Reliability assessment of laterally loaded single piles

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**NANYANG
TECHNOLOGICAL
UNIVERSITY**

**RELIABILITY ASSESSMENT OF
LATERALLY LOADED SINGLE PILES**

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SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING
2012**

RELIABILITY ASSESSMENT OF LATERALLY LOADED SINGLE PILES

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Abstract

Laterally-loaded piles play an important role in many modern civil engineering structures such as offshore platforms, bridge piers, and high-rise buildings. The deterministic solution of laterally-loaded piles is implicit and often requires an iterative numerical procedure to obtain. On the other hand, parametric uncertainty is inherent in geotechnical engineering. The uncertainties can be accounted for by adopting probabilistic methods. However, extending the deterministic laterally-loaded-pile analysis to a probabilistic study is a complicated and challenging task, given the implicit nature of the underlying limit state functions. The objective of this study is to present some practical approaches of performing reliability assessment of laterally-loaded piles.

An alternative approach of analyzing single laterally-loaded piles is developed based on the direct stiffness method. The deterministic direct stiffness analysis is able to couple nonlinear soil modulus with nonlinear pile flexural rigidity. The deterministic framework underlies the implicit limit state functions used in subsequent reliability analysis. Two failure modes are investigated: deflection failure and bending moment failure. The reliability index is evaluated using an efficient spreadsheet algorithm for the first-order reliability method (FORM, of which the Hasofer-Lind index is a special case) based on the alternative intuitive perspective of an expanding equivalent ellipsoid in the original space of the random variables. The spatial variation of soil properties is modeled using spatial autocorrelation and spatial averaging approaches. The single-failure-mode analyses are extended to system reliability analysis in which bi-modal system failure probability bounds are computed.

A practical procedure for the second-order reliability method (SORM) is proposed which considers the approach described in Der Kiureghian et al. 1987. A short program code is provided in this study for point-fitting a paraboloid to the limit-state surface in the neighborhood of the design point. Complex mathematical

operations associated with Cholesky factorization, Gram-Schmidt orthogonalization and inverse transformation are simplified via short function codes created in this study. The second-order failure probability is calculated by user-defined functions of the reliability index and principal curvatures of the limit-state surface. Comparisons are made with Monte Carlo simulations carried out based on importance sampling.

An efficient method to estimate probabilistic sensitivities is developed. The proposed sensitivity factors can be extended to compute reliability sensitivities with respect to changes in the mean values or standard deviations of the random variables. The method described herein is efficient because the sensitivity factors can be evaluated immediately after the design point and the direction cosines are obtained, without the need for further evaluations of the limit state function. Validation via numerical differentiation is carried out. Comparisons are made with omission sensitivity factors, regression coefficients and rank correlation coefficients.

The reliability approach can be integrated with standalone finite-element programs via the response surface or the artificial neural network methodologies. In this study an iterative procedure is carried out to construct progressively more accurate response surfaces or neural networks based on sample points in the vicinity of the design point. The extension of the reliability methods to any existing pile analysis package via such surrogate models helps to remove some of the computational barriers in reliability analysis and may encourage practitioners to attempt a paradigm shift in their way of performing pile design and analysis.

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List of Symbols and Abbreviations

The following list of symbols and abbreviations is provided for ease of reference for those symbols and abbreviations that are most frequently used in this thesis. It is therefore not exhaustive. Nonetheless all symbols and abbreviations used in this thesis are defined at their first mention in the main text.

Symbols:

C	=	Covariance matrix
c'	=	Cohesion of soil or rock
COV, Ω	=	Coefficient of variation
c_u	=	Undrained shear strength of clay
d	=	Diameter or width of pile
D	=	Vector of nodal displacements
E_p	=	Elastic modulus of pile material
$E_p I_p$	=	Flexural rigidity (or bending stiffness)
F	=	Vector of nodal forces
f	=	Nodal force
$F(x)$	=	Nonnormal CDF
$f(x)$	=	Nonnormal pdf
f_y	=	Tensile strength (or yield stress) of steel
$g(\mathbf{x})$	=	Performance function
I_p	=	Second moment of area of pile cross section
K	=	Stiffness matrix
k_h	=	Coefficient of subgrade reaction
k_{py}	=	Initial modulus of p -y criterion
k_{sec}	=	Secant modulus of p -y criterion
L	=	Length of pile
L	=	Cholesky matrix (lower triangular)
M	=	Bending moment
M_{max}	=	Maximum bending moment

M_{top}	=	Moment applied at top (head) of a pile
M_y	=	Yield bending moment
n	=	Number of pile segments
p	=	Lateral resistance of soil
p_f	=	Probability of failure
P_{top}	=	Horizontal force applied at top (head) of a pile
$\mathbf{R}$	=	Correlation matrix
$R(\mathbf{x})$	=	Response surface function
$r_{\Delta z}$	=	Common ratio of geometric progression of change in segmental length of pile
r_o	=	Autocorrelation distance
$\mathbf{S}$	=	Gram-Schmidt orthogonalization matrix
t	=	Thickness of pile wall (of pipe piles or tubular piles)
u, U	=	Standard normal variable
u'	=	Gram-Schmidt-rotated standard normal variable [$\mathbf{u}' = \mathbf{S}(\mathbf{u})$]
V	=	Shear force
$\mathbf{x}^*, \mathbf{u}^*$	=	Design point
x, X	=	Random variable
x'	=	Unrotated equivalent standard normal variable [$\Phi(x') = F(x)$]
y	=	Deflection of pile
y_0	=	Deflection of pile head
z	=	Depth (below ground surface)
Δz	=	Segmental length of pile
α	=	Direction cosine
β	=	Reliability index
γ	=	Total unit weight of soil
γ'	=	Effective unit weight of soil
δ	=	Scale of fluctuation
θ	=	Angle of rotation of pile
κ	=	Principal curvature
κ_p	=	Curvature of pile

κ_y	=	Yield curvature
μ	=	Mean value
ρ	=	Correlation coefficient
σ	=	Standard deviation
φ	=	Angle of friction of soil or rock
$\phi(\cdot)$	=	Standard normal pdf
$\Phi(\cdot)$	=	Standard normal CDF

Abbreviations:

FORM	=	First-order reliability method
SORM	=	Second-order reliability method
MCS	=	Monte Carlo simulation
RSM	=	Response surface method
ANN	=	Artificial neural network
FEM	=	Finite element method
FDM	=	Finite difference method
DSM	=	Direct stiffness method

Chapter 1 Introduction

1.1. Background

The conventional lumped factor-of-safety approach is widely used in the analysis and design of many geotechnical engineering systems, including laterally-loaded piles. This global safety factor approach, though convenient and straightforward, cannot explicitly account for the uncertainties of the underlying parameters. The more rational alternative is to use probabilistic or reliability methods. However, the implementation of such stochastic approaches in the analysis and design of laterally-loaded piles is often perceived to be technically complex and mathematically challenging, given the implicit nature of the underlying limit state functions. As a result for a long time probabilistic and reliability methods remain somewhat unpopular among practitioners. The development of new design codes that prescribe partial factors is expected to progressively bring about a change in the status quo. There will be an increase in demand for stochastic techniques in practice because the partial factors are in fact results of calibration using reliability methods to achieve target probabilities of non-failure under specific limit states.

First-order second-moment methods (including the simple mean-value first-order second-moment method and the advanced first-order reliability method) have been adopted in recent studies on laterally-loaded piles. Key failure modes associated with deflection and bending moment have been investigated and the performance of piles assessed using reliability indices and probabilities of failure (Ruiz 1984; Folse 1989; Tandjiria et al. 2000; Low et al. 2001). Studies have also been conducted on system reliability, which offers a high-level perspective of the overall system performance and its dependency on the various failure mechanisms under respective limit states (Wang et al. 1997; Elloseily et al. 2002).

Deterministic methods for analyzing laterally-loaded piles include: (1) subgrade reaction method; (2) p - y method; (3) elasticity method; and (4) finite-element

method. The p - y method is perhaps the most widely-used among practitioners and designers. Over the years extensive research has been devoted to the derivation and recommendation of p - y curves for different soils (for example, Matlock 1970, Reese et al. 1975, Dunnavant and O'Neill 1989, Gabr et al. 1994, Liang et al. 2009). Although the p - y method has been compared with the elasticity method and criticized for ignoring soil continuum effects, it was argued that such effects have been indirectly accounted for in the p - y curves that are formulated empirically based on field tests (Reese 1997). In recent years, the finite-element method has gained recognition as a versatile and rigorous technique, capable of modeling soil nonlinearity, soil continuum and soil-structure interaction for both 2- and 3-dimensional problems.

Nonlinearity of pile behavior in flexure needs to be considered when the pile is subjected to substantial lateral loads, but may not have received enough attention. Nakai and Kishida (1982) presented a finite-element approach that incorporated pile nonlinearity using moment-curvature relationships. Kramer and Heavey (1988) developed a finite-difference technique, and adopted a bilinear elastic-plastic model for the pile. Wang and Reese (1988) studied the influence of varying bending stiffness on reinforced concrete pile behavior, and suggested that the bending stiffness be progressively reduced for the entire pile corresponding to the maximum bending moment in a linear analysis. Reese (1997) introduced an analytical procedure to determine the flexural rigidity as a function of bending moment and axial load. Zhang (2003) investigated the influence of loading direction on the lateral nonlinear response of large-section rectangular piles or barrettes using the finite-element method.

The complexity to reliability analysis for laterally-loaded piles is brought to another level because the limit-state function is often not available as an explicit, closed-form function of the input variables. In such cases, the derivatives of the limit-state function with respect to the random variables required for locating the most probable failure point and for assessing the curvature of the failure surface are not readily available. Such difficulties can be dealt with using a number of

numerical approaches, such as the response surface method and the artificial neural network, through which stand-alone deterministic numerical packages can be coupled with reliability analysis based on the concept of approximating the limit-state function using simple closed-form models.

1.2. Objectives and scope of research

The objectives of this study are as follows:

- (i) To develop a deterministic framework for analyzing the behavior of laterally-loaded piles, coupling nonlinear stress-strain behavior of the soil (using p - y curves) with nonlinear flexural rigidity of the pile (using moment-curvature relationships).
- (ii) To develop a probabilistic framework for carrying out reliability assessment of laterally-loaded piles with respect to pile deflection failure (Serviceability Limit State) and bending moment failure (Ultimate Limit State) which may be extended to perform sensitivity analysis, system reliability analysis, and reliability-based design.
- (iii) To study different approaches of developing surrogate models for implicit performance functions, including the response surface method and the artificial neural network, and demonstrate the use of surrogate models in the coupling of reliability approaches with stand-alone numerical packages for laterally-loaded piles.

1.3. Outline of thesis

This thesis is organized into 7 chapters.

Chapter 1 presents the background, objectives and scope of the study, and an outline of this thesis.

Chapter 2 reviews existing methods for the deterministic analysis of a single pile under lateral loading and some of the commonly used probabilistic approaches, and provides a review of the current design philosophies used in geotechnical engineering. The link between reliability-based design and load-and-resistance-factor design based on API code and Eurocode is discussed.

Chapter 3 introduces a direct stiffness method in which the pile is idealized as one-dimensional beam elements and the soil as nonlinear spring elements. The proposed direct stiffness method is illustrated using a number of examples including cases involving nonlinear soil behavior and nonlinear pile flexural behavior. This chapter relates to objective (i).

Chapter 4 presents practical procedures for performing reliability analysis. These includes: (1) a practical second-order reliability method; (2) efficient Monte Carlo simulations with importance sampling; (3) probabilistic sensitivity factors for correlated random variables and (4) improved system failure probability estimates. A laterally-loaded pile is analyzed probabilistically with respect to serviceability and ultimate limit states, considering correlated non-normal variables and one-dimensional spatial variation of soil properties. Simple reliability-based design is demonstrated. This chapter relates to objective (ii).

Chapter 5 demonstrates reliability analysis of laterally-loaded piles based on surrogate models of the implicit performance functions in the form of response surfaces and neural networks. This chapter relates to objective (iii).

Chapter 6 shows a case study of laterally loaded piles at Roosevelt Bridge and demonstrates how reliability-based analysis and design may be carried out in a practical context.

Chapter 7 contains a summary of the findings of this study, with recommendations for future studies.

Chapter 2 Literature Review

2.1. Introductory comments

This chapter is divided into two parts. The first part, comprising sections 2.2 and 2.3, is devoted to the review of existing deterministic methods of analyzing laterally-loaded single piles and pile groups, respectively. The second part, which consists of sections 2.4, 2.5, 2.6 and 2.7, presents the concept of uncertainty in geotechnical engineering, describes the most commonly used probabilistic methods, and highlights the important role that reliability approach can play in the calibration and optimization of design codes.

2.2. Deterministic analysis of laterally-loaded single piles

2.2.1. 'Rigid-pile' methods

Brinch Hansen (1961) considered an infinitely free-head rigid pile which, at failure, was assumed to rotate as a *rigid body* about a point at some depth below the ground surface. The lateral soil resistance was then computed as the resultant of active and passive earth pressures acting on opposite sides of the pile. Charts were provided for calculating the ultimate lateral resistance of a pile founded in a generic c - ϕ soil. Based on a similar theory, Meyerhof et al. (1981) extended the analysis to free-head rigid piles embedded in *two-layered* soils.

In Broms' (1964a; b) approach, the ultimate lateral resistance of a pile was governed by either the yield strength of the pile section or by the ultimate lateral resistance of the soil. Homogeneous cohesive and cohesionless soils were treated separately, and lateral earth pressure distributions along the pile were idealized for simplified failure mechanisms. The failure mode depends on the restraints (*free-head* or *fixed-head*) and relative stiffness of the pile section compared to the supporting soil. If the pile section is relatively stiffer than the soil, and the soil

yields before the pile, the pile is considered a ‘*short*’ pile. On the other hand, if the pile yields before the soil then the pile is considered ‘*intermediate*’ or ‘*long*’. After ascertaining the appropriate failure mode, the ultimate lateral resistance is obtained using simple statics based on the assumed lateral earth pressure diagram.

2.2.2. Depth-to-fixity method

The *depth-to-fixity method* (Davisson 1970) uses a highly simplistic model that ignores the soil between the ground surface and the depth of fixity, and considers the soil below this depth to have infinite strength. The foundation is then analyzed using simple beam theory assuming there is a rigid connection at the depth of fixity and zero soil resistance above that point. This method easily fits into conventional structural analysis and was popular during the 1970s when engineers had difficulty modeling soil resistance properly.

2.2.3. Subgrade reaction method

A key soil parameter in the design of a laterally-loaded pile is the *modulus of subgrade reaction* E_{py} , defined as the soil lateral resistance p per unit length of the pile divided by the pile deflection y at that point:

$$E_{py} = \frac{p}{y} \quad (2.1)$$

The modulus of subgrade reaction E_{py} is related to the *coefficient of subgrade reaction* k_h via the pile diameter or width d :

$$E_{py} = k_h d \quad (2.2)$$

Terzaghi (1955) provided one of the early recommendations for the coefficient of subgrade reaction. He proposed a constant k_h value for a pile in *stiff clay*, and a value of k_h that varied linearly with depth for a pile in *sand*:

$$k_h = n_h \frac{z}{d} \quad (2.3)$$

where n_h = coefficient of subgrade reaction for a long strip with unit width at unit depth; and z = depth. It was assumed that the linear relationship between p and y was valid as long as p did not exceed half of the ultimate lateral resistance. Broms (1964a; b) calculated the ground-line deflections of piles in clay and sand based on Terzaghi's suggestions and made comparisons with test data. It was shown that the proposed method gave conservative estimates of pile deflection in sand, but might underestimate the pile deflections in clay of high unconfined compressive strength. A generic expression of the coefficient of subgrade reaction was used by Meyerhof et al. (1981):

$$k_h = k_0 + n_h \frac{z}{d} \quad (2.4)$$

where k_0 = coefficient of subgrade reaction for a long strip with unit width at ground surface. The calculated ground-line deflections were in fair agreement with model tests and some field cases.

2.2.4. 'p-y' method

The subgrade reaction method assumes a *linear* p - y relationship that is valid for problems involving small deformations only. At large deformations, p increases at a decreasing rate with increasing y . This *nonlinear* p - y relationship is a reflection of the inherent nonlinear stress-strain characteristic of the soil.

McClelland and Focht (1956) were among the first to present the concept of *p-y curves* based on full-scale lateral load tests on instrumented piles and triaxial tests on soil samples. Their paper provided conclusive evidence that the coefficient of subgrade reaction k_h is a parameter of soil properties, as well as of pile diameter and deflection. Since then, numerous studies on *p-y curves* have been carried out over the years, resulting in an extremely rich *p-y* literature. The ensuing paragraphs summarize some of the most important findings and recommendations.

2.2.4.1. Recommendations for various soils

Nonlinear *p-y* curves have been proposed for *clays* of different stiffness. These include *p-y* curves for: soft clay below the ground water level (Matlock 1970); stiff clay above (Reese and Welch 1975) and below the ground water level (Reese et al. 1975; Dunnavant and O'Neill 1989); very soft peaty and silty clay (Lee and Gilbert 1979); hard clay (Bhushan et al. 1979); and medium to stiff fissured clay (Wu et al. 1993). Sullivan et al. (1980) developed a hybrid of Matlock's and Reese et al.'s *p-y* curves and presented a *unified p-y* formulation for all clays below the ground water level. Gaziglu and O'Neill (1984) suggested *integrated p-y* curves that considered relative pile-soil stiffness and soil degradability effects, as well as improvements recommended in an earlier study by Stevens and Audibert (1979). Both Gabr et al. (1994) and Dewaikar and Patil (2006) proposed *hyperbolic p-y* models for clays.

The *p-y* curves for *sand*, both above and below the ground water level, were formulated by Reese et al. (1974). Borgard and Matlock (1980) later simplified the procedure so that the *p-y* curves could be constructed quickly by hand. Scott (1980) suggested a *bilinear p-y* model but did not impose a limiting value on *p*. O'Neill and Murchison (1983) described a *hyperbolic tangent p-y* model, which was based on Parker and Reese's (1970) earlier proposal. The American Petroleum Institute (1993; 2000) has adopted the hyperbolic tangent model as its recommendation for designing piles in sand.

The above p - y procedures assume that the soil can be characterized as either clay or sand, and should be used with caution for soils that exhibit both cohesion and friction (c - ϕ soils). Ismael (1990) carried out a study on piles in cemented sand, and suggested parabolic p - y curves that were similar to those for soft clay (Matlock 1970). On the other hand, Reese and Van Impe (2001) proposed using Reese et al.'s (1974) method to develop p - y curves for c - ϕ soils. Juirnarongrit and Ashford (2004) observed that p - y curves for sand underestimated the resistance of small-diameter piles, but overestimated the resistance of large-diameter piles in weakly cemented sand.

The attention given to p - y analysis of piles in rock has been little, but a number of important studies have been published in recent years. Reese (1997) developed a set of rational equations for a pile in *weak rock* but called the method 'interim' due to limited experimental data available for validation. Later studies by several other authors focused on drilled shafts in *weathered rock*. Gabr et al. (2002), Nusairat et al. (2006) and Liang et al. (2009) proposed *hyperbolic* p - y models. Cho et al. (2007) calculated the resistances of a drilled shaft in weathered rock based on Reese's weak rock model, and observed that the computations overestimated the resistances measured by full-scale instrumented tests.

2.2.4.2. Recommendations for specific ground, pile and loading conditions

The p - y method is highly versatile and adaptable to different ground and pile conditions. Some researchers have studied *slope effects* and suggested procedures to modify p - y curves for a pile in sand, clay and even rock-fill slopes (Reese and Van Impe 2001; McCullough and Dickenson 2004). Zhang et al. (1999a) developed p - y curves for *battered piles* embedded at any angle in sand of any density.

Seismic p - y approach, or p - y curves that are designed for earthquake loading, is gaining considerable research interest in recent years, with emphasis on developing p - y relationships that characterize *sand liquefaction* and *lateral*

spreading (Ashford and Juirnarongrit 2006; Gerber and Rollins 2006; Juirnarongrit and Ashford 2006; Zheng and Luna 2006).

2.2.4.3. Solution strategies

The standard procedure of p - y analysis employs the Winkler (beam-on-elastic-foundation) concept, in which the pile is treated as a vertical beam divided into segments, and the soil is modeled using a number of discrete, nonlinear springs characterized by p - y curves. Assuming that the pile is not subjected to axial load, the governing differential equation is (Hetenyi 1974):

$$E_p I_p \frac{d^4 y}{dz^4} + p(y) = 0 \quad (2.5)$$

where E_p = elastic modulus of pile material; and I_p = moment of inertia of pile section. The product $E_p I_p$ is commonly referred to as the *flexural rigidity* of the pile. The above nonlinear fourth-order differential equation may be solved via a finite-difference technique, and imposing boundary conditions at the pile head and toe according to the physical problem. The finite-difference procedure is described in, for example, Reese and Van Impe (2001) and Salgado (2008). COM624 and LPILE are two of the most well-known finite-difference computer programs for p - y analysis.

Low and Teh (1999) introduced a technique that performs p - y analysis using Microsoft EXCEL. Strain compatibility and equilibrium among pile segments, as well as between the segments and the soil springs, are formulated based only on simple concepts of statics and basic calculus. The optimization routine that resides in the spreadsheet searches for the pile deflection profile that satisfies force and moment equilibrium. The numerical procedure was validated for linear p - y (Low and Teh 1999) and nonlinear p - y models (Low et al. 2001). The latest refinement to this method incorporates nonlinear pile flexural rigidity (Chan and Low 2009).

The most serious criticism against the p - y method is that the soil resistance is represented by a series of discrete, uncoupled springs that fail to model *continuum effects*. However it was argued that such effects had been indirectly accounted for in the p - y curves that were formulated based on instrumented experiments (Reese 1997).

2.2.5. Characteristic load method (CLM)

The characteristic *load method* (Duncan et al. 1994) is a simplified version of Evans and Duncan's (1982) procedure for predicting nonlinear behavior of single piles under lateral loading. In CLM, a series of dimensionless load-deflection and load-moment curves are available for designing free-head and fixed-head piles in cohesive and cohesionless soils. These charts are constructed using nonlinear p - y analysis and they provide simple and fast estimates of ground-line deflections and maximum bending moments. The charts have been normalized with the so-called *characteristic load* or *characteristic moment*, which depend on parameters such as the diameter, elastic modulus and moment of inertia of the pile as well as the shear strength and unit weight of the soil. Brettmann and Duncan (1996) later formulated closed-form exponential equations for these charts so that the CLM could be easily programmed on the computer. Endley et al. (1997) developed similar equations for piles that penetrated only a short distance into the ground as well as for long piles. Ooi et al. (2004) proposed a modification that accounted for the increased lateral resistance of fixed-head piles due to embedment below ground.

The CLM is limited to soils of uniform strength with depth, and applies only to instances in which the pile head coincides with the ground surface. The charts were obtained using p - y analysis assuming static conditions and may not be accurate for piles subjected to cyclic loading.

2.2.6. Strain wedge method

The limitations of the *p-y* method was discussed earlier, and the fact that *p-y* curves do not explicitly account for soil continuity and pile properties such as pile flexural rigidity, pile cross-section shape and pile-head conditions was highlighted. The *strain wedge model* attempts to address these issues by characterizing the one-dimensional beam-on-elastic-foundation parameters in terms of three-dimensional pile-soil interaction behavior. A passive wedge of soil is conceptually envisioned to develop in front of the pile. This allows the stress-strain behavior of the soil to be correlated with one-dimensional beam-on-elastic-foundation parameters.

The strain wedge model was initially developed to analyze a flexible free-head pile in uniform soil (Norris 1986). Later research has enhanced the model for piles in multiple soil layers (Ashour et al. 1998; Ashour et al. 2004) and liquefied sand (Ashour and Norris 2003).

2.2.7. Elastic continuum method

The *elastic continuum method* assumes the soil to be an ideal, homogeneous, isotropic, semi-infinite elastic half-space with elastic modulus and Poisson's ratio unaffected by the presence of the pile. The soil at the back of the pile near the ground surface is also assumed to remain adhered to the pile. The pile is divided into a number of elements. The lateral earth pressure on each element is taken as uniform and constant across the pile width or diameter. Under a perfectly elastic condition, the lateral displacements of the soil and the pile are equal, and may be evaluated by integrating the Mindlin equation (Mindlin 1936; Douglas and Davis 1964). The pile deflections are then determined using the differential equation for bending of a thin beam, solved with finite-difference formulations (Poulos 1971a; 1972). The elastic model can be modified to account for: (1) local yield of the soil (Spillers and Stoll 1964); (2) variation of soil modulus with depth (George and

Cha 1974; Randolph 1981; Horvath 1984); (3) pile-soil separation near the ground surface (Poulos and Davis 1980); (4) reduction of soil modulus due to cyclic loading (Poulos 1982); (5) layered-soil system (Pise 1984; Sun 1994; Zhang et al. 2000); and (6) 3D effects (Verruijt and Kooijman 1989; Abedzadeh and Pak 2004).

More efficient elastic-continuum solution strategies have been proposed in recent years. Shen and Teh (2002) proposed a numerical solution for pile groups under lateral loads using a variational approach. Later on, Shen and Teh (2004) developed a variational solution for a one-layered soil profile where soil stiffness increases with depth. Yang and Liang (2006) presented a further development of the variational approach to two-layered soil profiles with varied soil stiffness. The soil is modeled using a subgrade reaction method. The displacements and reaction pressures of the piles are represented by finite series, and no discretization of the pile shaft is required. The principle of minimum potential energy is applied to determine the response of the piles. Basu and Salgado (2008), and Basu et al. (2009) developed an advanced continuum-based method to calculate the response of laterally-loaded piles in multilayered elastic media. The displacement fields are products of independent functions that vary in the vertical, radial and circumferential directions. The governing differential equations for the pile deflections in different soil layers are obtained using the principle of minimum potential energy. There is potential in extending the method to the case of a nonlinear soil, and even nonlinear pile.

2.2.8. Finite-element method (FEM)

The *finite-element method* is a method for numerical solution of field problems, which may be described by partial differential equations or integral equations. In FEM, the structure is discretized into a *mesh* of finite elements connected at *nodes*. In each element the field quantity is approximated by a simple function so that a fully continuous field is represented by *piecewise interpolation* of a finite number of nodal quantities – a special case of the *Rayleigh-Ritz method*. The *principle of*

stationary potential energy is then applied, resulting in a system of simultaneous equations with which the unknown nodal quantities may be solved:

$$\mathbf{KD} = \mathbf{F} \quad (2.6)$$

where $\mathbf{K}$ = global stiffness matrix; $\mathbf{D}$ = global nodal displacement vector; and $\mathbf{F}$ = global nodal load vector. Details on the concepts and applications of FEM can be found in many textbooks (e.g. Cook et al. 2002).

The FEM has several advantages over many other numerical methods. It can be applied to any field problem of any geometric shape and any boundary and loading conditions. A finite-element model closely resembles the actual structure, and a variety of elements (bar, beam, plate, etc.) with different material properties may be combined within a single model. Although the FEM is not exact, the solution can be improved by using more elements to model the structure, particularly where field gradients are high and where more resolution is desired.

Most finite-element analyses are conducted using ready-to-use finite-element formulations contained in general-purpose or specialized computer programs (e.g. ABAQUS, FLPIER and PLAXIS). With one of these programs, even an inept user can obtain a result. Reliable results, however, can only be obtained when the user understands the problem, models it properly (with considerations in dimensionality, element shapes and connections, mesh grading as well as boundary and loading conditions) and checks for possible pitfalls (e.g. ill-conditioned equations, locking of elements and structural instability) in the results.

In a laterally-loaded pile problem, the FEM faces additional challenges, not all of which have been satisfactorily resolved at the point of writing: selection of an appropriate nonlinear constitutive model for the soil, modeling of the separation between pile and soil during repeated loading, collapse of sand against the back of the pile and changes in soil characteristics associated with various types of loading, to name a few. Nonetheless numerous studies have demonstrated the potential

and versatility of the FEM in this field, with examples from two-dimensional (Bransby 1999) to three-dimensional studies (Rajagopal et al. 2006), and from linear elastic (Randolph 1981) to nonlinear soil models, including p - y (Zhang 2003; Hutchinson et al. 2005; Guo 2006) and elastic-plastic elements (Kooijman 1989; Brown and Shie 1990b; Yang and Jeremic 2005). Some models incorporated interface elements that allow for slippage at the pile-soil interface and gapping in the space behind the pile (Kooijman 1989; Brown and Shie 1990b; Trochanis et al. 1991). More recently the FEM have been used to analyze piles subjected to lateral soil movement (Goh et al. 2003; Martin and Chen 2005; Miao et al. 2006) and seismic loading (Jang et al. 2005; Kimiaei et al. 2006; Tuladhar et al. 2008).

2.2.9. Boundary-element method (BEM)

The *boundary-element method* is a numerical procedure of solving linear partial differential equations formulated as integral equations applicable on the boundary of the solution domain. The governing partial differential equations are transformed into *boundary integral equations*, which relate displacements and tractions at the boundary, using the reciprocal work theorem. The boundary that is enclosing the solution domain is divided into small segments or boundary elements. Numerical integration is performed over each element to obtain a system of linear algebraic equations that can be solved subject to the boundary conditions.

A close relation exists between the boundary-element method and the elastic continuum method: the soil is modeled as boundary elements formulated based on Mindlin's equation (Poulos 1968; Banerjee and Davies 1978). Two applications in the analysis of laterally-loaded piles are worth noting: (1) dynamic loading (Sen et al. 1985; Mamoon et al. 1990); and (2) lateral soil movement (Chen et al. 1999; Xu and Poulos 2000). In recent years, the hybrid numerical method has gained much attention (Poulos and Chen 1996; Kucukarslan and Banerjee 2004; Filho et

al. 2005), in which the pile is modeled as linear finite elements and the soil half-space as boundary elements.

Comparisons are often made between the boundary-element method (BEM) and the finite-element method (FEM). The most important advantage of BEM over FEM concerns discretization. In BEM discretization is needed only at the boundary, whereas in FEM the entire domain has to be discretized. Hence BEM can be more efficient than FEM in terms of computation time and storage. Other advantages of BEM include better accuracy in problems involving stress concentration, as well as infinite and semi-infinite domains. On the other hand one cannot avoid discretizing the interior domain when using BEM for nonlinear problems. In addition, BEM is also not suitable for problems involving large surface-to-volume ratios, such as the analysis of thin shells. And unlike the larger but sparsely populated solution matrix of FEM, the BEM solution matrix is non-symmetric and fully populated, and may trade off some of the savings in storage.

2.2.10. A short critical review

The different deterministic approaches for analyzing the laterally-loaded pile are each valuable in its own right. No single method is perfect for all problems; each method has its strengths and limitations. The Broms methods, for example, are able to provide quick assessment of pile capacity despite the idealized soil pressure distribution and the somewhat ambiguous classification of piles into short, intermediate and long categories. The widely-used p - y method (as well as the closely-related characteristic load method) is based on the beam-on-elastic foundation concept in which the soil springs, albeit nonlinear, are essentially independent and do not explicitly model the continuous nature of the soil. Moreover appropriate p - y curves must be selected and a p - y criterion remains specific to the particular type of soil for which it is developed. Fortunately a substantial amount of research has gone into establishing p - y curves for different soil and loading conditions. Many p - y criteria are based on full-scale instrumented lateral load tests and therefore reflect to some extent the continuum characteristics of the soil. At present, the p - y method is widely used to analyze

and design laterally-loaded piles and has been adopted in the American Petroleum Institute (API 1993, 2000) recommended practice for offshore structures. Computer codes implementing the p-y method continue to be in use by many practitioners. The finite-element method is based on the elastic continuum theory and is able to model pile-soil interaction more realistically than the p-y method. Salient features of the finite-element method include the ability to consider complex loading conditions and combinations, and the ability to incorporate various soil stress-strain relationships. Some of the commonly-used soil constitutive models are the Mohr-Coulomb, Cam-clay, Drucker-Prager and Duncan-Chang models. The selection of the right stress-strain model requires serious contemplation, which may not be any easier than choosing a suitable set of p-y curves, not to mention proper modeling of pile-soil interface elements. A full three-dimensional finite-element analysis typically takes greater time and effort to prepare, run and interpret. More involvement by the user is to be expected. This author thinks that the semi-empirical p-y method and the more sophisticated finite-element method can contribute in their complementary roles. Already there are several finite-element studies that produce p-y curves, such as in Brown and Shie (1990b), Bransby (1999), Yang and Jeremic (2005), Georgiadis and Georgiadis (2010), and Kim and Jeong (2011). The p-y curves so obtained are based on more rigorous finite-element analyses that directly account for three-dimensional pile-soil behavior, and hence are able to overcome the limitations of traditional p-y curves. The practitioner may then perform pile design and analysis more efficiently using the finite-element-based p-y curves without having to repeat the entire finite-element procedure.

2.3. Deterministic analysis of laterally-loaded pile groups

Piles are normally constructed in groups. The behaviour of a group of closely-spaced piles under lateral loading usually differs from that of a single pile because of interference between adjacent piles or *pile-soil-pile interaction*. As a result of pile-soil-pile interaction, and associated *shadow effects* and *edge effects*, a pile that is part of a group deflects *more* than a single pile under the same average load

per pile. The maximum bending moment in a pile group is also *larger* than that for a single pile due to reduced *efficiency*.

Pile group interaction effect is conveniently expressed by the *group efficiency*, which is defined as:

$$e = \frac{\text{Ultimate lateral resistance of pile group}}{\text{Ultimate lateral resistance of single pile} \times \text{number of piles}} \leq 1 \quad (2.7)$$

Reese and Van Impe (2001) reported a number of past experiments involving closely-spaced in-line and side-by-side piles (Prakash 1962; Schmidt 1981; Cox et al. 1984; Schmidt 1985; Wang and Reese 1986; Franke 1988; Shibata et al. 1989) under lateral loading and gathered some important observations:

- (i) The efficiency increased with pile spacing (indicated by spacing-to-diameter ratio s/d).
- (ii) The values of s/d at which the efficiency approached unity were larger for in-line piles than for side-by-side piles (in-line s/d ranged from 5 to 8 and side-by-side s/d was about 3 to 4).
- (iii) The leading pile in an in-line group showed a higher efficiency than the trailing piles. The values of s/d at which the efficiency approached unity were smaller for the leading pile than for the trailing piles (leading s/d about 4 and trailing s/d about 7).
- (iv) The efficiencies of side-by-side piles were not significantly influenced by their relative positions in the group.

2.3.1. 'p-y' method

Brown et al. (1988) described a procedure to reduce the p -values of p -y curves to reflect the loss of efficiency of closely-spaced piles in a group. The p -values are adjusted using p -multipliers, which depend on the pile spacing and the positions of the pile within the group, and are estimated from past experiments. The

procedure has the distinct advantage of leveraging on the well-established and versatile p -y method for the single pile.

In practice, the same p -multiplier is applied to the same row of (side-by-side) piles, consistent with the observations by Brown and Reese (1988), Morrison and Reese (1988), and McVay et al. (1995). Ruesta and Townsend (1997) and Rollins et al. (1998) also adopted the same approach. For the same pile, the p -multiplier is assumed constant with depth. The validity of this assumption was investigated and supported by full-scale instrumented tests (Brown et al. 1988). Experiments also showed that the lateral capacity of a pile in a group was most influenced by its row position (leading row, first trailing row, second trailing row and so on) apart from the pile spacing. The leading row was observed to carry more load than the trailing rows, and hence was associated with the largest p -multiplier. Tests by McVay et al. (1998) indicated that the same p -multiplier could be applied to the trailing rows. It is clear that the p -multiplier and the efficiency are closely related, and hence Reese and Van Impe (2001) proposed to use efficiency values as p -multipliers directly. More recent recommendations on p -multipliers have been given by Law and Lam (2001), Mostafa and El Naggar (2002), Rollins et al. (2003), Rollins et al. (2005), and Rollins et al. (2006).

2.3.2. Elastic continuum method

The elastic continuum method for pile groups extends from the elastic solution for the single pile and incorporates *interaction factors* α_y and α_θ , which address the additional deflection and rotation at the pile head (Poulos 1971b):

$$\alpha_y = \frac{\text{Additional deflection caused by adjacent pile}}{\text{Deflection of pile caused by its own loading}} \quad (2.8)$$

$$\alpha_\theta = \frac{\text{Additional rotation caused by adjacent pile}}{\text{Rotation of pile caused by its own loading}} \quad (2.9)$$

The interaction factor for rotation α_θ only applies to free-head piles. Poulos and Davis (1980) presented a series of charts that plot values of α_y and α_θ for various loading and pile-head conditions, for different length-to-diameter ratios, spacing-to-diameter ratios, and for both stiff and flexible piles. The deflection of a pile in a group of piles is then computed by applying the *principle of superposition*. The group deflection may be expressed in terms of ratios Λ_A or Λ_T , in which:

$$\Lambda_A = \frac{\text{Deflection of pile group}}{\text{Deflection of single pile carrying same } \textit{average} \text{ load}} \quad (2.10)$$

$$\Lambda_T = \frac{\text{Deflection of pile group}}{\text{Deflection of single pile carrying same } \textit{total} \text{ load}} \quad (2.11)$$

$$\Lambda_A = \Lambda_T \times \text{number of piles} \quad (2.12)$$

Values of Λ_T have been determined and published in chart forms (Poulos and Davis 1980). Although these solutions were based on linear elastic soil assumption, they provided valuable insight into pile group behaviour under different conditions of relative stiffness, pile-head fixity, length-to-diameter ratio, spacing-to-diameter ratio and breadth-to-diameter ratio. Notable studies on laterally loaded pile groups with the elastic continuum method have been carried by Leung and Chow (1987), Chen and Poulos (1997), and Ni and Chuang (2000).

2.3.3. Finite-element method (FEM)

The popularity of the *finite-element method* for pile groups subjected to lateral loading is evident from the numerous studies conducted and published (e.g. Zhang et al. 1999b; McVay et al. 2000; Law and Lam 2001). The soil may be modelled using p -y (Ng et al. 2001; Won et al. 2006) or elastic-plastic elements (Brown and Shie 1990a; 1991; Bransby and Springman 1999). Pile-soil-pile interaction is often accounted for using the elastic continuum method, as in Zhao and Stolarski (1999), and Kitiyodom and Matsumoto (2003). Fan and Long (2007) adopted a modulus-multiplier approach, which reduced the modulus of p -y curves (rather than the p -values), to account for group effect.

A number of computer programs are available for carrying out finite-element analysis on laterally loaded pile groups. Among the more well-known are general-purpose programs such as ABAQUS (Budiman and Ahn 2005), as well as special-purpose programs such as FLPIER (Hoit et al. 1996; Zhang et al. 1999b).

2.4. Uncertainty in soil property estimates

The uncertainty associated with the estimated value of a soil property (obtained either in the field or in the laboratory) is due to several different sources (Baecher 1987b; Christian et al. 1994; Phoon and Kulhawy 1999a). The key components are shown in Fig. 2.1 and elaborated in the following paragraphs.

- (i) **Data scatter** is a combination of inherent or real spatial variation of the soil property and random testing errors. *Real spatial variability* is caused by natural geologic processes that produced and continually modify the soil mass in situ, resulting in variation of the soil property from point to point. (Spatial variation of soil properties will be further discussed later.) *Random testing errors* can be attributed to equipment, procedural, operator or random testing effects.
- (ii) **Systematic error** originates from statistical error in the trend and bias in the measurement procedures. *Statistical error* in the trend (or mean value) of a soil property is due to insufficient sampling or limited data. A classic example of *bias in the measurement procedure* is the vane shear testing of undrained shear strength of cohesive soil. The measured strength needs to be corrected using an empirical factor to account for the plasticity index of the clay (Bjerrum 1972). Another source of bias, known as model uncertainty, is introduced when field or laboratory measurements are transformed into design values via correlation models.

The distinction between data scatter and systematic error was best described by Christian et al. (1994) who suggested that one may think of spatial variation as the

scatter of the soil property about the mean trend; and consider systematic error as uncertainty in the location of the mean trend itself.

Fig. 2.1. Conceptual separation of uncertainty into its components for geotechnical applications (after Christian et al. 1994)

2.4.1. Quantifying uncertainty in soil properties

The uncertainty in the estimated value of a soil property needs to be expressed quantitatively using mathematical formulation of probability theory before it could be incorporated in any engineering analysis or design. This section shall review the most fundamental and essential concepts in statistics.

If a soil property is represented by a random variable X , the *mean* or expected value of X calculated based on N observations or data points is:

$$\text{Mean} = E[X] = \mu_X = \frac{1}{N} \sum_{i=1}^N x_i \quad (2.13)$$

The dispersion in the data about the mean is measured by the *variance* of X :

$$\text{Variance} = \text{Var}[X] = \sigma_X^2 = \frac{1}{N-1} \sum_{i=1}^N (x_i - \mu_X)^2 \quad (2.14)$$

where σ_X is known as the standard deviation of X . The mean and variance are also called the first and second moments of X , respectively. Another useful statistic is the *coefficient of variation*, the degree of dispersion relative to the mean:

$$\text{COV}[X] = \frac{\sigma_X}{\mu_X} \quad (2.15)$$

The third and fourth moments of X are known as the *skewness* and the *kurtosis*. They measure the asymmetry and ‘peakedness’ of X , respectively.

$$\text{Skewness} = \frac{n}{(n-1)(n-2)} \frac{\sum_{i=1}^N (x_i - \mu_X)^3}{\sigma_X^3} \quad (2.16)$$

$$\text{Kurtosis} = \frac{n(n+1)}{(n-1)(n-2)(n-3)} \frac{\sum_{i=1}^N (x_i - \mu_X)^4}{\sigma_X^4} - \frac{3(n-1)^2}{(n-2)(n-3)} \quad (2.17)$$

If X and Y denote two correlated soil properties, then the degree of correlation between them can be expressed by the *correlation coefficient*:

$$\rho[X, Y] = \rho_{XY} = \frac{1}{N-1} \sum_{i=1}^N \frac{(x_i - \mu_X)}{\sigma_X} \frac{(y_i - \mu_Y)}{\sigma_Y} \quad (2.18)$$

A value of ‘1’ for ρ_{XY} indicates perfect linear correlation between X and Y ; ‘0’ indicates no correlation; and ‘-1’ indicates perfect inverse linear correlation. When the correlation coefficient is multiplied by the respective standard deviations of X and Y , we obtain the corresponding *covariance*:

$$Cov[X, Y] = \frac{1}{N-1} \sum_{i=1}^N (x_i - \mu_x)(y_i - \mu_y) \quad (2.19)$$

Researchers and practitioners have devoted much effort in estimating these soil property statistics. Reported values of the mean, standard deviation and coefficient of variation of soil properties can be found in many published papers. Table 2.1 shows the ranges of typical *COV* of some soil strength properties gathered from various sources. Strength properties such as undrained shear strength (c_u) are often correlated from field methods such as the standard penetration test (SPT) and cone penetration test (CPT). These in situ methods tend to introduce considerable random testing errors, and it is not unusual for the c_u so obtained to display a *COV* as high as 50%. On the other hand, the friction angle (ϕ) measured in a laboratory shows much less variability and its *COV* is typically not more than 15%. Bulk properties such as the unit weight (γ) can be measured with a reasonably good precision and its *COV* is usually less than 10%.

Phoon and Kulhawy (1999a, b) applied second-moment probabilistic techniques to combine inherent soil variability, measurement error, and transformation uncertainty in a consistent manner. The variability of the design properties is given in terms of the *COV* at a point and the *COV* for a spatial average.

Baecher and Christian (2003) recognized that many measures of soil strength appear to be well-modeled by normal distributions or by more flexible distributions such as the four-parameter beta, which mimic the normal within the high probability density region of the distribution.

Most soil strength measurements, especially those obtained from in situ tests, reflect the influence of moderate to large volumes of soils within which the effects of individual elements are aggregated to produce the macroscopic property. Thus, from a random process model point of view, soil strength is an averaging process, and is expected to display a tendency toward normal distributions (Baecher and Christian 2003).

Table 2.1. Typical coefficients of variation of common soil strength properties

Soil property	Range of COV	Source
ϕ'	0.02 – 0.05	Lacasse and Nadim (1996)
	0.03 – 0.22	Kulhawy et al. (1991); Cherubini et al. (1993)
		Singh (1971); Hoeg and Murarka (1974);
	0.05 – 0.15	Lumb (1974); Schultze (1975);
		Phoon and Kulhawy (1999a)
		Schultze (1972); Harr (1977); Kulhawy (1992);
	0.05 – 0.25	Manoliu and Marcu (1993);
		Meyerhof (1993; 1995); Becker (1996b)
	0.07 – 0.20	Phoon and Kulhawy (1999b)
	0.15 – 0.25	Barker et al. (1991)
$\tan\phi'$	0.05 – 0.15	Lumb (1966; 1974); Schneider (1999)
	0.07 – 0.15	Schultze (1972); Alonso (1976)
c'	0.20 – 0.50	Lumb (1966)
	0.25 – 0.30	Lumb (1974)
	0.30 – 0.50	Schneider (1999)
c_u	0.05 – 0.35	Lacasse and Nadim (1996)
	0.10 – 0.55	Phoon and Kulhawy (1999a; b)
		Alonso (1976); Fredlund and Dahlman (1977);
	0.12 – 0.85	Meyerhof (1993; 1995); Cherubini et al. (1993); Kulhawy (1992); Becker (1996b)
	0.20 – 0.40	Schneider (1999)
	0.20 – 0.50	Lumb (1966; 1974); Singh (1971)

The behavior of laterally-loaded piles is mainly governed by the ultimate soil resistance and the modulus of lateral subgrade reaction (and how the modulus varies with pile deflection determines nonlinear behavior). The ultimate resistance p_u is usually computed as a function of the soil/rock strength property:

the undrained shear strength c_u in the case of pile in clay, the angle of internal friction ϕ in the case of pile in sand, and the compressive strength q_u in the case of pile in rock. The unit weight of the soil/rock γ also has an effect on the ultimate resistance but the uncertainty associated with the unit weight is relatively small. The soil strength property is also often used to estimate the initial modulus of lateral subgrade reaction k_{py} , which is the initial slope of the p - y curve. Many parameters that define the p - y curve are expressed as empirical functions of the soil strength. For example, the initial modulus of the Reese et al. (1975) p - y curve for submerged stiff clay is a function of c_u . Another important parameter is ε_{50} , which is the soil strain at 50% of the deviatoric stress at failure. A direct correlation between ε_{50} and c_u is also given. On the other hand, the initial modulus of the Reese et al. (1974) p - y curve for sand is correlated with γ .

2.4.2. Spatial variation of soil properties

Soils are the products of the weathering of rocks. Physical weathering agents such as erosion, temperature changes, freezing and thawing disintegrate rocks into smaller particles. Chemical weathering processes such as oxidation, reduction and carbonation decompose the minerals in the rocks. Soils at a particular site may be residual or transported to their present location. Therefore soil properties are often highly variable from point to point within a soil mass, in other words, they exhibit spatial variability.

In practice the observed or measured value of a soil property x at location $\mathbf{r}$ can be expressed as a sum of a deterministic trend $t(\mathbf{r})$ and a residual component about the trend $\omega(\mathbf{r})$ (Fig. 2.2):

$$x(\mathbf{r}) = t(\mathbf{r}) + \omega(\mathbf{r}) \quad (2.20)$$

where $\mathbf{r}$ is a position vector specifying the location. This is the basic spatial variation model. In one-dimensional models, $\mathbf{r}$ may be replaced by the horizontal

or vertical distance (e.g. depth). The trend is characterized deterministically by an equation. The residual is normally characterized as a random variable with zero mean and variance given by:

$$\text{Var}[\omega] = \sigma_{\omega}^2 = E[\{x(\mathbf{r}) - t(\mathbf{r})\}^2] \quad (2.21)$$

The characterization of spatial soil properties using random variables *does not* imply that soil properties are random. Rather, the uncertainty lies in the difference between the true value of the soil properties and the mathematically modeled trend that is inferred based on a limited set of data.

The division of measurements of soil properties between trend and residuals is an artifact of modeling spatial variability. The selection of a certain trend function is a decision on the amount of the spatial variation to be treated as a deterministic function of space, and the amount to be accounted for statistically (Baecher 1987a). The variance of the residual will vary according to the trend function being chosen.

Fig. 2.2. Components of soil variability

2.4.2.1. Trend analysis

Trends are often established by using *regression analysis*, the fitting of well-defined mathematical functions to data points in space. The relationship between the dependent variable (i.e. the soil property) and the independent variable (i.e. distance) can be modeled using either a linear or nonlinear function (such as polynomial, exponential or power function). Parameters that define the functions are estimated so as to provide a best-fit of the given data, such that the residual is minimized. The *method of least squares* is the most commonly used approach for this purpose, and its procedure can be found in many statistics textbooks (Haldar and Mahadevan (2000a); Ayyub and McCuen (2003); Montgomery and Runger (2007), etc.). Also available is an alternative regression method based on *Bayesian statistics*, which is well-explained in Box and Tiao (1992); Congdon (2001); and Press (2005), among others.

2.4.2.2. Residual variations

Trend fitting using regression techniques involve several important underlying assumptions pertaining to the residual term. Firstly, the residual is assumed to be a normally distributed random variable with zero mean and constant variance; this assumption is usually valid for geotechnical properties. Secondly, the residuals are uncorrelated; this assumption is seldom true. It has been shown that two adjacent soil elements tend to possess similar properties. This characteristic is called spatial autocorrelation because it describes the correlation between a particular property and itself over space. Spatial autocorrelation is a function of distance. Correlations are stronger between soil elements that are closer together, and vice versa.

2.4.2.3. Autocorrelation and autocovariance functions

If ω_i and ω_j represent the residuals at locations i and j respectively, where r is the distance between them, their spatial relationship can be expressed by the following *autocorrelation function*:

$$\rho(r) = \frac{E[\omega_i \omega_j]}{Var[\omega]} = \frac{E[\omega_i \omega_{i+r}]}{Var[\omega]} \quad (2.22)$$

By the above definition, the autocorrelation at zero separation is $\rho(r = 0) = 1.0$; and the autocorrelation at extremely large separation is $\rho(r = \infty) = 0$. When $\rho(r)$ is multiplied by $Var[\omega]$ we obtain the *autocovariance function*:

$$C(r) = E[\omega_i \omega_{i+r}] = Cov[\omega_i, \omega_{i+r}] \quad (2.23)$$

Several methods of estimating the autocovariance function are available. These include: (1) moment estimation; (2) maximum likelihood estimation; (3) Bayesian estimation. The most commonly used approach is *moment estimation* or *method of moments*. Moment estimators are conceptually simple, easy to implement, non-parametric and consistent. However, as DeGroot and Baecher (1993) pointed out, the moment estimator is only asymptotically unbiased and tends to perform poorly when the sample size is small. This deficiency can be overcome with the *maximum likelihood estimator*, which is asymptotically normal, consistent and has minimum variance. *Bayesian estimation* of autocovariance has not been widely used in the area of geotechnical engineering, and is less well-developed as moment estimation. Discussions on the Bayesian approach can be found in Zellner (1971), Kitanidis (1986; 1997), Cressie (1993), Berger (1993) and Berger et al. (2001). Common autocorrelation functions are shown in Table 2.2.

**Table 2.2. One-dimensional autocorrelation functions
(after Baecher and Christian 2003)**

Model	Equation (Note: r_o = autocorrelation distance)
White noise	$\rho(r) = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases}$
Linear	$\rho(r) = \begin{cases} 1 - r /r_o & \text{if } r \leq r_o \\ 0 & \text{otherwise} \end{cases}$
Exponential	$\rho(r) = \exp\left(-\frac{ r }{r_o}\right)$
Squared exponential (Gaussian)	$\rho(r) = \exp\left(-\frac{r^2}{r_o^2}\right)$
Spherical	$\rho(r) = 1 - \frac{3r}{2r_o} + \frac{r^3}{2r_o^3}$
Power	$\rho(r) = 1 + \left(\frac{ r }{r_o}\right)^{-2\alpha} \quad \alpha > 0$

2.4.2.4. Spatial averaging and variance reduction

Spatial averaging of random fields is an important concept in many modes of geotechnical performance. For instance, the engineer is often required to evaluate the spatial average of soil strength along the failure surface for each segment of the discretized soil mass. Vanmarcke (1977) pointed out that although the mean of the strength averages (evaluated over intervals of width b) remained constant, their standard deviations (and variances) would tend to decrease as b increases. When b is large, the fluctuations in the observations tend to cancel out more in the process of spatial averaging, resulting in *variance reduction*. If σ_x denotes the standard deviation of spatial property x , and σ_b is the standard deviation of the spatial average of x taken over an interval b , then:

$$\sigma_b = \Gamma(b)\sigma_x \quad (2.24)$$

where $\Gamma(b)$ = reduction factor with value between 0 and 1. An approximate equation of the reduction factor is as follows (Vanmarcke 1977; 1980):

$$\Gamma(b) = \begin{cases} 1 & \text{if } b \leq \delta \\ \sqrt{\delta/b} & \text{if } b \geq \delta \end{cases} \quad (2.25)$$

where δ = scale of fluctuation. Vanmarcke (1983) also derived two models of variance reduction that are commonly used:

$$\Gamma(b) = \begin{cases} 1 & \text{if } b \leq \delta \\ \delta/b & \text{if } b \geq \delta \end{cases} \quad (2.26)$$

$$\Gamma(b) = \begin{cases} 1 & \text{if } b \leq \frac{\delta}{2} \\ \frac{\delta}{b} \left(1 - \frac{\delta}{4b}\right) & \text{if } b \geq \frac{\delta}{2} \end{cases} \quad (2.27)$$

The variance reduction function can be related to the autocorrelation function via the following equation (Vanmarcke 1983). Table 2.3 lists the variance reduction models deduced from the autocorrelation counterparts.

$$\Gamma(b) = \frac{2}{b} \int_0^b \left(1 - \frac{r}{b}\right) \rho(r) dr \quad (2.28)$$

**Table 2.3. One-dimensional variance reduction functions
(after Vanmarcke 1983)**

Model	Equation
White noise	$\Gamma(b) = \begin{cases} 1 & \text{if } b = 0 \\ 0 & \text{otherwise} \end{cases}$
Linear	$\Gamma(b) = \begin{cases} 1 - \frac{b}{3r_o} & \text{if } b \leq r_o \\ \frac{r_o}{b} \left(1 - \frac{r_o}{3b}\right) & \text{otherwise} \end{cases}$
Exponential	$\Gamma(b) = \frac{2r_o^2}{b^2} \left[\exp\left(-\frac{b}{r_o}\right) + \frac{b}{r_o} - 1 \right]$
Gaussian	$\Gamma(b) = \frac{r_o^2}{b^2} \left[\sqrt{\pi} \frac{b}{r_o} \operatorname{erf}\left(\frac{b}{r_o}\right) + \exp\left(-\frac{b^2}{r_o^2}\right) + 1 \right]$ $\operatorname{erf}(\cdot) = \text{error function}$
Spherical	$\Gamma(b) = 1 - \frac{b}{2r_o} + \frac{b^3}{20r_o^3}$
Power	$\Gamma(b) = 1 + \frac{1}{(1-\alpha)(1-2\alpha)} \left(\frac{b}{r_o}\right)^{-2\alpha} \quad \alpha > 0$

2.4.2.5. Variograms and geostatistics

The term ‘*geostatistics*’ refers to the application of Matheron’s (1971) theory to problems in mineral exploration and mining. In the mining industry, the concept of autocorrelation is particularly important for estimating ore reserves. In geostatistics the *variogram* is more commonly used to characterize spatial structure:

$$2\gamma(r) = E[(x_i - x_{i+r})^2] = \operatorname{Var}[x_i - x_{i+r}] \quad (2.29)$$

where $\gamma(r) = \text{semivariogram}$; x_i and x_{i+r} = observed values of the soil property x at locations i and $i+r$ respectively. By definition the variogram is a function of only the increments of the spatial property, not their absolute values. This implies that the variogram requires a less restrictive statistical assumption on stationarity than the autocovariance function, i.e. only the increment of the spatial property needs to be stationary and not the mean. A stochastic process is stationary if its probability distribution (including mean and variance) remains the same regardless of location.

