Crack-inclusion interaction problems with dugdale plastic zone correction

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**NANYANG
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**CRACK-INCLUSION INTERACTION PROBLEMS WITH
DUGDALE PLASTIC ZONE CORRECTION**

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SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING
2012**

CRACK-INCLUSION INTERACTION PROBLEMS WITH DUGDALE PLASTIC ZONE CORRECTION

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List of Symbols

a	Half crack length
b	Burgers vector
CTOD, δ	Crack tip opening displacement
E	Young's modulus
K_I, K_{II}	Stress intensity factor due to applied loads
K_ρ	Stress intensity factor due to closure stresses
PZS, ρ	Plastic zone size
κ_i	Kolosov's constant for material i
λ_i	Lame's first parameter of material i
μ_i	Lame's second parameter, also the shear modulus of material i
ν_i	Poisson's ratio of material i
σ	Stress component
δ_y^L	y-displacement CTOD for the left tip
δ_y^R	y-displacement CTOD for the right tip
δ_x^L	x-displacement CTOD for the left tip
δ_x^R	x-displacement CTOD for the right tip
K_I^L	K_I for the left tip
K_I^R	K_I for the right tip
K_{II}^L	K_{II} for the left tip
K_{II}^R	K_{II} for the right tip

Summary

The plastic zone correction technique is commonly used to more accurately analyse the fracture behaviour of cracks in ductile materials (such as metal-matrix-composite (MMC) materials). Previous research work on crack tip plastic zone correction restricts to such cases that the crack should be in homogeneous materials.

In the current PhD study, the Dugdale plastic zone correction was applied to solve various crack-inclusion interaction problems. In the Dugdale model, the plastic zone is modelled as a thin yielded strip (at yield stress σ_Y) ahead of each crack tip. This is achieved by ensuring that the Dugdale's condition of vanishing stress intensity factors at the each crack tip is fulfilled.

The first problem investigated in our study was a Griffith crack interacting with a circular inclusion with plastic zone correction added at both crack tips. The distributed dislocation technique is used to model the crack, using the solution for a single dislocation interacting with an inclusion as the Green's function. The problem is then formulated into a set of singular integral equations, which are subsequently solved numerically using Erdogan and Gupta's method for the stress intensity factors. Using the applied tensile load gives the stress intensity factors K_I , while using the closure stresses (of yield stress σ_Y) gives closure stress intensity factors K_ρ . Initial values of the plastic zone sizes ρ_L and ρ_R (at the left and right crack tips, respectively) are assumed, following which K_I and K_ρ are calculated. ρ_L and ρ_R are adjusted by iteration until K_I and K_ρ are balanced at both crack tips. The obtained ρ_L and ρ_R values are subsequently used to calculate the crack tip opening displacement (CTOD) values. The influence of various parameters on the CTOD is studied. Our numerical examples show that as fracture criterion, using K_I underestimates fracture-initiation conditions compared to using CTOD.

The second problem looked at a Griffith crack interacting with a coated circular inclusion under various loading conditions, with plastic zones added at both crack tips. In this problem, three different materials are involved: the matrix, the inclusion and the coating layer. The influence of the coating phase on fracture behaviour was the main consideration. The coating phase, if “harder” than the matrix material, will shield (retard) the crack propagation by reducing the CTOD, and if it is “softer” than the matrix material, will cause anti-shielding effect on the crack by increasing the CTOD. A thicker coating phase increases the shielding (if “harder”) or anti-shielding (if “softer”) effect.

A Griffith crack in a three-phase composite material was investigated under various loading conditions. In this problem, the composite material reinforced by circular inclusions is simulated by the three-phase model: the crack is near a single inclusion but embedded purely in the matrix material, while the rest (the outer phase) is an equivalent composite phase. With this three-phase composite model, the influence of other inclusions on the crack is also considered. Our study indicates that increasing the inclusion concentration promotes the effect of the inclusions on the CTOD and plastic zone size (PZS). Hence if the inclusion is “harder” than the matrix, the shielding effect increases with inclusion concentration, while if the inclusion is “softer” than the matrix, the anti-shielding effect increases with inclusion concentration.

The Zener-Stroh crack is complementary to the Griffith crack, with a blunt tip and a sharp tip, with the crack always propagating from the sharp crack tip. Two Zener-Stroh crack problems were solved in our study. The first is a Zener-Stroh crack interacting with a circular inclusion. The effect of various parameters on the sharp crack tip was looked into. This is followed by the Zener-Stroh crack in a three-phase composite, with both the blunt-sharp and sharp-blunt cracks studied. The influence of the shear modulus, load-to-yield-stress ratio,

Poisson's ratio, crack size, inclusion size and distance on the fracture behaviour of the crack was studied in detail.

We used two different ways to verify our obtained results. One method was to reduce our current models to simpler case where the crack locates in a homogeneous material without inclusion. The three-phase problems can be reduced to the two-phase ones, and can also be further reduced to the homogeneous case. The second method for verification is to use finite element analysis.

CHAPTER 1

Introduction

1.1. Background

For failure analysis and prevention of engineering structures and materials, mechanical engineers need to consider two basic forms of fracture failure: brittle and ductile fracture.

Early failure analysis stipulated that the maximum stress should be below material strength, failing which yielding or ductile fracture occurs. Hence the maximum stress was only compared to the maximum material strength, which was an independent material property.

However, this approach is not valid under multi-axial loads. Hence other criteria based on stresses, strains or energy principle were proposed to address such situations. The applied stress was checked against the proposed fracture criteria that accounted for multi-axial loads. Failure analysis based on those criteria was still insufficient in predicting brittle failure, and safety factors were utilized to offset these concerns. In some cases structures still failed below the design capacity, prompting the need for more accurate methods.

In one of Leonardo da Vinci's notebooks, he describes an experiment where the tensile strengths of various lengths of metal wire were measured. He concluded that longer wires were weaker than shorter wires [1]. This gave a hint as to how material geometry, specifically flaws could influence failure too.

The starting point of fracture mechanics was Griffith's work [2], which made use of energy considerations to formulate a fracture theory for the problem of unstable propagation of a crack. For a perfectly brittle material, crack extension occurs only when the energy

release rate G is greater than G_c , the critical energy release rate required to form new surfaces during crack propagation. This condition can be written as:

$$G = \pi\sigma^2 a/E > G_c = \pi\sigma_f^2 a_c/E \quad (1.1)$$

where a represents the half-crack length, a_c the critical crack length, σ the applied stress and σ_f the failure stress,

However, the equation in this form was not practical. Irwin [3] later made use of the Westergaard [4] approach to show that the stresses and displacements near the crack tip can be described by a single parameter, now commonly known as the stress intensity factor, K_I :

$$K_I = \sigma\sqrt{\pi a} \quad (1.2)$$

where K_I and G are related by:

$$K_I = \sqrt{GE} \quad (1.3)$$

This parameter related load and flaw size to the stress intensity factor, making it possible to characterize the stress fields in the neighbourhood of the crack tip. However, Griffith's approach was only relevant to ideally brittle solids such as glass, but not ductile materials like metals. This is because linear elastic fracture mechanics ceases to be valid when significant plastic deformation precedes failure.

To correct for yielding at the crack tip, Irwin [5], Dugdale [6] and Barenblatt [7] proposed different models of crack tip plasticity. Meanwhile Wells [8] proposed an alternative fracture criterion when plastic deformation precedes failure. Wells noticed that in ductile materials, crack faces moved apart with plastic deformation. This led to the crack tip opening displacement (CTOD) as a fracture criterion. Rice [9] developed the J-integral to characterize nonlinear material behaviour ahead of a crack.

1.2. Motivations and Objectives

A pile-up of dislocations can form a crack. The study of cracks in composite materials has gained importance due to the wide applications the composite materials possess. However, there has been little work on small-scale yielding in composite materials. The current research work aims to characterize crack behaviour with small-scale yielding in composite materials using the CTOD, and at the same time determine the plastic zone size.

The objectives of the current work are to:

- To understand how a pile-up of dislocations form a crack.
- To develop a generalized methodology for evaluating plastic zone size and crack tip opening displacement with Dugdale model when the fracture behaviour of a crack is influenced by near-by inclusions.
- To solve several crack-inclusion interaction problems with small-scale yielding.

Our work starts with a classical Griffith crack interacting with an inclusion, with small-scale yielding (crack tip plastic zone) at both crack tips. The effects of a coated inclusion on a near-by crack (with plastic zone correction) are investigated further. The basic unit cell case (the first case) can then be expanded to the three-phase composite model, where an equivalent composite phase is introduced.

These methods are also used to analyse the cases of a Zener-Stroh crack interacting with a circular inclusion, and further extended to three-phase composite model.

1.3. Overview

In this thesis, Chapter 2 reviews previous works on dislocation-inclusion interaction, as well as crack-inclusion interactions. Dislocations are used to simulate the Griffith crack and Zener-Stroh cracks. The numerical method used to solve the singular equations is also reviewed.

Chapter 3 solves the basic “single cell” problem of a crack interacting with a circular inclusion, considering Dugdale small-scale yielding at crack tips. Chapter 4 looks at the physical problem on the interaction between a crack and a coated circular inclusion with plastic zone correction. Chapter 5 solves the Griffith crack-inclusion interaction in a three-phase composite model.

Chapter 6 solves the basic “single cell” problem again, but this time for a Zener-Stroh crack interacting with a circular inclusion, considering Dugdale small-scale yielding. This will be expanded to the three-phase composite model in Chapter 7.

Chapter 8 discusses the limitations of the analysis, and verifies some of the results using finite element method. The final chapter will be the conclusions and recommendations for future work.

2.1. Dislocation Concepts

In the current research, the distributed dislocation method is used for our crack-inclusion investigations. It is worth reviewing some basic concepts about dislocations. A physical dislocation is a lattice line defect within the crystal structure, while a crack can be made equivalent to a large number of dislocations along the crack line [10]. The Burgers vector associated with a dislocation describes the distortion of the lattice caused by the defect, and can be used as a measure of the amount of dislocations present in the structure. There are two main types of dislocation, namely the screw and the edge dislocations. Dislocations found in actual materials are usually mixed. In our work, we will be dealing mainly with edge dislocations.

2.1.1. The edge dislocation

Figure 2.1 The edge dislocation [11]

An edge dislocation is characterised by the presence of an extra half-plane in the crystal, causing distortion to nearby planes. The dislocation line is shown by AB in Figure 2.1, where the additional plane causes the formation of the edge dislocation. In the resulting structure with the dislocation, the upper half of the lattice (containing the extra plane) is

compressed while there is dilatation in the lower half. In Figure 2.1, the edge dislocation has a Burgers vector b perpendicular to the dislocation line.

By convention, the edge dislocation with the extra half plane at the top shown is defined as a positive edge dislocation. Similarly, a negative edge dislocation occurs when the extra half plane is in the lower half of the crystal.

2.2. Mathematical Simulation of a Crack by Dislocations

2.2.1. Griffith crack

Mathematically, a Griffith crack can be simulated by a continuous distribution of dislocations [10] along the crack plane. When a large number of discrete dislocations are present along a line, each single dislocation of Burgers vector b , this distribution can be represented by a distribution density function $B(x)$ of infinitesimal continuous dislocations (Figure 2.2). The total Burgers vector of the dislocations that lie between x and δx are given by $B(x)\delta x$.

Figure 2.2 Distribution of schematic infinitesimal dislocations [11]

There are three basic modes of loading for a crack. Consider the crack in Figure 2.3, with its crack plane along the x -axis, crack tips at $x = a$ and $x = -a$.

Figure 2.3 The Mode I crack with climb edge dislocations [11]

If the crack is loaded in tension as in Figure 2.3, it is called a mode I load, and the crack is opened up in the y-direction. The dislocations used to simulate a mode I crack are climb edge dislocations as shown in Figure 2.3, with the Burgers vector perpendicular to the crack plane. Displacement occurs by crack opening, orthogonal to the crack plane.

Figure 2.4 The Mode II crack with glide edge dislocations [11]

A mode II crack is loaded by in-plane shear as in Figure 2.4. The upper and lower surfaces of the crack slide against each other. In this mode glide edge dislocations are used for crack simulation, where the Burgers vector is parallel to the crack plane. This allows displacement via sliding along the crack plane.

And in mode III loading, an out-of-plane load shear is present. A mode III crack can be modelled by distributed screw dislocations, with its Burgers vector (and hence displacement) in an out-of-plane direction. In addition, a crack can be loaded in a mix of these modes, appropriately called the mixed-mode loading.

Using distributed dislocations to simulate a crack, the displacement across the crack can be related to the Burgers vector by the relation from Weertman [10]:

$$D(x) = \int_x^a B(x)dx. \quad (2.1)$$

The stress field due to the existence of a Griffith crack can be obtained from the integration of the stress field components $\sigma_{ij}(x, y)$ caused by the infinitesimal “discrete dislocation” $B(x)dx$ along the crack line:

$$\sigma_{ij}(x, y) = \int_{-a}^a B(x_0)\sigma_{ij}(x - x_0, y)dx. \quad (2.2)$$

Knowing that the stresses along the crack surface are zero, the resultant of the applied stress and the traction stresses arising from dislocations must equal zero along the crack surface as well.

For a Griffith crack, the crack faces must physically come together at both crack ends. Hence $D(-a) = D(a) = 0$, which indicates that there is no net dislocation [12] resulting in zero sum of the Burgers vector. This condition can be written as:

$$\int_{-a}^a B(x)dx = 0. \quad (2.3)$$

2.2.2. Zener-Stroh crack

The Griffith crack does not explain how crack is nucleated in the first place. This is where the Zener-Stroh crack comes in, at the initial stage of a crack.

Figure 2.5 Mode I Zener-Stroh crack by dislocation pile-up at a grain boundary [13]

Zener [13] was the first to propose that a pile-up of edge dislocations stopped at an obstacle could coalesce into a crack nucleus, as depicted in Figure 2.5. Koehler [14] also independently studied such crack initiation phenomenon in various materials. Figure 2.6 shows an experimental SEM photo of a crack initiated by this physical mechanism. The crack nucleation mechanism in homogeneous materials was first theoretically investigated by Stroh [15-17], studying the number of dislocations needed to form a microcrack, concluding that the initiation of a Zener-Stroh crack in a solid is possible when the energy for crack initiation is lower than that for other mechanisms. Hence, this crack type is also known as the Zener-Stroh crack.

Figure 2.6 Pile-up of dislocations coalesce into a crack (Courtesy of Dr. Zhou Wei)

Other mechanisms of crack initiation are possible. Cottrell [18] suggested that piled up dislocations on two intersecting planes could coalesce into a microcrack, as shown in Figure 2.7. Meanwhile Kikuchi et al. [19] suggested the anti-Zener-Stroh crack shown in

Figure 2.8. Here the crack is nucleated at concentrated stress fields near the ends of the particle, when dislocations of one sign move away from the region, leaving stationary dislocations of the opposite sign to form the crack.

Figure 2.7 Cottrell's crack initiation mechanism [18]

Figure 2.8 Crack initiation mechanism suggested by Kikuchi et al. [19]

A Zener-Stroh crack is complementary to the well-known Griffith crack. This is because physical parameters that are symmetric for the Griffith crack are antisymmetric for the Zener-Stroh crack, and vice-versa. The sum of the Burgers vectors of the dislocations b^T for the Zener-Stroh crack is nonzero, giving a displacement loaded crack. The dislocation enters the crack through the blunt tip, then propagating through the other (sharp) tip [20]. This type of crack is important as its physical mechanism is representative of microcracks [21, 22], with distributed Zener-Stroh cracks resulting in a Griffith crack [23]. This fracture mechanism has also been used to explain fatigue fracture procedure in various engineering materials, such as silicon carbide [24] and metal fatigue [25].

Zener-Stroh cracks may be modelled using climb edge dislocations, such as the blunt-sharp crack (Figure 2.9a) as well as the sharp-blunt crack (Figure 2.9b). The b_y^T net dislocation loads determine the sharp and blunt crack tips of the Zener-Stroh crack.

Figure 2.9 Zener-Stroh cracks with climb edge dislocations

When a crack is loaded by both net dislocations and external mechanical loading, the crack becomes a combined Griffith-Zener-Stroh crack.

2.3. Dislocation-Inhomogeneity Interaction

The current work makes use of the distributed dislocation method [12] to simulate cracks in composite materials, investigating crack-inclusion (inhomogeneity) interaction in detail. An understanding of the previous work on dislocation-inclusion interaction is important in laying the foundation for crack-inclusion (inhomogeneity) interaction.

The interaction of dislocations with inhomogeneities is important in understanding the strengthening and hardening mechanism of materials. This is because a pile-up of dislocations could coalesce into a crack nucleus [13]. The earliest work on this is by Head [26, 27], where a dislocation near an interface between dissimilar materials was considered. Subsequently, Dundurs and Mura [28] looked at the dislocation interacting with a circular inclusion, while Dundurs and Sendeckyj [29] studied a dislocation placed inside a circular inclusion. The solution for a dislocation near a surface layer [30, 31] was obtained too. This was followed by the solutions for the dislocation interacting with an elliptical inclusion [32, 33]. Fukuzaki and Shioya [34, 35] then solved for the interaction between a dislocation and two inclusions. The problem of a dislocation in a multi-layered composite was analysed by

Erdogan and Gupta [36], while Worden and Keer [37] solved for the Green's function of a dislocation in an annular region.

Dislocation-inclusion interaction involving more than two dissimilar materials have also been investigated by various researchers. The stress solution for the dislocation in a three-phase composite was solved by Luo and Chen [38], while that for a screw or an edge dislocation in a coated fibre was solved by Xiao and Chen [39, 40]. An edge dislocation interacting with a slightly wavy interface was solved by Xiao, Chen and Steele [41]. Recent works include the study of a dislocation in a three-quarter plane by Churchman, Korsunsky and Hills [42]. Meanwhile Fang and Liu [43] solved the problem for an edge dislocation interacting with a nanoscale inhomogeneity, while Fang et al. [44] studied an edge dislocation inside a nanoscale cylindrical inclusion.

2.4. Crack-Inhomogeneity Interaction

The elastic analysis of crack-inclusion interaction behaviour has long been in literature, such as the interaction between a circular inclusion with a crack by Tamate [45]. The interaction between a circular inclusion and an arbitrarily oriented crack using the distributed dislocation technique was considered by Erdogan, Gupta and Ratwani [46]. Sendeckyj [47] studied the problem of a crack located between two rigid inclusions. There has also been attention on interaction of an inclusion with multiple cracks [48]. An antiplane shear crack in an infinite plate with a circular inclusion was analysed by Theocaris and Demakos [49]. The interaction between a planar crack and a flat inclusion in an elastic solid was studied by Liu and Erdogan [50]. The effect of a rigid elliptical inclusion on a crack was observed in ceramic composites [51].

Subsequently, other composite configurations were given attention as well, for example: a crack located in the matrix of a composite material was studied with the three-phase composite cylinder model using the distributed dislocation technique [52]. Interaction

between a coated inclusion with a Griffith crack, and with a Zener-Stroh crack were both studied by Xiao and Chen [53, 54]. This line of study has also been extended into three dimensions as well, such as the interaction between a penny-shaped crack and a circular inclusion by Xiao and Pae [55]. Work has also been done on non-linear crack shapes, such as curved crack with elastic inclusion interaction [56] and ring-shaped crack surrounded by spherical inclusions [57]. Yang, Chen and Li [58] used the Eshelby inclusion method to study crack-inclusion interaction in Mode II loading. Churchman and Hills [59] obtained the crack solution in a three-quarter plane.

Work on the Zener-Stroh crack in composite materials began with the interfacial Zener-Stroh crack [60]. Later studies that built on this include: use of the contact zone for an interfacial Zener-Stroh crack [20, 21], for a crack near a coated circular inclusion [54], for a crack in a fibre-reinforced composite material [61] and for an inclined crack near a free surface [62].

2.5. Numerical Solution of Singular Integral Equations

The mathematical formulation of crack-inclusion interaction is a large class of mixed boundary value problems with a system of singular integral equations [63] of the general form:

$$\frac{1}{\pi} \int_{-1}^1 \sum_{j=1}^M a_{ij} \varphi_j(t) \frac{dt}{t-x} + \int_{-1}^1 \sum_{j=1}^M k_{ij}(x, t) \varphi_j(t) dt = g_i(x), \quad (2.4)$$

$$-1 < x < 1, i = 1, \dots, M,$$

where the generalized Cauchy kernel $k_{ij}(x, t), i, j = 1, \dots, M$ are bounded in the closed domain $-1 < x, y < 1$ with unknown functions $\varphi_j(t), j = 1, \dots, M$ and known input functions $g_i(t), i = 1, \dots, M$. The constant matrix a_{ij} is non-singular. The integral end points $t = \pm 1$ are points of geometric singularity. The singular behaviours of the functions

$\varphi_j(t), j = 1, \dots, M$ at $t = \pm 1$ may be obtained by analyzing the dominant part of the integral equations through the use of function-theoretic methods [64].

Generally, to solve the problem, an additional condition is required, given by:

$$\int_{-1}^1 \varphi_j(t) dt = c_j, \quad (2.5)$$

where $c_j, j = 1, \dots, M$ are known constants.

Multiplying equation (2.4) by $Z = (a_{ij})^{-1}$ as in Muskhelishvili [65], the fundamental function can be expressed as:

$$R_j(t) = (1+t)^{-\frac{1}{2}+a}(1-t)^{\frac{1}{2}+b}. \quad (2.6)$$

The index of the integral equation can be written as $l = -(a+b) = \pm 0, 1$. There are several possible cases to this, depending on the behaviour of $\varphi_j(t)$ at the end points.

- a) If $\varphi_j(t)$ is bounded at the singular points $t = \pm 1, a = 1, b = 0, l = -1$ then the fundamental function becomes

$$R_j(t) = (1-t^2)^{\frac{1}{2}}. \quad (2.7)$$

- b) When $\varphi_j(t)$ has integral singularities at $t = \pm 1, a = 0, b = -1, l = 1$ then the fundamental function becomes

$$R_j(t) = (1-t^2)^{-\frac{1}{2}}. \quad (2.8)$$

For the case of $l = 1$, the solution of equation (2.8) contains M arbitrary constants that need to be determined by physical conditions

$$\int_{-1}^1 \varphi_j(t) dt = c_j, \quad (2.9)$$

where c_j are known constants.

c) Two other cases are possible, with $\varphi_j(t)$ singular at one end and bounded at the other.

Hence, if bounded at $t = 1$, singular at $t = -1$, while $a = b = 0, l = 0$:

$$R_j(t) = \left(\frac{1-t}{1+t} \right)^{\frac{1}{2}}. \quad (2.10)$$

d) Similarly, with opposite singularity; singular at $t = 1$, bounded at $t = -1$, while

$a = 1, b = -1, l = 0$:

$$R_j(t) = \left(\frac{1+t}{1-t} \right)^{\frac{1}{2}}. \quad (2.11)$$

The fundamental functions $R_j(t)$ in equations (2.7) and (2.8) are the weights of Chebyshev polynomials, while those in equations (2.10) and (2.11) are weights of Jacobi polynomials.

Erdogan and Gupta [66] have developed the Gaussian integration formula to obtain the solution for singular integral equations. For the basic singular integral

$$S(x) = \frac{1}{\pi} \int_{-1}^1 \frac{\varphi(t)}{t-x} dt, \quad (2.12)$$

where $\varphi(t) = R(t)F(t)$, $-1 < t < 1$, and $F(t)$ is a function bounded in $-1 < t < 1$. By choosing nodal points t_k and x_k within $-1 < t < 1$, the system can be treated as a system of Fredholm equations, and $F(t)$ can be determined using the conventional collocation method.

Thus the singular behaviour of the solution is dependent on $R(t)$.

The Chebyshev polynomials are defined by:

$$T_n(x) = \cos n\theta, U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \cos\theta = x. \quad (2.13)$$

2.5.1. Chebyshev polynomials of the first kind

Chebyshev polynomials of the first kind $T_j(t)$ have weights given by $R_j(t) = (1 - t^2)^{-\frac{1}{2}}$. This occurs when $l = 1$. The unknown function $F(t)$ can be approximated by

$$F(t) \approx \sum_0^p A_j T_j(t). \quad (2.14)$$

Using the Erdogan and Gupta method, the Cauchy singular integral in equation (2.12) can be discretised as

$$\frac{1}{\pi} \int_{-1}^1 \frac{\varphi(t)}{t - x} dt = \sum_{k=1}^n \frac{F(t_k)}{n(t_k - x_r)}, \quad (2.15)$$

where

$$\begin{aligned} T_n(t_k) &= 0, & t_k &= \cos \frac{\pi(2k-1)}{2n}, & k &= 1, \dots, n, \\ U_{n-1}(x_r) &= 0, & x_r &= \cos \frac{\pi r}{n}, & r &= 1, \dots, n-1. \end{aligned} \quad (2.16)$$

Equation (2.15) is identical to the standard Gauss-Chebyshev integration formula for a bounded function $g(x, t)$:

$$\frac{1}{\pi} \int_{-1}^1 \frac{g(x, t)}{(1 - t^2)^{\frac{1}{2}}} dt = \sum_{k=1}^n \frac{g(x, t_k)}{n}, \quad t_k = \cos \frac{\pi(2k-1)}{2n}, \quad k = 1, \dots, n. \quad (2.17)$$

The important difference between the two is that equation (2.17) is valid for any x while equation (2.15) is valid only for a discrete set of x_r , given by equation (2.16).

2.5.2. Chebyshev polynomials of the second kind

Chebyshev polynomials of the second kind $U_j(t)$ have weights given by $R(t) = (1 - t^2)^{\frac{1}{2}}$, when $l = -1$. The unknown function $F(t)$ are then approximated by

$$F(t) \approx \sum_0^p B_j U_j(t). \quad (2.18)$$

The discretised form of the Cauchy singular integral is

$$\frac{1}{\pi} \int_{-1}^1 \frac{\varphi(t)}{t-x} dt = \sum_{k=1}^n \frac{(1-t_k^2)F(t_k)}{(n+1)(t_k-x_r)}, \quad (2.19)$$

where

$$t_k = \cos \frac{k\pi}{n+1}, \quad k = 1, \dots, n, \\ x_r = \cos \frac{\pi(2r-1)}{2n-1}, \quad r = 1, \dots, n+1.$$

For a bounded function $g(x, t)$:

$$\frac{1}{\pi} \int_{-1}^1 \frac{g(x, t)}{(1-t^2)^{-\frac{1}{2}}} dt = \sum_{k=1}^n \frac{(1-t^2)^{\frac{1}{2}} g(x, t_k)}{n+1}, \quad t_k = \cos \frac{k\pi}{n+1}, \quad k = 1, \dots, n. \quad (2.20)$$

2.5.3. Jacobi polynomials

For Jacobi polynomials $P_j^{(\frac{1}{2}, -\frac{1}{2})}(t)$, the weight is given by $R_j(t) = \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}}$, when $l =$

0. The unknown function $F(t)$ can then be approximated by

$$F(t) \approx \sum_0^p C_j P_j^{(\frac{1}{2}, -\frac{1}{2})}(t). \quad (2.21)$$

The discretised form of the Cauchy singular integral is

$$\frac{1}{\pi} \int_{-1}^1 \frac{\varphi(x, t)}{t-x} dt = \frac{1}{\pi} \int_{-1}^1 \frac{F(t)}{t-x} \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}} dt = \sum_{k=1}^n \frac{2(1-t_k)F(t_k)}{(2n+1)(t_k-x_r)}, \quad (2.22)$$

where

$$t_k = \cos \frac{2k\pi}{2n+1}, \quad k = 1, \dots, n, \\ x_r = \cos \frac{\pi(2r-1)}{2n+1}, \quad r = 1, \dots, n.$$

For a bounded function $g(x, t)$:

$$\frac{1}{\pi} \int_{-1}^1 g(x, t) \left(\frac{1-t}{1+t} \right)^{\frac{1}{2}} dt = \sum_{k=1}^n \frac{2(1-t_k)g(x, t_k)}{2n+1}, t_k = \cos \frac{2k\pi}{n+1}, k = 1, \dots, n. \quad (2.23)$$

Similarly, with weight function $R_j(t) = \left(\frac{1+t}{1-t} \right)^{\frac{1}{2}}$, the discretised form of equation

(2.12) is given by

$$\frac{1}{\pi} \int_{-1}^1 \frac{\varphi(x, t)}{t-x} dt = \frac{1}{\pi} \int_{-1}^1 \frac{F(t)}{t-x} \left(\frac{1+t}{1-t} \right)^{\frac{1}{2}} dt = \sum_{k=1}^n \frac{2(1+t_k)F(t_k)}{(2n+1)(t_k-x_r)}, \quad (2.24)$$

in which

$$t_k = \cos \frac{2(k-1)\pi}{2n+1}, \quad k = 1, \dots, n,$$

$$x_r = \cos \frac{2\pi r}{2n+1}, \quad r = 1, \dots, n.$$

2.6. Small-Scale Yielding and the Dugdale Model

Linear elastic fracture mechanics predicts infinite stresses at crack tips of sharp cracks. However, stresses in real materials are finite as the crack-tip radius is finite [1]. Inelastic deformation, such as plasticity in metals and crazing in polymers, leads to further relaxation of tip stresses. Hence linear elastic fracture mechanics gets less accurate as the inelastic region at the crack tip grows.

