon^2 + (v-u)^2)^2} du &= -\mathcal{H} \int_{-1}^1 \frac{\psi(u)}{(v-u)^2} du \text{ for } -1 < v < 1, \\ \int_0^\infty \frac{\xi}{\xi^2 + \eta^2} \cos(a\xi) d\xi &= -\frac{1}{2} \cosh(a\eta) (\text{Ei}(a\eta) - E_1(a\eta)) \\ &\quad + \text{Shi}(a\eta) \sinh(a\eta) \quad (a\eta > 0), \\ \int_0^\infty \frac{\xi}{\xi^2 + \eta^2} \sin(a\xi) d\xi &= \frac{\pi}{2} \text{sgn}(a) [\cosh(|a\eta|) - \sinh(|a\eta|)]. \end{aligned} \quad (6.32)$$

Note that $\text{Ei}(x) - E_1(x)$ tend to $2 \ln(x)$ as $x \rightarrow 0^+$. This explains the presence of the Cauchy principal integral in (6.25).

If the cracks are electrically impermeable, the functions $\Delta \hat{U}_K^{(q)}(u, s)$ ($q = 1, 2, \dots, N$) in (6.23) are to be determined by solving the hypersingular integral equations in (6.25). If we make the approximation (as in Kaya and Erdogan [43])

$$\Delta \hat{U}_K^{(q)}(u, s) \simeq \sqrt{1-u^2} \sum_{j=1}^J \omega_K^{(qj)}(s) U^{(j-1)}(u), \quad (6.33)$$

where $U^{(j)}(x) = \sin([j+1] \arccos(x)) / \sin(\arccos(x))$ is the j^{th} order Chebyshev polynomial of the second kind and $\omega_P^{(nj)}(s)$ are unknown coefficients, then (6.25) can be used to set up a system of linear algebraic equations to determine $\omega_P^{(nj)}(s)$ for any fixed value of s .

6.4.3 Electrically Permeable Cracks

From (6.5) and (6.23), $\Delta \widehat{U}_4^{(q)}(u, s) = 0$ for $-1 < u < 1$ and $q = 1, 2, \dots, N$, if the cracks are electrically permeable. According to (6.3), the unknown functions $\Delta \widehat{U}_1^{(q)}(u, s)$, $\Delta \widehat{U}_2^{(q)}(u, s)$ and $\Delta \widehat{U}_3^{(q)}(u, s)$ that can be approximated as above by (6.33) are governed by (6.25) (with $\Delta \widehat{U}_4^{(q)}(u, s) = 0$) for $I = 1, 2, 3$ (instead of $I = 1, 2, 3, 4$).

6.5 Stress and Electric Displacement Intensity Factors

The dynamic stress and electric displacement intensity factors at the tips of the n -th crack $\gamma^{(n)}$ are defined as follows:

$$\begin{aligned}
 & K_I(X_1^{(n)}(-1), X_2^{(n)}(-1), t) \\
 = & \lim_{u \rightarrow -1^-} \sqrt{-2\ell^{(n)}(u+1)(S_{1j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_1^{(n)} \\
 & + S_{2j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_2^{(n)})m_j^{(n)},} \\
 & K_{II}(X_1^{(n)}(-1), X_2^{(n)}(-1), t) \\
 = & \lim_{u \rightarrow -1^-} \sqrt{-2\ell^{(n)}(u+1)(S_{1j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_2^{(n)} \\
 & - S_{2j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_1^{(n)})m_j^{(n)},} \\
 & K_{III}(X_1^{(n)}(-1), X_2^{(n)}(-1), t) \\
 = & \lim_{u \rightarrow -1^-} \sqrt{-2\ell^{(n)}(u+1)S_{3j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_j^{(n)},} \\
 & K_{IV}(X_1^{(n)}(-1), X_2^{(n)}(-1), t) \\
 = & \lim_{u \rightarrow -1^-} \sqrt{-2\ell^{(n)}(u+1)S_{4j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_j^{(n)},}
 \end{aligned}$$