There is a direct relationship between the variogram and the autocovariance (as well as the autocorrelation) function. If the spatial property x has stationary mean and covariance, the variogram equation can be expressed as:

$$2\gamma(r) = \text{Var}[x_i] - \text{Var}[x_{i+r}] - 2\text{Cov}[x_i, x_{i+r}] = 2[C(0) - C(r)] \quad (2.30)$$

$$\gamma(r) = C(0) - C(r) \quad (2.31)$$

where $C(.)$ = covariance function. At infinitely large separation (i.e. $r \rightarrow \infty$) the semivariogram levels off at a *sill* value equal to the variance of the spatial variable. Common semivariogram models are shown in Table 2.4.

2.4.2.6. Estimating autocorrelation or autocovariance

The following steps for estimating autocorrelation or autocovariance based on site investigation data are summarized from DeGroot and Baecher (1993), Christian et al. (1994), Jaksa et al. (2000), and Baecher and Christian (2004).

Table 2.4. One-dimensional normalized semivariogram models (after Baecher and Christian 2003)

Model	Equation
Nugget	$\gamma(r) = \begin{cases} 0 & \text{if } r = 0 \\ 1 & \text{otherwise} \end{cases}$
Linear	$\gamma(r) = \zeta r \quad \zeta = \text{gradient of semivariogram}$
Exponential	$\gamma(r) = 1 - \exp\left(-\frac{r}{r_o}\right)$
Gaussian	$\gamma(r) = 1 - \exp\left(-\frac{r^2}{r_o^2}\right)$
Spherical	$\gamma(r) = \begin{cases} 3r/2r_o - r^3/2r_o^3 & \text{if } r < r_o \\ 1 & \text{otherwise} \end{cases}$
Power	$\gamma(r) = r^\alpha \quad 0 < \alpha < 2$

Step 1: Conduct site characterization program based on a suitable sampling plan to gather sufficient data for quantifying vertical spatial variability. A sampling plan is a program of action for collecting data from a sampled population. For example, SPT, PMT, CPT or DMT may be performed based on a simple systematic sampling plan in which readings are taken at regular depth intervals.

Step 2: Plot soil investigation data against depth below the ground surface. In many practical instances, plots of in-situ test measurement, e.g. CPT q_c or SPT N , with depth have been used to examine vertical spatial variability directly. Otherwise, design soil properties such as c_u or ϕ , which are correlated from in-situ test measurement via established transformation models may be plotted instead.

Step 3: Remove trend from data to make data set stationary. The test data or soil property profile may exhibit a trend that can usually be described by some linear or quadratic function. Ordinary least-squares method is used to determine the most appropriate trend function, which is then subtracted from the original data.

Step 4: Evaluate the empirical autocorrelation (or autocovariance) function for the residuals off the trend. The most common method of estimating autocorrelation (or autocovariance) functions is the method of moments.

For example, consider N in-situ measurements $\{x_1, \dots, x_N\}$ made at equally spaced locations $\{z_1, \dots, z_N\}$ along a vertical borehole such that $\Delta z = z_{i+1} - z_i$. The sample autocorrelation of the measurements for any separation (or lag distance) $k\Delta z$ is:

$$\rho(k\Delta z) = \frac{1}{(n-k-1)s^2} \sum_{i=1}^{n-k} [x(z_i) - t(z_i)][x(z_{i+k}) - t(z_{i+k})] \quad (2.32)$$

where $\rho(k\Delta z)$ = moment estimator of the autocorrelation; $t(z_i)$ = trend removed from the data value $x(z_i)$ at location z_i ; and s^2 = sample variance.

Step 5: Obtain the theoretical best fit for the empirical autocorrelation function by using regression analysis based on a least-squares error approach. This can be done by fitting one of a number of existing models to the sample autocorrelation function (Fig. 2.3), followed by determining the autocorrelation distance (or scale of fluctuation) based on the best-fit model.

Fig. 2.3. Empirical and fitted autocorrelation functions

2.5. Review of fundamental probabilistic methods

2.5.1. First-order reliability methods

The mean-value first-order second-moment (MVFOSM) method is based on a first-order Taylor's series approximation of the limit state function linearized at the mean values of the random variables, and it uses only second-moment statistics (means and covariances) of the random variables. The MVFOSM is described in detail in, for example Ang and Tang (1984), Madsen et al. (1986), Melchers (1999), Haldar and Mahadevan (2000a), and Baecher and Christian (2003). The original formulation by Cornell (1969) assumed two statistically-independent normally-distributed random variables R and S such that the performance function $Z = R - S$ was also normally distributed. An alternative formulation by Rosenbleuth and Esteva (1972) assumed that R and S were statistically-independent lognormal random variables in which $Z = \ln R - \ln S$ was normally distributed. For the general case in which $Z = g(x_1, x_2, \dots, x_n)$, the MVFOSM defines the reliability index as (Cornell 1969):

$$\beta = \frac{\mu_Z}{\sigma_Z} \quad (2.33)$$

where μ_Z and σ_Z are respectively the first-order approximations of the mean value and standard deviation of Z , defined as (Cornell 1969):

$$\mu_Z = g(\mu_1, \mu_2, \dots, \mu_n) \quad (2.34)$$

$$\sigma_Z^2 = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} \text{Cov}(x_i, x_j) \quad (2.35)$$

where μ_i is the mean value of x_i ; and $\text{Cov}(x_i, x_j)$ is the covariance of x_i and x_j . The partial derivatives in Eq. (2.35) are evaluated at the mean values.

The MVFOSM is simple to use, but it suffers from a number of deficiencies (Haldar and Mahadevan 2000a). It does not consider the probability distributions of the random variables. A first-order approximation of $g(\mathbf{x})$ linearized at the mean values may not be accurate if $g(\mathbf{x})$ is nonlinear. There is a lack of invariance in the reliability index β defined by Eq. (2.33) under different but mechanically-equivalent formulations of the performance function (Ditlevsen 1973).

In a review on first-order second-moment reliability methods, USACE (1999) rightly noted that a potential problem with the MVFOSM is its lack of invariance for nonlinear performance functions. The document suggested the more general Hasofer-Lind reliability index (Hasofer and Lind 1974) as a better alternative. The matrix formulation of the Hasofer-Lind index for correlated normal random variables is (Veneziano 1974; Ditlevsen 1981):

$$\beta = \min_{g(\mathbf{x})=0} \sqrt{(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{C}^{-1} (\mathbf{x} - \boldsymbol{\mu})} \quad (2.36a)$$

or, equivalently

$$\beta = \min_{g(\mathbf{x})=0} \sqrt{\left[\frac{x_i - \mu_i}{\sigma_i} \right]^T \mathbf{R}^{-1} \left[\frac{x_i - \mu_i}{\sigma_i} \right]} \quad (2.36b)$$

where μ_i is the mean value of x_i ; σ_i is the standard deviation of x_i ; $\mathbf{C}$ is the covariance matrix; and $\mathbf{R}$ is the correlation matrix. For correlated non-normal random variables, μ_i and σ_i should be replaced by μ_i^N and σ_i^N – the equivalent normal mean and standard deviation of the non-normal x_i , respectively. The values of μ_i^N and σ_i^N may be estimated based on two-parameter equivalent normal transformation (Rackwitz and Fiessler 1978) or three-parameter equivalent transformation (Chen and Lind 1983; Wu and Wirsching 1987). The extension of the Hasofer-Lind index to correlated non-normals is known as the first-order reliability method (FORM).

Most algorithms (e.g., Rackwitz 1976; Liu and Der Kiureghian 1986) for evaluating the Hasofer-Lind reliability index carry out a search for the design

point (or checking point or most probable failure point) in a rotated and transformed space spanned by uncorrelated normal random variables u_i with zero means and unit standard deviations. This standard normal space is obtained from appropriate transformation of the original space of the basic random variables x_i (Rosenblatt 1952; Nataf 1962; Shinozuka 1983; Der Kiureghian and Liu 1986).

The design point $\mathbf{u}^*$ is defined as the point of minimum distance from the origin to the transformed performance function $G(\mathbf{u}) = 0$; the minimum distance so computed is the Hasofer-Lind reliability index:

$$\beta = \sqrt{\mathbf{u}^{*T} \mathbf{u}^*} = |\mathbf{u}^*| \quad (2.37)$$

The above classical FORM procedure is elegant by virtue of rotational symmetry of the standard normal space, in which the objective of finding $\mathbf{u}^*$ is equivalent to finding the smallest hypersphere tangent to the transformed limit state surface. The hypersphere meets the limit state surface at the point $\mathbf{u}^*$; its radius is simply the reliability index β (Fig. 2.4).

An alternative interpretation of the Hasofer-Lind reliability index was described in Low and Tang (1997) based on the perspective of an expanding equivalent dispersion ellipsoid in the original space of the basic random variables. The procedure is then of finding the smallest hyperellipsoid (which has the same aspect ratio as the one-standard-deviation hyperellipsoid) that is tangent to the original limit state surface. This hyperellipsoid is β times the size of the $1\text{-}\sigma$ hyperellipsoid and touches the limit state surface at the design point $\mathbf{x}^*$. The ellipsoid perspective is simpler and more intuitive than the classical approach because it does not involve concepts of transformed space (Fig. 2.5).

Fig. 2.4. Classical FORM approach in rotated space

The Hasofer-Lind index and the FORM are more consistent than the MVFOSM because the former method is based on a first-order Taylor's series approximation of the limit state function linearized at the design point, and so does not depend on how the limit state function is formulated. However, the MVFOSM appears to be more popular than the FORM. USACE (1999) conceded that "many published analyses of geotechnical problems have not used the Hasofer-Lind method, probably due to its complexity, especially for implicit functions such as those in slope stability analysis," and that "the most common method used in Corps practice is the Taylor's series method, based on a Taylor's series expansion of the performance function about the expected values." Low (2008), in his survey of recent papers on geotechnical reliability analysis, also found that the Taylor's series mean-value method was more often used than the Hasofer-Lind index. It is therefore encouraging to note that among recent probabilistic studies on laterally-loaded piles, those which adopted the FORM (e.g., Folse 1989; Wang et al. 1997; Elsoeily et al. 2002; Haldar and Sivakumar Babu 2009) outnumber those which used the MVFOSM (e.g., Ruiz 1984).

Fig. 2.5. Dispersion ellipsoids in original space

In most cases of practical relevance, the Hasofer-Lind method makes it possible to compare between the reliabilities of different structures. With reference to time-invariant reliability computation, Rackwitz (2001) observed that, “for 90% of all applications this simple first-order theory fulfils all practical needs. Its numerical accuracy is usually more than sufficient.” The Hasofer-Lind index generally provides a satisfactory ordering of reliabilities, consistent with the ordering of the actual probabilities of non-failure. Ditlevsen (1979a) showed that for a nonlinear limit state surface, the ordering might not be reasonable because the Hasofer-Lind index is independent of the curvature of the failure surface at the design point. Another shortcoming of the Hasofer-Lind index is observed in cases with multiple design points (Dolinski 1983; Der Kiureghian and Dakessian 1998). The optimization algorithm, depending on the initial values, may converge to a local design point. Even if the global design point is found, the Hasofer-Lind index cannot account for any significant contribution to the failure probability from the local design points. A more consistent and selective alternative is the generalized reliability index (Ditlevsen 1979). The Ditlevsen index is obtained by integrating a weight function over the safe domain in transformed space. The generalized reliability index is consistent and sufficiently selective provided the weight

function is rotationally symmetric and positive. The evaluation of the generalized reliability index is more tedious than the evaluation of the Hasofer-Lind index because of the difficulty in computing the integral of the weight function.

Where the limit state surface has a unique design point and is not extremely nonlinear, the failure probability may be well estimated by assuming a second-order approximation to the failure surface at the design point. The second-order reliability method is discussed in the next section.

2.5.2. Second-order reliability methods

The classical computation procedure of the second-order reliability method (SORM) involves transforming the basic random variables into independent standard normal variables, and approximating the limit state surface by a quadratic expansion at the design point, so that the probability of failure can be estimated from the probability of quadratic forms in normal variables.

Fiessler et al. (1979) were the first to investigate SORM in detail. They studied second-order Taylor expansions to approximate the failure surface and used some results of the statistical theory of quadratic forms in normal variables to calculate improved estimates of the failure probability. Breitung (1984) obtained an asymptotic approximation for multi-normal integrals by applying a quadratic Taylor expansion at the points on the boundary of the integration domain with minimal distance to the origin. However, there were concerns with regards to the accuracy of probability estimates obtained according to Breitung's approximation formula, especially in cases where the reliability index was relatively small. Nevertheless, Hohenbichler and Rackwitz (1988) showed that accurate probability estimates could be obtained if importance sampling techniques were applied. Meanwhile, Tvedt (1983) presented two second-order approximations for the failure probability, and subsequently derived the exact result for the probability content of the failure domain obtained from a parabolic approximation (Tvedt

1988), as well as the full second-order Taylor expansion (Tvedt 1990), of the failure function at the design point.

In the above approximations, the paraboloid was fitted to the principal curvatures of the limit state surface, defined in terms of the eigenvalues of the matrix of the second-order derivatives of the surface. Der Kiureghian et al. (1987) proposed a simpler method based on an approximating paraboloid fitted to the limit state surface at discrete points around the point with minimal distance from the origin. Der Kiureghian and De Stefano (1991) developed an efficient algorithm for computing the principal curvatures in an iterative manner using the gradient of the limit state function.

Koyluoglu and Nielsen (1994) suggested a set of simple closed-form approximations for SORM integrals that was free from singularities and remained valid even for negative reliability indices. Cai and Elishakoff (1994) devised an analytical approximate formula, and showed that their proposed method gave accurate results even when the failure surface possessed large curvatures and was relatively close to the origin. Zhao and Ono (1997) introduced the concept of average principal curvature radius and obtained an empirical second-order reliability index, which was shown to be sufficiently accurate in cases with relatively small numbers of random variables and/or relatively small principal curvatures. An improved formulation of the empirical index that gave good approximations even for relatively large numbers of random variables and/or relatively large principal curvatures was obtained later (Zhao and Ono 1999a). For the case of extremely large principal curvatures or the case where a parabolic approximation of the limit state surface might encounter difficulty, Zhao and Ono (1999b) proposed an inverse fast Fourier transformation (IFFT) method for computing the failure probability. Hong (1999) also derived two simple correction factors to be used in conjunction with the Hohenbichler-Rackwitz (1988) formula.

Several approximations for the probability integral bounded by a quadratic surface have been suggested. These formulas were developed by Tvedt (1983), Breitung (1984), Hohenbichler and Rackwitz (1988), Koyluoglu and Nielsen (1994), Cai and Elishakoff (1994) and Hong (1999). In one way or another, the formulas are functions of the Hasofer-Lind index and a set of principal curvatures κ_i evaluated at the design point. The Hasofer-Lind index β is obtained using an optimization algorithm as discussed in the previous chapter. The principal curvatures κ_i are defined in terms of eigenvalues of the matrix of second-order derivatives of the limit state surface (Breitung 1984; Tvedt 1984, 1985). The procedure for computing the curvatures using the eigenvalue method is illustrated in Haldar and Mahadevan (2000a).

When the number of random variables is large, and when the performance function involves complicated numerical algorithms, computation of the second-order derivatives can be prohibitively costly. Der Kiureghian et al. (1987) pointed out that a paraboloid fitted to the curvatures of a limit state surface containing ‘noise’ might give erroneous results unless proper smoothing was carried out in computing the curvatures. In some instances, such as when the design point is an inflection point, the curvatures at the design point may provide misleading information about the limit state surface. Der Kiureghian et al. (1987) proposed an approximating paraboloid fitted to a set of discrete points on the limit state surface at some prescribed distances from the design point. The point-fitted paraboloid is simpler to implement and requires less computation than the curvature-fitted paraboloid. It is also more stable, and is insensitive to noise in the limit state surface. Furthermore, it approximately accounts for the effect of higher-order terms of the limit state surface.

Three different stages of approximations and errors can be identified in SORM (Adhikari 2004): (1) Type-1 error due to approximating a general nonlinear failure surface by a quadratic hypersurface in the neighborhood of the design point; (2) Type-2 error due to approximating a general PDF by a Gaussian PDF in the

neighborhood of the design point; and (3) Type-3 error due to approximations in evaluating the probability content.

Type-1 error is fundamental to SORM and cannot be improved unless higher order derivatives of the failure surface are used (Breitung and Richter 1996). Type-2 error can be avoided provided the basic variables are already Gaussian or transformed to Gaussian using numerical transformations (although equivalent normal transformations may introduce an uncertain amount of error). Extensive work has been done, for example by Tvedt (1983, 1988, 1990), Naess (1987), Hohenbichler and Rackwitz (1988), Köyliöglu and Nielsen (1994), Cai and Elishakoff (1994), Zhao and Ono (1999a, b), Polidori et al. (1999), and Hong (1999), to improve the reliability assessment by dealing with Type-3 error.

Köyliöglu and Nielsen (1994) showed that the Breitung (1984) asymptotic and Tvedt (1983) three-term approximations carried singularities for negative reliability indices at relatively large positive curvatures, for positive reliability indices at relatively large negative curvatures, and also for cases with relatively large curvatures of opposing signs. Only cases with three random variables were considered. Zhao and Ono (1999a) examined the accuracy of the Tvedt (1983), Breitung (1984), Köyliöglu and Nielsen (1994), and Cai and Elishakoff (1994) formulas using several examples for a wide range of curvatures as well as for different numbers of random variables. The general observation was that these four formulas work well for cases with small numbers of random variables at large curvature radius (i.e., small curvatures). All produced significant errors for limit state surfaces with unevenly distributed curvatures of opposing signs.

Another means to reducing all three types of errors is to use simulation methods, which, with the development of importance sampling techniques are becoming very efficient. Further discussions on developments in this area can be found in the next section.

2.5.3. Monte Carlo method

A fundamental problem in reliability analysis is the computation of the probability of failure:

$$p_f = \int_{g(\mathbf{x}) \leq 0} f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \quad (2.38)$$

in which $f_{\mathbf{x}}(\mathbf{x})$ is the joint probability density function of the random variables $\mathbf{x}$; $g(\mathbf{x})$ is the limit state function defined such that $g(\mathbf{x}) \leq 0$ denotes the failure domain. Direct evaluation of the probability of failure via integration is not feasible for the reliability analysis of laterally-loaded piles because of the implicit nature of the performance function and the difficulty in identifying the joint probability density function in simple closed form. Traditionally, the probability density function of the values of the limit state function is estimated by carrying out a Monte Carlo simulation (Rubinstein 1981; Rubinstein and Kroese 2008). The limit state function is evaluated repeatedly based on a large number of realizations of the random variables defining the function. If $I[g(\mathbf{x}) \leq 0]$ is an indicator function defined such that $I[\cdot] = 1$ if $g(\mathbf{x}) \leq 0$ and $I[\cdot] = 0$ if otherwise, Eq. (2.38) can be rewritten as:

$$p_f = \int_{\text{all } \mathbf{x}} I[g(\mathbf{x}) \leq 0] f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \approx \frac{1}{N} \sum_{k=1}^N I[g(\mathbf{x}_k) \leq 0] \quad (2.39)$$

The ordinary (or brute-force or crude or direct) Monte Carlo method can be prohibitively costly for cases with very small failure probabilities, and where the deterministic analysis for each simulation trial is computationally intensive. The efficiency of the Monte Carlo method can be improved by using variance-reduction techniques (not to be confused with variance-reduction methods related to spatial averaging). Many of these techniques, including antithetic variables, control variables, stratified sampling, importance sampling and adaptive sampling are described in Rubinstein (1981), Melchers (1999), Haldar and Mahadevan

(2000a), Baecher and Christian (2003) and Rubinstein and Kroese (2008), for example. In one way or another, the variance-reduction techniques leverage on some known information about the problem, so as to estimate the probability of failure at the same level of accuracy with fewer trials, or at a higher level of accuracy with the same number of trials.

It has been shown that importance sampling can dramatically improve the efficiency of performing Monte Carlo simulation (Schueller and Stix 1987; Engelund and Rackwitz 1993). Eq. (2.39) needs to be rewritten as (Shinozuka 1983):

$$p_f = \int_{\text{all } \mathbf{v}} I[g(\mathbf{x}) \leq 0] \frac{f_{\mathbf{x}}(\mathbf{v})}{h_{\mathbf{v}}(\mathbf{v})} d\mathbf{v} \approx \frac{1}{N} \sum_{k=1}^N I[g(\mathbf{v}_k) \leq 0] \frac{f_{\mathbf{x}}(\mathbf{v}_k)}{h_{\mathbf{v}}(\mathbf{v}_k)} \quad (2.40)$$

where $h_{\mathbf{v}}(\mathbf{v})$ is the importance sampling probability density function; and $\mathbf{v}$ is a vector of sample values taken from $h_{\mathbf{v}}$.

Several methods have been developed for selecting the appropriate importance sampling density such that the variance of the failure probability estimate can be reduced (e.g., Melchers 1984; Harbitz 1986; Fujita and Rackwitz 1988; Ibrahim 1991). As opposed to the ordinary Monte Carlo method, importance sampling performs sampling in the region that contributes most to the failure probability, usually identified as the neighborhood of the design point. Excellent reviews on many of these importance sampling methods can be found in Schueller and Stix (1987), Engelund and Rackwitz (1993), Melchers (1999), Rubinstein and Kroese (2008), among others.

There are more advanced simulation methods. Two of which are briefly described below:

- (i) *Directional simulation* is a conditional expectation simulation method. The application of directional simulation to structural reliability is as

follows (Bjerager 1988): the n -dimensional standard normal vector $\mathbf{u}$ is expressed as $\mathbf{u} = R\mathbf{A}$ for $R \geq 0$ where R^2 is a chi-square-distributed random variable with n degrees of freedom and is independent of the random unit vector $\mathbf{A}$, which is uniformly distributed on the n -dimensional unit sphere Ω_n . Conditioning on $\mathbf{A} = \mathbf{a}$,

$$p_f = \int_{\mathbf{a} \in \Omega_n} P(g(R\mathbf{A}) \leq 0 | \mathbf{A} = \mathbf{a}) f_A(\mathbf{a}) d\mathbf{a} = \int_{\mathbf{a} \in \Omega_n} P[g(R\mathbf{a}) \leq 0] f_A(\mathbf{a}) d\mathbf{a} \quad (2.41)$$

where $f_A(\mathbf{a})$ is the uniform density of $\mathbf{A}$ on the unit sphere. The efficiency of directional simulation can be improved by means of directional importance sampling based on FORM and SORM results. Sampling densities for various types of failure surfaces are available.

- (ii) The basic idea of *subset simulation* is to express the failure probability as a product of larger conditional failure probabilities by introducing intermediate failure events (Au and Beck 2001). Given a failure event F , let $F_1 \supset F_2 \supset \dots \supset F_m = F$ be a decreasing sequence of failure events so that $F_k = {}^k\cap_{i=1} F_i$ for $k = 1, \dots, m$. By definition of conditional probability:

$$p_f = P(F_1) \prod_{i=1}^{m-1} P(F_{i+1} | F_i) \quad (2.42)$$

The conditional failure probabilities can be made sufficiently large so that they can be estimated using the Monte Carlo method with a small number of samples. A Markov chain Monte Carlo technique based on the Metropolis algorithm is used to estimate the conditional probabilities. The subset simulation approach was shown to be robust to the number of random variables and efficient in computing small failure probabilities.

2.5.4. Finite-element reliability methods

2.5.4.1. Stochastic finite-element method

The term stochastic finite-element method (SFEM) refers to a finite-element method which accounts for uncertainties in the geometry or material properties of

a structure, as well as the applied loads (Der Kiureghian and Ke 1988). Such uncertainties are usually spatially distributed over the region of the structure and modeled as random or stochastic fields. Prominent pioneer works are summarized below. Contreras (1980) proposed a generic stochastic finite-element method in which stochastic differential and difference equation theory was applied in structures discretized with the finite-element methodology. Transient structural loads were idealized as stochastic processes. Vanmarcke and Grigoriu (1983) evaluated the second-moment statistics of the deflection of a beam of randomly-varying rigidity along its axis, based on a stochastic finite-element formulation in which the elastic characteristics of each finite element are assumed equal to the local average over the element length. Shinozuka and Deodatis (1988) examined the response variability of finite-element systems arising from one-dimensional spatial randomness of the material properties. The problem was analyzed with a first-order Neumann expansion of the stiffness matrix of the system. Yamazaki et al. (1988) developed a Neumann expansion technique for deriving the finite-element solution for the response variability within the framework of the Monte Carlo method. The spatial variability of material properties were modeled as two-dimensional stochastic fields. Spanos and Ghanem (1989) used a Karhunen-Loeve expansion to represent the random material property modeled as a stochastic process.

A stochastic finite-element method based on the FORM was described in Der Kiureghian and Ke (1988). The main effort for finite-element implementation was in computing the gradient of response variables in terms of basic load, material property or geometry variables. The response gradient vector can be evaluated through sensitivity analysis using either finite-difference or perturbation methods (Haldar and Mahadevan 2000b). In their book, Haldar and Mahadevan (2000b) present SFEM-based reliability analysis in the context of displacement-based and stress-based deterministic finite-element approaches, considering linear, nonlinear, static and dynamic problems.

A few notable geotechnical applications of SFEM are mentioned here. Baecher and Ingra (1981) applied the stochastic finite-element method in settlement predictions. Ishii and Suzuki (1987) applied the stochastic finite-element method to earth slope stability analysis. Brzakala and Pula (1996) performed settlement analysis of shallow foundations resting on layered subsoil based on the perturbation and the Neumann expansion approaches for the stochastic finite-element method. Yeh and Rahman (1998) adapted and evaluated some available formulations of the stochastic finite-element method for the analysis of seismic response of soils. El Gonnouni et al. (2005) used the stochastic finite-element method to assess the influence of uncertainties related to soil variability in the numerical modeling of settlements caused by underground work. Chaudhuri and Sekhar (2005) applied the stochastic finite-element method to analyze one-dimensional pollutant migration in a soil liner. Chaudhuri and Chakraborty (2006) utilized the perturbation-based stochastic finite-element method in deriving time-varying unconditional reliability evaluation of linear-elastic multi-degree-of-freedom structures subjected to non-stationary earthquake ground motion. Ghanem et al. (2007) treated an embankment dam problem using a new spectral stochastic finite-element method based on stochastic model reduction for polynomial chaos representation.

2.5.4.2. Random finite-element method

The random finite-element method (RFEM) involves generating a random field of material properties which is mapped onto a finite-element mesh (Griffiths and Fenton 2009). A conventional finite-element analysis using these properties is then performed to compute the response variable, after which the process is repeated many times using Monte Carlo simulation. Prominent pioneer works are largely due to Griffiths and Fenton. The RFEM was used to study the effect of stochastic soil permeability on confined seepage beneath water-retaining structures (Griffiths and Fenton 1993). Paice et al. (1996) studied the effect of random and spatially-correlated soil stiffness on the total settlement under the

center of a uniformly-loaded flexible strip footing. Fenton and Vanmarcke (1998) assessed the spatial variability of liquefaction risk.

More recent geotechnical applications are summarized here. The bearing capacities of cohesive soil (Griffiths and Fenton 2001; Griffiths et al. 2002) and c - ϕ soil (Fenton and Griffiths 2003) were investigated. The soils were modeled as random media with spatially-varying shear strength. The reliability of shallow foundations on spatially-random two-dimensional soil (Fenton and Griffiths 2002) and three-dimensional soil (Fenton and Griffiths 2005) against excessive differential settlement was assessed. Griffiths et al. (2009) investigated the failure probability of slopes. Huang et al. (2010) investigated the consolidation behavior of soil deposits with spatially-varying compressibility and permeability.

2.5.4.3. Comparisons involving SFEM and RFEM

Sudret and Der Kiureghian (2002) compared two methods for finite-element reliability analysis. One is based on direct coupling of finite-element and reliability codes, and the other involves post-processing of results obtained from the spectral stochastic finite-element method (SSFEM). The two procedures are applied to the reliability analysis of the settlement of a foundation on randomly-heterogeneous soil. The direct-coupling approach was found to be robust and efficient. The SSFEM approach gave fair results only with high-order polynomial chaos expansions, which lead to prohibitively high computational costs.

Griffiths and Fenton (2009) compared results obtained in a probabilistic settlement analysis of a strip footing between SFEM based on MVFOSM approximations and RFEM based on Monte Carlo simulations. The shortcoming of the SFEM being unable to directly model the influence of spatial variability and non-symmetric lognormal distribution was highlighted. Furthermore the SFEM consistently underestimated the standard deviation of the footing settlement and hence inevitably leads to unconservative estimates of the probability of failure.

2.6. Review of advanced probabilistic methods

2.6.1. Parametric sensitivity

A parametric sensitivity factor is defined as the change in failure probability (or reliability index) per unit change in a certain parameter, i.e., the partial derivative of the failure probability $\partial p_f / \partial \Theta$ (or of the reliability index $\partial \beta / \partial \Theta$) with respect to the parameter Θ . The parameter Θ could be a deterministic physical parameter that enters the problem via the limit state function $g(\mathbf{x})$, or a statistical parameter related to any of the random variables $\mathbf{x}$. The parametric sensitivity factor is also known as ‘sensitivity coefficient’, ‘sensitivity index’, ‘sensitivity measure’ or ‘importance measure’ in the literature. For simplicity the term ‘sensitivity factor’ will be used consistently throughout this study.

The sensitivity factors are useful in problem formulation as they can help define the relative importance of the random variables and identify the most crucial ones that may require further investigation (Karamchandani et al. 1988). These factors have practical applications within reliability-based design, in optimization of structural design, construction, maintenance and inspection under reliability constraints, in parametric studies of reliability, and in reliability updating (Bjerager and Krenk 1987; 1989).

The most direct way of obtaining sensitivity information is by evaluating the partial derivative $\partial p_f / \partial \Theta$ numerically, considering a small change $\Delta \Theta$ in the parameter Θ :

$$\frac{\partial p_f}{\partial \Theta} \approx \frac{\Delta p_f}{\Delta \Theta} = \frac{(p_f)_{\Theta + \Delta \Theta} - (p_f)_{\Theta}}{\Delta \Theta} \quad (2.43)$$

where $(p_f)_{\Theta}$ = probability of failure based on the original value of parameter Θ ; and $(p_f)_{\Theta + \Delta \Theta}$ = probability of failure based on the new value of parameter Θ after introducing a small perturbation $\Delta \Theta$. The values of $(p_f)_{\Theta}$ and $(p_f)_{\Theta + \Delta \Theta}$ may be

obtained from Monte Carlo simulation, but to carry out duplicate simulation runs for every varied parameter is computationally expensive. A more efficient approach using modified probability density functions for sampling was presented by Karamchandani et al. (1988). An adaptive importance sampling method was proposed by Wu (1994). Conditional sampling using subset simulation was described by Au (2005). A line sampling technique was used by Lu et al. (2008). The above procedures essentially make use of variance reduction techniques to achieve the desired precision in Monte Carlo estimates with fewer numbers of samples.

Sensitivity factors in terms of the reliability index are given by:

$$\frac{\partial \beta}{\partial \Theta} \approx \frac{\Delta \beta}{\Delta \Theta} = \frac{\beta_{\Theta+\Delta\Theta} - \beta_{\Theta}}{\Delta \Theta} \quad (2.44)$$

where β_{Θ} = reliability index based on the original value of parameter Θ ; and $\beta_{\Theta+\Delta\Theta}$ = reliability index based on the new value of parameter Θ after introducing a small perturbation $\Delta\Theta$. The above reliability indices are computed using either FORM or SORM. It is possible to infer the sensitivity of the failure probability from the sensitivity of the reliability index β based on the hyperplane assumption $p_f = 1 - \Phi(\beta)$ by using the following equations:

$$\frac{\partial p_f}{\partial \Theta} = -\phi(\beta) \frac{\partial \beta}{\partial \Theta} \quad (2.45)$$

where $\phi(\beta)$ = standard normal probability density function evaluated at β .

As an alternative to numerical analysis, analytical sensitivity factors may instead be obtained as by-products of FORM. The unit design-point vector α is an integral part of the classical FORM computation procedure, and is generally regarded as a measure of the sensitivity of the reliability index to inaccuracies in the value of $\mathbf{u}$ at the point of minimum distance $\mathbf{u}^*$ (Madsen 1986):

$$\alpha_i = \frac{\partial \beta}{\partial u_i} \quad (2.46)$$

where α_i = direction cosine, a component of α in the direction of u_i (for $i = 1, 2, \dots, n$). It is implicit that $\partial \beta / \partial u_i$ is evaluated at $\mathbf{u}^*$. In the case of uncorrelated original random variables x_i , a one-to-one mapping between x_i and u_i occurs such that α_i is also the sensitivity factor of the reliability index with respect to x_i . However, if the x_i 's are correlated, the α_i 's have no direct physical meaning as a result of orthogonal transformation from x -space to u -space (Melchers 1999, page 101). Nevertheless it has been shown that the sensitivity of β with respect to Θ can be obtained using chain rule (Bjægerager and Krenk 1989):

$$\frac{\partial \beta}{\partial \Theta} = \alpha^T \frac{\partial \mathbf{u}^*}{\partial \Theta} \quad (2.47)$$

where $\mathbf{u}^*$ = design-point vector which is equal to $\beta \alpha$. The advantage of the analytical approach is that the direction cosines are already computed in the process of obtaining $\mathbf{u}^*$ and β , and hence no further evaluation of the limit state function is required. Sensitivity factors with respect to means and standard deviations of the random variables are not dimensionless and therefore not immediately comparable. In such cases, scaled sensitivity factors can be computed by multiplying $\partial \beta / \partial \mu_i$ or $\partial \beta / \partial \sigma_i$ by their respective standard deviations (Der Kiureghian and Ke 1985; Liu et al. 1989).

Other methods of performing parametric sensitivity analysis are as follows: Karamchandani and Cornell (1992) proposed a method that resulted in accurate sensitivities of FORM and SORM failure estimates. The change in failure probability was computed based on response surfaces constructed using points that were transformed according to parameter being varied. Melchers and Ahammed (2004) demonstrated that duplicate Monte Carlo simulations were not required to evaluate parametric sensitivities. This was achieved via constructing a hyperplane response surface based on simulation, and then performing the FORM

procedure. Sues and Cesare (2005) presented a hybrid approach that used FORM to find the most probable failure point and then conducted simulation based on response surfaces fitted at the failure point. Millwater (2009) derived universal properties of kernel functions needed for probabilistic sensitivity analysis. These properties are then used to develop analytical sensitivity results for linear and quadratic response surface functions. Lu et al. (2010) introduced a moment-based reliability sensitivity method, in which the partial derivative of the failure probability is related to partial derivatives of moments of the performance function. The point estimate method was used to approximate the partial derivatives of moments.

There are also a few other ways of defining parametric sensitivity apart from the conventional partial derivatives of the failure probability. Madsen (1988) defined the omission sensitivity factor as the inverse ratio between the original reliability index and the reliability index obtained under the condition that one of the basic random variables was replaced by a deterministic value. Gupta and Manohar (2004) proposed a set of importance measures for ranking basic random variables in terms of their relative importance with respect to the failure probability. These measures incorporate the geometry of the failure surface over regions that contribute significantly to the failure probability.

Among the approaches mentioned above, quite a number of them can be extended to obtain system sensitivity factors, i.e., the partial derivative of the system reliability or system failure probability with respect to some specified parameter (Karamchandani and Cornell 1992; Wu 1994; Gupta and Manohar 2004; Sues and Cesare 2005; Lu et al. 2008, 2010).

2.6.2. System reliability

Engineering problems often involve multiple failure modes. There may be several potential modes of failure, in which the occurrence of any one of the potential failure modes will constitute failure or nonperformance of the system or

component (Ang and Tang 1984). A laterally-loaded pile may fail due to excessive deflection or excessive flexure or both; the two failure modes are usually correlated. The calculation of the failure probability of a system by direct simulation method is generally time-consuming. In this regard lower and upper bounds of the failure probability are useful. The review below concerns reliability of series systems.

For positively-correlated failure modes the uni-modal bounds on the system failure probability $p_{f,sys}$ are (Cornell 1967):

$$\max_i p_{f,i} \leq p_{f,sys} \leq 1 - \prod_{i=1}^m (1 - p_{f,i}) \quad (2.48)$$

where $p_{f,i}$ is the failure probability of the i th mode out of m potential failure modes. The values of $p_{f,i}$ can be assessed via a variety of methods, such as FORM, SORM and Monte Carlo simulations. The uni-modal bounds may be too wide for practical use, but can be improved by taking into account the correlation between pairs of failure modes. The resulting bi-modal bounds require the probabilities of joint events. The following bi-modal bounds based on Kounias (1968), Hunter (1976), and Ditlevsen (1979b) are also summarized in Ang and Tang (1984), Madsen et al. (1986), Melchers (1999), and Haldar and Mahadevan (2000a):

$$\left\{ p_{f,1} + \sum_{i=2}^m \max \left[p_{f,i} - \sum_{j=1}^{i-1} p_{f,ij}, 0 \right] \right\} \leq p_{f,sys} \leq \left\{ p_{f,1} + \sum_{i=2}^m \left[p_{f,i} - \max_{j < i} p_{f,ij} \right] \right\} \quad (2.49)$$

where $p_{f,ij}$ is the probability that the system fails in the i th and j th mode simultaneously. Calculating $p_{f,ij}$, which is the intersection of two failure events, is generally not easy. The joint probability of failure can be calculated as (Madsen et al. 1986):

$$p_{f,ij} = \Phi_2(-\beta_i, -\beta_j, \rho_{ij}) \quad (2.50)$$

where Φ_2 refers to the bivariate standard normal cumulative distribution function. The computation of the joint failure probability as explained in Madsen et al. (1986), Hohenbichler and Rackwitz (1983), and Gollwitzer and Rackwitz (1988) remains difficult.

A weakened version of Eq. (2.49) due to Ditlevsen (1979b) suggested that the lower and upper bounds of $p_{f,ij}$ can be calculated as follows:

$$\max[p(a), p(b)] \leq p_{f,ij} \leq p(a) + p(b) \quad (2.51)$$

where $p(a)$ and $p(b)$ are defined as:

$$p(a) = \Phi(-\beta_i) \Phi\left(-\frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}}\right) \quad (2.52a)$$

$$p(b) = \Phi(-\beta_j) \Phi\left(-\frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1-\rho_{ij}^2}}\right) \quad (2.52b)$$

in which β_i and β_j are the reliability indices corresponding to failure modes i and j , respectively; and ρ_{ij} is the correlation coefficient between failure modes i and j . Substituting the bounds of Eq. (2.52) into Eq. (2.49), the upper and lower bounds of $p_{f,sys}$ can be estimated. In this way, the complex problem of estimating system failure probability is reduced to problems of estimating failure probabilities corresponding to various component limit states as well as the correlation coefficients among various failure modes. The correlation coefficient between failure modes i and j can be estimated from the following established equation (e.g., Ang and Tang 1984):

$$\rho_{ij} = \alpha_i^T \alpha_j = \frac{\mathbf{u}_i^{*T} \mathbf{u}_j^*}{\beta_i \beta_j} \quad (2.53)$$

where α_i and α_j are the direction cosine vectors corresponding to failure modes i and j , respectively; and $\mathbf{u}_i^*$ and $\mathbf{u}_j^*$ are the design points corresponding to failure modes i and j , respectively.

In a first-order system reliability analysis the failure set is approximated by the polyhedral set bounded by the tangent hyperplanes at the design points for the individual failure modes. The failure probabilities $p_{f,i}$ corresponding to respective component modes are based on the first-order reliability indices (i.e., FORM). The first-order approximation to $p_{f,ij}$ is obtained by estimating the joint failure set by the set bounded by the tangent hyperplanes at the design points for the two failure modes.

A second-order system reliability analysis (Madsen 1985) is based on a more accurate approximation of the failure surface. The failure probabilities $p_{f,i}$ corresponding to individual component modes are computed by approximating the failure surface by a quadratic surface with the same curvature at the design point (i.e., SORM). The second-order approximation to $p_{f,ij}$ is obtained by estimating the joint failure set by the set bounded by the tangent hyperplanes at the point $\mathbf{u}_{ij}^*$ on the joint failure surface closest to the origin. The solution of the joint design point $\mathbf{u}_{ij}^*$ necessarily requires a new optimization search based on two constraints simultaneously. Alternatively, one may first obtain quadratic approximations of the two failure surfaces at their respective design points, locate the intersection point of the two approximate surfaces closest to the origin, and then estimate $p_{f,ij}$ by using the hyperplanes at the intersection point tangent to the approximate surfaces. Yet another alternative (Low et al. 2011) computes the second-order approximation to $p_{f,ij}$ based on the second-order reliability indices and assumes that the direction cosine vector of the second-order index is the same as that of the first-order index. In this way $p_{f,ij}$ is obtained by estimating the joint failure set by the set bounded by the tangent hyperplanes at modified design points for the two failure modes. The modified design point is in the same direction as the original

design point, only that the distance between the modified design point and the origin is now given by the second-order index.

2.7. Review of design approaches and reliability-based design

2.7.1. Allowable stress design (ASD)

Allowable stress design (ASD), also known as *working stress design* (WSD) or *permissible stress design*, is a civil engineering design philosophy that ensures that the stresses developed in a structure (or the soil mass supporting the structure, in the context of geotechnical engineering) due to applied service loads do not exceed the specified allowable value. Uncertainties in loads and resistances are accounted for by a single lumped or *global* factor of safety F_s .

$$F_s = \frac{R}{S} = \frac{\sum R_i}{\sum S_j} \quad (2.54)$$

where R = total resistance; S = total loads; R_i = individual resistances; and S_j = individual loads. There are two common interpretations for F_s : (1) central factor of safety; and (2) nominal factor of safety. The *central factor of safety* is defined as:

$$F_s = \frac{E[R]}{E[S]} = \frac{\mu_R}{\mu_S} \quad (2.55)$$

where $E[R]$ and $E[S]$ = expected values of R and S , respectively; and μ_R and μ_S = mean values of R and S , respectively. The *nominal factor of safety* is defined as:

$$F_s = \frac{R_N}{S_N} = \frac{\mu_R - k_R \sigma_R}{\mu_S + k_S \sigma_S} = \frac{\mu_R / B_R}{\mu_S / B_S} \quad (2.56)$$

where R_N and S_N = nominal or characteristic values of R and S , respectively; σ_R and σ_S = standard deviations of R and S , respectively; k_R and k_S = constants; and B_R and B_S = biases of R and S , respectively.

It is customary to use F_s values of 2 to 3 for design of foundations (Terzaghi et al. 1996). API RP 2A-WSD's (API 2000) recommends F_s values are 1.5 and 2 for design and operating environmental conditions, respectively.

2.7.2. Limit state design (LSD)

Limit state design (LSD) is an alternative design methodology that is not based on allowable stress, but on predictions of how designs actually perform near failure. In civil engineering, a limit state is defined as a condition beyond which a structure or a structural member is deemed unsafe due to one or more loads or load effects. The two most critical limit states are: (1) ultimate limit state and (2) serviceability limit state. In geotechnical engineering, *ultimate limit state* usually pertains to the ultimate bearing capacity of the soil, and *serviceability limit state* to performance of the design under service loads (such as deflections or settlements). The design is deemed to have reached *failure* when it is unable to satisfy one or both limit states. The generic criteria governing limit state design are as follows:

Ultimate limit state (ULS):

$$\text{Factored resistance} \geq \text{Factored load effects} \quad (2.57)$$

Serviceability limit state (SLS):

$$\text{Limiting deformation} \geq \text{Calculated deformation (due to nominal loads)} \quad (2.58)$$

In allowable stress design, limit state compliance is only implicitly accounted for with the global factor of safety. In contrast, limit state design more directly relates the uncertainties of the loads and resistances to the probabilities of failure.

2.7.3. Load and resistance factor design (LRFD)

A notable extension of the limit state design philosophy is the *Load and resistance factor design* (LRFD). In LRFD, the factor of safety equation is substituted by an inequality relation of the following generic form:

$$\psi_R R_N \geq \psi_S S_N \quad (2.59)$$

where ψ_R = resistance factor; and ψ_S = load factor. In general the value of ψ_R is less than 1 whereas the value of ψ_S is greater than 1. In API RP 2A-LRFD (API 1993), the geotechnical design is checked for its adequacy under different loading conditions, such as gravity loads, wind actions and wave actions. For each loading condition, the nominal loads are multiplied by the prescribed load factors. The resulting factored load combination must be less than the factored resistance, which is the nominal resistance multiplied by the prescribed resistance factor for the type of geotechnical design at hand. On the other hand, Eurocode 7 (CEN 2004) provides recommended values for partial factors on actions (or loads) or their effects, partial factors for soil parameters, and also partial resistance factors for various geotechnical design problems (e.g. spread foundations, pile foundations, retaining structures). Therefore, Eurocode 7 is similar to API RP 2A-LRFD in the sense that both design codes specify some form of load factors and resistance factors, although there are some differences. In Eurocode 7, partial factors may be applied on the total or combined resistance or on individual component resistances. For example, partial factors are given for the pile-base resistance and pile-shaft resistance, as well as for the pile total resistance in the case of an axially-loaded pile. It is also noted that the factored resistance in Eurocode 7 is obtained by dividing (instead of multiplying) the resistance by the partial factor whose value is greater than or equal to one. The major difference between API RP 2A-LRFD and Eurocode 7 is that the latter prescribes partial factors for soil parameters such as friction angle, cohesion, undrained shear strength and unit weight. The design value of a soil parameter is obtained by

dividing its characteristic value by the partial factor which is greater than or equal to one. The resistance is then calculated based on the design values. In this way, the uncertainty due to variability of soil properties is directly taken into consideration.

2.7.4. Reliability-based design

The design philosophy based on probability theory and reliability concepts is closely linked with reliability methods of analysis as described earlier (Section 3.4 and 3.5). *Reliability-based design* relates directly to the probability of failure and reflects parameter uncertainty in an explicit manner. It is hence more realistic, rational and consistent than allowable stress design using the global factor of safety. The most critical obstacle that is holding back the geotechnical profession from widely accepting probabilistic design is the lack of proper statistical data on the parameters and the necessary knowledge in probability and statistics to make good use of the data. In addition it is also much more complicated to implement reliability-based design than traditional deterministic methods.

Reliability-based design can be generally categorized into three levels (Becker 1996a): (1) fully probabilistic methods; (2) approximate probabilistic methods; and (3) semi-probabilistic methods. *Fully probabilistic methods* (Level III) require full information on the probability distributions of all parameters. For most projects it is usually not possible to adopt such methods due to practical difficulties and limited resources. *Approximate probabilistic methods* (Level II) make assumptions on the probability distributions of the parameters based on available data. The load and resistance are usually treated as statistically independent variables with either normal or lognormal distributions. Second-moment reliability methods (as described in Section 3.4 and 3.5) come under this category. *Semi-probabilistic methods* (Level I) provide assurance of safety in the form of load and resistance partial factors, which may be calibrated from Level II methods. Design codes such as the API RP 2A-LRFD and Eurocode 7 are

examples of Level I methods, which offer a simple and convenient basis to introduce reliability concepts in geotechnical limit state design.

A simple first-order reliability-based design can be carried out by considering the probability that a load S exceeds a resistance R , i.e., the probability of failure is $P(S > R)$. If S and R are mutually independent and normally distributed, the performance function $g = R - S$ is also normally distributed. The probability of failure is computed as (e.g., Gilbert 1997):

$$P(S > R) = P(g < 0) = \Phi\left(-\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right) \quad (2.60)$$

If S and R are mutually independent and are lognormal variables, the performance function $g = \ln(R/S)$ is normally distributed. The probability of failure is computed as (e.g., Gilbert et al. 2005; Gilbert et al. 2008):

$$P(S > R) = P(g < 0) = \Phi\left(-\frac{\ln \mu_R - \ln \mu_S}{\sqrt{\Omega_R^2 + \Omega_S^2}}\right) \quad (2.61)$$

Using either equations Eq. (2.60) or Eq (2.61), and by defining the load factor as $\psi_S = S^*/\mu_S$ and the resistance factor as $\psi_R = R^*/\mu_R$, where S^* and R^* are the design load and resistance respectively, one can obtain combinations of load and resistance factors that satisfy a certain level of reliability.

The advantages of using a reliability-based approach versus the factor-of-safety approach are (Gilbert 1997):

- (i) The factor of safety is not a consistent measure of the level of reliability of a design: two different designs based on the same factor of safety may have different reliabilities. On the other hand, the reliability index relates directly to the level of safety. Load and resistance factors developed for

the same reliability index will ensure consistent levels of reliability among different designs.

- (ii) The factor-of-safety approach is based on experience with similar designs. In the absence of experience, there is no assurance that an arbitrarily chosen factor of safety will achieve a certain level of reliability. The reliability-based approach does not rely on past experience and develops both new as well as traditional designs based simply on the specified level of safety.
- (iii) In design optimization, cost-benefit analyses are carried out to balance construction costs against the risk of failure. The reliability-based approach, which can quantify risk explicitly, is therefore more suited for optimization than the factor-of-safety approach.

The limitation of the reliability-based approach in the form of load and resistance factors is that it does not allow the designer to specify his/her target failure probability (which may be different from the one used to derive the factors), or explicitly account for uncertainties particular to the design problem at hand (Gilbert 1997).

A more general and versatile form of reliability-based design criteria is to specify a design factor for each design variable, known also as *partial factors* (Ang and Tang 1984; Baecher and Christian 2003). The determination of the partial factors is essentially a problem of determining the most probable failure point using either the Hasofer-Lind method for correlated normals or the FORM for correlated non-normals. The partial factor, given by the ratio x_i^*/μ_{xi} (or its inverse), is akin to the partial factor of Eurocode 7. Therefore the reliability-based approach can play a supplementary verification and comparison role to a design based on Eurocode 7, particularly when the designer or code developer would like to explicitly incorporate his own estimates of parametric uncertainty and correlation into the design so as to achieve a target probability of non-failure (Low 2005). The advantage is in obtaining partial factors that reflect parametric sensitivities from

case to case in a way rigid partial factors cannot (Youssef Abdel Massih et al. 2008).

A rational framework for developing LRFD factors for geotechnical engineering, using reliability-based design tools, is presented in Foye et al. (2006a; b). The framework comprises three major steps: (1) identify component variables of the design equations; (2) quantify uncertainties of the component variables; and (3) determine resistance factors that correspond to a certain target reliability index based on the reliability approach. The algorithm for the FORM proposed by Low and Tang (1997) is used to evaluate the resistance factors. The framework is illustrated for shallow foundation design in Foye et al. (2006a, b) and extended to the design of driven pipe piles in sand in Foye et al. (2009).

2.8. Concluding remarks

In this chapter, literature review was presented for: (1) deterministic methods for the analysis of laterally-loaded piles; and (2) a number of the most commonly-used probabilistic methods.

The conventional factor-of-safety approach for designing laterally loaded piles based on deterministic analysis cannot explicitly account for the uncertainties of the underlying parameters. A more rational approach is to use a reliability index, which can be directly related to the probability of failure (or non-failure). As mentioned in Christian (2004), reliability theory does not invalidate calculations of factors of safety. It extends them by giving them a context and by giving additional information to help the engineer interpret the results. It is therefore not true that there is an inherent conflict between approaches that use a factor of safety and those that use the reliability index. The reliability index quantifies the risk that is associated with a given factor of safety for the problem at hand, and hence imparts more practical meaning to the conventional factor of safety.

Chapter 3 Deterministic Analysis of Laterally Loaded Piles based on Direct Stiffness Method

3.1. Introductory comments

This chapter develops a two-dimensional direct stiffness method for analyzing laterally-loaded single vertical piles. The details of the deterministic method are covered in section 3.2. The proposed direct stiffness algorithm is illustrated using a number of worked examples in sections 3.3, 3.4 and 3.5 – it is first applied to a case with a linear p-y soil model, and then extended to a case with a nonlinear hyperbolic tangent p-y soil model, followed by two cases involving nonlinear soil behavior as well as nonlinear pile flexural behavior.

3.2. Direct stiffness method for laterally-loaded piles

The direct stiffness method is the most common implementation of the finite-element method, and forms the basis for most commercial and free-source finite-element software. The direct stiffness algorithm developed in this study is akin to finite-element analysis with one-dimensional beam elements and nonlinear soil-spring elements.

The pile is discretized into finite elastic beam elements. A typical beam element has one node at each end. Each node has two degrees of freedom, namely, translation y and rotation θ restricted to the y - z plane in accordance with elementary beam theory. In pile analysis y and θ are simply the pile deflection and the angle of rotation, respectively. For simplicity it is assumed that the pile does not carry any axial load so that axial translations can be ignored. If we denote E_p as the Young's modulus, I_p as the second moment of area, and Δz as the segmental length, then the stiffness matrix of a pile segment between node i and node $i + 1$ is identical to that of a beam element (as given in Chandrupatla and

Belegundu 2002, Cook et al. 2002, West and Geschwindner 2002, Hibbeler 2009, and many others):

Fig. 3.1. Laterally-loaded pile in soil and discretization of pile-soil system

Fig. 3.2. Equivalent nodal loads on pile segment due to soil pressure

$$[\mathbf{k}] = \begin{bmatrix} C_1 & C_2 & -C_1 & C_2 \\ C_2 & C_3 & -C_2 & C_4 \\ -C_1 & -C_2 & C_1 & -C_2 \\ C_2 & C_4 & -C_2 & C_3 \end{bmatrix} \begin{Bmatrix} y_i \\ \theta_i \\ y_{i+1} \\ \theta_{i+1} \end{Bmatrix} \quad (3.1)$$

where $C_1 = 12E_p I_p / \Delta z^3$; $C_2 = 6E_p I_p / \Delta z^2$; $C_3 = 4E_p I_p / \Delta z$; and $C_4 = 2E_p I_p / \Delta z$. The matrix $[\mathbf{k}]$ operates on the displacement vector $\{y_i; \theta_i; y_{i+1}; \theta_{i+1}\}^T$ to give the load vector $\{f_i; m_i; f_{i+1}; m_{i+1}\}^T$ where f and m represent the nodal force and nodal moment, respectively.

The soil pressure p acting on the pile shaft is a function of the pile deflection y and may be idealized using trapezoidal-distributed loads between the two nodes of each beam element. Based on the classical beam-on-elastic-foundation concept, the soil resistance is modeled using a series of independent ‘soil springs’. The spring constants can be obtained from the equivalent nodal forces and moments corresponding to the trapezoidal-distributed load, and inserted into the appropriate entries of $[\mathbf{k}]$. The derivation of the equivalent nodal forces and moments corresponding to a trapezoidal-distributed load acting on a beam element is as follows.

