Optimized dynamic policy for robust receding horizon control

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OPTIMIZED DYNAMIC POLICY FOR ROBUST RECEDING HORIZON CONTROL

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Summary

As an on-line-optimization-based control technique, receding horizon control (RHC) has been a prominent control method for real-time control applications. Since this control approach relies on a model of the system being controlled, the presence of uncertainties in the system description has to be addressed with robust algorithms which, if designed naively, may lead to conservative results even with complex on-line computations, thus limiting the wider applicability of the method. The research in this thesis is aimed at developing RHC algorithms that allow to achieve a suitable tradeoff among control performance, applicability and on-line computational complexity, for control problems that require a systematic handling of uncertainties and constraints with low-complexity on-line computations.

With a focus on (possibly uncertain) linear time-varying systems with a polytopic system description and with (possibly unmeasurable) bounded additive disturbances, this thesis studies a class of admissible controller dynamics, and proposes a dynamic control policy that is computationally attractive and offers reduced conservativeness. The proposed policy uses time-varying controller dynamics with controller matrices that need not be explicitly determined on-line but only assumed to follow the same convex combination as the plant matrices, and with a disturbance feedforward term that does not require the disturbance to be measured. Essentially, the proposed policy incorporates all the ‘uncertain’ information into the controller dynamics and this reduces the conservativeness in the assessment of feasible control inputs and hence the feasible invariant set for the controlled system. Furthermore, this policy allows the control optimization problem to be split into two separate problems: one to determine the convex hull of the controller matrices and the other to compute the controller initial state. With the former carried out off-line, the on-line computations involving the latter part are considerably simplified. The dynamics of the proposed policy can also be optimized such that the resulting RHC law ensures a control performance with a suitable H_∞ performance bound.

This thesis also studies the use of the proposed policy as the terminal control policy for a class of time-varying but certain systems with a polytopic system description and with bounded additive disturbances. The proposed policy is a particularly suitable policy to terminate a finite-horizon disturbance based input parameterization for this class of time-varying systems. As a terminal control law, it offers an enlarged terminal feasibility domain, thus allowing a significant reduction in the required horizon length and the on-line computational complexity.

RHC schemes based on the use of the proposed dynamic policy, both as the main policy and as the terminal policy, are formulated in the context of the nominal and the (H_∞ -like) min-max cost minimization approaches. Control performances are assessed in terms of the ℓ_∞ - and ℓ_2 -stability of the controlled system and also demonstrated with several numerical examples. These results are also extended to develop computationally efficient output-feedback RHC algorithms through the use of a suitable state observer for situations in which the state information is not fully available.

This thesis also explores an RHC-based solution to a class of consensus-related coordinated control problems involving dynamically decoupled subsystems with time-varying dynamics and with external disturbances. Suitable near-consensus conditions are defined, and computationally efficient RHC schemes which ensure that a desired near-consensus condition is eventually reached are explored. Furthermore, control problems that arise in satellite formation flying are considered for an assessment of the practical applicability of the RHC schemes developed in the thesis. Problems involving relative position control in the context of formation keeping and formation reconfiguration are studied in detail with realistic simulations in order to assess the effectiveness of the performance of the proposed control schemes.

Notations

Sets of numbers

$\mathbb{Z}$	Set of integers
$\mathbb{Z}_+$	Set of non-negative integers
$\mathbb{Z}_{++}$	Set of strictly positive integers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of non-negative real numbers
$\mathbb{R}^n$	Set of n -dimensional vectors of real numbers
$\mathbb{R}^{m \times n}$	Set of $m \times n$ matrices of real numbers
$\mathbb{D}^n$	Set of $n \times n$ diagonal matrices of real numbers
$\mathbb{D}_+^n$	Set of $n \times n$ diagonal matrices of non-negative real numbers

Vectors and matrices

$\mathbf{1}$	Vector (of appropriate dimension) with all components one.
I	Identity matrix (of size $n \times n$ if a subscript n is added).
X^T	Transpose of matrix X .
$\text{Tr}(X)$	Trace of matrix X .
$\det(X)$	Determinant of matrix X .
$\lambda_{\min}(X)$	Minimum eigenvalue of a symmetric matrix X .
$\lambda_{\max}(X)$	Maximum eigenvalue of a symmetric matrix X .
$\rho(X)$	Spectral radius of a square matrix X , i.e., $\max_{i \in \{1, \dots, n\}} \lambda_i(X) $ where $\lambda_1(X), \dots, \lambda_n(X)$ are the eigenvalues of $X \in \mathbb{R}^{n \times n}$.
$x_{[i]}$	i th element of vector x (x_i denotes $x_{[i]}$ if not stated otherwise).
$X_{[i,j]}$	Element in the i th row and j th column of matrix X .
$X_{[i,:]}$	i th row of matrix X .
$X_{[i:j,:]}$	Matrix comprising of i th to j th rows of matrix X .
$X \otimes Y$	Kronecker product of matrices X and Y .
$ X $	Componentwise absolute value of X .
$\ x\ _p$	p -norm of vector x (2-norm if p is omitted).
$\ X\ $	Spectral norm (maximum singular value) of matrix X .
$\text{diag}(x)$	Diagonal matrix with the components $x_{[1]}, \dots, x_{[n]}$ of vector $x \in \mathbb{R}^n$ as diagonal entries.
$\text{diag}(X_1, \dots, X_n)$	Block diagonal matrix with diagonal blocks $X_1, \dots, X_n$
$\text{Co}\{X_1, \dots, X_n\}$	Convex hull of the set $\{X_1, \dots, X_n\}$ of matrices of the same dimension, i.e., $\left\{ \sum_{i=1}^n \lambda_i X_i \mid \lambda_i \in \mathbb{R}_+, \sum_{i=1}^n \lambda_i = 1 \right\}$.

Set operations

$U \times V$	Cartesian product of sets U and V .
U^n	Cartesian product $U \times U \times \dots \times U$ (n times) of set U .
$U \cup V$	Union of sets U and V .
$U \cap V$	Intersection of sets U and V .
$U \oplus V$	Minkowski sum of sets U and V , i.e., $\{u + v u \in U, v \in V\}$.
$U \ominus V$	Minkowski difference of sets U and V , i.e., $\{x x + v \in U, \forall v \in V\}$.
$U \setminus V$	Complement of set V in set U , i.e., $\{u \in U u \notin V\}$.
$\text{Co}\{U\}$	Convex hull of set $U \in \mathbb{R}^n$.

Functions

$f : U \rightarrow V$	f is a function on the set U into the set V .
$g \circ f$	Composition of functions $f : U \rightarrow V$ and $g : X \rightarrow Y$, i.e., $U \rightarrow Y$ defined as $g \circ f(u) = g(f(u))$ for $u \in U$.
$\mathcal{K}, \mathcal{K}_\infty, \mathcal{KL}$	Classes of strictly increasing functions (see Definition 2.6).

Sequences

$x(t)$	Value of the sequence x (i.e., $\{x(t)\}_{t \in \mathbb{Z}_+}$) at $t \in \mathbb{Z}_+$.
ℓ_p^n ($p \in [1, \infty)$)	$\{x \in \mathbb{S} \mid (\sum_{t=0}^{\infty} \ x(t)\ ^p)^{1/p} \text{ exists}\}$, where $\mathbb{S}$ is the set of $\mathbb{R}^n$ sequences. (The superscript n is usually omitted and is understood from the context.)
$\ x\ _{\ell_p}$	ℓ_p -norm of the sequence $x \in \ell_p$, i.e., $(\sum_{t=0}^{\infty} \ x(t)\ ^p)^{1/p}$.

Inequalities and inclusions

$\leq, \geq$ ($<, >$)	Componentwise or scalar inequalities (strict inequalities).
$\preceq, \succeq$ ($\prec, \succ$)	Matrix inequalities (strict matrix inequalities) (e.g., $P \succ 0$ implies the symmetric matrix P is positive definite, $Q \preceq P$ implies the symmetric matrix $Q - P$ is negative semidefinite.)
$\subseteq, \supseteq$ ($\subset, \supset$)	Set inclusion (strict set inclusion).

Others

*	(a) In a symmetric matrix: An off-diagonal block defined by symmetry, (b) As a superscript to a variable or entity: Optimal value.
★	In a symmetric matrix: A block that can be determined from accompanying definitions.

Abbreviations

ADTM	Approximate Discrete-Time Model
AUV	Autonomous Underwater Vehicles
BMI	Bilinear Matrix Inequality
DEM	Digital Elevation Model
DLR	German Aerospace Center
ECI	Earth-Centered Inertial
ESA	European Space Agency
FRP	Formation Reference Point
GPS	Geographical Positioning System
HCW	Hill-Clohessy-Wiltshire
ISS	Input-to-State Stability
L/F	Leader/Follower
LMI	Linear Matrix Inequality
LP	Linear Programming
LQ	Linear Quadratic
LQR	Linear Quadratic Regulator
LTI	Linear Time-Invariant
LTV	Linear Time-Varying
LVLH	Local-Vertical Local-Horizontal
MAS	Maximal Admissible Set
MIMO	Multiple-Input Multiple-Output
MPC	Model Predictive Control
NASA	National Aeronautics and Space Administration
NLP	Nonlinear Programming
PD	Proportional Derivative
QP	Quadratic Programming
RHC	Receding Horizon Control
RPI	Robust Positive Invariant
RRHC	Robust Receding Horizon Control
SAR	Synthetic Aperture Radar
SDP	Semidefinite Programming
SFF	Satellite Formation Flying
s.t.	such that
UAV	Unmanned Air Vehicle
VS	Virtual Structure
w.r.t.	with respect to

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Chapter 1

Introduction

1.1 Motivation

In the recent years, advances in the fields of computing, sensing and communications have motivated a range of applications that require a real-time control of complex systems under challenging circumstances. These applications, arising from diverse areas of engineering, such as aerospace and transportation systems, robotics, sensing and exploration systems to name a few, are marked by their demand of a guaranteed operation under constrained conditions in a possibly dynamically changing or uncertain environment [1,2]. Reliable algorithms for real-time control based on on-line sensing and decision making are at the heart of the success of these applications.

Although the mathematical and computational tools required for the design of efficient and reliable real-time control schemes have improved over the years, several emerging applications present more challenging control requirements in terms of performance and optimality. For example, in the recent years, there has been an immense interest in applications requiring cooperative behavior in networked systems, mostly comprising of autonomously actuated mobile subsystems, operating together to achieve a shared goal [3–8]. In many of these cases, apart from the fact that the subsystem dynamics may be complicated and may involve uncertainties, subsystems usually need to satisfy hard constraints on the state as well as the input because of operational, physical and resource-related limitations. Moreover, information about the state of the subsystems may not be available as desired because of noisy, non-ideal or incomplete sensing and/or inter-subsystem communications. Similarly, several novel applications in manufacturing and biomedical fields come with challenging control requirements in terms of guaranteed reliability and high performance under uncertain and constrained conditions [9–11].

A fundamental aspect of efficient control schemes that can offer guaranteed performance under challenging conditions involving uncertainties and constraints is the ability to be regularly aware of the prevailing situation through sensing, measurements and/or information-exchange mechanisms and to determine and apply control actions best suited

for the existing situation. Suitable sensing capabilities combined with a scheme for frequently evaluating the situation and updating the control course usually provides immunity from failures and improves performance. A framework suitable for a frequent situation evaluation and a corresponding control-course correction is the optimization-based control framework which allows one to use a suitable performance criterion to optimize future behavior of systems through appropriate choice of control actions.

Optimization-based control schemes rely on a model of the system being controlled for the determination of optimal control actions. Their control performance, in terms of stability, optimality and robustness against uncertainties, largely depends on how closely the assumed model matches with the actual system, how far into the future the optimization of the control actions is carried out and how thoroughly the uncertainties are accounted for in the optimization process. These factors affect the computational complexity of the control algorithms and hence the frequency with which the control computations can be updated and the way in which control actions can be implemented. A common trend in the optimization-based control framework is to optimize control actions over a fixed finite time horizon at a time and implement the first part of the computed optimal control actions until the control computations for a further similar time horizon are repeated at a later time and updated values of control actions become ready for further implementation. Widely known as *receding horizon control* (RHC), this approach offers promising features that motivate the design of efficient and reliable control algorithms suitable for challenging real-time control applications. This thesis studies several aspects of the RHC approach and explores the development of computationally efficient RHC schemes that address the issues of challenging control requirements in uncertain and constrained circumstances that arise in several existing and emerging control applications.

1.2 Receding Horizon Control

RHC, also popularly known as model predictive control (MPC), is a real-time control approach that employs an on-line solution of (usually) a finite-horizon optimal control problem for a plant to be controlled in order to determine optimal control actions for the desired operation of the plant. It uses a model of the plant to predict its future behavior over the considered time horizon and computes an optimal control input sequence based on some suitable performance criterion. Control actions are applied in a receding horizon fashion, meaning that, at a given time, only the initial portion of the computed control action sequence is implemented, and at a later time, the computations are carried out for a shifted time horizon and a new sequence of control actions is determined based on new measurements [12].

RHC has been successfully applied in various forms in the process industries especially for the constrained control of slow processes. However, continuous advances in computational hardware and optimization algorithms have made it a promising and feasible control technique for a range of demanding control applications including those that deal

with systems with faster dynamics [12]. RHC is known to perform better than the conventional control methods when high performance is required under constrained conditions. This is due to the fact that RHC can systematically handle all kinds of system constraints and give a solution that is optimal in the existing conditions. Moreover, the receding horizon implementation inherently provides robustness against modeling imperfections and uncertainties through indirect feedback [13].

1.2.1 Basic RHC algorithm

1.2.1.1 System description

RHC computations are based on a model of the dynamical system being controlled. Dynamics models based on an impulse/step response or a transfer function may be used but state-space models are increasingly commonly adopted for constrained multivariable systems. So, system dynamics are usually expressed using a discrete-time state-space model of the form:

$$x(t+1) = f(x(t), u(t)) \quad (1.1a)$$

$$y_c(t) = g_c(x(t)) \quad (1.1b)$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the state, $u(t) \in \mathbb{R}^{n_u}$ denotes the control input and $y_c(t) \in \mathbb{R}^{n_y}$ denotes a controlled output. $f(\cdot)$ defines the dynamics of the system and satisfies $f(0, 0) = 0$. When the system in (1.1) is linear and time-invariant, it is expressed as:

$$x(t+1) = A x(t) + B u(t) \quad (1.2a)$$

$$y_c(t) = C_c x(t). \quad (1.2b)$$

The system is constrained in the state as well as the input. The constraints can be collectively expressed as

$$(x(t), u(t)) \in \Xi \quad (1.3)$$

where Ξ is a closed convex set.

1.2.1.2 Control objective

The purpose of the control mechanism is to drive the output $y_c(t)$ to a constant reference r . This requires the system state to reach an equilibrium point x_e such that $g_c(x_e) = r$. Essentially, this problem can be readily transformed, by a suitable change of coordinates, into one in which the control mechanism is required to steer the system state to the origin. Therefore, we usually consider the control objective of driving the system state to the origin in an optimal way. The ‘optimality’ of the control mechanism is determined by the

value of a performance criterion defined at each time $t \in \mathbb{Z}_+$ as

$$J(t) = \sum_{i=0}^{\infty} \ell(x(t+i), u(t+i)) \quad (1.4)$$

where $\ell(x(t+i), u(t+i))$ represents the stage cost which is usually defined in terms of a norm of a performance output variable $z(t+i) = \begin{bmatrix} Q^{\frac{1}{2}}x(t+i) \\ R^{\frac{1}{2}}u(t+i) \end{bmatrix}$ with $Q \succeq 0$ and $R \succ 0$.

In an RHC scheme, future control actions are optimized at each time step to minimize the cost (1.4) evaluated in terms of the known value of the current state and the predicted values of future inputs and states. To make the optimization problem tractable, either the control actions are assumed to have a certain form (such as a constant state-feedback) or, typically, they are optimized only over a finite time horizon N to minimize a predicted cost of the form

$$J(t) = \sum_{i=0}^{N-1} \ell(x(t+i|t), u(t+i|t)) + \bar{h}(x(t+N|t)) \quad (1.5)$$

where $\bar{h}(x(t+N|t))$ equals, upper bounds or approximates the predicted terminal cost $\sum_{i=N}^{\infty} \ell(x(t+i|t), u(t+i|t))$, based on some assumption on the form of control actions for times $t+i, i \geq N$. Here, $x(t+i|t)$ and $u(t+i|t)$ respectively represent the values of the state and the input at time $t+i$ predicted at time t . It is ensured that the constraint (1.3) is satisfied by all predicted state-input pairs, and this is typically achieved by imposing the constraint

$$(x(t+i|t), u(t+i|t)) \in \Xi \quad (1.6)$$

for $i = 0, \dots, N-1$ and applying a terminal constraint

$$x(t+N|t) \in \mathcal{S}_{x_N} \quad (1.7)$$

where $\mathcal{S}_{x_N}$ is a convex compact set suitably chosen such that (1.6) is satisfied for all $i \geq N$.

1.2.1.3 Control computation and implementation

Fig. 1.1 shows the typical computation steps in a standard RHC scheme. The horizon length N , the terminal cost function $\bar{h}(\cdot)$ and the terminal set $\mathcal{X}_{x_N}$ are usually determined off-line so that the on-line computation of the control actions and their receding horizon implementation lead to a desired control performance. As mentioned above, the current and the future control actions are predicted at each time step t by minimizing the predicted cost function (1.5). Only the current part of the computed optimal control inputs, i.e., $u^*(t) = u^*(t|t)$, is actually applied to the system at time t and the computations are repeated at the subsequent time step $t+1$. In a typical RHC setup, the stage cost is chosen to be the 2-norm of the performance variable $z(t)$ and therefore, for a linear system,

Figure 1.1: Steps in a standard RHC scheme.

the on-line optimization problem becomes a quadratic programming problem which can be efficiently solved.

1.2.2 Features of RHC

- Generality:** RHC offers a general setup that can handle control problems associated with a broad range of dynamical systems. With state-space-based models, it naturally handles multivariable systems with linear or non-linear dynamics. This generality of the RHC approach is one of the reasons behind its widespread popularity.
- Constraint handling:** One of the distinguishing features of RHC is its ability to directly handle all system constraints. While one needs to resort to indirect techniques, such as the proper tuning of the cost function, in order to manage constraints in conventional optimal control approaches, RHC systematically incorporates all constraints in the on-line computation of control actions and even allows the plant to operate on the constraint boundary when such an operation is the most desired one.
- Adaptation to changing conditions:** Since RHC involves resolving the control optimization problem after new measurements become available, control actions are easily adapted to changing situations in RHC as compared with other control techniques. This also makes RHC well suited for time-varying systems and for tracking problems. Moreover, this feature of RHC provides it with an inherent robustness against some uncertainties in system dynamics.
- Computational features:** Because of the fact that RHC solves a finite-horizon optimal control problem, the computations at each time are simpler than the computations for an off-line solution of an infinite-horizon optimal control problem in which the optimal control law is to be determined. However, the RHC problem needs to be solved at each time step and it is essential that the computations are sufficiently simple so that they can be completed in real-time. Typical RHC optimization problems for linear systems are quadratic programming (QP) or linear programming (LP) problems and they are easily solved using efficient algorithms.

Moreover, for non-linear systems, efficient algorithms that solve non-linear programming (NLP) problems are increasingly becoming common.

5. **Practical applicability:** RHC is one of the most successful control techniques employed in industries. In fact, theoretical investigations into the performance of RHC schemes were prompted by their huge success in the process industries. Although the applicability of the approach was limited to the control of slow processes in the petrochemical industries in the early years (1970s and 80s), with the advances in the speed of computing hardware, it is gradually becoming a promising control approach for any kind of system dynamics - slow or fast [12].
6. **Stability and robustness:** Even though standard finite-horizon RHC algorithms offer inherent robustness against some uncertainties in system dynamics, a theoretical guarantee of robustness with standard algorithms may not be straightforward, especially when the system is constrained. Indeed, this is the reason why much of the academic research on RHC has focussed on investigating the theoretical bases of these issues and developing enhanced algorithms that promise a theoretical guarantee of robust control performance. It has now been understood that, for most systems, with the suitable use of a terminal cost, a terminal constraint or both, stability can generally be ensured [13]. For uncertain systems, it is essential to guarantee robust stability in the presence of uncertainties. There has been extensive research on this line [13–15] and it remains an important area of current research.

1.2.3 Brief review of RHC literature

Historical background: The RHC approach has a long history of research and practical application in industries. Coming from several decades of research, literature on RHC is extensive and includes results for various kinds of system models and for various performance criteria, obtained with the use of different optimization tools, and explored for different kinds of applications. The purpose of this section is to briefly mention some important research trends that have over the years established RHC as a mature control technique with a rich set of tools for analysis and design. Extensive reviews of specific trends can be found in several survey papers that have appeared over the years such as [13–20]. Moreover, detailed treatment of various aspects of the RHC approach can be found in a number of books such as [12, 21–26].

Initially envisaged in the context of open-loop optimal control in the 1960s and 1970s by Propoi [27], Lee et al. [28], Thomas [29] and Kwon et al. [30], RHC or MPC gained prominence as a practical industrial process control technique through the years since the late 1970s. During these years, MPC algorithms addressing specific demands of the process industry appeared in the process control literature [19]. These included several variants such as model predictive heuristic control [31], dynamic matrix control [32] and

quadratic dynamic matrix control [33], and were mostly based on the impulse or step response models of the plant. Several related algorithms based on input-output models appeared in the adaptive control literature during these years (e.g., [34, 35]) and they led to a formulation that was called generalized predictive control (GPC) [36, 37]. In spite of their industrial success, most versions of RHC algorithms proposed in the 1970s and 1980s did not automatically ensure stability and required a proper tuning of cost function parameters and cost and control horizons. This incompleteness in the understanding and the formulation of the original RHC algorithms was gradually addressed by the academic research community in the years starting in the late 1980s. Research in this area has since continued in various directions and several new versions of the algorithm have been developed [13].

Stability in RHC: Contributions in the area of stabilizing RHC have their roots in the early works of Kleinman [38] and Thomas [29] who considered the finite-horizon optimal (minimum input energy) control problem for linear time-invariant (LTI) systems and showed that the optimal controller, that implicitly constrains the terminal state to be at the origin, is linear and stabilizing. Extensions of these ideas to the case of linear time-varying (LTV) systems and general quadratic costs were explored in later works such as [30] and [39]. Notable among the works on stabilizing RHC in the subsequent years is [40] in which the authors proved and used the idea that the cost function of the finite-horizon optimal control problem, with a fixed terminal equality constraint, can be used as a Lyapunov function to establish stability of the controlled system - a central idea that was used in the stability proofs of various RHC algorithms developed in the later years. Extended results on this line for non-linear systems were presented in papers [41] and [42] which respectively dealt with discrete-time and continuous-time dynamics.

Early propositions on stable RHC that relied on an explicit or an implicit use of the fixed terminal equality constraint $x(N) = 0$ had their drawbacks arising from the ‘artificiality’ of such a constraint in the RHC context and its computational consequences. Research in the 1990s looked for alternative approaches, and, although the use of a terminal cost instead of a terminal equality constraint was found to be sufficient to ensure stability with RHC algorithms in the case of unconstrained [21, 43, 44] or stable [45] systems, the use of a suitable terminal constraint set instead of the terminal equality constraint was established as a standard and preferred approach of handling stability for general constrained systems [13]. In this approach, sometimes referred to as dual-mode control approach (e.g., [46, 47]), the finite-horizon control actions are optimized on-line to drive the terminal state $x(N)$ inside a terminal constraint set $\mathcal{X}_{x_N}$ that is associated with a permissible terminal control law. With the terminal constraint set suitably chosen, a terminal cost, equaling or approximating the infinite-horizon optimal cost for the terminal state, can be included in the optimization problem in order to obtain the qualities of infinite-horizon optimal control. [48] was likely the earliest work to contribute in this direction. In this work, the authors have considered a linear system with polytopic state and input constraints, and chosen a terminal cost and a permissible terminal constraint set (maximal output admissible set [49]) that correspond to the closed

loop system under a terminal control law which is optimal for the unconstrained infinite-horizon LQ problem. Several later works have further explored this approach in the context of linear (e.g., [50, 51]) and non-linear (e.g., [52–54]) systems.

Robustness in RHC: Robustness against uncertainties, which usually exist in practical system descriptions because of modeling imperfections and other external and internal factors, has been another key concern for the developers of RHC algorithms for the last several years. Although RHC is known to possess inherent robustness (see, e.g., [55]) to some extent because of the receding horizon implementation of the control law, it becomes necessary to thoroughly consider uncertainties when state and input constraints are present. Early RHC schemes based on impulse/step response or input-output models considered the effects of uncertainties, particularly disturbances, in their own ways; however, systematic handling of uncertainties was explored in the state-space framework only in the years starting in the late 1990s. Since a guarantee of robustness against uncertainties for constrained systems comes with its implications in the feasibility as well as the computational complexity of the control problem to be solved on-line, research in this area has focussed on developing algorithms that promise less conservative handling of uncertainties with low-complexity on-line computations [13–15]. Most of the works in this area have considered linear systems with either polytopic model uncertainty [56–62] or bounded additive disturbances [51, 63–72], and only a few of them have considered uncertainties in a more general setting (e.g., [73–78]). There have been some suggestions for using an open-loop worst-case optimization of control actions (e.g., [46]); however, it has been recognized that the traditional MPC approach of optimizing an open-loop free input sequence can be too conservative and may lead to unrealistic infeasibility problems for constrained uncertain systems. The non-conservative alternative requires that the optimization be done over a sequence of admissible feedback control laws rather than free control inputs [13]. However, the approaches that explicitly consider such an optimization (e.g., [51, 73]) end up being computationally prohibitive for most problems [70].

Several computationally simpler alternatives have been proposed over the years to deal with problems involving systems with uncertainties, and they include various combinations of off-line and on-line optimizations. In [74], the authors presented a method based on an off-line computation of the explicit solution - a piecewise linear control law - of the linear MPC optimization problem. Although the on-line computation required in this method consists of a lookup for a solution followed by a simple function evaluation, the storage requirements and the search complexity can grow exponentially with the order of the system and the number of constraints [74], [79], [60]. In [56, 80], the authors have considered an on-line optimization of an identical linear or affine state-feedback law over the infinite-horizon for a linear system with a polytopic or a structured uncertainty. Their approaches offer elegant solutions but are conservative and computationally demanding [13, 70].

One common approach employed in several other works (e.g., [57], [64], [71], [62]) involves an optimization over a finite-length free sequence as perturbations to a pre-

determined constant (optimal) state-feedback. This approach significantly simplifies the on-line computational burden. The perturbations which are optimized on-line and ‘open-loop’ for the pre-stabilized system, handle feasibility of the control problem whereas the feedback law limits the spread of state trajectories (because of uncertainties) within an invariant tube. Several later works have further explored these issues with what is known as the ‘tube approach’ [75], [67], [68], [77], [78] in which (the shape of) the cross-section of the invariant ‘tube’ of possible state trajectories under uncertainties is chosen or optimized off-line, usually associating it with a feedback control component involving, for example, a constant static feedback [67] or a time-varying feedback [68]. The overall control input (including a feedforward component) is then obtained on-line with simplified computations by minimizing a suitable cost for an average trajectory.

A more recent trend in the robust MPC literature is to explore less conservative schemes that employ tractable methods to optimize future control inputs which are directly tuned to the particular realization of uncertainties affecting the system until their respective times of execution. For a linear system with polytopic model uncertainty, the authors in [60] use a dynamically evolving sequence of predicted future input perturbations represented by the future states of a dynamic controller with a suitably chosen structure and with uncertain polytopic dynamics similar to the system dynamics. This allows one to avoid a conservative, deterministic prediction of input perturbations and hence significantly enlarge the size of the feasible set over which an efficient on-line optimization of the cost can be carried out. In [70, 81–83], the authors consider the case of additive disturbances and optimize inputs or input perturbations parameterized as affine functions of the previous disturbances by solving a tractable problem. The outcome is a reduced conservativeness in the application of control inputs and hence an enlarged feasibility domain. RHC schemes with uncertainty based control parameterizations appear to be promising schemes that ensure less conservative results, both in terms of the control performance and the size of the feasibility domain. We will explore this approach in much detail in this thesis and present several results on computationally efficient RHC schemes based on this approach.

RHC with partially measured state: Standard RHC schemes usually assume that the system state can be fully measured at each time step. However, in many practical systems, full state information may not always be available through measurement. RHC schemes based on a measured output have been explored in several works. [84] discusses a output-feedback scheme that combines a stable state estimator with a stabilizing controller for undisturbed linear systems. Such a scheme may not be applicable when the system is constrained and there are uncertainties in the system model. Several works have explored robust output-feedback schemes dealing with various forms of uncertainties in the system model, such as persistent disturbances and noises (e.g., [85–90]), structured or unstructured model uncertainties (e.g., [91], [92]) and both model uncertainties and disturbances (e.g., [93]). Most output-feedback schemes explored in the literature are based on their standard state-feedback counterparts and share similar features. Approaches used in output-feedback RHC include optimization of perturbations to a standard static

feedback ([86], [87], [89], [92]), on-line selection of a static feedback control law from a sequence of off-line determined laws ([91]), joint estimation and control [88] and on-line optimization of a dynamic output-feedback law [93]. We will explore some computationally attractive RHC schemes for systems with state disturbances and measurement noises in this thesis.

RHC in Applications: Since RHC has been widely used in industrial process control, works related to the application of RHC in control problems are extensive. Moreover, with continuous advances in the speed and availability of computing hardware, RHC is becoming increasingly popular in a wide range of control problems arising in diverse fields, and its applicability in several emerging applications are being thoroughly investigated. One such area relevant to this thesis is that of coordinated control problems involving dynamic agents interacting with each other for a common goal, such as problems in formation flying of spacecrafts [4] or unmanned air vehicles (UAVs) [94], ocean sampling by autonomous underwater vehicles (AUVs) [95], cooperative air traffic control [96] and so on. Distributed or decentralized RHC techniques have been investigated for applications in this and related areas [97–99]. Such techniques are presented for consensus problems involving dynamically decoupled subsystems in a number of works such as [100, 101] and [102]. We will explore the use of computationally efficient RHC schemes for the solution of a class of consensus related coordinated control problems, with a focus on the problems arising in satellite formation flying, in this thesis.

1.3 Contributions of the Thesis

This thesis explores RHC algorithms for control problems that require systematic handling of uncertainties and constraints with low on-line computations in various contexts. The studies are motivated by control applications in which the on-line control problem has to be solved with limited computational resources for a system in an uncertain, and possibly, dynamically changing environment. Examples of such applications include several multi-vehicle systems operating in remote locations, such as satellite formations. The results presented in this thesis, however, are not specific to the particular application that we have considered in detail.

Broadly, a major part of the studies in this thesis is aimed at understanding the use of the system ‘uncertainty’ information in the prediction of future control inputs for constrained systems with possibly multiple sources of uncertainty, and at developing novel techniques to address the fundamental issue of the tradeoff among feasibility, performance and computational complexity in the context of robust RHC. As we have mentioned in the last section, the effectiveness and optimality of any RHC policy is indicated by the correctness of the predicted system behaviour on which the computed control inputs are based. For a constrained system, the closeness between the predicted and the actual system behaviour depends on the chosen horizon length, and, when uncertainties are present,

it also depends on whether and how the information about the future uncertainty of the system is incorporated in the prediction of future controls. Although it has been recognized that an optimal (and not conservative) prediction can only be made by employing an optimization over a sequence of admissible feedback laws rather than a sequence of control actions, computational difficulties in doing so have led to ongoing investigations aimed at developing alternative strategies which are computationally simpler yet capable of ensuring the desired applicability and performance (see Section 2.2.3). The results on robust RHC that we present in this thesis are novel and important results in the context of these investigations.

Our results on robust RHC rely on the direct feeding of the information about system uncertainties into the control policy rather than through a (possibly time-varying) feedback from the state — an approach that allows us to not only avoid conservativeness (in terms of the size of the feasibility domain and the control performance) but also ensure computational tractability for a certain class of systems. These results demonstrate how all the system ‘uncertainty’ information can be fed into the controller in an infinite-horizon setting when multiple sources of uncertainty are present and how the receding horizon paradigm makes the resulting RHC policy implementable even without the knowledge of the uncertain quantities or disturbances. The study is focussed on a class of LTV systems with a polytopic system description and with (possibly unmeasurable) additive disturbances, for which extensive synthesis results are presented. For this class of systems, most of the available robust RHC techniques are not readily applicable and those that are applicable end up being either computationally intensive or too conservative. Although the focus of our study has been on a particular class of systems, the underlying concept is general and can be potentially employed in other relevant classes of systems. In a broader sense, the resulting approach can be seen as a kind of ‘virtual gain-scheduling’ where the (possibly uncertain) time varying parameters or quantities in the system description may never be known in real-time (in the way they are known in the case of actual gain-scheduling) but are only comfortably assumed to become known in the future so as to avoid conservativeness in the prediction of future inputs.

The basic robust RHC approach proposed in this thesis and its computational efficiency open up interesting possibilities in the control of certain classes of systems and in certain control applications. Results on some extensions of the basic approach and their application in some practical problems have also been included in this thesis. Some of the results of the study have been reported in our papers [103–110]. In the following, we summarize the specific contributions of the thesis.

1. For the class of (possibly uncertain) LTV systems with a polytopic system description and with (possibly unmeasurable) external disturbances, we have presented and analyzed a computationally attractive class of admissible dynamics for a dynamic RHC policy. The dynamics of the proposed policy involve time-varying controller matrices that need not be explicitly determined but just assumed to follow the

same convex combination as the plant matrices, and a disturbance feedforward term that does not require the disturbance to be really measured. The proposed policy, essentially, results in a sequence of dynamically evolving perturbations to the constant optimal static feedback input where all ‘uncertain’ system information is incorporated into the perturbation dynamics. This inclusion of the ‘uncertainty’ information into the perturbation dynamics leads to a less conservative utilization of the available control resources and a less conservative estimate of the feasibility domain. Furthermore, the proposed policy allows us to optimize the policy dynamics off-line so that the on-line computations, which only involve the optimization of the initial perturbation, are significantly simplified.

2. For a class of polytopic (but certain) systems with (possibly unmeasurable) external disturbances, the proposed dynamic policy provides a novel and very effective way to properly terminate the finite-horizon disturbance-based RHC policy (in which the control inputs are parameterized as affine functions of past disturbances) which offers computational tractability as well as reduced conservativeness (see e.g., [70,81]). Indeed, the proposed dynamic policy can be considered as an infinite-horizon counterpart of the disturbance-based control policy for this class of systems. When compared with the case of the conventional termination with a fixed optimal static feedback law, the use of the proposed policy as the terminal control law significantly increases the size of the terminal invariant set while allowing a mechanism to minimize the terminal cost. Consequently, the horizon length of the RHC problem and hence the on-line computational burden can be greatly reduced.
3. Apart from studying the off-line computation of the controller matrices of the proposed dynamic policy, we have presented stability results for various RHC schemes that use the proposed optimized dynamic policy for nominal as well as min-max (H_∞ -like) control performances. The performances of the RHC schemes have been studied in terms of ℓ_∞ or ℓ_2 stability of the controlled system. The min-max approach based on the proposed dynamic policy allows us to tune the gain parameter off-line so as to obtain a tradeoff between the size of the (terminal) feasible invariant set associated with the (terminal) controller matrices and the steady-state control performance. It also greatly simplifies the otherwise complex min-max or H_∞ RHC problem.
4. We have also investigated computationally efficient RHC algorithms based on an optimized dynamic policy for cases in which the state of the system is not fully known but has to be estimated using an observer. In particular, we have considered the use of a Luenberger observer, which estimates the state of the system from a measured output, in conjunction with an off-line-optimized policy whose dynamics include the feedforward of the output error signal besides ‘uncertain’ time-varying controller matrices. The dynamic policy reduces the conservativeness that would be associated with a deterministic prediction of future control actions feasible for all allowable plant disturbances, measurement noises and state estimation errors. We

have considered both the nominal and min-max cost minimization schemes based on the dynamic policy and presented stability results for both the cases.

5. On the application side, we have studied the use of the dynamic-policy-based RHC schemes in coordinated control problems, particularly a class of consensus problems in which the subsystems, whose dynamics are time-varying and include external disturbances, need to agree on a point of consensus in a decentralized way. The overall control strategy is required to achieve a suitably defined near-consensus condition, which is based either on the steady-state set inclusion or on the ℓ_2 gain bound from disturbances to the steady-state deviation of the members from the ideal consensus condition. RHC schemes are used as local control laws in the members and the consensus point is optimized distributedly using subgradient methods. Low computational complexity of these RHC schemes enable them to be used in consensus problems with a large feasibility domain for each of the members.
6. A final contribution of the thesis is related to the satellite formation control application. Formation flying of small satellites is a promising application being considered for various missions. We have considered the case of relative position control of satellites for formation reconfigurations as well as for formation keeping. System descriptions representing relative motion dynamics in satellite formations are usually time-varying and include disturbances which arise from external factors as well as modeling imperfections. We have explored the use of the dynamic-policy-based RHC algorithms for various relative position control problems arising in this application. Simulation studies show the effectiveness of the proposed control algorithms in providing a good performance with low complexity on-line computations.

1.4 Organization of the Thesis

The rest of the thesis is organized as follows. In Chapter 2, we introduce the robust RHC problem in a general setting. In the first part of the chapter, we give a general description of the system along with the uncertainty models and introduce the control objective as well as the ‘open-loop’ and ‘closed-loop’ RHC problems. In the second part of the chapter, we introduce some theoretical concepts and definitions related to feasibility, stability and set invariance, which will be extensively used in the rest of the thesis. We also discuss the features of the control law that ensure stability and robustness and broadly introduce various stabilizing robust RHC approaches.

In Chapter 3, we present our results on dynamic-policy-based RHC for a class of LTV systems. We introduce the proposed control parameterization and explore nominal-cost-based schemes which ensure ℓ_∞ -stability of the controlled system or an exponential stability of a minimal disturbance invariant set for the controlled system. Numerical examples are presented to assess the performance of the proposed schemes.

Min-max RHC schemes based on the proposed control parameterization are presented in Chapter 4. The performance of the schemes are assessed in terms of ℓ_2 stability of the controlled system and also studied with numerical examples.

We consider the case of output-feedback RHC in Chapter 5. The results in this chapter are based on the results in Chapters 3 and 4 and techniques similar to those used in the earlier chapters are used to assess the performance of the control schemes that we present.

In Chapter 6, we explore the dynamic-policy-based RHC schemes for their use in applications dealing with consensus-related coordinated control problems involving a class of subsystems with time-varying dynamics. We introduce two near-consensus conditions and propose computationally efficient control schemes which ensure that the subsystems eventually reach the desired near-consensus conditions.

In Chapter 7, we explore RHC-based solutions to the control problems in the area of satellite formation flying. We introduce the relative satellite dynamics and the control problems related to satellite formation keeping and formation reconfiguration. The use of dynamic-policy-based RHC schemes for the solution of the control problems is discussed and the results of detailed realistic nonlinear simulations of the control problems are presented.

Finally, in Chapter 8, we present the conclusions of this thesis. We also discuss a number of possible research problems for further investigation.

Chapter 2

Receding Horizon Control for Systems with Uncertainties

Receding horizon control laws usually perform nicely when the system models used in the design of control laws perfectly match the actual system dynamics. However, practical systems are often difficult to model accurately. Uncertainties do exist in system descriptions used for the design of control laws because of several factors such as imperfection in modeling and unavoidable external interferences. When hard constraints on the state and the input are present, it is necessary to make sure that the control problem remains feasible in the future for any allowable realization of uncertainties and that the control performance is satisfactory.

In this chapter, we discuss various aspects of the RHC problem when dealing with systems described by uncertain models. We introduce the robust RHC problem in a general setting, discuss the control objectives and mention some specific variants. We introduce and discuss some theoretical concepts and tools that are used in assessing the features of RHC algorithms and also give a brief overview of various approaches employed to ensure robustness with these algorithms.

2.1 Robust RHC Problem in a General Setting

2.1.1 System description

We express the dynamics of a general non-linear system using a discrete-time state-space model of the form:

$$x(t+1) = f(x(t), u(t), w(t), \varphi(t)) \quad (2.1a)$$

where $t \in \mathbb{Z}_+$ represents the discrete time steps, $x(t) \in \mathbb{R}^{n_x}$ denotes the state, $u(t) \in \mathbb{R}^{n_u}$ denotes the control input, and $w(t)$ and $\varphi(t)$ represent exogenous inputs. The exogenous

input $w(t)$, assumed to be bounded in a compact set $\mathcal{S}_w \subseteq \mathbb{R}^{n_w}$, represents external disturbances which may be unknown and unmeasurable. The other input $\wp(t)$, which satisfies $\wp(t) \in \mathcal{S}_\wp(t) \subseteq \Omega$ where Ω is a compact set, represents the (possibly uncertain) time-varying parameters describing f as a function of $x(t)$, $u(t)$ and $w(t)$ such that $f(0, 0, 0, \wp) = 0, \forall \wp \in \Omega$. The two exogenous input vectors are considered separately only for practical reasons. They can be expressed together as a combined exogenous input $\psi(t) = (\wp(t), w(t))$.

Together with the system dynamics, a performance output is usually considered:

$$z(t) = g_z(x(t), u(t)) \quad (2.1b)$$

where the function g_z satisfies $g_z(0, 0) = 0$. Other assumptions on g_z will be considered while defining the cost function in Section 2.1.4.

The dynamical model of the form (2.1a) can represent most practical dynamical systems. However, a system description of the form (2.1) is too general to be practically considered for control computations. More so in the case of RHC in which control computations are to be carried out in real time. It is therefore reasonable to consider some approximate, possibly conservative but relatively simpler, models instead of the actual system dynamics in the design of control laws. Indeed, most systems with common types of non-linearities and uncertainties can usually be approximated by uncertain forms of simpler linear models at least within a region around the operating point. A general linear time-varying (LTV) system can be described in the state-space form as

$$x(t+1) = A(t)x(t) + B_u(t)u(t) + B_w(t)w(t) \quad (2.2a)$$

where the variables $x(t)$, $u(t)$ and $w(t)$ have usual meanings as mentioned above. The system matrices in (2.2a) are time-varying and their time variation may be uncertain but they are assumed to satisfy the inclusion

$$\wp(t) = \begin{bmatrix} A(t) & B_u(t) & B_w(t) \end{bmatrix} \in \mathcal{S}_\wp(t) \subseteq \Omega \quad (2.2b)$$

where Ω is a compact set. Similarly, the performance output in the linear case can be expressed as

$$z(t) = C_z x(t) + D_{zu} u(t) \quad (2.2c)$$

where the matrices C_z and D_{zu} are appropriately defined.

2.1.2 Uncertainty models

In the system description of (2.1) or (2.2), uncertainties may arise from two sources - unknown external disturbances represented by $w(t)$ and model variations caused by the time-varying parameters represented by $\wp(t)$.

2.1.2.1 External disturbances

We usually model the external disturbances that are independent of the system state and the system input as $w(t)$. In the case of linear systems, they represent additive disturbances entering the system because of unavoidable external factors. $w(t)$ may be deterministic or stochastic and may be vanishing or non-vanishing. We usually assume $w(t)$ to be unknown and unmeasurable but bounded in a compact set, i.e., $w(t) \in \mathcal{S}_w$.

2.1.2.2 Model variations

The other possible source of uncertainty in (2.1) or (2.2) is the time variation of the system model defined by the model parameters which are represented by $\varphi(t)$. We assume that the parameters vary within defined limits and satisfy the inclusion $\varphi(t) \in \mathcal{S}_\varphi(t) \subseteq \Omega$, where $\mathcal{S}_\varphi(t)$ is either a known singleton in the set Ω or equal to the set Ω . In the latter case, we consider $\varphi(t)$ as uncertain and the resulting uncertainty in the system description as model uncertainty.

In the case of linear systems, two specific forms of model uncertainty characterized by the nature of the uncertainty set Ω in (2.2b) are commonly considered.

- a) Polytopic model uncertainty:** The system is known as a polytopic system when the uncertainty set Ω is a polytope. The polytope is usually defined as

$$\Omega = \text{Co} \left\{ \begin{bmatrix} A^{(j)} & B_u^{(j)} & B_w^{(j)} \end{bmatrix}, j = 1, \dots, n_\Omega \right\}. \quad (2.3)$$

Any particular realization of the system matrices belonging to the set Ω can be written as

$$\begin{bmatrix} A(t) & B_u(t) & B_w(t) \end{bmatrix} = \sum_{j=1}^{n_\Omega} \lambda_j \begin{bmatrix} A^{(j)} & B_u^{(j)} & B_w^{(j)} \end{bmatrix}$$

where

$$\sum_{j=1}^{n_\Omega} \lambda_j = 1; \quad \lambda_j \geq 0, j = 1, \dots, n_\Omega.$$

- b) Feedback uncertainty:** This kind of uncertainty arises when a feedback loop consists of an uncertain time-varying feedback gain. The system can be described as:

$$x(t+1) = Ax(t) + B_\varphi \varphi(t) + B_u u(t) + B_w w(t) \quad (2.4a)$$

$$\phi(t) = C_\phi x(t) + D_{\phi\varphi} \varphi(t) + D_{\phi u} u(t) + D_{\phi w} w(t) \quad (2.4b)$$

$$\varphi(t) = \Delta(t) \phi(t) \quad (2.4c)$$

$$z(t) = C_z x(t) + D_{zu} u(t) \quad (2.4d)$$

Figure 2.1: Schematic of feedback uncertainty in a linear system.

where $\Delta(t) = \text{diag}(\Delta_1(t), \dots, \Delta_r(t))$ is an uncertain time-varying block diagonal matrix with norm bounded blocks satisfying $\|\Delta_i(t)\| \leq 1 \forall t \in \mathbb{Z}_+$ for each block $\Delta_i(t)$. A schematic of the system is shown in Fig. 2.1. The uncertainty set Ω of the linear system (2.4) can be expressed as

$$\Omega = \left\{ \tilde{A} + B_\varphi \Delta (I - D_{\phi\varphi} \Delta)^{-1} \tilde{C} \mid \Delta = \text{diag}(\Delta_1, \dots, \Delta_r), \|\Delta_i\| \leq 1, i = 1, \dots, r \right\} \quad (2.5)$$

where

$$\tilde{A} = \begin{bmatrix} A & B_u & B_w \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C_\phi & D_{\phi u} & D_{\phi w} \end{bmatrix} \quad (2.6)$$

where we assume that $I - D_{\phi\varphi} \Delta$ is invertible for all allowable Δ . Each block in Δ can either be a full block or a repeated scalar block of the form $\delta_i(t) I_{r_i}$ with $|\delta_i(t)| \leq 1, \forall i \in \mathbb{Z}_+$. When there are repeated scalar blocks, the set Ω is accordingly defined and if Δ is diagonal, the convex hull of Ω is a polytope [111].

Various uncertain and/or non-linear systems arising in practical applications can often be modeled approximately, and possibly conservatively, as linear systems with polytopic uncertainty (2.3) or with feedback uncertainty (2.4). These uncertainty models have their own specific features and the choice of a particular model to represent a system will depend on various factors such as the actual or available information about the physical model of the system, the conservativeness of approximation and the computational complexity of the resulting model [56, 111, 112].

2.1.3 System constraints

Constraints arise in control problems because of operational and physical limitations that exist in practical systems. System constraints are usually expressed in the form of state and input constraints. We write the constraints collectively as

$$(x(t), u(t)) \in \Xi \quad (2.7)$$

where Ξ is a closed convex set. When no combined constraints are present, state and input constraints can be separately written as

$$x(t) \in \mathbb{X}, \quad u(t) \in \mathbb{U} \quad (2.8)$$

where $\mathbb{X}$ is a closed convex set and $\mathbb{U}$ is a compact convex set. It is usually common to consider a simple polyhedral representation of the constraint set Ξ expressed with linear inequalities in the form

$$M_x x(t) + M_u u(t) \leq \mathbf{1} \quad (2.9)$$

where M_x and M_u are matrices of suitable dimensions.

2.1.4 Control objective

As mentioned in Section 1.2.1.2, we ideally focus on the regulation problem, i.e., we consider that the objective of the control mechanism is to steer the system state to the origin in an optimal way. In cases where the actual control problem requires a controlled output $y_c(t) = g_c(x(t))$ to track a non-zero constant or time-varying reference signal $r(t)$, we assume that the reference signal is such that the problem can be transformed, through an appropriate change of coordinates and/or augmentation of the system dynamics with the reference signal dynamics, into a regulation problem with allowable constraints.

The optimal control strategy is to be obtained by minimizing a suitably defined cost function which, in some way, represents the cost of steering the state to the origin. In the RHC framework, therefore, ideally, the objective is to compute, at each time step t , the control inputs for the current and the future time steps such that they are optimal in the sense of minimizing a predicted cost function, which is of the form

$$J(t) = \sum_{i=0}^{\infty} \ell(z(t+i|t)) \quad (2.10)$$

where $z(t+i|t)$ is the predicted performance output and $\ell(z(t+i|t))$ represents the predicted stage cost for time step $t+i$. The stage cost function $\ell(z(t)) = \ell(g(x(t), u(t)))$ is assumed to satisfy

$$c_1(\|(x(t), u(t))\|) \leq \ell(g_z(x(t), u(t))) \leq c_2(\|(x(t), u(t))\|) \quad (2.11)$$

where c_1 and c_2 are class $\mathcal{K}$ functions (cf. Definition 2.6). The performance output $z(t)$ is usually defined as $z(t) = (Q^{\frac{1}{2}}x(t), R^{\frac{1}{2}}u(t))$, where Q and R are positive definite matrices selected as per the design criteria and the stage cost function $\ell(z(t))$ is commonly defined as the square of the 2-norm of $z(t)$.

The objective of driving the system state to the origin with control computations based on the cost function (2.10) may be realizable when the system dynamics are exactly

known and there are no external disturbances in the system description. However, when there are uncertainties in system dynamics, the cost in (2.10) will be uncertain, and, if the disturbance $w(t)$ is non-vanishing, the cost may not be finite and it may not be possible to guarantee the asymptotic regulation of the state to the origin. The alternative, therefore, is to efficiently compute the control inputs by evaluating and minimizing the cost in (2.10) for a nominal (disturbance-free) or a min-max (worst-case) performance and to apply the control inputs to the system so that the controlled system state is steered to a close neighbourhood of the origin.

- a) Nominal performance:** The nominal performance that we consider is characterized by a performance index that assumes a disturbance-free system. We define the nominal performance index as

$$\bar{J}(t) = \sum_{i=0}^{\infty} \ell(\bar{z}(t+i|t)) \quad (2.12)$$

where $\bar{z}(t+i|t)$ represents the predicted performance output for a disturbance-free system $\bar{x}(t+1) = (\bar{x}(t), u(t), 0, \varphi(t))$ for which $\{\bar{x}(t+i|t)\}_{i=0}^{\infty}$ and $\{\bar{u}(t+i|t)\}_{i=0}^{\infty}$ are the predicted state and input trajectories with $\bar{x}(t|t) = x(t)$. If the time variation of $\varphi(t)$ is uncertain, we either consider an upper bound of the right-hand side of (2.12) for all possible realizations of $\varphi(t) \in \Omega$, $t \in \mathbb{Z}_+$, or evaluate the right-hand side of (2.12) for an average or nominal value $\bar{\varphi}$ of $\varphi(t)$.

- b) Min-max performance:** The min-max performance that we consider is characterized by an H_{∞} -like performance index defined as

$$\hat{J}(t) = \max_{\substack{\varphi(t+i) \in \mathcal{S}_{\varphi}(t+i) \\ w(t+i) \in \mathcal{S}_w, i \in \mathbb{Z}_+}} \sum_{i=0}^{\infty} \{ \ell(z(t+i|t)) - \gamma^2 \ell_w(w(t+i)) \}, \quad \ell(\cdot) = \ell_w(\cdot) = \|\cdot\|^2 \quad (2.13)$$

where γ is a constant gain chosen *a-priori*.

In either of the cases of nominal and min-max performances, the goal is to compute and apply the control inputs in the receding horizon fashion in order to ensure that the system state is driven close to the origin. While disturbances are neglected in the nominal cost, they are accounted for in the case of the min-max performance index. In both the cases, however, it is ensured that the system constraints are strictly satisfied at all times for any allowable realization of uncertainties.

2.1.5 Control parameterization and computation

Although the cost function defined in (2.12) or (2.13) may be finite for any possible future realization of uncertainties, as mentioned in Section 1.2.1.2, it is not practical to explicitly

optimize the future control inputs for all time steps in the infinite time horizon. To make the problem amenable to practical solution, usually, the control inputs for only a finite time horizon are explicitly optimized on-line and the rest of the predicted control inputs are assumed to be based on a certain allowable terminal control law $\kappa_N(\cdot)$. The control law $\kappa_N(\cdot)$ is determined, either off-line or on-line, such that it minimizes an approximation or an upper bound of the terminal part of the relevant cost beyond the finite horizon, and also ensures that the predicted state and input trajectories of the controlled terminal system necessarily satisfy all system constraints provided the terminal state lies in an associated constraint admissible set. The cost function that is minimized on-line to obtain the finite-horizon control inputs usually include, apart from the finite-horizon cost, a terminal cost that approximates or upper bounds the predicted cost for the times after the finite time horizon. Therefore, typically, the overall performance function for a nominal performance is defined as

$$\bar{J}(t) = \sum_{i=0}^{N-1} \ell(\bar{z}(t+i|t)) + \bar{h}(\bar{x}(t+N|t); t) \quad (2.14)$$

where $\bar{h}(\bar{x}(t+N|t); t)$ approximates the predicted terminal nominal cost $\sum_{i=N}^{\infty} \ell(\bar{z}(t+i|t))$ for the nominal system under a terminal control law $\kappa_N(\cdot; t)$ for times $t+i, i \geq N$. And, for a min-max performance, the performance function has the form

$$\hat{J}(t) = \max_{\substack{\varphi(t+i) \in \mathcal{S}_{\varphi}(t+i) \\ w(t+i) \in \mathcal{S}_w, i=1, \dots, N-1}} \sum_{i=0}^{N-1} \{ \ell(z(t+i|t)) - \gamma^2 \ell_w(w(t+i)) \} + \bar{h}(x(t+N|t); t),$$

$$\ell(\cdot) = \ell_w(\cdot) = \|\cdot\|^2 \quad (2.15)$$

where $\bar{h}(x(t+N|t); t)$ upper bounds the predicted terminal cost for the uncertain system based on a terminal control law $\kappa_N(\cdot; t)$ for times $t+i, i \geq N$.

While minimizing the cost function (2.14) or (2.15), we need to ensure the satisfaction of all system constraints by the predicted state and input trajectories. This is typically done by explicitly considering the constraint

$$(x(t+i|t), u(t+i|t)) \in \Xi \quad (2.16)$$

for $i = 0, \dots, N-1$ and applying a terminal constraint

$$x(t+N|t) \in \mathcal{S}_{x_N, t} \quad (2.17)$$

where $\mathcal{S}_{x_N, t}$ is a convex compact set suitably chosen such that (2.16) is satisfied for all $i \geq N$ by the predicted state and input trajectories of the terminal controlled system under the control law $\kappa_N(\cdot; t)$ for all allowable realization of uncertainties.

In the cost functions (2.14) and (2.15), the predicted control inputs for time steps beyond the finite horizon are assumed to be parameterized as $u(t+i|t) = \kappa_N(x(t+i|t); t)$

$\forall i \geq N$. However, the predicted inputs for times $t = 1, \dots, N - 1$ can be freely optimized. The conventional way is to optimize the sequence of finite-horizon control actions $\{u(t + i|t)\}_{i=0}^{N-1}$ ‘open-loop’ without considering any feedback from the state. The open-loop robust RHC (RRHC) problem that assumes no feedback from the state can be stated in the following from.