$$\begin{aligned}
& K_I(X_1^{(n)}(1), X_2^{(n)}(1), t) \\
&= \lim_{u \rightarrow 1^+} \sqrt{2\ell^{(n)}(u-1)} (S_{1j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_1^{(n)} \\
&\quad + S_{2j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_2^{(n)})m_j^{(n)}, \\
& K_{II}(X_1^{(n)}(1), X_2^{(n)}(1), t) \\
&= \lim_{u \rightarrow 1^+} \sqrt{2\ell^{(n)}(u-1)} (S_{1j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_2^{(n)} \\
&\quad - S_{2j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_1^{(n)})m_j^{(n)}, \\
& K_{III}(X_1^{(n)}(1), X_2^{(n)}(1), t) \\
&= \lim_{u \rightarrow 1^+} \sqrt{2\ell^{(n)}(u-1)} S_{3j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_j^{(n)}, \\
& K_{IV}(X_1^{(n)}(1), X_2^{(n)}(1), t) \\
&= \lim_{u \rightarrow 1^+} \sqrt{2\ell^{(n)}(u-1)} S_{4j}(X_1^{(n)}(u), X_2^{(n)}(u), t)m_j^{(n)}. \tag{6.34}
\end{aligned}$$

Once the coefficients $\omega_P^{(nj)}(s)$ in (6.33) are determined, the above intensity factors can be approximately calculated in the Laplace transform domain using

$$\begin{aligned}
\widehat{K}_I(X_1^{(n)}(-1), X_2^{(n)}(-1), s) &\simeq \frac{1}{\sqrt{\ell^{(n)}}} (D_{P1}^{(n)}m_1^{(n)} + D_{P2}^{(n)}m_2^{(n)}) \\
&\quad \times \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(-1), \\
\widehat{K}_{II}(X_1^{(n)}(-1), X_2^{(n)}(-1), s) &\simeq \frac{1}{\sqrt{\ell^{(n)}}} (D_{P1}^{(n)}m_2^{(n)} - D_{P2}^{(n)}m_1^{(n)}) \\
&\quad \times \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(-1), \\
\widehat{K}_{III}(X_1^{(n)}(-1), X_2^{(n)}(-1), s) &\simeq -\frac{1}{\sqrt{\ell^{(n)}}} D_{P3}^{(n)} \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(-1), \\
\widehat{K}_{IV}(X_1^{(n)}(-1), X_2^{(n)}(-1), s) &\simeq -\frac{1}{\sqrt{\ell^{(n)}}} D_{P4}^{(n)} \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(-1),
\end{aligned}$$

$$\begin{aligned}
\hat{K}_I(X_1^{(n)}(1), X_2^{(n)}(1), s) &\simeq \frac{1}{\sqrt{\ell^{(n)}}} (D_{P1}^{(n)} m_1^{(n)} + D_{P2}^{(n)} m_2^{(n)}) \\
&\quad \times \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(+1), \\
\hat{K}_{II}(X_1^{(n)}(1), X_2^{(n)}(1), s) &\simeq \frac{1}{\sqrt{\ell^{(n)}}} (D_{P1}^{(n)} m_2^{(n)} - D_{P2}^{(n)} m_1^{(n)}) \\
&\quad \times \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(+1), \\
\hat{K}_{III}(X_1^{(n)}(1), X_2^{(n)}(1), s) &\simeq -\frac{1}{\sqrt{\ell^{(n)}}} D_{P3}^{(n)} \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(+1), \\
\hat{K}_{IV}(X_1^{(n)}(1), X_2^{(n)}(1), s) &\simeq -\frac{1}{\sqrt{\ell^{(n)}}} D_{P4}^{(n)} \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(+1). \quad (6.35)
\end{aligned}$$

The dynamics stress and electric displacement intensity factors at any time t may be recovered by using the Stehfest's algorithm [82] as explained in Section 2.6.

6.6 Specific Problems

In this Section, the dynamics crack tip stress and electric displacement intensity factors are computed for some specific cases of the problem.