The lateral soil pressure acting on each beam element that made up the pile shaft may be idealized using a distributed load that varies linearly from p_a to p_b from one end to the other, where p_a and p_b are functions of the pile deflection y_a and y_b , respectively. The equivalent nodal forces and moments may be derived by superposing the following two loading conditions (Fig. 3.3):

(a) Uniformly distributed load p_a

$$\{\mathbf{r}_e\}_u = \{p_a L/2; -p_a L^2/12; p_a L/2; p_a L^2/12\}^T \quad (3.2)$$

(b) Triangular distributed load that varies from 0 to $(p_b - p_a)$

$$\{\mathbf{r}_e\}_t = \{3(p_b - p_a)L/20; -(p_b - p_a)L^2/30; 7(p_b - p_a)L/20; (p_b - p_a)L^2/20\}^T \quad (3.3)$$

The force and moment expressions above are obtained from books on matrix structural analysis (e.g., Kassimali 1999; McGuire et al. 2000). These expressions adopt the following sign convention in this paper: positive upward force and clockwise moment, which are consistent with the sign convention used in the direct stiffness pile analysis method discussed in the main text (where the beam element is vertical instead of horizontal). Adding $\{\mathbf{r}_e\}_u$ and $\{\mathbf{r}_e\}_t$ gives the equivalent nodal forces and moments for the case of a trapezoidal-distributed load:

$$\{\mathbf{r}_e\} = \left\{ (7p_a + 3p_b)L/20; \quad -(3p_a + 2p_b)L^2/60; \quad (3p_a + 7p_b)L/20; \quad (2p_a + 3p_b)L^2/60 \right\}^T \quad (3.4)$$

Note that the derived expression $\{\mathbf{r}_e\}$ is reduced to $\{\mathbf{r}_e\}_u$ where $p_a = p_b$, and is reduced to $\{\mathbf{r}_e\}_t$ when $p_a = 0$. Also worth mentioning is that the equivalent nodal forces and moments are exactly equal but opposite in directions to the support reactions of an identically loaded fixed-end beam.

The modified element stiffness matrix of the pile segment is:

$$[\mathbf{k}]_{new} = \begin{bmatrix} C_1 + \frac{7k_{h,i}d\Delta z}{20} & C_2 & -C_1 + \frac{3k_{h,i+1}d\Delta z}{20} & C_2 \\ C_2 - \frac{3k_{h,i}d\Delta z^2}{60} & C_3 & -C_2 - \frac{2k_{h,i+1}d\Delta z^2}{60} & C_4 \\ -C_1 + \frac{3k_{h,i}d\Delta z}{20} & -C_2 & C_1 + \frac{7k_{h,i+1}d\Delta z}{20} & -C_2 \\ C_2 + \frac{2k_{h,i}d\Delta z^2}{60} & C_4 & -C_2 + \frac{3k_{h,i+1}d\Delta z^2}{60} & C_3 \end{bmatrix} \quad (3.5)$$

assuming uniform pile diameter d , and that the soil pressure varies linearly from $p_i = -k_{h,i}dy_i$ from one end to $p_{i+1} = -k_{h,i+1}dy_{i+1}$ to the other end of the beam element, where k_h is known as the modulus of subgrade reaction. The negative sign is due to the soil reaction acting in opposite direction to the pile deflection.

Fig. 3.3. Equivalent nodal loads for a beam carrying trapezoidal-distributed load

In the proposed direct stiffness method, the main equations are:

$$Ay_{i-1} + By_i + Cy_{i+1} + D\theta_{i-1} + F\theta_i + G\theta_{i+1} = 0 \quad \text{for } i = 1 \text{ to } n - 1 \quad (3.6a)$$

where

$$\begin{aligned} A &= -\frac{12(EI)_i}{\Delta z_i^3} + \frac{3(k_{\text{sec}}d)_{i-1}\Delta z_i}{20} \\ B &= \frac{12(EI)_i}{\Delta z_i^3} + \frac{12(EI)_{i+1}}{\Delta z_{i+1}^3} + \frac{7(k_{\text{sec}}d)_i(\Delta z_i + \Delta z_{i+1})}{20} \\ C &= -\frac{12(EI)_{i+1}}{\Delta z_{i+1}^3} + \frac{3(k_{\text{sec}}d)_{i+1}\Delta z_{i+1}}{20} \\ D &= \frac{6(EI)_i}{\Delta z_i^2} \\ F &= \frac{6(EI)_i}{\Delta z_i^2} - \frac{6(EI)_{i+1}}{\Delta z_{i+1}^2} \\ G &= \frac{6(EI)_{i+1}}{\Delta z_{i+1}^2} \end{aligned}$$

and

$$Qy_{i-1} + Ry_i + Sy_{i+1} + T\theta_{i-1} + U\theta_i + V\theta_{i+1} = 0 \quad \text{for } i = 1 \text{ to } n - 1 \quad (3.6b)$$

where

$$\begin{aligned} Q &= -\frac{6(EI)_i}{\Delta z_i^2} + \frac{2(k_{\text{sec}}d)_{i-1}\Delta z_{i-1}^2}{60} \\ R &= \frac{6(EI)_i}{\Delta z_i^2} - \frac{6(EI)_{i+1}}{\Delta z_{i+1}^2} + \frac{3(k_{\text{sec}}d)_i(\Delta z_i^2 - \Delta z_{i+1}^2)}{60} \\ S &= \frac{6(EI)_{i+1}}{\Delta z_{i+1}^2} + \frac{2(k_{\text{sec}}d)_{i+1}\Delta z_{i+1}^2}{60} \\ T &= \frac{2(EI)_i}{\Delta z_i} \end{aligned}$$

$$U = \frac{4(EI)_i}{\Delta z_i} + \frac{4(EI)_{i+1}}{\Delta z_{i+1}}$$

$$V = \frac{2(EI)_{i+1}}{\Delta z_{i+1}}$$

At the pile head, the two equations are:

$$\frac{12(EI)_1}{\Delta z_1^3}(y_0 - y_1) + \frac{\Delta z_1}{20}[7(k_{\text{sec}}d)_0 y_0 + 3(k_{\text{sec}}d)_1 y_1] - \frac{6(EI)_1}{\Delta z_1^2}(\theta_0 + \theta_1) = P_{\text{top}} \quad (3.7)$$

$$\frac{6(EI)_1}{\Delta z_1^2}(-y_0 + y_1) - \frac{\Delta z_1^2}{60}[3(k_{\text{sec}}d)_0 y_0 + 2(k_{\text{sec}}d)_1 y_1] + \frac{2(EI)_1}{\Delta z_1}(2\theta_0 + \theta_1) = M_{\text{top}} \quad (3.8)$$

At the pile toe, the two equations are:

$$\frac{12(EI)_n}{\Delta z_n^3}(-y_{n-1} + y_n) + \frac{\Delta z_n}{20}[3(k_{\text{sec}}d)_{n-1} y_{n-1} + 7(k_{\text{sec}}d)_n y_n] + \frac{6(EI)_n}{\Delta z_n^2}(\theta_{n-1} + \theta_n) = 0 \quad (3.9)$$

$$\frac{6(EI)_n}{\Delta z_n^2}(-y_{n-1} + y_n) + \frac{\Delta z_n^2}{60}[2(k_{\text{sec}}d)_{n-1} y_{n-1} + 3(k_{\text{sec}}d)_n y_n] + \frac{2(EI)_n}{\Delta z_n}(\theta_{n-1} + \theta_n) = 0 \quad (3.10)$$

In the equations above, y = deflection; $EI = E_p I_p$ = flexural rigidity; k_{sec} = secant modulus; d = diameter; and Δz = segmental length.

For the direct stiffness method, there are $2(n + 1)$ equations, which are used to obtain $(n + 1)$ nodal displacements and $(n + 1)$ nodal rotations. These $2(n + 1)$ equations can be rearranged to form the governing equation $\mathbf{KD} = \mathbf{F}$ (just like in finite-element method), with reference to the pile-soil system, $\mathbf{K}$ is the global stiffness matrix; $\mathbf{D}$ is the global nodal displacement vector; and $\mathbf{F}$ is the global nodal load vector. The procedure for assembling the global stiffness matrix is standard and is given in Chandrupatla and Belegundu (2002), Cook et al. (2002), West and Geschwindner (2002), and Hibbeler (2009), for example.

The equations presented above for the direct stiffness method are for a free head pile. They can be easily extended to analyze a fixed head pile by replacing the stiffness coefficients associated with the rotation θ at the pile head with some large constant values to simulate a very stiff rotation spring. If the shear force and bending moment at the pile toe is not negligible, as in the case of relatively short piles, the values of the shear force and bending moment should be entered into the equations for the pile toe on the right hand side. One may refer to Gabr and Borden (1988) for a practical approach to account for base effects in such cases.

It would be interesting to compare the direct stiffness method with the traditional finite difference approach (as described in Reese and Van Impe 2011, for example), whose main equation is:

$$Ay_{i-2} + By_{i-1} + Cy_i + Dy_{i+1} + Fy_{i+2} \quad (3.11)$$

where

$$\begin{aligned} A &= (EI)_{i-1} \\ B &= -2[(EI)_{i-1} - (EI)_i] \\ C &= [(EI)_{i-1} + 4(EI)_i + (EI)_{i+1} + (k_{\text{sec}}d)_i \Delta z^4] \\ D &= -2[(EI)_i - (EI)_{i+1}] \\ F &= (EI)_{i+1} \end{aligned}$$

Note that 4 imaginary nodes are introduced, two above the pile head (with deflections y_{-1} and y_{-2}) and two below the pile toe (with deflections y_{n+1} and y_{n+2}). For a free-head pile that has negligible shear and moment at the pile toe, the additional equations are:

$$\frac{(EI)_0}{\Delta z^3} (-0.5y_{-2} + y_{-1} - y_1 + 0.5y_2) = P_{\text{top}} \quad (3.12)$$

$$\frac{(EI)_0}{\Delta z^2} (y_{-1} - 2y_0 + y_1) = M_{top} \quad (3.13)$$

$$y_{n-1} - 2y_n + y_{n+1} = 0 \quad (3.14)$$

$$-y_{n-2} + 2y_{n-1} - 2y_{n+1} + y_{n+2} = 0 \quad (3.15)$$

Fig. 3.4 compares the matrices based on the equations for the finite difference method and that based on the equations for the direct stiffness method. For simplicity, a pile of length 10 m is divided into 5 uniform segments. The pile has a diameter of 0.4 m and a constant flexural rigidity of 50,000 kNm². The modulus of subgrade reaction k_h is assumed to be 10,000 kN/m³.

For the finite difference method, there are in total $n + 5$ equations based on Eqs. (3.11) – (3.15). These are used to solve for $n + 5$ nodal displacements. In contrast, for the direct stiffness method, there are $2(n + 1)$ equations based on Eqs. (3.6) – (3.10), which are used to obtain $n + 1$ nodal displacements and $n + 1$ nodal rotations.

The direct stiffness method described above is illustrated using a number of worked examples in the following sections. The proposed algorithm is first applied to a case with a linear p - y soil model, and then extended to a case with a nonlinear hyperbolic tangent p - y soil model, followed by two cases involving nonlinear soil behavior as well as nonlinear pile flexural behavior.

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Finite Difference Matrix:

i	-2	-1	0	1	2	3	4	5	6	7	
-2	-3125	6250	0	-6250	3125	0	0	0	0	0	y_{-2}
-1	0	12500	-25000	12500	0	0	0	0	0	0	y_{-1}
0	50000	-200000	364000	-200000	50000	0	0	0	0	0	y_0
1	0	50000	-200000	364000	-200000	50000	0	0	0	0	y_1
2	0	0	50000	-200000	364000	-200000	50000	0	0	0	y_2
3	0	0	0	50000	-200000	364000	-200000	50000	0	0	y_3
4	0	0	0	0	50000	-200000	364000	-200000	50000	0	y_4
5	0	0	0	0	0	50000	-200000	364000	-200000	50000	y_5
6	0	0	0	0	0	0	1	-2	1	0	y_{n+1}
7	0	0	0	0	0	-1	2	0	-2	1	y_{n+2}

Direct Stiffness Matrix:

i	0	1	2	3	4	5	0	1	2	3	4	5	
0	77800	-73800	0	0	0	0	-75000	-75000	0	0	0	0	y_0
1	-73800	155600	-73800	0	0	0	75000	0	-75000	0	0	0	y_1
2	0	-73800	155600	-73800	0	0	0	75000	0	-75000	0	0	y_2
3	0	0	-73800	155600	-73800	0	0	0	75000	0	-75000	0	y_3
4	0	0	0	-73800	155600	-73800	0	0	0	75000	0	-75000	y_4
5	0	0	0	0	-73800	77800	0	0	0	0	75000	75000	y_5
0	-75800	74466.7	0	0	0	0	100000	50000	0	0	0	0	θ_0
1	-74467	0	74466.7	0	0	0	50000	200000	50000	0	0	0	θ_1
2	0	-74467	0	74466.7	0	0	0	50000	200000	50000	0	0	θ_2
3	0	0	-74467	0	74466.7	0	0	0	50000	200000	50000	0	θ_3
4	0	0	0	-74467	0	74466.7	0	0	0	50000	200000	50000	θ_4
5	0	0	0	0	-74467	75800	0	0	0	0	50000	100000	θ_5

E_p 2.5E+07 kN/m²
 I_p 0.002 m⁴
 $E_p I_p$ 50000 kNm²
 k_h 10000 kN/m³
 d 0.4 m
 $k_h d$ 4000 kN/m²
 L 10 m
 Δz 2 m
 n 5 segments

y_i = deflection at node i

θ_i = rotation at node i

Fig. 3.4. Comparison of matrices for finite difference method and direct stiffness method

3.3. Linear and uniform modulus of subgrade reaction

The direct stiffness method used in this study is first illustrated in Microsoft Excel with a case of a free-headed pile and constant k_h with depth. More realistic scenarios involving nonlinear soil behavior (nonlinear p - y curves) and nonlinear pile flexural behavior (nonlinear $E_p I_p$) will be discussed after that. A vertical pile of diameter $d = 0.4$ m and length $L = 20$ m is loaded by a horizontal force $P_{\text{top}} = 300$ kN and bending moment $M_{\text{top}} = 100$ kNm, which are applied at the pile head. The lateral soil pressure at any depth is related to the pile deflection at that depth via k_h , which is assumed to be $10,000 \text{ kN/m}^3$. The Young's modulus E_p of the pile material is $25,000,000 \text{ kN/m}^2$ and the second moment of area I_p of the pile section is 0.002 m^4 . The flexural rigidity $E_p I_p$ of the pile is therefore $50,000 \text{ kNm}^2$. The 20-m long vertical pile is divided into 15 segments of varying length Δz_i (Fig. 3.5). The segmental lengths increase in geometric progression by the common ratio $r_{\Delta z}$. The spreadsheet is set up based on the equations shown in Fig. 3.5.

A function named *StiffMat*, which stands for stiffness matrix, is created in the Visual Basic programming environment in Microsoft Excel. The function code, as shown in Fig. 3.6, automatically assembles the global stiffness matrix $\mathbf{K}$ of the pile, based on information contained in the three columns " z_i ", " $(E_p I_p)_i$ ", and " $(k_h d)_i$ " of Fig. 3.5. A short function named *Load*, also shown in Fig. 3.6, constructs the load vector $\mathbf{F}$ based on the given externally-applied horizontal force " P_{top} " and bending moment " M_{top} " at the pile head. The two columns designated for the pile deflection " y_i " and angle of rotation " θ_i " contain array formulas involving the user-defined functions *StiffMat* and *Load*, and compute the global nodal displacement vector based on $\mathbf{D} = \mathbf{K}^{-1}\mathbf{F}$.

Δz_1	$r_{\Delta z}$	L	E_p	I_p	k_h	d	P_{top}	M_{top}
0.2	1.268	20	2.5E+07	0.002	10,000	0.4	300	100
(m)		(m)	(kN/m ²)	(m ⁴)	(kN/m ³)	(m)	(kN)	(kNm)

Boxed cells contain formulas:

$$z_2 = z_1 + \Delta z_1 \quad z_i = z_{i-1} + r_{\Delta z} (z_{i-1} - z_{i-2}) \quad \text{for } i > 2$$

$$(E_p I_p)_i = E_p I_p \quad (k_h d)_i = k_h d \quad p_i = -(k_h d)_i y_i$$

$$V_i = V_{i+1} - 0.5(p_{i+1} + p_i)(z_{i+1} - z_i)$$

$$M_i = M_{i+1} - V_{i+1}(z_{i+1} - z_i) + (2p_{i+1} + p_i)(z_{i+1} - z_i)^2 / 6$$

y_i and θ_i contain Function *StiffMat* and *Load*

i	z_i	$(E_p I_p)_i$	$(k_h d)_i$	y_i	θ_i	p_i	V_i	M_i
	(m)	(kNm ²)	(kN/m ²)	(m)	(rad)	(kN/m)	(kN)	(kNm)
1	0	50000	4000	0.0634	0.0266	-253.45	300.00	100.00
2	0.2	50000	4000	0.0581	0.0261	-232.37	251.42	155.07
3	0.45	50000	4000	0.0516	0.0251	-206.36	195.78	211.64
4	0.78	50000	4000	0.0437	0.0236	-174.96	134.46	264.48
5	1.18	50000	4000	0.0346	0.0213	-138.31	70.58	305.78
6	1.70	50000	4000	0.0244	0.0180	-97.69	9.55	325.60
7	2.36	50000	4000	0.0140	0.0137	-56.13	-40.90	313.83
8	3.19	50000	4000	0.0047	0.0089	-18.71	-72.02	264.71
9	4.24	50000	4000	-0.0021	0.0042	8.21	-77.56	183.33
10	5.58	50000	4000	-0.0049	0.0006	19.69	-58.91	90.35
11	7.28	50000	4000	-0.0041	-0.0011	16.34	-28.36	17.14
12	9.43	50000	4000	-0.0016	-0.0010	6.22	-4.09	-13.86
13	12.15	50000	4000	0.0001	-0.0003	-0.45	3.77	-10.16
14	15.61	50000	4000	0.0002	0.0001	-0.96	1.33	-0.83
15	20.00	50000	4000	-0.0001	0.0001	0.35	0	0

Fig. 3.5. Deterministic analysis of laterally-loaded pile based on direct stiffness method involving linear soil behavior (linear p-y)

```

Function StiffMat(z, EI, kd)
If z.Count = EI.Count Then
    If z.Count = kd.Count Then n = z.Count
End If
ReDim Kmat(1 To 2 * n, 1 To 2 * n) As Double, L(1 To n - 1) As Double, EL_(1 To n - 1) As Double
For i = 1 To n - 1
    L(i) = z(i + 1) - z(i): EL_(i) = 0.5 * (EI(i + 1) + EI(i))
Next i
Kmat(1, 1) = 12 * EL_(1) / L(1) ^ 3 + 7 * kd(1) * L(1) / 20
Kmat(1, 2) = -12 * EL_(1) / L(1) ^ 3 + 3 * kd(2) * L(1) / 20
Kmat(1, n + 1) = -6 * EL_(1) / L(1) ^ 2
Kmat(1, n + 2) = -6 * EL_(1) / L(1) ^ 2
Kmat(n + 1, 1) = -6 * EL_(1) / L(1) ^ 2 - 3 * kd(1) * L(1) ^ 2 / 60
Kmat(n + 1, 2) = 6 * EL_(1) / L(1) ^ 2 - 2 * kd(2) * L(1) ^ 2 / 60
Kmat(n + 1, n + 1) = 4 * EL_(1) / L(1)
Kmat(n + 1, n + 2) = 2 * EL_(1) / L(1)
For i = 2 To n - 1
    Kmat(i, i - 1) = -12 * EL_(i - 1) / L(i - 1) ^ 3 + 3 * kd(i - 1) * L(i - 1) / 20
    Kmat(i, i) = 12 * EL_(i - 1) / L(i - 1) ^ 3 + 12 * EL_(i) / L(i) ^ 3 + 7 * kd(i) * (L(i - 1) + L(i)) / 20
    Kmat(i, i + 1) = -12 * EL_(i) / L(i) ^ 3 + 3 * kd(i + 1) * L(i) / 20
    Kmat(i, n + i - 1) = 6 * EL_(i - 1) / L(i - 1) ^ 2
    Kmat(i, n + i) = 6 * EL_(i - 1) / L(i - 1) ^ 2 - 6 * EL_(i) / L(i) ^ 2
    Kmat(i, n + i + 1) = -6 * EL_(i) / L(i) ^ 2
    Kmat(n + i, i - 1) = -6 * EL_(i - 1) / L(i - 1) ^ 2 + 2 * kd(i - 1) * L(i - 1) ^ 2 / 60
    Kmat(n + i, i) = 6 * EL_(i - 1) / L(i - 1) ^ 2 - 6 * EL_(i) / L(i) ^ 2 + 3 * kd(i) * (L(i - 1) ^ 2 - L(i) ^ 2) / 60
    Kmat(n + i, i + 1) = 6 * EL_(i) / L(i) ^ 2 - 2 * kd(i + 1) * L(i) ^ 2 / 60
    Kmat(n + i, n + i - 1) = 2 * EL_(i - 1) / L(i - 1)
    Kmat(n + i, n + i) = 4 * EL_(i - 1) / L(i - 1) + 4 * EL_(i) / L(i)
    Kmat(n + i, n + i + 1) = 2 * EL_(i) / L(i)
Next i
Kmat(n, n - 1) = -12 * EL_(n - 1) / L(n - 1) ^ 3 + 3 * kd(n - 1) * L(n - 1) / 20
Kmat(n, n) = 12 * EL_(n - 1) / L(n - 1) ^ 3 + 7 * kd(n) * L(n - 1) / 20
Kmat(n, 2 * n - 1) = 6 * EL_(n - 1) / L(n - 1) ^ 2
Kmat(n, 2 * n) = 6 * EL_(n - 1) / L(n - 1) ^ 2
Kmat(2 * n, n - 1) = -6 * EL_(n - 1) / L(n - 1) ^ 2 + 2 * kd(n - 1) * L(n - 1) ^ 2 / 60
Kmat(2 * n, n) = 6 * EL_(n - 1) / L(n - 1) ^ 2 + 3 * kd(n) * L(n - 1) ^ 2 / 60
Kmat(2 * n, 2 * n - 1) = 2 * EL_(n - 1) / L(n - 1)
Kmat(2 * n, 2 * n) = 4 * EL_(n - 1) / L(n - 1)
StiffMat = Kmat
End Function

Function Load(Ptop, Mtop, n)
ReDim F(1 To 2 * n, 1 To 1) As Double
F(1, 1) = Ptop: F(n + 1, 1) = Mtop
Load = F
End Function

```

Fig. 3.6. User-defined functions for direct stiffness method

Fig. 3.7. Comparison of direct stiffness method with constrained optimization approach and Hetenyi equations (linear p-y)

Closed-form solutions for simple cases of loading on beams of finite length can be found in Hetenyi (1974, Equations 39a, b, c, d and 40a, b, c, d). Hetenyi's solutions for y_i , V_i and M_i are plotted in Fig. 3.7, for comparison with values from the direct stiffness method just presented. In addition, solutions obtained using the spreadsheet-based constrained optimization approach for p -y analyses of laterally-loaded piles (Low and Teh 1999) are also included in the comparison. The three sets of plots are practically identical.

The succinctness of the proposed numerical procedure is evident from Fig. 3.5, in which neither the global stiffness matrix $\mathbf{K}$ nor the load vector $\mathbf{F}$ need to be constructed by the user. For the interested reader, matrix $\mathbf{K}$ and vector $\mathbf{F}$ can be displayed by simply using array formulas containing just the user-defined functions *StiffMat* or *Load*. One can then see that, for the present example, the matrix $\mathbf{K}$ will occupy a rather sizeable, albeit sparsely populated, 30×30 grid on the worksheet. Also the 30×1 vector $\mathbf{F}$ contains mainly zeros except for the first

and the $(n + 1)^{\text{th}}$ entry. As the pile shaft discretization becomes more refined, matrix $\mathbf{K}$ and vector $\mathbf{F}$ can be expected to become even more unwieldy, and are best shown only when necessary in order to preserve a neat and compact user interface.

It is also worth mentioning that unlike the constrained optimization approach of Low and Teh (1999) and Low et al. (2001), the proposed direct stiffness method does not require the calling of Solver (Microsoft Excel's built-in optimization routine) to search for the pile deflection that satisfies force and moment equilibrium.

3.4. Nonlinear and non-uniform modulus of subgrade reaction

Gabr et al. (1994) presented a procedure for constructing p - y curves from dilatometer test data, and analyzed the Haga test pile that was reported in the literature. The p - y curves were modeled by the hyperbolic tangent function, and a specially-written program was used. The cylindrical test pile was 4.15 m long and 0.153 m in diameter, with wall thickness equal to 4.5 mm. The Young's modulus E_p of the pile material is 200,000,000 kN/m² and the second moment of area I_p of the pile section is 0.0000058 m⁴. The flexural rigidity $E_p I_p$ of the pile is therefore 1,158 kNm². The pile was closed ended and jacked into the ground. The point of application was about 0.30 m above the top of the pile.

The Haga test pile is analyzed with the proposed direct stiffness method using the hyperbolic tangent p - y curves (Fig. 3.8). The hyperbolic parameters p_u and k_{ho} as reported in Gabr et al. (1994) have been used. The ultimate soil reaction p_u (kN/m) and the initial tangent modulus k_{ho} (kN/m³) vary with depth according to the equations shown in Fig. 3.8. The distributions of p_u and k_{ho} with depth were obtained by linear regression of the p_u and k_{ho} values shown in Gabr et al. (1994).

Δz_1	$r_{\Delta z}$	d	L	E_p	I_p	P_{top}	M_{top}	r_1	r_2	f_y
0.04	1.151	0.153	4.15	2.0E+08	5.8E-06	20	6	0.0720	0.0765	417000
(m)		(m)	(m)	(kN/m ²)	(m ⁴)	(kN)	(kNm)	(m)	(m)	(kN/m ²)

Parameters for hyperbolic tangent p - y curves (Gabr et al. 1994):

$$k_{ho} = \begin{cases} 0.000001 & \text{for } z < 0.2 \text{ m} \\ 16,964 + 115,545(z + 0.3) & \text{for } 0.2 \text{ m} \leq z < 2.1 \text{ m} \\ 294,272 & \text{for } z \geq 2.1 \text{ m} \end{cases}$$

$$p_u = \begin{cases} 0.000001 & \text{for } z < 0.2 \text{ m} \\ 13 + 16.67(z + 0.3) & \text{for } 0.2 \text{ m} \leq z < 1.2 \text{ m} \\ 38 & \text{for } z \geq 1.2 \text{ m} \end{cases}$$

$(k_{sec}d)_i$ contains Function $ksecd_tanh$ for hyperbolic tangent p - y curves

i	z_i	$(E_p I_p)_i$	$k_{ho,i}$	$p_{u,i}$	$(k_{sec}d)_i$	y_i	$y_{prev,i}$	p_i	V_i	M_i
	(m)	(kNm ²)	(kN/m ³)	(kN/m)	(kN/m ²)	(m)	(m)	(kN/m)	(kN)	(kNm)
1	0	1158	1E-06	1E-06	0.00	0.02026	0.02026	0.00	20.00	6.00
2	0.04	1158	1E-06	1E-06	0.00	0.01940	0.01940	0.00	20.00	6.80
3	0.09	1158	1E-06	1E-06	0.00	0.01841	0.01841	0.00	20.00	7.72
4	0.14	1158	1E-06	1E-06	0.00	0.01729	0.01729	0.00	20.00	8.78
5	0.20	1158	74742	21.34	1331	0.01603	0.01603	-21.34	19.35	9.99
6	0.27	1158	82855	22.51	1539	0.01462	0.01462	-22.51	17.81	11.29
7	0.35	1158	92195	23.85	1827	0.01306	0.01306	-23.86	15.94	12.66
8	0.44	1158	102945	25.40	2239	0.01134	0.01134	-25.41	13.64	14.03
9	0.55	1158	115319	27.19	2861	0.00950	0.00950	-27.19	10.83	15.35
10	0.67	1158	129563	29.24	3861	0.00757	0.00757	-29.25	7.35	16.47
11	0.82	1158	145958	31.61	5620	0.00562	0.00562	-31.59	3.03	17.21
12	0.98	1158	164830	34.33	9103	0.00374	0.00374	-34.06	-2.33	17.27
13	1.17	1158	186554	37.47	16638	0.00207	0.00207	-34.39	-8.76	16.23
14	1.38	1158	211559	38	28622	0.00075	0.00075	-21.34	-14.79	13.63
15	1.63	1158	240341	38	36644	-0.0001	-0.0001	3.87	-16.97	9.55
16	1.92	1158	273471	38	38529	-0.0005	-0.0005	17.92	-13.84	5.03
17	2.25	1158	294272	38	41310	-0.0004	-0.0004	18.27	-7.87	1.45
18	2.63	1158	294272	38	43836	-0.0002	-0.0002	10.58	-2.39	-0.41
19	3.07	1158	294272	38	44954	-0.0001	-0.0001	2.60	0.49	-0.70
20	3.57	1158	294272	38	45012	0.00002	0.00002	-1.08	0.87	-0.28
21	4.15	1158	294272	38	44985	0.00004	0.00004	-1.93	0	0

Fig. 3.8. Deterministic analysis based on direct stiffness method involving nonlinear soil behavior (nonlinear p - y), Haga test pile

Hyperbolic tangent p - y curves (Gabr et al. 1994):

$$p = p_u \tanh(k_{ho} dy / p_u)$$

```
Function ksecd_tanh(pu, kho, d, y)
y = Abs(y)
If y < 10 ^ (-6) Then y = 10 ^ (-6)
w = kho * d * y / pu
If w > 10 Then w = 10: If w < -10 Then w = -10
tanhx = (Exp(w) - Exp(-w)) / (Exp(w) + Exp(-w))
p = pu * tanhx: ksecd_tanh = p / y
End Function
```

Fig. 3.9. User-defined function for hyperbolic tangent p - y curves

A short function named *ksecd_tanh* (Fig. 3.9), which stands for (secant k)* d , is created in this study based on the equations for the hyperbolic tangent p - y curves. The function code reads its parameter y from the newly-added column “ $y_{prev,i}$ ” and is used in column “ $(k_{sec}d)_i$ ” of Fig. 3.8. The values in column “ $y_{prev,i}$ ” were initially set to zeros. The final result displayed in Fig. 3.8 was obtained by the iteration-of-secant-modulus approach, in which the values in column “ y_i ” were automatically updated with the values in column “ $y_{prev,i}$ ” over a number of iterations so that the procedure converged to a deflection-compatible soil modulus. The iteration-of-secant-modulus approach has been found to be computationally efficient, and a necessary measure for p - y curves that contain kinks, such as those of Matlock (1970) which are nonlinear and exhibit strain softening.

Fig. 3.10 shows the measured and computed pile-head deflections (y_0) versus the applied loads (P_{top}). Reasonable agreement is observed between the deflections obtained by the direct stiffness method (DSM) and the results obtained by Gabr et al. (1994) using the finite-difference program LTBASE. The DSM curve (solid line) and the LTBASE curve (dotted line) both compare well with the measured pile deflections. The LTBASE results are generally closer to the measured responses than the DSM results.

Fig. 3.10. Computed and measured pile-head deflections at different applied loads, Haga test pile

Fig. 3.11. Computed and measured bending-moment profiles at different applied loads, Haga test pile

The plots in Fig. 3.11 compares the bending moments based on DSM with those based on the Gabr et al. (1994) analysis using LTBASE. Due to the differences between DSM results and LTBASE results, a separate finite difference analysis was performed, using the finite difference equations in Reese and Van Impe (2001). The pile was discretized into 55 elements, as was done by Gabr et al. (1994). The hyperbolic tangent p - y model was used, and the same p_u (ultimate soil reaction) and k_{ho} (coefficient of subgrade reaction) distributions with depth was reported in Gabr et al. (1994) were adopted. The FDM load versus pile-head displacement curve, and the bending moment versus depth curves are in excellent agreement with those computed by DSM, as well as with the results obtained using the Low et al. (2001) constrained optimization method (COM). In Figs. 3.10 and 3.11, these results are represented by the continuous curve.

This shows that given the same input (in terms of pile geometry, soil characteristics and p - y model), the three different numerical methods yield consistent results for the case in hand. It is therefore unlikely that the discrepancy between the FDM/DSM/COM results and the LTBASE results are due to computational inaccuracy.

The difference between the FDM and LTBASE results, especially for the bending moment versus depth profile at 20 kN lateral load, is somewhat surprising (Fig. 3.11). This may suggest two possibilities: (i) that the finite difference algorithm used by LTBASE is different from the finite difference formulation used by classical FDM; and/or (ii) that the input used by Gabr et al. (1994) is different from that used in the thesis.

These possibilities can be adequately studied by entering the same input used in the thesis into LTBASE, and then comparing the results with those of FDM. If the results are identical, it is most likely a case of difference in input. Further investigations are then needed to identify the input parameters that cause the discrepancy. If the result is the same as that reported in Gabr et al. (1994), then

there could be a difference in the finite difference code between LTBASE and FDM.

3.5. Nonlinear soil modulus and nonlinear pile flexural rigidity

3.5.1. Hypothetical offshore steel pipe pile in clay

This problem is described in Tomlinson (1994, Example 8.2). A steel tubular pile, having an outside diameter d of 1.3 m and a wall thickness of 0.03 m, forms part of a pile group in a breasting dolphin. The flexural rigidity $E_p I_p$ of the pile is 4829082 kNm². The pile is embedded 23 m in the stiff overconsolidated clay with undrained shear strength $c_u = 150$ kN/m², and protrudes 26 m above the seabed. A cyclic force is applied at the pile head. The Matlock p - y curves for clays, as summarized in Equations 6.66-6.70 of Tomlinson (1994), have been used.

For the present case, the Matlock p - y curves start from $p = 0$ at $y = 0$, and increase as cubical parabolas with the deflection y to reach a maximum p value at $y = 0.0975$ m. For y between 0.0975 m and 0.4875 m, the p - y curve decreases linearly, and remains constant beyond $y = 0.4875$ m. Hence the p - y curves are nonlinear, exhibit strain-softening, and vary with depth.

The offshore pile is analyzed using the proposed direct stiffness method considering nonlinear soil behavior as well as nonlinear pile behavior (Fig. 3.12). A short function named *ksecd* is created to compute the secant modulus of the Matlock (1970) p - y curves (Fig. 3.13). The function is used in column “ $(k_{sec}d)_i$ ” of Fig. 3.12. The closed-form moment-curvature (M - κ) expressions for a tubular section suggested by Chen (1971) and Duan et al. (1993) are used to characterize nonlinear flexural rigidity of the pile. The column “ $(E_p I_p)_i$ ” in Fig. 3.12 contains the short function *Eplpsec* (Fig. 3.14), which is created based on the above moment-curvature expressions.

The displayed numerical results in Fig. 3.12 were obtained by executing a simple program *Iterate* (Fig. 3.15) that automatically updates the secant soil modulus of the p - y curves and the secant flexural rigidity of the pile in tandem with the columns labeled “ $y_{\text{prev},i}$ ” and “ $M_{\text{prev},i}$ ”, based on the iteration-of-secant-modulus approach. The subroutine contains two For-Next loops one nested in the other, and simultaneously obtains the deflection-compatible soil modulus as well as the bending moment-compatible pile flexural rigidity, thus resulting in a program that couples two aspects of nonlinearity in the behavior of the laterally-loaded pile. The column “ $(E_p I_p)_i$ ” in Fig. 3.12 shows that, for an applied load $P_{\text{top}} = 600$ kN and an applied moment $M_{\text{top}} = 15,600$ kNm, the flexural rigidity of the pile has been somewhat reduced between $z = 0$ m to $z = 2.85$ m along the pile shaft since the bending moment developed along this portion of the pile has exceeded the yield bending moment $M_y = 15,490$ kNm. At greater depths up to $z = 23$ m, the flexural rigidity of the pile remained unchanged.

It can be seen from Fig. 3.16a that at the applied load of 720 kN, the pile-head deflection is about 0.14 m for the case of linear flexural rigidity (i.e. constant bending stiffness) and about 0.18 m for the case of nonlinear flexural rigidity. The percentage difference is about 29%. On the other hand, at the same applied load, the maximum bending moment is about 19346 kNm for the case of linear $E_p I_p$, and about 19405 kNm for the case of nonlinear $E_p I_p$ (see Fig. 3.16b). The percentage difference is only about 0.3%.

Δz_1	$r_{\Delta z}$	L	P_{top}	M_{top}
0.2	1.160	23	600	15600
(m)		(m)	(kN)	(kNm)

Parameters for Matlock (1970) p - y curves:

γ'	c_u	J	ε_{50}
11.8	150	0.25	0.01
(kN/m ³)	(kPa)		

$(k_{sec}d)_i$ contains Function $ksecd$ for Matlock p - y curves

Parameters for Chen (1971) & Duan et al. (1993) M - κ function:

f_y	E_p	I_p	d
417000	2.0E+08	0.02415	1.3
(kN/m ²)	(kN/m ²)	(m ⁴)	(m)

$(E_p I_p)_i$ contains Function $EpIpsec$ for M - κ function

i	z_i	$(E_p I_p)_i$	$(k_{sec}d)_i$	y_i	$y_{prev,i}$	p_i	V_i	M_i	$M_{prev,i}$
	(m)	(kNm ²)	(kN/m ²)	(m)	(m)	(kN/m)	(kN)	(kNm)	(kNm)
1	0	4799762	3998.04	0.10368	0.10368	-414.53	600.0	15600.0	15600.0
2	0.2	4768156	4336.26	0.09861	0.09861	-427.60	515.8	15711.6	15711.6
3	0.43	4735767	4642.97	0.09289	0.09289	-431.31	416.2	15819.7	15819.7
4	0.70	4705377	4983.3	0.08649	0.08649	-431.01	300.2	15916.1	15916.1
5	1.01	4681473	5416.91	0.07937	0.07937	-429.94	165.9	15988.8	15988.8
6	1.37	4670701	5979.77	0.07153	0.07153	-427.72	10.8	16020.7	16020.7
7	1.79	4681833	6727.7	0.06299	0.06299	-423.80	-167.9	15987.7	15987.7
8	2.28	4724467	7751.87	0.05385	0.05385	-417.42	-372.5	15856.1	15856.1
9	2.85	4805249	9211.51	0.04423	0.04423	-407.46	-605.3	15579.9	15579.9
10	3.50	4829082	11409.8	0.03438	0.03438	-392.23	-866.9	15097.7	15097.7
11	4.26	4829082	14996.5	0.02462	0.02462	-369.21	-1155.9	14328.9	14328.9
12	5.14	4829082	21648.9	0.01543	0.01543	-334.13	-1465.4	13173.0	13173.0
13	6.16	4829082	37534.4	0.00741	0.00741	-278.11	-1777.9	11512.8	11512.8
14	7.34	4829082	136518	0.00118	0.00118	-161.15	-2037.9	9240.5	9240.5
15	8.72	4829082	83694.2	-0.0027	-0.0027	229.17	-1991.2	6413.4	6413.4
16	10.31	4829082	67991	-0.0042	-0.0042	285.57	-1581.4	3557.4	3557.4
17	12.15	4829082	81881.9	-0.0036	-0.0036	294.65	-1045.8	1129.2	1129.2
18	14.30	4829082	136564	-0.0019	-0.0019	260.43	-451.4	-460.8	-460.8
19	16.78	4829082	412066	-0.0004	-0.0004	172.47	86.1	-869.2	-869.2
20	19.66	4829082	1180906	7.9E-05	7.9E-05	-93.11	200.4	-273.1	-273.1
21	23.00	4829082	1276300	2.1E-05	2.1E-05	-26.87	0	0	0

Fig. 3.12. Deterministic analysis based on direct stiffness method involving nonlinear soil behavior and nonlinear pile flexural behavior, hypothetical offshore pile

Matlock (1970) p - y curves:

$$p = 0.5 p_u (y / y_{50})^{1/3}$$

$$y_{50} = 2.5 \epsilon_{50} d$$

$$z_r = 6 c_u d / (\gamma' d + J c_u)$$

$$p_u = \begin{cases} 3 c_u d + \gamma' z d + J c_u z & \text{for } z < z_r \\ 9 c_u d & \text{for } z \geq z_r \end{cases}$$

```

Function ksecd(gamma, cu, J, z, d, epsilon, y)
pu = 3 * cu * d + gamma * z * d + J * cu * z
If pu > 9 * cu * d Then pu = 9 * cu * d
yc = 2.5 * epsilon * d
prest = 0.72 * pu
zr = 6 * cu * d / (gamma * d + J * cu)
If z < zr Then prest = prest * z / zr
y = Abs(y)
If y < 10 ^ (-4) Then y = 10 ^ (-4)
If y <= 3 * yc Then p = 0.5 * pu * (y / yc) ^ (1 / 3)
If y > 3 * yc And y < 15 * yc Then
tem = (15 * yc - y) / (12 * yc) * (0.72 * pu - prest)
p = (prest + tem)
End If
If y >= 15 * yc Then p = prest
ksecd = p / y
End Function

```

Fig. 3.13. User-defined function for Matlock (1970) p - y curves

Fig. 3.17a and 3.17b show the pile deflection and bending moment profiles with depth at different applied loads, respectively. When $P_{\text{top}} = 700$ kN, significant difference between the pile deflection profile computed based on linear $E_p I_p$ and that computed based on nonlinear $E_p I_p$ can be observed; however there is hardly any difference between the bending moment profiles.

Chen (1971) & Duan et al. (1993) moment-curvature function:

$$\kappa = \begin{cases} M/E_p I_p & \text{for } M < M_y \\ \sqrt{0.273\kappa_y^2 / (1.273 - M/M_y)} & \text{for } M_y < M < 1.273M_y \end{cases}$$

$$M_y = 2f_y I_p / d \quad \kappa_y = 2f_y / E_p d$$

where M_y = yield bending moment; κ_y = yield curvature;
 f_y = yield stress of steel

```
Function Eplpsec(M, fy, Ep, Ip, d)
M = Abs(M)
My = 2 * fy * Ip / d
ky = 2 * fy / Ep / d
If M >= 0.999 * 1.273 * My Then
    M = 0.999 * 1.273 * My
End If
k = Sqr(0.273 * ky ^ 2 / (1.273 - M / My))
Eplpsec = M / k
If M <= My Then Eplpsec = Ep * Ip
End Function
```

Fig. 3.14. User-defined function for moment-curvature function for tubular section

```
Sub Iterate()
'y, yprev, M, Mprev, maxdiff_y and maxdiff_M
'are pre-defined cells/blocks in spreadsheet.
del_y = 0.000001: del_M = 0.001
Range("Mprev").Value = 0
For j = 1 To 200
    Range("yprev").Value = 0
    For i = 1 To 200
        Range("yprev").Value = Range("y").Value
        If Range("maxdiff_y") < del_y Then Exit For
    Next i
    Range("Mprev").Value = Range("M").Value
    If Range("maxdiff_M") < del_M Then Exit For
Next j
End Sub
```

Fig. 3.15. User-defined program for iteration-of-secant-modulus approach

Fig. 3.16. Computed pile responses for linear and nonlinear pile flexural rigidity, hypothetical offshore pile:
(a) Pile-head deflections; (b) Maximum bending moments

Fig. 3.17. Computed pile responses for linear and nonlinear pile flexural rigidity, hypothetical offshore pile:
(a) Pile-deflection profiles; (b) Bending-moment profiles

3.5.2. Reinforced-concrete pile in weak rock, San Francisco

In 1992 the California Department of Transportation performed lateral load tests on bored piles in San Francisco. The case was reported in Reese (1997). One of the reinforced-concrete pile was 2.25 m in diameter. The reinforcement was comprised of 28 steel bars with a diameter of 43 mm surrounded by 0.18 m of concrete cover. The pile penetrated 13.8 m of sandstone and protruded 1.24 m above the ground surface. The initial pile flexural rigidity $E_p I_p$ was 35,150,000 kNm^2 . The concrete compressive strength f'_c was 34.5 MPa, and the steel tensile strength f_y was equal to 496 MPa.

Pressuremeter tests were performed on the site. The compressive strength q_{ur} profile of the rock was: 0 – 3.9 m, 1,860 kPa; 3.9 – 8.8 m, 6,450 kPa, and below 8.8 m, 16,000 kPa. The initial modulus of the rock E_{ir} was: 0 – 3.9 m, 186 MPa; 3.9 – 8.8 m, 645 MPa and below 8.8 m, 1,600 MPa. The strength reduction factor α_r was 1.0. The dimensionless stiffness constant k_{rm} had a value of 0.00005.

The behavior of the reinforced-concrete pile is analyzed with the proposed direct stiffness method, using the Reese (1997) interim p - y curves for weak rock (Fig. 3.18). A short function `ksecd_rock` is created to compute the secant modulus based on the p - y curves (Fig. 3.19).

Reese (1997) described two methods for modeling the moment-curvature relationship of a reinforced-concrete section: (1) the analytical method based on the mechanics of steel and concrete under flexure; and (2) the approximate method based on experimental observations (ACI 1989). Both methods have been tested together with the direct stiffness method.

Δz_1	$r_{\Delta z}$	d	L	P_{top}	M_{top}
0.1	1.175	2.25	13.8	9000	11160
(m)		(m)	(m)	(kN)	(kNm)

Parameters for p - y curves for rock (Reese 1997):

α_r	k_{rm}
1	0.00005

$(k_{sec}d)_i$ contains Function $ksecd_rock$ for p - y curves for rock

Parameters for ACI (1989) approximate $E_p I_p$ function:

f'_c	I_{cr}	$E_p I_p$
34500	0.068	3.5E+07
(kN/m ²)	(m ⁴)	(kN/m ²)

$(E_p I_p)_i$ contains Function $EpIp_ACI$ for ACI $E_p I_p$ function

i	z_i	$(E_p I_p)_i$	$(k_{sec}d)_i$	y_i	$y_{prev,i}$	p_i	V_i	M_i	$M_{prev,i}$
	(m)	(kNm ²)	(kN/m ²)	(m)	(m)	(kN/m)	(kN)	(kNm)	(kNm)
1	0	3.6E+06	158436	0.02642	0.02641	-4185.1	9000.0	11160.0	11160.0
2	0.1	3217860	180697	0.0246	0.0246	-4445.6	8568.5	12038.6	12038.6
3	0.22	2944702	210991	0.02252	0.02252	-4751.5	8028.2	13014.0	13014.0
4	0.36	2727162	253625	0.02015	0.02015	-5111	7347.4	14075.9	14075.9
5	0.52	2558546	316275	0.0175	0.0175	-5533.4	6484.1	15198.6	15198.6
6	0.71	2432242	413707	0.01457	0.01457	-6029.8	5382.2	16330.9	16330.9
7	0.93	2342781	577284	0.01146	0.01145	-6612.9	3966.7	17380.0	17380.0
8	1.20	2287340	883057	0.00826	0.00826	-7298.1	2136.6	18186.9	18186.9
9	1.50	2269193	1553516	0.00522	0.00522	-8103.3	-243.9	18485.9	18485.9
10	1.87	2309820	3456706	0.00262	0.00262	-9049.3	-3359.0	17841.9	17841.9
11	2.29	2508998	1E+07	0.00083	0.00083	-8375.1	-7077.2	15604.7	15604.7
12	2.80	3520291	4.9E+07	5.8E-05	5.8E-05	-2872.6	-9897.2	11233.7	11233.7
13	3.39	1.7E+07	5.6E+07	-7E-05	-7E-05	3829.15	-9615.4	5291.7	5291.7
14	4.08	3.5E+07	2.2E+08	-5E-05	-5E-05	10038.6	-4815.5	48.9	48.9
15	4.89	3.5E+07	2.5E+08	-1E-05	-1E-05	2466.83	270.2	-1382.1	-1382.1
16	5.85	3.5E+07	2.9E+08	3.4E-06	3.4E-06	-991.54	975.2	-523.8	-523.8
17	6.97	3.5E+07	3.2E+08	1.4E-06	1.4E-06	-456.44	162.2	58.5	58.5
18	8.29	3.5E+07	3.2E+08	-3E-07	-3E-07	93.5661	-77.1	34.9	34.9
19	9.84	3.5E+07	8E+08	-1E-08	-1E-08	7.61821	1.3	-6.7	-6.7
20	11.66	3.5E+07	8E+08	7.5E-09	7.5E-09	-6.0348	2.7	0.7	0.7
21	13.80	3.5E+07	8E+08	-4E-09	-4E-09	3.48705	0	0	0

Fig. 3.18. Deterministic analysis based on direct stiffness method involving nonlinear soil behavior and nonlinear pile flexural behavior, San Francisco test pile

p - y curves for rock (Reese 1997):

$$p = \begin{cases} K_{ir} y & \text{for } y \leq y_A \\ 0.5 p_{ur} (y / y_{rm})^{0.25} & \text{for } y_A < y \leq 16 y_{rm} \\ p_{ur} & \text{for } y > 16 y_{rm} \end{cases}$$

$$K_{ir} = k_{ir} E_{ir}$$

$$k_{ir} = \begin{cases} 100 + 400z/3d & \text{for } 0 \leq z < 3d \\ 500 & \text{for } z \geq 3d \end{cases}$$

$$y_{rm} = k_{rm} d$$

$$y_A = [p_{ur} / (2 y_{rm}^{0.25} K_{ir})]^{1.333}$$

$$p_{ur} = \begin{cases} \alpha_r q_{ur} d (1 + 1.4z/d) & \text{for } 0 \leq z < 3d \\ 5.2 \alpha_r q_{ur} d & \text{for } z \geq 3d \end{cases}$$

where k_{ir} , k_{rm} = dimensionless constants;

E_{ir} = initial modulus of rock; p_{ur} = ultimate resistance;

α_r = strength reduction factor; q_{ur} = compressive strength

```
Function ksecd_rock(z, d, y, alphas, krm, qur, Eir)
y = Abs(y)
If y < 10 ^ (-6) Then y = 10 ^ (-6)
kir = 100 + 400 * z / (3 * d)
Nc = alphas * (1 + 1.4 * z / d)
If z > 3 * d Then
    kir = 500
    Nc = 5.2 * alphas
End If
kir_ = kir * Eir
pur = Nc * qur * d
yrm = krm * d
yA = (pur / (2 * yrm ^ 0.25 * kir_)) ^ (4 / 3)
If y < yA Then p = kir_ * y
If y >= yA And y < 16 * yrm Then p = pur / 2 * (y / yrm) ^ 0.25
If y >= 16 * yrm Then p = pur
ksecd_rock = p / y
End Function
```

Fig. 3.19. User-defined function for p - y curves in rock

For the analytical method, the moment-curvature relationship is obtained based on a method described in Reese and Van Impe (2001), by considering a reinforced-concrete section subject to bending, and numerically integrating the induced stresses across the section. For the empirical ACI method, the original equations for estimating the effective moment of inertia are shown in Fig. 3.20. The function $EpIp_ACI$ is used to compute the approximate flexural rigidity based on the bending moment.

Fig. 3.21 compares the computed pile-head deflections and maximum bending moments with the field measurements at various applied loads. Three computed curves are presented: (1) nonlinear $E_p I_p$ using the ACI method; (2) nonlinear $E_p I_p$ using the analytical method; and (3) linear $E_p I_p$. They are practically the same as those reported by Reese (1997). It is evident from the load-deflection curve (Fig. 3.21a) that the linear $E_p I_p$ assumption seriously underestimates the pile deflection. Both the ACI method and the analytical method provide better predictions. At very large applied loads, it appears that the ACI method is more accurate than the analytical method.

For instance, when the applied load is equal to 8000 kN, the measured pile-head deflection is about 17 mm. The analysis that assumed linear $E_p I_p$ predicted a pile-head deflection of only about 2 mm. The analyses that considered nonlinear $E_p I_p$ based on the ACI method and the analytical method yielded predictions of 18 mm and 7 mm, respectively. Between the case of linear $E_p I_p$ and the case of nonlinear $E_p I_p$, the difference is clearly very large.

The load-bending moment curves (Fig. 3.21b) show that the maximum bending moment is much less sensitive to pile nonlinearity, and remains relatively unchanged be it linear or nonlinear $E_p I_p$. All the computed load-moment curves match the field measurements reasonably well. Fig. 3.22 shows the pile-deflection and bending-moment profiles with depth at different applied loads. It can be seen that when $P_{top} = 9,000$ kN, there is a very significant difference between the pile-deflection profile computed based on linear $E_p I_p$ and that

computed based on nonlinear $E_p I_p$. At the same applied load, the difference in the bending-moment profiles between linear and nonlinear $E_p I_p$ appears to be less pronounced.

ACI (1989) flexural rigidity function for reinforced concrete:

$$I_e = (M_{cr}/M)^3 I_g + [1 - (M_{cr}/M)^3] I_{cr}$$

$$M_{cr} = f_r I_g / y_c \quad f_r = 19.7(f'_c)^{0.5}$$

$$E_p I_p = E_c I_e \quad E_c = 151,000(f'_c)^{0.5}$$

where I_e = effective moment of inertia; I_g = moment of inertia of gross concrete section about centroidal axis, neglecting reinforcement; I_{cr} = moment of inertia of cracked section; M_{cr} = cracking bending moment; f_r = modulus of rupture of concrete; y_c = distance from centroidal axis of gross section, neglecting reinforcement, to extreme fibers in tension; f'_c = compressive strength of concrete; E_c = modulus of elasticity of concrete

```
Function Eplp_ACI(M, d, fc, lcr, Eplp)
M = Abs(M)
If M < 10 ^ (-6) Then M = 10 ^ (-6)
Ec = 151000 * Sqr(fc)
fr = 19.7 * Sqr(fc)
lg = Application.Pi() * d ^ 4 / 64
yc = d / 2
Mcr = fr * lg / yc
le = (Mcr / M) ^ 3 * lg + (1 - (Mcr / M) ^ 3) * lcr
Eplp_ACI = Ec * le
If Eplp_ACI > Eplp Then Eplp_ACI = Eplp
End Function
```

Fig. 3.20. User-defined function for moment-curvature function for reinforced-concrete section

Fig. 3.21. Computed and measured pile responses, San Francisco test pile:
(a) Pile-head deflections; (b) Maximum bending moments

For instance, when the applied load is 8000 kN, the maximum bending moment obtained based on linear $E_p I_p$ is about 16000 kNm. In comparison, the predictions based on nonlinear $E_p I_p$ using the ACI method and the analytical method are 15800 kNm and 15100 kNm respectively. Between the case of linear $E_p I_p$ and the case of nonlinear $E_p I_p$, the difference is only in the range of 1 to 2%.

Fig. 3.22. Computed pile responses for linear and nonlinear pile flexural rigidity, San Francisco test pile:
(a) Pile-deflection profiles; (b) Bending-moment profiles

3.6. Comparison of direct stiffness method and finite-element method

A comparison is made between the proposed spreadsheet-based direct stiffness method and the finite-element method. The finite-element analysis is carried out using a dedicated program called FB-MultiPier (BSI 2009). FB-MultiPier, the successor to the Florida Pier analysis program, is a nonlinear, finite-element analysis program designed for analyzing bridge interior bent structures composed of nonlinear interior bent columns and caps supported on a linear pile cap and nonlinear piles/shafts with nonlinear soil. This analysis program couples nonlinear structural finite-element analysis with nonlinear static soil models for axial, lateral, and torsional soil behavior to provide a robust system of analysis for coupled bridge interior bent structures and foundation systems. FB-MultiPier performs the generation of the finite-element model internally, given the geometric definition of the structure and foundation system as input graphically by the designer. This feature allows the engineer to work directly with the design parameters and decreases the data management necessary to create and interpret a model. FB-MultiPier contains an analysis program that is coupled with a graphical pre-processor and post-processor. These programs allow the user of FB-MultiPier to view the structure while generating the model and to view the resulting deflections, bi-axial and uni-axial interaction diagrams, and internal forces in a graphical environment. A free demo copy of FB-MultiPier version 4.15 can be downloaded from the Bridge Software Institute at bsi-web.ce.ufl.edu.

