Local structure theory and the Ershov hierarchy

Fang, Chengling

2012

Fang, C. L. (2012). Local structure theory and the Ershov hierarchy. Doctoral thesis,
Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/48687>

<https://doi.org/10.32657/10356/48687>

**NANYANG
TECHNOLOGICAL
UNIVERSITY**

**LOCAL STRUCTURE THEORY AND
THE ERSHOV HIERARCHY**

FANG CHENGLING

School of Physical and Mathematical Sciences

2012

LOCAL STRUCTURE THEORY AND THE ERSHOV HIERARCHY

FANG CHENGLING

School of Physical and Mathematical Sciences

A thesis submitted to the Nanyang Technological University
in partial fulfillment of the requirement for the degree of
Doctor of Philosophy

2012

Acknowledgements

First of all, I would like to express my sincere gratitude to my supervisor, Prof. Wu Guohua, for his instructive advices and helpful suggestions on my research. His creativity, excellent writing and wonderful teaching skills greatly affect my academic idea. I am also grateful to him for his kindness.

I would like to express my heartfelt gratitude to Prof. Zhang Dexue from Sichuan University, who led me into the world of logic.

I owe a special acknowledgement to Prof. Chong Chi-Tat, Prof. Feng Qi, Prof. Frank Stephan and Prof. Yang Yue from National University of Singapore (NUS). I sincerely appreciate their support of my attending the weekly logic seminars, annual workshops and summer schools at NUS and Chinese Academy of Sciences (CAS). I also thank Prof. Theodore A. Slaman from University of California at Berkeley for his invaluable advices and talks. My thanks also go to the Institute for Mathematical Sciences of NUS and Academy of Mathematics and Systems Science of CAS for their financial support of my attending the annual workshops and summer schools.

Thanks are due to Prof. Oggier Elise Frédérique, Prof. Khoo Khoong Ming, Prof. Yeo Sze Ling and Prof. Michael Kaminski, for their wonderful lectures given in graduate courses. I would like to thank Prof. Xing Chaoping for his helpful suggestions for studying algebra. I also thank Graduate Studies Office and Financial Office of NTU for providing me such a good chance to study at NTU with financial support.

I want to thank Dr. George Barmpalias from Beijing and Dr. Mars Yamaleev from NTU for their careful reading of my thesis draft. I wish to thank Wang Shenling for fruitful discussions, and to Ralph Bottesch for his wonderful presentations on computable algebra and complexity theory. I would also like to thank Zhang Yun, Lin Huiling, Li Zhen and Zhang Li for their friendship. Additionally, I thank my

friend Huang Chi for his words of encouragement.

Especially, I am deeply grateful to my parents and sister for their devoted love and perpetual encouragement through all these years. Last but not least, I also would like to thank my beloved husband, Jiang, for his sustained encouragement, continuous love and enthusiastic support through my doctoral study.

Contents

ACKNOWLEDGEMENTS	iii
SUMMARY	ix
1 INTRODUCTION	1
1.1 General Introduction	1
1.2 Nonhemimaximal Degrees and the High/Low Hierarchy	7
1.3 The Distribution of Simple c.e. Reals in the High/Low Hierarchy	10
1.4 Cupping and Diamond Embeddings: A Unifying Approach	12
2 NONHEMIMAXIMAL DEGREES AND THE HIGH/LOW HIER- ARCHY	17
2.1 Introduction	17
2.2 Basic strategies	18
2.2.1 The $\mathcal{G}$ -strategy	18
2.2.2 The $\mathcal{L}$ -requirement and the corresponding strategies	19
2.2.3 A $\mathcal{P}_e$ -strategy	21
2.2.4 An $\mathcal{N}_e$ -strategy	21
2.2.5 An $\mathcal{R}_e$ -strategy	22
2.2.6 A $\mathcal{T}_{e,i}$ -strategy	22
2.3 The construction	25
2.3.1 The construction of the priority tree	26
2.3.2 The full construction	27
2.4 The verification	30

3	THE DISTRIBUTION OF SIMPLE LEFT C.E. REALS IN THE HIGH/LOW HIERARCHY	37
3.1	Introduction	37
3.2	Requirements, Basic strategies and The interactions	39
3.2.1	Requirements	39
3.2.2	Basic $\mathcal{P}_e$ -strategy	39
3.2.3	Basic $\mathcal{N}_e$ -strategy	40
3.2.4	The interaction between $\mathcal{N}$ - and $\mathcal{P}$ -strategies	42
3.2.5	The interaction among two $\mathcal{N}$ -strategies and one $\mathcal{P}$ -strategy	43
3.2.6	Permitting below a high c.e. degree	44
3.2.7	Modified $\mathcal{P}$ -strategy	44
3.2.8	Modified $\mathcal{N}$ -strategy	46
3.3	The Construction	50
3.3.1	The priority tree	50
3.3.2	The full construction	51
3.4	The Verification	56
4	CUPPING AND DIAMOND EMBEDDINGS: A UNIFYING APPROACH	69
4.1	Introduction	69
4.2	Basic strategies and interactions	71
4.2.1	A $\mathcal{G}$ -strategy	71
4.2.2	An $\mathcal{N}$ -strategy	71
4.2.3	An $\mathcal{R}$ -strategy	75
4.2.4	The interaction between two $\mathcal{R}$ -strategies	77
4.2.5	The interaction between one $\mathcal{R}$ -strategy and one $\mathcal{G}$ -strategy	79
4.2.6	The interaction among several $\mathcal{G}$ -strategies and one $\mathcal{R}$ -strategy	81
4.2.7	The interaction between one $\mathcal{N}$ -strategy and one $\mathcal{R}$ -strategy	83
4.2.8	The interaction among two $\mathcal{R}$ -strategies and two $\mathcal{G}$ -strategies	84
4.3	The construction	85
4.3.1	The construction of the priority tree	85
4.3.2	The full construction	87
4.4	The verification	90

Summary

This thesis is concerned with three special properties of Turing degree structure and the Ershov hierarchy. We study the distributions of the nonhemimaximal c.e. degrees, noncomputable left c.e. reals with only computable presentations, and the cupping property in the Ershov hierarchy.

A general introduction is presented in Chapter 1, wherein we review some related work to the topics in this thesis. Our novel results are also mentioned in the introduction chapter.

In Chapter 2, we study the distribution of nonhemimaximal c.e. degrees. After showing the downwards density of nonhemimaximal degrees, Downey and Stob continued to prove that the existence of a low_2 , but not low, nonhemimaximal c.e. degree, and their proof uses the fact that incomplete m -topped degrees are low_2 but not low. As commented in their paper, the construction of such a nonhemimaximal degree is actually a primitive $0'''$ -priority argument. Here, we give an alternative proof of the existence of a low_2 , but not low, nonhemimaximal c.e. degree, where the technique used is a $0''$ -priority argument.

In Chapter 3, we investigate left c.e. reals and prove that below any high c.e. degree, there is a noncomputable left c.e. real with only computable presentations. The proof of this result utilizes the machinery developed by Shore and Slaman.

In Chapter 4, we study the complements of cappable c.e. degrees. We prove that for any nonzero cappable c.e. degree $\mathbf{c}$, there is an almost universal cupping d.c.e. degree $\mathbf{d}$ and a c.e. degree $\mathbf{b} < \mathbf{d}$ such that (i) $\mathbf{b}$ and $\mathbf{c}$ form a minimal pair; and (ii) $\mathbf{b}$ bounds all c.e. degrees below $\mathbf{d}$. Here a d.c.e. degree $\mathbf{d}$ is said to have the almost universal cupping property if it cups all c.e. degrees not below $\mathbf{d}$ to $\mathbf{0}'$. As a consequence, any nonzero cappable c.e. degree $\mathbf{c}$ has a d.c.e. complement with almost universal cupping property. Furthermore, by applying this result to $\mathbf{b}$, we

obtain another almost universal cupping d.c.e. degree $\mathbf{e}$ which is a complement of $\mathbf{b}$, and hence a complement of $\mathbf{d}$. This implies that each nonzero d.c.e. degree can be cupped to $\mathbf{0}'$ by either $\mathbf{d}$ or $\mathbf{e}$. Our result implies several well-known theorems, such as Arslanov's Cupping Theorem, Downey's Diamond Embedding Theorem, Downey-Li-Wu's Complementation Theorem and Li-Yi's Cupping Theorem.

Chapter 1

INTRODUCTION

1.1 General Introduction

Classical Recursion Theory studies unsolvability of functions from ω to ω , where ω is the set of all natural numbers. In [16], Dedekind introduced the study of functions definable over the set ω by recursion, and pulled the trigger of “Recursion Theory”. Although the Recursion Theory is very helpful to define functions, it has its own limitations, which was analyzed in detail by Skolem [77], Peter [63] and Hilbert and Bernays [40]. The identification of the most general form of the recursion principle and recursive functions were led by the collective work of some prominent recursion theory pioneers, including Post [65], Church [7], Gödel [36], Kleene [43] and Turing [83]. In [8], Church claimed the equivalence of the notions of “effectively calculable functions” and “computable functions”.

A partial function f from ω to ω is *partial computable* if there is an algorithm such that the algorithm halts in finitely many steps with output $f(n)$ if and only if the input n is in the domain of f . In particular, f is said to be *(total) computable* if f is partial computable with the domain ω . By Gödel numbering, each natural number $e \in \omega$ codes a partial computable function, denoted as φ_e , and conversely, for each partial computable function f , there is a natural number e coding f , i.e., $f = \varphi_e$, where e is called an *index* of f . Note that there are infinitely many indices for any given partial computable function. In the following, any set is a subset of natural numbers if there is no special explanation.

By using computable functions, in 1944, Post [64] first proposed the many-one

reduction. Given two sets A and B , say that A is *many-one reducible to B* (denoted as $A \leq_m B$) if there is a computable function f such that $n \in A$ if and only if $f(n) \in B$. For example, the set of even numbers is many-one reducible to the set of odd numbers. We are mainly concerned with the Turing reduction, a reduction induced by generally computable functionals over 2^ω (the power set of ω). Turing reduction was firstly proposed by Alan Turing in [84] in terms of Turing machines with oracles.

Given two sets A and B , A is called Turing reducible to B , denoted as $A \leq_T B$, if there is an oracle Turing machine Φ_e for some $e \in \omega$ such that A is computed from B (or B -computable) via Φ_e , i.e., $A = \Phi_e(B)$. Moreover, we say that A is *Turing equivalent to B* (denoted as $A \equiv_T B$) if and only if $A \leq_T B$ and $B \leq_T A$. It is easy to see that the relation $\equiv_T$ is an equivalence relation and under this equivalence relation, the equivalence classes are called Turing degrees. That is, for a given set A , the equivalence class of A , $\{B : B \equiv_T A\}$ is called the *Turing degree* (degree for short) of A , and we denote it as $\deg_T(A)$. Conventionally, we use boldface letters $\mathbf{a}, \mathbf{b}, \dots$ to denote degrees.

Turing reduction induces a partial order on the Turing degrees: $\mathbf{a} \leq \mathbf{b}$ if there are sets $A \in \mathbf{a}$ and $B \in \mathbf{b}$ such that $A \leq_T B$, and $\mathbf{a} < \mathbf{b}$ if there are sets $A \in \mathbf{a}$ and $B \in \mathbf{b}$ such that $A \leq_T B$ and $B \not\leq_T A$. Let $\mathbf{D}$ be the class of all Turing degrees, then $\mathbf{D}$ has a least element $\mathbf{0}$, the degree of empty set. The *jump operator*, denoted by $'$, read as *jump*, is defined by $A' = \{e : \Phi_e(A; e) \downarrow\}$ for a set A . Here $\Phi_e(A; e) \downarrow$ means that the e -th oracle Turing machine with oracle A runs on input e halts in finitely many steps. By applying a standard diagonalization argument, we can prove $A <_T A'$. Given a degree $\mathbf{a}$, we define the jump of $\mathbf{a}$, as the degree of A' , for any set $A \in \mathbf{a}$. This operation is well-defined.

A set A is called a *computably enumerable* (c.e. for short) set if it is the domain of a partial computable function from ω to ω . It is easy to see that a set A is *computable* if both of A and $\bar{A}$, the complement of A , are c.e. sets. A degree $\mathbf{a}$ is called a c.e. degree if it contains a c.e. set. The class of c.e. degrees is denoted as $\mathcal{R}$, and this class contains a greatest c.e. element, $\mathbf{0}'$, the degree of the halting problem set $K = \{e : \varphi_e(e) \downarrow\}$. Any c.e. set is Turing reducible to K , and those c.e. sets equivalent to K are called *Turing complete*.

Up to 1944, people only know two c.e. degrees $\mathbf{0}$ and $\mathbf{0}'$ and Post [64] asked the

question whether there is some other c.e. degree $\mathbf{c}$ with $\mathbf{0} < \mathbf{c} < \mathbf{0}'$. Post attempted to solve this problem by constructing a coinfinite c.e. set A such that $\bar{A}$ contains no infinite c.e. set, i.e. A intersects with any infinite c.e. set. Such a set A is called a *simple set*. Unfortunately, the simple sets constructed by Post are proved to be Turing complete. After seeing this, Post introduced several even stronger notions, such as hypersimple sets and hyperhypersimple sets [64]. The investigation of Post's problem speeds up the development of Recursion Theory, and this problem was eventually solved in 1950s, by Friedberg [34] and Muchnik [60] independently.

Theorem 1.1 (Friedberg [34], Muchnik [60]). *There exist two incomparable c.e. sets A and B , i.e. $A \not\leq_T B$ and $B \not\leq_T A$.*

This result definitely is a landmark in the Classical Recursion Theory, not just because it solves Post's Problem, but also because of the priority argument developed by Friedberg and Muchnik. The priority argument has been developed widely by the recursion theorists, and now become a powerful technique in Degree Theory.

The infinite injury priority argument was proposed by Shoenfield [74] and by Sacks ([68],[70],[71]) independently, where Sacks proved that the c.e. degrees are dense.

Theorem 1.2 (Sacks [71]). *For any two c.e. sets $A <_T B$, there is a c.e. set C such that $A <_T C <_T B$.*

Given two degrees $\mathbf{a}$ and $\mathbf{b}$, fix two sets $A \in \mathbf{a}$ and $B \in \mathbf{b}$, let $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$, then the degree of $A \oplus B$ is the least degree above $\mathbf{a}$ and $\mathbf{b}$, i.e. the supremum of $\mathbf{a}$ and $\mathbf{b}$ in the partially ordered set $(\mathbf{D}, \leq)$, denoted as $\mathbf{a} \cup \mathbf{b}$. Hence $(\mathbf{D}, \leq)$ is an upper semi-lattice. In fact, if $\mathbf{a}$ and $\mathbf{b}$ are c.e. degrees then the degree of $A \oplus B$ is the least upper bound of $\mathbf{a}$ and $\mathbf{b}$ in the partially ordered set $(\mathbf{R}, \leq)$. Hence $(\mathbf{R}, \leq)$ is also an upper semi-lattice.

From Sacks Density Theorem, it looks that $\mathcal{R}$ is a dense structure as an upper semi-lattice analogous to the rationals being a dense structure as a linearly ordered set. Shoenfield [75] conjectured that for every pair $\mathbf{P} \hookrightarrow \mathbf{Q}$ of finite upper semi-lattices with 0, 1 and every embedding f from $\mathbf{P}$ into $\mathbf{R}$, there is an extension g of f as an embedding from $\mathbf{Q}$ into $\mathbf{R}$. From Shoenfield's Conjecture, the following two consequences can be deduced:

C1. If $\mathbf{a}, \mathbf{b} \in \mathbf{R}$ are incomparable then they do not have infimum in $\mathbf{R}$.

C2. Given any c.e. degrees $\mathbf{0} < \mathbf{b} < \mathbf{a}$, there is a c.e. degree $\mathbf{c} < \mathbf{a}$ such that $\mathbf{a} = \mathbf{b} \cup \mathbf{c}$.

Unfortunately, Shoenfield's Conjecture was refuted by Lachlan [44] and independently by Yates [90] by showing the existence of minimal pairs of c.e. degrees (hence C1 is not true). That is, they proved that there are two nonzero c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ with infimum $\mathbf{0}$ in $\mathbf{R}$, denoted as $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$. In fact, Lachlan [44] and Yates [90] showed that there is a pair of c.e. degrees without infimum, and hence $(\mathbf{R}, \leq)$ is not a lattice.

In this upper semi-lattice, a degree $\mathbf{a}$ is called *cappable* if $\mathbf{a} = \mathbf{0}$ or $\mathbf{a}$ is a part of a minimal pair. $\mathbf{a}$ is called *noncappable*, if it is not cappable. Dually, a c.e. degree $\mathbf{c}$ is *cuppable* if there is an incomplete c.e. degree $\mathbf{b}$ such that $\mathbf{c} \cup \mathbf{b} = \mathbf{0}'$. $\mathbf{c}$ is called *noncuppable*, if it is not cuppable.

Sacks Splitting Theorem [69] indicates the existence of incomplete cuppable c.e. degrees. On the other hand, Yates (unpublished, see Cooper [11]) proved the existence of nonzero noncuppable c.e. degrees. Thus the second consequence, C2, of Shoenfield's Conjecture also fails. Harrington [39] (see Miller [59]) proved that below any high c.e. degree, there is a high noncuppable degree.

Although Shoenfield's Conjecture is not true, the study of it has been playing an important role in the development of Recursion Theory. In particular, it motivates a thorough investigation of the structure of $\mathbf{R}$, such as lattice embeddings into it. The following Lachlan's Nondiamond Theorem shows that the four-element lattice cannot be embedded into $\mathbf{R}$ preserving 0 and 1.

Theorem 1.3 (Lachlan [44]). *If $\mathbf{a}$ and $\mathbf{b}$ are nonzero c.e. degrees with $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$ then there is a nonzero c.e. degree $\mathbf{c}$ such that $\mathbf{c} \leq \mathbf{a}$ and $\mathbf{c} \leq \mathbf{b}$.*

Cuppability and cappable of c.e. degrees are related closely to each other in terms of the high/low hierarchy. For any number n and any degree $\mathbf{a}$, $\mathbf{a}^{(n)}$ is the n -th jump of $\mathbf{a}$. Given a natural number $n > 0$, a degree $\mathbf{a} \leq \mathbf{0}'$ is called *low_n* (*high_n*) if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$ ($\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$, respectively), and a set $A \leq_T \emptyset'$ is *low_n* (*high_n*) if $\deg_T(A)$ is *low_n* (*high_n*, respectively). For simplicity, *low₁* (*high₁*) is called *low* (*high*). A c.e. degree $\mathbf{a}$ is *low_n-cuppable* if there is a *low_n* c.e. degree $\mathbf{b}$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$. Let $LC_n = \{\mathbf{a} : \mathbf{a} \text{ is c.e. and low}_n\text{-cuppable}\}$ and LC_1 be denoted as LC . Let M denote the set of all cappable c.e. degrees and NC denote the set of all noncappable c.e.

degrees. Ambos-Spies, Jockusch, Shore and Soare obtained a relation among these classes in the following theorem.

Theorem 1.4 (Ambos-Spies, Jockusch, Shore and Soare [1]). *M is an ideal of $\mathbf{R}$, $LC = NC$, and $M \cup LC = \mathbf{R}$.*

Any nonzero c.e. degree, $\mathbf{a}$ say, bounds a nonzero cappable c.e. degree, and Cooper [10] showed that any high c.e. degree indeed bounds a minimal pair. However, Lachlan showed that there are nonzero c.e. degrees bounding no minimal pairs.

Theorem 1.5 (Lachlan [47]). *There is a c.e. degree $\mathbf{a} > \mathbf{0}$ with no c.e. minimal pair below it. Such c.e. degree $\mathbf{a}$ is called a nonbounding degree.*

Note that nonbounding degrees are all cappable. Hence, every nonzero c.e. degree below a nonbounding degree, $\mathbf{a}$ say, has its all partners of minimal pair not below $\mathbf{a}$. Seetapun extracted this property and proposed the concept of locally noncappable as follows.

Definition 1.6 (Seetapun [73]). *A c.e. degree $\mathbf{a}$ is called locally noncappable if there is a c.e. degree $\mathbf{c}$ above $\mathbf{a}$ such that no nonzero c.e. degree $\mathbf{w}$ below $\mathbf{c}$ forms a minimal pair with $\mathbf{a}$. Here $\mathbf{c}$ is a witness for the local noncappability of $\mathbf{a}$.*

Seetapun proved in [73] that every nonzero incomplete c.e. degree is locally noncappable. This property can imply that there are no maximal nonbounding c.e. degrees. The proof that there is no maximal nonbounding degree is in Welch's PhD

Thesis [85] under Jockusch. Recently, Stephan and Wu [82] showed that such witnesses can always be chosen as $high_2$ degrees.

Theorem 1.7 (Stephan and Wu [82]). *Given any nonzero incomplete c.e. degree $\mathbf{a}$, there exists a $high_2$ c.e. degree $\mathbf{c} > \mathbf{a}$ witnessing that $\mathbf{a}$ is locally noncappable.*

The proof of this theorem combines Seetapun's construction and $high_2$ construction developed by Lerman [50], Downey, Lempp and Shore [22], with some technique to cope with the interactions between two kinds of strategies. This theorem implies several well-known results, such as Downey, Lempp and Shore's result [22] that there is a $high_2$ nonbounding c.e. degree. This theorem can also imply that there are no maximal plus-cupping degrees and the existence of a $high_2$ plus-cupping degree, in terms of Harrington. The latter was proved recently by Li in [51].

On the other hand, we can show the existence of such incomparable witnesses with a high join.

Theorem 1.8 (Fang, Wang and Wu [31]). *For any nonzero incomplete c.e. degree $\mathbf{a}$, there exist two incomparable c.e. degrees $\mathbf{c}, \mathbf{d} > \mathbf{a}$ both witnessing that $\mathbf{a}$ is locally noncappable, and $\mathbf{c} \cup \mathbf{d}$ is high.*

As a direct corollary, we have that the nonbounding c.e. degrees do not form an ideal, and that the plus-cupping degrees do not form an ideal. Both results were proved by Li and Zhao [55] as two separated results.

Slaman and Soare [78] showed the existence of incomparable c.e. degrees $\mathbf{a}$ and $\mathbf{c}$ such that $\mathbf{c}$ cups the c.e. degrees below $\mathbf{a}$, but not below $\mathbf{c}$, to $\mathbf{0}'$. We refer such a pair $(\mathbf{a}, \mathbf{c})$ as *Slaman-Soare pair*. As a consequence, $\mathbf{c}$ cups $\mathbf{a}$ to $\mathbf{0}'$. Such $\mathbf{c}$ bounds a nonzero noncuppable c.e. degree, i.e. $\mathbf{c}$ can not be plus-cupping.

Note that by using Theorem 1.4, Li, Wu and Zhang showed an approach of constructing a low_2 , but not low, c.e. degree.

Theorem 1.9 (Li, Wu and Zhang [53]). $LC \subset LC_2$.

1.2 Nonhemimaximal Degrees and the High/Low Hierarchy

In Chapter 2, we consider how to apply Li-Wu-Zhang's technique to the construction of a low_2 , but not low, nonhemimaximal degree, a result first proved by Downey and Stob. For this, we need to have some background of the lattice of c.e. sets, maximal and hemimaximal sets.

For two c.e. sets A, B , $A \cup B = \{x : x \in A \text{ or } x \in B\}$ and $A \cap B = \{x : x \in A \text{ and } x \in B\}$ are also c.e. sets. Let $\mathcal{C}$ be the set of all c.e. sets. $(\mathcal{C}, \subseteq)$ is a lattice, which is also denoted as $\mathcal{C}$ for short. Let $\mathcal{C}^*$ denote the quotient lattice $\mathcal{C}/\mathcal{F}$, where $\mathcal{F}$ is the ideal of the finite sets. For any set $X \in \mathcal{C}$, we define $X^* = \{B : X =^* B\}$, where $X =^* B$ means that $(X - B) \cup (B - X)$ is finite. We call a c.e. set X *maximal* if X^* is a coatom of $\mathcal{C}^*$, i.e. if $\bar{X}$ is infinite and for any c.e. set W , either $W \cap \bar{X}$ or $\bar{W} \cap \bar{X}$ is finite. Friedberg proved in [35] the existence of maximal sets, which actually gave the first proof of the existence of hyperhypersimple sets, as maximal sets are hyperhypersimple. Here a c.e. set A is hyperhypersimple if it is coinfinite and for any computable disjoint sequence $\{W_{f(n)}\}_{n \in \omega}$ with $|W_{f(n)}| < \infty$, there is some n such that $W_{f(n)} \subseteq A$. If the c.e. set W , in the definition of maximal sets, is restricted to be computable, then the corresponding maximal sets are called *r-maximal*. It is obvious that maximal sets are r-maximal sets. However its inverse is not true, which can be proved from Proposition 1.10 and Theorem 1.11 to follow. For simplicity, we

make use of the following notations. $A \subset_\infty B$ if $A \subseteq B$ and $B - A$ is infinite. $A \subset_m B$ if $A \subset_\infty B$ and for any c.e. set W , if $B \cup W = \omega$, then $\omega - (A \cup W)$ is finite.

Proposition 1.10 (Soare [81, Chapter X 4.3]). *If B is r -maximal and $A \subset_m B$, then A is r -maximal but not maximal.*

Theorem 1.11 (Lachlan [46]). *For any noncomputable c.e. set B , there is a c.e. set A such that $A \subset_m B$.*

In Theorem 1.11, suppose B is a maximal set, then B is also an r -maximal set, and hence A is r -maximal but not maximal by Proposition 1.10. Contrast to Theorem 1.11, it is natural to ask whether there is some r -maximal set which has no maximal superset. Conventionally, we call a coinfinite c.e. set A *atomless* (see [81, Chapter X 5.1]) if A has no maximal superset. Lachlan [46] and Robinson [67] proved the existence of an atomless r -maximal set.

An investigation of $\mathcal{C}$ is to study the actions of the automorphism group $Aut(\mathcal{C})$, where $f \in Aut(\mathcal{C})$ is an *automorphism* (see Soare [81, Chapter X]) of $\mathcal{C}$ if f is a bijection from $\mathcal{C}$ to $\mathcal{C}$ preserving inclusion $\subseteq$. For $A, B \in \mathcal{C}$, we say that A is *automorphic* to B (see Soare [81, Chapter XV 1.3]), denoted as $A \cong_{\mathcal{C}} B$, if there is an $f \in Aut(\mathcal{C})$ such that $f(A) = B$. The orbit of $A \in \mathcal{C}$, denoted as $orbit(A)$ (see Soare [81, Chapter XV 1.3]), is the $\mathcal{C}$ -equivalent class $\{B : A \cong_{\mathcal{C}} B\}$ determined by A . Harrington proved that the creative sets form an orbit (see Soare [81, Chapter XV 1.4]) by showing that the property of being creative is elementary lattice theoretic. This orbit only contains m -complete sets by the equivalence of creativeness and m -completeness. Soare proved in [80] that the maximal sets form an orbit, which is a degree theoretically larger orbit, since the maximal sets realize all the high c.e. degrees [58, 25]. Is there some orbit containing sets not only from high c.e. degrees? To answer this question, Downey and Stob introduced the notion of hemimaximal sets in [24]. A c.e. set H is called *hemimaximal* if H is noncomputable and there is a noncomputable c.e. set A disjoint from H such that $H \cup A$ is maximal. Accordingly, a degree $\mathbf{a}$ is *hemimaximal* if there is a hemimaximal set in $\mathbf{a}$. Otherwise, $\mathbf{a}$ is called *nonhemimaximal*. Downey and Stob [25] proved that the hemimaximal sets form an orbit and any high c.e. degree contains a hemimaximal set. Moreover, they showed that there are some low c.e. hemimaximal degrees. Thus, the orbit formed by hemimaximal degree is a degree theoretically larger orbit than the two orbits mentioned above. It is known by

Downey and Harrington [19] that there is no orbit containing all nonzero degrees. By using hemimaximal sets, partial positive answer to Soare's Conjecture, which claimed that all noncomputable c.e. sets are automorphic to complete sets, can be obtained. For example, Downey and Stob [25] showed that all promptly simple sets of low degree are automorphic to complete sets by using hemimaximal sets. In the same paper they studied hemimaximal degrees, nonhemimaximal degrees and their distributions in the high/low hierarchy. Wherein, they showed that hemimaximal degrees are downwards dense and nonhemimaximal degrees are dense in the low c.e. degrees. Downey, Lempp and Shore [22] proved that nonhemimaximal sets can be high_2 . Furthermore, Downey and Stob [24] showed that jump inversion holds for hemimaximal sets.

Theorem 1.12 (Downey and Stob [24]). *If S is c.e. in and above $\emptyset'$, then there is a hemimaximal set C such that $C' \equiv_T S$.*

From the above theorem, Downey and Stob [24] obtained a result that there is an elementarily definable orbit of $\text{Aut}(\mathcal{C})$ containing members of all possible jumps. In [24], they also showed that the nonhemimaximal sets can have low_2 but not low degrees. To obtain this result, Downey and Stob used the property of incomplete m -topped degrees, which was introduced in [20] by Downey and Jockusch, that they are low_2 but not low. Here, a c.e. degree $\mathbf{a}$ is called *m -topped* if there is a c.e. set $A \in \mathbf{a}$ such that for any c.e. set $B \in \mathbf{a}$, $B \leq_m A$. Downey and Stob's idea was to construct an m -topped nonhemimaximal c.e. set. The combinatorics in their construction was quite complicated: as they pointed out in the paper, the construction is in some sense a primitive $0'''$ -priority argument.

In Chapter 2, we will provide another construction of a low_2 , but not low, nonhemimaximal degree, and the complexity of our construction is much simpler than the one given in [24], as it is a standard $0''$ -priority argument.

Theorem 1. [Fang and Wu [33]] *There are a nonhemimaximal low_2 c.e. degree $\mathbf{l}$ and nonzero c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ satisfying $\mathbf{a} \cup \mathbf{l} = \mathbf{0}'$ and $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$.*

This is joint work with Wu and will be published in the "Journal of Symbolic Logic".

1.3 The Distribution of Simple c.e. Reals in the High/Low Hierarchy

Chapter 3 is devoted to the study of presentations of left computably enumerable reals. A real number a is called a *left computably enumerable real* (left c.e. real for short) [79] if there is a computable increasing sequence of rational numbers $\{a_i : i \in \omega\}$ converging to a . Moreover, a left c.e. real is called a *computable real* if there is a computable function f such that $a - a_i \leq 2^{-j}$ for all i, j with $i \geq f(j)$. Another equivalent definition of computable reals is: let a be a real, define $f(0) = 0$ and $f(n) =$ the n -th digit in the binary expansion of a for any $n > 0$, then a is computable if and only if f is computable.

Fix a real number a , $L(a) = \{q \in \mathbb{Q} : q \leq a\}$ is the Dedekind cut of a . It is easy to see that a is a left c.e. real if and only if $L(a)$ is a computably enumerable set of rational numbers.

For any set A , there is a real $a = \sum_{n \in A} 2^{-n}$. Accordingly, if A is c.e., then a is a left c.e. real. Jockusch observed that there is some left c.e. real a such that $A \subseteq N$ is not a c.e. set, where $a = \sum_{n \in A} 2^{-n}$.

Let $\{0, 1\}^*$ denote the set of finite strings over $\{0, 1\}$. A set $W \subseteq \{0, 1\}^*$ is *prefix-free* if for any $\sigma \in W$, no extension of σ is in W . In [6], Calude, Hertling, Khoussainov and Wang provided several characterizations of left c.e. reals.

Theorem 1.13 (Calude, Hertling, Khoussainov and Wang [6]). *The following are equivalent for a real a :*

- (1) *a is a left c.e. real.*
- (2) *The left Dedekind cut of a is computable enumerable.*
- (3) *There is an infinite computably enumerable prefix-free set $W \subset \{0, 1\}^*$ such that $a = \sum_{\sigma \in W} 2^{-|\sigma|}$.*
- (4) *There is a computable prefix-free set $W \subset \{0, 1\}^*$ such that $a = \sum_{\sigma \in W} 2^{-|\sigma|}$.*
- (5) *There is a computable function $f(x, y)$ of two variables such that*
 - (a) *for all k, s , if $f(k, s) = 1$ and $f(k, s + 1) = 0$, then there exists $m < k$ such that $f(m, s) = 0$ and $f(m, s + 1) = 1$.*
 - (b) *$a = 0.a_1a_2\cdots$, where $a_i = \lim_s f(i, s)$*
- (6) *There is a computable increasing sequence of rationals with limit a .*

The set W in Theorem 1.13 (3) is called *a presentation* of the left c.e. real a , according to Downey and LaForte [21]. By Theorem 1.13 (4), every left c.e. real has a computable presentation. Downey and LaForte proved in [21] that there are noncomputable left c.e. reals with only computable presentations. According to Wu [88], left c.e. reals with only computable presentations are called *simple reals*.

Theorem 1.14 (Downey and LaForte [21]). *There exists a simple noncomputable left c.e. real a .*

Furthermore, Downey and LaForte [21] proved that if a has promptly simple degree, then a has noncomputable presentations. They also proved that there is a simple left c.e. real with high Turing degree. Wu [86] proved that the simple left c.e. reals are not dense in $\mathbf{R}$ by using a gap-cogap method. Recently, Stephan and Wu [82] prove the same result by using the notion of K -triviality from algorithmic randomness.

In Chapter 3, we prove that below any high c.e. degree, there is a noncomputable simple left c.e. real. We use the machinery of Shore and Slaman developed in [76] to work below a high c.e. degree.

Theorem 2. [Fang and Wu [32]] *For any high c.e. set H , there is a noncomputable simple left c.e. real $a \leq_T H$.*

This is joint work with Wu.

1.4 Cupping and Diamond Embeddings: A Unifying Approach

In Chapter 4, we study the cappable and cuppable in the Ershov hierarchy. Putnam [66] and Gold [37] proposed the notion of n -c.e. sets. A set $A \subseteq \omega$ is called an n -c.e. set if there is a computable approximation $\{A_s\}_{s \in \omega}$ such that the following conditions are satisfied:

- (1) $A_0(x) = 0$, for any $x \in \omega$;
- (2) $|\{s : A_{s+1}(x) \neq A_s(x)\}| \leq n$ for any $x \in \omega$;
- (3) $A(x) = \lim_s A_s(x)$ for any $x \in \omega$.

This hierarchy was extended by Ershov [27, 28, 29] into the transfinite level so as to include all Δ_2^0 sets. Especially, a set A is called ω -c.e. if the condition (2) is changed to (2') as follows:

- (2') there is a computable function f such that $|\{s : A_{s+1}(x) \neq A_s(x)\}| \leq f(x)$ for any $x \in \omega$;

Accordingly, a degree is n -c.e. (ω -c.e.) if it contains an n -c.e. (ω -c.e.) set. Let $\mathbf{D}_n$ ($\mathbf{D}_\omega$) denote the set of all n -c.e. (ω -c.e.) degrees. By definition, a set is a 1-c.e. set if and only if it is a c.e. set, so $\mathbf{D}_1$ is exactly $\mathbf{R}$. Obviously, $\mathbf{D}_m \subseteq \mathbf{D}_n$ for $0 < m < n$. In fact, Cooper [9] first showed that $\mathbf{D}_m \subset \mathbf{D}_n$ for $0 < m < n$. $\cup_n \mathbf{D}_n$ is also a proper subclass of $\mathbf{D}_\omega$ by the existence of ω -c.e. minimal degrees.

The structure of degrees below $\mathbf{0}'$ has been studied by many recursion theorists. Lachlan [45], Epstein [26], Lerman [49], Harrington and Shelah [38], showed that the the first-order theories of $\mathcal{D} = (\mathbf{D}, \leq)$ and $\mathcal{R} = (\mathbf{R}, \leq)$ are all undecidable. Recently, Cai, Shore and Slaman [5] proved the theory of $\mathcal{D}_n$ are undecidable for every $n \in \omega$.

Since $\mathbf{R} = \mathbf{D}_1 \subset \mathbf{D}_2 \cdots \subset \mathbf{D}_m \cdots \subset \mathbf{D}(\leq \mathbf{0}')$, it is natural to ask whether the partial order structures $\mathcal{D}_m$ and $\mathcal{D}_n$ are elementarily equivalent for $m \neq n$. For $m = 1$ and $n = 2$, Arslanov first gave a negative answer.

Theorem 1.15 (Arslanov [2]). *For any nonzero d.c.e. degree $\mathbf{d}$, there is a low d.c.e. degree $\mathbf{c}$ such that $\mathbf{d} \cup \mathbf{c} = \mathbf{0}'$.*

This theorem, Arslanov's Cupping Theorem, together with the existence of a non-cupppable c.e. degree, implies a Σ_3 -difference between the partial ordered structure $\mathcal{R} = (\mathbf{R}, \leq)$ and $\mathcal{D}_2 = (\mathbf{D}_2, \leq)$. Moreover, from Lachlan's observation that there is a nonzero c.e. (hence d.c.e.) degree below any nonzero n-c.e. degree, then for any nonzero n-c.e. degree $\mathbf{d}$, there is a low d.c.e. (hence n-c.e.) degree $\mathbf{c}$ such that $\mathbf{d} \cup \mathbf{c} = \mathbf{0}'$, thus there is also a Σ_3 -difference between $\mathcal{R}$ and $\mathcal{D}_n$ for any $n > 2$. Arslanov's result was extended by Cooper, Lempp, and Watson [13] to prove that every nonzero d.c.e. degree cups to any high c.e. degree above it by a low d.c.e. degree.