2.1.5.1 Open-loop robust RHC problem

Nominal performance: Solve

$$\min_{\{u(t+i|t)\}_{i=0}^{N-1}, \kappa_N(\cdot; t)} \bar{J}(t) \quad \text{with } \bar{J}(t) \text{ defined in (2.14)} \quad (2.18a)$$

such that (2.16) and (2.17) are satisfied for any allowable realization of uncertainties.

Min-max performance: Solve

$$\min_{\{u(t+i|t)\}_{i=0}^{N-1}, \kappa_N(\cdot; t)} \hat{J}(t) \quad \text{with } \hat{J}(t) \text{ defined in (2.15)} \quad (2.18b)$$

such that (2.16) and (2.17) are satisfied for any allowable realization of uncertainties.

The open-loop RRHC problem is simple in its statement and the problem may be usually tractable. However, any solution to the problem, which is solely based on the knowledge of the current system state, requires that a single sequence of predicted future control inputs ensure the satisfaction of system constraints for all possible realizations of uncertainties in the future. This is artificial because, in reality, the actual control actions that will be implemented in the future time steps will be computed in the respective time steps with the full knowledge of the system state at those time instants. Indeed, any solution to the open-loop problem is conservative and the problem may unrealistically become unsolvable in many cases. As pointed out in [13], the non-conservative alternative to the open-loop solution of the constrained RRHC problem is to adopt a closed-loop approach and optimize over a sequence of feedback control laws $\{\kappa_i(\cdot; t)\}_{i=0}^{N-1}$ instead of the sequence of control actions $\{u(t + i|t)\}_{i=0}^{N-1}$. Since $x(0)$ is fully known, the first input $u(t|t)$ can, however, be directly optimized as a control action without determining the control law $\kappa_0(\cdot; t)$.

We state the closed-loop robust RHC problem in the following:

2.1.5.2 Closed-loop robust RHC problem

Nominal performance: Solve

$$\min_{u(t|t), \kappa_i(\cdot; t)_{i=1}^N} \bar{J}(t) \quad \text{with } \bar{J}(t) \text{ defined in (2.14)} \quad (2.19a)$$

such that, with $u(t+i|t)$ parameterized as $u(t+i|t) = \kappa_i(x(t+i|t); t)$, $i = 1, \dots, N-1$, (2.16) and (2.17) are satisfied for any allowable realization of uncertainties.

Min-max performance: Solve

$$\min_{u(t|t), \{\kappa_i(\cdot; t)\}_{i=1}^N} \hat{J}(t) \quad \text{with } \hat{J}(t) \text{ defined in (2.15)} \quad (2.19b)$$

such that, with $u(t+i|t)$ parameterized as $u(t+i|t) = \kappa_i(x(t+i|t); t)$, $i = 1, \dots, N-1$, (2.16) and (2.17) are satisfied for any allowable realization of uncertainties.

The closed-loop RRHC problems stated above consider an optimization over a set of permissible control policies in order to minimize either a nominal or a min-max cost. But solving such a problem on-line at each time step for a general uncertain system may be computationally prohibitive. To make the problem efficiently solvable, some approximations on the form of the control policies are often considered. Most RRHC schemes employ a combination of off-line and on-line computations to solve the RRHC problem. Typically, the terminal control law $\kappa_N(\cdot)$, the associated terminal constraint set $\mathcal{S}_{x_N}$ and the terminal cost function $\bar{h}(\cdot)$ or $\hat{h}(\cdot)$ are determined off-line whereas the control strategies for the finite horizon are predicted on-line. In any case, the terminal control law $\kappa_N(\cdot)$, the terminal constraint set $\mathcal{S}_{x_N}$ and the terminal cost function $\bar{h}(\cdot)$ or $\hat{h}(\cdot)$, along with the finite-horizon control policies, must satisfy several conditions in order to ensure that control objectives are met without violating any constraints. We will discuss these issues in the next section.

2.2 Solution of the Robust RHC Problem

As discussed in the previous section, an RRHC scheme usually involves both off-line and on-line computations. Off-line computations are basically meant to simplify the computations that will be carried out on-line in real time. In most RRHC schemes, off-line computations usually deal with the terminal part of the RHC problem, i.e., the terminal control law and the terminal constraint set. These factors play an important role in the theoretical assessment of the performance of the RRHC scheme. On-line computations, on the other hand, mostly deal with the finite-horizon part of the problem and determine

the control actions to be applied at each time step. A reliable RRHC scheme must ensure, with a suitable combination of off-line and on-line computations, that the scheme achieves the control objectives with computable and implementable control actions for a desired set of initial conditions of the system state.

As with any other control scheme, the primary concern with an RRHC scheme is the stability of the controlled system under the scheme in the presence of uncertainties. Some other issues of concern include the steady-state performance, the size of the set of initial system states for which the control law is applicable and the on-line computational complexity of the algorithm. In this section, we first introduce some terminology and concepts that are used in the assessment of the performance of an RHC control scheme, and then present some common approaches to the efficient solution of the RRHC problem, especially for linear systems with uncertainties.

2.2.1 Preliminaries

In this section, we introduce some background information on feasibility, invariant sets and stability.

2.2.1.1 Feasibility

Since an RHC scheme determines the control actions to be applied in real time by solving an optimization problem, whether or not a solution to the optimization problem (say, (2.18) or (2.19)) exists is matter of importance. Feasibility of an RHC problem refers to the existence of a solution to the problem and hence the existence of implementable control actions.

Definition 2.1 (Feasibility or Solvability). An RHC optimization problem is said to be feasible or solvable at time t if there exists a solution to the problem for the current system initial state $x(t)$. $\square$

The applicability of an RHC scheme depends not only on the feasibility of the control problem at a given time t but also on the feasibility of the control problem for the predicted future states of the controlled system.

Definition 2.2 (Recursive feasibility or Resolvability). An RHC optimization problem is said to be recursively feasible or resolvable for a system $x(t+1) = f(x(t), u(t), \psi(t))$ if, given the problem is feasible at time t for the current state $x(t)$, it is necessarily feasible at time $t+1$ for any possible value of the subsequent state $x(t+1)$ that results with any feasible control input $u(t)$ and any allowable uncertainty $\psi(t)$ at time t . $\square$

Recursive feasibility, as defined in Definition 2.2, ensures future feasibility in the presence of uncertainties. So, it is sometimes also referred to as robust feasibility or robust resolvability. It plays an important role in proving stability of RHC and RRHC laws.

2.2.1.2 Set invariance

The concept of set invariance is closely related with the idea of stability and Lyapunov functions. Invariant sets are extensively used in the solution of RHC problems in order to establish a theoretical guarantee of recursive feasibility and stability of the control law. In this section, we introduce invariant sets with formal definitions.

Definition 2.3 (Positive invariance). A set $\mathcal{S}_x \subseteq \mathbb{R}^{n_x}$ is said to be positively invariant for a system $x(t+1) = f(x(t))$ if for all $x(0) \in \mathcal{S}_x$, the inclusion $x(t) \in \mathcal{S}_x$ holds for all $t \in \mathbb{Z}_+$.

A set $\mathcal{S}_x \subseteq \mathbb{R}^{n_x}$ is said to be positively controlled invariant for a system $x(t+1) = f(x(t), u(t))$ if for all $x(0) \in \mathcal{S}_x$, there exists a control $u(t) = \kappa(x(t))$ such that the inclusion $x(t) \in \mathcal{S}_x$ holds for all $t \in \mathbb{Z}_+$. $\square$

Clearly, a set $\mathcal{S}_x$ is positively invariant for the system $x(t+1) = f(x(t))$ if $f(x) \in \mathcal{S}_x \forall x \in \mathcal{S}_x$, and it is positively controlled invariant for the system $x(t+1) = f(x(t), u(t))$ if there exists a control $u(t) = \kappa(x(t))$ such that $f(x, \kappa(x)) \in \mathcal{S}_x \forall x \in \mathcal{S}_x$.

When uncertainties are present in the system model, we use the concept of robust invariant sets.

Definition 2.4 (Robust positive invariance). A set $\mathcal{S}_x \subseteq \mathbb{R}^{n_x}$ is said to be robust positively invariant for a system $x(t+1) = f(x(t), \psi(t))$ in the presence of uncertainties $\psi(t) \in \mathcal{S}_\psi \subseteq \mathbb{R}^{n_\psi}$ if for all $x(0) \in \mathcal{S}_x$ and for all $\psi(t) \in \mathcal{S}_\psi$ ($t \in \mathbb{Z}_+$), the inclusion $x(t) \in \mathcal{S}_x$ holds for all $t \in \mathbb{Z}_+$.

A set $\mathcal{S}_x \subseteq \mathbb{R}^{n_x}$ is said to be robust positively controlled invariant for a system $x(t+1) = f(x(t), u(t), \psi(t))$ in the presence of uncertainties $\psi(t) \in \mathcal{S}_\psi \subseteq \mathbb{R}^{n_\psi}$ if for all $x(0) \in \mathcal{S}_x$ and for all $\psi(t) \in \mathcal{S}_\psi$ ($t \in \mathbb{Z}_+$), there exists a control $u(t) = \kappa(x(t), \psi(t))$ such that the inclusion $x(t) \in \mathcal{S}_x$ holds for all $t \in \mathbb{Z}_+$. $\square$

Also in this case, the set $\mathcal{S}_x$ is robust positively invariant for $x(t+1) = f(x(t), \psi(t))$ if $f(x, \psi) \in \mathcal{S}_x \forall x \in \mathcal{S}_x$ and $\forall \psi \in \mathcal{S}_\psi$, and it is robust positively controlled invariant for the system $x(t+1) = f(x(t), u(t), \psi(t))$ if there exists a control $u(t) = \kappa(x(t), \psi(t))$ such that $f(x, \kappa(x, \psi), \psi) \in \mathcal{S}_x \forall x \in \mathcal{S}_x$ and $\forall \psi \in \mathcal{S}_\psi$.

A closely related concept relevant to the solution of constrained RHC problems is that of maximal admissible sets [49].

Definition 2.5 (Maximal admissible set). The maximal admissible set for a constrained system $x(t+1) = f(x(t), u(t), \psi(t))$, $(x(t), u(t)) \in \Xi$ under a control law $u(t) = \kappa(x(t), \psi(t))$ in the presence of uncertainties $\psi(t) \in \mathcal{S}_\psi \subseteq \mathbb{R}^{n_\psi}$ is defined as the set of all initial states $x(0) \in \mathbb{R}^{n_x}$ such that for all $\psi(t) \in \mathcal{S}_\psi$ ($t \in \mathbb{Z}_+$), the corresponding state and input sequences satisfy the constraint $(x(t), u(t)) \in \Xi$, $\forall t \in \mathbb{Z}_+$. $\square$

For a discrete-time system, the maximal admissible set is the maximal robust positively controlled invariant set [49].

Invariant sets form an important set of tools widely used in the solution of RHC problems. They allow the design of RHC schemes which can provide a theoretical guarantee of stability and robustness. Invariant sets are usually represented in the form of polyhedral sets or ellipsoidal sets. Both representations have their own advantages and disadvantages. A polyhedral set can be described using a number of linear constraints in the form

$$\mathcal{S}_{\mathcal{P}} = \{x \mid \mathcal{M}x \leq \mathbf{1}\} \quad (2.20)$$

where $\mathcal{M} \in \mathbb{R}^{n_{\mathcal{M}} \times n_x}$.

An ellipsoidal set, on the other hand, can be represented as

$$\mathcal{S}_{\mathcal{E}} = \{x \mid x^T W^{-1} x \leq 1\} \quad (2.21)$$

where W is a positive definite matrix.

Polyhedral sets usually give more natural representations of the invariant sets for most practical systems but they may be more complex in computational aspects. Ellipsoidal sets, on the other hand, are conservative approximations of invariant sets in most cases but are usually suitable for efficient computations. See [113] for the details on various aspects of invariant sets.

2.2.1.3 Stability

Stability is an important aspect of any dynamical system. It is studied in various ways in different contexts. For systems without exogenous inputs, asymptotic or exponential stability of the equilibrium point is usually studied using Lyapunov techniques. When there are exogenous inputs, stability concepts such as input-to-state stability or input-output stability are usually considered. Although the various techniques are closely related, they offer different insights about the stability of the underlying dynamical system.

Various tools are available to assess the stability of linear and non-linear systems (see e.g., [114]). In the case of RHC, the presence of system constraints usually makes the controlled system behave in a non-linear way even if the open-loop system is linear. So, stability properties of constrained RHC schemes are generally studied using techniques which are suitable for the study of non-linear systems. In the following, we introduce a few stability definitions and some relevant results that will be used in the assessment of the stability properties of RHC schemes that we explore in this thesis. We first introduce some special comparison functions which will be used in the stability definitions to follow.

Definition 2.6 (Comparison functions). A continuous function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be *positive definite* if $\alpha(0) = 0$ and $\alpha(r) > 0$ for all $r > 0$.

A continuous function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to belong to *class* $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if, in addition, $\gamma(r) \rightarrow \infty$ as $r \rightarrow \infty$.

A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to belong to *class* $\mathcal{KL}$ if for each fixed k , $\beta(r, k)$ belongs to class $\mathcal{K}$ w.r.t. r and for each fixed r , $\beta(r, k)$ is decreasing w.r.t. k with $\beta(r, k) \rightarrow 0$ as $k \rightarrow \infty$. $\square$

Definition 2.7 (Asymptotic / exponential stability). The equilibrium point $x = 0$ of a system

$$x(t+1) = f(x(t), t) \quad (2.22)$$

is asymptotically stable with a region of attraction $\mathcal{S}_x$ if, with any initial state $x(0) \in \mathcal{S}_x$, $x(t) \rightarrow 0$ as $t \rightarrow \infty$. The equilibrium point $x = 0$ of (2.22) is exponentially stable with a region of attraction $\mathcal{S}_x$ if there exists a $c > 0$ and a $\rho \in (0, 1)$ such that with any initial state $x(0) \in \mathcal{S}_x$, the solution $x(t)$ of the system satisfies $\|x(t)\| \leq c\rho^t x(0)$ for all $t \geq 0$. $\square$

Lemma 2.1 (Lyapunov function and stability). The equilibrium point $x = 0$ of system (2.22) is asymptotically stable with a region of attraction $\mathcal{S}_x \in \mathbb{R}^{n_x}$ if there exists a function $V : \mathcal{S}_x \times \mathbb{Z}_+ \rightarrow \mathbb{R}_+$, called a Lyapunov function, satisfying, for all $\vartheta \in \mathcal{S}_x$ and for all $t \in \mathbb{Z}_+$,

$$\begin{aligned} \gamma_1(\|\vartheta\|) &\leq V(\vartheta, t) \leq \gamma_2(\|\vartheta\|) \\ V(f(\vartheta, t), t+1) - V(\vartheta, t) &\leq -\gamma_3(\|\vartheta\|) \end{aligned} \quad (2.23)$$

for some positive definite functions γ_1, γ_2 and γ_3 .

The equilibrium point $x = 0$ of (2.22) is exponentially stable with a region of attraction $\mathcal{S}_x \in \mathbb{R}^{n_x}$ if there exists a function $V : \mathcal{S}_x \times \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ satisfying, for all $\vartheta \in \mathcal{S}_x$ for all $t \in \mathbb{Z}_+$,

$$\begin{aligned} \gamma_1\|\vartheta\|^\rho &\leq V(\vartheta, t) \leq \gamma_2\|\vartheta\|^\rho \\ V(f(\vartheta, t), t+1) - V(\vartheta, t) &\leq -\gamma_3\|\vartheta\|^\rho \end{aligned} \quad (2.24)$$

for some constants $\gamma_1, \gamma_2, \gamma_3 > 0$ and $\rho > 1$. $\square$

When disturbance inputs are present, one cannot guarantee the asymptotic stability of the origin as in the case of systems without disturbance inputs. In such cases, stability is usually assessed using the notion of input-to-state stability or input-output stability. In the latter case, a suitably defined output signal is considered for analysis. To be specific, we consider a system

$$x(t+1) = f(x(t), w(t), t) \quad (2.25a)$$

$$z(t) = g(x(t), w(t), t) \quad (2.25b)$$

where $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \times \mathbb{Z}_+ \rightarrow \mathbb{R}^{n_x}$ and $h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \times \mathbb{Z}_+ \rightarrow \mathbb{R}^{n_y}$ are continuous with f satisfying $f(0, 0, t) = 0, \forall t \in \mathbb{Z}_+$.

Definition 2.8 (Input-to-state stability - ISS). The system (2.25a) is said to be input-to-state stable with a region of attraction $\mathcal{S}_x \in \mathbb{R}^{n_x}$ if there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for all initial states $x(0) \in \mathcal{S}_x$, and for all disturbance sequences $\{w(t)\}_{t=0}^\infty$ satisfying $w(t) \in \mathcal{S}_w$, $t \in \mathbb{Z}_+$, the solution $x(t)$ of the system, for all $t \in \mathbb{Z}_+$, satisfies $x(t) \in \mathcal{S}_x$ and

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma\left(\sup_{i \in \{0, \dots, t-1\}} \|w(i)\|\right). \quad (2.26)$$

The ISS property stated by (2.26) implies that the system state trajectories will remain bounded for all allowable disturbance sequences $\{w(t)\}_{t=0}^\infty$ and that the equilibrium point $x = 0$ of the undisturbed system $x(t+1) = f(x(t), 0, t)$ is uniformly asymptotically stable with a region of attraction $\mathcal{S}_x$ [70, 114, 115]. In order to establish the ISS property of a system, ISS Lyapunov functions are commonly used. The following lemma (see [115, Lemma 3.5]) gives a Lyapunov-like-function-based sufficient condition for ISS.

Lemma 2.2 (ISS Lyapunov function). The system (2.25a) is input-to-state stable with a region of attraction $\mathcal{S}_x \in \mathbb{R}^{n_x}$ if $\mathcal{S}_x$ is positively invariant for the system and if there exists a function $V : \mathcal{S}_x \times \mathbb{Z}_+ \rightarrow \mathbb{R}_+$, called an ISS Lyapunov function, satisfying, for all $\vartheta \in \mathcal{S}_x$ for all $t \in \mathbb{Z}_+$,

$$\begin{aligned} \gamma_1(\|\vartheta\|) &\leq V(\vartheta, t) \leq \gamma_2(\|\vartheta\|) \\ V(f(\vartheta, t), t+1) - V(\vartheta, t) &\leq -\gamma_3(\|\vartheta\|) + \gamma(\|w\|) \end{aligned} \quad (2.27)$$

for some class $\mathcal{K}_\infty$ functions γ_1, γ_2 and γ_3 , and some class $\mathcal{K}$ function γ . $\square$

When a suitable output signal $z(t)$ is defined and the system has to be assessed by studying inputs and outputs, one usually considers input-output stability. Broadly speaking, if, for a ‘well-behaved’ input to a system, the output of the system is ‘well-behaved’ in the sense that it belongs to the same normed space (albeit possibly of a different dimension) as the input, then the system is considered to be input-output stable. We have a formal definition in the following.

Definition 2.9 (ℓ_p -Stability). A system described by the state-update and output equations in (2.25) is ℓ_p -stable, if, for each input sequence $w \in \ell_p$, there exist a $\mathcal{K}$ function γ and a nonnegative constant β such that the output sequence $z \in \ell_p$ satisfies

$$\|z\|_{\ell_p} \leq \gamma(\|w\|_{\ell_p}) + \beta. \quad (2.28)$$

And, it is finite gain ℓ_p - stable if there exist nonnegative constants β and γ such that

$$\|z\|_{\ell_p} \leq \gamma\|w\|_{\ell_p} + \beta. \quad (2.29)$$

As in the case of ISS, ℓ_p stability of a system can be established with the use of a suitable Lyapunov-like function. Here we present the following result (see [114, Chap 4] for a similar result in the continuous-time case) which deals with the case of ℓ_∞ -stability.

Lemma 2.3 (ℓ_∞ -Lyapunov function). Let $\mathcal{S}_x \in \mathbb{R}^{n_x}$ be a positively invariant set for the state of the system in (2.25) for all $w(t) \in \mathcal{S}_w$ and let it contain the origin. Suppose that the origin is a uniformly asymptotically stable equilibrium point of the undisturbed system

$$\bar{x}(t+1) = f(\bar{x}(t), 0, t) \quad (2.30)$$

and there exists a Lyapunov function $V(\vartheta, t)$ satisfying, for all $\vartheta \in \mathcal{S}_x$ and for all $t \in \mathbb{Z}_+$,

$$\gamma_1(\|\vartheta\|) \leq V(\vartheta, t) \leq \gamma_2(\|\vartheta\|) \quad (2.31a)$$

$$V(f(\vartheta, 0, t+1)) - V(\vartheta, t) \leq -\gamma_3(\|\vartheta\|) \quad (2.31b)$$

$$|V(\vartheta_1, t) - V(\vartheta_2, t)| \leq \gamma_4(\|\vartheta_1 - \vartheta_2\|) \quad (2.31c)$$

for some class $\mathcal{K}$ functions $\gamma_1, \gamma_2, \gamma_3$ and γ_4 . Also, suppose that the functions f and h satisfy, for all $\vartheta \in \mathcal{S}_x, \omega \in \mathcal{S}_w$ and $t \in \mathbb{Z}_+$,

$$\|f(\vartheta, \omega, t) - f(\vartheta, 0, t)\| \leq \gamma_5(\|\omega\|) \quad (2.32a)$$

$$\|h(\vartheta, \omega, t)\| \leq \gamma_6(\|\vartheta\|) + \gamma_7(\|\omega\|) + \varsigma \quad (2.32b)$$

for some class $\mathcal{K}$ functions γ_5, γ_6 and γ_7 and a non-negative constant ς . Then, for all $x(0) \in \mathcal{S}_x$, system (2.25) is ℓ_∞ -stable. $\square$

PROOF. From relations (2.31b), (2.31c) and (2.32a), we have

$$\begin{aligned} V(f(x(t), w(t), t), t+1) - V(x(t), t) \\ &= V(f(x(t), 0, t), t+1) - V(x(t), t) \\ &\quad + V(f(x(t), w(t), t), t+1) - V(f(x(t), 0, t), t+1) \\ &\leq -\gamma_3(\|x(t)\|) + \gamma_8(\|w(t)\|) \end{aligned}$$

where $\gamma_8 = \gamma_4 \circ \gamma_5$ is a class $\mathcal{K}$ function. Then, following the result of Lemma 2.2, we note that the closed-loop system in (2.25) is input-to-state stable and satisfies condition (2.26) for a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ . From condition (2.26), using the relation (2.32b), and noting the fact that $\gamma(a+b) \leq \gamma(2a) + \gamma(2b)$ for any class $\mathcal{K}$ function γ , we have

$$\|z(t)\| \leq \gamma_6(2\beta(\|x(0)\|, t)) + \gamma_6\left(2\gamma\left(\sup_{i \in \{0, \dots, t-1\}} \|w(i)\|\right)\right) + \gamma_7(\|w(t)\|) + \varsigma$$

from which we conclude

$$\|z\|_{\ell_\infty} \leq \gamma_0(\|w\|_{\ell_\infty}) + \beta_0$$

where $\beta_0 = \gamma_6(2\beta(\|x(0)\|, 0)) + \varsigma$, and $\gamma_0 = \gamma_6 \circ 2\gamma + \gamma_7$. $\blacksquare$

2.2.2 Features of a stabilizing RRHC solution

In RHC schemes, the cost function is normally used as the Lyapunov function to establish stability. Stability is achieved by effecting a monotonicity condition in the receding horizon cost with the use of a suitable combination of a terminal cost function and a terminal constraint set. In this section, we discuss the features of the terminal cost function and the terminal constraint set employed in a stabilizing robust RHC scheme. Since we are dealing with systems with uncertainties, we only consider the closed-loop RRHC schemes.

2.2.2.1 Disturbance-free systems

We first consider the case of a disturbance-free system. When there are no disturbances (i.e., $w(t) = 0, \forall t \in \mathbb{Z}_+$), assuming that the nominal cost in (2.19a) is an upper bound of the cost in case of uncertain $\wp(t) \in \Omega$, the min-max cost function (2.19b) is the same as the nominal cost function (2.19a). Let us consider the following assumptions:

$$1 : (x, \kappa_N(x; t)) \in \Xi, \forall x \in \mathcal{S}_{x_N, t} \quad (2.33a)$$

$$2 : f(x, \kappa_N(x; t), 0, \wp) \in \mathcal{S}_{x_N, t}, \forall x \in \mathcal{S}_{x_N, t} \ \& \ \forall \wp = \Omega \quad (2.33b)$$

$$3 : \bar{h}(f(x, \kappa_N(x; t), 0, \wp); t) - \bar{h}(x; t) + \ell(g_z(x, \kappa_N(x; t))) \leq 0, \\ \forall x \in \mathcal{S}_{x_N, t} \ \& \ \forall \wp = \Omega. \quad (2.33c)$$

Here, assumptions (2.33a-b) imply that $\mathcal{S}_{x_N, t}$ is a feasible robust invariant set for the uncertain system $x(t+1) = f(x(t), u(t), 0, \wp(t))$, $\wp(t) \in \Omega$ under the control law $u(t) = \kappa_N(x(t); t)$. (2.33c) implies that $\bar{h}(\cdot, \cdot)$ is a local Lyapunov function for the uncertain system. These conditions are generally sufficient to establish the asymptotic stability of the closed loop system under RHC based on the problem (2.19a).

Proposition 2.1. Let $w(t) = 0, \forall t \in \mathbb{Z}_+$ in (2.1a) or (2.2a). Also, let $\mathcal{S}_x$ be the set of all initial states $x(0)$ for which the optimization problem (2.19a) is feasible under the conditions in (2.33). Then, for any initial state $x(0) \in \mathcal{S}_x$, the receding horizon control of the form $u^*(t) = u^*(t|t)$ obtained by solving the problem (2.19a) under the conditions in (2.33) at each time step t ensures that the problem remains feasible for all successive times and that the closed loop system is asymptotically stable with a region of attraction $\mathcal{S}_x$. $\square$

PROOF. The proof follows from arguments based on standard cost monotonicity conditions (see, e.g., [13]). Conditions (2.33a-b) ensure that if $\{u^*(t|t), \{\kappa_i^*(\cdot; t)\}_{i=1}^N\}$ is an optimal control policy obtained from the solution of the problem at time t , then a policy $\{\kappa_1^*(x(t+1); t), \{\kappa_i^*(\cdot, t)\}_{i=2}^N, \kappa_N^*(\cdot, t)\}$ will be feasible at time $t+1$ satisfying conditions in (2.33) with $\bar{h}(\cdot; t+1) = \bar{h}(\cdot; t)$ and $\mathcal{S}_{x_N, t+1} = \mathcal{S}_{x_N, t}$. This fact, combined with the condition (2.33c), ensures that the optimal costs at times t and $t+1$ satisfy

$$\bar{J}^*(t+1) \leq \bar{J}^*(t) - \ell(z(t)). \quad (2.34)$$

Clearly, the above recursive relation implies that for an initial state $x(0) \in \mathcal{S}_x$, $\bar{J}^*(t)$ converges to a constant value and $\ell(z(t)) \rightarrow 0$ as $t \rightarrow \infty$. It then follows from (2.11) that $z(t) \rightarrow 0$ and $x(t) \rightarrow 0$ as $t \rightarrow \infty$. $\blacksquare$

Conditions (2.33a-c) are the key conditions that ensure a theoretical guarantee of the stability of the RHC problem we have considered. Also relevant is the definition of the stage cost function $\ell(\cdot)$. If (2.11) is strengthened to

$$c_1 \|(x(t), u(t))\|^2 \leq \ell(g_z(x(t), u(t))) \leq c_2 \|(x(t), u(t))\|^2 \quad (2.35)$$

where c_1 and c_2 are positive constants, the closed loop system under the RHC scheme considered in Proposition 2.1 can be shown to be exponentially stable.

2.2.2.2 Systems with disturbances

We next consider the case of systems with non-zero disturbances. As mentioned in an earlier section, when there are non-vanishing disturbances, it is usually not possible to guarantee asymptotic regulation of the system state to the origin. We therefore consider robust RHC schemes based on nominal or min-max cost functions and assess their performance using ISS or ℓ_p -stability definitions.

Let us consider the following set of assumptions:

$$1 : (x, \kappa_N(x; t)) \in \Xi, \forall x \in \mathcal{S}_{x_N, t} \quad (2.36a)$$

$$2 : f(x, \kappa_N(x; t), w, \wp) \in \mathcal{S}_{x_N, t}, \forall x \in \mathcal{S}_{x_N, t}, \forall w \in \mathcal{S}_w \text{ \& } \forall \wp = \Omega \quad (2.36b)$$

$$3 : \bar{h}(f(x, \kappa_N(x; t), 0, \wp); t) - \bar{h}(x; t) + \ell(g_z(x, \kappa_N(x; t))) \leq 0, \\ \forall x \in \mathcal{S}_{x_N, t} \text{ \& } \forall \wp = \Omega \quad (2.36c)$$

$$4 : \bar{h}(f(x, \kappa_N(x; t), w, \wp); t) - \bar{h}(x; t) + \ell(g_z(x, \kappa_N(x; t))) - \gamma^2 \ell_w(w) \leq 0, \\ \ell(\cdot) = \ell_w(\cdot) = \|\cdot\|^2, \quad \forall x \in \mathcal{S}_{x_N, t}, \forall w \in \mathcal{S}_w \text{ \& } \forall \wp = \Omega. \quad (2.36d)$$

Proposition 2.2. Let $\mathcal{S}_x$ be the convex compact set of all initial states $x(0)$ for which the optimization problem (2.19a) is feasible under the conditions in (2.36a-c) and has a unique optimal solution. Then, for any initial state $x(0) \in \mathcal{S}_x$, the receding horizon control of the form $u^*(t) = u^*(t|t)$ obtained by solving the problem (2.19a) under the conditions in (2.36a-c) at each time step t ensures that the problem remains feasible for all successive times and that the closed loop system is input-to-state (from $w(t)$ to $x(t)$) stable with the region of attraction $\mathcal{S}_x$.

Moreover, with the output $z(t)$ defined as in (2.1b) or (2.2c), the controlled system is ℓ_∞ -stable with $w(t)$ as input and $z(t)$ as output. $\square$

PROOF. Conditions (2.36a-b) ensure that, if $\{u^*(t|t), \{\kappa_i^*(\cdot; t)\}_{i=1}^N\}$ is an optimal control policy obtained from the solution of the problem at time t , then a policy $\{\kappa_1^*(x(t+1); t), \{\kappa_i^*(\cdot, t)\}_{i=2}^N, \kappa_N^*(\cdot, t)\}$ will be feasible at time $t+1$ satisfying

conditions in (2.36a-c) with $\bar{h}(:, t+1) = \bar{h}(:, t)$ and $\mathcal{S}_{x_N, t+1} = \mathcal{S}_{x_N, t}$. So, since the optimization problem is feasible at time $t = 0$, it will be feasible for all times $t \in \mathbb{Z}_{++}$. It, therefore, follows from (2.11) and (2.36c) that the cost function $\bar{J}^*(t)$ is a Lyapunov function for the undisturbed system and that the conditions (2.31a) and (2.31b) are satisfied for some class $\mathcal{K}$ functions $\gamma_1, \gamma_2, \gamma_3$. Moreover, since $\mathcal{S}_x$ is convex and compact and the optimization problem has a unique solution for any $x(t)$ in $\mathcal{S}_x$, condition (2.31c) is satisfied for some class $\mathcal{K}$ function γ_4 , and conditions (2.32a) and (2.32b) are satisfied by the controlled closed-loop system for some class $\mathcal{K}$ functions γ_5, γ_6 and γ_7 and some constant $\varsigma \geq 0$. Proposition 2.2, then follows from the result of Lemma 2.3. $\blacksquare$

Proposition 2.3. Let $\mathcal{S}_x$ be the convex compact set of all initial states $x(0)$ for which the optimization problem (2.19b) is feasible under the conditions in (2.36a-b,d). Then, for any initial state $x(0) \in \mathcal{S}_x$, the receding horizon control of the form $u^*(t) = u^*(t|t)$ obtained by solving the problem (2.19b) under the conditions in (2.36a-b,d) at each time step t ensures that the problem remains feasible for all successive times and that the closed loop system satisfies

$$\sum_{t=0}^{\infty} \{ \|z(t)\|^2 - \gamma^2 \|w(t)\|^2 \} \leq J^*(0) \quad (2.37)$$

so that it is ℓ_2 -stable (from $w(t)$ to $z(t)$) with a finite gain γ and a region of attraction $\mathcal{S}_x$. $\square$

PROOF. Since $x(0) \in \mathcal{S}_x$, the optimization problem (2.19b) subject to conditions (2.36a-b,d) is feasible at time $t = 0$. As mentioned in the proof of Proposition 2.2, conditions (2.36a-b) ensure that given a feasible control policy at some time t , there exists a feasible solution at time $t+1$ satisfying conditions (2.36a-b,d) with $\bar{h}(:, t+1) = \bar{h}(:, t)$ and $\mathcal{S}_{x_N, t+1} = \mathcal{S}_{x_N, t}$. This fact, combined with the condition (2.36d), ensures that the optimal costs at times t and $t+1$ satisfy

$$\bar{J}^*(t+1) \leq \bar{J}^*(t) - (\|z(t)\|^2 - \gamma^2 \|w(t)\|^2). \quad (2.38)$$

Applying the above recursive relation for $t = 0$ to $t = \infty$, we get (2.37) and hence ℓ_2 -stability of the closed-loop system with a finite gain γ . $\blacksquare$

2.2.3 Practical approaches to the solution of RRHC problem

As mentioned in Section 2.1.5, the closed-loop RRHC problems stated in (2.19) consider an optimization over a set of permissible control policies. The closed-loop approach is chosen over the open-loop approach of (2.18) in the case of systems with uncertainties because the open-loop approach can be too conservative and may cause unrealistic unsolvability problems. Moreover, it may be difficult to establish stability results employing

cost monotonicity conditions with an open-loop RHC scheme in the case of uncertain systems. The closed-loop approach allows realistic predictions of future control and state trajectories and makes the RHC solution feasible over a larger set of initial conditions.

The nice performance of the closed-loop approach combined with its applicability over a larger domain of feasibility comes with the complexity of optimizing the control policy over a set of admissible policies at each time step. The computational complexity of such an optimization is usually prohibitive for most systems including linear systems and some approximations need to be made for practically implementable closed-loop RHC schemes. Most practical schemes consider an optimization over linear state-feedback control laws and, even with linear feedback, consider some assumptions to make the problem computationally tractable. In particular, a part of the optimization problem is solved off-line so that simplified on-line computations can be employed to achieve a reasonably good control performance. In the following, we discuss some of the approaches that allow computationally implementable solutions to the closed-loop RRHC problem.

1. **Explicit off-line solution:** In RRHC schemes which consider an explicit off-line solution, the optimization problem is solved off-line for a given range of initial conditions using multi-parametric programming methods and the optimal solution is expressed as an explicit function of the state (see [20] for a survey of explicit RHC methods). The on-line computation required, therefore, is expected to be considerably simplified, since it simply consists of a lookup for a solution – which is usually a piece-wise affine control law in the case of uncertain linear systems [74] – followed by a simple function evaluation. With this approach, the simplification of the on-line computations can be significant in the case of low order deterministic systems. With higher order systems and multiple sources of uncertainty, however, the form of the explicit solution may be significantly complex, possibly increasing the storage requirements and the on-line search complexity exponentially with the order of the system and the number of constraints [74], [79], [60] unless some suboptimal approximations are considered.
2. **Solution with identical state-feedback:** The closed-loop RRHC problem can be simplified if an identical state-feedback is considered for the infinite horizon. Such an approach does away with the finite-horizon part of the problem and only requires the determination of the control law $\kappa_N(\cdot; t)$ along with $\mathcal{S}_{x_N, t}$ and $\bar{h}(\cdot; t)$ with $N = 0$ at each time step t . In particular, for linear systems with polytopic or structured model uncertainty, assuming a quadratic stage cost, an identical linear state-feedback law

$$u(t + i|t) = \kappa_0(x(t + i|t); t) = K_t x(t + i|t), \quad \forall i \in \mathbb{Z}_+ \quad (2.39)$$

and an ellipsoidal feasible invariant set $\mathcal{S}_{x_N, t}$, the closed loop worst-case RHC problem can be expressed as a linear matrix inequality (LMI) problem which can be

solved on-line [56]. Even though this approach gives elegant solutions at least for linear systems, it can be conservative because of the use of identical control laws and ellipsoidal sets, while also being computationally demanding [13, 70]. Some reduction in conservativeness can be expected with the use of affine state-feedback law including a feedforward component in (2.39) for a finite horizon [80], or with the use of polyhedral invariant sets instead of ellipsoidal sets in the controller synthesis [116]; however, this will require significantly complex on-line computations. A modified version of the approach based on an off-line determination of a set of the control laws, invariant sets and cost functions simplifies the on-line computations [58] but is still conservative in spite of being suboptimal.

- 3. Tube-based solution:** A popular approach employed for simplifying the closed-loop RRHC problem involves an optimization over a control policy of the form

$$u(t+i|t) = \begin{cases} \kappa_i(x(t+i|t); t) = K_{i,t}x(t+i|t) + v(t+i|t), & i = 0, \dots, N-1 \\ \kappa_N(x(t+i|t); t) = K_{N,t}x(t+i|t), & i \geq N \end{cases} \quad (2.40)$$

where the state-feedback gains $K_{i,t}$, $i = 0, \dots, N$, are determined off-line whereas the sequence of control actions $\{v(t+i|t)\}_{i=0}^{N-1}$ is optimized on-line at each-time step t to minimize the predicted nominal cost while ensuring robust feasibility. This approach is sometimes known as the ‘tube approach’ as the feedback control law limits the spread of system trajectories within an invariant tube around a nominal trajectory defined by the sequence of control actions $\{v(t+i|t)\}_{i=1}^{N-1}$. The ‘tube approach’ simplifies the on-line computational burden significantly and has been popularly used for linear systems (e.g., [57], [64], [67], [68], [71], [62]) as well as for certain classes of nonlinear systems such as systems with matched nonlinearities [77] and, systems with uncertain nonlinear terms that result in polytopic or norm bounded dynamics of the error state (actual state minus the nominal state) [78], [117]. However, it can still be conservative as the input perturbations (also referred to as the feedforward parts of the control inputs), which handle feasibility, are actually optimized ‘open-loop’ for the pre-stabilized system. Moreover, the selection of the pre-stabilizing feedback control law, which affects the on-line performance as well as the (the shape of) the cross-section of the invariant ‘tube’ of possible state trajectories under uncertainties is not so obvious. [75] offers a more flexible tube-based solution but it may require substantial on-line computations to reduce conservativeness when multiple sources of uncertainties are present.

- 4. Solution based on enlarged terminal constraint set:** Most RRHC schemes that employ an optimization of a finite sequence of control actions on top of a pre-determined feedback control law usually assume a terminal control law that is optimal in some sense. The terminal control law also defines the terminal invariant set for the controlled system. While the optimal feedback gain tuned for an optimal performance usually results in a small feasible invariant set, a detuned feed-

back gain gives a larger feasible invariant set but with a degraded performance [59]. An approach that can allow the use of a larger terminal constraint set without a corresponding degradation in performance has the benefits of simplified on-line computations since smaller horizon lengths are usually sufficient. In such an approach, multiple feedback laws are computed off-line along with the corresponding feasible invariant sets. During the on-line determination of control actions, one of the feedback inputs or a convex combination of them is selected as the terminal control input depending on the current state of the system [59, 61, 72].

5. Solution with uncertainty-based input parameterization: Uncertainty-based input parameterization has been a recent trend in the robust MPC literature. For a class of systems, this approach allows one to employ tractable methods to optimize future control actions that are directly tuned to the particular realization of uncertainties or disturbances affecting the system until their respective times of execution. The approaches in works such as [60] and [70, 81–83] fall in this category. For systems with polytopic model uncertainty, [60] has considered predicted future input perturbations represented by the future states of a stable dynamical system, whose initial state is optimized on-line, and whose dynamics are allowed to remain polytopic and uncertain. This allows a less conservative use of the control input and hence an enlarged feasibility domain.

For systems with bounded additive disturbances, the authors in works [70, 81, 82, 118] have considered inputs or input perturbations parameterized as affine functions of the previous disturbances. This parameterization further reduces the conservativeness in the use of the control input and hence enlarges the set of feasible initial conditions. It has also been shown in [70] that for an LTI system, the disturbance-based input parameterization

$$u(t+i|t) = \begin{cases} \tilde{v}(t+i|t) + \sum_{j=0}^{i-1} \tilde{K}_j(t+i|t) w(t+j), & i = 0, \dots, N-1 \\ K_N x(t+i|t), & i \geq N \end{cases} \quad (2.41)$$

is equivalent to an affine state-feedback parameterization

$$u(t+i|t) = \begin{cases} v(t+i|t) + \sum_{j=0}^i K_j(t+i|t) x(t+j|t), & i = 0, \dots, N-1 \\ K_N x(t+i|t), & i \geq N \end{cases} \quad (2.42)$$

in the sense that both policies lead to the same feasibility domain. However, while the policy in (2.42) leads to a non-convex control optimization problem in variables $v(t+i|t)$ and $K_j(t+i|t)$, $i = 0, \dots, N-1$, the disturbance-based policy (2.41), which parameterizes the control inputs as affine functions of only the uncertain quantities $w(t+i)$, $i = 1, \dots, N-1$, allows a convex optimization of the control inputs. The technique of optimizing the ‘adjustable’ decision variables parameterized as affine functions of uncertain parameters of the optimization problem has been explored

in more general forms in the context of using adjustable robust counterparts of uncertain problems in the optimization literature (see, e.g., [118–120]). While the approach of optimizing future inputs or input perturbations as affine functions of previous disturbances gives less conservative results through the solution of an elegantly tractable problem, the on-line computational complexity is significantly high - at least of the order $\mathcal{O}(N^3)$ when solving a relevant quadratic optimization problem with the perturbations $v(t + i|t)$ and gains $\tilde{K}_j(t + i|t)$, $i = 0, \dots, N - 1$, $j = 0, \dots, i - 1$ as variables [121]. Theoretical guarantee of feasibility and stability require that a terminal invariant constraint set be used [81], preferably one that is associated with an optimal static linear feedback, and this leads to the requirement of a longer horizon length and hence more complex on-line computations just to ensure feasibility over a larger set of initial conditions.

Various approaches to the practically implementable solution of the RRHC problem rely on several assumptions on the nature of the system and the control policy considered. The motivation behind the exploration of various approaches to the solution of the problem is usually the same: to obtain a suitable tradeoff among three important features of the RHC approach - control performance, size of the feasible domain of initial states and on-line computational complexity. These features in turn determine the extent of applicability of the control scheme in a range of applications. In the next few chapters, we explore a set of RHC schemes that address this tradeoff issue for a class of uncertain systems.

2.3 Conclusion

In this chapter, we have presented an overview of the robust RHC problem in a general setting. We have discussed various aspects of the problem: uncertainty descriptions, constraints, control objectives and control parameterizations. Background information on various relevant concepts of stability and invariant sets have been included. Moreover, we have presented a few relevant stability definitions and proved a few results that will be used in the next chapters. Finally, we have discussed some important approaches employed in the solution of the closed-loop robust RHC problem.

Chapter 3

Optimized Dynamic Policy for RHC of Linear Systems with Uncertainties

3.1 Introduction

As mentioned in Chapter 2, solutions to the closed-loop robust RHC need to achieve a suitable tradeoff among three key objectives: better performance, reduced on-line computational complexity and larger feasibility domain. In this chapter, we explore the development of efficient RHC algorithms to address this tradeoff issue for linear time-varying (LTV) systems with bounded additive disturbances. The general framework that we present is applicable for systems with uncertain time variation, specifically for systems with polytopic model uncertainty as well as bounded disturbances. We also explore some additional results with a focus on systems with deterministic time variation, which are common in practical applications (see, e.g., [109, 122]).

The results in this chapter are in line with a recent trend in the robust RHC literature - a trend to use uncertainty-based control policy and to employ tractable methods to optimize future control actions that are directly tuned to the particular realization of uncertainties affecting the system until their respective times of execution. Such an approach is known to be less conservative when one has to deal with systems with uncertainties and disturbances [60, 70, 81, 118]. In particular, for the class of (possibly uncertain) LTV systems described by polytopes and with (possibly unmeasurable) external disturbances, we explore a computationally attractive class of admissible dynamics for a dynamic RHC policy. It involves the use of time-varying controller matrices, which are not required to be explicitly determined on-line, and a disturbance feedforward term which does not require the measurement of the disturbances. The controller matrices are allowed to remain uncertain but within a convex hull defined by a set of matrices in the same way

as the plant matrices. Moreover, the proposed dynamic policy offers a natural way to separate the optimization of the vertices of convex hull of controller matrices from the optimization of controller initial state. The former can be carried out off-line whereas the latter on-line, thus significantly simplifying the on-line computational complexity.

We also study the use of the proposed dynamic policy as the terminal control policy for a class of polytopic (but certain) systems with (possibly unmeasurable) external disturbances. This policy is certainly one of the most appropriate and computationally attractive policies to be used to terminate the finite-horizon disturbance-based control law of the form in (2.41) for the class of systems mentioned. It can be considered as an infinite-horizon counterpart of the disturbance-based control policy (e.g., [70,81]) for this class of systems. It also allows a mechanism to minimize the cost over a significantly large feasible invariant set. So, when it is used as the terminal control policy, appended to a finite-horizon input parameterization of the form in (2.41), which involves $\mathcal{O}(N^2)$ decision variables and leads to an on-line computational complexity of the order $\mathcal{O}(N^3)$ [121], the horizon length N and hence the computational complexity can be greatly reduced.

Besides presenting results on the off-line computation of the controller matrices of the dynamic policy, we also discuss low-complexity RHC schemes based on the nominal cost optimization approach (e.g., [64, 67, 70, 72]), and present results on the performance of the proposed control schemes in terms of asymptotic/exponential or ℓ_∞ -stability of the controlled system. We also present several numerical examples in order to illustrate the features of the proposed RHC schemes.

3.2 Problem Description

We consider an LTV system described by

$$x(t+1) = A(t)x(t) + B_u(t)u(t) + B_w(t)w(t) \quad (3.1)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input and $w(t) \in \mathbb{R}^{n_w}$ is an unknown disturbance.

We make the following assumptions about the system:

A3.1: The time-varying system matrices $A(t)$, $B_u(t)$ and $B_w(t)$ are known to lie within a polytope Ω , i.e.,

$$[A(t) \ B_u(t) \ B_w(t)] \in \Omega = \text{Co} \left\{ [A^{(j)} \ B_u^{(j)} \ B_w^{(j)}], \ j = 1, \dots, n_\Omega \right\}, \forall t \in \mathbb{Z}_+ \quad (3.2)$$

where $[A^{(j)} \ B_u^{(j)} \ B_w^{(j)}]$, $j = 1, 2, \dots, n_\Omega$ are the vertices of the polytope Ω .

A3.2: The disturbance $w(t)$ is componentwise bounded, i.e.,

$$w(t) \in \mathcal{W} = \{w \in \mathbb{R}^{n_w} \mid \|w\|_\infty \leq 1\}. \quad (3.3)$$

A3.3: The constraints are such that the state-input combine $(x(t), u(t))$ lies in a compact polyhedral set Ξ that contains the origin in its interior¹. The set Ξ is expressed as

$$\Xi = \{(x, u) \mid M_x x + M_u u \leq \mathbf{1}\} \quad (3.4)$$

where $M_x \in \mathbb{R}^{m \times n_x}$ and $M_u \in \mathbb{R}^{m \times n_u}$.

A3.4: The polytopic system (3.1-3.2) is quadratically stabilizable².

The general control objective is to drive the state of the system (3.1) close to the origin with inputs computed, at each time step t , by minimizing the predicted infinite-horizon quadratic cost function

$$J(t) = \sum_{i=0}^{\infty} \left\{ \|Q^{\frac{1}{2}} x(t+i|t)\|^2 + \|R^{\frac{1}{2}} u(t+i|t)\|^2 \right\} \quad (3.5)$$

where Q and R are positive definite matrices selected as per the design criteria, and $\{x(t+i|t)\}_{i=0}^{\infty}$ and $\{u(t+i|t)\}_{i=0}^{\infty}$ are the state and control input sequences predicted at time t . Because of system uncertainties, however, an exact evaluation of the cost in (3.5) is not possible and an optimization over open-loop control actions is undesirable. In this chapter, we consider an objective of efficiently computing the closed-loop control strategies that minimize the cost in (3.5) for a nominal performance but ensure robust feasibility for any allowable realization of uncertainties. The nominal performance index is a disturbance-free cost function defined as

$$\bar{J}(t) = \sum_{i=0}^{\infty} \left\{ \|Q^{\frac{1}{2}} \bar{x}(t+i|t)\|^2 + \|R^{\frac{1}{2}} \bar{u}(t+i|t)\|^2 \right\} \quad (3.6)$$

where $\{\bar{x}(t+i|t)\}_{i=0}^{\infty}$ and $\{\bar{u}(t+i|t)\}_{i=0}^{\infty}$ are the predicted state trajectory and control input sequence for the undisturbed system $\bar{x}(t+1) = A(t)\bar{x}(t) + B_u(t)\bar{u}(t)$ with $\bar{x}(t|t) = x(t)$.

We wish to compute and implement the optimal control actions in the receding horizon fashion, and evaluate the control performance by analyzing the stability of the controlled system

¹A convex and compact set with the origin in its interior is often referred to as a C -set.

²A linear polytopic system is said to be quadratically stabilizable (via linear state-feedback) if there exists a linear state-feedback with a constant gain such that the closed-loop system is quadratically stable [111, Chap. 7].

$$x(t+1) = f(x(t), w(t), t) = A(t)x(t) + B_u(t)u^*(t) + B_w(t)w(t) \quad (3.7a)$$

$$z(t) = h(x(t), t) = \begin{bmatrix} Q^{\frac{1}{2}}x(t) \\ R^{\frac{1}{2}}u^*(t) \end{bmatrix} \quad (3.7b)$$

where $u^*(t)$ is the control input obtained through optimization at time t . The choice of $z(t)$ is such that $\|z(t)\|^2 = z(t)^\top z(t)$ represents the stage cost of time t considered in the cost function (3.5).

In the following sections, we discuss the details of the efficient control schemes that address the objectives mentioned above and present relevant results on the performance of these schemes.

3.3 RHC Based on Optimized Dynamic Policy

In this section, we present the proposed control parameterization and discuss the details of the off-line and on-line optimization steps required for the computation of the desired input sequence.

3.3.1 Control parameterization

Motivated by the advantages of using a control parameterization based on past disturbances, we wish to parameterize the predicted control inputs in the form

$$\left. \begin{aligned} u(t+i|t) &= Kx(t+i|t) + v(t+i|t), \quad i \in \mathbb{Z}_+ \\ \text{with } v(t+i|t) &= \bar{v}(t+i|t) + \sum_{j=0}^{i-1} E_j(t+i|t)w(t+j) \end{aligned} \right\} \quad (3.8)$$

where K is a constant static feedback gain that ensures quadratic stability of the polytopic system (3.1-3.2) and $\bar{v}(t+i|t)$ and $E_j(t+i|t)$, $i \in \mathbb{Z}_+$ are vectors and matrices of suitable dimensions. The gain K is such that it is optimal in some sense, typically satisfying unconstrained LQ or H_∞ optimality for the nominal or average model (e.g., the center of Ω). The quantities $\bar{v}(t+i|t)$, $i \in \mathbb{Z}_+$, parameterized as affine functions of past disturbances, therefore, can be considered as predicted input perturbations, or deviations from the optimal values, which ensure the satisfaction of constraints. The perturbations are ‘tuned’ to past disturbances, and when the system matrices are uncertain, we would want them to ‘follow’ the system model uncertainty as well. However, it will be impractical to compute such perturbations (i.e., $\bar{v}(t+i|t)$ and $E_j(t+i|t)$) for the infinite horizon (i.e., $\forall i \in \mathbb{Z}_+$) at each time t . In order to simplify the computations, we employ a dynamically evolving sequence of perturbations by expressing $v(t+i|t)$ as $v(t+i|t) = H\xi(t+i|t)$, and letting $\xi(t+i|t) \in \mathbb{R}^{n_\xi}$, $i \in \mathbb{Z}_+$ be the predicted future states, starting from an initial

state $\xi(t|t) = \xi(t) = \bar{\xi}(t)$, of a time-varying controller described by the dynamics

$$\xi(t+1) = G(t)\xi(t) + F(t)w(t) \quad (3.9a)$$

where, considering the fact that the system (3.1) is polytopic, the controller matrices $G(t)$ and $F(t)$ are allowed to lie within polytopes of as many vertices as the system matrices, i.e.,

$$\begin{bmatrix} G(t) & F(t) \end{bmatrix} \in \text{Co} \left\{ \begin{bmatrix} G^{(j)} & F^{(j)} \end{bmatrix}, j = 1, \dots, n_\Omega \right\}, \quad (3.9b)$$

and are assumed to follow the same convex combination as the system matrices, i.e., $\begin{bmatrix} G(t) & F(t) \end{bmatrix} = \sum_{j=1}^{n_\Omega} \lambda_{[j]}(t) \begin{bmatrix} G^{(j)} & F^{(j)} \end{bmatrix}$ where $\lambda(t) \in \mathbb{R}_+^{n_\Omega}$ satisfies $\lambda(t)^\top \mathbf{1} = 1$ and defines the actual realization of the system matrices at time t so that $\begin{bmatrix} A(t) & B_u(t) & B_w(t) \end{bmatrix} = \sum_{j=1}^{n_\Omega} \lambda_{[j]}(t) \begin{bmatrix} A^{(j)} & B_u^{(j)} & B_w^{(j)} \end{bmatrix}$.

With the perturbation sequence defined in the way mentioned above, the predicted input for each time $t+i$

$$u(t+i|t) = Kx(t+i|t) + H\xi(t+i|t) \quad (3.9c)$$

comprises of the standard feedback $Kx(t+i|t)$ plus an ‘uncertain’ perturbation represented by $H\xi(t+i|t) = H\bar{\xi}(t+i|t) + \sum_{j=0}^{i-1} H\tilde{F}_j(t+i|t)w(t+j)$ where $\bar{\xi}(t+i|t) = (G(t+i-1)..G(t))\xi(t)$ for $i \geq 1$, and $\tilde{F}_j(t+i|t) = F(t+i-1)$ for $i \geq 1$, $j = i-1$ and $\tilde{F}_j(t+i|t) = (G(t+i-1)..G(t+j+1))F(t+j)$ for $i \geq 2$, $0 \leq j \leq i-2$.

Note that the dynamic policy (3.9a,c) with a suitable n_ξ and arbitrary matrices $G(t)$ and $F(t)$ would be more general than the parameterization in (3.8) but it would be computationally intractable. Therefore, we restrict the time-variation of $G(t)$ and $F(t)$ such that they satisfy (3.9b) and follow the same convex combination as the uncertain system matrices, and, as we will see later, this leads to a tractable control problem. By allowing the controller dynamics to be polytopic and to include the disturbance, we incorporate all ‘uncertain’ system information into the controller dynamics. This results in a less conservative estimate of the feasible set for the controlled system and also allows a less conservative prediction of feasible control inputs. But the execution of the predicted control actions based on (3.9) would violate the assumption that the time-varying plant is uncertain and the disturbance is unmeasurable. However, this will be permissible in our control framework since the implementation of the control law will be essentially in the receding horizon fashion implying that the predicted future control actions will never be executed and they need not be computable. Indeed, it suffices to prove that an admissible control sequence exists for any possible realization of uncertainties and disturbances in the future.