The proposed direct stiffness method is compared with the finite-element method based on FB-MultiPier using a case study by Meyer (1979). A series of lateral load tests were carried out on a steel pipe pile that was 319 mm in diameter, with a wall thickness of 12.7 mm, and a length of 12.8 m (Fig. 3.23). The bending stiffness was 31280 kNm^2 . The bending moment at which the extreme fibers would first yield was computed to be 231 kNm, and the bending moment for the formation of a fully plastic hinge was computed to be 304 kNm. The pile was driven into clays near Sabine that was slightly overconsolidated. The undrained shear strength was found to be almost constant with depth and averaged 14.4 kPa.

The value of ε_{50} was found from triaxial tests and averaged 0.02. The submerged unit weight was determined at several points below the mudline and the average value was found to be 5.5 kN/m^3 . The load was applied at 0.305 m above the groundline. The Matlock (1970) p - y criterion for submerged soft clay is used for the analyses.

Fig. 3.23. Sabine test pile

Fig. 3.24a and 3.24b shows the pile-head deflection and the maximum bending moment, respectively, as functions of the applied lateral load. The results obtained by the proposed direct stiffness analyses compare very well with those obtained by the finite-element analyses using FB-MultiPier. The computed pile responses are also similar to the measured values. Fig. 3.25a and 3.25b shows that the pile-deflection profiles and bending-moment profiles computed by the proposed direct stiffness method are practically the same as those computed by the finite-element method.

Fig. 3.24. Computed and measured pile responses, Sabine test pile:
(a) Pile-head deflections; (b) Maximum bending moments

Fig. 3.25. Comparison of direct stiffness method (DSM) with finite-element method (FEM), Sabine test pile:
(a) Pile-deflection profiles; (b) Bending-moment profiles

3.7. Concluding remarks

In this chapter, a simple direct stiffness method was developed based on the beam-on-elastic-foundation model. The proposed technique could be easily implemented in the spreadsheet platform, and was illustrated for a number of examples. It was shown that the direct stiffness algorithm presented could be extended to analyze nonlinear soil behavior (by incorporating nonlinear p - y criteria) and nonlinear pile flexural behavior (by incorporating nonlinear moment-curvature criteria). The results obtained by the proposed direct stiffness method agree well with those obtained by the specialized finite-element program FB-MultiPier. The deterministic model based on the direct stiffness method underlies the performance functions of subsequent reliability analyses in the next chapter.

Chapter 4 Reliability Analysis illustrated for a Laterally Loaded Pile

4.1. Introductory comments

This chapter is divided into two parts. The first part describes the various reliability analysis methodologies. Procedures of these methods are covered in associated appendices at the end of this thesis. The Hasofer-Lind index and an established algorithm for the first-order reliability method are presented in section 4.2. The theories and equations underlying the second-order reliability method (in section 4.3) and the Monte Carlo simulation with importance sampling (in section 4.4) are no doubt classical and established. This study contributes by providing alternative efficient and convenient procedures to overcome the tediousness faced by practicing engineers in using these methods. The probabilistic sensitivity factors involving correlated random variables described in section 4.5 are original contributions in terms of concept and formulation. In section 4.6, a simple but new improvement is introduced to the Kounias-Ditlevsen bimodal system failure probability bounds. The second part of this chapter (section 4.7) is an illustration of the application of the various reliability analysis approaches to a laterally-loaded pile.

4.2. Hasofer-Lind index and first-order reliability method (FORM) via ellipsoid perspective

For the purpose of this study, the first-order reliability index reinterpreted based on the ellipsoid perspective (Low and Tang 1997) is used. The reliability index is evaluated using a robust and efficient constrained optimization approach (Low and Tang 2007). The matrix formulation of the Hasofer-Lind index is:

$$\beta = \min_{g(\mathbf{x})=0} \sqrt{\mathbf{x}'^T \mathbf{R}^{-1} \mathbf{x}'} \quad (4.1)$$

where $\mathbf{x}'$ is the vector of $(x_i - \mu_i)/\sigma_i$; μ_i is the mean value of x_i ; σ_i is the standard deviation of x_i ; and $\mathbf{R}$ is the correlation matrix. For correlated non-normal random variables, $\mathbf{x}'$ in Eq. (4.1) becomes the vector of $(x_i - \mu_i^N)/\sigma_i^N$ where μ_i^N and σ_i^N are the equivalent normal mean and standard deviation of the non-normal x_i , respectively. The procedure does not require coordinate transformation and frame-of-reference rotation. Correlation is accounted for by setting up the quadratic form in Eq. (4.1) directly. The reliability index is obtained by minimizing the quadratic form subject to $g(\mathbf{x}) = 0$. During constrained optimization, the equivalent standard normal vector $\mathbf{x}'$ is varied as dimensionless numbers; $\mathbf{x}$ is backcalculated from $\mathbf{x}'$ by the inverse-transform method:

$$F(x_i) = \Phi(x'_i) \quad (4.2)$$

where $F(x_i)$ is the non-normal cumulative distribution function at x_i ; and $\Phi(x'_i)$ is the standard normal cumulative distribution function at x'_i . Equations for obtaining x_i from x'_i , based on Eq. (4.2), are given in Low and Tang (2007) for various non-normal distributions. Formulas for the inverse-transform method for some common probability distributions are also provided in Table 4.1.

Low and Tang (1997; 2004; 2007) demonstrated the use of an optimization tool that resides in spreadsheet software like Microsoft Excel (where it is named Solver) for carrying out constrained optimization. The Solver tool uses the generalized reduced gradient (GRG2) nonlinear optimization code. More information about the algorithms used in Excel Solver can be found at www.solver.com. The constrained optimization can also be implemented in MATLAB using the *fmincon* solver under the Optimization Toolbox. The *fmincon* solver uses a sequential quadratic programming (SQP) method and a subspace trust region method for medium-scale and large-scale optimization, respectively. Details on these algorithms can be found at www.mathworks.com. The use of existing optimization routines saves the user from manual iterative searching and partial derivatives, and is therefore attractive for cases with

complicated or non-explicit performance functions. The procedure for the FORM based on Low and Tang (2007) is given in Appendix A.

Doubts have been raised with regards to the stability of constrained optimization in the original space (Phoon and Nadim 2004; Phoon 2008). Phoon and Nadim (2004) showed that Solver returned an erroneous reliability index for a case where there were several orders of magnitude difference between random variables; the formulation in Eq. (2.36a) was used and Solver's 'automatic scaling' option was not activated. This concern was investigated thoroughly in Low and Tang (2007). The finding was that with the formulation in Eq. (2.36a) and by varying the vector $\mathbf{x}$ directly, Solver might not find the correct reliability index at the first attempt, but would normally converge to the right answer upon reinvoking Solver. Had 'automatic scaling' been used, Solver would have obtained the correct solution immediately. An even better option is to use the formulation in Eq. (4.1) and by varying the vector $\mathbf{x}'$ as dimensionless numbers. With this approach Solver is made to operate on scaled numbers, and so the use of 'automatic scaling' is not necessary. Phoon (2008) commented that "under some circumstances, Solver will produce different results from different initial trial values." Low and Tang (2007) showed that Solver was able to obtain the same design point and reliability index despite using randomized initial values. Should Solver fail to converge at a unique solution but instead alternate between two or more solutions, it is most likely a case of multiple design points. In this case, the global minimum and the local minima can be located by repeating Solver search using different randomized initial points.

Having obtained the design point $\mathbf{x}^*$ and the reliability index β by the Low and Tang 2007 method, one can proceed to use them as the basis of further analysis if desired, including importance sampling for probability of failure and second-order reliability method for investigating the curvature of the limit state surface at the design point. If information pertaining to rotated and standardized space is desired, it can be done after the design point has been obtained, for example, by the following orthogonal transformation (Halдар and Mahadevan 2000a):

$$\mathbf{u} = \mathbf{L}^{-1} \mathbf{x}' \quad (4.3)$$

where $\mathbf{L}$ is the lower triangular matrix obtained by Cholesky factorization of the correlation matrix $\mathbf{R}$.

Table 4.1. Formulas for the inverse-transform method for common probability distributions

Distribution	Formula for computing x from x'
Lognormal	$F(x) = \Phi[(\ln x - \lambda)/\zeta]$ $x = \exp(\lambda + \zeta x')$ where $\lambda = \ln \mu - 0.5\zeta^2$ and $\zeta^2 = \ln[1 + (\sigma/\mu)^2]$
Shifted exponential	$F(x) = 1 - \exp[\lambda(\zeta - x)]$ $x = \zeta - \ln[1 - \Phi(x')]/\lambda$ where $\lambda = \sigma^{-1}$ and $\zeta = \mu - \sigma$
Frechet	$F(x) = \exp[-(\zeta/x)^\lambda]$ $x = \zeta / \{-\ln[\Phi(x')]\}^{1/\lambda}$ where $\mu = \zeta\Gamma(1 - \lambda^{-1})$ and $\sigma^2 = \zeta^2[\Gamma(1 - 2\lambda^{-1}) - \Gamma^2(1 - \lambda^{-1})]$
Gamma	$F(x) = \gamma(\lambda, x/\zeta)/\Gamma(\lambda)$ $f(x) = x^{\lambda-1} \exp(-x/\zeta)/[\zeta^\lambda \Gamma(\lambda)]$ where $\mu = \lambda\zeta$ and $\sigma^2 = \lambda\zeta^2$; $\Gamma(\cdot)$ = gamma function; $\gamma(\cdot)$ = lower incomplete gamma function. Newton-Raphson method is used to obtain x from x' .
Gumbel	$F(x) = \exp\{-\exp[-(x - \lambda)/\zeta]\}$ $x = \lambda - \zeta \ln\{-\ln[\Phi(x')]\}$ where $\lambda = \mu - 0.5772\zeta$ and $\zeta = \sqrt{6}\sigma/\pi$
Shifted Rayleigh	$F(x) = 1 - \exp\{-0.5[(x - \zeta)/\lambda]^2\}$ $x = \{\zeta - 2\lambda^2 \ln[1 - \Phi(x')]\}^{0.5} + \zeta$ where $\lambda = \sigma\sqrt{2/(4 - \pi)}$ and $\zeta = \mu - \lambda\sqrt{\pi/2}$

Type-I smallest	$F(x) = 1 - \exp\{-\exp[(x - \lambda)/\zeta]\}$ $x = \lambda + \zeta \ln\{-\ln[1 - \Phi(x')]\}$ <p>where $\lambda = \mu + 0.5772\zeta$ and $\zeta = \sqrt{6}\sigma/\pi$</p>
Uniform	$F(x) = \begin{cases} 0 & \text{for } x < \min \\ (x - \min)/(\max - \min) & \text{for } \min \leq x \leq \max \\ 1 & \text{for } x > \max \end{cases}$ $x = \min + (\max - \min)\Phi(x')$ <p>where 'min' and 'max' = minimum and maximum x values, respectively.</p>
Weibull	$F(x) = 1 - \exp[-(x/\zeta)^\lambda]$ $x = \zeta\{-\ln[1 - \Phi(x')]\}^{1/\lambda}$ <p>where $\mu = \zeta\Gamma(1 + \lambda^{-1})$ and $\sigma^2 = \zeta^2\Gamma(1 + 2\lambda^{-1}) - \mu^2$</p>
Triangular	$F(x) = \begin{cases} (x - \min)^2 / [(\max - \min)(\text{mode} - \min)] & \text{for } \min \leq x \leq \text{mode} \\ 1 - (\max - x)^2 / [(\max - \min)(\max - \text{mode})] & \text{for } \text{mode} \leq x \leq \max \end{cases}$ $x = \begin{cases} \min + [\Phi(x')(\text{mode} - \min)(\max - \min)]^{0.5} & \text{for } \min \leq x \leq \text{mode} \\ \max - [(1 - \Phi(x'))(\max - \text{mode})(\max - \min)]^{0.5} & \text{for } \text{mode} \leq x \leq \max \end{cases}$ <p>where 'min', 'mode' and 'max' = minimum, mode and maximum x values, respectively.</p>
Beta	$f(x) = \frac{1}{B(\lambda, \zeta)} \frac{(x - \min)^{\lambda-1} (\max - x)^{\zeta-1}}{(\max - \min)^{\lambda+\zeta-1}}$ <p>where $\mu = \min + (\max - \min) \frac{\lambda}{\lambda + \zeta}$ and $\sigma = \frac{\max - \min}{\lambda + \zeta} \sqrt{\frac{\lambda\zeta}{\lambda + \zeta + 1}}$; $B(\cdot)$ = beta function; 'min' and 'max' = minimum and maximum x values, respectively. Newton-Raphson method is used to obtain x from x'.</p>

4.3. Practical second-order reliability method (SORM)

In general, the SORM attempts to assess the curvatures of the limit state surface in the vicinity of the design point, and then provide a refined second-order

approximation to the failure probability based on the curvatures. Typically a number of points are selected in the neighborhood of the design point for this purpose. A practical SORM procedure is proposed in this thesis.

The SORM procedure proposed in this study adopts the point-fitted paraboloid method (Der Kiureghian et al. 1987) of obtaining curvature information. Based on this method, the paraboloid is established in a standardized normal space in which the n th axis coincides with the design point. That is, after obtaining the design point in u -space, the frame-of-reference is further rotated such that the n th axis is in the direction of $\mathbf{u}^*$. The new coordinate system is henceforth referred to as the u' -space, in which the design point $\mathbf{u}'^*$ is of the form $(0, 0, \dots, \beta)$. The mapping of u -coordinates to u' -coordinates, and vice versa, is realized via the orthogonal transformation:

$$\mathbf{U}' = \mathbf{S}\mathbf{U} \quad (4.4a)$$

or inversely,

$$\mathbf{U} = \mathbf{S}^{-1}\mathbf{U}' \quad (4.4b)$$

where $\mathbf{S}$ = rotation matrix determined using the Gram-Schmidt orthogonalization method (e.g., as described in Haldar and Mahadevan 2000a):

$$\mathbf{s}_n = \mathbf{s}_{0n} \quad (4.5a)$$

$$\mathbf{s}_i = \mathbf{s}_{0i} - \sum_{j=i+1}^n \frac{\mathbf{s}_j \mathbf{s}_{0i}^T}{\mathbf{s}_j \mathbf{s}_j^T} \mathbf{s}_j \quad i = 1, \dots, n-1 \quad (4.5b)$$

where $\mathbf{s}_i$ = i th row vector of rotation matrix $\mathbf{S}$; and $\mathbf{s}_{0i}$ = i th row vector of an n by n identity matrix whose last row is replaced by the direction cosines α . The fitting points for the paraboloid approximation are chosen in the neighborhood of $\mathbf{u}'^*$, based on a specified format in which only the last coordinate (i.e., the value of u'_n) is unknown. If the total number of random variables is n then $2(n-1)$ fitting points are selected, such as:

$$n = 2, \quad \begin{bmatrix} -k\beta \\ u'_{n,1-} \end{bmatrix}, \begin{bmatrix} +k\beta \\ u'_{n,1+} \end{bmatrix}$$

2 fitting points

$$n = 3, \quad \begin{bmatrix} -k\beta \\ 0 \\ u'_{n,1-} \end{bmatrix}, \begin{bmatrix} +k\beta \\ 0 \\ u'_{n,1+} \end{bmatrix}, \begin{bmatrix} 0 \\ -k\beta \\ u'_{n,2-} \end{bmatrix}, \begin{bmatrix} 0 \\ +k\beta \\ u'_{n,2+} \end{bmatrix}$$

4 fitting points

$$n = 4, \quad \begin{bmatrix} -k\beta \\ 0 \\ 0 \\ u'_{n,1-} \end{bmatrix}, \begin{bmatrix} +k\beta \\ 0 \\ 0 \\ u'_{n,1+} \end{bmatrix}, \begin{bmatrix} 0 \\ -k\beta \\ 0 \\ u'_{n,2-} \end{bmatrix}, \begin{bmatrix} 0 \\ +k\beta \\ 0 \\ u'_{n,2+} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -k\beta \\ u'_{n,3-} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ +k\beta \\ u'_{n,3+} \end{bmatrix}$$

6 fitting points

and so on.

The sampling coefficient k is selected based on β : if β is less than 3 then k is equal to 1; if β is greater than 3 then k is equal to $3/\beta$. In other words, $k\beta = \beta$ for $\beta < 3$, and $k\beta = 3$ for $\beta \geq 3$.

The values of $u'_{n,i-}$ and $u'_{n,i+}$ (or collectively $u'_{n,i\pm}$) have to be determined such that each fitting point would lie on the limit state surface. This is equivalent to imposing that the corresponding x -values of the fitting points satisfy $g(\mathbf{x}) = 0$. The unknown coordinates $u'_{n,i\pm}$ can be determined using the secant-rule method (Der Kiureghian et al. 1987). If the performance function is available in closed form, Microsoft Excel's built-in Goal Seek program can be used to solve for $u'_{n,i\pm}$ automatically and efficiently (Chan and Low 2011).

With the $u'_{n,i\pm}$ determined, each pair of fitting points in the u'_i - u'_n plane is fitted by a semi-parabola with vertex at $(0, 0, \dots, \beta)$. The curvatures of the semi-parabolas are computed as:

$$\kappa_{i\pm} = 2(u'_{n,i\pm} - \beta) / (k\beta)^2 \quad i = 1, \dots, n-1 \quad (4.6)$$

The two semi-parabolas are replaced by one intersecting parabola with a weighted average curvature κ_i computed in accordance with Breitung's (1984) approximation:

$$(1 + \beta\kappa_i)^{-0.5} = 0.5[(1 + \beta\kappa_{i-})^{-0.5} + (1 + \beta\kappa_{i+})^{-0.5}] \quad i = 1, \dots, n-1 \quad (4.7)$$

for which κ_i may be expressed as:

$$\kappa_i = (R^2 - 1)/\beta \quad (4.8)$$

where $R = 2/[(1 + \beta\kappa_{i+})^{-0.5} + (1 + \beta\kappa_{i-})^{-0.5}]$. For n random variables, there are $n - 1$ principal curvatures κ_i , based upon which a second-order approximation of the failure probability may now be computed. Several closed-form SORM formulas are available, and seven of which are listed in Table 4.2. It can be easily shown that when κ_i tends to zero (i.e. zero curvature) all listed SORM equations reduce to the FORM approximation of $\Phi(-\beta)$.

In Chan and Low (2011), the proposed SORM procedure was demonstrated in the Microsoft Excel spreadsheet platform, and point-fits a paraboloid to the limit state surface in the neighborhood of the design point by means of the built-in Goal Seek program. Complex mathematical operations associated with Cholesky factorization, Gram-Schmidt orthogonalization, and inverse transformation were computed automatically via short open-source function codes created in the study. The failure probability is calculated based on the reliability index and principal curvatures inferred from the second-order approximation of the limit state surface using established SORM formulas. For convenience and simplicity, all selected formulas are in closed-form and do not require any probability integrals to be solved, and a single user-created function is provided to calculate multiple SORM formulas automatically. Four common foundation engineering examples were analyzed using the proposed method and discussed. The proposed SORM computational approach was able to quickly and accurately assess the curvature of

the limit state surface at the design point and can therefore play a very useful complementary role to FORM in pointing out any significant deviation in the limit state surface from the hyperplane assumption, and whether the FORM estimate is likely to overestimate or underestimate the true failure probability. The procedure for the SORM based on Chan and Low (2011) is given in Appendix B.

Table 4.2. Formulas for second-order failure probability approximations

Originator	SORM formula
	$p_f = T_1 + T_2 + T_3$ $T_1 = \Phi(-\beta) \prod_{i=1}^{n-1} (1 + \beta \kappa_i)^{-0.5}$ $T_2 = [\beta \Phi(-\beta) - \phi(-\beta)] \cdot \left\{ \prod_{i=1}^{n-1} (1 + \beta \kappa_i)^{-0.5} - \prod_{i=1}^{n-1} [1 + (\beta + 1) \kappa_i]^{-0.5} \right\}$
Tvedt (1983)	$T_3 = (\beta + 1) [\beta \Phi(-\beta) - \phi(-\beta)] \cdot \left\{ \prod_{i=1}^{n-1} (1 + \beta \kappa_i)^{-0.5} - \text{Re} \prod_{i=1}^{n-1} [1 + (\beta + \sqrt{-1}) \kappa_i]^{-0.5} \right\}$ <p>where $\phi(\cdot)$ & $\Phi(\cdot)$ = probability density function and cumulative distribution function of the standard normal variable, respectively.</p>
Breitung (1984)	$p_f = \Phi(-\beta) \prod_{i=1}^{n-1} (1 + \beta \kappa_i)^{-0.5}$
Hohenbichler & Rackwitz (1988)	$p_f = \Phi(-\beta) \prod_{i=1}^{n-1} [1 + \kappa_i \psi(\beta)]^{-0.5}$ <p>where $\psi(\beta) = \phi(\beta)/\Phi(-\beta)$</p>
Koyluoglu & Nielsen (1994)	$p_f = \Phi(-\beta) - \Phi(-\beta) \left(\prod_{i=m}^{n-1} [1 - 0.5 \kappa_i \xi(\beta)]^{-0.5} \cdot \left\{ 1 - \prod_{i=1}^{m-1} [1 + \kappa_i \psi(\beta)]^{-0.5} \right\} \right)$ $+ \Phi(\beta) \left(\prod_{i=1}^{m-1} [1 + 0.5 \kappa_i \psi(\beta)]^{-0.5} \cdot \left\{ 1 - \prod_{i=m}^{n-1} [1 - \kappa_i \xi(\beta)]^{-0.5} \right\} \right)$ <p>where $\xi(\beta) = \phi(\beta)/\Phi(\beta)$</p>

		$p_f = \Phi(-\beta) - \phi(\beta)(D_1 + D_2 + D_3)$
Cai & Elishakoff (1994)		$D_1 = \sum_{i=1}^{n-1} \lambda_i ; D_2 = -\frac{1}{2} \beta \left(3 \sum_{i=1}^{n-1} \lambda_i^2 + \sum_{i \neq j}^{n-1} \lambda_i \lambda_j \right) ;$ $D_3 = \frac{1}{6} (\beta^2 - 1) \left(15 \sum_{i=1}^{n-1} \lambda_i^3 + 9 \sum_{i \neq j}^{n-1} \lambda_i^2 \lambda_j + \sum_{i \neq j \neq k}^{n-1} \lambda_i \lambda_j \lambda_k \right)$ <p>where $\lambda_i = \kappa_i / 2$</p>
Hong (1999)		$p_f = C_1 P_0 ; P_0 = \Phi(-\beta) \prod_{i=1}^{n-1} [1 + 2\psi(\beta)\lambda_i]^{-0.5}$ $C_1 = \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{\Phi(-\beta - \{[(n-1)\lambda_i] / [1 + 2\psi(\beta)\lambda_i]\})}{\Phi(-\beta)} \cdot \exp \left[\frac{(n-1)\psi(\beta)\lambda_i}{1 + 2\psi(\beta)\lambda_i} \right]$
Hong (1999)		$p_f = C_2 P_0 ; P_0 = \Phi(-\beta) \prod_{i=1}^{n-1} [1 + 2\psi(\beta)\lambda_i]^{-0.5}$ $C_2 = 1 - \frac{n-1}{3} + \frac{1}{3} \sum_{i=1}^{n-1} \frac{\Phi(-\beta - \{3\lambda_i / [1 + 2\psi(\beta)\lambda_i]\})}{\Phi(-\beta)} \cdot \exp \left[\frac{3\psi(\beta)\lambda_i}{1 + 2\psi(\beta)\lambda_i} \right]$

4.4. Efficient Monte Carlo simulations with importance sampling

In this study, Monte Carlo simulations are performed to provide results for assessing the accuracy of FORM, SORM, system reliability bounds and sensitivity factors. Obtaining a satisfactory level of accuracy requires that a large number of simulation trials be carried out, especially if the probability of failure is very small and the number of random variables very large. The so-called variance reduction techniques can be employed to reduce the computational burden or, equivalently, to increase the accuracy for the same number of simulation trials.

The direct method of importance sampling proposed by Melchers (1984) based on Shinozuka (1983) and Rubinstein (1981) is a variance reduction technique that is used in this study:

$$p_f \approx \frac{1}{N} \sum_{k=1}^N I[G(\mathbf{u}_k) \leq 0] \frac{\phi_n(\mathbf{u}_k)}{\psi_n(\mathbf{u}_k)} \quad (4.9)$$

$$\phi_n(\mathbf{u}) = \prod_{i=1}^n \phi_i(u_i) \quad (4.10)$$

$$\psi_n(\mathbf{u}) = \prod_{i=1}^n f_i(u_i; u_i^*, 1) \quad (4.11)$$

where $G(\mathbf{u})$ is the limit state function in the rotated and transformed space of uncorrelated standard normal variables $\mathbf{u}$; $I[G(\mathbf{u}) \leq 0]$ is an indicator function defined such that $I[\cdot] = 1$ if $G(\mathbf{u}) \leq 0$ and $I[\cdot] = 0$ if otherwise; $\phi_i(u_i)$ is the probability density function of the standard normal variable u_i ; and $f_i(u_i; u_i^*, 1)$ is the probability density function of the normal variables u_i with mean value u_i^* and unit variance.

It can be seen from Eq. (4.9) that $\psi_n(\mathbf{u})$ is the importance sampling probability density function. It is essentially the product of n numbers of ‘translated’ standard normal probability density function. The ‘translation’ here refers to a shift in the mean value of the standard normal distribution by an amount that is equal to the design-point value. Before this shift the mean value is at zero; after the shift the mean value is equal to u_i^* . That $\psi_n(\mathbf{u})$ (i.e. a joint pdf) may be expressed as a product of n separate pdfs is due to the fact that the u_i ’s are all independent of each other.

The procedure for the direct importance-sampling method based on Rubinstein (1981), Shinozuka (1983), and Melchers (1984) is given in Appendix C.

The $100(1 - \alpha)\%$ confidence interval for the probability of failure obtained by Monte Carlo simulation can be estimated from the following equation (Shooman 1968):

$$\varepsilon = z_{\alpha/2} \sqrt{\frac{1-p_f}{np_f}} \quad (4.12)$$

where $z_{\alpha/2}$ is the z -score of the standard normal cdf defined as $P(Z \leq z_{\alpha/2}) = 1 - \alpha/2$; p_f is the probability of failure estimated using Monte Carlo simulations; ε is the error; and n is the number of simulation trials. For example, it is 95% likely that the actual probability of failure will be within $p_f \pm \varepsilon$, given that $z_{\alpha/2} = 1.96$.

On whether the above equation can be used to compute the confidence interval when p_f is estimated using Importance Sampling, Melchers (1989) noted that the ‘relevant’ p_f to enter the above equation is the ‘likely success/failure rate’ of 0.5, assuming that the above equation remains valid for p_f that is not small. A better alternative is to repeat the Importance Sampling N times, and treat the sample mean of p_f as a t -distribution random variable with $N - 1$ degrees of freedom. The error is given by:

$$\varepsilon = t_{\alpha/2, N-1} \frac{s}{\sqrt{N}} \quad (4.13)$$

where $t_{\alpha/2, N-1}$ is the upper $100\alpha/2$ percentage point of the t -distribution with $N - 1$ d.o.f.; and s is the sample standard deviation. For example, there is a 95% chance that the interval $m \pm \varepsilon$ will contain the true mean value of p_f , given that $t_{\alpha/2, N-1} = 2.78$ (assuming that $N = 5$). The following equations are used to compute the sample mean m and the sample standard deviation s :

$$m = \frac{\sum_{i=1}^N (p_f)_i}{N} \quad (4.14)$$

$$s = \frac{\sqrt{N \sum_{i=1}^N (p_f)_i^2 - \left[\sum_{i=1}^N (p_f)_i \right]^2}}{N(N-1)} \quad (4.15)$$

4.5. Probabilistic sensitivity analysis involving correlated random variables

A simple sensitivity factor that is able to relate directly to the original correlated random variables x_i is given by $\partial\beta/\partial x'_i$ where x'_i is the dimensionless standard equivalent normal vector of which $x'_i = (x_i - \mu_i^N)/\sigma_i^N$, where μ_i^N and σ_i^N are the equivalent normal mean and standard deviation of non-normal variable x_i , respectively. If x_i is normal, x'_i is computed simply based on the original normal mean and standard deviation. A very useful relationship between u_i and x'_i is given by (Haldar and Mahadevan 2000a; Baecher and Christian 2003) $\mathbf{u} = \mathbf{L}^{-1}\mathbf{x}'$ where $\mathbf{L}$ = lower triangular matrix that is obtained from the correlation matrix $\mathbf{R}$ (that is, $\mathbf{R} = \mathbf{L}\mathbf{U}$, in which $\mathbf{U} = \mathbf{L}^T$). The Cholesky decomposition approach is used to evaluate $\mathbf{L}$ because it is simple, efficient and well-suited for the symmetric and positive-definite $\mathbf{R}$ matrix. Based on chain rule the partial derivative $\partial\beta/\partial x'_i$ may be expressed as:

$$\frac{\partial\beta}{\partial x'_i} = \frac{\partial\beta}{\partial u_1} \frac{\partial u_1}{\partial x'_i} + \frac{\partial\beta}{\partial u_2} \frac{\partial u_2}{\partial x'_i} + \dots + \frac{\partial\beta}{\partial u_n} \frac{\partial u_n}{\partial x'_i} = \alpha_1 \frac{\partial u_1}{\partial x'_i} + \alpha_2 \frac{\partial u_2}{\partial x'_i} + \dots + \alpha_n \frac{\partial u_n}{\partial x'_i} \quad (4.16)$$

A more general form of Eq. (4.16) is derived as follows. For illustration, the relationship $\mathbf{u} = \mathbf{L}^{-1}\mathbf{x}'$, when expanded for a case with $n = 3$ random variables, may be written as:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \quad (4.17)$$

where $c_{ji} = c_{11}, c_{12}, \dots, c_{33}$ are components of $\mathbf{L}^{-1}$ matrix ($i = 1, 2, 3$ and $j = 1, 2, 3$). The standard normal variable u_j is essentially a linear function of x'_i such that:

$$u_j = c_{j1}x'_1 + c_{j2}x'_2 + c_{j3}x'_3 \quad (4.18)$$

Therefore the partial derivative of u_j with respect to x'_i is given by:

$$\frac{\partial u_j}{\partial x'_i} = c_{ji} \quad (4.19)$$

Substituting Eq. (4.19) into Eq. (4.16), one obtains:

$$\frac{\partial \beta}{\partial x'_i} = \alpha_1 c_{1i} + \alpha_2 c_{2i} + \alpha_3 c_{3i} \quad (4.20)$$

As a result,

$$\begin{bmatrix} \partial \beta / \partial x'_1 \\ \partial \beta / \partial x'_2 \\ \partial \beta / \partial x'_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 c_{11} & \alpha_2 c_{21} & \alpha_3 c_{31} \\ \alpha_1 c_{12} & \alpha_2 c_{22} & \alpha_3 c_{32} \\ \alpha_1 c_{13} & \alpha_2 c_{23} & \alpha_3 c_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad (4.21)$$

or, equivalently

$$\frac{\partial \beta}{\partial \mathbf{x}'} = (\mathbf{L}^{-1})^T \boldsymbol{\alpha} \quad (4.22)$$

where $(\mathbf{L}^{-1})^T$ = the transpose of the inverse of the lower Cholesky matrix. Eq. (4.22) is true for any number of random variables. In the case of uncorrelated x_i where $\mathbf{L} = \mathbf{I}$ = identity matrix, $\partial \beta / \partial x'_i$ reduces to α_i .

For any continuous probability distribution the following expressions compute the sensitivity of β with respect to μ_i and σ_i of x_i :

$$\frac{\partial \beta}{\partial \mu_i} = \frac{\partial \beta}{\partial x'_i} \frac{\partial x'_i}{\partial \mu_i} \quad (4.23a)$$

$$\frac{\partial \beta}{\partial \sigma_i} = \frac{\partial \beta}{\partial x'_i} \frac{\partial x'_i}{\partial \sigma_i} \quad (4.23b)$$

where $\partial \beta / \partial x'_i$ is evaluated based on Eq. (4.16). For normally-distributed x_i the terms $\partial x'_i / \partial \mu_i$ and $\partial x'_i / \partial \sigma_i$ can be obtained analytically based on $x'_i = (x_i - \mu_i) / \sigma_i$

such that $\partial x'_i/\partial \mu_i = -1/\sigma_i$ and $\partial x'_i/\partial \sigma_i = -x'_i/\sigma_i$. Furthermore for uncorrelated normal x_i , $\partial \beta/\partial x'_i = \alpha_i$, and this leads to:

$$\frac{\partial \beta}{\partial \mu_i} = -\frac{\alpha_i}{\sigma_i} \quad (4.24a)$$

$$\frac{\partial \beta}{\partial \sigma_i} = -\frac{\beta \alpha_i^2}{\sigma_i} \quad (4.24b)$$

which are exactly the same sensitivity factors as established in the literature (Madsen et al. 1986, p.122).

For non-normal x_i whose $\partial x'_i/\partial \mu_i$ and $\partial x'_i/\partial \sigma_i$ cannot be easily obtained in closed form, an approximation may be computed based on numerical differentiation such that $\partial x'_i/\partial \mu_i$ is estimated using $\Delta x'_i/\Delta \mu_i$ and $\partial x'_i/\partial \sigma_i$ is estimated using $\Delta x'_i/\Delta \sigma_i$. The value of $\Delta x'_i$ is evaluated based on the inverse-transform method assuming that the perturbations $\Delta \mu_i$ and $\Delta \sigma_i$ are small relative to the original μ_i and σ_i , respectively, such that the corresponding change in the design-point value x_i^* may be deemed negligible.

Two-parameter probability distributions that are not directly defined in terms of μ and σ need additional treatment. For example the lognormal random variable x is typically described in terms of $\lambda = \ln \mu - 0.5\zeta^2$ and $\zeta = \{\ln[1 + (\sigma/\mu)^2]\}^{0.5}$. For a perturbation $\Delta \mu$ (or $\Delta \sigma$) the corresponding $\Delta \lambda$ and $\Delta \zeta$ need to be computed before using the inverse-transform method for the lognormal distribution $x' = (\ln x - \lambda)/\zeta$.

Probability distributions with three (e.g., triangular distribution) or four parameters (e.g., beta distribution) may be dealt with using the same approach for two-parameter probability distributions if reasonable assumptions can be made with respect to the shape and/or bounds of the distribution. For example, a general triangular distribution is defined by three parameters, namely lower bound a , upper bound b and mode c . The values of μ and σ are functions of all three parameters a , b and c . For a perturbation $\Delta \mu$ (or $\Delta \sigma$) the corresponding Δa , Δb and

Δc need to be computed. Ambiguity arises because only two equations (of μ and σ in terms of a , b and c) are available. Assuming that $c = \mu$ such that $\Delta c = \Delta \mu$ the ambiguity is removed, the remaining Δa and Δb can now be solved and the corresponding value of $\Delta x'$ is evaluated based on the inverse-transform method. Similarly for the 4-parameter beta distribution, the values of μ and σ are functions of four parameters λ , ζ , a and b . If the shape parameters λ and ζ can be assumed to be constant and independent of μ and σ then for a perturbation $\Delta \mu$ (or $\Delta \sigma$) the corresponding Δa and Δb can be solved based on the two equations that express μ and σ in terms of λ , ζ , a and b .

A more direct approach that is also meaningful is to evaluate the sensitivity of β with respect to a , b and c for the triangular distribution, and with respect to λ , ζ , a and b for the beta distribution, resulting in so-called sensitivity factors of specific parameters θ_j (c.f. statistical-moment parameters μ_i and σ_i). The computation is straightforward:

$$\frac{\partial \beta}{\partial \theta_j} = \frac{\partial \beta}{\partial x'_i} \frac{\partial x'_i}{\partial \theta_j} \quad (4.25)$$

It is possible to infer the sensitivity of the failure probability p_f from the sensitivity of the reliability index β based on the hyperplane assumption $p_f = 1 - \Phi(\beta)$ by using the following equations:

$$\frac{\partial p_f}{\partial \mu_i} = -\phi(\beta) \frac{\partial \beta}{\partial \mu_i} \quad (4.26a)$$

$$\frac{\partial p_f}{\partial \sigma_i} = -\phi(\beta) \frac{\partial \beta}{\partial \sigma_i} \quad (4.26b)$$

where $\phi(\beta)$ = standard normal probability density function evaluated at β .

The procedure for probabilistic sensitivity analysis involving correlated random variables is given in Appendix D. It considers commonly-used two-parameter probability distributions, as well as the 3-parameter triangular distribution and the 4-parameter beta distribution.

4.6. Improved system failure probability estimates

In cases where there are only two failure modes, the Kounias-Ditlevsen system failure probability bounds reduced to the following:

$$p_{f,1} + \max(p_{f,2} - p_{f,12}, 0) \leq p_{f,\text{sys}} \leq p_{f,1} + p_{f,2} - p_{f,12} \quad (4.27)$$

Provided there are no more than two failure modes, one may safely assume that $(p_{f,2} - p_{f,12}) \geq 0$, such that the lower and upper bounds of $p_{f,\text{sys}}$ are coincidental and equal to $p_{f,1} + p_{f,2} - p_{f,12}$ (which is consistent with set theory). However, since the intersection $p_{f,12}$ as calculated based on Ditlevsen (1979) is given in terms of lower and upper bounds, the resulting Kounias-Ditlevsen bounds are *not* coincidental.

The Ditlevsen (1979b) method calculates bounds for the bimodal joint failure probability $p_{f,ij}$ as an alternative to solving the bivariate standard normal cumulative distribution function Φ_2 . The bounds can be very narrow and even coincidental for single-mode failure probabilities of order 10^{-4} (Ditlevsen 1979b). However, the bounds could be wide if the single-mode failure probabilities are all of order 10^{-2} (Ang and Tang 1984) and this ultimately affects the precision of the system reliability bounds $p_{f,\text{sys}}$. The difficulty in solving Φ_2 by numerical integration can be overcome by using the approximate closed-form formulas in Owen (1956):

$$\Phi_2(h, k; \rho) = \begin{cases} \frac{1}{2} \Phi(h) - T\left(h, \frac{k - \rho h}{h\sqrt{1 - \rho^2}}\right) + \frac{1}{2} \Phi(k) - T\left(k, \frac{h - \rho k}{k\sqrt{1 - \rho^2}}\right) & \text{if } hk > 0 \text{ or if } hk = 0, h \text{ or } k \geq 0 \\ \frac{1}{2} \Phi(h) - T\left(h, \frac{k - \rho h}{h\sqrt{1 - \rho^2}}\right) + \frac{1}{2} \Phi(k) - T\left(k, \frac{h - \rho k}{k\sqrt{1 - \rho^2}}\right) - \frac{1}{2} & \text{if } hk < 0 \text{ or if } hk = 0, h \text{ or } k < 0 \end{cases} \quad (4.28)$$

where

$$T(h, a) = \frac{\arctan a}{2\pi} - \frac{1}{2\pi} \sum_{j=0}^{\infty} c_j a^{2j+1} \quad \text{if } 0 < |a| \leq 1 \quad (4.29)$$

$$c_j = (-1)^j \frac{1}{2j+1} \left[1 - \exp\left(-\frac{1}{2} h^2\right) \sum_{i=0}^j \frac{h^{2i}}{2^i i!} \right] \quad (4.30)$$

$$T(h, a) = \frac{1}{2} \Phi(h) + \frac{1}{2} \Phi(ah) - \Phi(h) \Phi(ah) - T\left(ah, \frac{1}{a}\right) \quad \text{if } 1 < |a| < \infty \quad (4.31)$$

In Eq. (4.31), the last term is evaluated using Eq. (4.29). Values for negative a or h may be obtained by using $T(h, -a) = -T(h, a)$ and $T(-h, a) = T(h, a)$.

Therefore, computation of the bimodal joint failure probability $p_{f,ij}$ is now equivalent to the calculation of $\Phi_2(-\beta_i, -\beta_j, \rho_{ij})$ based on Eq. (4.28) where β_i and β_j are the reliability indices corresponding to failure modes i and j , respectively; and ρ_{ij} is the correlation coefficient between failure modes i and j . The correlation coefficient ρ_{ij} is obtained based on the unit design point vectors of the two failure modes.

The procedure for system reliability analysis based on Kounias (1968), Hunter (1976), and Ditlevsen (1979b) is given in Appendix E. It considers the Ditlevsen (1979b) method as well as the Owen (1956) formulas for calculating the bimodal joint failure probability.

4.7. Application to laterally-loaded pile in marine clay at Sabine

Fig. 4.1. Sabine test pile

Meyer (1979) examined the soil test results at Sabine and reported that the clay was a slightly overconsolidated marine deposit, had an undrained shear strength of 14.4 kPa, and a submerged unit weight of 5.5 kN/m³. The p - y curves for the analyses were obtained by using the criteria for soft clay with free water (Matlock 1970). Computations were made with $J = 0.5$ and $\varepsilon_{50} = 0.02$.

The lateral loads were applied at 0.305 m above the ground-line (Fig. 4.1), on a steel pipe pile that was 319 mm in diameter, with a wall thickness of 12.7 mm, and a length of 12.8 m. The bending stiffness (flexural rigidity) was 31,280 kNm². The bending moment at which the extreme fibers would first yield (i.e., yield moment) was computed to be 231 kNm, and the bending moment for the formation of a fully plastic hinge was computed to be 304 kNm.

Δz_1	$r_{\Delta z}$	d	L	E_p	I_p	P_{top}	M_{top}	r_1	r_2
0.125	1.148	0.319	12.495	2.2E+08	0.00014	79.593	22.702	0.1468	0.1595
(m)		(m)	(m)	(kN/m ²)	(m ⁴)	(kN)	(kNm)	(m)	(m)
	γ'	J	ε			$y-y_{prev}$	$M-M_{prev}$		
	5.5	0.5	0.02			1E-05	0.00909		
	(kN/m3)					(m)	(kNm)		
i	z_i	$(E_p I_{p,sec})_i$	$(k_{sec} d)_i$	y_i	$y_{prev,i}$	p_i	V_i	M_i	$M_{prev,i}$
	(m)	(kNm ²)	(kN/m ²)	(m)	(m)	(kN/m)	(kN)	(kNm)	(kNm)
1	0	31280	0	0.29992	0.29991	0.00	79.59	22.70	22.70
2	0.125	31280	1	0.29216	0.29215	-0.42	79.57	32.65	32.65
3	0.27	31280	3	0.28327	0.28326	-0.97	79.47	44.06	44.06
4	0.43	31280	6	0.27310	0.27309	-1.71	79.25	57.13	57.13
5	0.62	31280	10	0.26149	0.26148	-2.69	78.83	72.08	72.08
6	0.84	31280	16	0.24827	0.24826	-3.98	78.11	89.12	89.12
7	1.09	31280	26	0.23325	0.23324	-6.01	76.86	108.44	108.44
8	1.37	31280	42	0.21628	0.21627	-9.11	74.70	130.13	130.13
9	1.70	31280	65	0.19723	0.19722	-12.81	71.10	154.10	154.10
10	2.08	31280	97	0.17601	0.17600	-17.10	65.46	179.89	179.89
11	2.51	31280	143	0.15267	0.15266	-21.88	57.03	206.46	206.46
12	3.01	31180	213	0.12740	0.12740	-27.09	44.87	231.88	231.87
13	3.58	27635	271	0.10074	0.10074	-27.30	29.37	253.05	253.04
14	4.23	23889	371	0.07385	0.07385	-27.39	11.47	266.41	266.41
15	4.98	23583	565	0.04867	0.04867	-27.49	-9.14	267.30	267.29
16	5.85	28354	848	0.02720	0.02720	-23.07	-30.95	249.73	249.73
17	6.84	31280	1602	0.01069	0.01069	-17.13	-50.85	208.76	208.75
18	7.97	31280	27739	-0.0002	-0.0002	4.19	-58.20	144.50	144.50
19	9.28	31280	2577	-0.0054	-0.0054	13.88	-46.42	74.89	74.88
20	10.78	31280	2304	-0.0064	-0.0064	14.80	-24.94	21.29	21.29
21	12.50	31280	2531	-0.0056	-0.0056	14.21	0	0	0.00

Fig. 4.2. Deterministic analysis based on direct stiffness method, Sabine test pile

The pile was instrumented internally, at close spacings, with electrical-resistance strain gauges for the measurement of bending moment. The pile was tested both under static and cyclic loading. The discussions in this study pertain only to cyclic loading.

Probabilistic analyses are carried out here with the following basic statistical assumptions. The stochastic load parameters P_{top} and M_{top} are described by the Gumbel distribution. The Gumbel distribution belongs to the family of extreme-

value distributions, and is often used to represent wind loads or wave loads. The applied load P_{top} has a mean value of 60 kN and standard deviation of 12 kN; the applied moment M_{top} has a mean value of 20 kNm and standard deviation of 4 kNm. This amount of uncertainty (COV 20%) is within the range of typical load statistics as mentioned in Phoon and Kulhawy (1999b). Values of P_{top} and M_{top} are positively correlated with a correlation coefficient of 0.5. The undrained shear strength of the clay c_u is a lognormal random variable with mean value 14.4 kPa and standard deviation 2.88 kPa. Based on data gathered by Phoon and Kulhawy (1999a), the COV of the undrained shear strength of clay can vary from 10 to 60%, depending on whether it is measured directly or correlated empirically with certain field measurements. For the case in hand, it is not known how the value of c_u was determined. Hence, only inherent soil variability is considered (measurement error and transformation uncertainty are ignored), and assuming that the value of c_u is obtained from unconsolidated-undrained triaxial compression tests in the laboratory, the COV is between 10 and 30%. An average of 20% is adopted herein. A lognormal distribution is assigned so that negative values of c_u (which do not make physical sense) are excluded from the reliability analysis.

The two most critical failure mechanisms of the laterally-loaded pile are investigated: failure under serviceability limit state (SLS) occurs when the pile-head deflection y_0 exceeds a certain permissible value y_{lim} , and failure under ultimate limit state (ULS) is said to have taken place if the maximum bending moment M_{max} anywhere along the pile exceeds the plastic moment M_p . These failure criteria translate to the following limit state equations:

$$g_1(\mathbf{x}) = y_{\text{lim}} - y_0 \quad (4.32a)$$

$$g_2(\mathbf{x}) = M_p - M_{\text{max}} \quad (4.32b)$$

where $\mathbf{x}$ represents the vector of random variables, which includes P_{top} , M_{top} and c_u . Both limit state functions above depend on the direct stiffness method introduced

in Chapter 3 for the computed values of y_0 and $M_{\max}$, which in turn vary according to $\mathbf{x}$. For the present example, we set $y_{\lim} = 0.3$ m and $M_p = 300$ kNm. The direct stiffness method for the Sabine test pile is shown in Fig. 4.2.

Fig. 4.3. Reliability analysis with respect to pile deflection failure and bending moment failure, Sabine test pile

Reliability analysis is carried out by incorporating into Fig. 4.2 the additional input shown in Fig. 4.3. Two sets of solutions are as shown in Fig. 4.3. They were obtained using the Low and Tang 2007 algorithm for FORM. The reliability indices (β) corresponding to $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ are 1.5568 and 1.7907, respectively. The inferred probabilities of failure (p_F) are 5.98% and 3.67%, respectively, based on $p_F = 1 - \Phi(\beta)$. Note that the inference above is a first-order approximation. Ordinary Monte Carlo (OMS) simulations and advanced Monte Carlo simulations with Direct Importance Sampling (DIS) were conducted for verification. The OMC method samples around the mean-value point of the random variables in the original x -space; whereas the DIS method samples around the design point in the transformed u -space, based on the design point information obtained in Fig. 4.3.

note that while the Low and Tang 2007 method gives only the design point in x -space ($\mathbf{x}^*$), the corresponding design point in u -space ($\mathbf{u}^*$) can be easily obtained based on $\mathbf{u}^* = \mathbf{L}^{-1}\mathbf{x}^*$.

Five sets of simulations (*Sim* 1, ..., 5) were carried out for OMC and DIS. Table 4.3 shows the probabilities of failure for each set of simulation, as well as the mean p_F and the COV of p_F . It can be seen that DIS yields practically the same mean p_F as OMC, at comparable levels of precision as indicated by the COV of p_F , even though the number of samples required by DIS is considerably less than that required by OMC. For the case in hand, 500 – 1000 samples were used for DIS, compared with 20,000 – 40,000 samples for OMC.

In the remainder of this thesis, all results reports as ‘MCS’ are obtained based on DIS, unless otherwise stated. The number of samples used differs from case to case, depending on the value of the failure probability, in order to achieve a more or less consistent COV of about 5%.

4.7.1. Effect of spatial variability

In the spatial-autocorrelation approach, the vertical variation of c_u of the marine clay is modeled by a number of discrete points that are spatially auto-correlated according to the exponential autocorrelation function:

$$\rho_{ij} = \exp\left\{-\frac{|z_i - z_j|}{r_o}\right\} \quad (4.33)$$

Table 4.3. Probabilities of failure obtained by ordinary Monte Carlo method and direct Importance Sampling method

(a) Pile deflection failure

Method	Probability of Failure p_F (%):					Mean	COV
	Sim 1	Sim 2	Sim 3	Sim 4	Sim 5		
OMC	5.824	6.576	6.351	5.592	6.359	6.14	0.067
DIS	6.124	5.790	6.545	6.232	6.058	6.15	0.045

Note: OMC: Ordinary Monte Carlo method (40,000 samples per simulation)
DIS: Direct Importance Sampling method (1000 samples per simulation)
Mean = Average p_F based on 5 simulations (Sim 1 to Sim 5)
COV = Coefficient of variation of p_F based on 5 simulations

(b) Bending moment failure

Method	Probability of Failure p_F (%):					Mean	COV
	Sim 1	Sim 2	Sim 3	Sim 4	Sim 5		
OMC	3.751	3.416	3.599	3.457	3.587	3.56	0.037
DIS	3.408	3.683	3.844	3.679	3.488	3.62	0.048

Note: OMC: Ordinary Monte Carlo method (40,000 samples per simulation)
DIS: Direct Importance Sampling method (1000 samples per simulation)
Mean = Average p_F based on 5 simulations (Sim 1 to Sim 5)
COV = Coefficient of variation of p_F based on 5 simulations

General case (Vanmarcke 1983)

$$\rho_{ij} = \frac{b_0^2 \gamma(b_0) - b_1^2 \gamma(b_1) + b_2^2 \gamma(b_2) - b_3^2 \gamma(b_3)}{2b_i b_j [\gamma(b_i) \gamma(b_j)]^{0.5}}$$

Special case: $b_i = b_j = b$

$$\rho_{ij} = \frac{b_0^2 \gamma(b_0) - b_1^2 \gamma(b_1) + b_2^2 \gamma(b_2) - b_3^2 \gamma(b_3)}{2b^2 \gamma(b)}$$

Fig. 4.4. Realization of one-dimensional random field of undrained shear strength with depiction of local averages over intervals b_i and b_j

where z_i and z_j are the depths of discrete point i and j , respectively; and r_o is the autocorrelation distance, which is assumed to be 2.5 m. The vertical scale of fluctuation δ for c_u is in the order of 1 to 2 m, although it can be as high as 6 m (Phoon and Kulhawy 1999a). So the assumed r_o of 2.5 m, which is half of δ , is within reasonable range. The value of c_u at any location within the random field can be linearly interpolated based on the values of c_u at the discrete points, each being a lognormal random variable with mean value 14.4 kPa and standard deviation 2.88 kPa. The topmost discrete point is at $z = 0$ m; the bottommost discrete point is at $z = L = 12.5$ m, where L is the length of the pile. For simplicity the discrete points are spaced uniformly. The separation distance s between adjacent discrete points is equal to $L/(n_p - 1)$ where n_p is the number of discrete points.

In the spatial-averaging approach, the vertical variation of c_u of the marine clay is modeled by a number of local averages. The local average c_u – akin to the sample mean of c_u over a certain vertical distance b – is a lognormal random variable with mean 14.4 kPa and reduced standard deviation $2.88[\gamma(b)]^{0.5}$ kPa, where $\gamma(b)$ is the variance function. If the averaging interval b is constant and uniform, the correlation coefficient between any two local averages is given by:

$$\rho_{ij} = \frac{b_0^2\gamma(b_0) - b_1^2\gamma(b_1) + b_2^2\gamma(b_2) - b_3^2\gamma(b_3)}{2b^2\gamma(b)} \quad (4.34)$$

where the lengths b_0 , b_1 , b_2 and b_3 are as defined in Fig. 4.4, and the analytical variance function corresponding to the exponential autocorrelation function is (Vanmarcke 1983):

$$\gamma(b) = 2\left(\frac{\delta}{b}\right)^2 \left[\frac{b}{\delta} - 1 + \exp\left(-\frac{b}{\delta}\right) \right] \quad (4.35)$$

The averaging interval b is equal to L/n_a where n_a is the number of local averages.

The following points may be noted:

- (a) Fig. 4.5a and 4.5b show the computed reliability indices β as a function of the normalized separation distance s/r_o or normalized averaging interval b/r_o . Note that the x -axis is plotted in log-scale. A small value of s/r_o and/or b/r_o means a large number of discrete points n_p and/or local averages n_a , and vice versa. It can be seen that the reliability index β obtained based on the spatial-autocorrelation method is consistent with that obtained based on the spatial-averaging method, at small values of s/r_o and b/r_o , for both serviceability limit state [$g_1(\mathbf{x}) = 0$, Fig. 4.5a] and ultimate limit state [$g_2(\mathbf{x}) = 0$, Fig. 4.5b]. When the number of local averages n_a approaches infinity, the averaging interval b theoretically vanishes such that each local average reduces to a discrete point. At the

same time the correlations between the local averages given by Eq. (4.34) also approach the correlations between the discrete points given by Eq. (4.33). Rightfully, the variances of the discrete points ought to be reduced for large values of s/r_o (say, $s/r_o > 1$). That is why the spatial-autocorrelation curve shows a dotted line for $s/r_o > 1$, which indicates that comparison between the two methods is meaningful only for $s/r_o < 1$. That the spatial-averaging method yields a higher estimate of the reliability index than the spatial-autocorrelation method is to be expected. This is because the variance of the random variables in the spatial-autocorrelation method is equal to the point variance of the random field, whereas in the spatial-averaging method the variance is reduced through a variance function. Naturally the lower variance of the random variables produces a higher value of the reliability index. In Haldar and Mahadevan (2000b), it is noted that the spatial-autocorrelation method tends to over-represent the variability while the spatial-averaging method tends to under-represent the same variability. An important point that is also raised in Haldar and Mahadevan (2000b) pertains to the probability distributions of the random variables. For the spatial-autocorrelation method, the distribution of the random variables remains the same as that of the underlying random field. However, for the spatial-averaging method, this is true only if the underlying random field is normally distributed, because each spatial average is a linear integration of the random field. For the case in hand, the underlying random field (of undrained shear strength c_u) is given a lognormal distribution. Therefore for the spatial-averaging method, a certain amount of error may be due to the assumption that the random variables are also of the same lognormal distribution (albeit of reduced variance).