Let $\mathbf{a}$ be a nonzero noncupppable c.e. degree in $\mathbf{R}$. Applying Arslanov's Cupping Theorem, there is a low d.c.e. degree $\mathbf{c}$ such that $\mathbf{a} \cup \mathbf{c} = \mathbf{0}'$. It is easy to see that there is no c.e. degree strictly between such $\mathbf{c}$ and $\mathbf{0}'$, otherwise, $\mathbf{a}$ must be cupppable in $\mathbf{R}$. Furthermore, does there exist a d.c.e. degree $\mathbf{c}$ such that there is no d.c.e. degree strictly between $\mathbf{c}$ and $\mathbf{0}'$? The following theorem gives a positive solution to this question.

Theorem 1.16 (Cooper, Harrington, Lachlan, Lempp and Soare [12]). *There is a d.c.e. degree $\mathbf{c}$ such that there is no d.c.e. degree strictly between $\mathbf{c}$ and $\mathbf{0}'$.*

Such a d.c.e. degree $\mathbf{c}$ in Theorem 1.16 is called a *maximal d.c.e. degree*. This and Sacks Density Theorem show that $\mathbf{R}$ and $\mathbf{D}_2$ differ at the Σ_2 -level. In fact, the Σ_2 -difference was firstly obtained by Downey who showed that the four-element lattice can be embedded into the d.c.e. degrees preserving 0 and 1.

Theorem 1.17 (Downey [18]). *There exist d.c.e. degrees $\mathbf{a}, \mathbf{b} > \mathbf{0}$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$.*

In [87], Wu gave an another approach of such lattice embeddings by using the isolation phenomenon, and this approach has been improved a lot in his recent work with his coauthors.

In [18], Downey also proposed the following conjecture.

Conjecture 1.18 (Downey [18]). *The structures $\mathbf{D}_m$ and $\mathbf{D}_n$ are elementarily equivalent for any natural numbers $m, n > 1$.*

Recently, Arslanov, Kalimullin and Lempp [3] gave a negative answer to this Conjecture 1.18 by showing that the degree structures of $\mathbf{D}_2$ and $\mathbf{D}_3$ are not elementarily equivalent. People are working to show that $\mathbf{D}_m$ and $\mathbf{D}_n$ are not elementarily equivalent for any m and n , if $m \neq n$.

Notice that any elementary differences among $\mathbf{D}_n$ would not occur at the Σ_1 -level. People ask whether $\mathbf{D}_n$ is a substructure of $\mathbf{D}_m$ if $n < m$. According to Kaddah [42], Slaman was able to show that $\mathbf{R}$ is not a Σ_1 -substructure of $\mathbf{D}_2$, but Slaman never published this result. Recently, Yang and Yu [89] provided a proof showing that $\mathbf{R}$ is not a Σ_1 -substructure of $\mathbf{D}_2$. Cai, Shore and Slaman [5] completely answered this question by proving that $\mathbf{D}_n$ is not a Σ_1 -substructure of $\mathbf{D}_m$ for any $n < m$.

Isolation phenomenon is widely used for studying the structural relation between $(\mathbf{R}, \leq)$ and $(\mathbf{D}_2, \leq)$. A d.c.e degree $\mathbf{d}$ is called *an isolated degree* if there is some c.e. degree $\mathbf{b}$ such that $\mathbf{b}$ is the greatest c.e. degree below $\mathbf{d}$. We also say that $\mathbf{b}$ *isolates* $\mathbf{d}$. This notion was proposed by Cooper and Yi in [15].

Ding and Qian [17], and independently LaForte [48], showed that the isolated degrees as well as the isolating degrees are dense in the c.e. degrees. Especially, Salts [72] proved that there is an interval of c.e. degrees inside which each c.e. degree is an isolating degree, so the nonisolating c.e. degrees are not dense. Cooper, Salts and Wu [14] proved that non-isolating degrees are also upwards dense in the c.e. degrees. On the other hand, Arslanov, Lempp and Shore [4] proved that the non-isolated degrees are also dense in the c.e. degrees, and that showed the nonisolating degrees occur in every jump class.

In [41], Ishmukhametov and Wu proved that there is a high d.c.e. degree which is isolated by a low c.e. degree. Li, Wu and Yang [52] improved this result by showing that any nonzero c.e. degree bounds a c.e. degree $\mathbf{b}$ which isolates a high d.c.e. degree $\mathbf{d}$. This result implies the existence of a high d.c.e. nonbounding degree.

The construction of maximal d.c.e. degree is fairly complicated, and it is not easy to combine such a construction with some other properties. Note that a maximal d.c.e. degree cups every nonzero c.e. degree not below it to $\mathbf{0}'$. Liu and Wu [57] studied the d.c.e degrees with this property and called it *almost universal cupping property*. Formally, a d.c.e. degree $\mathbf{d}$ is said to have almost universal cupping property if it

cups all c.e. degrees not below $\mathbf{d}$ to $\mathbf{0}'$, i.e. $\mathbf{d} \cup \mathbf{c} = \mathbf{0}'$ for any c.e. degree $\mathbf{c} \not\leq \mathbf{d}$. Liu and Wu proved that the d.c.e. degrees with almost universal cupping property can be isolated degrees.

Theorem 1.19 (Liu and Wu [57]). *There is an incomplete almost universal cupping d.c.e. degree $\mathbf{d}$ which is isolated by a c.e. degree $\mathbf{b}$.*

Recently, Liu and Wu further showed that the d.c.e. degrees in [56] can be cappable in the d.c.e. degrees. That is, there are an incomplete almost universal cupping d.c.e. degree $\mathbf{d}$, isolated by a c.e. degree $\mathbf{b}$, and a nonzero c.e. degree $\mathbf{c}$ such that $\mathbf{b}$ and $\mathbf{c}$ form a minimal pair. This implies that any nonzero c.e. degree below $\mathbf{c}$ and any incomplete d.c.e. degree above $\mathbf{d}$, together with $\mathbf{d}$, $\mathbf{0}$ and $\mathbf{0}'$, form a diamond lattice, greatly improving Downey's Diamond Embedding Theorem.

Recall that in [23], Downey, Li and Wu showed that each cappable c.e. degree $\mathbf{c}$ has a d.c.e. degree $\mathbf{d}$ such that $\mathbf{c} \cap \mathbf{d} = \mathbf{0}$ and $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$. Such $\mathbf{d}$ is called a *complement* of $\mathbf{c}$ in $\mathbf{D}_2$. We can prove that such a d.c.e. degree $\mathbf{d}$ can be chosen as one with almost universal cupping property.

Theorem 3 (Fang, Liu and Wu [30]). *Let $\mathbf{c}$ be a nonzero cappable c.e. degree. Then there is an almost universal cupping d.c.e. degree $\mathbf{d}$ and a c.e. degree $\mathbf{b} < \mathbf{d}$ such that $\mathbf{c}$ cups $\mathbf{d}$ to $\mathbf{0}'$, caps $\mathbf{b}$ to $\mathbf{0}$ and $\mathbf{b}$ isolates $\mathbf{d}$.*

The proof of Theorem 3 is given in Chapter 4. Obviously, $\mathbf{d}$ cups $\mathbf{c}$ to $\mathbf{0}'$, as $\mathbf{d}$ has almost universal cupping property and that $\mathbf{c}$ is not below $\mathbf{d}$. We claim that $\mathbf{c}$ and $\mathbf{d}$ form a minimal pair in the d.c.e. degrees, as otherwise, there is a nonzero d.c.e. degree $\mathbf{v}$ below both $\mathbf{c}$ and $\mathbf{d}$. By Lachlan's observation, we can assume that $\mathbf{v}$ is c.e. As $\mathbf{b}$ is the greatest c.e. degree below $\mathbf{d}$, $\mathbf{v}$ is below $\mathbf{b}$, which contradicts the condition that $\mathbf{b}$ and $\mathbf{c}$ form a minimal pair. Thus $\mathbf{0}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{0}'$ form a diamond in d.c.e. degrees. In other words, $\mathbf{c}$ has an almost universal cupping d.c.e. degree $\mathbf{d}$ as its complement.

Now we apply Theorem 3 to $\mathbf{b}$, and obtain another almost universal cupping d.c.e. degree $\mathbf{f}$ and a c.e. degree $\mathbf{e}$, which isolates $\mathbf{f}$ and form a minimal pair with $\mathbf{b}$. Thus, for any nonzero d.c.e. degree, $\mathbf{x}$ say, if it is below $\mathbf{d}$, then $\mathbf{x}$ cups $\mathbf{f}$ to $\mathbf{0}'$, and if it is below $\mathbf{f}$, then $\mathbf{x}$ cups $\mathbf{d}$ to $\mathbf{0}'$. On the other hand, if $\mathbf{x}$ is c.e. and is below neither $\mathbf{d}$ nor $\mathbf{f}$, then it cups both $\mathbf{d}$ and $\mathbf{f}$ to $\mathbf{0}'$. Because of this, we can say that $\mathbf{b}$ and $\mathbf{f}$ above have exact cupping property.

Theorem 3 has several well-known theorems as direct corollaries, such as Arslanov's Cupping Theorem, Downey's Diamond Embedding Theorem, Downey-Li-Wu's Complementation Theorem and Li-Yi's Cupping Theorem. We comment here that the degrees in Li-Yi's cupping theorem do not have the exact cupping property.

Theorem 3 is joint work with Liu and Wu and was published in "Lecture Notes in Computer Science", volume 6735, 71-80.

Our notations and terminologies are standard and generally follow Soare [81] and Odifreddi [61, 62]. We will use upper-case Greek letters, such as Φ, Ψ, Θ , etc. to denote functionals and lower-case Greek letters, such as φ, ψ, θ , etc. to denote partial computable functions. Our constructions are based on priority trees which grows downwards. In the course of construction, a fresh number x means that $x - 1$ is larger than any number we have mentioned before.

Chapter 2

NONHEMIMAXIMAL DEGREES AND THE HIGH/LOW HIERARCHY

The work in this chapter is joint with Wu and will be published in the “Journal of Symbolic Logic”.

2.1 Introduction

A c.e. set H is called *hemimaximal* if H is noncomputable and there is a noncomputable c.e. set A disjoint from H such that $H \cup A$ is maximal. Accordingly, a degree $\mathbf{a}$ is *hemimaximal* if there is a hemimaximal set in $\mathbf{a}$. $\mathbf{a}$ is nonhemimaximal if it is not hemimaximal.

Downey and Stob used the property of incomplete m -topped degrees, introduced by Downey and Jockusch in [20], that they are low_2 but not low to prove the existence of a nonhemimaximal low_2 c.e. but not low c.e. degree by the following theorem. The combinatorics in their construction is fairly complicated: as they pointed out in the paper, the construction is in some sense a primitive $0'''$ -priority argument. In this chapter, we give a proof of Downey and Stob’s result in an alternative approach by the following theorem.

Theorem 1 (Fang and Wu [33]) *There are a nonhemimaximal low_2 c.e. degree $\mathbf{l}$ and nonzero c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ satisfying $\mathbf{a} \cup \mathbf{l} = \mathbf{0}'$ and $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$.*

From the complexity point of view, our construction is a standard $0''$ -argument, which is much simpler than the one used in Downey and Stob's construction.

To prove Theorem 1, it suffices to construct c.e. sets A, B, L and a p.c. functional satisfying the following requirements:

$$\mathcal{G}: K = \Gamma(A \oplus L),$$

$$\mathcal{L}: Tot^L \leq_T \emptyset'',$$

$$\mathcal{P}_e: B \neq \varphi_e,$$

$$\mathcal{N}_e: \Phi_e(A) = \Phi_e(B) = h \text{ total} \rightarrow h \text{ is computable},$$

$$\mathcal{R}_e: \Delta_e(L) = U_e \ \& \ \Psi_e(U_e) = L \ \& \ U_e \cap V_e = \emptyset \rightarrow U_e \cup V_e \text{ is not maximal},$$

where $Tot(L) = \{e : \Phi_e(L) \text{ is total}\}$ is a Π_2^L -complete set, and $\{(\Delta_e, \Psi_e, U_e, V_e) : e \in \omega\}$ is an effective list of quadruples such that Δ_e, Ψ_e are partial computable functionals, U_e, V_e are c.e. sets, and $\{\varphi_e : e \in \omega\}$ is an effective list of all partial computable functions.

Let $\mathbf{a} = \deg_T(A)$, $\mathbf{b} = \deg_T(B)$, $\mathbf{l} = \deg_T(L)$. The $\mathcal{G}$ -requirement guarantees that $\mathbf{0}' = \mathbf{a} \cup \mathbf{l}$. The $\mathcal{L}$ -requirements guarantee that $\mathbf{l}$ is low_2 . So from these two kinds of requirements, $\mathbf{a}$ is low_2 -cuppable. The $\mathcal{P}$ - and $\mathcal{N}$ -requirements guarantee that $\mathbf{a}$ is cuppable. Thus $\mathbf{a}$ is not low -cuppable, which implies that $\mathbf{l}$ is not low. The $\mathcal{R}$ -requirements guarantee that $\mathbf{l}$ is nonhemimaximal. Therefore, if all the requirements can be satisfied, then $\mathbf{l}$ is the degree wanted in Theorem 1.

2.2 Basic strategies

In the following, we describe the basic strategies for satisfying these requirements.

2.2.1 The $\mathcal{G}$ -strategy

$\mathcal{G}$ is a global requirement. The basic idea of the $\mathcal{G}$ -strategy is to code K into $A \oplus L$ via a partial computable functional Γ (*constructed by us*). If $\Gamma(A \oplus L; x)$ is defined and $K(x)$ changes, then we correct $\Gamma(A \oplus L; x)$ by putting a number $\leq \gamma(x)$ into A or

L . In the construction, if we are forced to do such a correction, we always enumerate $\gamma(x)$ into L , due to the constraints posed by minimal pair strategies.

1. Find the least x , if it exists, such that $\Gamma(A \oplus L; x)$ is defined and not equal to $K(x)$. Enumerate $\gamma(x)$ into L .
2. If there is no such x , then find the least x such that $\Gamma(A \oplus L; x)$ is not defined. Define $\Gamma(A \oplus L; x)$ as $K(x)$ with use $\gamma(x)$ a fresh number.

We will guarantee that for each x , $\Gamma(A \oplus L; x)$ is defined, which will make $\Gamma(A \oplus L)$ total and computing K correctly.

2.2.2 The $\mathcal{L}$ -requirement and the corresponding strategies

Our approach to satisfy $Tot^L \leq_T \emptyset''$ follows the one introduced in [53]. That is, we will make $Tot^L \leq_T TP$, and then by the fact that the true path TP is $\emptyset''$ -computable, we will have $Tot^L \leq_T \emptyset''$. So, to satisfy the $\mathcal{L}$ -requirement, we need to satisfy the following $\mathcal{L}_e$ -requirements:

$\mathcal{L}_e$: Check whether $\Phi_e(A)$ is total or not.

Let η be an $\mathcal{L}_e$ strategy on the priority tree, which will be introduced later in the construction. Define the length of convergence function as follows:

$$l_\eta(e, s) = \max\{x : \forall y < x [\Phi_e(L; y)[s] \downarrow \text{ \& } s \text{ is an } \eta\text{-stage}]\}.$$

A stage s is η -expansionary if $s = 0$ or $l_\eta(e, s) > \max\{e+1, l_\eta(e, t) : t \text{ is an } \eta\text{-stage} < s\}$. η has two outcomes ∞ and f with $\infty <_L f$, where ∞ denotes that there are infinitely many η -expansionary stages, and f for finitely many.

To guarantee $Tot^L \leq_T TP$, it is enough to ensure that on the true path, if η has outcome ∞ , then $\Phi_e(L)$ is total, i.e. $e \in Tot^L$, and if η has outcome f , then $\Phi_e(L)$ is not total, $e \notin Tot^L$.

Assume that η has outcome f , then there is a stage s large enough and a number x such that after the stage s , $\Phi_e(L; x)$ never converges at any η -stage, and hence $\Phi_e(L)$ is not total, $e \notin Tot^L$.

Now assume that η has outcome ∞ and we will ensure that $\Phi_e(L)$ is total. That is, for a given number x , a computation of $\Phi_e(L; x)$ will be preserved forever. There is a direct conflict between this idea and the $\mathcal{G}$ -strategy, as the $\mathcal{G}$ -strategy can enumerate numbers into L at any time during the construction. To avoid such a conflict, we use the idea of “threshold strategy” to lift the γ -uses to bigger numbers. That is, we fix a number k_η first, and when we see that $\Phi_e(L; x) \downarrow$ at a stage s , we enumerate $\gamma(k_\eta)$ into A to undefine $\Gamma(A \oplus L; y)$ for numbers $y \geq k_\eta$ and if $\gamma(y)$ is defined later, we let it bigger than $\varphi_e(L; x)[s]$. Of course, in the construction, whenever a number $z \leq k_\eta$ enters K , to rectify $\Gamma(A \oplus L; z)$, we still enumerate the corresponding $\gamma(z)$ into L , which can injure the computation $\Phi_e(L; x)$, the one we want to preserve. As k_η is fixed, such a process can repeat at most $k_\eta + 1$ many times, and after a stage large enough, such a process will never happen again, and our action described above can preserve the computation $\Phi_e(L; x)$ forever.

Note that it is impossible for η to preserve computations $\Phi_e(L; x)$ for infinitely many $x \in \omega$, as infinite restraint would be imposed at η . To avoid this, we introduce substrategies $\{\mathcal{S}_{e,i}\}_{i \in \omega}$ for η such that each of them, $\mathcal{S}_{e,i}$ say, only needs to protect computations $\Phi_e(L)$ on its initial (finite) segment i , and these substrategies work below η 's outcome ∞ . Let ρ be an $\mathcal{S}_{e,i}$ -strategy below η 's outcome ∞ . Here, η is called the *mother node* of ρ , and correspondingly, ρ is called a *child node* of η .

ρ works as follows:

1. Choose a fresh number k_ρ as a threshold of ρ , if it is not selected yet.
2. Wait for a stage s_0 such that $\Phi_e(L; y)[s_0] \downarrow$ for any $y < i + 1$.
(s_0 exists as we are assuming that η has outcome ∞ .)
3. Enumerate $\gamma(k_\rho)$ into A , and protect these computations from injuries from other strategies.
4. Wait for $K \upharpoonright (k_\rho + 1)$ to change.
5. If $K \upharpoonright (k_\rho + 1)$ changes, then go back to step 2.

As k_ρ is fixed, it can happen at most finitely many times to go from step 5 back to step 2.

ρ has one outcome 1.

2.2.3 A $\mathcal{P}_e$ -strategy

A $\mathcal{P}_e$ -strategy, α say, is a diagonalization argument. α works as follows:

1. Choose a fresh witness x_α at a stage s_0 ;
2. Wait for a stage s_1 such that $\varphi_{e,s_1}(x_\alpha) \downarrow = 0$.
3. Put x_α into B and stop.

The outcomes of $\mathcal{P}_e$ -strategy are $d <_L w$, where w denotes the case that we wait at step 2 forever, and d denotes the case that step 3 is reached.

2.2.4 An $\mathcal{N}_e$ -strategy

An $\mathcal{N}_e$ -strategy, β say, is the standard minimal pair argument. Define the length of agreement function as follows:

$$l(\beta, s) = \max\{x : \forall y < x [\Phi_e(A; y)[s] \downarrow = \Phi_e(B; y)[s] \downarrow \text{ and } s \text{ is a } \beta\text{-stage}]\}.$$

A stage s is β -expansionary if $s = 0$ or $l(\beta, s) > \max\{l(\beta, t) : t < s\}$. β has two outcomes ∞ and f with $\infty <_L f$, where ∞ denotes that there are infinitely many β -expansionary stages, and f for finitely many.

The basic idea of the minimal pair argument is to preserve at least one side of $\Phi_e(A) \upharpoonright l(\beta, s)$ and $\Phi_e(B) \upharpoonright l(\beta, s)$ up to the next β -expansionary stage. That is, small numbers can be enumerated into A or B only at β -expansionary stages, and at each such a stage, numbers can be enumerated into only one of A and B . If s is a β -expansionary stage, then we define $h(y) = \Phi_e(A; y) = \Phi_e(B; y)$ for those $y < l(\beta, s)$ with $h(y)$ currently not defined. If there are infinitely many β -expansionary stages, then h is totally defined and computable. Furthermore, if $\Phi_e(A) = \Phi_e(B)$ are both total, then h computes them correctly.

On the other hand, if there are only finitely many β -expansionary stages, then there is a largest β -expansionary stage s after which no bigger β -expansionary stage exists. This shows that “ $\Phi_e(A; y) = \Phi_e(B; y)$ is total” is not true, and hence $\mathcal{N}_e$ is satisfied vacuously.

2.2.5 An $\mathcal{R}_e$ -strategy

Let ζ be an $\mathcal{R}_e$ -strategy. We construct two disjoint c.e. sets P_ζ and Q_ζ such that both $P_\zeta - (U_e \cup V_e)$ and $Q_\zeta - (U_e \cup V_e)$ are infinite, which is enough to show that $U_e \cup V_e$ is not maximal, as $\overline{U_e \cup V_e}$ is infinite and $P_\zeta \cap \overline{U_e \cup V_e}$ and $\overline{P_\zeta} \cap \overline{U_e \cup V_e}$ are both infinite.

To construct P_ζ and Q_ζ , we construct a computable sequence of disjoint c.e. sets $\{T_{e,i}\}_{i \in \omega}$ such that $P_\zeta = \cup_{k \in \omega} T_{e,2k}$ and $Q_\zeta = \cup_{k \in \omega} T_{e,2k+1}$, and if $U_e = \Delta_e^L$, $L = \Psi_e^{U_e}$ and $U \cap V_e = \emptyset$, then each $T_{e,i}$ contains a number not in $U_e \cup V_e$. Furthermore, if i is even, then $T_{e,i}$ contains a number neither in $\cup_{k < i} T_{e,k}$ nor in Q_ζ and if i is odd, then $T_{e,i}$ contains a number neither in $\cup_{k < i} T_{e,k}$ nor in P_ζ . This ensures that P_ζ and Q_ζ are also disjoint. Define the length of agreement function for ζ as follows:

$$l(\zeta, s) = \max\{x : (\forall y < x)[\Psi_e(U_e; y)[s] = L_s(y) \ \& \ (\forall z \leq \psi_e(U_e; y)[s]) \\ [\Delta_e(L; z)[s] = U_{e,s}(z)]] \text{ and } s \text{ is a } \zeta\text{-stage}\}$$

A stage s is an ζ -expansionary stage if $s = 0$ or $l(\zeta, s) > l(\zeta, t)$ for all $t < s$ and $U_{e,s} \cap V_{e,s} = \emptyset$. ζ has two outcomes $\infty <_L f$, where ∞ denotes that there are infinitely many ζ -expansionary stages, and f for finitely many.

Obviously, if ζ has outcome f , then $\mathcal{R}_e$ is satisfied. So, assume that ζ has outcome ∞ . Under this outcome, we will construct $\{T_{e,i}\}_{i \in \omega}$ with the properties described above. We introduce substrategies $\{\mathcal{T}_{e,i}\}_{i \in \omega}$ of ζ , where each $\mathcal{T}_{e,i}$ builds a finite set $T_{e,i}$ such that

$$\mathcal{T}_{e,i} : T_{e,i} \not\subseteq U_e \cup V_e.$$

In the construction, if we cannot find such an $T_{e,i}$, we will show that either $L \neq \Psi_e(U_e)$ or $U_e \neq \Delta_e(L)$ or $U_e \cap V_e \neq \emptyset$, and again, the requirement $\mathcal{R}_e$ is satisfied vacuously.

2.2.6 A $\mathcal{T}_{e,i}$ -strategy

Assume that ζ has outcome ∞ , and below this outcome, there are infinitely many substrategies $\{\mathcal{T}_{e,i}\}_{i \in \omega}$, τ say. Then, ζ is the *mother node* of τ , and τ is a *child node* of ζ .

We first describe the basic idea of how τ works (from Downey and Stob's original construction given in [24]), and then consider how to modify it to make sure that τ is consistent with other $\mathcal{T}$ -strategies.

Fix a number k_τ as a threshold for τ and reset τ whenever a number $\leq k_\tau$ enters K . Again, when we want to protect computations $\Delta_e(L) \upharpoonright \psi_e(U_e; x)$ for some x , we enumerate $\gamma(k_\tau)$ into A first, to lift the γ -uses to bigger numbers. Let $T_{\tau,s}$ be the set constructed by τ by the stage s . τ works as follows:

(1) Select its threshold k_τ as a fresh number.

(2) Choose a fresh number x_τ and create a link between τ and ζ .

(Since then, a ζ -stage t is called ζ -expansionary only if $l(\zeta, t) > l(\zeta, s)$ for all $s < t$ and in addition, $l(\zeta, t) > x_\tau$.)

(3) At the next ζ -expansionary stage s , we travel the link to construct

$$T_{\tau,s} = \{y : y \notin U_{e,s} \cup V_{e,s} \ \& \ y < \psi_\zeta(U_e; l(\zeta, s))[s]\},$$

and enumerate $\gamma(k_\tau)[s]$ into A . Cancel the link. Consider whether $T_{\tau,s}$ is empty or not.

(3.1) If $T_{\tau,s} = \emptyset$, then put x_τ into L . This is called a *disagreement attack* at x_τ .

There are two possibilities: U_e can either have or have no change below $\psi_\zeta(U_e; l(\zeta, s))$.

If U_e has no change below $\psi_\zeta(U_e; l(\zeta, s))$, then we have

$$\Psi_\zeta(U_e; x_\tau) = \Psi_\zeta(U_e; x_\tau)[s] = L_s(x_\tau) = 0 \neq 1 = L(x_\tau).$$

If U_e does have change below $\psi_\zeta(U_e; l(\zeta, s))$ later, a number $y \leq \psi_\zeta(U_e; x_\tau)[s_2]$ enters U_e after the stage s . This makes $U_e \cap V_e \neq \emptyset$ as according to the stage s , y is already in V_e .

(3.2) Otherwise, go to the step (4). *(Note that enumerating $\gamma(k_\tau)$ into A lifts the γ -uses to bigger numbers, which enables us to protect $L \upharpoonright s$, and hence computations seen at the stage s .)*

- (4) Wait for a τ -stage s' with $T_{\tau,s} \subseteq U_{e,s'} \cup V_{e,s'}$ and $U_{e,s'} \cap V_{e,s'} = \emptyset$.
- (5) Put x_τ into L to make a *disagreement*. Now by a similar argument as in (3.1), either

$$\Psi_\zeta(U_e; x_\tau) = \Psi_\zeta(U_e; x_\tau)[s'] = L_{s'}(x_\tau) \neq L(x_\tau),$$

or $U_e \cap V_e$ is not empty. Either one of these two cases satisfies $\mathcal{R}_e$.

(Note that as s' is also a ζ -expansionary stage, we have that no number in $T_{\tau,s}$ can now be in $U_{e,s'}$, as otherwise we would not have agreement between $\Delta_e(L)$ and U_e at the stage s' . This means that all numbers in $T_{\tau,s}$ are now in $V_{e,s'}$. This ensures that after putting x_τ into L , if we cannot make a disagreement, then some small number enters U_e after the stage s' and such a number should be in V_e already at the stage s' .)

τ has one outcome, 1, which denotes the case that τ waits at step (4) forever. In this case, we succeed in constructing $T_{\tau,s}$ to witness $T_{\zeta,i} \not\subseteq U_e \cup V_e$. If τ stops at step (3.1) or reaches to step (5), then there must be a disagreement either between $\Delta_e(L)$ and U_e or between $\Psi_e(U_e)$ and L or $U_e \cap V_e$ is not empty. In this case, ζ has outcome f .

Now we consider the consistency between τ and other $\mathcal{T}$ -strategies. We don't need to worry about the possibility that τ 's enumerations into A (step 3) or L (step 3.1 or step 5) can affect other strategies, because whenever a $\mathcal{T}$ -strategy acts as above, all the strategies with lower priority are initialized, and because between step 2 and step 3, no strategy (in particular, no $\mathcal{N}$ -strategy) between ζ and τ is visited. The main concern is when we define $T_{\tau,s}$ at step 3, we see that all the numbers in $T_{\tau,s}$ is already in some $T_{\tau',s}$, where τ' is another $\mathcal{T}$ -strategy. This will defeat our purpose of making T_τ having an element not contained in $T_{\tau'}$, where τ' is any $\mathcal{T}$ -strategy being visited so far. With this in mind, we modify (2) to (2') as follows:

- (2') Enumerate $\gamma(k_\tau)$ into A , choose a fresh number x_τ and create a link between τ and ζ .

At step 3, we will do almost the same thing, but with the modified definition of $T_{\tau,s}$ considered. We let $P_\zeta \cup Q_\zeta[s]$ be the collection of elements in those $T_{\tau',t}$, where

$t < s$ and τ' is any substrategy of ζ being visited so far, no matter whether τ' has priority higher or lower than τ . Define $T_{\tau,s}$ as the set

$$\{y : y \notin U_{e,s} \cup V_{e,s} \cup (P_\zeta \cup Q_\zeta)[s] \ \& \ y < \psi_\zeta(U_e; l(\zeta, s))[s]\}.$$

Now we check whether $T_{\tau,s}$ is empty or not.

If it is empty, again, we do as in step 3.1 by putting the number x_τ into L and show that this will create a disagreement. Without loss of generality, suppose that U_e does have a change, at y say, below $\psi_\zeta(l(\zeta, s))$, then as y is not in $T_{\tau,s}$, then either $y \in U_e \cap V_e$, or y is in $(P_\zeta \cup Q_\zeta)[s]$. Assume that $y \in T_{\tau',s'}$. The crucial point is that at step $2'$, $\gamma(k_\tau)$ is enumerated into A , and this enumeration lifts the γ -uses to big numbers and hence preserves the computation $\Delta_e(L; y)$, till the stage we see that y enters U_e . This actually provides a disagreement between $\Delta_e(L)$ and U_e , and hence satisfies the requirement $\mathcal{R}_e$.

If $T_{\tau,s}$ is not empty, then, without loss of generality, suppose that step 5 is reached, and x_τ is enumerated into L . Again, we assume that U_e does have a change below $\psi_\zeta(l(\zeta, s))$, at y . If y is not in $T_{\tau,s}$, then by the same argument given above, we have that if U_e and V_e are disjoint, then a disagreement between $\Delta_e(L)$ and U_e is created, due to the enumeration at step $2'$. So we assume y is in $T_{\tau,s}$. In this case, note that at step 3, at the stage s , $\gamma(k_\tau)[s]$ is enumerated into A , and the computation $\Delta_e(L; y)[s]$ is preserved (the value is 0) till the stage s' (step 4), when we see that $T_{\tau,s} \subseteq U_{e,s'} \cup V_{e,s'}$, $U_{e,s'} \cap V_{e,s'} = \emptyset$. This implies that $T_{\tau,s} \subseteq V_{e,s'}$, so after we put x_τ into L at step 5 and U_e changes at y , y is in both U_e and V_e , making $U_e \cap V_e \neq \emptyset$. Again, the requirement $\mathcal{R}_e$ is satisfied.

2.3 The construction

We now give the whole construction. First, we need the priority tree T on which the construction will proceed.

2.3.1 The construction of the priority tree

For simplicity, for $n = \langle e, i \rangle$, we let $\mathcal{S}_n$ and $\mathcal{T}_n$ denote the strategies $\mathcal{S}_{e,i}$ and $\mathcal{T}_{e,i}$ respectively. We assign priority to the (sub)requirements as follows:

$$\mathcal{G} < \mathcal{P}_0 < \mathcal{N}_0 < \mathcal{L}_0 < \mathcal{S}_0 < \mathcal{R}_0 < \mathcal{T}_0 < \cdots < \mathcal{P}_n < \mathcal{N}_n < \mathcal{L}_n < \mathcal{S}_n < \mathcal{R}_n < \mathcal{T}_n < \cdots$$

Where $\mathcal{X} < \mathcal{Y}$ means that $\mathcal{X}$ -requirement has priority higher than $\mathcal{Y}$ -requirement.

The priority tree T grows downwards and is constructed as follows. $\mathcal{G}$ is a global requirement, so we do not put it on T . Given a node $\xi \in T$, we assign to ξ a (sub)requirement with the highest priority (as in the priority ranking list given above) that has not been assigned to any node $\theta \subset \xi$, subject to the following two provisos:

- (i) Below $\eta \frown f$, where η is an $\mathcal{L}_e$ -strategy, no subrequirement $\mathcal{S}_{e,i}$ is assigned.
- (ii) Below $\zeta \frown f$, where ζ is an $\mathcal{R}_e$ -strategy, no subrequirement $\mathcal{T}_{e,i}$ is assigned.

After we assign a requirement to ξ , we let ξ have the corresponding outcomes: a $\mathcal{P}$ -strategy has two outcomes $d <_L w$; an $\mathcal{L}$ -strategy, or an $\mathcal{N}$ -strategy, or an $\mathcal{R}$ -strategy, has two outcomes $\infty <_L f$; an $\mathcal{S}$ -strategy, or a $\mathcal{T}$ -strategy has one outcome 1.

In the construction, when a strategy ξ is initialized, then all the strategies with lower priority will be also initialized automatically, and all parameters of ξ will be cancelled. When a strategy ξ is reset, then all the strategies with lower priority will be also initialized automatically, and all the parameters of ξ , except for k_ξ , will be cancelled.

Also, in the construction, when a link between a $\mathcal{T}$ -strategy, τ say, and its mother node, η , is created or cancelled at a stage s , we will let $\sigma_s = \tau$, and so, for any strategy ξ between η and τ , we have $\xi \subset \sigma_s$. With this in mind, we say that the stage s is a genuine stage for a strategy, if there is a substage t such that $\xi = \sigma_{s,t}$, i.e. ξ is not crossed over at the stage s . Hence, when ξ is an $\mathcal{L}$, or an $\mathcal{N}$, or an $\mathcal{R}$ -strategy, we say that s is a genuine ξ -expansionary stage, we mean that s is a genuine ξ -stage and that s is ξ -expansionary, comparing the previous genuine ξ -stage. Note that it can happen that s is not a genuine ξ -stage, but $\xi \frown \infty \subseteq \sigma_s$, even though s is actually not ξ -expansionary.

2.3.2 The full construction

We assume that K is enumerated at odd stages, and that exactly one element can be enumerated into K at each odd stage. We will construct a p.c. functional Γ at odd stages and strings σ_s to approximate the true path at even stages. At every stage $s+1$, if we do not specify $A_{s+1}, B_{s+1}, L_{s+1}, T_{\tau, s+1}$, then they are equal to $A_s, B_s, L_s, T_{\tau, s}$ respectively.

The full construction is as follows.

Stage 0: Let $A_0 = B_0 = L_0 = \emptyset$, and for any $\mathcal{T}$ -strategy τ , let $T_{\tau, 0} = \emptyset$. Initialize all the nodes on T .

Stage $s+1 = 2n+1$: Let $k \in K_{s+1} - K_s$. For any strategy ξ , if k_ξ is defined and $k \leq k_\xi$, then we reset ξ . Otherwise, there are two cases.

1. If $\Gamma(A \oplus L; k)[s] \downarrow$, then enumerate $\gamma(k)[s]$ into L . Undefine all the $\Gamma(A \oplus L; x)[s]$ for $x \geq k$. Go to the next stage.
2. Otherwise, find the least x such that $\Gamma(A \oplus L; x)[s] \uparrow$. Define $\Gamma(A \oplus L; x)[s+1] = K_{s+1}(x)$ with $\gamma(x)[s+1]$ fresh, go to the next stage.

Stage $s+1 = 2n+2$: This stage is divided into several substages.

Substage 0: Let $\sigma_{s+1} \upharpoonright 0 = \lambda$, the root of the priority tree.

Substage t : Given $\sigma_{s+1} \upharpoonright t = \xi$. If $t = s+1$, then let $\sigma_{s+1} = \xi$ and initialize all the nodes with priority lower than σ_{s+1} , and go to the next stage. If $t < s+1$, there are six cases:

Case 1. $\xi = \alpha$ is a $\mathcal{P}_e$ -strategy. There are four subcases.

- ($\alpha 1$) If x_α is undefined, then we choose a fresh number as x_α . Let $\sigma_{s+1} = \alpha$, initialize all the nodes with priority lower than σ_{s+1} and go to the next stage.
- ($\alpha 2$) If x_α is defined, $\varphi_{e, s+1}(x_\alpha) \downarrow = 0$ but $x_\alpha \notin B_s$, then put x_α into B . Let $\sigma_{s+1} = \alpha$, initialize all the nodes with priority lower than σ_{s+1} and go to the next stage.
- ($\alpha 3$) If x_α is defined and $x_\alpha \in B_s$, then let $\sigma_{s+1} \upharpoonright (t+1) = \alpha \wedge d$. Go to the next substage.

($\alpha 4$) Otherwise, let $\sigma_{s+1} \upharpoonright (t+1) = \alpha \hat{\ } w$, and go to the next substage.