Remark 3.1. The fact that the controller matrices are allowed to follow a convex combination of a set of matrices in the same way as the plant matrices is, in some way, conceptually related to the gain-scheduling methods applied for parameter-varying systems (see, e.g., [112]) in which the controller dynamics are assumed to have the same

kind of parameter variation as the plant dynamics and the parameters are assumed to be known in real-time. In the RHC framework, with a special structure in the controller dynamics, this concept is permissible, as well as less conservative, even for uncertain time variation of the parameters, because the control actions to be implemented at each time step are optimized in real-time whereas only the future control predictions are allowed to remain uncertain depending on the uncertainty affecting the system. $\square$

The features of the control law (3.9) are determined by the controller matrices K , H , $G^{(j)}$, $F^{(j)}$ ($j = 1, \dots, n_\Omega$) as well as the initial perturbation $\xi(t)$. Since we select an optimal K *a-priori*, a suitable input perturbation sequence can be obtained by optimizing the matrices H , $G^{(j)}$, $F^{(j)}$ ($j = 1, \dots, n_\Omega$) and the initial perturbation $\xi(t)$. While optimizing these quantities, we need to ensure that the resulting control input is recursively feasible and that the controlled system behaves in a desired way. For this, we ensure that the predicted closed-loop system always satisfies the system constraints and behaves in the desired way. The predicted closed-loop system can be assessed by considering the augmented system

$$\chi(t+1) = \Psi(t)\chi(t) + \mathcal{B}_w(t)w(t) \quad (3.10)$$

where $\chi(t) = [x(t)^T \ \xi(t)^T]^T$ is the augmented state. The augmented system matrices defined as

$$\Psi(t) = \begin{bmatrix} \mathcal{A}(t) & B_u(t)H \\ 0 & G(t) \end{bmatrix}, \quad \mathcal{B}_w(t) = \begin{bmatrix} B_w(t) \\ F(t) \end{bmatrix}$$

with $\mathcal{A}(t) = A(t) + B_u(t)K$ lie in a polytope, i.e., $[\Psi(t) \ \mathcal{B}_w(t)] \in \text{Co}\{[\Psi^{(j)} \ \mathcal{B}_w^{(j)}], j = 1, \dots, n_\Omega\}$ where

$$\Psi^{(j)} = \begin{bmatrix} \mathcal{A}^{(j)} & B_u^{(j)}H \\ 0 & G^{(j)} \end{bmatrix}, \quad \mathcal{B}_w^{(j)} = \begin{bmatrix} B_w^{(j)} \\ F^{(j)} \end{bmatrix}$$

and $\mathcal{A}^{(j)} = A^{(j)} + B_u^{(j)}K$.

Clearly, an optimal control input sequence for system (3.1) can be computed at any time t by optimizing the controller matrices H , $G^{(j)}$, $F^{(j)}$ ($j = 1, \dots, n_\Omega$) and the controller initial state $\xi(t)$ to minimize the predicted cost (3.6) while ensuring that the augmented state $\chi(t)$ lies in a feasible invariant set $\mathcal{S}_\chi$ for the system in (3.10). However, this control optimization problem is computationally not so simple and solving it on-line may not be practical for several systems. Therefore, in order to reduce the on-line computational burden, we choose to split the control problem into two parts — one to optimize the controller matrices and the other to optimize $\xi(t)$ — and then solve the first part off-line while allowing the second part to be carried out on-line. We carry out the off-line part of the problem such that certain desired features of the resulting control law including the size of the feasible invariant set $\mathcal{S}_x$ for the controlled system are guaranteed. Once the controller matrices are computed, the resulting feasible invariant set for the augmented system (3.10) can be easily determined. The on-line computation then involves only the optimization of $\xi(t)$ in the receding horizon fashion in order to minimize a nominal cost

Figure 3.1: Basic steps in a typical RHC scheme based on an optimized dynamic policy.

function. Fig. 3.1 shows a summary of the basic steps involved in a typical RHC scheme based on this approach. In the following, we present the details of the off-line and on-line computations that constitute this control approach.

3.3.2 Computation of controller matrices and feasible invariant set

We wish to compute the controller matrices $G^{(j)}, F^{(j)}$ ($j = 1, \dots, n_\Omega$) and H in (3.9) such that we can determine, off-line, for the controlled system, an invariant set of a desired (usually the largest possible) size, which is feasible w.r.t. the constraints in (3.4). A feasible invariant set $\mathcal{S}_x$ for the controlled system is given by the projection, on the x -subspace of a positively invariant set $\mathcal{S}_\chi$ for the state of (3.10) on which the constraints in (3.4) are satisfied. Since the constraint set Ξ and the disturbance set $\mathcal{W}$ are compact polyhedral sets, the largest feasible invariant set for the augmented system (3.10) and its projection on the x -subspace will, in general, be polyhedral (see Appendix A). An optimization of the controller matrices with an objective directly related to the size of the projection $\mathcal{X}_x$ on the x -subspace of the corresponding polyhedral feasible invariant set $\mathcal{X}_\chi$ for the augmented system (3.10) is usually computationally challenging (see, e.g., [113, Chap. 7]); so we consider the ellipsoidal inner approximations $\mathcal{E}_\chi$ and $\mathcal{E}_x$ so that the size of $\mathcal{E}_x$ can be directly optimized by solving a tractable problem.

We now introduce the concept of disturbance invariance of a set for a linear polytopic system and then present a lemma which describes the condition for the invariance of an ellipsoidal set for such a system.

Definition 3.1 (Robust positively invariant set for a polytopic system). A convex set $\mathcal{S}_x$ is robust positively invariant for a dynamical system

$$x(t+1) = \Phi(t)x(t) + B_w(t)w(t) \quad (3.11)$$

where $[\Phi(t) \ B_w(t)] \in \bar{\Omega} = \text{Co} \left\{ [\Phi^{(j)} \ B_w^{(j)}], j = 1, \dots, n_\Omega \right\}$, in the presence of disturbances $w(t) \in \mathcal{S}_w$ if, $\Phi^{(j)}x + B_w^{(j)}w \in \mathcal{S}_x$, $j = 1, \dots, n_\Omega$ for all $x \in \mathcal{S}_x$ and for all $w \in \mathcal{S}_w$. $\square$

Lemma 3.1. For the dynamical system (3.11), an ellipsoidal set $\mathcal{E}_x = \{x \mid x^T W^{-1} x \leq 1\}$ is robust positively invariant in the presence of disturbances $w(t) \in \mathbb{R}^{n_w}$ that satisfy $\|w(t)\|_\infty \leq 1$ if there exist positive constants $\alpha^{(j)}$ and non-negative matrices $R_w^{(j)} \in \mathbb{D}_+^{n_w}$, satisfying

$$\begin{bmatrix} W & \Phi^{(j)}W & B_w^{(j)} \\ * & \alpha^{(j)}W & 0 \\ * & * & R_w^{(j)} \end{bmatrix} \succeq 0, \quad \text{Tr}(R_w^{(j)}) \leq 1 - \alpha^{(j)}, \quad j = 1, \dots, n_\Omega. \quad (3.12)$$

PROOF. See Appendix A.1. ■

Remark 3.2. The invariance condition (3.12) is not a necessary but only a sufficient condition for the invariance of the ellipsoidal set $\mathcal{E}_x$ in the presence of componentwise bounded disturbance $w(t)$. The set $\mathcal{E}_x$, therefore, is only a conservative approximation. Further, considering the system's polytopic uncertainty, variables α and R_w are allowed to be polytopic in (3.12). However, since the matrix inequality in (3.12) is bilinear in $\alpha^{(j)}$ and W , using a single $\alpha = \alpha^{(1)} = \dots = \alpha^{(n_\Omega)}$ can simplify computations when searching for a W satisfying (3.12) at the risk of some additional conservativeness. □

Next, following Lemma 3.1, a set $\mathcal{E}_\chi = \{\chi \mid \chi^T \mathcal{P}^{-1} \chi \leq 1\}$ will be invariant for the system (3.10) if

$$\begin{bmatrix} \mathcal{P} & \Psi^{(j)}\mathcal{P} & \mathcal{B}_w^{(j)} \\ * & \alpha^{(j)}\mathcal{P} & 0 \\ * & * & R_w^{(j)} \end{bmatrix} \succ 0, \quad j = 1, \dots, n_\Omega \quad (3.13)$$

for some positive $\alpha^{(j)}$ and diagonal matrices $R_w^{(j)} \in \mathbb{D}_+^{n_w}$, $j = 1, \dots, n_\Omega$ such that $\text{Tr}(R_w^{(j)}) \leq 1 - \alpha^{(j)}$.

Further, the set $\mathcal{E}_\chi$ will also be feasible w.r.t. the constraints in (3.4) if (see, e.g., [111], [60])

$$\begin{bmatrix} Z & [M_x + M_u K & M_u H] \mathcal{P} \\ * & \mathcal{P} \end{bmatrix} \succ 0, \quad Z_{[j,j]} \leq 1, \quad j = 1, \dots, m. \quad (3.14)$$

The invariance condition (3.13) and the feasibility condition (3.14), in variables $\mathcal{P}$, $G^{(j)}$, $F^{(j)}$ ($j = 1, \dots, n_\Omega$) and H , must be applied while obtaining a feasible invariant set for the controlled system, viz., $\mathcal{E}_x = \{x \mid x^T (T \mathcal{P} T^T)^{-1} x \leq 1\}$, where $T = [I_{n_x} \ 0]$. It is known that, for $n_\xi \geq n_x$, with a suitable change of variables [123], [60], the non-linear inequalities in (3.13) and (3.14) can be expressed as equivalent LMI conditions for constant $\alpha^{(j)}$, $j = 1, \dots, n_\Omega$.

Proposition 3.1. Let $n_\xi \geq n_x$. There exist Z , $\mathcal{P}$, H , $G^{(j)}$, $F^{(j)}$, $\alpha^{(j)}$ and $R_w^{(j)}$ ($j = 1, \dots, n_\Omega$) satisfying (3.13) and (3.14) if and only if the following matrix inequalities are

feasible in Z , $\alpha^{(j)}$ and $R_w^{(j)}$ ($j = 1, \dots, n_\Omega$) for some W , X , $\mathcal{H}$, $\Theta^{(j)}$, $\mathcal{F}^{(j)}$ ($j = 1, \dots, n_\Omega$):

$$\begin{bmatrix} W & I & \mathcal{A}^{(j)}W + B_u^{(j)}\mathcal{H} & \mathcal{A}^{(j)} & B_w^{(j)} \\ * & X & \Theta^{(j)} & X\mathcal{A}^{(j)} & X B_w^{(j)} + \mathcal{F}^{(j)} \\ * & * & \alpha^{(j)}W & \alpha^{(j)}I & 0 \\ * & * & * & \alpha^{(j)}X & 0 \\ * & * & * & * & R_w^{(j)} \end{bmatrix} \succ 0, \quad R_w^{(j)} \in \mathbb{D}_+^{n_w}, \text{Tr}(R_w^{(j)}) \leq 1 - \alpha^{(j)}, \quad j = 1, \dots, n_\Omega \quad (3.15a)$$

$$\begin{bmatrix} Z & (M_x + M_u K)W + M_u \mathcal{H} & M_x + M_u K \\ * & W & I \\ * & * & X \end{bmatrix} \succ 0, \quad Z_{[j,j]} \leq 1, \quad j = 1, \dots, m. \quad (3.15b)$$

PROOF. The proof follows the standard approach (see, e.g., [123], [60]) of partitioning $\mathcal{P}$ and $\mathcal{P}^{-1}$ into components as $\mathcal{P} = \begin{bmatrix} W & U \\ U^T & \star \end{bmatrix}$ and $\mathcal{P}^{-1} = \begin{bmatrix} X & V \\ V^T & \star \end{bmatrix}$ with $W, X \in \mathbb{R}^{n_x \times n_x}$ so that $\mathcal{P}\mathcal{P}^{-1} = I$ implies $WX + UV^T = I$. Then, for a matrix $\mathcal{T} = \begin{bmatrix} I & X \\ 0 & V^T \end{bmatrix}$, $\mathcal{P}\mathcal{T} = \begin{bmatrix} W & I \\ U^T & 0 \end{bmatrix}$.

Now, pre- and post-multiplying the inequality in (3.13) by $\text{diag}(\mathcal{T}^T, \mathcal{T}^T, I)$ and $\text{diag}(\mathcal{T}, \mathcal{T}, I)$, and the inequality in (3.14) by $\text{diag}(I, \mathcal{T}^T)$ and $\text{diag}(I, \mathcal{T})$, we get the conditions (3.15a) and (3.15b) where $\mathcal{H}$, $\Theta^{(j)}$ and $\mathcal{F}^{(j)}$ ($j = 1, \dots, n_\Omega$) are defined as

$$\mathcal{H} = H U^T; \quad \Theta^{(j)} = V G^{(j)} U^T + X B_u^{(j)} \mathcal{H} + X \mathcal{A}^{(j)} W; \quad \mathcal{F}^{(j)} = V F^{(j)}. \quad (3.16)$$

Clearly, the feasibility of (3.15a) and (3.15b) is necessary for the existence of some $\mathcal{P}$, H , $G^{(j)}$ and $F^{(j)}$ ($j = 1, \dots, n_\Omega$) satisfying (3.13) and (3.14). On the other hand, since $n_\xi \geq n_x$, conditions (3.15a) and (3.15b) are also sufficient for the feasibility of (3.13) and (3.14) as the solutions to H , $G^{(j)}$ and $F^{(j)}$ ($j = 1, \dots, n_\Omega$) exist in (3.16) with values of U and V satisfying $UV^T = I - WX$. ■

From the results of Proposition 3.1, an ellipsoidal feasible invariant set for the controlled system is given by $\mathcal{E}_x = \{x \mid x^T W^{-1} x \leq 1\}$ where W satisfies the conditions in (3.15). Note that the conditions (3.15a) and (3.15b) of Proposition 3.1 remain the same for any choice of n_ξ larger than or equal to n_x . So, it is sufficient to consider $n_\xi = n_x$ for obtaining any possible feasible invariant set for the controlled system under the control law of the form (3.9).

3.3.2.1 Maximal feasible invariant set

It is often desirable to optimize the controller matrices such that the associated ellipsoidal feasible invariant set $\mathcal{E}_x$ for the controlled system is as large as possible. For this, we solve the problem

$$\max_{W, X, Z, \mathcal{H}, \Theta^{(j)}, \mathcal{F}^{(j)}, \alpha^{(j)}, R_w^{(j)} (j=1, \dots, n_\Omega)} \log \det(W) \quad (3.17)$$

subject to conditions (3.15a) and (3.15b). Additional conditions may be applied if other features (such as a fixed nominal cost bound for initial states over the set) are desired. Problem (3.17) can be solved using efficient algorithms applicable for solving semidefinite programming (SDP) problems [124, 125].

Note that K is not a variable in the optimization problem (3.17) since the proposed control law assumes that K is fixed *a priori*. Instead of having an arbitrary state-feedback $K_f x(t)$, the proposed control law adds a dynamic perturbation to the prescribed constant optimal state-feedback $Kx(t)$. This, however, does not make the proposed control law restrictive as far as the size of the feasible invariant sets that can be achieved with the control law is concerned. To see this, let us consider an arbitrary static feedback control $u(t) = K_f x(t)$ for the system in (3.1). The conditions for the invariance and feasibility of an ellipsoidal set $\mathcal{E}_x = \{x \mid x^T W^{-1} x \leq 1\}$ for the closed-loop system under such a control can be written in terms of W and a variable $Y = K_f W$ as (see, e.g., [111])

$$\begin{bmatrix} W & A^{(j)}W + B_u^{(j)}Y & B_w^{(j)} \\ * & \alpha^{(j)}W & 0 \\ * & * & R_w^{(j)} \end{bmatrix} \succ 0, \quad R_w^{(j)} \in \mathbb{D}_+^{n_w}, \text{Tr}(R_w^{(j)}) \leq 1 - \alpha^{(j)}, \quad j = 1, \dots, n_\Omega \quad (3.18a)$$

$$\begin{bmatrix} Z & M_x W + M_u Y \\ * & W \end{bmatrix} \succ 0, \quad Z_{[j,j]} \leq 1, \quad j = 1, \dots, m. \quad (3.18b)$$

The following proposition shows that any large ellipsoidal feasible invariant set $\mathcal{E}_x = \{x \mid x^T W^{-1} x \leq 1\}$ achievable with the static state feedback gain $K_f = YW^{-1}$ under condition (3.18) can also be obtained with the control law (3.9) (and vice versa) since the solution W is indeed common to both (3.15) and (3.18) under certain conditions on $\alpha^{(j)}, j = 1, \dots, n_\Omega$.

Proposition 3.2. Let K be fixed and let $\alpha^{(j)} \in (\alpha_m^{(j)}, 1)$, $j = 1, \dots, n_\Omega$, where $\alpha_m^{(j)} = (\rho(\mathcal{A}^{(j)}))^2 = \max_{i \in \{1, \dots, n_x\}} |\lambda_i(\mathcal{A}^{(j)})|^2$. Then, for any solution $W, Y, Z, \alpha^{(j)}, R_w^{(j)}$ ($j = 1, \dots, n$) to conditions (3.18a) and (3.18b), there exist $X, \mathcal{H}, \mathcal{F}^{(j)}, \Theta^{(j)}$ ($j = 1, \dots, n$) satisfying conditions (3.15a) and (3.15b). Conversely, for any solution to (3.15a) and (3.15b), there exists a Y , namely $Y = \mathcal{H} + KW$ satisfying (3.18a) and (3.18b). $\square$

PROOF. Here we present the outline of the proof for $n_\Omega = 1$. The proof for $n_\Omega > 1$ will be a straightforward generalization. For $n_\Omega = 1$, denoting the variables $(\cdot)^{(1)}$ by $(\cdot)$, we write (3.15a) as

$$\begin{bmatrix} W & AW + B_u \mathcal{H} & \mathcal{A} & I & B_w \\ * & \alpha W & \alpha I & \Theta^T & 0 \\ * & * & \alpha X & \mathcal{A}^T X & 0 \\ * & * & * & X & XB_w + \mathcal{F} \\ * & * & * & * & R_w \end{bmatrix} \succ 0. \quad (3.19)$$

It can be shown that the optimization variable $\mathcal{F}$ in (3.19) can be eliminated to express (3.19) equivalently by

$$\begin{bmatrix} W & \mathcal{A}W + B_u\mathcal{H} & \mathcal{A} & I \\ * & \alpha W & \alpha I & \Theta^T \\ * & * & \alpha X & \mathcal{A}^T X \\ * & * & * & X \end{bmatrix} \succ 0, \quad \begin{bmatrix} W & \mathcal{A}W + B_u\mathcal{H} & \mathcal{A} & B_w \\ * & \alpha W & \alpha I & 0 \\ * & * & \alpha X & 0 \\ * & * & * & R_w \end{bmatrix} \succ 0. \quad (3.20)$$

Here, necessity is obvious from the fact that the condition pair (3.20) is obtained by eliminating certain row/column pairs from (3.19) whereas sufficiency follows from the equivalence of the pair (3.20) with (3.19) for a choice of $\mathcal{F}$ that makes the Schur complement of

$$\begin{bmatrix} W & \mathcal{A}W + B_u\mathcal{H} & \mathcal{A} \\ * & \alpha W & \alpha I \\ * & * & \alpha X \end{bmatrix} \succ 0 \quad (3.21)$$

in (3.19) block diagonal. Next, with a similar reasoning, the first inequality in (3.20) can be equivalently expressed without the variable Θ by (3.21) and

$$\begin{bmatrix} W & \mathcal{A} & I \\ * & \alpha X & \mathcal{A}^T X \\ * & * & X \end{bmatrix} \succ 0 \quad (3.22)$$

where (3.22) can be shown to be equivalent to the pair

$$\begin{bmatrix} W & I \\ I & X \end{bmatrix} \succ 0 \quad (3.23a)$$

$$\alpha X - \mathcal{A}^T X \mathcal{A} \succ 0. \quad (3.23b)$$

Here, (3.21) and (3.23a) are implied by the second inequality in (3.20). Finally, expressing the second inequality in (3.20) and the constraint feasibility condition (3.15b) in their respective equivalent forms

$$\begin{bmatrix} W & \mathcal{A}W + B_u\mathcal{H} & B_w \\ * & \alpha W & 0 \\ * & * & R_w \end{bmatrix} - \begin{bmatrix} \mathcal{A} \\ \alpha I \\ 0 \end{bmatrix} \frac{X^{-1}}{\alpha} \begin{bmatrix} \mathcal{A}^T & \alpha I & 0 \end{bmatrix} \succ 0, \quad X \succ 0 \quad (3.24a)$$

$$\begin{bmatrix} Z & (M_x + M_u K)W + M_u \mathcal{H} \\ & W \end{bmatrix} - \begin{bmatrix} M_x + M_u K \\ I \end{bmatrix} X^{-1} \begin{bmatrix} (M_x + M_u K)^T & I \end{bmatrix} \succ 0, \quad Z_{[j,j]} \leq 1, \quad j = 1, \dots, m \quad (3.24b)$$

we note that condition pair (3.15) is feasible for $n_\Omega = 1$ iff (3.23b) and the pair (3.24) are feasible. Next, for any solution W, Y, Z, α and R_w to the condition pair (3.18)

(for $n_\Omega = 1$) with $\alpha \in (\alpha_m, 1)$, an X_α that satisfies (3.23b) exists and the feasibility of the pair (3.24) can be established by choosing $\mathcal{H} = Y - KW$ and $X = \sigma X_\alpha$ with an arbitrarily large σ . On the other hand, for any $W, \mathcal{H}, Z, \alpha$ and R_w satisfying (3.23b) and the pair (3.24), the condition pair (3.18) (for $n_\Omega = 1$) is readily satisfied with $Y = \mathcal{H} + KW$. Finally, we note that the generalization to the case of a polytopic system is obvious when both $\mathcal{F}$ and Θ are assumed to be polytopic. ■

Remark 3.3. Proposition 3.2 considers the closed-loop system with a static gain K_f satisfying the invariance and feasibility conditions in (3.18) with $\alpha^{(j)} > \alpha_m^{(j)} = (\rho(\mathcal{A}^{(j)}))^2$, $j = 1, \dots, n_\Omega$. This can be interpreted as allowing the static feedback control $u(t) = K_f x(t)$ to place the closed-loop eigenvalues of the system more towards the boundary of the unit circle than the optimal feedback $u(t) = Kx(t)$ does, which typically results in a larger feasibility set. However, Proposition 3.2 says that a set feasible with such a static feedback control $u(t) = K_f x(t)$ can also be made feasible by adding a dynamic perturbation to $Kx(t)$ as indicated in the control law (3.9). □

3.3.2.2 Maximal feasible invariant set without disturbance feedforward

In (3.17), we optimize the controller matrices with the sole objective of maximizing the size of the feasible invariant set. We allow the dynamic controller state to depend on the disturbance entering the system. While this feedforward of disturbance essentially allows us to get rid of the conservativeness associated with an open-loop prediction of input perturbations, it may also affect the steady-state performance of the controlled system. Therefore, in some situations, it may be desirable to consider a dynamic controller that does not depend on the disturbance inputs. In order to determine the controller matrices in this case, we solve the problem

$$\max_{W, X, Z, \mathcal{H}, \Theta^{(j)}, \alpha^{(j)}, R_w^{(j)} (j=1, \dots, n_\Omega)} \log \det(W) \quad (3.25)$$

subject to conditions (3.15a) and (3.15b) with $\mathcal{F}^{(j)} = 0$, $j = 1, \dots, n_\Omega$.

The ellipsoidal feasible set $\mathcal{E}_\chi$ obtained from (3.17) or (3.25) is usually a conservative inner ellipsoidal approximation of the largest feasible invariant set associated with the controller matrices that are determined in (3.17) or (3.25). The largest feasible invariant set is (better approximated as) a polyhedral set when Ξ and $\mathcal{W}$ are bounded polyhedral sets, and such a polyhedral set $\mathcal{X}_\chi = \{\chi \mid M_\chi \chi \leq \mathbf{1}\}$ can be computed by using some iterative algorithms (see, e.g., [126], [113, Chap. 5]). While $\mathcal{E}_\chi$ and hence $\mathcal{E}_x$ are conservative in size, $\mathcal{X}_\chi$ and its projection $\mathcal{X}_x$ on the x -subspace may have comparatively more complex representations. See Appendix A for the details on the computation of polyhedral invariant sets.

We now consider an example to illustrate the ideas presented in this section.

Figure 3.2: (a) Feasible invariant sets for the polytopic system of (3.26) corresponding to control parameterization without (dotted curve) and with (solid curve) disturbance feedforward. (b) Ranges of feasible inputs for initial state conditions $[x_{01} \ 0]^T, 0 \leq x_{01} \leq 10$.

Illustration 3.1. We consider a second order LTV system with additive disturbances, described by the system matrices

$$A(t) = \begin{pmatrix} 1.1 & 1 \\ 0 & a(t) \end{pmatrix}; \quad B_u(t) = \begin{pmatrix} b(t) \\ b(t) \end{pmatrix}; \quad B_w(t) = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} \quad (3.26)$$

where $a(t) = 1.3 + 0.05\rho(t)$, $b(t) = 0.9 + 0.1\rho(t)$, $|\rho(t)| \leq 1$ and $\|w\|_\infty \leq 1$. Here, the elements of the system matrices $A(t)$ and $B(t)$ depend linearly on $\rho(t)$, and since $\rho(t)$ varies between -1 and 1, $n_\Omega = 2$ and

$$A^{(1)} = \begin{pmatrix} 1.1 & 1 \\ 0 & 1.25 \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} 0.8 \\ 0.8 \end{pmatrix}, \quad D^{(1)} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},$$

$$A^{(2)} = \begin{pmatrix} 1.1 & 1 \\ 0 & 1.35 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix}, \quad D^{(2)} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

We also consider an input constraint $|u(t)| \leq 1$. Moreover, we choose $Q = I$ and $R = 0.01$, and select $K = [-0.82 \ -1.21]$, which is the unconstrained LQ optimal gain for the nominal system (that corresponds to $\rho(t) = 0$)³.

We compute the feasible invariant sets for the polytopic system of (3.26) under the control of the form (3.9) in two cases: one with no disturbance feedforward and the other with optimized disturbance feedforward. Fig. 3.2a shows the ellipsoidal feasible invariant sets $\bar{\mathcal{E}}_x$ and $\mathcal{E}_x$ obtained by solving the problems in (3.25) and (3.17) respectively. $\bar{\mathcal{X}}_x$ and $\mathcal{X}_x$ are the corresponding polyhedral invariant sets for the controlled system. Clearly, there is a substantial increase in the size of the feasible sets - both ellipsoidal and polyhedral - when the disturbance feedforward is included in the control parameterization. We note that the ellipsoidal set $\mathcal{E}_x$ is the same as the maximal ellipsoidal invariant set under static feedback, $\mathcal{E}_{xS} = \{x \mid x^T W^{-1} x \leq 1\}$, which is obtained by maximizing $\log \det(W)$ under the conditions in (3.18) with variables W , Y , $\alpha^{(j)}$ and R_w^j ($j = 1, 2$).

Next, we consider a group of system states of the form $x_0 = [x_{01} \ 0]^T$, $0 \leq x_{01} \leq 10$, and, for each state x_0 , compute the ranges of the feasible input $u_0 = Kx_0 + H\xi_0$ in the two

³In the examples throughout this thesis, whenever an LQ optimal gain is considered, the optimal gain K is computed for a nominal system defined by the numerical mean of all vertices of the polytopic system matrices (unless otherwise mentioned).

cases, in which we constrain the augmented state $[x_0^\top \ \xi_0^\top]^\top$ to belong respectively to the sets $\bar{\mathcal{X}}_\chi$ and $\mathcal{X}_\chi$ to ensure the feasibility. Fig. 3.2b shows these ranges in both the cases, and it can be seen from the figure that, for each system state, the range of the feasible input is larger (or at least equal) in the second case than in the first – a fact that indicates a less conservative use of control inputs when the disturbance feedforward term is included in the controller dynamics.

3.3.3 On-line computations and control performance

In this section, we discuss the computation of the suitable control input $u(t)$ at each time t in the receding horizon fashion. Given the off-line-determined feasible invariant sets $\mathcal{S}_x$ and $\mathcal{S}_\chi$, respectively for the controlled system and the augmented system (3.10), a feasible control input for any system state $x(t) \in \mathcal{S}_x$ at time t can be written as $u(t) = \kappa(x(t))$ where $\kappa(x(t))$ satisfies

$$\kappa(x(t)) \in \left\{ Kx(t) + H\xi(t) \mid [x(t)^\top \ \xi(t)^\top]^\top \in \mathcal{S}_\chi \right\}. \quad (3.27)$$

Because of the robust invariance of the set $\mathcal{S}_\chi$, any control input $u(t) = \kappa(x(t))$ satisfying (3.27) at time t will ensure the existence of feasible control inputs at successive times.

We now discuss efficient nominal cost minimization schemes based on the predicted nominal cost (3.6) and analyze the performance of the scheme in terms of the stability of the closed-loop system (3.7). Note that because of system constraints, the optimal input $u^*(t) = \kappa^*(x(t))$, and hence the successor state $x(t+1)$ and the output $z(t)$ in (3.7) are usually related to $x(t)$ in a nonlinear way.

Given the controller matrices, a nominal (disturbance-free) cost bound for an initial state $x(t)$ of the uncertain system can be represented by a cost function

$$\bar{J}(t) = \left\| \bar{\mathcal{P}}^{\frac{1}{2}} [x(t)^\top \ \xi(t)^\top]^\top \right\|^2$$

where $\bar{\mathcal{P}}$ is a positive definite matrix obtained from the minimization

$$\min_{\bar{\mathcal{P}}} \text{Tr}(\bar{\mathcal{P}}) \quad (3.28)$$

$$\text{s.t.} \quad \bar{\mathcal{P}} - \Psi^{(j)\top} \bar{\mathcal{P}} \Psi^{(j)} \succeq C_z^\top C_z, \quad j = 1, \dots, n_\Omega, \quad C_z = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ R^{\frac{1}{2}} K & R^{\frac{1}{2}} H \end{bmatrix}$$

and $\xi(t)$ is a suitably-chosen, constraint-admissible initial controller state (that defines the initial input perturbation). The feasibility of (3.13) ensures the existence of a solution to (3.28).

We first consider a nominal cost minimization scheme for the case in which the controller matrices are optimized off-line allowing disturbance feedforward, as in (3.17). We

have the following result on the performance of the nominal cost minimization scheme in this case.

Proposition 3.3. Let $\mathcal{S}_\chi$ be a compact feasible invariant set for the augmented system (3.10) with controller matrices obtained from (3.17), and let $\mathcal{S}_x$ be its projection on the x -subspace. Also, let $\bar{\mathcal{P}}$ be the nominal cost matrix obtained from (3.28). Then, for any $x(0) \in \mathcal{S}_x$, the receding horizon control $u^*(t) = \kappa^*(x(t)) = Kx(t) + H\xi^*(t)$ where $\xi^*(t)$ is obtained by solving the problem

$$\left. \begin{aligned} & \min_{\xi(t)} \bar{J}(t) \\ & \text{with } \bar{J}(t) = \left\| \bar{\mathcal{P}}^{\frac{1}{2}} \begin{bmatrix} x(t)^T & \xi(t)^T \end{bmatrix}^T \right\|^2 \end{aligned} \right\} \quad (3.29)$$

such that $\begin{bmatrix} x(t)^T & \xi(t)^T \end{bmatrix}^T \in \mathcal{S}_\chi$ at each time t , ensures that the controlled system (3.7) is ℓ_∞ -stable, while satisfying the system constraints at all times. $\square$

PROOF. For any $x(0) \in \mathcal{S}_x$, there exists some $\xi(0)$ satisfying $\begin{bmatrix} x(0)^T & \xi(0)^T \end{bmatrix}^T \in \mathcal{S}_\chi$ so that problem (3.31) is feasible at time $t = 0$. Further, since $\mathcal{S}_\chi$ is invariant, the problem will be feasible at all subsequent times $t \in \mathbb{Z}_{++}$. Also, system constraints are satisfied for all times $t \in \mathbb{Z}_{++}$ as they are satisfied for all points in the set $\mathcal{S}_\chi$. Next, since the set $\mathcal{S}_\xi = \{\xi(t) \mid \begin{bmatrix} x(t)^T & \xi(t)^T \end{bmatrix}^T \in \mathcal{S}_\chi\}$ is convex and compact for any $x(t) \in \mathcal{S}_x$, and the cost function $\bar{J}(t)$ is strictly convex in $x(t)$ and $\xi(t)$, it follows that the optimal input $u^*(t) = \kappa^*(x(t))$ and hence the output $z(t)$ in (3.7) and the optimal cost $\bar{J}^*(t)$ are Lipschitz on $\mathcal{S}_x$ for all $x(t) \in \mathcal{S}_x$. Therefore, $f(x(t), w(t), t)$ in (3.7) is Lipschitz on $\mathcal{S}_x \times \mathcal{W}$. Clearly, $\bar{J}^*(t)$ is a Lyapunov function for the undisturbed system with $\gamma_1(r) = \gamma_3(r) = \lambda_{\min}(Q)\|r\|^2$, and a suitable γ_2 . Proposition 3.3 then directly follows from Lemma 2.3. $\blacksquare$

Remark 3.4. As discussed in Section 2.2.1.3, the notion of ℓ_∞ -stability is closely related to the concept of *input-to-state stability* (ISS). Indeed, the proof of ℓ_∞ -stability in Proposition 3.3 is based on the fact that the closed loop system is input-to-state stable. The ISS property implies that the system state trajectories will remain bounded for all allowable disturbances and that the system will be asymptotically stable in the absence of disturbances. Proposition 3.3 implies that the stage cost $\|z(t)\|^2$ will remain bounded in the presence of disturbances and will go down to zero as $t \rightarrow \infty$ when there are no disturbances. $\square$

Although the nominal cost minimization of Proposition 3.3 ensures that the stage cost will remain bounded, the characterization of the asymptotic behaviour of the closed-loop system is not so attractive as the gain function γ is not explicitly known. Alternatively, we can consider a special case with no disturbance feedforward in (3.9) and solve problem (3.25) in order to obtain the controller matrices. Nominal cost minimization in this case leads to the convergence of the state into the smallest set that is invariant for system (3.1) under the stabilizing static feedback input $u(t) = Kx(t)$. Such a set (or its outer

ellipsoidal or polytopic approximation) $\mathcal{R}^K$ satisfies

$$\mathcal{A}^{(j)}\mathcal{R}^K \oplus B_w^{(j)}\mathcal{W} \subseteq \mathcal{R}^K, \quad j = 1, \dots, n_\Omega \quad (3.30)$$

and is commonly referred to as the disturbance reachability set of the closed-loop system $x(t+1) = (A(t) + B(t)K) + D(t)w(t)$. It can be easily computed using the result of Lemma 3.1 or some iterative algorithm [68] [127] (See Section A.2.2 in Appendix A).

In order to ensure that the state converges into the set $\mathcal{R}^K$, we consider optimizing $\xi(t)$ for any state $\bar{x}(t) = x(t) - d(t)$, which we call the nominal state, under the nominal dynamics (without disturbances) such that $d(t) \in \mathcal{R}^K$. In other words, we optimize $\xi(t)$ along with an offset $d(t) \in \mathcal{R}^K$ to the initial state $x(t)$ such that the cost of driving $\bar{x}(t) = x(t) - d(t)$ to the origin under the nominal dynamics is minimized.

We have the following result on the performance of the nominal cost minimization scheme when there is no disturbance feedforward.

Proposition 3.4. Let $\mathcal{S}_\chi$ be a compact feasible invariant set for the augmented system (3.10) with the controller matrices obtained from (3.25), and let $\mathcal{S}_x$ be its projection on the x -subspace. Also, let $\bar{\mathcal{P}}$ be the corresponding nominal cost matrix obtained from (3.28). Then, for any $x(0) \in \mathcal{S}_x$, the receding horizon control $u^*(t) = \kappa^*(x(t)) = Kx(t) + H\xi^*(t)$ where $\xi^*(t)$ is obtained by solving the problem

$$\left. \begin{aligned} & \min_{\xi(t), d(t)} \bar{J}(t) \\ & \text{with } \bar{J}(t) = \left\| \bar{\mathcal{P}}^{\frac{1}{2}} \begin{bmatrix} (x(t) - d(t))^T & \xi(t)^T \end{bmatrix}^T \right\|^2 \end{aligned} \right\} \quad (3.31)$$

such that $\begin{bmatrix} x(t)^T & \xi(t)^T \end{bmatrix}^T \in \mathcal{S}_\chi$ and $d(t) \in \mathcal{R}^K$ at each time t , ensures that the state of the controlled system (3.7) converges to the set $\mathcal{R}^K$ while satisfying the system constraints at all times. $\square$

PROOF. Let $x(t) - d(t)$ be denoted by $\bar{x}(t)$. Since $x(0) \in \mathcal{S}_x$, problem (3.31) will be feasible at time t . Let $\{\xi^*(t), d^*(t)\}$ form the optimal solution at time t so that $\bar{x}^*(t) = x(t) - d^*(t)$ and $u^*(t) = Kx(t) + H\xi^*(t)$. Then, the actual state at time $t+1$ will be $x(t+1) = \mathcal{A}(t)x(t) + B_u(t)H\xi^*(t) + B_w(t)w(t)$, $w(t) \in \mathcal{W}$. Also let $\bar{x}(t+1) = x(t+1) - d(t+1) = \mathcal{A}(t)\bar{x}^*(t) + B_u(t)H\xi^*(t)$ be the predicted nominal state at time $t+1$. Since $\mathcal{X}_\chi$ is disturbance-invariant for the polytopic augmented system (3.10), there exists $\xi(t+1) = G(t)\xi^*(t)$ with some $G(t) \in \text{Co}\{G^{(j)}, j = 1, \dots, n_\Omega\}$ such that the augmented state $\begin{bmatrix} x(t+1)^T & \xi(t+1)^T \end{bmatrix}^T$ lies within the set $\mathcal{S}_\chi$. Further, since $d^*(t) = x(t) - \bar{x}^*(t) \in \mathcal{R}^K$, the invariance property (3.30) of $\mathcal{R}^K$ implies $d(t+1) = x(t+1) - \bar{x}(t+1) = \mathcal{A}(t)(x(t) - \bar{x}^*(t)) + B_w(t)w(t) \in \mathcal{A}(t)\mathcal{R}^K \oplus B_w(t)\mathcal{W} \subseteq \mathcal{R}^K$, ensuring feasibility of $\{\xi(t+1), d(t+1)\}$ as the solution to (3.31) at time $t+1$.

Next, let $\bar{J}^*(t)$ denote the optimal cost at time t . Also, let $\bar{J}(t+1)$ be the cost corresponding to the feasible solution pair $\{\xi(t+1), d(t+1)\}$ at time $t+1$ and let

Figure 3.3: State trajectories for the polytopic system in (3.26) under the control scheme mentioned in Proposition 3.3: (a) Trajectories originating from the points around the boundary of the feasible set $\mathcal{X}_x$, (b) Trajectories starting from an initial state $[-10.2 \ 2.2]^T$.

$\bar{J}^*(t+1)$ be the optimal cost at time $t+1$. Then, we have,

$$\begin{aligned} \bar{J}^*(t+1) - \bar{J}^*(t) &\leq \bar{J}(t+1) - \bar{J}^*(t) \\ &= \left\| \bar{\mathcal{P}}^{\frac{1}{2}} [\bar{x}(t+1)^T \ \xi(t+1)^T]^T \right\|^2 - \left\| \bar{\mathcal{P}}^{\frac{1}{2}} [\bar{x}^*(t)^T \ \xi^*(t)^T]^T \right\|^2. \end{aligned}$$

But, since $[\bar{x}(t+1)^T \ \xi(t+1)^T]^T = \Psi(t) [\bar{x}^*(t)^T \ \xi^*(t)^T]^T$ where $\Psi(t) \in \text{Co}\{\Psi^{(j)}, j = 1, \dots, n_\Omega\}$, and since $\Psi(t)^T \bar{\mathcal{P}} \Psi(t) - \bar{\mathcal{P}} \preceq -C_z^T C_z$ from (3.28), we have

$$\bar{J}^*(t+1) - \bar{J}^*(t) \leq -\left\| C_z [\bar{x}^*(t)^T \ \xi^*(t)^T]^T \right\|^2 = -\left\| Q^{\frac{1}{2}} \bar{x}^*(t) \right\|^2 - \left\| R^{\frac{1}{2}} \bar{u}^*(t) \right\|^2.$$

Clearly, $\bar{J}^*(t)$ is a Lyapunov function for the ‘nominal’ system whose state is $\bar{x}^*(t)$. Since $Q \succ 0$, the conditions in (2.24) are satisfied for $\rho = 2$, $\gamma_1 = \gamma_3 = \lambda_{\min}(Q)$ and a suitable γ_2 , implying an exponential stability of the origin of the ‘nominal’ system. Since the ‘nominal’ state $\bar{x}^*(t)$ goes down to zero exponentially as $t \rightarrow \infty$, the constraint $d(t) = x(t) - \bar{x}(t) \in \mathcal{R}^K$ in (3.31) guarantees that $x(t)$ converges into the set $\mathcal{R}^K$ exponentially as $t \rightarrow \infty$. ■

Illustration 3.2. In order to illustrate the robust performance of the nominal cost minimization schemes discussed above, we again consider the system described in (3.26). Fig. 3.3a shows the state trajectories under the receding horizon control of Proposition 3.3 for initial states near the boundary of the disturbance-invariant polyhedral feasible set $\mathcal{X}_x$ for the system in (3.26). A worst-case disturbance realization $w(t) = \text{sgn}(x(t))$ and random values of $\rho(t)$ (uniformly distributed in $[-1, 1]$) are used in each case. As is seen from the figure, all the trajectories converge to a neighbourhood of the origin. Fig. 3.3b, on the other hand, shows a set of 50 state trajectories originating from an initial state $[-10.2 \ 2.2]^T$ when the system is under a similar control but with random realizations (uniformly distributed) of the disturbance $w(t)$. It is observed that the trajectories in all the cases get converged to a neighbourhood of the origin in a finite number of steps.

Similarly, Fig. 3.4a shows the state trajectories under the receding horizon control of Proposition 3.4 for initial states near the boundary of the disturbance-invariant polyhedral feasible set $\bar{\mathcal{X}}_x$ for the system in (3.26). Further, Fig. 3.4b shows a set of 50 state trajectories originating from $[-7.80 \ 1.25]^T$ under the same control scheme. While uniformly

Figure 3.4: State trajectories for the polytopic system in (3.26) under the control scheme mentioned in Proposition 3.4: (a) Trajectories originating from the points around the boundary of the feasible set $\bar{\mathcal{X}}_x$, (b) Trajectories starting from an initial state $[-7.8 \ 1.25]^T$.

distributed random values $\rho(t)$ are used in all cases, a worst-case disturbance realization $w(t) = \text{sgn}(x(t))$ is considered for the trajectories in Fig. 3.4a whereas uniformly distributed random realizations of $w(t)$ are considered for the trajectories in Fig. 3.4b. As is seen from the figure, all the trajectories converge into the set $\mathcal{R}^K$ in a finite number of steps.

Remark 3.5. While computing the matrix $\bar{\mathcal{P}}$ defining the cost function $\bar{J}(t)$ employed in Propositions 3.3 and 3.4, we have considered a minimum-trace solution for $\bar{\mathcal{P}}$ from (3.28). Although it is fairly common to use an off-line-computed minimum-trace cost matrix $\bar{\mathcal{P}}$ for the simplification of on-line computations, it is possible, and may also be preferable from the optimality point of view, to allow the computation of the cost matrix $\bar{\mathcal{P}}$ satisfying the condition in (3.28) on-line along with the other variables, when the system is polytopic. This will, however, increase the complexity of on-line computations since the on-line problem in this case will be an SDP problem with several LMI constraints. Note that if the system is time-invariant, the exact cost matrix can be computed off-line by solving an appropriate Lyapunov equation. $\square$

3.4 RHC with Optimized Dynamic Terminal Policy

The optimized dynamic control policy discussed in the last section provides an efficient, robust solution for the control of linear systems with both polytopic uncertainty as well as additive disturbances. In several cases of LTV systems, the time variation of the system matrices is not really uncertain and can be known in advance. The proposed control approach can be used for such systems if they satisfy assumption A3.1; however, it may lead to conservative results since the actual knowledge of the system matrices is not used in the calculation of control actions. In such situations, an improved performance can be achieved by employing the disturbance-based input parameterization (3.8) for a short horizon N and appending it with the dynamic control policy of the last section. Indeed, the proposed dynamic policy forms an effective formulation to terminate a finite horizon

parameterization of the form (3.8) for this class of systems. In the following, we discuss the details of this scheme.

We first make the following further assumption about the system.

A3.5: The time-varying system matrices in (3.1) are known and satisfy assumption A3.1.

3.4.1 Control parameterization

Considering N steps of free input perturbations before using (3.9) as the terminal control policy, we have the following form of input for the prediction of the cost at each time step:

$$u(t+i|t) = \begin{cases} Kx(t+i|t) + v(t+i|t), & i = 0, 1, \dots, N-1 \\ Kx(t+i|t) + H\xi(t+i|t), & i \geq N. \end{cases} \quad (3.32)$$

where K is chosen as before and $v(t+i|t) = \bar{v}(t+i|t) + \sum_{j=0}^{i-1} E_j(t+i|t)w(t+j)$. Similarly, $H\xi(t+i|t), i \geq N$ form the sequence of terminal input perturbations generated by the evolution of $\xi(t+N|k) = \bar{\xi}(t+N|t) + \sum_{j=0}^{N-1} \tilde{F}_j(t+N|t)w(t+j)$ as per the dynamics (3.9a) for all $i \geq N$.

3.4.2 Feasible solutions and recursive feasibility

Now, we discuss the determination of feasible control inputs for a given system state for any possible realization of disturbances. Let us consider the sequences of the predicted state and the control input in the finite horizon defined as

$$\mathbf{x}(t) = \begin{bmatrix} x(t|t) \\ x(t+1|t) \\ \vdots \\ x(t+N-1|t) \\ x(t+N|t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u(t|t) \\ u(t+1|t) \\ \vdots \\ u(t+N-1|t) \end{bmatrix}.$$

With the control policy defined in (3.32), we can express $\mathbf{x}(t)$ and $\mathbf{u}(t)$ as

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{A}(t) x(t) + \mathbf{B}_u(t) \bar{\mathbf{v}}(t) + (\mathbf{B}_w(t) + \mathbf{B}_u(t)\mathbf{E}(t)) \mathbf{w}(t) \\ \mathbf{u}(t) &= \check{\mathbf{A}}(t) x(t) + \check{\mathbf{B}}_u(t) \bar{\mathbf{v}}(t) + (\check{\mathbf{B}}_w(t) + \check{\mathbf{B}}_u(t)\mathbf{E}(t)) \mathbf{w}(t) \end{aligned}$$

where

$$\bar{\mathbf{v}}(t) = \begin{bmatrix} \bar{v}(t|t) \\ \bar{v}(t+1|t) \\ \vdots \\ \bar{v}(t+N-1|t) \end{bmatrix}, \quad \mathbf{w}(t) = \begin{bmatrix} w(t) \\ w(t+1) \\ \vdots \\ w(t+N-1) \end{bmatrix}.$$

Here, the constant matrices $\mathbf{A}(t)$, $\mathbf{B}_u(t)$, $\mathbf{B}_w(t)$, $\check{\mathbf{A}}(t)$, $\check{\mathbf{B}}_u(t)$ and $\check{\mathbf{B}}_w(t)$ are as defined in Appendix B and $\mathbf{E}(t)$ is defined as

$$\mathbf{E}(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ E_0(t+1|t) & 0 & \cdots & 0 \\ E_0(t+2|t) & E_1(t+2|t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_0(t+N-1|t) & E_1(t+N-1|t) & \cdots & 0 \end{bmatrix}. \quad (3.33)$$

If we use the polyhedral invariant set $\mathcal{X}_\chi = \{\chi \mid M_\chi \chi \leq \mathbf{1}\}$, computed off-line as mentioned in the last section, as the terminal constraint set, the constraints for the finite horizon ($i = 0, \dots, N-1$) can be combined with the terminal constraint and expressed as:

$$\mathcal{M}_x x(t) + \mathcal{M}_u \begin{bmatrix} \bar{\mathbf{v}}(t) \\ \bar{\xi}(t+N|t) \end{bmatrix} + \max_{\mathbf{w}(t) \in \mathcal{W}^N} \left(\mathcal{M}_w + \mathcal{M}_u \begin{bmatrix} \mathbf{E}(t) \\ \tilde{\mathbf{F}}(t) \end{bmatrix} \right) \mathbf{w}(t) \leq \mathbf{1} \quad (3.34)$$

where the matrices $\mathcal{M}_x$, $\mathcal{M}_u$ and $\mathcal{M}_w$ are as defined in Appendix B and $\tilde{\mathbf{F}}(t) = [\tilde{F}_0(t+N|t) \quad \tilde{F}_1(t+N|t) \quad \dots \quad \tilde{F}_{N-1}(t+N|t)]$.

Now, for a given initial condition $x(t)$, the set of constraint-admissible control decision variables $\bar{\mathbf{v}}(t)$, $\bar{\xi}(t+N|t)$, $\mathbf{E}(t)$, $\tilde{\mathbf{F}}(t)$ in the control policy (3.32) can be defined as

$$\mathfrak{X}_t^N(x(t)) = \left\{ (\bar{\mathbf{v}}(t), \bar{\xi}(t+N|t), \mathbf{E}(t), \tilde{\mathbf{F}}(t)) \mid (3.34) \text{ is satisfied.} \right\}. \quad (3.35)$$

We define the overall feasible set with the control policy of (3.32) as $\mathcal{X}_{x,t}^N = \{x \mid \mathfrak{X}_t^N(x) \neq \emptyset\}$. The following lemma defines the nature of this feasible set.

Lemma 3.2. The set $\mathcal{X}_{x,t}^N$ is a bounded polyhedron. Moreover, the feasible sets obtained with increasing horizon lengths N satisfy the successive inclusion $\mathcal{X}_x \subseteq \mathcal{X}_{x,t}^1 \subseteq \mathcal{X}_{x,t}^2 \dots$ at each time t . $\square$

PROOF. That $\mathcal{X}_{x,t}^N$ is a bounded polyhedron follows directly from the fact that the sets $\mathcal{W}$ and Ξ are bounded polyhedral sets and that the constraint in (3.34) is linear. To prove the relation on successive set inclusions, we show that $x(t) \in \mathcal{X}_{x,t}^N$ implies $x(t) \in \mathcal{X}_{x,t}^{N+1}$. Let $x(t) \in \mathcal{X}_{x,t}^N$ so that there exist decision variables $\bar{\mathbf{v}}(t)$, $\bar{\xi}(t+N|t)$, $\mathbf{E}(t)$ and $\tilde{\mathbf{F}}(t)$ such that the predicted future state-input pairs $(x(t+i|t), u(t+i|t))$, $i = 0, \dots, N-1$ lie in Ξ and the predicted terminal augmented state $[x(t+N|t)^\top \quad \xi(t+N|t)^\top]^\top$ at time $t+N$ lies in $\mathcal{X}_\chi$. Then, if we consider a finite horizon of length $N+1$, because of the fact that $\mathcal{X}_\chi$ is invariant for the terminal augmented polytopic system, the extended variables $\check{\mathbf{v}}(t) = [\bar{\mathbf{v}}(t)^\top \quad (H \bar{\xi}(t+N|t))^\top]^\top$, $\check{\xi}(t+N+1|t) = G(t+N)\bar{\xi}(t+N|t)$, $\check{\mathbf{E}}(t) = \begin{bmatrix} \mathbf{E}(t) & 0 \\ H \tilde{\mathbf{F}}(t) & 0 \end{bmatrix}$ and $\check{\tilde{\mathbf{F}}}(t) = [G(t+N)\tilde{\mathbf{F}}(t) \quad F(t+N)]$ with some $[G(t+N) \quad F(t+N)] \in \text{Co} \{ [G^{(j)} \quad F^{(j)}], j = 1, \dots, n_\Omega \}$ will ensure that

the predicted future state-input pairs $(x(t+i|t), u(t+i|t)), i = 0, \dots, N$ lie in Ξ and the predicted terminal augmented state at $t + N + 1$, $[x(t + N + 1|t)^T \ \bar{\xi}(t + N + 1|t)^T]^T$ lies in $\mathcal{X}_\chi$, thus implying $\mathfrak{X}_t^{N+1}(x(t)) \neq \emptyset$ and hence $x(t) \in \mathcal{X}_{x,t}^{N+1}$. ■

Next, given an initial state $x(t)$, a constraint admissible control input at time t can be written as $u(t) = \kappa(x(t), t)$ where $\kappa(x(t), t)$ satisfies

$$\kappa(x(t), t) \in \left\{ Kx(t) + \bar{v}(t) \mid (\bar{\mathbf{v}}(t), \bar{\xi}(t + N|t), \mathbf{E}(t), \tilde{\mathbf{F}}(t)) \in \mathfrak{X}_t^N(x(t)) \right\}. \quad (3.36)$$

The following lemma discusses the existence of constraint admissible control inputs at successive time steps given a constraint admissible control input at time t .