Problem 1

Consider the case of a single electrically impermeable crack in an infinite piezoelectric space. The crack lies in the region $-a < x_1 < a$, $x_2 = 0$, and the only non-zero uniform load acting on it is given by $S_{22} = -H(t)\sigma_0$, where $H(t)$ is the unit-step Heaviside function. (Note that S_{12} , S_{32} and S_{42} are taken to be zero on the crack.)

The electrical poling direction is taken to be along the x_2 direction.

The piezoelectric material is PZT-BaTiO₃ with material constants A , N , F , C , L , e_1 , e_2 , e_3 , ϵ_1 and ϵ_2 and density ρ given by

$$\begin{aligned} A &= 15.0 \times 10^{10}, \quad N = 7.78 \times 10^{10}, \quad F = 6.6 \times 10^{10}, \\ C &= 14.6 \times 10^{10}, \quad L = 4.4 \times 10^{10}, \quad e_1 = 11.4, \\ e_2 &= -4.35, \quad e_3 = 17.5, \quad \epsilon_1 = 98.7 \times 10^{-10}, \\ \epsilon_2 &= 112 \times 10^{-10}, \quad \rho = 5800. \end{aligned} \quad (6.36)$$

The values of A , N , F , C and L above are in N/m², e_1 , e_2 and e_3 are in C/m², ϵ_1 and ϵ_2 are in C/(Vm) and ρ in kg/m³.

Figure 6.2: Plots of $K_I/(\sigma_0\sqrt{a})$ against the normalized time $t\sqrt{L/(\rho a^2)}$.

The numerical stress intensity factor $K_I/(\sigma_0\sqrt{a})$ and electric displacement intensity factor $CK_{IV}/(e_3\sigma_0\sqrt{a})$ at the crack tip $(a, 0)$ are plotted against the normal-

Figure 6.3: Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against the normalized time $t\sqrt{L/(\rho a^2)}$.

ized time $t\sqrt{L/(\rho a^2)}$ in Figures 6.2 and 6.3 respectively. The results are obtained by using $J = 10$ in the numerical solution of the hypersingular integral equations and $M = 4$ (8 terms) in the Stehfest's formula for inverting Laplace transform. In Figures 6.2 and 6.3, the numerical values of $K_I/(\sigma_0\sqrt{a})$ and $CK_{IV}/(e_3\sigma_0\sqrt{a})$ are also compared with those extracted from Shindo *et al* [75] and García-Sánchez, Zhang, Sládek and Sládek [32]. (Numerical values of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ are not given in Shindo *et al* [75].) All the plots of $K_I/(\sigma_0\sqrt{a})$ in Figure 6.2 are quite close to one another, exhibiting the same general trends and reaching peak values at about the same time. So are the plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ in Figure 6.3. As pointed out earlier on, 8 terms ($M = 4$) are used in the Stehfest's formula for obtaining the plots in Figure 6.2. Convergence is observed in the numerical values of the intensity factors when the number of terms in the formula is increased to 10. The numerical inversion

of Laplace transform starts becoming unstable when the number of terms is increased beyond 10. To use more terms, it is necessary to refine the calculation to obtain more a more accurate solution of the hypersingular integral equations.

Problem 2

Consider a pair of coplanar cracks, each of length $2a$, as shown in Figure 6.4. The distance between the inner tips of the cracks is $2d$. The uniform tractions acting on the crack faces are given by $S_{22} = -H(t)\sigma_0$. Here electrically impermeable and permeable cracks will be examined separately. For the case in which the coplanar cracks are electrically impermeable, the condition $S_{42} = -H(t)D_0$, where D_0 is a constant, applies.

Figure 6.4: A pair of coplanar cracks.

The electrical poling is along the x_2 direction. For the purpose of obtaining some numerical results for the dynamic stress and electric displacement intensity factors at the inner and outer tips of the coplanar cracks, the material constants of PZT- BaTiO₃ in (6.36) are used and the load ratio σ_0/D_0 for electrically impermeable cracks is taken to be 10^{10} NC^{-1} .