Case 2. $\xi = \beta$ is an $\mathcal{N}_e$ -strategy. There are two subcases.

($\beta 1$) If $s+1$ is a β -expansionary stage, let $\sigma_{s+1} \upharpoonright (t+1) = \beta \hat{\ } \infty$. Define $h(x) = \Phi_e(A; x)[s+1] = \Phi_e(B; x)[s+1]$ for all $x < l(\beta, s+1)$ with $h(x)[s]$ not defined. Go to the next substage.

($\beta 2$) If $s+1$ is not a β -expansionary stage, let $\sigma_{s+1} \upharpoonright (t+1) = \beta \hat{\ } f$. Go to the next substage.

Case 3. $\xi = \eta$ is an $\mathcal{L}_e$ -strategy. There are two subcases.

($\eta 1$) If $s+1$ is an η -expansionary stage, let $\sigma_{s+1} \upharpoonright (t+1) = \eta \hat{\ } \infty$. Go to the next substage.

($\eta 2$) If $s+1$ is not an η -expansionary stage, let $\sigma_{s+1} \upharpoonright (t+1) = \eta \hat{\ } f$. Go to the next substage.

Case 4. $\xi = \rho$ is an $\mathcal{S}_{e,i}$ -strategy. Let η be the mother node of ρ . There are four subcases.

($\rho 1$) If k_ρ is undefined, then define it as a fresh number. Let $\sigma_{s+1} = \rho$, initialize all the nodes with priority lower than σ_{s+1} and go to the next stage.

($\rho 2$) If k_ρ is defined, and there is an $x < i+1$ such that $\Phi_e(L; x)[s+1] \uparrow$, then let $\sigma_{s+1} = \rho$, initialize all the nodes with priority lower than σ_{s+1} . Go to the next stage.

($\rho 3$) If k_ρ is defined, ρ is not satisfied yet, $\Phi_e(L; x)$ converges for each $x < i+1$, then put $\gamma(k_\rho)$ into A , declare that ρ is satisfied at the stage $s+1$. Let $\sigma_{s+1} = \rho$, initialize all the nodes with priority lower than σ_{s+1} , and go to the next stage.

($\rho 4$) If k_ρ is defined, and ρ is satisfied at a previous stage, then let $\sigma_{s+1} \upharpoonright (t+1) = \rho \hat{\ } 1$, and go to the next substage.

Case 5. $\xi = \zeta$ is an $\mathcal{R}_e$ -strategy.

- (ζ 1) If $s + 1$ is not ζ -expansionary, let $\sigma_{s+1} \upharpoonright (t + 1) = \zeta \cap f$ and go to the next substage.
- (ζ 2) If $s + 1$ is ζ -expansionary, check whether a child node τ requests for the selection of x_τ , or whether there is a link between ζ and a child node τ . If there is no such a τ or such a link, then let $\sigma_{s+1} \upharpoonright (t + 1) = \tau \cap \infty$, and go to the next substage. Otherwise, we have:

If there is a τ requesting for the selection of x_τ , then select x_τ as a fresh number and enumerate $\gamma(k_\tau)[s + 1]$ into A . Create a link between ζ and τ . Let $\sigma_{s+1} = \tau$, initialize all the nodes with priority lower than σ_{s+1} and go to the next stage.

If there is a link between ζ and a child node τ , then define $T_{\tau, s+1}$ as

$$\{y : y \notin U_{e, s+1} \cup V_{e, s+1} \cup (P_\zeta \cup Q_\zeta)[s + 1] \ \& \ y < \psi_\zeta(U_e; l(\zeta, s + 1))[s + 1]\}.$$

Cancel the link, and enumerate $\gamma(k_\tau)[s + 1]$ into A . Furthermore, if $T_{\tau, s+1}$ is empty, then enumerate x_τ into L (*we redefine all $\Gamma(A \oplus L; y)$ with the same value and the same uses as before if $x_\tau \leq \gamma(y)[s]$*), and say that a *disagreement attack* is made at the stage $s + 1$.

Let $\sigma_{s+1} = \tau$, initialize all the nodes with priority lower than σ_{s+1} . Go to the next stage.

Case 6. $\xi = \tau$ is a $\mathcal{T}_{e, i}$ -strategy. Let ζ be the mother node of τ .

- (τ 1) If k_τ is not defined yet, then define it as a fresh number. Say that τ requests the selection of x_τ at the stage $s + 1$. Let $\sigma_{s+1} = \tau$, initialize all the nodes with priority lower than σ_{s+1} and go to the next stage.
- (τ 2) If k_τ, x_τ are defined, we consider whether $T_{\tau, s}$ is contained in $U_{e, s+1} \cup V_{e, s+1}$ or not.

If yes, then put x_τ into L (we redefine all $\Gamma(A \oplus L; y)$ with the same value and the same uses as before if $x_\tau \leq \gamma(y)[s]$) and declare that ζ is satisfied at the stage $s + 1$. Let $\sigma_{s+1} = \tau$, initialize all the nodes with priority lower than σ_{s+1} and go to the next stage.

If not, then let $\sigma_{s+1} \upharpoonright (t + 1) = \tau \smallfrown 1$. Go to the next substage.

This completes the full construction.

2.4 The verification

In this section, we prove that the constructed c.e. sets A, B, L and p.c. functional Γ satisfy all the requirements. Define the true path $TP = \liminf_s \sigma_{2s}$ and we first prove that TP is infinite.

Lemma 2.1. *Let σ be any node on TP , then*

- (1) σ can only be initialized or reset at most finitely often;
- (2) σ has an outcome $\mathcal{O}$ such that $\sigma \smallfrown \mathcal{O} \in TP$;
- (3) σ can act and initialize all the strategies with lower priority at most finitely many times.

Therefore, TP is infinite.

Proof. We prove this lemma by induction on the length of σ .

First, λ , the root node of the priority tree, is on TP , as it is visited at all even stages. As λ is a $\mathcal{P}_0$ -strategy, it can never be initialized or reset at anytime during the construction. (1) is true. If λ enumerates a number into B at stage s , then after this stage, every even stage is a $\lambda \smallfrown d$ -stage. Otherwise, all even stages are $\lambda \smallfrown w$ -stages. Therefore, (2) is true. λ initializes other strategies at most twice, when it acts for diagonalization. (3) is true.

For any non-root node σ on TP , let σ^- be the immediate predecessor, and we let $\sigma = \sigma^- \frown \mathcal{O}$. By the induction hypothesis, suppose that (1)-(3) hold for σ^- . That is, there is a (least) stage s_0 after which σ^- can neither be initialized nor be reset nor initialize σ . In the following, we will prove that (1)-(3) also hold for σ .

(1) is true as after the stage s_0 , σ^- has only finitely many outcomes, and by the definition of TP , σ^- can have outcome $\mathcal{O}' < \mathcal{O}$ at most finitely many times, which implies the existence of a stage $s_1 > s_0$ after which σ cannot be initialized. If σ is an $\mathcal{S}$ or a $\mathcal{T}$ -strategy, then after the stage s_1 and after k_σ is defined, it becomes fixed, and hence σ can be reset for at most $k_\sigma + 1$ times.

Now we show that (2) and (3) are also true for σ . By (1), we assume that after a stage s' , σ can never be initialized or reset again. There are six cases.

Case 1. $\sigma = \alpha$ is a $\mathcal{P}_e$ -strategy.

If $\varphi(x_\alpha)$ never converges to 0, then each σ -stage is a $\sigma \frown w$ -stage, which gives that $\sigma \frown w$ is on TP . If $\varphi(x_\alpha)$ does converge to 0, then when σ sees this converges (after the stage s'), σ enumerates x_α into B , and after this, each σ -stage is a $\sigma \frown d$ -stage. This gives that $\sigma \frown d$ is on TP . (2) is true. After the stage s' , σ initializes other strategies at most twice. (3) holds.

Case 2. $\sigma = \beta$ is an $\mathcal{N}_e$ -strategy.

If there are infinitely many β -expansionary stages, then $\beta \frown \infty$ is on TP . Otherwise, $\beta \frown f$ is on TP . (2) is true.

Note that in the construction, it can happen that a stage s is η -expansionary, where $\eta \frown \infty \subseteq \beta$ is an $\mathcal{R}$ -strategy, and $\tau = \sigma_s$ requests for a selection of x_τ at the last η -expansionary stage, or a link between η and τ is created at the last η -expansionary stage. In both cases, the stage s is not necessary to be β -expansionary (the crucial point is that at these stages, the definition of h is not extended), even though $\beta \frown \infty \subset \sigma_s$. It is not a problem here, as at any stage, at most one such request or one link exists, and at an η -expansionary stage, once x_τ is selected, a link between τ and η is created, and at the next η -expansionary stage at which T_τ is constructed, the link is cancelled automatically. After this, a stage when a new request is raised by a $\mathcal{T}$ -strategy below $\beta \frown \infty$ must be β -expansionary. This says that if β has f as the true outcome, then the scenario described above can happen at most finitely many times. Or equivalently, this kind of crosses over β do not affect the true outcome of β and

also the definition of h .

This explanation also applies to those $\mathcal{L}$ and $\mathcal{R}$ -strategies.

In the construction, $\mathcal{N}$ -strategies never act to initialize other strategies. (3) is true.

Case 3. $\sigma = \eta$ is an $\mathcal{L}_e$ -strategy.

The argument is similar to the one given in Case 2 for $\mathcal{N}_e$ -strategies.

Case 4. $\sigma = \rho$ is an $\mathcal{S}_e$ -strategy with $e = \langle i, j \rangle$. Then σ^- is an $\mathcal{L}_e$ -strategy. Let η be the mother node of ρ . After the stage s' (i.e. ρ cannot be reset later), when ρ is visited again, it first defines k_ρ , and later, at a ρ -stage s_1 say, Φ_i^L converges on all $y \leq j$, ρ enumerates $\gamma(k_\rho)$ into A , and also initializes all strategies with lower priority. s_1 exists, as ρ knows that η has outcome ∞ . These computations are preserved forever, by the choice of s' and by the enumeration of $\gamma(k_\rho)$ into A . This ensures that after this stage, every ρ -stage is also a $\rho^\wedge 1$ -stage. $\rho^\wedge 1$ is on TP . (2) and (3) are both true.

Case 5. $\sigma = \zeta$ is an $\mathcal{R}_e$ -strategy.

The argument is the same as the one given in Case 2 for $\mathcal{N}_e$ -strategies. Here, at a ζ -expansionary stage $s > s_0$, if x_τ is selected as requested or a link is cancelled, then σ_s is defined as the corresponding τ , and $\zeta^\wedge \infty$ is visited at the stage s . The discussion about the outcomes given in Case 2 also applies to this strategy.

Case 6. $\sigma = \tau$ is a $\mathcal{T}_e$ -strategy for $e = \langle i, j \rangle$. Let ζ be the mother node of τ .

After the stage s' , it first defines k_τ and requests for the selection of x_τ , which is done at the next ζ -expansionary stage, s_1 say. Note that at the stage s_1 , a link between τ and ζ is created. At the next ζ -expansionary stage s_2 with $l(\zeta, s_2) > x_\tau$, the link is cancelled, and all the strategies with priority lower than τ are initialized. Note that T_{τ, s_2} cannot be empty, as otherwise, x_τ is enumerated into L and after this, each ζ -stage is a $\zeta^\wedge f$ -stage, $\zeta^\wedge f$ is on TP , a contradiction. Also note that after the stage s_2 , τ cannot initialize lower priority strategies again, as it happens only when T_{τ, s_2} becomes a subset of $U_i \cup V_i$, and at this stage x_τ is enumerated into L . Again, this will lead to a consequence that $\zeta^\wedge f$ is on TP , which is impossible. Thus after stage s_2 , any τ -stage is also a $\tau^\wedge 1$ -stage. $\tau^\wedge 1$ is on TP . (2) and (3) are both true.

This completes the proof of Lemma 2.1. $\square$

Now we show that all the requirements are satisfied.

Lemma 2.2. *All the $\mathcal{P}$ -requirements are satisfied by the corresponding $\mathcal{P}$ -strategies on TP .*

Proof. Fix e and let α be the $\mathcal{P}_e$ -strategy on the true path TP . The proof of Lemma 2.1 shows that $\mathcal{P}_e$ -requirement is satisfied. $\square$

Lemma 2.3. *All the $\mathcal{N}$ -requirements are satisfied by the corresponding $\mathcal{N}$ -strategies on TP .*

Proof. Fix e and let β be the $\mathcal{N}_e$ -strategy on the true path TP . By Lemma 2.1, let s^- be the least stage after which β can not be initialized any more.

If $\beta \smallfrown f \in TP$, then the $\mathcal{N}_e$ -requirement is satisfied vacuously, as either $\Phi_e(A) \neq \Phi_e(B)$, or at least one of $\Phi_e(A)$ and $\Phi_e(B)$ does not converge at an argument.

If $\beta \smallfrown \infty \in TP$, then $\beta \smallfrown \infty \subseteq \sigma_s$ for infinitely many stages s . Note that if a stage s is β -expansionary, a necessary condition is that $\beta = \sigma_s \upharpoonright t$ for some substage t . That is, s is a genuine β -expansionary stage. For the case that β is crossed over at a stage s , from $\eta \subset \beta$ to $\tau \supset \beta$, if this link is just created, then β can be crossed over for one more time, at the next η -expansionary stage (and the link is cancelled), and after this, the next stage when $\beta \smallfrown \infty$ is on TP again is a genuine β -expansionary stage. This shows that there are infinitely many genuine β -expansionary stages if and only if $\beta \smallfrown \infty$ is on σ_s for infinitely many s . This ensures that the function h is defined as a total function (extension is done only at genuine β -expansionary stages).

Also note that at a stage s , when τ requests for selecting x_τ , s is a genuine β -expansionary stage, and $\gamma(k_\tau)[s]$ is enumerated into A , and before the next genuine β -expansionary stage, no $\mathcal{P}$ -strategy below $\beta \smallfrown \infty$ can be visited and enumerate a number into B .

Now assume that $\Phi_e(A) = \Phi_e(B)$ is total, then according to the discussion above, there are infinitely many genuine β -expansionary stages and h is defined as a total function. It suffices to show that h computes $\Phi_e(A) = \Phi_e(B)$ correctly. For any $x \in \omega$, let t_0 be the least genuine β -expansionary stage $> s^-$ such that $l(\beta, t_0) > x$. Then $h(x)$ is defined at the stage t_0 with $h(x) = \Phi_e(A; x)[t_0] = \Phi_e(B; x)[t_0] = y$. Let

$$t_0 < t_1 < t_2 < \cdots < t_n < \cdots$$

be the sequence of all genuine β -expansionary stages. We will show $\Phi_e(A; x)[t_j] = \Phi_e(B; x)[t_j] = y$ for any $j \in \omega$. This will be proved by induction. Assume that

$\Phi_e(A; x)[t_k] = \Phi_e(B; x)[t_k] = y$, then $\Phi_e(A; x)[t] = \Phi_e(A; x)[t_k] = y$ or $\Phi_e(B; x)[t] = \Phi_e(B; x)[t_k] = y$ for $t_k \leq t \leq t_{k+1}$, since at least one side of the computations $\Phi_e(A; x)[t_k]$ and $\Phi_e(B; x)[t_k]$ will be preserved up to the stage t_{k+1} by our construction. Thus h witnesses that $\Phi_e(A) = \Phi_e(B)$ is computable, and the $\mathcal{N}_e$ -requirement is satisfied. $\square$

Lemma 2.4. *All the $\mathcal{L}$ -requirements are satisfied by the corresponding $\mathcal{L}$ -strategy on TP .*

Proof. To prove that $Tot^L \leq_T \emptyset''$, it suffices to show that (1) $TP \leq_T \emptyset''$, and (2) $Tot^L \leq_T TP$.

(1) follows from the definition of true path. To see that (2) is correct, TP can compute the outcome of the $\mathcal{L}_e$ -strategy η on TP , for each $e \in \omega$. If η^∞ is on TP , then $\Phi_e(L)$ is not total (i.e. $e \notin Tot^L$) as there are only finitely many η -expansionary stages. If η^∞ is on TP , then we want to show that $e \in Tot^L$. This follows from the assumption that there are infinitely many η -expansionary stages, and for a given y , the $\mathcal{S}_{e,y+1}$ -strategy ρ on TP will be visited at a stage large enough, and at this stage, $\gamma(k_\rho)$ is enumerated into A , protecting computations $\Phi_e(L; z)$, $z \leq y$, forever. $\square$

Lemma 2.5. *Let ζ be an $\mathcal{R}_e$ -strategy on TP such that ζ^∞ is also on TP , and let τ be a child node of ζ being visited at a stage s in the construction. Then $T_{\tau,s}$ is defined and contains numbers not in $(U_e \cup V_e)[s]$, and if τ is on TP , then T_τ contains numbers not in $U_e \cup V_e$. This shows that if ζ is an $\mathcal{R}_e$ -strategy with ζ^∞ on TP , then all substrategies are satisfied.*

Proof. Let τ be a child node of ζ being visited at the stage s . Then according to the construction, k_τ is defined and requests the selection of x_τ . At the next ζ -expansionary stage s_1 say, $\gamma(k_\tau)$ is enumerated into A and x_τ is defined as a big number. After this, at the next ζ -expansionary stage s_2 say, $\gamma(k_\tau)$ is enumerated into A and T_{τ,s_2} is defined, as a set of numbers disjoint from all $T_{\tau',s'}$ defined by some τ' at the stage $s' < s$, according to the construction.

T_{τ,s_2} cannot be empty, as otherwise, x_τ is enumerated into L , leading to $\Delta_e(L) \neq U_e$ or $\Psi_e(U_e) \neq L$ or $U_e \cap V_e \neq \emptyset$, which cannot happen, as we are assuming that ζ^∞ is on TP . This implies that if T_{τ_1,s_1} is defined at a stage s_1 and T_{τ_2,s_2} is defined at a stage s_2 , where $s_1 < s_2$, then T_{τ_2,s_2} contains numbers not in T_{τ_1,s_1} .

If τ is on TP , then T_τ cannot be a subset of $U_e \cup V_e$. To see this, without loss of generality, assume that $T_\tau = T_{\tau, s_2}$. That is, after the stage s_2 , τ cannot be reset or initialized. If at a τ -stage $s_3 > s_2$, $T_\tau \subseteq (U_e \cup V_e)[s_3]$, then $T_\tau \subseteq V_e[s_3]$, as the L -computations seeing at the stage s_2 are preserved. x_τ is now enumerated into L , and we will have one of $\Delta_e(L) \neq U_e$, $\Psi_e(U_e) \neq L$, $U_e \cap V_e \neq \emptyset$ true, which again, is impossible. $\square$

Lemma 2.6. *All the $\mathcal{R}_e$ -requirements are satisfied by the corresponding $\mathcal{R}_e$ -strategies on the true path TP .*

Proof. Fix e and let ζ be the $\mathcal{R}_e$ -strategy on the true path TP . By Lemma 2.1, let s_0 be the least stage after which ζ cannot be initialized any more.

If $\zeta \frown f \in TP$, then there are only finitely many ζ -expansionary stages, then the $\mathcal{R}_e$ -requirement is satisfied vacuously.

If $\zeta \frown \infty \in TP$ and $\Delta_e(L) = U_e, \Psi_e(U_e) = L, U_e \cap V_e = \emptyset$, we need to prove that $U_e \cup V_e$ is not maximal. By $\Delta_e(L) = U_e, \Psi_e(U_e) = L, U_e \cap V_e = \emptyset$, we know from Lemma 2.5 that in the construction, each substrategy τ on TP will have x_τ selected, and no x_τ is put into L , because otherwise, one of these three clauses is false. This ensures that if τ is on TP , then T_τ contains numbers not in $U_e \cup V_e$, and furthermore, if $\tau_1 \subset \tau_2$ are both on TP , then T_{τ_2} contains elements not in T_{τ_1} and $T_{\tau_1} \cap T_{\tau_2} = \emptyset$. Therefore, $P_\zeta = \bigcup \{T_{\tau, s} : \tau \text{ is a } \mathcal{T}_{\zeta, 2i}\text{-strategy below } \zeta \frown \infty, i \in \omega, s \in \omega\}$, $Q_\zeta = \bigcup \{T_{\tau, s} : \tau \text{ is a } \mathcal{T}_{\zeta, 2i+1}\text{-strategy below } \zeta \frown \infty, i \in \omega, s \in \omega\}$ are both c.e., infinite and disjoint. This implies that $P_\zeta \cap \overline{U_e \cup V_e}$ and $\overline{P_\zeta} \cap \overline{U_e \cup V_e} = \overline{P_\zeta \cup (U_e \cup V_e)} \supseteq Q_\zeta - (P_\zeta \cup (U_e \cup V_e)) = Q_\zeta - (U_e \cup V_e)$ are infinite, and hence $U_e \cup V_e$ is not maximal. Therefore, the $\mathcal{R}_e$ -requirement is satisfied at ζ . $\square$

Lemma 2.7. *The $\mathcal{G}$ -requirement is satisfied.*

Proof. By the construction, if $\Gamma(A \oplus L; y)$ is defined, then it must compute $K(y)$ correctly. So it suffices to show that $\Gamma(A \oplus L; y)$ is defined for each $y \in \omega$.

Note that once $\Gamma(A \oplus L; y)$ is defined, it can be undefined only when one of the following appears:

- (i) a small number $\leq y$ enters K , or
- (ii) $\gamma(k)$ is enumerated into A by an $\mathcal{S}$ -strategy or a $\mathcal{R}$ -strategy, and $k \leq y$. In this case, k is chosen as a threshold, or

- (iii) some x with $x \leq \gamma(y)$ is enumerated into L by some $T_{e,i}$ -strategy to perform its disagreement attack.

However, as we will see in what follows each of them can happen at most finitely many times:

- (a) At most $y + 1$ many numbers $\leq y$ can enter K , so $\Gamma(A \oplus L; y)$ can only be undefined at most $y + 1$ many times by (i).
- (b) By the construction, each $\mathcal{S}_{e,i}$ or $\mathcal{T}_{e,i}$ -strategy can enumerate $\gamma(k)$ into A at most finitely often on the same k . Thus (ii) can happen at most finitely many times for y .
- (c) The action (iii) never increase the value of $\gamma(y)$ by our construction. From (a) and (b), the value of $\gamma(y)$ is finite eventually. So (iii) can happen at most finitely often.

Therefore, $\Gamma(A \oplus L; y)$ must be defined eventually, for any y . As a consequence, $\Gamma(A \oplus L)$ is total and computes K correctly. $\square$

This completes the proof of Theorem 1.

From the verification part, the construction is in level $\mathbf{0}''$, since it only requires a $\emptyset''$ to compute exactly how each requirement was satisfied. The construction given by Downey and Stob is in level $\mathbf{0}'''$, and hence our construction is simpler than Downey and Stob's construction.

Chapter 3

THE DISTRIBUTION OF SIMPLE LEFT C.E. REALS IN THE HIGH/LOW HIERARCHY

The work in this chapter is joint with Wu.

3.1 Introduction

A real a is called *left computably enumerable* (left c.e. for short) if there is a computable increasing sequence of rational numbers $\{a_i : i \in \omega\}$ which converges to a . Moreover, if there is a computable function f such that $a - a_i \leq 2^{-j}$ for all i, j with $i \geq f(j)$, then a is called *computable*.

In the following, we just consider the reals in $(0, 1]$. Given $a \in (0, 1]$, $a(i)$ presents the i -th digit of the dyadic expansion of a for $i > 0$, i.e. $a = \sum_{i=1}^{\infty} a(i)2^{-i}$. First, we have the following observation.

Fact 3.1. *If a is a computable real, then there is a computable function g from natural numbers ω to set $\{0, 1\}$ such that for any $i > 0$, $g(i) = a(i)$.*

Proof. There are two cases.

Case (1): If a is a rational number, then there is a canonical way to compute $a(i)$ for all $i \in \omega$. That is, $a(i)$ is computable. Let $g(i) = a(i)$ for $i > 0$ and $g(0) = 0$. Then g is a desired computable function.

Case (2): If a is an irrational number, then there must exist infinitely many i such that $a(i) = 0$. Let $\{a_i : i \in \omega\}$ be a computable increasing sequence of rational numbers converging to a (note that such sequence exists as a is a left c.e. real). Let f be a witness of a being computable, then $f(j) \rightarrow \infty$ as $j \rightarrow \infty$. Now we construct a function g as follows: first let $g(0) = 0$. For any $i > 0$, search for the least $n > i$ such that there is an integer m strictly between i and n with $a_{f(n)}(m) = 0$, and set $g(i) = a_{f(n)}(i)$ (note that such n exists, since a is irrational and $\{a_i : i \in \omega\}$ converges to a). Then g is a computable function, since $\{a_i : i \in \omega\}$ and f are computable. Now it suffices to show $g(i) = a(i)$. If there is an i such that $g(i) \neq a(i)$, then $a(i) \neq a_{f(n)}(i)$ for the above n . For this n , let $k \leq i$ be the least number such that $a(k) \neq a_{f(n)}(k)$, then $a(k) = 1$ and $a_{f(n)}(k) = 0$, since $\{a_i : i \in \omega\}$ is an increasing sequence. By the choice of n , there is some m with $i < m < n$ such that $a_{f(n)}(m) = 0$. Hence $a - a_{f(n)} \geq 2^{-k} + 2^{-m} - \sum_{j=1}^{\infty} 2^{-k-j} = 2^{-m} > 2^{-n}$, since $a - \sum_{j < k} a(j)2^{-j} \geq 2^{-k}$, $a_{f(n)} - \sum_{j < k} a_{f(n)}(j)2^{-j} \leq \sum_{j=1}^{\infty} 2^{-k-j} - 2^{-m}$ and $a_{f(n)}(j) = a(j)$ for any $j < k$. It is a contradiction to $a - a_{f(n)} \leq 2^{-n}$.

□

Before the main theorem, we recall some other basic definitions we will use in this chapter.

Definition 3.2 (Downey and LaForte [21]). *For any $W \subset 2^{<\omega}$, W is called a presentation of a left c.e. real a if W is a prefix-free c.e. set such that $a = \sum_{\sigma \in W} 2^{-|\sigma|}$, where W is prefix-free if for any $\sigma_1, \sigma_2 \in W$ with $\sigma_1 \subseteq \sigma_2$ we have $\sigma_1 = \sigma_2$, $|\sigma|$ means the length of the string σ .*

Definition 3.3 (Wu [86]). *A left c.e. real a is called simple if any presentation of a is computable.*

In this chapter, we will prove the following theorem.

Theorem 2 (Fang and Wu [32]) *For any high c.e. set H , there is a noncomputable simple left c.e. real $a \leq_T H$.*

Fix a one-to-one computable function τ from ω onto $2^{<\omega}$. In the following, for any natural number x , $|x|$ means the length of $\tau(x)$.

3.2 Requirements, Basic strategies and The interactions

3.2.1 Requirements

To prove Theorem 2, we will construct a computable increasing sequence of rational numbers $\{a_i : i \in \omega\}$ such that $\lim_{i \rightarrow \infty} a_i = a$, $a \leq_T H$ and the following requirements are also satisfied.

$$\mathcal{P}_e: \exists x > 0 (\varphi_e(x) \neq a(x)),$$

$$\mathcal{N}_e: \sum_{x \in W_e} 2^{-|x|} = a \rightarrow W_e \text{ is computable,}$$

where $\{\varphi_e\}_{e \in \omega}$ is an effective list of all p.c. functions, $\{W_e\}_{e \in \omega}$ is an effective list of all c.e. sets.

The real number a we will construct is a left c.e. real, since a is a limit of a computable increasing sequence of rational numbers. All $\mathcal{P}$ -requirements guarantee that a is noncomputable. All $\mathcal{N}$ -requirements ensure a to be simple. Hence a is a noncomputable simple left c.e. real.

In the construction, a_s is defined at the stage s . Its definition depends on the value of a_{s-1} and the strategies considered at the stage s . To guarantee $\{a_i : i \in \omega\}$ increasing, a_s will be set as the sum of a_{s-1} and a nonnegative number. In the following, a fresh number x means that $x - 1$ is larger than any number we have mentioned before. Let us introduce the basic strategies first. The construction is proceeded on a priority tree T .

3.2.2 Basic $\mathcal{P}_e$ -strategy

The strategy α for $\mathcal{P}_e$ is a slight variant of the diagonal argument. Its basic idea is to choose a fresh number $x \in \omega^{[e]} = \{\langle e, y \rangle : y \in \omega\}$ as a potential witness to satisfy $\mathcal{P}_e$ -requirement. The basic strategy works as follows:

- (1) Choose a fresh number $x \in \omega^{[e]}$, and set $a_s = a_{s-1}$, hence $a_s(x) = 0$ currently;
- (2) Wait for a stage $s' > s$ such that $\varphi_{e,s'}(x) \downarrow = 0$; (At any stage t with $s \leq t < s'$, if α has not been initialized after the stage s , we preserve $a_t(x) = 0$).

- (3) Set $a_{s'} = a_{s'-1} + 2^{-x}$, then $a_{s'}(x) = 1$. (At any stage $t \geq s'$, if α has not been initialized after stage s' , we preserve $a_t(x) = 1$).

Here α has two outcomes: $d <_L w$. The outcome d means that the strategy can reach step (3) and $a(x) = 1 \neq 0 = \varphi_e(x)$; the outcome w means that the strategy waits at step (2) for ever and $a(x) = 0 \neq \varphi_e(x)$, since $\varphi_e(x) \uparrow$ or $\varphi_e(x) \downarrow \neq 0$. Notice that $a(x) = \lim_{i \rightarrow \infty} a_i(x)$ for any $x > 0$. Thus the $\mathcal{P}_e$ -requirement is satisfied.

3.2.3 Basic $\mathcal{N}_e$ -strategy

Let β be an $\mathcal{N}_e$ -strategy. Its basic idea is to construct a such that $\sum_{x \in W_e} 2^{-|x|} \neq a$ or W_e is computable.

In the construction, once β discovers a stage t such that $\sum_{x \in W_{e,t}} 2^{-|x|} > a_{t-1}$, it sets a restraint $r(e, t)$ as the least number k satisfying $\sum_{x \in W_{e,t}} 2^{-|x|} - a_{t-1} > 2^{-k}$. By this restraint, β makes the difference between a and a_{t-1} less than $2^{-r(e,t)}$ to force $a < \sum_{x \in W_e} 2^{-|x|}$. To achieve this goal, β only allows the lower priority $\mathcal{P}$ -nodes to choose numbers bigger than $r(e, t)$. As different $\mathcal{P}$ -nodes have different witnesses, $a - a_{t-1} \leq \sum_{j=1}^{\infty} 2^{-r(e,t)-j} = 2^{-r(e,t)} < \sum_{x \in W_{e,t}} 2^{-|x|} - a_{t-1}$. Therefore $a < \sum_{x \in W_{e,t}} 2^{-|x|} \leq \sum_{x \in W_e} 2^{-|x|}$, and $\mathcal{N}_e$ is satisfied. Note that for any stage $u \geq t$, $a_{u-1} \leq a < \sum_{x \in W_{e,t}} 2^{-|x|} \leq \sum_{x \in W_{e,u}} 2^{-|x|}$.

On the other hand, there is no stage $t \in \omega$ such that $\sum_{x \in W_{e,t}} 2^{-|x|} > a_{t-1}$, i.e. $\sum_{x \in W_{e,t}} 2^{-|x|} \leq a_{t-1}$ for any stage t . Since the enumeration of W_e cannot be controlled by us, it is possible that $\sum_{x \in W_{e,s}} 2^{-|x|}$ can keep up with a whenever any number is added to a . In this case, it is hard to make $\sum_{x \in W_e} 2^{-|x|} \neq a$. The corresponding strategy is to force W_e to be computable by controlling the increment of the increasing sequence $\{a_i\}_{i \in \omega}$. Its basic idea is in what follows. When $\sum_{x \in W_{e,s}} 2^{-|x|}$ gets closer to a_{s-1} from below, we impose a restraint $r_{\infty}(e, s)$ (which is related to the first different digit of number a_{s-1} and $\sum_{x \in W_{e,s}} 2^{-|x|}$) to prevent the strings of length $< k(e, s)$ from entering W_e , where $r_{\infty}(e, s) - k(e, s)$ is a constant for any s , (we will explain it later). After the stage s , once some string of length $< k(e, s)$ enters W_e at some stage $t > s$, $\sum_{x \in W_{e,t}} 2^{-|x|} > a_{t-1}$ must hold at the stage t . It contradicts to the assumption. Thus, W_e is computable. To formulate the above idea, we define the length agreement function as follows:

$$l(e, s) = \begin{cases} \min\{n : a_{s-1} - \sum_{x \in W_{e,s}} 2^{-|x|} > 2^{-n}\}, & \sum_{x \in W_{e,s}} 2^{-|x|} < a_{s-1}, \\ s, & \sum_{x \in W_{e,s}} 2^{-|x|} = a_{s-1}, \\ 0, & \text{otherwise.} \end{cases}$$

If $s > 0$, let $m(e, s) = \max\{l(e, t) : t < s\}$. A stage s is an e -expansionary stage if $s = 0$ or $l(e, s) > m(e, s)$ for $s > 0$.

At any e -expansionary stage s , define $r_\infty(e, s) = m(e, s)$ (temporary restraint). We only add at most $2^{-r_\infty(e, s)}$ to a at the e -expansionary stage s , and wait for the next e -expansionary stage t . Then there is no y with length $|y| < r_\infty(e, s) - 1$ entering W_e between stages s and t . Otherwise, $\sum_{x \in W_{e,t}} 2^{-|x|} - \sum_{x \in W_{e,s}} 2^{-|x|} \geq 2^{-|y|} > 2^{-r_\infty(e, s)+1} = 2^{-r_\infty(e, s)} + 2^{-r_\infty(e, s)} \geq a_{s-1} - \sum_{x \in W_{e,s}} 2^{-|x|} + 2^{-r_\infty(e, s)} \geq a_{t-1} - \sum_{x \in W_{e,s}} 2^{-|x|}$, hence $\sum_{x \in W_{e,t}} 2^{-|x|} > a_{t-1}$. Thus t is not e -expansionary, and it is a contradiction. (*Note that here we have not considered the action of other strategies.*)

If there are infinitely many e -expansionary stages, β has outcome i . Otherwise, there are finitely many e -expansionary stages, β has outcome f . We assign the priority of the outcomes as $i <_L f$.

In the following, we denote $e(\xi)$ as ξ , where ξ is some $\mathcal{P}_e$ or $\mathcal{N}_e$ strategy. For any strategy ξ , a stage s is called a ξ -stage if β is visited at the stage s .

For an $\mathcal{N}_e$ strategy β , at any β -stage s , we define the length agreement function as follows:

$$l(\beta, s) = \begin{cases} \min\{n : a_{s-1} - \sum_{x \in W_{\beta,s}} 2^{-|x|} > 2^{-n}\}, & \sum_{x \in W_{\beta,s}} 2^{-|x|} < a_{s-1}, \\ s, & \sum_{x \in W_{\beta,s}} 2^{-|x|} = a_{s-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $m(\beta, s) = \max\{0, l(\beta, t) : t < s \text{ \& } t \text{ is a } \beta\text{-stage}\}$. A stage s is called a real β -expansionary stage if $s = 0$ or $s > 0$ is a β -stage and $l(\beta, s) > m(\beta, s)$ (here we use the notion, real expansionary stage, since later we will define another kind of expansionary stage). It is easy to see that $a_{s-1} - \sum_{x \in W_{\beta,s}} 2^{-|x|} \leq 2^{-m(\beta, s)}$ for any real β -expansionary stage $s > 0$, since $m(\beta, s) < l(\beta, s)$.

In the following, we discuss the interaction among various strategies.

3.2.4 The interaction between $\mathcal{N}$ - and $\mathcal{P}$ -strategies

Assume that β is an $\mathcal{N}_m$ -strategy and α is a $\mathcal{P}_e$ -strategy with $\beta \smallfrown i \subseteq \alpha$. Let s be an α -stage (it is a real β -expansionary stage naturally) with $x_\alpha < m(\beta, s) = r_\infty(\beta, s)$ such that $a_{s-1}(x_\alpha) = 0$ and $\varphi_{e,s}(x_\alpha) \downarrow = 0$, where x_α is the potential witness of α . Since α has lower priority than β , it should respect the restraint $r_\infty(\beta, s)$ imposed by β . Thus α cannot add 2^{-x_α} to a_{t-1} at any stage $t \geq s$. This makes α to choose a new value for the witness x_α . However, if $a = \sum_{x \in W_{m,s}} 2^{-|x|}$, then β will impose infinitely many restraints $r_\infty(\beta, s)$ approaching to infinity. As a consequence, α may update its witness x_α infinitely many times and none of them gets a successful diagonal argument. That is, the $\mathcal{P}_e$ -requirement cannot be satisfied. We solve this problem by modifying $\mathcal{P}$ -strategy α and $\mathcal{N}$ -strategy β as follows.