Lemma 3.3. If there exists a feasible input $u(t) = \kappa(x(t), t)$ satisfying (3.36) at time t , then the application of the control input $u(t)$ at time t guarantees that a feasible control input $u(t+1) = \kappa(x(t+1), t+1)$ satisfying (3.36) with $t = t+1$ will exist at time $t+1$. □

PROOF. Let a control input $u(t) = \kappa(x(t), t) = Kx(t) + \bar{v}(t|t)$ be feasible at time t such that $(\bar{\mathbf{v}}(t), \bar{\xi}(t + N|t), \mathbf{E}(t), \tilde{\mathbf{F}}(t)) \in \mathfrak{X}_t^N(x(t))$. Now, let $\mathbf{E}_{\mathbf{I}\mathbf{r}}(t)$ be the matrix formed by removing the first block row and the first block column of $\mathbf{E}(t)$ and let $\mathbf{E}_{\mathbf{II}}(t)$ be the first block column of $\mathbf{E}(t)$ without its first row. Also, let $\tilde{\mathbf{F}}_{\mathbf{r}}(t) = [\tilde{F}_1(t + N|t) \dots \tilde{F}_{N-1}(t + N|t)]$ and let $\bar{\mathbf{v}}_{\mathbf{r}}(t) = [\bar{v}(t + 1|t)^T \dots \bar{v}(t + N - 1|t)^T]^T$. Then, it follows from the invariance property of the set $\mathcal{X}_\chi$ that there exists some $[G(t + N) \ F(t + N)] \in \text{Co} \{ [G^{(j)} \ F^{(j)}], j = 1, \dots, n_\Omega \}$ such that the decision variables $(\hat{\bar{\mathbf{v}}}(t), \hat{\bar{\xi}}(t + N + 1|t), \hat{\mathbf{E}}(t), \hat{\tilde{\mathbf{F}}}(t))$ with

$$\hat{\bar{\mathbf{v}}}(t) = \begin{bmatrix} \bar{\mathbf{v}}_{\mathbf{r}}(t) + \mathbf{E}_{\mathbf{II}}(t) w(t) \\ H\bar{\xi}(t + N|t) + H\tilde{F}_0(t + N|t) w(t) \end{bmatrix}, \quad \hat{\mathbf{E}}(t) = \begin{bmatrix} \mathbf{E}_{\mathbf{I}\mathbf{r}}(t) & 0 \\ H\mathbf{F}_{\mathbf{r}}(t) & 0 \end{bmatrix},$$

$\hat{\bar{\xi}}(t + N + 1|t) = G(t + N)(\bar{\xi}(t + N|t) + \tilde{F}_0(t + N|t) w(t))$ and $\hat{\tilde{\mathbf{F}}}(t) = [G(t + N)\tilde{\mathbf{F}}_{\mathbf{r}}(t) \ F(t + N)]$ are feasible at time $t+1$ satisfying $(\hat{\bar{\mathbf{v}}}(t), \hat{\bar{\xi}}(t + N + 1|t), \hat{\mathbf{E}}(t), \hat{\tilde{\mathbf{F}}}(t)) \in \mathfrak{X}_{t+1}^N(x(t + 1))$. Hence the existence of a feasible input $u(t + 1) = \kappa(x(t + 1), t + 1)$ satisfying (3.36) with $t = t + 1$ at time $t + 1$. ■

3.4.3 On-line computations and control performance

Now, we consider the nominal cost minimization schemes based on the cost function in (3.6) for the control parameterization considered in (3.32). For a predicted terminal nominal state $\bar{x}(t + N|t)$, the terminal nominal cost bound for the system can be represented by a cost function $\bar{\chi}(t + N|t)^T \bar{\mathcal{P}} \bar{\chi}(t + N|t)$ with $\bar{\chi}(t + N|t) = [\bar{x}(t + N|t)^T \ \bar{\xi}(t + N|t)^T]^T$, where $\bar{\mathcal{P}}$ is the nominal cost matrix obtained from (3.28) and $\bar{\xi}(t + N|t)$ is the predicted terminal controller initial state (nominal). The terminal cost bound can be combined with the finite-horizon cost to give the total cost

$$\bar{J}(t) = \left\| \mathbf{H}(t)^{\frac{1}{2}} \begin{bmatrix} x(t)^T & \bar{\mathbf{v}}(t)^T & \bar{\xi}(t + N|t)^T \end{bmatrix}^T \right\|^2 \quad (3.37)$$

where $\mathbf{H}(t)$ is defined as

$$\begin{aligned} \mathbf{H}(t) = & \begin{pmatrix} \mathbf{A}(t) & \mathbf{B}_u(t) & 0 \\ 0 & 0 & I_{n_\xi} \end{pmatrix}^T \begin{pmatrix} \mathbf{Q} & 0 \\ 0 & \bar{\mathcal{P}} \end{pmatrix} \begin{pmatrix} \mathbf{A}(t) & \mathbf{B}_u(t) & 0 \\ 0 & 0 & I_{n_\xi} \end{pmatrix} \\ & + \begin{pmatrix} \check{\mathbf{A}}(t) & \check{\mathbf{B}}_u(t) & 0 \end{pmatrix}^T \mathbf{R} \begin{pmatrix} \check{\mathbf{A}}(t) & \check{\mathbf{B}}_u(t) & 0 \end{pmatrix} \end{aligned} \quad (3.38)$$

in which $\mathbf{Q} = I_N \otimes Q$ and $\mathbf{R} = I_N \otimes R$.

We need to minimize the predicted cost $\bar{J}(t)$ under the constraint condition (3.34) at each time t . Note that (3.34) requires the satisfaction of the constraints for any allowable disturbance sequence $\mathbf{w}(t)$. The maximization over $\mathbf{w}(t)$ in (3.34) is a row-wise maximization. Considering the fact that $\mathcal{W}$ is a hypercube under assumption A3.2, we have, for any $\mathcal{M}$,

$$\max_{\mathbf{w}(t) \in \mathcal{W}^N} \mathcal{M}_{[i,:]} \mathbf{w}(t) = \|\mathcal{M}_{[i,:]} \|_1 = |\mathcal{M}_{[i,:]}| \mathbf{1}.$$

Therefore, defining a new variable matrix $\mathbf{L} \in \mathbb{R}^{m_1 \times m_2}$, where m_1 is the number of rows in $\mathcal{M}_x$ and $m_2 = N n_w$, constraint (3.34) can be expressed as

$$\mathcal{M}_x x(t) + \mathcal{M}_u \begin{bmatrix} \bar{\mathbf{v}}(t) \\ \bar{\xi}(t + N|t) \end{bmatrix} + \mathbf{L} \mathbf{1} \leq \mathbf{1}, \quad -\mathbf{L} \leq \left(\mathcal{M}_w + \mathcal{M}_u \begin{bmatrix} \mathbf{E}(t) \\ \tilde{\mathbf{F}}(t) \end{bmatrix} \right) \leq \mathbf{L}. \quad (3.39)$$

Now, we have the following result on the performance of a nominal cost minimization scheme for the control parameterization considered in (3.32).

Proposition 3.5. If the optimization problem

$$\left. \begin{aligned} & \min_{\bar{\mathbf{v}}(t), \bar{\xi}(t+N|t), \mathbf{L}, \mathbf{E}(t), \tilde{\mathbf{F}}(t)} \bar{J}(t) \\ & \text{with } \bar{J}(t) = \left\| \mathbf{H}(t)^{\frac{1}{2}} \begin{bmatrix} x(t)^T & \bar{\mathbf{v}}(t)^T & \bar{\xi}(t+N|t)^T \end{bmatrix}^T \right\|^2 \end{aligned} \right\} \quad (3.40)$$

is feasible under the constraint condition (3.39) for an initial state $x(0)$ at time $t = 0$, then the receding horizon control $u^*(t) = \kappa^*(x(t), t) = K x(t) + \bar{v}^*(t)$ where $\bar{v}^*(t)$ is obtained from the solution of (3.40) at each time $t \in \mathbb{Z}_+$ ensures that the problem remains feasible at subsequent times, and guarantees that the controlled system (3.7) is ℓ_∞ -stable, while satisfying the system constraints at all times. $\square$

PROOF. Given the optimization problem is feasible at time $t = 0$, the fact that the problem remains feasible for all times $t \in \mathbb{Z}_{++}$ follows from Lemma 3.3. The proof of the proposition, then, follows from the strict convexity of the control problem and the compactness of the overall feasible set $\mathcal{X}_{x,t}^N$ with the lines of reasoning used in the proof of Proposition 3.3. $\blacksquare$

Proposition 3.5, like Proposition 3.3, does not provide any explicit information about the asymptotic performance of the control algorithm in the presence of non-vanishing

disturbances. If we consider a special case with no disturbance feedforward in (3.32), i.e., consider $\mathbf{E}(t) = 0, \tilde{\mathbf{F}}(t) = 0$ and $F^{(j)} = 0$ ($j = 1, \dots, n_\Omega$), and optimize the other parameters for a nominal cost minimization, we can guarantee the convergence of the state into the smallest set $\mathcal{R}^K$ as mentioned before. In this case, the combined constraint condition (3.39) can be expressed as

$$\bar{\mathcal{M}}_x x(t) + \bar{\mathcal{M}}_u \begin{bmatrix} \bar{\mathbf{v}}(t) \\ \bar{\xi}(t+N|t) \end{bmatrix} + |\bar{\mathcal{M}}_w| \mathbf{1} \leq \mathbf{1} \quad (3.41)$$

where the matrices $\bar{\mathcal{M}}_x$, $\bar{\mathcal{M}}_u$ and $\bar{\mathcal{M}}_w$ differ from $\mathcal{M}_x$, $\mathcal{M}_u$ and $\mathcal{M}_w$ from (3.39) only due to the fact that the terminal constraint set considered in obtaining the constraint (3.41) is different from that used in (3.39).

Proposition 3.6. Let the optimization problem

$$\left. \begin{aligned} & \min_{\mathbf{v}(t), \bar{\xi}(t+N|t), d(t)} \bar{J}(t) \\ & \text{with } \bar{J}(t) = \left\| \mathbf{H}(t)^{\frac{1}{2}} \begin{bmatrix} (x(t) - d(t))^T & \mathbf{v}(t)^T & \xi(t+N|t)^T \end{bmatrix} \right\|^2 \end{aligned} \right\} \quad (3.42)$$

be feasible under the constraint condition (3.41) such that $d(t) \in \mathcal{R}^K$ for an initial state $x(0)$ at time $t = 0$. Then, the receding horizon control $u^*(t) = \kappa^*(x(t), t) = Kx(t) + \bar{v}^*(t)$ where $\bar{v}^*(t)$ is obtained from the solution of (3.42) under condition (3.41) such that $d(t) \in \mathcal{R}^K$ at each time $t \in \mathbb{Z}_+$ ensures that the problem remains feasible at subsequent times, and that the state of the controlled system (3.7) converges to the set $\mathcal{R}^K$ while satisfying the system constraints at all times. $\square$

PROOF. Let $x(t) - d(t)$ be the ‘nominal’ state denoted by $\bar{x}(t)$. Let $\bar{\mathbf{v}}^*(t) = [\bar{v}^*(t|t)^T \dots \bar{v}^*(t+N-1|t)^T]^T$, $\bar{\xi}^*(t+N|t)$ and $d^*(t)$ form the optimal solution at time t so that $\bar{x}^*(t) = x(t) - d^*(t)$ and $u^*(t) = Kx(t) + \bar{v}^*(t|t)$. Then, the actual state at time $t+1$ will be $x(t+1) = \mathcal{A}(t)x(t) + B_u(t)\bar{v}^*(t|t) + B_w(t)w(t)$, $w(t) \in \mathcal{W}$. Also let $\bar{x}(t+1) = x(t+1) - d(k+1) = \mathcal{A}(t)\bar{x}^*(t) + B_u(t)\bar{v}^*(t|t)$ be the predicted ‘nominal’ state at time $t+1$. Clearly, a sequence $\tilde{\mathbf{v}}(t+1) = [\bar{v}^*(t+1|t)^T \dots \bar{v}^*(t+N-1|t)^T (H\bar{\xi}^*(t+N|t))^T]^T$ will be feasible at time $t+1$. Also, since the terminal constraint set $\mathcal{X}_\chi$ is invariant, there exists $\bar{\xi}(t+N+1|t) = G(t+N)\bar{\xi}^*(t+N|t)$ with some $G(t+N) \in \text{Co}\{G^{(j)}, j = 1, \dots, n_\Omega\}$ such that the augmented state $[x(t+N+1|t)^T \bar{\xi}(t+N+1|t)^T]^T$ where $x(t+N+1|t)$ is the predicted actual state at time $t+N+1$, will lie within the set $\mathcal{X}_\chi$. Further, since $d^*(t) = x(t) - \bar{x}^*(t) \in \mathcal{R}^K$, the invariance property (3.30) of $\mathcal{R}^K$ implies $d(t+1) = x(t+1) - \bar{x}(t+1) = \mathcal{A}(t)(x(t) - \bar{x}^*(t)) + B_w(t)w(t) \in \mathcal{A}(t)\mathcal{R}^K \oplus B_w(t)\mathcal{W} \subseteq \mathcal{R}^K$, ensuring feasibility of $\{\tilde{\mathbf{v}}(t+1), d(k+1), \bar{\xi}(k+N+1|t)\}$ as the solution to (3.42) at time $t+1$.

Next, let $\bar{J}^*(t)$ denote the optimal cost at time t . Also, let $\bar{J}(t+1)$ be the cost corresponding to the feasible solution $\tilde{\mathbf{v}}(t+1)$, $d(k+1)$ and $\bar{\xi}(k+N+1|t)$ at time

Figure 3.5: Time variation of states under the control schemes in Proposition 3.5 and Proposition 3.6 for various values of N . (a) Under the scheme in Proposition 3.5 for an initial state $[-10.2 \ 2.2]^T$, and (b) under the scheme in Proposition 3.6 for an initial state $[-7.80 \ 1.25]^T$.

$t + 1$, and let $\bar{J}^*(t + 1)$ be the optimal cost at time $t + 1$. Then, we have,

$$\begin{aligned}
\bar{J}^*(t + 1) - \bar{J}^*(t) &\leq \bar{J}(t + 1) - \bar{J}^*(t) \\
&= \left\| \bar{\mathcal{P}}^{\frac{1}{2}} \begin{bmatrix} \bar{x}(t + T + 1|t)^T & \bar{\xi}(t + N + 1|t)^T \end{bmatrix}^T \right\|^2 \\
&\quad + \left\| C_z \begin{bmatrix} \bar{x}^*(t + N|t)^T & \bar{\xi}^*(t + N|t)^T \end{bmatrix}^T \right\|^2 \\
&\quad - \left\| \bar{\mathcal{P}}^{\frac{1}{2}} \begin{bmatrix} \bar{x}^*(t + N|t)^T & \bar{\xi}^*(t + N|t)^T \end{bmatrix}^T \right\|^2 \\
&\quad - \left\| Q^{\frac{1}{2}} \bar{x}^*(t) \right\|^2 - \left\| R^{\frac{1}{2}} \bar{u}^*(t) \right\|^2.
\end{aligned}$$

But, since $\begin{bmatrix} \bar{x}(t + N + 1|t)^T & \bar{\xi}(t + N + 1|t)^T \end{bmatrix}^T = \Psi(t + N) \begin{bmatrix} \bar{x}^*(t + N|t)^T & \bar{\xi}^*(t + N|t)^T \end{bmatrix}^T$ where $\Psi(t + N) \in \text{Co}\{\Psi^{(j)}, j = 1, \dots, n_\Omega\}$, and since $\Psi(t + N)^T \bar{\mathcal{P}} \Psi(t + N) + C_z^T C_z - \bar{\mathcal{P}} \preceq 0$ from (3.28), we have the cost monotonicity condition

$$\bar{J}^*(t + 1) - \bar{J}^*(t) \leq -\left\| Q^{\frac{1}{2}} \bar{x}^*(t) \right\|^2 - \left\| R^{\frac{1}{2}} \bar{u}^*(t) \right\|^2.$$

Clearly, as in Proposition 3.4, $\bar{J}^*(t)$ is a Lyapunov function for the ‘nominal’ system whose state is $\bar{x}^*(t)$ satisfying the conditions in (2.24) for $\rho = 2$, $\gamma_1 = \gamma_3 = \lambda_{\min}(Q)$ and a suitable γ_2 . Hence, if the optimization problem (3.42) is feasible at time $t = 0$, the application of the control input obtained from its solution ensures that the problem remains feasible at subsequent times and that the ‘nominal’ state $\bar{x}^*(t)$ goes down to zero exponentially as $t \rightarrow \infty$. Moreover, the constraint $d(t) = x(t) - \bar{x}(t) \in \mathcal{R}^K$ guarantees that the state $x(t)$ converges into the set $\mathcal{R}^K$ exponentially as $t \rightarrow \infty$. ■

Illustration 3.3. We again consider the system mentioned in (3.26) in order to illustrate the performance of the control schemes discussed in Proposition 3.5 and Proposition 3.6. Here, we assume $\rho(t) = \sin \frac{2\pi t}{T_a}$ with $T_a = 25$.

This set of simulations is carried out to assess the performance of the control schemes in Proposition 3.5 and Proposition 3.6 for various values of N . Fig. 3.5a shows the state

trajectories starting from $[-10.2 \ 2.2]^\top$ under the control actions calculated as mentioned in Proposition 3.5 with $N = 0, 1, 2, 3$ and 4. A worst-case disturbance $w(t) = \text{sgn}(x(t))$ is used in each case. While the control response is relatively sluggish for smaller values of N , it becomes less and less conservative and even starts to overshoot with increasing values of N .

Fig. 3.5b, similarly, shows the state trajectories starting from $[-7.80 \ 1.25]^\top$ under the control scheme in Proposition 3.6 with $N = 0, 1, 2, 3$ and 4 for a worst-case disturbance $w(t) = \text{sgn}(x(t))$ in each case. As in the case of Fig. 3.5a, the response is sluggish for smaller values of N . The performance improves with increasing values of N , albeit in a more conservative way than in the case of Fig. 3.5a. Since this improvement saturates at small values of N , it is reasonable to choose a small N that gives a reasonably good performance. $\square$

3.5 Computational Complexity

The RHC schemes presented in this chapter use significant off-line computations that simplify the computational complexity in the on-line phase. The off-line computations include an optimization problem for finding the (terminal) controller matrices and this problem is an SDP problem in which the constraints are LMI's when the values of the scalars $\alpha^{(j)}$, $j = 1, \dots, n_\Omega$ are fixed. So the problem can be efficiently solved with available LMI solvers by employing a search over the scalars $\alpha^{(j)}$, $j = 1, \dots, n_\Omega$. We also need to determine the polyhedral feasible invariant set for a stable polytopic system with an infinity norm bound disturbance and this requires the use of an iterative algorithm which terminates in a finite number of steps for an arbitrary non-zero numerical tolerance [49, 113]. The complexity of the algorithm, however, grows exponentially with the system dimension, and, for higher order systems, some approximation techniques (see, e.g., [128]) can be used to obtain a lower-complexity inner approximation of the actual invariant set. See Appendix A for details.

The complexity of on-line computations substantially depends on the length N of the chosen finite horizon. In the optimization problem considered in Proposition 3.5, there are $n_u N$ free variables in $\bar{\mathbf{v}}(t)$, $n_\xi (= n_x)$ in $\bar{\xi}(t + N|t)$, $n_u n_w N(N - 1)/2$ in $\mathbf{E}(t)$, $n_x n_w N$ in $\tilde{\mathbf{F}}(t)$, $(mN + n_P)n_w N$ in $\mathbf{L}$. The total number of decision variables is of the order $\mathcal{O}(N^2)$. In Proposition 3.6, however, the optimization problem has only $\mathcal{O}(N)$ decision variables including those in $\bar{\mathbf{v}}(t)$, $\bar{\xi}(t + N|t)$ and $d(t) \in \mathbb{R}^{n_x}$. The optimization problem is a QP problem in both cases.

The computational advantages of the proposed control schemes over corresponding schemes that use a static feedback as the terminal control law arise from the fact that the proposed schemes offer feasibility over a larger set of initial conditions with an optimization over a shorter (possibly zero) horizon length N . Though the proposed schemes involve relatively more complex terminal constraint sets (with linear constraints in $2n_x$ variables) and require an on-line optimization of the variable $\bar{\xi}(t)$ (or variables $\bar{\xi}(t + N|t)$ and $\tilde{\mathbf{F}}(t)$), it is reasonable to expect significantly reduced on-line computations with them

Table 3.1: Average computation times and costs with the RHC law of Proposition 3.5

	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
Average computation time (ms)	0.8	14.6	42.7	86.7	149.0
Average normalized cost	100.0	95.0	94.0	93.6	93.4

as a much shorter horizon length N is usually sufficient. The numerical examples that follow illustrate the comparative computational efficiency of the proposed schemes.

Illustration 3.4. We again consider the example system in (3.26) in order to assess the computational complexity of the proposed control schemes and to compare their performances with those of the schemes that use the optimal static feedback as the terminal control law (e.g., [64, 70]). For this study, we assume that the system is time-invariant with $a(t) = 1.3$ and $b(t) = 1$, and choose the LQ optimal gain $K = [-0.74 \ -1.09]$ for the time-invariant system.

Fig. 3.6a shows the polyhedral invariant set $\mathcal{X}_x$ for system (3.26) with $a(t) = 1.3$ and $b(t) = 1$ corresponding to the controller matrices obtained using (3.17). We first consider a set of 100 initial conditions chosen randomly along the boundary of the set $\mathcal{X}_x$, and apply the control scheme of Proposition 3.5 with $N = 0, 1, 2, 3$ and 4 for the worst-case disturbance $w(t) = \text{sgn}(x(t))$ in each case. Table 3.1 shows the average computation times observed for the different horizon lengths when the computations are carried out with the BPMPD solver [129] in the Matlab 7.4 (R2007a) environment in a dual-core 2.26 GHz machine with 3GB RAM. The table also shows the corresponding average normalized regulation cost in each case. The average of the costs for each value of N is taken over the individual costs expressed as a percentage of the corresponding cost with $N = 0$.

Further, Fig. 3.6a also shows the terminal invariant set for the system under the optimal (LQ) static feedback, $\mathcal{X}_{x_S}$. It can be observed that the domain of attraction for the system with the static terminal feedback for $N = 18$, $\mathcal{X}_{x_S}^{18}$ is almost the same in size as that for the system with a dynamic terminal policy for $N = 3$, $\mathcal{X}_x^3$. Fig. 3.6b shows the control performances of the nominal cost minimization schemes with disturbance feedforward, using $\mathcal{X}_{x_S}$ and $\mathcal{X}_x$ as the terminal sets. Because of more degrees of freedom in the longer finite horizon, the response of the scheme with a static terminal feedback policy is less conservative as compared to that of the scheme with a dynamic terminal feedback policy. However, since the cost minimization scheme ignores predicted inputs parameterized as linear functions of disturbances, the response is also too optimistic and overshoots more in the case of static terminal feedback than that in the case of dynamic terminal feedback which uses only a small horizon length.

A comparison, in terms of computational complexity and performance, between the two nominal-cost-based schemes with a worst-case disturbance realization of $w(t) = \text{sgn}(x(t))$, for 500 random initial conditions lying inside both $\mathcal{X}_{x_S}^{18}$ and $\mathcal{X}_x^3$, shows that the computation times are significantly higher, by about 20 times or higher on the average (1.82s as against 87ms), in the terminal-static-feedback-based scheme,⁴ clearly due to significantly higher number of decision variables. Moreover, even the average regulation costs are about 15% higher for this scheme than for the one using the proposed dynamic terminal policy.

⁴Computations times are based on the computing setup as mentioned above.

Figure 3.6: (a) Feasibility sets for an LTI system - (3.26) with $a(t) = 1.3$ and $b(t) = 1$ - for control parameterizations involving disturbance feedforward. (b) Time variation of states for an initial state $[-14.0 \ 2.5]^T$ when the system is controlled with nominal cost minimization schemes based on static and dynamic terminal policies.

Figure 3.7: (a) Feasibility sets for an LTI system - (3.26) with $a(t) = 1.3$ and $b(t) = 1$ - for control parameterizations without disturbance feedforward. (b) Time variation of states for an initial state $[-12.0 \ 1.6]^T$ when the system is controlled with nominal cost minimization schemes based on static and dynamic terminal policies.

We next consider the performance and the computational aspects of the scheme considered in Proposition 3.6. Fig. 3.7a shows the polyhedral invariant set $\bar{\mathcal{X}}_x$ for system (3.26) with $a(t) = 1.3$ and $b(t) = 1$, associated with the controller matrices obtained using (3.25). Again, we consider a set of 100 initial conditions chosen randomly along the boundary of the set $\bar{\mathcal{X}}_x$ and apply the scheme of Proposition 3.6 with $N = 0, 1, 2, 3$ and 4 for the worst-case realization of the disturbance as before. Table 3.2 shows the average computation times⁵ observed for different horizon lengths and the corresponding average normalized regulation costs. Clearly, the increase in computation times with increasing N in this scheme is far smaller than the corresponding growth in the scheme of Proposition 3.5 (cf. Table 3.1).

Next, we compare the performance of the scheme of Proposition 3.6 with that of a similar scheme that is based on a static terminal feedback (e.g., [64, 67]). It can also be observed that the domain of attraction for the system with the static terminal feedback for $N = 18$, $\bar{\mathcal{X}}_{xS}^{18}$ is almost the same in size as that for the system with a dynamic terminal policy for $N = 5$, $\bar{\mathcal{X}}_x^5$. Fig. 3.6b shows the control performances in the two cases for an initial state $[-12.0 \ 1.6]^T$, using $\mathcal{X}_{xS}$ and $\bar{\mathcal{X}}_x$ as the terminal sets. It can be seen from the plots that the two schemes have almost the same performance with the one with static terminal feedback being slightly better in this example. A comparison, in terms of computational requirements and performance, between two schemes with a worst-case disturbance realization as men-

⁵The computation times are based on the computing environment mentioned above.

Table 3.2: Average computation times and costs with the RHC law of Proposition 3.6

	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
Average computation time (ms)	0.8	1.3	1.4	1.7	2.0
Average cost (normalized)	100.0	97.3	96.1	95.3	95.0

tioned above for 500 random initial conditions lying inside both $\bar{\mathcal{X}}_{xS}^{18}$ and $\bar{\mathcal{X}}_x^5$ shows that the average regulation costs are almost the same, with the one in the scheme using the static terminal feedback being slightly lower by about 1.0%. Per step computation time is however significantly higher for the static-feedback-based scheme, by more than 2.4 times (5.6ms as compared to 2.3ms for the dynamic-feedback-based scheme), on the average, when solved with a solver and a machine as mentioned above.

Illustration 3.5. In this example, we consider a higher dimensional system defined by system matrices $A(t) = \Phi(\tau, t)_{\tau=T_s}$, $B_u(t) = \left(\int_0^{\sigma T_s} \Phi(T_s - \tau, t) d\tau \right) \begin{bmatrix} 0 \\ I_2 \end{bmatrix}$ and $B_w(t) = \left(\int_0^{T_s} \Phi(T_s - \tau, t) d\tau \right) \begin{bmatrix} 0 \\ I_2 \end{bmatrix} \bar{T}$, where

$$\Phi(\tau, t) = \begin{bmatrix} 1 & 0 & \tau & \omega(t)\tau^2 \\ 0 & 1 & -\omega(t)\tau^2 & \tau \\ 0 & 0 & 1 & 2\omega(t)\tau \\ 0 & 0 & -2\omega(t)\tau & 1 \end{bmatrix}. \quad (3.43)$$

Here, $\omega(t)$ is a time-varying parameter. This system approximately represents the dynamics of relative motion between satellites on a low Earth orbit. The state, the input and the disturbance input are expressed as $x(t) = [r_1 \ r_2 \ v_1 \ v_2]^T$, $u(t) = [u_1 \ u_2]^T$ and $w(t) = [w_1 \ w_2]^T$. The details of the system with more accurate modeling will be considered in Chapter 7. For an illustrative purpose, we consider the following data in this example: $T_s = 30$, $\sigma = 1/3$, $0.0008 \leq \omega(t) \leq 0.0012$, $\bar{T} = 10^{-4}I_2$. The constraints considered are: $-2 \times 10^{-3} \leq u_1, u_2 \leq 2 \times 10^{-3}$, $-500 \leq r_1, r_2 \leq 500$ and $-0.5 \leq v_1, v_2 \leq 0.5$. Since the system matrices depend linearly on $\omega(t)$, the system polytope is defined by $n_\Omega = 2$ vertices that correspond to $\omega(t) = 0.0008$ and $\omega(t) = 0.0012$. We choose $Q = \text{diag}(I_2, 10^4 I_2)$, $R = 10^6 I_2$ and the corresponding LQ optimal feedback gain K for the average model that corresponds to $\omega(t) = 0.001$.

Because of the higher dimension and the uncertain nature of the system of this example, the polyhedral invariant sets corresponding to the controller matrices computed by solving (3.17) or (3.25) have complex representations with a large number of linear constraints; so some lower-complexity inner approximations are needed for practical purposes (see Appendix A). We only consider the case with no disturbance feedforward in this illustration. Fig. 3.8 shows the projections $\bar{\mathcal{X}}_r$ and $\bar{\mathcal{X}}_v$, on the $r_1 - r_2$ and $v_1 - v_2$ planes, of an inner approximation of the polyhedral feasible invariant set $\bar{\mathcal{X}}_x$ which corresponds to the controller matrices obtained by solving problem (3.25) with $F^{(j)} = 0$ ($j = 1, 2$). These projections are compared with the corresponding projections of the maximal admissible set $\mathcal{X}_{xS}$ for the controlled system with the optimal static feedback. Clearly, a control scheme using $\mathcal{X}_{xS}$ as the terminal set will require a large horizon length to ensure feasibility for initial conditions lying in $\bar{\mathcal{X}}_x$.

It is found that for initial conditions on the boundary of $\bar{\mathcal{X}}_x$, using $\mathcal{X}_{xS}$ as the terminal set requires a horizon length of 40 on the average. Comparing the performance of the

Figure 3.8: Projections of the polyhedral feasible invariant sets for the system of Illustration 3.5 - (a) on the r -space, and, (b) on the v -space.

proposed scheme with $N = 10$ to that of the control scheme using the terminal static feedback (e.g., [64]) with $N = 50$ — for 500 randomly chosen initial conditions feasible for both the schemes and with a worst-case disturbance realization — we find that the former approach offers a remarkable computational efficiency requiring, on the average, only about a fifth of the computation time required by the latter (53ms as compared to 261ms for the latter)⁶, while also having a comparable performance in terms of the average regulation cost which is about 3.5% lower in the latter scheme. $\square$

In the illustrative examples presented above, we have not detailed the computation times of the off-line part of the control problem. The reason is that for the off-line part of the problem, the actual computation times are not so important providing the problem can be solved in a reasonable time with available solvers. However, to give a rough idea of the off-line computation times that may be expected for the kind of example problems discussed above, we note that for the example system in (3.26), solving problem (3.17) or (3.25) is found to take a few seconds with the LMI control toolbox [130] in MATLAB in the computing setup mentioned in Illustration 3.4. Further, the computation of the polyhedral set $\mathcal{X}_x$ or $\bar{\mathcal{X}}_x$ with the algorithm mentioned in Appendix A using the BPMPD solver is found to take time in the order of 10 seconds in the same computing setup. In the case of the system of Illustration 3.5, the corresponding computations are found to take longer times with the computation of the polyhedral sets taking up to a few minutes.

3.6 Conclusion

In this chapter, we have presented a unified, uncertainty-based, infinite-horizon control parameterization based on a dynamic policy for a class of LTV systems with a polytopic system description and with bounded disturbances. We have proposed an off-line approach to optimize the parameters defining the dynamic control policy and also explored various nominal-cost-based RHC schemes that are built on the proposed policy.

⁶The computation times are based on computing environment mentioned in Illustration 3.4.

The use of the off-line-optimized dynamic policy allows us to compute the feasible invariant set for the controlled system off-line and to efficiently optimize the infinite-horizon cost on-line over the feasible set. We have also presented a generalized parameterization for a deterministically time-varying system by appending the dynamic policy to a standard finite-horizon MPC parameterization, and explored efficient nominal-cost-based RHC schemes for this case. We have analyzed the control performance of the proposed schemes and presented several numerical examples to illustrate various aspects of the proposed input parameterization and the proposed RHC schemes.

The results on uncertainty-based RHC policies that we have presented in this chapter and will subsequently be exploited in the coming chapters apply to linear polytopic systems with componentwise bounded disturbances. They can be easily extended to cases where the additive disturbances are bounded in polytopes of any shape. Therefore, it is possible to apply these results to any linear system with uncertainties provided that the model uncertainty set and the disturbance set can be expressed, albeit possibly in a conservative way, as polytopes.

Chapter 4

Min-max RHC based on Optimized Dynamic Policy

4.1 Introduction

The robust RHC schemes explored in Chapter 3, like many other robust RHC schemes in the literature, are based on the minimization of a nominal cost that does not account for the presence of disturbances. While such a scheme essentially simplifies control computations and may have a satisfactory performance in many cases, it may be desirable to have a guarantee of a robust control performance for any possible realization of disturbances affecting the system. In order to guarantee a robust control performance, we often consider the min-max RHC problem, which is based on a worst-case performance criterion.

Min-max RHC in the presence of disturbances has been explored in a number of existing works (see, e.g., [51, 65, 71, 73, 81, 83, 131–133]) and most of these works consider a game theoretical approach based on an H_∞ -like cost function. Solving a min-max RHC problem is significantly more complicated than solving the nominal RHC problem and ordinary approaches may lead to excessively conservative solutions [18, 81]. In this chapter, we explore computationally efficient min-max RHC schemes that employ the dynamic-policy-based control parameterization considered in Chapter 3. We optimize the parameters of the dynamic policy off-line in order to determine the corresponding feasible invariant set in such a way that, under a suitable RHC scheme, the ℓ_2 gain from the disturbance to a certain performance output can be guaranteed to be bounded by a desired value. Besides presenting results on the computation of the parameters of the dynamic policy and the corresponding feasible invariant set for the controlled system, we discuss on-line optimization schemes for uncertain and deterministic LTV systems and present results on the performance of these schemes in terms of the ℓ_2 stability of the controlled system.

4.2 Problem Description

As in Chapter 3, we consider an LTV system described by (3.1) and satisfying assumptions A3.1-A3.4 mentioned in Section 3.2. The control objective is to efficiently compute closed loop control strategies by minimizing the cost in (3.5) for a min-max performance. The min-max (worst-case) criterion that we consider is an H_∞ -like performance index defined as

$$\hat{J}(t) = \max_{w(t+i) \in \mathcal{W}, i \in \mathbb{Z}_+} \sum_{i=0}^{\infty} \left\{ \|Q^{\frac{1}{2}}x(t+i|t)\|^2 + \|R^{\frac{1}{2}}u(t+i|t)\|^2 - \gamma^2 \|w(t+i)\|^2 \right\} \quad (4.1)$$

where γ is a gain bound chosen *a-priori*.

We compute and implement the control actions at each time step in the receding horizon fashion. The control performance is evaluated by analyzing the stability of the controlled system (3.7).

4.3 Min-max RHC Based on Optimized Dynamic Policy

We have seen in Chapter 3 that the input parameterization of Section 3.3 employing a dynamic policy based on disturbances (and model uncertainties when present) allows the use of efficient RHC schemes with reduced conservativeness and enlarged feasibility domain. In this section, we explore the use of this parameterization for an RHC scheme based on the min-max performance index of (4.1). Since we parameterize control inputs in terms of disturbances (and model uncertainties), the right hand side of (4.1) is a function of the parameters that define the dynamic policy, viz., the matrices H , $G^{(j)}$ and $F^{(j)}$, $j = 1, \dots, n_\Omega$ as well as the initial state $\xi(t)$ of the dynamic controller. An on-line optimization over the parameters of all constraint admissible dynamic policies is, however, computationally intensive. So, following the approach used in Chapter 3, we compute the parameter matrices H , $G^{(j)}$ and $F^{(j)}$, $j = 1, \dots, n_\Omega$ off-line in such a way that the min-max optimization can be efficiently carried out on-line.

4.3.1 Computation of controller matrices and feasible invariant set

While determining the controller matrices off-line, we wish to ensure that the volume of the invariant set, over which the resulting min-max control policy is feasible, is as large as possible. For this, we define the augmented state $\chi(t)$ as described in (3.10) and consider maximizing the volume of the projection $\mathcal{S}_x$, on the x -subspace, of a feasible invariant set $\mathcal{S}_\chi$ for the augmented system. As mentioned in Section 3.3.2, we consider an ellipsoidal feasible invariant set $\mathcal{E}_\chi = \{\chi \mid \chi^T \mathcal{P}^{-1} \chi \leq 1\}$ for the augmented system (3.10) while determining the controller matrices so that the size of its projection $\mathcal{E}_x$ on the

x -subspace can be directly optimized by solving a tractable problem, namely, maximizing the determinant of $T \mathcal{P} T^T$ where $T = [I_{n_x} \ 0]$.

The invariance condition (3.13) and the feasibility condition (3.14) must be applied while determining the controller matrices in order to ensure that the resulting dynamic policy is constraint admissible. However, these two conditions alone do not ensure a control performance based on the performance function (4.1) for all initial conditions in the set $\mathcal{E}_x$. To ensure such a performance, the predicted state and input sequences must be such that the right hand side of (4.1) is bounded. For this, the controller matrices must satisfy a condition that guarantees boundedness. We next note the following lemma (see, e.g., [26]) which states such a condition.

Lemma 4.1. For a dynamical system described by

$$x(t+1) = \Phi(t)x(t) + B_w(t)w(t), \quad z(t) = C_z(t)x(t) + D_z(t)w(t) \quad (4.2)$$

where $\begin{bmatrix} \Phi(t) & B_w(t) \\ C_z(t) & D_z(t) \end{bmatrix} \in \tilde{\Omega} = \text{Co} \left\{ \begin{bmatrix} \Phi^{(j)} & B_w^{(j)} \\ C_z^{(j)} & D_z^{(j)} \end{bmatrix}, j = 1, \dots, n_{\tilde{\Omega}} \right\}$ with stable $\Phi^{(j)}$ ($j = 1, \dots, n_{\tilde{\Omega}}$), if there exists a matrix $P \succeq 0$ that satisfies the LMIs

$$\begin{bmatrix} P & \Phi^{(j)\top} P & C_z^{(j)\top} & 0 \\ * & P & 0 & P B_w^{(j)} \\ * & * & I & D_z^{(j)} \\ * & * & * & \gamma^2 I \end{bmatrix} \succeq 0, \quad j = 1, 2, \dots, n_{\tilde{\Omega}} \quad (4.3)$$

for a positive scalar γ , then, for all $t \in \mathbb{Z}_+$,

$$\|P^{\frac{1}{2}}x(t+1)\|^2 - \|P^{\frac{1}{2}}x(t)\|^2 \leq -\|z(t)\|^2 + \gamma^2\|w(t)\|^2 \quad (4.4)$$

and hence

$$\sum_{t=0}^{\infty} (\|z(t)\|^2 - \gamma^2\|w(t)\|^2) \leq \|P^{\frac{1}{2}}x(0)\|^2 \quad (4.5)$$

so that the system satisfies $\frac{\|z\|_{\ell_2}}{\|w\|_{\ell_2}} \leq \gamma$ for all $w \in \ell_2$ and $x(0) = 0$. $\square$

Now, in order to ensure that a boundedness condition of the form (4.5) is satisfied for a suitable γ by the closed loop controlled system (3.7), we can apply condition (4.3) for the augmented system (3.10) while determining the controller matrices. That is, while determining the controller matrices and the feasible invariant set, we solve the problem

$$\max_{\mathcal{P}, \mathcal{P}, Z, H, G^{(j)}, F^{(j)}, \alpha^{(j)}, R_w^{(j)} (j=1, \dots, n_{\Omega})} \log \det(T \mathcal{P} T^T) \quad (4.6)$$

subject to conditions (3.13) and (3.14), and the boundedness condition

$$\begin{bmatrix} \mathcal{P} & \Psi^{(j)\top} \mathcal{P} & C_z^\top & 0 \\ * & \mathcal{P} & 0 & \mathcal{P} \mathcal{B}_w^{(j)} \\ * & * & I & 0 \\ * & * & * & \gamma^2 I \end{bmatrix} \succeq 0, \quad j = 1, 2, \dots, n_\Omega \quad (4.7)$$

where $C_z = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ R^{\frac{1}{2}} K & R^{\frac{1}{2}} H \end{bmatrix}$.

The constraints applied in problem (4.6) are bilinear matrix inequalities (BMIs) and can be transformed into simpler conditions as in Proposition 3.1 if we express (4.7) in terms of $\tilde{\mathcal{P}} = \mathcal{P}^{-1}$ and consider $\tilde{\mathcal{P}} = \mathcal{P}$. However, using the same Lyapunov matrix for both (3.13) and (4.7) will lead to conservative results. So, alternatively, we can directly optimize the controller matrices under the BMI conditions using available BMI solvers such as PENBMI [134], or using an alternating SDP optimization approach, making use of the special structure of the problem. We briefly outline the second approach that we use to solve this problem.

Algorithm 4.1: Computation of controller matrices

1. Choose Q, R and γ for a desired control performance, and compute the infinite-horizon unconstrained LQ or H_∞ optimal state-feedback gain K for the nominal or average system.
2. Express $\mathcal{P}$ in (3.13) and (3.14) as $\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{12}^\top & \mathcal{P}_{22} \end{bmatrix}$ and express (4.7) in terms of $\tilde{\mathcal{P}} = \begin{bmatrix} \tilde{\mathcal{P}}_{11} & \tilde{\mathcal{P}}_{12} \\ \tilde{\mathcal{P}}_{12}^\top & \tilde{\mathcal{P}}_{22} \end{bmatrix} = \mathcal{P}^{-1}$, where $\mathcal{P}_{11}, \tilde{\mathcal{P}}_{11} \in \mathbb{R}^{n_x \times n_x}$. Then, when either $H, G^{(j)}, \alpha^{(j)}$ or $\mathcal{P}_{12}, \mathcal{P}_{22}, \tilde{\mathcal{P}}_{12}, \tilde{\mathcal{P}}_{22}, \alpha^{(j)}$ are fixed, the conditions in (3.13), (3.14) and (4.7) become LMIs in the rest of the variables.
3. Initialize H and $G^{(j)}$ ($j = 1, \dots, n_\Omega$) such that the conditions in (3.13), (3.14) and (4.7) (expressed in terms of $\tilde{\mathcal{P}}$) are satisfied for some $\mathcal{P}_{11} = \mathcal{P}_{11}^0$ and some values of the other variables.
4. Fix H and $G^{(j)}$ ($j = 1, \dots, n_\Omega$) at their current values and maximize $\log \det(\mathcal{P}_{11})$ over the other variables in the three sets of conditions.
5. Fix $\mathcal{P}_{12}, \mathcal{P}_{22}, \tilde{\mathcal{P}}_{12}$ and $\tilde{\mathcal{P}}_{22}$ at their current values and maximize $\log \det(\mathcal{P}_{11})$ over the other variables in the three sets of conditions.
6. If $\det(\mathcal{P}_{11}) - \det(\mathcal{P}_{11}^0) \geq TOL$, a predefined tolerance, set $\mathcal{P}_{11}^0 = \mathcal{P}_{11}$ and go to Step 4.

Algorithm 4.1 will terminate when $\det(\mathcal{P}_{11})$ reaches a local maximum that depends on the starting conditions used in Step 3. A suitable selection of the initial conditions will lead to a solution close to the global maximum. Matrices $\mathcal{P}$ and $\mathcal{P}_{11}^0$ obtained from

Algorithm 4.1 define the ellipsoidal inner approximations $\mathcal{E}_\chi$ and $\mathcal{E}_x$ of the largest feasible invariant sets for the augmented system state and the controlled system state respectively. The corresponding polyhedral sets $\mathcal{X}_\chi$ and $\mathcal{X}_x$ can be obtained using available algorithms as mentioned in Chapter 3.

4.3.2 On-line computations and control performance

Now, we discuss the on-line determination of constraint-admissible control actions for a desired min-max performance. As discussed in Section 3.3.3, given off-line-determined feasible invariant sets $\mathcal{S}_x$ and $\mathcal{S}_\chi$, respectively for the controlled system and the augmented system (3.10), a feasible control input for any system state $x(t) \in \mathcal{S}_x$ at time t can be written as $u(t) = \kappa(x(t))$ where $\kappa(x(t))$ satisfies (3.27). Further, because of the robust invariance of the set $\mathcal{S}_\chi$, any such control input $u(t) = \kappa(x(t))$ satisfying (3.27) at time t will ensure the existence of feasible control inputs at successive times.

Given the controller matrices, we wish to determine the controller initial state $\xi(t)$ on-line at each time step t such that it minimizes the predicted min-max performance index (4.1) of the closed loop system under the control law of (3.9). An upper bound of the min-max performance index for the uncertain polytopic system is represented by the function

$$\hat{J}(t) = \left\| \mathcal{P}^{\frac{1}{2}} \begin{bmatrix} x(t)^T & \xi(t)^T \end{bmatrix}^T \right\|^2$$

where $\mathcal{P}$ is a positive definite cost matrix obtained by solving, under condition (4.7), the problem

$$\min_{\mathcal{P}} \text{Tr}(\mathcal{P}) \quad (4.8)$$

where $\xi(t)$ is a suitably-chosen, constraint-admissible controller initial state. The following proposition deals with the performance of an RHC scheme based on the minimization of the min-max performance index.

Proposition 4.1. Let $\mathcal{S}_\chi$ be a compact feasible invariant set for the augmented system (3.10) with controller matrices obtained from (4.6) and let $\mathcal{S}_x$ be its projection on the x -subspace. Also, let $\mathcal{P}$ be the cost matrix obtained from (4.8). Then, for any initial state $x(t) \in \mathcal{S}_x$, the receding horizon control $u^*(t) = \kappa^*(x(t)) = Kx(t) + H\xi^*(t)$ where $\xi^*(t)$ is obtained by solving

$$\left. \begin{array}{l} \min_{\xi(t)} \hat{J}(t) \\ \text{with } \hat{J}(t) = \left\| \mathcal{P}^{\frac{1}{2}} \begin{bmatrix} x(t)^T & \xi(t)^T \end{bmatrix}^T \right\|^2 \end{array} \right\} \quad (4.9)$$

such that $\begin{bmatrix} x(t)^T & \xi(t)^T \end{bmatrix}^T \in \mathcal{S}_\chi$ at each time t , ensures the satisfaction of constraints for all times, and guarantees that the cost monotonicity condition

$$\hat{J}^*(x(t+1), t+1) \leq \hat{J}^*(x(t), t) - \left(\|Q^{\frac{1}{2}}x(t)\|^2 + \|R^{\frac{1}{2}}u^*(t)\|^2 - \gamma^2\|w(t)\|^2 \right) \quad (4.10)$$

Figure 4.1: State and control trajectories for the polytopic system in (3.26) under the control scheme mentioned in Proposition 4.1: (a) State trajectories starting from an initial state $[4.5 \ 1.5]^T$, and (b) the corresponding control trajectories.

and hence the boundedness condition

$$\sum_{t=0}^{\infty} \left(\|Q^{\frac{1}{2}}x(t)\|^2 + \|R^{\frac{1}{2}}u^*(t)\|^2 - \gamma^2\|w(t)\|^2 \right) \leq \beta_0 \quad (4.11)$$

with $\beta_0 = \|\mathcal{P}^{\frac{1}{2}} [x(0)^T \ \xi^*(0)^T]^T\|^2$ are satisfied, so that the controlled system (3.7) is ℓ_2 -stable with a finite gain γ . $\square$

PROOF. The proof is immediate as the predicted future states and inputs are all feasible because of the invariance of the set $\mathcal{S}_X$, and satisfy the monotonicity condition

$$\begin{aligned} \left\| \mathcal{P}^{\frac{1}{2}} [x(t+1|t)^T \ \xi(t+1|t)^T]^T \right\|^2 - \left\| \mathcal{P}^{\frac{1}{2}} [x(t)^T \ \xi^*(t)^T]^T \right\|^2 \\ \leq - \left(\|Q^{\frac{1}{2}}x(t)\|^2 + \|R^{\frac{1}{2}}u^*(t)\|^2 - \gamma^2\|w(t)\|^2 \right) \end{aligned}$$

for all allowable $w(t)$. $\blacksquare$

Illustration 4.1. In order to illustrate the robust performance of the min-max cost minimization scheme discussed above, we again consider the system described in (3.26). We choose the same values of Q , R and the corresponding LQ optimal feedback gain K as in Illustration 3.1, and determine the controller matrices by solving (4.6) with $\gamma = 0.5$ using Algorithm 4.1. Fig. 4.1a shows the resulting polyhedral invariant set $\mathcal{X}_x$ for the controlled system. It also shows a set of 50 state trajectories originating from an initial state $[4.5 \ 1.5]^T$ when the system is under the control scheme of Proposition 4.1. Random realizations of the time-varying element $\rho(t)$ as well as the disturbance signal $w(t)$ within assumed bounds are considered in the simulations. It is observed that the trajectories in all the cases converge to a neighbourhood of the origin in a finite number of steps. Fig. 4.1b shows the set of control trajectories corresponding to the state trajectories in Fig. 4.1a. The control constraint $|u(t)| \leq 1$ is respected in all cases.

Proposition 4.1 states that if the control problem (4.9) is feasible at time $t = 0$, then the controlled system is guaranteed to satisfy the H_∞ performance condition with the given gain bound γ . Clearly, γ specifies the level of disturbance attenuation guaranteed

for the closed-loop predicted system and hence affects the system control performance. In the next subsection, we further discuss the role of γ in shaping the features of the min-max dynamic policy with regard to its guaranteed performance and the size of the feasibility domain it offers.

4.3.3 Guaranteed-performance-versus-feasibility tradeoff

The problem of determining the controller matrices in (4.6) involves the H_∞ performance condition (4.7) in addition to the invariance and feasibility conditions (3.13) and (3.14) considered in (3.17). Clearly, the choice of the gain bound γ in condition (4.7) will affect the outcome of (4.6). While it obviously determines the guaranteed performance of the resulting control policy, it will also affect the size of feasibility domain for the controlled system. With a very large γ , condition (4.7) will likely have no impact and the resulting controller matrices will give a feasibility domain that is close to the maximal one obtained with (3.17). A very small value of γ , on the other hand, may result in the infeasibility of the optimization problem. In this section, we explore a practical lower bound on γ that can be considered in the proposed min-max RHC framework and discuss the role of γ as a tuning parameter that can be adjusted for a tradeoff between the guaranteed H_∞ performance and the size of the resulting feasibility domain.

To motivate the definition of a practical lower bound on γ , let us consider the case of a static feedback control $u(t) = Kx(t)$ with the feedback gain K for the system in (3.1). The closed-loop system under such an input, along with an output $z(t)$ defined such that $\|z(t)\|^2$ represents the stage cost $\|Q^{\frac{1}{2}}x(t)\|^2 + \|R^{\frac{1}{2}}u(t)\|^2$ can then be written as

$$x(t+1) = (A(t) + B_u(t)K)x(t) + B_w(t)w(t), \quad z(t) = \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}K \end{bmatrix} x(t). \quad (4.12)$$

Clearly, a constant γ will be an upper bound of the ℓ_2 gain of the closed loop system (4.12) if there exists a $P \succeq 0$ satisfying the LMIs

$$\begin{bmatrix} P - Q - K^T R K & \mathcal{A}^{(j)} P & 0 \\ * & P & P B_w^{(j)} \\ * & * & \gamma^2 I \end{bmatrix} \succeq 0, \quad j = 1, \dots, n_\Omega. \quad (4.13)$$

Now, let us first note the following result.

Proposition 4.2. Let $F^{(j)} = 0$, $j = 1, \dots, n_\Omega$ in condition (4.7) and let $G^{(j)}$, $j = 1, \dots, n_\Omega$ satisfy $\tilde{X} - G^{(j)\top} \tilde{X} G^{(j)} \succ 0$ for some $\tilde{X} \succ 0$. Then, there exists a $\mathcal{P} \succeq 0$ satisfying (4.7) with a given γ for any H if γ satisfies condition (4.13) for some $P \succeq 0$. $\square$

PROOF. Since $F^{(j)} = 0$, $j = 1, \dots, n_\Omega$, a $\mathcal{P} \succeq 0$ satisfying condition (4.7) with a given

γ exists if there exist $P, X \succeq 0$ such that

$$\begin{bmatrix} P & 0 & \mathcal{A}^{(j)\top}P & 0 & Q^{\frac{1}{2}} & K^{\top}R^{\frac{1}{2}} & 0 \\ * & X & H^{\top}B_u^{(j)\top}P & G^{(j)\top}X & 0 & H^{\top}R^{\frac{1}{2}} & 0 \\ * & * & P & 0 & 0 & 0 & PB_w^{(j)} \\ * & * & * & X & 0 & 0 & 0 \\ * & * & * & * & I & 0 & 0 \\ * & * & * & * & * & I & 0 \\ * & * & * & * & * & * & \gamma^2 I \end{bmatrix} \succeq 0, \quad j = 1, \dots, n_{\Omega}. \quad (4.14)$$

Here, condition (4.14) is obtained by letting $\mathcal{P} = \text{diag}(P, X)$ in (4.7) with $F^{(j)} = 0, j = 1, \dots, n_{\Omega}$. This condition is satisfied if and only if

$$\begin{bmatrix} X & G^{(j)\top}X \\ * & X \end{bmatrix} \succeq 0, \quad j = 1, \dots, n_{\Omega} \quad (4.15a)$$

$$\text{and} \quad \begin{bmatrix} P - Q & \mathcal{A}^{(j)\top}P & 0 & K^{\top}R^{\frac{1}{2}} \\ * & P & PB_w^{(j)} & 0 \\ * & * & \gamma^2 I & 0 \\ * & * & * & I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ PB_u^{(j)}H & 0 \\ R^{\frac{1}{2}}H & 0 \\ * & 0 \end{bmatrix} \times \begin{bmatrix} X & G^{(j)\top}X \\ * & X \end{bmatrix}^{-1} \begin{bmatrix} 0 & H^{\top}B_u^{(j)\top}P & H^{\top}R^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \succeq 0, \quad j = 1, \dots, n_{\Omega}. \quad (4.15b)$$

Since $\tilde{X} - G^{(j)\top}\tilde{X}G^{(j)} \succ 0, j = 1, \dots, n_{\Omega}$ for some $\tilde{X} \succ 0$, (4.15a) is satisfied with any $X = \sigma\tilde{X}$, where σ is a scalar greater than 1. Moreover, if (4.13) is feasible, we can always find a P satisfying (4.15b) by choosing an $X = \sigma\tilde{X}$ with arbitrarily large σ . ■

Proposition 4.2 implies that if the gain considered in (4.7) is not smaller than the smallest γ satisfying (4.13) for a given K , condition (4.7) is satisfied for any H and any quadratically stable $G^{(j)}, j = 1, \dots, n_{\Omega}$ for $F^{(j)} = 0, j = 1, \dots, n_{\Omega}$. To further discuss the result, let us denote by $\mathcal{G}$ all vertices $G^{(j)}, j = 1, \dots, n_{\Omega}$ of the polytopic matrix G and by $\mathcal{F}$ all vertices $F^{(j)}, j = 1, \dots, n_{\Omega}$ of F . Moreover, let us define the set $\mathcal{G}$ as $\mathcal{G} = \{ \{G^{(j)}, j = 1, \dots, n_{\Omega}\} \mid \exists X \text{ s.t. } X - G^{(j)\top}XG^{(j)} \succ 0, j = 1, \dots, n_{\Omega} \}$. Now, we present the following result which relates the gain bounds in the cases of static and dynamic control policies.

Proposition 4.3. Let $\gamma_s(P, K)$ denote any γ satisfying condition (4.13) with the given P and K , and, let $\gamma_d(\mathcal{P}, K, H, \mathcal{G}, \mathcal{F})$ denote any γ satisfying condition (4.7) with the given matrices $\mathcal{P}, K, H, \mathcal{G}$ and $\mathcal{F}$ where $\mathcal{G}$ and $\mathcal{F}$ respectively denote all vertices $G^{(j)}, j = 1, \dots, n_{\Omega}$ and $F^{(j)}, j = 1, \dots, n_{\Omega}$ of the polytopic matrices G and F . Then

$$\sup_{H, \mathcal{G} \in \mathcal{G}} \inf_{\mathcal{P}} \gamma_d(\mathcal{P}, K, H, \mathcal{G}, 0) = \inf_P \gamma_s(P, K). \quad (4.16)$$

PROOF. It follows from Proposition 4.2 that

$$\sup_{H, \mathcal{G} \in \mathcal{G}} \inf_{\mathcal{P}} \gamma_d(\mathcal{P}, K, H, \mathcal{G}, 0) \leq \inf_P \gamma_s(P, K). \quad (4.17)$$

On the other hand, setting $H = 0$ and $G^{(j)} = 0, F^{(j)} = 0, j = 1, \dots, n_\Omega$ in (4.7), and deleting certain row-column pairs, we get condition (4.13) with the leading $n_x \times n_x$ block of $\mathcal{P}$ as P . Therefore, we have,

$$\begin{aligned} \inf_P \gamma_s(P, K) &\leq \inf_{\mathcal{P}} \gamma_d(\mathcal{P}, K, 0, 0, 0) \\ &\leq \sup_{H, \mathcal{G} \in \mathcal{G}} \inf_{\mathcal{P}} \gamma_d(\mathcal{P}, K, H, \mathcal{G}, 0). \end{aligned} \quad (4.18)$$

Hence, from (4.17) and (4.18), we have

$$\begin{aligned} \sup_{H, \mathcal{G} \in \mathcal{G}} \inf_{\mathcal{P}} \gamma_d(\mathcal{P}, K, H, \mathcal{G}, 0) &\leq \inf_P \gamma_s(P, K) \\ &\leq \inf_{\mathcal{P}} \gamma_d(\mathcal{P}, K, 0, 0, 0) \leq \sup_{H, \mathcal{G} \in \mathcal{G}} \inf_{\mathcal{P}} \gamma_d(\mathcal{P}, K, H, \mathcal{G}, 0) \end{aligned}$$

from which (4.16) directly follows. ■

Proposition 4.3 states that the smallest upper bound of the ℓ_2 gain, from the disturbance w to the output z , provable via quadratic Lyapunov functions, of system (3.1) under the dynamic control (3.9) without disturbance feedforward is the same as the smallest upper bound of the ℓ_2 gain of the closed-loop system (4.12) under the static feedback input $u(t) = Kx(t)$, where K is the same feedback gain as the one used in the dynamic control policy (3.9).