As this work deals primarily with small-scale yielding, the inelastic region will be referred to as the plastic region. To estimate the size of the small plastic region, two methods are possible; the Irwin [5] method, and the Dugdale [6] model. The theory of the Dugdale model was later developed more completely by Bilby, Cottrell and Swinden [67]. Hayes and Williams [68] makes use of finite element methods to solve for the Dugdale crack. This work focuses on the Dugdale model for several reasons. By considering the strip-yield model, the influence of the plastic zone on the fracture behaviour of the crack is revealed. The Dugdale

model can be further generalized to the cohesive zone model, and related to the J-integral as well. The Irwin plastic zone correction does not allow these. The Dugdale model also occurs for a number of materials, especially metals.

For mode I cracks there exist solutions for centre and edge cracked plates, using Green's functions [69]. Bueckner and Rice introduced the weight function technique, which was then used as a method to solve for the Dugdale crack [70-72]. Galatenko [73] developed a generalized model of a Dugdale crack which allows for different yield criteria. Theocaris [74] solved for the Dugdale model for two collinear unequal cracks. There are also other works that also provide analytical solutions to Dugdale crack problem [75-77]. Wang and Shen [78] studied the Dugdale crack in a bimaterial interface. Unger [79] compares the results of the Dugdale model to an elastic-perfectly plastic crack under the Tresca yield condition.

Figure 2.10 The Dugdale crack model

The Dugdale model of small-scale yielding is used to approximate plastic zones ahead of the crack tips. In this model, the plastic zone is treated as a short line segment ahead of the crack tips. The solution is obtained by fulfilling the condition of vanishing stress intensity factors at the crack tips: the applied load stress intensity factor K_I cancelling the one due to closure stresses, K_ρ at the crack tips [1]:

$$K_I + K_{\rho} = 0. \quad (2.25)$$

The relevant plastic zone size (PZS), ρ for a homogeneous material under loads much smaller than yield strength, $\sigma_A \ll \sigma_Y$ is approximated by [1]:

$$\rho_0 = \frac{\pi}{8} \left(\frac{K_I}{\sigma_Y} \right)^2, \quad (2.26)$$

where the subscript 0 refers to the homogeneous case. Work by Weertman [10] on the Dugdale model, using the Muskhelishvili equations give:

$$\rho_{0G} = 2a \frac{\sin^2(\pi\sigma_A/4\sigma_Y)}{\cos(\pi\sigma_A/2\sigma_Y)}, \quad (2.27)$$

where the subscript G refers to Griffith to differentiate this from the Zener-Stroh crack results. Note that equation (2.27) reduces to (2.26) for $\sigma_A \ll \sigma_Y$.

For a Zener-Stroh crack, Weertman gives the PZS as:

$$\rho_{0Z} = \sqrt{\left[\frac{\mu b_y^T}{(\kappa + 1)\sigma_Y} \right]^2 + a^2} - a. \quad (2.28)$$

where subscript Z stands for “Zener”. Note that for $\frac{2\mu b_y^T}{(\kappa+1)\pi a} \ll \sigma_Y$, this simplifies to:

$$\rho_{0zs} = \left[\frac{\mu b_y^T}{(\kappa + 1)\sigma_Y} \right]^2 / 2a. \quad (2.29)$$

where the subscript s refers to “simplified”.

2.7. The Crack Tip Opening Displacement

The crack tip opening displacement (CTOD) was proposed independently by Wells [8] and Cottrell [80] in the early 1960s as a fracture criterion where significant plastic deformation precedes fracture [81]. Burdekin and Stone [82] subsequently tested the CTOD on materials that exhibit yielding before fracture. Many other works also make use of the CTOD as a measure of fracture [83-85]. The CTOD of the right tip of a Dugdale model crack

is shown in Figure 2.11, using the definition of Burdekin and Stone [82], taken at the open end of the plastic zone.

Figure 2.11 CTOD of the Dugdale model crack [82]

Based on the original Dugdale model crack in a homogeneous material, the CTOD [82], δ is given by:

$$\delta_0 = \frac{\kappa + 1}{\mu} \frac{\sigma_Y a}{\pi} \ln \sec \left(\frac{\pi \sigma_A}{2 \sigma_Y} \right), \quad (2.30)$$

where a is the half-crack length. This also applies to the tangential displacement in Mode II [75]. To find the crack tip opening displacement, Weertman [10] used:

$$D(\xi) = \int_{\xi}^{\infty} B(\xi) d\xi = - \int_{-\infty}^{\xi} B(\xi) d\xi, \quad (2.31)$$

which gives the displacement along the crack length. This equation, with dislocation densities $B(\xi)$ solved by Muskhelishvili equations, gives the CTOD of the Dugdale model crack as:

$$\delta_{0G} = \frac{\kappa + 1}{\mu} \frac{\sigma_Y a}{\pi} \ln \left(\frac{\rho_{0G}}{a} \right). \quad (2.32)$$

The above equation reduces to (2.30) under the $\sigma_A \ll \sigma_Y$ condition.

In the case of a Zener-Stroh crack, solving (2.31) for the CTODs [10] gives:

$$\begin{aligned}
\delta_{0Z_blunt} &= \frac{\kappa + 1}{\mu} \frac{\sigma_Y a}{\pi} \left[\ln \left(\frac{a + \rho_{0Z}}{a} \right) + \frac{\sqrt{(a + \rho_{0Z})^2 - a^2}}{a} \cos^{-1} \left(\frac{-a}{a + \rho_{0Z}} \right) \right], \\
\delta_{0Z_sharp} &= \frac{\kappa + 1}{\mu} \frac{\sigma_Y a}{\pi} \left[-\ln \left(\frac{a + \rho_{0Z}}{a} \right) + \frac{\sqrt{(a + \rho_{0Z})^2 - a^2}}{a} \cos^{-1} \left(\frac{a}{a + \rho_{0Z}} \right) \right].
\end{aligned} \tag{2.33}$$

2.8. Chapter Summary

This chapter reviewed works on the dislocation influence functions, as well as how these functions can be used to simulate a Griffith or Zener-Stroh crack via the distributed dislocation method. The Erdogan-Gupta numerical method used to solve the system of singular integral equations was discussed. The basic idea behind the Dugdale model was presented, which is used mainly for analysis in homogeneous materials.

There is a lack of works that study small-scale yielding in composite materials, and this thesis aims to bridge this gap by making use of the Dugdale model in composite materials. The current work makes use of the dislocation influence functions to simulate a crack, thus following the method of several works [12, 63], and extends it by making use of the Dugdale model to evaluate the PZS and CTOD with small-scale yielding correction at both crack tips. To the best knowledge of the author, small-scale yielding model has been introduced for cracks only in homogeneous materials, prior to our current work.

CHAPTER 3

Griffith Crack Interacting with a Circular Inclusion

3.1. Introduction

The importance of studying the unit cell composite material has been highlighted in Chapter 1. Although Tamate [45] solved the linear elastic problem of a crack interacting with a circular inclusion, the solution does not take yielding into account. To more accurately predict the elastic-perfectly plastic fracture behaviour of cracks in composite materials, in addition to the interaction between the crack and the inclusion (the reinforcing phase of composites), the effect of plastic zones at crack tips should also be taken into consideration. Here, the case of a crack interacting with a circular inclusion under uniaxial and uniform tensile stresses is investigated, with the Dugdale plastic zone introduced at both crack tips.

This problem was chosen as the first problem to be solved, as this is basic and fundamental in depicting a unit cell of composite materials. When the composite material reinforcing phase volume fraction is low, the interaction among inclusions can be neglected, leading to this two-phase case. It represents a crack interacting with a circular inclusion in dilute inclusions. This physical problem also serves as preparation for work on the three-phase model, where the volume fraction of the reinforcing phase is not negligible.

3.2. Problem Formulation

The problem to be solved is shown in Figure 3.1a, where a Dugdale model line crack is present in the infinite solid (the matrix) near a circular inclusion (the inhomogeneity). The crack is oriented along the radial direction of the circular inclusion and the whole body is assumed to be in plane stress. The matrix (phase 1) has elastic properties κ_1 and μ_1 ,

occupying the region $r_0 > 1$. The inclusion (phase 2), with elastic properties κ_2 and μ_2 , occupies $r_0 \leq 1$.

Figure 3.1 Interaction between a Dugdale model crack and a circular inclusion

The Dugdale model crack is situated along the x-axis, in the matrix phase (phase 1), with the plastic zones on the left and right tips having lengths of ρ_L and ρ_R respectively. The left tip locates at $t_1 = c_1 - \rho_L$ while the right tip is at $t_2 = c_2 + \rho_R$. Hence the Dugdale model crack has an effective length of $\rho_L + 2a + \rho_R$. Without the plastic zones, the crack simplifies to a crack of length $2a$ from $t_1 = c_1$ to $t_2 = c_2$, where $a = (c_2 - c_1)/2$ is the half crack length.

The load conditions considered here are the uniaxial tension and biaxial tension applied on the matrix at infinity. The boundary conditions for the crack problem are given next. In the far field the applied load is given by the stresses:

$$\begin{aligned} \sigma_{yy}^{\infty} &= \sigma_A, \sigma_{xx}^{\infty} = \sigma_A \text{ for biaxial tension, or} \\ \sigma_{yy}^{\infty} &= \sigma_A, \sigma_{xx}^{\infty} = 0 \text{ for uniaxial tension.} \end{aligned} \quad (3.1)$$

At the interface of phase 1 and 2:

$$[\sigma_{rr}] = 0, \quad [\sigma_{r\theta}] = 0, \quad [u_r] = 0, \quad [u_{\theta}] = 0, \quad \text{at } r = r_0, \quad (3.2)$$

where $[f]$ is the jump of function f , while r and θ refer to polar coordinates with origin 0.

Following the Dugdale model, the plastic zones are yielded with yield stress σ_Y . Thus at the crack surface:

$$\begin{aligned}\sigma_{xy} &= 0, \sigma_{yy} = 0, & c_1 \leq x \leq c_2, \\ \sigma_{yy} &= \sigma_Y, & c_1 - \rho_L \leq x \leq c_1, c_2 \leq x \leq c_2 + \rho_R,\end{aligned}\tag{3.3}$$

where the applied load σ_A is related to the yield stresses by:

$$\sigma_A/\sigma_Y = T,\tag{3.4}$$

and T is a known parameter relating applied load to yield strength.

3.3. Solution Procedure

To obtain the Dugdale model solution for the plastic zone sizes (PZS) and crack tip opening displacement (CTOD), the condition of vanishing stress intensity factors applies, but this time at both crack tips due to asymmetry:

$$\left. \begin{aligned}K_I^L + K_{I\rho}^L &= 0 \\ K_I^R + K_{I\rho}^R &= 0\end{aligned} \right\}\tag{3.5}$$

To portray the Dugdale model crack, superposition of the stresses applied on the crack is used. The use of superposition is justified here, as the study is still based on linear elastic fracture mechanics, despite the presence of small-scale yielding. Hence, the Dugdale model crack (Figure 3.1a) can be represented by a crack with closure stresses (of magnitude equal to yield stress) at the plastic zone regions (Figure 3.1b), and a crack of effective length $\rho_L + 2a + \rho_R$ (Figure 3.1c) where ρ_L and ρ_R are the plastic zones of differing sizes on the left and right respectively.

Equations (3.5) are represented by Figure 3.1. $K_{I\rho}^L$ and $K_{I\rho}^R$ can be obtained by solving problem Figure 3.1b while K_I^L and K_I^R are gathered from solving Figure 3.1c. To obtain the plastic zone sizes of the Dugdale model for Figure 3.1a, the problems of Figures 3.1b and 3.1c must be solved. The distributed dislocation technique [12] is used to obtain K at the crack tips in both applied load and closure stress. Section 3.4 solves for K_ρ , while Section 3.5

solves for the K values. The ρ_L and ρ_R are found by ensuring that the two solutions cancel each other.

3.4. Solving for K_ρ

With reference to Figure 3.1b, the stresses at the crack tips of the crack should remain bounded and equal to the yield stress σ_Y of the matrix material, as the material is assumed elastic-perfectly plastic. Thus closing the crack requires closure stresses equal to σ_Y applied to the left and right of the crack tips as shown. Hence the closure stresses are:

$$D_y(x) = \begin{cases} -\sigma_Y & \text{if } c_1 - \rho_L < x < c_1, c_2 < x < c_2 + \rho_R \\ 0 & \text{else} \end{cases} \quad (3.6)$$

3.4.1. Crack simulation with distributed dislocations

In this section, the equivalence between a crack and a dislocation pile-up is applied to Figure 3.1b. Using $B_y(x)$ as the climb dislocation density in the crack zone, while utilising the single dislocation solution [12] shown in Figure 3.2, the dislocation density $B_y(x)$ is related to the crack surface by:

$$\sigma_{yy}(x, 0) = -\frac{2\mu_1}{\pi(\kappa_1 + 1)} \left[\int_{t_1}^{t_2} \frac{B_y(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_1(x, \xi) B_y(\xi) d\xi \right]. \quad (3.7)$$

Figure 3.2 An edge dislocation interacting with a circular inclusion

In order to satisfy the conditions of (3.3) at the plastic zones, $\sigma_{yy}(x, 0) = -D_y(x)$ is used as the corrective traction for uniaxial and biaxial tension.

$$\frac{1}{\pi} \left[\int_{t_1}^{t_2} \frac{B_y(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_1(x, \xi) B_y(\xi) d\xi \right] = \frac{\kappa_1 + 1}{2\mu_1} D_y(x). \quad (3.8)$$

For a Griffith crack, the dislocation densities must also satisfy:

$$\int_{t_1}^{t_2} B_y(\xi) d\xi = 0. \quad (3.9)$$

where equations (3.8) and (3.9) form the set of singular integral equations to be solved.

Meanwhile, the relevant influence functions can be obtained from Hills [12]:

$$\begin{aligned} k_1 = & \frac{x_1}{r_1^4} [x_1^2 + 3y^2] + \frac{x_2}{r_2^2} \left[\frac{2Ax_2^2}{r_2^2} - \frac{B + 5A}{2} \right] - \frac{x}{r^2} \left[\frac{2Ax^2}{r^2} - \frac{B + 5A}{2} \right] \\ & + \frac{2Ax_2L}{r_2^4 d^2} \left[\frac{4x_2^2}{r_2^2} \left(\frac{L}{2} - x_2 \right) + \left((5 - d^2)x_2 - \frac{3L}{2} \right) \right] \\ & - \frac{r_0 x}{r^4} \left[r_0 A \left(\frac{4x^2}{r^2} - 3 \right) + \frac{x}{d} (A(2d^2 - 1) + Q - 1) \right] - \frac{AL}{r_2^2} \left[\frac{2}{d^2} - 1 \right] \\ & + \frac{r_0 [A(2d^2 - 1) + Q - 1]}{2r^2 d}. \end{aligned} \quad (3.10)$$

with

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ r_1 &= \sqrt{x_1^2 + y^2}, \quad x_1 = x - \xi, \quad L = \frac{r_0(d^2 - 1)}{d}, \\ r_2 &= \sqrt{x_2^2 + y^2}, \quad x_2 = x_1 + L, \quad d = \frac{\xi}{r_0}. \end{aligned}$$

Meanwhile, the elastic constants A , B and Q are given by

$$A = \frac{\beta - \alpha}{1 + \beta}, \quad B = -\frac{\alpha + \beta}{1 - \beta}, \quad Q = \frac{(1 + \alpha)(1 - \alpha)}{(1 - \beta)(1 + \alpha - 2\beta)}, \quad (3.11)$$

in which α and β are the Dundurs' parameters, defined as:

$$\alpha = \frac{\mu_2(\kappa_1 + 1) - \mu_1(\kappa_2 + 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)}, \quad \beta = \frac{\mu_2(\kappa_1 - 1) - \mu_1(\kappa_2 - 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)}. \quad (3.12)$$

3.4.2. Numerical procedure for solving singular integral equations

The singularities on both crack tips are inverse square root because the whole crack is located in the matrix material. Using the fundamental functions [66], let

$$B_y(x) = w(x)F_y(x), \quad (3.13)$$

where $F_y(x)$ is bounded functions in $t_1 \leq x \leq t_2$, with

$$w(x) = (x - t_1)^{-\frac{1}{2}}(t_2 - x)^{-\frac{1}{2}}. \quad (3.14)$$

To shift the interval of the integral along the crack line from (t_1, t_2) to $(-1, 1)$, let

$$\begin{aligned} x &= \frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \\ \xi &= \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2}. \end{aligned} \quad (3.15)$$

Equations (3.8) and (3.9) can then be rewritten in terms of s and t as:

$$\begin{aligned} \frac{1}{\pi} \left[\int_{-1}^1 \frac{B_y(s)}{s - t} ds + \int_{-1}^1 k_{11}(t, s) B_y(s) ds \right] &= \frac{\kappa_1 + 1}{2\mu_1} D_y(x), \\ \int_{-1}^1 B_y(s) ds &= 0, \end{aligned} \quad (3.16)$$

where

$$k_{11}(t, s) = \frac{t_2 - t_1}{2} k_1 \left(\frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2} \right). \quad (3.17)$$

Following the numerical method of Erdogan and Gupta [66], the discretised form of equations (3.16) are obtained:

$$\begin{aligned}
\sum_{k=1}^n \frac{1}{n} F_y(s_k) \left[\frac{1}{s_k - u_r} + k_{11}(u_r, s_k) \right] &= \frac{\kappa_1 + 1}{2\mu_1} D_y(x_r), \\
\sum_{k=1}^n \frac{1}{n} F_y(s_k) &= 0,
\end{aligned} \tag{3.18}$$

where n is the number of terms used in the numerical procedure, and:

$$\begin{aligned}
s_k &= \cos \frac{\pi(2k-1)}{2n}, & k &= 1..n, \\
x_r &= \frac{t_2 - t_1}{2} u_r + \frac{t_2 + t_1}{2}, & u_r &= \cos \frac{\pi r}{n}, \quad r = 1..n-1.
\end{aligned}$$

The Mode I stress intensity factors [10] are:

$$\begin{aligned}
K_{I\rho}^L &= \lim_{\xi \rightarrow t_1} \frac{-2\mu_1 \sqrt{2\pi}}{\kappa_1 + 1} \sqrt{\xi - t_1} B_y(x) = \frac{-2\mu_1 \sqrt{\pi(t_2 - t_1)/2}}{\kappa_1 + 1} F_y(-1) \\
&= -\sigma_A \sqrt{\pi(t_2 - t_1)/2} F_{0y}(-1), \\
K_{I\rho}^R &= \lim_{\xi \rightarrow t_2} \frac{2\mu_1 \sqrt{2\pi}}{\kappa_1 + 1} \sqrt{t_2 - \xi} B_y(x) = \frac{2\mu_1 \sqrt{\pi(t_2 - t_1)/2}}{\kappa_1 + 1} F_y(1) \\
&= \sigma_A \sqrt{\pi(t_2 - t_1)/2} F_{0y}(1),
\end{aligned} \tag{3.19}$$

where $F_{0y}(s)$ is solved from equations (3.18) by letting $F_y(s) = \frac{\sigma_A(\kappa_2+1)}{2\mu_2} F_{0y}(s)$ for tensile loading, using equation (3.4) to relate applied load to yield strength.

3.5. Solving for K

Figure 3.3 The crack interacting with a circular inclusion

Solving for the stress intensity factors K of Figure 3.1c (repeated in Figure 3.3a) is identical to the problem solved by Tamate [45], with the left crack tip at $t_1 = c_1 - \rho_L$ and the right at $t_2 = c_2 + \rho_R$. This sub-problem is similar to Figure 3.1a, but without the plastic strips, with boundary conditions (3.1) and (3.2) still applicable, but (3.3) is replaced by:

$$\sigma_{xy} = 0, \quad \sigma_{yy} = 0, \quad t_1 \leq x \leq t_2. \quad (3.20)$$

To transform the far-field load of Figure 3.3a onto the crack surfaces, the problem is decomposed into Figures 3.3b and 3.3c. The problem in Figure 3.3b shows the circular inclusion under far-field loading with no crack. The problem in Figure 3.3c shows the inclusion with the near-by crack, but with different loading conditions: normal stress $p_y(x)$ (or shear stress $p_x(x)$) is applied on the top and bottom surfaces of the crack but without far-field loading. By superposition, the sum of the stress fields of Figures 3.3b and 3.3c give the stress field in Figure 3.3a.

To obtain the stress intensity factors of Figure 3.3a, we follow the distributed dislocation technique outlined in Sections 3.4.1 and 3.4.2, with some modifications. The corrective tractions are now $\sigma_{yy}(x, 0) = -p_y(x)$ for tensions where $p_y(x)$ represent stresses at the crack surface transformed from the far-field load. Hence, we replace $D_y(x)$ by $p_y(x)$ in Sections 3.4.1 and 3.4.2.

3.5.1. Transforming far-field loads to surface loads

The stress fields for the different load conditions of biaxial tension and uniaxial tension can be easily obtained, whether from elasticity texts, or from previous work [11]. For biaxial tension ($\sigma_{xx}^\infty = \sigma_{yy}^\infty = \sigma_A$ in Figure 3.1b), the stresses along the crack surface at $\theta = 0$, $y = 0$ are:

$$\begin{aligned}
p_y(x) &= \sigma_A \left[1 + \frac{\mu_1}{\lambda_1 + \mu_1} \left(\frac{r_0}{x} \right)^2 a_2 \right], & (a) \\
p_x(x) &= 0, & (b)
\end{aligned} \tag{3.21}$$

where a_2 is related to material constants, obtained by solving Appendix equations (A1).

Similarly, for uniaxial tension ($\sigma_{xx}^\infty = 0$ and $\sigma_{yy}^\infty = \sigma_A$), obtained tensile stress along the crack surface at $\theta = 0, y = 0$ is:

$$\begin{aligned}
p_y(x) &= \sigma_A \left[1 + \frac{\mu_1}{2(\lambda_1 + \mu_1)} \left(\frac{r_0}{x} \right)^2 a_2 - \frac{3}{2} \left(\frac{r_0}{x} \right)^2 b_3 \right], & (a) \\
p_x(x) &= 0, & (b)
\end{aligned} \tag{3.22}$$

where again a_2 and b_3 are obtained from Appendix equations (A1).

Hence by replacing $K_{I\rho}^L$ and $K_{I\rho}^R$ in equation (3.19) with K_I^L and K_I^R respectively, the solutions for K_I^L and K_I^R can be calculated.

3.6. Solving for PZS and CTOD

The material is modelled to be elastic-perfectly plastic. Thus when yielding occurs at the crack tips, the stresses there will be equal to the yielding stress, not as predicted by linear elastic fracture mechanics. This model is suitable for studying small-scale yielding in composite materials.

The procedure to solve for the PZS and CTOD requires solving for the PZS first, which can then be used to extract the CTOD. The CTOD here refers to the vertical displacement at the open end of the plastic zones, as defined in Section 2.7. Initial values of ρ_L and ρ_R are assumed, and then using an iteration procedure to satisfy the equations of (3.5), plastic zone sizes ρ_L and ρ_R can be determined.

The iteration consists of determining whether $|K_I^L|$ or $|K_{I\rho}^L|$ is larger on the left, and $|K_I^R|$ or $|K_{I\rho}^R|$ is on the right. ρ_L is increased if $|K_I^L|$ is larger (decreased if $|K_{I\rho}^L|$ is larger), as increasing ρ_L increases $|K_{I\rho}^L|$ (and also $|K_{I\rho}^R|$ slightly). Similarly ρ_R is increased if $|K_I^R|$ is

larger (and decreased if $|K_{I\rho}^R|$ is larger), and the stress intensity factors are re-evaluated. The procedure is repeated until the condition is sufficiently satisfied. The detailed procedure is described in Appendix B1.

The PZS is an important measure of the extent of small-scale yielding present in the composite material, and is useful to calculate the CTOD. To find the CTOD, we make use of dislocation densities in equation (2.31). The opening displacement for the right and left tips (using the Dugdale model definition in Figure 2.11) can thus be given by:

$$\begin{aligned}\delta_L = D(c_1) &= - \int_{c_2 + \rho_R}^{c_1} B_y(\xi) d\xi, \\ \delta_R = D(c_2) &= \int_{c_2}^{c_1 - \rho_L} B_y(\xi) d\xi.\end{aligned}\tag{3.23}$$

To solve for the CTODs, equations (3.23) are normalised using (3.14) and (3.15):

$$\begin{aligned}\delta_L &= - \frac{t_2 - t_1}{2} \int_{-1}^s B_y(s) ds, \\ \delta_R &= \frac{t_2 - t_1}{2} \int_s^1 B_y(s) ds.\end{aligned}\tag{3.24}$$

Then applying the trapezoidal rule of numerical integration gives:

$$\begin{aligned}\delta_L &= - \frac{t_2 - t_1}{2} \left\{ \frac{s_{L\rho} - s_L}{2} [B_y(s_{L\rho}) - B_y(s_L)] \right. \\ &\quad \left. + \sum_{k=L}^n \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] + \frac{s_n - (-1)}{2} [B_y(s_n) + 0] \right\}, \\ \delta_R &= \frac{t_2 - t_1}{2} \left\{ \frac{1 - s_1}{2} [0 + B_y(s_1)] + \sum_{k=1}^{R-1} \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] \right. \\ &\quad \left. + \frac{s_R - s_{R\rho}}{2} [B_y(s_R) + B_y(s_{R\rho})] \right\},\end{aligned}\tag{3.25}$$

where $B_y(s_k) = F_y(s_k)(1 - s_k^2)^{-1/2}$ is from equation (3.13), $s_k = \cos \frac{\pi(2k-1)}{2n}$ from equation (3.18) while the terms s_L , $s_{L\rho}$, s_R and $s_{R\rho}$ (which denote locations near the open end of the plastic zones) are, using equation (3.15):

$$s_{L\rho} = \frac{c_1 - (t_2 + t_1)/2}{(t_2 - t_1)/2}, s_L = s_{\text{ceil}(L\rho)},$$

$$s_{R\rho} = \frac{c_2 - (t_2 + t_1)/2}{(t_2 - t_1)/2}, s_R = s_{\text{floor}(R\rho)}.$$

Both $B_y(s_{L\rho})$ and $B_y(s_{R\rho})$ tend to infinity, but are interpolated to:

$$B_y(s_{L\rho}) = \frac{s_{L\rho} - s_L}{s_{\text{floor}(L\rho)} - s_L} [B_y(s_{\text{floor}(L\rho)}) - B_y(s_L)] + B_y(s_L)$$

$$B_y(s_{R\rho}) = \frac{s_{R\rho} - s_{\text{ceil}(R\rho)}}{s_R - s_{\text{ceil}(R\rho)}} [B_y(s_R) - B_y(s_{\text{ceil}(R\rho)})] + B_y(s_{\text{ceil}(R\rho)})$$

3.7. Numerical Results and Discussion

Crack behaviour is affected by factors such as: relative shear modulus μ_2/μ_1 , Poisson's ratio ν , relative crack-inclusion size r_0/a , crack-inclusion distance t_1/a , and also the load σ_A/σ_Y . The influence of these factors on the stress intensity factor K_I and plastic zone sizes ρ_L and ρ_R are studied and discussed in the following sections. The results have been normalised to the homogeneous case, with $n = 200$.

3.7.1. The influence of load ratio

The influence of load ratios is particularly important, as stress intensity factors are affected by the load ratios or yield values $T = \sigma_A/\sigma_Y$. The CTOD values are calculated for $\sigma_A/\sigma_Y = 0.04$ to 0.33 , $r_0/c_1 = 0.5, 0.7$ and $c_2/c_1 = 2, 10$, and normalised by using the CTOD for $\sigma_A/\sigma_Y = 0.1$ shown in Figure 3.4 for the left tip and in Figure 3.6 for the right tip (denoted normalised δ_y^L and δ_y^R respectively). Meanwhile, the K_I values are also normalised by using K_I for $\sigma_A/\sigma_Y = 0.1$, and are shown in Figures 3.5 and 3.7 for the left (as normalised K_I^L) and right tips (as normalised K_I^R) respectively.