If α sees $a_{s-1}(x_\alpha) = 0$ and $\varphi_{e,s}(x_\alpha) \downarrow = 0$ with $x_\alpha < m(\beta, s) = r_\infty(\beta, s)$ at an α -stage s , then it creates a link, denoted by $\ell(\beta, \alpha)$, between α and β . After the stage s , we shall add to a_{t-1} at most $2^{-r_\infty(e,s)}$ at each real β -expansionary stage t until a stage t_0 such that $a_{t_0}(x_\alpha) = 1$ at some stage t_0 . To achieve this, at each real β -expansionary stage $t \in (s, t_0]$, (i) β keeps its temporary restraint as $r_\infty(e, s)$ instead of increasing it to $m(e, t)$; (ii) the stage t stopped at β instead of going to its outcome i ; (iii) β adds a number $2^{-r_\infty(e,s)}$ to a_{t-1} instead of adding 2^{-x_α} to a_{t-1} . Intuitively, α distributes its mission of adding 2^{-x_α} into several stages instead of one stage. In what follows, we present a formal description for such process.

- (1) At an α -stage s_0 , α chooses a fresh number $x \in \omega^{[e]}$ if x_α is undefined, let $x_\alpha = x$. Such x_α is called *potential witness*.

Establishing Link $\ell(\beta, \alpha)$ and Define Counter Function

- (2) At an α -stage $s_1 > s_0$, $\varphi_{\alpha,s_1}(x_\alpha) \downarrow = 0$ and $a_{s_1-1}(x_\alpha) = 0$ while β imposes a restraint $r_\infty(\beta, s_1)$. (*Unfortunately, $r_\infty(\beta, s_1) > x_\alpha$, β prevents adding 2^{-x_α} to a_{s_1-1} at the stage s_1 , while α wants to increase 2^{-x_α} to a_{s_1-1} in order to make $a(x_\alpha) = 1$.) Then α sets up a link $\ell(\beta, \alpha)$ between α and β , and define a counter function $C(\beta, s_1) = 2^{r_\infty(\beta, s_1) - x_\alpha}$. Here we call link $\ell(\beta, \alpha)$ is associated with (α, x) . Wait for the next real β -expansionary stage.*

For $j \in (1, 2^{r_\infty(\beta, s_1) - x_\alpha} + 2]$, let s_j be the next real β -expansionary stage after s_{j-1} .

Addition Process

- (3) If $C(\beta, s_j - 1) > 0$, then β sets $r_\infty(\beta, s_j) = r_\infty(\beta, s_1)$ and $C(\beta, s_j) = C(\beta, s_{j-1}) - 1 = C(\beta, s_1) - j + 1$. Now β let $a_{s_j} = a_{s_{j-1}} + 2^{-r_\infty(\beta, s_{j-1})}$ and stops the stage s_j . (We say β takes action associated with (α, x) at the stage s_j). Wait for the next real β -expansionary stage.

Cancelling Link $\ell(\beta, \alpha)$

- (4) If $C(\beta, s_j - 1) = 0$, then β cancels the link $\ell(\beta, \alpha)$. Let $\beta \smallfrown i$ be accessible at the stage s_j , and set $r_\infty(\beta, s_j) = m(\beta, s_j)$.

In this process, if $k = 2^{r_\infty(\beta, s_1) - x} + 2$, then $C(\beta, s_k - 1) = 0$ and s_k is the latest $\beta \smallfrown i$ -stage after the stage s_1 . For any stage v between s_1 and s_k , let $r_\infty(\beta, v) = r_\infty(\beta, v - 1)$ and $C(\beta, v) = C(\beta, v - 1)$ if we did not define it in particular. Let $z = \max(\{y : a_{s_1-1}(y) = 1\} \cup \{x_\alpha\})$, in fact, $a_{s_k} = a_{s_1-1} + d$ with $2^{-x_\alpha} \leq d < 2^{-x_\alpha} + 2^{-z-1}$, since any witness chosen by the lower priority $\mathcal{P}$ -strategies to the right of $\beta \smallfrown i$ after the stage s_1 must be greater than $z + 1$ by the definition of the fresh number. Thus the increment cannot affect the n -th digit for any $n < z + 1$ except the x_α -th digit, hence $a_{s_k}(x_\alpha) = 1$. Furthermore, by using the similar argument, $a(x_\alpha) = 1$, and $\mathcal{P}_e$ is satisfied.

Note that we only increase the value of $r_\infty(\beta, s)$ to $m(\beta, s)$ at the real β -expansionary stage s such that $C(\beta, s - 1) = 0$. Similarly, $\beta \smallfrown i$ can be accessible at the real β -expansionary stage s only when $C(\beta, s - 1) = 0$.

3.2.5 The interaction among two $\mathcal{N}$ -strategies and one $\mathcal{P}$ -strategy

Let β' and β be $\mathcal{N}$ -strategies, α be a $\mathcal{P}$ -strategy such that $\beta' \smallfrown i \subseteq \beta \smallfrown i \subseteq \alpha$. Let s be a real β -expansionary stage (it is a real β' -expansionary stage naturally) with $C(\beta, s - 1) > 0$ and $r_\infty(\beta', s) > r_\infty(\beta, s - 1)$. Then there is a link $\ell(\beta, \alpha)$ associated with (α, x) at the stage s . By the above modified β -strategy, β wants to add $2^{-r_\infty(\beta, s-1)}$ to a_{s-1} . However, β has priority lower than β' . β should consider the temporary restraint $r_\infty(\beta', s)$ imposed by β' . Here β' prevents β from adding $2^{-r_\infty(\beta, s-1)}$ to a_{s-1} at the stage s . For this problem, we use the similar method as above to cope with it. The only difference which needs to be specified is: We call the link $\ell(\beta', \beta)$ created at the stage s associated with (α, x) .

3.2.6 Permitting below a high c.e. degree

In the following, we will make use of the technique for “working below high” introduced by Shore and Slaman [76], to force $a \leq_T H$. In fact, the history of high permitting goes back to Martin and Cooper. This method introduced by Shore and Slaman makes us to redefine the outcomes of strategies in order to enable H to enumerate the nodes ξ and the stages s such that no nodes to the left of ξ can be visited after the stage s . The basic idea is described in what follows.

First of all, we fix an effective enumeration $\{H_s\}_{s \in \omega}$ of H . As H is high, there is a computable functional Γ such that $\emptyset''(e) = \lim_{t \rightarrow \infty} \Gamma(H; e, t)$ for any e . Based upon the approximations of H and Γ , we define a new functional $\hat{\Gamma}$ corresponding to Γ as follows: at any stage s , for any $t < s$, define $\hat{\Gamma}(H; e, t)[s] = \Gamma(H; e, t)[s]$ if $\Gamma(H; e, t)[s]$ has not changed since the last stage s' at which $\mathcal{N}_e$ was accessible, i.e. $\Gamma(H; e, t)[s'] \downarrow$ and $H_{s'} \upharpoonright \gamma(H; e, t)[s'] = H_s \upharpoonright \gamma(H; e, t)[s']$, and $\hat{\Gamma}(H; e, t)[s] \uparrow$ otherwise. It is easy to prove that $\lim_{s \rightarrow \infty} \Gamma(H; e, t)[s] = \lim_{s \rightarrow \infty} \hat{\Gamma}(H; e, t)[s]$.

A stage s is H -true if $H \upharpoonright h_s = H_s \upharpoonright h_s$, where h_s is the least number entering H at the stage s . Let s be an H -true stage. If $\hat{\Gamma}(H; e, t)[s]$ is defined, then $\hat{\Gamma}(H; e, t)[s] = \Gamma(H; e, t)$. For any stage s , suppose $t_{\max}$ is the largest t such that $\hat{\Gamma}(H; e, u)[s] \downarrow$ for any $u \leq t$. The value $\hat{\Gamma}(H; e, t_{\max})[s]$ is called *the current limiting value*. Without loss of generality, we assume that $t_{\max}$ exists at any stage, since $\Gamma(H)$ is total. Let v be the least number such that for any u with $v \leq u \leq t_{\max}$, $\hat{\Gamma}(H; e, u)[s] \downarrow = \hat{\Gamma}(H; e, t_{\max})[s]$. Let $n = \max\{\hat{\Gamma}(H; e, t)[s] : t \leq v\}$, and $\sigma = H_s \upharpoonright (n + 1)$. We call σ the *current true use*. In order to guarantee $a \leq_T H$, we use $\hat{\Gamma}(H)$ to guess the true outcome of strategies on the priority tree. For any strategy ξ , let $H_{\xi, s} = H_s$, if s is a ξ -stage and $H_{\xi, s} = H_{\xi, s-1}$ otherwise. By convention, if we write $\varphi_{e, s}(x) \downarrow$, then $s > x$. We modify the basic $\mathcal{P}$ -strategy α and $\mathcal{N}$ -strategy β as follows.

3.2.7 Modified $\mathcal{P}$ -strategy

Let α be a $\mathcal{P}_e$ -requirement. The basic α -strategy produces an explicit disagreement between φ_e and a at a particular argument. The modified α -strategy creates some disagreement between φ_e and a by changing some digit, say x -th digit, of a from 0 to 1 with $\varphi_e(x) \downarrow = 0$ if there are infinitely many $x \in \omega^{[e]}$ such that $\varphi_e(x) \downarrow = 0$; otherwise, we create some disagreement by choosing some $x > \max\{y \in \omega^{[e]} : \varphi_e(y) \downarrow = 0\}$

at some stage s with $a_{s-1}(x) = 0$ and keeping the value $a(x) = 0$ for ever. Here the question that “whether there are infinitely many $x \in \omega^{[e]}$ such that $\varphi_e(x) \downarrow = 0$ ” is a Π_2 question. In the construction, we use the property of the highness of H to approximate the answer of the Π_2 -question. We uniformly give a construction in parameter r . By the recursion theorem, there is a parameter r_0 such that the r_0 -th construction works. Thus, there is a reduction f of the question to $\mathbf{0}''$ uniformly in α , i.e. the answer to the above Π_2 -question is “yes” if and only if $\lim_{s \rightarrow \infty} \hat{\Gamma}(H; f(\alpha), s) = 0$. (We will give the detail of applying the recursion theorem to get such a computable function f after we introduce the modified $\mathcal{N}_e$ -strategy). In the following, we just consider the r_0 -th construction.

In the construction, we arrange the outcomes of α as follows. The type of outcomes is (o, σ) , which is called *apparent outcome*, where $o = i$ if the current limiting value of $\hat{\Gamma}(H; g(e), t)$ is 0, $o = f$ otherwise, and σ is the current true use. Their priority is arranged as in what follows.

Assume that (o, σ) and (o', σ') are apparent outcomes of ξ . Let t (or t') be the maximal number such that for any $x \leq t$ (or $x \leq t'$), $\hat{\Gamma}(\sigma; f(\xi), x)[|\sigma|] \downarrow$ (or $\hat{\Gamma}(\sigma'; f(\xi), x)[|\sigma'|] \downarrow$). We assign the priority to the outcomes of ξ as follows: $(o, \sigma) <_L (o', \sigma')$ if and only if one of the following is satisfied: (1) there is some x such that $x < |\sigma|, |\sigma'|$ and $\sigma(x) = 1$ & $\sigma'(x) = 0$; (2) $\sigma \subset \sigma'$ and there is some x with $t < x \leq t'$ such that $\hat{\Gamma}(\sigma; f(\xi), t)[|\sigma|] \downarrow \neq \hat{\Gamma}(\sigma'; f(\xi), x)[|\sigma'|] \downarrow$. *Note that it is reasonable to assign the priority as $(o, \sigma) <_L (o', \sigma')$ if one of the above condition is satisfied. Since otherwise if we assign $(o', \sigma') <_L (o, \sigma)$, it is possible that there is no true path on the priority tree, or the requirement can not be satisfied at the corresponding node on the true path, we can see it from the verification.*

Now the α -strategy is modified as follows:

Case I If the true outcome of α is (i, σ) , then there are infinitely many $z \in \omega^{[e]}$ such that $\varphi_\alpha(z) \downarrow = 0$. In this case, α must make $a(z) = 1$ for a such z . To achieve this, α attaches the numbers z with $\varphi_\alpha(z) \downarrow = 0$ to its edge $\alpha^\frown(i, \sigma)$, and picks one of them to apply the disagreement argument and label it with $\checkmark$, at the same time, impose a restraint and initialize all lower priority strategies to ensure $a(x) = 1$ finally. *To consider the interaction of $\mathcal{P}$ and $\mathcal{N}$ -strategies, we create some link $\ell(\beta, \alpha)$ which is associated with some ordered pair (α, x) if it*

necessary, it is similar as in section 3.2.4.

Note: in Case I, when we create the disagreement argument on x at the stage s , we consider to add the number $d = 2^{-x} - \sum_{n>x} a_{s-1}(n)2^{-n}$ to a . It is one key point to make the modified construction working.

Case II If the outcome of α is (f, σ) , then it means that there are only finitely many $x \in \omega^{[e]}$ such that $\varphi_\alpha(x) \downarrow = 0$. Notice that it is possible that $\varphi_\alpha(x) \downarrow = 1$ and $a(x) = 1$ for all $x \in \omega^{[e]}$ such that $\varphi_\alpha(x) \downarrow \neq 0$, since other $\mathcal{P}$ -strategies can make $a(x) = 1$ for such x even though α itself does not do so. That is, α cannot straightly employ $x \in \omega^{[e]}$ such that $\varphi_\alpha(x) \downarrow \neq 0$ without any action, as its successful witness. Thus, the original α -strategy must be modified to satisfy $\mathcal{P}_e$ -requirement. Its basic idea is to choose one big enough number $x \in \omega^{[e]}$ with $a(x) = 0$ and attach it to the edge $\alpha^\wedge(f, \sigma)$, impose a restraint and initialize all the lower priority strategies to ensure $a(x) = 0$ finally.

Note that the essential difference, between original and modified α -strategies, is the choice of the witnesses. For instance, α only chooses x with $\varphi_e(x) \downarrow = 0$ already for its diagonal argument associated with edges $\alpha^\wedge(i, \sigma)$ in the modified α -strategy, while α chooses x as its diagonal argument before $\varphi_e(x) \downarrow = 0$ in the original α -strategy.

3.2.8 Modified $\mathcal{N}$ -strategy

Let β be an $\mathcal{N}_e$ -strategy. The problem, whether there are infinitely many real β -expansionary stages, is a Π_2 -question. So this question can also be effectively approached by using H and Γ . It is similar to the $\mathcal{P}_e$ -strategy. There is a computable function f such that there are infinitely many real β -expansionary stages if and only if $\lim_{s \rightarrow \infty} \hat{\Gamma}(H; f(\beta), s) = 0$.

First of all, the outcomes of β are of form (o, σ) that is resulted from the approximation of Π_2 -question via H , Γ and f , and is defined similarly to ones in the $\mathcal{P}$ -strategy. If β has outcome (o, σ) with $o = i$, then s is called *an apparent β -expansionary stage*. Now, there are two sorts of expansionary stages: (i) apparent β -expansionary stages and (ii) real β -expansionary stages defined in section 3.2.3. However, these two sorts of expansionary stages may happen to take place simultaneously at most finitely many times. Since we are using H , Γ and f to approach the Π_2 -question, it is unreasonable

to infer that there are infinitely many stages which is a real expansionary stage and meanwhile an apparent expansionary stage. On the other hand, if the construction works, and there are infinitely many apparent β -expansionary stages, then there must be infinitely many real β -expansionary stages. Thus β makes these two sorts of expansionary stages working together as follows. For each outcome (i, σ) , β believes it only when this outcome is confirmed by a real β -expansionary stage. *To confirm an outcome (i, σ) at a stage s , β just ensures that there is a real β -expansionary stage t with $t \leq s$ which is attached to β and is not canceled until the stage s .* As a consequence, the stage s is stopped at β if the apparent outcome of β is (i, σ) but (i, σ) is not confirmed at the stage s . There are two cases:

Case I If there are infinitely many real β -expansionary stages, then for any apparent expansionary stage there is a real β -expansionary stage after it. Thus, all links $\ell(\beta, \xi)$ with $\xi \in TP$ can be canceled eventually, and the original β -strategy applies essentially here.

Case II If there are only finitely many real β -expansionary stages, then the true outcome of β is (f, σ) for σ being some initial segment of H .

Formally, we describe the modified single β -strategy as follows.

Step 1. Check whether s is a real β -expansionary stage, if it is, then attach s to the left most edge leaving β with outcome $(i, -)$ which has been visited but there is no number attached to it until now, or the number was canceled at some stage before s and after it is canceled there is no new number was attached to the edge. Otherwise, do nothing at this step.

(Note: to consider the interaction of $\mathcal{P}$ and $\mathcal{N}$ -strategies, we create some link associate with some ordered pair (α, x) if it is necessary)

Step 2. Check whether β is satisfied or $\sum_{x \in W_{\beta, s}} 2^{-|x|} > a_{s-1}$.

2.1 If β is satisfied, then go to step 3.

2.2 If β is not satisfied, but $\sum_{x \in W_{\beta,s}} 2^{-|x|} > a_{s-1}$, let $m = \max\{n : a_u(n) = 1 \text{ for } u < s\}$, and define y to be the least natural number $> \max\{m, s, -\log_2(\sum_{x \in W_{\beta,s}} 2^{-|x|} - a_{s-1})\}$. Declare that β is satisfied and attach y to the node β . Let $r(\beta, s) = y$. Let the stage s stop at node β .

2.3 Otherwise, go to step 3 directly.

Step 3. Now β computes its outcome (o, σ) based on $\hat{\Gamma}(H; e, t)[s]$ and performs actions based on the outcome o as follows.

β -1 If $o = i$, there are three cases.

β -1.1: If β is satisfied, then there is some number y attached to β , define $r(\beta, s) = y$. Let the stage s stop at node β .

Note. In this cases (β -1.1), we can guarantee that the antecedent of $\mathcal{N}_e$ is not true. Then there are only finitely many real β -expansionary stages. By using of the recursion theorem and the highness of c.e. set H , we know (i, σ) is not the true outcome of β . It is reasonable to wait for the true outcome.

β -1.2: If β -1.1 fails, and there is some number $t \leq s$ attached to the edge leaving β with outcome (i, σ) , that is, the outcome (i, σ) is confirmed. There are two cases as follows.

β -1.2.a: If there is no link $\ell(\beta, \xi)$ for any $\xi \supseteq \beta^\frown(i, \sigma)$, or $C^{(i, \sigma)}(\beta, s-1) = 0$, then detach the number on the edge, and let $r_\infty^{(i, \sigma)}(\beta, s) = m(\beta, u)$, where u is the last real β -expansionary stage which is less than or equal to s . Cancel link $\ell(\beta, \xi)$ for $\xi \supseteq \beta^\frown(i, \sigma)$ if it exists. Let $r(\beta, s)$ be the maximal stage at which an edge leaving β to the left of (i, σ) was accessible. Impose the restraint $r(\beta, s)$.

β -1.2.b: If there is a link $\ell(\beta, \xi)$ for some $\xi \supseteq \beta^\frown(i, \sigma)$ and $C^{(i, \sigma)}(\beta, s-1) > 0$, then $\ell(\beta, \xi)$ must be associated with some ordered pair (α, x) . Let $r(\beta, s)$ be the maximal stage at which an edge leaving β to the left of

(i, σ) was accessible and impose the restraint $r(\beta, s)$. If $a_{s-1}(x) = 1$, then cancel all links associated with (α, x) , and set all corresponding counter function as 0 at the stage s . Let the stage s stop at node β . If $a_{s-1}(x) = 0$, first let $C^{(i, \sigma)}(\beta, s) = C^{(i, \sigma)}(\beta, s-1) - 1$.

(i) If there is some $\mathcal{N}$ -strategy β' with $\beta' \smallfrown (i, \sigma') \subseteq \beta$ such that $r_{\infty}^{(i, \sigma')}(\beta', s) > r_{\infty}^{(i, \sigma)}(\beta, s-1)$, then we find the longest such node, β' say, and create a link $\ell(\beta', \beta)$. We call $\ell(\beta', \beta)$ associated with ordered pair (α, x) . Define $C^{(i, \sigma')}(\beta', s) = 2^{r_{\infty}^{(i, \sigma')}(\beta', s) - r_{\infty}^{(i, \sigma)}(\beta, s-1)}$. Let the stage s stop at node β .

(ii) If there is no such node, let $d = 2^{-x} - \sum_{n > x} a_{s-1}(n)2^{-n}$, and define $a_s = a_{s-1} + \min\{d, 2^{-r_{\infty}^{(i, \sigma)}(\beta, s-1)}\}$. Let the stage s stop at node β . Furthermore, if $a_s(x) = 1$, then cancel all links associated with (α, x) and set the corresponding counter function to be 0 at the stage s .

Note: if there is some strategy will take actions associated with (α, x) at the stage s , we always need to consider the number $2^{-x} - \sum_{n > x} a_{s-1}(n)2^{-n}$.

β -1.3: If β -1.1 and β -1.2 fail, then the outcome (i, σ) is not confirmed. Let $r(\beta, s)$ be the maximal stage at which an edge leaving β to the left of (i, σ) was accessible and impose the restraint $r(\beta, s)$. Let the stage s stop at β , and wait for the next real β -expansionary stage.

β -2 If $o = f$ and β acts based on the cases.

β -2.1: If β is satisfied, then there is some number y attached to the node β . Let $r(\beta, s) = \max\{y, u\}$, where $u < s$ is the last stage at which some node to the left of $\beta \smallfrown (f, \sigma)$ was visited. Go to the next substage of s .

β -2.2: If β is not satisfied, define $r(\beta, s) = u$, where $u < s$ is the last stage at which some node to the left of $\beta \smallfrown (f, \sigma)$ was visited.

Note: For an $\mathcal{N}$ -node β , we construct counter function $C^{(i, \sigma)}(\beta, -)$ and temporary restraint function $r_{\infty}^{(i, \sigma)}(\beta, -)$ corresponding to the outcome (i, σ) of β which can be

visited. It is a difference between the original strategy and the modified strategy. This is an important point to make the modified strategy and the whole construction working.

Now, we are ready to show how to obtain the total computable function f . First, let Σ be the set of all finite strings over set $\{(i, \sigma), (f, \sigma) \mid \sigma \in 2^{<\omega}\}$. Note that the elements of Σ can be effectively enumerated. Assume q is a computable function from Σ to ω . In the construction, we uniformly construct Q_r as follows: we put $\langle q(\xi), x \rangle$ into Q_r at a stage s if one of the following is satisfied, (1) ξ is a $\mathcal{P}$ -strategy which is visited at the stage s in the r -th construction and $\varphi_{\xi,s}(x) \downarrow = 0$ and $x \in \omega^{[e]}$; (2) ξ is an $\mathcal{N}$ -strategy and x is a real ξ -expansionary stage in the r -th construction. Thus Q_r is a c.e. set. In the r -th construction, we proceed as if $Q_r = W_r$. By s-m-n theorem, there is a computable function h such that $W_{h(q(\xi), r)} = \{x : \langle q(\xi), x \rangle \in W_r\}$. Let g be a computable function witnessing the following m -reduction: $\text{Inf} = \{e : |W_e| = \infty\} \leq_m (\omega - \emptyset'')$, g exists since the index set Inf is Π_2 and $\emptyset''$ is Σ_2 -complete. Then $|W_{h(q(\xi), r)}| = \infty$ iff $h(q(\xi), r) \in \text{Inf}$ iff $gh(q(\xi), r) \in \omega - \emptyset''$ iff $\lim_t \hat{\Gamma}(H; gh(q(\xi), r), t) = 0$. By the recursion theorem, there is some construction, the r_0 -th construction say, such that the assumption $Q_{r_0} = W_{r_0}$ is true, then the r_0 -th construction works. Based on the r_0 -th construction, we define the function f from Σ to ω as follows: $f(\xi) = gh(q(\xi), r_0)$ for any $\xi \in \Sigma$. Hence, $\lim_t \hat{\Gamma}(H; f(\xi), t) = 0$ if and only if the answer to the above Π_2 questions in the $\mathcal{P}_e$ and $\mathcal{N}_e$ -strategy is “yes”.

3.3 The Construction

We now describe the whole construction. First, we need the priority tree T on which the construction proceeds.

3.3.1 The priority tree

For any node ξ on the priority tree, ξ is labeled $\mathcal{P}_e$ if $|\xi| = 2e$; ξ is labeled $\mathcal{N}_e$ if $|\xi| = 2e + 1$. We assign the order about the nodes on the priority tree as follows. For any two nodes ζ and ξ , we say ζ has higher priority than ξ if one of the following is satisfied: (1) $\zeta \subset \xi$; (2) there is some node η with $\eta^\wedge(o, \sigma) \subseteq \zeta$ and $\eta^\wedge(o', \sigma') \subseteq \xi$ such that $(o, \sigma) <_L (o', \sigma')$. If ζ has higher priority than ξ , we denote it as $\zeta < \xi$ or $\xi > \zeta$.

We use $\zeta \leq \xi$ to denote $\zeta = \xi$ or $\zeta < \xi$.

In the construction, we define $r_\infty^{(i,\sigma)}(\beta, -)$ and $C^{(i,\sigma)}(\beta, -)$ only for the node β labeled $\mathcal{N}_e$ for some e with $\beta^\wedge(i, \sigma)$ accessible at some stage in the construction. If we do not define $r_\infty^{(i,\sigma)}(\beta, s)$ particularly at the stage s , while $r_\infty^{(i,\sigma)}(\beta, s-1)$ converges and β can not be initialized at this stage, then $r_\infty^{(i,\sigma)}(\beta, s) = r_\infty^{(i,\sigma)}(\beta, s-1)$ naturally. For the counter function $C^{(i,\sigma)}(\beta, -)$, it satisfies the similar rule of $r_\infty^{(i,\sigma)}(\beta, -)$. For a_s , if we did not define a_s particularly at the stage s , then $a_s = a_{s-1}$. To initialize all the strategies associated with the node ξ at stage s means that: (1) if ξ is labeled $\mathcal{P}_e$, remove all numbers attached to the edge $\xi^\wedge(f, -)$, and cancel the label $\checkmark$ on the numbers attached to $\xi^\wedge(i, -)$; if link $\ell(\beta, \eta)$ is associated with (ξ, x) for any potential witness x , then cancel $\ell(\beta, \eta)$ and set $C^{(i,\sigma')}(\beta, s) = 0$, where $\beta^\wedge(i, \sigma') \subseteq \eta$; (2) if ξ is labeled $\mathcal{N}_e$, then cancel the number attached to it and cancel the number attached to the edge out of it, set $C^{(i,\sigma)}(\xi, s) = 0$; (3) let all the functions corresponding to ξ diverge except for the counter function.

3.3.2 The full construction

In the following, we give the full construction.

Stage 0: Let $\delta_0 = \lambda$ be the root node of the priority tree.

Stage $s > 0$: We divide this stage into some substages.

Substage 0: Let $\delta_s \upharpoonright 0 = \lambda$, and go to the next substage.

Substage $t + 1$: Assume that $\delta_s \upharpoonright t$ has been defined. If $t = s$, let $\delta_s = \delta_s \upharpoonright t$ and initialize all strategies associated with any node $> \delta_s$, then go to the next stage. If $t < s$, there are two cases:

Case 1: $\delta_s \upharpoonright t = \alpha$ is a $\mathcal{P}_e$ -strategy.

(1) If there is some $x \in \omega^{[e]}$ with $\varphi_{\alpha,s}(x) \downarrow = 0$ such that x has never been attached to any edge out of α , attach all such x to all the edges labeled $(i, -)$ leaving α which has been accessible.

(2) Calculate the outcomes (o, σ) . Remove all numbers attached to the edge $\alpha^\wedge(f, -) >_L \alpha^\wedge(o, \sigma)$ and cancel the label $(\checkmark)$ on the number attached to the edge $\alpha^\wedge(i, -) >_L \alpha^\wedge(o, \sigma)$. There are two cases depending on the outcome o .

($\alpha 1$) For $o = f$, there are two cases.

($\alpha 1.1$) If there is some number x attached to the edge leaving α with outcome (f, σ) and it has not been canceled until now, moreover, there is no $y \geq x$ in $\omega^{[e]}$ such that $\varphi_{\alpha, s}(y) \downarrow = 0$, then let t be the stage at which x was attached to the edge, $m = \max\{n : a_u(n) = 1 \text{ for } u < t\}$, then $x > t, m$, (*which can be seen from the choice of x in $\alpha 1.2$.*) We define $r(\alpha, s) = x$. Let $\delta_s \upharpoonright t + 1 = \alpha^\wedge(f, \sigma)$ and go to the next substage.

($\alpha 1.2$) Otherwise, find the least fresh number $x \in \omega^{[e]}$ greater than s with $a_{s-1}(x) = 0$ and attach x to the edge leaving α with outcome (f, σ) to replace the number (if it exists) attached to (f, σ) . Let $m = \max\{n : a_u(n) = 1 \text{ for } u < s\}$, then $x > s, m$. We define $r(\alpha, s) = x$. Let $\delta_s = \alpha$, $a_s = a_{s-1}$ and initialize all strategies associated with any node $\geq \alpha^\wedge(f, \sigma)$. Go to the next stage.

($\alpha 2$) For $o = i$, there are four cases.

($\alpha 2.1$) If there is some x attached to the edge leaving α with outcome (i, σ) with label $(\checkmark)$, then let t be the stage at which we labeled x with $(\checkmark)$. Let $m = \max\{n : a_u(n) = 1 \text{ for } u < t\}$. Define $r(\alpha, s)$ to be the maximal number of m, t and x . Let $\delta_s \upharpoonright t + 1 = \alpha^\wedge(i, \sigma)$. Go to the next substage.

($\alpha 2.2$) ($\alpha 2.1$) fails. If there is some x attached to the edge leaving α with outcome (i, σ) such that $\varphi_{\alpha, s}(x) = 0$ and $a_{s-1}(x) = 1$, then choose the largest such x and label it with $(\checkmark)$. Let $m = \max\{n : a_u(n) = 1 \text{ for } u < s\}$. Define $r(\alpha, s) = \max\{m, s, x\}$. Let $\delta_s = \alpha$, $a_s = a_{s-1}$, and initialize all strategies associated with any node $\geq \alpha^\wedge(i, \sigma)$. Go to the next stage.

($\alpha 2.3$) ($\alpha 2.1$) and ($\alpha 2.2$) fail. If for any x attached to the edge leaving α with outcome (i, σ) such that $a_{s-1}(x) = 0$, $x \leq R(\alpha, s) + 1$, where $R(\alpha, s) = \max(\{r(\xi, t) : t \leq s \text{ is an } \alpha \text{ stage with } \xi \subset \alpha\} \cup \{r(\xi, t) : t < s \text{ is a } \xi \text{ stage with } \xi <_L \alpha \text{ or is an } \eta^\wedge(o', \sigma') \text{ stage with } \xi \subset \eta^\wedge(o', \sigma') <_L \alpha^\wedge(i, \sigma)\} \cup \{r_\infty^{(i, \sigma')}(\beta, t) : t < s \text{ is a } \beta^\wedge(i, \sigma') \text{ stage with } \beta^\wedge(i, \sigma') <_L \alpha \text{ and } \beta \text{ is an } \mathcal{N}\text{-strategy}\})$, then let $\delta_s \upharpoonright t + 1 = \alpha^\wedge(i, \sigma)$ and define $r(\alpha, s)$ to be

the maximal stage at which some edge leaving α to the left of (i, σ) was accessible. Go to the next substage.

($\alpha 2.4$) ($\alpha 2.1$), ($\alpha 2.2$) and ($\alpha 2.3$) fail. There is some x attached to the edge leaving α with outcome (i, σ) such that $a_{s-1}(x) = 0$, $x > R(\alpha, s) + 1$. Choose the largest such x and label it with $(\checkmark)$. Let $m = \max\{n : a_u(n) = 1 \text{ for } u < s\}$. Define $r(\alpha, s) = \max\{m, s, x\}$. Let $\delta_s = \alpha$, initialize all strategies associated with any node $\geq \alpha^\wedge(i, \sigma)$. Let $d = 2^{-x} - \sum_{n > x} a_{s-1}(n)2^{-n}$. There are two different cases as follows.

($\alpha 2.4.1$) If there is some β labeled $\mathcal{N}_m$ for some m such that $\beta^\wedge(i, \sigma') \subseteq \alpha$ and $2^{-r_\infty^{(i, \sigma')}(\beta, s)} < d$, then find the longest such β and build a link $\ell(\beta, \alpha)$, we say this link is associated with (α, x) . Let $C^{(i, \sigma')}(\beta, s) = \lceil \frac{d}{2^{-r_\infty^{(i, \sigma')}(\beta, s)}} \rceil$, (for any real number y , $\lceil y \rceil$ means the least integer $m \geq y$). Go to the next stage.

($\alpha 2.4.2$) Otherwise, let $a_s = a_{s-1} + d$, (we say that α takes action associated with (α, x) at the stage s). Go to the next stage.

($\alpha 2.5$) If all of the above cases do not happen, then there is no $x \in \omega^{[e]}$ with $x < s$ such that $\varphi_{e,s}(x) \downarrow = 0$. Let $r(\alpha, s)$ be the maximal stage at which some edge leaving α to the left of (i, σ) was accessible. Let $\delta_s = \alpha$, and initialize all strategies associated with any node $\geq \alpha^\wedge(i, \sigma)$.

Case 2: $\delta_s \upharpoonright t = \beta$ is an $\mathcal{N}_e$ -strategy.

First, if s is a real β -expansionary stage, then attach it to the leftmost edge leaving β with outcome $(i, -)$ which has been visited before the stage s and has no number attached to it currently (if this kind of edge exists).

($\beta 1$) If $\mathcal{N}_e$ has already been permanently satisfied at some stage $u < s$ and it was not initialized at any stage $w > u$ with $w < s$, then calculate the outcome (o, σ) . If $o = i$, let $\delta_s = \beta$. We define $r(\beta, s)$ to be the number attached to β (see $\beta 2$). Go to the next stage and initialize all the strategies $>_L \beta^\wedge(i, \sigma)$ and cancel all numbers attached to edge $\beta^\wedge(i, \sigma') >_L \beta^\wedge(i, \sigma)$. If $o = f$, it is a part of the case ($\beta 3.1$).

($\beta 2$) If ($\beta 1$) fails and $\sum_{x \in W_{\beta,s}} 2^{-|x|} > a_{s-1}$, let $m = \max\{n : a_u(n) = 1 \text{ for } u < s\}$. We define y to be the least number greater than $\max\{m, s, -\log_2(\sum_{x \in W_{\beta,s}} 2^{-|x|} - a_{s-1})\}$ and attach y to the node β . Set $r(\beta, s) = y$. Declare that $\mathcal{N}_e$ is permanently satisfied unless it is initialized. Let $\delta_s = \beta$. Initialize all strategies associated with any node with lower priority than β , and cancel all numbers attached to edge $\beta^\wedge(i, \sigma')$. (*Note that if there is some link $\ell(\beta, \xi)$ which is associated with some (α, x) , it will be canceled, since $\beta \subset \alpha$ and α is initialized at the stage s . Hence after the stage s , by ($\beta 1$), there is no link $\ell(\beta, \xi)$ for any $\xi \supset \beta$ unless β is initialized*). Go to the next stage.

($\beta 3$) If ($\beta 1$), ($\beta 2$) fail, then $\sum_{x \in W_{\beta,s}} 2^{-|x|} \leq a_{s-1}$. We calculate the outcome (o, σ) . Cancel all numbers attached to the edge $\beta^\wedge(i, \sigma'') >_L \beta^\wedge(o, \sigma)$. There are two cases depending on the outcome o .

($\beta 3.1$) $o = f$. Let r be the maximal stage at which some edge leaving β to the left of (f, σ) was accessible. If β is permanently satisfied at a stage $v < s$ and it was not initialized between stages v and s , then define $r(\beta, s) = \max\{r, y\}$, where y is the number attached to β at the stage v . Otherwise, define $r(\beta, s) = r$. Let $\delta_s \upharpoonright t + 1 = \beta^\wedge(f, \sigma)$ and go to the next substage.

($\beta 3.2$) For $o = i$, there are two cases depending on the value of counter function $C^{(i, \sigma)}(\beta, s - 1)$.

If $C^{(i, \sigma)}(\beta, s - 1) = 0$. At first, if there is some link $\ell(\beta, \xi)$ with $\beta^\wedge(i, \sigma) \subseteq \xi$, then cancel link $\ell(\beta, \xi)$. Second, there are two cases as follows:

($\beta 3.2.1$) If there is a number attached to edge $\beta^\wedge(i, \sigma)$ currently, i.e. the outcome (i, σ) is confirmed, then cancel the number attached to the edge $\beta^\wedge(i, \sigma)$, and define $r_\infty^{(i, \sigma)}(\beta, s) = m(\beta, t')$, where t' is the last real β -expansionary stage $\leq s$. Let $r(\beta, s)$ be the maximal stage at which some edge leaving α to the left of (i, σ) was accessible. Let $\delta_s \upharpoonright t + 1 = \beta^\wedge(i, \sigma)$ and go to the next substage.

($\beta 3.2.2$) Otherwise, the outcome (i, σ) is not confirmed. Let $r(\beta, s)$ be the maximal stage at which some edge leaving α to the left of (i, σ) was accessible. Let

$\delta_s = \beta$. Initialize all the strategies associated with the nodes $>_L \beta^\wedge(i, \sigma)$ and go to the next stage.