Figure 6.5: Plots of $K_I/(\sigma_0\sqrt{a})$ against the normalized time $t\sqrt{L/(\rho a^2)}$ at inner and outer crack tips of electrically impermeable cracks for selected values of d/a .

In Figure 6.5, the normalized stress intensity factor $K_I/(\sigma_0\sqrt{a})$ at the inner and outer crack tips are plotted against the normalized time $t\sqrt{L/(\rho a^2)}$ for $d/a = 0.125$, 0.50 and 0.25 . The plots of $K_I/(\sigma_0\sqrt{a})$ for electrically permeable and impermeable cracks are almost indistinguishable. Thus, only the plots electrically impermeable

case are given in Figure 6.5. In each of the plots, $K_I/(\sigma_0\sqrt{a})$ increases rapidly to a peak value before settling down to the corresponding value of the static stress intensity factor. For all the values of d/a in Figure 6.5, the peak values of $K_I/(\sigma_0\sqrt{a})$ at the inner and the outer crack tips are significantly different and the peak value at the inner tips is larger than that at the outer tips. As may be expected, the peak value of $K_I/(\sigma_0\sqrt{a})$ at each crack tip is larger if the cracks are closer to each other. Further calculations show that the plot of $K_I/(\sigma_0\sqrt{a})$ at the inner tips is almost identical as that at the outer tips for $d/a > 3$.

Figure 6.6: Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against the normalized time $t\sqrt{L/(\rho a^2)}$ at inner and outer crack tips of electrically impermeable cracks for selected values of d/a .

Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ at the inner and the outer crack tips against $t\sqrt{L/(\rho a^2)}$ for $d/a = 0.125, 0.50$ and 0.25 are given in Figures 6.6 and 6.7 for electrically im-

permeable and impermeable cracks respectively. The plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ for electrically impermeable cracks are distinct from those for electrically permeable cracks. For a fixed d/a , the peak value of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ at each crack tip is apparently higher for the electrically permeable cracks than that for the impermeable cracks.

Figure 6.7: Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against the normalized time $t\sqrt{L/(\rho a^2)}$ at inner and outer crack tips of electrically permeable cracks for selected values of d/a .

Problem 3

Consider two equal length parallel electrically impermeable cracks as sketched in Figure 6.8. The half length of each crack is given by a . The centers of the cracks lie on a vertical line and are separated by a distance denoted by d . The non-zero constant

loads acting on the crack faces are given by $S_{22} = -H(t)\sigma_0$ and $S_{42} = -H(t)D_0$, with $\sigma_0/D_0 = 10^{10} \text{ NC}^{-1}$.

Figure 6.8: Two parallel cracks.

The electrical poling is along the x_2 direction. Using the material constants of PZT- BaTiO₃ in (6.36), for selected values of d/a , we plot $K_I/(\sigma_0\sqrt{a})$, $CK_{II}/(F\sigma_0\sqrt{a})$ and $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against the normalized time $t\sqrt{L/(\rho a^2)}$ in Figures 6.9, 6.10 and 6.11 respectively.

In Figure 6.9, for a given d/a , the normalized stress intensity factor $K_I/(\sigma_0\sqrt{a})$ rises to a peak (maximum) value and then drops to a trough (minimum) value before gradually settling down to approach its static value. Both the trough and the peak values decrease in magnitude as d/a decreases. A similar observation may be made

of the normalized electric displacement intensity factor $CK_{IV}/(e_3\sigma_0\sqrt{a})$ in Figure 6.11.

Figure 6.9: Plots of $K_I/(\sigma_0\sqrt{a})$ against the normalized time for selected values of d/a .

As d/a tends to infinity, the $CK_{II}/(F\sigma_0\sqrt{a})$ vanishes. Nevertheless, when the cracks come close to each other, there is an increase in the magnitude $CK_{II}/(F\sigma_0\sqrt{a})$ due to larger differences in the stress distribution on opposite crack faces. This is shown in Figure 6.10. For $d/a = 10$, the magnitude of $CK_{II}/(F\sigma_0\sqrt{a})$ is very small at all time.