Fig. 4.5. Reliability indices computed using spatial-autocorrelation and spatial-averaging approaches:
(a) serviceability limit state; (b) ultimate limit state

- (b) When a single discrete point is used to represent the entire random field, the marine clay is assumed to have the same value of c_u everywhere within the deposit, and that c_u is a random value based on the lognormal distribution with a mean of 14.4 kPa and a standard deviation of original value 2.88 kPa. Perfect correlation of c_u between soil elements within the random field is implied, and the autocorrelation distance r_o is equal to infinity. The reliability indices β obtained this way tend to be conservative: for $r_o = 2.5$ m, $\beta_1 = 1.497$ [corresponding to $g_1(\mathbf{x}) = 0$] and $\beta_2 = 1.781$ [corresponding to $g_2(\mathbf{x}) = 0$]. On the other hand, when the random field is represented by a single local average, the value of c_u is still the same everywhere within the deposit. While the mean value of the lognormal random variable c_u remains unchanged, the standard deviation is now reduced based on the averaging interval b (which is equal to the pile length L) and the autocorrelation distance r_o . Unless r_o is equal to infinity, the single-local-average approach yields β indices that are more optimistic than those obtained by the single-discrete-point approach. For $r_o = 2.5$ m, the reliability indices are $\beta_1 = 1.610$ [corresponding to $g_1(\mathbf{x}) = 0$] and $\beta_2 = 1.818$ [corresponding to $g_2(\mathbf{x}) = 0$] when only one local average is used.
- (c) The variation of the design-point values of c_u^* with depth z is shown in Fig. 4.6a and 4.6b for the spatial-autocorrelation method, and in Fig. 4.7a and 4.7b for the spatial-averaging method. The design-point profile is nonlinear, and hence the number of discrete points or local averages needs to exceed a certain minimum in order to capture the nonlinear profile. If only two or three discrete points or local averages are used, the resulting linear or bilinear design-point profiles are obviously inaccurate. For the spatial-autocorrelation method, the least number of discrete points required is about five; for the spatial-averaging method, the least number of local averages required is greater than six. This suggests that a more refined discretization of the random field is needed for the spatial-averaging method compared with the spatial-autocorrelation method.

Fig. 4.6. Design undrained shear strength values with depth computed using spatial-autocorrelation approach:
(a) serviceability limit state; (b) ultimate limit state

Fig. 4.7. Design undrained shear strength values with depth computed using spatial-averaging approach:
(a) serviceability limit state; (b) ultimate limit state

- (d) Fig. 4.8a and 4.8b show the computed reliability indices β as a function of autocorrelation distance r_o for the spatial-autocorrelation method. The results obtained based on 6, 11, 21 and 41 discrete points are shown. The curve corresponding to 41 discrete points is taken as the benchmark for comparison. Provided that the autocorrelation distance r_o is greater than the separation distance s , the spatial-autocorrelation method obtains the correct β index. For instance, the discretization based on 6 discrete points has a uniform separation distance of $s = 2.5$ m; the 6-discrete-point curve practically coincides with the 41-discrete-point curve for $r_o \geq 2.5$ m. This is consistent with the findings in Low (2001). For $r_o < 2.5$ m the 6-discrete-point curve falls below the 41-discrete-point curve, suggesting that variance reduction may be needed in order to obtain the correct β index. The extent of variance reduction depends on the autocorrelation distance r_o .
- (e) Fig. 4.9a and 4.9b show the computed reliability indices β as a function of autocorrelation distance r_o for the spatial-averaging method. The results obtained based on 5, 10, 20 and 40 local averages are shown. The curve corresponding to 40 local averages is taken as the benchmark for comparison. The accuracy of the spatial averaging method is (unlike the spatial-autocorrelation method) less dependent on the autocorrelation distance r_o than the number of local averages n_a used. For instance, increasing n_a from 5 to 20 local averages brings the computed β indices to be virtually identical to the 40-local-average curve. That the spatial-averaging method requires a more refined discretization of the random field compared with the spatial-autocorrelation method for the case in hand was also noted and discussed in paragraph (c) based on observations on the design-point profile.

Fig. 4.8. Reliability indices as functions of autocorrelation distance computed using spatial-autocorrelation approach: (a) serviceability limit state; (b) ultimate limit state

Fig. 4.9. Reliability indices as functions of autocorrelation distance computed using spatial-averaging approach:
(a) serviceability limit state; (b) ultimate limit state

4.7.2. Effect of probability distribution of random variables

The probability distributions of the underlying random variables may affect the behavior of the limit-state surface in the rotated and transformed u -space. Limit state functions that are linear in the original x -space may exhibit nonlinearity in u -space in cases where the basic random variables are non-normal. A classical example in structural mechanics is the failure in a plastic collapse mechanism of a one-bay frame, in which the limit state function is linear but contains lognormal variables. On the other hand, a nonlinear limit state function in x -space may become somewhat linear in u -space depending on the assumptions made with regards to the probability distributions of the basic random variables.

Probabilistic analyses were carried out on the laterally-loaded pile in marine clay at Sabine for four scenarios with different assumptions on the probability distributions of P_{top} , M_{top} (load parameters) and c_u (resistance parameters): (1) load parameters – Gumbel, and resistance parameters – Lognormal; (2) load parameters – Normal, and resistance parameters – Lognormal; (3) load parameters – Gumbel, and resistance parameters – Normal; (4) load parameters – Normal, and resistance parameters – Normal. The vertical variation of c_u of the marine clay is modeled by six discrete points that are spatially auto-correlated according to the exponential autocorrelation function with $r_o = 2.5$ m. The following points may be noted:

- (a) In scenario ‘1’ the probability distributions of the basic random variables are non-normal (Fig. 4.10). The first-order reliability indices are $\beta_{F,1} = 1.5568$ and $\beta_{F,2} = 1.7907$ corresponding to $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$, respectively. The inferred probabilities of failure, $(p_f)_{F,1} = 0.0598$ and $(p_f)_{F,2} = 0.0367$, are comparable with the average probabilities of failure, $(p_f)_1 = 0.0615$ and $(p_f)_2 = 0.0362$, obtained based on five Monte Carlo simulations with importance sampling (five sets of 1000 samples were drawn for each limit state). This implies that the limit-state surfaces in u -space are more or less linear at their respective design points.

Fig. 4.10. Failure probabilities computed based on various assumptions with respect to probability distributions for load and resistance parameters: (a) serviceability limit state; (b) ultimate limit state

- (b) The behavior of the limit state surfaces at the design points in x -space can be determined with scenario '4' in which the probability distributions of the basic random variables are normal (Fig. 4.10a, b). The first-order reliability indices are $\beta_{F,1} = 1.6783$ and $\beta_{F,2} = 2.0877$ corresponding to $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$, respectively. The inferred probabilities of failure, $(p_f)_{F,1} = 0.0466$ and $(p_f)_{F,2} = 0.0184$, are compared with the average probabilities of failure, $(p_f)_1 = 0.0522$ and $(p_f)_2 = 0.0191$, obtained based on five Monte Carlo simulations with importance sampling (five sets of 1000 samples were drawn for each limit state). For $g_1(\mathbf{x})$ the first-order approximation based on the hyperplane assumption has a significant error [compare $(p_f)_{F,1}$ and $(p_f)_1$, the relative percentage error is -10.5%]. This implies that the limit-state surface $g_1(\mathbf{x})$ in original space is nonlinear at the design point. On the other hand, for $g_2(\mathbf{x})$ the first-order approximation is reasonably accurate. This implies that the limit-state surface $g_2(\mathbf{x})$ in original space is close to linear at the design point.
- (c) Additional scenarios '2' and '3' are useful for understanding the relative influence of the probability distributions of the load and resistance parameters with respect to the behavior of the limit-state surface $g_1(\mathbf{x})$ (Fig. 4.10a). In scenario '2' only the load parameters are normally distributed, whereas in scenario '3' only the resistance parameters are normally distributed. For scenario '2' the first-order reliability index is $\beta_{F,1} = 1.6446$ and the inferred probability of failure is $(p_f)_{F,1} = 0.0500$, which agrees well with the average probability of failure $(p_f)_1 = 0.0495$ obtained based on five Monte Carlo simulations (each with a sample size of 1000 based on importance sampling). The implied linearity of the limit-state surface is consistent with the observation based on scenario '1'. For scenario '3' the first-order reliability index is $\beta_{F,1} = 1.5890$ and the inferred probability of failure is $(p_f)_{F,1} = 0.0560$, which is significantly lower than the average probability of failure $(p_f)_1 = 0.0636$ obtained based on five Monte Carlo simulations (each with a sample size of 1000 based on importance sampling). The implied nonlinearity of the limit-state surface

is consistent with the observation based on scenario '4'. The outcomes of scenarios '2' and '3' show that the change in behavior of the limit-state surface (from nonlinear in x -space to almost linear in u -space) is mainly due to the lognormal resistance parameters.

- (d) The behavior of the limit-state surface due to non-symmetric (skewed) probability distributions is examined, by representing c_u using a variety of beta distributions and assuming that P_{top} and M_{top} are normally distributed. The general beta distribution is described by four parameters: λ , ζ , min and max. The shape parameters λ and ζ control the skewness of the distribution: for $\lambda = \zeta$, the probability density is symmetric; for $\lambda < \zeta$, the probability density is positively skewed; and for $\lambda > \zeta$, the probability density is negatively skewed. The lower and upper bounds of the distribution are given by 'min' and 'max' respectively. Therefore, the five scenarios A to E show a progressive shift from positive skew to negative skew in the probability distribution of c_u , with scenario C being a special case in which c_u is normally distributed (Fig. 4.11). The failure probability obtained based on FORM is higher than that obtained based on Monte Carlo simulations in scenarios A and B, but is lower than that obtained based on Monte Carlo simulation in scenarios C, D and E. This implies that the general curvature of the limit-state surface in u -space is convex with respect to the origin for the two scenarios with positively-skewed beta distributions, and is concave with respect to the origin for the two scenarios with negatively-skewed beta distributions. When c_u is normally distributed the limit-state surface is also concave with respect to the origin.
- (e) In all the scenarios investigated in Fig. 4.10 and Fig. 4.11 the probabilities of failure obtained based on SORM show excellent agreement with those obtained based on the Monte Carlo method with importance sampling.

Scenario	ProbDist of c_u
A	$\text{Beta}(1.5, 5, \mu - 1.5\sigma, \mu + 5\sigma)$
B	$\text{Beta}(2, 4, \mu - 2\sigma, \mu + 4\sigma)$
C	Normal
D	$\text{Beta}(4, 2, \mu - 4\sigma, \mu + 2\sigma)$
E	$\text{Beta}(5, 1.5, \mu - 5\sigma, \mu + 1.5\sigma)$

Fig. 4.11. Failure probabilities computed based on various assumptions with respect to skewness of probability distributions:
(a) serviceability limit state; (b) ultimate limit state

4.7.3. Effect of coefficient of variation of random variables

Probabilistic analyses were repeated for the laterally-loaded pile in marine clay at Sabine using different coefficient of variation (COV) values for the underlying random variables. The original statistical assumptions on the underlying random variables are used: Gumbel distributions for P_{top} and M_{top} , and lognormal distributions for c_u (which is modeled by six discrete points that are spatially auto-correlated according to the exponential autocorrelation function with $r_o = 2.5$ m). The mean values of P_{top} , M_{top} and c_u remain as 60 kN, 20 kNm and 14.4 kPa, respectively. The default COV is 0.2, except for the changing parameter whose COV is varied over the range of 0.1 – 0.5.

The following points may be noted:

- (a) That the probability of failure increases with the coefficient of variation is to be expected because the COV is a measure of the dispersion of the probability distribution and a high COV leads to more uncertainty underlying the parameter than a low COV.
- (b) The increase in the probability of failure is almost proportional with the COV of P_{top} for both $g_1(\mathbf{x})$ (Fig. 4.12a) and $g_2(\mathbf{x})$ (Fig. 4.12b). The failure probabilities obtained based on FORM and SORM are practically indistinguishable from those obtained based on the Monte Carlo method with importance sampling. This implies that the limit-state surfaces in u -space remain more or less linear even though the COV of P_{top} may have increased.

Fig. 4.12. Failure probabilities as functions of coefficient of variation of applied load: (a) serviceability limit state; (b) ultimate limit state

Fig. 4.13. Failure probabilities as functions of coefficient of variation of applied moment: (a) serviceability limit state; (b) ultimate limit state

- (c) The probability of failure also increases proportionately with the COV of M_{top} . Corresponding to $g_1(\mathbf{x})$ (Fig. 4.13a), the failure probabilities obtained based on Monte Carlo simulations are consistently higher than those obtained based on FORM (the disparity is in fact quite small but exaggerated due to the vertical scale adopted in the figure). The difference between the failure probabilities increases with the COV of M_{top} . This suggests a progressive increase in the curvature of the limit-state surface $g_1(\mathbf{x})$ in u -space with the COV of M_{top} . Corresponding to $g_2(\mathbf{x})$ (Fig. 4.13b), the failure probabilities obtained based on Monte Carlo simulations are lower than those obtained based on FORM for COV of M_{top} less than 0.3, and are higher than those obtained based on FORM for COV of M_{top} greater than 0.3. This suggests that the sign of the curvature of the limit-state surface $g_2(\mathbf{x})$ in u -space changes with the COV of M_{top} .
- (d) The probability of failure increases with the COV of c_u with respect to $g_1(\mathbf{x})$ (Fig. 4.14a). There is a gradual increase in the curvature of the limit-state surface $g_1(\mathbf{x})$ in u -space with the COV of c_u because the difference between the failure probabilities obtained based on Monte Carlo simulations and those obtained based on FORM increases with the COV of c_u . The Monte Carlo failure probability tends to be higher than the FORM failure probability. On the other hand, the curvature of the limit-state surface $g_2(\mathbf{x})$ in u -space is observed to increase rapidly with the COV of c_u (Fig. 4.14b) as shown by the divergence of the curves representing FORM and Monte Carlo simulation results. The Monte Carlo failure probability tends to be lower than the FORM failure probability.
- (e) In all the cases examined in Fig. 4.12, Fig. 4.13 and Fig. 4.14 the probabilities of failure obtained based on SORM show excellent agreement with those obtained based on the Monte Carlo method.

Fig. 4.14. Failure probabilities as functions of coefficient of variation of undrained shear strength: (a) serviceability limit state; (b) ultimate limit state

4.7.4. Effect of correlation of random variables

Probabilistic analyses were performed for the laterally-loaded pile in marine clay at Sabine for different correlation coefficients between P_{top} and M_{top} . The original statistical assumptions on the underlying random variables are used: Gumbel distributions for P_{top} and M_{top} , and lognormal distributions for c_u (which is modeled by six discrete points that are spatially auto-correlated according to the exponential autocorrelation function with $r_o = 2.5$ m). The mean values of P_{top} , M_{top} and c_u remain as 60 kN, 20 kNm and 14.4 kPa, respectively. The default COV is 0.2. The correlation between P_{top} and M_{top} is varied within the range of -1 and 1 .

The following points may be noted:

- (a) The probability of failure increases with the correlation between P_{top} and M_{top} almost proportionately for both $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ (Fig. 4.15a and 4.15b). Positive correlations in the load parameters result in lower reliability indices (and higher failure probabilities) than zero or negative correlations.
- (b) For all permissible values of the correlation coefficient, the failure probabilities obtained based on the Monte Carlo method are consistently higher than those obtained based on FORM for $g_1(\mathbf{x})$; the failure probabilities obtained based on the Monte Carlo method are consistently lower than those obtained based on FORM for $g_2(\mathbf{x})$. This implies that the sign of the curvature of the limit-state surface in u -space is not influenced by the correlation between P_{top} and M_{top} .
- (c) In all the cases examined in Fig. 4.15 the probabilities of failure obtained based on SORM show excellent agreement with those obtained based on the Monte Carlo method with importance sampling.

Fig. 4.15. Failure probabilities as functions of correlation between applied load and applied moment: (a) serviceability limit state; (b) ultimate limit state

If desired, the original correlation matrix ρ_{ij} can be modified to ρ'_{ij} in line with the equivalent normal transformation, as suggested in Der Kiureghian and Liu (1986). In their paper, a set of semi-empirical formulas for the ratio ρ'_{ij}/ρ_{ij} has been developed for selected two-parameter distributions. Some tables of the ratio ρ'_{ij}/ρ_{ij} are also given in Appendix B2 of Melchers (1999).

According to Der Kiureghian and Liu (1986), two-parameter distributions that are reducible to standard forms are denoted as Group 1 distributions; those that are not reducible are denoted as Group 2 distributions. Group 1 distributions include uniform, shifted exponential, shifted Rayleigh, Type-I largest value (Gumbel) and Type-I smallest value distributions. Group 2 distributions include lognormal, gamma, Type-II largest value (Frechet) and Type-III smallest value (Weibull) distributions. Five categories of formulas were developed as follows:

- (i) For normal x_i and x_j belonging to Group 1 distributions, the ratio ρ'_{ij}/ρ_{ij} is a constant;
- (ii) For normal x_i and x_j belonging to Group 2 distributions, the ratio ρ'_{ij}/ρ_{ij} is a function of COV_j ;
- (iii) For x_i and x_j both belonging to Group 1 distributions, the ratio ρ'_{ij}/ρ_{ij} is a function of ρ_{ij} ;
- (iv) For x_i belonging to Group 1 distributions and x_j belonging to Group 2 distributions, the ratio ρ'_{ij}/ρ_{ij} is a function of ρ_{ij} and COV_j ;
- (v) For x_i and x_j both belonging to Group 2 distributions, the ratio ρ'_{ij}/ρ_{ij} is a function of ρ_{ij} , COV_i and COV_j ,

where COV_i and COV_j are the coefficients of variation of x_i and x_j , respectively. The formulas established by Der Kiureghian and Liu (1986) were derived based on least-square fitting of polynomial expressions to exact values of ρ'_{ij}/ρ_{ij} obtained by numerical integration. The range of coefficients of variation used in generating the formulas is 0.1 to 0.5. Applying the formulas outside this range is generally not advisable.

The ‘worst-case’ scenario of each category can be identified by trial-and-error. The ‘worst case’ is defined as the combination of the probability distributions of x_i and x_j , and associated values of ρ_{ij} , COV_i and COV_j , that maximize the modification ratio ρ'_{ij}/ρ_{ij} in each category:

- (i) Normal x_i and shifted exponential x_j ;
- (ii) Normal x_i and Frechet x_j , with $\text{COV}_j = 0.5$;
- (iii) Shifted exponential x_i and shifted exponential x_j , with $\rho_{ij} = -0.6$;
- (iv) Type-I smallest value x_i and Frechet x_j , with $\rho_{ij} = 0.6$ and $\text{COV}_j = 0.5$;
- (v) Frechet x_i and Frechet x_j , with $\rho_{ij} = -0.5$, $\text{COV}_i = 0.5$ and $\text{COV}_j = 0.5$.

Although the case of the laterally-loaded pile at Sabine does not involve any of the probability distributions mentioned in the ‘worst cases’ above, a series of probabilistic analyses were carried out nonetheless to compare the failure probabilities computed based on original correlation matrix and those based on the modified correlation matrix. The probability distributions for P_{top} , M_{top} and c_u and their mean values remain as per the original statistical assumptions. The COV is increased to 0.5. The (original) correlation between P_{top} and M_{top} is varied within the range of -1 and 1 .

The following points may be noted:

- (a) The probabilities of failure, as obtained based on FORM and SORM, are generally larger as a result of the high COV used in all the random variables (Fig. 4.16a and Fig. 4.16b). The value of the COV also affects the modified correlation coefficients among the six spatially-auto-correlated lognormal random variables representing c_u such that the modification ratio increases with the COV. On the other hand, the modified correlation coefficient between the Gumbel random variables representing P_{top} and M_{top} is a function of the original correlation coefficient. The modification ratio is higher for negatively-correlated than for positively-correlated P_{top} and M_{top} . Note that no solution is obtained

for the case where the original correlation coefficient is -0.9 because the modified correlation coefficient is less than -1 .

- (b) The failure probabilities obtained based on the modified correlation matrices are practically the same as those obtained based on the original correlation matrices because the modification ratios associated with Gumbel and lognormal distributions are relatively low. Had P_{top} and M_{top} been represented by the shifted exponential distribution and c_u been represented by the Frechet distribution, the outcome could have been otherwise. The modified correlation coefficients have almost negligible effect on the probabilities of failure inferred based on FORM. Some difference is observed in the SORM-approximated probabilities of failure between those obtained based on the original correlation coefficients and those obtained based on the modified correlation coefficients but the disparity is quite insignificant.

4.7.5. Sensitivity analysis

Sensitivity analyses were performed for the laterally-loaded pile in marine clay at Sabine based on the proposed sensitivity factors for correlated two-parameter probability distributions. The procedures are illustrated in Appendix D. The original statistical assumptions on the underlying random variables are used: Gumbel distributions for P_{top} and M_{top} , and lognormal distributions for c_u (which is modeled by six discrete points that are spatially auto-correlated according to the exponential autocorrelation function with $r_o = 2.5$ m). The mean values of P_{top} , M_{top} and c_u remain as 60 kN, 20 kNm and 14.4 kPa, respectively. The default COV is 0.2.

Fig. 4.16. Failure probabilities as functions of correlation between applied load and applied moment computed based on original and modified correlation matrices: (a) serviceability limit state; (b) ultimate limit state

The following points may be noted:

- (a) The sensitivity factor of the reliability index $\partial\beta/\partial x'_i$ in Eq. (4.16) is validated by performing numerical differentiation on the following expression for the reliability index (Low and Tang 2007): $\beta = [(\mathbf{x}'^*)^T \mathbf{R}^{-1}(\mathbf{x}'^*)]^{0.5}$. Table 4.4 shows that the proposed sensitivity factor and that based on numerical differentiation are identical up to the fourth decimal place. For the present example it can be shown that the maximum absolute relative error of $\partial\beta/\partial x'_i$ is only about 0.002%.
- (b) The advantage of the sensitivity factor $\partial\beta/\partial x'_i$ is in two aspects: (1) that the unrotated standard equivalent normal variable x'_i is related directly to the original physical quantity x_i means that $\partial\beta/\partial x'_i$ has more physical meaning than $\partial\beta/\partial u_i$ for correlated random variables; (2) that the standard equivalent normal variable x'_i is dimensionless means that the $\partial\beta/\partial x'_i$'s are comparable even if the original random variables x_i 's are of different units. The evaluation of $\partial\beta/\partial x'_i$ by Eq. (4.16) is also simple because computation of the equivalent normal means and standard deviations are not required.
- (c) The sensitivities of the reliability index with respect to the means and standard deviations of the stochastic parameters are computed using Eq. (4.23a) and (4.23b), and compared with numerical differentiation based the actual changes in the reliability index corresponding to small perturbations in the means and standard deviations. In other words the numerical differentiation involves additional Solver runs for obtaining new design points and reliability indices. The proposed sensitivity factors, namely $\partial\beta/\partial\mu_i$ and $\partial\beta/\partial\sigma_i$ are observed to agree very well with the sensitivities obtained by numerical differentiation based on FORM, as shown in Table 4.5. The strong consistency is due to $\partial\beta/\partial x'_i$, whose computation [by Eq. (4.16)] involves only first-order reliability results in the form of the direction cosines α_i . The excellent agreement with FORM implies that the approximations of $\partial x'_i/\partial\mu_i$ and $\partial x'_i/\partial\sigma_i$ by $\Delta x'_i/\Delta\mu_i$ and

$\Delta x'_i/\Delta \sigma_i$ respectively introduce very little error, even though the computation of the two latter quantities assumes that the design point remains unchanged. Eq. (4.23a) and (4.23b) are also efficient because they provide sensitivity information based on the original design point and reliability index without the need for further reliability computations.

- (d) The sensitivity factors $\partial\beta/\partial\mu_i$ and $\partial\beta/\partial\sigma_i$ computed based on the proposed approximate method are not dimensionless; they carry the same units (or rather the inverse of the same units) as the basic random variables x_i . The sensitivity factors $\partial\beta/\partial\mu_i$ and $\partial\beta/\partial\sigma_i$ of P_{top} , M_{top} and c_u are measured in different units. Comparison is therefore of little meaning (if any) between sensitivity factors that are specified in different units. A sensitivity factor that is measured in terms of kN^{-1} , for example, is not comparable with another that is measured in terms of kPa^{-1} . The exception is when the two underlying random variables have identical base units. If, for some reason, two c_u quantities are specified such that $c_{u,1}$ is in kPa and $c_{u,2}$ is in ksi, then the sensitivity factor 0.013 kPa^{-1} of $c_{u,1}$ is comparable with the sensitivity factor 475.755 ksi^{-1} of $c_{u,2}$ when the latter value is converted to 0.069 kPa^{-1} ($1 \text{ ksi}^{-1} = 6895 \text{ kPa}^{-1}$).
- (e) The sensitivity factors $\partial\beta/\partial\mu_i$ and $\partial\beta/\partial\sigma_i$ can arbitrarily be rendered dimensionless by multiplying with the respective σ_i (Der Kiureghian and Ke 1985; Liu et al. 1989) to allow random variables with dissimilar units to be compared.
- (f) If the sensitivity factor $\partial\beta/\partial\mu_i$ or $\partial\beta/\partial\sigma_i$ of a certain random variable x_i is positive, it is implied that the reliability index increases when μ_i or σ_i of x_i is increased. The reverse is also true. In general, the sensitivity factors $\partial\beta/\partial\mu_i$ are negative for P_{top} and M_{top} but are positive for $c_{u,k}$ ($k = 1, 2, \dots, 6$). This shows that β decreases when the mean value of a load parameter is increased, but increases when the mean value of a resistance parameter is increased. A higher mean load (or mean strength) usually results in a

lower (or higher) reliability index. All the sensitivity factors $\partial\beta/\partial\sigma_i$ are negative which indicate that β decreases when the standard deviation of any (regardless of load or resistance) parameter is increased. A higher variance in the underlying random variables usually leads to a lower reliability index.

- (g) It is possible to infer the sensitivity of the failure probability p_f from the sensitivity of the reliability index β based on Eq. (4.26). One would expect that, unless the performance function is perfectly planar, Eq. (4.26) is at best approximate because: (1) the direction cosines α_i 's used in Eq. (4.16) convey no information with regards to the curvature of the performance function at the design point, and (2) the hyperplane assumption that underlies Eq. (4.26) holds for linear performance functions only. The accuracy of the inferred sensitivity factors of the failure probability is examined using Monte Carlo simulations with importance sampling. By repeating the simulation twice with correlated samples one may estimate the actual changes in the failure probability p_f corresponding to small perturbations in the means and standard deviations and obtain values for $\Delta p_f/\Delta\mu_i$ and $\Delta p_f/\Delta\sigma_i$.
- (h) The sensitivities of the failure probability with respect to the means and standard deviations of the stochastic parameters are compared in Table 4.6. The errors of $\partial p_f/\partial\mu_i$ relative to $\Delta p_f/\Delta\mu_i$ are fairly small, and the average absolute relative error is about 6%. On the other hand, the errors of $\partial p_f/\partial\sigma_i$ relative to $\Delta p_f/\Delta\sigma_i$ are significant, with an average absolute relative error of about 40%. Such discrepancies are mainly due to changes in the non-linearity of the performance function corresponding to changes in the means and standard deviations of the random variables. That larger percentage errors are observed for $\partial p_f/\partial\sigma_i$ compared to $\partial p_f/\partial\mu_i$ implies that unit changes in the σ_i 's tend to alter the non-linearity of the performance function more than unit changes in the μ_i 's. More discussion on this issue is given as follows.

Further investigation is carried out below pertaining to the observations described in paragraph (h). Let p_f denote the probability of failure and $p_{f,FORM}$ be the first-order approximation of p_f . Unless the limit state function is linear and the random variables are all normally distributed, there will be a discrepancy between p_f and $p_{f,FORM}$ given by the error term e_f , which may be regarded as a measure of the nonlinearity of the limit state surface at the design point:

$$e_f = p_{f,FORM} - p_f \quad (4.26)$$

The sensitivity of p_f with respect to some statistical parameter Θ (where Θ can be either μ or σ) can be obtained by the central difference:

$$\frac{\partial p_f}{\partial \Theta} \approx \frac{(p_f)_{\Theta+\Delta\Theta} - (p_f)_{\Theta-\Delta\Theta}}{2\Delta\Theta} \quad (4.27)$$

where $\Delta\Theta$ is a small perturbation of Θ . Based on Eq. (4.26), Eq. (4.27) can be re-expressed as:

$$\begin{aligned} \frac{\partial p_f}{\partial \Theta} &\approx \frac{[(p_{f,FORM}) - e_f]_{\Theta+} - [(p_{f,FORM}) - e_f]_{\Theta-}}{2\Delta\Theta} \\ &= \frac{(p_{f,FORM})_{\Theta+} - (p_{f,FORM})_{\Theta-}}{2\Delta\Theta} - \frac{(e_f)_{\Theta+} - (e_f)_{\Theta-}}{2\Delta\Theta} \end{aligned} \quad (4.28)$$

where $\Theta+ = \Theta + \Delta\Theta$ and $\Theta- = \Theta - \Delta\Theta$. The first term on the right hand side of Eq. (4.28) is simply the sensitivity of p_f estimated based on the hyperplane assumption and is equivalent to $-\phi(\beta) \cdot \partial\beta/\partial\Theta$. The second term is naturally the error term of the approximate sensitivity of p_f based on FORM. If the nonlinearity of the limit state surface at the design point does not change significantly due to the small perturbation $\Theta \pm \Delta\Theta$, then the numerator of the second term $(e_f)_{\Theta+} - (e_f)_{\Theta-}$ is practically zero. In this case, the true sensitivity of p_f is well estimated by FORM. On the other hand, if the nonlinearity of the limit state surface *does* change

significantly, the second term in Eq. (4.28) will not be zero. In this case, one would observe discrepancy between the true sensitivity of p_f and the approximation by FORM.

Therefore, Eq. (4.28) can be used to explain the phenomenon observed from Table 4.6 and described in paragraph (h). The error of the FORM β -inferred $\partial p_f / \partial \mu_i$ relative to $\Delta p_f / \Delta \mu_i$ obtained by MCS is small. This implies that for small perturbations of μ_i , the change in nonlinearity of the limit state surface is small. On the other hand, the large error of the FORM β -inferred $\partial p_f / \partial \sigma_i$ relative to $\Delta p_f / \Delta \sigma_i$ obtained by MCS must be due to significant changes in nonlinearity of the limit state surface corresponding to small perturbations of σ_i . That being said, Eq. (4.28) does not in any way suggest that for any limit state surface, unit changes in the σ_i 's will always cause larger changes to its nonlinearity compared to unit changes in the μ_i 's.

- (i) Numerical differentiation based on SORM is an alternative to provide a better approximation for the sensitivity of the failure probability. The sensitivities $\partial p_f / \partial \sigma_i$ for $c_{u,1}$ were 4.48×10^{-4} for $g_1(x)$ and 2.86×10^{-4} for $g_2(x)$. The above values, when compared with the true sensitivities obtained based on Monte Carlo simulations (2.78×10^{-4} and 1.70×10^{-4} , respectively, which are computed based on importance sampling with 100,000 samples in each case), had very large errors of 61% and 68%, respectively. The sensitivities computed by numerical differentiation based on SORM are 2.84×10^{-4} and 1.76×10^{-4} for $g_1(x)$ and $g_2(x)$, respectively, which have relatively much smaller errors of 2% and 4%.

Table 4.4. Comparison between sensitivities obtained using the proposed method and those obtained using numerical differentiation

I. Sensitivity factors for $g_1(\mathbf{x})$

Sensitivity	x_i							
factors for $g_1(\mathbf{x})$	P_{top}	M_{top}	$c_{u,1}$	$c_{u,2}$	$c_{u,3}$	$c_{u,4}$	$c_{u,5}$	$c_{u,6}$
$\partial\beta/\partial x'_i$	0.9037	0.0307	-0.0302	-0.2266	-0.2164	-0.0304	-0.0234	-0.0244
$\Delta\beta/\Delta x'_i$	0.9037	0.0307	-0.0302	-0.2266	-0.2164	-0.0304	-0.0234	-0.0244

II. Sensitivity factors for $g_2(\mathbf{x})$

Sensitivity	x_i							
factors for $g_2(\mathbf{x})$	P_{top}	M_{top}	$c_{u,1}$	$c_{u,2}$	$c_{u,3}$	$c_{u,4}$	$c_{u,5}$	$c_{u,6}$
$\partial\beta/\partial x'_i$	0.9520	0.0449	-0.0290	-0.1568	-0.0871	-0.0108	-0.0054	-0.0077
$\Delta\beta/\Delta x'_i$	0.9520	0.0449	-0.0290	-0.1568	-0.0871	-0.0108	-0.0054	-0.0077

Table 4.5. Comparison between sensitivities w.r.t. mean and standard deviation obtained using the proposed method and those obtained using numerical differentiation based on FORM

I. Sensitivity factors for $g_1(\mathbf{x})$

Sensitivity factors for $g_1(\mathbf{x})$	x_i							
	P_{top}	M_{top}	$c_{u,1}$	$c_{u,2}$	$c_{u,3}$	$c_{u,4}$	$c_{u,5}$	$c_{u,6}$
$\partial\beta/\partial\mu_i$	-0.0475	-0.0064	0.0115	0.0908	0.0866	0.0116	0.0088	0.0090
$\Delta\beta/\Delta\mu_i$	-0.0475	-0.0064	0.0115	0.0909	0.0866	0.0116	0.0088	0.0091
$\partial\beta/\partial\sigma_i$	-0.0801	-0.0043	-0.0045	-0.0569	-0.0539	-0.0047	-0.0027	-0.0024
$\Delta\beta/\Delta\sigma_i$	-0.0801	-0.0043	-0.0045	-0.0569	-0.0538	-0.0047	-0.0027	-0.0024

II. Sensitivity factors for $g_2(\mathbf{x})$

Sensitivity factors for $g_2(\mathbf{x})$	x_i							
	P_{top}	M_{top}	$c_{u,1}$	$c_{u,2}$	$c_{u,3}$	$c_{u,4}$	$c_{u,5}$	$c_{u,6}$
$\partial\beta/\partial\mu_i$	-0.0460	-0.0088	0.0109	0.0610	0.0334	0.0040	0.0020	0.0028
$\Delta\beta/\Delta\mu_i$	-0.0460	-0.0088	0.0110	0.0610	0.0334	0.0039	0.0020	0.0027
$\partial\beta/\partial\sigma_i$	-0.0955	-0.0075	-0.0037	-0.0301	-0.0142	-0.0012	-0.0005	-0.0006
$\Delta\beta/\Delta\sigma_i$	-0.0955	-0.0075	-0.0038	-0.0301	-0.0142	-0.0012	-0.0005	-0.0006

Table 4.6. Comparison between inferred sensitivities of failure probability w.r.t. mean and standard deviation, and those obtained using numerical differentiation based on Monte Carlo simulation

I. Sensitivity factors for $g_1(\mathbf{x})$

Sensitivity factors for $g_1(\mathbf{x})$	x_i							
	P_{top}	M_{top}	$c_{u,1}$	$c_{u,2}$	$c_{u,3}$	$c_{u,4}$	$c_{u,5}$	$c_{u,6}$
$\partial p_f / \partial \mu_i$	4.72 $\times 10^{-3}$	6.31 $\times 10^{-4}$	-1.14 $\times 10^{-3}$	-9.01 $\times 10^{-3}$	-8.60 $\times 10^{-3}$	-1.15 $\times 10^{-3}$	-8.68 $\times 10^{-4}$	-8.95 $\times 10^{-4}$
$\Delta p_f / \Delta \mu_i$	5.17 $\times 10^{-3}$	6.68 $\times 10^{-4}$	-1.24 $\times 10^{-3}$	-9.79 $\times 10^{-3}$	-9.23 $\times 10^{-3}$	-1.19 $\times 10^{-3}$	-9.25 $\times 10^{-4}$	-9.00 $\times 10^{-4}$
$\partial p_f / \partial \sigma_i$	7.95 $\times 10^{-3}$	4.26 $\times 10^{-4}$	4.48 $\times 10^{-4}$	5.64 $\times 10^{-3}$	5.34 $\times 10^{-3}$	4.68 $\times 10^{-4}$	2.68 $\times 10^{-4}$	2.35 $\times 10^{-4}$
$\Delta p_f / \Delta \sigma_i$	8.37 $\times 10^{-3}$	5.55 $\times 10^{-4}$	2.78 $\times 10^{-4}$	5.26 $\times 10^{-3}$	7.96 $\times 10^{-3}$	7.54 $\times 10^{-4}$	4.00 $\times 10^{-4}$	5.01 $\times 10^{-4}$

II. Sensitivity factors for $g_2(\mathbf{x})$

Sensitivity factors for $g_2(\mathbf{x})$	x_i							
	P_{top}	M_{top}	$c_{u,1}$	$c_{u,2}$	$c_{u,3}$	$c_{u,4}$	$c_{u,5}$	$c_{u,6}$
$\partial p_f / \partial \mu_i$	3.52 $\times 10^{-3}$	6.76 $\times 10^{-4}$	-8.37 $\times 10^{-4}$	-4.67 $\times 10^{-3}$	-2.56 $\times 10^{-3}$	-3.08 $\times 10^{-4}$	-1.52 $\times 10^{-4}$	-2.16 $\times 10^{-4}$
$\Delta p_f / \Delta \mu_i$	3.46 $\times 10^{-3}$	7.12 $\times 10^{-4}$	-8.74 $\times 10^{-4}$	-4.84 $\times 10^{-3}$	-2.28 $\times 10^{-3}$	-2.80 $\times 10^{-4}$	-1.58 $\times 10^{-4}$	-1.97 $\times 10^{-4}$
$\partial p_f / \partial \sigma_i$	7.31 $\times 10^{-3}$	5.75 $\times 10^{-4}$	2.86 $\times 10^{-4}$	2.31 $\times 10^{-3}$	1.08 $\times 10^{-3}$	9.07 $\times 10^{-5}$	3.62 $\times 10^{-5}$	4.64 $\times 10^{-5}$
$\Delta p_f / \Delta \sigma_i$	6.90 $\times 10^{-3}$	7.00 $\times 10^{-4}$	1.70 $\times 10^{-4}$	1.33 $\times 10^{-3}$	1.96 $\times 10^{-3}$	2.24 $\times 10^{-4}$	8.49 $\times 10^{-5}$	1.15 $\times 10^{-4}$

4.7.6. Comparison with regression and correlation coefficients

The Excel add-in @RISK (www.palisade.com) can perform sensitivity analysis using Monte Carlo simulations. Two methods are used, namely multivariate stepwise regression and rank order correlation. In multivariate stepwise regression the input data is fitted to a planar equation that can produce the output data. The sensitivity values returned by @RISK are normalized regression coefficients associated with each input. In rank order correlation the relationship between two sets of data is investigated by comparing the rank of each value in the data sets. The sensitivity values returned by @RISK are Spearman's rank correlation coefficients. Both regression coefficients and rank correlation coefficients measure the sensitivity of the performance function with respect to the random variables, and are therefore fundamentally different from the sensitivity of the reliability index. And since the coefficients are computed based on a large number of samples, they represent the global behavior of the performance function and not the local behavior at the design point. While it is not expected that the proposed sensitivity factor $\partial\beta/\partial x'_i$ would be numerically similar to the regression or correlation coefficients, it is both interesting and useful to observe any similarities between the different measures. In Table 4.7 the numbers in square brackets indicate the rank for the particular coefficient. The ranking is carried out based on descending order of the absolute coefficient values. The regression and correlation coefficients are obtained by @RISK based on 100,000 Latin hypercube samples.

Table 4.7 compares the values of the regression and correlation coefficients with the values of $\partial\beta/\partial x'_i$. In terms of absolute numerical values, there are more similarities between the regression coefficients and $\partial\beta/\partial x'_i$, than between the correlation coefficients and $\partial\beta/\partial x'_i$. In terms of ranking for $g_1(\mathbf{x})$, the ranks based on the $\partial\beta/\partial x'_i$ values more closely resemble those based on the correlation coefficients than those based on the regression coefficients. The opposite case is for $g_2(\mathbf{x})$, where in terms of ranking, the ranks based on the $\partial\beta/\partial x'_i$ values more

closely resemble those based on the regression coefficients than those based on the correlation coefficients.

Table 4.7. Comparison between the proposed sensitivity factors and regression and correlation coefficients

I. Sensitivity factors for $g_1(\mathbf{x})$

Sensitivity factors for $g_1(\mathbf{x})$	x_i							
	P_{top}	M_{top}	$c_{u,1}$	$c_{u,2}$	$c_{u,3}$	$c_{u,4}$	$c_{u,5}$	$c_{u,6}$
$\partial\beta/\partial x'_i$	0.9037 [1]	0.0307 [4]	-0.0302 [6]	-0.2266 [2]	-0.2164 [3]	-0.0304 [5]	-0.0234 [8]	-0.0244 [7]
Regress. Coeff.	-0.977 [1]	-0.494 [2]	0.087 [5]	0.174 [3]	0.121 [4]	0.049 [6]	0.026 [7]	0.018 [8]
Correl. Coeff.	-0.747 [1]	-0.394 [4]	0.240 [5]	0.531 [2]	0.450 [3]	0.207 [6]	0.117 [7]	0.074 [8]

II. Sensitivity factors for $g_2(\mathbf{x})$

Sensitivity factors for $g_2(\mathbf{x})$	x_i							
	P_{top}	M_{top}	$c_{u,1}$	$c_{u,2}$	$c_{u,3}$	$c_{u,4}$	$c_{u,5}$	$c_{u,6}$
$\partial\beta/\partial x'_i$	0.9520 [1]	0.0449 [4]	-0.0290 [5]	-0.1568 [2]	-0.0871 [3]	-0.0108 [6]	-0.0054 [8]	-0.0077 [7]
Regress. Coeff.	-0.921 [1]	-0.054 [3]	0.030 [5]	0.163 [2]	0.049 [4]	0.009 [6]	0.005 [8]	0.006 [7]
Correl. Coeff.	-0.792 [1]	-0.034 [6]	0.051 [4]	0.406 [2]	0.230 [3]	0.040 [5]	0.032 [7]	0.030 [8]

4.7.7. Comparison with omission sensitivity factors

The omission sensitivity factor (Madsen 1988) gives the relative error in the reliability index when a random variable is replaced by its deterministic mean value:

$$\gamma_{osf}(\mu_i) = \frac{\beta(x_i = \mu_i)}{\beta} \quad (4.36)$$

where β = reliability index obtained when all uncertain parameters are treated as random variables; and $\beta(x_i = \mu_i)$ = reliability index obtained when one of the uncertain parameter x_i is replaced by its mean value. Unlike the regression coefficients and the rank correlation coefficients shown in the previous section, the omission sensitivity factor is a direct measure of the reliability index, and hence in a way more comparable with the proposed sensitivity factor $\partial\beta/\partial x'_i$. However, by definition the value of γ_{osf} is always greater than or equal to one and therefore always positive. Therefore one cannot, based on γ_{osf} alone, deduce the direction of change of β due to small changes of x_i in the vicinity of the design point, but should instead compare the magnitudes of γ_{osf} and determine from such comparison the relative importance of x_i .

For obvious reasons there is little meaning in comparing the numerical values of the two sensitivity factors per se, as shown in Table 4.8, but it is encouraging to observe that for most x_i 's the rankings (as given in the square brackets) given by the two sets of sensitivity factors agree with each other.

Table 4.8. Comparison between the proposed sensitivity factors and omission sensitivity factors**I. Sensitivity factors for $g_1(\mathbf{x})$**

Sensitivity factors for $g_1(\mathbf{x})$	x_i							
	P_{top}	M_{top}	$c_{u,1}$	$c_{u,2}$	$c_{u,3}$	$c_{u,4}$	$c_{u,5}$	$c_{u,6}$
$\partial\beta/\partial x'_i$	0.9037 [1]	0.0307 [4]	-0.0302 [6]	-0.2266 [2]	-0.2164 [3]	-0.0304 [5]	-0.0234 [8]	-0.0244 [7]
OSF	1.8402 [1]	1.0101 [4]	1.0060 [5]	1.0663 [2]	1.0528 [3]	1.0057 [6]	1.0031 [7]	1.0024 [8]

II. Sensitivity factors for $g_2(\mathbf{x})$

Sensitivity factors for $g_2(\mathbf{x})$	x_i							
	P_{top}	M_{top}	$c_{u,1}$	$c_{u,2}$	$c_{u,3}$	$c_{u,4}$	$c_{u,5}$	$c_{u,6}$
$\partial\beta/\partial x'_i$	0.9520 [1]	0.0449 [4]	-0.0290 [5]	-0.1568 [2]	-0.0871 [3]	-0.0108 [6]	-0.0054 [8]	-0.0077 [7]
OSF	4.9636 [1]	1.0161 [3]	1.0042 [5]	1.0318 [2]	1.0133 [4]	1.0012 [6]	1.00046 [8]	1.00051 [7]

4.7.8. System reliability analysis

System reliability analysis is performed on the laterally-loaded pile in marine clay at Sabine. The first-order and second-order reliability indices for the two failure modes are shown in Fig. 4.17. Failure mode 1 refers to failure due to pile-head deflection exceeding 0.3 m; failure mode 2 refers to failure due to maximum bending moment exceeding 300 kNm. The failure-mode correlation matrix at the top-left corner of Fig. 4.17 is computed based on the direction cosine vectors α_1 and α_2 .

Fig. 4.17 shows system reliability results based on FORM as well as SORM reliability indices. The difference between the two sets of results is small, but as the SORM β leads to solutions that are slightly more comparable with Monte Carlo simulations than the FORM β , the following discussions are made with reference to SORM β .

Based on the second-order reliability indices $\beta_1 = 1.5430$ and $\beta_2 = 1.7953$, and failure-mode correlation coefficient $\rho_{12} = 0.98445$, the value of $p_{f,12} = \Phi_2(-\beta_1, -\beta_2, \rho_{12}) = 0.03571$. By Monte Carlo method the value of $p_{f,12}$ is estimated as 0.0358 (based on five importance sampling trials each with 1000 samples). For comparison the Low et al. (2011) function *ProbFiFj* evaluates the lower and upper bounds of $p_{f,12}$ as 0.03265 and 0.03620, respectively.

The improved bounds based on $p_{f,12}$ computed directly using the Owen (1956) formulas and the second-order reliability indices are coincidental such that $p_{f,sys} = 0.06201$, which is remarkable when compared with the failure probability of 0.0616 from Monte Carlo simulation (based on five importance sampling trials each with 1000 samples). For comparison the Low et al. (2011) functions *SysProbLowerBound* and *SysProbUpperBound* (with *ProbFiFj*) evaluates the lower and upper bounds of $p_{f,sys}$ as 0.06152 and 0.06508, respectively. For cases with more than two failure modes the improved bounds, though not coincidental, are expected to be narrower than the conventional Kounias-Ditlevsen bounds. An example which involves three failure mechanisms of plastic collapse of a plane-frame structure is analyzed in Appendix E. The interesting outcome is that the improved bounds, though very sharp and precise, fail to encompass the true system failure probability, unlike the Kounias-Ditlevsen bounds. It thus brings to light some relative merits and limitations of the two computational methods.

Fig. 4.17. Comparison of system failure probability bounds computed for Sabine test pile considering two failure modes

The proposed bivariate standard normal integral evaluated following Owen (1956) improves the precision of the solution of $p_{f,sys}$ by eliminating the uncertainty associated with the joint failure probability $p_{f,ij}$. The value of $p_{f,ij}$ so obtained is exact if the limit-state surfaces of failure modes i and j are hyperplanes in the vicinity of the design points. Violation of the hyperplane assumption by one or both limit-state surfaces requires more accurate computation of $p_{f,ij}$ based on a second-order approximation, such as Madsen (1985). In other words, the narrower improved bounds may not necessarily provide a more accurate solution (when the limit-state functions are nonlinear) because the fundamental assumption that the joint failure probability is bounded by hyperplanes may not be satisfied, and has little to do with how the bivariate standard normal integral is estimated, by Owen (1956) or others. Therefore to obtain accurate system bounds the non-linearity of the limit-state functions need to be accounted for in the computation of the component failure probabilities as well as the joint failure probabilities. Otherwise the more precise solution may be misconstrued as a more accurate

solution. From this perspective the less precise Kounias-Ditlevsen bounds appear to have a practical advantage even though the uncertainty in the solution does not exactly reflect the nonlinearity of the limit-state functions.

Note that the intermodal correlation coefficient is high ($\rho_{12} = 0.984$) and very close to 1.0. As explained in Low et al. (2011), this implies that the design point vectors are nearly parallel for the two failure modes. Also note that the value of $p_{f,12}$ is 0.0357 (or 3.57%), which is very nearly the same as the value of $p_{f,2}$ (i.e. the less critical of the two failure modes) of 0.0363 (or 3.63%). Statistically, this means that out of 10,000 events, there are about 363 instances of failure in mode 2. And of these 363 instances, 357 instances also register failure in mode 1. This is an indication of a very significant overlapping of the failure probability contents for the two failure modes. Due to the fact that there are few instances in which failure occurs *only* in mode 2 (and *not* in mode 1), the system failure probability $p_{f,sys}$ of 0.0620 (or 6.20%) is only marginally higher than the value of $p_{f,1}$ (i.e. the more critical of the two failure modes) of 0.0614 (or 6.14%).

By setting the test limiting pile-head deflection at 0.1 m, a wider difference is introduced to the state of reliability of the two failure modes. The effect of a much larger difference between the reliability indices of the two failure modes on system reliability is investigated in Fig. 4.18. The SORM reliability index for failure mode 1 is recomputed as 0.5466, which translates to a probability of failure of $p_{f,1} = 0.2923$ (or 29.23%). The value of $p_{f,2} = 0.0363$ (or 3.63%) is unchanged. The intermodal correlation coefficient ρ_{12} is now equal to 0.965. Interestingly, for this case the Low et al. (2011) function *ProbFiFj* yields the same value for the upper and lower bounds for the joint probability of failure $p_{f,12}$ at 0.0363 (or 3.63%). The Owen (1956) equations obtain the same result. Note that the value of $p_{f,12}$ is now exactly equal to the value of $p_{f,2}$. It follows that the system failure probability $p_{f,sys}$ is now equal to the value of $p_{f,1}$, which is 0.2923 (or 29.23%).

Fig. 4.18. Comparison of system failure probability bounds computed for Sabine test pile considering two failure modes

4.7.9. Reliability-based design

In this section, a variant of the case of the Sabine test pile is examined. The modifications are introduced to create a more realistic example. The undrained shear strength c_u profile is now non-uniform, and varies with depth as shown in Fig. 4.19. The limiting pile-head deflection y_{lim} is now set at 0.05 m, which is a more reasonable value for design purposes.

The variation of the reliability index with the mean lateral load is plotted in Fig. 4.20 for pile deflection failure (Perfunc 1) and bending moment failure (Perfunc 2). The mean lateral force to mean applied moment ratio ($\mu_{P_{top}}:\mu_{M_{top}}$) is consistently maintained at 3:1 to reflect the close association between the lateral force P_{top} and the applied moment M_{top} . For example, for $\mu_{P_{top}} = 60$ kN, the corresponding $\mu_{M_{top}} = 20$ kNm; and for $\mu_{P_{top}} = 30$ kN, the corresponding $\mu_{M_{top}} = 10$

kNm. All the while P_{top} and M_{top} are regarded as lognormal random variables with a fixed COV of 20%.

A plot such as Fig. 4.20 can be used to determine the maximum mean load that the pile can withstand such that the state of reliability is maintained at some specified target reliability index. For example, if a target reliability index of not less than 3.0 is desired for pile deflection failure, the maximum allowable mean lateral force and mean applied moment should not exceed 18.2 kN and 6.1 kNm, respectively. And if the same criterion is imposed for bending moment failure, the maximum allowable mean lateral force and mean applied moment should not exceed 38.5 kN and 12.8 kNm, respectively. These values are obtained from Fig. 4.20, by noting the abscissa values that correspond to the intersecting points of the respective curves and the horizontal line that passes through $\beta = 3.0$.

Fig. 4.19. Modified undrained shear strength profile for Sabine test pile

Fig. 4.20. Variation of reliability index with mean lateral load and mean applied moment for pile deflection failure and bending moment failure

A multi-criteria reliability-based design can be carried out to optimize the dimensions of the pile, as was done in Low et al. (2001) and Chan and Low (2009). Some target reliability indices were specified. The appropriate pile wall thickness and pile diameter combination that achieved the desired level of reliability with respect to both ultimate and serviceability limit states was found by plotting the t versus d curves corresponding to the two limit states and locating the point of intersection.

Fig. 4.21 shows a plot of pile wall thickness against pile diameter that is prepared for the case in hand, in which $\mu_{P_{top}} = 30$ kN and $\mu_{M_{top}} = 10$ kNm. The curves represent combinations of the two geometries that correspond to the same reliability index. The curve on top is for $\beta = 3$ with respect to serviceability limit state; the 3 other curves below are for $\beta = 3, 4$ and 5 with respect to ultimate limit state. It can be seen that no intersection takes place between the SLS curve and

the ULS curve for $\beta = 3$ due to the fact that one limit state is much more critical than the other. In this case, SLS is the governing limit state because the pile is more likely to fail as a result of exceeding the limiting pile-head deflection. Therefore for the case in hand, a configuration that yields $\beta = 3$ with respect to both SLS and ULS does not exist but if the SLS curve for $\beta = 3$ is used to provide guidance in selecting a suitable sizing option, it would also ensure that β is greater than 3 with respect to ULS.

A plot of pile length against pile diameter is shown in Fig. 4.22. Again a sizing option that achieves $\beta = 3$ with respect to both SLS and ULS cannot be found, as explained in the previous paragraph. In this case the SLS curve for $\beta = 3$ may be used to determine the appropriate dimensions of the pile.

By increasing the pile diameter from $d = 0.319$ m to $d = 0.46$ m, the target reliability index of 3 can be fulfilled with regards to the serviceability limit state. The design-point values corresponding to $\beta = 3$, when divided by the original mean values, produces the partial factors for the stochastic parameters. The partial factors so obtained are:

$$P_{\text{top}}: 1.914; M_{\text{top}}: 1.349;$$

$$c_{u,1}: 0.870; c_{u,2}: 0.868; c_{u,3}: 0.890; c_{u,4}: 0.939; c_{u,5}: 0.962; c_{u,6}: 0.972$$

The partial factors are able to reflect the relative influence of the applied load P_{top} and applied moment M_{top} . In addition the partial factors for the undrained shear strength indicate the relative importance of the spatially-autocorrelated discrete points at 6 different depths. For practical reasons design codes will usually specify a consistent partial factor for the strength parameter.

The API RP 2A-LRFD suggests that the given load and resistance factors be applied to *nominal* loads and *nominal* resistances, respectively. In the same spirit the Eurocode 7 has also defined the *characteristic* value for a geotechnical parameter and the *characteristic* value for a load effect. The nominal and characteristic values may or may not be equal to the mean values; that decision is

left to the designer. In the case where the nominal/characteristic value deviates from the mean value, the ratio of the mean value to the nominal/characteristic value is called the bias. The selection of the bias is mainly based on the designer's experience and judgment, and his/her perception of how adequate the load and resistance factors or partial factors prescribed by the design code are, with reference to the engineering problem and site conditions at hand. In this way the design code caters for some flexibility in the otherwise constant and rigid load and resistance factors or partial factors. In Eurocode 7 the characteristic value of a geotechnical parameter is "selected as a cautious estimate of the value affecting the occurrence of the limit state" and "if statistical methods are used, the characteristic value should be derived such that the calculated probability of a worse value governing the occurrence of the limit state under consideration is not greater than 5%.". The characteristic value of an action may be specified "as a mean value, an upper or lower value, or a nominal value". Concepts of nominal and characteristic values are not required in reliability-based design. Furthermore, load and resistance factors, partial factors and biases need not be specified beforehand. The load and resistance factors or partial factors are directly related to the most probable failure point which is automatically obtained using a constrained optimization algorithm. An important point to note is that all load and resistance factors and partial factors are derived based on some analytical and statistical models, code-based design and reliability-based design alike. The difference is that the analytical and statistical assumptions underlying code-based factors are usually not obvious to the designer, while reliability-based partial factors explicitly reflect the probability distributions, standard deviations, correlation structure and relative significance of the load and resistance parameters.