Now, let us define a constant γ_m as $\gamma_m = \inf_{P, K} \gamma_s(P, K)$, and let K_m be the feedback gain K that minimizes $\gamma_s(P, K)$. The following relation follows from the result of Proposition 4.3.

Corollary 4.1.

$$\begin{aligned} \inf_{\mathcal{P}, K, H, \mathcal{G}, \mathcal{F}} \gamma_d(\mathcal{P}, K, H, \mathcal{G}, \mathcal{F}) &= \sup_{H, \mathcal{G} \in \mathcal{G}} \inf_{\mathcal{P}, K} \gamma_d(\mathcal{P}, K, H, \mathcal{G}, 0) \\ &= \gamma_m. \end{aligned} \quad (4.19)$$

PROOF. We first note that

$$\begin{aligned} \inf_{\mathcal{P}, K, H, \mathcal{G}, \mathcal{F}} \gamma_d(\mathcal{P}, K, H, \mathcal{G}, \mathcal{F}) &\leq \sup_{H, \mathcal{G} \in \mathcal{G}} \inf_{\mathcal{P}, K} \gamma_d(\mathcal{P}, K, H, \mathcal{G}, 0) \\ &\leq \sup_{H, \mathcal{G} \in \mathcal{G}} \inf_{\mathcal{P}} \gamma_d(\mathcal{P}, K_m, H, \mathcal{G}, 0) = \gamma_m \end{aligned}$$

where the last equality follows from Proposition 4.3. However, considering the fact that no closed-loop system specifications based on quadratic Lyapunov functions can

be improved for a linear system by the use of a dynamic feedback instead of a static feedback (see, e.g., [111, Sec. 7.5.3]), we have,

$$\gamma_m \leq \inf_{\mathcal{P}, K, H, G, \mathcal{F}} \gamma_d(\mathcal{P}, K, H, G, \mathcal{F})$$

and hence the relation in (4.19). ■

In the discussion above, γ_m denotes the smallest upper bound of the ℓ_2 gain, provable via quadratic Lyapunov functions, of the closed-loop system (4.12) under an arbitrary static feedback input $u(t) = Kx(t)$. Proposition 4.1 shows it to be equal to the smallest upper bound of the ℓ_2 gain of system (3.1) under the dynamic policy (3.9) of any allowable dynamics for an equivalent output signal. Clearly, γ_m would be an ideal lower bound of γ that could be considered in the performance index (4.1) and hence in the control policy formulation. However, the dynamic control policy that ensures the satisfaction of condition (4.7) with the gain bound γ_m may not necessarily be feasible in the sense of satisfying the invariance and feasibility conditions (3.13) and (3.14) for a $\mathcal{P} \succ 0$ and hence ensuring a non-empty feasible invariant set.

To look for a gain bound that actually ensures a non-empty feasible invariant set, let us again consider the closed-loop system (4.12) under a static feedback input $u(t) = Kx(t)$. The conditions for the invariance and feasibility of an ellipsoidal set $\mathcal{E}_x = \{x \mid x^T W^{-1} x \leq 1\}$, $W \succ 0$ for the closed-loop system (4.12) are given by the LMIs in (3.18).

Now, let $\mathcal{K}$ be a set defined as $\mathcal{K} = \{K \mid (3.18a) \text{ and } (3.18b) \text{ are feasible}\}$. Then, a lower bound on γ that also allows a non-empty feasible invariant set with an arbitrary static feedback can be defined as

$$\gamma_m^o = \inf_{P, K \in \mathcal{K}} \gamma_s(P, K). \quad (4.20)$$

Here, clearly, $\gamma_m^o = \gamma_m$ if $K_m \in \mathcal{K}$. Next, since there obviously exists a dynamic policy, defined by some matrices K , H , $G^{(j)}$ and $F^{(j)}$, $j = 1, \dots, n_\Omega$, that ensures the feasibility of conditions (3.13), (3.14) and (4.7) for $\gamma = \gamma_m^o$, we can consider γ_m^o as a practical lower bound on γ in the control policy formulation. Furthermore, we can use, as the gain K in the dynamic policy, the feedback gain matrix K_m^o that minimizes γ_s in (4.20). In that case, it follows from Proposition 4.2 that condition (4.7) is really irrelevant for any choice of $\gamma \geq \gamma_m^o$ if we consider $F^{(j)} = 0$, $j = 1, \dots, n_\Omega$, and the controller matrices H and $G^{(j)}$, $j = 1, \dots, n_\Omega$ can be freely optimized under conditions (3.13) and (3.14) only in order to maximize the volume of the feasible set. As discussed in Section 3.3.2, this optimization problem (i.e., problem (4.6) without condition (4.7)) can be transformed into a simpler one when $n_\xi \geq n_x$ and can be easily solved using LMI solvers. Its solution obtained for $F^{(j)} = 0$, $j = 1, \dots, n_\Omega$ can be used as the initial solution while solving the full problem (4.6) using the alternating SDP approach mentioned earlier. When we allow disturbance feedforward (i.e., non-zero $F^{(j)}$, $j = 1, \dots, n_\Omega$), the imposition of condition

Figure 4.2: Feasible invariant sets for the polytopic system of (3.26) obtained by solving (4.6) for two values of γ : $\gamma_1 = 0.25$ and $\gamma_2 = 1$.

Table 4.1: Volumes of ellipsoidal feasible invariant sets for various γ (m^4)

$F^{(j)} = 0$	$\gamma = 0.75$	$\gamma = 1$	$\gamma = 2$	$\gamma = 5$	$\gamma = 10$
2.49×10^3	3.48×10^3	4.26×10^3	8.06×10^3	3.83×10^4	1.34×10^5

(4.7) with a desired $\gamma \geq \gamma_m^o$ will be necessary while solving (4.6), and the volume of the feasible invariant set obtained from the solution will be affected by the choice of γ — a larger γ will typically allow a larger feasible invariant set.

Note that while γ_m can be easily determined by solving an LMI problem, the determination of γ_m^o as defined in (4.20) will require the solution of a problem with BMI constraints if $K_m \notin \mathcal{K}$. In such a case, γ_m^o can be closely approximated by testing the feasibility of the BMI constraints (3.18a), (3.18b) and (4.13) for a range of values of γ above γ_m by employing methods such as cone-complementarity linearization [135] or concave minimization [136].

Illustration 4.2. For the example system described in (3.26), $\gamma_m^o = \gamma_m = 0.24$ whereas $\inf_P \gamma_s(P, K_{LQ}) = 0.306$ where $K_{LQ} = [-0.82 \ -1.21]$ is the unconstrained LQ optimal gain as considered in Illustration 4.1. Using $K = K_m = [-1.37 \ -1.07]$, we solve problem (4.6) in order to obtain the controller matrices for two choices of γ : $\gamma_1 = 0.25$ and $\gamma_2 = 1$. The feasible invariant sets $\mathcal{E}_x^{\gamma_1}$ and $\mathcal{E}_x^{\gamma_2}$ obtained are shown in Fig. 4.2. The figure also shows the corresponding polyhedral sets $\mathcal{X}_x^{\gamma_1}$ and $\mathcal{X}_x^{\gamma_2}$. Similarly, for the system considered in Illustration 3.5, with the given details, $\gamma_m = 0.466$, and $\inf_P \gamma_s(P, K_{LQ}) = 0.598$. Using the LQ optimal K , we solve problem (4.6) for increasing values of γ . Table 4.1 shows the volumes of the ellipsoidal feasible invariant sets for the controlled system obtained for these various values of γ . The first entry of the table, however, corresponds to the case of no disturbance feedforward and is obtained solving (3.25) with $F^{(j)} = 0$, $j = 1, \dots, n_\Omega$. Clearly, we obtain larger-volume feasible invariant sets with larger values of γ in both the examples.

It is clear from the above discussion and the observations in Illustration 4.2 that, in the proposed min-max RHC framework, we can practically consider a γ as small as γ_m^o , and the choice of a larger γ typically results in a control policy that offers a larger feasibility domain. A smaller (but allowable) γ , on the other hand, implies the guarantee

of a better closed-loop performance of the controlled system with regard to disturbance attenuation. The bound γ can, therefore, be seen as a tuning parameter which can be appropriately chosen in order to achieve a desired tradeoff between the guaranteed H_∞ control performance and the volume of the feasibility domain.

Remark 4.1. The determination of the policy dynamics in (4.6) is based on the maximization of the volume of a relevant ellipsoidal approximation of the feasible set under the invariance condition (3.13) which is a conservative sufficient condition. Therefore, the maximal feasible invariant set that we compute and use for a min-max RHC scheme will likely be conservative to some extent. It may typically be more so when the feedforward of disturbance is restricted (by choosing a small γ) or is not allowed. As we consider larger values of γ and hence allow more freedom in the choice of the feedforward gain, the conservativeness will decrease and when γ is very large and irrelevant, the ellipsoidal invariant set that we obtain will be equal to the largest such set achievable with an arbitrary static feedback control (cf. Proposition 3.2). On the optimality front, however, the fact that we optimize the policy dynamics off-line in order to maximize the size of the feasibility domain means that the resulting RHC policy will be suboptimal. Furthermore, when larger values of γ are used, controller matrices may be adjusted to their limits to enlarge the feasibility domain, thus further reducing the optimality of the RHC policy. The min-max RHC scheme can, therefore, be expected to improve the performance when a smaller value of γ is considered. However, since a smaller value of γ may also mean an increased conservativeness of the control policy as mentioned above, and since the guaranteed H_∞ performance is based on the predicted closed-loop system, the actual improvement in the performance of the controlled system with the lowering of the value of γ may not necessarily match with the expected or predicted improvement. $\square$

Illustration 4.3. For an illustration, we consider a system defined by the system matrices $A(t) = \Phi(T_s, t)$, $B_u(t) = \left(\int_0^{T_s} \Phi(T_s - \tau, t) d\tau \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $B_w(t) = \begin{bmatrix} d_{m1} & 0 \\ 0 & d_{m2} \end{bmatrix}$, where $\Phi(\tau, t) = f(\bar{\omega}, \tau) + (\omega(t) - \bar{\omega}) \frac{\partial f}{\partial \bar{\omega}}(\bar{\omega}, \tau)$ and

$$f(\omega, \tau) = \begin{bmatrix} \cos(\psi\tau) & (1/\psi)\sin(\psi\tau) \\ -\psi\sin(\psi\tau) & \cos(\psi\tau) \end{bmatrix}, \quad \psi = \bar{\omega}^{1/4}\omega^{3/4}. \quad (4.21)$$

The dynamical model of this example corresponds to the approximate discrete-time dynamics representing the relative motion of an earth-orbiting satellite, in a direction normal to a reference orbit, w.r.t. a rotating reference point on the reference orbit (see, Chapter 7). The time variation of the parameter ω is caused by the non-zero eccentricity of the reference orbit. Furthermore, the effects of unmodeled dynamics including gravity harmonics and other disturbances are included as componentwise bounded external disturbances. We consider the case of a reference orbit for which $\omega(t)$ lies in the interval $[0.0008 \ 0.0012]$ and $\bar{\omega} = 0.001$. Assuming $T_s = 30s$ and $c = 1/3$, we let $B_w(t) = \text{diag}(0.15, 0.002)$ so that the disturbance signal $w(t)$ can be considered to be componentwise bounded by 1 (in m and m/s). Moreover, we also consider the constraints: $|u(t)| \leq 3 \text{ mm/s}^2$, $|x_{[1]}(t)| \leq 3000\text{m}$, $|x_{[2]}(t)| \leq 3\text{m/s}$. We let $Q = \text{diag}(10^{-4}, 1)$ and $R = 10^2$, and for these values $\gamma_m^o = \gamma_m = 0.006$. We choose the gain K_m as K and consider four values of the gain bound — $\gamma_1 = 0.006$, $\gamma_2 = 0.05$, $\gamma_3 = 0.1$ and $\gamma_4 = 1$ — and compute the controller matrices and the feasible invariant sets $\mathcal{X}_x^{\gamma_j}$,

Figure 4.3: Feasible invariant sets for the system of Illustration 4.3 for various values of γ .

$j = 1, 2, 3, 4$. The feasible sets are shown in Fig. 4.3 where the displacement and velocity components $x_1 = x_{[1]}$ and $x_2 = x_{[2]}$ of the state are shown in km and m/s respectively.

For the comparison of control performances, we consider the min-max RHC schemes with the dynamic control policies $\kappa_2(\cdot)$ and $\kappa_4(\cdot)$, which are respectively associated with $\gamma_2 = 0.05$ and $\gamma_4 = 1$. We also randomly choose a set of 30 initial states lying within the set $\mathcal{X}_x^{\gamma_2}$. The plots in Fig. 4.4 show the time-variations of the displacement and velocity components of the system states under control policies $\kappa_2(\cdot)$ and $\kappa_4(\cdot)$ for these initial conditions. In the simulations, we consider the time-variation of $\omega(t)$ as in the actual orbital motion and choose the worst-case disturbance given by $w(t) = \text{sgn}(x(t))$. Clearly, the performance of the control policy $\kappa_2(\cdot)$ is significantly better than that of $\kappa_4(\cdot)$. Fig. 4.5 compares the control trajectories in the two control schemes for a few of the regulation cases shown in Fig. 4.4. It can be observed that the control input remains at the boundaries for longer times before quickly stabilizing to steady-state levels in the case of the control policy $\kappa_2(\cdot)$ than in the case of $\kappa_4(\cdot)$. This indicates an increased suboptimality in the control policy $\kappa_4(\cdot)$ resulting in a poorer performance as compared with the case of $\kappa_2(\cdot)$ (cf. Remark 4.1).

4.4 Min-max RHC with Optimized Dynamic Terminal Policy

In this section, we consider the case of deterministically time-varying systems with additive disturbances and explore a min-max RHC scheme that is based on the use of the optimized dynamic policy considered in the last section as the terminal control policy. As mentioned in Section 3.4, such a scheme allows us to use more degrees of freedom in the finite-horizon control parameterization and thus improve the control performance in this case. In the following, we discuss the details of this scheme.

4.4.1 Control parameterization and feasible solutions

We consider that assumption A3.5 of Section 3.4 is satisfied, and follow the control parameterization discussed in that section. The characterization of the overall feasible set

Figure 4.4: Regulation of the displacement and velocity components of the system states under the control policies $\kappa_2(\cdot)$ and $\kappa_4(\cdot)$.

as well as the feasible inputs considered for the control scheme of this section are as discussed in Section 3.4.1. The only difference here is the use of the terminal set obtained by solving (4.6) in place of (3.17).

4.4.2 On-line cost minimization and control performance

With the control parameterization of (3.32), a min-max performance index of the form (4.1) can be expressed as

$$\hat{J}(t) = \max_{\mathbf{w}(t) \in \mathcal{W}^N} \left\{ \left\| \mathcal{Q}^{\frac{1}{2}} (\mathcal{C}_x + \mathcal{G}_x \mathbf{w}(t)) \right\|^2 + \left\| \mathbf{R}^{\frac{1}{2}} (\mathcal{C}_u + \mathcal{G}_u \mathbf{w}(t)) \right\|^2 - \gamma^2 \|\mathbf{w}(t)\|^2 \right\} \quad (4.22)$$

where

$$\begin{aligned} \mathcal{C}_x &= \begin{bmatrix} \mathbf{A}(t) \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} \mathbf{B}_u(t) & 0 \\ 0 & I_{n_\xi} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{v}}(t) \\ \bar{\xi}(t + N|t) \end{bmatrix}, \quad \mathcal{G}_x = \begin{bmatrix} \mathbf{B}_w(t) + \mathbf{B}_u(t) \mathbf{E}(t) \\ \bar{\mathbf{F}}(t) \end{bmatrix} \\ \mathcal{C}_u &= \check{\mathbf{A}}(t) x(t) + \check{\mathbf{B}}_u(t) \bar{\mathbf{v}}(t), \quad \mathcal{G}_u = \check{\mathbf{B}}_w(t) + \check{\mathbf{B}}_u(t) \mathbf{E}(t) \end{aligned} \quad (4.23)$$

and $\mathcal{Q} = \text{diag}(\mathbf{Q}, \mathcal{P})$. Here, we have included, in the cost (4.22), an upper bound of the terminal cost represented by $[x(t + N|t)^\top \quad \xi(t + N|t)^\top]^\top \mathcal{P} [x(t + N|t)^\top \quad \xi(t + N|t)^\top]^\top$ where $\mathcal{P}$ is obtained from (4.8). The cost $\hat{J}(t)$ in (4.22) can be minimized by minimizing

Figure 4.5: Control inputs applied for a few of the cases shown in Fig. 4.4 under the control policies $\kappa_2(\cdot)$ (dotted) and $\kappa_4(\cdot)$ (solid).

a scalar $g(t)$ such that, for all $\mathbf{w}(t) \in \mathcal{W}^N$,

$$g(t) - \hat{J}(t) \geq 0. \quad (4.24)$$

Applying the S -procedure [111, Sec. 2.6.3], (4.24) is satisfied for all $\mathbf{w}(t) \in \mathcal{W}^N$ if there exists a matrix $\Upsilon \in \mathbb{D}_+^{n_w N}$ such that

$$g(t) - \left(\|\mathcal{Q}^{\frac{1}{2}}(\mathcal{C}_x + \mathcal{G}_x \mathbf{w}(t))\|^2 + \|\mathbf{R}^{\frac{1}{2}}(\mathcal{C}_u + \mathcal{G}_u \mathbf{w}(t))\|^2 - \gamma^2 \|\mathbf{w}(t)\|^2 \right) - (\text{Tr}(\Upsilon) - \mathbf{w}(t)^T \Upsilon \mathbf{w}(t)) \geq 0 \quad (4.25)$$

for all $\mathbf{w}(t)$. Further, (4.25) is satisfied for all $\mathbf{w}(t)$ if and only if

$$\begin{bmatrix} g(t) - \text{Tr}(\Upsilon) & 0 \\ 0 & \gamma^2 I + \Upsilon \end{bmatrix} - [\mathcal{C}_x \ \mathcal{G}_x]^T \mathcal{Q} [\mathcal{C}_x \ \mathcal{G}_x] - [\mathcal{C}_u \ \mathcal{G}_u]^T \mathbf{R} [\mathcal{C}_u \ \mathcal{G}_u] \succeq 0 \quad (4.26)$$

which can be written as

$$\begin{bmatrix} g(t) - \text{Tr}(\Upsilon) & 0 & \mathcal{C}_x^T & \mathcal{C}_u^T \\ 0 & \gamma^2 I + \Upsilon & \mathcal{G}_x^T & \mathcal{G}_u^T \\ * & * & \mathcal{Q}^{-1} & 0 \\ * & * & * & \mathbf{R}^{-1} \end{bmatrix} \succeq 0. \quad (4.27)$$

Remark 4.2. Condition (4.27) is an LMI in $g(t), \bar{\mathbf{v}}(t), \bar{\xi}(t+N|t), \mathbf{E}(t), \tilde{\mathbf{F}}(t)$ and Υ . In general, (4.27) is only a sufficient but not a necessary condition for (4.24) to hold when the disturbance $\mathbf{w}(t)$ is only componentwise bounded. So, minimizing the worst-case cost bound $g(t)$ under condition (4.27) in place of condition (4.24) can be conservative to some extent. Nevertheless, as discussed in [81], it provides a close and computationally attractive approximation to the otherwise complex min-max optimization problem. $\square$

The next proposition deals with the performance of an RHC scheme based on the minimization of the performance index of (4.22).

Proposition 4.4. If the optimization problem

$$\min_{g(t), \bar{\mathbf{v}}(t), \bar{\xi}(t+N|t), \mathbf{E}(t), \tilde{\mathbf{F}}(t), \mathbf{L}, \Upsilon} g(t) \quad (4.28)$$

under constraint conditions (3.39) and (4.27) is feasible for an initial state $x(0)$ at time $t = 0$, then the receding horizon control $u^*(t) = \kappa^*(x(t), t) = Kx(t) + \bar{v}^*(t)$ where $\bar{v}^*(t)$ is obtained from the solution of (4.28) at each time $t \in \mathbb{Z}_+$ ensures that the problem remains feasible at subsequent times, and guarantees that the cost monotonicity condition

$$g^*(t+1) - g^*(t) \leq - \left(\|Q^{\frac{1}{2}}x(t)\|^2 + \|R^{\frac{1}{2}}u^*(t)\|^2 - \gamma^2\|w(t)\|^2 \right) \quad (4.29)$$

and hence the boundedness condition

$$\sum_{t=0}^{\infty} \left\{ \|Q^{\frac{1}{2}}x(t)\|^2 + \|R^{\frac{1}{2}}u^*(t)\|^2 - \gamma^2\|w(t)\|^2 \right\} \leq g^*(0) \quad (4.30)$$

are satisfied. $\square$

PROOF. The fact that the problem remains feasible for all times $t \in \mathbb{Z}_{++}$ given that it is feasible at time $t = 0$, follows from Lemma 3.3. To see that the monotonicity condition (4.29) is satisfied, we note that, by the definition of $\mathcal{P}$,

$$\begin{aligned} & \left\| \mathcal{P}^{\frac{1}{2}} \begin{bmatrix} x(t+N+1|t)^T & \xi(t+N+1|t)^T \end{bmatrix}^T \right\|^2 - \left\| \mathcal{P}^{\frac{1}{2}} \begin{bmatrix} x(t+N|t)^T & \xi(t+N|t)^T \end{bmatrix}^T \right\|^2 \\ & \leq - \left(\|Q^{\frac{1}{2}}x(t+N|t)\|^2 + \|R^{\frac{1}{2}}u(t+N|t)\|^2 - \gamma^2\|w(t+N)\|^2 \right). \end{aligned}$$

So, (4.25), or, equivalently, (4.27) implies that there exist some $\hat{\mathbf{v}}(t), \hat{\xi}(t+N+1|t), \hat{\mathbf{E}}(t), \hat{\mathbf{F}}(t)$ so that $g^*(t)$ satisfies, for all $w(t)$ and $\hat{\mathbf{w}}(t) = [w(t+1)^T \ w(t+2)^T \ \dots \ w(t+N)^T]^T$,

$$\begin{aligned} g^*(t) - & \left(\|Q^{\frac{1}{2}}x(t)\|^2 + \|R^{\frac{1}{2}}u(t)\|^2 - \gamma^2\|w(t)\|^2 \right) - \left(\text{Tr}(\Upsilon_0) - \|\Upsilon_0^{\frac{1}{2}}w(t)\|^2 \right) \\ & - \left(\|\mathcal{Q}^{\frac{1}{2}}(\hat{\mathcal{C}}_x + \hat{\mathcal{G}}_x\hat{\mathbf{w}}(t))\|^2 + \|\mathbf{R}^{\frac{1}{2}}(\hat{\mathcal{C}}_u + \hat{\mathcal{G}}_u\hat{\mathbf{w}}(t))\|^2 - \gamma^2\|\hat{\mathbf{w}}(t)\|^2 \right) \\ & - \left(\text{Tr}(\hat{\Upsilon}) - \|\hat{\Upsilon}^{\frac{1}{2}}\hat{\mathbf{w}}(t)\|^2 \right) \geq 0 \end{aligned}$$

Figure 4.6: Time variation of states under the control scheme of Proposition 4.4 for various values of N for an initial state $[-9.6 \ 1.7]^T$.

where $\dot{\mathcal{C}}_x, \dot{\mathcal{G}}_x, \dot{\mathcal{C}}_u$ and $\dot{\mathcal{G}}_u$ are defined in terms of $\dot{\mathbf{v}}(t), \dot{\xi}(t + N + 1|t), \dot{\mathbf{E}}(t), \dot{\mathbf{F}}(t)$ as in (4.23), and $\dot{\Upsilon} = \text{diag}(\dot{\Upsilon}, 0)$ such that $\Upsilon = \text{diag}(\Upsilon_0, \tilde{\Upsilon})$. Clearly, there exists a $g(t + 1)$ at time $t + 1$ satisfying

$$g(t + 1) - \left(\|\mathcal{Q}^{\frac{1}{2}}(\dot{\mathcal{C}}_x + \dot{\mathcal{G}}_x \dot{\mathbf{w}}(t))\|^2 + \|\mathbf{R}^{\frac{1}{2}}(\dot{\mathcal{C}}_u + \dot{\mathcal{G}}_u \dot{\mathbf{w}}(t))\|^2 - \gamma^2 \|\dot{\mathbf{w}}(t)\|^2 \right) - \left(\text{Tr}(\dot{\Upsilon}) - \|\dot{\Upsilon}^{\frac{1}{2}} \dot{\mathbf{w}}(t)\|^2 \right) \geq 0$$

such that

$$g(t + 1) \leq g^*(t) - \left(\|Q^{\frac{1}{2}}x(t)\|^2 + \|R^{\frac{1}{2}}u^*(t)\|^2 - \gamma^2 \|w(t)\|^2 \right)$$

as $(\text{Tr}(\Upsilon_0) - w(t)^T \Upsilon_0 w(t)) \geq 0$ for all $w(t) \in \mathcal{W}$. Hence the condition (4.29) and hence (4.30). ■

Illustration 4.4. We again consider the system mentioned in (3.26) with the in order to illustrate the performance of the control scheme discussed in Proposition 4.4. We assume that $\rho(t) = \sin \frac{2\pi t}{T_a}$ with $T_a = 25$. In the design of the controller, we consider the LQ optimal K as in Illustration 4.1. Fig. 4.6 shows the state trajectories starting from $[-9.6 \ 1.7]^T$ under the control actions calculated as in Proposition 4.4 with $N = 0, 1, 2, 3$ and 4 . A worst-case disturbance $w(t) = \text{sgn}(x(t))$ is used in each case. While the control response is relatively sluggish for smaller values of N , it improves with increasing values of N . However, because of the min-max strategy, unlike in the case of the control scheme of Proposition 3.5, the response does not overshoot for large values of N . □

4.5 Computational Complexity

As in the case of nominal RHC schemes discussed in Chapter 3, the RHC schemes presented in this chapter require both off-line and on-line computations. For the schemes considered in this chapter, the off-line optimization problem for finding the (terminal) controller matrices is an SDP problem with bilinear matrix inequality (BMI) constraints.

Algorithm 4.1 presents a method for solving the problem with an alternative SDP optimization approach. The algorithm will generally converge to a local solution and may require to be run for a number of times with different feasible initial values in order to obtain a desired solution. Moreover, we also need to determine the polyhedral feasible invariant sets once the (terminal) controller matrices are determined, and, as mentioned in Chapter 3, this requires the use of some iterative algorithms [49, 113] as well as possibly some approximation techniques (e.g., [128]). See Appendix A for details.

Similar to the case of the optimization problem in Proposition 3.5, the complexity of the optimization problem considered in Proposition 4.4 depends on the length of the chosen finite horizon N . In the optimization problem, there are $n_u N$ free variables in $\bar{\mathbf{v}}(t)$, $n_\xi (= n_x)$ in $\bar{\xi}(t + N|t)$, $n_u n_w N(N - 1)/2$ in $\mathbf{E}(t)$, $n_x n_w N$ in $\tilde{\mathbf{F}}(t)$, $(n_M N + n_P) n_w N$ in $\mathbf{L}$ and $n_w N$ in Υ . The total number of decision variables is roughly of the order $\mathcal{O}(N^2)$. The problem is a QP problem for $N = 0$ (which corresponds to the problem in Proposition 4.1) and, for $N \geq 1$, it is an SDP problem, which is significantly more complex.

As in the case of nominal RHC schemes discussed in Chapter 3, the computational advantages of the proposed min-max RHC schemes over corresponding schemes that use a static feedback as the terminal control law arise from the fact that the proposed schemes offer feasibility over a larger set of initial conditions with an optimization over a shorter (possibly zero) horizon length N . We present a comparison in the following example.

Illustration 4.5. We again consider the example system in (3.26) in order to assess the computational aspects of the proposed min-max scheme in Proposition 4.4 and to compare them with those of the corresponding min-max schemes employing a static-feedback-based terminal control law (e.g. [83]) and those of the nominal schemes employing the dynamic-policy-based terminal control law. For this study, we assume that the system is time-invariant with $a(t) = 1.3$ and $b(t) = 1$ and choose the corresponding LQ optimal feedback gain K .

Fig. 4.7a shows the polyhedral invariant set $\mathcal{X}_x$ for the LTI system corresponding to the controller matrices obtained using (4.6) with $\gamma = 0.5$. We consider a set of 100 initial conditions chosen randomly along the boundary of the set $\mathcal{X}_x$, and apply the control scheme of Proposition 4.4 with $N = 0, 1, 2, 3$ and 4 for the worst-case disturbance $w(t) = \text{sgn}(x(t))$ in each case. Table 4.2 shows the average computation times observed for the different horizon lengths when the computations are carried out with the SeDuMi solver [137] in the Matlab 7.4 (R2007a) environment in a dual-core 2.26 GHz machine with 3GB RAM. These times are larger than the corresponding average computation times observed in the case of the nominal cost scheme of Proposition 3.5 (cf. Table 3.1), and this is expected since the optimization problem of Proposition 4.4 is an SDP problem (except when $N = 0$) and is more complex than the QP problem of Proposition 3.5. The table also shows the average normalized regulation costs in all cases.

Further, Fig. 4.7a also shows the terminal invariant set for the system under the optimal static feedback, $\mathcal{X}_{xS}$. It can be observed that the domain of attraction for the system with the static terminal feedback for $N = 18$, $\mathcal{X}_{xS}^{18}$ is almost the same in size as that for the system with a dynamic terminal policy for $N = 4$, $\mathcal{X}_x^4$. Clearly, the computational requirements for a min-max scheme using $\mathcal{X}_{xS}$ as the terminal set will be much higher than those for

Figure 4.7: (a) Feasibility sets for an LTI system - (3.26) with $a(t) = 1.3$ and $b(t) = 1$ - for control parameterizations involving disturbance feedforward for min-max performance with $\gamma = 0.5$. (b) Time variation of states for an initial state $[-14.0 \ 2.9]^T$ when the system is controlled as per various cost minimization schemes based on static and dynamic terminal policies.

Table 4.2: Average computation times and costs with the RHC law of Proposition 4.4

	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
Average computation time (ms)	70	114	165	220	307
Average cost (normalized)	100.0	87.9	85.2	83.9	83.3

the proposed scheme that uses $\mathcal{X}_x$ as the terminal set. Fig. 3.6b compares the performance of the min-max scheme with nominal schemes employing static or dynamic terminal policy. The control response is better in the case of the min-max scheme as compared to those in the nominal schemes which ignore the presence of disturbance while minimizing the cost.

We have mentioned above that the off-line computation of the controller matrices using Algorithm 4.1 may require several runs to ensure the desired result. In this context, we note that in the illustrative examples presented in this chapter, we have used the solution of problem (3.25) as the initial feasible solution and the alternating SDP algorithm is found to converge after a few tens of iterations through the steps of the algorithm. Each such iteration is found to take time in the order of 10 seconds when solved with the LMI control toolbox [130] in the computing setup mentioned in Illustration 3.4. Clearly, the computation of the controller matrices for the min-max RHC scheme of this chapter is much more complex and takes much more time than that for the nominal schemes of Chapter 3. The computation of the polyhedral sets, however, are carried out with the same algorithm in both the cases and require computation times of the same order.

4.6 Conclusion

We have presented min-max RHC schemes that employ an uncertainty-based infinite-horizon control parameterization for the control of uncertain linear systems. The proposed schemes allow the computation of the (terminal) controller matrices and feasible

invariant set off-line such that an upper bound of the min-max cost for the infinite horizon can be expressed in a simple form and can be minimized on-line in a simple way. The schemes are applicable over a larger feasibility domain with relatively simpler on-line computations. In the case of a deterministically time-varying system, the infinite-horizon control parameterization can be appended to a finite-horizon disturbance-based parameterization and the overall min-max problem can be formulated as an SDP problem which can efficiently be solved on-line. The advantages of this scheme over corresponding min-max schemes using a static-feedback-based terminal control law come from the fact that, with this scheme, a good control performance is achieved for larger feasible sets with an on-line optimization over relatively smaller horizon lengths. The min-max scheme, indeed, is computationally more complex than the nominal schemes considered in Chapter 3. However, this scheme is less conservative than the nominal scheme without disturbance feedforward and offers a guaranteed control performance which is not the case with the nominal scheme with disturbance feedforward. The simulation results that we have presented verify the effective control performance of the proposed min-max schemes.

Chapter 5

Output-Feedback RHC Based on Optimized Dynamic Policy

5.1 Introduction

The RHC schemes explored in Chapter 3 and Chapter 4 are based on the assumption that the state information is fully available at each time instant. In many situations, however, not all states can be exactly measured at each time step. In such cases, the control strategy should be based on the value of a measured output. Output-feedback RHC schemes have been extensively explored in the last several years in order to extend the applicability of the standard RHC schemes to cases in which the states are not fully measurable. Robust schemes need to be considered when uncertainties are present in the system model and such schemes usually tend to be computationally complex as well as conservative. In this chapter, we explore output-feedback counterparts of the computationally efficient RHC schemes considered in Chapter 3 and 4, for a class of linear time-varying (LTV) systems with bounded disturbances and measurement noises.

Output-feedback RHC problems are generally easily solved for undisturbed linear systems by combining a stable state estimator with a stabilizing controller [84]. However, when the system is constrained and uncertainties are present, such a combination may not readily ensure stability of the closed-loop system [89]. Existing works on output-feedback RHC for linear systems are mainly based on set-membership estimation [138] and those presenting robust schemes have dealt with various forms of uncertainties in the system model, such as persistent disturbances and noises (e.g., [85–90]) structured or unstructured model uncertainties (e.g., [91], [92]) and both model uncertainties and disturbances (e.g., [93]).

Like their standard state-feedback counterparts, most robust output-feedback RHC schemes are usually aimed at offering guaranteed control performance without being too conservative or computationally too complex. The additional uncertainty arising from

the lack of the full knowledge of the state adds to the conservativeness and computational complexity of the algorithms in the output-feedback case. As is the case in a full state-feedback robust RHC scheme, a robust output-feedback RHC scheme needs to consider an optimization over allowable output feedback policies in order to minimize the conservativeness of the scheme. But such an optimization is not easy, and computationally viable alternatives are often sought. For example, the authors in works such as [86], [87], [89], [92] have employed the popular approach of optimizing perturbations to a standard, possibly optimal, static feedback control law in order to simplify the on-line computations. Similarly, [91] has employed an on-line selection of a suitable control law from a sequence of explicit feedback control laws determined off-line. Some other works have focussed more on performance than on computational complexity. For instance, [88] has considered joint state estimation and control in the framework of min-max MPC, and the authors in [93] have explored a robust scheme based on the on-line optimization of a dynamic output-feedback law.

In this chapter we present output-feedback RHC schemes which offer a good performance over a larger feasibility domain with simplified on-line computations. We present and use a control parameterization that can be considered as an infinite-horizon counterpart of the output-error-based parameterization considered in [90, 118], and is applicable to systems with a polytopic system description and additive disturbances and noises. It involves the use of a dynamically evolving sequence of input perturbations that have an uncertain evolution through the use of polytopic dynamics along with a feedforward of the output error. As in Chapter 3 and Chapter 4, we explore nominal and min-max cost minimization schemes based on the proposed control parameterization and analyse the performance of the proposed schemes in terms of the ℓ_∞ - and ℓ_2 -stability of the controlled system. Finally, we present some numerical examples that illustrate the performance of the proposed RHC schemes.

5.2 Problem Description

We consider an LTV system described by

$$x(t+1) = A(t)x(t) + B_u(t)u(t) + B_w(t)w(t) \quad (5.1a)$$

$$y(t) = Cx(t) + De(t) \quad (5.1b)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is an unknown but bounded disturbance, $y(t) \in \mathbb{R}^{n_y}$ is the measured output and $e(t) \in \mathbb{R}^{n_e}$ is a bounded measurement noise.

The assumptions that we make about the system are similar to those considered in Chapter 3. We list them in the following:

A5.1: The time-varying system matrices $A(t)$, $B_u(t)$ and $B_w(t)$ are known and lie within a polytope Ω , i.e.,

$$[A(t), B_u(t), B_w(t)] \in \Omega = \text{Co}\left\{[A^{(j)}, B_u^{(j)}, B_w^{(j)}], j = 1, \dots, n_\Omega\right\}, \forall t \in \mathbb{Z}_+ \quad (5.2)$$

where $[A^{(j)}, B_u^{(j)}, B_w^{(j)}]$, $j = 1, 2, \dots, n_\Omega$ are the vertices of the polytope Ω .

A5.2: The disturbance $w(t)$ and the noise $e(t)$ are componentwise bounded, i.e.,

$$w(t) \in \mathcal{W}_w = \{w \in \mathbb{R}^{n_w} \mid \|w\|_\infty \leq 1\} \quad (5.3a)$$

$$e(t) \in \mathcal{W}_e = \{e \in \mathbb{R}^{n_e} \mid \|e\|_\infty \leq 1\}. \quad (5.3b)$$

A5.3: The constraints are such that the state-input combine $(x(t), u(t))$ lies in a compact polyhedral set Ξ that contains the origin in its interior. The set Ξ is expressed as

$$\Xi = \{(x, u) \mid M_x x + M_u u \leq \mathbf{1}\} \quad (5.4)$$

where $M_x \in \mathbb{R}^{m \times n_x}$ and $M_u \in \mathbb{R}^{m \times n_u}$.

A5.4: The polytopic system (5.1a-5.2) is quadratically stabilizable.

A5.5: There exists a gain L such that the polytopic system $\vartheta(t+1) = (A(t) - LC)\vartheta(t)$ is quadratically stable.

The control objective is to employ an efficient RHC scheme based on an output-feedback of the form

$$\hat{x}(t+1) = g(\hat{x}(t), y(t), t), \quad u(t) = \kappa(\hat{x}(t), t), \quad (5.5)$$

where $\hat{x}(t)$ is a suitably defined state estimate, in order to drive the actual state of system (3.1) close to the origin. We wish to achieve this by computing, at each time step, a sequence of control strategies that minimize a predicted quadratic cost function of the form

$$J(t) = \sum_{i=0}^{\infty} \left\{ \|Q^{\frac{1}{2}} x(t+i|t)\|^2 + \|R^{\frac{1}{2}} u(t+i|t)\|^2 \right\}, \quad Q \succ 0, R \succ 0 \quad (5.6)$$

for either a nominal performance based on the nominal (disturbance-free) dynamics of the estimated state, or a min-max performance based on worst-case disturbances, noises and state estimation errors. For the estimation of the state, we consider a Luenberger observer defined by

$$\hat{x}(t+1) = A(t) \hat{x}(t) + B_u(t) u(t) + L(y(t) - C\hat{x}(t)) \quad (5.7)$$

so that the state estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$ has the dynamics

$$\tilde{x}(t+1) = \tilde{A}(t) \tilde{x}(t) + B_w(t) w(t) - LDe(t) \quad (5.8)$$

with $\tilde{A}(t) = A(t) - LC$, where the output injection gain L is chosen such that the polytopic system (5.8) is quadratically stable.

We wish to compute and implement the optimal control actions in the receding horizon fashion, and assess the control performance by analyzing the stability of the controlled composite system

$$\zeta(t+1) = f(\zeta(t), w(t), t) = \mathcal{A}(t)\zeta(t) + \mathcal{B}_u(t)u^*(t) + \mathcal{B}_w(t)\omega(t) \quad (5.9a)$$

$$z(t) = h(t) = \begin{bmatrix} \mathcal{Q}\zeta(t) \\ \mathcal{R}u^*(t) \end{bmatrix} \quad (5.9b)$$

where

$$\mathcal{A}(t) = \begin{bmatrix} A(t) & LC \\ 0 & \tilde{A}(t) \end{bmatrix}, \quad \mathcal{B}_u(t) = \begin{bmatrix} B_u(t) \\ 0 \end{bmatrix}, \quad \mathcal{B}_w(t) = \begin{bmatrix} 0 & L \\ B_w(t) & -L \end{bmatrix},$$

$\mathcal{Q} = [Q^{\frac{1}{2}} \quad Q^{\frac{1}{2}}]$, $\mathcal{R} = R^{\frac{1}{2}}$, $\zeta(t) = [\hat{x}(t)^T \tilde{x}(t)^T]^T$, $\omega(t) = [w(t)^T e(t)^T]^T$ and $u^*(t)$ is the control input obtained through optimization at time t . The choice of $z(t)$ is such that $\|z(t)\|^2 = z(t)^T z(t)$ represents the stage cost of time t considered in the cost function (5.6).

In the following sections, we discuss the details of the proposed control schemes that address the objectives mentioned above.

5.3 Output-Feedback RHC Based on Optimized Dynamic Policy

In this section, we present the details of the control parameterization as well as the details of the off-line and on-line computations required for the proposed control schemes. Since we assume that the time-varying system matrices are actually known, we discuss the general parameterization in which the optimized dynamic policy is used as the terminal control parameterization.

5.3.1 Control parameterization

Following the control parameterizations used in Chapters 3 and 4, we consider the following form of control policy for the prediction of the cost at each time step:

$$u(t+i|t) = \begin{cases} K \hat{x}(t+i|t) + v(t+i|t), & i = 0, 1, \dots, N-1 \\ K \hat{x}(t+i|t) + H\xi(t+i|t), & i \geq N. \end{cases} \quad (5.10a)$$

where K is a precomputed constant static feedback gain and $v(t+i|t)$, $i \in \mathbb{Z}_+$ are quantities that are parameterized as $v(t+i|t) = \bar{v}(t+i|t) + \sum_{j=0}^{i-1} E_j(t+i|t)\eta(t+j|t)$, where

$\eta(t+j|t) = (C\tilde{x}(t+j|t) + D e(t+j))$ represents the predicted output error $y(t+j|t) - C\hat{x}(t+j|t)$, and $\bar{v}(t+i|t)$ and $E_j(t+i|t)$ denote vectors and matrices of suitable dimensions. Moreover, $\xi(t+i|t)$, $i \geq N$ are the predicted states of a dynamic terminal controller starting from an initial terminal state $\xi(t+N|t) = \bar{\xi}(t+N|t) + \sum_{j=0}^{N-1} \tilde{F}_j(t+N|t) \eta(t+j|t)$ and having LTV dynamics

$$\xi(t+1) = G(t)\xi(t) + F(t)(C\tilde{x}(t) + D e(t)) \quad (5.10b)$$

where considering the fact that the system matrices lie in a polytope Ω , the controller dynamics are allowed to remain uncertain by assuming that the matrices $G(t)$ and $F(t)$ lie within a polytope of as many vertices as the system matrices, i.e.,

$$[G(t) \ F(t)] \in \text{Co} \left\{ [G^{(1)} \ F^{(1)}], [G^{(2)} \ F^{(2)}], \dots, [G^{(n_\Omega)} \ F^{(n_\Omega)}] \right\}. \quad (5.10c)$$

We choose the feedback gain K off-line such that it is optimal in some sense (e.g., LQ optimal for a nominal or average model under full state-feedback). As discussed in Chapter 3, The quantities $v(t+i|t)$, $i = 1, 2, \dots, N-1$ and the quantities $H\xi(t+i|t)$, $i \geq N$ can be considered as predicted input perturbations which basically ensure the satisfaction of system constraints.

Note that the parameterization of the finite-horizon part of the input in (5.10a) is similar to that used in [90] and related to other parameterizations considered in [70, 81, 82] in the case of full state information. The terminal part of the input in (5.10a) approximately extends this parameterization to the infinite horizon and allows an efficient prediction of the future inputs for time steps beyond the finite horizon by the use of an uncertain time-varying dynamic policy. As discussed in Chapter 3, the permissibility of such an uncertain prediction of future inputs follows from the nature of the control computation and implementation which only requires that an admissible control sequence exists for any possible realization of uncertainties in the future.

Among the different parameters in (5.10a-5.10c), we optimize the terminal controller matrices H , $G^{(j)}$ and $F^{(j)}$, ($j = 1, \dots, n_\Omega$) off-line in order to ensure a desired feasible terminal invariant set for the estimated state $\hat{x}(t)$ of the controlled system. The quantities $\bar{v}(k+i)$ ($i = 0, \dots, N-1$), $\bar{\xi}(t+N|t)$, $E_j(t+i|t)$ ($i = 1, \dots, N-1$, $j = 0, \dots, i-1$) and $\tilde{F}_j(t+N|t)$ ($j = 0, \dots, N-1$) are optimized at each time step for minimizing a suitably chosen cost function.

5.3.2 Computation of terminal control law and invariant set

Augmenting the estimated state dynamics (5.7) with the terminal controller dynamics (5.10b), we get the autonomous system:

$$\chi(t+1) = \Psi(t)\chi(t) + \mathcal{B}_\nu(t)\nu(t) \quad (5.11)$$

where $\chi(t) = [\hat{x}(t)^T \ \xi(t)^T]^T$ and $\nu(t) = [\tilde{x}(t)^T \ e(t)^T]^T$. The system matrices

$$\Psi(t) = \begin{bmatrix} \mathcal{A}(t) & B_u(t)H \\ 0 & G(t) \end{bmatrix}, \quad \mathcal{B}_\nu(t) = \begin{bmatrix} LC & LD \\ F(t)C & F(t)D \end{bmatrix}$$

with $\mathcal{A}(t) = A(t) + B_u(t)K$ lie in a polytope, i.e., $[\Psi(t) \ \mathcal{B}_\nu(t)] \in \text{Co}\left\{[\Psi^{(j)} \ \mathcal{B}_\nu^{(j)}], j = 1, \dots, n_\Omega\right\}$ where

$$\Psi^{(j)} = \begin{bmatrix} \mathcal{A}^{(j)} & B_u^{(j)}H \\ 0 & G^{(j)} \end{bmatrix}, \quad \mathcal{B}_\nu^{(j)} = \begin{bmatrix} LC & LD \\ F^{(j)}C & F^{(j)}D \end{bmatrix}$$

and $\mathcal{A}^{(j)} = A^{(j)} + B_u^{(j)}K$.

Clearly, a feasible invariant set $\mathcal{S}_{\tilde{x}}$ for the estimated state of the controlled system is given by the projection, on the $\hat{x}$ -subspace, of a positively invariant set $\mathcal{S}_\chi$ for the state of (5.11) on which the constraints in (5.4) are satisfied. While finding a desired feasible invariant set $\mathcal{S}_\chi$, we first consider an ellipsoidal set $\mathcal{E}_\chi = \{\chi \mid \chi^T \mathcal{P}^{-1} \chi \leq 1\}$ so that the size of its projection $\mathcal{E}_{\tilde{x}}$ on the $\hat{x}$ -subspace can be optimized by solving a tractable problem. The set $\mathcal{E}_\chi$ must be invariant for the polytopic system (5.11) in the presence of the uncertain additive quantity $\nu(t)$. With L determined off-line as mentioned in the last subsection, there exists a minimal robust positively invariant set $\mathcal{R}^{min}$ for the estimation error $\tilde{x}(t)$ which follows the dynamics (5.8). The minimal invariant set is a polyhedral set usually represented by a (possibly infinitely) large number of linear inequalities [126]. However, close outer polyhedral approximations, represented by a tractable number of linear inequalities, of the minimal invariant set can be easily computed [68, 127] (See Appendix A). We consider an ellipsoidal outer bound of $\mathcal{E}_{\tilde{x}} = \{\tilde{x} \mid \tilde{x}^T Q_{\tilde{x}} \tilde{x} \leq 1\}$ of $\mathcal{R}^{min}$ in the computation of the terminal controller matrices and the terminal ellipsoidal feasible invariant set.

We now present some results that will be used in the determination of the terminal controller matrices.

Lemma 5.1 (Invariance). Let $\tilde{x}(t) \in \mathcal{E}_{\tilde{x}} = \{\tilde{x} \mid \tilde{x}^T Q_{\tilde{x}} \tilde{x} \leq 1\}$ in (5.11). An ellipsoidal set $\mathcal{E}_\chi = \{\chi \mid \chi^T \mathcal{P}^{-1} \chi \leq 1\}$ will be invariant for the augmented system in (5.11) if

$$\begin{bmatrix} \mathcal{P} & \Psi^{(j)}\mathcal{P} & \mathcal{B}_\nu^{(j)} \\ * & \alpha_1^{(j)}\mathcal{P} & 0 \\ * & * & \begin{bmatrix} \alpha_2^{(j)}Q_{\tilde{x}} & 0 \\ 0 & R_w^{(j)} \end{bmatrix} \end{bmatrix} \succ 0, \quad j = 1, \dots, n_\Omega \quad (5.12)$$

for some scalars $\alpha_1^{(j)}, \alpha_2^{(j)} > 0$ and diagonal matrices $R_w^{(j)} \in \mathbb{D}_+^{n_e}$ $j = 1, \dots, n_\Omega$ such that $\text{Tr}(R_w^{(j)}) \leq 1 - \alpha_1^{(j)} - \alpha_2^{(j)}$. $\square$

PROOF. The proof follows the lines of reasoning used in the proof of Lemma 3.1. $\blacksquare$

Lemma 5.2 (Feasibility). The ellipsoidal set $\mathcal{E}_\chi$ will be feasible w.r.t. the constraints in (5.4) if

$$\begin{bmatrix} Z & \begin{bmatrix} M_x + M_u K & M_u H \end{bmatrix} \\ * & \mathcal{P} \end{bmatrix} \succ 0, \quad Z_{[j,j]} \leq b_{[j]}^2, \quad j = 1, \dots, n_M \quad (5.13)$$

where $b_{[j]} = \min_{\varsigma \in \mathcal{E}_{\hat{x}}} (1 - M_x[j,:]\varsigma)$ □

PROOF. Noting that the actual system state $x(t) = \hat{x}(t) + \tilde{x}(t)$, the proof follows from the standard results in [111], [56] etc. ■

Now, in order to determine the controller matrices for a maximal feasible invariant set, we solve the problem

$$\max_{\mathcal{P}, Z, H, G^{(j)}, F^{(j)}, \alpha_1^{(j)}, \alpha_2^{(j)}, R_w^{(j)} (j=1, \dots, n_\Omega)} \log \det(T \mathcal{P} T^T) \quad (5.14)$$

where $T = [I_{n_x} \ 0]$, subject to conditions (5.12) and (5.13). The constraints applied in problem (5.14) are bilinear matrix inequalities (BMIs) but, as mentioned in Chapter 3, for $n_\xi \geq n_x$, with a suitable change of variables, the BMIs in (5.12) and (5.13) can be expressed equivalently as simpler matrix inequalities (LMIs when $\alpha_1^{(j)}$ and $\alpha_2^{(j)}$ are constants) and solved efficiently.

The optimization in (5.14) determines the dynamics of the perturbations to the prescribed static feedback $K\hat{x}(t)$ where K is fixed *a priori*. The choice of a fixed K does not make the proposed terminal policy more conservative as far as the size of the feasible invariant set that can be obtained from (5.14) is concerned. Indeed, as discussed in Chapter 3, any feasible invariant set that is possible with an arbitrary static feedback from the estimated state can usually be also obtained from (5.14).

The set $\mathcal{E}_\chi$ obtained from (5.14) is usually a conservative inner ellipsoidal approximation of the largest feasible invariant set associated with the controller matrices that are determined in (5.14). Since $\mathcal{W}_w$ and $\mathcal{W}_e$ are compact polyhedral sets, the largest feasible invariant set is a polyhedral set and this set (or its inner invariant approximation) $\mathcal{X}_{\hat{x}}$ can be computed using some iterative algorithms as mentioned in Appendix A.

Illustration 5.1. We consider an example of the control of an LTV system described by

$$\left. \begin{aligned} A(t) &= \begin{bmatrix} 1.05 & 1 \\ 0 & a(t) \end{bmatrix}; \quad B_u(t) = \begin{bmatrix} b(t) \\ b(t) \end{bmatrix}; \\ B_w(t) &= 0.12 I_2; \quad C = [1 \ 1]; \quad D = 0.1 \end{aligned} \right\} \quad (5.15)$$

with $a(t) = 1.2 + 0.05\rho(t)$, $b(t) = 0.9 + 0.1\rho(t)$ and $|\rho(t)| \leq 1$. $Q = I$, $R = 0.01$ are considered and $K = [-0.75 \ -1.18]$ — the LQ optimal gain for the unconstrained average system — is selected. Also, $L = [1 \ 1]^T$ is chosen.

Fig. 5.1 shows the terminal feasible invariant set $\mathcal{X}_{\hat{x}}$ for the estimated state of the system in (5.15) under an input constraint $|u(t)| \leq 4$, corresponding to terminal controller parameters obtained by solving (5.14). The figure also shows the terminal feasible invariant set

Figure 5.1: Terminal feasible invariant sets for the system in (5.15) under different terminal policies: with output error feedforward ($\mathcal{X}_{\hat{x}}$), without such feedforward ($\bar{\mathcal{X}}_{\hat{x}}$), and with full state feedback ($\mathcal{X}_x^{sf}$).

$\bar{\mathcal{X}}_{\hat{x}}$ that would be obtained with a terminal control law without output error feedforward, i.e., with $F^{(j)} = 0$. The set $\mathcal{X}_{\hat{x}}$ is also compared with $\mathcal{X}_x^{sf}$, the feasible invariant set for the actual state obtained for the case of full state-feedback. $\square$

5.3.3 On-line computations and control performance

We now consider the determination of control inputs that are feasible for the actual system for any allowable realization of disturbances, noises and state estimation errors, and optimal for the nominal dynamics (one without disturbances) of the estimated state.