Problem 4

Consider now the two pairs of electrically impermeable cracks, which are of equal length $2a$, in the piezoelectric space, as sketched in Figure 6.12. The faces of the

Figure 6.10: Plots of $CK_{II}/(F\sigma_0\sqrt{a})$ against the normalized time for selected values of d/a .

Figure 6.11: Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against the normalized time for selected values of d/a .

horizontal cracks are subject to internal uniform loads given by $S_{12} = S_{22} = 0$, $S_{32} = -H(t)\tau_0$ and $S_{42} = -H(t)D_0$, such that $\tau_0/D_0 = 10^{10} \text{ NC}^{-1}$. The internal loads on the faces of the vertical cracks are given by $S_{K1} = 0$ (for $K = 1, 2, 3, 4$).

Figure 6.12: Two pairs of cracks.

The electrical poling is in the x_3 direction. As in the problems above, the material occupying the piezoelectric space is taken to be PZT-BaTiO₃ with material constants as given in (6.36).

The deformation of the cracks is antiplane. The normalized crack tip stress and electrical displacement intensity factors $K_{III}/(\tau_0\sqrt{a})$, $K_{IV}/(D_0\sqrt{a})$ and normalized energy release rate are of interest here. For $b/a = 1.25$ and a few selected values of d/a , these intensity factors at the lower tip of the left vertical crack are plotted against the normalized time $t\sqrt{L/(\rho a^2)}$ in Figures 6.13 and 6.14. For the same

Figure 6.13: Plots of $K_{III}/(\tau_0\sqrt{a})$ at the upper tip of the left vertical crack against the normalized time for $b/a = 1.25$ and selected values of d/a .

values of b/a and d/a , plots of the intensity factors at the left tip of the upper horizontal crack are given in Figures 6.15 and 6.16. As shown in Figures 6.14 and 6.16, $K_{IV}/(D_0\sqrt{a})$ at both crack tips does not vary with time. In Figures 6.13 and 6.15, for a fixed b/a and d/a , $K_{III}/(\tau_0\sqrt{a})$ rises quickly to a peak value, drops to a trough value and gradually approaches the corresponding static value. As expected, as d/a increases, both $K_{III}/(\tau_0\sqrt{a})$ and $K_{IV}/(D_0\sqrt{a})$ for the vertical cracks decrease in magnitude, becoming closer to zero. Figures 6.17 and 6.18 gives plots of the energy release rate $G/(4\tau_0^2 a/L)$ against time for selected values of d/a .

Figure 6.14: Plots of $K_{IV}/(D_0\sqrt{a})$ at the upper tip of the left vertical crack against the normalized time for $b/a = 1.25$ and selected values of d/a .

Figure 6.15: Plots of $K_{III}/(\tau_0\sqrt{a})$ at the left tip of the upper horizontal crack against the normalized time for $b/a = 1.25$ and selected values of d/a .

Figure 6.16: Plots of $K_{IV}/(D_0\sqrt{a})$ at the left tip of the upper horizontal crack against the normalized time for $b/a = 1.25$ and selected values of d/a .

Figure 6.17: Plots of G/G_0 at the upper tip of the left vertical crack against the normalized time for $b/a = 1.25$ and selected values of d/a .

Figure 6.18: Plots of G/G_0 at the left tip of the upper horizontal crack against the normalized time for $b/a = 1.25$ and selected values of d/a .

6.7 Summary

Through the use of Laplace and exponential Fourier transforms, a semi-analytic solution is derived for an electro-elastodynamic problem involving an arbitrary number of arbitrarily located planar cracks in a piezoelectric space. The problem is eventually reduced to solving a system of hypersingular integral equations. The unknown functions in the hypersingular integral equations are the Laplace transforms of the jumps in the displacements and electric potential across opposite crack faces. Once they are determined, the crack tip stress and electric displacement intensity factors can be easily computed in the Laplace transform domain. A numerical technique for inverting Laplace transforms is employed to recover the intensity factors in the physical domain.