Fig. 4.21. Combinations of pile diameter and pile wall thickness for reliability index of 3 with respect to serviceability limit state and reliability indices of 3, 4 and 5 with respect to ultimate limit state

Fig. 4.22. Combinations of pile length and pile diameter for reliability index of 3 with respect to serviceability limit state and reliability indices of 3, 4 and 5 with respect to ultimate limit state

4.8. Concluding remarks

In this chapter, the key reliability analysis methodologies employed in this thesis were described. These included well-established methods: first-order reliability method (FORM), second-order reliability method (SORM), and Monte Carlo simulations with importance sampling (MCS). Practical procedures for implementing these methods in the spreadsheet platform are presented in associated appendices at the end of this thesis. The probabilistic sensitivity factors for correlated random variables are original contributions of this study. The alternative approach for obtaining more precise bimodal joint probability of failure is also a new initiative. The above procedures were applied to the reliability analysis of a laterally-loaded pile embedded in soft marine clay whose undrained shear strength was modeled as a one-dimensional random field, and subjected to uncertain loads which were modeled as correlated non-normal random variables. The spatial-autocorrelation and spatial-averaging approaches of accounting for spatial variation were compared. The influences of the probability distributions, coefficients of variation, and correlation structure of the underlying random variables on probabilistic results were discussed.

A simple sensitivity factor of the reliability index is developed for correlated random variables. The evaluation of this new sensitivity factor is efficient and based only on the direction cosines and the correlation matrix. The sensitivities of the reliability index with respect to the mean and standard deviation are also derived. The proposed sensitivity factors agreed very well with the sensitivities obtained by numerical differentiation based on FORM. The inferred sensitivity factors of the failure probability were compared with results obtained by MCS with importance sampling. The inferred sensitivities with respect to mean values were generally quite accurate, but those with respect to standard deviations had significant errors. The reason was that variations in the standard deviations of the random variables tend to cause more change in the nonlinearity of the performance function than variations in the mean values. It was demonstrated that more accurate estimates of the sensitivity of the failure probability can be obtained

by numerical differentiation based on SORM. The proposed sensitivity factor was compared with existing sensitivity measures including regression coefficients, rank correlation coefficients and omission sensitivity factors. In many instances the different sensitivity measures had similar rankings of the relative significance of the stochastic parameters.

The Kounias-Ditlevsen bimodal bounds were used to approximate the system failure probability of a laterally-loaded pile considering two failure modes. In general the bi-modal bounds, which take into account correlation between failure modes, provide a narrower range than the uni-modal bounds. In addition more accurate system bounds were obtained based on second-order reliability indices, compared with first-order reliability indices, for cases with nonlinear limit-state surfaces. The joint failure probability between any two failure modes is typically approximated by the bivariate standard normal cumulative distribution function which is also expressed in terms of bounds due to mathematical difficulty. It is possible to replace the bounds by a single value based on simple and reliable methods of evaluating the bivariate standard normal integral, and consequently improve the precision of the system bounds. It was found that one limitation of this approach was that the very narrow improved bounds could not encompass the true system failure probability as easily as the wider Kounias-Ditlevsen bounds for nonlinear limit-state surfaces, and therefore might be of less practical use. To obtain system bounds that are both accurate and precise would require better approximations for the joint failure probability that address the nonlinearity of the limit-state surfaces, and additional computational effort is to be expected.

The geotechnical engineering community has, for a long time, been using the allowable stress design approach which is based on a single lumped factor of safety. With the development of new LRFD design codes in the United States and in Europe, future designs for geotechnical systems will adopt the same limit-state design concepts as the current designs for structural systems. The basis of the load and resistance factors recommended in the API LRFD code and the partial factors in Eurocode 7 is reliability-based design. The intention of the prescribed

factors is to ensure that the final design will fulfill a certain level of reliability (or to achieve a target probability of non-failure). Therefore the reliability-based design demonstrated in this chapter is in line with the spirit of load-and-resistance-factor design of API and partial-factor design of Eurocode 7. In reliability-based design no assumption is made with regards to the load and resistance factors, partial factors or the characteristic values of the underlying parameters. The designer is allowed to incorporate his/her estimates of parametric uncertainty and correlation in the analysis. The load and resistance factors or partial factors are calculated as ratios of the design-point values and the mean values, and they provide useful information on the relative importance of the parameters. Therefore reliability-based design and design based on partial factors can play complementary roles. A clear understanding of reliability-based design is essential for developing more refined design codes and more instructive recommendations for future designers.

Chapter 5 Reliability Analysis based on Response Surface Method and Artificial Neural Network

5.1. Introductory comments

In the previous chapter, reliability analyses of a laterally loaded pile were carried out based on the direct stiffness method, which provided direct access to the underlying implicit performance functions, and allowed probabilistic methods to be applied within the same platform as the deterministic model. Nonetheless, reliability analyses can still be conducted with other deterministic approaches (e.g. a standalone FEM package) by first approximating the performance function by a closed-form surrogate via numerical approaches such as the response surface method (RSM) or the artificial neural network (ANN). In this chapter, stochastic analyses of laterally-loaded piles based RSM and ANN are performed. For illustration, the deterministic pile responses are evaluated using the finite-element program FB-MultiPier. A sequential reliability approach is adopted in which response surfaces and neural networks are constructed in the neighborhood of the mean-value point for the first trial and around tentative design points for subsequent trials. For the response surface approach, the effects of cross terms, experimental designs and sampling parameters with reference to the accuracy and efficiency of the quadratic response surface model are examined. Conventional deterministic and advanced stochastic response surface methods are compared and discussed. For the neural network approach, the number of hidden neurons, the learning-rate parameter, and the momentum constant are optimized with respect to the error function. The number of training cycles is controlled using cross-validation. The accuracy of the artificial neural network is examined for various sizes of the training data.

5.2. Response surface methodology

Response surface methodology (RSM) is a collection of experimental strategies, mathematical methods and statistical inference, which enable an experimenter to make efficient empirical exploration of the system of interest. RSM first appeared in a paper by Box and Wilson (1951) that investigated the problem of experimental attainment of optimum conditions in a chemical process. For the past sixty years there have been many new developments in RSM and a dramatic spread of its applications into many diverse areas, including chemical and process industries, physical and engineering sciences, biological and clinical sciences, food science and social sciences, just to name a few. This section shall not attempt to cover the many aspects of RSM but only to explain the fundamental ideas and how we may adopt them in reliability assessment. Accordingly for the rest of this thesis the term '*response surface method*' (RSM) should be appreciated within this confined scope. More comprehensive reviews on the methodology in its widest sense can be found in these principal papers: Hill and Hunter (1966), Myers et al. (1989), Myers (1999) and Myers et al. (2004).

The basic concept of RSM involves the approximation of the unknown implicit performance function by using a simple and explicit function (often a polynomial). The construction of a response surface is based on a set of designed experiments or numerical analyses performed at selected sampling points. Regression analysis can then be carried out to determine the coefficients that define the response surface. An appropriately chosen response surface can suffice as a cost-effective surrogate model and replace computationally expensive experiments or direct analyses. With the response surface one is able to predict system response and carry out reliability analysis more efficiently and economically (Romero et al. 2004).

The application of RSM in geotechnical engineering began only in recent years. Some instances in the area of laterally-loaded piles are mentioned here. Tandjiria et al. (2000) used three different response surface models in the reliability of

laterally-loaded piles and obtained promising results. The response surface method was also used by Pula (2007) in a reliability analysis on the lateral capacity of rigid piles. Chan and Low (2009) presented a procedure for reliability analysis based on the response surface method in the spreadsheet platform, which is given in Appendix F.

The analyses presented in the following sections are performed on response surfaces constructed based on deterministic pile responses evaluated using a commercially-available finite-element program FB-MultiPier. FB-MultiPier (current version 4.12) is developed by the Bridge Software Institute at the University of Florida (<http://bsi-web.ce.ufl.edu>) for analyzing multiple bridge pier structures interconnected by bridge spans. The program is capable of coupling nonlinear structural finite-element analysis (including nonlinear pile behavior) with nonlinear soil models characterized by p - y curves. Over the past half century numerous studies on p - y curves have been carried out, resulting in an extremely rich p - y literature. In FB-MultiPier geotechnical practitioners who are already familiar with the conventional p - y approach are able to subscribe to a more rigorous computational algorithm (that can be easily extended to perform 3D analysis) integrated with options to either select from a list of standard p - y models or import their own custom p - y curves.

5.2.1. Comparison of two response surface models

A commonly-used response surface model is the quadratic polynomial without cross (or mixed or interaction) terms (Bucher and Bourgund 1990):

$$R(\mathbf{x}) = a_0 + \sum_{i=1}^n a_i x_i + \sum_{i=1}^n a_{ii} x_i^2 \quad (5.1)$$

The unknown coefficients a_0 , a_i and a_{ii} can be determined using linear algebra, matrix inversion or other methods so long as there are at least $2n + 1$ sample points. Each sample point is essentially a systematically-selected point in the

space of the basic variables x_i at which the value of the performance function $g(\mathbf{x})$ needs to be evaluated. For $n = 2$, the vector of unknown coefficients $\mathbf{a} = \{a_0 \ a_1 \ a_2 \ a_{11} \ a_{22}\}^T$ is related to the response value g and the random variables x_1 and x_2 :

$$g = \{1 \quad x_1 \quad x_2 \quad x_1^2 \quad x_2^2\} \{a_0 \quad a_1 \quad a_2 \quad a_{11} \quad a_{22}\}^T = \mathbf{v} \cdot \mathbf{a} \quad (5.2)$$

If 5 observations of g are made at point (x_1, x_2) and its immediate vicinity, then the 5 unknown coefficients comprising vector $\mathbf{a}$ can be solved:

$$\mathbf{a} = \mathbf{V}^{-1} \mathbf{g} \quad (5.3)$$

where

$$\mathbf{V} = \begin{bmatrix} 1 & x_1 & x_2 & x_1^2 & x_2^2 \\ 1 & x_1 - h\sigma_1 & x_2 & (x_1 - h\sigma_1)^2 & x_2^2 \\ 1 & x_1 + h\sigma_1 & x_2 & (x_1 + h\sigma_1)^2 & x_2^2 \\ 1 & x_1 & x_2 - h\sigma_2 & x_1^2 & (x_2 - h\sigma_2)^2 \\ 1 & x_1 & x_2 + h\sigma_2 & x_1^2 & (x_2 + h\sigma_2)^2 \end{bmatrix} \quad (5.4)$$

$$\mathbf{g} = \{ \quad g(x_1, x_2) \quad g(x_1 - h\sigma_1, x_2) \quad g(x_1 + h\sigma_1, x_2) \quad g(x_1, x_2 - h\sigma_2) \quad g(x_1, x_2 + h\sigma_2) \}^T \quad (5.5)$$

The prescribed multiplier h is an arbitrarily chosen sampling parameter. It is common practice to use a constant and uniform value for h although in some cases there are advantages for using different h values for different sample points.

Reliability analyses based on the response surface model in Eq. (5.1) were performed by Tandjiria et al. (2000), Xu and Low (2006), and Chan and Low (2009). A procedure for implementing the response surface methodology in Microsoft Excel is described in the appendix of Chan and Low (2009). The first trial response surface is based on points sampled around the mean-value point. The design point and the reliability index computed based on the first trial response surface are unlikely to be very accurate. An iterative process, in which the next trial response surface is built on points sampled around the tentative design point from the previous trial response surface, is carried out to obtain more accurate solutions.

An alternative formulation of the quadratic response surface is (Li 2000):

$$R(x) = b_0 + \sum_{i=1}^n b_i \frac{(x_i - x_{ic})}{\sigma_i} + \sum_{i=1}^n b_{ii} \frac{(x_i - x_{ic})^2}{\sigma_i^2} \quad (5.6)$$

For $n = 2$, the vector of unknown coefficients is $\mathbf{b} = \{b_0 \ b_1 \ b_2 \ b_{11} \ b_{22}\}^T$. The point denoted by (x_{1c}, x_{2c}) is the center point for sampling. In Li's (2000) procedure the center point is initially set at the mean-value point (μ_1, μ_2) . Sample points are selected based on $x_i = \mu_i \pm h\sigma_i$ for the first iteration. In subsequent iterations, the design point (x_1^*, x_2^*) based on the trial response surface constructed in the previous iteration becomes the new center point for sampling. Sample points are then selected based on $x_i = x_i^* \pm h\sigma_i$. The unknown vector $\mathbf{b}$ is solved as follows:

$$\mathbf{b} = \mathbf{W}^{-1}\mathbf{g} \quad (5.7)$$

where

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -h & 0 & h^2 & 0 \\ 1 & +h & 0 & h^2 & 0 \\ 1 & 0 & -h & 0 & h^2 \\ 1 & 0 & +h & 0 & h^2 \end{bmatrix} \quad (5.8)$$

$$\mathbf{g} = \left\{ \begin{array}{ccccc} g(x_{1c}, x_{2c}) & g(x_{1c} - h\sigma_1, x_{2c}) & g(x_{1c} + h\sigma_1, x_{2c}) & g(x_{1c}, x_{2c} - h\sigma_2) & g(x_{1c}, x_{2c} + h\sigma_2) \end{array} \right\}^T \quad (5.9)$$

It is clear that matrix $\mathbf{W}$ is much simpler to work with than matrix $\mathbf{V}$. The advantage of Li's (2000) formulation is that matrix $\mathbf{W}$ is dependent only on the h value. Also, by virtue of the uncoupling of equations due to matrix $\mathbf{W}$, the coefficient vector $\mathbf{b}$ can be quite easily solved by hand. Li (2000) pointed out that the coefficients b_0 , b_i and b_{ii} have physical meanings with respect to the performance function $g(\mathbf{x})$: (i) b_0 is the value of $g(\mathbf{x})$ at the center point for sampling (x_{1c}, x_{2c}) , i.e., $b_0 = g(x_{1c}, x_{2c})$; (ii) b_i is the first-order numerical derivative

of $g(\mathbf{x})$ at (x_{1c}, x_{2c}) ; (iii) b_{ii} is one-half the approximate second-order numerical derivative of $g(\mathbf{x})$ at (x_{1c}, x_{2c}) . Another potential advantage in using reduced variables $(x_i - x_{ic})/\sigma_i$ is to overcome the undesirable effect of large differences in magnitude between the original variables x_i .

Probabilistic analyses were performed for the laterally-loaded pile in marine clay at Sabine based on the two response surface methods. Response surface method 1 refers to the conventional quadratic polynomial of Eq. (5.1); Response surface method 2 refers to the alternative quadratic polynomial of Eq. (5.6). For $n (= 8)$ parameters the above response surface models each contain $2n + 1 (= 17)$ unknown coefficients, which are solved using the axial-point experimental design with exactly the same number of sample points. FORM, SORM as well as Monte Carlo simulations were carried out on the final response surface, which is found when the reliability index based on the current trial response surface and that based on the previous trial response surface do not differ by more than 1%.

The following points may be noted:

- (a) Table 5.1 shows that the reliability indices and failure probabilities computed based on the two response surface models are practically the same. The solutions are identical because Eq. (5.1) and Eq. (5.6) are in fact the same polynomial function that has been formulated in two different ways. It is also encouraging to note that the reliability indices and failure probabilities obtained based on the surrogate models compare well with those based on the limit state functions as defined by the direct stiffness method.

Table 5.1. Results of reliability analysis based on two response surface methods with respect to: serviceability limit state (top) and ultimate limit state (bottom)

(a) Pile deflection failure (SLS)

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Response surface method 1	1.579	5.71	1.558	5.96	6.05
Response surface method 2	1.579	5.71	1.558	5.96	6.06
Direct stiffness method	1.557	5.98	1.543	6.14	6.15

(b) Bending moment failure (ULS)

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Response surface method 1	1.798	3.61	1.805	3.55	3.54
Response surface method 2	1.798	3.61	1.805	3.55	3.54
Direct stiffness method	1.791	3.66	1.795	3.63	3.62

Note: β_{SORM} is computed based on $-\Phi^{-1}(p_{f,\text{SORM}})$.

Response surface method 1 is based on Eq. (5.1).

Response surface method 2 is based on Eq. (5.6).

All response surface models are constructed based on FEM using FB-MultiPier.

- (b) Fig. 5.1 shows that the final response surface is found after just three iterations, regardless of the response surface method used, since the first-order reliability indices at the second and third iterations are virtually indistinguishable. The initial trial response surface (at the first iteration) is highly inaccurate because the sample points are in the neighborhood of the mean-value point. The selection of sample points from the vicinity of the tentative design point in subsequent iterations proved to be an accurate and efficient approach.

Fig. 5.1. Convergence of reliability indices based on two response surface methods with respect to: (a) serviceability limit state; (b) ultimate limit state

5.2.2. Effect of cross terms

The complete second-order polynomial containing the full suite of cross terms (i.e., terms of the form $x_i x_j$ where $i \neq j$) can also be used as a response surface model, at the expense of an additional $n(n - 1)/2$ sample points. It has the following form (Bucher and Bourgund 1990):

$$R(\mathbf{x}) = a_0 + \sum_{i=1}^n a_i x_i + \sum_{i=1}^n a_{ii} x_i^2 + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_i x_j \quad (5.10)$$

The advantage of including cross terms was demonstrated in the analysis of a nonlinear single-degree-of-freedom oscillator subjected to a rectangular load pulse with random duration and amplitude (Bucher and Bourgund 1990). There were 6 random variables, so up to 15 cross terms could be added, but it was shown that the computed failure probability improved considerably after including only 3 cross terms.

The ideal location for selecting a sample point for determining the coefficient a_{ij} depends on whether x_i and x_j are resistance or load parameters (Rajashekhar and Ellingwood 1993). It was recommended that the sample point ought to have coordinates that lie on the negative side of the resistance axes and positive side of the load axes. This selection criterion was applied to the nonlinear single-degree-of-freedom system previously analyzed by Bucher and Bourgund (1990) and it gave the best result, in terms of reliability index and failure probability, among various selection methods. The same selection procedure was used by Xu and Low (2006) in their reliability analyses of the stability of embankments.

Not every cross term is as important. Cross terms that are relatively insignificant can be excluded from the response surface model without compromising the accuracy of the reliability index (Rajashekhar and Ellingwood 1993). An assessment is made at each iteration step and relatively unimportant cross terms

are discarded. Computation efficiency is improved as the number of sample points reduces progressively. Alternatively one may begin with a response surface model that contains no cross terms. Potentially-important cross terms are then added to the response surface model in successive iteration steps until the reliability index converges (Lee and Kwak 2006).

Substituting the original variable x_i in Eq. (5.10) for the reduced variable $(x_i - x_{ic})/\sigma_i$ (as suggested by Li 2000) gives the following alternative model:

$$R(\mathbf{x}) = b_0 + \sum_{i=1}^n \frac{b_i(x_i - x_{ic})}{\sigma_i} + \sum_{i=1}^n \frac{b_{ii}(x_i - x_{ic})^2}{\sigma_i^2} + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{b_{ij}(x_i - x_{ic})(x_j - x_{jc})}{\sigma_i \sigma_j} \quad (5.11)$$

Probabilistic analyses were performed for the laterally-loaded pile in marine clay at Sabine based on the complete second-order polynomial response surface model. For $n (= 8)$ parameters the response surface model contains $(n + 1)(n + 2)/2 (= 45)$ unknown coefficients, which are solved using the axial-point experimental design augmented by the selection procedure of Rajashekhar and Ellingwood (1993) to make up exactly 45 sample points. On the other hand, the truncated second-order polynomial response surface model has only 17 terms.

The following points may be noted:

- (a) Table 5.2 shows that the reliability indices and failure probabilities computed based on the complete second-order polynomial response surface model (with cross terms) are not significantly more accurate than those computed based on the truncated second-order polynomial response surface model (without cross terms). This indication that the effect of cross terms is almost negligible on the reliability index and the probability of failure was also observed by Li (2000), Tandjiria et al. (2000) and Xu and Low (2006).

**Table 5.2. Results of reliability analysis based on complete and truncated quadratic response surface models with respect to:
serviceability limit state (top) and ultimate limit state (bottom)**

(a) Pile deflection failure (SLS)

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Complete response surface	1.579	5.71	1.562	5.91	5.99
Truncated response surface	1.579	5.71	1.558	5.96	6.06
Direct stiffness method	1.557	5.98	1.543	6.14	6.15

(b) Bending moment failure (ULS)

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Complete response surface	1.798	3.61	1.803	3.57	3.57
Truncated response surface	1.798	3.61	1.805	3.55	3.54
Direct stiffness method	1.791	3.66	1.795	3.63	3.62

Note: Complete response surface model (with cross terms) is based on Eq. (5.10).

Truncated response surface model (without cross terms) is based on Eq. (5.1).

All response surface models are constructed based on FEM using FB-MultiPier

- (b) Fig. 5.2 shows that the final response surface is found by the third iteration for both the complete as well the truncated second-order polynomial response surface model. For three iterations, the complete model requires 135 (= 3 times of 45) evaluations of the limit state function while the truncated model requires 51 (= 3 times of 17) evaluations.

Fig. 5.2. Convergence of reliability indices based on complete and truncated quadratic response surface models with respect to:
(a) serviceability limit state; (b) ultimate limit state

5.2.3. Effect of experimental designs

In an earlier discussion (Section 5.2.1), the response surface model characterized by the quadratic polynomial without cross terms was constructed using exactly $2n + 1$ points sampled according to $x_i = \mu_i \pm h\sigma_i$ for the first iteration, and $x_i = x_i^* \pm h\sigma_i$ in subsequent iterations. Such a response surface design is known as an ‘axial-point’ design. More elaborate response surface designs are available, including classical quadratic designs such as the central-composite design (CCD, Box and Wilson 1951) and the Box-Behnken design (BBD, Box and Behnken 1960). Fig. 5.3 shows (a) the CCD and (b) the BBD for three factors. The boundary of the 3-dimensional sampled space is outlined by a cube. The CCD is made up of the center point, the axial points and the points at the corners of the cube; the BBD comprises the center point and the mid-points along the edge of the cube.

Fig. 5.3. Response surface designs for 3 factors with 3 levels

Probabilistic analyses were performed for the laterally-loaded pile in marine clay at Sabine based on CCD and BBD. The CCD has 273 sample points and the BBD has 113 sample points. The second-order polynomial without cross terms is chosen as the response surface model because it was shown in the previous section that the cross terms had little effect on the reliability index and the probability of failure. As the number of sample points is greater than the number

of unknown coefficients to be solved, the coefficient vector **a** (or **b**) is solved using the method of least squares (Hill and Hunter 1966):

$$\mathbf{a} = (\mathbf{V}^T \mathbf{V})^{-1} \mathbf{V}^T \mathbf{g} \quad (5.12)$$

$$\mathbf{b} = (\mathbf{W}^T \mathbf{W})^{-1} \mathbf{W}^T \mathbf{g} \quad (5.13)$$

The following points may be noted:

- (a) Table 5.3 shows that the reliability indices and failure probabilities computed based on the three response surface designs are practically the same.
- (b) Fig. 5.4 shows that the final response surface is found by the third iteration for all three response surface designs. For three iterations, the CCD requires 819 (= 3 times of 273) evaluations of the limit state function while the BBD requires 339 (= 3 times of 113) evaluations.
- (c) The CCD used in the above study is the face-centered central-composite design (CCF). Two other variations of the CCD exist: the circumscribed central-composite design (CCC) and the inscribed central-composite design (CCI). Fig. 5.5 compares the three types of CCD for a two-factor, three-level case. The CCC and the CCI are rotatable designs; the CCF is not. A design is said to be rotatable if upon rotating the sample points about the center point, the moments of the distribution of the design remain unchanged. Design rotatability is advantageous for estimating coefficients in second-order response surface models because it allows the second-order derivatives to be determined with equal precision (Box and Wilson 1951). Investigations with CCC and CCI are deemed unnecessary since the CCF already provides a sufficiently accurate solution.

**Table 5.3. Results of reliability analysis based on three response surface designs with respect to:
serviceability limit state (top) and ultimate limit state (bottom)**

(a) Pile deflection failure (SLS)

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Central-composite design	1.5794	5.71	1.5578	5.96	6.06
Box-Behnken design	1.5794	5.71	1.5579	5.96	6.06
Axial-point design	1.5794	5.71	1.5579	5.96	6.06

(b) Bending moment failure (ULS)

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Central-composite design	1.7983	3.61	1.8054	3.55	3.54
Box-Behnken design	1.7983	3.61	1.8053	3.55	3.54
Axial-point design	1.7983	3.61	1.8053	3.55	3.54

Note: β_{SORM} is computed based on $-\Phi^{-1}(p_{f,\text{SORM}})$.

Response surface model of Eq. (5.1) is used.

All response surface models are constructed based on FEM using FB-MultiPier.

Fig. 5.4. Convergence of reliability indices based on three response surface designs with respect to:
(a) serviceability limit state; (b) ultimate limit state

Fig. 5.5. Variations of central-composite response surface design

5.2.4. Effect of sampling parameter

At the centre of the experimental design introduced by Bucher and Bourgund (1990) and later adopted by many is the sampling parameter h . The originators used a constant value of $h = 3$ in their numerical examples. Rajashekhar and Ellington (1993) noted that if progressively diminishing h is used in an indefinite sequential procedure, then one would at some stage end up with an ill-conditioned system of equations. Starting with $h = 2$ and repeating with $h = 1$ seemed to give satisfactory results. Guan and Melchers' (2001) detailed parametric study provide evidence that h strongly influences the estimated probability of failure and the stability of convergence, depending on the shape of the actual limit state surface.

In particular, the estimated probability of failure was observed to fluctuate considerably for a non-symmetric and/or non-smooth limit state surface. No clear guidance for the selection of h can be given in any single case, but a value between 1 and 3 appears to work in most typical circumstances. Wong et al. (2005) proposed that h be chosen according to the coefficients of variation of the random variables: low h values for high c.o.v. and vice versa. In addition, an

adaptive algorithm which identified unrealistic sampling points and adjusted h accordingly was described.

Probabilistic analyses were repeated for the laterally-loaded pile in marine clay at Sabine for different values of the sampling parameter h . The second-order polynomial without cross terms was chosen as the response surface model. The highest value of h that was investigated was capped at 3; higher values were not tested because the finite-element method was unable to return a solution for very large P_{top} and M_{top} and for very small c_u . The following points may be noted:

- (a) Fig. 5.6 shows that the failure probabilities, especially those obtained based on FORM and SORM, are quite insensitive to the sampling parameter. When the value of h is decreased from 1.0 to 0.1, the failure probability inferred from FORM/SORM remains virtually unchanged; when the of h is increased from 1.0 to 3.0, the failure probability inferred from FORM/SORM may either increase or decrease but the amount of change is quite insignificant. The failure probability computed based on Monte Carlo simulation gradually increases with the value of h . The upward trend of the 'MCSim' curve is consistent with that of the 'SORM' curve but is in conflict with that of the 'FORM' curve.
- (b) Fig. 5.7 shows that the first-order reliability index based on the initial trial response surface generally decreases when the sampling parameter increases. However, the final reliability index is relatively insensitive to the sampling parameter. For $h = 3.0$, the final response surface is found after four iterations, which is one iteration slower than for $h = 0.1, 0.2, 0.5, 1.0$ and 2.0 . This implies that the efficiency of the response surface method may somewhat deteriorate when the sampling parameter is increased.

Fig. 5.6. Failure probabilities as functions of sampling parameter:
(a) serviceability limit state; (b) ultimate limit state

Fig. 5.7. Convergence of reliability indices based on various sampling parameters with respect to:
(a) serviceability limit state; (b) ultimate limit state

- (c) For the present case, variation of the sampling parameter h between 0.1 and 3.0 has little effect on the accuracy and efficiency of the response surface method. This is probably due to the fact the limit state surfaces are not highly nonlinear for the case in hand.

5.2.5. Comparison of deterministic and stochastic response surface methods

The conventional response surface method that employs a polynomial approximation fitted to sample points designated according to some experimental design is essentially a deterministic approach. This is because the response surface is constructed in the original space without considering the probability distributions and the correlation structure of the basic random variables. The stochastic response surface method is a newer and more advanced approach in which the response surface is defined in the rotated and transformed space of uncorrelated standard normal random variables. The response surface model is formulated based on a multi-dimensional Hermite polynomial chaos approximation, of which the second-order and third-order forms are as follows (Isukapalli 1999):

$$R(\mathbf{u}) = c_0 + \sum_{i=1}^n c_i u_i + \sum_{i=1}^n c_{ii} (u_i^2 - 1) + \sum_{i=1}^{n-1} \sum_{j>i}^n c_{ij} u_i u_j \quad (5.14)$$

$$R(\mathbf{u}) = c_0 + \sum_{i=1}^n c_i u_i + \sum_{i=1}^n c_{ii} (u_i^2 - 1) + \sum_{i=1}^n c_{iii} (u_i^3 - 3u_i) \\ + \sum_{i=1}^{n-1} \sum_{j>i}^n c_{ij} u_i u_j + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n c_{ijj} (u_i u_j^2 - u_i) + \sum_{i=1}^{n-2} \sum_{j>i}^{n-1} \sum_{k>j}^n c_{ijk} u_i u_j u_k \quad (5.15)$$

In general, $N_2 (= 1 + 2n + n(n-1)/2)$ and $N_3 (= 1 + 3n + 3n(n-1)/2 + n(n-1)(n-2)/6)$ terms are required for the second-order and third-order expansions, respectively (Isukapalli 1999).

If the unknown coefficients in the Hermite polynomials in Eq. (5.14) and Eq. (5.15) are evaluated using a collocation method (Tatang 1995; Isukapalli 1999; Huang et al. 2007) then it is known as a collocation-based stochastic response surface method. The collocation method places the sample points at the roots of the next higher-order Hermite polynomial so as to capture regions of high probability leading to fewer function evaluations for higher accuracy (Tatang 1995). One-dimensional Hermite polynomials are given by:

$$H_0(u) = 1 \quad (5.16)$$

$$H_1(u) = u \quad (5.17)$$

$$H_{k+1}(u) = uH_k(u) - kH_{k-1}(u) \quad \text{for integer } k \geq 2 \quad (5.18)$$

For the laterally-loaded pile in marine clay at Sabine, to consider 8-parameter stochastic response surfaces using Hermite polynomials will be computationally costly. For a second-order expansion, the collocation samples are placed at the roots of the third-order Hermite polynomial, i.e., 0 and $\pm\sqrt{3}$. The number of available collocation points is $3^8 = 6561$ while there are 45 terms in the response surface. For a third-order expansion, the roots of the fourth-order Hermite polynomial are $\pm\sqrt{3} \pm \sqrt{6}$. The number of available collocation points is $4^8 = 65536$ while there are 165 terms in the response surface. Hence fairly tedious algebraic expressions are incurred even at a third-order expression. However, there appears to be no simple method to manage the proliferation of Hermite expansions for higher random dimensions (Huang et al. 2009). In general, for a r^{th} -order expansion of a n -parameter stochastic response surface, there are $(r + 1)^n$ candidate points available for constructing the response surface. Huang et al. (2009) acknowledged that the selection of the appropriate collocation points from the large number of potential candidates is a practical issue. It is uneconomical to use all the candidate points. On the other hand, using just enough collocation points to solve for the unknown coefficients can result in ‘unstable solution’, regardless of whether the points are selected randomly or systematically (Isukapalli 1999). This compares interestingly with the reliability analyses carried

out in Section 4.3 based on the conventional response surface method, where $2n + 1$ axial points selected around the design point are sufficient for constructing a second-order polynomial response surface (without cross terms) that yields a stable and accurate reliability index. Huang et al. (2009) recommended the unknown coefficients be solved with more collocation points using regression. They also highlighted two important criteria in the selection of collocation points. The selected points should (i) capture regions of high probability and (ii) be symmetrically distributed with respect to the origin. Li et al. (2011) investigated the effect of the number of collocation points and concluded that the number of collocation points selected should ensure a full-rank Hermite polynomial matrix.

An effective way of limiting the number of terms in the Hermite polynomial and the number of possible collocation points is to reduce the number of random variables by excluding those stochastic parameters that are relatively insignificant and regard them as deterministic parameters as suggested by Madsen (1988). The measure of relative significance is the omission sensitivity factor, defined as the inverse ratio between the value of the first-order reliability index considering all random variables and the first-order reliability index with one of the random variables replaced by its mean value (Madsen 1988). When an important random variable is omitted the reliability index changes considerably leading to an omission sensitivity factor that is much greater than one; when an unimportant random variable is omitted the reliability index is unchanged or changes slightly such that the omission sensitivity factor is more or less equal to one.

With respect to $g_1(\mathbf{x})$, the omission sensitivity factors for P_{top} , M_{top} , $c_{u,1}$, $c_{u,2}$, $c_{u,3}$, $c_{u,4}$, $c_{u,5}$ and $c_{u,6}$ are 1.8402, 1.0101, 1.0060, 1.0663, 1.0528, 1.0057, 1.0031 and 1.0024, respectively. With respect to $g_2(\mathbf{x})$, the omission sensitivity factors for

P_{top} , M_{top} , $c_{u,1}$, $c_{u,2}$, $c_{u,3}$, $c_{u,4}$, $c_{u,5}$ and $c_{u,6}$ are 4.9636, 1.0161, 1.0042, 1.0318, 1.0133, 1.0012, 1.000046 and 1.00051, respectively. A new set of probabilistic analyses are carried out with only the first five random variables namely, P_{top} , M_{top} , $c_{u,1}$, $c_{u,2}$ and $c_{u,3}$; the parameters $c_{u,4}$, $c_{u,5}$ and $c_{u,6}$ are now regarded as deterministic

and equated to their mean values because the omission sensitivity factors for these parameters are very close to one. For $n = 5$ random variables, the second-order stochastic response surface contains 21 terms (in terms of u_i) and up to 243 collocation points. The conventional response surface based on the second-order polynomial with cross terms also contains 21 terms (in terms of x_i) for which $n = 5$.

The following points may be noted:

- (a) Table 5.4 shows that the reliability indices and failure probabilities computed based on the stochastic response surface method is almost identical to those computed based on the conventional response surface method. The solutions obtained based on the response surface methods compare well with those based on the direct stiffness method. For consistency the direct stiffness method is also carried out using $n = 5$ random variables. The deterministic or conventional response surface method appears to be more efficient than the stochastic response surface method simply because the conventional approach requires only 63 (3 iterations of 21 sample points each) evaluations of the limit-state function compared with the more advanced stochastic approach which requires 243 evaluations (single iteration with the entire set of collocation points).

**Table 5.4. Results of reliability analysis based on conventional and stochastic response surface methods with respect to:
(a) serviceability limit state; (b) ultimate limit state**

(a) Pile deflection failure (SLS)

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Stochastic response surface	1.5860	5.64	1.5770	5.74	5.75
Conventional response surface	1.5860	5.64	1.5781	5.73	5.73
Direct stiffness method	1.5634	5.90	1.5564	5.98	5.98

(b) Bending moment failure (ULS)

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Stochastic response surface	1.7978	3.61	1.8014	3.58	3.58
Conventional response surface	1.7978	3.61	1.8017	3.58	3.56
Direct stiffness method	1.7910	3.66	1.7952	3.63	3.62

Note: β_{SORM} is computed based on $-\Phi^{-1}(p_{f,\text{SORM}})$.

Conventional response surface method is based on Eq. (5.10).

Stochastic response surface method is based on Eq. (5.14).

All response surface models are constructed based on FEM using FB-MultiPier.

Fig. 5.8. Reliability indices as functions of:
(a) pile-head deflection and (b) maximum bending moment

Fig. 5.9. Cumulative probabilities as functions of:
(a) pile-head deflection and (b) maximum bending moment

- (b) Fig. 5.8 and Fig. 5.9 show the reliability indices and the cumulative distribution functions (cdf) of the limiting (or permissible) pile-head deflection y_{lim} and the plastic moment M_p . The curves obtained based on the direct stiffness method are representative of the true responses based on the actual limit-state function. The black circles of the stochastic response surface method (SRSM) are obtained based on the same Hermite polynomial constructed earlier in this chapter, and they show excellent agreement with the true response curve. The accuracy of SRSM can be attributed to the large number of sample points that cover a considerably wide domain of the sample space. The stochastic response surface so constructed is therefore a good approximation of the limit state function $g(\mathbf{x})$ in general. The results obtained based on the conventional response surface method (CRSM), indicated by crosses in Fig. 5.8 and 5.9, also agree very well with the direct stiffness method. In using CRSM, one needs to construct a new response surface for each value of y_{lim} or M_p desired. This is because CRSM provides a good approximation to $g(\mathbf{x}) = 0$, but not to $g(\mathbf{x})$ in general. The comparison between CRSM and SRSM in this section shows their relative merits and limitations with respect to the objective of the probabilistic analysis at hand. If the objective is to obtain the reliability index for specific values of y_{lim} or M_p , CRSM is more efficient because it requires fewer sample points than SRSM. However, if the objective is to compute the variation of the reliability index for different values of y_{lim} or M_p , SRSM is more advantageous than CRSM since SRSM does not involve an iterative process to compute and also does not need to be repeated for different y_{lim} or M_p values.

5.3. Artificial neural network

Traditionally the term ‘neural network’ refers to a group of real biological neurons that are connected within the nervous system in order to perform a specific physiological function. An ‘artificial neural network’ (ANN) is a computational mechanism that is able to “acquire, represent, and compute a mapping from multivariate space of information to another, given a set of data representing that mapping” (Garrett 1994).

The basic architecture of ANN has been covered widely (e.g., Haykin 1999; Duda et al. 2001; Bishop 2006). A typical neural network (Fig. 5.10) consists of a number of interconnected artificial neurons where each neuron is a processing element that mimics the properties of the biological neuron. The neurons are logically arranged in layers. Information or signals are passed from the input layer, through one or more hidden layers, to the output layer. In between layers the neurons are linked via connectors called synapses. These synapses are assigned with numerical weights. The neurons in the respective layers have somewhat different functions: the input neuron normalizes the input signal; the hidden neuron receives a number of normalized and weighted signals from the input layer and applies a transfer function to their aggregate sum; the output neuron receives weighted signals from the hidden layer, applies a transfer function to their aggregate sum and produces an output signal which is then subject to inverse normalization. The universal approximation theorem states that a neural network with just a single layer of hidden neurons is able to provide an approximation of any continuous function (Cybenko 1989; Funahashi 1989; Hornik et al. 1989). The advantage is that no prior assumption on the nature of the function is needed (c.f. response surface method).

Fig. 5.10. Typical artificial neural network comprising 3 input neurons, 4 hidden neurons and 2 output neurons

The weights associated with the synapses are initially unknown and randomly assigned. The recursive process of adjusting the weights is called training. During the training phase the neural network progressively learns and adapts based on a set of data that is representative of the actual input-output mapping of the function to be approximated. At the beginning differences between the target outputs and the neural-network predicted outputs are large because of the assumed weights, but the errors provide the basis for correcting the weights over a number of training cycles. The neural network is considered sufficiently trained when the errors of its predicted outputs reach a certain acceptable magnitude. It is common practice to subject the trained neural network to a separate set of data that is not used for training, in order to test the ability of the neural network to generalize. A well-trained neural network is not only capable of reproducing the outputs of the training data, but is also able to offer reasonably accurate outputs given any set of inputs outside of the training data.

For the purpose of this study, the backpropagation learning algorithm (Rumelhart et al. 1986) is used. The highly popular backpropagation learning algorithm is a form of supervised learning that is based on the error-correction rule. A typical backpropagation training cycle consists of two passes through the respective layers of the neural network: a forward pass and a backward pass. In the forward pass, the input data is presented to the network such that the effect of the input data propagates forward through the network (the direction is from the input layer to the output layer) resulting in an output that represents the response of the network. During the forward pass the synaptic weights are fixed. In the backward pass, the response predicted by the network is compared with desired response to produce an error signal that is then propagated backward through the network (the direction is from the output layer to the input layer). During the backward pass the synaptic weights are adjusted in accordance with the error-correction rule so that the new predicted response converges toward the desired response. A simplified explanation of the backpropagation learning algorithm is given below. More detailed descriptions are found in Rumelhart and McClelland (1986), Eberhart and Dobbins (1990), and Haykin (1999), for example.

Before presenting data to the neural network some data preparation in the form of scaling or normalizing is usually required to minimize the influence of absolute scale. In this study the data is normalized to fall within the range (0, 1) based on:

$$x_{norm} = \frac{x_{actual} - x_{min}}{x_{max} - x_{min}} \quad (5.19a)$$

$$g_{norm} = \frac{g_{actual} - g_{min}}{g_{max} - g_{min}} \quad (5.19b)$$

where x_{actual} = actual input parameter; g_{actual} = actual output parameter; x_{norm} = normalized input parameter; g_{norm} = normalized output parameter; x_{min} = minimum x value; g_{min} = minimum g value; x_{max} = maximum x value; and g_{max} = maximum g value. For an input parameter x_i , the normalized input parameter $x_{norm,i}$ is simply the output from the input layer o_{pi} , as shown in Fig. 5.10.

The output from the hidden layer o_{pj} is the outcome of applying a transfer function to the weighted sum of the outputs from the input layer o_{pi} :

$$o_{pj} = f_{sig} \left(\sum_i (w_1)_{ji} o_{pi} + (w_1)_{bias} \right) \quad (5.20)$$

where $(w_1)_{ji}$ = numerical weight of the synaptic connection between input node i and hidden node j ; and $(w_1)_{bias}$ = numerical weight of the bias introduced between the input and hidden layers. In this study the commonly-used sigmoid function f_{sig} is used (with v representing a generic argument):

$$f_{sig}(v) = [1 + \exp(-v)]^{-1} \quad (5.21)$$

The output from the output layer o_{pk} is obtained by applying the sigmoid function to the weighted sum of the outputs from the hidden layer o_{pj} :

$$o_{pk} = f_{sig} \left(\sum_j (w_2)_{kj} o_{pj} + (w_2)_{bias} \right) \quad (5.22)$$

where $(w_2)_{kj}$ = numerical weight of the synaptic connection between hidden node j and output node k ; and $(w_2)_{bias}$ = numerical weight of the bias introduced between the hidden and output layers.

The initial values of the numerical weights $(w_1)_{ji}$, $(w_1)_{bias}$, $(w_2)_{kj}$ and $(w_2)_{bias}$ are randomly assigned. The guesswork most certainly will lead to an error e_p between the computed output o_{pk} and the target or desired output t_{pk} :

$$e_p = 0.5 \sum_k (t_{pk} - o_{pk})^2 \quad (5.23)$$

For an output parameter g_k , the target output t_{pk} is simply the normalized output parameter $g_{\text{norm},k}$, as calculated by Eq. (5.19b).

The backpropagation algorithm specifies that the synaptic weights of the neural network are updated iteratively so as to minimize the error function, which is usually defined as the sum of e_p for all training data. The synaptic weights are adjusted through the following equations:

$$\Delta_{c+1}(w_2)_{kj} = \eta \delta_{pk} o_{pj} + \alpha \Delta_c(w_2)_{kj} \quad (5.24)$$

$$\Delta_{c+1}(w_1)_{ji} = \eta \delta_{pj} o_{pi} + \alpha \Delta_c(w_1)_{ji} \quad (5.25)$$

in which

$$\delta_{pk} = (t_{pk} - o_{pk}) o_{pk} (1 - o_{pk}) \quad (5.26)$$

$$\delta_{pj} = o_{pj} (1 - o_{pj}) \sum_k \delta_{pk} (w_2)_{kj} \quad (5.27)$$

where $\Delta_c(w_2)_{kj}$ = correction to synaptic weight connecting hidden node j and output node k at training cycle c ; $\Delta_{c+1}(w_2)_{kj}$ = correction to synaptic weight connecting hidden node j and output node k at training cycle $c + 1$; $\Delta_c(w_1)_{ji}$ = correction to synaptic weight connecting input node i and hidden node j at training cycle c ; $\Delta_{c+1}(w_1)_{ji}$ = correction to synaptic weight connecting input node i and hidden node j at training cycle $c + 1$; δ_{pk} = error signal at output node k ; and δ_{pj} = error signal at hidden node j .

The updating of the synaptic weights is controlled by the learning-rate parameter η and the momentum constant α . The learning rate η affects the speed at which the neural network arrives at the minimum solution. The learning-rate parameter of backpropagation learning is analogous to the step-size parameter of the gradient-descent algorithm. If the step size is too large, the system will either oscillate about the true solution, or will diverge completely. If the step size is too small, the system will take a long time to converge to the final solution. The momentum constant α forces the search to take into accounts its movement from the previous

iteration. By doing so, the system may avoid local minima or saddle points, and approach the global minimum. A high momentum constant can also help to increase the speed of convergence of the system. However, setting the momentum constant too high may lead to greater risk of overshooting the minimum, which can cause the system to become unstable. A momentum constant that is too low cannot reliably avoid local minima, and may also slow down the training of the system.

At the end of the training the approximated output in absolute scale is obtained by an inverse-normalization as follows:

$$g_{approx} = g_{min} + g_{norm}(g_{max} - g_{min}) \quad (5.28)$$

where g_{approx} = approximated output parameter based on the artificial neural network.

While simple to implement, the backpropagation algorithm has a number of limitations. One serious pitfall is known as *over-fitting*, which is when the network learns too many specific input-output relations and fails to generalize. This may be overcome with cross-validation, i.e. using a separate test set of data to evaluate the performance of the network and imposing a stopping criterion to training. Another disadvantage is *slow convergence*, but backpropagation modifications such as the delta-bar-delta learning rule (Jacobs 1988) and alternative second-order conjugate-gradient (Fletcher and Reeves 1964) method or Levenberg-Marquardt algorithm (Levenberg 1944; Marquardt 1963) are available to improve the convergence rate. During the process of minimizing the error, the backpropagation algorithm may be trapped in *local minima* and miss the desired global minimum. A trial and error approach, such as repeated training with different initial weights each time, seems to be the only way to identify local minima.

A procedure for reliability analysis based on the artificial neural network in the spreadsheet platform is given in Appendix G. The analyses presented in the following sections are performed on artificial neural networks constructed based on deterministic pile responses evaluated using a commercially-available finite-element program FB-MultiPier.

Two other types of neural networks that are also suitable for function approximation and prediction are briefly described below:

- (i) A *radial-basis function* (RBF) network (Broomhead and Lowe 1988; Moody and Darken 1989) has three layers: an input layer, a hidden layer and an output layer. The mapping from the input space to the hidden space is *nonlinear*, whereas the transformation from the hidden space to the output space is *linear*. The hidden space of the RBF network is typically of high dimension due to Cover's (1965) theorem on the *separability of patterns*. The theorem states that a complex pattern classification problem cast in high-dimensional space nonlinearly is likely to be more linearly separable than in low-dimensional space. Likewise, we may simplify a difficult nonlinear interpolation problem through nonlinear mapping of the input space into a new space of sufficiently high dimension. The nonlinear mapping is realized using *radial-basis functions* (RBF) as the activation functions of the hidden neurons. An RBF is a nonlinear function whose value depends only on the distance from the origin, or alternatively on the distance from some other point called the centre. The output of the network is then linearly mapped from the hidden neurons and expressed as a weighted sum of RBF. The design of a RBF network is influenced by three parameters: *centers* and *deviations* of the RBF associated with the hidden neurons, and the linear *weights* of the output neuron. Several learning strategies are available. The centers of the RBF can either be *fixed* at randomly selected positions (Lowe 1989); or allowed to *self-organize* according to the *k*-nearest-neighbor rule (Moody and Darken 1989); or obtained in a *supervised* manner using the gradient

descent procedure. The deviation is usually uniform and fixed based on the spread of the centers (Haykin 1999). The weights are optimized using the *pseudo-inverse* method (Broomhead and Lowe 1988).

- (ii) A *generalized regression neural network* (GRNN) (Specht 1991) is a network that provides estimates of continuous variables using a one-pass learning algorithm that enables fast convergence to the underlying regression surface. It is made up of four layers: an input layer, a pattern layer, a summation layer and an output layer. The *pattern* layer allocates one neuron for each training pattern and each neuron is assigned input values of that pattern. When the training patterns are presented to the network one after another, each pattern neuron will compute its distance from the presented pattern except when the presented pattern is that to which the neuron has been assigned to, in which case the distance must clearly be zero. The *summation* layer comprises a numerator neuron and a denominator neuron, which compute the numerator and denominator components of the output. The learning process involves selecting a smoothing parameter that minimizes the mean squared error. One may notice that the above expression is in fact the regression of v on $\mathbf{x}$. Specht (1991) compared the GRNN with other nonlinear regression techniques and noted the following advantages. The GRNN can be trained in a single pass and is therefore a more efficient learning algorithm than backpropagation. The GRNN provides smooth transitions from one observation to another, even with sparse data in a multidimensional space. And when more data is present, the GRNN converges to the conditional mean regression surface. One is also assured that the converged estimate will not be a local minimum. The main disadvantage of the GRNN is that it takes much computation for the trained network to estimate a new output vector, but clustering versions of GRNN are able to overcome this problem (Specht 1991; 2006).

5.3.1. Effect of hidden neurons

The artificial neural network approach was implemented on the laterally-loaded pile in marine clay at Sabine. The axial-point design with a sampling parameter of one was used to select 17 samples for the training data set. The first trial neural network was established based on 17 sample points in the neighborhood of an arbitrarily chosen center point. For $g_1(\mathbf{x})$ the starting center point was $P_{\text{top}} = 74.66$ kN, $M_{\text{top}} = 24.89$ kNm and $c_{u,i} = 11.57$ kPa. For $g_2(\mathbf{x})$ the starting center point was $P_{\text{top}} = 81.30$ kN, $M_{\text{top}} = 27.10$ kNm and $c_{u,i} = 10.63$ kPa. The above starting center points fulfill the conditions $g_1(\mathbf{x}) = 0$ and $g_2(\mathbf{x}) = 0$, respectively.

The neural network consists of 8 input neurons and 1 output neuron. A single layer of hidden neurons is furnished. The determination of the optimal number of hidden neurons is important with regards to the accuracy of the neural network. General guidelines on finding the optimal number are lacking except that the number should not be too small (Eberhart and Dobbins 1990). Hence the most appropriate number of hidden neurons is usually ascertained through a trial-and-error process. The aim of this section is to determine the smallest number of hidden neurons that minimizes the overall error of the neural network. Backpropagation training is repeated on the neural network starting with one hidden neuron and then progressively increasing the number of hidden neurons. The accuracy of the trained neural network is measured by the mean-squared error with respect to all sample points in the training data.

Fig. 5.11. Root-mean-squared errors as functions of number of hidden neurons: (a) serviceability limit state; (b) ultimate limit state

Fig. 5.11 shows the root-mean-squared error (i.e., the square root of the mean-squared-error) of the neural network as a function of the number of hidden neurons after a training phase of 10,000 cycles. The root-mean-squared error decreases rapidly at the beginning of the trial-and-error process, when the number of hidden neurons is small and increasing by one neuron each time. For both $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ the root-mean-squared error approaches a minimum value when the number of hidden neurons is in the range of 5 to 12 and does not substantially decrease further when more hidden neurons are introduced. The optimal number of hidden neurons also appears to be smaller for higher learning-rate parameter η .

5.3.2. Effect of learning-rate parameter and momentum constant

As mentioned in Section 5.2, the backpropagation learning algorithm is a form of supervised learning in which the pace of learning is controlled by the learning-rate parameter η and the moment constant α via the generalized delta rule.

The values of η and α may vary between 0 and 1. If the value of η is too small, it may take numerous cycles to train the neural network. For practical purposes the value of η should be as large as possible without leading to oscillation in the mean-squared error during training. The incorporation of the momentum term in Eq. (5.24) and (5.25) helps to prevent oscillation and allows larger values of η to be used. Furthermore it also provides the backpropagation algorithm some ability to overcome the problem of local minima. Rumelhart et al. (1987) have employed $\eta = 0.25$ or smaller and $\alpha = 0.9$ in most of their analyses without encountering difficulty. Eberhart and Dobbins (1990) often achieve good results with $\eta = 0.15$ and $\alpha = 0.075$. At other times they have used $\eta = 0.04$ and $\alpha = 0.02$ for larger networks. Haykin (1999) observed that for $\eta \rightarrow 0$, the use of $\alpha \rightarrow 1$ increased the speed of convergence, whereas for $\eta \rightarrow 1$, the use of $\alpha \rightarrow 0$ was needed to ensure stability. The use of $\eta \rightarrow 1$ and $\alpha \rightarrow 1$ caused oscillation and a higher mean-squared error at convergence.

Fig. 5.12. Root-mean-squared errors as functions of momentum constant for various learning rates: (a) serviceability limit state; (b) ultimate limit state

The artificial neural network approach was implemented on the laterally-loaded pile in marine clay at Sabine for different combinations of the learning-rate parameter η and the momentum constant α . The aim is to determine the optimal combination of η and α that minimizes the overall error of the neural network. The neural network used in the trial-and-error process contains 8 hidden neurons, based on the findings in the previous section.

Fig. 5.12 shows the root-mean-squared error of the neural network for various combinations of the learning-rate parameter η and the momentum constant α after a training phase of 10,000 cycles. That a higher η results in a lower root-mean-squared error, even in the absence of the momentum term (i.e., $\alpha = 0$), is intuitively satisfying. The trends appear to favor large values of η and α , such that the optimal combination is $\eta = 0.9$ and $\alpha = 0.9$. Higher values are unlikely to further improve the accuracy of the neural network. For $g_1(\mathbf{x})$ the root-mean-squared error decreases more rapidly with α for lower values of η ; for $g_2(\mathbf{x})$ the root-mean-squared error decreases more rapidly with α for higher values of η .

5.3.3. Effect of training cycles

In backpropagation training, the mean-squared error decreases continuously with the increase in the number of training cycles. The mean-squared error may start off at a large value, decrease rapidly, and then continue to decrease gradually as the neural network approaches a state of convergence. Therefore the mean-squared error with reference to the training data itself provides no useful information on the optimal training duration at which the neural network so obtained will be able to generalize. A neural network is considered to generalize well when the output response computed by the network is reasonably accurate even for test data that is never used to train the network. The risk of excessive training is that the neural network may end up memorizing the training data and lose the ability to generalize different but similar data. Such a phenomenon is called overtraining or over-fitting.

A statistical method known as cross-validation (Haykin 1999) can be used to guard against overtraining. The training session is paused periodically at which the neural network is tested based on a separate validation data set. In a typical situation the mean-squared error with reference to the validation data decreases monotonically to a minimum and then starts to increase as the training continues. The number of training cycles corresponding to the minimum position is an ideal point to terminate the training. This procedure is referred to as the early-stopping rule based on cross-validation.

The artificial neural network approach was implemented on the laterally-loaded pile in marine clay at Sabine using two sets of data: a training data set that contains 17 samples selected based on the axial-point design with a sampling parameter of one; a validation data set that contains another 17 samples randomly drawn from the same population used to generate the training data. Backpropagation training is carried out with learning-rate parameter $\eta = 0.9$ and momentum constant $\alpha = 0.9$ on a neural network that contains 8 hidden neurons, based on the findings in previous sections.