The normalised CTOD and K_I at the left crack tip are plotted against σ_A/σ_Y in Figures 3.4 and 3.5, respectively. It is observed that the normalised CTOD values increase exponentially, while the normalised K_I increase in linear fashion. It is obvious that the normalised CTOD values will increase to higher values (from 8 to 10 at $\sigma_A/\sigma_Y = 0.3$) compared to the normalised K_I (slightly below 3 at $\sigma_A/\sigma_Y = 0.3$). This shows K_I underestimates fracture compared to CTOD, especially when σ_A/σ_Y is high.

Figure 3.4 Normalised δ_y^L against σ_A/σ_Y

Figure 3.5 Normalised K_I^L against σ_A/σ_Y

Figure 3.6 Normalised δ_y^R against σ_A/σ_Y

Figure 3.7 Normalised K_I^R against σ_A/σ_Y

Similarly, for the right crack tip, results are given in Figures 3.6 and 3.7, for the normalised CTOD and K_I respectively. Like the left tip, exponential behaviour in normalised δ_y^R and linear in normalised K_I^R is observed. Hence normalised CTOD values increases to higher values (about 11 to 12 at $\sigma_A/\sigma_Y = 0.3$) than normalised K_I (about 3 at $\sigma_A/\sigma_Y = 0.3$). As expected with the right tip less shielded by the inclusion than the left tip, the results are higher than the corresponding ones at the left crack tip.

3.7.2. The influence of shear modulus ratio

Figure 3.8 Normalised δ_y^L against $\mu_2/\mu_1 > 1$

Figure 3.9 Normalised K_I^L against $\mu_2/\mu_1 > 1$

Setting the load ratio $\sigma_A/\sigma_Y = 0.1$ and Poisson's ratio $\nu_1 = \nu_2 = 1/3$, we vary the shear modulus ratio μ_2/μ_1 . The geometric parameters are $r_0/c_1 = 0.5, 0.7$ and $c_2/c_1 = 2, 10$. The results for normalised CTOD are shown for both the left and right tips (Figures 3.8 and 3.10, respectively), while curves for the normalised K_I are shown in Figures 3.9 and 3.11.

Figure 3.10 Normalised δ_y^R against $\mu_2/\mu_1 > 1$

Figure 3.11 Normalised K_I^R against $\mu_2/\mu_1 > 1$

Comparing the results of both normalised CTOD (Figure 3.8) and K_I (Figure 3.9) at the left tips when $\mu_2/\mu_1 > 1$, it is visible that the normalised CTOD values predict more shielding effect (the inclusion on the crack) than K_I values, and the shielding effect naturally increases with increased inclusion stiffness. This difference shows the influence of considering small-scale plasticity on judging the overall fracture behaviour of the crack. At $r_0/c_1 = 0.7$, the normalised K_I levels off at about 0.81, while the normalised CTOD levels

off at 0.63. Meanwhile at $r_0/c_1 = 0.5$, the normalised K_I levels off at about 0.93, while the normalised CTOD levels off at 0.86.

Naturally, a larger inclusion ($r_0/c_1 = 0.7$) shields the crack more than a smaller one ($r_0/c_1 = 0.5$). This is the case for brittle materials considering stress intensity factors too.

At $r_0/c_1 = 0.7$, the near-tip (the left crack tip which is nearer to the inclusion than the right crack tip) of a smaller crack ($c_2/c_1 = 2$) is better shielded by the inclusion than that of a larger ($c_2/c_1 = 10$) crack. However, at inclusion sizes of $r_0/c_1 = 0.5$ and smaller, the near-tips are similarly shielded, with similar behaviour displayed for both normalised CTOD and K_I .

Results for the right tip in various crack-inclusion sizes are shown in Figure 3.10 for CTOD and Figure 3.11 for K_I . Again the shielding difference of the inclusion on the crack for the two parameters is observed: larger shielding effect occurs when small-scale plasticity is considered. At $r_0/c_1 = 0.7, c_2/c_1 = 2$, the normalised K_I levels off at about 0.93, while the normalised CTOD levels off at 0.86. Meanwhile at $r_0/c_1 = 0.5, c_2/c_1 = 2$, the normalised K_I levels off at about 0.97, while the normalised CTOD levels off at 0.95. For the larger cracks ($c_2/c_1 = 10$) the difference is less obvious.

As expected, the larger inclusion ($r_0/c_1 = 0.7$) shields the crack more, while the crack tip further away ($c_2/c_1 = 10$) is less shielded.

Figure 3.12 Normalised δ_y^L against $\mu_2/\mu_1 < 1$

Figure 3.13 Normalised K_I^L against $\mu_2/\mu_1 < 1$

Similarly, results at the left tip for $\mu_2/\mu_1 < 1$ are shown in Figures 3.12 and 3.13. Anti-shielding is involved here, but as evident when comparing the normalised CTOD and K_I , anti-shielding is more prominent with plasticity considered. For example, for $r_0/c_1 = 0.7$, the normalised K_I reaches 1.4, but the CTOD goes up to about 2.0, while for $r_0/c_1 = 0.5$, the normalised K_I reaches 1.15, but the CTOD goes up to about 1.35.

At the right crack tip, similar behaviour is observed in Figures 3.14 and 3.15. Anti-shielding is again more obvious when plasticity is considered.

Other factors, such as inclusion size and crack size, yield similar effects for both normalised CTOD and K_I .

Figure 3.14 Normalised δ_y^R against $\mu_2/\mu_1 < 1$

Figure 3.15 Normalised K_I^R against $\mu_2/\mu_1 < 1$

3.7.3. The effect of Poisson's ratios and distance

The effect of Poisson's ratio is observed by studying cases $\nu_1 = \nu_2 = 0.1, 0.2, 0.33$ and 0.45 , with results shown in Table 3.1. The Poisson's ratios are also set as $\nu_1 = 0.1, \nu_2 = 0.45$ and $\nu_1 = 0.45, \nu_2 = 0.1$ in Table 3.2. Other variables are taken as $\mu_2/\mu_1 = 10$, $r_0/c_1 = 0.5$, $c_2/c_1 = 10$ and $\sigma_A/\sigma_Y = 0.1$.

Table 3.1 Normalised CTOD for some values of Poisson's ratio $\nu_1 = \nu_2$

$\nu_1 = \nu_2$	Normalised δ_y^L	Normalised δ_y^R
0.10	0.8197	0.9188
0.20	0.8463	0.9329
0.33	0.8839	0.9527
0.45	0.9163	0.9708

Table 3.1 shows the normalised CTOD values obtained for some Poisson's ratios where $\nu_1 = \nu_2$. In this set of simplified results, the Poisson's ratios of the matrix and inclusion are set equal. From these results, it can be observed that the higher the Poisson's ratio, the higher the normalised CTOD. This is because a lower Poisson's ratio indicates that the material has less transverse contraction, leading to lower normalised CTOD as well. However, it can be seen that overall, the effect of the Poisson's ratio on CTOD is small, more likely because of the small range of Poisson's ratios.

Table 3.2 Normalised CTOD for different values of Poisson's ratio ν_1 and ν_2

ν_1	ν_2	Normalised δ_y^L	Normalised δ_y^R
0.10	0.10	0.8197	0.9188
0.10	0.45	0.7975	0.9078
0.45	0.10	0.9304	0.9783
0.45	0.45	0.9163	0.9708

Table 3.2 shows the normalised CTOD values for differing values of ν_1 and ν_2 , with the results for $\nu_1 = \nu_2 = 0.10, 0.45$ included for comparison. When $\nu_1 = 0.10$ the higher $\nu_2 = 0.45$ leads higher shielding of the inclusion on the crack, compared to the case where $\nu_2 = 0.10$. Conversely, when $\nu_2 = 0.45$, there is less shielding for $\nu_1 = 0.10$ than $\nu_1 = 0.45$. This result is also observed for the K_I .

3.8. Chapter Summary

In this chapter, a crack near a circular inclusion with Dugdale plastic zones at both crack tips under uniaxial and uniform tension is investigated, using the superposition

principle and the solution of an edge dislocation interacting with a circular inclusion as the Green's function.

The current study gives accurate analysis for the case a Dugdale model crack in dilute inclusions. The influence of the crack-inclusion distance, materials constants, and load-to-yield-stress-ratio on the CTOD and K_I are studied. Numerical examples are given to demonstrate the variation trends of CTOD and K_I . The inclusion either shields or anti-shields the crack tips, depending on the material properties relative to the matrix. However, it is shown that the Dugdale model predicts higher normalised CTOD values compared to the normalised K_I values, especially at higher load to yield ratios. Also, the Dugdale model predicts more shielding effect when the inclusion is “harder” than the matrix, and more anti-shielding when the inclusion is “softer” than the matrix. For common Poisson's ratio values, the effect on fracture was not significant.

CHAPTER 4

A Crack Interacting with a Coated Circular Inclusion

4.1. Introduction

Upon obtaining the results of the plastic zone and crack tip opening displacement (CTOD) using the Dugdale model for the interaction between a crack and a circular inclusion in the previous chapter, the problem is now extended to the case of a crack interacting with a coated circular inclusion. This case adds a coating layer to the inclusion in a dilute composite problem solved in Chapter 3. This will aid understanding the effect of coating layers in materials on fracture when plasticity is considered. The coating phase is an important phase in composite materials that functions to transfer stress to the fibre and improve the interface bonding between the inclusion and the matrix. It may represent either a reinforcing coating phase, or a layer that was created during manufacturing. It can also function to prevent inclusion-matrix chemical reaction, such as nickel coating of carbon fibres in an aluminium matrix to prevent formation of a brittle and water-soluble compound Al_4C_3 [86].

This problem makes use of the solution of an edge dislocation near a coated fibre derived by Xiao and Chen [53, 54] as the Green's function, to solve for a crack near a coated circular inclusion. The Dugdale model is applied to obtain solutions for the plastic zone and the CTOD.

4.2. Problem Formulation

The problems in consideration are depicted in Figures 4.1a under tension and 4.3a under shear. The two problems have different loading conditions, and are otherwise identical. The Dugdale model crack is located in phase 1 (the matrix material), with phase 2 being the coating layer, and phase 3 the circular inclusion. The circular inclusion of material properties

κ_3 and μ_3 occupies the region $r < r_a$ while the coating layer (properties κ_2 and μ_2 ,) occupies $r_a < r < r_b$. The matrix (material properties κ_1 and μ_1) occupies the area in $r > r_b$.

Figure 4.1 The Dugdale model crack and coated circular inclusion problem under tensile loads

Figure 4.2 The Dugdale model crack and coated circular inclusion problem under shear

The Dugdale model crack is located in the matrix phase along the x-axis, with plastic zones on the left and right tips of lengths ρ_L and ρ_R , respectively. The left crack tip is at $t_1 = c_1 - \rho_L$ while the right tip is at $t_2 = c_2 + \rho_R$. The Dugdale model crack has an effective length of $\rho_L + 2a + \rho_R$. Without the plastic zones, the crack simplifies to a crack of length $2a$ from $t_1 = c_1$ to $t_2 = c_2$, where $a = (c_2 - c_1)/2$ is the half crack length.

The load conditions considered here are the uniaxial tension, biaxial tension and shear. Meanwhile, the boundary conditions for the crack problem are described next. In the far field the applied load is given by one of the stress conditions:

$$\begin{aligned}
\sigma_{yy}^\infty &= \sigma_{xx}^\infty = \sigma_A, \tau_{xy}^\infty = 0 \text{ for biaxial tension, or} \\
\sigma_{yy}^\infty &= \sigma_A, \sigma_{xx}^\infty = \tau_{xy}^\infty = 0 \text{ for uniaxial tension or} \\
\tau_{xy}^\infty &= \tau_A, \sigma_{yy}^\infty = \sigma_{xx}^\infty = 0 \text{ for pure shear.}
\end{aligned} \tag{4.1}$$

At the interfaces of phases 1 and 2, and of phases 2 and 3:

$$[\sigma_{rr}] = 0, \quad [\sigma_{r\theta}] = 0, \quad [u_r] = 0, \quad [u_\theta] = 0, \quad \text{at } r = r_a, r_b, \tag{4.2}$$

where $[f]$ is the jump of function f , while r and θ are polar coordinates with origin 0. At the crack surface, the Dugdale model requires that the plastic zones loaded to yielding stress σ_Y :

$$\begin{aligned}
\tau_{xy} &= 0, \sigma_{yy} = 0, & c_1 \leq x \leq c_2, \\
\sigma_{yy} &= \sigma_Y, & c_1 - \rho_L \leq x \leq c_1, c_2 \leq x \leq c_2 + \rho_R, \\
\tau_{xy} &= \tau_Y, & c_1 - \rho_L \leq x \leq c_1, c_2 \leq x \leq c_2 + \rho_R.
\end{aligned} \tag{4.3}$$

where the applied stress σ_A (or τ_A) is related to the yield stresses through:

$$\begin{aligned}
\sigma_A/\sigma_Y &= T \text{ or} \\
\tau_A/\tau_Y &= S.
\end{aligned} \tag{4.4}$$

in which T and S are known parameters relating applied load to yield strength.

4.3. Solution Procedure

The solution procedures for the two problems are similar. Hence in outlining the solution, the procedure to solve for Figure 4.1 is described, while to solve for Figure 4.2, the shear terms in the brackets are used instead. Meanwhile, subsequent equations labelled (a) will apply to the tensile problems, while (b) are for the shear case.

To obtain the Dugdale model solution for the plastic zone sizes (PZS) and CTOD, the condition of vanishing stress intensity factors again applies at both tips, repeated from (3.5) but for both tension and shear:

$$\begin{aligned}
&\left. \begin{aligned} K_I^L + K_{I\rho}^L &= 0 \\ K_I^R + K_{I\rho}^R &= 0 \end{aligned} \right\} & (a) \\
&\left. \begin{aligned} K_{II}^L + K_{II\rho}^L &= 0 \\ K_{II}^R + K_{II\rho}^R &= 0 \end{aligned} \right\} & (b)
\end{aligned} \tag{4.5}$$

Superposition procedure is applied to solve this problem, where the Dugdale model of Figure 4.1a can be decomposed into the two simpler problems of Figures 4.1b and 4.1c (Figure 4.2a decomposes into Figures 4.2b and 4.2c for shear), both with cracks of $\rho_L + 2a + \rho_R$ length. Figure 4.1b (Figure 4.2b) shows a crack with closure stresses at yield stress for the plastic zones, while Figure 4.1c (Figure 4.2c) depicts the crack with far-field loads but without plastic zones.

Equations (4.5) are visualised by Figure 4.1 (Figure 4.2 for shear). $K_{I\rho}^L$ and $K_{I\rho}^R$ used in tensile problems are obtained from Figure 4.1b ($K_{II\rho}^L$ and $K_{II\rho}^R$ for shear are from Figure 4.2b), while K_I^L and K_I^R are gathered from Figure 4.1c (K_{II}^L and K_{II}^R from Figure 4.2c). To obtain the plastic zone size of the Dugdale model for Figure 4.1a (Figure 4.2a), the problems of Figures 4.1b and 4.1c (Figures 4.3b and 4.3c) must be solved. The distributed dislocation technique [12] is used to obtain stress intensity factors at the crack tips in both applied load and closure stress. Section 4.4 solves for K_ρ , while Section 4.5 solves for the stress intensity factors.

4.4. Solving for K_ρ

In Figure 4.1b (Figure 4.2b for shear), the stresses at the crack tips should remain bounded and equal to the yield stress σ_Y of the matrix material, as the material is assumed elastic-perfectly plastic. Hence closing the crack requires closure stresses of σ_Y -magnitude (or τ_Y) applied to the left and right of the crack tips:

$$\begin{aligned} D_y(x) &= \begin{cases} -\sigma_Y & \text{if } c_1 - \rho_L < x < c_1, c_2 < x < c_2 + \rho_R \\ 0 & \text{else} \end{cases} & (a) \\ D_x(x) &= \begin{cases} -\tau_Y & \text{if } c_1 - \rho_L < x < c_1, c_2 < x < c_2 + \rho_R \\ 0 & \text{else} \end{cases} & (b) \end{aligned} \quad (4.6)$$

4.4.1. Crack simulation with distributed dislocations

The equivalence between a crack and a dislocation pile-up is applied to Figure 4.1b (Figure 4.2b in shear). Using $B_y(x)$ and $B_x(x)$ as the climb and glide dislocation densities in the crack, while applying the single dislocation solution [53] of Figure 4.3, the dislocation densities are related to the crack surface stress by:

$$\begin{aligned} \sigma_{yy}(x, 0) &= -\frac{2\mu_1}{\pi(\kappa_1 + 1)} \left[\int_{t_1}^{t_2} \frac{B_y(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_1(x, \xi) B_y(\xi) d\xi \right], \quad (a) \\ \sigma_{xy}(x, 0) &= -\frac{2\mu_1}{\pi(\kappa_1 + 1)} \left[\int_{t_1}^{t_2} \frac{B_x(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_2(x, \xi) B_x(\xi) d\xi \right]. \quad (b) \end{aligned} \quad (4.7)$$

Figure 4.3 An edge dislocation interacting with a coated circular inclusion

For the resultant stress at the crack surfaces in Figure 4.1a (Figure 4.2a) to meet the conditions of (4.3) at the plastic zones, $\sigma_{yy}(x, 0) = -D_y(x)$ is applied as the corrective traction for uniaxial and biaxial tension ($\sigma_{xy}(x, 0) = -D_x(x)$ for pure shear):

$$\begin{aligned} \frac{1}{\pi} \left[\int_{t_1}^{t_2} \frac{B_y(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_1(x, \xi) B_y(\xi) d\xi \right] &= \frac{\kappa_1 + 1}{2\mu_1} p_y(x), \quad (a) \\ \frac{1}{\pi} \left[\int_{t_1}^{t_2} \frac{B_x(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_2(x, \xi) B_x(\xi) d\xi \right] &= \frac{\kappa_1 + 1}{2\mu_1} p_x(x). \quad (b) \end{aligned} \quad (4.8)$$

The dislocation densities must also satisfy:

$$\int_{t_1}^{t_2} B_y(\xi) d\xi = 0, \quad (a)$$

$$\int_{t_1}^{t_2} B_x(\xi) d\xi = 0, \quad (b)$$
(4.9)

where equations (4.9) and (4.10) form a set of singular integral equations. The relevant

Green's functions [53] for Mode I and a Mode II crack are $k_1(x, \xi)$ and $k_2(x, \xi)$ respectively:

$$\begin{aligned} k_1(x, \xi) = & \frac{C + D}{2(x - b^2/\xi)} + C \frac{\xi^2 - b^2}{\xi^3} \frac{b^2}{(x - b^2/\xi)^2} \left[\left(\frac{\xi}{b} \right)^2 - \frac{\xi^2 - b^2}{x\xi - b^2} \right] - \frac{C + D}{2x} \\ & - \frac{1 - CD}{2(1 - C)} a_0 \frac{b}{x^2} - \frac{1}{2\xi} \left(\frac{b}{x} \right)^2 \left[2C \left(\frac{\xi}{b} \right)^2 - C - 1 \right] \\ & + \frac{1 - CD}{2b(1 - D)} \left[\sum_{n=1}^{\infty} n a_{-n} \left(\frac{b}{x} \right)^n - \sum_{n=1}^{\infty} (n + 1) a_{-n} \left(\frac{b}{x} \right)^{n+2} \right] - C \frac{b^2}{x^3} \\ & - \frac{1 - CD}{b(1 - D)} \sum_{n=1}^{\infty} a_{-n} \left(\frac{b}{x} \right)^n - \frac{1 - CD}{2b(1 - C)} \sum_{n=1}^{\infty} a_n \left(\frac{b}{x} \right)^{n+2}, \\ k_2(x, \xi) = & \frac{C + D}{2(x - b^2/\xi)} + C \frac{b^2}{(x - b^2/\xi)^2} \left[\frac{x - \xi}{\xi^2} \frac{\xi^2 - b^2}{x\xi - b^2} \right] - \frac{C + D}{2x} - \frac{1 - C}{2\xi} \left(\frac{b}{x} \right)^2 \\ & + \frac{1 - CD}{2(1 - C)} a'_0 \frac{b}{x^2} + C \frac{b^2}{x^3} \\ & + \frac{1 - CD}{2b(1 - D)} \left[\sum_{n=1}^{\infty} n a'_{-n} \left(\frac{b}{x} \right)^n - \sum_{n=1}^{\infty} (n + 1) a'_{-n} \left(\frac{b}{x} \right)^{n+2} \right] \\ & + \frac{1 - CD}{2b(1 - C)} \sum_{n=1}^{\infty} a'_{-n} \left(\frac{b}{x} \right)^{n+2}, \end{aligned} \quad (4.10)$$

where $C = \frac{\mu_3 - \mu_2}{\mu_2 \kappa_3 + \mu_3}$ and $D = \frac{\mu_3 \kappa_2 - \mu_2 \kappa_3}{\mu_3 \kappa_2 + \mu_2}$. Meanwhile, constants a_n , b_n , a'_n and b'_n are found

from solving the equations of (A2).

4.4.2. Numerical solutions for singular integral equations

The singularities on both crack tips are inverse square root because the whole crack is located in the matrix material. Thus, using the fundamental functions [63], let

$$\begin{aligned} B_y(x) &= w(x) F_y(x), \\ B_x(x) &= w(x) F_x(x), \end{aligned} \quad (4.11)$$

where $F_y(x)$ and $F_x(x)$ are bounded functions in $t_1 \leq x \leq t_2$, with

$$w(x) = (x - t_1)^{-\frac{1}{2}}(t_2 - x)^{-\frac{1}{2}}. \quad (4.12)$$

To shift the interval of the crack integral from (t_1, t_2) to $(-1, 1)$, let

$$\begin{aligned} x &= \frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \\ \xi &= \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2}. \end{aligned} \quad (4.13)$$

Equations (4.8) and (4.9) can then be rewritten in terms of s and t as:

$$\left. \begin{aligned} \frac{1}{\pi} \left[\int_{-1}^1 \frac{B_y(s)}{s - t} ds + \int_{-1}^1 k_{11}(t, s) B_y(s) ds \right] &= \frac{\kappa_1 + 1}{2\mu_1} p_y(x), \\ \int_{-1}^1 B_y(s) ds &= 0, \end{aligned} \right\} \quad (a) \quad (4.14)$$

$$\left. \begin{aligned} \frac{1}{\pi} \left[\int_{-1}^1 \frac{B_x(s)}{s - t} ds + \int_{-1}^1 k_{22}(t, s) B_x(s) ds \right] &= \frac{\kappa_1 + 1}{2\mu_1} p_x(x), \\ \int_{-1}^1 B_x(s) ds &= 0, \end{aligned} \right\} \quad (b)$$

where

$$\begin{aligned} k_{11}(t, s) &= \frac{t_2 - t_1}{2} k_1 \left(\frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2} \right), \\ k_{22}(t, s) &= \frac{t_2 - t_1}{2} k_2 \left(\frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2} \right). \end{aligned} \quad (4.15)$$

Following the numerical method of Erdogan [63], the discretised forms of equations (4.14) can be rewritten:

$$\left. \begin{aligned} \sum_{k=1}^n \frac{1}{n} F_y(s_k) \left[\frac{1}{s_k - u_r} + k_{11}(u_r, s_k) \right] &= \frac{\kappa_1 + 1}{2\mu_1} p_y(x_r), \\ \sum_{k=1}^n \frac{1}{n} F_y(s_k) &= 0, \end{aligned} \right\} \quad (a) \quad (4.16)$$

$$\left. \begin{aligned} \sum_{k=1}^n \frac{1}{n} F_x(s_k) \left[\frac{1}{s_k - u_r} + k_{22}(u_r, s_k) \right] &= \frac{\kappa_1 + 1}{2\mu_1} p_x(x_r), \\ \sum_{k=1}^n \frac{1}{n} F_x(s_k) &= 0, \end{aligned} \right\} \quad (b)$$

where n is the number of terms used, while:

$$\begin{aligned} s_k &= \cos \frac{\pi(2k-1)}{2n}, & k &= 1..n, \\ x_r &= \frac{t_2 - t_1}{2} u_r + \frac{t_2 + t_1}{2}, & u_r &= \cos \frac{\pi r}{n}, \quad r = 1..n-1. \end{aligned}$$

Hence, the Mode I (and Mode II) stress intensity factors [10] are:

$$\left. \begin{aligned} K_I^L &= \lim_{\xi \rightarrow t_1} \frac{-2\mu_1 \sqrt{2\pi}}{\kappa_1 + 1} \sqrt{\xi - t_1} B_y(x) = \frac{-2\mu_1 \sqrt{\pi(t_2 - t_1)/2}}{\kappa_1 + 1} F_y(-1) \\ &= -\sigma_A \sqrt{\pi(t_2 - t_1)/2} F_{0y}(-1), \\ K_I^R &= \lim_{\xi \rightarrow t_2} \frac{2\mu_1 \sqrt{2\pi}}{\kappa_1 + 1} \sqrt{t_2 - \xi} B_y(x) = \frac{2\mu_1 \sqrt{\pi(t_2 - t_1)/2}}{\kappa_1 + 1} F_y(1) \\ &= \sigma_A \sqrt{\pi(t_2 - t_1)/2} F_{0y}(1), \\ K_{II}^L &= \lim_{\xi \rightarrow t_1} \frac{-2\mu_1 \sqrt{2\pi}}{\kappa_1 + 1} \sqrt{\xi - t_1} B_x(x) = \frac{-2\mu_1 \sqrt{\pi(t_2 - t_1)/2}}{\kappa_1 + 1} F_x(-1) \\ &= -\tau_A \sqrt{\pi(t_2 - t_1)/2} F_{0x}(-1), \\ K_{II}^R &= \lim_{\xi \rightarrow t_2} \frac{2\mu_1 \sqrt{2\pi}}{\kappa_1 + 1} \sqrt{t_2 - \xi} B_x(x) = \frac{2\mu_1 \sqrt{\pi(t_2 - t_1)/2}}{\kappa_1 + 1} F_x(1) \\ &= \tau_A \sqrt{\pi(t_2 - t_1)/2} F_{0x}(1), \end{aligned} \right\} \quad \begin{matrix} (a) \\ (b) \end{matrix} \quad (4.17)$$

where $F_{0y}(s)$ and $F_{0x}(s)$ are solved from equations (4.16) by letting $F_y(s) = \frac{\sigma_A(\kappa_2+1)}{2\mu_2} F_{0y}(s)$

for tensile loading or $F_x(s) = \frac{\tau_A(\kappa_2+1)}{2\mu_2} F_{0x}(s)$ for shear, using equation (4.4) to relate the applied load to yield strength.

4.5. Solving for K

To solve the sub-problem given in Figure 4.1c (or Figure 4.2c for shear), repeated in Figure 4.4a (Figure 4.5a), we note its similarity to Figure 4.1a (Figure 4.2a) but with no plastic zone, leading to the linear elastic problem solved by Luo and Chen [52], with the left

crack tip at $t_1 = c_1 - \rho_L$ and the right at $t_2 = c_2 + \rho_R$. Boundary conditions (4.1) and (4.2) still apply, while (4.3) is replaced by simply:

$$\sigma_{xy} = 0, \quad \sigma_{yy} = 0, \quad t_1 \leq x \leq t_2. \quad (4.18)$$

By superposition, Figure 4.4a (or Figure 4.5a for shear) is broken down into two sub-problems given in Figures 4.4b and 4.4c (Figures 4.5b and 4.5c), respectively. The problem in Figure 4.1b (Figure 4.2b) refers to the coated inclusion under far-field tension (or shear) without any crack present. The problem in Figure 4.4c is similar to Figure 4.4a (Figure 4.5c is similar to Figure 4.5a under shear), but with loading conditions: normal stress $p_y(x)$ (or shear stress $p_x(x)$) are applied on the top and bottom surfaces of the crack, instead of far-field loading. By superposition, the sum of Figures 4.4b and 4.4c equals Figure 4.4a (while Figures 4.5b and 4.5c equals Figure 4.5a).

Figure 4.4 Crack interacting with a coated circular inclusion under tensile loads

Figure 4.5 Crack interacting with a coated circular inclusion under shear

To calculate the stress intensity factors of Figure 4.4a (or Figure 4.5a for shear), the distributed dislocation method described in Sections 0 and 4.4.2 is applied, with minor adjustments. The corrective tractions are now $\sigma_{yy}(x, 0) = -p_y(x)$ for tensions (or $\sigma_{xy}(x, 0) = -p_x(x)$ for shear) where $p_y(x)$ (or $p_x(x)$) represent stresses at the crack surface transformed from the far-field load. $D_y(x)$ (or $D_x(x)$) are replaced by $p_y(x)$ (or $p_x(x)$) respectively for the formulations in Sections 0 and 4.4.2.