5.3.3.1 Feasible solutions and recursive feasibility

Let us consider the sequences of the estimated state, the control input and the estimation error in the finite horizon defined as

$$\hat{\mathbf{x}}(t) = \begin{bmatrix} \hat{x}(t|t) \\ \hat{x}(t+1|t) \\ \vdots \\ \hat{x}(t+N|t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u(t|t) \\ u(t+1|t) \\ \vdots \\ u(t+N-1|t) \end{bmatrix}, \quad \tilde{\mathbf{x}}(t) = \begin{bmatrix} \tilde{x}(t|t) \\ \tilde{x}(t+1|t) \\ \vdots \\ \tilde{x}(t+N-1|t) \end{bmatrix}.$$

With the control policy defined in (5.10a), we can express $\hat{\mathbf{x}}(t)$ and $\mathbf{u}(t)$ as

$$\begin{aligned} \hat{\mathbf{x}}(t) &= \mathbf{A}(t) \hat{x}(t) + \mathbf{B}_u(t) \bar{\mathbf{v}}(t) + (\mathbf{B}_\eta(t) + \mathbf{B}_u(t) \mathbf{E}(t)) \boldsymbol{\eta}(t) \\ \text{and } \mathbf{u}(t) &= \check{\mathbf{A}}(t) \hat{x}(t) + \check{\mathbf{B}}_u(t) \bar{\mathbf{v}}(t) + (\check{\mathbf{B}}_\eta(t) + \check{\mathbf{B}}_u(t) \mathbf{E}(t)) \boldsymbol{\eta}(t) \end{aligned}$$

where

$$\bar{\mathbf{v}}(t) = \begin{bmatrix} \bar{v}(t|t) \\ \bar{v}(t+1|t) \\ \vdots \\ \bar{v}(t+N-1|t) \end{bmatrix}, \quad \boldsymbol{\eta}(t) = \begin{bmatrix} \eta(t|t) \\ \eta(t+1|t) \\ \vdots \\ \eta(t+N-1|t) \end{bmatrix} = \mathbf{C}\tilde{\mathbf{x}}(t) + \mathbf{D}\mathbf{e}(t)$$

with $\tilde{\mathbf{x}}(t) = \tilde{\mathbf{A}}(t)\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}_{\mathbf{w}}(t)\mathbf{w}(t) + \tilde{\mathbf{B}}_{\mathbf{e}}(t)\mathbf{e}(t)$ and $\mathbf{w}(t) = [w(t)^T \ w(t+1)^T \ \cdots \ w(t+N-1)^T]^T$ and $\mathbf{e}(t) = [e(t)^T \ e(t+1)^T \ \cdots \ e(t+N-1)^T]^T$. Here the constant matrices are defined in the same way as the constant matrices in Section 3.4.2 and are given in Appendix B for completeness. The variable matrix $\mathbf{E}(t)$ is defined as

$$\mathbf{E}(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ E_0(t+1|t) & 0 & \cdots & 0 \\ E_0(t+2|t) & E_1(t+2|t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_0(t+N-1|t) & E_1(t+N-1|t) & \cdots & 0 \end{bmatrix}. \quad (5.16)$$

Now we consider the optimization of the variables defining the inputs for a given estimated state $\hat{x}(t)$ and an assumed polyhedral estimation error set $\mathcal{R}(t)$. While optimizing the variables, it must be ensured that the constraints in (5.4) are satisfied in the finite horizon ($t = 0, \dots, N-1$) and that the terminal control law is feasible at time $t+N$. Moreover, the solvability of the control problem at successive time instants for future estimation error sets $\mathcal{R}(t+i)$ should also be ensured. Let us consider that we use, as the terminal set for the augmented state, a polyhedral invariant set $\mathcal{X}_\chi = \{\chi \mid M_\chi \chi \leq \mathbf{1}\}$, computed off-line considering an invariant outer polyhedral approximation $\hat{\mathcal{R}} \in \mathcal{E}_{\tilde{x}}$ of $\mathcal{R}^{min}$ as mentioned in the last section. Moreover, we assume that the state estimation error set $\mathcal{R}(t)$ is admissible in the sense that the predicted estimation error set at time $t+N$ belongs to $\hat{\mathcal{R}}$, i.e., $\mathcal{R}(t+N|t) \subseteq \hat{\mathcal{R}}$, where the predicted estimation error sets are defined recursively as

$$\mathcal{R}(t+i+1|t) = \tilde{A}(t+i)\mathcal{R}(t+i|t) \oplus B_w(t+i)\mathcal{W}_w \oplus (-LD)\mathcal{W}_e \quad (5.17)$$

with $\mathcal{R}(t|t) = \mathcal{R}(t)$. Then, the constraints for the finite horizon ($i = 0, \dots, N-1$) combined with the terminal constraint $\chi(t+N|t) \in \mathcal{X}_\chi$ can be expressed as:

$$\mathcal{M}_{\hat{x}}\hat{x}(t) + \mathcal{M}_u \begin{bmatrix} \bar{\mathbf{v}}(t) \\ \bar{\xi}(t+N|t) \end{bmatrix} + \max_{\boldsymbol{\eta}(t) \in \mathfrak{W}(t)} \left(\mathcal{M}_\eta + \mathcal{M}_u \begin{bmatrix} \mathbf{E}(t) \\ \tilde{\mathbf{F}}(t) \end{bmatrix} \right) \boldsymbol{\eta}(t) \leq \mathbf{b}(t) \quad (5.18)$$

where $\mathfrak{W}(t) = \mathbf{C}\mathfrak{R}(t) \oplus \mathbf{D}\mathcal{W}_e^N$, with $\mathfrak{R}(t) = \mathcal{R}(t|t) \times \mathcal{R}(t+1|t) \times \dots \times \mathcal{R}(t+N-1|t)$, $\tilde{\mathbf{F}}(t) = [\tilde{F}_0(t+N|t) \ \tilde{F}_1(t+N|t) \dots \tilde{F}_{N-1}(t+N|t)]$ and $\mathbf{b}(t)_{[j]} = \min_{\tilde{\mathbf{x}}(t) \in \mathfrak{R}(t)} (1 - \mathcal{M}_{\tilde{x}[j,:]} \tilde{\mathbf{x}}(t))$. Here, the constant matrices $\mathcal{M}_{\hat{x}}$, $\mathcal{M}_u$ and $\mathcal{M}_\eta$ and $\mathcal{M}_{\tilde{x}}$ are defined in Appendix B.

Now, for a given initial estimated state $\hat{x}(t)$ and an estimation error set $\mathfrak{R}(t)$, the set of constraint-admissible control decision variables $\bar{\mathbf{v}}(t), \bar{\xi}(t+N|t), \mathbf{E}(t), \tilde{\mathbf{F}}(t)$ in the control

policy (5.10a) can be defined as

$$\mathfrak{X}_t^N(\hat{x}(t), \mathfrak{R}(t)) = \{(\bar{\mathbf{v}}(t), \bar{\xi}(t+N|t), \mathbf{E}(t), \tilde{\mathbf{F}}(t)) \mid (5.18) \text{ is satisfied.}\}.$$

We define the overall feasible set with the control policy of (5.10a) as $\mathcal{X}_{\hat{x},t}^N(\mathfrak{R}(t)) = \{x \mid \mathfrak{X}_t^N(x, \mathfrak{R}(t)) \neq \emptyset\}$. The following lemma defines the nature of this feasible set.

Lemma 5.3. Let the set $\mathcal{R}(t)$ be admissible in the sense that $\mathcal{R}(t+N|t) \subseteq \hat{\mathcal{R}}(t)$. The set $\mathcal{X}_{\hat{x},t}^N(\mathfrak{R}(t))$ is a bounded polyhedron. Moreover, the feasible sets obtained with increasing horizon lengths satisfy the successive inclusion

$$\mathcal{X}_{\hat{x},t}^N(\mathfrak{R}(t)) \subseteq \mathcal{X}_{\hat{x},t}^{N+1}(\mathfrak{R}(t)) \subseteq \mathcal{X}_{\hat{x},t}^{N+2}(\mathfrak{R}(t)) \dots$$

at each time t . □

PROOF. The proof follows the lines of reasoning used in the proof of Lemma 3.2 in Chapter 3. ■

In an RHC scheme, we determine and apply, at each time instant t , a suitable constraint-admissible control input $u(t) = \kappa(\hat{x}(t), \mathfrak{R}(t), t)$ which satisfies

$$\kappa(\hat{x}(t), \mathfrak{R}(t), t) \in \left\{ K\hat{x}(t) + \bar{v}(t) \mid (\bar{\mathbf{v}}(t), \bar{\xi}(t+N|t), \mathbf{E}(t), \tilde{\mathbf{F}}(t)) \in \mathfrak{X}_t^N(\hat{x}(t), \mathfrak{R}(t)) \right\}. \quad (5.20)$$

Now we discuss the computation of a constraint-admissible input $u(t) = \kappa(\hat{x}(t), \mathfrak{R}(t), t)$ at each time t for an estimated state $\hat{x}(t)$ and some error set $\mathcal{R}(t)$, and the existence of constraint-admissible control inputs at successive time steps given that there exists an admissible control input $u(t)$ at time t . It is clear that the set $\mathfrak{R}(t)$ considered in (5.18) can become too complex if it is computed exactly. It may therefore be reasonable to consider simpler (e.g., rectangular) outer approximations of the predicted sets $\mathcal{R}(t+i|t)$, $i = 1, \dots, N-1$ so as to reduce the complexity of the constraint (5.18). Moreover, since we know that the state estimation error will ultimately be bounded in the set $\mathcal{R}^{min}$, it is desirable to consider a close outer invariant approximation $\mathcal{R}$ of $\mathcal{R}^{min}$ and assume this set $\mathcal{R}$ as the estimation error set once the real estimation error enters this set. Therefore, for each $i \in 0, 1, 2, \dots, N-1$, let $\hat{\mathcal{R}}(t+i|t)$ represent a close but simpler outer approximation of the set $\mathcal{R}(t+i|t) \cup \mathcal{R}$, and let $\hat{\mathfrak{R}}(t)$ and $\hat{\mathfrak{W}}(t)$ be the corresponding set combinations.

Now, given an initial estimated state $\hat{x}(0)$, and an initial estimation error set $\mathcal{R}(0)$, we consider the computation of control inputs of the form

$$u(t) = \kappa(\hat{x}(t), \hat{\mathfrak{R}}(t), t) \quad (5.21)$$

where $\hat{\mathfrak{R}}(t)$ comprises of the outer bounds $\hat{\mathcal{R}}(t+i|t)$, $i = 1, \dots, N-1$ of the predicted estimation error sets $\mathcal{R}(t+i|t) \cup \mathcal{R}$ computed with $\hat{\mathcal{R}}(t|t) = \mathcal{R}(t)$ at $t = 0$ and with

$\hat{\mathcal{R}}(t|t) = \hat{\mathcal{R}}(t)$ at $t \in \mathbb{Z}_{++}$. Here, $\hat{\mathcal{R}}(t)$ is an outer approximation of the estimation error set obtained via set membership estimation at time t , viz., the set $(\mathcal{R}^p(t) \cap \mathcal{R}^m(t)) \cup \mathcal{R}$, where $\mathcal{R}^p(t) = \tilde{A}(t-1)\hat{\mathcal{R}}(t-1) \oplus B_w(t-1)\mathcal{W}_w \oplus (-LD)\mathcal{W}_e$ is the set predicted from $\hat{\mathcal{R}}(t-1)$ whereas $\mathcal{R}^m(t) = \{x \mid y(t) - Cx \in D\mathcal{W}_e\} \oplus \{-\hat{x}(t)\}$ is the set determined from the current measurement.

The following lemma deals with the existence of constraint-admissible control inputs at times $t \in \mathbb{Z}_{++}$ provided an admissible input exists at time $t = 0$.

Lemma 5.4. Given $\hat{x}(0)$ and an admissible $\mathcal{R}(0)$, if there exists a feasible input $u(0)$ of the form (5.21) at time $t = 0$, then the application of the control input $u(t)$ from (5.21) at each time $t \in \mathbb{Z}_+$ guarantees that system constraints are satisfied and feasible control inputs exist at all times $t \in \mathbb{Z}_{++}$. $\square$

PROOF. Given the existence of a feasible input $u(t) = \kappa(\hat{x}(t), \hat{\mathfrak{R}}(t), t)$ at time t , the fact that there exists a feasible input $\kappa(\hat{x}(t+1), \tilde{\mathfrak{R}}(t+1), t+1)$ with $\tilde{\mathfrak{R}}(t+1) = \hat{\mathcal{R}}(t+1|t) \times \dots \times \hat{\mathcal{R}}(t+N-1|t) \times \hat{R}$ at time $t+1$ follows directly from the satisfaction of (5.18) with $\mathfrak{R}(t)$ replaced by $\hat{\mathfrak{R}}(t)$ as well as the fact that the terminal constraint set $\mathcal{X}_\chi$ is invariant for any estimation error in the set $\hat{\mathcal{R}}$ and that the estimation error $\tilde{x}(k+N|t)$ at time $t+N$ belongs to the set $\hat{\mathcal{R}}$. Clearly, since $\hat{\mathfrak{R}}(t+1) \subseteq \tilde{\mathfrak{R}}(t+1)$, a feasible input $u(t+1) = \kappa(\hat{x}(t+1), \hat{\mathfrak{R}}(t+1), t+1)$ will exist at time $t+1$. Therefore, given a feasible input $u(0)$ at time $t = 0$, system constraints are satisfied and feasible control inputs mentioned in (5.21) exist at all times $t \in \mathbb{Z}_{++}$. $\blacksquare$

5.3.3.2 Control performance with nominal cost minimization

For the on-line computation of control actions that ensure a nominal performance, we consider the minimization of a nominal (disturbance-free) cost function

$$\bar{J}(t) = \sum_{i=0}^{\infty} \left\{ \|Q^{\frac{1}{2}} \bar{x}(t+i|t)\|^2 + \|R^{\frac{1}{2}} \bar{u}(t+i|t)\|^2 \right\} \quad (5.22)$$

where $\{\bar{x}(t+i|t)\}_{i=0}^{\infty}$ and $\{\bar{u}(t+i|t)\}_{i=0}^{\infty}$ are the predicted state trajectory and the control input sequence for the undisturbed system $\bar{x}(t+1) = A(t)\bar{x}(t) + B_u(t)\bar{u}(t)$ with $\bar{x}(t|t) = \hat{x}(t)$.

With the input sequence as in (5.10a), $\bar{J}(t)$ will be the sum of the predicted cost in the finite horizon and the predicted terminal cost. We consider a terminal cost bound of the form $\|\bar{\mathcal{P}}^{\frac{1}{2}} \bar{\chi}(t+N|t)\|^2$ where $\bar{\mathcal{P}}$ is a positive definite matrix obtained from the minimization

$$\min_{\bar{\mathcal{P}}} \text{Tr } \bar{\mathcal{P}} \quad (5.23)$$

subject to

$$\bar{\mathcal{P}} - \Psi^{(j)\top} \bar{\mathcal{P}} \Psi^{(j)} \succeq C_z^\top C_z, \quad C_z = \begin{pmatrix} Q^{\frac{1}{2}} & 0 \\ R^{\frac{1}{2}} K & R^{\frac{1}{2}} H \end{pmatrix}$$

for $j = 1, \dots, n_\Omega$, $\bar{\chi}(t + N|t) = [\bar{x}(t + N|t)^T \ \bar{\xi}(t + N|t)^T]^T$ and $\bar{\xi}(t + N|t)$ is the terminal controller initial state that is be optimized on-line.

The terminal cost bound $\|\bar{\mathcal{P}}^{\frac{1}{2}} \bar{\chi}(t + N|t)\|^2$ can be combined with the finite-horizon cost to give the total nominal cost

$$\bar{J}(t) = \left\| \mathbf{H}(t)^{\frac{1}{2}} \begin{bmatrix} \hat{x}(t)^T & \bar{\mathbf{v}}(t)^T & \bar{\xi}(t + N|t)^T \end{bmatrix}^T \right\|^2 \quad (5.24)$$

where

$$\begin{aligned} \mathbf{H}(t) = & \begin{pmatrix} \mathbf{A}(t) & \mathbf{B}_u(t) & 0 \\ 0 & 0 & I_{n_\xi} \end{pmatrix}^T \begin{pmatrix} \mathbf{Q} & 0 \\ 0 & \bar{\mathcal{P}} \end{pmatrix} \begin{pmatrix} \mathbf{A}(t) & \mathbf{B}_u(t) & 0 \\ 0 & 0 & I_{n_\xi} \end{pmatrix} \\ & + \begin{pmatrix} \check{\mathbf{A}}(t) & \check{\mathbf{B}}_u(t) & 0 \end{pmatrix}^T \mathbf{R} \begin{pmatrix} \check{\mathbf{A}}(t) & \check{\mathbf{B}}_u(t) & 0 \end{pmatrix}. \end{aligned} \quad (5.25)$$

We minimize $\bar{J}(t)$ subject to (5.18) considering the set $\hat{\mathfrak{W}}(t)$ instead of the set $\mathfrak{W}(t)$. The maximization in (5.18) is a row-wise maximization. Since the set $\hat{\mathfrak{W}}(t)$ is polyhedral, let us consider that the condition $\boldsymbol{\eta}(t) \in \hat{\mathfrak{W}}(t)$ is expressed as $\hat{\mathbf{M}}_{\mathfrak{W}} \boldsymbol{\eta}(t) \leq \mathbf{1}$. Then dualizing the maximization operations, condition (5.18) can be equivalently written in terms of a new variable matrix $\mathbf{L} \in \mathbb{R}^{m_1 \times m_2}$, where m_1 and m_2 are the number of rows in $\mathcal{M}_\eta$ and $\hat{\mathbf{M}}_{\mathfrak{W}}$ respectively, as

$$\left. \begin{aligned} & \mathcal{M}_{\hat{x}} \hat{x}(t) + \mathcal{M}_u \begin{bmatrix} \bar{\mathbf{v}}(t) \\ \bar{\xi}(t + N|t) \end{bmatrix} + \mathbf{L} \mathbf{1} \leq \mathbf{b}(t), \\ & \mathbf{L} \geq 0, \ \mathbf{L} \hat{\mathbf{M}}_{\mathfrak{W}} = \mathcal{M}_\eta + \mathcal{M}_u \begin{bmatrix} \mathbf{E}(t) \\ \tilde{\mathbf{F}}(t) \end{bmatrix} \end{aligned} \right\}. \quad (5.26)$$

Now, we present the following result on the the performance of an RHC scheme based on the minimization of the nominal cost.

Proposition 5.1. If the optimization problem

$$\left. \begin{aligned} & \min_{\bar{\mathbf{v}}(t), \bar{\xi}(t+N|t), \mathbf{L}, \mathbf{E}(t), \tilde{\mathbf{F}}(t)} \bar{J}(t) \\ & \text{with } \bar{J}(t) = \left\| \mathbf{H}(t)^{\frac{1}{2}} \begin{bmatrix} \hat{x}(t)^T & \bar{\mathbf{v}}(t)^T & \bar{\xi}(t + N|t)^T \end{bmatrix}^T \right\|^2 \end{aligned} \right\} \quad (5.27)$$

is feasible under the constraint condition (5.26) for an initial estimated state $\hat{x}_0$ and an initial estimation error set $\mathcal{R}(0)$ at time $t = 0$, then the receding horizon control $u^*(t) = \kappa^*(\hat{x}(t), \hat{\mathfrak{R}}(t), t) = K \hat{x}(t) + \bar{v}^*(t)$ obtained from the solution of (5.27) as per the rule (5.21) at each time $t \in \mathbb{Z}_+$ guarantees that the closed-loop controlled system (5.9) is ℓ_∞ -stable, while satisfying the system constraints at all times. $\square$

PROOF. Since the optimization problem (5.27) is feasible at time $t = 0$, the feasibility of the problem for the controlled system at all times $t \geq 1$ follows from Lemma 5.4.

Figure 5.2: State trajectories of the system in (5.15) under the control scheme of Proposition 5.1 with $N = 5$: (a) A pair of actual and estimated state trajectories for $\hat{x}(0) = [-85 \ 10]^T$, $\tilde{x}(0) = [-0.5 \ 2]^T$ and $\mathcal{R}(0) = [-0.5, 0.5] \times [-2, 2]$; (b) Actual state trajectories for $\hat{x}(0) = [-85 \ 10]^T$, $\mathcal{R}(0) = [-1, 1] \times [-2, 2]$ and randomly chosen state estimation errors in $\mathcal{R}(0)$.

Next, since the constraint set Ξ is compact, a feasible input $u(t) = \kappa(\hat{x}(t), \hat{\mathfrak{R}}(t), t)$ takes values from a compact set for any $\hat{x}(t) \in \mathcal{X}_{\hat{x},t}^N(\hat{\mathfrak{R}}(t))$ at each time t . Moreover, the cost function $\bar{J}(t)$ is strictly convex in $\bar{\mathbf{v}}(t)$ and $\bar{\xi}(t + N|t)$. It, therefore, follows that $u^*(t)$ and hence $\bar{J}^*(t)$ are Lipschitz continuous on $\mathcal{X}_{\hat{x},t}^N$ for all $\hat{x}(t) \in \mathcal{X}_{\hat{x},t}^N$. So, $\hat{f}(\hat{x}(t), \nu(t), t) = A(t)\hat{x}(t) + B(t)\kappa^*(\hat{x}(t), \hat{\mathfrak{R}}(t), t) + L[C \ D]\nu(t)$ and $z(t) = \hat{h}(\hat{x}(t), \nu(t), t) = \begin{bmatrix} Q^{\frac{1}{2}}\hat{x}(t) \\ R^{\frac{1}{2}}\kappa^*(\hat{x}(t), \hat{\mathfrak{R}}(t), t) \end{bmatrix} + \begin{bmatrix} Q^{\frac{1}{2}} \ 0 \\ 0 \ 0 \end{bmatrix} \nu(t)$, where $\nu(t) = [\tilde{x}(t)^T \ e(t)^T]^T$, are Lipschitz on $\mathcal{X}_{\hat{x},t}^N \times \mathcal{W}_\nu$ where $\mathcal{W}_\nu = \hat{\mathcal{R}}(t) \times \mathcal{W}_e$. Clearly, $\bar{J}^*(t)$ is a Lyapunov function for the undisturbed system $\bar{x}(t+1) = A(t)\bar{x}(t) + B(t)\kappa^*(\hat{x}(t), \hat{\mathfrak{R}}(t), t)$, $\bar{x}(0) = \hat{x}(0)$ with $\gamma_1(r) = \gamma_3(r) = \lambda_{\min}(Q)\|r\|^2$, and a suitable γ_2 . It therefore follows from Lemma 2.3 that there exist a class $\mathcal{K}$ function γ_ν and a non-negative constant β_ν so that $\|z\|_{\ell_\infty} \leq \gamma_\nu(\|\nu\|_{\ell_\infty}) + \beta_\nu$.

Moreover, since the linear system $\tilde{x}(t+1) = \tilde{A}(t)\tilde{x}(t) + [B_w \ -LD]\omega(t)$, $\nu(t) = \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 0 \\ I_{n_e} \end{bmatrix} \omega(t)$ satisfies the ℓ_∞ -stability condition, there exist a class $\mathcal{K}$ function γ_ω and a non-negative constant β_ω so that $\|\nu\|_{\ell_\infty} \leq \gamma_\omega(\|\omega\|_{\ell_\infty}) + \beta_\omega$. Therefore, we have $\|z\|_{\ell_\infty} \leq \gamma_0(\|\omega\|_{\ell_\infty}) + \beta_0$ with $\gamma_0 = \gamma_\nu \circ 2\gamma_\omega$ and $\beta_0 = \gamma_\nu(2\beta_\omega) + \beta_\nu$. Hence the ℓ_∞ -stability of the controlled composite system (5.9). ■

Remark 5.1. If we consider an input parameterization with $\mathbf{E}(t) = 0$, $\tilde{\mathbf{F}}(t) = 0$ and $F^{(j)} = 0$ ($j = 1, \dots, n_\Omega$), we can guarantee that the system state ultimately converges into a small set in the neighbourhood of the origin - in particular, the set $\mathcal{S}_{\hat{x}}^K \oplus \mathcal{R}_{\hat{x}}^{\min}$ where $\mathcal{S}_{\hat{x}}^K$ is the smallest invariant set for the estimated state under the static feedback input $u(t) = K\hat{x}(t)$ (see e.g., [89], [104]). Considering $\mathbf{E}(t) = 0$, $\tilde{\mathbf{F}}(t) = 0$ and $F^{(j)} = 0$ ($j = 1, \dots, n_\Omega$) implies a reduction in the on-line computational complexity as well. However, it may lead to significant conservativeness in the size of the feasible set as well as the control performance. □

Illustration 5.2. We again consider the system in (5.15) with all the details mentioned in Illustration 5.1 in order to assess the control performance of the nominal cost minimization scheme of Proposition 5.1. The plots in Fig. 5.2 show the control performance of the proposed

Figure 5.3: (a) Performance of the nominal output-feedback control schemes (with $N = 0$) with and without the output-error feedforward; (b) Performance of proposed scheme of Proposition 5.1 with $N = 0, 1, 2, 3$ for an estimated state $\hat{x}(0) = [30 \ 7]^T$ and uncertainty set $\mathcal{R}_0 = \mathcal{R}$.

control scheme with $N = 5$ when we consider $\rho(t) = \sin(2\pi k/25)$ and random realizations of $w(t)$ and $e(t)$ within the mentioned bounds. Fig. 5.2a shows a pair of actual and estimated state trajectories obtained for $\hat{x}(0) = [-85 \ 10]^T$, $\tilde{x}(0) = [-0.5 \ 2]^T$ and $\mathcal{R}(0) = [-0.5, 0.5] \times [-2, 2]$. Fig. 5.2b, on the other hand, shows the actual state trajectories for an initial estimated state $\hat{x}_0 = [-85 \ 9]^T$ and random estimation errors distributed uniformly in the set $\mathcal{R}_0 = [-1, 1] \times [-2, 2]$. It can be observed that all the trajectories converge to the neighbourhood of the origin in a finite number of steps.

The plots in Fig. 5.3 compare the control performances of the nominal output feedback schemes with and without the output-error feedforward and with various horizon lengths. The time variations of the states under the proposed scheme with $N = 0$ and under a similar scheme but without disturbance feedforward (i.e., with $F^{(j)} = 0$, $(j = 1, \dots, n_\Omega)$) are shown in Fig. 5.3a for an initial state estimate $[40 \ 4]^T$. Here, $\rho(t)$, $w(t)$ and $e(t)$ are chosen as mentioned above, and $\mathcal{R}_0 = \mathcal{R}$ is considered. Clearly, the response in the latter case is more conservative as compared to that in the former. Similarly, the plots in Fig. 5.3b compare the control performances of the proposed scheme for various horizon lengths N with an initial state estimate $[30 \ 7]^T$. The response improves with increasing values of N but a small value of N is sufficient to ensure a good performance. $\square$

Remark 5.2. Note that the treatment of the output-feedback RHC of this section parallels the treatment of the nominal-cost RHC in Section 3.4 of Chapter 3. The essential difference is the presence of an additional uncertainty arising from the state-estimation error. Since the state is not known completely even in the steady state, it is necessary to account for the presence of this uncertainty in the design of the terminal controller and as we have seen in Illustration 5.1, one consequence of this is that the terminal feasible set is significantly smaller than the one that would otherwise be possible (in the case of full state feedback). Furthermore, additional difficulties arise in handling the feasibility when the state estimation is in the transient phase and the estimation error is not already bounded in the set $\mathcal{R}^{min}$. As discussed above, since the state estimation error remains uncertain in the successive time instants, such a situation requires the computation of (possibly conservative) outer bounds of the state estimation error and the resulting output error for each future instant in the finite horizon considered. This allows us to avoid additional conservativeness while ensuring the recursive feasibility of the problem. Clearly, the need

to deal with time-varying uncertainty sets and to compute their outer bounds for each instant of the considered time horizon makes on-line computations more complex when compared with the case of full state feedback. Nevertheless, these difficulties are typical of most output-feedback RHC schemes and the proposed dynamic-policy-based approach of this chapter allows us to limit the complexities by effectively reducing the required horizon length for on-line computations. $\square$

5.4 Min-max Output-Feedback RHC Based on Optimized Dynamic Policy

In the previous section, we discussed the off-line and on-line computations required for an output-feedback RHC scheme that employs an on-line optimization of a nominal cost and ensures the ℓ_∞ stability of the controlled composite system (5.9). As discussed in Chapter 4, RHC schemes based on a nominal cost minimization may not always ensure a desired control performance especially when the control inputs are parameterized in terms of disturbances and noises. In this section, we discuss a min-max output-feedback RHC scheme, based on the input parameterization presented in the last section, which provides a theoretical guarantee of a desired control performance. Ideally, we wish to consider a predicted performance function of the form

$$\hat{J}(t) = \max_{\substack{\tilde{x}(t) \in \mathcal{R}(t) \\ \omega(t+i) \in \mathcal{W}_\omega, i \in \mathbb{Z}_+}} \sum_{i=0}^{\infty} \left\{ \|Q^{\frac{1}{2}} x(t+i|t)\|^2 + \|R^{\frac{1}{2}} u(t+i|t)\|^2 - \gamma^2 \|\omega(t+i)\|^2 \right\} \quad (5.28)$$

where $\omega(t+i) = [w(t+i)^T \ e(t+i)^T]^T$ and $\mathcal{W}_\omega = \mathcal{W}_w \times \mathcal{W}_e$, so that we can ensure a bound on the ℓ_2 gain from ω to z for the composite system (5.9). However, when we have an output-error-based input parameterization, it is difficult to ensure recursive feasibility and cost monotonicity with a minimization of the performance index of the form (5.28). Since we treat the state estimation error as an additional disturbance constituting the output error (a combination of the state estimation error $\tilde{x}(t)$ and the measurement noise $e(t)$) and since our input parameterization is based on the feedforward of the output error, we consider a modified version of the performance function which assumes a maximization over the sequence of state estimation errors for the time horizon considered rather than over just the initial estimation error $\tilde{x}(t)$. In particular, we define the modified min-max performance function as

$$\hat{J}(t) = \max_{\substack{\tilde{x}(t+i|t) \in \mathcal{R}(t+i|t) \\ e(t+i) \in \mathcal{W}_e, i \in \mathbb{Z}_+}} \sum_{i=0}^{\infty} \left\{ \|Q^{\frac{1}{2}} x(t+i|t)\|^2 + \|R^{\frac{1}{2}} u(t+i|t)\|^2 - \gamma_s^2 \|\tilde{x}(t+i|t)\|^2 - \gamma_e^2 \|e(t+i)\|^2 \right\} \quad (5.29)$$

where $\mathcal{R}(t + i|t)$, $i \in \mathbb{Z}_+$ are the predicted state estimation error sets and γ_s and γ_e are suitably chosen gain bounds. In the following, we discuss the computations required to implement an RHC scheme based on the function (5.29) and analyze the control performance of the min-max RHC scheme in terms of the ℓ_2 stability of the controlled composite system (5.9).

5.4.1 Control parameterization and computation of terminal controller matrices

We consider the control input parameterization presented in the last section and determine the terminal controller matrices off-line in a similar way. As discussed in Chapter 4, an off-line optimization of the controller matrices under the invariance and feasibility conditions (5.12) and (5.13) alone does not ensure a control performance based on the min-max performance function. To ensure such a performance, the predicted state and input sequences must be such that the right hand side of (5.29) has a bound. We use the result of Lemma 4.1 to present a boundedness condition which is sufficient to ensure such a performance.

For the augmented system (5.11), under the terminal control input parameterization, the output signal $z(t)$ which defines the stage cost $z(t)^T z(t)$ can be written as $z(t) = C_z \chi(t) + D_z \nu(t)$, where

$$C_z = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ R^{\frac{1}{2}}K & R^{\frac{1}{2}}H \end{bmatrix}, \quad D_z = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}$$

So, following the result of Lemma 4.1, the LMI conditions

$$\begin{bmatrix} \mathcal{P} & \Psi^{(j)T} \mathcal{P} & C_z^T & 0 \\ * & \mathcal{P} & 0 & \mathcal{P} \mathcal{B}_w^{(j)} \\ * & * & I & D_z \\ * & * & * & \begin{bmatrix} \gamma_s^2 I_{n_x} & 0 \\ 0 & \gamma_e^2 I_{n_e} \end{bmatrix} \end{bmatrix} \succeq 0, \quad j = 1, 2, \dots, n_\Omega \quad (5.30)$$

with $\mathcal{P} \succeq 0$ ensure, for all $t \in \mathbb{Z}_+$,

$$\|\mathcal{P}^{\frac{1}{2}} \chi(t+1)\|^2 - \|\mathcal{P}^{\frac{1}{2}} \chi(t)\|^2 \leq -z(t)^T z(t) + \gamma_s^2 \|\tilde{x}(t)\|^2 + \gamma_e^2 \|e(t)\|^2 \quad (5.31)$$

and hence

$$\sum_{t=0}^{\infty} \left\{ \|z(t)\|^2 - \gamma_s^2 \|\tilde{x}(t)\|^2 - \gamma_e^2 \|e(t)\|^2 \right\} \leq \|\mathcal{P}^{\frac{1}{2}} \chi(0)\|^2. \quad (5.32)$$

We optimize the controller matrices off-line under the invariance, feasibility and bound-

Figure 5.4: Feasible invariant sets for the polytopic system of (5.15) obtained by solving (5.33) for $\gamma_1 = (2, 0.75)$ and $\gamma_2 = (4, 1.5)$.

edness conditions (5.12), (5.13) and (5.30). In particular, we solve the problem

$$\max_{\mathcal{P}, \mathcal{P}, Z, H, G^{(j)}, F^{(j)}, \alpha_1^{(j)}, \alpha_2^{(j)}, R_w^{(j)} (j=1, \dots, n_\Omega)} \log \det(T \mathcal{P} T^T) \quad (5.33)$$

subject to conditions (5.12), (5.13) and (5.30). As mentioned in Chapter 4, using the same Lyapunov matrix for the conditions allows us to express the conditions in simpler forms such as in Chapter 3, but this may lead to conservative results. For this reason, we consider a separate Lyapunov matrix in the boundedness condition (5.30). Problem (5.33) is therefore a BMI problem which can be solved with the algorithm presented in Chapter 4 (cf. Algorithm 4.1). The result of Proposition 4.2 is valid in this case as well. So, if we select γ_s and γ_e such that they satisfy the condition,

$$\begin{bmatrix} P - K^T R K & \mathcal{A}^{(j)T} P & Q^{\frac{1}{2}} & 0 & 0 \\ * & P & 0 & PLC & PLD \\ * & * & I & Q^{\frac{1}{2}} & 0 \\ * & * & * & \gamma_s^2 I & 0 \\ * & * & * & 0 & \gamma_e^2 I \end{bmatrix} \succeq 0, \quad j = 1, 2, \dots, n_\Omega \quad (5.34)$$

for some $P \succeq 0$, then we can use the solution of (5.14) obtained with $F^{(j)} = 0, j = 1, \dots, n_\Omega$ in order to initialize the algorithm to solve (5.33).

Illustration 5.3. Here, we again consider example system (5.15) for an illustration. With the LQ optimal gain K chosen in the dynamic policy, we solve problem (5.33) in order to obtain the controller matrices for two choices of the gain bounds: $\gamma_1 = (2, 0.75)$ and $\gamma_2 = (4, 1.5)$. The feasible invariant sets $\mathcal{X}_x^{\gamma_1}$ and $\mathcal{X}_x^{\gamma_2}$ in the two cases are shown in Fig. 5.4. Evidently, the set $\mathcal{X}_x^{\gamma_2}$ has larger size than $\mathcal{X}_x^{\gamma_1}$.

5.4.2 On-line computations and control performance

We now discuss the on-line computations to be considered for min-max RHC scheme based on the cost (5.29). The characterizations of the overall feasible set and the feasible

control inputs follow those mentioned in Section 5.3.3. However, unlike in Section 5.3.3, here we consider the state estimation error sets and measurement error sets separately in the constraint equation. In particular, we write the constraint equation (5.18) as

$$\mathcal{M}_{\hat{x}} \hat{x}(t) + \mathcal{M}_u \begin{bmatrix} \bar{\mathbf{v}}(t) \\ \bar{\xi}(t+N|t) \end{bmatrix} + \max_{\substack{\tilde{\mathbf{x}}(t) \in \mathfrak{R}(t) \\ \mathbf{e}(t) \in \mathcal{W}_e^N}} \left(\mathcal{M}_\eta + \mathcal{M}_u \begin{bmatrix} \mathbf{E}(t) \\ \tilde{\mathbf{F}}(t) \end{bmatrix} \right) \times \\ (\mathbf{C} \tilde{\mathbf{x}}(t) + \mathbf{D} \mathbf{e}(t)) \leq \mathbf{b}(t) \quad (5.35)$$

When employing an RHC scheme, because of computational considerations, we need to consider simple outer bounds of the current and predicted future state estimation error sets. In particular, for each predicted estimation error set $\mathcal{R}(t+i|t)$, we consider its close outer bound $\hat{\mathcal{R}}(t+i|t)$ such that $\hat{\mathcal{R}}(t+i|t) = \mathcal{T}(t+i|t) \mathcal{W}_s$ where $\mathcal{W}_s = \{w \mid \|w\|_\infty \leq 1\}$ and $\mathcal{T}(t+i|t)$ is a suitable matrix. Then, defining a new variable $\mathbf{s}(t) = [s(t)^\top s(t+1)^\top \dots s(t+N-1|t)^\top]^\top$, and a new matrix $\mathbf{T}(t) = \text{diag}(\mathcal{T}(t|t), \mathcal{T}(t+1|t), \dots, \mathcal{T}(t+N-1|t))$, we write a simplified but conservative form of the constraint (5.35) as

$$\mathcal{M}_{\hat{x}} \hat{x}(t) + \mathcal{M}_u \begin{bmatrix} \bar{\mathbf{v}}(t) \\ \bar{\xi}(t+N|t) \end{bmatrix} + \max_{\substack{\mathbf{s}(t) \in \mathcal{W}_s^N \\ \mathbf{e}(t) \in \mathcal{W}_e^N}} \left(\mathcal{M}_\eta + \mathcal{M}_u \begin{bmatrix} \mathbf{E}(t) \\ \tilde{\mathbf{F}}(t) \end{bmatrix} \right) \times \\ (\mathbf{CT}(t)\mathbf{s}(t) + \mathbf{De}(t)) \leq \mathbf{b}(t) \quad (5.36)$$

where $\mathbf{T}(t)$ is such that $\hat{\mathfrak{R}}(t) = \mathbf{T}(t) \mathcal{W}_s^N$. Condition (5.36) can be equivalently written in terms of two new variable matrices $\mathbf{L}_s \in \mathbb{R}^{m_1 \times m_s}$ and $\mathbf{L}_e \in \mathbb{R}^{m_1 \times m_e}$, where $m_s = N n_x$ and $m_e = N n_e$, as

$$\left. \begin{aligned} & \mathcal{M}_{\hat{x}} \hat{x}(t) + \mathcal{M}_u \begin{bmatrix} \bar{\mathbf{v}}(t) \\ \bar{\xi}(t+N|t) \end{bmatrix} + \mathbf{L}_s \mathbf{1} + \mathbf{L}_e \mathbf{1} \leq \mathbf{b}(t), \\ & -[\mathbf{L}_s \quad \mathbf{L}_e] \leq \left(\mathcal{M}_\eta + \mathcal{M}_u \begin{bmatrix} \mathbf{E}(t) \\ \tilde{\mathbf{F}}(t) \end{bmatrix} \right) [\mathbf{CT}(t) \quad \mathbf{D}] \leq [\mathbf{L}_s \quad \mathbf{L}_e]. \end{aligned} \right\} \quad (5.37)$$

Now, as discussed in Section 4.4.2, with the control parameterization of (5.10a), a min-max performance index of the form (5.29) can be expressed (possibly conservatively) as

$$\hat{J}(t) = \max_{\substack{\mathbf{s}(t) \in \mathcal{W}_s^N \\ \mathbf{e}(t) \in \mathcal{W}_e^N}} \left\{ \left\| \mathbf{Q}^{\frac{1}{2}} \left(\mathcal{C}_x + \mathcal{G}_{xs} \mathbf{s}(t) + \mathcal{G}_{xe} \mathbf{e}(t) \right) \right\|^2 + \left\| \mathbf{R}^{\frac{1}{2}} \left(\mathcal{C}_u + \mathcal{G}_{us} \mathbf{s}(t) + \mathcal{G}_{ue} \mathbf{e}(t) \right) \right\|^2 \right. \\ \left. - \gamma_s^2 \|\mathbf{T}(t) \mathbf{s}(t)\|^2 - \gamma_e^2 \|\mathbf{e}(t)\|^2 \right\} \quad (5.38)$$

where

$$\begin{aligned}
\mathcal{C}_x &= \begin{bmatrix} \mathbf{A}(t) \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} \mathbf{B}_u(t) & 0 \\ 0 & I_{n_\xi} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{v}}(t) \\ \bar{\xi}(t + N|t) \end{bmatrix}, \quad \mathcal{C}_u = \check{\mathbf{A}}(t) x(t) + \check{\mathbf{B}}_u(t) \bar{\mathbf{v}}(t), \\
\mathcal{G}_x &= \begin{bmatrix} \mathbf{B}_\eta(t) + \mathbf{B}_u(t) \mathbf{E}(t) \\ \tilde{\mathbf{F}}(t) \end{bmatrix}, \quad G_{xs} = (\mathcal{G}_x \mathbf{C} + I) \mathbf{T}(t), \quad G_{xe} = G_x \mathbf{D}, \\
\mathcal{G}_u &= \check{\mathbf{B}}_\eta(t) + \check{\mathbf{B}}_u(t) \mathbf{E}(t), \quad \mathcal{G}_{us} = \mathcal{G}_u \mathbf{C} \mathbf{T}(t), \quad \mathcal{G}_{ue} = \mathcal{G}_u \mathbf{D},
\end{aligned} \tag{5.39}$$

and $\mathcal{Q} = \text{diag}(\mathbf{Q}, \mathcal{P})$. Here, the cost in (5.38) includes an upper bound of the terminal cost represented by $\left\| \mathcal{P}^{\frac{1}{2}} [x(t + N|t)^\top \quad \xi(t + N|t)^\top]^\top \right\|^2$ where $\mathcal{P}$ is obtained by solving the problem

$$\min_{\mathcal{P}} \text{Tr} \mathcal{P} \tag{5.40}$$

subject to condition (5.30). It also follows from the discussion in Section 4.4.2, that a scalar $g(t)$ upper bounds (possibly conservatively) the cost $\hat{J}(t)$ in (5.38) if it satisfies the LMI condition

$$\begin{bmatrix} g(t) - \text{Tr}(\Upsilon_s) - \text{Tr}(\Upsilon_e) & 0 & 0 & \mathcal{C}_x^\top & \mathcal{C}_u^\top \\ * & \gamma_s^2 \Lambda + \Upsilon_s & 0 & \mathcal{G}_{xs}^\top & \mathcal{G}_{us}^\top \\ * & 0 & \gamma_e^2 I + \Upsilon_e & \mathcal{G}_{xe}^\top & \mathcal{G}_{ue}^\top \\ * & * & * & \mathcal{Q}^{-1} & 0 \\ * & * & * & * & \mathbf{R}^{-1} \end{bmatrix} \succeq 0 \tag{5.41}$$

with $\Lambda = \mathbf{T}(t)^\top \mathbf{T}(t)$ for some $\Upsilon_s \in \mathbb{D}_+^{Nn_x}$ and $\Upsilon_e \in \mathbb{D}_+^{Nn_e}$.

Now, before we proceed to discuss an RHC scheme based on the min-max optimization, let us again consider a system with the dynamics of (5.8), i.e.,

$$\left. \begin{aligned} \tilde{x}(t+1) &= \tilde{A}(t) \tilde{x}(t) + [B_w(t) \quad -LD] \omega(t) \\ \tilde{z}(t) &= \tilde{x}(t) \end{aligned} \right\}. \tag{5.42}$$

Since L is chosen such that the origin of the polytopic undisturbed system $\tilde{x}(t+1) = \tilde{A}(t) \tilde{x}(t)$ is quadratically stable, system (5.42) is finite-gain ℓ_2 -stable. An upper bound on the ℓ_2 gain of the system can be obtained minimizing γ subject to the condition

$$\begin{bmatrix} P_{\tilde{x}} - I & \tilde{A}^{(j)\top} P_{\tilde{x}} & 0 \\ * & P_{\tilde{x}} & P_{\tilde{x}} [B_w^{(j)} \quad -LD] \\ * & * & \gamma I \end{bmatrix} \succeq 0, \quad j = 1, 2, \dots, n_\Omega \tag{5.43}$$

for some $P_{\tilde{x}} \succeq 0$.

The next proposition deals with the performance of an RHC scheme based on the min-max optimization.

Proposition 5.2. Let γ_ω be the smallest γ for which (5.43) is feasible for some $P_{\tilde{x}} \succeq 0$. Then, if the optimization problem

$$\min_{g(t), \bar{\mathbf{v}}(t), \bar{\xi}(t+N|t), \mathbf{E}(t), \bar{\mathbf{F}}(t), \mathbf{L}_s, \mathbf{L}_e, \Upsilon_s, \Upsilon_e} g(t) \quad (5.44)$$

under constraint conditions (5.37) and (5.41) is feasible for an initial estimated state $\hat{x}(0)$ at time $t = 0$, then the receding horizon control $u^*(t) = \kappa^*(\hat{x}(t), \hat{\mathbf{R}}(t), t) = K \hat{x}(t) + \bar{v}^*(t)$ where $\bar{v}^*(t)$ is obtained from the solution of (5.44) at each time $t \in \mathbb{Z}_+$ ensures that the problem remains feasible at subsequent times, and guarantees that the boundedness condition

$$\sum_{t=0}^{\infty} \left\{ \|Q^{\frac{1}{2}} x(t)\|^2 + \|R^{\frac{1}{2}} u^*(t)\|^2 - \gamma^2 \|\omega(t)\|^2 \right\} \leq \theta^*(0) \quad (5.45)$$

is satisfied with $\gamma^2 = \gamma_s^2 \gamma_\omega^2 + \gamma_e^2$ and $\theta^*(0) = g^*(0) + \gamma_s^2 \|P_{\tilde{x}}^{\frac{1}{2}} \tilde{x}(0)\|^2$. $\square$

PROOF. The proof immediately follows from the lines of reasoning in the proof of Proposition 4.4 and the fact that $\sum_{t=0}^{\infty} (\|\tilde{x}(t)\|^2 - \gamma_\omega^2 \|\omega(t)\|^2) \leq \|P_{\tilde{x}}^{\frac{1}{2}} \tilde{x}(0)\|^2$, which is implied by the satisfaction of (5.43) with $\gamma = \gamma_\omega$ for some $P_{\tilde{x}} \succeq 0$. $\blacksquare$

Remark 5.3. The treatment of the min-max output-feedback RHC in this section parallels the treatment of the standard min-max RHC in Section 4.4 of Chapter 4. The proposed min-max output-feedback RHC scheme shares the computational difficulties of the nominal-cost-based output-feedback scheme of Section 5.3 arising from the need to ensure the recursive feasibility of the control problem without much conservativeness (cf. Remark 5.2). Additionally, the min-max scheme needs to ensure the cost monotonicity condition for the worst-case realization of uncertainties in the time-varying uncertainty sets. Clearly, this further increases the complexity of the min-max scheme. However, as mentioned in Remark 5.2, these complexities are again typical of min-max output-feedback RHC schemes and the proposed min-max scheme keeps them low by effectively reducing the required horizon length for on-line computations. $\square$

5.5 Computational Complexity

As in the case of state-feedback RHC schemes explored in Chapters 3 and 4, the RHC schemes presented in this chapter require off-line computations for the determination of the (terminal) controller matrices and the associated (largest) feasible invariant sets. As mentioned in the earlier chapters, the computation of the controller matrices involves the solution of an SDP problem with LMI constraints in the case of the nominal cost scheme and that of a problem with BMI constraints in the case of the min-max cost scheme. The determination of the (largest) polyhedral feasible invariant set requires an iterative procedure involving LP problems as discussed in Appendix A.

The complexity of on-line computations substantially depends on the cost minimization criterion as well as the length of the chosen finite horizon. As in the case of the

standard state-feedback schemes in Chapter 3, for a horizon length N , the number of decision variables in the nominal-cost-based optimization problem (5.27) is of the order $\mathcal{O}(N^2)$ when $\mathbf{E}(t)$ and $\tilde{\mathbf{F}}$ are optimized on-line and of the order $\mathcal{O}(N)$ when they are assumed to be zero. The optimization problem is a QP problem with $\mathcal{O}(N^2)$ and $\mathcal{O}(N)$ linear constraints respectively in the two cases. Similarly, as with the min-max scheme of Chapter 4, the output-feedback min-max RHC scheme of this chapter requires an on-line solution of an SDP problem with $\mathcal{O}(N^2)$ variables, $\mathcal{O}(N^2)$ linear constraints and an LMI constraint of size $(2n_x + n_u + n_e + 2)N + 1$. The presence of the LMI constraint makes the min-max problem (5.44) significantly more complex (for $N \geq 1$) than the problem (5.27). Nevertheless, the min-max scheme of Proposition 5.2 allows us to characterize the control performance in terms of an ℓ_2 bound γ , albeit at the expense of some additional on-line computational complexity.

Unlike the standard state-feedback schemes, the output-feedback RHC schemes need to handle additional uncertainty arising from the state estimation error. Unless the estimation error is already known to be bounded in the set $\mathcal{R}$ (which is a close outer approximation of the set $\mathcal{R}_{min}$), the current and the predicted estimation error sets $\hat{\mathcal{R}}(t + i|t)$, $i = 1, \dots, N - 1$ may need to be computed or updated and the matrix $\hat{\mathbf{M}}_{\mathfrak{M}}$ redetermined on-line after obtaining the output measurement at each time step. This will mean some additional on-line computations related to set operations apart from the usual computations required in the case of the state-feedback schemes.

As discussed in Remark 5.2 and Remark 5.2, the additional set-related on-line computations mentioned above will be required in any output-feedback RHC scheme. So, as in the case of standard RHC schemes, the schemes discussed in this chapter will clearly have computational advantages over corresponding schemes that use a static feedback as the terminal control law. This is because of the fact that the use of the off-line optimized terminal controller allows the schemes to offer feasibility over a larger set of initial conditions with an optimization over a shorter horizon length N . A smaller finite horizon length clearly means lower additional computations arising from the set-related operation apart from the standard optimization of the control moves. In the following, we present a numerical example to illustrate the computational advantage of the dynamic-policy-based output-feedback RHC approach over the standard approach using the static feedback law as the terminal control law.

Illustration 5.4. To illustrate the computational efficiency of the dynamic-policy-based output-feedback RHC schemes, we consider a nominal-cost-based scheme with an LTI system obtained by choosing $a(t) = 1.2$ and $b(t) = 1$ in (5.15). We assume the other details as in Illustration 5.1 and choose $K = [-0.68 \ -1.07]$ — the unconstrained LQ feedback gain for the time-invariant system. Fig. 5.5a makes a comparison between the size of the overall feasible set $\mathcal{X}_x^6$ that can be achieved with the scheme of Proposition 5.1 for $\mathcal{R}_0 = \mathcal{R}$ and $N = 6$ and the size of the feasible set $\mathcal{X}_{xS}^{24}$ that can be achieved with a corresponding scheme that uses a static feedback as the terminal control law appended to a finite-horizon output-error based parameterization of length $N = 24$. It can be observed that the two sets $\mathcal{X}_x^6$ and $\mathcal{X}_{xS}^{24}$ have almost the same size. The difference in the horizon lengths required in the two

Figure 5.5: Comparison of (a) feasible sets and (b) control performances with static and dynamic output-feedback terminal policies.

cases implies a significant difference in the on-line computational complexity. A comparison made with simulations for 100 initial estimated states randomly selected from inside both $\mathcal{X}_x^6$ and $\mathcal{X}_{xS}^{24}$ shows that the computation time required for solving the optimization problem at each time step in the latter case is, on the average, more than 15 times larger than the corresponding time required in the former case (1.8s as against 77ms in the former case)¹ The total regulations costs are, however, almost identical in the two cases. Fig. 5.5b shows the time variation of the states under the two schemes for an initial state $[-90 \ 6]^T$ as an example.

5.6 Conclusion

In this chapter, we have presented nominal and minmax output-feedback RHC schemes for a class of LTV systems with bounded disturbances and measurement noises. Employing a suitably estimated state in place of the unknown actual state, this schemes largely follow the set of steps in their standard state-feedback counterparts discussed in Chapter 3 and Chapter 4. They employ a dynamic policy that involves the feedforward of the output error, and the time-varying dynamics of the terminal part of the policy is optimized off-line to enlarge the terminal feasible invariant set for the estimated state. These schemes, therefore, allow the computation of control actions over large feasibility domains with an on-line optimization over small horizon lengths. When compared with the full state-feedback schemes, the output-feedback schemes need additional mechanisms to handle feasibility in the presence of state estimation errors. This requires additional on-line computations and also results in additional conservativeness in the control policy. However, the computational advantages of the proposed schemes over other relevant schemes such as those using the static-feedback-based terminal control law still remain the same. Numerical examples that illustrate the effective performance of the proposed schemes have been presented.

¹The computation times are based on simulations using the BPMPD solver in a computing setup mentioned in Illustration 3.4.

Chapter 6

Dynamic-Policy-Based RHC for Coordinated Control Problems

6.1 Introduction

In recent years, there have been considerable research interests in the area of cooperative behavior in networked systems. One important problem in this area is that of the coordinated control of the subsystems, called agents, of a large multi-agent system in such a way that the subsystems reach an agreement or consensus in a certain quantity of interest in order to achieve a shared goal [139]. Such a problem arises in various forms in the context of applications related to multi-vehicle systems, such as formation flight [140], rendezvous in space [141], cooperative air traffic control [7] and vehicle platooning in automated highways [8]. For instance, in an automated highway, vehicles in the platoon need to agree on the common speed for the platoon. Similarly, in formation flying of spacecrafts, the members need to agree on the point around which the formation is to be configured. This chapter presents an RHC-based solution to the consensus-related coordinated control problems in which the subsystems have decoupled but time-varying and possibly uncertain dynamics affected by external bounded disturbances, and are constrained in their states and inputs.

Recent works on consensus-related coordinated control problems have employed approaches that are inspired by natural models of consensus such as those found in schools of fish and flocks of birds (e.g., [142], [143]). In [139], the authors have presented an extensive survey of consensus algorithms presented and used in diverse areas of science and engineering and summarized their convergence characteristics under various inter-subsystem information exchange scenarios. More results on information consensus algorithms considered for multi-vehicle applications have been presented in [144]. In their basic forms, these algorithms are simple, decentralized (linear) interaction rules for one-dimensional first-order agents exchanging information only with the neighboring members. The consensus point finally reached is a weighted average of the initial conditions in the member

agents. Research results on these consensus algorithms have basically focussed on analyzing their convergence characteristics under various inter-subsystem communications or information exchange conditions, such as time-varying communications structure and delayed, noisy or intermittent communications (see, e.g., [141, 145–148]). Generalizations of the algorithms to subsystems with more complicated dynamics have also been investigated. [149] and [144], for instance, have explored consensus algorithms for subsystems with double integrator dynamics – the latter has also considered actuator saturation. Similarly, papers such as [150] and [151] have explored cooperative control for subsystems with general multi-input multi-output dynamics. While [150] analyzed the stability of multi-vehicle formations using tools from graph theory and robust control theory, [151] presented a framework for the analysis and design of cooperative control based on matrix theory under time-varying sensing/communications conditions. A few works such as [152], [153] and [154] have dealt with the consensus synthesis problem. Consensus algorithms were explored in [152] and [153] for the fastest average consensus with first-order scalar subsystems whereas [154] considered a convex synthesis for consensus among subsystems with general dynamics, possibly with inter-subsystem dynamic coupling.