Fig. 5.13 shows the learning curves based on the training data and the validation data. The training session was paused every 10 cycles for the root-mean-squared errors to be computed with reference to both sets of data. As expected the neural network performs better on the training data than on the validation data. For both $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ the validation learning curves register their minimum points at about 100 training cycles.

Fig. 5.13. Root-mean-squared errors as functions of number of training cycles: (a) serviceability limit state; (b) ultimate limit state

5.3.4. Effect of size of training data

The size of the training data was small at the beginning of this study: 17 samples were selected based on the axial-point design with a sampling parameter of one. The accuracy of the neural network should logically improve by increasing the number of sample points, which inevitably, will require additional evaluation of the performance function. Without altering the architecture of the neural network or the learning-rate parameter and the momentum constant, the neural network is retrained based on a 33-sample dataset and a 49-sample dataset. The 33-sample dataset contains the original 17 samples and 16 new samples selected based on the axial-point design with a sampling parameter of 0.5. The 49-sample dataset contains the original 17 samples and 32 new samples selected based on the axial-point design with sampling parameters of 1/3 and 2/3. In other words all the additional samples are confined within the same one-standard-deviation space and this should ensure a certain level of comparability between the neural networks. All three neural networks are subjected to the same amount of learning at 100 training cycles, based on observations from the previous section that this is the optimum training duration at which the neural network would retain its ability to generalize.

Table 5.5 compares the probabilistic results obtained based on the three neural networks. The top table is for the serviceability limit state given by $g_1(\mathbf{x})$, and the bottom table is for the ultimate limit state given by $g_2(\mathbf{x})$. In each table, the first two rows of numbers are the reliability indices computed by FORM and SORM, respectively. The last row shows the failure probabilities obtained using Monte Carlo simulation with importance sampling. In general the reliability indices and failure probabilities provided by the neural networks are comparable with those given by the direct stiffness method. The results of the 33-sample and 49-sample neural networks resemble those of the response surface method discussed in the previous chapter (compare Table 5.1 and Table 5.5). The resemblance may be due to the fact that both the response surface approach and the neural network approach rely on the same finite-element program for calculating the deterministic

pile responses. If the accuracies of the three neural networks may be assessed with reference to the response surface method, there is noticeable improvement in the neural network approach when the number of training data is increased from 17 samples to 33 samples. That improvement, however, is considerably reduced when the number of training samples is further increased to 49.

5.4. Concluding remarks

In this chapter the response surface method and the artificial neural network were employed to provide surrogates for the limit-state surface on which stochastic analyses of laterally-loaded piles could be carried out. For the RSM, the coupling of probabilistic methods with an existing deterministic pile analysis program was realized with the construction of a quadratic response surface model in the vicinity of the design point. The influences of cross terms, experimental designs and sampling parameters with reference to the accuracy and efficiency of the quadratic response surface model were discussed. There are relative merits and limitations between the conventional deterministic response surface method (based on simple polynomials) and the more advanced stochastic response surface method (based on Hermite polynomials). For the ANN, reliability approaches were coupled with a standalone deterministic pile analysis program using a multilayer perceptron neural network. The dataset for training the neural network consists of samples that were taken from the neighborhood of the design point. The well-established backpropagation learning algorithm was used to provide training to the neural network. The optimum number of hidden neurons, as well as appropriate values for the learning-rate parameter and the momentum constant could not be known beforehand and required a trial-and-error process to be ascertained. An early-stopping rule based on cross-validation was also incorporated to prevent overtraining of the neural network. It was found that the probabilistic results of the neural network could be improved by increasing the number of samples in the training dataset, but the amount of improvement would reduce significantly once a certain threshold number of samples were reached.

Table 5.5. Results of reliability analysis based on three artificial neural networks trained with different numbers of samples with respect to: serviceability limit state (top) and ultimate limit state (bottom)

(a) Pile deflection failure (SLS)

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
17-sample training dataset	1.5484	6.08	1.5288	6.32	6.29
33-sample training dataset	1.5735	5.78	1.5556	5.99	5.97
49-sample training dataset	1.5745	5.77	1.5563	5.98	5.96
Direct stiffness method	1.5568	5.98	1.5430	6.14	6.15

(b) Bending moment failure (ULS)

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
17-sample training dataset	1.7789	3.76	1.7884	3.69	3.70
33-sample training dataset	1.7940	3.64	1.8007	3.59	3.60
49-sample training dataset	1.8000	3.59	1.8075	3.53	3.55
Direct stiffness method	1.7910	3.66	1.7952	3.63	3.62

Note: β_{SORM} is computed based on $-\Phi^{-1}(p_{f,\text{SORM}})$.

All neural network models are constructed based on FEM using FB-MultiPier.

Chapter 6 Case Study: Laterally Loaded Piles at Roosevelt Bridge

6.1. Introductory comments

This chapter studies a case that was previously reported by Ruesta and Townsend (1997). An isolated single pile and a pile group of 16 piles arranged in a 4-by-4 configuration were subjected to static loading. Each pile in the group was of identical geometry and material as the single pile. The foundation consisted of sand overlying a partially cemented sand at the Roosevelt Bridge replacement, Stuart, Florida. Based on the lateral load test data and in-situ test data in that paper, reliability analyses are carried out in the following sections.

6.2. Site description

The test site as shown in Fig. 6.1 is submerged under 2 m of water and has two well-defined layers of cohesionless soil: a loose fine sand underlain by partially cemented sand that is difficult to penetrate. The 4-m thick surface layer of slightly silty fine sand contains occasional shell fragments, with standard penetration test (SPT) N values ranging between 2 and 8, and estimated ϕ values between 30° and 38° . Underlying the loose layer exists a partially cemented fine sand with discontinuous layers of weakly cemented to cemented sandy limestone with SPT N values ranging between 11 and 35, and estimated ϕ values between 40° and 46° .

The process of pile installation consisted of jetting to 7.6 m, placing the piles, and subsequently driving them with an air hammer. Predriving and postdriving in-situ tests were performed to establish relationships at the site. The predriving SPT N blow counts are shown in Fig. 6.1, and were used to correlate soil properties for computing p - y curves. The friction angle ϕ , unit weight γ' , and subgrade modulus k_{py} , used in the Reese et al. (1974) p - y curve were estimated through SPT tests as

32° , 8.9 kN/m^3 , and 16.3 MNm^3 , respectively for the upper layer, and as 42° , 11.1 kN/m^3 , and 34 MNm^3 for the lower layer.

The lateral load test program consisted of three fully instrumented lateral load tests: a single isolated 0.76-m square pile and two 16-pile groups. Each pile is about 12.1 m long with 2.1 m exposed above the mudline, and is of composite material with a 35-cm steel pipe (of 9.5-mm wall thickness) encased in concrete, to form a final width of 0.76 m. As a result, the pile flexural rigidity needs to be back-analyzed in this study, by using the information from the load-deflection curves in Ruesta and Townsend (1997). It is found that the flexural rigidity does not stay constant, but instead reduces with increasing lateral load. A linear approximation is fitted to the load versus rigidity variation, and is then used in the direct stiffness method proposed earlier in this thesis. The pile responses computed below are based on the direct stiffness method.

Fig. 6.2 shows the measured and computed load-deflection curves. The results show that the agreement between predicted and observed load-deflection relationships is reasonable. Fig. 6.3 presents the measured and computed bending moments along the pile for a lateral load of 138 kN. At this load, the predicted bending moment profile appears to be accurate. The above validation lends credence to the use of the proposed model in obtaining the response of laterally-loaded piles for the case in hand. The load versus maximum bending moment curve was not reported in Ruesta and Townsend (1997), and hence no comparison is made in that respect.

6.3. Uncertainty in soil properties

The variability in the design soil property ξ_d is a function of inherent soil variability, measurement error and transformation uncertainty. The combined point coefficient of variation (COV) of ξ_d is given by (Phoon & Kulhawy 1999b):

$$\sigma_{\xi_d}^2 \approx \left(\frac{\partial T}{\partial w} \right) \sigma_w^2 + \left(\frac{\partial T}{\partial e} \right) \sigma_e^2 + \left(\frac{\partial T}{\partial \varepsilon} \right) \sigma_\varepsilon^2 \quad (6.1)$$

where T is the transformation function, σ_w , σ_e and σ_ε are the standard deviations of inherent soil variability w , measurement error e , and transformation uncertainty ε , respectively. The design soil property ξ_d is estimated from a test measurement ξ_m based on the following probabilistic transformation model (Phoon & Kulhawy 1999b):

$$\xi_d = T(\xi_m, \varepsilon) = T(t + w + e, \varepsilon) \quad (6.2)$$

where t is the deterministic trend. For the case in hand, the design soil property ξ_d is the friction angle ϕ , which is estimated from SPT N based on the lower-right table in Fig. 6.1. Kishida's (1967) empirical formula is found to be somewhat comparable with the above SPT correlation:

$$\phi(^{\circ}) \approx \sqrt{20N} + 15 \quad (6.3)$$

We may regard $N = t + w + e$, and following the procedure illustrated in Phoon and Kulhawy (1999a), determine the mean and variance of ϕ as follows:

$$m_\phi \approx \sqrt{20t} + 15 \quad (6.4a)$$

$$\sigma_\phi^2 \approx m_\phi^2 [0.5^2 (COV_w^2 + COV_e^2) + (\ln 10)^2 \sigma_\varepsilon^2] \quad (6.4b)$$

The COV of ϕ is given by:

$$COV_\phi^2 \approx 0.5^2 (COV_w^2 + COV_e^2) + (\ln 10)^2 \sigma_\varepsilon^2 \quad (6.5)$$

The COV of inherent variability (COV_w) and measurement error (COV_e) for N are estimated to be between 25 and 40% and 15 and 35%, respectively, based on

Phoon and Kulhawy (1999a). The standard deviation of transformation uncertainty (σ_ϵ) of the empirical relationship by Kishida (1967) is unknown. Using these numerical data, the total COV (COV_φ) for φ correlated from N is between 14 and 26%, without considering transformation uncertainty. To prevent the value of φ from falling in the negative realm, a lognormal distribution is commonly adopted. The vertical scale of fluctuation for φ was not reported for the present case, but it is likely to lie between 2 and 6 m, as discussed in Phoon and Kulhawy (1999a). Assuming that the exponential autocorrelation function is used, the autocorrelation distance for φ should lie between 1 and 3 m.

6.4. Reliability analysis of a single pile

The vertical variation of φ is modeled by 12 discrete points uniformly spaced at 1 m apart. The upper fine sand layer has 5 discrete points ($\varphi_{U,1}$ to $\varphi_{U,5}$), and the lower cemented sand layer has 7 ($\varphi_{L,1}$ to $\varphi_{L,7}$). The point values of $\varphi_{U,k}$ have mean values of 32° ; while the point values of $\varphi_{L,k}$ have mean values of 42° . The COV for these point values is between 14 and 26%. The point values that belong to the same sand layer are spatially autocorrelated according to the established exponential model:

$$\rho_{ij} = \exp\left\{-\frac{|z_i - z_j|}{r_o}\right\} \quad (6.6)$$

where ρ_{ij} is the correlation coefficient between any two point values of φ at depths z_i and z_j ; and r_o is the autocorrelation distance. For the case in hand, the value of r_o is between 1 and 3 m.

The performance functions with respect to pile-head deflection failure and maximum bending moment failure are:

$$g_1(\mathbf{x}) = y_{\text{lim}} - y_0 \quad (6.7a)$$

$$g_2(\mathbf{x}) = M_p - M_{\max} \quad (6.7b)$$

The limiting pile-head deflection $y_{\lim}$ and the plastic bending moment M_p are taken as 0.025 m and 900 kNm, respectively. The values of pile-head deflection y_0 and maximum bending moment $M_{\max}$ are computed using the direct stiffness method.

A series of reliability analysis was carried out, considering the condition in which $P_{\text{top}} = 200$ kN and $M_{\text{top}} = 420$ kN. Several cases were analyzed, assuming different values for COV_ϕ (0.14, 0.20 or 0.26) and r_o (1, 2 or 3 m). The results are summarized in Table 6.1(a) for pile deflection failure and Table 6.1(b) for bending moment failure. The first-order reliability index (FORM β) is computed based on the Low and Tang (2007) method. The first-order probability of failure ($p_{f, \text{FORM}}$) is then approximated as:

$$p_{f, \text{FORM}} \approx 1 - \Phi(\beta_{\text{FORM}}) = \Phi(-\beta_{\text{FORM}}) \quad (6.8)$$

The second-order probability of failure ($p_{f, \text{SORM}}$) is obtained based on the procedure of Chan and Low (2011). The second-order reliability index (SORM β) is then backcalculated as follows:

$$\beta_{\text{SORM}} = \Phi^{-1}(1 - p_{f, \text{SORM}}) = -\Phi^{-1}(p_{f, \text{SORM}}) \quad (6.9)$$

The probabilities of failure estimated by FORM and SORM are checked against the probability of failure obtained by Monte Carlo simulations ($p_{f, \text{MCS}}$) with importance sampling. In Table 6.1, the MCS are performed with 1000 – 5000 trials. The second-order approximations are seen to provide significant improvement over the first-order approximations.

Table 6.2(a) and 6.2(b) show three cases of reliability analyses obtained from the response surface and neural network approaches, for pile deflection failure and bending moment failure, respectively. For comparison, results based on the direct

approach are also presented. The difference between the direct approach and the response surface/neural network approaches is that the former carries out reliability analysis directly on the implicit performance functions evaluated using the direct stiffness method, whereas the latter two conducts reliability analysis on some approximate surrogate counterparts of the performance functions. It is observed that the solutions obtained from the response surface and neural network approaches are generally in good agreement with the solutions obtained from the direct approach. The second-order polynomial function with no cross terms is adopted here for the response surface approach; while a multilayer perceptron with a 12-6-1 structure is employed for the neural network approach. In both approaches, the response surface and the neural network are fitted to the same design point.

Table 6.3 shows the probabilities of failure for the two component failure modes (pile deflection failure and bending moment failure) and the system probabilities of failure. The lower and upper bounds (LB and UB) of the system failure probability as shown are Kounias-Ditlevsen bimodal bounds, obtained based on the FORM and SORM reliability index of each failure mode as well as the correlation coefficient between the two failure modes. Five Monte Carlo simulations each with importance sampling of size 1000 – 5000 provide estimates for the *true* system failure probability in each case. In all three cases, the system failure probability bounds based on SORM are much more accurate and consistent than the system failure probability bounds based on FORM that do not account for any nonlinearity of the performance functions.

A comparison is also made between the Kounias-Ditlevsen approach and the proposed system reliability approach which attempts to provide a more precise result by obtaining the joint failure probability based on the bivariate standard normal cumulative distribution function directly. Table 6.4 shows the Kounias-Ditlevsen result in terms of (lower and upper) bounds, and the single value approximation yielded by proposed approach. (Note that the proposed approach gives a point-value result *only* for cases with *two* failure modes; for cases with

more than two failure modes, the proposed approach gives a bounded result which is expected to be narrower than the Kounias-Ditlevsen bounds). The consistency between the proposed approach (based on SORM reliability indices as inputs) and Monte Carlo simulations is encouraging.

The sensitivity factors of the random variables are computed based on the formulation derived in this study (i.e. $\partial\beta/\partial x_i'$) and are shown in Table 6.5. For both performance functions, the values of $\partial\beta/\partial x_i'$ of $\varphi_{L,3}$, $\varphi_{L,4}$, $\varphi_{L,5}$, $\varphi_{L,6}$ and $\varphi_{L,7}$ are comparatively smaller than the rest of the random variables. This implies that, for the cases in hand, the reliability index is not sensitive to these random variables with small values of $\partial\beta/\partial x_i'$, and one can probably omit them and regard them as deterministic. It would be interesting to see whether this omission leads to any significant error in the resulting reliability indices and probabilities of failure. In Table 6.6(a) and 6.6(b), two scenarios are considered for each case. The first scenario ('without omission') retains all 12 of the random variables; the second scenario ('with omission') retains only the first 7 random variables. The difference between the two scenarios is practically negligible in all of the three cases.

From the direct stiffness model presented earlier, the lateral load that will just produce an allowable pile-head deflection of 0.025 m is found to be 219 kN, and the lateral load that will just produced an allowable bending moment of 900 kNm is found to be 225 kN. By applying a global factor of safety of, say 3.0, on these two loads, the allowable loads are estimated as 73 kN and 75 kN. Usually the lesser of the above two values will be adopted as the allowable load for the system, as per the conventional factor-of-safety approach. Alternatively, allowable loads can be specified corresponding to a target reliability index such as $\beta = 3.0$. This is achieved by plotting the variation of the reliability index (FORM β and SORM β) for different lateral loads on the pile, as shown in Figs. 6.4(a) and 6.4(b). The allowable loads are the values at which the respective curves intersect with the horizontal line corresponding to $\beta = 3.0$. Again, three cases are considered for each of the two failure modes. The allowable loads are summarized in Table 6.7.

One may note that in each case, the allowable load with respect to bending moment failure is higher than that with respect to pile deflection failure, and that for the same failure mode, the allowable load based on SORM β is higher than that based on FORM β . For the cases in hand, the allowable loads corresponding to $\beta = 3.0$ fall in the range of about 150 – 200 kN, which are much higher than the allowable loads of 73 and 75 kN corresponding to $FS = 3.0$.

Fig. 6.1. Single pile and site characteristics at Roosevelt Bridge (after Ruesta & Townsend 1997)

Fig. 6.2. Measured and computed (direct stiffness method, DSM) load-deflection curves for single pile

Fig. 6.3. Measured and computed (direct stiffness method, DSM) bending moment profiles for single pile at a lateral load of 138 kN

Table 6.1. Results of reliability analysis for different values of COV and r_o

(a) Pile deflection failure

COV	r_o	FORM		SORM		MCS
		β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
0.14	1	1.593	5.56	1.787	3.70	3.78
0.14	2	1.364	8.63	1.520	6.42	6.37
0.14	3	1.247	10.62	1.375	8.46	8.38
0.20	1	1.011	15.60	1.282	9.99	10.06
0.20	2	0.866	19.32	1.087	13.86	13.79
0.20	3	0.792	21.42	0.975	16.48	16.19
0.26	1	0.672	25.08	1.017	15.45	15.32
0.26	2	0.576	28.24	0.855	19.62	19.36
0.26	3	0.526	29.93	0.763	22.26	23.40

(b) Bending moment failure

COV	r_o	FORM		SORM		MCS
		β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
0.14	1	2.616	0.44	2.725	0.32	0.33
0.14	2	2.235	1.27	2.315	1.03	1.06
0.14	3	2.049	2.02	2.111	1.74	1.78
0.20	1	1.775	3.79	1.915	2.78	2.80
0.20	2	1.523	6.38	1.614	5.32	5.37
0.20	3	1.398	8.11	1.460	7.22	7.28
0.26	1	1.310	9.51	1.459	7.22	7.24
0.26	2	1.120	13.13	1.219	11.15	11.18
0.26	3	1.018	15.44	1.111	13.34	13.14

Table 6.2. Results of reliability analysis for direct, response surface and neural network approaches

(a) Pile deflection failure

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Case 1: COV = 0.14, $r_o = 1$ m					
Direct Approach	1.593	5.56	1.787	3.70	3.78
Response Surface Approach	1.588	5.61	1.799	3.60	3.50
Neural Network Approach	1.588	5.61	1.800	3.59	3.47
Case 2: COV = 0.20, $r_o = 2$ m					
Direct Approach	0.866	19.32	1.087	13.86	13.79
Response Surface Approach	0.865	19.36	1.091	13.77	13.41
Neural Network Approach	0.865	19.36	1.092	13.75	13.63
Case 3: COV = 0.26, $r_o = 3$ m					
Direct Approach	0.526	29.93	0.763	22.26	23.40
Response Surface Approach	0.525	29.96	0.756	22.49	21.96
Neural Network Approach	0.525	29.96	0.758	22.41	22.06

(b) Bending moment failure

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Case 1: COV = 0.14, $r_o = 1$ m					
Direct Approach	2.616	0.44	2.725	0.32	0.33
Response Surface Approach	2.570	0.51	2.640	0.41	0.42
Neural Network Approach	2.570	0.51	2.645	0.41	0.42
Case 2: COV = 0.20, $r_o = 2$ m					
Direct Approach	1.523	6.38	1.614	5.32	5.37
Response Surface Approach	1.495	6.75	1.563	5.90	5.93
Neural Network Approach	1.495	6.75	1.556	5.98	6.04
Case 3: COV = 0.26, $r_o = 3$ m					
Direct Approach	1.018	15.44	1.111	13.34	13.14
Response Surface Approach	1.001	15.84	1.061	14.44	14.34
Neural Network Approach	1.001	15.84	1.060	14.46	14.43

Table 6.3. Results of reliability analysis for component and system reliability

Method	Component		System	
	$p_{F1}(\%)$	$p_{F2}(\%)$	$p_{F\text{ sys}}(\%)$	
Case 1: COV = 0.14, $r_o = 1$ m				
FORM	5.56	0.44	<i>LB:</i> 5.61	<i>UB:</i> 5.71
SORM	3.70	0.32	<i>LB:</i> 3.75	<i>UB:</i> 3.82
MCS	3.78	0.33	3.88	
Case 2: COV = 0.20, $r_o = 2$ m				
FORM	19.32	6.38	<i>LB:</i> 19.76	<i>UB:</i> 21.62
SORM	13.86	5.32	<i>LB:</i> 14.65	<i>UB:</i> 16.24
MCS	13.79	5.37	15.42	
Case 3: COV = 0.26, $r_o = 3$ m				
FORM	29.93	15.44	<i>LB:</i> 30.25	<i>UB:</i> 35.36
SORM	22.26	13.34	<i>LB:</i> 23.65	<i>UB:</i> 28.27
MCS	23.40	13.14	26.90	

Table 6.4. Results of system reliability analysis for two different approaches

	<u>Kounias-Ditlevsen approach</u>	<u>Proposed approach</u>
Method	$p_{F\text{ sys}}(\%)$	$p_{F\text{ sys}}(\%)$
Case 1: COV = 0.14, $r_o = 1$ m		
FORM	5.61 ~ 5.71	5.66
SORM	3.75 ~ 3.82	3.79
(MCS approx. $p_{F\text{ sys}} = \mathbf{3.88\%}$)		
Case 2: COV = 0.20, $r_o = 2$ m		
FORM	19.76 ~ 21.62	20.75
SORM	14.65 ~ 16.24	15.43
(MCS approx. $p_{F\text{ sys}} = \mathbf{15.42\%}$)		
Case 3: COV = 0.26, $r_o = 3$ m		
FORM	30.25 ~ 35.36	32.35
SORM	23.65 ~ 28.27	25.78
(MCS approx. $p_{F\text{ sys}} = \mathbf{26.90\%}$)		

Table 6.5. Sensitivity factors for the random variables

z (m)	Variable	$\partial\beta/\partial x'_i$ w.r.t. $g_1(x)$			$\partial\beta/\partial x'_i$ w.r.t. $g_2(x)$		
		Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
0	$\varphi_{U,1}$	-0.1067	-0.0860	-0.0755	-0.1403	-0.1159	-0.0957
1	$\varphi_{U,2}$	-0.4808	-0.4089	-0.3753	-0.5495	-0.4754	-0.4215
2	$\varphi_{U,3}$	-0.5463	-0.4694	-0.4340	-0.5701	-0.4941	-0.4551
3	$\varphi_{U,4}$	-0.3756	-0.3134	-0.2832	-0.2739	-0.2241	-0.2300
4	$\varphi_{U,5}$	-0.1232	-0.0998	-0.0860	-0.0085	-0.0084	-0.0205
4	$\varphi_{L,1}$	-0.3029	-0.2613	-0.2377	-0.3425	-0.3254	-0.2889
5	$\varphi_{L,2}$	-0.1779	-0.1746	-0.1577	-0.0398	-0.0705	-0.0822
6	$\varphi_{L,3}$	-0.0024	-0.0065	-0.0077	-0.0051	-0.0031	-0.0006
7	$\varphi_{L,4}$	-0.0004	-0.0026	-0.0031	0.0000	-0.0028	-0.0005
8	$\varphi_{L,5}$	-0.0001	-0.0009	-0.0020	0.0000	-0.0005	-0.0002
9	$\varphi_{L,6}$	0.0000	-0.0003	-0.0014	0.0000	-0.0003	-0.0002
10	$\varphi_{L,7}$	0.0000	-0.0001	-0.0006	0.0000	-0.0002	0.0000

Table 6.6. Results of reliability analysis with and without omission of insignificant random variables

(a) Pile deflection failure

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Case 1: COV = 0.14, $r_o = 1$ m					
Without omission (using 12 r.v.)	1.593	5.56	1.787	3.70	3.78
With omission (using 7 r.v.)	1.593	5.56	1.792	3.65	3.68
Case 2: COV = 0.20, $r_o = 2$ m					
Without omission (using 12 r.v.)	0.866	19.32	1.087	13.86	13.79
With omission (using 7 r.v.)	0.868	19.26	1.082	13.96	13.86
Case 3: COV = 0.26, $r_o = 3$ m					
Without omission (using 12 r.v.)	0.526	29.93	0.763	22.26	23.40
With omission (using 7 r.v.)	0.527	29.90	0.768	22.12	21.85

(b) Bending moment failure

Method	FORM		SORM		MCS
	β	$p_F(\%)$	β	$p_F(\%)$	$p_F(\%)$
Case 1: COV = 0.14, $r_o = 1$ m					
Without omission (using 12 r.v.)	2.616	0.44	2.725	0.32	0.33
With omission (using 7 r.v.)	2.616	0.44	2.724	0.32	0.37
Case 2: COV = 0.20, $r_o = 2$ m					
Without omission (using 12 r.v.)	1.523	6.38	1.614	5.32	5.37
With omission (using 7 r.v.)	1.523	6.38	1.613	5.33	5.25
Case 3: COV = 0.26, $r_o = 3$ m					
Without omission (using 12 r.v.)	1.018	15.44	1.111	13.34	13.14
With omission (using 7 r.v.)	1.018	15.44	1.107	13.41	13.70

(a) Pile deflection failure

(b) Bending moment failure

Fig. 6.4. Variation of reliability index with lateral load for different cases on single pile (Case 1: $COV = 0.14$, $r_o = 1$ m; Case 2: $COV = 0.20$, $r_o = 2$ m; Case 3: $COV = 0.26$, $r_o = 3$ m)

Table 6.7. Allowable loads from probabilistic and deterministic approaches (single pile)

Method	Allowable load corresponding to $\beta = 3.0$ w.r.t.:	
	Pile deflection failure	Bending moment failure
Case 1: COV = 0.14, $r_o = 1$ m		
FORM	187 kN	197 kN
SORM	189 kN	198 kN
Case 2: COV = 0.20, $r_o = 2$ m		
FORM	170 kN	184 kN
SORM	173 kN	185 kN
Case 3: COV = 0.26, $r_o = 3$ m		
FORM	154 kN	175 kN
SORM	158 kN	176 kN
Note: Allowable load corresponding to Factor of Safety = 3.0 is:		
73 kN (w.r.t. pile deflection failure)		
75 kN (w.r.t. bending moment failure)		

6.5. Reliability analysis considering group effect

Ruesta and Townsend (1997) noted that the concept of p -multipliers applied to the single pile test results seemed to work well to account for group effects, and was able to predict the pile group behavior accurately. The p -multipliers were determined by trial and error for each pile. The total group has an overall average p -multiplier of 0.55 and an efficiency of 0.8. The average p -multiplier for the leading, middle leading, middle trailing, and trailing row is 0.8, 0.7, 0.3, and 0.3, respectively. Fig. 6.5 presents the pile driving sequence and p -multiplier distribution within the group. The piles are spaced at 2.28 m (i.e. 3 times the pile width) apart. The overall width of the pile group is 7.6 m.

The direct stiffness model for the single pile is modified for pile group analysis, by multiplying all the soil reactions of the p - y curves by the average p -multiplier of 0.55. Fig. 6.6 compares the measured load-deflection curve of the pile group with that predicted using the modified direct stiffness method. The results show good agreement between predicted and observed load-deflection relationships. The load versus maximum bending moment curve was not available in Ruesta and Townsend (1997) for comparison.

It is well established that a ‘group’ pile (i.e., a pile that forms part of a group) will deflect more than a single pile under the same average load per pile. Also, the maximum bending moment in the ‘group’ pile is higher than that in the single pile. Therefore, it is expected that the allowable loads for the ‘group’ pile will be lower than those for the single pile, regardless of whether deterministic or probabilistic approach is used. Based on the direct stiffness model, the lateral load that will just produce an allowable pile-head deflection of 0.025 m in the average pile is found to be 145 kN, and the lateral load that will just produced an allowable bending moment of 900 kNm in the average pile is found to be 203 kN. By applying a global factor of safety of 3.0 on these two loads, the allowable loads are estimated as 48 kN and 68 kN per pile. The lesser value usually governs the design decision, thus the allowable load for the 16-pile group at hand is 16×48 kN or 768 kN.

As was done earlier for the single isolated pile, allowable loads can also be determined corresponding to a target reliability index such as $\beta = 3.0$, now for the average ‘group’ pile based on a p -multiplier of 0.55. The variation of the reliability index (FORM β and SORM β) for different lateral loads on the average ‘group’ pile is shown in Figs. 6.7(a) and 6.7(b). The allowable loads are summarized in Table 6.8. For the three cases in hand, the allowable loads corresponding to $\beta = 3.0$ fall in the range of 96 – 181 kN per pile. These are still higher than the allowable loads of 48 – 68 kN corresponding to $FS = 3.0$. In making a design-related decision, the practitioner is more likely to select an allowable load that is determined with respect to the dominant failure mode. In

the present case, the pile group performance is governed by pile deflection failure, and so the range of allowable loads per pile is between 96 and 121 kN, or between 1536 and 1936 kN per group, depending on the level of uncertainty.

The above discussion considers the average pile group effect and therefore assumes that the average p -multiplier of 0.55 applies to every pile within the group. In reality, the p -multiplier may be higher than the average value for the leading row of piles and lower for the trailing row of piles. There is also a tendency for the p -multiplier to be higher for the corner piles than the interior piles. As mentioned before, the average backcalculated p -multipliers for the leading, middle leading, middle trailing, and trailing row is 0.8, 0.7, 0.3, and 0.3, respectively. Based on this information, additional but similar reliability analyses can be carried out for any specific pile within the group.

Fig. 6.5. Pile driving sequence and p -multiplier distribution (after Ruesta & Townsend 1997)

Fig. 6.6. Measured and computed (direct stiffness method, DSM) load-deflection curves for group pile

(a) Pile deflection failure

(b) Bending moment failure

Fig. 6.7. Variation of reliability index with lateral load for different cases on group pile (Case 1: COV = 0.14, r_o = 1 m; Case 2: COV = 0.20, r_o = 2 m; Case 3: COV = 0.26, r_o = 3 m)

Table 6.8. Allowable loads from probabilistic and deterministic approaches (pile group)

Method	Allowable load corresponding to $\beta = 3.0$ w.r.t.:	
	Pile deflection failure	Bending moment failure
Case 1: COV = 0.14, $r_o = 1$ m		
FORM	120 kN / pile	181 kN / pile
SORM	121 kN / pile	181 kN / pile
Case 2: COV = 0.20, $r_o = 2$ m		
FORM	107 kN / pile	172 kN / pile
SORM	109 kN / pile	172 kN / pile
Case 3: COV = 0.26, $r_o = 3$ m		
FORM	96 kN / pile	165 kN / pile
SORM	98 kN / pile	165 kN / pile
Note:	Allowable load corresponding to Factor of Safety = 3.0 is:	
	48 kN / pile (w.r.t. pile deflection failure)	
	68 kN / pile (w.r.t. bending moment failure)	

6.6. Concluding remarks

A realistic example of a single pile and a 4-by-4 pile group at Roosevelt Bridge was presented in this chapter. While most of the discussion was made with reference to the single pile, all of the reliability analysis approaches that were illustrated for the single pile can also be applied to the pile group. It should be noted that the probabilistic methods can be used in conjunction with any standalone pile analysis program (such as the FEM software FB-MultiPier or FDM software LPile) via the response surface method or artificial neural network. For convenience the direct stiffness method was used as it can be integrated with the probabilistic methods quite easily. Nonetheless the accuracy of the direct stiffness method was checked to ensure that it was consistent with the measured pile responses before transiting from deterministic to stochastic analysis.

The objective of this chapter is to show how reliability-based analysis and design may be carried out in a practical context. Comparison between the probability of failure obtained from first-order/second-order reliability methods (FORM/SORM) and those obtained from Monte Carlo simulations (MCS) indicated that FORM was not always in agreement with MCS. This was because the limit state surface is nonlinear at the design point. In these cases, SORM provided more accurate estimates which were closer to MCS. As SORM generally requires fewer evaluations of the limit state function than MCS, there is a relatively higher degree of efficiency and feasibility in using SORM over MCS. Nevertheless it would be prudent to carry out MCS selectively to ensure that SORM does not differ significantly from MCS. The number of simulation trials can be substantially reduced with the use of a variance reduction method such as importance sampling.

A large number of random variables translate to high computational cost. Less important random variables can be identified using the proposed sensitivity factors. The practicing engineer may use this as a tool to decide whether some of these random variables may be excluded or omitted from subsequent reliability analysis.

In practice the overall performance of the laterally-loaded pile is controlled by pile deflection failure and bending moment failure. The two failure modes are correlated. For the present example the pile deflection failure mode is more critical than the bending moment failure mode, and therefore contributes to the main bulk of the system probability of failure.

The reliability-based design approach was illustrated first for the single pile, and then for the average 'group' pile that forms part of the 4-by-4 pile group. This approach permits a rational choice of allowable lateral loads depending on the uncertainty level and spatial variability of the soil. The probabilistic allowable load corresponds to some specified reliability index, which can be directly interpreted in terms of the probability of failure.

Chapter 7 Conclusions and Recommendations

7.1. Summary and conclusions

A summary of the findings of this study and of the conclusions drawn is as follows:

A simple direct stiffness method was developed based on the beam-on-elastic-foundation model. The pile is idealized as one-dimensional beam elements and the soil as nonlinear spring elements. The direct stiffness algorithm presented is able to analyze nonlinear soil behavior (by incorporating nonlinear p - y criteria) and nonlinear pile flexural behavior (by incorporating nonlinear moment-curvature criteria). Results obtained by the proposed direct stiffness method agreed well with those obtained by the specialized finite-element program FB-MultiPier.

Reliability analyses were performed on a laterally-loaded single pile using the first-order reliability method (FORM), second-order reliability method (SORM) and Monte Carlo simulations (MCS). The design points and reliability indices corresponding to deflection failure and bending moment failure were computed by the Low & Tang (2007) approach. As an extension of the FORM approach, a practical procedure for SORM based on the concepts of Der Kiureghian et al. (1987) was developed in this study. The procedure was rendered efficient by a short program created in this study to calculate the principal curvatures of the limit-state surface. A user-defined function was also given to calculate the second-order failure probability automatically. It was shown that SORM was generally more accurate than FORM for cases involving nonlinear limit-state functions. A simple procedure for implementing the importance sampling method based on Rubinstein (1981), Shinozuka (1983) and Melchers (1984) was also presented and was shown to be much more efficient than the ordinary Monte Carlo method.

A simple sensitivity factor of the reliability index, as given by $\partial\beta/\partial x'_i$, was developed in Chapter 6 for correlated random variables. The evaluation of $\partial\beta/\partial x'_i$ was based only on the direction cosines and the correlation matrix. The sensitivities of the reliability index with respect to the mean and standard deviation, as given by $\partial\beta/\partial\mu_i$ and $\partial\beta/\partial\sigma_i$, were also derived. The proposed sensitivity factors agreed very well with the sensitivities obtained by numerical differentiation based on FORM. The inferred sensitivity factors of the failure probability $\partial p_f/\partial\mu_i$ and $\partial p_f/\partial\sigma_i$ could be inaccurate when compared with results obtained by Monte Carlo simulations because variations in the statistical parameters of the random variables might cause changes in the nonlinearity of the limit-state function. More accurate estimates of the sensitivity of the failure probability could be obtained by numerical differentiation based on SORM. The proposed sensitivity factors, when compared with regression coefficients, rank correlation coefficients and omission sensitivity factors, had similar rankings of the relative significance of the stochastic parameters. The above are observations for the case in hand. Further studies are certainly needed to develop more general conclusions or recommendations for the proposed sensitivity factors.

The Kounias-Ditlevsen bimodal bounds were used to approximate the system failure probability of a laterally-loaded pile considering two failure modes. More accurate system bounds could be obtained based on SORM than FORM for cases with nonlinear limit-state surfaces. The joint failure probability between any two failure modes was typically approximated by the bivariate standard normal cumulative distribution function which was also expressed in terms of bounds. By replacing the bounds by a single value of the bivariate standard normal integral, the precision of the system bounds could be improved. It was found that one limitation of this approach was that the very narrow improved bounds might miss the true system failure probability for nonlinear limit-state surfaces. However, this is due to the fundamental assumption that the joint failure probability is bounded by hyperplanes being invalid, rather than the numerical accuracy of the bivariate standard normal integral.

The reliability-based design approach was in line with the spirit of load-and-resistance-factor design. In reliability-based design no assumption is made with regards to the partial factors or the characteristic values of the underlying parameters. The designer is allowed to incorporate his/her estimates of parametric uncertainty and correlation in the analysis. The partial factors are calculated as ratios of the design-point values and the mean values, and they provide useful information on the relative importance of the parameters. Reliability-based design and design based on partial factors can play complementary roles, as a clear understanding of reliability-based design is essential for developing more refined design codes and more instructive recommendations for future designers.

To overcome the complexity of carrying out reliability analysis of laterally-loaded piles based on standalone deterministic programs, the response surface method and the artificial neural network were used to provide approximations of the implicit limit-state functions. For the response surface approach, the simplified quadratic polynomial without cross terms was found to be a sufficiently accurate model. Variations in the sampling parameter and experimental design had little influence on the final results. Between the conventional and the stochastic response surface methods, the conventional approach was able to approximate $g(x) = 0$ based on a small number of samples, whereas the stochastic approach could approximate $g(x)$ in general, but required a much large number of samples. For the neural network approach, the multilayer perceptron was trained by the backpropagation algorithm to learn the behavior of the limit-state function. An optimization phase was needed to find the correct number of hidden neurons, as well as the appropriate learning-rate and momentum parameters. Increasing the number of samples in the training dataset could improve the accuracy of the artificial neural network but the improvement appeared to diminish as more samples were added.

7.2. Recommendations for future work

Some suggestions for future research are given below:

- (i) The analyses presented in this thesis were mainly on single piles subjected to lateral loads. Future studies may extend the deterministic and stochastic methods described herein to laterally-loaded pile groups.
- (ii) The spatial variability of soil properties was modeled by one-dimensional random fields because only analyses of laterally-loaded single piles were carried out. For laterally-loaded pile groups, two- or even three-dimensional random fields may be required to represent the behavior of the soil medium.
- (iii) Comparisons of reliability analysis based on FEM via surrogate models with stochastic/random FEM are expected to be of much value and would provide interesting insights into their relative strengths and limitations.
- (iv) The incorporation of importance sampling was shown to increase the efficiency of the Monte Carlo method. More advanced simulation techniques such as directional simulation and subset simulation may be used to further improve the efficiency of probabilistic analysis.
- (v) The radial-basis function neural network and the generalized regression neural network are possible alternatives to the artificial neural network architecture based on multilayer perceptrons used in this study. Future work may also consider other machine learning approaches such as support vector machines and genetic algorithms.
- (vi) The ordinary backpropagation algorithm was used to provide training to the artificial neural networks due to its simplicity. More advanced algorithms such as the Levenberg-Marquardt algorithm may be used to accelerate the learning process.

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The appendices of this thesis contain details of methods that have been used to obtain the results shown in the main chapters. While the underlying theories and equations of these methods are classical and established, most of the procedures presented are original to this doctoral research. Original work include (1) practical procedures for the second-order reliability method (Appendix B) and for the Monte Carlo method with importance sampling (Appendix C); (2) sensitivity factors of the reliability index with respect to statistical parameters for various probability distributions (Appendix D); (3) a simple improvement on the precision of system failure probability bounds (Appendix E); and (4) sequential reliability approaches based on the response surface method (Appendix F) and the artificial neural network (Appendix G);

Appendix A

Procedure for first-order reliability method

The procedure described below is based on the first-order reliability method using the Low and Tang (2007) algorithm. A commonly-cited example pertains to one plastic collapse failure mechanism of a one-bay frame (Der Kiureghian et al. 1987; Tvedt 1990; Hong 1999; Zhao and Ono 1999):

$$g(\mathbf{x}) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6 \quad (\text{A.1})$$

where the x_i 's are statistically independent lognormal random variables with means $\mu_1 = \dots = \mu_4 = 120$, $\mu_5 = 50$, $\mu_6 = 40$, and standard deviations $\sigma_1 = \dots = \sigma_4 = 12$, $\sigma_5 = 15$ and $\sigma_6 = 12$. A first-order reliability analysis is carried out in Microsoft Excel as shown in Fig. A.1. The quadratic form in Eq. (4.1) — $\beta = (\mathbf{x}'^T \mathbf{R}^{-1} \mathbf{x}')^{0.5}$ — is used to compute the reliability index β . For the present case the correlation matrix $\mathbf{R}$ is a 6×6 identity matrix. The equivalent standard normal vector $\mathbf{x}'$ is varied as six dimensionless numerical values; the corresponding $\mathbf{x}$ values are calculated from $\mathbf{x}'$ based on $x_i = \exp(\lambda_i + \zeta_i x'_i)$, where $\lambda_i = \ln \mu_i - 0.5 \zeta_i^2$ and $\zeta_i^2 = \ln[1 + (\sigma_i/\mu_i)^2]$. If x_i is of a different non-normal distribution, the inverse-transform method $x = F^{-1}[\Phi(x')]$ is used to obtain x_i from x'_i . The β index is obtained using Solver, by minimizing the quadratic form subject to $g(\mathbf{x}) = 0$. The first-order reliability index so obtained is 2.348; the inferred failure probability is 0.0094 based on $\Phi(-\beta)$. The design point in the original space is $\mathbf{x}^* = \{117.269, 115.240, 115.240, 117.269, 83.651, 55.449\}$.

$$g(\mathbf{x}) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6$$

$$\beta = \sqrt{\mathbf{x}'^T \mathbf{R}^{-1} \mathbf{x}'}$$

$g(\mathbf{x}^*)$	β
0	2.348

<i>ProbDist</i>		μ	σ	λ	ζ	$\mathbf{x}^*$	$\mathbf{x}'^*$
Lognormal	x_1	120	12	4.783	0.100	117.27	-0.181
Lognormal	x_2	120	12	4.783	0.100	115.24	-0.356
Lognormal	x_3	120	12	4.783	0.100	115.24	-0.356
Lognormal	x_4	120	12	4.783	0.100	117.27	-0.181
Lognormal	x_5	50	15	3.869	0.294	83.651	1.900
Lognormal	x_6	40	12	3.646	0.294	55.449	1.259

Correlation matrix $\mathbf{R}$

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

Boxed columns above

contain formulas:

$$\lambda_i = \ln \mu_i - 0.5 \zeta_i^2$$

$$\zeta_i^2 = \ln \left[1 + (\sigma_i / \mu_i)^2 \right]$$

$$x_i = \exp(\lambda_i + \zeta_i x'_i)$$

Fig. A.1. Reliability analysis of the plastic collapse of a one-bay frame using the first-order reliability method (FORM)

Appendix B

Procedure for second-order reliability method

The procedure described below is based on the point-fitted paraboloid method of Der Kiureghian et al. (1987). A commonly-cited example pertains to one plastic collapse failure mechanism of a one-bay frame (Der Kiureghian et al. 1987; Tvedt 1990; Hong 1999; Zhao and Ono 1999):

$$g(\mathbf{x}) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6 \quad (\text{B.1})$$

where the x_i 's are statistically independent lognormal random variables with means $\mu_1 = \dots = \mu_4 = 120$, $\mu_5 = 50$, $\mu_6 = 40$, and standard deviations $\sigma_1 = \dots = \sigma_4 = 12$, $\sigma_5 = 15$ and $\sigma_6 = 12$. A second-order reliability analysis is carried out in Microsoft Excel as shown in Fig. B.1, and is an extension of the first-order reliability analysis as shown in Fig. A.1. The first-order reliability index is 2.348; the inferred failure probability is 0.0094 based on $\Phi(-\beta)$. The exact failure probability is 0.0121 (Der Kiureghian et al. 1987; Tvedt 1990; Hong 1999; Zhao and Ono 1999); the relative percentage error of the FORM estimate is therefore about -22%.

The design point in the original space and in the reduced space are $\mathbf{x}^* = \{117.269, 115.240, 115.240, 117.269, 83.651, 55.449\}$ and $\mathbf{x}'^* = \{-0.181, -0.356, -0.356, -0.181, 1.900, 1.259\}$, respectively. The design point in the standard normal space $\mathbf{u}^*$ is equal to $\mathbf{x}'^*$ because the random variables $\mathbf{x}$ are statistically independent. If $\mathbf{x}$ are correlated, $\mathbf{u}^*$ can be obtained from $\mathbf{x}'^*$ by Cholesky factorization $\mathbf{u}^* = \mathbf{L}^{-1}\mathbf{x}'^*$ where $\mathbf{L}$ is the lower Cholesky matrix computed based on the correlation matrix $\mathbf{R}$. The user-defined matrix function *Cholesky* (Fig. B.2) can be used to obtain the lower Cholesky matrix. The VBA code in Fig. B.2 is a translation of the C++ program code found in Press et al. (2007).

The second-order approximation is carried out in a rotated standard normal space $\mathbf{u}'$ in which the u'_n axis coincides with $\mathbf{u}^*$ based on Gram-Schmidt orthogonalization. The Gram-Schmidt rotation matrix $\mathbf{S}$ is computed based on $\mathbf{u}^*$. The user-defined matrix function *GramSchmidt* (Fig. B.3) can be used to obtain the Gram-Schmidt orthogonalization matrix. For $n = 6$ random variables, $2(n - 1) = 10$ points are needed to construct the point-fitted paraboloid. According to Der Kiureghian et al. (1987) the points to be selected are as follows:

$$\begin{bmatrix} -k\beta & +k\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k\beta & +k\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k\beta & +k\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k\beta & +k\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k\beta & +k\beta \\ u'_{n,1-} & u'_{n,1+} & u'_{n,2-} & u'_{n,2+} & u'_{n,3-} & u'_{n,3+} & u'_{n,4-} & u'_{n,4+} & u'_{n,5-} & u'_{n,5+} \end{bmatrix} \quad (\text{B.2})$$

The coefficient k is equal to 1 since $\beta < 3$. The unknown coordinates $u'_{n,i\pm}$ (for $i = 1, 2, \dots, 5$) can be determined using the secant-rule method (Der Kiureghian et al. 1987) or by the proposed simplified approach described below.

In Fig. B.1, a template is set up to transform the fitting points from u' -space to x -space. As a start, the column vector $\mathbf{u}'$ is set to $\{0, 0, \beta\}$. The vector $\mathbf{u}$ is computed based on $\mathbf{u} = \mathbf{S}^{-1}\mathbf{u}'$, where $\mathbf{S}$ is the Gram-Schmidt rotation matrix. Since the random variables are statistically independent, $\mathbf{x}'$ is equal to $\mathbf{u}$; otherwise $\mathbf{x}'$ is obtained based on $\mathbf{x}' = \mathbf{L}\mathbf{u}$, where $\mathbf{L}$ is the lower Cholesky matrix. For lognormal-distributed random variables, the vector $\mathbf{x}$ is computed based on $x_i = \exp(\lambda_i + \zeta_i x'_i)$. If x_i is of a different non-normal distribution, the inverse-transform method $x = F^{-1}[\Phi(x')]$ is used to obtain x_i from x'_i .

The cell labeled as $g(\mathbf{x})$ contains the limit state function and depends on the $\mathbf{x}$ values obtained from the template set up above. The program *CurvatureSORM* (Fig. B.4), when run, automatically determines the unknown coordinates $u'_{n,i\pm}$ by calling Microsoft Excel's Goal Seek program and displays the computed principal

curvatures in column κ . The user-defined function *ProbFailSORM* (Fig. B.5) allows the user to compute second-order failure probabilities based on the principal curvatures obtained above and the reliability index β .

The principal curvatures of the point-fitted paraboloid are $\kappa = \{4.36 \times 10^{-3}, 2.83 \times 10^{-3}, 4.49 \times 10^{-3}, 5.31 \times 10^{-3}, -1.66 \times 10^{-1}\}$. The last value is obviously the dominant curvature, and by neglecting the other four values, Breitung's (1984) simple asymptotic formula $p_f \approx \Phi(-\beta)(1 + \beta\kappa_5)^{-0.5}$ gives a failure probability of 0.01207, which is almost the same as the exact answer.

		$g(\mathbf{x}^*)$		β	
		0		2.348	

<i>ProbDist</i>		μ	σ	λ	ζ	$\mathbf{x}^*$	$\mathbf{x}'^*$
Lognormal	x_1	120	12	4.783	0.100	117.27	-0.181
Lognormal	x_2	120	12	4.783	0.100	115.24	-0.356
Lognormal	x_3	120	12	4.783	0.100	115.24	-0.356
Lognormal	x_4	120	12	4.783	0.100	117.27	-0.181
Lognormal	x_5	50	15	3.869	0.294	83.651	1.900
Lognormal	x_6	40	12	3.646	0.294	55.449	1.259

	$\mathbf{u}^*$	$\mathbf{u}'$	$\mathbf{u}$	$\mathbf{x}'$	$\mathbf{x}$	κ
Boxed columns on the	-0.181	0	-0.390	-0.390	114.84	0.00436
right contain formulas:	-0.356	0	-0.768	-0.768	110.60	0.00283
$\mathbf{u}^* = \mathbf{x}'^*$	-0.356	0	-0.768	-0.768	110.60	0.00449
$\mathbf{u} = \mathbf{S}^{-1} \mathbf{u}'$	-0.181	0	-0.390	-0.390	114.84	0.00531
$\mathbf{x}' = \mathbf{u}$	1.900	-2.3482	0.103	0.103	49.364	-0.1659
$x_i = \exp(\lambda_i + \zeta_i x'_i)$	1.259	1.833	2.717	2.717	85.055	

Gram-Schmidt orthogonalization matrix $\mathbf{S}$						$g(\mathbf{x})$
0.990	0.000	0.000	0.000	0.000	0.142	0 (perfunc)
-0.038	0.963	0.000	0.000	0.000	0.267	
-0.036	-0.070	0.966	0.000	0.000	0.248	
-0.017	-0.034	-0.034	0.991	0.000	0.121	
0.106	0.209	0.209	0.106	0.588	-0.738	
-0.077	-0.152	-0.152	-0.077	0.809	0.536	

Fig. B.1. Reliability analysis of the plastic collapse of a one-bay frame using the second-order reliability method (SORM)

```

Function Cholesky(Corrmatrix)
'For positive-definite & symmetric Corrmatrix
'Based on original C++ code by Press et al.
'in "Numerical Recipes" 3rd ed., 2007, p.100

n = Corrmatrix.Rows.Count
ReDim L(1 To n, 1 To n) As Double, p(1 To n) As Double
For i = 1 To n
    For j = i To n
        sum = Corrmatrix(i, j)
        For k = 1 To i - 1
            sum = sum - L(i, k) * L(j, k)
        Next k
        If i = j Then
            p(i) = Sqr(sum): L(j, i) = p(i)
        Else: L(j, i) = sum / p(i)
        End If
    Next j
Next i
Cholesky = L
End Function

```

Fig. B.2. User-defined function for computing lower triangular matrix based on Cholesky factorization

```

Function GramSchmidt(ustar)

n = ustar.Count
ReDim dircos(1 To n) As Double, Tem(1 To n) As Double
ReDim Bef(1 To n, 1 To n) As Double
ReDim Aft(1 To n, 1 To n) As Double
SqSum = 0
For i = 1 To n
    SqSum = SqSum + (ustar(i)) ^ 2
Next i
beta = Sqr(SqSum)
For i = 1 To n
    dircos(i) = ustar(i) / beta
Next i
For i = 1 To n
    For j = 1 To n
        Bef(i, j) = 0
        If i = j Then Bef(i, j) = 1: If i = n Then Bef(i, j) = dircos(j)
    Next j
Next i
For i = 1 To n
    c = n - i + 1
    If c = n Then
        For j = 1 To n
            Aft(c, j) = dircos(j)
        Next j
    End If
    If c < n Then
        sumsq = 0
        For k = 1 To n
            Tem(k) = 0
            For H = 1 To n - c
                numer = Aft(c + H, c)
                For m = 1 To n
                    Tem(m) = Tem(m) + numer * Aft(c + H, m)
                Next m
            Next H
            Aft(c, k) = Bef(c, k) - Tem(k): sumsq = sumsq + Aft(c, k) ^ 2
        Next k
        For j = 1 To n
            Aft(c, j) = Aft(c, j) / Sqr(sumsq)
        Next j
    End If
Next i
GramSchmidt = Aft

End Function

```

Fig. B.3. User-defined function for Gram-Schmidt orthogonalization

```

Sub CurvatureSORM()
'beta, curvatures, perfunc, uprime are pre-defined cells/blocks in spreadsheet.

n = Range("uprime").Count
beta = Range("beta").Value
If beta < 3 Then k = 1 Else k = 3 / beta
Range("curvatures").Value = 0
For i = 1 To (n - 1)
    Range("uprime").Value = 0
    For j = -1 To 1
        Range("uprime").Cells(i).Value = -j * k * beta
        Range("perfunc").GoalSeek Goal:=0, ChangingCell:=Range("uprime").Cells(n)
        If j = -1 Then kappa1 = 2 * (Range("uprime").Cells(n) - beta) / (k * beta) ^ 2
        If j = 1 Then kappa2 = 2 * (Range("uprime").Cells(n) - beta) / (k * beta) ^ 2
        j = j + 1
    Next j
    tem1 = 1 / Sqr(1 + beta * kappa1): tem2 = 1 / Sqr(1 + beta * kappa2)
    Range("curvatures").Cells(i).Value = 1 / beta * (4 / (tem1 + tem2) ^ 2 - 1)
Next i
End Sub

```

Fig. B.4. User-defined procedure for determining principal curvatures of limit-state surface

```

Function ProbFailSORM(FormulaName, beta, curvatures)
Dim kappa() As Double, temp3re(1 To 2) As Double, temp3im(1 To 2) As Double
CurNum = WorksheetFunction.CountA(curvatures): CellNum = curvatures.Count: k = 1
ReDim kappa(1 To CurNum) As Double
For i = 1 To CellNum
    If curvatures(i) <> "" Then kappa(k) = curvatures(i): k = k + 1
Next i
With Application.WorksheetFunction
Select Case UCase(Trim(FormulaName)) 'trim leading/trailing spaces & convert to uppercase
Case "TVEDT": temp1 = 1: temp2 = 1: temp3re(1) = 1: temp3im(1) = 0
    For i = 1 To CurNum
        temp1 = temp1 / Sqr(1 + beta * kappa(i)): temp2 = temp2 / Sqr(1 + (beta + 1) * kappa(i))
        temp3re(2) = ComplexQuotient(temp3re(1), temp3im(1), 1 + beta * kappa(i), kappa(i), 1)
        temp3im(2) = ComplexQuotient(temp3re(1), temp3im(1), 1 + beta * kappa(i), kappa(i), 2)
        temp3re(1) = temp3re(2): temp3im(1) = temp3im(2): temp3 = ComplexRoot(temp3re(1), temp3im(1),
1)
    Next i
    T1 = .NormSDist(-beta) * temp1: T2 = (beta * .NormSDist(-beta) - .NormDist(-beta, 0, 1, False)) *
(temp1 - temp2)
    T3 = (beta + 1) * (beta * .NormSDist(-beta) - .NormDist(-beta, 0, 1, False)) * (temp1 - temp3)
    ProbFailSORM = T1 + T2 + T3
Case "BREITUNG": temp = 1
    For i = 1 To CurNum
        temp = temp / Sqr(1 + beta * kappa(i))
    Next i
    ProbFailSORM = .NormSDist(-beta) * temp
Case "HOHENBICHLER_RACKWITZ": temp = 1
    For i = 1 To CurNum
        temp = temp / Sqr(1 + kappa(i) * .NormDist(beta, 0, 1, False) / .NormSDist(-beta))
    Next i
    ProbFailSORM = .NormSDist(-beta) * temp
Case "KOYLUOGLU_NIElsen": temp1 = 1: temp2 = 1: temp3 = 1: temp4 = 1
    For i = 1 To CurNum
        If kappa(i) > 0 Then
            temp1 = temp1 / Sqr(1 + kappa(i) * .NormDist(beta, 0, 1, False) / .NormSDist(-beta))
            temp2 = temp2 / Sqr(1 + 0.5 * kappa(i) * .NormDist(beta, 0, 1, False) / .NormSDist(-beta))
        ElseIf kappa(i) < 0 Then
            temp3 = temp3 / Sqr(1 - kappa(i) * .NormDist(beta, 0, 1, False) / .NormSDist(beta))
            temp4 = temp4 / Sqr(1 - 0.5 * kappa(i) * .NormDist(beta, 0, 1, False) / .NormSDist(beta))
        End If
    Next i
    ProbFailSORM = .NormSDist(-beta) - .NormSDist(-beta) * temp4 * (1 - temp1) + .NormSDist(beta) *
temp2 * (1 - temp3)
Case "CAI_ELISHAKOFF": temp1 = 0: temp2 = 0: temp3 = 0
    For i = 1 To CurNum
        temp1 = temp1 + kappa(i): temp2 = temp2 + kappa(i) ^ 2: temp3 = temp3 + kappa(i) ^ 3
    Next i
    D1 = temp1 / 2: D2 = -0.5 * beta * (3 * temp2 / 4 + (temp1 ^ 2 - temp2) / 4)
    D3 = (beta ^ 2 - 1) / 6 * (15 * temp3 / 8 + 9 * (temp1 * temp2 - temp3) / 8 + (temp1 ^ 3 + 2 * temp3 - 3 *
temp1 * temp2) / 8)
    ProbFailSORM = .NormSDist(-beta) - .NormDist(beta, 0, 1, False) * (D1 + D2 + D3)
Case "HONG(P3)": temp = 1: m = .NormDist(beta, 0, 1, False) / .NormSDist(-beta): C1 = 0
    For i = 1 To CurNum
        temp = temp / Sqr(1 + kappa(i) * m): lamda = kappa(i) / 2
        C1 = C1 + (1 / CurNum) * (.NormSDist(-beta - CurNum * lamda / (1 + 2 * m * lamda)) / .NormSDist(-
beta)) * Exp(CurNum * m * lamda / (1 + 2 * m * lamda))
    Next i
    ProbFailSORM = .NormSDist(-beta) * temp * C1

```

```

Case "HONG(P4)": temp = 1: m = .NormDist(beta, 0, 1, False) / .NormSDist(-beta): C2 = 1 - CurNum / 3
  For i = 1 To CurNum
    temp = temp / Sqr(1 + kappa(i) * m): lamda = kappa(i) / 2
    C2 = C2 + (1 / 3) * (.NormSDist(-beta - 3 * lamda / (1 + 2 * m * lamda)) / .NormSDist(-beta)) * Exp(3 *
m * lamda / (1 + 2 * m * lamda))
  Next i
  ProbFailSORM = .NormSDist(-beta) * temp * C2
End Select
End With
End Function

Function ComplexQuotient(Re1, Im1, Re2, Im2, code)
If code = 1 Then ComplexQuotient = (Re1 * Re2 + Im1 * Im2) / (Re2 ^ 2 + Im2 ^ 2)
If code = 2 Then ComplexQuotient = (Im1 * Re2 - Re1 * Im2) / (Re2 ^ 2 + Im2 ^ 2)
End Function

Function ComplexRoot(Re, Im, code)
r = Sqr(Re ^ 2 + Im ^ 2)
If code = 1 Then ComplexRoot = Sqr((r + Re) / 2): If code = 2 Then ComplexRoot = Im / Sqr(2 * (r + Re))
End Function

```

Fig. B.5. User-defined function for computing failure probability using SORM formulas. Functions ComplexQuotient and ComplexRoot are required by the Tvedt formula for complex number operations

Appendix C

Procedure for Monte Carlo method with importance sampling

The procedure described below is based on the importance-sampling method proposed by Melchers (1984) based on Shinozuka (1983) and Rubinstein (1981). A commonly-cited example pertains to one plastic collapse failure mechanism of a one-bay frame (Der Kiureghian et al. 1987; Tvedt 1990; Hong 1999; Zhao and Ono 1999):

$$g(\mathbf{x}) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6 \quad (\text{C.1})$$

where the x_i 's are statistically independent lognormal random variables with means $\mu_1 = \dots = \mu_4 = 120$, $\mu_5 = 50$, $\mu_6 = 40$, and standard deviations $\sigma_1 = \dots = \sigma_4 = 12$, $\sigma_5 = 15$ and $\sigma_6 = 12$. The exact failure probability is 0.0121 (Der Kiureghian et al. 1987; Tvedt 1990; Hong 1999; Zhao and Ono 1999). Monte Carlo simulation with importance sampling is carried out in Microsoft Excel as shown in Fig. C.1, and is an extension of the first-order reliability analysis as shown in Fig. A.1.