4.5.1. Transforming far-field loads to surface loads

The stress fields for the different load conditions of biaxial tension, uniaxial tension and pure shear are found in previous work [11]. For biaxial tension ($\sigma_{xx}^\infty = \sigma_{yy}^\infty = \sigma_A$ in Figure 4.1b), the stresses along the crack surface at $\theta = 0, y = 0$ are:

$$\begin{aligned} p_y(x) &= \sigma_A \left[1 + \frac{\mu_1}{\lambda_1 + \mu_1} \left(\frac{c}{x} \right)^2 a_4 \right], \\ p_x(x) &= 0, \end{aligned} \quad (4.19)$$

where a_4 is a material-related constant, obtained through solving the equations of (A3).

Under uniaxial tension ($\sigma_{xx}^\infty = 0$ and $\sigma_{yy}^\infty = \sigma_A$), the problem is solved by superimposing a biaxial tension load and a tension-compression load. The tensile stress along the crack surface at $\theta = 0, y = 0$ is:

$$\begin{aligned} p_y(x) &= \sigma_A \left[1 + \frac{\mu_1}{2(\lambda_1 + \mu_1)} \left(\frac{c}{x} \right)^2 a_4 - \frac{3}{2} \left(\frac{b}{x} \right)^2 b_7 \right], \\ p_x(x) &= 0, \end{aligned} \quad (4.20)$$

where a_4 and b_7 can again be obtained by solving the equations of (A3).

For pure shear, the shear traction along the crack surface is given by:

$$\begin{aligned} p_y(x) &= 0, \\ p_x(x) &= \tau_A \left[1 + 3b_7 \left(\frac{r_b}{x} \right)^4 + 2b_8 \left(\frac{r_b}{x} \right)^2 \right], \end{aligned} \quad (4.21)$$

where b_7 and b_8 are from (A3).

4.6. Solving for PZS and CTOD

In the Dugdale crack model, the crack tips are loaded with the yield stress of the matrix material. In our solution procedure, the plastic zone sizes are evaluated first. The results are then used to calculate the CTOD. The plastic zone sizes are obtained by an iteration procedure: first assuming initial values of ρ_L and ρ_R and then following an iteration procedure to satisfy the equations of (4.5).

By assuming initial values ρ_L and ρ_R , the equations of (4.5) are used to determine whether $|K_I^L|$ or $|K_{I\rho}^L|$ is larger on the left crack tip, and whether $|K_I^R|$ or $|K_{I\rho}^R|$ is larger on the right tip. ρ_L is increased if $|K_I^L|$ is larger (decreased if $|K_{I\rho}^L|$ is larger), as increasing ρ_L increases $|K_{I\rho}^L|$ (and also $|K_{I\rho}^R|$ slightly). And similarly ρ_R is increased if $|K_I^R|$ is larger (and decreased if $|K_{I\rho}^R|$ is larger), and the stress intensity factors are evaluated again. A similar procedure is used for Mode II problems as well. The procedure is repeated until the conditions in (4.5) are sufficiently satisfied, with ρ_L and ρ_R obtained. The detailed procedure is elaborated upon in Appendix B1.

And with the evaluated ρ_L and ρ_R , the CTOD is computed next. To calculate the CTOD, the dislocation densities in equation (2.31) are applied. The opening displacement for the right and left tips (using the Dugdale model definition in Figure 2.11) can thus be given by:

$$\begin{aligned}\delta_L = D(c_1) &= - \int_{c_2 + \rho_R}^{c_1} B_y(\xi) d\xi, \\ \delta_R = D(c_2) &= \int_{c_2}^{c_1 - \rho_L} B_y(\xi) d\xi.\end{aligned}\tag{4.22}$$

To solve for the CTODs, equation (3.23) is normalised using (3.14) and (3.15):

$$\begin{aligned}\delta_L &= -\frac{t_2 - t_1}{2} \int_{-1}^s B_y(s) ds, \\ \delta_R &= \frac{t_2 - t_1}{2} \int_s^1 B_y(s) ds.\end{aligned}\tag{4.23}$$

The trapezoidal rule of numerical integration gives:

$$\begin{aligned}\delta_L &= -\frac{t_2 - t_1}{2} \left\{ \frac{s_{L\rho} - s_L}{2} [B_y(s_{L\rho}) - B_y(s_L)] \right. \\ &\quad \left. + \sum_{k=L}^n \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] + \frac{s_n - (-1)}{2} [B_y(s_n) + 0] \right\} \\ \delta_R &= \frac{t_2 - t_1}{2} \left\{ \frac{1 - s_1}{2} [0 + B_y(s_1)] + \sum_{k=1}^{R-1} \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] \right. \\ &\quad \left. + \frac{s_R - s_{R\rho}}{2} [B_y(s_R) + B_y(s_{R\rho})] \right\}\end{aligned}\tag{4.24}$$

where $B_y(s_k) = F_y(s_k)(1 - s_k^2)^{-1/2}$ is from equation (4.11), $s_k = \cos \frac{\pi(2k-1)}{2n}$ from equation (4.16) while the terms s_L , $s_{L\rho}$, s_R and $s_{R\rho}$ (which denotes locations at the open end of the plastic zones) are, using equation (4.13):

$$\begin{aligned}s_{L\rho} &= \frac{c_1 - (t_2 + t_1)/2}{(t_2 - t_1)/2}, \quad s_L = s_{\text{ceil}(L\rho)}, \\ s_{R\rho} &= \frac{c_2 - (t_2 + t_1)/2}{(t_2 - t_1)/2}, \quad s_R = s_{\text{floor}(R\rho)},\end{aligned}$$

Both $B_y(s_{L\rho})$ and $B_y(s_{R\rho})$ are interpolated to:

$$\begin{aligned}B_y(s_{L\rho}) &= \frac{s_{L\rho} - s_L}{s_{\text{floor}(L\rho)} - s_L} [B_y(s_{\text{floor}(L\rho)}) - B_y(s_L)] + B_y(s_L) \\ B_y(s_{R\rho}) &= \frac{s_{R\rho} - s_{\text{ceil}(R\rho)}}{s_R - s_{\text{ceil}(R\rho)}} [B_y(s_R) - B_y(s_{\text{ceil}(R\rho)})] + B_y(s_{\text{ceil}(R\rho)})\end{aligned}$$

Similarly, to obtain the x-component CTOD for shear, replace B_y by B_x and F_y by F_x from equations (4.22) onwards.

4.7. Numerical Results and Discussion

Results in the previous chapter looked at the crack interacting with a single inclusion. The results are useful for dilute composites, and will be used as a basic unit cell to study fracture behaviour of general composite materials.

This chapter also studies the crack interacting with a single inclusion, with the inclusion coated by the third material. So here we will focus more on the effects of the coated phase on the fracture behaviour in general. The parameters affecting the CTOD are the coating thickness, crack-inclusion distance, material properties of all the three phases (matrix material, inclusion and coating layer), and load-to-yield-stress ratio. As other factors have been studied for the 2-phase problem [87] and in the previous chapter, our current investigation concentrates on the parameters of the coating material: the thickness, and shear modulus of the coating layer. Since the effect of the coated inclusion on the closer tip of the crack (the left crack tip) is stronger [45, 53], only the results of the left tip are presented. All results are obtained using $n = 200$ and $\nu_1 = \nu_2 = \nu_3 = 1/3$.

4.7.1. Effect of coating phase shear modulus

Here, the inclusion is set to be stiffer than the matrix, testing the coating phase in cases $\mu_3 > \mu_2 > \mu_1$ and $\mu_2 > \mu_3 > \mu_1$. Also, $\mu_2 = \mu_3$ is included as comparison to the two-phase case in the previous chapter. Hence the parameters are set as: $\mu_2/\mu_1 = 2, 10, 25$, $T = S = 0.1$, $\mu_3/\mu_1 = 10$, $r_b/a = 4$ and $r_b/r_a = 1.1$ while the crack distance c_1/a is varied.

For uniaxial tension, results for the normalised CTOD and K_I with changes of distance for the left tip (as normalised δ_y^L and normalised K_I^L) are shown in Figures 4.6 and 4.7, respectively. Similarly, results for uniform tension are shown in Figures 4.8 and 4.9, and for pure shear in Figures 4.10 and 4.11 (using normalised δ_x^L and normalised K_{II}^L instead).

First, note that again the normalised CTOD values are lower than the K_I values, showing that to use CTOD to judge the fracture behaviour with small-scale yielding is more advantageous, as the K_I underestimates the shielding/anti-shielding effects of the inclusion and coating phase.

Figure 4.6 Normalised δ_y^L vs distance c_1/a for various μ_2/μ_1 under uniaxial tension

Figure 4.7 Normalised K_I^L vs distance c_1/a for various μ_2/μ_1 under uniaxial tension

Considering the effect of distance; the normalised CTOD and K_I for both uniform tension and uniaxial tension increase with distance, to a limit of 1. Note that at closer distances $c_1/a < 9$, the normalised CTOD values under uniform tension are higher than that

for the uniaxial tension. This is due to the uniform tension having a “hydrostatic” state effect which is less affected by the shielding effect compared to uniaxial tension. However, farther away where the effect of the inclusion is less felt, the values for uniaxial tension are larger. Meanwhile for pure shear, the CTOD decreases at the beginning (very close to the inclusion and coating) and then increases with distance. This trend is also reported for K_{II} [53].

Figure 4.8 Normalised δ_y^L vs distance c_1/a for various μ_2/μ_1 under uniform tension

Figure 4.9 Normalised K_I^L vs distance c_1/a for various μ_2/μ_1 under uniform tension

Meanwhile, for the tensile cases, comparing the effects of shear modulus of the coating phase for the three values $\mu_2/\mu_1 = 2, 10, 25$, the larger coating phase shear modulus

gives smaller CTOD values. This is simply because a larger value gives more shielding effect to the left crack tip. This is also the case for the pure shear case when the crack is far from the inhomogeneity. However, at closer distances, the opposite is observed.

Figure 4.10 Normalised δ_x^L vs distance c_1/a for various μ_2/μ_1 under pure shear

Figure 4.11 Normalised K_{II}^L vs distance c_1/a for various μ_2/μ_1 under pure shear

4.7.2. Effect of coating phase thickness

The influence of coating thickness r_b on CTOD and K_I is shown in Figures 4.12 and 4.13 for uniaxial tension loading case, Figures 4.14 and 4.15 for biaxial tension, and Figures 4.16 and 4.17 for pure shear, respectively. The shear modulus of the coating phase is set as $\mu_2/\mu_1 = 2, 10, 25$ while other parameters are $T = S = 0.1$, $r_a/a = 4$, $\mu_3/\mu_1 = 10$ and $c_1/a = 7.8$.

Figure 4.12 Normalised δ_y^L vs distance r_b/a for various μ_2/μ_1 under uniaxial tension

Figure 4.13 Normalised K_I^L vs distance r_b/a for various μ_2/μ_1 under uniaxial tension

Figure 4.14 Normalised δ_y^L vs distance r_b/a for various μ_2/μ_1 under uniform tension

Figure 4.15 Normalised K_I^L vs distance r_b/a for various μ_2/μ_1 under uniform tension

Since the coating and inclusion phases in actual materials are usually “harder” (higher Young’s or shear modulus) than the matrix phase, increased coating thickness will usually result in decreasing CTOD and K_I . The “harder” coating phase provides more shielding effect to the crack tip. Thus depending on the properties of the coating material, adjustments can be made to provide more shielding (when the coating is “harder”) or less anti-shielding (when the coating is “softer”) to improve the fracture toughness of the composite material as a

whole. However, another exception occurs in shear loading case: the CTOD values increase after a certain coating thickness. The increased coating thickness reduces the distance separating the crack and the coating layer, and the left tip is now in the region where the “softer” coating provides more shielding instead, just as in Figures 4.16 and 4.17.

Figure 4.16 Normalised δ_x^L vs distance r_b/a for various μ_2/μ_1 under pure shear

Figure 4.17 Normalised K_{II}^L vs distance r_b/a for various μ_2/μ_1 under pure shear

4.8. Chapter Summary

In this chapter, the interaction between a crack and a coated circular inclusion with Dugdale plastic zone correction was investigated. The major coating parameters affecting the CTOD and K_I at the crack tips were studied. Numerical examples for the different crack-inclusion distance, coating shear modulus and coating thickness were given and discussed. These results indicate how the presence of the coating layer affects the CTOD more than K_I .

Combined with other parameters, the distance between the crack and inclusion can influence the CTOD and K_I by causing either shielding or anti-shielding effects to the crack. Generally the effects of shielding and anti-shielding are amplified when CTOD is considered, compared to K_I . These effects are become less visible at distances further away from the inclusion. As for the coating phase shear modulus, a “harder” coating phase generally shields the crack better, the exception being in pure shear loading for cracks very close to the inclusion. Meanwhile for the coating thickness, a thicker coating layer usually results in a higher shielding effect on the crack as well, due to the fact that the coating phase in most materials are made to be “harder” than the matrix phases. However, an exception again occurs in shear loading case; when the coating gets too thick, it causes close proximity between the crack and the coating.

CHAPTER 5

Crack in a Three-Phase Model Composite Material

5.1. Introduction

The two-phase model of Chapter 3 is not a realistic representation of the actual material as it shows only a single inclusion interacting with the crack [45]. Instead, a more accurate representation of fibre-reinforced composite material is possible by application of the three-phase cylindrical composite model. This model was first introduced by Hashin and Rosen [88] and later elaborated upon by many other researchers [89-92]. The study of the crack in a three-phase composite material is important as it looks at the plastic zone in an actual composite. This is relevant to materials such as the SiC-Al alloy metal matrix or SiC-Ti alloy metal matrix composites [93, 94], which may be matrix-damage dominant.

In this chapter, using the three-phase model, the fracture behaviour of a crack in a fibre-reinforced composite is investigated. The problem can be simulated as such: the crack is near a circular inclusion (a fibre), and both the crack and the inclusion are embedded in their immediate surrounding pure matrix, while the other inclusions and matrix parts are replaced by an equivalent homogeneous media – the composite material.

5.2. Problem Formulation

Our physical problem with Dugdale plastic zones at both crack tips in a three-phase cylinder model are shown in Figure 5.1a for tensile loads and Figure 5.2a for shear. The two problems are similar, having only different loading conditions. Phase 1 is the equivalent homogeneous media (the composite phase), phase 2 is the pure matrix material and phase 3 the circular inclusion (the reinforcing fibre). The circular inclusion (phase 3) has material properties κ_3 and μ_3 , occupying the region $r < r_a$ while the matrix phase (phase 2) has

properties κ_2 and μ_2 , in $r_a < r < r_b$. The equivalent homogeneous composite (phase 1) occupies the area in $r > r_b$ with material properties κ_1 and μ_1 .

Figure 5.1 The Dugdale model of a crack in a three-phase cylindrical composite under tensile loads

Figure 5.2 The Dugdale model of a crack in a three-phase cylindrical composite under shear load

The material properties of the equivalent homogeneous phase (composite) can be found using the three-phase cylindrical model [95] (Appendix equations (A4) for the plane strain bulk modulus K_{23} and (A5) for the transverse shear modulus μ_{23}).

The Dugdale model crack is situated along the x-axis, embedded in the matrix phase (phase 2), with the plastic zones on the left and right tips having lengths of ρ_L and ρ_R respectively. The left tip is at $t_1 = c_1 - \rho_L$ and right at $t_2 = c_2 + \rho_R$. Hence the current Dugdale model crack has an effective length of $\rho_L + 2a + \rho_R$, and the left open end of the crack is at a distance $d = c_1 - r_a$ away from the inclusion. Without the plastic zones, the

crack becomes as a Griffith crack of length $2a$ from $t_1 = c_1$ to $t_2 = c_2$, where $a = (c_2 - c_1)/2$ is the half Griffith crack length.

The loading conditions considered here are uniaxial tension, biaxial tension or pure shear. In the far field the applied loads can be denoted by:

$$\begin{aligned} \sigma_{yy}^{\infty} &= \sigma_A, \sigma_{xx}^{\infty} = \sigma_{xy}^{\infty} = 0 \text{ for uniaxial tension, or} \\ \sigma_{yy}^{\infty} &= \sigma_{xx}^{\infty} = \sigma_A, \sigma_{xy}^{\infty} = 0 \text{ for biaxial tension, or} \\ \sigma_{xy}^{\infty} &= \tau_A, \sigma_{yy}^{\infty} = \sigma_{xx}^{\infty} = 0 \text{ for pure shear,} \end{aligned} \quad (5.1)$$

where the subscript A refers to the applied load. At the interface of phase 1 and 2:

$$[\sigma_{rr}] = 0, \quad [\sigma_{r\theta}] = 0, \quad [u_r] = 0, \quad [u_{\theta}] = 0, \quad \text{at } r = r_a, r_b, \quad (5.2)$$

where $[f]$ is the jump of function f , while r and θ refer to polar coordinates with origin 0. And at the crack surface:

$$\begin{aligned} \sigma_{xy} &= 0, \sigma_{yy} = 0 \quad c_1 \leq x \leq c_2 \text{ and} \\ \sigma_{yy} &= \sigma_Y, c_1 - \rho_L \leq x \leq c_1, c_2 \leq c_2 + \rho_R \text{ for tensile loads, or} \\ \sigma_{xy} &= \tau_Y, c_1 - \rho_L \leq x \leq c_1, c_2 \leq c_2 + \rho_R \text{ for shear loads,} \end{aligned} \quad (5.3)$$

in which the applied loads are related to the yield strengths by:

$$\begin{aligned} \sigma_A / \sigma_Y &= T \text{ or} \\ \tau_A / \tau_Y &= S, \end{aligned} \quad (5.4)$$

where T and S are known parameters relating applied load to yield strength.

5.3. Solution Procedure

The solution procedures for the two problems are closely related. Thus the procedure to solve for Figure 5.1 is described. Meanwhile to solve for Figure 5.2, the shear terms in the brackets are used to replace the tensile terms. Subsequent equations labelled (a) apply to the tensile problems, while (b) apply to the shear case.

To solve for the plastic zone sizes (PZS) and crack tip opening displacements (CTOD) with Dugdale plastic zone correction, the condition of vanishing stress intensity

factors applies, but at both tips due to the asymmetry of the problem. Hence the conditions to be fulfilled for Mode I and Mode II are:

$$\left. \begin{aligned} K_I^L + K_{I\rho}^L &= 0 \\ K_I^R + K_{I\rho}^R &= 0 \end{aligned} \right\} \quad (a)$$

$$\left. \begin{aligned} K_{II}^L + K_{II\rho}^L &= 0 \\ K_{II}^R + K_{II\rho}^R &= 0 \end{aligned} \right\} \quad (b)$$
(5.5)

Visually, these equations are depicted in Figure 5.1 (or Figure 5.2 under shear), where the Dugdale model problem of Figure 5.1a is decomposed into Figures 5.1b and 5.1c (Figure 5.2a is broken down into Figures 5.2b and 5.2c). Solving Figure 5.1b gives $K_{I\rho}^L$ and $K_{I\rho}^R$ (Figure 5.2b gives $K_{II\rho}^L$ and $K_{II\rho}^R$ used in shear) values used in tensile problems at the crack tips, while solving Figure 5.1c gives K_I^L and K_I^R (Figure 5.2c gives K_{II}^L and K_{II}^R for shear).

The distributed dislocation technique [12] is used to obtain the stress intensity factors in equations (5.5) at the crack tips, for both applied load and closure stress. Solving Figure 5.1b (Figure 5.2b) for K_ρ is outlined in Section 5.4. Meanwhile the K for the applied load in Figure 5.1c (Figure 5.2c) is solved with the distributed dislocation technique outlined in Section 5.5, following Luo and Weng [52].

5.4. Solving for K_ρ

This sub-problem is shown in Figure 5.1b (Figure 5.2b under shear load). The stress at the tip of the effective crack should remain bounded and equal to the yield stress σ_Y (or τ_Y for shear) of the matrix material. Hence to “close” the crack, closure stresses equal to σ_Y (or τ_Y) are applied to the left and right of the crack tip along the plastic zones of the crack, as shown in Figure 5.1b (Figure 5.2b). Thus the closure stresses are:

$$D_y(x) = \begin{cases} -\sigma_Y & \text{if } c_1 - \rho_L < x < c_1, c_2 < x < c_2 + \rho_R \\ 0 & \text{else} \end{cases} \quad (a)$$

$$D_x(x) = \begin{cases} -\tau_Y & \text{if } c_1 - \rho_L < x < c_1, c_2 < x < c_2 + \rho_R \\ 0 & \text{else} \end{cases} \quad (b)$$
(5.6)

5.4.1. Crack simulation with distributed dislocations

Figure 5.3 An edge dislocation in a three-phase cylindrical composite

Equivalence between a crack and dislocation pile-up is applied to the problem in Figure 5.1b (Figure 5.2b). With $B_y(x)$ (and $B_x(x)$) as the climb and glide dislocation densities in the crack, and applying the solution of a single dislocation in three-phase composite (Figure 5.3) as the Green's function [96], the dislocation densities are related to the crack surface stress by:

$$\sigma_{yy}(x, 0) = -\frac{2\mu_2}{\pi(\kappa_2 + 1)} \left[\int_{t_1}^{t_2} \frac{B_y(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_1(x, \xi) B_y(\xi) d\xi \right], \quad (a)$$

$$\sigma_{xy}(x, 0) = -\frac{2\mu_2}{\pi(\kappa_2 + 1)} \left[\int_{t_1}^{t_2} \frac{B_x(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_2(x, \xi) B_x(\xi) d\xi \right]. \quad (b)$$

For the resultant stress at the crack surfaces in Figure 5.1a (Figure 5.2a for shear) to meet the conditions of (6.3) at the plastic zones, we use $\sigma_{yy}(x, 0) = -D_y(x)$ as the corrective traction for uniaxial and biaxial tension ($\sigma_{xy}(x, 0) = -D_x(x)$ for pure shear):

$$\frac{1}{\pi} \left[\int_{t_1}^{t_2} \frac{B_y(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_1(x, \xi) B_y(\xi) d\xi \right] = \frac{\kappa_2 + 1}{2\mu_1} p_y(x), \quad (a)$$

$$\frac{1}{\pi} \left[\int_{t_1}^{t_2} \frac{B_x(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_2(x, \xi) B_x(\xi) d\xi \right] = \frac{\kappa_2 + 1}{2\mu_2} p_x(x). \quad (b)$$

The dislocation densities must also satisfy:

$$\int_{t_1}^{t_2} B_y(\xi) d\xi = 0, \quad (a)$$

$$\int_{t_1}^{t_2} B_x(\xi) d\xi = 0, \quad (b)$$

where equations (5.8) and (5.9) form the set of singular integral equations to be solved. The relevant Green's functions [38, 96] for Mode I and Mode II are given by $k_1(x, \xi)$ and $k_2(x, \xi)$ respectively:

$$\begin{aligned} k_1(x, \xi) = & \frac{A + B}{2(x - r_a^2/\xi)} + A \frac{\xi^2 - r_a^2}{\xi^3} \frac{r_a^2}{(x - r_a^2/\xi)^2} \left[\left(\frac{\xi}{r_a} \right)^2 - \frac{\xi^2 - r_a^2}{x\xi - r_a^2} \right] - \frac{A + B}{2x} \\ & - Nb_0 \frac{r_a}{x^2} - A \frac{r_a^2}{x^3} - \frac{1}{2\xi} \left(\frac{r_a}{x} \right)^2 \left[A \left(\frac{2\xi^2}{r_a^2} - 1 \right)^2 + B - 2N \right] \\ & + \frac{1}{2r_a} \left\{ \sum_{n=1}^{\infty} b_n(n+2) \left(\frac{x}{r_a} \right)^n - \sum_{n=1}^{\infty} [b_n(n-1) - b_{-n}] \left(\frac{x}{r_a} \right)^{n-2} \right\} \\ & + \frac{1}{r_a} \left\{ \sum_{n=1}^{\infty} Ab_{-n}(n-2) \left(\frac{r_a}{x} \right)^n - \sum_{n=1}^{\infty} [Bb_n + A(n+1)b_{-n}] \left(\frac{r_a}{x} \right)^{n+2} \right\}, \quad (5.10) \\ k_2(x, \xi) = & \frac{A + B}{2(x - r_a^2/\xi)} + A \frac{r_a^2}{(x - r_a^2/\xi)^2} \left[\frac{x - \xi}{\xi^2} \frac{\xi^2 - r_a^2}{x\xi - r_a^2} \right] - \frac{A + B}{2x} - \frac{B - A}{2\xi} \left(\frac{r_a}{x} \right)^2 \\ & + A \frac{r_a^2}{x^3} - \frac{1}{2r_a} \left\{ \sum_{n=1}^{\infty} nb'_n \left(\frac{x}{r_a} \right)^n - \sum_{n=1}^{\infty} [(n-1)b'_n - b'_{-n}] \left(\frac{x}{r_a} \right)^{n-2} \right\} \\ & + \frac{1}{2r_a} \left\{ \sum_{n=1}^{\infty} Anb'_{-n} \left(\frac{r_a}{x} \right)^{n+2} - \sum_{n=1}^{\infty} [Ab'_n(n+1) - Bb'_n] \left(\frac{r_a}{x} \right)^{n+2} \right\}, \end{aligned}$$

where $A = \frac{\mu_2 - \mu_3}{\mu_2 + \mu_3 \kappa_2}$, $B = \frac{\kappa_3 \mu_2 - \mu_3 \kappa_2}{\mu_2 \kappa_3 + \mu_3}$, $N = \frac{\mu_2(\kappa_3 - 1) - \mu_3(\kappa_2 - 1)}{\mu_2(\kappa_3 - 1) + 2\mu_3}$ while constants a_n , b_n , a'_n and

b'_n can be found [38] in the Appendix equations (A6) and (A7).

5.4.2. Numerical solutions for singular integral equations

The singularities on both crack tips are inverse square root because the whole crack is located in the matrix material. Hence, using the fundamental functions [66], let

$$\begin{aligned} B_y(x) &= w(x)F_y(x), \\ B_x(x) &= w(x)F_x(x), \end{aligned} \quad (5.11)$$

where $F_y(x)$ and $F_x(x)$ are bounded functions in $t_1 \leq x \leq t_2$, with

$$w(x) = (x - t_1)^{-\frac{1}{2}}(t_2 - x)^{-\frac{1}{2}}. \quad (5.12)$$

The interval of the crack integral is shifted from (t_1, t_2) to $(-1, 1)$, by letting

$$\begin{aligned} x &= \frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \\ \xi &= \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2}. \end{aligned} \quad (5.13)$$

Equations (5.8) and (5.9) can then be rewritten in terms of s and t as:

$$\left. \begin{aligned} \frac{1}{\pi} \left[\int_{-1}^1 \frac{B_y(s)}{s-t} ds + \int_{-1}^1 k_{11}(t, s) B_y(s) ds \right] &= \frac{\kappa_2 + 1}{2\mu_2} p_y(x), \\ \int_{-1}^1 B_y(s) ds &= 0, \end{aligned} \right\} \quad (a) \quad (5.14)$$

$$\left. \begin{aligned} \frac{1}{\pi} \left[\int_{-1}^1 \frac{B_x(s)}{s-t} ds + \int_{-1}^1 k_{22}(t, s) B_x(s) ds \right] &= \frac{\kappa_2 + 1}{2\mu_2} p_x(x), \\ \int_{-1}^1 B_x(s) ds &= 0, \end{aligned} \right\} \quad (b)$$

where

$$\begin{aligned} k_{11}(t, s) &= \frac{t_2 - t_1}{2} k_1 \left(\frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2} \right), \\ k_{22}(t, s) &= \frac{t_2 - t_1}{2} k_2 \left(\frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2} \right). \end{aligned} \quad (5.15)$$

Following the numerical method of Erdogan and Gupta [66], the discretised form of equations (5.14) are obtained:

$$\left. \begin{aligned} & \sum_{k=1}^n \frac{1}{n} F_y(s_k) \left[\frac{1}{s_k - u_r} + k_{11}(u_r, s_k) \right] = \frac{\kappa_2 + 1}{2\mu_2} p_y(x_r), \\ & \sum_{k=1}^n \frac{1}{n} F_y(s_k) = 0, \end{aligned} \right\} \quad (a) \\
\left. \begin{aligned} & \sum_{k=1}^n \frac{1}{n} F_x(s_k) \left[\frac{1}{s_k - u_r} + k_{22}(u_r, s_k) \right] = \frac{\kappa_2 + 1}{2\mu_2} p_x(x_r), \\ & \sum_{k=1}^n \frac{1}{n} F_x(s_k) = 0, \end{aligned} \right\} \quad (b)
\tag{5.16}$$

where n is the number of terms used in the numerical procedure, and:

$$\begin{aligned} s_k &= \cos \frac{\pi(2k-1)}{2n}, & k &= 1..n, \\ x_r &= \frac{t_2 - t_1}{2} u_r + \frac{t_2 + t_1}{2}, & u_r &= \cos \frac{\pi r}{n}, \quad r = 1..n-1. \end{aligned}$$

The Mode I (and Mode II) stress intensity factors [10] are given as:

$$\left. \begin{aligned} K_{I\rho}^L &= \lim_{\xi \rightarrow t_1} \frac{-2\mu_2\sqrt{2\pi}}{\kappa_2 + 1} \sqrt{\xi - t_1} B_y(x) = \frac{-2\mu_2\sqrt{\pi(t_2 - t_1)/2}}{\kappa_2 + 1} F_y(-1) \\ &= -\sigma_A \sqrt{\pi(t_2 - t_1)/2} F_{0y}(-1), \\ K_{I\rho}^R &= \lim_{\xi \rightarrow t_2} \frac{2\mu_2\sqrt{2\pi}}{\kappa_2 + 1} \sqrt{t_2 - \xi} B_y(x) = \frac{2\mu_2\sqrt{\pi(t_2 - t_1)/2}}{\kappa_2 + 1} F_y(1) \\ &= \sigma_A \sqrt{\pi(t_2 - t_1)/2} F_{0y}(1), \end{aligned} \right\} (a) \\
\left. \begin{aligned} K_{II\rho}^L &= \lim_{\xi \rightarrow t_1} \frac{-2\mu_2\sqrt{2\pi}}{\kappa_2 + 1} \sqrt{\xi - t_1} B_x(x) = \frac{-2\mu_2\sqrt{\pi(t_2 - t_1)/2}}{\kappa_2 + 1} F_x(-1) \\ &= -\tau_A \sqrt{\pi(t_2 - t_1)/2} F_{0x}(-1), \\ K_{II\rho}^R &= \lim_{\xi \rightarrow t_2} \frac{2\mu_2\sqrt{2\pi}}{\kappa_2 + 1} \sqrt{t_2 - \xi} B_x(x) = \frac{2\mu_2\sqrt{\pi(t_2 - t_1)/2}}{\kappa_2 + 1} F_x(1) \\ &= \tau_A \sqrt{\pi(t_2 - t_1)/2} F_{0x}(1), \end{aligned} \right\} (b)
\tag{5.17}$$

where $F_{0y}(s)$ and $F_{0x}(s)$ are solved from equations (5.16) by letting $F_y(s) = \frac{\sigma_A(\kappa_2+1)}{2\mu_2} F_{0y}(s)$

for tensile loading, $F_x(s) = \frac{\tau_A(\kappa_2+1)}{2\mu_2} F_{0x}(s)$ for shear. Equation (5.4) is used to relate the

applied load to the yield strength.