On-line-optimization-based control approaches that allow the optimization of the consensus point along with the subsystem control inputs provide an alternative to biologically inspired algorithms for consensus-related coordinated control problems. Such approaches can systematically incorporate the state and input constraints in the optimization process – something that traditional consensus algorithms do not readily offer. RHC-based approaches have been employed in the context of distributed or coordinated control of a large system with dynamically decoupled subsystems in several works such as [97–99]. [97] has considered subsystems coupled in the cost function alone and guaranteed stability with parallel local RHC computations by adding a move suppression term to each local cost function. The authors in [99] have considered subsystems with coupling constraints and employed a scheme that solves local robust MPC optimization subproblems sequentially in order to ensure future feasibility and convergence of the overall control problem. Subsystems coupled in the constraints as well as the cost function have been considered [98] in which local cost minimizations are carried out in parallel in the subsystems ensuring, through the imposition of certain additional conditions, that the local cost functions behave as Lyapunov functions. Consensus-related coordinated control problems have also been explored in the framework of on-line-optimization-based control in a few works such as [102], [101] and [100]. In [101], the authors have considered the use of predictive mechanism with unconstrained inputs, both in centralized and decentralized settings, for improved consensus in having a collective behavior among agents. [102] has considered systems with single or double integrator dynamics and proved asymptotic convergence of a distributed-MPC-based consensus algorithm using geometric properties of the optimal path. In [100], a finite-horizon predictive consensus problem with terminal equality constraints for agents with general dynamics has been implemented in a distributed way through negotiations between agents by employing an incremental subgradient method with primal decomposition. In these works, the authors have considered solutions based

on finite-horizon optimization for subsystems with time-invariant dynamics and without disturbances. In practical situations, disturbance are often present and it may become necessary to ensure robust constraint satisfaction at all times in order to account for the presence of disturbances.

In this chapter, we explore computationally efficient RHC-based schemes for a class of consensus-related coordinated control problems involving systems with autonomously actuated subsystems that have a general multi-dimensional time-varying dynamics including (possibly unmeasurable) additive disturbances. The subsystems are required to satisfy hard constraints on the state and the input. In order to deal with the situation arising from the presence of additive disturbances in subsystem dynamics, we consider suitably defined near-consensus conditions – one based on a steady state set inclusion and the other based on an H_∞ performance – as the control objectives. While we consider a scheme involving the minimization of the nominal cost in the case of the set-based near-consensus problem, we employ the H_∞ -like min-max cost approach in order to achieve the H_∞ -based near-consensus condition. We explore the use of the computationally efficient RHC schemes developed in Chapters 3 and 4 as the local control schemes in the subsystems, and, in each case, the overall control optimization problem, which aggregates the local nominal or min-max RHC problems of the subsystems, can be decomposed into convex subproblems with the consensus point as the coupling variable. In order to solve the overall optimization problem in a distributed way, we consider the use of the subgradient method [155] such as in [100]. Employing an approximate subgradient method based on dual decomposition, we first determine a feasible and (near-) optimal consensus point with a decentralized solution of the subproblems in the subsystems, and then employ the RHC schemes independently in the subsystems to attain the near-consensus condition considered. We also extend the results to develop schemes in which the consensus point is also updated on a regular basis. Some simulation results that we present exemplify the performance of the proposed control schemes.

6.2 Problem Description

We consider a system comprising of set of $\mathcal{N}$ subsystems or agents labeled $j = 1, 2, \dots, \mathcal{N}$ described by the state dynamics

$$x_j(t+1) = A_j(t)x_j(t) + B_{u_j}(t)u_j(t) + B_{w_j}(t)w_j(t), \quad j \in \mathcal{N} = \{1, 2, \dots, \mathcal{N}\} \quad (6.1)$$

where $x_j(t) \in \mathbb{R}^{n_{x_j}}$ is the state of subsystem j , $u_j(t) \in \mathbb{R}^{n_{u_j}}$ is the control input applied to it and $w_j(t) \in \mathbb{R}^{n_{w_j}}$ is the external disturbance affecting it. The subsystems are dynamically decoupled and may have dissimilar dimensions. We define the overall system state $x(t) \in \mathbb{R}^{n_x}$ where $n_x = \sum_{j=1}^{\mathcal{N}} n_{x_j}$, the overall control input $u(t) \in \mathbb{R}^{n_u}$ where

$n_u = \sum_{j=1}^{\mathcal{N}} n_{u_j}$, and the overall disturbance $w(t) \in \mathbb{R}^{n_w}$ where $n_w = \sum_{j=1}^{\mathcal{N}} n_{w_j}$ as

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{\mathcal{N}}(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{\mathcal{N}}(t) \end{bmatrix} \text{ and } w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_{\mathcal{N}}(t) \end{bmatrix}.$$

We assume that the subsystems considered here have the models of dynamics and other features similar to those of the system considered in Chapter 3. For completeness, we state the assumptions about the subsystems in the following.

A6.1: The time-varying subsystem matrices $A_j(t)$, $B_{u_j}(t)$ and $B_{w_j}(t)$ are known to lie within a polytope Ω_j , i.e.,

$$[A_j(t) \ B_{u_j}(t) \ B_{w_j}(t)] \in \Omega_j = \text{Co} \left\{ [A_j^{(r)} \ B_{u_j}^{(r)} \ B_{w_j}^{(r)}], \ r = 1, \dots, n_{\Omega_j} \right\}, \forall j \in \mathcal{N}, t \in \mathbb{Z}_+. \quad (6.2)$$

where $[A_j^{(r)} \ B_{u_j}^{(r)} \ B_{w_j}^{(r)}]$, $r = 1, 2, \dots, n_{\Omega_j}$ are the vertices of the polytope Ω_j .

A6.2: For each subsystem $j \in \mathcal{N}$, the disturbance $w_j(t)$ is componentwise bounded, i.e.,

$$w_j(t) \in \mathcal{W}_j = \{w \in \mathbb{R}^{n_{w_j}} \mid \|w\|_{\infty} \leq 1\}. \quad (6.3)$$

A6.3: The constraints for each subsystem $j \in \mathcal{N}$ are such that the state and the input lie in bounded polyhedra containing the origin in their interior. These constraints are expressed as:

$$x_j(t) \in \mathbb{X}_j = \{x \mid M_{x_j} x \leq \mathbf{1}\}, \quad u_j(t) \in \mathbb{U}_j = \{u \mid M_{u_j} u \leq \mathbf{1}\} \quad (6.4)$$

where $M_{x_j} \in \mathbb{R}^{m_{x_j} \times n_{x_j}}$ and $M_{u_j} \in \mathbb{R}^{m_{u_j} \times n_{u_j}}$.

A6.4: For each $j \in \mathcal{N}$, the polytopic subsystem (6.1-6.2) is quadratically stabilizable.

We are interested in the control of the overall system in such a way that the subsystems ultimately reach a ‘consensus condition’ in some optimal way. The consensus condition is expressed in terms of the ‘consensus components’ of subsystem states, namely,

$$s_j(t) = S_j^T x_j(t), \quad j = 1, \dots, \mathcal{N} \quad (6.5)$$

where S_j , $j = 1, \dots, \mathcal{N}$ are suitably defined constant matrices.

Definition 6.1 (Ideal Consensus). The subsystems in (6.1) are said to reach consensus at time t if, for all $i \geq 0$,

$$\left. \begin{aligned} s_1(t+i) &= s_2(t+i) = \dots = s_{\mathcal{N}}(t+i) = \varsigma, \\ x_j(t+i) &= x_j^e, \quad u_j(t+i) = u_j^e, \quad j = 1, \dots, \mathcal{N} \end{aligned} \right\} \quad (6.6)$$

where $\varsigma \in \mathbb{R}^{n_\varsigma}$ is the consensus point reached and (x_j^e, u_j^e) , $j = 1, \dots, \mathcal{N}$ are equilibrium state-input pairs for the subsystems. $\square$

The ideal control objective is to drive the subsystems to the consensus condition (6.6) with optimal inputs, computed, at each time instant t , by minimizing the predicted infinite-horizon cost function of the form

$$J(t) = \sum_{j=1}^{\mathcal{N}} J_j(t) \quad \left. \vphantom{\sum_{j=1}^{\mathcal{N}}} \right\} \quad (6.7)$$

$$\text{with } J_j(t) = \sum_{i=0}^{\infty} \left\{ \left\| Q_j^{\frac{1}{2}} (x_j(t+i|t) - x_j^e) \right\|^2 + \left\| R_j^{\frac{1}{2}} (u_j(t+i|t) - u_j^e) \right\|^2 \right\}$$

where Q_j and R_j , $j = 1, \dots, \mathcal{N}$ are positive definite matrices selected as per the design criteria and (x_j^e, u_j^e) , $j = 1, \dots, \mathcal{N}$ are equilibrium state-input pairs for the subsystems such that $\bar{s}_j = S_j^T x_j^e = \varsigma$, $\forall j \in \mathcal{N}$. The ideal performance function (6.7), however, can hardly be evaluated and the ideal consensus condition (6.6) can hardly be reached when there are non-vanishing disturbances. Therefore, alternatively, we wish to evaluate the cost in (6.7) for a nominal (disturbance-free) or an H_∞ -based min-max performance so that a suitably defined near-consensus condition can be ultimately reached. We now define two relevant near-consensus conditions.

Definition 6.2 (Set-based near-consensus). The subsystems in (6.1) are said to reach a near-consensus condition specified by a given set $\mathcal{X}^0$ at time t if, for all $i \in \mathbb{Z}_+$,

$$x(t+i) - x^e \in \mathcal{X}^0 \quad (6.8)$$

where $x^e = [x_1^{eT} \dots x_{\mathcal{N}}^{eT}]^T$ is an overall equilibrium point such that $\bar{s}_j = S_j^T x_j^e = \varsigma$, and $\mathcal{X}^0$ is a predefined small set. $\square$

Definition 6.3 (H_∞ -based near-consensus). The subsystems in (6.1) are said to reach a near-consensus condition specified by a given ℓ_2 bound γ as time $t \rightarrow \infty$ if the subsystem states satisfy the condition

$$\sum_{t=0}^{\infty} \|x(t) - x^e\|^2 \leq \gamma^2 \sum_{t=0}^{\infty} \|w(t)\|^2 + \Theta_0 \quad (6.9)$$

where $x^e = [x_1^{eT} \dots x_{\mathcal{N}}^{eT}]^T$ is an overall equilibrium point satisfying $\bar{s}_j = S_j^T x_j^e = \varsigma$, and Θ_0 is a constant depending on the overall state $x(0)$. $\square$

We wish to achieve the set-based consensus condition of Definition 6.2 by employing a control scheme that computes control actions which are optimal in the sense of minimizing

a nominal (disturbance-free) version of the performance function in (6.7), defined as

$$\left. \begin{aligned} \bar{J}(t) &= \sum_{j=1}^{\mathcal{N}} \bar{J}_j(t) \\ \text{with } \bar{J}_j(t) &= \sum_{i=0}^{\infty} \left\{ \left\| Q_j^{1/2} (\bar{x}_j(t+i|t) - x_j^e) \right\|^2 + \left\| R_j^{1/2} (\bar{u}_j(t+i|t) - u_j^e) \right\|^2 \right\} \end{aligned} \right\} \quad (6.10)$$

where $\{\bar{x}_j(t+i|t)\}_{i=0}^{\infty}$ and $\{\bar{u}_j(t+i|t)\}_{i=0}^{\infty}$ are the predicted state and input trajectories of the disturbance-free subsystem $\bar{x}_j(t+1) = A_j(t) \bar{x}_j(t) + B_{u_j}(t) \bar{u}_j(t)$. On the other hand, in order to achieve the H_{∞} -based near-consensus condition of Definition 6.3, we wish to employ a scheme that involves the minimization of the cost in (6.7) for a min-max performance defined with an H_{∞} -like performance index

$$\left. \begin{aligned} \hat{J}(t) &= \sum_{j=1}^{\mathcal{N}} \hat{J}_j(t) \\ \text{with } \hat{J}_j(t) &= \max_{\substack{w_j(t+i) \in \mathcal{W}_j, \\ i \in \mathbb{Z}_+}} \sum_{i=0}^{\infty} \left\{ \left\| Q_j^{\frac{1}{2}} (x_j(t+i|t) - x_j^e) \right\|^2 + \left\| R_j^{\frac{1}{2}} (u_j(t+i|t) - u_j^e) \right\|^2 \right. \\ &\quad \left. - \gamma_j^2 \|w_j(t+i)\|^2 \right\} \end{aligned} \right\} \quad (6.11)$$

with suitable ℓ_2 bounds γ_j , $j = 1, \dots, \mathcal{N}$.

Computing the consensus point, the equilibrium points and the optimal control inputs that minimize the infinite-horizon performance index (6.10) or (6.11) requires the solution of a large distributed optimization problem. Such a problem is computationally challenging if the computations are to be carried out in a receding horizon fashion for a traditional MPC optimization for each subsystem such that the system constraints are guaranteed to be satisfied at all times. In the rest of this chapter, we limit our discussion to cases where the consensus point ς lies within a compact set $\mathcal{X}_{\varsigma}$ such that the corresponding subsystem equilibrium points require no steady state control inputs, i.e., $u_j^e = 0$, $j = 1, \dots, \mathcal{N}$. In this regard, we make a further assumption about the system:

A6.5: The subsystem dynamics are such that

$$A_j S_j = S_j, \quad j = 1, \dots, \mathcal{N}. \quad (6.12)$$

With assumption A6.5 satisfied, either naturally or through a suitable closing of the inner loops in the subsystems, we consider equilibrium points of the form $(x_j^e, u_j^e) = (S_j \varsigma, 0)$ where $\varsigma \in \mathcal{X}_{\varsigma}$. We then wish to employ computationally efficient RHC methods along with a distributed optimization algorithm in order to efficiently find subsystem control actions that are optimal in the sense of minimizing the performance index (6.10) or (6.11) and hence to achieve the near-consensus condition (6.8) or (6.9) with $x^e = S\varsigma$, ($u^e = 0$), where $S = [S_1^T \ S_2^T \ \dots \ S_{\mathcal{N}}^T]^T$ and ς is the consensus point.

When we consider the minimization of the performance index (6.10) or (6.11), the quantities to be determined include $u_j(t+i|t)$, $i \in \mathbb{Z}_+$, $j = 1, \dots, N$, which are predicted local control inputs in the subsystems, and the consensus point ς , which is a global variable common to all subsystems. Given a value of the global variable ς , suitable RHC policies can be used locally in the subsystems to compute optimal control actions. Optimizing the value of ς , which the local policies treat as an external reference input, however, requires information from all subsystems and can be better done in distributed manner.

6.3 Determination of Local-level RHC Policies

In this section, we present the details of the local RHC policies to be employed in the subsystems. Since each subsystem is constrained in the state and the input, the presence of disturbances in subsystem dynamics necessitates a ‘closed-loop’ approach in the prediction of future control inputs. Otherwise, the resulting control scheme will be unnecessarily conservative and computationally intensive. For this reason, we wish to employ the class of RHC policies discussed in Chapters 3 and 4 as the subsystems’ local RHC policies in the context of nominal and min-max cost approaches.

6.3.1 Control parameterization

We parameterize the predicted control inputs for each of the subsystems $j = 1, \dots, \mathcal{N}$ as

$$u_j(t+i|t) = K_j \tilde{x}_j(t+i|t) + H_j \xi_j(t+i|t), \quad i \in \mathbb{Z}_+ \quad (6.13)$$

where $\tilde{x}_j(t+i|t) = x_j(t+i|t) - S_j \varsigma$ is the error state of the j th subsystem. K_j is a constant static feedback gain for subsystem (6.1), which satisfies some optimality condition, typically the unconstrained LQ or H_∞ optimality. The quantities $\xi_j(t+i|t)$, $i \in \mathbb{Z}_+$ are parameterized as the predicted states, starting from an initial state $\xi_j(t|t) = \xi_j(t) = \bar{\xi}_j(t)$ of a time-varying controller described by the dynamics

$$\xi_j(t+1) = G_j(t) \xi_j(t) + F_j(t) w_j(t) \quad (6.14a)$$

where the matrices $G_j(t)$ and $F_j(t)$ satisfy

$$[G_j(t) \quad F_j(t)] \in \text{Co} \left\{ [G_j^{(r)} \quad F_j^{(r)}], \quad r = 1, \dots, n_{\Omega_j} \right\}. \quad (6.14b)$$

As discussed in Chapter 3, the motivation behind the use of the particular form of the controller dynamics (6.14) is to avoid conservativeness. For the simplification of on-line computations, we determine the controller matrices H_j , $G_j^{(r)}$ and $F_j^{(r)}$ and the associated (largest) feasible invariant set for the controlled subsystem error state $\tilde{x}_j(t)$ off-line whereas the initial controller state $\xi_j(t)$ is optimized on-line to minimize the relevant cost.

6.3.2 Computation of subsystem controller matrices and invariant set

As discussed above, we compute the controller matrices $G_j^{(r)}, F_j^{(r)}$ ($r = 1, \dots, n_{\Omega_j}$) and H_j for each subsystem j such that we can ensure a feasible invariant set (of usually the largest possible size) for the subsystem error state. In order to present the conditions that are to be applied while determining the terminal controller matrices, we consider the augmented subsystem error-state dynamics

$$\chi_j(t+1) = \Psi_j(t)\chi_j(t) + \mathcal{B}_{wj}(t)w_j(t) \quad (6.15)$$

where $\chi_j(t) = [\tilde{x}_j(t)^T \ \xi_j(t)^T]^T$ is the augmented subsystem error state. The augmented subsystem matrices can be written as

$$\Psi_j(t) = \begin{bmatrix} \mathcal{A}_j(t) & B_{uj}(t)H_j \\ 0 & G_j(t) \end{bmatrix}, \quad \mathcal{B}_{wj}(t) = \begin{bmatrix} B_{wj}(t) \\ F_j(t) \end{bmatrix}$$

and they lie in a polytope, i.e., $[\Psi_j(t) \ \mathcal{B}_{wj}(t)] \in \text{Co} \left\{ [\Psi_j^{(r)} \ \mathcal{B}_{wj}^{(r)}], r = 1, \dots, n_{\Omega_j} \right\}$ where

$$\Psi_j^{(r)} = \begin{bmatrix} \mathcal{A}_j^{(r)} & B_{uj}^{(r)}H_j \\ 0 & G_j^{(r)} \end{bmatrix}, \quad \mathcal{B}_{wj}^{(r)} = \begin{bmatrix} B_{wj}^{(r)} \\ F_j^{(r)} \end{bmatrix}$$

and $\mathcal{A}_j^{(r)} = A_j^{(r)} + B_{uj}^{(r)}K_j$. Clearly, a feasible invariant set $\mathcal{S}_{\tilde{x}_j}$ for the controlled subsystem error-state is given by the projection, on the $\tilde{x}_j$ -subspace, of a positively invariant set $\mathcal{S}_{\chi_j}$ for the state of (6.15) on which the constraints in (6.4) are satisfied. As in Chapters 3 and 4, while finding a desired feasible invariant set $\mathcal{S}_{\chi_j}$, we first consider an ellipsoidal set $\mathcal{E}_{\chi_j}$ so that the size of its projection $\mathcal{E}_{\tilde{x}_j}$ on the $\tilde{x}_j$ -subspace can be directly optimized by solving a tractable problem.

6.3.2.1 Controller matrices for nominal cost approach

For a nominal-cost-based scheme to be employed for attaining the set-based consensus of Definition 6.2, we consider no disturbance feedforward in the control policy (6.13) (i.e., we assume $F_j^{(r)} = 0$, $r = 1, \dots, n_{\Omega_j}$) and optimize the controller matrices H_j and $G_j^{(r)}$, $r = 1, \dots, n_{\Omega_j}$ to maximize the size of the set $\mathcal{E}_{\tilde{x}_j}$. For this we solve the problem

$$\max_{\mathcal{P}_j, Z_j, H_j, G_j^{(r)}, \alpha_j^{(r)}, R_{wj}^{(r)} (r=1, \dots, n_{\Omega_j})} \log \det(T_j \mathcal{P}_j T_j^T) \quad (6.16)$$

with $T_j = [I_{n_{x_j}} \ 0]$, subject to the invariance condition

$$\begin{bmatrix} \mathcal{P}_j & \Psi_j^{(r)} \mathcal{P}_j & \mathcal{D}_j^{(r)} \\ * & \alpha_j^{(r)} \mathcal{P}_j & 0 \\ * & * & R_{wj}^{(r)} \end{bmatrix} \succ 0, \quad \text{Tr}(R_{wj}^{(r)}) \leq 1 - \alpha_j^{(r)}, \quad r = 1, \dots, n_{\Omega_j} \quad (6.17)$$

with $F_j^{(r)} = 0$, $r = 1, \dots, n_{\Omega_j}$, and the feasibility condition

$$\begin{bmatrix} Z_j & \begin{bmatrix} \tilde{M}_{x_j} & 0 \\ M_{u_j} K & M_{u_j} H_j \end{bmatrix} \mathcal{P}_j \\ * & \mathcal{P}_j \end{bmatrix} \succ 0, \quad Z_{i[r,r]} \leq 1, \quad r = 1, \dots, m_{u_j} + m_{x_j}. \quad (6.18)$$

where $\tilde{M}_{x_j} = (\text{diag}(b))^{-1} M_{x_j}$ with $b_{[j]} = \min_{\varsigma \in \mathcal{X}_\varsigma} (1 - M_{x_j[j,:]} S_j \varsigma)$. Here, the feasibility condition (6.18) follows from the fact that the subsystem error state $\tilde{x}_j(t) = x_j(t) - S_j \varsigma$ lies in the set $\mathbb{X}_j \ominus S_j \mathcal{X}_\varsigma$. Problem 6.16 is an SDP problem involving BMI constraints. But it can be easily solved after transforming it into a problem with simpler matrix inequalities as mentioned in Chapter 3. Matrix $\mathcal{P}_j$ and its leading $n_{x_j} \times n_{x_j}$ block $\mathcal{P}_j^{11}$ obtained from the solution of problem (6.16) define the ellipsoidal inner approximations $\bar{\mathcal{E}}_{\chi_j} = \{\chi \mid \chi^\top \mathcal{P}_j^{-1} \chi \leq 1\}$ and $\bar{\mathcal{E}}_{\tilde{x}_j} = \{x \mid x^\top (\mathcal{P}_j^{11})^{-1} x \leq 1\}$ of the feasible invariant sets for the augmented subsystem error state and the controlled subsystem error state respectively. Since $\bar{\mathcal{E}}_{\chi_j}$ is usually a conservative inner ellipsoidal approximation of the largest feasible invariant set associated with the controller matrices that are determined from (6.16), we compute the corresponding polyhedral invariant set $\{\chi \mid M_{\chi_j} \chi \leq \mathbf{1}\}$, which we denote by $\bar{\mathcal{X}}_{\chi_j}$, and employ it in the control schemes.

6.3.2.2 Controller matrices for min-max cost approach

For the min-max-cost-based scheme to be employed for an H_∞ -based near-consensus, the subsystem controller matrices need to satisfy a boundedness condition to ensure that the min-max cost in (6.11) remains bounded. Therefore, while determining the subsystem controller matrices, we solve the problem

$$\max_{\mathcal{P}_j, \mathcal{Q}_j, Z_j, H_j, G_j^{(r)}, F_j^{(r)}, \alpha_j^{(r)}, R_{wj}^{(r)} (r=1, \dots, n_{\Omega_j})} \log \det(T_j^\top \mathcal{P}_j T_j) \quad (6.19)$$

subject to the invariance condition (6.17), the feasibility condition (6.18) and the boundedness condition

$$\begin{bmatrix} \mathcal{P}_j & \Psi_j^{(r)\top} \mathcal{P}_j & C_{z_j}^\top & 0 \\ * & \mathcal{P}_j & 0 & \mathcal{P}_j \mathcal{B}_{wj}^{(r)} \\ * & * & I & 0 \\ * & * & * & \gamma_j^2 I \end{bmatrix} \succeq 0, \quad r = 1, 2, \dots, n_{\Omega_j} \quad (6.20)$$

where $C_{z_j} = \begin{bmatrix} Q_j^{1/2} & 0 \\ R_j^{1/2} K_j & R_j^{1/2} H_j \end{bmatrix}$ and γ_j is the ℓ_2 bound selected *a-priori*. Here, condition (6.20) ensures that the augmented subsystem (6.15) satisfies, for all $t \in \mathbb{Z}_+$,

$$\left\| \mathcal{P}_j^{\frac{1}{2}} \chi_j(t+1) \right\|^2 - \left\| \mathcal{P}_j^{\frac{1}{2}} \chi_j(t) \right\|^2 \leq -\|z_j(t)\|^2 + \gamma_j^2 \|w_j(t)\|^2 \quad (6.21)$$

and hence

$$\sum_{t=0}^{\infty} \{ \|z_j(t)\|^2 - \gamma_j^2 \|w_j(t)\|^2 \} \leq \left\| \mathcal{P}_j^{\frac{1}{2}} \chi_j(0) \right\|^2 \quad (6.22)$$

for the output signal $z_j(t) = C_{z_j} \chi_j(t)$ so that we have $\frac{\|z_j\|_{\ell_2}}{\|w_j\|_{\ell_2}} \leq \gamma_j$ for all $w_j \in \ell_2$ and $\chi_j(0) = 0$. Problem (6.19) can be solved using the alternative semidefinite programming algorithm mentioned in Chapter 4. The feasible invariant sets $\mathcal{X}_{\chi_j}$ and $\mathcal{X}_{\tilde{x}_j}$ associated with the resulting controller matrices can be easily computed as mentioned earlier.

6.3.3 Local-level computation of subsystem control actions

Once the controller matrices are determined for each subsystem, we can determine simpler expressions upper bounding the performance functions (6.10) and (6.11). Given a consensus point ς , the optimal control actions to reach the corresponding equilibrium for each subsystem $j \in \mathcal{N}$ can be computed, at each time t , by minimizing the upper bound of the performance function in (6.10) or (6.11).

6.3.3.1 Nominal performance function

With the nominal-cost-based RHC scheme, for each subsystem, we intend to drive the subsystem error state $\tilde{x}_j(t) = x_j(t) - S_j \varsigma$ into the smallest disturbance invariant set $\mathcal{R}^{K_j}$ under the optimal static feedback $u_j(t) = K_j \tilde{x}_j(t)$. Such a set satisfies the relation

$$(A_j^{(r)} + B_{u_j}^{(r)} K_j) \mathcal{R}^{K_j} + B_{w_j}^{(r)} \mathcal{W}_j \subseteq \mathcal{R}^{K_j}, \quad r = 1, \dots, n_{\Omega_j}. \quad (6.23)$$

So, as discussed in Section 3.3.3, we would like to minimize the performance index in (6.10) for ‘nominal’ initial subsystem states $\bar{x}_j(t) = x_j(t) - d_j(t)$, $j = 1, \dots, \mathcal{N}$, with $d_j(t)$ allowed to lie in $\mathcal{R}^{K_j}$, rather than for the actual states $x_j(t)$, $j = 1, \dots, \mathcal{N}$. Given the subsystem controller matrices, an upper bound of the nominal (disturbance-free) cost for an initial state $x_j(t)$ of the j th subsystem is given by

$$\bar{J}_j(t) = \left\| \bar{\mathcal{P}}_j^{\frac{1}{2}} \begin{bmatrix} (x_j(t) - d_j(t) - S_j \varsigma)^T & \xi_j(t)^T \end{bmatrix}^T \right\|^2 \quad (6.24)$$

where $\bar{\mathcal{P}}_j$ is a positive definite matrix obtained from the minimization

$$\min_{\bar{\mathcal{P}}_j} \text{Tr}(\bar{\mathcal{P}}_j) \quad (6.25)$$

$$\text{s.t.} \quad \bar{\mathcal{P}} - \Psi_j^{(r)\top} \bar{\mathcal{P}}_j \Psi_j^{(r)} \succeq C_{z_j}^\top C_{z_j}, \quad r = 1, \dots, n_{\Omega_j}$$

and $\xi_j(t)$ is the controller initial state such that $[(x_j(t) - S_j \varsigma)^\top \quad \xi_j(t)^\top]^\top \in \bar{\mathcal{X}}_{\chi_j}$ and $d_j(t)$ is the initial state offset satisfying $d_j(t) \in \mathcal{R}^{K_j}$. Therefore, given the value of ς , the subsystem-level control action $u_j(t) = K_j(x_j(t) - S_j \varsigma(t)) + H_j \xi_j(t)$ can be computed by solving the problem

$$\left. \begin{aligned} & \min_{d_j(t), \xi_j(t)} \bar{J}_j(t) \\ & \text{with } \bar{J}_j(t) = \left\| \bar{\mathcal{P}}_j^{\frac{1}{2}} [(x_j(t) - d_j(t) - S_j \varsigma)^\top \quad \xi_j(t)^\top]^\top \right\|^2 \end{aligned} \right\} \quad (6.26)$$

such that $[(x_j(t) - S_j \varsigma)^\top \quad \xi_j(t)^\top]^\top \in \bar{\mathcal{X}}_{\chi_j}$.

In problem (6.26), ς is assumed to be known and $\xi_j(t)$ and $d_j(t)$ are the only decision variable. So the problem can be solved locally in each subsystem $j \in \mathbb{N}_N$. Therefore, we refer to (6.26) as the nominal local control problem.

6.3.3.2 H_∞ -based min-max performance function

In the case of the H_∞ -based min-max performance function (6.11), given the subsystem controller matrices, an upper bound of the subsystem performance index for an initial state $x_j(t)$ of the uncertain subsystem can be represented by

$$\hat{J}_j(t) = \left\| \mathcal{P}_j^{\frac{1}{2}} [(x_j(t) - S_j \varsigma)^\top \quad \xi_j(t)^\top]^\top \right\|^2 \quad (6.27)$$

where $\mathcal{P}_j$ is a positive definite matrix obtained from the minimization

$$\min_{\mathcal{P}_j} \text{Tr}(\mathcal{P}_j) \quad (6.28)$$

subject to condition (6.20), and $\xi_j(t)$ is the controller initial state such that $[(x_j(t) - S_j \varsigma)^\top \quad \xi_j(t)^\top]^\top \in \mathcal{X}_{\chi_j}$. Therefore, as in the case of the nominal cost approach, given the value of ς , the subsystem-level control action $u_j(t) = K_j(x_j(t) - S_j \varsigma(t)) + H_j \xi_j(t)$ can be computed by solving the problem

$$\left. \begin{aligned} & \min_{\xi_j(t)} \hat{J}_j(t) \\ & \text{with } \hat{J}_j(t) = \left\| \mathcal{P}_j^{\frac{1}{2}} [(x_j(t) - S_j \varsigma)^\top \quad \xi_j(t)^\top]^\top \right\|^2 \end{aligned} \right\} \quad (6.29)$$

such that $[(x_j(t) - S_j \varsigma)^\top \quad \xi_j(t)^\top]^\top \in \mathcal{X}_{\chi_j}$.

We refer to problem (6.29) as the min-max local control problem.

6.4 Distributed Optimization of the Consensus Point

In this section, we discuss the determination of the consensus point ς which is optimal in the sense of minimizing the predicted performance index (6.10) or (6.11) with the form of local control policies presented in the last section. Following the control parameterization and the subsequent discussion in the last section, we can state the global nominal-cost-based near-consensus control problem as

$$\min_{\substack{\varsigma \in \mathcal{X}_\varsigma, \\ d_j(t) \in \mathcal{R}^{K_j}, \\ \xi_j(t), j=1, \dots, \mathcal{N}}} \sum_{j=1}^{\mathcal{N}} \bar{J}_j(t) \quad (6.30)$$

with $\bar{J}_j(t)$ as defined in (6.24), such that $[(x_j(t) - S_j \varsigma)^\top \ \xi_j(t)^\top]^\top \in \bar{\mathcal{X}}_{\chi_j}$, $j = 1, \dots, \mathcal{N}$. Similarly, the global min-max-cost-based control problem can be stated as

$$\min_{\substack{\varsigma \in \mathcal{X}_\varsigma, \\ \xi_j(t), j=1, \dots, \mathcal{N}}} \sum_{j=1}^{\mathcal{N}} \hat{J}_j(t) \quad (6.31)$$

with $\hat{J}_j(t)$ as defined in (6.27), such that $[(x_j(t) - S_j \varsigma)^\top \ \xi_j(t)^\top]^\top \in \mathcal{X}_{\chi_j}$. Problem (6.30) or (6.31) is a convex problem involving convex constraints and is efficiently solvable provided the subsystem state information is fully available and the computational resources are not limited. However, in a practical situation possibly with large number of geographically separated subsystems a centralized solution at one of the subsystems or some other central location may not be appealing. If the subsystems are all equipped with some computational resources, it is natural to let the subsystems share the computational burden. It is obvious that problems (6.30) and (6.31) are of the form

$$\min_{(\varphi_j, \varsigma) \in \mathcal{S}_{\varphi_j}, j=1, \dots, \mathcal{N}} \sum_{j=1}^{\mathcal{N}} f_j(\varphi_j, \varsigma) \quad (6.32)$$

where $\mathcal{S}_{\varphi_j}$, $j = 1, \dots, \mathcal{N}$ are convex sets. Also, for each subsystem $j \in \mathcal{N}$, φ_j which is equal to $(\xi_j(t), d_j(t))$ in the case of problem (6.30) and equal to $\varphi_j = \xi_j(t)$ in the case of problem (6.31), represents the local variables associated with the subsystem only whereas ς is a global variable common to all. For a distributed solution of the optimization problem of the form (6.32), techniques based on primal or dual decompositions [156, Chap. 6] are commonly used. In this work, we consider a subgradient-based solution employing the decomposition of the dual function. Using a local instance of the global variable ς in each subproblem, problem (6.32) can be written as

$$\min_{\varsigma, (\varphi_j, s_j) \in \mathcal{S}_{\varphi_j}, j=1, \dots, \mathcal{N}} \sum_{j=1}^{\mathcal{N}} f_j(\varphi_j, s_j) \quad (6.33)$$

subject to the condition $s_1 = s_2 = \dots = s_{\mathcal{N}} = \varsigma$. The dual function for the problem in (6.33) can be written as [156]

$$q(\lambda) = \min_{\varsigma, (\varphi_j, s_j) \in \mathcal{S}_{\varphi_j}, j=1, \dots, \mathcal{N}} \left\{ \sum_{j=1}^{\mathcal{N}} f_j(\varphi_j, s_j) + \lambda^T (s - (\mathbf{1} \otimes I_{n_\varsigma}) \varsigma) \right\} \quad (6.34)$$

where $s = [s_1^T \ s_2^T \ \dots \ s_{\mathcal{N}}^T]^T$ and $\lambda = [\lambda_1^T \ \lambda_2^T \ \dots \ \lambda_{\mathcal{N}}^T]^T$. The minimum of (6.34) w.r.t. ς occurs when $(\mathbf{1} \otimes I_{n_\varsigma})^T \lambda = 0$. So, the dual problem can be written as

$$\max_{\lambda \in \mathcal{S}_\lambda} h(\lambda) \quad \text{with} \quad h(\lambda) = \sum_{j=1}^{\mathcal{N}} h_j(\lambda_j) \quad (6.35a)$$

where $h_j(\lambda_j)$ is defined as the optimal value of the problem

$$\min_{(\varphi_j, s_j) \in \mathcal{S}_{\varphi_j}} f_j(\varphi_j, s_j) + \lambda_j^T s_j \quad (6.35b)$$

and $\mathcal{S}_\lambda = \{\lambda \mid (\mathbf{1} \otimes I_{n_\varsigma})^T \lambda = 0\}$. Here, (6.35a) is the dual master-problem and (6.35b) is the subproblem associated with the j th subsystem. We solve the master-problem using the projected subgradient method. The use of subgradient methods in solving various optimization problems has been extensively explored in literature and their suitability for solving certain ‘decomposable’ large-scale problems with distributed/decentralized computations has been well recognized (see e.g., [157], [155], [156], [158]). In (6.35), clearly, a subgradient of the function $-h_j(\lambda_j)$ at λ_j is given by $-s_j^*(\lambda_j)$, where $s_j^*(\lambda_j)$ is an optimal value of s_j in the subproblem (6.35b) [156, Chap. 6]. Since the constraint set $\mathcal{S}_\lambda$ is affine, the dual variable λ can be updated using the projection of the overall subgradient $-s^*(\lambda) = -[s_1^*(\lambda_1)^T \ \dots \ s_{\mathcal{N}}^*(\lambda_{\mathcal{N}})^T]^T$ on the set $\mathcal{S}_\lambda$. The projection of $s^*(\lambda)$ on $\mathcal{S}_\lambda$ is given by $g_p(\lambda) = s^*(\lambda) - \bar{s}(\lambda)$ with $\bar{s}(\lambda) = (\mathbf{1} \otimes I_{n_\varsigma}) \hat{\varsigma}(\lambda)$ where $\hat{\varsigma}(\lambda) = \frac{1}{\mathcal{N}} (\mathbf{1} \otimes I_{n_\varsigma})^T s^*(\lambda)$ represents the component-wise average of the optimal values $s_j^*(\lambda_j), j = 1, \dots, \mathcal{N}$ of the local instances of the coupling variable. We present the outline of the algorithm for the solution of the dual problem (6.35) in the following.

Algorithm 6.1: Distributed computation of the consensus point

1. Initialization: Set $k = 0$. Choose $\lambda^{(0)} \in \mathcal{S}_\lambda$.

2. Repeat for $k = 0, 1, 2, \dots$

(a) For $j = 1, \dots, \mathcal{N}$, solve (6.35b) to obtain

$$(\varphi_j^*(\lambda_j^{(k)}), s_j^*(\lambda_j^{(k)})) = \arg \min_{(\varphi_j, s_j) \in \mathcal{S}_{\varphi_j}} f_j(\varphi_j, s_j) + \lambda_j^{(k)T} s_j$$

(b) Compute $\bar{s}(\lambda^{(k)})$ and $g_p(\lambda^{(k)})$

$$\begin{aligned} \text{i. } \bar{s}(\lambda^{(k)}) &= \frac{1}{\mathcal{N}} (\mathbf{1} \otimes I_{n_\varsigma}) (\mathbf{1} \otimes I_{n_\varsigma})^T s^*(\lambda^{(k)}) \\ \text{where } s^*(\lambda^{(k)}) &= [s_1^*(\lambda_1^{(k)})^T \ \dots \ s_{\mathcal{N}}^*(\lambda_{\mathcal{N}}^{(k)})^T]^T \end{aligned}$$

- ii. $g_p(\lambda^{(k)}) = s^*(\lambda^{(k)}) - \bar{s}(\lambda^{(k)})$
- (c) Update λ

$$\lambda^{(k+1)} = \lambda^{(k)} + a_k g_p(\lambda^{(k)})$$

Here, Step 2a of Algorithm 6.1 involves the solution of subproblems which are convex QP problems. The dual variable is updated in Step 2c where the quantities $a_k, k \in \mathbb{Z}_+$ are positive step-sizes that are suitably chosen *a-priori* and satisfy $\sum_{k=0}^{\infty} a_k \rightarrow \infty$. Clearly, Step 2a and Step 2c of Algorithm 6.1 can be carried out in the subsystems in a decentralized way. Step 2b, however, involves averaging of the optimal values $s_j^*, j = 1, \dots, \mathcal{N}$ of the local instances of the coupling variable in order to compute the projected subgradient at each iteration instant k . This requires an exchange of information among the subsystems. Depending on the nature of the inter-subsystem communications architecture, the averaging step can be carried out suitably either in a decentralized manner or sequentially with the other two steps using the incremental variant of the subgradient method [156, Chap. 6].

We consider distributed averaging with linear iterations (see [152], [153]) for Step 2b(i) in Algorithm 6.1. Let us consider that the inter-subsystem communication architecture is represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ comprising of a set of nodes $\mathcal{N}$ and a set of edges $\mathcal{E}$. This means that each subsystem $j \in \mathcal{N}$ can directly communicate only with the set of its neighbours denoted by $\mathcal{N}_j = \{l \mid \{j, l\} \in \mathcal{E}\}$. Since the averaging of the quantities $s_j^*, j = 1, \dots, \mathcal{N}$ has to be carried out component-wise, we consider, for each component, iterations of the form

$$\zeta(i+1) = \mathcal{W} \zeta(i) \quad (6.36)$$

where $\mathcal{W}$ is a sparse weight matrix satisfying

$$\mathcal{W} \in \{W \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}} \mid W_{[i,j]} = 0 \ \forall \{i, j\} \notin \{\mathcal{E}, \{1, 1\}, \{2, 2\}, \dots, \{\mathcal{N}, \mathcal{N}\}\}\}. \quad (6.37)$$

We require that the iterates in (6.36) converge to the average of the initial values in ζ , i.e.,

$$\lim_{i \rightarrow \infty} \zeta(i) = \lim_{i \rightarrow \infty} \mathcal{W}^i \zeta(0) = \frac{1}{\mathcal{N}} \mathbf{1} \mathbf{1}^T \zeta(0) = \bar{\zeta}. \quad (6.38)$$

Clearly, $\mathcal{W}$ defines the convergence properties of the averaging iterations (6.36). In particular, a quantity of interest is the per-step convergence factor [152] defined as

$$c = \sup_{\zeta(i) \neq \bar{\zeta}} \frac{\|\zeta(i+1) - \bar{\zeta}\|}{\|\zeta(i) - \bar{\zeta}\|}. \quad (6.39)$$

We next note the following result from [152] regarding the requirements on the weight matrix $\mathcal{W}$ for convergent averaging.

Lemma 6.1. Convergence condition (6.38) holds for any initial vector $\zeta(0)$ if and only if $\mathcal{W}$ has $\mathbf{1}$ as one of its left as well as one of its right eigenvectors, and the spectral radius

of $\mathcal{W} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top$ is less than unity. Moreover, the per-step convergence factor c is given by the spectral norm of $\mathcal{W} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top$, i.e., $c = \|\mathcal{W} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top\|$. $\square$

A weight matrix $\mathcal{W}$ satisfying the condition in Proposition 6.1 exists if the graph $\mathcal{G}$ is connected, and one that leads to the fastest convergence can be obtained by minimizing the spectral norm of $\mathcal{W} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top$ subject to the sparsity constraint (6.37) specified by the graph $\mathcal{G}$. This minimization problem can be expressed as an SDP problem and can be easily solved [152]. With $\mathcal{W}$ determined *a-priori*, we compute $\bar{s}(\lambda^{(k)})$ in Step 2b(i) of Algorithm 6.1 using the iterations

$$\bar{s}(\lambda^{(k)}, l+1) = (\mathcal{W} \otimes I_{n_c})\bar{s}(\lambda^{(k)}, l) \quad (6.40)$$

starting from $\bar{s}(\lambda^{(k)}, 0) = s^*(\lambda^{(k)})$. These iterations are completely decentralized and involve trivial computations in the subsystems. However, since the convergence to the average is only asymptotic, it is necessary that the iterations (6.40) be carried out to an accuracy that is sufficient to ensure the desired convergence of the subgradient algorithm. We now present the following result on the convergence of Algorithm 6.1 with the averaging in Step 2b(i) approximated by decentralized averaging (6.40).

Proposition 6.1. Let λ^* be an optimal point that maximizes $h(\lambda)$ in (6.35a) and h^* be the optimal value of h . Consider Algorithm 6.1 with $g_p(\lambda^{(k)})$ in Step 2 approximated by $\hat{g}_p(\lambda^{(k)}) = s^*(\lambda^{(k)}) - \tilde{s}(\lambda^{(k)}, \eta_k)$ where $\tilde{s}(\lambda^{(k)}, \eta_k)$ is obtained from (6.40) with η_k , $k \geq 0$ satisfying

$$\eta_k \geq \max \left(1, \frac{\ln(\epsilon_k^{-1} \|g_p(\lambda^{(k)})\| \|\lambda^* - \lambda^{(k)}\|)}{\ln c^{-1}} \right) \quad (6.41)$$

for a sequence of given small positive quantities $\{\epsilon_k\}$, $k \geq 0$. Let $\{\lambda^{(k)}\}$ be the sequence of λ generated by the algorithm. Then, we have

$$\liminf_{k \rightarrow \infty} (h^* - h(\lambda^{(k)})) \leq \delta \quad (6.42)$$

where $\delta = \limsup_{k \rightarrow \infty} (\frac{1}{2}a_k(1 + c^{2\eta_k})\|g(\lambda^{(k)})\|^2 + \epsilon_k)$. Moreover, if $a_k \rightarrow 0$ and $\epsilon_k \rightarrow \epsilon$, we have

$$\liminf_{k \rightarrow \infty} (h^* - h(\lambda^{(k)})) \leq \epsilon \quad (6.43)$$

PROOF. For any k , we have

$$\begin{aligned} \|\lambda^* - \lambda^{(k+1)}\|^2 &= \|\lambda^* - \lambda^{(k)} - a_k \hat{g}_p(\lambda^{(k)})\|^2 \\ &= \|\lambda^* - \lambda^{(k)}\|^2 + a_k^2 \|\hat{g}_p(\lambda^{(k)})\|^2 - 2a_k \hat{g}_p(\lambda^{(k)})^\top (\lambda^* - \lambda^{(k)}) \\ &= \|\lambda^* - \lambda^{(k)}\|^2 + a_k^2 \|\hat{g}_p(\lambda^{(k)})\|^2 - 2a_k s^*(\lambda^{(k)})^\top (\lambda^* - \lambda^{(k)}) \\ &\quad + 2a_k (\tilde{s}(\lambda^{(k)}, \eta_k) - \bar{s}(\lambda^{(k)}))^\top (\lambda^* - \lambda^{(k)}). \end{aligned}$$

Since

$$\begin{aligned}\tilde{s}(\lambda^{(k)}, \eta_k) - \bar{s}(\lambda^{(k)}) &= ((\mathcal{W} \otimes I_{n_\zeta})^{\eta_k} - \mathbf{1}\mathbf{1}^\top / \mathcal{N} \otimes I_{n_\zeta})(s^*(\lambda^{(k)}) - \bar{s}(\lambda^{(k)})) \\ &= ((\mathcal{W} - \mathbf{1}\mathbf{1}^\top / \mathcal{N})^{\eta_k} \otimes I_{n_\zeta})g_p(\lambda^{(k)}), \\ \|\tilde{s}(\lambda^{(k)}, \eta_k) - \bar{s}(\lambda^{(k)})\| &\leq c^{\eta_k} \|g_p(\lambda^{(k)})\| \text{ and } \|\hat{g}_p(\lambda^{(k)})\| = \|g_p(\lambda^{(k)}) + \bar{s}(\lambda^{(k)}) - \tilde{s}(\lambda^{(k)}, \eta_k)\| \\ &\leq (1 + c^{\eta_k}) \|g_p(\lambda^{(k)})\|. \text{ Moreover, since } -(s^*(\lambda^{(k)})) \text{ is a subgradient of } -h(\lambda^{(k)}), \\ -(h^* - h(\lambda^{(k)})) &\geq -s^*(\lambda^{(k)})^\top (\lambda^* - \lambda^{(k)}). \text{ So, we have,}\end{aligned}$$

$$\begin{aligned}\|\lambda^* - \lambda^{(k+1)}\|^2 &\leq \|\lambda^* - \lambda^{(k)}\|^2 + a_k^2(1 + c^{\eta_k})^2 \|g_p(\lambda^{(k)})\|^2 - 2a_k(h^* - h(\lambda^{(k)})) \\ &\quad + 2a_k c^{\eta_k} \|g_p(\lambda^{(k)})\| \|\lambda^* - \lambda^{(k)}\| \\ &\leq \|\lambda^* - \lambda^{(k)}\|^2 + a_k^2(1 + c^{\eta_k})^2 \|g_p(\lambda^{(k)})\|^2 - 2a_k(h^* - h(\lambda^{(k)})) + 2a_k \epsilon_k\end{aligned}$$

where the last inequality follows from the assumption in (6.41). Applying this inequality successively for $k = k - 1$ until $k = 0$, we get

$$\begin{aligned}\|\lambda^* - \lambda^{(k+1)}\|^2 &\leq \|\lambda^* - \lambda^{(0)}\|^2 + \sum_{i=0}^k a_i^2(1 + c^{\eta_i})^2 \|g_p(\lambda^{(i)})\|^2 \\ &\quad - 2 \sum_{i=0}^k a_i(h^* - h(\lambda^{(i)})) + 2 \sum_{i=0}^k a_i \epsilon_i.\end{aligned}$$

Since $\|\lambda^* - \lambda^{(k+1)}\|^2 \geq 0$, we can write this as

$$\frac{\sum_{i=0}^k a_i(h^* - h(\lambda^{(i)}))}{\sum_{i=0}^k a_i} \leq \frac{\frac{1}{2}\|\lambda^* - \lambda^{(0)}\|^2 + \sum_{i=0}^k a_i(\epsilon_i + \frac{1}{2}a_i(1 + c^{\eta_i})^2 \|g_p(\lambda^{(i)})\|^2)}{\sum_{i=0}^k a_i}. \quad (6.44)$$

Since $\sum_{i=0}^k a_i \rightarrow \infty$ as $k \rightarrow \infty$, using the properties of the Cesàro averages of sequences [159] and the facts that $h^* - h(\lambda^{(k)})$ is convex and $\|g_p(\lambda^{(k)})\|$ is bounded, conclusions (6.42) and (6.43) directly follow from (6.44). $\blacksquare$

Remark 6.1. The use of an approximate projection of the subgradient $-s^*(\lambda^{(k)})$ of $-h(\lambda^{(k)})$ on the set $\mathcal{S}_\lambda$ in Step 2b of Algorithm 6.1 can be considered as using the actual projection on $\mathcal{S}_\lambda$ of an approximate subgradient $\hat{g}(\lambda^{(k)}) = -(s^*(\lambda^{(k)}) - (\tilde{s}(\lambda^{(k)}, \eta_k) - \bar{s}(\lambda^{(k)})))$. It follows from condition (6.41) that $\hat{g}(\lambda^{(k)})$ belongs to the ϵ_k -subdifferential of $-h(\lambda^{(k)})$ at $\lambda^{(k)}$. The result of Proposition 6.1 can, therefore, be seen as a case of the convergence of an approximate subgradient projection method of optimization [159]. $\square$

Remark 6.2. Condition (6.41) on η_k can be satisfied for a given sequence $\{\epsilon_k\}$ with a sufficiently large η_k at each k . However, since both $\|g_p(\lambda^{(k)})\|$ and $\|\lambda^* - \lambda^{(k)}\|$ decrease with the converging values of the quantities $s_j^*, j = 1, \dots, \mathcal{N}$, a small η_k may usually be sufficient to ensure that a suitable sequence $\{\epsilon_k\}$ exists so that the algorithm converges with reasonable accuracy. Indeed, if $s_j^*(\lambda_j^{(k)}), j = 1, \dots, \mathcal{N}$ obtained from the minimizations in (6.35b) are unique, $\eta_k = 1$ would ensure convergence since, in such a case, $s^*(\lambda^{(k)})$ actually coincides with the gradient and the norm of the gradient error $((\mathcal{W} - \mathbf{1}\mathbf{1}^\top / \mathcal{N}) \otimes I_{n_\zeta})s^*(\lambda^{(k)})$ is smaller than the gradient norm so that the approximate gradient is still

Figure 6.1: Overall control scheme for achieving a near-consensus condition.

a direction of ascent. In any case, since smaller η_k means poorer ascent in each iteration and larger η_k implies more inter-subsystem communications and more delay between iterations of the main algorithm, a suitable choice of η_k will involve a tradeoff between these factors. $\square$

6.5 Overall Control Scheme for Achieving Near-Consensus

The overall RHC scheme for attaining the near-consensus condition requires a suitable real-time solution of the global control problem (6.30) or (6.31), which involves the determination of both the global decision variable ς and the local variables $\xi_j(t)$, $j = 1, \dots, \mathcal{N}$. In an ideal RHC framework, in order to utilize the full potential of the RHC paradigm, we would wish to compute all the decision variables at each time instant t and apply the computed optimal control actions in the subsystems. However, since the determination of the optimal value of ς is computationally intensive and requires extensive inter-subsystem communications, it may not be possible, and not even reasonable, to optimize ς together with the local control variables in the receding horizon fashion. The exact way in which the optimization of the consensus point ς is incorporated into the overall control scheme should be a design decision based on factors such as the nature of subsystem dynamics, uncertainties, system constraints and available computing resources.

In the following, we discuss the overall control schemes that employ either one-time or repeated optimization of the consensus point.

6.5.1 Near-consensus with one-time optimization of consensus point

Fig. 3.1 outlines the overall control scheme in which the consensus point is determined at the beginning before the subsystems start to apply the control inputs in order to achieve the set-based or the H_∞ -based near-consensus condition. We have the following results on the performance of the overall control scheme based on this approach.

6.5.1.1 Set-based near-consensus

While employing the overall control scheme outlined in Fig. 6.1 for set-based near-consensus, we determine the subsystem controller matrices and the feasible sets by solving (6.16). Similarly, for on-line optimization, we consider the nominal problem (6.26) in the receding horizon fashion. The following lemma summarizes the performance of the overall control scheme of Fig. 6.1 when employed with a nominal cost approach.

Proposition 6.2. If a feasible consensus point ς is determined from the solution of the global problem (6.30) for the given subsystem initial conditions at time $t = 0$, then the nominal-cost-based scheme employing the solution of the optimization problem (6.26) on-line at each time $t \in \mathbb{Z}_+$ as outlined in the scheme of Fig. 6.1 guarantees that the control problem remains feasible at all times $t \in \mathbb{Z}_+$ and that the nominal cost monotonicity condition

$$\bar{J}^*(t+1) - \bar{J}^*(t) \leq - \sum_{j=1}^{\mathcal{N}} \left\{ \|Q^{\frac{1}{2}} (\bar{x}_j^*(t) - S_j \varsigma)\|^2 + \|R^{\frac{1}{2}} \bar{u}_j^*(t)\|^2 \right\}, \quad (6.45)$$

where $\bar{x}_j^*(t) = x_j(t) - d_j^*(t)$, holds at each time t so that the subsystems ultimately satisfy the consensus condition (6.8) with $x^e = S\varsigma$ and $\mathcal{X}^0 = \mathcal{R}^{K_1} \times \dots \times \mathcal{R}^{K_{\mathcal{N}}}$ at some $t \in \mathbb{Z}_+$. $\square$

PROOF. Since a feasible consensus point is determined from the solution of the global problem (6.30) at time $t = 0$, the proof follows the lines of reasoning used in the proof of Proposition 3.4 for each subsystem. $\blacksquare$

6.5.1.2 H_∞ -based near-consensus

We consider the problems (6.16) and (6.29) during off-line and on-line phases respectively while employing the control scheme in Fig. 6.1 for achieving the H_∞ -based near-consensus condition. The following lemma explores the performance of such a control scheme.