The solution is applied to study some specific cases of the problem. For the case of a single crack under impact loadings, the computed crack tip stress and electric displacement intensity factors are found to be in reasonably good agreement with those published in the literature. Numerical results are also obtained for other cases which include one which involves four interacting planar cracks under antiplane deformations.

The numerical Green's function boundary element approach in Chapters 4 and 5 can be extended to solve the dynamic piezoelectric crack problem here. However, as mentioned earlier in Section 6.1, the evaluation of the fundamental solution of the time dependent governing partial differential equations may be an extremely tedious exercise. Thus, the computation of the numerical Green's functions for the dynamic problem is expected to be much more computationally intensive than that of the elastostatic problems in Chapters 4 and 5.

Chapter 7

Research Contributions and Extensions

7.1 Summary of Contributions

The research contributions of the thesis may be summarized as follows.

In Chapter 3, an electro-elastostatic Green's function which satisfies the conditions of vanishing traction and normal electric displacement on the edges of an infinitely-long piezoelectric strip is derived in explicit form. Such a Green's function is useful for analyzing thin piezoelectric plates found in the design of many electronic devices.

The Green's function above is used together with the direct boundary integral equations for electro-elastostatics to obtain a semi-analytic solution for multiple arbitrarily-located planar cracks in a piezoelectric strip. Explicit formulae are derived for the electro-elastostatic fields around the cracks.

In Chapter 4, special electro-elastostatic Green's functions are numerically constructed for stress-free and either electrically permeable or impermeable planar cracks in an infinite piezoelectric space.

The numerically constructed Green's functions are applied successfully to develop an boundary element method for analyzing planar cracks in a piezoelectric material

of finite extent. With the singular behaviors of the stress and electric displacement analytically built into the Green's functions, the crack faces do not have to be discretized into boundary elements and the crack tip stress and electric displacement intensity factors are accurately extracted from the boundary element solutions.

The hypersingular integral formulations of crack problems contain unknowns which have direct physical interpretations. The unknowns are the jumps in the displacement and electric potential across opposite crack faces. Consequently, the solutions here can be easily adapted to deal with the nonlinear boundary conditions for semi-permeable cracks. In Chapter 5, an iterative boundary element method based on the numerical Green's function for impermeable cracks is proposed for semi-permeable cracks in a piezoelectric materials of finite extent.

In Chapter 6, a semi-analytic solution based on the Laplace and the exponential Fourier transform techniques, is obtained for an electro-elastodynamic problem involving multiple planar cracks in a piezoelectric space. There are comparatively few works on dynamic piezoelectric crack problems in the literature (Kuna [46]).

For all the problems considered here, explicit formulae are given for the stress and the electric displacement intensity factors at the crack tips. For specific cases of the problems, the crack tip intensity factors calculated from the formulae are compared and found to agree with those published in the literature. Some new physically interesting results are also obtained for certain configurations of the planar cracks.

7.2 Extensions

Some possible extensions of the research works carried here are as follows.

The methods of solution in Chapters 3, 4 and 5 can be generalized to include arbitrarily curved cracks in piezoelectric media. The generalization can be achieved by using appropriate parametric equations to describe the geometry of each crack. The derivation of the hypersingular integral equations for the boundary conditions of the cracks will, however, be more tedious.

The formulation for each of the problems in Chapters 3, 4 and 5 may be extended to investigate edge cracks which intersect with the boundary of the solution domain. For edge cracks, the numerical method for solving the hypersingular integral equations, however, must be appropriately modified as explained in Nied [64].

The methodology based on exponential Fourier integral transformation developed in Chapter 6 for dynamic electro-elastic analysis of multiple cracks may be extended to piezoelectric half-spaces and strips and other dynamic multi-field crack problems.

For a very large number of cracks in the solid, the possibility of speeding up the numerical solution of the hypersingular integral equations here through the use of the fast multipole method may be investigated [60].

The methods here may also be adapted to solve the crack problems in the context of non-local theory of electro-elasticity.

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