5.5. Solving for K

Figure 5.4 The superposition process to transform loading to crack surface for tensile loads

Figure 5.5 The superposition process to transform loading to crack surface for shear

Now we try to solve the sub-problem of Figure 5.1c (or Figure 5.2c for shear), repeated in Figure 5.4a (Figure 5.5a). This problem is similar to Figure 5.1a (Figure 5.2a) but with no plastic zone, which leads to the linear elastic problem solved by Luo and Chen [52], with the left crack tip at $t_1 = c_1 - \rho_L$ and the right at $t_2 = c_2 + \rho_R$. Boundary conditions (5.1) and (5.2) still apply, while (5.3) is replaced by:

$$\sigma_{xy} = 0, \quad \sigma_{yy} = 0, \quad t_1 \leq x \leq t_2. \quad (5.18)$$

To solve this problem, we follow the distributed dislocation technique outlined in Sections 5.4.1 and 5.4.2 for Figure 5.4a (Figure 5.5a for shear), with the corrective tractions replaced by $\sigma_{yy}(x, 0) = -p_y(x)$ for tensions ($\sigma_{xy}(x, 0) = -p_x(x)$ for pure shear), where

$p_y(x)$ (and $p_x(x)$) represent the stresses at the crack surface transformed from the far-field load. As a result, we substitute $D_y(x)$ by $p_y(x)$ ($D_x(x)$ by $p_x(x)$) in Sections 5.4.1 and 5.4.2.

To transform the far-field loading onto the crack surfaces, the problem in Figure 5.4a is decomposed into two sub-problems given in Figures 5.4b and 5.4c (Figure 5.5a decomposed into Figures 5.5b and 5.5c under shear). Figure 5.4b (Figure 5.5b) shows the circular inclusion under far-field loading with no crack present. The problem in Figure 5.4c (Figure 5.5c) shows the inclusion with the near-by crack, but with different loading conditions: normal stress $p_y(x)$ (or shear stress $p_x(x)$) is applied on the top and bottom surfaces of the crack with no far-field loading. By superposition, the sum of the stress fields of Figures 5.4b and 5.4c give the stress field in Figure 5.4a (Figures 5.5b and 5.5c sums up to give Figure 5.5a).

The stress fields for the different loading conditions of uniform tension, pure shear and uniaxial tension are obtained from Chen [11] (or Luo [96], with some correction). For biaxial tension, the stresses are given by:

$$\begin{aligned} p_y(x) &= \frac{\sigma_A}{K_{23}} \left[(\lambda_2 + \mu_2) a_2 + \mu_2 \left(\frac{r_b}{x} \right)^2 a_3 \right], & (a) \\ p_x(x) &= 0, & (b) \end{aligned} \quad (5.19)$$

where λ and μ are Lamé's parameters; and a_2 and a_3 are material related constants solved from equation (A8) in the Appendix.

In uniaxial tension, the tensile stress along the crack surface is:

$$\begin{aligned} p_y(x) &= \frac{\sigma_A}{2K_{23}} \left[(\lambda_2 + \mu_2) a_2 + \mu_2 \left(\frac{r_b}{x} \right)^2 a_3 \right] & (a) \\ &\quad - \frac{\sigma_A \mu_2}{2\mu_{23}} \left[-b_3 + 12 \left(\frac{x}{r_b} \right)^2 b_4 + \left(\frac{r_b}{x} \right)^4 b_5 \right], & (b) \\ p_x(x) &= 0, \end{aligned} \quad (5.20)$$

where a_2 , a_3 , b_3 , b_4 and b_5 are obtained from Appendix equations (A8) and (A9).

In pure shear, the shear traction along the crack surface is:

$$\begin{aligned}
p_y(x) &= 0, & (a) \\
p_x(x) &= \frac{\mu_2 \tau_A}{\mu_{23}} \left[b_3 - 6 \left(\frac{x}{r_b} \right)^2 b_4 + 3 \left(\frac{r_b}{x} \right)^4 b_5 + 2b_6 \left(\frac{r_b}{x} \right)^2 \right], & (b)
\end{aligned} \tag{5.21}$$

where b_3 , b_4 , b_5 and b_6 are from equation (A9).

Hence K_I^L and K_{II}^R (K_{II}^L and K_{II}^R for shear) can be obtained, by replacing $K_{I\rho}^L$ and $K_{I\rho}^R$ ($K_{II\rho}^L$ and $K_{II\rho}^R$) in equation (5.17) with K_I^L and K_I^R (K_{II}^L and K_{II}^R), respectively.

5.6. Solving for PZS and CTOD

By assuming initial values for the plastic zone sizes ρ_L and ρ_R , followed by an iteration procedure which satisfies the equations of (5.5), ρ_L and ρ_R are determined.

The iteration consists of, for Mode I, determining whether $|K_I^L|$ or $|K_{I\rho}^L|$ is larger on the left, and $|K_I^R|$ or $|K_{I\rho}^R|$ is on the right. ρ_L is increased if $|K_I^L|$ is larger (decreased if $|K_{I\rho}^L|$ is larger) and ρ_R is increased if $|K_I^R|$ is larger (decreased if $|K_{I\rho}^R|$ is larger), and the stress intensity factors are re-evaluated. The procedure is repeated until the condition is sufficiently satisfied. A similar procedure applies for Mode II. The detailed procedure is elaborated in Appendix B1.

The PZS is an important measure of the extent of small-scale yielding present in the composite material, and useful to calculate the CTOD. To find the CTOD, we make use of dislocation densities in equation (2.31). The opening displacement for the right and left tips (using the Dugdale model definition in Figure 2.11) can thus be given by:

$$\begin{aligned}
\delta_L = D(c_1) &= - \int_{c_2 + \rho_R}^{c_1} B_y(\xi) d\xi, \\
\delta_R = D(c_2) &= \int_{c_2}^{c_1 - \rho_L} B_y(\xi) d\xi.
\end{aligned} \tag{5.22}$$

To solve for the CTODs, we first normalise equations (5.22) using (5.12) and (5.13):

$$\begin{aligned}\delta_L &= -\frac{t_2 - t_1}{2} \int_{-1}^s B_y(s) ds, \\ \delta_R &= \frac{t_2 - t_1}{2} \int_s^1 B_y(s) ds.\end{aligned}\tag{5.23}$$

Then applying the trapezoidal rule of numerical integration gives:

$$\begin{aligned}\delta_L &= -\frac{t_2 - t_1}{2} \left\{ \frac{s_{L\rho} - s_L}{2} [B_y(s_{L\rho}) - B_y(s_L)] \right. \\ &\quad \left. + \sum_{k=L}^n \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] + \frac{s_n - (-1)}{2} [B_y(s_n) + 0] \right\}, \\ \delta_R &= \frac{t_2 - t_1}{2} \left\{ \frac{1 - s_1}{2} [0 + B_y(s_1)] + \sum_{k=1}^{R-1} \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] \right. \\ &\quad \left. + \frac{s_R - s_{R\rho}}{2} [B_y(s_R) + B_y(s_{R\rho})] \right\},\end{aligned}\tag{5.24}$$

where $B_y(s_k) = F_y(s_k)(1 - s_k^2)^{-1/2}$ from equation (5.11), $s_k = \cos \frac{\pi(2k-1)}{2n}$ from equation (5.16) while the terms s_L , $s_{L\rho}$, s_R and $s_{R\rho}$ (which denote locations near the open end of the plastic zones) are, using equation (5.13):

$$\begin{aligned}s_{L\rho} &= \frac{c_1 - (t_2 + t_1)/2}{(t_2 - t_1)/2}, \quad s_L = s_{\text{ceil}(L\rho)}, \\ s_{R\rho} &= \frac{c_2 - (t_2 + t_1)/2}{(t_2 - t_1)/2}, \quad s_R = s_{\text{floor}(R\rho)},\end{aligned}$$

Both $B_y(s_{L\rho})$ and $B_y(s_{R\rho})$ are evaluated with interpolation technique:

$$\begin{aligned}B_y(s_{L\rho}) &= \frac{s_{L\rho} - s_L}{s_{\text{floor}(L\rho)} - s_L} [B_y(s_{\text{floor}(L\rho)}) - B_y(s_L)] + B_y(s_L), \\ B_y(s_{R\rho}) &= \frac{s_{R\rho} - s_{\text{ceil}(R\rho)}}{s_R - s_{\text{ceil}(R\rho)}} [B_y(s_R) - B_y(s_{\text{ceil}(R\rho)})] + B_y(s_{\text{ceil}(R\rho)}).\end{aligned}$$

Similarly, to obtain the x-component CTOD, replace B_y by B_x and F_y by F_x from equations (5.22) onwards.

5.7. Numerical Examples and Discussion

In the three-phase composite model, the main parameter to take note of is the concentration of the reinforcement phase (volume fraction of the inclusions), because this affects the properties of the equivalent outer phase. Similarly, the shear modulus of the reinforcement phase (reinforcing fibres) also affects the properties of the outer phase. Other factors are the Poisson's ratio, load-to-yield-stress ratio and the crack-inclusion distance. Since it has been found that the Poisson's ratio shows little effect in the 2-phase problem [87], the current work will focus on studying the influence of the volume fraction and shear modulus of the reinforcing phase. $n = 200$ terms are used, while the Laurent series is truncated at $n_L = 10$ and the Poisson's ratio is taken as $\nu_1 = \nu_2 = \nu_3 = 1/3$.

5.7.1. Influence of fibre shear modulus and volume fraction

To study the effect of fibre concentration (volume fraction) $c = (r_a/r_b)^2$ and fibre shear modulus μ_1 on the normalised CTOD and PZS we set $\mu_3/\mu_2 = 5,15$ and 23 , respectively. The crack-inclusion distance is set at $d/a = 4/3$, with the inclusion size at $r_a/a = 20/3$. The load-to-yield-stress-ratio $T = S = 0.1$ is used.

Since the results of the normalised PZS are similar to that of the normalised CTOD, only that of the results for the corresponding CTOD are shown, with K_I shown for comparison. Figures 5.6 and 5.8 depict the normalised CTOD for uniform tension for the left and right crack tips (as normalised δ_y^L and δ_y^R respectively), while Figures 5.7 (left) and 5.9 (right) show the normalised K_I (as normalised K_I^L and K_I^R , respectively). The results of normalised CTOD for pure shear are shown in Figures 5.10 (left as normalised δ_y^L) and 5.12 (right as normalised δ_y^R), with the normalised K_{II} shown in Figures 5.11 and 5.13 (as normalised K_{II}^L and K_{II}^R , respectively). Meanwhile results for uniaxial tension are given in Figures 5.14 (left) and 5.16 (right), with the normalised K_I displayed in Figures 5.15 and

5.17. The effect of a porous medium ($\mu_3/\mu_2 = 0$) under the loading conditions of uniform tension, pure shear and uniaxial tension are shown in Tables 5.1 and 5.2.

Figure 5.6 Normalised δ_y^L against c under uniform tension

Figure 5.7 Normalised K_I^L against c under uniform tension

Figure 5.8 Normalised δ_y^R against c under uniform tension

Figure 5.9 Normalised K_I^R against c under uniform tension

Figure 5.10 Normalised δ_x^L against c under pure shear

Figure 5.11 Normalised K_H^L against c under pure shear

Figure 5.12 Normalised δ_x^R against c under pure shear

Figure 5.13 Normalised K_H^R against c under pure shear

Figure 5.14 Normalised δ_y^L against c under uniaxial tension

Figure 5.15 Normalised K_I^L against c under uniaxial tension

Figure 5.16 Normalised δ_y^R against c under uniaxial tension

Figure 5.17 Normalised K_I^R against c under uniaxial tension

The results indicate that a higher concentration of fibres decreases the normalised CTOD when the fibre is “harder” than the matrix. With higher fibre volume fraction, the outer phase (the equivalent composite) is able to shield the crack tips as well. And at the extreme case of a very dilute composite, the results are reduced to those of the two-phase model. Hence the study in this chapter clearly shows that the shielding effect comes from both the near-by single fibre (phase 3), and the remote fibres represented by the equivalent media (phase 1).

Similarly, a higher fibre shear modulus of the fibres also results in decreased normalised CTOD due to better shielding. This is shown by the results that the CTODs for the case $\mu_3/\mu_2 = 5$ are higher than those for the case $\mu_3/\mu_2 = 15$ (and the CTOD results of $\mu_3/\mu_2 = 15$ are higher than those for $\mu_3/\mu_2 = 23$). Similar results are found for the plastic zone size PZS as well, for all the loading conditions.

Comparing the results for the normalised CTOD and K_I , the CTOD values predict more shielding compared to the stress intensity factors in all the figures. This difference occurs due to the consideration of small-scale yielding in materials, and shows how much more accurate the current analysis with plastic zone correction becomes in composite materials.

Table 5.1 Normalised δ_y^L for porous materials at different concentrations

r_b/r_a	10^{29}	4	3	2.2	1.7
$\begin{array}{c} c \\ \text{Load} \end{array}$	10^{-58}	0.0625	0.1111	0.2066	0.3460
Uniform tension	3.2088	3.6950	4.1740	5.4026	8.5623
Pure shear	1.3324	1.5733	1.8674	2.7078	5.1852
Uniaxial tension	2.7166	3.1676	3.6043	4.6661	5.9855

Table 5.2 Normalised δ_y^R at right tip for porous materials at different concentrations

r_b/r_a	10^{29}	4	3	2.2	1.7
$\begin{array}{c} c \\ \text{Load} \end{array}$	10^{-58}	0.0625	0.1111	0.2066	0.3460
Uniform tension	2.6312	3.0314	3.4306	4.4458	7.2354
Pure shear	1.7633	2.0732	2.4361	3.4449	5.8224
Uniaxial tension	2.1620	2.5367	2.8606	3.5894	3.7828

As a “by-product” of the current research, we can also study the case when the filler (reinforcing phase of the composite) is weaker than the matrix ($\mu_3 < \mu_2$). From Tables 5.1 and 5.2, we can see that an increased concentration of porosities ($\mu_3 = 0$) causes an increase

in CTOD (with similar increase in PZS, not shown) to the crack present in the material, as the porosities decrease the strength of the equivalent composite phase.

5.8. Chapter Summary

In this chapter, crack-inclusion interaction with plastic zone correction is investigated in fibre-reinforced composite material. The influence of other inclusions on crack in the composite is taken into consideration with the three-phase composite cylinder model. The effects of fibre concentration (volume fraction) and shear modulus ratio on the CTOD and stress intensity factor are studied and discussed in detail.

In general, a higher fibre concentration increases the influence of the inclusions on the crack behaviour. When the fibre is “harder” than the matrix, the fracture toughness of the composite increases, and the crack is shielded by the inclusions (it is more difficult for the crack to propagate). On the other hand, when the fibre is “softer” than the matrix (a weak filler phase in composites), the fracture toughness of the composite decreases, and the crack is anti-shielded by the inclusions (it is easier for the crack to propagate). The left tip, which is closer to the inclusion, is more affected than the right tip, which is influenced by only surrounding inclusions of the equivalent outer phase. Thus in a composite material, the tip closer to a reinforcing inclusion will be more shielded, while that closer to a void is more likely to propagate. The results show that CTOD is more affected by shielding and anti-shielding than K_I , as some plasticity consideration allows for more accurate prediction than pure linear elastic methods.

A Zener-Stroh Crack Interacting with a Circular Inclusion

6.1. Introduction

Zener [13] proposed that a pile-up of edge dislocations stopped at an obstacle can coalesce into a crack nucleus. Later work by Stroh [15, 16] led to the crack also being known as the Zener-Stroh crack. A Zener-Stroh crack differs from the conventional Griffith crack in that the sum of the Burgers vectors of the climb dislocations b_y^T , or b_x^T for the glide dislocations, is not zero along the crack line. This leads to a displacement loaded crack, even without applied external loading. The crack tip where the dislocation enters the crack is called the blunt tip, while the other is called the sharp tip, and this sharp tip is where crack propagation is always initiated. The homogeneous Zener-Stroh crack has also been studied in anisotropic materials [97].

Work on the Zener-Stroh crack in composite materials began with the interfacial Zener-Stroh crack [60]. Later studies that built on this include: use of the contact zone [20] for an interfacial Zener-Stroh crack, and [54] for a crack near a coated circular inclusion. However, the works above concentrate on the stress intensity factor K and are more appropriate for fracture behaviour analysis of brittle materials. While there are studies on small-scale yielding for a Zener-Stroh crack in homogeneous material [10, 98], there is a lack of similar literature for composite materials.

This chapter investigates the fracture behaviour of a Zener-Stroh crack in a composite material of dilute inclusions with plastic zone correction. This acts a study of a “unit cell” before looking at the three-phase composite model problem in the next chapter where the effects of the surrounding inclusions are no longer negligible.

6.2. Problem Formulation

The current physical problem of a Zener-Stroh crack interacting with a circular inclusion is depicted in Figure 6.1a. The crack is located in the matrix orientated along the radial direction of the circular inclusion, with plastic zone sizes (PZS) of ρ_L and ρ_R on the left and right, respectively. The left tip of the crack is at $t_1 = c_1 - \rho_L$ while the right tip is at $t_2 = c_2 + \rho_R$, for an effective Dugdale model crack length of $\rho_L + 2a + \rho_R$, where $a = (c_2 - c_1)/2$ is the half Zener-Stroh crack without plastic zone (when $\rho_L = \rho_R = 0$). For the current Zener-Stroh crack, the left crack tip is assumed to be the blunt tip while the right tip is the sharp tip.

Figure 6.1 A Zener-Stroh crack interacting with an inclusion with Dugdale plastic zone correction

The matrix (phase 1) has elastic properties κ_1 and μ_1 , occupying the region of $r_0 > 1$. The inclusion (phase 2), with elastic properties κ_2 and μ_2 , occupies the region of $r_0 < 1$.

To concentrate on studying the effect of the net dislocations inside the Zener-Stroh crack, the problem is studied here without external applied loading. Hence in our problem the external loads are:

$$\sigma_{yy}^{\infty} = 0, \quad \sigma_{xy}^{\infty} = 0. \quad (6.1)$$

At the interface of phase 1 and 2:

$$[\sigma_{rr}] = 0, \quad [\sigma_{r\theta}] = 0, \quad [u_r] = 0, \quad [u_{\theta}] = 0, \quad \text{at } r = r_0, \quad (6.2)$$

where $[f]$ is the jump of function f , while r and θ refer to polar coordinates with origin 0. At the crack surface:

$$\begin{aligned}
\sigma_{xy} &= 0, \sigma_{yy} = 0, & c_1 \leq x \leq c_2, \\
\sigma_{yy} &= -\sigma_Y, & c_1 - \rho_L \leq x \leq c_1, \\
\sigma_{yy} &= \sigma_Y, & c_2 \leq c_2 + \rho_R,
\end{aligned} \tag{6.3}$$

for compressive yield on the left tip and tensile yield at the right tip. The net (climb) dislocation load b_y^T is related to the yield strength by:

$$\frac{2\mu_1}{\kappa_1 + 1} \frac{b_y^T}{\sigma_Y \pi a} = T, \tag{6.4}$$

where T is a known parameter relating the net dislocation load with yield strength.

6.3. Solution Procedure

To solve for the PZS and crack tip opening displacement (CTOD) with Dugdale plastic zone correction, the condition of vanishing stress intensity factors applies at both tips due to the asymmetry of the problem:

$$\left. \begin{aligned} K_I^L + K_{I\rho}^L &= 0 \\ K_I^R + K_{I\rho}^R &= 0 \end{aligned} \right\} \tag{6.5}$$

Visually, equation (6.5) can be represented by Figure 6.1, where the Dugdale model problem (Figure 6.1a) is decomposed into Figures 6.1b (with closure stresses, but no net dislocation load) and 6.1c (with a net dislocation load). Solving Figure 6.1b gives $K_{I\rho}^L$ and $K_{I\rho}^R$ values at the crack tips, while solving Figure 6.1c gives the solutions of K_I^L and K_I^R .

The distributed dislocation technique [12] is used to obtain the stress intensity factor K in equations (6.5) at the crack tips, for both the net dislocation load and closure stress. The solution for the $K_{I\rho}$ is outlined in Section 6.4 while the solution for the stress intensity factor due to net dislocation load K_I is shown in Section 6.5.

6.4. Solving for K_ρ

The sub-problem to find K due to closure stresses is shown in Figure 6.1b. The stress at the tip of the effective crack should remain bounded and equal to the yield stress σ_Y

(whether compressive or tensile) of the matrix material. In order to “close” the crack, closure stresses equal to σ_Y are applied to the left (blunt) tip for compressive yield, and to the right (sharp) tip for tensile yield [10] along the plastic zones of the crack, as shown in Figure 6.1b.

Hence, the closure stresses are:

$$D_y(x) = \begin{cases} \sigma_Y & c_1 - \rho_L < x < c_1 \\ -\sigma_Y & c_2 < x < c_2 + \rho_R \\ 0 & \text{else} \end{cases} \quad (6.6)$$

It is assumed that the compressive yielding strength is equal in magnitude to the tensile strength, though different values could be easily used to reflect actual materials.

6.4.1. Formulation of singular integral equations

Figure 6.2 An edge dislocation interacting with a circular inclusion

For Mode I loading, only the climb dislocation $B_y(x)$ is considered. Setting $B_y(x)$ as the climb dislocation density in the crack zone, while making use of the single dislocation solution [12] and Figure 6.2, the traction due to these dislocations is:

$$\sigma_{yy}(x, 0) = -\frac{2\mu_1}{\pi(\kappa_1 + 1)} \left[\int_{t_1}^{t_2} \frac{B_y(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_1(x, \xi) B_y(\xi) d\xi \right]. \quad (6.7)$$

For the resultant stress at the crack surfaces in Figure 6.1a to meet the conditions of equation (6.3), we use $\sigma_{yy}(x, 0) = -D_y(x)$ as the corrective traction:

$$\frac{1}{\pi} \left[\int_{t_1}^{t_2} \frac{B_y(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_1(x, \xi) B_y(\xi) d\xi \right] = \frac{\kappa_1 + 1}{2\mu_1} D_y(x). \quad (6.8)$$

Knowing that the Zener-Stroh crack has a net dislocation load, the dislocation densities for Mode I must satisfy:

$$\int_{t_1}^{t_2} B_y(\xi) d\xi = b_y^T, \quad (6.9)$$

where b_y^T is the total Burgers vector of the net climb dislocations inside the Zener-Stroh crack. However, since we are now solving for only the K_ρ closure stresses (the net dislocation load $b_y^T \neq 0$ is considered when solving K_I in Section 6.5), we consider $b_y^T = 0$. The relevant influence function [12] is:

$$\begin{aligned} k_1 = & \frac{x_1}{r_1^4} [x_1^2 + 3y^2] + \frac{x_2}{r_2^2} \left[\frac{2Ax_2^2}{r_2^2} - \frac{B + 5A}{2} \right] - \frac{x}{r^2} \left[\frac{2Ax^2}{r^2} - \frac{B + 5A}{2} \right] \\ & + \frac{2Ax_2L}{r_2^4 d^2} \left[\frac{4x_2^2}{r_2^2} \left(\frac{L}{2} - x_2 \right) + \left((5 - d^2)x_2 - \frac{3L}{2} \right) \right] \\ & - \frac{r_0 x}{r^4} \left[r_0 A \left(\frac{4x^2}{r^2} - 3 \right) + \frac{x}{d} (A(2d^2 - 1) + Q - 1) \right] - \frac{AL}{r_2^2} \left[\frac{2}{d^2} - 1 \right] \\ & + \frac{r_0 [A(2d^2 - 1) + Q - 1]}{2r^2 d}, \end{aligned} \quad (6.10)$$

where $r = \sqrt{x^2 + y^2}$, $r_1 = \sqrt{x_1^2 + y^2}$ and $r_2 = \sqrt{x_2^2 + y^2}$; $x_1 = x - \xi$ and $x_2 = x_1 + L$;

$L = r_0(d^2 - 1)/d$ and $d = \xi/r_0$. The elastic constants are $A = \frac{\beta - \alpha}{1 + \beta}$, $B = -\frac{\alpha + \beta}{1 - \beta}$ and

$Q = \frac{(1 + \alpha)(1 - \alpha)}{(1 - \beta)(1 + \alpha - 2\beta)}$ while Dundurs' parameters are $\alpha = \frac{\mu_2(\kappa_1 + 1) - \mu_1(\kappa_2 + 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)}$ and $\beta =$

$$\frac{\mu_2(\kappa_1 - 1) - \mu_1(\kappa_2 - 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)}.$$

6.4.2. Numerical procedure

The singularities on both crack tips are inverse square root because the whole crack is located in the matrix material. Using the fundamental functions [63], let:

$$B_y(x) = w(x)F_y(x), \quad (6.11)$$

where $F_y(x)$ is bounded functions in $t_1 \leq x \leq t_2$, with

$$w(x) = (x - t_1)^{-\frac{1}{2}}(t_2 - x)^{-\frac{1}{2}}. \quad (6.12)$$

To normalise the interval of the crack integral from (t_1, t_2) to $(-1, 1)$, let

$$\begin{aligned} x &= \frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \\ \xi &= \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2}. \end{aligned} \quad (6.13)$$

Equations (6.8) and (6.9) can then be rewritten in terms of s and t as:

$$\begin{aligned} \frac{1}{\pi} \left[\int_{-1}^1 \frac{B_y(s)}{s - t} ds + \int_{-1}^1 k_{11}(t, s) B_y(s) ds \right] &= \frac{\kappa_1 + 1}{2\mu_1} D_y(x), \\ \int_{-1}^1 B_y(s) ds &= \frac{2b_y^T}{(t_2 - t_1)}, \end{aligned} \quad (6.14)$$

in which:

$$k_{11}(t, s) = \frac{t_2 - t_1}{2} k_1 \left(\frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2} \right). \quad (6.15)$$

Following the numerical method of Erdogan [63], the discretised forms of the equations (6.14) can then be written as:

$$\begin{aligned} \sum_{k=1}^n \frac{1}{n} F_y(s_k) \left[\frac{1}{s_k - u_r} + k_{11}(u_r, s_k) \right] &= \frac{\kappa_1 + 1}{2\mu_1} D_y(x_r), \\ \sum_{k=1}^n \frac{1}{n} F_y(s_k) &= \frac{2b_y^T}{\pi(t_2 - t_1)}, \end{aligned} \quad (6.16)$$

where n is the number of terms used in the numerical procedure, and:

$$s_k = \cos \frac{\pi(2k-1)}{2n} \text{ where } k = 1..n.$$

$$u_r = \cos \frac{\pi r}{n} \text{ where } r = 1..n-1, \text{ and } x_r = \frac{t_2 - t_1}{2} u_r + \frac{t_2 + t_1}{2}.$$

The Mode I stress intensity factors [10] are:

$$\begin{aligned}
K_{I\rho}^L &= \lim_{\xi \rightarrow t_1} \frac{-2\mu_1\sqrt{2\pi}}{\kappa_1 + 1} \sqrt{\xi - t_1} B_y(x) = \frac{-2\mu_1\sqrt{\pi(t_2 - t_1)/2}}{\kappa_1 + 1} F_y(-1) \\
&= \frac{-2\mu_1}{\kappa_1 + 1} \frac{b_y^T}{\sqrt{\pi(t_2 - t_1)/2}} F_{0y}(-1), \\
K_{I\rho}^R &= \lim_{\xi \rightarrow t_2} \frac{2\mu_1\sqrt{2\pi}}{\kappa_1 + 1} \sqrt{t_2 - \xi} B_y(x) = \frac{2\mu_1\sqrt{\pi(t_2 - t_1)/2}}{\kappa_1 + 1} F_y(1) \\
&= \frac{2\mu_1}{\kappa_1 + 1} \frac{b_y^T}{\sqrt{\pi(t_2 - t_1)/2}} F_{0y}(1),
\end{aligned} \tag{6.17}$$

where $F_{0y}(s)$ is solved from equations (6.16) by letting $F_y(s) = \frac{2b_y^T}{\pi(t_2 - t_1)} F_{0y}(s)$ for tensile loading, using equation (6.4) to relate applied load to yield strength.