The design point in the original space and in the reduced space are $\mathbf{x}^* = \{117.269, 115.240, 115.240, 117.269, 83.651, 55.449\}$ and $\mathbf{x}'^* = \{-0.181, -0.356, -0.356, -0.181, 1.900, 1.259\}$, respectively. The design point in the standard normal space $\mathbf{u}^*$ is equal to $\mathbf{x}'^*$ because the random variables $\mathbf{x}$ are statistically independent. If $\mathbf{x}$ are correlated, $\mathbf{u}^*$ can be obtained from $\mathbf{x}'^*$ by Cholesky factorization $\mathbf{u}^* = \mathbf{L}^{-1}\mathbf{x}'^*$ where $\mathbf{L}$ is the lower Cholesky matrix computed based on the correlation matrix $\mathbf{R}$. The user-defined matrix function *Cholesky* (Fig. B.2) can be used to obtain the lower Cholesky matrix.

In Fig. C.1, a template is set up to transform the random samples from u -space to x -space. The column vector $\mathbf{u}$ contains randomly-generated numbers based on the standard normal distribution. A simple way to quickly generate such random samples in Microsoft Excel is by entering this formula: =NormSInv(Rand()). The

vector $\mathbf{v}$ is computed based on $\mathbf{v} = \mathbf{u} + \mathbf{u}^*$. Since the random variables are statistically independent, $\mathbf{x}'$ is equal to $\mathbf{u}$; otherwise $\mathbf{x}'$ is obtained based on $\mathbf{x}' = \mathbf{L}\mathbf{u}$, where $\mathbf{L}$ is the lower Cholesky matrix. For lognormal-distributed random variables, the vector $\mathbf{x}$ is computed based on $x_i = \exp(\lambda_i + \zeta_i x'_i)$. If x_i is of a different non-normal distribution, the inverse-transform method $x = F^{-1}[\Phi(x')]$ is used to obtain x_i from x'_i .

The column $\phi(\mathbf{v})$ computes the value of the probability density function of the standard normal distribution $N(0,1)$ at v_i . The column $\psi(\mathbf{v})$ computes the value of the probability density function of the shifted standard normal distribution $N(u_i^*, 1)$ at v_i .

The cell labeled as $g(\mathbf{x})$ contains the limit state function and depends on the $\mathbf{x}$ values obtained from the template set up above. The cell labeled as $A(\mathbf{x}, \mathbf{v})$ contains the formula $I[g(\mathbf{x})][\phi(\mathbf{v})/\psi(\mathbf{v})]$, where $I[g(\mathbf{x})]$ is an indicator function defined such that $I[\cdot] = 1$ if $g(\mathbf{x}) \leq 0$ and $I[\cdot] = 0$ if otherwise. The program *ImportanceSampling* (Fig. C.2), when run, automatically computes the probability of failure based on the specified number of samples and displays the result in a dialog box.

For comparison the ordinary Monte Carlo method (without importance sampling) and the advanced Monte Carlo method (with importance sampling) were performed for 100, 500, 1000, 5000 and 10000 samples and each simulation was repeated ten times. The samples were generated using the commercial software @RISK based on Latin Hypercube sampling. Fig. E.3a and E.3b show the probabilities of failure obtained based on the ordinary and advance Monte Carlo method, respectively. The 'I'-shaped vertical line shows the maximum and minimum results among the ten simulations; the mean value of the ten simulations is represented by the dots near the middle of the vertical line. The upper and lower dashed lines trace the convergence of the simulation result; with 10000 samples the Monte Carlo method, with or without importance sampling, converges at a failure probability of 0.0121, which is virtually identical to the

exact probability obtained by direct integration. For both Monte Carlo methods the height of the vertical line, which correlates with the variance of the simulation results, diminishes with the increase in sample size. Notably the variance of the simulation results obtained based on the Monte Carlo method with importance sampling (Fig. C.3*b*) is consistently smaller than that without importance sampling (Fig. C.3*a*). The coefficient of variation of the probability of failure is calculated by based on the mean value and the standard deviation of the ten simulations. Fig. C.4 shows that the coefficient of variation for the advanced Monte Carlo method is significantly smaller than that for the ordinary Monte Carlo method when the sample size is small. This implies that the importance sampling technique is most effective, in terms of variance reduction, at small sample sizes; the advantage of the advanced Monte Carlo method over the ordinary Monte Carlo method is reduced when the sample size is increased.

		$g(\mathbf{x}^*)$		β				
		0		2.348				
<i>ProbDist</i>		μ	σ	λ	ζ	$\mathbf{x}^*$	$\mathbf{x}'^*$	$\mathbf{u}^*$
Lognormal	x_1	120	12	4.783	0.100	117.27	-0.181	-0.181
Lognormal	x_2	120	12	4.783	0.100	115.24	-0.356	-0.356
Lognormal	x_3	120	12	4.783	0.100	115.24	-0.356	-0.356
Lognormal	x_4	120	12	4.783	0.100	117.27	-0.181	-0.181
Lognormal	x_5	50	15	3.869	0.294	83.651	1.900	1.900
Lognormal	x_6	40	12	3.646	0.294	55.449	1.259	1.259

$\mathbf{u}$	$\mathbf{v}$	$\mathbf{x}'$	$\mathbf{x}$	$\phi(\mathbf{v})$	$\psi(\mathbf{v})$	$g(\mathbf{x})$	$A(\mathbf{x},\mathbf{v})$
1.532	1.351	1.351	136.63	0.160	0.123	-335	0.00075
-1.410	-1.766	-1.766	100.12	0.084	0.148		
-1.393	-1.749	-1.749	100.29	0.086	0.151		
-1.164	-1.345	-1.345	104.41	0.161	0.203		
0.640	2.539	2.539	100.930	0.016	0.325		
1.816	3.076	3.076	94.508	0.004	0.077		

$$A(\mathbf{x},\mathbf{v})=I[g(\mathbf{x})]\frac{\prod\phi(v_i)}{\prod\psi(v_i)}$$
$$I[g(\mathbf{x})]=\begin{cases} 0 & g(\mathbf{x})>0 \\ 1 & g(\mathbf{x})\leq 0 \end{cases}$$

Column $\mathbf{u}$ contains randomly-generated numbers based on standard normal distribution

Boxed columns above contain formulas: $\mathbf{v} = \mathbf{u} + \mathbf{u}^*$ $\mathbf{x}' = \mathbf{u}$

$x_i = \exp(\lambda_i + \zeta_i x'_i)$ $\phi(v_i) = \text{pdf of } N(0, 1) \text{ at } v_i$ $\psi(v_i) = \text{pdf of } N(u_i^*, 1) \text{ at } v_i$

Fig. C.1. Reliability analysis of the plastic collapse of a one-bay frame using Monte Carlo simulation with importance sampling

```
Sub ImportanceSampling()  
sample = 10000  
Sum = 0  
For i = 1 To sample  
    Calculate  
    Sum = Sum + Range("Afunc").Value  
Next i  
ProbFail = Sum / sample  
Msg = "Probability of failure is " & ProbFail _  
    & vbCr & "Total number of samples is " & sample  
Style = vbOKOnly + vbInformation + vbDefaultButton1  
Title = "Monte Carlo Simulation Result"  
Response = MsgBox(Msg, Style, Title)  
End Sub
```

Fig. C.2. User-defined procedure for importance sampling

Fig. C.3. Convergence of the failure probability:
(a) without importance sampling; (b) with importance sampling

Fig. C.4. Coefficient of variation of failure probability as function of simulation sample size

Appendix D

Validation of proposed sensitivity factors

The limit-state function below represents the failure in one plastic collapse mechanism of a simple one-bay portal frame:

$$g(\mathbf{x}) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6 \quad (\text{D.1})$$

Two-parameter probability distributions

In the original problem, the basic random variables are lognormal and uncorrelated. In the modified problem shown in Fig. D.1, the basic random variables take on an assortment of common two-parameter continuous probability distributions. Arbitrary correlations have also been introduced. The statistical information and correlation structure are as follows:

x_1 : Normal, $\mu_1 = 120$, $\sigma_1 = 12$

x_2 : Lognormal, $\mu_2 = 120$, $\sigma_2 = 12$

x_3 : Weibull, $\mu_3 = 120$, $\sigma_3 = 12$

x_4 : Gumbel, $\mu_4 = 120$, $\sigma_4 = 12$

x_5 : Uniform, $\mu_5 = 50$, $\sigma_5 = 15$

x_6 : Gamma, $\mu_6 = 40$, $\sigma_6 = 12$

$\rho_{12} = 0.4$; $\rho_{13} = 0.3$; $\rho_{14} = 0.2$; $\rho_{23} = 0.4$; $\rho_{24} = 0.3$; $\rho_{34} = 0.4$; $\rho_{56} = 0.5$

Reliability analysis based on FORM obtained the following results: the first-order reliability index β_F is 2.042; the inferred probability of failure $(p_f)_F$ is 0.02055.

Reliability analysis based on SORM obtained the following results: the average probability of failure $(p_f)_S$ is 0.01784; the back-calculated second-order reliability index β_S is 2.101.

Five Monte Carlo simulations based on importance sampling (each with a sample size of 1000) showed that the actual probability of failure is about 0.0175.

The design point in the standard normal space $\mathbf{u}^*$ is obtained from $\mathbf{x}'^*$ by Cholesky factorization $\mathbf{u}^* = \mathbf{L}^{-1}\mathbf{x}'^*$ where $\mathbf{L}$ is the lower Cholesky matrix computed based on the correlation matrix $\mathbf{R}$. The direction cosines $\boldsymbol{\alpha}$ are computed based on $\mathbf{u}^*/\beta$. The sensitivity factors $\partial\beta/\partial\mathbf{x}'$ are evaluated based on $(\mathbf{L}^{-1})^T\boldsymbol{\alpha}$ [Eq. (4.22)]. The sensitivity factors $\partial\beta/\partial\mu_i$ and $\partial\beta/\partial\sigma_i$ are evaluated based on Eq. (4.23) as products of $\partial\beta/\partial x'_i$ and the user-defined function *Sensitivity2p* (Fig. D.2, '2p' denotes two-parameter probability distributions). The function has arguments ProbDist, miu, sigma, x and code, and returns the ratio $\Delta x'_i/\Delta\mu_i$ or $\Delta x'_i/\Delta\sigma_i$ depending on code = 1 or code = 2, respectively. The argument 'x' is the design-point value x_i^* , and 'miu' and 'sigma' are the mean and standard deviation of x_i .

The values of $\partial\beta/\partial\mu_i$ and $\partial\beta/\partial\sigma_i$ obtained based on the above proposed method are found to agree very well with the ratios $\Delta\beta/\Delta\mu_i$ and $\Delta\beta/\Delta\sigma_i$ obtained based on Eq. (2.44), which computes the actual changes in the reliability index corresponding to small changes in μ_i and σ_i . The values of $\partial p_f/\partial\mu_i$ and $\partial p_f/\partial\sigma_i$ inferred from the proposed sensitivity factors are also comparable with the ratios $\Delta p_f/\Delta\mu_i$ and $\Delta p_f/\Delta\sigma_i$ obtained based on Eq. (2.43), which computes the actual differences in the probabilities of failure by repeating Monte Carlo simulations with the original values and perturbed values of μ_i and σ_i .

				$g(\mathbf{x}^*)$	β			
				-5E-13	2.04247			
<i>ProbDist</i>		μ	σ	$\mathbf{x}^*$	$\mathbf{x}'^*$	$\mathbf{u}^*$	α	$\partial\beta/\partial\mathbf{x}'$
Normal	x_1	120	12	113.60	-0.533	-0.533	-0.261	-0.1008
Lognormal	x_2	120	12	111.33	-0.702	-0.533	-0.261	-0.1865
Weibull	x_3	120	12	111.84	-0.756	-0.479	-0.235	-0.2348
Gumbel	x_4	120	12	113.06	-0.503	-0.141	-0.069	-0.0762
Uniform	x_5	50	15	71.632	1.381	1.381	0.676	0.3360
Gamma	x_6	40	12	62.966	1.732	1.203	0.589	0.6801

		Based on proposed sensitivity factors		Based on numerical differentiation	
Sensitivity of reliability index		$\partial\beta/\partial\mu$	$\partial\beta/\partial\sigma$	$\Delta\beta/\Delta\mu$	$\Delta\beta/\Delta\sigma$
	x_1	0.0084	-0.0045	0.0084	-0.0045
	x_2	0.0168	-0.0124	0.0168	-0.0124
	x_3	0.0169	-0.0124	0.0169	-0.0124
	x_4	0.0084	-0.0049	0.0084	-0.0049
	x_5	-0.0422	-0.0607	-0.0420	-0.0606
	x_6	-0.0400	-0.0872	-0.0400	-0.0872
Sensitivity of failure probability		$\partial p_f/\partial\mu$	$\partial p_f/\partial\sigma$	$\Delta p_f/\Delta\mu$	$\Delta p_f/\Delta\sigma$
	x_1	-0.0004	0.0002	-0.0004	0.0002
	x_2	-0.0008	0.0006	-0.0007	0.0006
	x_3	-0.0008	0.0006	-0.0007	0.0005
	x_4	-0.0004	0.0002	-0.0004	0.0002
	x_5	0.0021	0.0030	0.0019	0.0024
	x_6	0.0020	0.0043	0.0017	0.0040

Fig. D.1. Sensitivity analysis of the plastic collapse of a one-bay frame using the proposed sensitivity factors for two-parameter probability distributions

```

Function Sensitivity2p(ProbDist, miu, sigma, x, code)
ReDim num(-1 To 1) As Double
del = 0.01: para1 = miu: para2 = sigma
With Application.WorksheetFunction
For i = -1 To 1
    If code = 1 Then para1 = miu * (1 + del * i): div = miu
    If code = 2 Then para2 = sigma * (1 + del * i): div = sigma
Select Case UCase(Trim(ProbDist))
Case "NORMAL":
    num(i) = .NormSInv(.NormDist(x, para1, para2, True))
Case "LOGNORMAL":
    zeta = Sqr(Log(1 + (para2 / para1) ^ 2))
    lamda = Log(para1) - 0.5 * zeta ^ 2
    num(i) = .NormSInv(.LogNormDist(x, lamda, zeta))
Case "GUMBEL":
    zeta = 0.7797 * para2: lamda = para1 - 0.5772 * zeta
    num(i) = .NormSInv(Exp(-Exp(-(x - lamda) / zeta)))
Case "EXPONENTIAL":
    If code = 2 Then para1 = para2
    num(i) = .NormSInv(.ExponDist(x, 1 / para1, True))
Case "UNIFORM":
    Min = para1 - Sqr(3) * para2: Max = para1 + Sqr(3) * para2
    num(i) = .NormSInv((x - Min) / (Max - Min))
Case "WEIBULL":
    lamda = WeibullPara(para1, para2, 1)
    zeta = WeibullPara(para1, para2, 2)
    num(i) = .NormSInv(.Weibull(x, lamda, zeta, True))
Case "GAMMA":
    zeta = para2 ^ 2 / para1: lamda = para1 / zeta
    num(i) = .NormSInv(.GammaDist(x, lamda, zeta, True))
End Select
Next i
Sensitivity2p = (num(1) - num(-1)) / (2 * del * div)
End With
End Function

```

Fig. D.2. User-defined function for computing sensitivity factors for two-parameter probability distributions

Three-parameter and four-parameter probability distributions

In the original problem, the basic random variables are lognormal and uncorrelated. In the modified problem shown in Fig. D.3, x_1 , x_2 , x_3 and x_4 are characterized by 4-parameter beta distributions, whereas x_5 and x_6 are characterized by 3-parameter triangular distributions. Arbitrary correlations have also been introduced. The statistical information and correlation structure are as follows:

x_1 : Beta, $\theta_1 = 12.6$, $\theta_2 = 12.6$, $\theta_3 = 60$, $\theta_4 = 180$

x_2 : Beta, $\theta_1 = 4$, $\theta_2 = 4$, $\theta_3 = 84$, $\theta_4 = 156$

x_3 : Beta, $\theta_1 = 1.5$, $\theta_2 = 5$, $\theta_3 = 102$, $\theta_4 = 180$

x_4 : Beta, $\theta_1 = 5$, $\theta_2 = 1.5$, $\theta_3 = 60$, $\theta_4 = 138$

(For beta distributions, θ_1, θ_2 = shape factors, θ_3 = min, θ_4 = max)

x_5 : Triangular, $\theta_1 = 23$, $\theta_2 = 35$, $\theta_3 = 92$

x_6 : Triangular, $\theta_1 = 6.5$, $\theta_2 = 52$, $\theta_3 = 61.5$

(For triangular distributions, θ_1 = min, θ_2 = mode, θ_3 = max)

$\rho_{12} = 0.4$; $\rho_{13} = 0.3$; $\rho_{14} = 0.2$; $\rho_{23} = 0.4$; $\rho_{24} = 0.3$; $\rho_{34} = 0.4$; $\rho_{56} = 0.5$

Reliability analysis based on FORM obtained the following results: the first-order reliability index β_F is 2.031; the inferred probability of failure $(p_f)_F$ is 0.02113.

Reliability analysis based on SORM obtained the following results: the average probability of failure $(p_f)_S$ is 0.01451; the back-calculated second-order reliability index β_S is 2.183.

Five Monte Carlo simulations based on importance sampling (each with a sample size of 1000) showed that the actual probability of failure is about 0.0143.

The sensitivity factors $\partial\beta/\partial\theta_1$, $\partial\beta/\partial\theta_2$ and $\partial\beta/\partial\theta_3$ for the triangular distribution are evaluated based on Eq. (4.25) as products of $\partial\beta/\partial x'_i$ and the user-defined function *Sensitivity3p* (Fig. D.4, '3p' denotes three-parameter triangular distribution). The

function has arguments ProbDist, min, mode, max, x and code, and returns the ratio $\Delta x'_i/\Delta\theta_1$, $\Delta x'_i/\Delta\theta_2$ or $\Delta x'_i/\Delta\theta_3$ depending on code = 1, 2 or 3, respectively. The sensitivity factors $\partial\beta/\partial\theta_1$, $\partial\beta/\partial\theta_2$, $\partial\beta/\partial\theta_3$ and $\partial\beta/\partial\theta_4$ for the beta distribution are evaluated based on Eq. (4.25) as products of $\partial\beta/\partial x'_i$ and the user-defined function *Sensitivity4p* (Fig. D.5, '4p' denotes four-parameter beta distribution). The function has arguments ProbDist, lamda, zeta, min, max, x and code, and returns the ratio $\Delta x'_i/\Delta\theta_1$, $\Delta x'_i/\Delta\theta_2$, $\Delta x'_i/\Delta\theta_3$ or $\Delta x'_i/\Delta\theta_4$ depending on code = 1, 2, 3 or 4, respectively.

The values of $\partial\beta/\partial\theta_j$ obtained based on the above proposed method are found to agree very well with the ratios $\Delta\beta/\Delta\theta_j$ obtained based on Eq. (2.44), which computes the actual changes in the reliability index corresponding to small changes in θ_j . The values of $\partial p_f/\partial\theta_j$ inferred from the proposed sensitivity factors are also comparable with the ratios $\Delta p_f/\Delta\theta_j$ obtained based on Eq. (2.43), which computes the actual differences in the probabilities of failure by repeating Monte Carlo simulations with the original values and perturbed values of θ_j .

		$g(\mathbf{x}^*)$				β				
		2.8E-07				2.03094				
<i>ProbDist</i>		θ_1	θ_2	θ_3	θ_4	$\mathbf{x}^*$	$\mathbf{x}'^*$	$\mathbf{u}^*$	α	$\partial\beta/\partial\mathbf{x}'$
Beta	x_1	12.6	12.6	60	180	112.88	-0.5919	-0.5919	-0.2915	-0.1162
Beta	x_2	4	4	84	156	109.646	-0.8058	-0.6209	-0.3057	-0.2393
Beta	x_3	1.5	5	102	180	110.107	-0.7229	-0.3935	-0.1938	-0.1658
Beta	x_4	5	1.5	60	138	113.353	-0.6302	-0.2732	-0.1345	-0.149
Triangular	x_5	23	35	92		78.6865	1.69469	1.69469	0.83444	0.68204
Triangular	x_6	6.5	52	61.5		54.461	1.3116	0.53607	0.26395	0.30479

		Based on proposed sensitivity factors				Based on numerical differentiation			
		$\frac{\partial\beta}{\partial\theta_1}$	$\frac{\partial\beta}{\partial\theta_2}$	$\frac{\partial\beta}{\partial\theta_3}$	$\frac{\partial\beta}{\partial\theta_4}$	$\frac{\Delta\beta}{\Delta\theta_1}$	$\frac{\Delta\beta}{\Delta\theta_2}$	$\frac{\Delta\beta}{\Delta\theta_3}$	$\frac{\Delta\beta}{\Delta\theta_4}$
Sensitivity of reliability index	x_1	0.0247	-0.0220	0.0054	0.0042	0.0250	-0.0222	0.0054	0.0043
	x_2	0.1016	-0.0762	0.0124	0.0069	0.1026	-0.0770	0.0125	0.0069
	x_3	0.1565	-0.0283	0.0173	0.0020	0.1580	-0.0286	0.0174	0.0020
	x_4	0.0374	-0.1087	0.0030	0.0066	0.0378	-0.1098	0.0031	0.0067
	x_5	-0.0046	-0.0056	-0.038		-0.0047	-0.0057	-0.0383	
	x_6	-0.0031	-0.0179	-0.0273		-0.0031	-0.0180	-0.0275	

		$\frac{\partial p_f}{\partial\theta_1}$	$\frac{\partial p_f}{\partial\theta_2}$	$\frac{\partial p_f}{\partial\theta_3}$	$\frac{\partial p_f}{\partial\theta_4}$	$\frac{\Delta p_f}{\Delta\theta_1}$	$\frac{\Delta p_f}{\Delta\theta_2}$	$\frac{\Delta p_f}{\Delta\theta_3}$	$\frac{\Delta p_f}{\Delta\theta_4}$
Sensitivity of failure probability	x_1	-0.0013	0.0011	-0.0003	-0.0002	-0.0010	0.0008	-0.0002	-0.0002
	x_2	-0.0052	0.0039	-0.0006	-0.0003	-0.0038	0.0033	-0.0004	-0.0003
	x_3	-0.0079	0.0014	-0.0009	-0.0001	-0.0061	0.0013	-0.0007	-0.0001
	x_4	-0.0019	0.0055	-0.0002	-0.0003	-0.0013	0.0038	-0.0001	-0.0002
	x_5	0.0002	0.0003	0.0019		0.0002	0.0002	0.0016	
	x_6	0.0002	0.0009	0.0014		0.0002	0.0007	0.0012	

Fig. D.3. Sensitivity analysis of the plastic collapse of a one-bay frame using the proposed sensitivity factors for three- and four-parameter probability distributions

```

Function Sensitivity3p(ProbDist, min, mode, max, x, code)
ReDim num(-1 To 1) As Double, R(1 To 3) As Double
del = 0.01: a = min: b = max: c = mode
With Application.WorksheetFunction
For i = -1 To 1
    If code = 1 Then a = min * (1 + del * i): div = a
    If code = 2 Then b = max * (1 + del * i): div = b
    If code = 3 Then c = mode * (1 + del * i): div = c
    Select Case UCase(Trim(ProbDist))
    Case "TRIANGULAR":
        If x < a Then CDF = 0
        If (x >= a) And (x <= c) Then CDF = (x - a) ^ 2 / (b - a) / (c - a)
        If (x > c) And (x <= b) Then CDF = 1 - (b - x) ^ 2 / (b - a) / (b - c)
        If x > b Then CDF = 1
        num(i) = .NormSInv(CDF)
    End Select
Next i
Sensitivity3p = (num(1) - num(-1)) / (2 * del * div)
End With
End Function

```

Fig. D.4. User-defined function for computing sensitivity factors for the three-parameter triangular distribution

```

Function Sensitivity4p(ProbDist, lamda, zeta, min, max, x, code)
ReDim num(-1 To 1) As Double
del = 0.01: c1 = lamda: c2 = zeta: a = min: b = max
With Application.WorksheetFunction
For i = -1 To 1
    If code = 1 Then c1 = lamda * (1 + del * i): div = c1
    If code = 2 Then c2 = zeta * (1 + del * i): div = c2
    If code = 3 Then a = min * (1 + del * i): div = a
    If code = 4 Then b = max * (1 + del * i): div = b
    Select Case UCase(Trim(ProbDist))
    Case "BETA":
        num(i) = .NormSInv(.BetaDist(x, c1, c2, a, b))
    End Select
Next i
Sensitivity4p = (num(1) - num(-1)) / (2 * del * div)
End With
End Function

```

Fig. D.5. User-defined function for computing sensitivity factors for the four-parameter beta distribution

Appendix E

Procedure for system reliability evaluation

In Madsen et al. (1986, p.112) plastic hinge mechanisms leading to the collapse of a plane frame structure are analyzed by utilizing elastic-plastic stress-strain relations. Hinges are assumed to form at the end of elements or at points of load application only. Loads and plastic moment capacities are random variables which are assumed mutually independent and described by lognormal distributions. There are seven basic variables and their mean values and standard deviations are given in Fig. E.1. The principle of virtual work gives the following three linear safety margins:

$$g_1(x) = Z_1 + Z_2 + Z_4 + Z_5 - 5H \quad (\text{E.1})$$

$$g_2(x) = Z_1 + 2Z_3 + 2Z_4 + Z_5 - 5H - 5V \quad (\text{E.2})$$

$$g_3(x) = Z_2 + 2Z_3 + Z_4 - 5Z_7 \quad (\text{E.3})$$

The first-order and second-order reliability indices for the 3 failure modes are shown in Fig. E.2 and arranged in ascending orders of magnitude. The correlation coefficients between failure modes i and j are computed as $\rho_{12} = 0.844$, $\rho_{13} = 0.014$, and $\rho_{23} = 0.531$. The failure modes are all positively correlated with one another. The results obtained above are virtually identical to those reported in Madsen et al. (1986), which are based on manual calculations with round off.

The simple uni-modal bounds (Cornell 1967) based on first-order or second-order approximations of failure probabilities corresponding to the various failure modes are wide:

$$0.00335 \leq p_{f,\text{sys}} \leq 0.00560 \quad (\text{uni-modal bounds based on FORM})$$

$$0.00323 \leq p_{f,\text{sys}} \leq 0.00612 \quad (\text{uni-modal bounds based on SORM})$$

The Kounias-Ditlevsen (Kounias 1968, Hunter 1976, Ditlevsen 1979) bimodal bounds on the system failure probability are:

$$0.004512 \leq p_{f,sys} \leq 0.004987 \quad (\text{bi-modal bounds based on FORM})$$

$$0.004841 \leq p_{f,sys} \leq 0.005448 \quad (\text{bi-modal bounds based on SORM})$$

The FORM and SORM bimodal bounds above are exactly the same as those computed by the Low et al. (2011) spreadsheet algorithm for system reliability bounds. The bounds are consistent with the ‘exact’ value 0.00485 obtained by Madsen (1985) via extensive simulation using an importance sampling technique.

The Owen (1956) equations have been coded in the Visual Basic programming environment of spreadsheet software Microsoft Excel, as shown in Fig. E.3. The main function is *BiVariate*, with supporting functions *T_large*, *T_small* and *c_func*. For cases with 3 or more failure modes the function *BiVariate* can be used in conjunction with the Low et al. (2011) functions *SysProbLowerBound* and *SysProbUpperBound* (replacing *ProbFiFj*) to calculate the bimodal bounds of Eq. (2.49) more efficiently.

The new and improved bimodal bounds are very sharp:

$$0.004641 \leq p_{f,sys} \leq 0.004642 \quad (\text{new bi-modal bounds based on FORM})$$

$$0.004994 \leq p_{f,sys} \leq 0.004996 \quad (\text{new bi-modal bounds based on SORM})$$

because the joint failure probabilities $p_{f,ij}$ are evaluated directly based on the approximate closed-form equations of Owen (1956) rather than as lower and upper bounds following Ditlevsen (1979). One drawback of the improved bimodal bounds is that they may easily miss the actual system failure probability due to their high precision.

Variable	Mean value	Standard deviation
$Z_1, \dots, Z_5$	134.9 kNm	13.49 kNm
$H = Z_6$	50 kN	15 kN
$V = Z_7$	40 kN	12 kN

Fig. E.1. Statistical assumptions for stochastic analysis of plane frame structure under three failure mechanisms

(a)	ρ of failure modes		β_1	β_2	β_3											
	<table><tr><td>1</td><td>0.8438</td><td>0.0141</td></tr><tr><td>0.8438</td><td>1</td><td>0.5308</td></tr><tr><td>0.0141</td><td>0.5308</td><td>1</td></tr></table>	1	0.8438	0.0141	0.8438	1	0.5308	0.0141	0.5308	1		<table><tr><td>2.7118</td><td>2.8825</td><td>3.4375</td></tr></table>	2.7118	2.8825	3.4375	
1	0.8438	0.0141														
0.8438	1	0.5308														
0.0141	0.5308	1														
2.7118	2.8825	3.4375														
		(beta indices based on FORM)														
Kounias-Ditlevsen		Improved														
	$p_{f,sys}$ bounds		$p_{f,sys}$ bounds													
	Lower	<table><tr><td>0.004512</td></tr></table>	0.004512	Lower	<table><tr><td>0.004641</td></tr></table>		0.004641									
0.004512																
0.004641																
	Upper	<table><tr><td>0.004987</td></tr></table>	0.004987	Upper	<table><tr><td>0.004642</td></tr></table>		0.004642									
0.004987																
0.004642																
(b)	ρ of failure modes		β_1	β_2	β_3											
	<table><tr><td>1</td><td>0.8438</td><td>0.0141</td></tr><tr><td>0.8438</td><td>1</td><td>0.5308</td></tr><tr><td>0.0141</td><td>0.5308</td><td>1</td></tr></table>	1	0.8438	0.0141	0.8438	1	0.5308	0.0141	0.5308	1		<table><tr><td>2.7240</td><td>2.7917</td><td>3.4476</td></tr></table>	2.7240	2.7917	3.4476	
1	0.8438	0.0141														
0.8438	1	0.5308														
0.0141	0.5308	1														
2.7240	2.7917	3.4476														
		(beta indices based on SORM)														
Kounias-Ditlevsen		Improved														
	$p_{f,sys}$ bounds		$p_{f,sys}$ bounds													
	Lower	<table><tr><td>0.004841</td></tr></table>	0.004841	Lower	<table><tr><td>0.004994</td></tr></table>		0.004994									
0.004841																
0.004994																
	Upper	<table><tr><td>0.005448</td></tr></table>	0.005448	Upper	<table><tr><td>0.004996</td></tr></table>		0.004996									
0.005448																
0.004996																
<div> Monte Carlo simulation by Madsen (1985) yielded system failure probability of 0.00485. </div>																

Fig. E.2. Comparison of system failure probability bounds computed for plane frame structure considering three failure modes

```

Function BiVariate(h, k, rho)
With Application.WorksheetFunction
jmax = 100
If rho > 0.999999 Then rho = 0.999999
If rho < -0.999999 Then rho = -0.999999
If h = 0 And k = 0 Then BiVariate = .Asin(rho) / (2 * .Pi) + 0.25: Exit Function
If h = 0 Then h = 10 ^ -6
a = (k - rho * h) / (h * Sqr(1 - rho ^ 2))
If Abs(a) > 0 And Abs(a) <= 1 Then T_h = T_small(h, a, jmax)
If Abs(a) > 1 Then T_h = T_large(h, a, jmax)
If k = 0 Then k = 10 ^ -6
a = (h - rho * k) / (k * Sqr(1 - rho ^ 2))
If Abs(a) > 0 And Abs(a) <= 1 Then T_k = T_small(k, a, jmax)
If Abs(a) > 1 Then T_k = T_large(k, a, jmax)
If h * k > 0 Then BiVariate = (.NormSDist(h) + .NormSDist(k)) / 2 - T_h - T_k
If h * k < 0 Then BiVariate = (.NormSDist(h) + .NormSDist(k)) / 2 - T_h - T_k - 0.5
End With
End Function

Function T_large(v, a, jmax)
With Application.WorksheetFunction
T_large = .NormSDist(v) / 2 + .NormSDist(Abs(a) * v) / 2 - .NormSDist(v) *
.NormSDist(Abs(a) * v) - T_small(Abs(a) * v, 1 / Abs(a), jmax)
If a < 0 Then T_large = -T_large
End With
End Function

Function T_small(v, a, jmax)
Sum = 0
For j = 0 To jmax
    Sum = Sum + c_func(v, j) * Abs(a) ^ (2 * j + 1)
Next j
T_small = (Atn(Abs(a)) - Sum) / (2 * Application.Pi)
If a < 0 Then T_small = -T_small
End Function

Function c_func(v, j)
Sum = 0
For i = 0 To j
    factorial = 1
    For ii = 1 To i
        factorial = factorial * ii
    Next ii
    Sum = Sum + v ^ (2 * i) / (2 ^ i * factorial)
Next i
c_func = (-1) ^ j / (2 * j + 1) * (1 - Exp(-(v ^ 2) / 2) * Sum)
End Function

```

Fig. E.3. User-defined function for evaluating bivariate standard normal cumulative distribution function based on Owen (1956)

Appendix F

Procedure for response surface method

Fig. F.1 summarizes in a flowchart the procedure of performing sequential reliability analysis using the response surface method. A response surface approximated by the second-order polynomial function in Eq. (4.1) has $2n + 1$ unknown coefficients, where n is the number of random variables. To determine the coefficients, $2n + 1$ sampling points are selected and their corresponding limit state values obtained via deterministic analysis. This leads to $2n + 1$ simultaneous equations, which are then solved using simple matrix operations.

For illustration, consider a laterally-loaded free-head rigid pile in dry sand. The performance function is given by (Broms 1964):

$$g(\gamma, \phi') = FS - 1 = \frac{0.5\gamma d L^3 \tan^2(45^\circ + \phi'/2)}{P(e + L)} - 1 \quad (F.1)$$

where γ = unit weight of sand; ϕ' = effective friction angle of sand; d = diameter of pile; L = embedded length of pile; e = exposed length of pile; and P = applied horizontal load. The stochastic parameters are γ and ϕ' : γ is normally distributed with mean value 18 kN/m³ and c.o.v. 0.1; ϕ' is normally distributed with mean value 30° and c.o.v. 0.1; γ and ϕ' are correlated with $\rho = 0.5$. The remaining parameters are deterministic: $d = 1$ m; $L = 10$ m; $e = 1$ m; and $P = 1000$ kN. Detail explanations of the steps in Fig. F.1 are as follows:

Step 1: Identify random variables. Fig. F.2 shows the construction of the first trial response surface involving two random variables: γ and ϕ' .

Step 2: Prepare required sets of random variables at sampling points around mean-value point (first trial) or around tentative design points (later iterations). The response surface is of the form $R(\gamma, \phi') = a_0 + a_1\gamma + a_2\phi' + a_{11}\gamma^2 + a_{22}\phi'^2$ and has 5 coefficients. Five sample points are selected in the neighborhood

of the mean-value point based on the axial-point design with a sampling parameter of one. The 5-by-5 $\mathbf{V}$ matrix consists of: (1) a first column containing '1's; (2) five columns containing the sample points; and (3) five columns containing the square of the sample points.

Step 3: Evaluate performance function at sampling points. The real value of the performance function $g(\gamma, \phi')$ corresponding to each sample point is computed based on Eq. (F.1).

Step 4: Construct trial response surface. The 5 coefficients (collectively as vector $\mathbf{a}$) are solved simultaneously based on $\mathbf{V}$ and $\mathbf{g}$ using built-in matrix functions in Microsoft Excel. The constructed trial response surface is contained as a function in a single cell $R(\mathbf{x})$.

Step 5: Calculate tentative design point and reliability index. The setup for reliability analysis is shown at the bottom half of Fig. F.2. The response surface function $R(\mathbf{x})$ reads the values from $\mathbf{x}$ as inputs. The results displayed in Fig. F.2 are obtained after calling Solver to minimize β by changing the values under $\mathbf{x}'$ subject to $R(\mathbf{x}) = 0$. Better results can be obtained by selecting another 5 sample points in the vicinity of the tentative design point and repeating steps 3, 4 and 5. This technique concentrates the sample points in the vicinity of the tentative design point and results in a sequentially more accurate response surface because the sample points are taken progressively nearer the final design point with each repetition. It has been observed in most problems that only a few iterations are required to reach convergence.

In Fig. F.3 the first and final trial response surfaces $R(\gamma, \phi') = 0$ are compared with the true limit-state surface $g(\gamma, \phi') = 0$. The final trial response surface coincides with the true limit-state surface in the neighborhood of the design point (10.93, 19.64) but deviates away from the true limit-state surface in other regions. The final trial response surface may be adequate for obtaining the design point and the Hasofer-Lind reliability index, but as it fails to reproduce the general curvature of the true limit-state surface, the accuracy of the failure probability computed using second-order reliability method or direct Monte Carlo method based on the response surface may be questionable.

A more accurate response surface can be constructed by including the cross term $a_{12}\gamma\varphi'$ in $R(\gamma,\varphi')$. Fig. F.4 shows the construction of the first trial response surface based on the complete second-order polynomial that has 6 coefficients. The additional sample point is selected such that its coordinates are always less than the mean values or the design-point values. In Fig. F.5 the final trial response surface (which includes the cross term) provides a good approximation to the true limit-state surface over a relatively large region and is able to reflect the curvature of the true limit-state surface correctly. The importance of the cross term is likely due to the somewhat strong correlation ($\rho = 0.5$) between γ and φ' .

Five Monte Carlo simulations (each with a sample size of 300,000) based on importance sampling yield the following results: on the true limit-state surface, $p_f = 9.44 \times 10^{-6}$; on the response surface without the cross term, $p_f = 8.68 \times 10^{-6}$; on the response surface with the cross term, $p_f = 9.39 \times 10^{-6}$.

Fig. F.1. General procedure of sequential reliability analysis using response surface method

Fig. F.2. Reliability analysis of laterally-loaded rigid pile in sand using response surface method (quadratic model without cross term)

Fig. F.3. True limit state surface and response surfaces based on quadratic model without cross term

Fig. F.4. Reliability analysis of laterally-loaded rigid pile in sand using response surface method (quadratic model with cross term)

Fig. F.5. True limit state surface and response surfaces based on quadratic model with cross term

Appendix G

Procedure for artificial neural network

Fig. G.1 summarizes in a flowchart the procedure of performing sequential reliability analysis using the artificial neural network. The most widely used neural network approach is the multilayer perceptron with the backpropagation learning algorithm, which establishes an appropriate set of weights for the neural network connections based on a gradient-descent algorithm on the sum-squared error.

For illustration, consider a laterally-loaded free-head rigid pile in dry sand. The performance function is given by (Broms 1964):

$$g(\gamma, \varphi') = FS - 1 = \frac{0.5\gamma d L^3 \tan^2(45^\circ + \varphi'/2)}{P(e + L)} - 1 \quad (\text{G.1})$$

where γ = unit weight of sand; φ' = effective friction angle of sand; d = diameter of pile; L = embedded length of pile; e = exposed length of pile; and P = applied horizontal load. The stochastic parameters are γ and φ' : γ is normally distributed with mean value 18 kN/m³ and c.o.v. 0.1; φ' is normally distributed with mean value 30° and c.o.v. 0.1; γ and φ' are correlated with $\rho = 0.5$. The remaining parameters are deterministic: $d = 1$ m; $L = 10$ m; $e = 1$ m; and $P = 1000$ kN. Detail explanations of the steps in Fig. G.1 are as follows:

Step 1: Identify random variables. Fig. G.2 shows the determination of the first trial neural network involving two random variables: γ and φ' .

Step 2: Prepare required sets of random variables at sampling points around initial center point (first trial) or around tentative design points (later iterations). The initial center point is arbitrarily chosen as (11.25, 18.75). Other points may be used as long as $g(\gamma, \varphi') = 0$. Nine sample points are selected in the neighborhood of the initial center point based on a 3-level full-factorial design

with a sampling parameter of 0.5.

Step 3: Evaluate performance function at sampling points. The real value of the performance function $g(\gamma, \phi')$ corresponding to each sample point is computed based on Eq. (G.1). The preparation of the training data is now completed.

Step 4: Train trial neural network. The neural network has a '2-2-1' architecture, consisting of an input layer of two neurons, a hidden layer of two neurons and an output layer of one neuron. The backpropagation training is carried out in Microsoft Excel using a user-defined VBA subroutine *BackPropagation* (Fig. G.3). A learning-rate parameter of $\eta = 0.9$ and a momentum constant of $\alpha = 0.9$ are found to be suitable. All connections start with random weights uniformly distributed between -0.3 and 0.3 . The neural network is subjected to 5000 cycles of training based on the 9-sample training data. The process takes less than 5 seconds on an Intel Core CPU 2-Ghz personal computer. The trained trial neural network is contained as a function in a single cell $N(\mathbf{x})$, as given by the user-defined function *NeuralNetwork* (Fig. G.4).

Step 5: Calculate tentative design point and reliability index. The setup for reliability analysis is shown at the bottom half of Fig. G.2. The neural network function $N(\mathbf{x})$ reads the values from $\mathbf{x}$ as inputs. The results displayed in Fig. G.2 are obtained after calling Solver to minimize β by changing the values under $\mathbf{x}'$ subject to $N(\mathbf{x}) = 0$. Better results can be obtained by selecting another 9 sample points in the vicinity of the tentative design point and repeating steps 3, 4 and 5. This technique concentrates the sample points in the vicinity of the tentative design point and results in a sequentially more accurate neural network because the sample points are taken progressively nearer the final design point with each repetition. It has been observed in most problems that only a few iterations are required to reach convergence.

The Hasofer-Lind reliability index obtained based on the final trial neural network is 4.288. The design point is (10.92, 19.65). The above results are practically the same as those obtained based on the true limit-state function. Five Monte Carlo simulations (each with a sample size of 300,000) based on importance sampling

yield the following results: on the true limit-state surface, $p_f = 9.44 \times 10^{-6}$; on the neural network, $p_f = 9.15 \times 10^{-6}$. Fig. G.5 shows that the neural network provides good approximations of the limit-state function based not only on the 9-sample training data but also on the additional randomly-generated 9-sample test data.

Fig. G.1. General procedure of sequential reliability analysis using artificial neural network

$$g(\gamma, \phi') = \frac{0.5\gamma d L^3 \tan^2(45^\circ + \phi'/2)}{P(e + L)} - 1$$

Actual Approx.					Weights	
γ	ϕ'	$g(x)$	$g(x)$	%error		
11.25	18.75	0.00	0.00	-3%	-4.084	3.516
11.25	17.25	-0.06	-0.06	-3%	-3.101	2.278
11.25	20.25	0.05	0.05	-2%	7.580	0.087
10.35	18.75	-0.08	-0.09	6%		-9.86
10.35	17.25	-0.13	-0.13	-5%		7.812
10.35	20.25	-0.03	-0.03	-11%		2.159
12.15	18.75	0.08	0.08	3%	W_1	
12.15	17.25	0.02	0.01	-17%	W_2	
12.15	20.25	0.14	0.13	-3%		
					$\gamma : \text{kN/m}^3$	$N(\gamma, \phi')$
					$\phi' : \text{degrees}$	0
	μ	σ	x	x'	$[R]$	
γ	18	1.8	10.96	-3.91	1	0.5
ϕ'	30	3	19.59	-3.47	0.5	1
					β	
					4.286	

Fig. G.2. Reliability analysis of laterally-loaded rigid pile in sand using artificial neural network

```

Sub BackPropapation()
CycNum = 5000: eta = 0.9: alpha = 0.9
VarNum = 2: HidNum = 2: SamNum = 9: OutNum = 1

ReDim v(1 To SamNum)
ReDim W1new(1 To VarNum + 1, 1 To HidNum) As Double
ReDim W2new(1 To HidNum + 1, 1 To OutNum) As Double
ReDim W1old(1 To VarNum + 1, 1 To HidNum) As Double
ReDim W2old(1 To HidNum + 1, 1 To OutNum) As Double
ReDim W1delta(1 To VarNum + 1, 1 To HidNum) As Double
ReDim W2delta(1 To HidNum + 1, 1 To OutNum) As Double
ReDim Delj(1 To VarNum + 1, 1 To HidNum + 1) As Double
ReDim Delk(1 To HidNum + 1, 1 To OutNum) As Double
ReDim Oi(1 To SamNum, 1 To VarNum + 1) As Double
ReDim Oj(1 To SamNum, 1 To HidNum + 1) As Double
ReDim Ok(1 To SamNum, 1 To OutNum) As Double
ReDim Tk(1 To SamNum, 1 To OutNum) As Double

With Application.WorksheetFunction
  For i = 1 To VarNum
    vmin = .Min(Range("expv").Columns(i)): vmax = .Max(Range("expv").Columns(i))
    For p = 1 To SamNum
      v(p) = Range("expv").Cells(p, i)
      Oi(p, i) = (v(p) - vmin) / (vmax - vmin): Oi(p, VarNum + 1) = 1
    Next p
  Next i
  For k = 1 To OutNum
    vmin = .Min(Range("resv").Columns(k)): vmax = .Max(Range("resv").Columns(k))
    For p = 1 To SamNum
      v(p) = Range("resv").Cells(p, k): Tk(p, k) = (v(p) - vmin) / (vmax - vmin)
    Next p
  Next k
End With
For i = 1 To VarNum + 1
  For j = 1 To HidNum
    Randomize: W1new(i, j) = 0.6 * Rnd(-i * j) - 0.3
  Next j
Next i
For j = 1 To HidNum + 1
  For k = 1 To OutNum
    Randomize: W2new(j, k) = 0.6 * Rnd(-i * j) - 0.3
  Next k
Next j
W1old = W1new: W2old = W2new
For c = 1 To CycNum
  errorsum = 0
  For p = 1 To SamNum
    For j = 1 To HidNum
      lj = 0
      For i = 1 To VarNum + 1
        lj = lj + W1new(i, j) * Oi(p, i)
      Next i
      Oj(p, j) = 1 / (1 + Exp(-lj))
    Next j
    Oj(p, HidNum + 1) = 1
    For k = 1 To OutNum
      lk = 0
      For j = 1 To HidNum + 1
        lk = lk + W2new(j, k) * Oj(p, j)
      Next j
      Ok(p, k) = 1 / (1 + Exp(-lk))
      errorsum = errorsum + (Tk(p, k) - Ok(p, k)) ^ 2
    Next k
  Next p
Next c

```

```

For i = 1 To VarNum + 1
    For j = 1 To HidNum + 1
        For k = 1 To OutNum
            Delk(j, k) = Delk(j, k) + Ok(p, k) * (1 - Ok(p, k)) * (Tk(p, k) - Ok(p, k)) * Oj(p, j) / (VarNum + 1)
            Delj(i, j) = Delj(i, j) + Oj(p, j) * (1 - Oj(p, j)) * Ok(p, k) * (1 - Ok(p, k)) * (Tk(p, k) - Ok(p, k)) *
W2new(j, k) * Oi(p, i)
        Next k
    Next j
Next i
Next p
For i = 1 To VarNum + 1
    For j = 1 To HidNum
        W1delta(i, j) = W1new(i, j) - W1old(i, j)
        W1old(i, j) = W1new(i, j)
        W1new(i, j) = W1new(i, j) + eta * Delj(i, j) + alpha * W1delta(i, j)
        Delj(i, j) = 0
    Next j
Next i
For j = 1 To HidNum + 1
    For k = 1 To OutNum
        W2delta(j, k) = W2new(j, k) - W2old(j, k)
        W2old(j, k) = W2new(j, k)
        W2new(j, k) = W2new(j, k) + eta * Delk(j, k) + alpha * W2delta(j, k)
        Delk(j, k) = 0
    Next k
Next j
Application.StatusBar = "Training in progress... Cycle " & c & " of " & CycNum & "; Mean-squared error
is " & errorsum / SamNum
Next c
For i = 1 To VarNum + 1
    For j = 1 To HidNum + 1
        If j <= HidNum Then Range("Weights").Cells(i, j).Value = W1new(i, j)
        For k = 1 To OutNum
            Range("Weights").Cells(j, HidNum + k).Value = W2new(j, k)
        Next k
    Next j
Next i
End Sub

```

Fig. G.3. User-defined procedure for backpropagation learning


```

Function NeuralNetwork(x1, x2, W1, W2)
'expv and resv are predefined blocks in spreadsheet
Dim x(1 To 2) As Double
x(1) = x1: x(2) = x2
VarNum = 2: HidNum = 2: OutNum = 1

ReDim Oi(1 To VarNum + 1) As Double
ReDim Oj(1 To HidNum + 1) As Double
ReDim Ok(1 To OutNum) As Double

With Application.WorksheetFunction
  For i = 1 To VarNum
    vmin = .Min(Range("expv").Columns(i))
    vmax = .Max(Range("expv").Columns(i))
    v = x(i)
    Oi(i) = (v - vmin) / (vmax - vmin): Oi(VarNum + 1) = 1
  Next i
End With
For j = 1 To HidNum
  lj = 0
  For i = 1 To VarNum + 1
    lj = lj + W1(i, j) * Oi(i)
  Next i
  Oj(j) = 1 / (1 + Exp(-lj))
Next j
Oj(HidNum + 1) = 1
For k = 1 To OutNum
  lk = 0
  For j = 1 To HidNum + 1
    lk = lk + W2(j, k) * Oj(j)
  Next j
  Ok(k) = 1 / (1 + Exp(-lk))
Next k
With Application.WorksheetFunction
  For k = 1 To OutNum
    vmin = .Min(Range("resv").Columns(k))
    vmax = .Max(Range("resv").Columns(k))
    NeuralNetwork = vmin + (vmax - vmin) * Ok(k)
  Next k
End With
End Function

```

Fig. G.4. User-defined function for obtaining artificial neural network output

Fig. G.5. Comparison between actual responses and approximate responses based on artificial neural network: (a) training data; (b) test data

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