If $C^{(i, \sigma)}(\beta, s-1) > 0$, then there is a link $\ell(\beta, \xi)$ for some $\xi \supseteq \beta^\wedge(i, \sigma)$. We assume that $\ell(\beta, \xi)$ is associated with (α, x) , where $\alpha \supseteq \xi$ is some $\mathcal{P}$ -strategy and x is some witness attached to some edge leaving α with outcome (i, σ') and there is some label $(\checkmark)$ on x . There are also three cases.

($\beta 3.2.3$) If there is no number attached to the edge $\beta^\wedge(i, \sigma)$ currently, then let $r(\beta, s)$ be the maximal stage at which some edge leaving α to the left of (i, σ) was accessible. Define $\delta_s = \beta$, and initialize all nodes $>_L \beta^\wedge(i, \sigma)$. Go to the next stage.

($\beta 3.2.4$) If there is a number attached to the edge leaving β with outcome (i, σ) currently, then cancel it. If $a_{s-1}(x) = 1$, then cancel all links $\ell(\beta', \eta)$ associated with (α, x) with $\beta'^\wedge(i, \sigma') \subseteq \eta \subseteq \alpha$, and set $C^{(i, \sigma')}(\beta', s) = 0$. Let $r(\beta, s)$ be the maximal stage at which some edge leaving β to the left of (i, σ) was accessible. Define $\delta_s = \beta$, and initialize all nodes $>_L \beta^\wedge(i, \sigma)$. Go to the next stage.

($\beta 3.2.5$) If there is a number attached to the edge leaving β with outcome (i, σ) currently, then cancel it. If $a_{s-1}(x) = 0$, let $r(\beta, s)$ be the maximal stage at which some edge leaving α to the left of (i, σ) was accessible. There are two cases as follows:

($\beta 3.2.5a$) If there is some $\mathcal{N}$ -strategy β' with $\beta'^\wedge(i, \sigma') \subseteq \beta$ such that $r_\infty^{(i, \sigma')}(\beta', s) > r_\infty^{(i, \sigma)}(\beta, s-1)$, then choose the longest one, β' say, and build a link $\ell(\beta', \beta)$, we say $\ell(\beta', \beta)$ is associated with (α, x) . Set $C^{(i, \sigma')}(\beta', s) = 2^{r_\infty^{(i, \sigma')}(\beta', s) - r_\infty^{(i, \sigma)}(\beta, s-1)}$ and $C^{(i, \sigma)}(\beta, s) = C^{(i, \sigma)}(\beta, s-1) - 1$. Let $\delta_s = \beta$, and initialize all nodes $>_L \beta^\wedge(i, \sigma)$. Go to the next stage.

($\beta 3.2.5b$) If there is no such $\mathcal{N}$ -strategy, define $d = 2^{-x} - \sum_{n > x} a_{s-1}(n) 2^{-n}$, and $a_s = a_{s-1} + \min\{d, 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)}\}$. Let $\delta_s = \beta$, and initialize all nodes $>_L \beta^\wedge(i, \sigma)$. $C^{(i, \sigma)}(\beta, s) = C^{(i, \sigma)}(\beta, s-1) - 1$. Go to the next stage. (In this case, we say β takes action associated with (α, x) at the stage s .)

This completes the full construction.

3.4 The Verification

We now prove that the construction produces a noncomputable simple left c.e. real a below H . Hence, we need to prove that all the requirements are satisfied, and $a \leq_T H$.

At first, we prove that $TP = \liminf_{s \rightarrow \infty} \delta_s$ is infinite. We call TP the *true path* of the construction. For any node ξ , assume that t is the least number such that for any $t' \geq t$, $\hat{\Gamma}(H; f(\xi), t') = \hat{\Gamma}(H; f(\xi), t)$, and $m = \max\{\gamma(H; f(\xi), s) : s \leq t\}$. We call $\sigma_\xi = H \upharpoonright m$ the *true use* to compute the $\lim_{s \rightarrow \infty} \Gamma(H; f(\xi), s)$.

Lemma 3.4. *If $\xi \in TP$, o_ξ is the outcome corresponding to $\lim_{s \rightarrow \infty} \hat{\Gamma}(H; f(\xi), s)$ and σ_ξ is the true use to compute this limit, then the followings are satisfied.*

($\xi 1$) $\xi^\wedge(o_\xi, \sigma_\xi) \in TP$;

($\xi 2$) $\xi^\wedge(o_\xi, \sigma_\xi)$ can only be initialized at most finitely many times.

Proof. We prove it by induction. First, we know λ , the root node, must be on the true path TP , and it is a $\mathcal{P}_0$ -strategy. We need to prove ($\lambda 1$) and ($\lambda 2$). Assume that s_0 is the first stage such that $\sigma_\lambda \subseteq H_{\lambda, s_0}$ and $s_0 \geq |\sigma_\lambda|$. By the construction and the using of the recursion theorem and the highness of H , any $\lambda^\wedge(o, \sigma') <_L \lambda^\wedge(o_\lambda, \sigma_\lambda)$ cannot be visited after the stage s_0 .

(a) If $o_\lambda = f$, by the highness of H and the use of the recursion theorem, there are only finitely many $x \in \omega^{[0]}$ such that $\varphi_\alpha(x) \downarrow = 0$. Let $x_1, x_2, \dots, x_n$ be all such numbers and s_1 be the least stage $\geq s_0$ such that $\varphi_{\alpha, s_1}(x_i) \downarrow$ for $i = 1, 2, \dots, n$. Assume s_2 is the least H_λ -true stage $> s_1$. (a1) If there is some number z attached to the edge leaving λ with outcome (f, σ_λ) and it has not been canceled until now, moreover, $z > x_i$ for any $i = 1, 2, \dots, n$, then this number z cannot be canceled later by the choice of the stage s_1 . Then any H_λ -true stage $\geq s_2$ must be a $\lambda^\wedge(f, \sigma_\lambda)$ -stage. Hence $\lambda^\wedge(f, \sigma_\lambda) \in TP$, and $\lambda^\wedge(f, \sigma_\lambda)$ cannot be initialized after the stage s_2 , i.e. ($\lambda 1$) and ($\lambda 2$) are true. (a2) Otherwise, attach the least fresh number z to this edge, and the stage s_2 is stopped at node λ . By the construction, this number z can never be canceled later. Then any H_λ -true stage after s_2 must be a $\lambda^\wedge(f, \sigma_\lambda)$ -stage. Hence

$\lambda^\wedge(f, \sigma_\lambda) \in TP$, and $\lambda^\wedge(f, \sigma_\lambda)$ cannot be initialized after stage s_2 , i.e. $(\lambda 1)$ and $(\lambda 2)$ are true.

(b) If $o_\lambda = i$, by the highness of H and the use of the recursion theorem, there are infinitely many $x \in \omega^{[0]}$ such that $\varphi_\alpha(x) \downarrow = 0$. (b1) If there is some stage $s_1 \geq s_0$, such that there is some number attached to the edge leaving λ with outcome (i, σ_λ) which can be labeled with $(\checkmark)$, then this label cannot be canceled later, then by the construction, any H_λ -true stage $> s_1$ must be a $\lambda^\wedge(i, \sigma_\lambda)$ -stage, and $\lambda^\wedge(i, \sigma_\lambda)$ cannot be initialized after stage s_1 . Hence $(\lambda 1)$ and $(\lambda 2)$ are true. (b2) Otherwise, by the construction, any H_λ -true stage $> s_0$ must be a $\lambda^\wedge(i, \sigma_\lambda)$ -stage, and $\lambda^\wedge(i, \sigma_\lambda)$ cannot be initialized after stage s_1 . Hence $(\lambda 1)$ and $(\lambda 2)$ are true.

In what follows, we assume $\xi \in TP$. Let ξ^- be the immediate predecessor of ξ and s_0 be a stage after which ξ cannot be initialized. Let $s_1 \geq s_0$ be the first stage such that $\sigma_\xi \subseteq H_{\xi, s_1}$ and $s_1 \geq |\sigma_\xi|$, then by the construction, the priority of the tree order and the induction hypothesis, any edge or node to the left of $\xi^\wedge(o_\xi, \sigma_\xi)$ cannot be visited after the stage s_1 . At any H_ξ -true ξ -stage $t \geq s_1$, $\xi^\wedge(o_\xi, \sigma_\xi)$ will be visited unless the definition of accessible nodes is cut short at ξ . As $\xi \in TP$, there are infinitely many H_ξ -true stages such that ξ can be accessible. In the following, we will prove that $\xi^\wedge(o_\xi, \sigma_\xi)$ can be accessible infinitely many times after the stage s_1 , and there is some stage $s_2 \geq s_1$ after which $\xi^\wedge(o_\xi, \sigma_\xi)$ can never be initialized. There are two cases depending on the strategy of ξ worked for.

Case 1. $\xi = \alpha$ is a $\mathcal{P}_e$ -strategy. It is similar to the proof of $\xi = \lambda$. We do not give the detail here.

Case 2. $\xi = \beta$ is an $\mathcal{N}_e$ -strategy.

- (a) If $o_\beta = f$, due to the highness of H and the use of the recursion theorem, there are only finitely many real β -expansionary stages. (a1) If $\mathcal{N}_e$ can be permanently satisfied at some stage $s_2 > s_1$, then any H_β -true β -stage $s > s_2$ is a $\beta^\wedge(f, \sigma_\beta)$ -stage, and after the stage s_2 , $\beta^\wedge(f, \sigma_\beta)$ can never be initialized, i.e. $(\beta 1)$ and $(\beta 2)$ are true. (a2) Otherwise, any H_β -true β -stage $s > s_1$ is a $\beta^\wedge(f, \sigma_\beta)$ -stage, and after the stage s_1 , $\beta^\wedge(f, \sigma_\beta)$ can never be initialized, i.e. $(\beta 1)$ and $(\beta 2)$ are true.
- (b) If $o_\beta = i$, due to the highness of H and the use of the recursion theorem, there are infinitely many real β -expansionary stages. We know that there are

only finitely many $(i, -) <_L (i, \sigma_\beta)$ which has been visited before the stage s_1 . Assume $(i, \sigma_1) <_L (i, \sigma_2) <_L \cdots <_L (i, \sigma_n)$ are all such edges leaving β which has been visited before the stage s_1 such that there is no number attached to them at the stage s_1 . All the edges can be attached to a number at some stage after s_1 , since there are infinitely many real β -expansionary stages. The number cannot be canceled once it is attached to such edge after the stage s_1 due to the choice of the stage s_1 . Let $s_2 > s_1$ be a β -stage such that all such edges has been attached a number, then all such number can never be canceled later. The edge leaving β with outcome (i, σ_β) has infinitely many chances to attach a number after any stage $t > s_2$, since there are infinitely many real β -expansionary stages. Hence the counter function $C^{(i, \sigma)}(\beta, -)$ has infinitely many chances to change to 0 after any stage $t > s_2$. Moreover, since there are infinitely many H_β -true β -stages, there are infinitely many $\beta^\wedge(i, \sigma_\beta)$ -stages. Hence $(\beta 1)$ is true. By the construction, we know $\beta^\wedge(i, \sigma_\beta)$ can never be initialized after the stage s_1 . Thus $(\beta 2)$ is true.

From the above discussion, the lemma is true. □

Lemma 3.5. *Let β be an $\mathcal{N}_e$ -strategy, s_0 be a stage after which $\beta^\wedge(i, \sigma)$ will never be initialized. Let $s > s_0$ be a β -stage at which β has a confirmed outcome (i, σ) , $C^{(i, \sigma)}(\beta, s-1) > 0$ and there is a link $\ell(\beta, \eta)$ associated with (α, x) at the stage s . If t is the next β -stage with confirmed outcome (i, σ) , $a_{t-1}(x) = 0$ and $C^{(i, \sigma)}(\beta, t-1) = C^{(i, \sigma)}(\beta, s-1) - 1$, then $a_{t-1} - a_{s-1} \geq 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)}$.*

Proof. There are two cases:

- Case 1: There is no β' with $\beta'^\wedge(i, \sigma') \subseteq \beta$ such that $r_\infty^{(i, \sigma')}(\beta', s) > r_\infty^{(i, \sigma)}(\beta, s-1)$. then $a_s = a_{s-1} + 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)}$. Otherwise, if $a_s = a_{s-1} + d$, where $d = 2^{-x} - \sum_{n > x} a_{s-1}(n)2^{-n} < 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)}$, then $a_s(x) = 1$. Furthermore, $a_{t-1}(x) = 1$, since any other strategy will not affect the n -th digit of a_s for $n < x+1$ until the stage t . It is a contradiction to $a_{t-1}(x) = 0$. Thus $a_{t-1} - a_{s-1} \geq 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)}$.
- Case 2: Otherwise, choose the longest such β' and create a link $\ell(\beta', \beta)$ which associated with (α, x) at the stage s . By the construction, $C^{(i, \sigma')}(\beta', s) = 2^{r_\infty^{(i, \sigma')}(\beta', s) - r_\infty^{(i, \sigma)}(\beta, s-1)}$. By induction, $a_{t-1} - a_{s-1} \geq C^{(i, \sigma')}(\beta', s) * 2^{-r_\infty^{(i, \sigma')}(\beta', s)} = 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)}$.

□

Lemma 3.6. *For any ζ on the true path and any $\mathcal{P}$ -strategy α with $\alpha^\wedge(i, \sigma) \subseteq \zeta$, $\lim_{i \rightarrow \infty} R(\alpha, v_j) < \infty$, where $\{v_j\}_{j \in \omega}$ is the subsequence of stages at which ζ is accessible.*

Proof. First, we know that the strategy to the left of $\alpha^\wedge(i, \sigma)$ will not be visited after some stage t which is big enough. We only need to consider the restraint coming from the node $\xi \subset \alpha$. Assume that after some stage $s_0 \geq t$, α cannot be initialized.

(1) If ξ is a $\mathcal{P}$ -strategy, there are two cases.

(1.1) $\xi^\wedge(f, \sigma) \subseteq \alpha$. By the construction, there is some number x which can be attached to the edge leaving ξ with outcome (f, σ) at some stage $s_1 \geq s_0$, and it cannot be canceled later. Thus, for any $\xi^\wedge(f, \sigma)$ -stage $s > s_1$, $r(\xi, s) = x$.

(1.2) $\xi^\wedge(i, \sigma) \subseteq \alpha$, there are two subcases.

- a. If there is no number x can be labeled with $\checkmark$ at any stage $u > s_0$, then for any $\xi^\wedge(i, \sigma)$ -stage $s > s_0$, $r(\xi, s)$ is the maximal stage at which some edge leaving ξ to the left of (i, σ) was visited. It is a constant.
- b. If there is some number x attached to the edge leaving ξ with outcome (i, σ) which can be labeled at a stage $u > s_0$, then this label can never be canceled. Thus, $r(\xi, s) = \max\{m, u, x\}$ for any $\xi^\wedge(i, \sigma)$ -stage $s > u$, where $m = \max\{n : a_v(n) = 1 \text{ for } v \leq u\}$.

(2) If ξ is an $\mathcal{N}$ -strategy, there are two cases:

(2.1) If $\xi^\wedge(f, \sigma) \subseteq \alpha$, there are two subcases.

- a. If there is some stage $w > s_0$ at which ξ is satisfied, then we will attach some number y to ξ at the stage w . Assume that r is the maximal stage at which some edge leaving ξ to the left of (f, σ) was visited, then $r(\xi, s) = \max\{r, y\}$ for any $\beta^\wedge(f, \sigma)$ -stage $s > w$.
- b. Otherwise, $r(\xi, s) = r$ for any $\beta^\wedge(f, \sigma)$ -stage $s > s_0$.

(2.2) If $\xi^\wedge(i, \sigma) \subseteq \alpha$, by the construction, $r(\xi, s)$ is the maximal stage at which some edge leaving ξ to the left of (i, σ) was visited at any $\beta^\wedge(i, \sigma)$ -stage $s > s_0$.

By the above discussion, the lemma holds. □

Lemma 3.7. *Assume that α is a $\mathcal{P}_e$ -strategy and $\alpha^\wedge(i, \sigma) \in TP$. Suppose that s is an $\alpha^\wedge(i, \sigma)$ -stage after which $\alpha^\wedge(i, \sigma)$ cannot be initialized, and there is some number x attached to the edge leaving α with (i, σ) which is labeled at the stage s with $a_{s-1}(x) = 0$. Assume that s' is the next $\alpha^\wedge(i, \sigma)$ -stage, then $a_{s'}(x) = 1$. Furthermore, $a(x) = 1$.*

Proof. By the construction $r(\alpha, s) = \max\{m, s, x\}$ and $a_{s-1}(x) = 0$, where $m = \max\{n : a_u(n) = 1 \text{ for } u < s\}$ and $s > x$ by convention. By the construction, any $\alpha' >_L \alpha$ is initialized at the stage s . If there is some strategy which can take action associated with (α', y) for $\alpha' \neq \alpha$ between stages s and s' , then $\alpha' >_L \alpha$, and so y must be chosen after the stage s and $y > r(\alpha, s) + 1$. Let $s_1 < s_2 < \dots < s_m$ be all the stages with $s < s_k \leq s'$ such that there is some strategy ξ_k takes action related to (α_k, x_k) at the stage s_k , then $x_k > r(\alpha, s) + 1 > x$ if $\alpha_k \neq \alpha$, and $x_k = x$ if $\alpha_k = \alpha$. By the construction, $a_{s_1-1} = a_s$ since any strategy can not take action at any stage v with $s < v < s_1$ by the choice of s_1 . The action at the stage s_1 does not affect the n -th digit of a_{s_1-1} for $n < x_1$, i.e. $a_{s_1-1}(n) = a_{s_1}(n)$ for $n < x_1$, and hence $a_s(n) = a_{s_1}(n)$ for $n < x_1$. Furthermore, $a_{s_2-1} = a_{s_1}$, and the action at the stage s_2 does not affect the n -th digit of a_{s_2-1} for $n < x_2$, i.e. $a_{s_2-1}(n) = a_{s_2}(n)$ for $n < x_2$, and hence $a_{s_1}(n) = a_{s_2}(n)$ for $n < x_2$. So $a_{s_2}(n) = a_s(n)$ for any $n < \min\{x_1, x_2\}$. By using the same argument $a_{s_m}(n) = a_s(n)$ for any $n < \min\{x_1, x_2, \dots, x_m\}$. Since $a_{s'-1} = a_{s_m}$, then $a_{s'-1}(n) = a_s(n)$ for any $n < \min\{x_1, x_2, \dots, x_m\}$. Since $x \leq \min\{x_1, x_2, \dots, x_m\}$, then $a_{s'-1}(n) = a_s(n)$ for any $n < x$. Let $d = 2^{-x} - \sum_{n>x} a_{s-1}(n)2^{-n}$. There are two cases.

Case 1 : If for any $\mathcal{N}$ -strategy β' with $\beta'^\wedge(i, \sigma') \subseteq \alpha$, $d \leq 2^{-r_\infty^{(i, \sigma')}(\beta', s)}$, then $a_s = a_{s-1} + d$, and so $a_s(x) = 1$. Since $\{a_v\}_{v \in \omega}$ is increasing, and $a_{s'-1}(n) = a_s(n)$ for any $n < x$, hence $a_{s'-1}(x) = 1$.

Case 2 : Otherwise, if there is some $\mathcal{N}$ -strategy β' with $\beta'^\wedge(i, \sigma') \subseteq \alpha$ such that $d > 2^{-r_\infty^{(i, \sigma')}(\beta', s)}$, then the stage s stopped at node α and $a_s = a_{s-1}$. Let β' be the longest such $\mathcal{N}$ -strategy, then $C^{(i, \sigma')}(\beta', s) = \lceil \frac{d}{2^{-r_\infty^{(i, \sigma')}(\beta', s)}} \rceil$. If $a_{s'}(x) = 0$, then $a_v(x) = 0$ for $s-1 \leq v \leq s'$. There must be some β' -stage t with $s-1 < t < s'$ at which the outcome (i, σ') of β' is confirmed such that $C^{(i, \sigma')}(\beta', t -$

1) = 1. (Otherwise, there must be some stage v with $s \leq v \leq s'$ such that $C^{(i,\sigma')}(\beta', v) = 0$, $C^{(i,\sigma')}(\beta', v-1) > 1$ and it must be obtained from $\beta 3.2.4$ in the full construction. Then $a_{v-1}(x) = 1$. Since $a_{s-1}(n) = a_{s'}(n)$ for $n < x$, and $\{a_v\}$ is an increasing sequence, $a_{s'}(x) = 1$.) By the construction and Lemma 3.5, $a_{t-1} - a_{s-1} \geq (C^{(i,\sigma')}(\beta', s) - 1)2^{-r_{\infty}^{(i,\sigma')}(\beta', s)}$. Since $a_{t-1}(n) = a_{s-1}(n)$ for $n \leq x$, $\sum_{n>x} a_{t-1}(n)2^{-n} - \sum_{n>x} a_{s-1}(n)2^{-n} = a_{t-1} - a_{s-1} \geq (C^{(i,\sigma')}(\beta', s) - 1)2^{-r_{\infty}^{(i,\sigma')}(\beta', s)}$. Then $\sum_{n>x} a_{t-1}(n)2^{-n} \geq \sum_{n>x} a_{s-1}(n)2^{-n} + (C^{(i,\sigma')}(\beta', s) - 1)2^{-r_{\infty}^{(i,\sigma')}(\beta', s)}$. Let $d' = 2^{-x} - \sum_{n>x} a_{t-1}(n)2^{-n}$, then $d' \leq 2^{-x} - [\sum_{n>x} a_{s-1}(n)2^{-n} + (C^{(i,\sigma')}(\beta', s) - 1)2^{-r_{\infty}^{(i,\sigma')}(\beta', s)}] = d - (C^{(i,\sigma')}(\beta', s) - 1)2^{-r_{\infty}^{(i,\sigma')}(\beta', s)} \leq 2^{-r_{\infty}^{(i,\sigma')}(\beta', s)} = 2^{-r_{\infty}^{(i,\sigma')}(\beta', t)}$. If for any other $\mathcal{N}$ -strategy β'' with $\beta'' \hat{\ } (i, \sigma'') \subseteq \beta'$, $r_{\infty}^{(i,\sigma'')}(\beta'', t) \leq r_{\infty}^{(i,\sigma')}(\beta', t)$, then $a_t = a_{t-1} + d'$. Hence $a_t(x) = 1$, it is a contradiction. Otherwise, if there is some $\mathcal{N}$ -strategy β'' with $\beta'' \hat{\ } (i, \sigma'') \subseteq \beta'$, $r_{\infty}^{(i,\sigma'')}(\beta'', t) > r_{\infty}^{(i,\sigma')}(\beta', t)$, then choose the longest such $\mathcal{N}$ -strategy, β_1 say, and the corresponding outcome, (i, σ_1) say. We can use the similar proof as above to prove that there must be some β_1 -stage t_1 with $t-1 < t_1 < s'$ at which the outcome (i, σ_1) of β_1 is confirmed such that $C^{(i,\sigma_1)}(\beta_1, t_1 - 1) = 1$, and $d_1 = 2^{-x} - \sum_{n>x} a_{t_1-1}(n)2^{-n} \leq 2^{-r_{\infty}^{(i,\sigma_1)}(\beta_1, t)} = 2^{-r_{\infty}^{(i,\sigma_1)}(\beta_1, t_1)}$. By using the similar argument, we can find a list of stages $t_1 < t_2 < \dots < t_k$ with $t-1 < t_m < s'$ for $m = 1, 2, \dots, k$, a list of $\mathcal{N}$ -strategies $\beta_1 \supset \beta_2 \supset \dots \supset \beta_k$ with $\beta' \supseteq \beta_1 \hat{\ } (i, \sigma_1) \supset \beta_2 \hat{\ } (i, \sigma_2) \supset \dots \supset \beta_k \hat{\ } (i, \sigma_k)$ and a list of $d_1 > d_2 > \dots > d_k$ such that $d_m = 2^{-x} - \sum_{n>x} a_{t_m-1}(n)2^{-n} \leq 2^{-r_{\infty}^{(i,\sigma_m)}(\beta_m, t_m)}$, and for any $\mathcal{N}$ -strategy β'' with $\beta'' \hat{\ } (i, \sigma'') \subseteq \beta_k$, $r_{\infty}^{(i,\sigma'')}(\beta'', t_k) \leq r_{\infty}^{(i,\sigma_k)}(\beta_k, t_k)$. Hence $a_{t_k} = a_{t_k-1} + d_k$, $a_{t_k}(x) = 1$. Since $t_k \leq s'$, $a_{s'}(x) = 1$, this is a contradiction.

By using the similar argument, any actions associated with (α', x') after the stage s' cannot affect the x -th digit of a since $x' > x$. Then $a(x) = 1$. □

Lemma 3.8. *For any e , the $\mathcal{P}_e$ -requirement is satisfied.*

Proof. Fix an e . Let $\alpha \in TP$ be the node labeled $\mathcal{P}_e$ and $\alpha \hat{\ } (o, \sigma) \in TP$. There are two cases depending on the outcome of o .

(1) If $o = f$, by Lemma 3.4, there is a least stage t_0 such that $\alpha \hat{\ } (f, \sigma)$ cannot be initialized after the stage t_0 . By the highness of H and the use of the recursion theorem, there are only finitely many $x \in \omega^{[e]}$ such that $\varphi_{\alpha}(x) \downarrow = 0$. By Lemma 3.4,

there is a fixed number $x \in \omega^{[el]}$ which is attached permanently to the edge leaving α with outcome (f, σ) and $\varphi_\alpha(x) \uparrow$ or $\varphi_\alpha(x) \downarrow \neq 0$. If $\varphi_\alpha(x) \uparrow$ or $\varphi_\alpha(x) \downarrow \neq 0, 1$, then $\varphi_\alpha(x) \neq a(x)$. If $\varphi_\alpha(x) \downarrow = 1$, we will prove that $a(x) = 0$. Assume that x is attached to the edge $\alpha^\wedge(f, \sigma)$ at a stage $t > t_0$, then $a_t(x) = a_{t-1}(x) = 0$. By the restraint function given in the construction, any witness z chosen by $\alpha' \geq \alpha^\wedge(i, \sigma)$ at some stage $s' > t$ such that some strategy can take action associated with (α', z) must be bigger than x . By using the similar argument like in the Lemma 3.7, $a(x) = a_t(x) = 0$.

(2) If $o = i$, by Lemma 3.4, there is a least stage t_0 such that $\alpha^\wedge(i, \sigma)$ cannot be initialized after the stage t_0 . By the highness of H and the use of the recursion theorem, there are infinitely many $x \in \omega^{[el]}$ such that $\varphi_\alpha(x) \downarrow = 0$. By Lemma 3.6, $\lim_{j \rightarrow \infty} R(\alpha, v_j)$ exists, where $\{v_j\}_{j \in \omega}$ is the subsequence of stages at which $\alpha^\wedge(i, \sigma)$ is accessible. Then there is a stage $v_k \geq t_0$ at which a number $y > \lim_{j \rightarrow \infty} R(\alpha, v_j) + 1$ is attached to the edge $\alpha^\wedge(i, \sigma)$.

(2a) If there is some stage s with $t_0 \leq s \leq v_k$ at which some x attached to the edge $\alpha^\wedge(i, \sigma)$ was labeled with $\checkmark$. If $a_{s-1}(x) = 1$, then by using the similar argument in Lemma 3.7, we can prove $a(x) = 1$. If $a_{s-1}(x) = 0$, then by Lemma 3.7, $a(x) = 1$.

(2b) (2a) fails, if there is some x attached to the edge $\alpha^\wedge(i, \sigma)$ such that $a_{v_k-1}(x) = 1$, then choose x as the largest one at the stage v_k , by using the similar argument like in Lemma 3.7, $a(x) = 1$. Otherwise, we choose the largest x at the stage v_k such that $x > \lim_{j \rightarrow \infty} R(\alpha, v_j) + 1$ and labeled it with $(\checkmark)$ at the stage v_k , then $a_{v_k-1}(x) = 0$, and hence by Lemma 3.7, $a(x) = 1$.

From the above discussion, the $\mathcal{P}_e$ -requirement is satisfied. $\square$

Lemma 3.9. *Let β be an $\mathcal{N}$ -strategy, s_0 be a stage after which $\beta^\wedge(i, \sigma)$ cannot be initialized. Assume $u > s_0$ is a β -stage with confirmed outcome (i, σ) and $v > u$ is the next β -stage with confirmed outcome (i, σ) . Then $a_{v-1} - a_{u-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)+1}$.*

Proof. There are two cases as follows:

Case 1: If $C^{(i, \sigma)}(\beta, u-1) > 0$, then there is a link $\ell(\beta, \eta)$ which is associated with (α, x) , where $\beta^\wedge(i, \sigma) \subseteq \eta \subseteq \alpha$. There are two subcases.

Subcase 1.1 : If $a_{u-1}(x) = 1$, then $a_u = a_{u-1}$, and we cancel all links associated with (α, x) and reset all corresponding counter function as 0. If there is some

strategy can take action associated with some (α', x') between stages u and v , then $\alpha' >_L \beta^\wedge(i, \sigma)$, and so $x' > r_\infty^{(i, \sigma)}(\beta, u-1) + 1$. Hence all the actions between stages u and v cannot affect the n -th digit of a_u for $n \leq r_\infty^{(i, \sigma)}(\beta, u-1) + 1$. Hence $a_{v-1} - a_{u-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)+1}$.

Subcase 1.2 : If $a_{u-1}(x) = 0$.

1.2a : If for any $\mathcal{N}$ -strategy β' with $\beta'^\wedge(i, \sigma') \subseteq \beta$, $r_\infty^{(i, \sigma')}(\beta', u) \leq r_\infty^{(i, \sigma)}(\beta, u-1)$.

By the construction $a_u = a_{u-1} + \min\{d', 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)}\}$, where $d' = 2^{-x} - \sum_{n > x} a_{u-1}(n)2^{-n}$, there is no any other strategy can take actions associated with (α, x) until the stage v . If there is some strategy which can take action associated with (α', x') until the stage v , then $x' > r_\infty^{(i, \sigma)}(\beta, u) + 1$, then all the action cannot affect the n -th digit of a_u for $n \leq r_\infty^{(i, \sigma)}(\beta, u-1) + 1$. $a_{v-1} - a_u < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)}$. Hence $a_{v-1} - a_{u-1} \leq a_{v-1} - a_u + a_u - a_{u-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)} + 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)} = 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)+1}$.

1.2b : Otherwise, choose the longest $\mathcal{N}$ -strategy, β_0 say, with $\beta_0^\wedge(i, \sigma_0) \subseteq \beta$ such that $r_\infty^{(i, \sigma_0)}(\beta_0, u) > r_\infty^{(i, \sigma)}(\beta, u-1)$. Then link $\ell(\beta_0, \beta)$ which is associated with (α, x) is created at the stage $s_0 = u$. $C^{(i, \sigma_0)}(\beta_0, u) = 2^{r_\infty^{(i, \sigma_0)}(\beta_0, u) - r_\infty^{(i, \sigma)}(\beta, u-1)}$. Denote $C_0 = C^{(i, \sigma_0)}(\beta_0, u)$, and $C^{(i, \sigma)}(\beta, u) = C^{(i, \sigma)}(\beta, u-1) - 1$.

Let s_1 be the first stage with $v > s_1 \geq s_0$ satisfying that s_1 is a β_1 -stage with confirmed outcome (i, σ_1) for some β_1 with $\beta_1^\wedge(i, \sigma_1) \subseteq \beta$ such that $r_\infty^{(i, \sigma_1)}(\beta_1, s_1) > r_\infty^{(i, \sigma_0)}(\beta_0, s_0)$ and some link $\ell(\beta_1, \eta_1)$ associated with (α, x) is created at the end of the stage s_1 . Then the total increment due to strategies associated with (α', x') for $\alpha' >_L \alpha$ is less than $2^{-r_\infty^{(i, \sigma_0)}(\beta_0, s_0)-1}$ between stages u and $s_1 + 1$.

Let s_2 be the first stage with $v > s_2 \geq s_1$ satisfying that s_2 is a β_2 -stage with confirmed outcome (i, σ_2) for some β_2 with $\beta_2^\wedge(i, \sigma_2) \subseteq \beta$ such that $r_\infty^{(i, \sigma_2)}(\beta_2, s_2) > r_\infty^{(i, \sigma_1)}(\beta_1, s_1)$ and some link $\ell(\beta_2, \eta_2)$ associated with (α, x) is created at the end of the stage s_2 . Then the total increment due to strategies associated with (α', x') for $\alpha' >_L \alpha$ is less than $2^{-r_\infty^{(i, \sigma_1)}(\beta_1, s_1)-1}$ between stages s_1 and $s_2 + 1$.

We continue the process. Let s_k be the first stage with $v > s_k \geq s_{k-1}$ satisfying that s_k is a β_k -stage with confirmed outcome (i, σ_k) for some

β_k with $\beta_k \widehat{\cap} (i, \sigma_k) \subseteq \beta$ such that $r_\infty^{(i, \sigma_k)}(\beta_k, s_k) > r_\infty^{(i, \sigma_{k-1})}(\beta_{k-1}, s_{k-1})$ and some link $\ell(\beta_k, \eta_k)$ associated with (α, x) is created at the end of the stage s_k . Then the total increment due to strategies associated with (α', x') for $\alpha' >_L \alpha$ is less than $2^{-r_\infty^{(i, \sigma_{k-1})}(\beta_{k-1}, s_{k-1})-1}$ between stages s_{k-1} and $s_k + 1$.

Assume there is no stage s' with $v > s' \geq s_k$ such that s' is some β' -stage with confirmed outcome (i, σ') for some β' with $\beta' \widehat{\cap} (i, \sigma') \subseteq \beta$, $r_\infty^{(i, \sigma')}(\beta', s') > r_\infty^{(i, \sigma_k)}(\beta_k, s_k)$ and some link $\ell(\beta', \eta')$ associated with (α, x) is created at the end of the stage s' .

(1.2b-i) If there is some β_k -stage t which is between stages s_k and v with confirmed outcome (i, σ_k) at which β_k takes action associated with (α, x) to add value d' to a_t , i.e. $a_t = a_{t-1} + d'$, for $d' = 2^{-x} + \sum_{n>x} a_{t-1}(n)2^{-n} < 2^{-r_\infty^{(i, \sigma_k)}(\beta_k, s_k)}$, then the increment due to strategies associated with (α, x) is less than $2^{-r_\infty^{(i, \sigma)}(\beta, u-1)}$ between stages u and v , and the total increment due to strategies associated with (α', x') for $\alpha' >_L \alpha$ is less than $2^{-r_\infty^{(i, \sigma_k)}(\beta_k, s_k)-1}$ between stages s_k and $t + 1$, and the total increment due to strategies associated with (α', x') for $\alpha' >_L \alpha$ is less than $2^{-r_\infty^{(i, \sigma_k)}(\beta_k, s_k)-1}$ between stages t and v . By the above discussion, the total increment due to strategies associated with (α', x') for $\alpha' >_L \alpha$ is less than $2^{-r_\infty^{(i, \sigma_0)}(\beta_0, s_0)-1} + 2^{-r_\infty^{(i, \sigma_1)}(\beta_1, s_1)-1} + \dots + 2^{-r_\infty^{(i, \sigma_{k-1})}(\beta_{k-1}, s_{k-1})-1} + 2^{-r_\infty^{(i, \sigma_k)}(\beta_k, s_k)-1} + 2^{-r_\infty^{(i, \sigma_k)}(\beta_k, s_k)-1} \leq 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)-1}$ between stages u and v , since $r_\infty^{(i, \sigma)}(\beta, u-1) < r_\infty^{(i, \sigma_0)}(\beta_0, s_0) < r_\infty^{(i, \sigma_1)}(\beta_1, s_1) < \dots < r_\infty^{(i, \sigma_k)}(\beta_k, s_k)$. Hence $a_{v-1} - a_{u-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)} + 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)+1}$.

(1.2b-ii) Otherwise, the increment due to strategies associated with (α, x) is less than or equal to $2^{-r_\infty^{(i, \sigma)}(\beta, u-1)}$ between stages u and v , and the total increment due to strategies associated with (α', x') for $\alpha' >_L \alpha$ is less than $2^{-r_\infty^{(i, \sigma_k)}(\beta_k, s_k)-1}$ between stages s_k and v . By the above discussion, the total increment due to strategies associated with (α', x') for $\alpha' >_L \alpha$ is less than $2^{-r_\infty^{(i, \sigma_0)}(\beta_0, s_0)-1} + 2^{-r_\infty^{(i, \sigma_1)}(\beta_1, s_1)-1} + \dots + 2^{-r_\infty^{(i, \sigma_{k-1})}(\beta_{k-1}, s_{k-1})-1} + 2^{-r_\infty^{(i, \sigma_k)}(\beta_k, s_k)-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)-1}$ between stages u and v , since $r_\infty^{(i, \sigma)}(\beta, u-1) < r_\infty^{(i, \sigma_0)}(\beta_0, s_0) < r_\infty^{(i, \sigma_1)}(\beta_1, s_1) < \dots < r_\infty^{(i, \sigma_k)}(\beta_k, s_k)$. Hence $a_{v-1} - a_{u-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)} + 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)+1}$.