6.5. Solving for K

Figure 6.3 The crack interacting with a circular inclusion

To solve the sub-problem of Figure 6.1c (repeated in Figure 6.3), we note its similarity to Figure 6.1a, but without the plastic zones. It is a Zener-Stroh crack with its left tip at $t_1 = c_1 - \rho_L$ and right at $t_2 = c_2 + \rho_R$. The boundary conditions (6.1) and (6.2) are still relevant, while (6.3) is replaced by:

$$\sigma_{xy} = 0, \quad \sigma_{yy} = 0, \quad t_1 \leq x \leq t_2. \tag{6.18}$$

To solve this sub-problem, we follow the distributed dislocation technique outlined in Sections 6.4.1 and 6.4.2, but replacing $D_y(x) = 0$ since there is no external load. Also, the net dislocation load $b_y^T \neq 0$, indicating that it is a net dislocation loaded Zener-Stroh crack.

6.6. Solving for PZS and CTOD

The solution of the PZS has to be obtained first, which can then be used to calculate the CTOD. To solve for the plastic zone, we assume initial values for plastic zone sizes ρ_L and ρ_R , then follow an iteration procedure to ensure that equations (6.5) are satisfied.

The iteration consists of determining whether $|K_I^L|$ or $|K_{I\rho}^L|$ is larger on the left, and whether $|K_I^R|$ or $|K_{I\rho}^R|$ is larger on the right. ρ_L is increased if $|K_I^L|$ is larger (decreased if $|K_{I\rho}^L|$ is larger) and ρ_R is increased if $|K_I^R|$ is larger (decreased if $|K_{I\rho}^R|$ is larger), and the stress intensity factors are re-evaluated. This procedure is repeated until the condition of vanishing stress intensity factors of (6.5) is sufficiently satisfied, giving values of ρ_L and ρ_R . The steps are detailed in Appendix B2.

Next is to solve for the CTOD, making use of dislocation densities in equation (2.31). The opening displacement for the right and left tips (with the Dugdale model definition in Figure 2.11) can thus be given by:

$$\begin{aligned}\delta_L = D(c_1) &= \int_{c_1}^{c_2+\rho_R} B_y(\xi) d\xi \\ \delta_R = D(c_2) &= \int_{c_2}^{c_1+\rho_L} B_y(\xi) d\xi\end{aligned}\tag{6.19}$$

To solve for the CTODs, we normalise equations (6.19) using equation (6.13):

$$\begin{aligned}\delta_L &= \frac{t_2 - t_1}{2} \int_{s_{L\rho}}^1 B_y(s) ds, \\ \delta_R &= \frac{t_2 - t_1}{2} \int_{s_{R\rho}}^1 B_y(s) ds.\end{aligned}\tag{6.20}$$

Then applying the trapezoidal rule of numerical integration gives:

$$\begin{aligned}
\delta_L &= \frac{t_2 - t_1}{2} \left\{ \frac{1 - s_1}{2} [0 + B_y(s_1)] + \sum_{k=1}^{L-1} \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] \right. \\
&\quad \left. + \frac{s_L - s_{L\rho}}{2} [B_y(s_L) + B_y(s_{L\rho})] \right\}, \\
\delta_R &= \frac{t_2 - t_1}{2} \left\{ \frac{1 - s_1}{2} [0 + B_y(s_1)] + \sum_{k=1}^{R-1} \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] \right. \\
&\quad \left. + \frac{s_R - s_{R\rho}}{2} [B_y(s_R) + B_y(s_{R\rho})] \right\},
\end{aligned} \tag{6.21}$$

where $B_y(s_k) = F_y(s_k)(1 - s_k^2)^{-1/2}$ from equation (6.12), $s_k = \cos \frac{\pi(2k-1)}{2n}$ from equation (6.16) while the terms s_L , $s_{L\rho}$, s_R and $s_{R\rho}$ (which denote locations near the open end of the plastic zones) are, using equation (6.13):

$$\begin{aligned}
s_{L\rho} &= \frac{c_1 - (t_2 + t_1)/2}{(t_2 - t_1)/2}, \quad s_L = s_{\text{floor}(L\rho)}, \\
s_{R\rho} &= \frac{c_2 - (t_2 + t_1)/2}{(t_2 - t_1)/2}, \quad s_R = s_{\text{floor}(R\rho)}.
\end{aligned}$$

Both $B_y(s_{L\rho})$ and $B_y(s_{R\rho})$ tend to infinity, but are simplified by interpolation to:

$$\begin{aligned}
B_y(s_{L\rho}) &= \frac{s_{L\rho} - s_L}{s_{\text{ceil}(L\rho)} - s_L} [B_y(s_{\text{ceil}(L\rho)}) - B_y(s_L)] + B_y(s_L) \\
B_y(s_{R\rho}) &= \frac{s_{R\rho} - s_{\text{ceil}(R\rho)}}{s_R - s_{\text{ceil}(R\rho)}} [B_y(s_R) - B_y(s_{\text{ceil}(R\rho)})] + B_y(s_{\text{ceil}(R\rho)})
\end{aligned}$$

6.7. Results and Discussion

There are various factors influencing the fracture behaviour of the crack. The material parameters involved include the inclusion/matrix shear modulus ratios and Poisson's ratios. Physical parameters include relative crack-inclusion size and crack-inclusion distance. Another important parameter is T of equation (6.4), representing how much displacement loading is present relative to the yield strength.

6.7.1. The influence of crack-inclusion distance and shear modulus

Since the crack propagates along the sharp tip [20], which in our problem is the right tip, only the results at the right tip will be given. Also, as results for the normalised CTOD and normalised PZS are similar, only the normalised CTOD is displayed in this report.

To examine the influence of the crack-inclusion distance and shear modulus μ_2/μ_1 , other parameters are set as: $T = 0.10$ and $\nu_1 = \nu_2 = 1/3$. For our calculations from here onwards, the number of terms used to solve the integral equations is $n = 100$.

Figure 6.4 Normalised δ_y^R against c_l/a for $\mu_2/\mu_1 > 1$

Figure 6.5 Normalised K_I^R against c_l/a for $\mu_2/\mu_1 > 1$

Figure 6.6 Normalised δ_y^R against c_1/a for $\mu_2/\mu_1 < 1$

Figure 6.7 Normalised K_I^R against c_1/a for $\mu_2/\mu_1 < 1$

Figures 6.4 and 6.5 show the normalised values of CTOD and K_I as normalised δ_y^R and K_I^R respectively, for the right sharp tip, when $\mu_2/\mu_1 > 1$ (meaning that the inclusion is “harder” than the matrix). Meanwhile Figures 6.6 and 6.7 display the CTOD and K_I respectively when $\mu_2/\mu_1 < 1$, which is for the cases when the inclusion is “softer” than the matrix (at the limiting case, the inclusion becomes a hole or void).

The CTOD results show that as the distance increases, the influence of the inclusion on the crack decreases. The further away the crack is from the inclusion, the results become

more similar to the Zener-Stroh crack in a homogeneous material case described earlier. Hence the normalised results approach a limit of 1 with increasing distance. The normalised K_I exhibits similar behaviour, but at values lower than the corresponding normalised CTOD when $\mu_2/\mu_1 > 1$, and at higher values $\mu_2/\mu_1 < 1$. This again shows how the K_I underestimates the effect of the inclusion on fracture behaviour.

As for the shear modulus influence, when the sharp crack tip is away from the inclusion, a “harder” inclusion causes larger CTODs, and that a “softer” inclusion gives smaller CTODs. This is opposite of that for the Griffith crack, where a “harder” inclusion shields the crack. This is explained by the wedging effect of the inclusion’s compressive deformation of the left tip, increasing CTOD at the right tip.

6.7.2. The influence of Poisson’s ratio

Table 6.1 Normalised CTOD for some values of Poisson’s ratio ν_1 and ν_2

$\nu_1 = \nu_2$	Normalised δ_y^L	Normalised δ_y^R
0.20	1.0043	1.2136
1/3	1.0037	1.1837
0.45	1.0033	1.1599

To study the influence of Poisson’s ratio on the crack, other parameters are set as: $(c_1 - r_0)/a = 2$, $\mu_2/\mu_1 = 10$ and $T = 0.10$. The numerical results are obtained for $\nu_1 = \nu_2 = 0.2, 1/3$ and 0.45 , respectively, given in Table 6.1 for both the left and right tips of the Zener-Stroh crack.

From the results, we see for common values of Poisson’s ratio between 0.2 and 0.45, the CTOD at the right tip decreases slightly with the increasing Poisson’s ratio. However, It is found that overall, the effect of the Poisson’s ratio on CTOD is small for most materials.

6.7.3. The influence of displacement load on crack behaviour

Table 6.2 CTOD for some values of T , in $\sigma_Y \pi a (\kappa + 1) / 2\mu$

T	δ_y^L	δ_y^R
0.10	9.859E-02	2.103E-03
0.20	1.929E-01	1.036E-02
0.25	2.378E-01	1.778E-02

To study the influence of displacement load on the crack, other parameters are set as: $(c_1 - r_0)/a = 2$, $\mu_2/\mu_1 = 10$ and $\nu_1 = \nu_2 = 1/3$. Here we will look at the results (shown in Table 6.2) at both crack tips for $T = 0.10$, 0.20 and 0.25 (where $T = \frac{2\mu_1}{\kappa_1 + 1} \frac{b_y^T}{\sigma_Y \pi a}$ from equation (6.4)), which amounts to displacement loads of 10 to 25% of the yield strength.

As the displacement load increases, the CTOD for both the left and right tips increases as well. This is expected as the higher displacement load causes larger plastic zone sizes, and will thus cause larger crack openings as well.

6.8. Chapter Summary

This chapter introduces a procedure to obtain the PZS and CTOD for a Zener-Stroh crack interacting with a circular inclusion under small-scale yielding, focusing on the sharp tip where the crack will propagate for a Zener-Stroh crack. Various parameters affecting the fracture behaviour of the crack were investigated and discussed, such as the crack-inclusion distance, shear modulus ratio, Poisson's ratio and loading parameter T . Compared to the CTOD, the K_I underestimates the shielding or anti-shielding effects of the inclusion on fracture behaviour. The effect of the inclusion is stronger the closer it is to the crack. The Poisson's ratio had minimal effect for its common values. Increasing the loading parameter naturally results in higher CTOD and plastic zone sizes. Also, for the crack tip away from the inclusion, wedging is observed.

CHAPTER 7

A Zener-Stroh Crack in a Three-Phase Composite

7.1. Introduction

As the two-phase model for a Zener-Stroh crack interacting with a single inclusion given in Chapter 6 is not a realistic representation of an actual material, the problem is extended to the three-phase composite model. This model considers the effects of other inclusions in the problem as well. This is important as the Zener-Stroh crack is often observed in metal matrix composites [99]. Studying the problem with the Dugdale model allows plasticity to be considered.

Hence, this chapter looks at the Zener-Stroh crack in a three-phase composite with plastic zone correction. The Dugdale model of small-scale yielding is applied and the crack tip opening displacement (CTOD) is used as the fracture criterion.

7.2. Problem Formulation

Figure 7.1 The Dugdale model for a blunt-sharp Zener-Stroh crack in a three-phase composite

The physical problems of a Zener-Stroh crack in a three-phase cylinder with Dugdale plastic zone correction is shown in Figures 7.1a and 7.2a. Figure 7.1a has the crack with left

tip blunt and right tip sharp (blunt-sharp), while Figure 7.2a has the two tips vice-versa (sharp-blunt).

In both problems, phase 1 occupying $r > r_b$ is the equivalent homogeneous media (with material properties κ_1 and μ_1). Phase 2 is the pure matrix in $r_a < r < r_b$ (properties κ_2 and μ_2). Phase 3 is the circular inclusion (the reinforcing fiber) of region $r < r_a$ (properties κ_3 and μ_3). The material properties of the equivalent homogeneous phase follow the three-phase cylindrical model [95].

Figure 7.2 The Dugdale model for a sharp-blunt Zener-Stroh crack in a three-phase composite

As shown in the figure, the Zener-Stroh crack is located in the matrix phase along the x-axis, with plastic zones lengths on the left tip, ρ_L and right tip, ρ_R . The left crack tip is at $t_1 = c_1 - \rho_L$ and right at $t_2 = c_2 + \rho_R$, giving an effective Dugdale crack length of $\rho_L + 2a + \rho_R$. Without the added plastic zones, the crack simplifies to a Zener-Stroh crack of length $2a$ from $t_1 = c_1$ to $t_2 = c_2$, where $a = (c_2 - c_1)/2$ is the half crack length.

To better appreciate the effect of the net dislocations of the Zener-Stroh crack, the two problems are studied without external loading. Hence:

$$\sigma_{yy}^{\infty} = 0, \quad \sigma_{xy}^{\infty} = 0. \quad (7.1)$$

At the interface of phase 1 and 2:

$$[\sigma_{rr}] = 0, \quad [\sigma_{r\theta}] = 0, \quad [u_r] = 0, \quad [u_\theta] = 0, \quad \text{at } r = r_a, r_b, \quad (7.2)$$

where $[f]$ is the jump of function f , while r and θ refer to polar coordinates with origin 0. At the blunt-sharp crack surface of Figure 7.1a, with compressive yield on the left tip and tensile yield at the right tip:

$$\begin{aligned} \sigma_{xy} &= 0, \sigma_{yy} = 0, & c_1 \leq x \leq c_2, \\ \sigma_{yy} &= -\sigma_Y, & c_1 - \rho_L \leq x \leq c_1, \\ \sigma_{yy} &= \sigma_Y, & c_2 \leq c_2 + \rho_R. \end{aligned} \quad (7.3)$$

Similarly for the sharp-blunt crack case shown in Figure 7.2a, tensile yield is along the left tip and compressive yield at the right tip:

$$\begin{aligned} \sigma_{xy} &= 0, \sigma_{yy} = 0, & c_1 \leq x \leq c_2, \\ \sigma_{yy} &= \sigma_Y, & c_1 - \rho_L \leq x \leq c_1, \\ \sigma_{yy} &= -\sigma_Y, & c_2 \leq c_2 + \rho_R. \end{aligned} \quad (7.4)$$

The net (climb) dislocation load b_y^T is related to the yield strength by:

$$\frac{2\mu_1}{\kappa_1 + 1} \frac{b_y^T}{\sigma_Y \pi a} = T, \quad (7.5)$$

where T is a known parameter relating the net dislocation load to the yield strength.

7.3. Solution Procedure

The solution lies in fulfilling the condition of vanishing stress intensity factors at both crack tips:

$$\left. \begin{aligned} K_I^L + K_{I\rho}^L &= 0 \\ K_I^R + K_{I\rho}^R &= 0 \end{aligned} \right\} \quad (7.6)$$

For the blunt-sharp Zener-Stroh crack problem, equation (7.6) is represented by Figure 7.1, where the Dugdale model problem Figure 7.1a is decomposed into Figures 7.1b (with closure stresses, but no net dislocation load) and 7.1c (with a net dislocation load). Solving Figure 7.1b gives $K_{I\rho}^L$ and $K_{I\rho}^R$ values at the crack tips, while solving Figure 7.1c

gives K_I^L and K_I^R . The sharp-blunt Zener-Stroh crack is depicted in a similar fashion in Figure 7.2, where Figure 7.2b gives $K_{I\rho}^L$ and $K_{I\rho}^R$ while Figure 7.2c gives K_I^L and K_I^R .

The distributed dislocation technique [12] is used to calculate the stress intensity factors required in both problems, for both net dislocation load and closure stress. The solution for K_ρ is outlined in Section 7.4 while the procedure of solving K_I is shown in Section 7.5.

7.4. Solving for K_ρ

The stress intensity factor K due to closure stresses can be evaluated from solving the sub-problem given in Figure 7.1b. The stress at the effective crack tip remains bounded and equal to the yield stress σ_Y (compressive or tensile) of the matrix. In order to “close” the crack, closure stresses equal to σ_Y are applied to the left (blunt) tip for compressive yield, and to the right (sharp) tip for tensile yield [10] along the plastic zones, as shown in Figure 7.1b. Hence, the closure stresses are:

$$D_y(x) = \begin{cases} \sigma_Y & c_1 - \rho_L < x < c_1 \\ -\sigma_Y & c_2 < x < c_2 + \rho_R \\ 0 & \text{else} \end{cases} \quad (7.7)$$

The compressive yielding strength is assumed equal in magnitude to the tensile yielding strength, although different values could be used just as easily to reflect actual materials. Similarly, for the problem of a left sharp tip and right blunt tip in Figure 7.2b:

$$D_y(x) = \begin{cases} -\sigma_Y & c_1 - \rho_L < x < c_1 \\ \sigma_Y & c_2 < x < c_2 + \rho_R \\ 0 & \text{else} \end{cases} \quad (7.8)$$

7.4.1. Crack simulation with distributed dislocations

The equivalence between a crack and a dislocation pile-up is used in Figures 7.1b and 7.2b. Using dislocation densities $B_y(x)$ (for climb) ($B_x(x)$ for glide) in the crack zone, with the solution of a single dislocation in three-phase composite [38, 96] as the Green’s function

(Figure 7.3a for the blunt-sharp crack, and Figure 7.3b for the sharp-blunt crack), the dislocation densities can be related to crack surface stress by:

$$\sigma_{yy}(x, 0) = -\frac{2\mu_2}{\pi(\kappa_2 + 1)} \left[\int_{t_1}^{t_2} \frac{B_y(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_1(x, \xi) B_y(\xi) d\xi \right]. \quad (7.9)$$

Figure 7.3 An edge dislocation in a three-phase cylindrical composite

For the resultant stress at the crack surfaces in Figures 7.1a and 7.2a to meet the conditions of equation (7.3), $\sigma_{yy}(x, 0) = -D_y(x)$ is used as the corrective traction:

$$\frac{1}{\pi} \left[\int_{t_1}^{t_2} \frac{B_y(\xi)}{\xi - x} d\xi + \int_{t_1}^{t_2} k_1(x, \xi) B_y(\xi) d\xi \right] = \frac{\kappa_2 + 1}{2\mu_2} D_y(x). \quad (7.10)$$

And the dislocation densities must satisfy the Zener-Stroh's net dislocation load:

$$\int_{t_1}^{t_2} B_y(\xi) d\xi = b_y^T, \quad (7.11)$$

where b_y^T is the total Burgers vector of the net climb dislocations inside the crack. The relevant influence function [12] is:

$$\begin{aligned}
k_1(x, \xi) = & \frac{A+B}{2(x-r_a^2/\xi)} + A \frac{\xi^2-r_a^2}{\xi^3} \frac{r_a^2}{(x-r_a^2/\xi)^2} \left[\left(\frac{\xi}{r_a}\right)^2 - \frac{\xi^2-r_a^2}{x\xi-r_a^2} \right] - \frac{A+B}{2x} \\
& - Nb_0 \frac{r_a}{x^2} - A \frac{r_a^2}{x^3} - \frac{1}{2\xi} \left(\frac{r_a}{x}\right)^2 \left[A \left(\frac{2\xi^2}{r_a^2} - 1\right)^2 + B - 2N \right] \\
& + \frac{1}{2r_a} \left\{ \sum_{n=1}^{\infty} b_n(n+2) \left(\frac{x}{r_a}\right)^n - \sum_{n=1}^{\infty} [b_n(n-1) - b_{-n}] \left(\frac{x}{r_a}\right)^{n-2} \right\} \\
& + \frac{1}{r_a} \left\{ \sum_{n=1}^{\infty} Ab_{-n}(n-2) \left(\frac{r_a}{x}\right)^n - \sum_{n=1}^{\infty} [Bb_n + A(n+1)b_{-n}] \left(\frac{r_a}{x}\right)^{n+2} \right\},
\end{aligned} \tag{7.12}$$

where $A = \frac{\mu_2 - \mu_3}{\mu_2 + \mu_3 \kappa_2}$, $B = \frac{\kappa_3 \mu_2 - \mu_3 \kappa_2}{\mu_2 \kappa_3 + \mu_3}$, $N = \frac{\mu_2(\kappa_3 - 1) - \mu_3(\kappa_2 - 1)}{\mu_2(\kappa_3 - 1) + 2\mu_3}$ while constants a_n , b_n , a'_n and

b'_n can be found from [38] in Appendix equations (A6) and (A7).

7.4.2. Numerical solutions for singular integral equations

The singularities on both crack tips are inverse square root because the whole crack is located in the matrix material. Using the fundamental functions [63], let:

$$B_y(x) = w(x)F_y(x), \tag{7.13}$$

where $F_y(x)$ is a bounded function in $t_1 \leq x \leq t_2$, with:

$$w(x) = (x - t_1)^{-\frac{1}{2}}(t_2 - x)^{-\frac{1}{2}}. \tag{7.14}$$

To shift the interval of the crack integral from (t_1, t_2) to $(-1, 1)$, let:

$$\begin{aligned}
x &= \frac{t_2 - t_1}{2}t + \frac{t_2 + t_1}{2}, \\
\xi &= \frac{t_2 - t_1}{2}s + \frac{t_2 + t_1}{2}.
\end{aligned} \tag{7.15}$$

Equations (7.10) and (7.11) can then be rewritten in terms of s and t as:

$$\begin{aligned}
\frac{1}{\pi} \left[\int_{-1}^1 \frac{B_y(s)}{s-t} ds + \int_{-1}^1 k_{11}(t, s) B_y(s) ds \right] &= \frac{\kappa_2 + 1}{2\mu_2} D_y(x), \\
\int_{-1}^1 B_y(s) ds &= \frac{2b_y^T}{(t_2 - t_1)},
\end{aligned} \tag{7.16}$$

where:

$$k_{11}(t, s) = \frac{t_2 - t_1}{2} k_1 \left(\frac{t_2 - t_1}{2} t + \frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2} s + \frac{t_2 + t_1}{2} \right) \quad (7.17)$$

Following the numerical method [63], the discretised forms of the equations (7.16) can be written as:

$$\begin{aligned} \sum_{k=1}^n \frac{1}{n} F_y(s_k) \left[\frac{1}{s_k - u_r} + k_{11}(u_r, s_k) \right] &= \frac{\kappa_2 + 1}{2\mu_2} D_y(x_r), \\ \sum_{k=1}^n \frac{1}{n} F_y(s_k) &= \frac{2b_y^T}{\pi(t_2 - t_1)}. \end{aligned} \quad (7.18)$$

in which n represents the number of terms used in the numerical procedure, and:

$$s_k = \cos \frac{\pi(2k-1)}{2n} \text{ where } k = 1..n.$$

$$u_r = \cos \frac{\pi r}{n} \text{ where } r = 1..n-1, \text{ and } x_r = \frac{t_2 - t_1}{2} u_r + \frac{t_2 + t_1}{2}.$$

The Mode I stress intensity factors [10] for Figure 7.1 are thus given by:

$$\begin{aligned} K_{I\rho}^L &= \lim_{\xi \rightarrow t_1} \frac{-2\mu_2 \sqrt{2\pi}}{\kappa_2 + 1} \sqrt{\xi - t_1} B_y(x) = \frac{-2\mu_2 \sqrt{\pi(t_2 - t_1)/2}}{\kappa_2 + 1} F_y(-1) \\ &= -\frac{2\mu_2}{\kappa_2 + 1} \frac{b_y^T}{\sqrt{\pi(t_2 - t_1)/2}} F_{0y}(-1), \\ K_{I\rho}^R &= \lim_{\xi \rightarrow t_2} \frac{2\mu_2 \sqrt{2\pi}}{\kappa_2 + 1} \sqrt{t_2 - \xi} B_y(x) = \frac{2\mu_2 \sqrt{\pi(t_2 - t_1)/2}}{\kappa_2 + 1} F_y(1) \\ &= \frac{2\mu_2}{\kappa_2 + 1} \frac{b_y^T}{\sqrt{\pi(t_2 - t_1)/2}} F_{0y}(1), \end{aligned} \quad (7.19)$$

in which $F_{0y}(s)$ is solved from equations (7.18) by letting $F_y(s) = \frac{2b_y^T}{\pi(t_2 - t_1)} F_{0y}(s)$, using equation (7.5) to relate the net dislocation load to yield strength.

Similarly, the Mode I stress intensity factors for the problem in Figure 7.2 are:

$$\begin{aligned} K_{I\rho}^L &= \frac{2\mu_2}{\kappa_2 + 1} \frac{b_y^T}{\sqrt{\pi(t_2 - t_1)/2}} F_{0y}(-1), \\ K_{I\rho}^R &= -\frac{2\mu_2}{\kappa_2 + 1} \frac{b_y^T}{\sqrt{\pi(t_2 - t_1)/2}} F_{0y}(1), \end{aligned} \quad (7.20)$$

7.5. Solving for K

Figure 7.4 The crack in a three-phase composite

To solve the blunt-sharp crack sub-problem of Figure 7.1c (repeated in Figure 7.4a), we note its similarity to Figure 7.1a, but without the yielded plastic zones. It is a Zener-Stroh crack with its blunt left tip at $t_1 = c_1 - \rho_L$ and right sharp at $t_2 = c_2 + \rho_R$. The boundary conditions (7.1) and (7.2) are still relevant, while (7.3) is replaced by:

$$\sigma_{xy} = 0, \quad \sigma_{yy} = 0, \quad t_1 \leq x \leq t_2. \quad (7.21)$$

To solve this sub-problem, we follow the distributed dislocation technique outlined in Sections 7.4.1 and 7.4.2, but replacing $D_y(x) = 0$ since there is no external load. Also, the net dislocation load $b_y^T \neq 0$, indicating that it is a net dislocation loaded Zener-Stroh crack. Hence K_I^L , and K_I^R can be obtained, by replacing $K_{I\rho}^L$, and $K_{I\rho}^R$ in equation (7.19) with K_I^L and K_I^R respectively.

Meanwhile, to solve the sharp-blunt crack sub-problem of Figure 7.2c, (or Figure 7.4b), the boundary conditions (7.1) and (7.2) are still used while (7.4) is replaced by (7.21). Also, $D_y(x) = 0$ is used in Sections 7.4.1 and 7.4.2. Hence K_I^L , and K_I^R are obtained, by replacing $K_{I\rho}^L$, and $K_{I\rho}^R$ in equation (7.20) with K_I^L and K_I^R respectively.

7.6. Solution for the PZS and CTOD

For Figures 7.1c and 7.2c, the plastic zone size (PZS) is evaluated first, which is then used to obtain the crack tip opening displacement (CTOD). To solve for the PZS, we assume initial values of ρ_L and ρ_R , then follow an iteration procedure to ensure equations (7.6) are satisfied.

The iteration procedure, which applies for both problems, consists of determining whether $|K_I^L|$ or $|K_{I\rho}^L|$ is larger on the left crack tip, and whether $|K_I^R|$ or $|K_{I\rho}^R|$ is larger on the right crack tip. ρ_L is increased if $|K_I^L|$ is larger than $|K_{I\rho}^L|$ (decreased if $|K_{I\rho}^L|$ is larger) and ρ_R is increased if $|K_I^R|$ is larger than $|K_{I\rho}^R|$ (decreased if $|K_{I\rho}^R|$ is larger), and the stress intensity factors are re-evaluated correspondingly. This procedure is repeated until the condition of vanishing stress intensity factors of (7.6) is sufficiently satisfied. The steps are detailed in Appendix B2.

Next is to solve for the CTOD, making use of dislocation densities in equation (2.31). For the problem in Figure 7.1, with the crack propagating at the right tip, the opening displacement for the right and left tips (using the Dugdale model definition in Figure 2.11) can thus be given by:

$$\begin{aligned}\delta_L = D(c_1) &= \int_{c_1}^{c_2+\rho_R} B_y(\xi) d\xi \\ \delta_R = D(c_2) &= \int_{c_2}^{c_1} B_y(\xi) d\xi\end{aligned}\tag{7.22}$$

When the above equations are normalised using equation (7.15), we have:

$$\begin{aligned}\delta_L &= \frac{t_2 - t_1}{2} \int_{s_{L\rho}}^1 B_y(s) ds, \\ \delta_R &= \frac{t_2 - t_1}{2} \int_{s_{R\rho}}^1 B_y(s) ds.\end{aligned}\tag{7.23}$$

Applying the trapezoidal rule of numerical integration gives:

$$\begin{aligned}\delta_L &= \frac{t_2 - t_1}{2} \left\{ \frac{1 - s_1}{2} [0 + B_y(s_1)] + \sum_{k=1}^{L-1} \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] \right. \\ &\quad \left. + \frac{s_L - s_{L\rho}}{2} [B_y(s_L) + B_y(s_{L\rho})] \right\}, \\ \delta_R &= \frac{t_2 - t_1}{2} \left\{ \frac{1 - s_1}{2} [0 + B_y(s_1)] + \sum_{k=1}^{R-1} \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] \right. \\ &\quad \left. + \frac{s_R - s_{R\rho}}{2} [B_y(s_R) + B_y(s_{R\rho})] \right\},\end{aligned}\tag{7.24}$$

where $B_y(s_k) = F_y(s_k)(1 - s_k^2)^{-1/2}$ from equation (7.14), $s_k = \cos \frac{\pi(2k-1)}{2n}$ from equation (7.18) while the terms s_L , $s_{L\rho}$, s_R and $s_{R\rho}$ (which denote locations near the open end of the plastic zones) are, using equation (7.15):

$$\begin{aligned}s_{L\rho} &= \frac{c_1 - (t_2 + t_1)/2}{(t_2 - t_1)/2}, \quad s_L = s_{\text{floor}(L\rho)}, \\ s_{R\rho} &= \frac{c_2 - (t_2 + t_1)/2}{(t_2 - t_1)/2}, \quad s_R = s_{\text{floor}(R\rho)}.\end{aligned}$$

Both $B_y(s_{L\rho})$ and $B_y(s_{R\rho})$ tend to infinity, but are simplified by interpolation to:

$$\begin{aligned}B_y(s_{L\rho}) &= \frac{s_{L\rho} - s_L}{s_{\text{ceil}(L\rho)} - s_L} [B_y(s_{\text{ceil}(L\rho)}) - B_y(s_L)] + B_y(s_L), \\ B_y(s_{R\rho}) &= \frac{s_{R\rho} - s_{\text{ceil}(R\rho)}}{s_R - s_{\text{ceil}(R\rho)}} [B_y(s_R) - B_y(s_{\text{ceil}(R\rho)})] + B_y(s_{\text{ceil}(R\rho)}).\end{aligned}$$

Similarly, for Figure 7.2 where the crack propagates from the sharp left tip instead of the right blunt tip, the CTOD is given by:

$$\begin{aligned}
\delta_L = D(c_1) &= \int_{c_1 - \rho_L}^{c_1} B_y(\xi) d\xi, \\
\delta_R = D(c_2) &= \int_{c_1 - \rho_L}^{c_2} B_y(\xi) d\xi.
\end{aligned} \tag{7.25}$$

When the above equations are normalised, using the trapezoidal rule again, we have:

$$\begin{aligned}
\delta_L &= \frac{t_2 - t_1}{2} \left\{ \frac{1 - s_N}{2} [0 + B_y(s_N)] + \sum_{k=L+1}^{N-1} \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] \right. \\
&\quad \left. + \frac{s_{L\rho} - s_{ceil(L\rho)}}{2} [B_y(s_{L\rho}) + B_y(s_{ceil(L\rho)})] \right\}, \\
\delta_R &= \frac{t_2 - t_1}{2} \left\{ \frac{1 - s_N}{2} [0 + B_y(s_N)] + \sum_{k=R+1}^{N-1} \frac{s_k - s_{k+1}}{2} [B_y(s_k) - B_y(s_{k+1})] \right. \\
&\quad \left. + \frac{s_{R\rho} - s_{ceil(R\rho)}}{2} [B_y(s_{R\rho}) + B_y(s_{ceil(R\rho)})] \right\}.
\end{aligned} \tag{7.26}$$

7.7. Numerical Results and Discussion

In three-phase composite model, there are various parameters affecting the fracture behaviour of the crack. For example, material properties: the inclusion/matrix shear modulus ratio and Poisson's ratios. Physical parameters affecting the fracture behaviour of the crack are: volume fraction of the reinforcing phase, relative crack-inclusion size and crack-inclusion distance. Another important parameter is T from equation (7.5), representing the displacement load relative to the yield strength. For the numerical procedure, the number of terms in the Chebyshev series is set at $n = 100$ and Laurent series is set at $n_L = 10$.

The main factor to be studied here is the volume fraction of the reinforcing phase, as it is the main difference between the three-phase composite and the two-phase version given in Chapter 6. The other two parameters, crack-inclusion distance and load ratio, are also studied. However, as Poisson's ratio has little effect within the usual material range [87], it is not looked into in this chapter. We set them as $\nu_1 = \nu_2 = 1/3$.

7.7.1. The influence of inclusion volume fraction

The volume fraction of inclusions is important as it controls the overall mechanical properties of the composites. In the study of this section the distance between the crack and inclusion is set to be $(t_1 - r_a)/a = 4/3$ and inclusion size, $r_a/a = 20/3$ while $T = 0.10$. Since the crack propagation occurs along the sharp tip [20], only results at the sharp tips are shown.

Figure 7.5 The normalised CTOD and K_I for a sharp right tip, varying b/a for $r = 10$

Figure 7.6 The normalised CTOD and K_I for a sharp left tip, varying b/a for $r = 10$

For $\mu_3/\mu_2 = 10$, representing a reinforced composite, the normalised CTOD and K_I for the sharp right tip (denoted as normalised δ_y^R and K_I^R) are shown in Figure 7.5. The CTOD trends resemble that of the stress intensity factor, but with values ranging from about 1.215 to 1.22, compared to normalised K_I^R that goes from 1.084 to 1.0856. The results for the case when the sharp tip is at the left hand side are depicted in Figure 7.6. Again with similar trends, but here normalised δ_y^L is between 3.3 and 3.32 while the normalised K_I^L is from about

0.8747 to 0.8763. In both cases, the normalised CTOD is higher than the corresponding K_I . This result implies that using stress intensity factor criterion will underestimate crack propagation in practical cases.

Figure 7.7 The normalised CTOD and K_I for a sharp right tip, varying b/a for $r = 0$

Figure 7.8 The normalised CTOD and K_I for a sharp left tip, varying b/a for $r = 0$

Meanwhile, results for $\mu_3/\mu_2 = 0$, representing a porous material, are shown in Figures 7.7 and 7.8 for a sharp right crack tip and a sharp left crack tip, respectively. In Figure 7.7 the normalised δ_y^R drops from 1.15 to about 0.75 as the volume fraction of voids decreases, while normalised K_I^R decreases from 1.05 to 0.87. In Figure 7.8, the normalised δ_y^L increases from 3.5 to 6, while the normalised K_I^L shows little change (from 0.9 to 1.17). Again, the normalised δ_y^L is higher than the normalised K_I^L .

7.7.2. The influence of displacement load on crack behaviour

To study the influence of displacement load on the crack, other parameters are set as: $(t_1 - r_a)/a = 4/3$, $\mu_3/\mu_2 = 10$ and $\nu_1 = \nu_2 = 1/3$. Here we will look at the CTOD at both

crack tips for various values of T , for a blunt-sharp crack case (shown in Figure 7.9) and for a sharp-blunt crack case (Figure 7.10). As the displacement load increases, the CTOD for both the left and right tips increases too. This is expected as the higher displacement load causes larger plastic zone sizes, and will thus cause larger crack openings as well.

Figure 7.9 CTOD for varying loads for a blunt-sharp crack, in units of a

Figure 7.10 CTOD for varying loads for a sharp-blunt crack, in units of a

In both cases, as the load increases, CTOD increases exponentially at the sharp tip. At the blunt tips, the rate of increase slows down with increasing load. This is reasonable as the exponential increase indicates failure at higher loads.

Comparing Figures 7.9 and 7.10, we see that both the blunt and sharp tips of Figure 7.10 are more shielded than that of Figure 7.9.

7.8. Chapter Summary

In this chapter, a Zener-Stroh crack with plastic zone correction in a composite material with three-phase model was investigated. With a three-phase model, the effects of

other inclusions are taken into account as part of the equivalent composite outer phase. The problems were studied for both a blunt-sharp crack, and a sharp-blunt crack near a circular inclusion (the reinforcing fiber). The effects of inclusion volume fraction, load ratio and crack-inclusion distance, on the fracture behaviour of the crack were studied with numerical examples. The results show how K_I underestimates fracture compared to CTOD, and that the CTOD predicts the shielding (or anti-shielding) effects better.

Higher volume fraction of inclusion increases the effect of the shielding if the inclusion were harder, or anti-shielding if the inclusion were softer than the matrix. Increasing load ratio increases the plastic zone size and CTOD, making it more likely to crack. Also, increasing load ratio also makes obvious K_I how fracture is underestimated compared to CTOD. The effects of the inclusion on the crack are stronger closer to the crack.

Limitations and Validation of Analysis

8.1. Limitations of Analysis

There are some limitations to this method of estimating the PZS using the Dugdale model while simulating a crack using dislocations.

Several limitations are inherited from the Dugdale model. The first one is that the actual plastic zone shape in materials may not necessarily be a thin strip. Although the Dugdale model was first suggested to consider yielding in thin steel sheets, it does not always reflect the shape for this case, and instead reflects the shape in polymers [1]. Many polymers produce crack-tip craze zones that are similar to that of the strip-yield model used in this work. Another problem is that while this analysis is reasonable for cases of small-scale yielding, it does not solve large-scale plasticity problems. Also, the material is assumed to be elastic-perfectly-plastic, which of course does not reflect the actual material behaviour in most cases.

Another point to be mentioned is when the crack is very close to the inclusion. two cases of crack propagation are possible. First, the crack could propagate around the inclusion if the interface is weak in comparison to the matrix strength, leading to the inclusion debonding from the matrix (known as the Cook-Gordon effect [100]). This debonding could be modeled using interfacial crack considerations with Dugdale model plastic zones. Modifications to the method described in this work are required to take into account interfacial crack problems. Secondly, it could propagate through the inclusion if the interface is strong. For this case, the Dugdale plastic zone size estimation could still apply, but modifications taking into account the inclusion material properties are required.

Despite these limitations, the Dugdale model analysis does provide useful clues for elastic-plastic fracture analysis such as the plastic zone sizes and CTOD in composite materials. This method also allows an exact solution to the problem posed (of a crack in a composite material, with limited crack tip plasticity). Moreover, this method provides a useful method to calculate the CTOD fracture criterion, compared to a full elastic-plastic fracture analysis which will be computationally intensive and costly.

8.2. Validation of Analysis

Several approaches are utilized to verify the results. The first method is by reducing our cases to homogeneous single-material case for which the solution for plastic zone with Dugdale model is available. The second approach makes use of finite element computation.

8.2.1. Verification by reduction to simpler cases

For verification by reduction, our physical problems should be simplified to a simpler case. From the use of the Dugdale model, it is visible that the results obtained (CTOD and PZS) are affected only by the dislocation solution function used, as well as the applied stress solution. Hence to verify the results, if the dislocation influence function, applied stress solution, CTOD and PZS can be reduced to the simpler case, the results should be reasonable. The basic Dugdale model Griffith crack problem has plastic PZS given by equation (2.27), while the CTOD is given by (2.32) following Weertman [10].

8.2.1.1. Two-phase problem given in Chapter 3

For the case of a single crack interacting with a single circular inclusion in Chapter 3, verification is achieved by reducing the problem to the homogeneous case. From Tamate's work [45], by having material properties equal, the influence function of the two-phase problem; equation (3.10) is reduced to the homogeneous case. When $\mu_2 = \mu_1$, Dundurs'

parameters of (3.12) become $\alpha = \beta = 0$, while the elastic constants of (3.11) become $A = B = 0$ and $Q = 1$. This reduces equation (3.10) to the homogeneous influence function:

$$k_1 = \frac{x_1}{r_1^4} [x_1^2 + 3y^2] \quad (8.1)$$

where the symbols follow the notation of equations after equation (3.10).

Similarly, the applied stress solutions of equations (3.21) and (3.22) can be simplified as well. When $\mu_2 = \mu_1$ and $\lambda_2 = \lambda_1$, this gives $a_2 = b_3 = 0$ from Appendix equations (A1). Hence equations (3.21) and (3.22) both reduce to

$$\begin{aligned} p_y(x) &= \sigma_A, & (a) \\ p_x(x) &= 0, & (b) \end{aligned} \quad (8.2)$$

Meanwhile the CTOD and PZS for a dilute composite of Chapter 3 under uniaxial tension may be compared to Weertman's results by reducing the problem to a homogeneous one by having $\mu_2/\mu_1 = 1$. Other parameters are set $n = 1000$, $r_0/c_1 = 0.7$, $c_2/c_1 = 2$, $\nu_1 = \nu_2 = 1/3$ while T is varied, with the comparison shown in Table 8.1. We see that the results are close, especially at higher values of T . The errors can be further decreased by using more terms (setting n higher) in our calculations.

Table 8.1 Verification of reducing the two-phase case to homogeneous case, using CTOD and PZS

T	CTOD / $\sigma_Y \pi a (\kappa+1)/2\mu$			PZS / a		
	Chapter 3	Weertman [10]	% Error	Chapter 3	Weertman [10]	% Error
0.10	0.00251	0.00251	-0.07261	0.01222	0.01247	1.97950
0.20	0.01018	0.01017	-0.11242	0.05093	0.05146	1.03149
0.25	0.02916	0.02915	-0.05217	0.15366	0.15470	0.67256
0.33	0.04295	0.04295	-0.01620	0.23467	0.23607	0.59351
0.50	0.07020	0.07023	0.04261	0.41201	0.41421	0.53284

8.2.1.2. Coated inclusion problem of Chapter 4

Next, for the coated fibre case of Chapter 4, the results may be verified by reducing it to the two-phase version, and subsequently reducing it to the homogeneous case as well. For

this coated inclusion case, it is known from literature [11] that both the dislocation influence and applied stress functions both reduce to the simpler cases (two-phase and homogeneous material). By checking that the CTOD and PZS also reduce to these cases validate these results.

To reduce the problem to the two-phase one of Chapter 3, we set $\mu_3 = \mu_2 = 10\mu_1$, with other parameters $n = 200$, $n_L = 10$, $\nu_1 = \nu_2 = \nu_3 = 1/3$, $c_1 = 2r_b$, $c_2 = 4r_b$, and $r_b/r_a = 1.1$, while the load-to-yield-strength ratios T is varied under uniaxial tension. The CTOD and PZS results for the left crack tip are shown in Table 8.2, where nearly identical values are observed, validating the results. The slight difference may be attributed to the numerical method, such as the number of terms in the Laurent series, n_L .

Table 8.2 Verification of coated inclusion to two-phase case, using CTOD and PZS

T	$\delta_y^L / \sigma_Y \pi a (\kappa+1) / 2\mu$			ρ_L / a		
	Chapter 4	Chapter 3	% Error	Chapter 4	Chapter 3	% Error
0.10	0.00181	0.00182	0.58938	0.00994	0.00999	0.52327
0.20	0.00800	0.00804	0.49520	0.04299	0.04324	0.57808
0.25	0.01287	0.01293	0.45622	0.06931	0.06973	0.60768
0.33	0.02382	0.02392	0.41349	0.13062	0.13147	0.64277
0.50	0.05828	0.05841	0.21377	0.34564	0.34857	0.84076

Meanwhile, to reduce the coated fibre problem to the homogeneous case, we set the parameters as: $\mu_3 = \mu_2 = \mu_1$, $r_b/a = 4$ and $r_b/r_a = 1.1$ with T varied. The crack tips are set $c_1/a = 11$ and $c_2/a = 13$, though they can take any values. The reduction gives identical CTOD and PZS values to that shown in Table 8.1 and is thus not shown. Note that when the physical problem is reduced to the homogeneous case, the biaxial tension case gives very similar readings to the uniaxial tension case for PZS and CTOD. The pure shear results also give identical readings for PZS and CTOD, despite CTOD being the tangential displacement.

8.2.1.3. Three-phase problem of Chapter 5

The three-phase composite model of Chapter 5 can be reduced to the two-phase problem of Chapter 3 and subsequently to the homogeneous case for comparison too. Previous work [96] has shown that the influence functions and applied stress solution can be reduced to the simpler cases. Meanwhile, in order to compare the CTOD and PZS when the current case is reduced to the two-phase problem, we set $r_b/r_a \rightarrow \infty$, $\mu_1/\mu_2 = 10$, while other parameters are set $n = 200$, $n_L = 10$, $\nu_1 = \nu_2 = 1/3$, $c_1 = 2r_a$, $c_2 = 4r_a$. T is varied, with the results for reduction to a similar two-phase problem shown in Table 8.3.

Table 8.3 Verification of three-phase to two-phase case, using CTOD and PZS

T	$\delta_y^L / \sigma_Y \pi a (\kappa+1)/2\mu$			ρ_L / a		
	Chapter 4	Chapter 3	% Error	Chapter 4	Chapter 3	% Error
0.10	0.00182	0.00182	-7.420E-07	0.00999	0.00999	4.344E-08
0.20	0.00804	0.00804	-1.650E-06	0.04324	0.04324	4.207E-08
0.25	0.01293	0.01293	1.943E-06	0.06973	0.06973	3.012E-08
0.33	0.02392	0.02392	2.335E-07	0.13147	0.13147	5.259E-08
0.50	0.05841	0.05841	2.193E-07	0.34857	0.34857	3.212E-08

Once again, reduction to the homogeneous case delivers identical results to that shown in Table 8.1.

8.2.1.4. Two-phase Zener-Stroh crack problem of Chapter 6

Table 8.4 Verification of Zener-Stroh crack, two-phase to homogeneous case for CTOD

T	$\delta_y^L / \sigma_Y \pi a (\kappa+1)/2\mu$			$\delta_y^R / \sigma_Y \pi a (\kappa+1)/2\mu$		
	Chapter 6	Weertman [10]	% Error	Chapter 6	Weertman [10]	% Error
0.10	0.097572	0.09751	-0.06361	0.002428	0.00249	2.491186
0.20	0.189886	0.190158	0.143275	0.010114	0.009842	-2.76833
0.25	0.23384	0.234754	0.389025	0.01616	0.015246	-5.98997

Similarly, the Zener-Stroh crack problem in Chapter 6 can be reduced to the homogeneous case for agreement with Weertman [10]. Weertman's results are given by equations (2.28) for PZS and (2.33) for CTOD. In the homogeneous case, $\mu_2/\mu_1 = 1$, with

parameters are set as: $n = 1500$, $\nu_1 = \nu_2 = 1/3$ with T varied. The CTOD results are shown in Table 8.4.

8.2.1.5. Three-phase Zener-Stroh crack problem of Chapter 7

The Zener-Stroh crack in a three-phase case composite model in Chapter 7 can then be reduced to the two-phase of Chapter 6. For that purpose, we set $r_b/r_a \rightarrow \infty$, $\mu_1/\mu_2 = 10$, while other parameters are set $n = 200$, $n_L = 10$, $\nu_1 = \nu_2 = 1/3$, $c_1 = 2r_a$, $c_2 = 4r_a$ with T varied. The results are presented in Tables 8.5 and 8.6 for PZS and CTOD, respectively.

Table 8.5 Verification of Zener-Stroh crack, three-phase to two-phase case, with PZS

T	ρ_L / a			ρ_R / a		
	Chapter 7	Chapter 6	% Error	Chapter 7	Chapter 6	% Error
0.10	0.009329	0.009329	3.216E-08	0.010585	0.010585	0.000E+00
0.20	0.042932	0.042932	2.329E-08	0.048575	0.048575	0.000E+00
0.25	0.071859	0.071859	1.392E-08	0.080261	0.080261	0.000E+00
0.33	0.14602	0.14602	6.848E-08	0.162303	0.162303	0.000E+00
0.50	0.234702	0.234702	4.261E-08	0.261157	0.261157	0.000E+00

Table 8.6 Verification of Zener-Stroh crack, three-phase to two-phase case, with CTOD

T	$\delta_y^L / \sigma_Y \pi a (\kappa+1)/2\mu$			$\delta_y^R / \sigma_Y \pi a (\kappa+1)/2\mu$		
	Chapter 7	Chapter 6	% Error	Chapter 7	Chapter 6	% Error
0.10	0.098396	0.098396	2.337E-07	0.001828	0.001828	0.000E+00
0.20	0.192309	0.192309	-1.040E-07	0.008754	0.008754	1.074E-06
0.25	0.237372	0.237372	0.000E+00	0.014284	0.014284	0.000E+00
0.33	0.3095	0.3095	0.000E+00	0.027196	0.027196	1.103E-06
0.50	0.36423	0.36423	8.237E-08	0.041334	0.041334	-9.193E-07

The three-phase case also can be reduced to the homogeneous problem, and results identical to Table 8.4 are obtained.

8.2.2. Finite element validation

An independent finite element analysis using Abaqus was performed, using the cohesive zone model to study the Dugdale plastic zones. In previous work, Becker's finite element modelling and validation of the Dugdale model [101] compared CTOD without modelling the plastic zones.

Figure 8.1 The mesh model, showing the circular inclusion and the crack

Figure 8.2 Close-up of the left crack-tip Dugdale plastic zone

Using Abaqus 6.10, here we will model the crack near a circular inclusion under uniaxial tension with Dugdale plastic zones. The computed CTOD and PZS will be compared with our analytical solutions given in Chapter 3.

The COH2D4, a 4-node two-dimensional cohesive element was used to represent the plastic zone. The rest were modelled using elements CPS4R. This 4-node quadrilateral was used instead of the 8-node CPS8R for compatibility with the cohesive element COH2D4. Smaller mesh sizes were used at the crack tips, especially at the cohesive zones for more

precise and accurate values. The mesh model is shown in Figure 8.1, while Figure 8.2 shows a close-up of the left crack-tip and plastic zone.

The parameters are set as $n = 200$, $\sigma_A/\sigma_Y = 0.1$, $\nu_1 = \nu_2 = 1/3$, $t_1/r_0 = 1.2$, $t_2/r_0 = 1.5$, while μ_2/μ_1 is varied. The normalised PZS comparison is summarised in Table 8.7 while the normalised CTOD comparison is summarised in Table 8.8. From the tables, it is visible that the analytical results coincide well with the finite element values.

Table 8.7 Verification of PZS with finite element method

μ_2/μ_1	ρ_L/ρ_{0w}			ρ_R/ρ_{0w}		
	Analytical	FEM	% Difference	Analytical	FEM	% Difference
1	1	1	0	1	1	0
2	0.6740	0.6773	-0.4857	0.7724	0.7741	-0.2170
2.5	0.6054	0.6128	-1.2102	0.7195	0.7096	1.4055
5	0.4625	0.4676	-1.0820	0.6129	0.6127	0.0266
10	0.3879	0.4031	-3.7829	0.5538	0.5482	1.0216

Table 8.8 Verification of CTOD with finite element method

μ_2/μ_1	Normalised δ_y^L			Normalised δ_y^R		
	Analytical	FEM	% Difference	Analytical	FEM	% Difference
1	1	1	0	1	1	0
2	0.6679	0.6750	-1.0559	0.7657	0.7500	2.0973
2.5	0.5951	0.6000	-0.8161	0.7150	0.7000	2.1440
5	0.4491	0.4500	-0.1985	0.6017	0.5750	4.6464
10	0.3781	0.3750	0.8143	0.5450	0.5000	8.9985

Table 8.9 CPU time comparison between this work and finite element method

μ_2/μ_1	CPU time/ s	
	Analytical	FEM
1	49.937	2748
2	131.328	2748
2.5	133.706	2748
5	170.625	2857
10	118.954	2780

Meanwhile, the CPU comparison times are shown in Table 8.9, where the analytical method of this work is more than 20 times faster for the example cases here. For the

analytical method, using more terms would increase accuracy but slow it down, while the finite element method could be speeded up with rougher meshes, at the cost of accuracy.

8.3. Estimation of Maximum Error

To estimate the error, the iteration procedure of Appendix B1 or B2 has to be considered. An estimate for the maximum error of ρ_L and ρ_R will be calculated.

Figure 8.3 Crack with Dugdale model plastic zones showing x-axis and u-axis.

For the maximum error, the location of the open end of the plastic zone when normalised to $(-1,1)$ has to be identified. Figure 8.3 shows the crack with Dugdale model plastic zones, showing the relation between the x-axis and when (t_1, t_2) is normalised to $(-1,1)$ on the u-axis. Thus when c_1 and c_2 are normalised to the u-axis:

$$\begin{cases} u_{r_{c_1}} = \frac{-(c_2 - c_1) - (\rho_R - \rho_L)}{(c_2 - c_1) + (\rho_R + \rho_L)} \\ u_{r_{c_2}} = \frac{(c_2 - c_1) - (\rho_R - \rho_L)}{(c_2 - c_1) + (\rho_R + \rho_L)} \end{cases} \quad (8.3)$$

From $u_r = \cos \frac{\pi r}{n}$, $r = 1..n-1$ we obtain the equation $r = \frac{n}{\pi} [\cos^{-1}(u_r)]$. From this

r_{c_1} and r_{c_2} can be calculated. Thus an estimate of the maximum error is given by:

$$\begin{cases} \text{Error}_{\rho_L} = \frac{u_{\text{floor}(r_{c_1})} - u_{\text{ceil}(r_{c_1})}}{u_{r_{c_1}} + 1} \\ \text{Error}_{\rho_R} = \frac{u_{\text{floor}(r_{c_2})} - u_{\text{ceil}(r_{c_2})}}{1 - u_{r_{c_2}}} \end{cases} \quad (8.4)$$

Thus for example, in the case of $\rho_L = \rho_R = 0.012337a$ and $n = 200$ the maximum error is 19.16%, while at $n = 1000$, the maximum error is 3.99%.

Conclusions and Future Work

9.1. Conclusions

In short, five crack-inclusion interaction problems were solved, with consideration of the Dugdale plastic zone correction. The Dugdale plastic zone is modelled as a thin yielded strip ahead of each crack tip, by ensuring the Dugdale condition of vanishing stress intensity factors are fulfilled at both crack tips. The stress intensity factors K_I (due to applied stress) and K_p (due to closure stresses) are calculated by formulating the problems into a set of singular integral equations that are then solved using Erdogan and Gupta's approach [66]. This solution procedure can be used to solve other problems if the Green's function is known. This opens up possibilities for investigating other types of crack-inhomogeneity interaction problems such as the sub-interface crack, or cracks in half-spaces.

On the interaction between a Griffith crack and an inclusion, the problem is solved by balancing the stress intensity factors due to the applied load, K_I and stress intensity factors due to closure stresses, K_p at both crack tips. The shear modulus, load-to-yield ratio, crack size and inclusion size show various influences on the CTOD and PZS, while the Poisson's ratios have limited effect on the CTOD and PZS. The advantages of using CTOD instead of K_I as the fracture criterion are shown: compared to using CTOD, using the stress intensity factor K_I tends to underestimate crack propagation.

For the problem of a crack interacting with a coated circular inclusion, the simulation of the problem becomes more complicated as three different materials are involved. As a result, more parameters affect the overall fracture behaviour of the crack. The influence of the coating phase was the primary focus of this problem. A "harder" coating phase shields the

crack tips and reduces CTOD, while a “softer” coating phase causes anti-shielding effect on the crack. Also, a thicker coating phase increases the effect of shielding or anti-shielding.