Case 2: $C^{(i, \sigma)}(\beta, u-1) = 0$. There are two subcases.

Subcase 2.1: If there is no link $\ell(\beta', \eta)$ which is associated with (α, x) is created at the stage u for any $\beta' \subseteq \beta$ with $\beta'^\wedge(i, \sigma') \subseteq \beta^\wedge(i, \sigma)$, then there is no action associated with (α, x) which can be taken between stages u and v . Thus for any actions between stages u and v associated with (α', x') , $\alpha' >_L \beta^\wedge(i, \sigma)$, by the initialization $x' > r_\infty^{(i, \sigma)}(\beta, u) + 1$. Hence any actions between stages u and v cannot affect the n -th digit for $n \leq r_\infty^{(i, \sigma)}(\beta, u) + 1$, $a_{v-1} - a_u < 2^{-r_\infty^{(i, \sigma)}(\beta, u)-1}$. It is easy to see $a_u - a_{u-1} \leq 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)}$. Thus $a_{v-1} - a_{u-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)+1}$.

Subcase 2.2: Otherwise, there is some $\beta' \subseteq \beta$ with $\beta'^\wedge(i, \sigma') \subseteq \beta^\wedge(i, \sigma)$ such that a link $\ell(\beta', \eta)$ associated with some (α, x) is created at the stage u . If $\beta' = \beta$, it is easy to see that the result is true. If $\beta' \subset \beta$, we can use the similar argument in 1.2b to prove $a_{v-1} - a_{u-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)+1}$.

□

Lemma 3.10. *For any e , the $\mathcal{N}_e$ -requirement is satisfied.*

Proof. Fix an e , assume that $\beta \in TP$ is labeled $\mathcal{N}_e$ and $\beta^\wedge(o, \sigma) \in TP$. In the following, we will divide it into two cases to prove the lemma depending on the outcome of o .

(1) If $o = f$, by the highness of H and the use of the recursion theorem, there are only finitely many real β -expansionary stages. Hence $a \neq \sum_{x \in W_e} 2^{-|x|}$, then $\mathcal{N}_e$ is satisfied.

(2) If $o = i$, by the highness of H and the use of the recursion theorem, there are infinitely many real β -expansionary stages. Assume that s_0 is the least stage after which $\beta^\wedge(i, \sigma)$ cannot be initialized and for any $(i, \sigma') <_L (i, \sigma)$ leaving β which has been visited, there is some number attached to it before or at the stage s_0 and this number is not canceled until the stage s_0 .

Fix β , by the construction $r_\infty^{(i, \sigma)}(\beta, s)$ is increasing. For any y , find the least apparent β -expansionary stage $s > s_0$ such that $|y| \leq r_\infty^{(i, \sigma)}(\beta, s-1) - 3$ and $\beta^\wedge(i, \sigma)$ is visited at the stage s . Let $t_0 < t_1 < t_2 < \dots < t_n < \dots$ be all of the real β -expansionary stages after the stage s . We will prove that $y \in W_\beta$ if and only if $y \in W_{\beta, t_0}$.

At first, we will prove that for any $j \in \omega$, $a_{t_{j+1}-1} - a_{t_j-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, s)+2}$. Let u be the maximal β -stage $< t_j$ with a confirmed outcome (i, σ) , then $u \geq s$ and there is

no number attached to any $(i, \sigma') \geq_L (i, \sigma)$ at the end of the stage u . Let $v > u$ be the next β -stage with confirmed outcome (i, σ) and $w > v$ be the next β -stage with confirmed outcome (i, σ) . Then $w \geq t_{j+1}$.

By Lemma 3.9, $a_{v-1} - a_{u-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)+1}$ and $a_{w-1} - a_{v-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, v-1)+1}$. Hence $a_{t_{j+1}-1} - a_{t_j-1} \leq a_{w-1} - a_{u-1} < 2^{-r_\infty^{(i, \sigma)}(\beta, u-1)+2} \leq 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)+2}$.

Assume that $y \in W_{\beta, t_{j+1}} - W_{\beta, t_j}$, then $\sum_{x \in W_{\beta, t_{j+1}}} 2^{-|x|} \geq \sum_{x \in W_{\beta, t_j}} 2^{-|x|} + 2^{-|y|} \geq \sum_{x \in W_{\beta, t_j}} 2^{-|x|} + 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)+3} > \sum_{x \in W_{\beta, t_j}} 2^{-|x|} + 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)} + 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)+2} \geq a_{t_j-1} + 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)+2} > a_{t_{j+1}-1}$, thus $\sum_{x \in W_{\beta, t_{j+1}}} 2^{-|x|} > a_{t_{j+1}-1}$. Let $t \leq s-1$ be the stage such that $r_\infty^{(i, \sigma)}(\beta, s-1) = m(\beta, t)$. As $t_j > s-1 \geq t$ is a real β -expansionary stage, $r_\infty^{(i, \sigma)}(\beta, s-1) = m(\beta, t) \leq m(\beta, t_j)$ and $a_{t_j-1} - \sum_{x \in W_{\beta, t_j}} 2^{-|x|} \leq 2^{-m(\beta, t_j)} \leq 2^{-r_\infty^{(i, \sigma)}(\beta, s-1)}$, the fourth inequality is correct. By the above discussion, the fifth equality is correct. Hence t_{j+1} is not a real β -expansionary stage. It is a contradiction. Thus W_β is computable. $\square$

Lemma 3.11. *Let $A = \{\xi : (\exists \zeta \in TP)[\xi \leq \zeta] \ \& \ \xi \text{ can be visited at some stage}\}$, then A is c.e. in H . For any $\xi \in A$, we can H -enumerate one stage s_ξ such that after which no node or edge to the left of ξ can be accessible.*

Proof. (1) We construct a set B such that B is c.e. in H and $B = A$. At first, we enumerate λ into B . At the stage $s+1$, if $\xi^\wedge(o, \sigma)$ is accessible and $\xi \in B_s$, then H can check whether $\sigma \subseteq H$ or not. If $\sigma \subset H$, then we enumerate $\xi^\wedge(o, \sigma)$ and all nodes to the left of $\xi^\wedge(o, \sigma)$ which has been accessible before the stage s into B . Otherwise, do nothing. It is easy to see that B is c.e. in H . In the following, we prove that $B = A$.

By the construction of B , any node on the true path or to the left of the true path is in B . Thus $A \subseteq B$.

In the following, we will prove that the following property $(*)$ is true for any $\eta^\wedge(o, \sigma) \in TP$ inductively:

$(*)$: for any $\eta^\wedge(o', \sigma') >_L \eta^\wedge(o, \sigma)$, $\eta^\wedge(o', \sigma') \notin B$.

(a) Assume that $\lambda^\wedge(o, \sigma) \in TP$, $\lambda^\wedge(o', \sigma') >_L \lambda^\wedge(o, \sigma)$. If $\lambda^\wedge(o', \sigma') \in B$, then there must be a stage s and an outcome (o'', σ'') such that $\lambda^\wedge(o'', \sigma'')$ is visited at the stage s , $\lambda^\wedge(o'', \sigma'') >_L \lambda^\wedge(o', \sigma')$ or $\lambda^\wedge(o'', \sigma'') = \lambda^\wedge(o', \sigma')$ and $\sigma'' \subseteq H_s$ & $\sigma'' \subseteq H$. Hence $\lambda^\wedge(o'', \sigma'') >_L \lambda^\wedge(o, \sigma)$. By the definition of priority of edges and Lemma

3.4, $\sigma \subseteq \sigma''$ and there is some y between t_σ and $t_{\sigma''}$ such that $\hat{\Gamma}(\sigma; f(\lambda), t_\sigma)[[\sigma]] \neq \hat{\Gamma}(\sigma''; f(\lambda), y)[[\sigma'']]$, then at t_σ , the limit has not been reached, then $\lambda^\wedge(o, \sigma) \notin TP$, it is a contradiction.

(b) Assume that $\eta \in TP$, and for any $\xi \subset \eta$, the above property $(*)$ is true. Suppose $\eta^\wedge(o, \sigma) \in TP$, $\eta^\wedge(o', \sigma') >_L \eta^\wedge(o, \sigma)$, we will prove that $\eta^\wedge(o', \sigma') \notin B$. By the construction of B , if $\eta^\wedge(o', \sigma') \in B$, then there must be some $\sigma'' \subseteq H$ such that $\eta^\wedge(o'', \sigma'') >_L \eta^\wedge(o', \sigma')$ or $\eta^\wedge(o'', \sigma'') = \eta^\wedge(o', \sigma')$, then $\eta^\wedge(o'', \sigma'') >_L \eta^\wedge(o, \sigma)$ naturally. We can use the similar way as in (a) to prove that it is not possible. Thus if $\zeta \notin A$, then $\zeta \notin B$.

From the above discussion, $A = B$. Thus, A is c.e. in H .

(2) Assume that $\xi \in A$ and the corresponding stage is s_ξ . If $\xi^\wedge(o, \sigma) \in A$, then enumerate the stages as follows. We find some η -stage s such that there is an $\eta^\wedge(o', \sigma') \in A$ with $\xi^\wedge(o, \sigma) \leq \eta^\wedge(o', \sigma')$, and for any ζ with $\zeta^\wedge(o_\zeta, \sigma_\zeta) \subseteq \eta^\wedge(o', \sigma')$, $\sigma_\zeta \subseteq H$ and $\sigma_\zeta \subseteq H_s$, then let $s_{\xi^\wedge(o, \sigma)} = s$. By the construction, there is no node to the left of $\eta^\wedge(o', \sigma')$ can be accessible after the stage s , then it is true that there is no node to the left of $\xi^\wedge(o, \sigma)$ can be accessible after the stage s . $\square$

Lemma 3.12. $a \leq_T H$

Proof. For any x , we find a stage $s > x$ at which an edge of the form $\alpha^\wedge(i, \sigma)$ is visited after $\alpha^\wedge(i, \sigma)$ has been enumerated relative to H such that the followings are satisfied: (1) no node or edge to the left of $\alpha^\wedge(i, \sigma)$ will be visited after the stage s ; (2) for any α' labeled $\mathcal{P}_m$ with $\alpha'^\wedge(i, \sigma') \subseteq \alpha$, there is some labeled z attached to the edge $\alpha'^\wedge(i, \sigma')$ at the stage s ; (3) there is some labeled $y > x$ attached to the edge $\alpha^\wedge(i, \sigma)$ with $a_{s-1}(y) = 1$ at or before the stage s .

Assume that we can find such s . By the construction, (1) if α cannot be initialized after the stage s , then for any $z \leq y$, $a(z) = a_s(z)$; (2) otherwise, there is some ξ with $\xi < \alpha$ such that the action of ξ can initialize α after the stage s , then $\xi \subset \alpha$, since after the stage s no node or edge to the left of $\alpha^\wedge(i, \sigma)$ can be visited. Assume that t is the first stage after s such that α is initialized by some ξ , then $a_{t-1}(x) = a_s(x)$. There are two cases for this ξ : (a) $\xi = \alpha'$ is some $\mathcal{P}$ -strategy, by the construction and the choice of s and α , $\alpha'^\wedge(f, \sigma') \subseteq \alpha$. As α' initializes α at a stage $t > s$, then there is a $z > y$ with $a_{t-1}(z) = 0$ which is attached to the edge $\alpha'^\wedge(f, \sigma')$ at the stage t , if α' cannot be initialized after the stage t , then for any stage $t' \geq t$, $r(\alpha', t') \geq y > x$,

$a(x) = a_t(x) = a_{t-1}(x) = a_s(x)$. (b) $\xi = \beta$ is some $\mathcal{N}$ -strategy, by the construction, $\beta^\wedge(f, \sigma') \subseteq \alpha$. As β initializes α at the stage $t > s$, then β is permanently satisfied at the stage t . Hence there is a $z \geq y > x$ which is attached to the node β . If β cannot be initialized after the stage t , then $a(x) = a_t(x) = a_{t-1}(x) = a_s(x)$. Since there are only finitely many strategies above α , hence by induction, $a(x) = a_s(x)$.

In the following, we will prove that the existence of such a stage s . At first, we prove that (*) is true.

(*) There is some $\mathcal{P}$ -strategy α such that $\alpha^\wedge(i, \sigma) \in TP$ for some σ and there is some fixed labeled number $y > x$ attached to the edge $\alpha^\wedge(i, \sigma)$ such that the label on y cannot be canceled.

By the construction, if $\alpha^\wedge(i, \sigma) \in TP$, it is easy to see that there is some fixed labeled number y attached to $\alpha^\wedge(i, \sigma)$ such that the label on y cannot be canceled. We will prove that there are infinitely many $\mathcal{P}$ -strategies α with $\alpha^\wedge(i, \sigma) \in TP$ for some corresponding σ . Suppose that there are only finitely many such α , then there is some $\mathcal{P}$ -strategy α' such that for any $\mathcal{P}$ -strategy $\alpha'' \geq \alpha'$ with $\alpha'' \in TP$, $\alpha''^\wedge(f, \sigma_{\alpha''}) \in TP$. Assume $|\alpha'| = 2e'$, then for any $e'' \geq e'$, there are only finitely many $x \in \omega^{[e']}$ such that $\varphi_{e''}(x) \downarrow = 0$. By padding lemma, it is impossible, since the constant function $f = 0$ is computable and has infinitely many indices. All the labeled numbers attached to such edge $\alpha^\wedge(i, \sigma_\alpha) \in TP$ with α labeled $\mathcal{P}_m$ are pairwise different, so (*) is true.

Let $\alpha \in TP$ be a $\mathcal{P}_e$ -strategy with $\alpha^\wedge(i, \sigma) \in TP$ and s' be a stage such that there is some $y > x$ attached to the edge (i, σ) leaving α which is labeled at the stage s' and the label will not be canceled after the stage s' . Let $s > s'$ be the next H_α -true α -stages, then s satisfies the conditions in the first paragraph of the proof.

□

This completes the proof of Theorem 2.

Chapter 4

CUPPING AND DIAMOND EMBEDDINGS: A UNIFYING APPROACH

The work in this chapter is joint with Liu and Wu and one part of this chapter has been published in the “Lecture Notes in Computer Science”.

4.1 Introduction

A d.c.e. degree $\mathbf{d}$ is said to have almost universal cupping property, if it cups all c.e. degrees not below $\mathbf{d}$ to $\mathbf{0}'$. This notion was firstly proposed by Liu and Wu in their paper [57], where the existence of such degrees is proved. In this chapter, we prove that any nonzero cappable c.e. degree can have an almost universal cupping degree as a complement.

Theorem 3 (Fang, Liu and Wu [30]) *Let $\mathbf{c}$ be a nonzero cappable c.e. degree. Then there is an almost universal cupping d.c.e. degree $\mathbf{d}$ and a c.e. degree $\mathbf{b} < \mathbf{d}$ such that $\mathbf{c}$ cups $\mathbf{d}$ to $\mathbf{0}'$, caps $\mathbf{b}$ to $\mathbf{0}$ and $\mathbf{b}$ isolates $\mathbf{d}$.*

Obviously, Theorem 3 implies Downey-Li-Wu’s Complementation Theorem [23], which states that any cappable c.e. degree is complemented in the d.c.e. degrees, and also Arslanov’s Cupping Theorem [2] and Downey’s Diamond Embedding Theorem [18]. Furthermore, Theorem 3 implies Li-Yi’s Cupping Theorem [54], which states

that there are incomplete d.c.e. degrees $\mathbf{d}_1$ and $\mathbf{d}_2$ such that any nonzero c.e. degree cups one of them to $\mathbf{0}'$. Fix a nonzero cappable c.e. degree $\mathbf{c}$, apply Theorem 3 to obtain corresponding degrees $\mathbf{b}$ and $\mathbf{d}$, and then apply Theorem 3 once again, to $\mathbf{b}$ now, to obtain degrees $\mathbf{e}$ and $\mathbf{f}$, where $\mathbf{e} < \mathbf{f}$, $\mathbf{f}$ is an almost universal cupping degree, $\mathbf{e}$ isolates $\mathbf{f}$, and $\mathbf{b}$ and $\mathbf{e}$ (hence $\mathbf{d}$ and $\mathbf{f}$) form a minimal pair. Then, for any nonzero c.e. degree $\mathbf{w}$, it is easy to see that $\mathbf{w}$ cups one of $\mathbf{d}$ and $\mathbf{f}$ to $\mathbf{0}'$, as if $\mathbf{w}$ is below $\mathbf{b}$ (or below $\mathbf{e}$), then $\mathbf{w}$ cups $\mathbf{f}$ ($\mathbf{d}$ respectively) to $\mathbf{0}'$, and if $\mathbf{w}$ is below neither $\mathbf{b}$ nor $\mathbf{e}$, then $\mathbf{w}$ cups both $\mathbf{f}$ and $\mathbf{d}$ to $\mathbf{0}'$. We comment here that the latter property cannot be obtained from Li-Yi's Cupping Theorem. In fact, any d.c.e. degree $\mathbf{a}$ in $[\mathbf{d}, \mathbf{0}')$ and any d.c.e. degree $\mathbf{h}$ in $[\mathbf{f}, \mathbf{0}')$, together with $\mathbf{0}$ and $\mathbf{0}'$, form a diamond.

Fix a c.e. set C in the nonzero cappable c.e. degree $\mathbf{c}$ and fix an enumeration $\{C_s\}_{s \in \omega}$ of C . To prove Theorem 3, we will construct a c.e. set B , a d.c.e. set D and partial computable (p.c.) functionals $\{\Gamma_e\}_{e \in \omega}$, $\{\Delta_e\}_{e \in \omega}$, $\{\Theta_e\}_{e \in \omega}$ to satisfy the following requirements:

$$\mathcal{G}_e: K = \Gamma_e(B \oplus D \oplus W_e) \vee W_e = \Theta_e(B),$$

$$\mathcal{N}_e: \Phi_e(C) = \Phi_e(B) = f \text{ total} \rightarrow f \text{ is computable},$$

$$\mathcal{R}_e: W_e = \Lambda_e(B \oplus D) \rightarrow W_e = \Delta_e(B),$$

where $\{(W_e, \Lambda_e)\}$ is an effective list of pairs (W, Λ) such that W is a c.e. set and Λ is a p.c. functional, $\{\Phi_e\}_{e \in \omega}$ is an effective list of p.c. functionals.

Let $\mathbf{b}$ and $\mathbf{d}$ be the degree of B and the degree of $B \oplus D$ respectively. All the $\mathcal{N}$ -requirements and the assumption that $\mathbf{c}$ is nonzero guarantee that $\mathbf{b} < \mathbf{0}'$. This, together with all the $\mathcal{R}$ -requirements and the Sacks Density Theorem imply $\mathbf{d} < \mathbf{0}'$, otherwise, if $\mathbf{d} = \mathbf{0}'$ then there is some c.e. set A such that $\mathbf{b} < \deg_T(A) < \mathbf{0}' = \mathbf{d}$ by the Sacks Density Theorem, thus there is some e with $A = W_e = \Lambda_e(B \oplus D)$, but $W_e \neq \Delta(B)$ for any p.c. functional Δ , the $\mathcal{R}_e$ -requirement can not be satisfied. Moreover, since $\mathbf{d} < \mathbf{0}'$, then by the $\mathcal{G}$ -requirements and the Sacks Density Theorem, $\mathbf{b} < \mathbf{d}$, otherwise, if $\mathbf{b} = \mathbf{d}$ then there is some c.e. set A with $\mathbf{d} = \mathbf{b} < \deg_T(A) < \mathbf{0}'$ by the Sacks Density Theorem, hence there is some e such that $A = W_e$, the $\mathcal{G}_e$ -requirement can not be satisfied. Furthermore, by Lachlan's observation and the satisfaction of all the requirements, $\mathbf{b} > \mathbf{0}$, otherwise, if $\mathbf{b} = \mathbf{0}$ then there is some noncomputable c.e. set A with $\deg_T(A) < \mathbf{d}$, for this A , there is some e such that

$A = W_e = \Lambda_e(B \oplus D)$, but $W_e \neq \Delta(B)$ for any p.c. functional Δ , then the $\mathcal{G}_e$ -requirement can not be satisfied. Now, it is easy to see that if all the requirements can be satisfied, then $\mathbf{d}$ is an almost universal cupping degree. All the $\mathcal{N}$ -requirements guarantee that $\mathbf{c} \cap \mathbf{b} = \mathbf{0}$. And from all the $\mathcal{R}$ -requirements, $\mathbf{b}$ isolates $\mathbf{d}$.

4.2 Basic strategies and interactions

In this section, we describe all the basic strategies at first, and then consider the interactions among them. Strategies are nodes on a priority tree.

4.2.1 A $\mathcal{G}$ -strategy

Let η be a $\mathcal{G}_e$ -strategy. *For convenience, we use Γ_η to denote $\Gamma_{e(\eta)}$, use $\gamma_\eta(x)$ to denote $\gamma_{e(\eta)}(B \oplus D \oplus W_{e(\eta)}; x)$ and use W_η to denote $W_{e(\eta)}$.* In the following, we always omit the oracle in the expression of the use function if there is no confusion. A basic $\mathcal{G}_e$ -strategy η is to construct a p.c. functional Γ_η such that $\Gamma_\eta(W_\eta \oplus B \oplus D)$ can compute K correctly. The strategy η works as follows:

1. At an η -stage s , check whether there is some $x < s$ such that $\Gamma_\eta(B \oplus D \oplus W_\eta; x)[s-1] \downarrow \neq K_s(x)$. If so, then we choose the least such x , put $\gamma_\eta(x)[s-1]$ into D and undefine all $\Gamma_\eta(B \oplus D \oplus W_\eta; z)$ for $z \geq x$ with $\Gamma_\eta(B \oplus D \oplus W_\eta; z)[s-1] \downarrow$. If not, go to step 2.
2. Find the least $x < s$ such that $\Gamma_\eta(B \oplus D \oplus W_\eta; x)[s-1] \uparrow$. Then, we define $\Gamma_\eta(B \oplus D \oplus W_\eta; x)[s]$ with a proper use $\gamma_\eta(x)[s]$.

In the construction, the interactions between the $\mathcal{G}$ -strategies and the $\mathcal{R}$ -strategies will turn the definition of $\Gamma_\eta(B \oplus D \oplus W_\eta)$ even more subtle, as $\mathcal{R}$ -strategies may enumerate, or extract, numbers into, or from, D , and request the γ_η -uses big, or the same as before.

4.2.2 An $\mathcal{N}$ -strategy

An $\mathcal{N}$ -strategy is a standard gap-cogap argument, a good reference is from Soare's book [81].

Ambos-Spies et al. showed in [1] that a c.e. degree $\mathbf{c}$ is cappable iff it does not contain any promptly simple set. A c.e. set A has *promptly simple degree* if there is a partial computable function p such that for all s , $p(s) \geq s$, and for all $i \in \omega$,

$$|W_i| = \infty \Rightarrow \exists x \exists s [x \in W_{i, \text{at } s} \ \& \ A_s \upharpoonright x \neq A_{p(s)} \upharpoonright x],$$

where $W_{i, \text{at } s}$ is the set of numbers entering W_i at the stage s . The basic idea of an $\mathcal{N}_e$ -strategy, α say, is to build a computable function p_α threatening C to have promptly simple degree. We try to satisfy the following subrequirements:

$$\mathcal{S}_{\alpha, i} : |W_i| = \infty \rightarrow \exists x \exists s [x \in W_{i, \text{at } s} \ \& \ C_s \upharpoonright x \neq C_{p_\alpha(s)} \upharpoonright x]$$

A substrategy $\mathcal{S}_{\alpha, i}$ also constructs a computable function f_α^i such that if $\mathcal{S}_{\alpha, i}$ is not satisfied and $\Phi_\alpha(B) = \Phi_\alpha(C)$ is total, then f_α^i will be defined as a total function and computes $\Phi_\alpha(B) = \Phi_\alpha(C)$ correctly.

Say that an $\mathcal{S}_{\alpha, i}$ -requirement is satisfied at a stage s if p_α is constructed such that there are x and $t \leq s$ satisfying $x \in W_{i, \text{at } t} \ \& \ C_t \upharpoonright x \neq C_{p_\alpha(t)} \upharpoonright x$. As C is cappable, there must exist some i (least) such that $\mathcal{S}_{\alpha, i}$ -requirement cannot be satisfied.

In the following, we describe how α works. As usual, the length agreement functions are defined as follows:

$$l(\alpha, s) = \max\{x \mid \forall y < x [\Phi_\alpha(C; y)[s] \downarrow = \Phi_\alpha(B; y)[s] \downarrow]\}$$

and

$$m(\alpha, s) = \max\{0; l(\alpha, t) \mid t < s \text{ is an } \alpha \text{ stage}\}.$$

s is an α -expansionary stage if $s = 0$ or s is an α -stage with $l(\alpha, s) > m(\alpha, s)$, where a stage s is an α -stage if α is visited at this stage.

Let s be an α -expansionary stage. If there are some i and x such that

- (O1) $\mathcal{S}_{\alpha, i}$ is not satisfied yet;
- (O2) x enters W_i at a stage s' with $t < s' \leq s$, where $t = \max\{0, t'\}$ with t' the last α -expansionary stage before s at which a gap was opened for $\mathcal{S}_{\alpha, i}$ if there is such a stage t' ;
- (O3) $\exists y [y < l(\alpha, s) \ \& \ f_\alpha^i(y)[s-1] \uparrow \ \& \ x > \max\{\varphi_\alpha(C; y')[s] : y' \leq y\}]$,

then let i be the least one, y be the least number in (O3) and open a gap for $\mathcal{S}_{\alpha,i}$ as follows:

- (1) For those $y' \leq y$ with $f_\alpha^i(y')[s-1] \uparrow$, define $f_\alpha^i(y') = \Phi_\alpha(C; y')[s]$;
- (2) Set the restraint $r(\alpha, i, s) = 0$, and initialize all substrategies $\mathcal{S}_{\alpha,j}$ with $j > i$ by cancelling the function f_α^j .

Let v be the next α -expansionary stage and close the gap as follows:

- (C1) Define $p_\alpha(w) = v$ for any $t < w \leq s$;
- (C2) Set $r(\alpha, i, v) = v$.

During a gap, if C has a change below x , then $C_{s'} \restriction x \neq C_{p_\alpha(s')} \restriction x$, since $s' \leq s$ and $p_\alpha(s') = v$, declare that $\mathcal{S}_{\alpha,i}$ is satisfied, and say that the gap is closed successfully. If there is no such a C -change, then we say that the gap is closed unsuccessfully.

We assume that $\Phi_\alpha(B) = \Phi_\alpha(C) = f$ is total. Since C has a cappable degree, there is a least i such that $\mathcal{S}_{\alpha,i}$ cannot be satisfied. Then infinitely many gaps are opened for $\mathcal{S}_{\alpha,i}$, and each one is closed unsuccessfully. Let

$$s_0 < v_0 < \cdots < s_n < v_n < \cdots$$

be the stages at which gaps are opened and closed alternatively for $\mathcal{S}_{\alpha,i}$. We prove below that f_α^i computes f correctly.

Fix y and let $f_\alpha^i(y)$ be defined at the stage s_n . Then a gap for $\mathcal{S}_{\alpha,i}$ is opened at the stage s_n .

At stage v_n , we close this gap seeing at stage s_n and define $p_\alpha(s) = v_n$, where s is the stage at which the x corresponding to the gap entering W_i , restraining numbers less than v_n from entering B till the stage s_{n+1} . Since C has no change below $\varphi_\alpha(C; y)$ inside this gap (otherwise, $\mathcal{S}_{\alpha,i}$ will be satisfied), $\Phi_\alpha(C; y)[s_n] = \Phi_\alpha(C; y)[v_n]$, and hence

$$f_\alpha^i(y) = \Phi_\alpha(C; y)[s_n] = \Phi_\alpha(C; y)[v_n] = \Phi_\alpha(B; y)[v_n]$$

Now numbers less than v_n are restrained from entering B between stages v_n and s_{n+1} , i.e. inside the cogap, and as a result, the computation $\Phi_\alpha(B; y)[v_n]$ is preserved and hence

$$f_\alpha^i(y) = \Phi_\alpha(B; y)[v_n] = \Phi_\alpha(B; y)[s_{n+1}] = \Phi_\alpha(C; y)[s_{n+1}]$$

By induction, we have for all $m \geq n$,

$$f_\alpha^i(y) = \Phi_\alpha(B; y)[v_m] = \Phi_\alpha(B; y)[s_m]$$

since both $\Phi_\alpha(B; y)$ and $\Phi_\alpha(C; y)$ converge, we have

$$f_\alpha^i(y) = \Phi_\alpha(C; y) = \Phi_\alpha(B; y) = f(y)$$

Note that if $\Phi_\alpha(C)$ is partial, then (O3) may be prevented from opening a gap infinitely many times. In this case, the $\mathcal{N}$ -requirement is satisfied vacuously.

Say that $\mathcal{S}_{\alpha,i}$ *requires attention* at an α -expansionary stage s if one of the following holds:

- (1) $\mathcal{S}_{\alpha,i}$ is inside a gap.
- (2) $\mathcal{S}_{\alpha,i}$ is inside a cogap. There are two subcases:
 - (2A) There is some $y \in \text{dom}(f_\alpha^i)$ such that $C_s \upharpoonright \varphi_\alpha(C; y)[v] \neq C_v \upharpoonright \varphi_\alpha(C; y)[v]$, where v is the last α -expansionary stage.
 - (2B) $\mathcal{S}_{\alpha,i}$ is ready to open a gap.

In case (2), (2A) has priority higher than (2B). It may happen that (2A) prevents $\mathcal{S}_{\alpha,i}$ from opening a gap (2B) for almost all times. In this case, (f_α^i) can be defined finitely many times, and there is some $y \in \text{dom}(f_\alpha^i)$ with $\Phi_\alpha(C; y) \uparrow$. $\mathcal{S}_{\alpha,i}$ has two outcomes $g_i <_L d_i$, where g_i denotes the case that $\mathcal{S}_{\alpha,i}$ opens and closes infinitely many gaps during the construction (as a result, f_α^i is totally defined), and d_i denotes that there is some $y \in \text{dom}(f_\alpha^i)$ with $\Phi_\alpha(C; y) \uparrow$.

Say that $\mathcal{S}_{\alpha,i}$ *receives attention* at an α -expansionary stage s as follows if $\mathcal{S}_{\alpha,i}$ requires attention at this stage:

Case 1: (1) happens.

then close the gap, define $p_\alpha(v') = s$ for any $v' \leq v$ with $p_\alpha(v') \uparrow$ currently, where v is the last α -expansionary stage before the stage s at which the gap is opened, and

initialize all nodes with lower priority. Stop the stage s . If C has a change below x (x is defined in (O2) at stage v), then we say that the gap is closed successfully, and declare that $\mathcal{S}_{\alpha,i}$ is satisfied. Otherwise, the gap is closed unsuccessfully.

Case 2: (2A) happens. Then $\mathcal{S}_{\alpha,i}$ has outcome d_i .

Case 3: (2B) happens. Then $\mathcal{S}_{\alpha,i}$ opens a gap, extends the definition of f_α^i according to (O3). Now $\mathcal{S}_{\alpha,i}$ has outcome g_i .

In the construction, we don't put $\mathcal{S}_{\alpha,i}$ strategies on the priority tree. We just attach the outcomes of $\mathcal{S}_{\alpha,i}$ to α . Thus, α has outcomes

$$g_0 <_L d_0 <_L g_1 <_L d_1 <_L \cdots <_L d <_L f$$

which are described as follows:

g_i : g_i denotes the case in which the substrategy $\mathcal{S}_{\alpha,i}$ opens (and closes) gaps infinitely often. As described above, f_α^i is totally defined.

d_i : d_i denotes the case in which $\mathcal{S}_{\alpha,i}$ can open gaps only finitely often and there is some $y \in \text{dom}(f_\alpha^i)$ such that $\Phi_\alpha(C; y)$ diverges.

d : d denotes the case in which there are infinitely many α -expansionary stages, and α 's substrategies can require attention only finitely many times (if so, then $\Phi_\alpha(C)$ is not total because (O3) fails for almost all times). Let $y = \max \{\text{dom}(f_\alpha^i)\} + 1$, then $\Phi_\alpha(C; y')$ diverges for some $y' \leq y$. Here i is the least number such that $\mathcal{S}_{\alpha,i}$ can not be satisfied and $|W_i| = \infty$.

f : f denotes the case in which there are only finitely many α -expansionary stages (and hence, $\mathcal{N}_e$ is satisfied trivially).

4.2.3 An $\mathcal{R}$ -strategy

Let β be an $\mathcal{R}_e$ -strategy. We use Λ_β and W_β to denote $\Lambda_{e(\beta)}$ and $W_{e(\beta)}$ respectively. The basic idea of $\mathcal{R}$ -strategy is standard *isolation strategy* introduced by Wu in [87]. To approximate the equation $W_\beta = \Lambda_\beta(B \oplus D)$, we define the length agreement

functions at each β -stage s as follows:

$$l(\beta, s) = \max\{x : (\forall y < x)[W_{\beta,s}(y) = \Lambda_\beta(B \oplus D; y)[s] \downarrow]\},$$

and

$$m(\beta, s) = \max\{0; l(\beta, t) \mid t < s \text{ is a } \beta \text{ stage}\}.$$

A stage s is β -expansionary if $s = 0$ or s is a β -stage with $m(\beta, s) < l(\beta, s)$. The strategy β will extend the definition of Δ_β such that $\Delta_\beta(B) \upharpoonright l(\beta, s)$ is defined as same as $W_\beta \upharpoonright l(\beta, s)$ at β -expansionary stages; once a disagreement between $\Delta_\beta(B)$ and W_β appears, β will force a disagreement between W_β and $\Lambda_\beta(B \oplus D)$. We formulate this idea as follows.

1. Wait for a β -expansionary stage s_0 .
2. For any $y < l(\beta, s_0)$ with $\Delta_\beta(B; y) \uparrow$, define $\Delta_\beta(B; y)[s_0] = W_{\beta, s_0}(y)$ with use $\delta(y)[s_0] = s_0$. Go to step 1, at the same time, wait for a β -expansionary stage $s_1 > s_0$ such that $\Delta_\beta(B)$ and W_β differ at some y .
3. At the stage s_1 , extract numbers from D being enumerate into D between stages s_0 and s_1 to recover the computation $\Lambda_\beta(B \oplus D; y)$ to $\Lambda_\beta(B \oplus D; y)[s_0]$ in order to make a disagreement between W_β and $\Lambda_\beta(B \oplus D)$ at y . Restrain D from changing on small numbers. Then this disagreement will be kept from now on as W_β is a c.e. set.

In general, there are three outcomes of this strategy:

$$\infty <_L f <_L d$$

∞ : Go back from step 2 to step 1 infinitely many times, then $\Delta_\beta(B)$ will be defined as a total function, and it computes W_β correctly;

f: After some stage, β waits at step 1 for ever, then there are only finitely many β -expansionary stages, and hence $W_\beta \neq \Lambda_\beta(B \oplus D)$. *In this case, the $\mathcal{R}_e$ requirement is satisfied vacuously;*

- d: The strategy reaches step 3 eventually. Then a disagreement between W_β and $\Lambda_\beta(B \oplus D)$ is created and preserved. *In this case, the $\mathcal{R}_e$ requirement is satisfied vacuously.*

4.2.4 The interaction between two $\mathcal{R}$ -strategies

First, we consider the interaction between two $\mathcal{R}$ -strategies β_1 and β_2 , especially, how a lower priority strategy β_2 injures a higher priority strategy β_1 . If $\beta_1 \hat{\perp} d \subseteq \beta_2$, then the initialization by the strategy β_1 when it successfully builds a disagreement, any number enumerated into or extracted from D by β_2 is big enough to preserve the disagreement built by β_1 , and hence will not injure the strategy β_1 . If $\beta_1 \hat{\perp} f \subseteq \beta_2$, the strategy β_2 will not injure the strategy β_1 . The non-trivial case is $\beta_1 \hat{\perp} \infty \subseteq \beta_2$, there are two particular scenarios we need to carefully treat with.

Scenario 1:

- (1) At a β_1 -expansionary stage s_0 , $\Delta_{\beta_1}(B; y)$ is (re)defined with value of $W_{\beta_1, s_0}(y)$ and use of s_0 .
- (2) At a stage $s_1 > s_0$, there is some η -node labeled $\mathcal{G}_m$ for some m with $\eta \subset \beta_2$ which enumerates some $x = \gamma_\eta(k)[s_1]$ into D .
- (3) At a β_2 -expansionary stage $s_2 > s_1$, (it is also an β_1 -expansionary stage), $\Delta_{\beta_2}(B; z)$ is (re)defined with value of $W_{\beta_2, s_2}(z)$ and use of s_2 . (Assume that the definition of $\Delta_{\beta_1}(B)$ is right currently and $x < \lambda_{\beta_2}(B \oplus D; z)[s_2]$).
- (4) At a β_1 -expansionary stage $s_3 > s_2$, $\Delta_{\beta_1}(B; y)[s_3] = \Delta_{\beta_1}(B; y)[s_0] \neq W_{\beta_1, s_3}(y)$ with use $\delta_{\beta_1}(B; y)[s_3] = \delta_{\beta_1}(B; y)[s_0] = s_0$. By the basic strategy, we extract numbers entering D between stages s_0 and s_3 .
- (5) There is some stage $s_4 > s_3$, $\Delta_{\beta_2}(B; z)[s_4] = \Delta_{\beta_2}(B; z)[s_2] \neq W_{\beta_2, s_4}(z)$ with use $\delta_{\beta_2}(B; z)[s_4] = \delta_{\beta_2}(B; z)[s_2] = s_2$. But $\Lambda_{\beta_2}(B \oplus D; z)$ can not be recovered to $\Lambda_{\beta_2}(B \oplus D; z)[s_2]$ by extracting number entering into D between stages s_2 and s_4 , since $x \in D_{s_2}$ but $x \notin D_{s_4}$. We cannot put x into D after the stage s_3 , since D is d.c.e.

It is evident that there is a disagreement between W_{β_1} and $\Lambda_{\beta_1}(B \oplus D)$ which is built at the stage s_3 , then after the stage s_3 , β_2 can not be accessible (*that is, s_4 is not a β_2 -stage*) unless β_1 is initialized. So in this construction, such an injury is admissible.

Scenario 2:

(1) At a β_2 -expansionary stage s_0 , (it is also an β_1 -expansionary stage), $\Delta_{\beta_2}(B; z)$ is (re)defined with value of $W_{\beta_2, s_0}(z)$ and use of s_0 .

(2) At a stage $s_1 > s_0$, there is some η -node labeled $\mathcal{G}_m$ for some m with $\eta \subset \beta_2$ which enumerates some $x = \gamma_\eta(k)[s_1]$ into D .

(3) At a β_1 -expansionary stage $s_2 > s_1$, $\Delta_{\beta_1}(B; y)$ is (re)defined with value of $W_{\beta_1, s_2}(y)$ and use of s_2 . (Without loss of generality, assume that $x < \lambda_{\beta_1}(B \oplus D; y)[s_2]$).

(4) At a β_2 -expansionary stage $s_3 \geq s_2$, (it is a β_1 -expansionary stage too, and the definition of $\Delta_{\beta_1}(B)$ is correct currently). $\Delta_{\beta_2}(B; z)[s_3] = \Delta_{\beta_2}(B; z)[s_0] \neq W_{e, s_3}(z)$ with use $\delta_{\beta_2}(B; z)[s_3] = \delta_{\beta_2}(B; z)[s_0] = s_0$. By the basic strategy, we extract numbers entering D between stages s_0 and s_3 from D .

(5) At a β_1 -expansionary stage $s_4 > s_3$, $\Delta_{\beta_1}(B; y)[s_4] = \Delta_{\beta_1}(B; y)[s_2] \neq W_{e', s_4}(y)$ with use $\delta_{\beta_1}(B; y)[s_4] = \delta_{\beta_1}(B; y)[s_2] = s_2$. Now, we want to recover $\Lambda_{\beta_1}(B \oplus D; y)$ to $\Lambda_{\beta_1}(B \oplus D; y)[s_2]$, while this is impossible, since it requires to put x into D again but D is a d.c.e. set.

How to solve this problem? It is not able to build disagreement between $\Lambda_{\beta_1}(B \oplus D)$ and W_{β_1} at the stage s_4 , while $\Delta_{\beta_1}(B; y)$ disagrees with $W_{\beta_1}(y)$ currently. This problem is derived from the β_2 -strategy extracting certain small numbers from D to build a disagreement between W_{β_2} and $\Lambda_{\beta_2}(B \oplus D)$ at the stage s_3 . But there is no reason for β_2 to give up building such a disagreement. Then β_2 ought to prevent such problem at the stage s_3 . But it is impossible to predict the disagreement between $\Lambda_{\beta_1}(B \oplus D)$ and W_{β_1} appearing at the stage s_4 . On the other hand, we can undefine each $\Delta_{\beta_1}(B; y)$ with incorrect definition at the stage s_3 by controlling set B . Therefore, to get around the problem, β_2 just enumerates certain small number into B at the stage s_3 to undefine all $\Delta_{\beta_1}(B; y)$ (re)defined at the stage $s > s_0$. To achieve this goal, it suffices to enumerate s_0 into B . Since $\lambda_{\beta_2}(B \oplus D; z)[s_0] < s_0$, the enumeration of s_0 into B will not affect the computation of $\Lambda_{\beta_2}(B \oplus D; z)[s_0]$. At the same time, this action will not affect the computation of $\Delta_{\beta_1}(B; y)$ which is (re)defined before the stage s_0 . This is important to guarantee $\Delta_{\beta_1}(B)$ to be total if $W_{\beta_1} = \Lambda_{\beta_1}(B \oplus D)$.

On the priority tree, there are infinitely many $\mathcal{R}$ -strategies below $\beta_1 \hat{\ } \infty$. For any fixed y , we should prove that the computation of $\Delta_{\beta_1}(B; y)$ can be undefined finitely many times by the action of $\mathcal{R}$ -strategies with priority lower than β_1 . This is because if $\Delta_{\beta_1}(B; y)$ is defined at a stage t then it can be undefined later at most finitely many times only by certain strategy visited before or at the stage t .

4.2.5 The interaction between one $\mathcal{R}$ -strategy and one $\mathcal{G}$ -strategy

We now consider the interaction between an $\mathcal{R}$ -strategy β and a $\mathcal{G}$ -strategy η with $\eta \subset \beta$. To code K into $B \oplus D \oplus W_\eta$, η may enumerate infinitely many numbers into D . These enumerations might destroy the disagreements made by β -strategy infinitely often, and β fails to satisfy the $\mathcal{R}_e$ requirement. To solve this problem, β uses the threshold strategy to lift the γ -use to ensure the computations of λ are clear of the enumeration of γ -uses. The basic idea is as follows. When β is firstly visited, it chooses a threshold k_β , and in the remainder of the construction, whenever a number $m \leq k_\beta$ enters K , we reset β by canceling all the parameters of β , except k_β , so once the threshold k_β is fixed, the resetting process can happen at most $k_\beta + 1$ times.

Now β works by running cycles (infinitely), with cycle 0 started first, and each cycle n constructs a functional Δ_β^n correspondingly. Once it sees a difference between $\Delta_\beta^n(B)$ and W_β , cycle n initiates a new cycle $n + 1$. The cycle n works as follows:

1. Wait for a β -expansionary stage s_0^n .
2. For any $x < l(\beta, s_0^n)$ with $\Delta_\beta^n(B; x) \uparrow$ currently, define $\Delta_\beta^n(B; x)[s_0^n] = W_{\beta, s_0^n}(x)$ with use $\delta_\beta^n(B; x)[s_0^n] = s_0^n$. Go back to step 1, and simultaneously, wait for a β -expansionary stage $s_1^n > s_0^n$ such that $\Delta_\beta^n(B)$ and W_β differ at some y .
3. At a stage s_1^n , put $\gamma_\eta(k_\beta)[s_1^n]$ into D . For any $z \leq \gamma_\eta(k_\beta)[s_1^n]$ with $\Theta_{\beta\eta}(B; z) \uparrow$ currently, define $\Theta_{\beta\eta}(B; z)[s_1^n] = W_{\eta, s_1^n}(z)$ with use $\theta_{\beta\eta}(z)[s_1^n] = s_1^n$. Wait for a β -expansionary stage $s_2^n > s_1^n$ such that W_η changes below $\gamma_\eta(k_\beta)[s_1^n]$ (*we want to lift the γ_η -uses by this change instead of putting $\gamma_\eta(k_\beta)[s_1^n]$ into D*). Start cycle $n + 1$.
4. At a stage s_2^n , extract all the numbers which have been enumerated into D after the stage s_0^n out of D to make a disagreement between W_β and $\Lambda_\beta^{B \oplus D}$ at y , and put s_0^n into B to undefine all $\Delta_{\beta'}^k(B; y)$ which is defined between stages s_0^n and s_2^n , where β' is an $\mathcal{R}$ -strategy with $\beta' \smallfrown (k, \infty) \subset \beta$. Restrain B and D from changing on small numbers.

Note: The enumeration of s_0^n can also undefine $\Theta_{\beta'\eta'}(B; z)$ which is (re)defined after the stage s_0^n . Here $\Theta_{\beta'\eta'}(B; z)$ is (re)defined by a β' -strategy labeled $\mathcal{R}_{e'}$

with $\beta' \subset \beta$. It will be considered in the interaction when $\mathcal{R}$ and $\mathcal{G}$ -strategies are considered.

For the full β -strategy, there are four cases:

g_η : β runs infinitely many different cycles, and each cycle n eventually reaches step (3) and waits at step (3) forever. In this case, $\Theta_{\beta\eta}(B)$ is defined as total and it computes W_η correctly. Hence $\mathcal{G}_{e'}$ is satisfied at β as W_η is computable in B . Note that in this case, the $\mathcal{R}_e$ -requirement is not satisfied at β .

Note: In this case, $\mathcal{R}_e$ is not satisfied at β , and hence a backup strategy, β' say, for $\mathcal{R}_e$ -requirement ought to be set below $\beta \frown g_\eta$. As β' knows that $\gamma_\eta(k_\beta)$ -use goes to infinity, β' only believes a computation $\Lambda_{\beta'}(B \oplus D; y)[s]$ if $\gamma_\eta(k_\beta)[s] > \lambda_{\beta'}(B \oplus D; y)[s]$. We call such a computation a β' -believable computation. From now on, in the definition of length agreement function for an $\mathcal{R}_e$ strategy β' , we only use β' -believable computations. As η 's further enumeration into D can not affect the action of β' , β' works in the usual way.

(n, ∞): Cycle n goes back from step (2) to step (1) infinitely often. Then there are infinitely many β -expansionary stages, and $\Delta_\beta^n(B)$ and W_β agree at all these stages. $\Delta_\beta^n(B)$ is defined as total and it computes W_β correctly. In this case, the $\mathcal{R}_e$ -requirement is satisfied at β , and β enumerates $\gamma_\eta(k_\beta)$ into D only finitely often.

(n,f): There are only finitely many β -expansionary stages. Let s be the largest β -expansionary stage, and n be the largest cycle which has been started and has not been canceled until the stage s . Then after stage s , β will wait at cycle n for a β -expansionary stage forever. In this case, as there are only finitely many β -expansionary stages and hence the $\mathcal{R}_e$ -requirement is satisfied at β . Note that in this case, β enumerates $\gamma_\eta(k_\beta)$ into D only finitely often.

d: Some cycle n can reach step 4, a disagreement between $\Lambda_\beta(B \oplus D)$ and W_β is created and this disagreement will be preserved forever, since all $\gamma_\eta(k)$ for $k \geq k_\beta$ is lift to be greater than $\lambda_\beta(B \oplus D; y)[s_0^n]$. Hence $\mathcal{R}_e$ -requirement is satisfied at β . Again, β enumerates $\gamma_\eta(k_\beta)$ into D only finitely often.

The priority of these outcomes is as follows:

$$g_\eta <_L (0, \infty) <_L (0, f) <_L (1, \infty) <_L (1, f) <_L \cdots <_L (n, \infty) <_L (n, f) <_L \cdots <_L d$$

Note: Here we reconsider the $\mathcal{G}$ -strategy η . If there is some $\mathcal{R}$ -strategy β and some $\mathcal{G}$ -strategy η' such that $\beta \smallfrown g_{\eta'}$ is on the true path, and η is on the true path too. Then β will enumerate infinitely many $\gamma_{\eta'}(k_\beta)[s]$ into D for infinitely many s , we wish that the infinitely many enumerations do not injure η . Then the use $\gamma_\eta(x)[s]$ ought to be carefully chosen in order that $\Gamma_\eta(B \oplus D \oplus W_\eta)$ is total if $W_\eta \not\leq_T B$. So if we define $\Gamma_\eta(B \oplus D \oplus W_\eta; x)$ at the stage s , then let the use be a fresh number if one of the followings is satisfied: (1) $\gamma_\eta(x)$ has never been defined before; In the rest of this paragraph, t is the last stage at which $\gamma_\eta(x)$ was defined, (2) There is some $y < x$ with $\gamma_\eta(y)[s] > \gamma_\eta(x)[t]$; (3) There is some γ_η -uses $\leq \gamma_\eta(x)[t]$ enumerated into D or removed from D after the stage t , (*in the construction, if there is some γ_η -use z which is enumerated into D at the stage t_1 and is removed at stage t_2 , where $t \leq t_1 < t_2 \leq s$, then W_η must have a change below $z + 1$, hence we can redefine the corresponding γ_η -use as a big number*); (4) W_η gets a change below an active requesting number (z say, we will define it later) of η after the stage t , and $z \leq \gamma_\eta(x)[t]$. (Here we call $\gamma_\eta(x)$ is request to be big if one of the above holds). Otherwise, we define $\Gamma_\eta(B \oplus D \oplus W_\eta; x) = K_s(x)$ with use $\gamma_\eta(x)[s] = \gamma_\eta(x)[t]$.

4.2.6 The interaction among several $\mathcal{G}$ -strategies and one $\mathcal{R}$ -strategy

Let β be an $\mathcal{R}_e$ -strategy, for $i = 1, \dots, j$, η_i be $\mathcal{G}_{e_i}$ -strategy with $\eta_1 \subset \eta_2 \subset \dots \subset \eta_j \subset \beta$. Now, β is to make its computations clear of all the enumerations of γ_{η_i} -uses into D for $i \leq j$. The basic idea is to iteratively utilize the threshold strategy described in the previous section to deal with η_j first, and then $\eta_{j-1}, \dots, \eta_1$. As before, β will choose a threshold k_β when it is visited for the first time, and then start cycle 0. In this general case, cycle n works as follows.

1. Wait for a β -expansionary stage s_0^n .
2. For any $y < l(\beta, s_0^n)$ with $\Delta_\beta^n(B; y) \uparrow$ currently, define $\Delta_\beta^n(B; y)[s_0^n] = W_{\beta, s_0^n}(y)$

with use $\delta_\beta^n(y)[s_0^n] = s_0^n$. Go back to step 1, and simultaneously, wait for a β -expansionary stage $s_1^n > s_0^n$ such that $\Delta_\beta^n(B)$ and W_β differ at a number y , i.e. $(\exists y)[\Delta_\beta^n(B; y)[s_1^n] \downarrow = \Delta_\beta^n(B; y)[s_0^n] \neq W_{\beta, s_1^n}(y)]$ with $\delta_\beta^n(y)[s_1^n] = \delta_\beta^n(y)[s_0^n] = s_0^n$.

3. At the stage s_1^n , choose the least such y as y_n , and we call y_n a *potential witness* found by cycle n . Go to 3(j).

3(i). At the stage s_{j+1-i}^n , put $\gamma_{\eta_i}(k_\beta)[s_{j+1-i}^n]$ into D . For any $z \leq \gamma_{\eta_i}(k_\beta)[s_{j+1-i}^n]$ with $\Theta_{\beta\eta_i}(B; z) \uparrow$ currently, define $\Theta_{\beta\eta_i}(B; z)[s_{j+1-i}^n] = W_{\eta_i, s_{j+1-i}^n}(z)$ with use $\theta_{\beta\eta_i}(z)[s_{j+1-i}^n] = s_{j+1-i}^n$. Wait for a β -expansionary stage $s_{j+2-i}^n > s_{j+1-i}^n$ such that W_{η_i} changes below $\gamma_{\eta_i}(k_\beta)[s_{j+1-i}^n]$. ($\gamma_{\eta_i}(k_\beta)[s_{j+1-i}^n]$ is called an *active requesting number*). Simultaneously, start cycle $n+1$.

In this case, cycle n is called alive on η_i . In general, a cycle n is called alive if it is alive on some η_i . An alive cycle n means that the potential witness found by cycle n may possibly make a disagreement in the future.

4(i). At the stage s_{j+2-i}^n , cancel all cycle $n' > n$ by cancelling all parameters associated with cycle n' . We say that *cycle n pass the threshold k_β for η_i* . If $i > 1$, go to step 3(i-1) in cycle n . Otherwise, extract all numbers entering D after the stage s_0^n out of D , and put s_0^n into B to undefine all $\Delta_{\beta'}^k(B; y)$ (re)defined after the stage s_0^n , where β' is an $\mathcal{R}_{e'}$ -strategy with $\beta' \wedge (k, \infty) \subseteq \beta$. Restrain small numbers (i.e. numbers $\leq s_{j+1}^n$) being enumerated into B and D or being removed from D .

Note that in general, if cycle n passes the threshold k_β for η_i , then all cycles alive on $\eta_{i'}$ for $i' \geq i$ will automatically become not alive, since β has gotten a better potential witness which passes more thresholds than the witness associated with those cycles alive on $\eta_{i'}$.

There are four types of possible outcomes for a β -strategy:

g_i : After some stage, there are infinitely many cycles such that each of them is alive on η_i forever once it becomes alive, i.e. infinitely many cycles can reach step 3(i), but none of them reaches 4(i). In this case, $\Theta_{\beta\eta_i}(B)$ will be defined infinitely often. We can prove that for any x , $\Theta_{\beta\eta_i}(B; x)$ can be undefined at most finitely many times. Hence $\Theta_{\beta\eta_i}(B)$ is total and computes W_{η_i} correctly, and so the $\mathcal{G}_{e_i}$ -requirement is satisfied at β .

- (n,∞): Cycle n goes back from step 2 to step 1 infinitely often and after some large enough stage, cycle n can not reach step 3, hence $\Delta_\beta^n(B)$ is total and it computes W_β correctly. So the $\mathcal{R}_e$ -requirement is satisfied at β .
- (n,f): After some stage, β waits at step 1 of cycle n forever. Again, the $\mathcal{R}_e$ -requirement is satisfied at β .
- d: Eventually, β reaches step 4(1) via some cycle n . That is, cycle n passes the threshold k_β for all of $\eta_j, \dots, \eta_1$, and clear of all γ -uses. In this case, $\Lambda_\beta(B \oplus D)$ and W_β differ at y_n , and the computation $\Lambda_\beta(B \oplus D; y_n)$ is preserved forever. Thus $\mathcal{R}_e$ -requirement is satisfied at β . In this case, β 's actions have only finite impact on η_i for $1 \leq i \leq j$.

The priority of these outcomes is in what follows

$$g_1 <_L g_2 <_L \dots <_L g_j <_L (0, \infty) <_L (0, f)$$

$$<_L (1, \infty) <_L (1, f) <_L \dots <_L (n, \infty) <_L (n, f) <_L \dots <_L d$$

Note that numbers are enumerated into D when 3(i) is reached for some $1 \leq i \leq j$, at those stages when outcome g_i is true. So below these outcomes, strategies know that the related γ -uses go to infinite, and hence believable computations are used at these strategies. Numbers can be extracted from D only when step 4(1) is reached, where for the consistency between β and other strategies, a small number is enumerated into B . This strategy differs from that used in [57]. In [57] numbers are extracted from D whenever β changes outcome g_i for any $1 \leq i < j$, which requires to enumerate numbers into B at these stages. It is this subtle difference that allows us to combine the construction of almost universal cupping degrees with the gap-cogap argument to prove Theorem 3.

4.2.7 The interaction between one $\mathcal{N}$ -strategy and one $\mathcal{R}$ -strategy

We now consider the interaction between $\mathcal{N}$ -strategies and $\mathcal{R}$ -strategies. In fact, these two kinds of strategies are consistent, and the interactions between them are quite simple. To see this, let β and α be an $\mathcal{R}_e$ -strategy and an $\mathcal{N}_{e'}$ -strategy respectively

for some e and e' . (1) For $\beta \subset \alpha$, we know that β can enumerate at most one number into B (modulo initialization) when β removes some number from D to build the disagreement between W_β and $\Lambda_\beta(B \oplus D)$, and once β enumerates some number into B , it initializes all nodes with lower priority than β , so the enumeration of β can not injure α . (2) For $\alpha \subset \beta$, a nontrivial case is $\alpha \cap g_i \subseteq \beta$ for some $i \in \omega$. In this case, at any β -stage, α opens a gap for $\mathcal{S}_{\alpha,i}$, and at these stages, if β takes some action, then β either puts numbers into D , or puts numbers into B (i.e. β performs the disagreement argument by extracting numbers from D and putting some number into B simultaneously). The former actions do not have any impact on α , and the latter actions, enumerating numbers into B , is allowed as the gap $\mathcal{T}_{\alpha,i}$ is opened at the current stage. As β acts only when it is visited, β 's action does not injure α .

4.2.8 The interaction among two $\mathcal{R}$ -strategies and two $\mathcal{G}$ -strategies

Lastly, we consider the interaction between two $\mathcal{R}$ -strategies and two $\mathcal{G}$ -strategies. Let β and β' be $\mathcal{R}_m$ and $\mathcal{R}_{m'}$ -strategies respectively, η and η' be $\mathcal{G}_e$ and $\mathcal{G}_{e'}$ -strategies respectively. Here, we mainly focus on the interaction between two $\mathcal{R}$ -strategies involved in two $\mathcal{G}$ -strategies. Without loss of generality, let $\eta \subset \eta' \subset \beta' \cap g_{\eta'} \subseteq \beta$. In the following, we consider a problematic scenario.

- (1) At a β -expansionary stage s_0 , (it is also an β' -expansionary stage), $\Delta_\beta^n(B; z)$ is (re)defined with value of $W_{\beta,s_0}(z)$ and use of s_0 .
- (2) At a β -expansionary stage $s_1 > s_0$, $\Delta_\beta^n(B; z)[s_1] = \Delta_\beta^n(B; z)[s_0] \neq W_{\beta,s_1}(z)$, β enumerates $\gamma_\eta(k_\beta)[s_1]$ into D and wait for a W_η -change below $\gamma_\eta(k_\beta)[s_1]$.
- (3) At a β' -expansionary stage $s_2 > s_1$, $\Delta_{\beta'}^{n'}(B; y)$ is (re)defined with value of $W_{\beta',s_2}(y)$ and use of s_2 . (Assume that $\gamma_\eta(k_\beta)[s_1] < \lambda_{\beta'}(B \oplus D; y)[s_2]$).
- (4) At a β' -expansionary stage $s_3 > s_2$, $\Delta_{\beta'}^{n'}(B; y)[s_3] = \Delta_{\beta'}^{n'}(B; y)[s_2] \neq W_{\beta',s_3}(y)$, β' enumerates $\gamma_{\eta'}(k_{\beta'})[s_3]$ into D at this stage, and defines $\Theta_{\beta'\eta'}(B; x)[s_3] = W_{\eta',s_3}(x)$ with use of s_3 for any $x \leq \gamma_{\eta'}(k_{\beta'})[s_3]$ with $\Theta_{\beta'\eta'}(B; x) \uparrow$ currently. Wait for a $W_{\eta'}$ -change below $\gamma_{\eta'}(k_{\beta'})[s_3]$. Notice that β' has outcome $g_{\eta'}$ at this stage.

- (5) At a β -expansionary stage $s_4 \geq s_3$, (it is also a β' -expansionary stage, and the definition of $\Delta_{\beta'}^{n'}(B)$ is correct currently), we find a W_η -change below $\gamma_\eta(k_\beta)[s_1]$. Now β extracts numbers entering D between stages s_0 and s_4 from D (including $\gamma_\eta(k_\beta)[s_1]$), and put s_0 into B .

As $\gamma_\eta(k_\beta)[s_1] \in D_{s_2}$, $\gamma_\eta(k_\beta)[s_1] \notin D_{s_4+1}$, and D is d.c.e., $\gamma_\eta(k_\beta)[s_1]$ can not be enumerated into D after the stage s_4 . Thus after the stage s_4 , it is not possible to recover the computation to $\Lambda_{\beta'}(B \oplus D; y)[s_2]$ in order to build a disagreement between $\Lambda_{\beta'}(B \oplus D)$ and $W_{\beta'}$ at y . Fortunately, this problem can be solved by correcting the definition of $\Theta_{\beta'\eta'}(B; x)$ instead of making a disagreement point at y . Actually, we are able to use the enumeration of s_0 into B at the stage s_4 to undefine $\Theta_{\beta'\eta'}(B; x)[s_3]$ which is (re)defined after the stage s_0 in advance in order to correct it. In this case, we give up the diagonalization argument at y , and hence there is no need to wait for a $W_{\eta'}$ -change below $\gamma_{\eta'}(k_{\beta'})[s_3]$ since all computations of $\Theta_{\beta'\eta'}(B)$ which are (re)defined at the stage s_3 are undefined at the stage s_4 . Accordingly, cycle n' of β' will become *not alive* since the stage s_4 .

4.3 The construction

We now describe the whole construction. First, we need the priority tree T on which the construction will proceed.

4.3.1 The construction of the priority tree

We assign priority to the requirements as follows:

$$\mathcal{G}_0 < \mathcal{R}_0 < \mathcal{N}_0 < \cdots < \mathcal{G}_n < \mathcal{R}_n < \mathcal{N}_n < \cdots$$

Where $\mathcal{X} < \mathcal{Y}$ means that $\mathcal{X}$ -requirement has higher priority than $\mathcal{Y}$ -requirement.

The priority tree T grows downwards and is constructed recursively as follows. The top node on the tree T is labeled $\mathcal{G}_0$. Assume that τ is a node on the tree T . If τ is labeled $\mathcal{G}_e$, then there is only one edge leaving τ labeled 1. If τ is labeled $\mathcal{N}_e$, then there are infinitely many edges leaving τ labeled $g_0 <_L d_0 <_L g_1 <_L d_1 < \cdots < d < f$. If τ labeled $\mathcal{R}_e$, it is more complicated than the previous two cases. For this case,

we will introduce some notions first. Let σ be a path starting from root node λ and ending with τ . A node η labeled $\mathcal{G}_e$ on the path σ is called *active* at τ if for any node η' labeled $\mathcal{G}_{e'}$ with $\eta' \subset \eta$, there is no node β' labeled $\mathcal{R}_{m'}$ with $\eta \subset \beta' \subset \tau$, such that $\beta' \wedge g_{\eta'} \subseteq \tau$. A node $\eta \in \sigma$ labeled $\mathcal{G}_e$ is *satisfied* by some $\beta \supset \eta$ on the path σ if η is active at τ and $\beta \wedge g_\eta \in \sigma$. We say that a node $\zeta \in \sigma$ is *injured* by some $\beta \supset \zeta$ if there is some node $\eta \subset \zeta$ in σ such that $\beta \wedge g_\eta \in \sigma$. So if τ is labeled $\mathcal{R}_e$, then τ has outcomes $(0, \infty) <_L (0, f) <_L \dots <_L d$ and g_{η_i} on the left of $(0, \infty)$ for some $\eta_i \subset \tau$ labeled $\mathcal{R}_{e_i}$ such that η_i is active at τ and not satisfied on the path σ , where the priority of these outcomes are as follows: $g_{\eta_i} <_L g_{\eta_j}$ iff $\eta_i \subset \eta_j \subset \tau$.

We assign the order of nodes on the priority tree as follows, for any node $\sigma, \tau \in T$, (1) $\sigma <_L \tau$ if there is a node $\rho \in T$ and $\mathcal{O}_1 <_L \mathcal{O}_2$ such that $\rho \wedge \mathcal{O}_1 \subseteq \sigma$ and $\rho \wedge \mathcal{O}_2 \subseteq \tau$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are two outcomes of ρ ; (2) $\sigma \leq \tau$ if $\sigma <_L \tau$ or $\sigma \subseteq \tau$; (3) $\sigma < \tau$ if $\sigma \leq \tau$ and $\sigma \neq \tau$. We say that ξ_1 has higher priority than ξ_2 if $\xi_1 < \xi_2$.

Let σ be a finite path on T ending with the node τ . If $\mathcal{T}$ -requirement is a $\mathcal{G}$ or an $\mathcal{N}$ -requirement, we say that $\mathcal{T}$ is *represented* by some node $\xi \in \sigma$ if ξ is labeled $\mathcal{T}$ and can not be injured by any β with $\xi \subset \beta \in \sigma$. If $\mathcal{T}$ -requirement is an $\mathcal{R}$ -requirement, then we say that $\mathcal{T}$ is *represented* by some node $\xi \in \sigma$ if ξ is labeled by $\mathcal{T}$ and $\xi \wedge \mathcal{O} \in \sigma$ for $\mathcal{O} = (n, f), (n, \infty)$, or d . If σ is a finite path ending with τ , then label τ with the highest requirement which is not represented by any node in σ .

In the construction, if a strategy ξ is initialized, then all strategies with priority lower than ξ will be also initialized automatically, and all numbers and all functionals attached to ξ will be canceled, if ξ is an $\mathcal{R}$ -strategy, then all the cycles are canceled. And also, if a strategy ξ is reset, then all strategies with priority lower than ξ are initialized automatically.

It is easy to see that the following propositions are true.

Proposition 4.1. *For any infinite path P on the tree T and any requirement $\mathcal{G}_e$, there is a strategy η on P such that η is active at $P \upharpoonright n$ for any n with $n > |\eta|$.*

Proposition 4.2. *Given an infinite path P , for any requirement $\mathcal{U}$, $\mathcal{U}$ is either $\mathcal{N}_e$, $\mathcal{G}_e$ or $\mathcal{R}_e$ for some e , and there is a node ξ on P such that for any $n > |\xi|$, $\mathcal{U}$ is represented by ξ in $P \upharpoonright n$.*

4.3.2 The full construction

The full construction is as follows.

Stage 0: Let $B_0 = D_0 = \emptyset$, and initialize all nodes, let $\sigma_0 = \lambda$ which is the root node of the priority tree and go to the next stage.

Stage $s + 1$: This stage includes two phases.

Phase 1: If there is some $k \in K_{s+1} - K_s$, choose the least such k . Then find out all β -nodes with the threshold $k_\beta \geq k$ if any. Now, reset all such β and initialize all nodes with lower priority than β . Otherwise, do nothing.

Phase 2: Divide this stage into several substages.

Substage 0: Let $\sigma_{s+1} \upharpoonright 0 = \lambda$, the root node of the priority tree.

Substage $t + 1$: Assume that $\sigma_{s+1} \upharpoonright t = \xi$. If $t = s + 1$, then let $\sigma_{s+1} = \xi$ and initialize all the nodes with priority lower than σ_{s+1} , and go to the next stage.

If $t < s + 1$, there are three cases:

Case 1. $\xi = \eta$ is a $\mathcal{G}_e$ -strategy. There are two subcases:

($\eta 1$) There is some $x < s + 1$ such that $\Gamma_\eta(B \oplus D \oplus W_e; x)[s] \downarrow \neq K_{s+1}(x)$, then choose the least such x and put $\gamma_s(x)$ into D . Undefine all $\Gamma_\eta(B \oplus D \oplus W_e; y)$ for $y \geq x$. Let $\sigma_{s+1} \upharpoonright t + 1 = \eta^\wedge 1$, go to the next substage.

($\eta 2$) $\eta 1$ fails. Choose the least number $x < s + 1$ such that $\Gamma_\eta(B \oplus D \oplus W_e; x)[s] \uparrow$. If $\gamma_\eta(x)$ is requested to be defined big, then define $\Gamma_\eta(B \oplus D \oplus W_e; x)[s + 1] \downarrow = K_{s+1}(x)$ with use as a fresh number; Otherwise, define $\Gamma_\eta(B \oplus D \oplus W_e; x)[s + 1] \downarrow = K_{s+1}(x)$ with use $\gamma_\eta(x)[s + 1] = \gamma_\eta(x)[s']$, where s' is the last stage before $s + 1$ at which $\gamma_\eta(x)$ was defined. Let $\sigma_{s+1} \upharpoonright t + 1 = \eta^\wedge 1$, go to the next substage.

Case 2. $\xi = \alpha$ is an $\mathcal{N}_e$ -strategy. There are five subcases.

($\alpha 1$) If $s + 1$ is not α -expansionary, then let $\sigma_{s+1} \upharpoonright t + 1 = \alpha^\wedge f$. Go to the next substage.

($\alpha 2$) If $s + 1$ is α -expansionary, but for any $i < s + 1$, $\mathcal{S}_{\alpha,i}$ does not require attention, then $\sigma_{s+1} \upharpoonright t + 1 = \alpha \wedge d$, go to the next substage.

For the other subcases, the stage $s + 1$ is an α -expansionary stage, and there is some $\mathcal{S}_{\alpha,i}$ with $i < s + 1$ which can require attention. Let i be the least such number such that $\mathcal{S}_{\alpha,i}$ requires attention.

($\alpha 3$) If $\mathcal{S}_{\alpha,i}$ is inside a gap, then close the gap as follows, let v be the stage at which the gap is opened, for any $u \leq v$ with $p_\alpha(u) \uparrow$ currently, define $p_\alpha(u) = s + 1$. Let $\sigma_{s+1} = \alpha$, initialize all nodes to the right of $\alpha \wedge g_i$, and cancel all f_α^j for $j > i$. Go to the next stage. *(Let x be the number chosen by ($\alpha 5$) at the stage v . If $C_{s+1} \upharpoonright x \neq C_v \upharpoonright x$, then declare that the gap is closed successfully and $\mathcal{S}_{\alpha,i}$ is satisfied. Otherwise, declare that the gap is closed unsuccessfully.)*

($\alpha 4$) If $\mathcal{S}_{\alpha,i}$ is inside a cogap and there is some y such that $f_\alpha^i(y) \downarrow$ and $C_{s+1} \upharpoonright \varphi_\alpha(C; y)[v] \neq C_v \upharpoonright \varphi_\alpha(C; y)[v]$, where v is the last α -expansionary stage before the stage $s + 1$, then let $\sigma_{s+1} \upharpoonright t + 1 = \alpha \wedge d_i$, and cancel all f_α^j for $j > i$. Go to the next substage.

($\alpha 5$) If ($\alpha 3$) and ($\alpha 4$) fail, and $\mathcal{S}_{\alpha,i}$ is ready to open a gap, let v be the last $\alpha \wedge g_i$ -stage, then there is some x such that $x \in W_{i,s+1} - W_{i,v}$ and there is some $y < l(\alpha, s + 1)$ with $f_\alpha^i(y) \uparrow$ and $x > \max\{\varphi_\alpha(C; y')[s + 1] : y' \leq y\}$. Choose the least such x and y and define $f_\alpha^i(y') = \Phi_\alpha(C; y')[s + 1]$ for any $y' \leq y$ with $f_\alpha^i(y') \uparrow$ currently. Let $\sigma_{s+1} \upharpoonright t + 1 = \alpha \wedge g_i$, and cancel all f_α^j for $j > i$. Go to the next substage.

Case 3. $\xi = \beta$ is an $\mathcal{R}_e$ -strategy. There are some subcases.

($\beta 1$) If $k_\beta \uparrow$, then β chooses a fresh number as k_β and starts cycle 0. Let $\sigma_{s+1} = \beta$. Initialize all nodes with lower priority than σ_{s+1} and go to the next stage.

For the following subcases, $k_\beta \downarrow$. Let cycle n be the active cycle at the stage $s + 1$, and $n = 0$ if β is firstly visited (after β is reset or initialized). A cycle will keep active until it is explicitly declared to be inactive or it is canceled or β is either reset or initialized.

($\beta 2$) If β is satisfied at some stage $u < s + 1$ and it is not initialized at any stage v between u and $s + 1$, let $\sigma_{s+1} \upharpoonright t + 1 = \beta \wedge d$. Go to the next substage.

($\beta 3$) If $\beta 2$ fails and $s + 1$ is not a β -expansionary stage, let $\sigma_{s+1} \upharpoonright t + 1 = \beta^\wedge(n, f)$.
Go to the next substage.

($\beta 4$) If $s + 1$ is a β -expansionary stage, there are two cases.

($\beta 4.1$) If there is some η -strategy labeled $\mathcal{G}_e$ with $\eta \subset \beta$ which is active at β and not satisfied on the path starting from the root node and ending with the node β . Let $\eta_1 \subset \eta_2 \subset \dots \subset \eta_j \subset \beta$ be a list of all such $\mathcal{G}$ -strategies. Check whether there is some alive cycle $n' \leq n$ such that cycle n' can pass the threshold k_β for η_i for some $1 \leq i \leq j$ at this stage, and n' has passed the threshold k_β for η_k for any $k > i$ between stages v and $s + 1$. Here v is the last stage at which cycle n' was canceled if it exists, otherwise, $v = 0$.

($\beta 4.1.1$) If yes, then β first chooses the least such i . Now, β chooses the largest n' to pass the threshold if there are several such alive cycles on η_i . At last, β takes actions depending on the value of i as follows:

($\beta 4.1.1.1$) If $i > 1$, then β implements the threshold strategy as follows. (1) Cancel all cycles $> n'$ and let cycle $n' + 1$ be active; (2) Put $\gamma_{\eta_{i-1}}(k_\beta)[s]$ into D ; (3) Extend the definition of $\Theta_{\beta\eta_{i-1}}$, i.e. define $\Theta_{\beta\eta_{i-1}}(y)[s + 1] = W_{\eta_{i-1}, s+1}(y)$ with use of $s + 1$ for all $y \leq \gamma_{\eta_{i-1}}(k_\beta)[s]$ with $\Theta_{\beta\eta_{i-1}}(y) \uparrow$ currently, wait for $W_{\eta_{i-1}} \upharpoonright (\gamma_{\eta_{i-1}}(k_\beta)[s] + 1)$ -change (*if there is some number $\leq s$ entering B at some β -stage $u > s$, then we declare that cycle n' not alive after the stage u*); (4) Let $\sigma_{s+1} \upharpoonright t + 1 = \beta^\wedge g_{i-1}$. Go to the next substage.