The third physical problem looked at the crack in a three-phase composite material, where an equivalent composite outer phase is used. The volume fraction of the reinforcing fibres was the main parameter studied here. A very low fibre concentration of “harder” fibres reduces the problem to the two-phase case, while higher fibre concentration shields the crack tips more by further reducing CTOD.

Two Zener-Stroh crack problems with Dugdale plastic zone corrections were investigated. The first one was a single Zener-Stroh crack interacting with a single circular inclusion in a matrix, where the effect of various parameters on the sharp crack tip was studied. The second one had the Zener-Stroh crack in a three-phase composite. In this problem, we studied both the blunt-sharp and sharp-blunt Zener-Stroh cracks. The influence of the shear modulus, load-to-yield ratio, Poisson’s ratio, crack size, inclusion size and distance, on the fracture behaviour of the crack, was studied in detail. The Poisson’s ratios were found to have little influence for most materials.

Two independent ways to verify our obtained results was performed. One method is to reduce our current models to simpler case where the crack locates in a homogeneous material without inclusion. The results obtained from reduced formula can match well with those available in open literature. The second method for verification is to use finite element analysis. The discrepancies with finite element range from less than 1% to maximum about 9%.

9.2. Suggestions for Future Work

In order to understand crack tip plasticity better, other possible research directions could follow the investigations of this thesis.

The Dugdale model could be used to study various different cases of cracks in composite materials, such as sub-interface crack and interface crack [102], amongst others.

Another possibility would be to predict the shape of the plastic zone at larger scale yielding in composite materials. In homogeneous materials, this was achieved using the plastic zone lengths along with the stress and dislocation fields, amongst others for Mode III [10]. In Modes I and II, the plastic zone shape was predicted by ensuring that conditions for dislocation crack extension force and shielding are satisfied [10].

In the cohesive zone model [103], the relationship between the material separation is described by a traction-separation curve. In the Dugdale model, this traction-separation curve is a flat, perfectly yielded line, which assumes perfectly yielded materials. By using different forms of the traction-separation curve [104], it is possible to look into composite material problems not limited to perfectly yielded materials, and predict crack propagation or debonding of cohesive elements.

Appendix

A1. The Two-Phase Model

The constant a_2 and b_3 in equations (3.21) and (3.22) for the loading conditions are given by solving:

$$\begin{aligned} a_1 - a_2 &= 1 \\ \frac{\lambda_2 + \mu_2}{\lambda_1 + \mu_1} a_1 + \frac{\mu_1}{\lambda_1 + \mu_1} a_2 &= 1 \\ b_1 + (\kappa_2 - 3)b_2 - b_3 - (\kappa_1 + 1)b_4 &= 1 \\ -b_1 + (\kappa_2 + 3)b_2 - b_3 + (\kappa_1 - 1)b_4 &= -1 \\ \frac{\mu_2}{\mu_1} b_1 + 3b_3 + 4b_4 &= 1 \\ -\frac{\mu_2}{\mu_1} b_1 + 6\frac{\mu_2}{\mu_1} b_2 + 3b_3 + 4b_4 &= 1 \end{aligned} \tag{A1}$$

A2. The Coated Fibre Model

Constants a_n , b_n , a'_n and b'_n in equations (4.10) can be found by solving the equations:

$$\begin{aligned}
& \frac{(1-D)C}{1-C} a_n + b_n = -(1-D) \left(\frac{1}{\beta} \right)^{n+1} \\
& (1-D)a_{-n} - Ab_{-n} = 0 \\
& -\frac{(1-D)C}{1-C} (n-1)a_n + \left[\frac{(1-C)D}{1-D} - \frac{1-CD}{1-D} \delta_{n0} \right] a_{-n} \\
& \quad - [n-1 + (1-M)\delta_{n0}] \alpha^2 b_n + [1 - (1-M)\delta_{n0}] \alpha^{-2n+2} b_{-n} \\
& \quad = (1-C) \frac{1}{\beta} (2\beta^2 - 1) \delta_{n0} + (1-C) \delta_{n1} \\
& \quad + (n-1)(C-D) \left(\frac{1}{\beta} \right)^{n+1} (n-2)(1-C) \left(\frac{1}{\beta} \right)^{n-1} \\
& a_n + (n+1)a_{-n} - B\alpha^{2n+2} b_n - A(n+1)\alpha^2 b_{-n} = 0 \\
& \frac{(1-D)C}{1-C} a'_n + b'_n = (1-D) \left(\frac{1}{\beta} \right)^{n+1} \\
& (1-D)a'_{-n} - Ab'_{-n} = 0 \\
& -\frac{(1-D)C}{1-C} (n-1)a'_n - \left[\frac{(1-C)D}{1-D} - \frac{1-CD}{1-D} \delta_{n0} \right] a'_{-n} \\
& \quad - [n-1 + (1-M)\delta_{n0}] \alpha^2 b'_n \\
& \quad - [1 - (1-M)\delta_{n0}] \alpha^{-2n+2} b'_{-n} \\
& \quad = (1-C) \frac{1}{\beta} \delta_{n0} + (1-C) \delta_{n1} - (n-1)(C-D) \left(\frac{1}{\beta} \right)^{n+1} \\
& \quad + n(1-C) \left(\frac{1}{\beta} \right)^{n-1} \\
& -a'_n + (n+1)a'_{-n} - B\alpha^{2n+2} b'_n - A(n+1)\alpha^2 b'_{-n} = 0
\end{aligned} \tag{A2}$$

where $\alpha = \frac{r_a}{r_b}$ and $\beta = \frac{\xi}{r_b}$.

The constants a_4 , b_7 and b_8 from equations (4.19) to (4.21) are gathered by solving the equations of:

$$\begin{aligned}
a_1 - a_2 - \left(\frac{r_b}{r_a}\right)^2 a_3 &= 0 \\
\frac{\lambda_3 + \mu_3}{\lambda_2 + \mu_2} a_1 - a_2 + \frac{\mu_2}{\lambda_2 + \mu_2} \left(\frac{r_b}{r_a}\right)^2 a_3 &= 0 \\
a_2 + a_3 - a_4 &= 1 \\
a_2 - \frac{\mu_2}{\lambda_2 + \mu_2} a_3 + \frac{\mu_1}{\lambda_2 + \mu_2} a_4 &= \frac{\lambda_1 + \mu_1}{\lambda_2 + \mu_2} \\
b_1 + (\kappa_3 - 3)b_2 - b_3 - (\kappa_2 - 3) \left(\frac{r_a}{r_b}\right)^2 b_4 - \left(\frac{r_b}{r_a}\right)^4 b_5 - (\kappa_2 + 1) \left(\frac{r_b}{r_a}\right)^2 b_6 &= 0 \\
-b_1 + (\kappa_3 + 3)b_2 + b_3 - (\kappa_2 + 3) \left(\frac{r_a}{r_b}\right)^2 b_4 - \left(\frac{r_b}{r_a}\right)^4 b_5 + (\kappa_2 - 1) \left(\frac{r_b}{r_a}\right)^2 b_6 &= 0 \quad (\text{A3}) \\
\frac{\mu_3}{\mu_1} b_1 - b_3 + 3 \left(\frac{r_b}{r_a}\right)^4 b_5 + 4 \left(\frac{r_b}{r_a}\right)^2 b_6 &= 0 \\
-\frac{\mu_3}{\mu_1} b_1 + 6 \frac{\mu_3}{\mu_2} b_2 + b_3 - 6 \left(\frac{r_a}{r_b}\right)^2 b_4 + 3 \left(\frac{r_b}{r_a}\right)^4 b_5 + 2 \left(\frac{r_b}{r_a}\right)^2 b_6 &= 0 \\
b_3 + (\kappa_2 - 3)b_4 + b_5 + (\kappa_2 + 1)b_6 - b_7 - (\kappa_1 + 1)b_8 &= 1 \\
-b_3 + (\kappa_2 + 3)b_4 + b_5 - (\kappa_2 - 1)b_6 - b_7 + (\kappa_1 - 1)b_8 &= -1 \\
\frac{\mu_2}{\mu_1} b_3 - 3 \frac{\mu_2}{\mu_1} b_5 - 4 \frac{\mu_2}{\mu_1} b_6 + 3b_7 + 4b_8 &= 1 \\
-\frac{\mu_2}{\mu_1} b_3 + 6 \frac{\mu_2}{\mu_1} b_4 - 3 \frac{\mu_2}{\mu_1} b_5 - 2 \frac{\mu_2}{\mu_1} b_6 + 3b_7 + 2b_8 &= -1
\end{aligned}$$

A3. The Three-Phase Cylindrical Model

The longitudinal Young's modulus E_{11} , major Poisson's ratio ν_{12} , axial shear modulus μ_{12} and plane strain bulk modulus K_{23} can be determined exactly [95] as:

$$\begin{aligned} E_{11} &= cE_3 + (1-c)E_2 + \frac{4c(1-c)(\nu_3 - \nu_2)\mu_2^2}{[(1-c)\mu_2/(k_3 + \mu_3/3)] + [c\mu_2/(k_2 + \mu_2/3)] + 1} \\ \nu_{12} &= c\nu_3 + (1-c)\nu_2 + \frac{c(1-c)(\nu_3 - \nu_2)[\mu_2/(k_2 + \mu_2/3) - \mu_2/(k_3 + \mu_3/3)]}{[(1-c)\mu_2/(k_3 + \mu_3/3)] + [c\mu_2/(k_2 + \mu_2/3)] + 1} \\ K_{23} &= k_2 + \frac{\mu_2}{3} + \frac{c}{1/[k_3 - k_2 + (\mu_3 - \mu_2)/3] + (1-c)/[k_2 + 4\mu_2/3]} \\ \mu_{12} &= \frac{\mu_3(1+c) + \mu_2(1-c)}{\mu_3(1-c) + \mu_2(1+c)} \end{aligned} \quad (\text{A4})$$

where $c = (r_a/r_b)^2$ refers to the fibre concentration. However, for the transverse shear modulus μ_{23} many different formulations have been proposed. Using the solution of Christensen [90, 105] μ_{23} is solved from:

$$A \left(\frac{\mu_{23}}{\mu_2} \right)^2 + 2B \left(\frac{\mu_{23}}{\mu_2} \right) + C = 0 \quad (\text{A5})$$

where

$$\begin{aligned} A &= 3c(1-c)^2 \left(\frac{\mu_3}{\mu_2} - 1 \right) \left(\frac{\mu_3}{\mu_2} + \kappa_3 \right) + \left[\frac{\mu_3}{\mu_2} \kappa_2 + \kappa_2 \kappa_3 - \left(\frac{\mu_3}{\mu_2} \kappa_2 - \kappa_3 \right) c^3 \right] \left[c\kappa_2 \left(\frac{\mu_3}{\mu_2} - 1 \right) - \left(\frac{\mu_3}{\mu_2} \kappa_2 + 1 \right) \right] \\ B &= -3c(1-c)^2 \left(\frac{\mu_3}{\mu_2} - 1 \right) \left(\frac{\mu_3}{\mu_2} + \kappa_3 \right) + \frac{1}{2} \left[\frac{\mu_3}{\mu_2} \kappa_2 + \left(\frac{\mu_3}{\mu_2} - 1 \right) c + 1 \right] \left[(\kappa_2 - 1) \left(\frac{\mu_3}{\mu_2} + \kappa_3 \right) - 2 \left(\frac{\mu_3}{\mu_2} \kappa_2 - \kappa_3 \right) c^3 \right] \\ &\quad + \frac{c}{2} (\kappa_2 + 1) \left(\frac{\mu_3}{\mu_2} - 1 \right) \left[\frac{\mu_3}{\mu_2} + \kappa_3 + \left(\frac{\mu_3}{\mu_2} \kappa_2 - \kappa_3 \right) c^3 \right] \\ C &= 3c(1-c)^2 \left(\frac{\mu_3}{\mu_2} - 1 \right) \left(\frac{\mu_3}{\mu_2} + \kappa_3 \right) + \left[\frac{\mu_3}{\mu_2} \kappa_2 + \left(\frac{\mu_3}{\mu_2} - 1 \right) c + 1 \right] \left[\frac{\mu_3}{\mu_2} + \kappa_3 + \left(\frac{\mu_3}{\mu_2} \kappa_2 - \kappa_3 \right) c^3 \right] \end{aligned}$$

Constants for the Green's functions

Meanwhile, for the Green's functions of the three-phase composite in equation (5.10),

the constants a_n, b_n, a'_n and b'_n are solved [38] from:

$$\begin{aligned}
 -a_n + \frac{1}{D}b_n &= \frac{1}{\beta}\delta_{n0} + \beta^{n+1}\delta^{2n+2} + (n+1)(\beta^2\delta^2 - 1)\beta^{n-1}\delta^{2n}, \quad (\text{for } n \geq 0) \\
 \frac{1}{A}a_{-n} - b_{-n} &= \delta_{n1} - \beta^{1-n} + (n-1)(1 - \beta^{-2})\beta^{1-n}, \quad (\text{for } n \geq 1) \\
 D(n-1)a_n + [C\delta^{2n} - (1-C)\delta_{n0}]a_{-n} - [n-1 + (N-1)\delta_{n0}]\delta^2b_n \\
 &\quad + [1 - (N-1)\delta_{n0}]\delta^2b_{-n} \\
 &= \left[\frac{D-C}{\beta} + \frac{2N}{\beta}\delta^2 \right] \delta_{n0} + (C + 2D\beta^2\delta^2)\beta^{n-1}\delta^{2n} \\
 &\quad + 3D(n+1)(\beta^2\delta^2 - 1)\beta^{n-1}\delta^{2n} - D(n+1)\beta^{n+1}\delta^{2n+2} \\
 &\quad - 2(-1)^n D(\beta^2\delta^2 - 1)\beta^{n-1}\delta^{2n}C_{-3}^n + \frac{D(1-C)}{1-D}\delta^2\delta_{n1}, \quad (\text{for } n \geq 0) \\
 \delta^{-2n}a_n + (n+1)a_{-n} - B\delta^{2n+2}b_n - A(n+1)\delta^2b_{-n} \\
 &= A(n+1)\beta^{1-n}\delta^2 - B\beta^{-(n+1)}\delta^2 + 3A(n-1)(\beta^2 - 1)\delta^2\beta^{-(n+1)} \\
 &\quad + 2A\delta^2\delta_{n1} \\
 &\quad + 2A(-1)^{n+1}(\beta^2 - 1)\delta^2\beta^{-(n+1)}C_{-3}^{n-3}(1 - \delta_{n1})(1 - \delta_{n2}), \quad (\text{for } n \geq 1)
 \end{aligned} \tag{A6}$$

as well as the equations of:

$$\begin{aligned}
 -a'_n + \frac{1}{D}b'_n &= \frac{1}{\beta}\delta_{n0} - \beta^{n+1}\delta^{2n+2} + (n+1)(\beta^2\delta^2 - 1)\beta^{n-1}\delta^{2n}, \quad (\text{for } n \geq 0) \\
 \frac{1}{A}a'_{-n} - b'_{-n} &= -\delta_{n1} + \beta^{1-n} + (n-1)(1 - \beta^{-2})\beta^{1-n}, \quad (\text{for } n \geq 1) \\
 D(n-1)a'_n + [C\delta^{2n} + (1-C)\delta_{n0}]a'_{-n} - [n-1 + (N-1)\delta_{n0}]\delta^2b'_n \\
 &\quad + [1 - (N-1)\delta_{n0}]\delta^2b'_{-n} \\
 &= \left[\frac{D-C}{\beta} + \frac{2N}{\beta}\delta^2 \right] \delta_{n0} + (C + 2D\beta^2\delta^2)\beta^{n-1}\delta^{2n} \\
 &\quad + 3D(n+1)(\beta^2\delta^2 - 1)\beta^{n-1}\delta^{2n} - D(n+1)\beta^{n+1}\delta^{2n+2} \\
 &\quad - 2(-1)^n D(\beta^2\delta^2 - 1)\beta^{n-1}\delta^{2n}C_{-3}^n + \frac{D(1-C)}{1-D}\delta^2\delta_{n1}, \quad (\text{for } n \geq 0) \\
 -\delta^{-2n}a'_n + (n+1)a'_{-n} + B\delta^{2n+2}b'_n - A(n+1)\delta^2b'_{-n} \\
 &= A(n+1)\beta^{1-n}\delta^2 - B\beta^{-(n+1)}\delta^2 + 3A(n-1)(\beta^2 - 1)\delta^2\beta^{-(n+1)} \\
 &\quad - 2A\delta^2\delta_{n1} \\
 &\quad + 2A(-1)^{n+1}(\beta^2 - 1)\delta^2\beta^{-(n+1)}C_{-3}^{n-3}(1 - \delta_{n1})(1 - \delta_{n2}), \quad (\text{for } n \geq 1)
 \end{aligned} \tag{A7}$$

where $\delta = \frac{r_a}{r_b}$ and $\beta = \frac{\xi}{r_b}$, $C = \frac{\kappa_1\mu_2 - \mu_1\kappa_2}{\mu_2\kappa_1 + \mu_1}$, $D = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1\kappa_2}$, $C_{-3}^n = \frac{(-3)(-4)\dots(-3-n+1)}{n!}$.

Constants for loading conditions

For the loading conditions of uniaxial tension, biaxial tension and pure shear, the material constants a_2 and a_3 in equations (5.19) and (5.20) are solved [96] from:

$$\begin{aligned}
 a_1 - a_2 - \left(\frac{r_b}{r_a}\right)^2 a_3 &= 0 \\
 \frac{\lambda_3 + \mu_3}{\lambda_2 + \mu_2} a_1 - a_2 + \frac{\mu_2}{\lambda_2 + \mu_2} \left(\frac{r_b}{r_a}\right)^2 a_3 &= 0 \\
 a_2 + a_3 - a_4 &= 1 \\
 a_2 - \frac{\mu_2}{\lambda_2 + \mu_2} a_3 + \frac{\mu_1}{\lambda_2 + \mu_2} a_4 &= \frac{\lambda_1 + \mu_1}{\lambda_2 + \mu_2}
 \end{aligned} \tag{A8}$$

while b_3, b_4, b_5 and b_6 from equations (5.20) and (5.21) are obtained [96] below:

$$\begin{aligned}
 b_1 + (\kappa_3 - 3)b_2 - b_3 - (\kappa_2 - 3)\left(\frac{r_a}{r_b}\right)^2 b_4 - \left(\frac{r_b}{r_a}\right)^4 b_5 - (\kappa_2 + 1)\left(\frac{r_b}{r_a}\right)^2 b_6 &= 0 \\
 -b_1 + (\kappa_3 + 3)b_2 + b_3 - (\kappa_2 + 3)\left(\frac{r_a}{r_b}\right)^2 b_4 - \left(\frac{r_b}{r_a}\right)^4 b_5 + (\kappa_2 - 1)\left(\frac{r_b}{r_a}\right)^2 b_6 &= 0 \\
 \frac{\mu_3}{\mu_2} b_1 - b_3 + 3\left(\frac{r_b}{r_a}\right)^4 b_5 + 4\left(\frac{r_b}{r_a}\right)^2 b_6 &= 0 \\
 -\frac{\mu_3}{\mu_2} b_1 + 6\frac{\mu_3}{\mu_2} b_2 + b_3 - 6\left(\frac{r_a}{r_b}\right)^2 b_4 + 3\left(\frac{r_b}{r_a}\right)^4 b_5 + 2\left(\frac{r_b}{r_a}\right)^2 b_6 &= 0 \\
 b_3 + (\kappa_2 - 3)b_4 + b_5 + (\kappa_2 + 1)b_6 - b_7 - (\kappa_1 + 1)b_8 &= 1 \\
 -b_3 + (\kappa_2 + 3)b_4 + b_5 - (\kappa_2 - 1)b_6 - b_7 + (\kappa_1 - 1)b_8 &= -1 \\
 \frac{\mu_2}{\mu_1} b_3 - 3\frac{\mu_2}{\mu_1} b_5 - 4\frac{\mu_2}{\mu_1} b_6 + 3b_7 + 4b_8 &= 1 \\
 -\frac{\mu_2}{\mu_1} b_3 + 6\frac{\mu_2}{\mu_1} b_4 - 3\frac{\mu_2}{\mu_1} b_5 - 2\frac{\mu_2}{\mu_1} b_6 + 3b_7 + 2b_8 &= -1
 \end{aligned} \tag{A9}$$

B1. Detailed Iteration Procedure for a Griffith Crack

Figure B1.1 shows the crack with Dugdale model plastic zones, along with its position on the x-axis and when (t_1, t_2) are normalised to $(-1, +1)$ on the u-axis. The discretisation points on the u-axis are given by $u_r = \cos \frac{\pi r}{n}$ where $r = 1..n - 1$. The objective of the iteration is to interpolate $u_{\text{ceil}(r_{c_1})}$ and $u_{\text{floor}(r_{c_1})}$ to obtain $u_{r_{c_1}}$, and interpolate $u_{\text{ceil}(r_{c_2})}$ and $u_{\text{floor}(r_{c_2})}$ for $u_{r_{c_2}}$.

Figure B1.1 Crack with Dugdale model plastic zones showing x-axis and u-axis.

The procedure used to iterate for the plastic zone sizes consists of the steps below:

1. The variables are initialized. For step $i = 0$, $\rho_{L_i} = \rho_{R_i} = 0.01a$ was chosen as it is close to the Dugdale model $\rho = 0.012337a$ when the load ratio $T = 0.1$. Note that any arbitrary value could have been used.

2. Using the initial values, solve for K_I^L , $K_{I\rho}^L$, K_I^R , $K_{I\rho}^R$, and calculate

$$\begin{cases} \text{sum_}L_i = K_I^L + K_{I\rho}^L \\ \text{sum_}R_i = K_I^R + K_{I\rho}^R \end{cases}$$

3. Normalise ρ_{L_i} and ρ_{R_i} to the u-axis gives
$$\begin{cases} u_{r_{c_1-i}} = \frac{-(c_2-c_1) - (\rho_{R_i} - \rho_{L_i})}{(c_2-c_1) + (\rho_{R_i} + \rho_{L_i})} \\ u_{r_{c_2-i}} = \frac{(c_2-c_1) - (\rho_{R_i} - \rho_{L_i})}{(c_2-c_1) + (\rho_{R_i} + \rho_{L_i})} \end{cases}$$

4. Then $\begin{cases} \text{ratio_}L = K_I^L / K_{I\rho}^L \\ \text{ratio_}R = K_I^R / K_{I\rho}^R \end{cases}$ are calculated, from which

$$\begin{cases} u_{r_{c_1-i+1}} = \text{ratio_}L \cdot (1 + u_{r_{c_1-i}}) - 1 \\ u_{r_{c_2-i+1}} = 1 - \text{ratio_}R \cdot (1 - u_{r_{c_2-i}}) \end{cases} \text{ can be calculated.}$$

5. Then ρ_{L_i} and ρ_{R_i} can be calculated by
$$\begin{cases} \rho_{L_i+1} = \left(1 + u_{r_{c_1-i+1}}\right) \frac{c_2 - c_1}{u_{r_{c_2-i+1}} - u_{r_{c_1-i+1}}} \\ \rho_{R_i+1} = \left(1 - u_{r_{c_2-i+1}}\right) \frac{c_2 - c_1}{u_{r_{c_2-i+1}} - u_{r_{c_1-i+1}}} \end{cases}.$$
 6. Recalculate for $K_I^L, K_{I\rho}^L, K_I^R, K_{I\rho}^R$ and calculate sums
$$\begin{cases} sum_L_{i+1} = K_I^L + K_{I\rho}^L \\ sum_R_{i+1} = K_I^R + K_{I\rho}^R \end{cases},$$
 7. Check if the condition of
$$\begin{cases} \left| \frac{sum_L_{i+1} - sum_L_i}{sum_L_i} \right| < 0.10 \\ \left| \frac{sum_R_{i+1} - sum_R_i}{sum_R_i} \right| < 0.10 \end{cases}$$
 is true. If not, increase counter
- $i = i + 1$ and repeat steps 3 to 6 until it is true.
8. Next each neighbouring point is tested until four points are found such that $u_{\text{floor}(r_{c_1})}$ and $u_{\text{ceil}(r_{c_2})}$ give
$$\begin{cases} K_I^L + K_{I\rho}^L = sum_L_{low} < 0 \\ K_I^R + K_{I\rho}^R = sum_R_{low} < 0 \end{cases}$$
 and $u_{\text{ceil}(r_{c_1})}$ and $u_{\text{floor}(r_{c_2})}$ give
$$\begin{cases} K_I^L + K_{I\rho}^L = sum_L_{high} > 0 \\ K_I^R + K_{I\rho}^R = sum_R_{high} > 0 \end{cases}.$$
 9. With these points, $u_{r_{c_1}}$ can be found by interpolating $u_{\text{floor}(r_{c_1})}, u_{\text{ceil}(r_{c_1})}$, with sum_L_{low} and sum_L_{high} for $sum_L = 0$. Similarly $u_{r_{c_2}}$ is found through interpolating $u_{\text{floor}(r_{c_2})}, u_{\text{ceil}(r_{c_2})}$, with sum_R_{low} and sum_R_{high} for $sum_R = 0$.
 10. ρ_R and ρ_L is then found by converting $u_{r_{c_1}}$ and $u_{r_{c_2}}$ from the u-axis to the x-axis via
$$x = \frac{t_2 - t_1}{2}u + \frac{t_2 + t_1}{2}.$$

B2. Detailed Iteration Procedure for a Zener-Stroh Crack

The iteration procedure involved here is very similar to that of Appendix B1. The difference lies in the conditions of step 8, depending on whether the crack is a blunt left tip, sharp right tip (blunt-sharp) Zener-Stroh crack or a sharp left tip, blunt right tip (sharp-blunt) Zener-Stroh crack.

If the problem concerns a blunt-sharp crack, the iteration procedure follows Appendix B1, where step 8 should be modified to:

8. Next each neighbouring point is tested until four points are found such that for

$$u_{\text{floor}(r_{c_1})} \text{ and } u_{\text{ceil}(r_{c_2})} \begin{cases} K_I^L + K_{I\rho}^L = \text{sum_}L_{\text{low}} > 0 \\ K_I^R + K_{I\rho}^R = \text{sum_}R_{\text{low}} < 0 \end{cases} \text{ and for } u_{\text{ceil}(r_{c_1})} \text{ and } u_{\text{floor}(r_{c_2})}$$
$$\begin{cases} K_I^L + K_{I\rho}^L = \text{sum_}L_{\text{high}} < 0 \\ K_I^R + K_{I\rho}^R = \text{sum_}R_{\text{high}} > 0 \end{cases}.$$

One the other hand, if the problem concerns a sharp-blunt crack, the iteration procedure still follows Appendix B1, but step 8 is modified to:

8. Next each neighbouring point is tested until four points are found such that for

$$u_{\text{floor}(r_{c_1})} \text{ and } u_{\text{ceil}(r_{c_2})} \begin{cases} K_I^L + K_{I\rho}^L = \text{sum_}L_{\text{low}} < 0 \\ K_I^R + K_{I\rho}^R = \text{sum_}R_{\text{low}} > 0 \end{cases} \text{ and for } u_{\text{ceil}(r_{c_1})} \text{ and } u_{\text{floor}(r_{c_2})}$$
$$\begin{cases} K_I^L + K_{I\rho}^L = \text{sum_}L_{\text{high}} > 0 \\ K_I^R + K_{I\rho}^R = \text{sum_}R_{\text{high}} < 0 \end{cases}.$$

List of Publications

1. H. J. Hoh, Z. M. Xiao, and J. Luo, "On the plastic zone size and crack tip opening displacement of a Dugdale crack interacting with a circular inclusion," *Acta Mechanica*, pp. 1-10, 2009.
2. H. J. Hoh, Z. M. Xiao, and J. Luo, "Plastic zone size and crack tip opening displacement of a Dugdale crack interacting with a coated circular inclusion," *Philosophical Magazine*, vol. 90, pp. 3511 - 3530, 2010.
3. H. J. Hoh, Z. M. Xiao, and J. Luo, "Crack tip opening displacement of a Dugdale crack in a three-phase cylindrical model composite material," *International Journal of Engineering Science*, vol. 49, pp. 523-535, 2011.
4. H. J. Hoh, Z. M. Xiao, and J. Luo, "On the plastic zone size and CTOD study for a Zener–Stroh crack interacting with a circular inclusion," *Acta Mechanica*, pp. 1-11, 2011.
5. H. J. Hoh, Z. M. Xiao, and J. Luo, "On the fracture behavior of a Zener–Stroh crack with plastic zone correction in three-phase cylindrical composite material," *Mechanics of Materials*, vol. 45, pp. 1-9, 2012.

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