($\beta 4.1.1.2$) If $i = 1$, let $s_0^{n'}$ be the β expansionary stage at which $\Delta_\beta^{n'}(B; y_{n'})$ was lastly defined, where $y_{n'}$ is the potential witness chosen by cycle n' . Then β makes a disagreement as follows. (1) Extract the numbers being enumerated into D after the stage $s_0^{n'}$ to make a disagreement at $y_{n'}$; (2) Put $s_0^{n'}$ into B to undefine all $\Delta_{\beta'}^k(B; z)$ and $\Theta_{\beta'\eta'}(B; z)$ (re)defined after the stage $s_0^{n'}$, where $\beta' \wedge(k, \infty) \subseteq \beta$ or $\beta' \wedge g_{\eta'} \subseteq \beta$. Declare that β is satisfied unless it is reset or initialized later, let $\sigma_{s+1} = \beta$. Initialize all nodes with priority lower than σ_{s+1} and go to the next stage.

($\beta 4.1.2$) If no, check whether $\Delta_\beta^n(B)[s] \upharpoonright \ell = W_{\beta, s+1} \upharpoonright \ell$ on the domain ℓ of $\Delta_\beta^n(B)[s]$.

($\beta 4.1.2.1$) If yes, then extend the definition of $\Delta_\beta^n(B)$ as follows, for all $y < l(\beta, s + 1)$ with $\Delta_\beta^n(B; y) \uparrow$ currently, define $\Delta_\beta^n(B; y)[s + 1] = W_{\beta, s+1}(y)$ with use

$s + 1$. Let $\sigma_{s+1} \upharpoonright t + 1 = \beta^\wedge(n, \infty)$ and go to the next substage.

($\beta 4.1.2.2$) If no, choose the least y , y_n say, such that $\Delta_\beta^n(B; y) \neq W_\beta(y)$ currently.

We say that y_n is the *potential witness* chosen by cycle n . Now, β takes actions as follows. (1) Put $\gamma_{\eta_j}(k_\beta)[s]$ into D ; (2) Extend the definition of $\Theta_{\beta\eta_j}(B)$ as follows, for all $z \leq \gamma_{\eta_j}(k_\beta)[s]$ with $\Theta_{\beta\eta_j}(B; z) \uparrow$ currently, define $\Theta_{\beta\eta_j}(B; z)[s + 1] = W_{\eta_j, s+1}(z)$ with use of $s + 1$, wait for a W_{η_j} -change below $\gamma_{\eta_j}(k_\beta)[s]$, (*if there is some number $\leq s$ entering B at some β -stage $u > s$, then we declare that cycle n' not alive after the stage u*); (3) Let cycle $n + 1$ be active. Let $\sigma_{s+1} \upharpoonright t + 1 = \beta^\wedge g_j$, and go to the next substage.

($\beta 4.2$) Otherwise, β will have only one active cycle $n = 0$ in the whole construction. Check whether $\Delta_\beta^0(B)[s] \upharpoonright \ell = W_{\beta, s+1} \upharpoonright \ell$ on the domain ℓ of $\Delta_\beta^0(B)[s]$.

($\beta 4.2.1$) If yes, then extend the definition of $\Delta_\beta^0(B)$, i.e. for any $z \leq l(\beta, s + 1)$ such that $\Delta_\beta^0(B; z) \uparrow$ currently, define $\Delta_\beta^0(B; z)[s + 1] = W_{\beta, s+1}(z)$ with use $s + 1$. Let $\sigma_{s+1} \upharpoonright t + 1 = \beta^\wedge(0, \infty)$, go to the next substage.

($\beta 4.2.2$) If no, then there is some y such that $\Delta_\beta^0(B; y)[s] \downarrow \neq W_{\beta, s+1}(y)$. Choose the least such y , let s' be the last stage at which $\Delta_\beta^0(B; y)$ was (re)defined. Do the following things: (1) Remove all numbers being enumerated into D after the stage s' ; (2) Put s' into B to undefine all $\Theta_{\beta'\eta'_i}(B; z)$ and $\Delta_{\beta'}^n(B; z)$ (re)defined after the stage s' , where β' is an $\mathcal{R}$ -strategy such that $\beta'^\wedge g_{\eta'_i}$ or $\beta'^\wedge(n, \infty) \subset \beta$. Declare that β is satisfied unless it is initialized, let $\sigma_{s+1} = \beta$. Initialize all nodes with lower priority than σ_{s+1} and go to the next stage.

This completes the full construction.

4.4 The verification

At first, we will prove that the true path $TP = \liminf_s \sigma_s$ is infinite.

Lemma 4.3. *Let σ be any node on TP , then*

(1) *σ can only be initialized or reset at most finitely often;*

(2) σ has an outcome $\mathcal{O}$ such that $\sigma \smallfrown \mathcal{O} \in TP$;

(3) σ can initialize the strategy $\sigma \smallfrown \mathcal{O}$ at most finitely many times.

Therefore, TP is infinite.

Proof. We will prove this lemma by induction on the length of σ . At first, we consider $\sigma = \lambda \in TP$, the root node of the priority tree. As λ is a $\mathcal{G}_0$ -strategy, it is easy to see that the lemma is true for λ .

For any non-root node $\sigma \in TP$, assume that σ^- is the immediate predecessor of σ . Suppose that (1), (2) and (3) hold for σ^- . In the following, we will prove that they hold for σ too.

By the induction hypothesis, let s_0 be the least stage after which σ^- can neither be reset nor initialized and also will never initialize σ . So σ can not be initialized after the stage s_0 . If σ is labeled R_e for some e , assume that k_σ is chosen at a stage $u > s_0$, then k_σ will be held by σ forever. σ can be reset only if there is some $k \leq k_\sigma$ entering into K . As there are at most $k_\sigma + 1$ numbers $\leq k_\sigma$, σ can be reset at most $k_\sigma + 1$ times. If σ is labeled $\mathcal{N}_e$ or $\mathcal{G}_e$ for some e , then σ will not be reset. Thus (1) is true for σ .

In the following, we will prove that (2) and (3) are true for σ . There are three cases.

Case 1. $\sigma = \alpha$ is an $\mathcal{N}_e$ -strategy.

(1.1) If there are only finitely many α -expansionary stages, then there is some stage $s_1 \geq s_0$ such that after which there is no any α -expansionary stage. By the construction, any α -stage $s > s_1$ is an $\alpha \smallfrown f$ -stage. So $\alpha \smallfrown f \in TP$ and α can not initialize $\alpha \smallfrown f$ after the stage s_1 .

In the following items (1.2), (1.3) and (1.4), there are infinitely many α -expansionary stages.

(1.2) There is no substrategy of α can require attention after some stage $s_1 \geq s_0$. By the construction, $\alpha \smallfrown d$ can be visited infinitely many times and there is no node to the left of $\alpha \smallfrown d$ can be visited after the stage s_1 . Hence $\alpha \smallfrown d \in TP$ and α can not initialize $\alpha \smallfrown d$ after the stage s_1 .

- (1.3) There is no $\mathcal{S}_{\alpha,i'}$ with $i' < i$ which requires attention after some stage $s_1 \geq s_0$, $\mathcal{S}_{\alpha,i}$ requires attention infinitely many times, and requires attention only as (r2.1). By the construction, $\alpha \smallfrown d_i$ is visited infinitely often and there is no node to the left of $\alpha \smallfrown d_i$ can be visited after the stage s_1 . So $\alpha \smallfrown d_i \in TP$ and α can not initialize $\alpha \smallfrown d_i$ after the stage s_1 .
- (1.4) There is no $\mathcal{S}_{\alpha,i'}$ with $i' < i$ which requires attention after some stage $s_1 \geq s_0$, (1.3) fails, and $\mathcal{S}_{\alpha,i}$ requires attention infinitely many times as (r2.2). By the construction, $\alpha \smallfrown g_i$ can be visited infinitely many times and there is no node to the left of $\alpha \smallfrown g_i$ can be visited after the stage s_1 . So $\alpha \smallfrown g_i \in TP$ and α can not initialize $\alpha \smallfrown g_i$ after the stage s_1 .

Note that it is not possible that there are infinitely many i such that $\mathcal{S}_{\alpha,i}$ can require attention but any such $\mathcal{S}_{\alpha,i}$ only requires attention finitely many times, since C is cappable.

Hence (2) and (3) are true in this case.

Case 2. $\sigma = \eta$ is a $\mathcal{G}_e$ -strategy.

By the construction, every η -stage $s > s_0$ is an $\eta \smallfrown 1$ stage. So $\eta \smallfrown 1 \in TP$ and η can not initialize $\eta \smallfrown 1$ after the stage s_0 .

So (2) and (3) are true in this case.

Case 3. $\sigma = \beta$ is an $\mathcal{R}_e$ -strategy.

- (3.1) If there is some η -strategy labeled by $\mathcal{G}_e$ for some e such that η is active at β and not satisfied on the path starting from the root node λ and ending with β , then let $\eta_1 \subset \eta_2 \subset \dots \subset \eta_j \subset \beta$ be a list of all such $\mathcal{G}$ -strategies. Assume that β can not be reset or initialized after a stage s_1 .

- (3.1.1) If β is satisfied at some stage $s_2 \geq s_1$, then any β -stage $s > s_2$ is a $\beta \smallfrown d$ -stage. Hence $\beta \smallfrown d \in TP$ and β will not initialize $\beta \smallfrown d$ after the stage s_2 .

In the following two subcases (3.1.2) and (3.1.3) we consider that after the stage s_1 there are only finitely many potential witnesses found by β and none of them passes all thresholds. Then there must be a stage $s_2 \geq s_1$ and a cycle, cycle n say, such that after s_2 cycle n is always active. Let $s_3 \geq s_2$ be the last stage at which β passes

a threshold. Then after the stage s_3 , no cycle requires to pass threshold and cycle n will never be canceled, furthermore, cycle n never find any potential disagreement witness. Thus after the stage s_3 , no node to the left of (n, ∞) can be visited again.

(3.1.2) If there are infinitely many β -expansionary stages, then each β -expansionary stage $> s_3$ is also a $\beta^\wedge(n, \infty)$ -stage. So $\beta^\wedge(n, \infty) \in TP$ and β will never initialize $\beta^\wedge(n, \infty)$ after the stage s_3 .

(3.1.3) If there are only finitely many β -expansionary stages, let $s_4 \geq s_3$ be the largest β -expansionary stage. Then after the stage s_4 , each β -stage is a $\beta^\wedge(n, f)$ -stage. So $\beta^\wedge(n, f) \in TP$ and β will never initialize $\beta^\wedge(n, f)$ after the stage s_4 .

(3.1.4) At last, we consider the subcase that after the stage s_1 there are infinitely many potential witnesses are found by β and none of them can pass all thresholds. In this case, there must be a least i with $0 \leq i \leq j$ such that there are infinitely many alive cycles on η_i finally. Then there is a stage $s_2 \geq s_1$ such that after which no alive cycle can pass threshold associated with η_i , that is, no node to the left of $\beta^\wedge g_i$ can be visited again after the stage s_2 . So $\beta^\wedge g_i$ can not be initialized by β after the stage s_2 . Since there are infinitely many alive cycles on η_i , $\beta^\wedge g_i$ will be visited infinitely often. Thus $\beta^\wedge g_i \in TP$.

(3.2) Otherwise, assume that β can not be initialized or reset after a stage $s_1 \geq s_0$.

(3.2.1) If β is satisfied at some stage $s_2 \geq s_1$, then any β -stage $s > s_2$ is a $\beta^\wedge d$ -stage. Hence $\beta^\wedge d \in TP$ and β will not initialize $\beta^\wedge d$ after the stage s_2 .

(3.2.2) (3.2.1) fails. If there is no β -expansionary stage after the stage $s_1 \geq s_0$, then every β -stage $s > s_1$ is a $\beta^\wedge(0, f)$ stage. Thus $\beta^\wedge(0, f) \in TP$ and β will not initialize $\beta^\wedge(0, f)$ after the stage s_1 .

(3.2.3) If there are infinitely many β -expansionary stages, it is easy to see that every β -expansionary stage $s > s_1$ is a $\beta^\wedge(0, \infty)$ stage. Otherwise, there is some β -expansionary stage $s > s_1$ at which β is satisfied, then at any stage $t > s$, β has outcome d , and hence after the stage s there is no a β -expansionary stage. It is a contradiction.

Hence (2) and (3) are true in this case. □

Now we show that all the requirements are satisfied.

Lemma 4.4. *For the $\mathcal{N}_e$ -requirement, let α be the last node labeled $\mathcal{N}_e$ on the true path TP , then $\mathcal{N}_e$ is satisfied by α .*

Proof. By Lemma 4.3, there is a stage s_0 after which α can not be initialized any more. There are four cases.

Case 1: $\alpha \wedge f \in TP$. By the construction, there are only finitely many α -expansionary stages, then the condition “ $\Phi_e(C) = \Phi_e(B) = f$ total” is not right, hence $\mathcal{N}_e$ is satisfied by α .

Case 2: $\alpha \wedge d \in TP$. By the construction, there is a stage $s_1 > s_0$ such that for any k , $\mathcal{S}_{\alpha,k}$ can not require attention after the stage s_1 . Choose the least i such that $\mathcal{S}_{\alpha,i}$ is not satisfied and $|W_i| = \infty$. By the construction, the definition of f_α^i can not be extended after the stage s_1 , choose the least y such that $f_\alpha^i(y) \uparrow$, let $t_0 < t_1 < \dots < t_n < \dots$ be a list of α -expansionary stages after the stage s_1 such that $y < l(\alpha, t_0)$ and there is some $x_n \in W_{i,t_n} - W_{i,t_{n-1}}$ with $x_n \leq \max\{\varphi_\alpha(C; y')[t_n] : y' \leq y\}$ for any $n > 0$. Then $\sup\{x_n : n \in \omega\} = \infty$. There is a $y' \leq y$ such that $\lim_{s \rightarrow \infty} \varphi_\alpha(C; y')[s] = \infty$ since there are only finitely many $y' \leq y$. Thus $\Phi_\alpha(C; y') \uparrow$, $\mathcal{N}_e$ is satisfied.

Case 3: $\alpha \wedge d_i \in TP$. Assume that after a stage $s_1 \geq s_0$, $\alpha \wedge d_i$ can not be initialized. By the construction, there is some y with $f_\alpha^i(y) \downarrow$ such that $\lim_{s \rightarrow \infty} \varphi_\alpha(C; y)[s] = \infty$, then $\Phi_\alpha(C; y) \uparrow$. Hence $\mathcal{N}_e$ is satisfied.

Case 4: $\alpha \wedge g_i \in TP$. By the construction, there is a stage $s_1 \geq s_0$ such that after which $\alpha \wedge g_i$ will not be initialized. $\mathcal{S}_{\alpha,i}$ -gap can be opened infinitely often, and any $\mathcal{S}_{\alpha,i}$ -gap can not be closed successfully. By the construction, f_α^i is computable. Assume that $t_0 < u_0 < \dots < t_n < u_n < \dots$ are all the α -expansionary stages after the stage s_1 at which $\mathcal{S}_{\alpha,i}$ -gap is opened and closed alternatively. In the following, we will prove that if the condition of the $\mathcal{N}_e$ -requirement “ $\Phi_e(C) = \Phi_e(B) = f$ total” is true, then $f = f_\alpha^i$.

For any y , let t be the least α -expansionary stage after s_1 such that $f_\alpha^i(y)$ is defined at which, then $t = t_n$ for some n . In the following, we prove that $f(y) = f_\alpha^i(y)$.

(1) At the stage t_k for $k \geq n$, $\mathcal{S}_{\alpha,i}$ -gap is opened, so there is some $x > \varphi_e(C; y)[t_k]$ which enters W_i between stages t_{k-1} and t_k , where t_{-1} is the last α -expansionary stage before the stage t_0 . We know that $C_{t_k} \upharpoonright x = C_{u_k} \upharpoonright x$, otherwise, $\mathcal{S}_{\alpha,i}$ will be satisfied, it is a contradiction. Thus $\Phi_e(B; y)[t_k] \downarrow = \Phi_e(C; y)[t_k] \downarrow = \Phi_e(C; y)[u_k] \downarrow = \Phi_e(B; y)[u_k] \downarrow$.

(2) At the stage u_k for $k \geq n$, $\mathcal{S}_{\alpha,i}$ -gap is closed. All the nodes to the right of $\alpha \wedge g_i$ are initialized, and the stage u_k is stopped. For any stage s between u_k and t_{k+1} , there is no small number entering B , so $\Phi_e(C; y)[u_k] \downarrow = \Phi_e(B; y)[u_k] \downarrow = \Phi_e(B; y)[t_{k+1}] \downarrow = \Phi_e(C; y)[t_{k+1}] \downarrow$.

By (1) and (2), $f_\alpha^i(y) = \Phi_e(C; y)[t_k] = \Phi_e(C; y)[u_k]$ for any $k \geq n$. If $\Phi_e(C) = \Phi_e(B) = f$ total, then $f(y) = \Phi_e(C; y) \downarrow = \Phi_e(C; y)[t_n]$. Hence $f = f_\alpha^i$ is computable. $\square$

Lemma 4.5. *Let β be the last node labeled $\mathcal{R}_e$ on the true path TP , then $\mathcal{R}_e$ -requirement is satisfied at β .*

Proof. At first, there is no η labeled $\mathcal{G}_{e'}$ for some e' such that $\eta \subset \beta \wedge g_\eta \in TP$. Otherwise, there is some backup strategy $\beta' \in TP$ for $\mathcal{R}_e$, it contradicts the assumption on β . Hence there are only three cases.

Case 1: $\beta \wedge d \in TP$. Assume that $\beta \wedge d$ can not be initialized or reset after a stage s_0 . By the construction, there is a cycle n which can pass the threshold for all η_i for $i = 1, \dots, j$ after or at the stage s_0 , (where $\eta_1 \subset \eta_2 \subset \dots \subset \eta_j \subset \beta$ are $\mathcal{G}$ -strategies which are active at β and not satisfied on the path starting from the root node and ending with β), assume that $s \geq s_0$ is the stage such that cycle n passes the threshold for η_1 , y_n is the potential witness of strategy β found by cycle n and $\Delta_\beta^n(B; y_n)$ is lastly (re)defined at a stage t with $s_0 \leq t < s$. By the construction, we recover the computation $\Lambda_\beta(B \oplus D; y_n)$ to $\Lambda_\beta(B \oplus D; y_n)[t] = 0 \neq 1 = W_{\beta,s}(y_n)$, and the computation $\Lambda_\beta(B \oplus D; y_n)[s] = \Lambda_\beta(B \oplus D; y_n)[t]$ is preserved by the initialization. Hence $\mathcal{R}_e$ is satisfied.

Case 2: $\beta \wedge (n, f) \in TP$, then there are finitely many β -expansionary stages. $\Lambda_\beta(B \oplus D) \neq W_\beta$, so $\mathcal{R}_e$ is satisfied.

Case 3: $\beta \wedge (n, \infty) \in TP$, assume that $\beta \wedge (n, \infty)$ can not be initialized or reset after a stage s_0 , then the definition of $\Delta_\beta^n(B)$ and W_β will be consistent forever after the stage s_0 , otherwise, there is some stage $s > s_0$ such that there is some node to the left of $\beta \wedge (n, \infty)$ can be accessible. Then it suffices to prove that $\Delta_\beta^n(B; y)$ can be undefined finitely many times for any y , since $\Delta_\beta^n(B)$ can be defined infinitely many times. Due to the initialization action, we only need to consider the injury of β' -node labeled $\mathcal{R}_{e'}$ for some e' with $\beta \wedge (n, \infty) \subseteq \beta'$. Fix y , assume that $s_1 \geq s_0$ is the first stage at which $\Delta_\beta^n(B; y)$ is defined. At a stage $s_2 \geq s_1$, $\Delta_\beta^n(B; y)$ is firstly undefined

by the enumeration of some number $x < s_1$ into B due to some $\beta' \supseteq \beta^\wedge(n, \infty)$ labeled $\mathcal{R}_{e'}$ for some e' . Then at the next β -expansionary stage s_3 , $\Delta_\beta^n(B; y)$ is redefined with use s_3 . By the construction, all the nodes with priority lower than β' are initialized at the stage s_2 . After the stage s_2 , only the nodes with priority higher than β' which are visited before the stage s_2 can undefine the definition of $\Delta_\beta^n(B; y)$. The number of such nodes is finite, thus $\Delta_\beta^n(B; y)$ can be undefined finitely many times.

□

Lemma 4.6. *Suppose that $\eta \in TP$ is a $\mathcal{G}_e$ -strategy which is active on all path $TP \upharpoonright n$ for $n > |\eta|$, and there is an $\mathcal{R}$ -strategy $\beta \in TP$ such that $\eta \subset \beta^\wedge g_\eta \in TP$, then $\Theta_{\beta\eta}(B) = W_\eta$.*

Proof. By the construction, β will define $\Theta_{\beta\eta}$ infinitely often. First note that $\Theta_{\beta\eta}(B)$ is total. The proof for “ $\Theta_{\beta\eta}(B)$ is total” is similar to the argument for “ $\Delta_\beta^n(B)$ is total” in Lemma 4.5. Moreover, $\Theta_{\beta\eta}$ computes W_η correctly, and again, its proof is similar to the proof for “ $\Delta_\beta^n(B) = W_\beta$ ” in Lemma 4.5.

□

Lemma 4.7. *Suppose that η is the last node on TP labeled $\mathcal{G}_e$ for some fixed e .*

(1) *If $\Gamma_\eta(B \oplus D \oplus W_\eta)$ is total, then $\Gamma_\eta(B \oplus D \oplus W_\eta) = K$.*

(2) *If $\Gamma_\eta(B \oplus D \oplus W_\eta)$ is not total, then there is some β -node on TP labeled $\mathcal{R}_{e'}$ for some e' with $\eta \subset \beta^\wedge g_\eta \in TP$.*

Proof. (1) If $\Gamma_\eta(B \oplus D \oplus W_\eta)$ is total, then by the construction, for any z , $\Gamma_\eta(B \oplus D \oplus W_\eta; z) = K(z)$. The requirement $\mathcal{G}_e$ is satisfied at η via $\Gamma_\eta(B \oplus D \oplus W_\eta) = K$.

(2) If $\Gamma_\eta(B \oplus D \oplus W_\eta)$ is not total, then there is some k such that $\Gamma_\eta(B \oplus D \oplus W_\eta; k)$ is undefined infinitely often, let k be the least such number. By the construction, there is some β -node on TP labeled $\mathcal{R}_{e'}$ for some e' with $\eta \subset \beta^\wedge g_\eta \in TP$ such that $k = k_\beta$. β runs infinitely many cycles, and after some stage any cycle can not pass the threshold k_β for η , but there are infinitely many alive cycles can pass the threshold k_β for any $\eta' \supset \eta$ which is active and not satisfied at β . As η is the last node on TP labeled $\mathcal{G}_e$ for some fix e , then η is active on $TP \upharpoonright n$ for any $n > |\beta|$. By Lemma 4.6, $\Theta_{\beta\eta}(B)$ is total, and compute W_η correctly. So $\mathcal{G}_e$ is satisfied at β via $\Theta_{\beta\eta}(B) = W_\eta$.

In summary, $\mathcal{G}_e$ is satisfied on the true path.

□

This completes the proof of Theorem 3.

Bibliography

- [1] K. Ambos-Spies, Jr., C. G. Jockusch, R. A. Shore, and R. I. Soare. An algebraic decomposition of the recursively enumerable degrees and the coincidence of several classes with the promptly simple degrees. *Transactions of the American Mathematical Society*, 281:109–128, 1984.
- [2] M. M. Arslanov. Structural properties of the degrees below $\mathbf{0}'$. *Doklady Akademii Nauk SSSR*, 283:270–273, 1985.
- [3] M. M. Arslanov, I. Sh. Kalimullin, and S. Lempp. On Downey’s conjecture. *J. Symbolic Logic*, 75(2):401–441, 2010.
- [4] M. M. Arslanov, S. Lempp, and R. A. Shore. On isolating r.e. and isolated d-r.e. degrees. In *Computability, enumerability, unsolvability*, volume 224 of *London Math. Soc. Lecture Note Ser.*, pages 61–80. Cambridge Univ. Press, Cambridge, 1996.
- [5] M. Cai, R. A. Shore, and T. A. Slaman. The n-r.e. degrees: undecidability and Σ_1 substructures. to appear.
- [6] C. S. Calude, P. H. Hertling, B. Khoussainov, and Y. Wang. Recursively enumerable reals and Chaitin’s Ω number. *Lecture Notes in Computer Science*, 1373:596–606, 1998.
- [7] A. Church. A set of postulates for foundation of logic (second paper). *Annals of Mathematics*, 34:839–864, 1933.
- [8] A. Church. An Unsolvable Problem of Elementary Number Theory. *American Journal of Mathematics*, 58:345–363, 1936.

- [9] S. B. Cooper. *Degrees of unsolvability, Ph.D. thesis*. PhD thesis, Leicester University, Leicester, England, 1970.
- [10] S. B. Cooper. Minimal pairs and high recursively enumerable degrees. *The Journal of Symbolic Logic*, 39:655–660, 1974.
- [11] S. B. Cooper. On a theorem of C. E. M. Yates. handwritten notes, 1974.
- [12] S. B. Cooper, L. A. Harrington, A. H. Lachlan, S. Lempp, and R. I. Soare. The d.r.e. degrees are not dense. *Annals of Pure and Applied Logic*, 55:125–151, 1991.
- [13] S. B. Cooper, S. Lempp, and P. Watson. Weak density and cupping in the d-r.e. degrees. *Israel Journal of Mathematics*, 67:137–152, 1989.
- [14] S. B. Cooper, M. C. Salts, and G. Wu. The non-isolating degrees are upwards dense in the computably enumerable degrees. In *Theory and applications of models of computation*, volume 4978 of *Lecture Notes in Comput. Sci.*, pages 588–596. Springer, Berlin, 2008.
- [15] S. B. Cooper and X. Yi. Isolated d.r.e. degrees. Preprint Series, 1995.
- [16] R. Dedekind. Was sind und was sollen die zahlen. Braunschweig, 1888.
- [17] D. Ding and L. Qian. Isolated d.r.e. degrees are dense in r.e. degree structure. *Archive for Mathematical Logic*, 36:1–10, 1996.
- [18] R. G. Downey. D.r.e. degrees and the nondiamond theorem. *The Bulletin of the London Mathematical Society*, 21:43–50, 1989.
- [19] R. G. Downey and L. A. Harrington. There is no fat orbit. *Annals of Pure and Applied Logic*, 80:277–289, 1996.
- [20] R. G. Downey and Jr. C. G. Jockusch. T-degrees, jump classes, and strong reducibilities. *Transactions of the American Mathematical Society*, 301:103–136, 1987.
- [21] R. G. Downey and G. LaForte. Presentations of computably enumerable reals. *Theoretical Computer Science*, 284:539–555, 2002. Computability and complexity in analysis (Castle Dagstuhl, 1999).

- [22] R. G. Downey, S. Lempp, and R. A. Shore. Highness and Bounding Minimal Pairs. *Mathematical Logic Quarterly*, 39:475–491, 1993.
- [23] R. G. Downey, A. Li, and G. Wu. Complementing cappable degrees in the difference hierarchy. *Annals of pure and applied logic*, 125:101–118, 2004.
- [24] R. G. Downey and M. Stob. Jumps of hemimaximal sets. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 37:113–120, 1991.
- [25] R. G. Downey and M. Stob. Automorphisms of the lattice of recursively enumerable sets: orbits. *Advances in Mathematics*, 92:237–265, 1992.
- [26] R. L. Epstein. *Degrees of unsolvability: structure and theory*, volume 759 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.
- [27] Yu. L. Ershov. A hierarchy of sets, part i. *Algebra and Logic*, 7:24–43, 1968.
- [28] Yu. L. Ershov. A hierarchy of sets, part ii. *Algebra and Logic*, 7:212–232, 1968.
- [29] Yu. L. Ershov. A hierarchy of sets, part iii. *Algebra and Logic*, 9:20–31, 1970.
- [30] C. Fang, J. Liu, and G. Wu. Cupping and Diamond embeddings: A unifying approach. *Lecture Notes in Computer Science*, 6735:71–80, 2011.
- [31] C. Fang, S. Wang, and G. Wu. Highness and Locally Noncappability (accepted). *Lecture Notes in Computer Science*.
- [32] C. Fang and G. Wu. The Distribution of Simple c.e. Reals in the High/Low Hierarchy. in preparation.
- [33] C. Fang and G. Wu. Nonhemimaximal Degrees and the High/Low Hierarchy. *Journal of Symbolic Logic*, in Press.
- [34] R. M. Friedberg. Two recursively enumerable sets of incomparable degrees of unsolvability (solution of Post’s problem, 1944). *Proceedings of the National Academy of Sciences of the United States of America*, 43:236–238, 1957.
- [35] R. M. Friedberg. Three theorems on recursive enumeration. I. Decomposition. II. Maximal set. III. Enumeration without duplication. *The Journal of Symbolic Logic*, 23:309–316, 1958.

- [36] K. Gödel. On undecidable propositions of formal mathematical systems. *mimeographed notes*, pages 41–81, 1965.
- [37] E. M. Gold. Limiting recursion. *The Journal of Symbolic Logic*, 30:28–48, 1965.
- [38] L. Harrington and S. Shelah. The undecidability of the recursively enumerable degrees. *American Mathematical Society. Bulletin. New Series*, 6:79–80, 1982.
- [39] L. A. Harrington. On Cooper’s proof of a theorem of Yates. Parts I and II (handwritten notes), 1976.
- [40] D. Hilbert and P. Bernays. *Grundlagen der Mathematical*. Berlin, 1934.
- [41] S. Ishmukhametov and G. Wu. Isolation and the high/low hierarchy. *Archive for Mathematical Logic*, 41:259–266, 2002.
- [42] D. Kaddah. Infima in the d.r.e. degrees. *Annals of Pure and Applied Logic*, 62:207–263, 1993.
- [43] S. K. Kleene. General recursive functions of natural numbers. *Mathematische Annalen*, 112:727–742, 1936.
- [44] A. H. Lachlan. Lower bounds for pairs of recursively enumerable degrees. *Proceedings of the London Mathematical Society. Third Series*, 16:537–569, 1966.
- [45] A. H. Lachlan. Distributive initial segments of the degrees of unsolvability. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 14:457–472, 1968.
- [46] A. H. Lachlan. On the lattice of recursively enumerable sets. *Transactions of the American Mathematical Society*, 130:1–37, 1968.
- [47] A. H. Lachlan. Bounding minimal pairs. *The Journal of Symbolic Logic*, 44:626–642, 1979.
- [48] G. LaForte. The isolated d.r.e. degrees are dense in the r.e. degrees. *Mathematical Logic Quarterly*, 42:83–103, 1996.
- [49] M. Lerman. *Degrees of Unsolvability*. Springer-Verlag, Berlin, 1983.

- [50] M. Lerman. Degrees which do not bound minimal degrees. *Annals of Pure and Applied Logic*, 30:249–276, 1986.
- [51] A. Li. Elementary differences among jump classes. *Theoretical Computer Science*, 410:1042–1053, 2009.
- [52] A. Li, G. Wu, and Y. Yang. Bounding computably enumerable degrees in the Ershov hierarchys. *Annals of Pure and Applied Logic*, 141:79–88, 2006.
- [53] A. Li, G. Wu, and Z. Zhang. A hierarchy for cuppable degrees. *Illinois Journal of Mathematics*, 44:619–632, 2000.
- [54] A. Li and X. Yi. Cupping the recursively enumerable degrees by d.r.e. degrees. *Proceedings of the London Mathematical Society. Third Series*, 79:1–21, 1999.
- [55] A. Li and Y. Zhao. Plus cupping degrees do not form an ideal. *Science in China. Series F. Information Sciences*, 47:635–654, 2004.
- [56] J. Liu and G. Wu. Almost universal cupping and diamond embedding. In press.
- [57] J. Liu and G. Wu. An almost-universal cupping degree. *The Journal of Symbolic Logic*, 76:1137–1152, 2011.
- [58] D. A. Martin. Classes of recursively enumerable sets and degrees of unsolvability. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 12:295–310, 1966.
- [59] D. Miller. High recursively enumerable degrees and the anti-cupping property. In *Conference of the Applied Econometrics Association (AEA) on Exchange rates and Risk Econometrics*, 2006.
- [60] A. A. Muchnick. On the unsolvability of the problem of reducibility in the theory of algorithms. *Doklady Akademii Nauk SSSR*, 108:194–197, 1956.
- [61] P. Odifreddi. *Classical recursion theory I*. North-Holland, 1989.
- [62] P. Odifreddi. *Classical recursion theory II*. North-Holland, 1999.
- [63] R. Peter. Über den zusammenhang der verschiedenen begriffe der rekursiiven funktion. *Mathematische Annalen*, 110:612–632, 1934.

- [64] E. L. Post. Recursively enumerable sets of positive integers and their decision problems. *Bulletin of the American Mathematical Society*, 50:284–316, 1944.
- [65] E. L. Post. Absolutely unsolvable problems and relatively undecidable propositions. *Account of an anticipation*, in *Davis*, pages 340–433, 1965.
- [66] H. Putnam. Trial and error predicates and the solution to a problem of Mostowski. *The Journal of Symbolic Logic*, 30:49–57, 1965.
- [67] R. W. Robinson. Simplicity of recursively enumerable sets. *The Journal of Symbolic Logic*, 32:162–172, 1967.
- [68] G. E. Sacks. *Degrees of unsolvability*. Princeton University Press, Princeton, N.J., 1963.
- [69] G. E. Sacks. On the degrees less than $0'$. *Annals of Mathematics. Second Series*, 77:211–231, 1963.
- [70] G. E. Sacks. Recursive enumerability and the jump operator. *Transactions of the American Mathematical Society*, 108:223–239, 1963.
- [71] G. E. Sacks. The recursively enumerable degrees are dense. *Annals of Mathematics. Second Series*, 80:300–312, 1964.
- [72] M. C. Salts. An interval of computably enumerable isolating degrees. *Mathematical Logic Quarterly*, 45:59–72, 1999.
- [73] D. Seetapun. *Contributions to recursion theory*. PhD thesis, Trinity College, Cambridge, 1991.
- [74] J. R. Shoenfield. Undecidable and creative theories. *Polska Akademia Nauk. Fundamenta Mathematicae*, 49:171–179, 1961.
- [75] J. R. Shoenfield. An application of model theory to degrees of unsolvability. In L. Henkin J. W. Addison and A. Tarski, editors, *Symposium on the Theory of Models*, pages 359–363. North-Holland, Amsterdam, 1965.
- [76] R. I. Shore and T. A. Slaman. Working below a high recursively enumerable degree. *The Journal of Symbolic Logic*, 58:824–859, 1993.

- [77] T. Skolem. Begründung der elementaren arithmetik durch die rekurrirende denkweise ohne anwendung scheinbarer veränderlichen mit unendlichem ausdehnungsbereich. In From Frege to Gödel, editor, *Jean van Heijenoort*, pages 302–333. Harvard University Press, 1923.
- [78] T. A. Slaman and R. I. Soare. Extension of embeddings in the computably enumerable degrees. *Annals of Mathematics. Second Series*, 154:1–43, 2001.
- [79] R. I. Soare. Recursive theory and Dedekind cuts. *Transactions of the American Mathematical Society*, 140:271–294, 1969.
- [80] R. I. Soare. Automorphisms of the lattice of recursively enumerable sets. I. Maximal sets. *Annals of Mathematics. Second Series*, 100:80–120, 1974.
- [81] R. I. Soare. *Recursively enumerable sets and degrees*. Springer-Verlag, Heidelberg, 1987.
- [82] F. Stephan and G. Wu. Highness, Nonboundings and Presentations. to appear.
- [83] A. M. Turing. On computable numbers with an application to the entscheidungsproblem. *Proceedings London Mathematical Society*, 42:230–265, 1936.
- [84] A. M. Turing. Systems of logic based on ordinals. *Proceedings London Mathematical Society, Second Series*, 45:161–228, 1939.
- [85] L. Welch. *A hierachy of families of recursively enumerable degrees and a theorem on bounding minimal pairs*. PhD thesis, University of Illinois, Urbana-Champaign, Illinois, 1981.
- [86] G. Wu. Prefix-free languages and initial segments of computably enumerable degrees. In *Computing and combinatorics (Guilin, 2001)*, volume 2108 of *Lecture Notes in Comput. Sci.*, pages 576–585. Springer, Berlin, 2001.
- [87] G. Wu. Isolation and lattice embeddings. *The Journal of Symbolic Logic*, 67:1055–1064, 2002.
- [88] G. Wu. *Structure Properties of D. C. E. Degrees and Presentations of C. E. Reals*. PhD thesis, Victoria University of Wellington, 2002.

- [89] Y. Yang and L. Yu. On Σ_1 -structural differences among finite levels of the Ershov hierarchies. *The Journal of Symbolic Logic*, 71:1223–1236, 2006.
- [90] C. E. M. Yates. A minimal pair of recursively enumerable degrees. *The Journal of Symbolic Logic*, 31:159–168, 1966.