Proposition 6.3. If a feasible consensus point ς is determined from the solution of the global problem (6.31) for the given subsystem initial conditions at time $t = 0$, then the min-max-cost-based scheme employing the solution of the optimization problem (6.29) on-line at each time $t \in \mathbb{Z}_+$ as outlined in the scheme of Fig. 6.1 guarantees that the control problem remains feasible at all times $t \in \mathbb{Z}_+$ and that the monotonicity condition

$$\hat{J}^*(t+1) - \hat{J}^*(t) \leq - \sum_{j=1}^{\mathcal{N}} \left\{ \|Q^{\frac{1}{2}} (x_j(t) - S_j \varsigma)\|^2 + \|R^{\frac{1}{2}} u_j^*(t)\|^2 - \gamma_j^2 \|w_j(t)\|^2 \right\} \quad (6.46)$$

holds at all times $t \in \mathbb{Z}_+$ so that the subsystems satisfy the consensus condition (6.9) for

$x^e = S\varsigma$ with

$$\gamma = \max \left\{ \frac{\gamma_1}{\sqrt{\lambda_{\min}(Q_1)}}, \frac{\gamma_2}{\sqrt{\lambda_{\min}(Q_2)}}, \dots, \frac{\gamma_{\mathcal{N}}}{\sqrt{\lambda_{\min}(Q_{\mathcal{N}})}} \right\} \quad (6.47)$$

$$\Theta_0 = \sum_{j=0}^{\mathcal{N}} \frac{\hat{J}_j^*(0)}{\lambda_{\min}(Q_j)}$$

PROOF. If a feasible ς is determined at time $t = 0$ from Algorithm 6.1, the feasibility of the overall control problem at all times $t \in \mathbb{Z}_+$ follows from the invariance of the feasible sets $\mathcal{X}_{\chi_j}$, $j = 1, \dots, \mathcal{N}$. The monotonicity condition (6.46) follows directly from the fact that each augmented subsystem satisfies the relation (6.21). Furthermore, it follows from the relation in (6.22) that

$$\sum_{t=0}^{\infty} \{ \lambda_{\min}(Q_j) \| (x_j(t) - S_j \varsigma) \|^2 - \gamma_j^2 \| w_j(t) \|^2 \} \leq \| \mathcal{P}_j^{\frac{1}{2}} \chi_j(0) \|^2. \quad (6.48)$$

Therefore, if γ is defined as in (6.47)

$$\begin{aligned} \sum_{t=0}^{\infty} \{ \| (x_j(t) - S_j \varsigma) \|^2 - \gamma^2 \| w_j(t) \|^2 \} &\leq \sum_{t=0}^{\infty} \left\{ \| (x_j(t) - S_j \varsigma) \|^2 - \frac{\gamma_j^2}{\lambda_{\min}(Q_j)} \| w_j(t) \|^2 \right\} \\ &\leq \frac{\hat{J}_j^*(0)}{\lambda_{\min}(Q_j)} \end{aligned} \quad (6.49)$$

Summing the above inequality for all subsystems, we obtain (6.9) with $x^e = S\varsigma$ and γ as defined in (6.47). $\blacksquare$

The above result allows us to characterize the overall gain bound γ in the near-consensus condition (6.9) in terms of the gain bounds γ_j , $j = 1, \dots, \mathcal{N}$ considered in the min-max performance functions of individual subsystems. The subsystem gain bounds are assumed to be chosen *a-priori* such that they ensure desired control performances in the subsystems. The bounds, however, also affect the size of the off-line-optimized feasible invariant sets $\mathcal{X}_{\tilde{x}_j}$, $j = 1, \dots, \mathcal{N}$ for the closed-loop subsystem error states. A larger bound usually allows a larger size of the feasible invariant set. Therefore, when an allowable overall gain bound γ is specified *a-priori* for a desired overall control performance, it may be reasonable to choose $\gamma_j = \sqrt{\lambda_{\min}(Q_j)} \gamma$ in order to allow, for each subsystem $j \in \mathcal{N}$, as large $\mathcal{X}_{\tilde{x}_j}$ as possible.

6.5.2 Near-consensus with repeated optimization of consensus point

The overall control scheme outlined in Fig. 6.1 provides a simple solution to the near-consensus problem. Employing an optimization of the consensus point in the beginning, it does not envisage any change in the consensus point once the subsystems start implementing the control actions in the receding horizon fashion. However, in many situations

dealing with time-varying subsystems under uncertain circumstances, it may be reasonable to regularly update the consensus point in the course of the control of the subsystems in order to utilize the full potential of the control scheme based on the receding horizon paradigm. Since the consensus point and the predicted state and input trajectories for each subsystem that we determine are based on the specific forms of off-line-optimized infinite-horizon control policies, the state and input trajectories that the subsystems actually follow under the RHC scheme will likely be different from the predicted ones. Hence, it may be reasonable to expect that an overall control scheme employing an on-line update of the consensus point, apart from the on-line update of the control decision variables, can further enhance the optimality of the control scheme.

While an on-line update of the consensus point at each time step t is attractive from the optimality point of view, it significantly increases the computation and inter-subsystem communication requirements. Moreover, the iterative procedure to be applied for the distributed optimization of the consensus point may not lead to a converged solution within the allowed time between the successive time instants. For this reason, we consider a modified scheme in which the consensus point too is optimized on-line but not necessarily at each time step t . The details of the modified scheme are outlined in Fig. 6.2, and, in this scheme, we seek to optimize and update the consensus point every T_{cc} time steps and we allow $T_c < T_{cc}$ time steps for its determination in each update cycle.

In the on-line scheme detailed in Fig. 6.2, we treat the consensus point as a time-varying quantity. We assume that a feasible but not necessarily optimal consensus point $\varsigma(0)$ is determined at time $t = 0$ before the control actions are initiated. Further, we seek to optimize and update the consensus point at $t = T_c$ and then in every T_{cc} time steps. However, in order to avoid unnecessary optimizations being carried out in Step B3(c), we apply the checking condition in Step B3(b) which implies that a relatively more optimal consensus point is likely to exist. We assume that T_c is large enough to allow a sufficient convergence of the algorithm computing the consensus point in a distributed way. Once a new consensus point is computed, we apply further checks based on the aggregate costs before actually proceeding to update its value. The checking condition in Step B4(c) ensures that the consensus point obtained for subsystem states $x_j(t - T_c)$ is still relatively optimal for the current subsystem states at time t . Furthermore, even though we regularly seek to update the consensus point, we would want it to eventually converge to some constant value. In the presence of non-vanishing disturbances in the subsystems, $\varsigma(t)$ will not necessarily converge if we continue to optimize and update it. In order to ensure the convergence $\varsigma(t)$, we apply the checking condition in Step B4(d) which requires that the intended update in the consensus point $\varsigma^\Delta(t) = \varsigma^+(t) - \varsigma^-(t)$, where $\varsigma^-(t)$ and $\varsigma^+(t)$ are respectively the existing and the newly computed values of the consensus point, results in an aggregate cost which is lower than the one at the time of the last update by at least the amount $\|P_\varsigma^{\frac{1}{2}} \varsigma^\Delta(t)\|$ where P_ς is a suitably chosen positive definite matrix. The aggregate costs in Step B3(b) and B4(a,b) of the scheme of Fig. 6.2 can be estimated by employing an averaging of the local costs in a distributed way as

A. Off-line (Before time $t = 0$):

1. For each subsystem $j \in \mathcal{N}$, select the feedback gain K_j (and the ℓ_2 bound γ_j). Solve (6.16) (or (6.19)) and determine controller matrices H_j , $G_j^{(r)}$, $F_j^{(r)}$ ($r = 1, \dots, n_{\Omega_j}$). Also, determine the appropriate invariant set $\mathcal{X}_{\chi_j}$ (or $\mathcal{X}_{\chi_j}$) and the cost matrix $\mathcal{P}_j$ (or $\mathcal{P}_j$).
2. Define T_c and choose an appropriate $T_{cc} > T_c$.

B. On-line (At each time instant $t \in \mathbb{Z}_+$):

1. For each $j \in \mathcal{N}$, obtain the state measurement $x_j(t)$.
2. If $t = 0$, choose a feasible consensus point $\varsigma(0)$. Set $t_0 = 0$, $\iota = 1$.
3. If $t = nT_{cc}$, where $n \in \mathbb{Z}_+$, then
  - (a) Set $updateFlag = 1$.
  - (b) For each $j \in \mathcal{N}$, solve problem (6.26) (or (6.29)) with $\varsigma = \varsigma(t)$ (i.e., the existing value of $\varsigma(t)$) to obtain $\xi_j^*(t)$, $j = 1, \dots, \mathcal{N}$, and compute an estimate of the total cost $\bar{J}(t) = \sum_{j=1}^{\mathcal{N}} \bar{J}_j(t)$ (or $\hat{J}(t) = \sum_{j=1}^{\mathcal{N}} \hat{J}_j(t)$). If $t > 0$ and $\bar{J}(t) \geq \bar{J}(t_{\iota-1})$ (or $\hat{J}(t) \geq \hat{J}(t_{\iota-1})$), set $updateFlag = 0$.
  - (c) If $updateFlag = 1$, start a fresh run of Algorithm 1 to solve (6.30) (or (6.31)) with the current states $x_j(t)$, $j \in \mathcal{N}$ to obtain the value of ς .
4. If $t = nT_{cc} + T_c$, where $n \in \mathbb{Z}_+$ and if $updateFlag = 1$, then
  - (a) Let $\varsigma^-(t)$ denote the existing value of $\varsigma(t)$ which is equal to $\varsigma(t-1)$. For each $j \in \mathcal{N}$, solve problem (6.26) (or (6.29)) with $\varsigma = \varsigma^-(t)$ to obtain $\xi_j^*(t)$, $j = 1, \dots, \mathcal{N}$. Compute the corresponding total cost $\sum_{j=1}^{\mathcal{N}} \bar{J}_j(t)$ (or $\sum_{j=1}^{\mathcal{N}} \hat{J}_j(t)$) and denote it by $\bar{J}^-(t)$ (or $\hat{J}^-(t)$).
  - (b) Denote by $\varsigma^+(t)$ the value of ς obtained from the latest run of Algorithm 6.1 started at $t - T_c$. For each $j \in \mathcal{N}$, solve problem (6.26) (or (6.29)) with $\varsigma = \varsigma^+(t)$ to obtain $\xi_j^+(t)$, $j = 1, \dots, \mathcal{N}$. Compute the corresponding total cost $\sum_{j=1}^{\mathcal{N}} \bar{J}_j(t)$ (or $\sum_{j=1}^{\mathcal{N}} \hat{J}_j(t)$) and denote it by $\bar{J}^+(t)$ (or $\hat{J}^+(t)$).
  - (c) If $\bar{J}^+(t) < \bar{J}^-(t)$ (or $\hat{J}^+(t) < \hat{J}^-(t)$), set $updateFlag = 0$.
  - (d) If $\bar{J}^+(t) > \bar{J}(t_{\iota-1}) - \|P_{\varsigma}^{\frac{1}{2}} \varsigma^{\Delta}(t)\|$ (or $\hat{J}^+(t) > \hat{J}(t_{\iota-1}) - \|P_{\varsigma}^{\frac{1}{2}} \varsigma^{\Delta}(t)\|$), where $\varsigma^{\Delta}(t) = \varsigma^+(t) - \varsigma^-(t)$, set $updateFlag = 0$.
  - (e) If $updateFlag = 1$, set $\varsigma(t) = \varsigma^+(t)$, $\xi_j^*(t) = \xi_j^+(t)$, $j \in \mathcal{N}$, $\hat{J}(t) = \hat{J}^+(t)$, $t_{\iota} = t$, $\iota = \iota + 1$.
5. If $t = nT_{cc} + i$, $i \in \{1, \dots, T_c - 1, T_c + 1, \dots, T_{cc} - 1\}$ or if $t = nT_{cc} + T_c$ and $updateFlag = 0$, solve (6.26) (or (6.29)) with the existing value of $\varsigma(t)$ to obtain $\xi_j^*(t)$, $j \in \mathcal{N}$.
6. For each subsystem, $j \in \mathcal{N}$, apply the control input $u_j^*(t) = K_j(x_j(t) - S_j \varsigma(t)) + H_j \xi_j^*(t)$.
7. Set $\varsigma(t+1) = \varsigma(t)$.

Figure 6.2: Overall control scheme for achieving a near-consensus condition with repeated optimization of the consensus point.

discussed in the last section for Step 2(b) of Algorithm 6.1. We now present results on the performance of the overall control scheme of Fig. 6.2.

6.5.2.1 Set-based near-consensus

We have the following result on the performance of the nominal-cost-based overall control scheme of Fig. 6.2.

Proposition 6.4. If a feasible consensus point $\varsigma(0)$ is available for the given subsystem initial conditions at time $t = 0$, then the overall control scheme of Fig. 6.2 with the nominal-cost-based computations guarantees that (a) the control problem will remain

feasible at all times $t \in \mathbb{Z}_+$, (b) the nominal cost monotonicity condition

$$\bar{J}^*(t+1) - \bar{J}^*(t) \leq - \sum_{j=1}^{\mathcal{N}} \left\{ \|Q^{\frac{1}{2}} (\bar{x}_j^*(t) - S_j \varsigma(t))\|^2 + \|R^{\frac{1}{2}} \bar{u}_j^*(t)\|^2 \right\} \quad (6.50)$$

will hold at each time $t \in \mathbb{Z}_+$, and (c) the consensus point $\varsigma(t)$ will converge to a value ς_{fin} , so that the subsystems will ultimately satisfy the consensus condition (6.8) with $x^e = S\varsigma_{fin}$ and $\mathcal{X}^0 = \mathcal{R}^{K_1} \times \dots \times \mathcal{R}^{K_{\mathcal{N}}}$ at some $t \in \mathbb{Z}_+$. $\square$

PROOF. Given a feasible consensus point $\varsigma(0)$ at time $t = 0$, part (a) of the statement follows from the invariance of the feasible sets $\mathcal{X}_{\chi_j}$, $j \in \mathcal{N}$. Next, the cost monotonicity condition of part (b) of the statement follows from the standard reasoning based on the feasibility of the predicted inputs at all times when $\varsigma(t)$ remains unchanged at time $t + 1$. At times $t = nT_{cc} + T_c$ when $\varsigma(t)$ may be updated, the monotonicity condition directly follows from checking condition in Step B4(c) of the scheme. To see part (c) of the statement, let $\{t_\iota\}_{\iota=0}^{N_u}$ be the subsequence of time instants at which $\varsigma(t)$ is updated in Step B4(e). If this subsequence is finite, then, clearly, $\varsigma(t)$ will converge to a value $\varsigma_{fin} = \varsigma(0) + \sum_{\iota=1}^{N_u} \varsigma^\Delta(t_\iota)$. If the subsequence is infinite, $\varsigma(t)$ will converge if the series $\sum_{\iota=1}^{\infty} \varsigma^\Delta(t_\iota)$ converges, and then $\varsigma(t)$ will tend to $\varsigma_{fin} = \varsigma(0) + \sum_{\iota=1}^{\infty} \varsigma^\Delta(t_\iota)$. However, the checking condition in Step B4(d) guarantees the absolute convergence of this series since

$$\begin{aligned} \bar{J}(t_\iota) &\leq \bar{J}(t_{\iota-1}) - \|P_\varsigma^{\frac{1}{2}} \varsigma^\Delta(t_\iota)\| \\ \left(\text{or } \hat{J}(t_\iota) &\leq \hat{J}(t_{\iota-1}) - \|P_\varsigma^{\frac{1}{2}} \varsigma^\Delta(t_\iota)\| \right) \end{aligned}$$

implies (summing from $\iota = 1$ to ∞),

$$\begin{aligned} \sum_{\iota=1}^{\infty} \|P_\varsigma^{\frac{1}{2}} \varsigma^\Delta(t_\iota)\| &\leq \bar{J}(0) \\ \left(\text{or } \sum_{\iota=1}^{\infty} \|P_\varsigma^{\frac{1}{2}} \varsigma^\Delta(t_\iota)\| &\leq \hat{J}(0) \right). \end{aligned}$$

Hence, the convergence of $\varsigma(t)$ follows. Further, the fact that the subsystems will ultimately satisfy the consensus condition (6.8) with $x^e = S\varsigma_{fin}$ and $\mathcal{X}^0 = \mathcal{R}^{K_1} \times \dots \times \mathcal{R}^{K_{\mathcal{N}}}$ at some $t \in \mathbb{Z}_+$ follows from the convergence of $\varsigma(t)$ and the satisfaction of the monotonicity condition $\blacksquare$

6.5.2.2 H_∞ -based near-consensus

Before we present the result on the performance of the H_∞ -based overall control scheme with a repeated update of the consensus point, we first define a slightly modified version of the H_∞ -based near-consensus condition.

Definition 6.4 (Modified H_∞ -based near-consensus). The subsystems in (6.1) are said to reach a near-consensus condition specified by a given ℓ_2 bound γ as time $t \rightarrow \infty$ if the subsystem states satisfy the condition

$$\sum_{t=0}^{\infty} \|x(t) - x^e(t)\|^2 \leq \gamma^2 \sum_{t=0}^{\infty} \|w(t)\|^2 + \Theta_0 \quad (6.51)$$

where Θ_0 is a constant depending on $x(0)$, and $x^e(t) = [x_1^e(t)^\top \dots x_{\mathcal{N}}^e(t)^\top]^\top$ is an overall equilibrium point that satisfies $s_j(t) = S_j^\top x_j^e(t) = \varsigma(t)$, $\forall j \in \mathcal{N}$ such that $\varsigma(t)$ converges to some value ς_{fin} (and $x^e(t)$ converges to some value x_{fin}^e) as $t \rightarrow \infty$. $\square$

Clearly, the above definition allows us to properly deal with the real-time updates of the value of the consensus point. We now present the following result on the performance of the H_∞ -based overall control scheme as outlined in Fig. 6.2.

Proposition 6.5. If a feasible consensus point $\varsigma(0)$ is available for the given subsystem initial conditions at time $t = 0$, then the min-max-cost-based scheme employing the solution of the optimization problem (6.29) on-line at each time $t \in \mathbb{Z}_+$ as outlined in the scheme of Fig. 6.2 guarantees that (a) the control problem will remain feasible at all times $t \in \mathbb{Z}_+$, (b) the cost monotonicity condition

$$\hat{J}^*(t+1) - \hat{J}^*(t) \leq \sum_{j=1}^{\mathcal{N}} \left\{ \|Q^{\frac{1}{2}} (x_j(t) - S_j \varsigma(t))\|^2 + \|R^{\frac{1}{2}} u_j^*(t)\|^2 - \gamma_j^2 \|w_j(t)\|^2 \right\} \quad (6.52)$$

will hold at all times $t \in \mathbb{Z}_+$, (c) the consensus point $\varsigma(t)$ will converge to a constant value ς_{fin} , and d) the subsystems will achieve the near-consensus condition of Definition 6.4 with $\gamma = \max(\gamma_1, \gamma_2, \dots, \gamma_{\mathcal{N}}) / \sqrt{\lambda_{min}(\mathbf{Q})}$, $\Theta_0 = \hat{J}(0) / \lambda_{min}(\mathbf{Q})$, where $\mathbf{Q} = \text{diag}(Q_1, Q_2, \dots, Q_{\mathcal{N}})$, and $x_{fin}^e = S \varsigma_{fin}$. $\square$

PROOF. Part (a) of the statement follows obviously from the invariance of the feasible sets $\mathcal{X}_{x_j}$, $j \in \mathcal{N}$. To see part (b), note that at times when $\varsigma(t+1)$ remains unchanged from $\varsigma(t)$, the monotonicity relation directly follows from (6.21), which is satisfied by the predicted successor state of each subsystem $j \in \mathcal{N}$. At times when $\varsigma(t+1)$ is updated and is different from $\varsigma(t)$, the relation will still hold since the checking condition in Step B4(c) ensures that the new value of $\varsigma(t+1)$ is accepted only if it further reduces the aggregate cost $\hat{J}(t+1)$. Next, part (c) of the statement follows from the lines of reasoning used in the proof of part (c) of Proposition 6.4. Finally, since,

$$\begin{aligned} & \sum_{t=0}^{\infty} \left\{ \lambda_{min}(\mathbf{Q}) \|x(t) - S \varsigma(t)\|^2 - \left(\max_{j \in \mathcal{N}} \gamma_j^2 \right) \|w(t)\|^2 \right\} \\ & \leq \sum_{t=0}^{\infty} \left\{ \sum_{j=1}^{\mathcal{N}} \|Q_j^{\frac{1}{2}} (x_j(t) - S_j \varsigma(t))\|^2 - \sum_{j=1}^{\mathcal{N}} \gamma_j^2 \|w_j(t)\|^2 \right\} \leq \hat{J}(0). \end{aligned} \quad (6.53)$$

where the last inequality is implied by the monotonicity relation (29), condition (6.51) will be satisfied with $\gamma = \max(\gamma_1, \gamma_2, \dots, \gamma_N)/\sqrt{\lambda_{\min}(\mathbf{Q})}$, $\Theta_0 = \hat{J}/\lambda_{\min}(\mathbf{Q})$ and with $x^e(t)$ converging to $x_{fin}^e = S_{\zeta_{fin}}$. Hence, we have part (d) of the statement. ■

Remark 6.3. Proposition 6.4 and Proposition 6.5 hold without the checking condition in Step B3(b) before the optimization of ς and the checking condition in Step B4(c) after its optimization in the scheme of Fig. 6.2. Indeed, the posterior condition in Step B3(d) alone would be sufficient to guarantee the convergence of the consensus point. The prior condition in Step B3(b) indicates the existence of a relatively more optimal value of the consensus point. Hence the introduction of this condition in Step B3(b) is solely motivated by computational concerns since it allows us to avoid too many unnecessary optimizations being carried out in Step B3(c). Further, the posterior checking condition in Step B4(c) is meant to ensure the relative optimality of the newly computed consensus point for the current subsystem states since there is a delay of T_c time steps in the computation of the consensus point. Indeed this posterior optimality check can be avoided if computational delays are taken into account in the overall RHC formulation. One of the straightforward ways to do so is to first project the state information of each subsystem forward into an appropriate future time instant and then to optimize the future value of the global variable based on this projected information. However, because of disturbances and possible uncertainties in subsystem dynamics, projecting the state information of each subsystem forward into a future time instant using the actual uncertain model will result in a set of possible values, rather than an exact value, of the future state, and solving the global control problem for sets of possible subsystem states at the chosen future time instant will likely result in conservativeness or even infeasibility unless we find some suitable way to parameterize the future subsystem control variables as functions of the uncertain future subsystem states. Alternatively, we may consider the computation of the global variable based on the current subsystem state information along with a suitable mechanism to project the optimization result forward into the future. However, it is not usually straightforward to design such a mechanism to project the optimization result into a future time instant. A practically feasible way to optimize future values of the consensus point using the current subsystem state information might be to consider time-delayed versions of the subsystem controller matrices with special structures and to incorporate $\varsigma(t)$ as a global state in the augmented dynamics of each subsystem. However, we do not further explore this aspect of the problem in this chapter. □

Illustration 6.1. To illustrate the performance of the control schemes discussed in this section, we consider an example of a system with 5 subsystems, $j = 1, \dots, 5$, each having a second order LTV system dynamics described by the matrices

$$A_j(t) = \begin{pmatrix} 1.0 & 1.0 \\ 0.0 & 1.2 + 0.05\rho_j(t) \end{pmatrix}; \quad B_{u_j}(t) = \begin{pmatrix} 0 \\ 0.9 + 0.1\rho_j(t) \end{pmatrix}; \quad B_{w_j}(t) = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.1 \end{pmatrix} \quad (6.54)$$

where $|\rho_j(t)| \leq 1$. We also consider that an input constraint $|u_j(t)| \leq 1$ is applied in each subsystem. In the cost expression, $Q_j = I$ and $R_j = 1$ are considered for subsystems $j = 1, 2$,

Figure 6.3: Inter-subsystem communications structure for the example system in Illustration 6.1 (left) and for that in Illustration 6.2 (right).

Figure 6.4: Feasible invariant sets for the example subsystems in Illustration 6.1 obtained for the nominal and the min-max cost schemes.

and $Q_j = 2I$ and $R_j = 2$ for subsystems $j = 3, 4, 5$. The consensus components considered are defined by $S_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for all the subsystems. We consider both the nominal and the H_∞ -based min-max RHC schemes, and select an LQ optimal K_j for each subsystem $j \in \mathcal{N}$. For the min-max scheme we choose $\gamma_j = 2$ for all the subsystems. Also, we consider bidirectional communications between interacting subsystems. The communications structure considered is shown in Fig. 6.3 (left). We compute the controller matrices for each subsystem for both the nominal and the min-max schemes. The corresponding feasible invariant sets for the subsystems obtained for the nominal and the min-max schemes ($\bar{\mathcal{X}}_{\bar{x}_j}, j = 1, \dots, 5$ and $\mathcal{X}_{\bar{x}_j}, j = 1, \dots, 5$ respectively) are shown in Fig. 6.4.

We first consider the nominal overall control scheme based on a one-time or a repeated optimization of the consensus point for a set of randomly chosen subsystem initial states. The initial states chosen are given in the first row of Table 6.1. Fig. 6.5a shows the convergence of the dual variables $\lambda_j, j = 1, \dots, \mathcal{N}$ to their final values in the distributed optimization of the consensus point ς , using Algorithm 6.1 for the nominal-cost-based scheme. A step-size rule of the form $a_k = \kappa_0 / (b_0 + k)$, with suitably chosen $\kappa_0 > 0$ and $b_0 \geq 1$, is considered in the algorithm. Moreover, we use $\eta_k = 5$ for all k for distributed averaging in Step 2b of the algorithm. Also, we assume $d_j(t) = 0$ for each subsystem during the distributed optimization

Table 6.1: Initial subsystem states for examples in Illustration 6.1

Member 1	Member 2	Member 3	Member 4	Member 5
$[-42.0 \ -1.8]^T$	$[33.0 \ -1.0]^T$	$[-104.0 \ 1.0]^T$	$[50.0 \ -1.3]^T$	$[-80.0 \ -2.0]^T$
$[81.6 \ 0.9]^T$	$[-82.4 \ 1.2]^T$	$[152.0 \ -1.6]^T$	$[-40.0 \ -0.8]^T$	$[120.0 \ 1.6.0]^T$

Figure 6.5: Convergence of dual variables $\lambda_j, j = 1, \dots, \mathcal{N}$ (a) and convergence of instances of the coupling variable $s_j, j = 1, \dots, \mathcal{N}$ with Algorithm 6.1 (b) for the nominal-cost-based scheme in Illustration 6.1.

of ς so that the subsystem nominal cost $f_j(\varphi_j, s_j) = \bar{J}_j(t)$ remains strictly convex in s_j for each subsystem $j \in \mathcal{N}$. The corresponding convergence of the local instances $s_j, j = 1, \dots, 5$ of the consensus component of the subsystem states to the common consensus point ς is shown in Fig. 6.5b. The nominal-cost-based RHC algorithm is then applied in each of the subsystems to drive them to the common consensus point. Fig. 6.6 shows the convergence of the subsystem state trajectories from their initial values to the set $\mathcal{R}^K \otimes \{(\varsigma_0, 0)\}$, where $\varsigma_0 = -39.8$, under the nominal RHC scheme based on the outline in Fig. 6.1. For the same set of initial subsystem states, we also consider the scheme employing the repeated optimization of the consensus point. Fig. 6.6 also shows the convergence of the subsystem trajectories to the set $\mathcal{R}^K \otimes \{(\varsigma_{fin}, 0)\}$, where $\varsigma = \varsigma_{fin} = -28.8$, when controlled with the nominal RHC scheme of Fig. 6.2 as mentioned in Proposition 6.4. The successive consensus points obtained at intervals of 5 times steps in this case are shown in Fig. 6.8a (small circles). In both the cases we consider the same sets of uniformly distributed random realizations of $\rho_j(t)$ and $w_j(t)$ within the allowable bounds. We also compare the total nominal regulation costs in the two cases with the corresponding costs when the subsystems are driven to the neighbourhood of the origin. It is found that when compared with the costs in the case of the regulation of the subsystems to the origin, the total nominal regulation cost and the cost of the control input are about 36% and 16% lower in the case of the scheme considered in Proposition 6.2 and about 40% and 13% lower in the case of the scheme considered in Proposition 6.4.

Next, we consider a similar set of simulations for the min-max scheme. The initial subsystem states chosen are given in the second row of Table 6.1. Algorithm 6.1 gives the consensus point $\varsigma = \varsigma_0 = 53.2$ at time $t = 0$. The control of the subsystems as per the scheme mentioned in Proposition 6.3 results in about 36% reduction in the total cost when compared with the corresponding regulation to the origin. When a repeated update of the consensus point as mentioned in Proposition 6.5 is considered, there is about 40% reduction in the total cost with about 7% reduction in the control input requirement. Fig. 6.7 shows the convergence of the consensus components of subsystem states when controlled with the scheme mentioned in Proposition 6.5. The consensus points obtained in the successive update cycles are shown in Fig. 6.8b (denoted by small circles). Allowable random realizations of the uncertain components of the system matrices and the disturbance signal have been considered in the simulations.

Figure 6.6: Convergence of the state trajectories of subsystems in Illustration 6.1 to the sets $\mathcal{R}^{K_j} = \mathcal{R}^K$, $j = 1, \dots, \mathcal{N}$ centered at the consensus point $(\varsigma_0, 0)$ under the RHC scheme mentioned in Proposition 6.2 (blue lines), and to the one centered at $(\varsigma_{fin}, 0)$ under the RHC scheme mentioned in Proposition 6.4 (brown lines).

Figure 6.7: (a) Convergence of the consensus components of subsystems in Illustration 6.1 with Algorithm 6.1 for the min-max scheme; (b) Convergence of the actual consensus components of subsystem states under the RHC scheme of Proposition 6.5.

Illustration 6.2. We next consider another example with identical subsystems, each defined by the LTV model considered in Illustration 3.5 where the state, the input and the disturbance input are expressed as $x_j(t) = [r_j^T \ v_j^T]^T = [(r_j)_1 \ (r_j)_2 \ (v_j)_1 \ (v_j)_2]^T$, $u_j(t) = [(u_j)_1 \ (u_j)_2]^T$ and $w_j(t) = [(w_j)_1 \ (w_j)_2]^T$. We consider the following details in this example: $T_s = 30$, $\sigma = 1/3$, $0.0008 \leq \omega_j(t) \leq 0.0012$, $\bar{T} = 0.00025I_2$. We also assume the constraints: $\|u_j(t)\| \leq 0.005$, $\|r_j(t)\| \leq 2000$ and $\|v_j(t)\| \leq 2$. The consensus components considered are defined by $S_j = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}$. We consider a problem in which a set of 8 subsystems with given initial states $x_j(0)$, $j = 1, \dots, 8$ are required reach a condition in which $(r_{j+1})_2(t) - (r_j)_2(t) = \delta$, $j = 1, \dots, 7$, where $\delta = 50$ units. We choose the min-max performance index with $Q_j = \text{diag}(I_2, 10^4 I_2)$ and $R_j = 10^6$ for all the members, $j = 1, \dots, 8$. Moreover, the gain K_j is chosen to minimize γ_j in (4.13). Considering $\gamma_1 = \dots = \gamma_8 = \gamma = 2.5$, we determine the controller parameters and low-complexity inner approximations of the terminal feasible invariant sets for each subsystem.

The inter-member communications structure considered in this example is shown in Fig. 6.3(right). Considering $\eta_k = 3$ and a step size rule of the form $a_k = \kappa_0/(b_0 + k)$ as in the previous example, we apply Algorithm 6.1 for a set of randomly chosen initial conditions $x_j(0)$, $j = 1, \dots, 8$ given in Table 6.2. The member cost functions used Algorithm 6.1 are appropriately modified for the final desired configuration of the members mentioned

Figure 6.8: Successive consensus points obtained with the scheme of Fig. 6.2 for the nominal-cost-based scheme (a) and for the min-max-cost-based scheme (b) in Illustration 6.1

Figure 6.9: (a) Convergence of the coupling variables s_j 's for subsystems in Illustration 6.2 with Algorithm 6.1 at time $t = 0$; (b) Trajectories in the r -space of the members converging around the consensus point ζ under the min-max control scheme based on the outline of Fig. 6.1.

above. Fig. 6.9a shows the convergence of the set of local instances of the consensus point to the optimal consensus point $\zeta = (-310, -282)$ as per the control law considered for each member. The actual trajectories of the members on the consensus subspace, when they are controlled under the RHC scheme of Fig. 6.1 are shown in Fig. 6.9b. Here, for each member uniformly randomly distributed disturbances have been considered. A comparison of the overall tracking cost for all the members shows that there is about 30% reduction in the overall cost and about 6% reduction in the fuel cost in the proposed scheme than in the default case in which the members are aligned about the origin. We note that using the mean position of the members as the consensus point resulted in the RHC problem being infeasible for two of the members.

Table 6.2: Initial subsystem states in Illustration 6.2

Mem. 1	Mem. 2	Mem. 3	Mem. 4	Mem. 5	Mem. 6	Mem. 7	Mem. 8
$\begin{bmatrix} -698 \\ -786 \\ -0.20 \\ 0.08 \end{bmatrix}$	$\begin{bmatrix} -1100 \\ -768 \\ 0.08 \\ 0.07 \end{bmatrix}$	$\begin{bmatrix} -917 \\ -910 \\ -0.16 \\ 0.16 \end{bmatrix}$	$\begin{bmatrix} 258 \\ -1035 \\ -0.23 \\ 0.00 \end{bmatrix}$	$\begin{bmatrix} -698 \\ 901 \\ 0.16 \\ 0.16 \end{bmatrix}$	$\begin{bmatrix} 560 \\ -984 \\ -0.16 \\ 0.10 \end{bmatrix}$	$\begin{bmatrix} -796 \\ 509 \\ 0.16 \\ 0.16 \end{bmatrix}$	$\begin{bmatrix} 202 \\ 461 \\ -0.08 \\ 0.25 \end{bmatrix}$

6.6 Consensus Problem with Deterministically Time-Varying Subsystems

When the subsystem dynamics considered in a consensus-related coordinated control problem are time-varying and satisfy (6.2) but are not uncertain, the schemes outlined in Sections 3.4 and 4.4 can be employed as the local RHC methods in order to have less conservative results. In this section, we briefly discuss RHC-based solution to the consensus problem for this class of time-varying subsystems, i.e., for the class of time-varying subsystems which satisfy the following additional assumption:

A6.6: The time-varying subsystem matrices in (6.1) are not uncertain but known.

For this class of subsystems, we can parameterize the predicted control inputs for each of the subsystems $j = 1, \dots, \mathcal{N}$ as

$$u_j(t+i|t) = \begin{cases} K_j \tilde{x}_j(t+i|t) + v_j(t+i|t), & i = 0, \dots, N-1 \\ K_j \tilde{x}_j(t+i|t) + H_j \xi_j(t+i|t), & i \geq N. \end{cases} \quad (6.55)$$

where quantities $v_j(t+i|t)$, $i = 1, \dots, N-1$ are parameterized as affine functions of previous disturbances, i.e., as $v_j(t+i|t) = \bar{v}_j(t+i|t) + \sum_{l=0}^{j-1} E_{lj}(t+i|t)w_j(t+l)$, where $\bar{v}_j(t+i|t)$ and $E_{lj}(t+i|t)$ denote vectors and matrices of suitable dimensions. The control law of (6.13) is now used as the terminal control policy whose dynamics are determined as discussed in the previous sections.

Given a consensus point ς , the control actions for each subsystem $j \in \mathcal{N}$ can be obtained by optimizing the variables $\bar{\mathbf{v}}_j(t) = [\bar{v}_j(t|t)^\top \ \bar{v}_j(t+1|t)^\top \ \dots \ \bar{v}_j(t+N-1|t)^\top]^\top$, $\xi_j(t+N|t)$,

$$\mathbf{E}_j(t) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ E_{0j}(t+1|t) & 0 & \dots & 0 \\ E_{0j}(t+2|t) & E_{1j}(t+2|t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_{0j}(t+N-1|t) & E_{1j}(t+N-1|t) & \dots & 0 \end{bmatrix}$$

and $\tilde{\mathbf{F}}_j(t) = [\tilde{F}_{0j}(t+N|t) \ \tilde{F}_{1j}(t+N|t) \ \dots \ \tilde{F}_{(N-1)j}(t+N|t)]$ such that they satisfy

$$\left. \begin{aligned} & \mathcal{M}_{x_j} x_j(t) + \mathcal{M}_{s_j} \varsigma + \mathcal{M}_{u_j} \begin{bmatrix} \bar{\mathbf{v}}_j(t) \\ \xi_j(t+N|t) \end{bmatrix} + \mathbf{L}_j \mathbf{1} \leq \mathbf{1} \\ & -\mathbf{L}_j \leq \left(\mathcal{M}_{w_j} + \mathcal{M}_{u_j} \begin{bmatrix} \mathbf{E}_j(t) \\ \tilde{\mathbf{F}}_j(t) \end{bmatrix} \right) \leq \mathbf{L}_j \end{aligned} \right\} \quad (6.56)$$

where the vector $\bar{\mathbf{v}}_j(t)$ and the matrices $\mathcal{M}_{x_j}$, $\mathcal{M}_{u_j}$ and $\mathcal{M}_{w_j}$ for each subsystem j are defined as in Section 3.4 and $\mathcal{M}_{s_j} = \begin{bmatrix} I_{N \otimes M_{x_j}} S_j \\ 0 \end{bmatrix}$. $\mathbf{L}_j \in \mathbb{R}^{m_{j1} \times n_{w_j} N}$, where $m_{j1} = (m_{x_j} + m_{u_j})N + m_{j2}$ with m_{j2} being the number of rows of matrix M_{χ_j} .

We consider both the nominal and the min-max-cost-based schemes for the overall control.

6.6.1 Nominal-cost-based scheme for set-based near-consensus

For the nominal-cost-based scheme to be employed for a set-based near-consensus, we consider $\mathbf{E}_j(t) = 0$, $\tilde{\mathbf{F}}_j(t) = 0$ and $F_j^{(r)} = 0$, $r = 1, \dots, n_{\Omega_j}$. The performance function for each subsystem can be written as

$$\bar{J}_j(t) = \left\| \mathbf{H}_j(t)^{\frac{1}{2}} \begin{bmatrix} (x_j(t) - d_j(t) - S_j \varsigma)^T & \bar{\mathbf{v}}_j(t)^T & \bar{\xi}_j(t + N|t)^T \end{bmatrix}^T \right\|^2 \quad (6.57)$$

where $\mathbf{H}_j(t)$ is defined as in (3.4.3) for each $j \in \mathcal{N}$, and $d_j \in \mathcal{R}^{K_j}$. With the performance function defined as in (6.57), the optimization of the consensus point ς can be carried out in a distributed way using Algorithm 6.1 as mentioned in the previous sections. We may choose $d_j = 0$ in (6.57) to keep the cost function strictly convex in ς while optimizing it in a distributed way. In any case, given an optimized consensus point ς , each subsystem $j \in \mathcal{N}$ needs to solve the problem

$$\left. \begin{aligned} & \min_{d_j(t), \mathbf{v}_j(t), \xi_j(t)} \bar{J}_j(t) \\ & \text{with } \bar{J}_j(t) = \left\| \mathbf{H}_j(t)^{\frac{1}{2}} \begin{bmatrix} (x_j(t) - d_j(t) - S_j \varsigma)^T & \bar{\mathbf{v}}_j(t)^T & \bar{\xi}_j(t + N|t)^T \end{bmatrix}^T \right\|^2 \end{aligned} \right\} \quad (6.58)$$

subject to condition (6.56) with $\mathbf{L}_j = |\mathcal{M}_{w_j}|$ such that $d_j \in \mathcal{R}^{K_j}$. Therefore, the RHC schemes outlined in Fig. 6.1 and Fig. 6.2 can be easily employed for the overall control of the subsystems with the control of the form in (6.55) and the performance results of Propositions 6.2 and 6.4 can be directly established.

6.6.2 Minmax-cost-based scheme for H_∞ -based near-consensus

In the case of the min-max-cost-based scheme to be employed for an H_∞ -based near-consensus, we follow the approach discussed in Section 4.4.2. The performance function for each subsystem $\hat{J}_j(t)$ can be upper bounded by a scalar $\theta_j(t)$ such that

$$\begin{bmatrix} \theta_j(t) - \text{Tr}(\Upsilon_j) & 0 & \mathcal{C}_{x_j}^T & \mathcal{C}_{u_j}^T \\ 0 & \gamma_j^2 I + \Upsilon_j & \mathcal{G}_{x_j}^T & \mathcal{G}_{u_j}^T \\ * & * & \mathcal{Q}_j^{-1} & 0 \\ * & * & * & \mathbf{R}_j^{-1} \end{bmatrix} \succeq 0 \quad \Upsilon_j \in \mathbb{D}_+^{n_{w_j} N} \quad (6.59)$$

where

$$\begin{aligned} \mathcal{C}_{x_j} &= \begin{bmatrix} \mathbf{A}_j(t) \\ 0 \end{bmatrix} (x_j(t) - S_j \varsigma) + \begin{bmatrix} \mathbf{B}_j(t) & 0 \\ 0 & I_{n_\xi} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{v}}_j(t) \\ \bar{\xi}_j(t + N|t) \end{bmatrix}, \quad \mathcal{G}_{x_j} = \begin{bmatrix} \mathbf{D}_j(t) + \mathbf{B}_j(t) \mathbf{E}_j(t) \\ \tilde{\mathbf{F}}_j(t) \end{bmatrix} \\ \mathcal{C}_{u_j} &= \check{\mathbf{A}}_j(t) (x_j(t) - S_j \varsigma) + \check{\mathbf{B}}_j(t) \bar{\mathbf{v}}_j(t), \quad \mathcal{G}_{u_j} = \check{\mathbf{D}}_j(t) + \check{\mathbf{B}}_j(t) \mathbf{E}_j(t). \end{aligned} \quad (6.60)$$

Here, the matrices $\mathbf{A}_j(t)$, $\mathbf{B}_j(t)$, $\mathbf{D}_j(t)$, $\check{\mathbf{A}}_j(t)$, $\check{\mathbf{B}}_j(t)$, $\check{\mathbf{D}}_j(t)$, $\mathcal{Q}_j$ and $\mathbf{R}_j$ are defined as in Section 4.4.2 for each subsystem $j \in \mathcal{N}$. With $\hat{J}_j(t)$ replaced by $\theta_j(t)$ for each $j \in \mathcal{N}$, the overall control problem is convex with linear and LMI constraints. The optimization of the consensus point ς can be carried out to minimize the aggregate cost bound $\sum_{j=1}^{\mathcal{N}} \theta_j(t)$ in a distributed way using Algorithm 6.1 as mentioned in the previous sections. Given an optimized consensus point ς , each subsystem $j \in \mathcal{N}$ needs to solve the problem

$$\min_{\theta_j(t), \bar{\mathbf{v}}_j(t), \bar{\xi}_j(t+N|t), \mathbf{E}_j(t), \tilde{\mathbf{F}}_j(t), \mathbf{L}_j, \Upsilon_j} \theta_j(t) \quad (6.61)$$

subject to conditions (6.56) and (6.59). As in the case of the nominal-cost-based approach, we employ the RHC schemes outlined in Fig. 6.1 and Fig. 6.2 with the min-max approach for the overall control of the subsystems. It is straightforward to establish the performance results of Propositions 6.3 and 6.5 with the min-max performance functions considered in this section.

Illustration 6.3. We again consider the system considered in Illustration 6.1 for a further illustration. The subsystem dynamics and the communications structure remain the same. We assume that $\rho(t)$ is known to follow $\rho(t) = \sin(\frac{2\pi t}{T_a})$ with $T_a = 25$. For the nominal-cost-based scheme, we consider the control law in (6.55) with $N = 4$. For the same set of initial conditions given in the first row of Table 6.1, the consensus point obtained at time $t = 0$ is -38.2 as against -39.8 obtained with $N = 0$. The RHC scheme based on one-time minimization of the consensus point then results in about 37% and 16% reduction in the total and control costs when compared with the corresponding costs in the case of the regulation to the origin. Repeated update of the consensus points as per the scheme in Fig. 6.2 results in about 39% reduction in the total cost. The successive consensus points obtained at intervals of 5 time steps are shown in Fig. 6.8a (denoted by small stars).

We next consider the simulations for the min-max scheme. For the set of initial states given in the second row of Table 6.2, we obtain $\varsigma_0 = 48.1$ at time $t = 0$ with $N = 2$ while the one obtained with $N = 0$ is $\varsigma_0 = 53.1$. Successive consensus points obtained at intervals of 5 times steps are shown in Fig. 6.8b (denoted by small stars). As in the case of the nominal-cost-based scheme, there is a significant reduction in the regulation cost when compared with the corresponding cost of regulation to the neighbourhood of the origin. $\square$

6.7 Conclusion

We have presented RHC schemes for a class of consensus-related coordinated control problems that involve autonomously actuated subsystems with time-varying dynamics

including additive disturbances. The proposed schemes employ dynamic-policy-based local RHC laws in conjunction with an approximate subgradient method suitable for distributed implementation in order to optimize the consensus point as well as the necessary control actions for the subsystems. The overall control schemes have been designed to guarantee a suitably defined set-based or H_∞ -based near-consensus condition. We have explored schemes that employ either a one-time or a repeated optimization of the consensus point. The performance of the control schemes have been assessed with several simulation examples. We have observed that a control scheme which allows the optimization of the consensus point leads to significant reductions in both the total and the control costs when compared with the one that uses a predetermined consensus point. Moreover, schemes that allow repeated optimization of the consensus point further enhance the optimality of the overall control scheme.

Chapter 7

Dynamic-Policy-Based RHC in Satellite Formation Flying

In this chapter, we study the application of the RHC algorithms developed in this thesis to the control problem related to formation flying of micro- or nano-satellites in orbits around the Earth. Formation flying missions require efficient real-time control algorithms for their successful completion, and on-line-optimization-based control algorithms are arguably among the most suitable algorithms for the said purpose. Since the satellites in a formation flying mission are expected to operate reliably in constrained conditions in the presence of disturbance forces and other uncertainties, it is necessary that the algorithms are robust while being simple enough for the onboard computing hardware to implement in real-time. The RHC algorithms developed in this thesis have the desired features to be successfully used for this kind of application. In the following, we first introduce the concept of formation flying, discuss the control problems arising in the application and briefly mention the relevant works. The subsequent sections will deal with the application of our RHC algorithms in various problems related to translational control of member satellites in a mission especially in a low Earth orbit. These problems arise during formation configuration/reconfiguration as well as during formation keeping.

7.1 Introduction

7.1.1 Satellite formation flying (SFF)

A satellite formation refers to a set of spatially distributed satellites flying in a particular pattern and maintaining a precise (possibly time-varying) relative position, velocity, attitude and angular velocity with each other. Formation flying of several small spacecrafts with distributed sensing and measurement capabilities is thought to open new avenues in space-based observation/exploration of the Earth and the universe. It has been identified

Figure 7.1: Illustrations of satellite formations: (a) a circular formation and (b) a cartwheel formation.

as an enabling technology for many current and future space-science missions, such as NASA's Terrestrial Planet Finder (TPF) [160], ESA's Darwin [161], DLR's TerraSAR-X/TanDEM-X [162] and so on. These missions have been considered for a number of topographical, meteorological and astronomical applications that require space-based sensors with large aperture sizes. Examples include interferometric imaging of the Earth and the heavenly bodies, collection of atmospheric data, detection of gravity waves etc. Use of a fleet of small satellites instead of a single large satellite has several potential advantages that include simple and low-cost design, increased modularity and redundancy allowing graceful degradation in case of single satellite failure, and reconfigurable aperture shape and size enabling flexible mission objectives [163].

Fig. 7.1 shows conceptual illustrations of two simple configurations of satellite formations in low Earth orbits. Such configurations typically form distributed sensor platforms that can be used for a number of sensing applications, typically high resolution topography by the generation of high-precision digital elevation model (DEM) of the Earth's terrain [164]. Particular configurations can properly employ along-track and cross-track interferometry forming an interferometric synthetic aperture radar (SAR) for a number of other sensing applications such as ground moving target indication, measurement of ocean currents and sea ice-drifts, and monitoring of vegetation growth or loss, soil erosion, fire destruction and so on [162]. More complex formations with more member satellites and with autonomous operation abilities can be potentially used in many other applications such as high data rate secure communications [165].

7.1.2 Control of satellite formations

While satellite formations may be remarkably better in terms of achieving the application objectives when compared with their monolithic counterpart, a number of challenges, especially in the areas of coordination and control, arise from their operational complexity.

Formation flying requires that the satellites fly in a particular pattern, maintaining a precise (possibly time-varying) relative position, velocity, attitude and angular velocity with each other. As noted in [166], in formation flying, the dynamic states of the satellites are coupled only through a common control law that binds them for a cooperative objective. This coupling can be in translational and/or rotational degrees of freedom. To maintain the formation, each member must track a desired state profile relative to another member or any other virtual reference so that the control law employed depends upon the state of at least one neighboring spacecraft in the formation.

The conceptual study of spacecraft formations was first made in [167] in 1977 considering an Earth-orbiting multiple spacecraft interferometer. Following the first extensive study of SFF coordination and control made in 1996 in [168], a number of works have further analyzed formation relative dynamics and formation patterns, formation initialization and keeping control laws, relative attitude alignment and fleet coordination.

7.1.2.1 Formation configuration and relative dynamics

The choice of the formation pattern and the selection of an appropriate model of the relative motion are two important aspects of SFF. In the Earth-orbiting environment, strong gravitational effects make dynamics more complicated and hence the selection of appropriate formation geometry that is supported by natural dynamics is essential for a formation to be maintainable. There has been considerable research in the past few years in developing accurate mathematical models representing satellite relative motion and designing fuel optimal trajectories, referred to as the passive relative orbits, for Earth-orbiting formations. A survey of the results in this area can be found in [169]. The book [170] gives a more comprehensive treatment of the subject and presents the details of the various approaches proposed in the literature.

7.1.2.2 Formation coordination and control

A proper modeling of the relative dynamics and the selection of an appropriate formation configuration greatly reduce control requirements for formation initializing and keeping. However, control, along with proper coordination among member satellites, becomes essential in formation initialization or reconfiguration and in long-term formation keeping. The control inputs to be applied on the member satellites for this purpose are determined by the choice of the overall control architecture as well as by the local control algorithm used in individual satellites. Therefore, a satellite formation control scheme comprises a proper combination of a control architecture and a local level control algorithm.

Formation control architectures are sometimes expressed with graph-theoretical approach in which a formation is represented by a (directed) graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, 2, \dots, \mathcal{N}\}$ is the set of nodes representing the member spacecrafts and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is the set of (directed) edges representing the interconnection between them, each such

edge being expressed as ε_{ij} , where $\text{tail}(\varepsilon_{ij}) = i$ and $\text{head}(\varepsilon_{ij}) = j$. If $\varepsilon_{ij} \in \mathcal{E}$, then nodes i and j are adjacent. A *path* of length l in a digraph is a sequence of distinct vertices $i_0 i_1 \dots i_l$ such that for every $n \in [1, l]$, $(i_{n-1}, i_n) \in \mathcal{E}$. A digraph is called *connected* if any two vertices can be joined with a path. A path $i_0 i_1 \dots i_l$ of at least 3 vertices in which the last vertex i_l is the first vertex i_0 is called a *cycle*. A digraph is called *cyclic* if it contains a cyclic path and is called *acyclic* otherwise.

The control architectures commonly used in SFF include: (a) Hierarchical or Leader-Follower (L/F), (b) Non-hierarchical or cyclic, and (c) Multiple-Input Multiple-Output (MIMO) [166]. In an L/F architecture, the followers simply track a leader and maintain definite distances from the leader. It is generally represented by an acyclic graph. While inter-spacecraft communications requirement is low and establishing formation stability is quite straightforward in this architecture, lack of global optimality and the complete dependence on the functioning of the leader are its drawbacks. A non-hierarchical control setup has no specific leaders or followers, and is generally represented by cyclic formation graphs. Non-hierarchical setups may range from a single directed cycle to one in which each member tracks a number of its neighbouring members. Non-hierarchical architectures easily lend to distributed or decentralized control and are generally robust to local failures. However, stability analysis is more difficult and is usually confirmed by simulations. In an MIMO architecture, typically, complete information about the states of all members of the formation will be available to each spacecraft, thus allowing a globally optimal control. It is represented by a completely connected formation graph and it supports global optimality and stability. However, this architecture suffers from a number of drawbacks including the requirement of a high information exchange and the lack of scalability.

While the control architecture defines the interaction and coordination among the formation members, local level control algorithms in each member handle the member's movement relative to the other members for configuring or maintaining the formation. Depending on the choice of the dynamical model, the local level control algorithms may be based on any suitable control approach such as PD, linear quadratic, nonlinear, impulsive, model predictive, adaptive, etc.

7.1.3 Existing works on SFF control

Problems related to SFF control have been extensively investigated over the past several years and research work dealing with them is extensive. A survey of this work can be found in [166]. Most of the works dealing with Earth-orbiting formations have analyzed control issues related to formation initialization/reconfiguration or formation keeping in the leader-follower framework. Also, as mentioned in the survey paper [166], most of the papers consider only translational aspects of control and present a range of follower control techniques. Linear quadratic control algorithms are considered with pulse-based actuators in [171], with decentralized estimators in [172] and with reduced inputs in [173].

Lyapunov-based nonlinear output feedback controller is designed in [174]. A finite-horizon MPC for a formation in elliptical reference orbit is considered in [175] and a predictive command governor is used in [176]. Some other control techniques used in SFF in the L/F setup include adaptive control [177], [178], μ -control [179] and nonlinear numerical optimization [180].

Non-hierarchical control architectures are used in a number of works including [181], [4], [182], [3], [183], [184], [185] etc. In [181], line-of-sight-based feedback control law is developed using a cyclic topology and formation stability under this law is analyzed. A number of works consider non-hierarchical architecture in a virtual structure (VS) framework, in which member spacecrafts are considered as rigid bodies embedded in a larger virtual rigid structure. The overall motion of the formation as a VS along with the specified states of the member spacecrafts within it is used to calculate the desired reference trajectories of the spacecrafts. VS coordination is achieved in [3] through the use of a coordination variable throughout the formation whereas in [183], VS approach is applied in the decentralized form. In [4] and [182], control design is based on VS-based non-hierarchical control architectures and fuel optimal control algorithms are used in the local level. The former uses a finite-horizon MPC based on linear programming that minimizes the 1-norm of the control input (fuel use), whereas, in the latter, LQR-based optimal control is used. Both works consider circular or near-circular reference orbits and, at each time step, determine the center (and orientation) of the virtual structure that best matches with the current positions of the member satellites. A decentralized control algorithm based on graph-theoretic methods and using dynamic feedback local controllers is presented in [184] in which asymptotic stability along with the consensus on formation reference point as the virtual leader is discussed for various communications/sensing topologies. Considering the problem as a dynamic team problem with delayed information-exchange, [185] presents optimal decentralized control algorithms that are applied for the control of formation missions in elliptical reference orbits.

MIMO architecture has been used in a few works including [186], in which the architecture expressed in the form of a digraph is encoded in LQR formulation for an Earth-orbiting formation.

7.2 Satellite Formation Dynamics

Orbital motion of satellites is primarily governed by the gravitational laws and is non-linear in nature. It is also affected by disturbances such as aerodynamic drag and solar radiation pressure. Since the satellites in a formation are required to maintain a definite spatial pattern for long periods of time, an accurate model of the relative motion among them is required for devising efficient mechanisms to be applied for initialization, keeping and reconfiguration of the formation. In this section, we briefly describe the modeling of relative dynamics of satellites flying in formation in orbits around the Earth. We also

discuss formation patterns that are supported by the satellites' natural relative dynamics and mention initialization conditions that allow us to achieve particular formation patterns with a number of member satellites. Note that, unlike elsewhere in the thesis, the use of the variable t in this section is in the continuous-time sense, i.e., in this section, $t \in \mathbb{R}_+$ and represents the actual time in time units.

7.2.1 Model of relative motion

The dynamical model commonly used in SFF control expresses the relative motion of a member satellite in a non-inertial rotating coordinate frame, usually called the Local-Vertical-Local-Horizontal (LVLH) frame or the Hill's frame, centered at a suitably chosen reference point [4]. Such a rotating coordinate frame is shown in Fig. 7.2. The formation reference point (FRP) is described by a virtual (or real) satellite moving along a reference orbit and is represented by a set of standard orbital elements — a_{ref} , e_{ref} , i_{ref} , Ω_{ref} , ω_{ref} , θ_{ref} — which correspond respectively to the semi-major axis, eccentricity, inclination, right ascension of the ascending node, argument of the periapsis and true anomaly.

Several models of satellite relative motion for formation applications have been explored by researchers and they vary from a simple LTI model to more accurate nonlinear models (see, e.g., [170] and references therein). For close formations in circular and eccentric reference orbits, linearized models closely approximate the actual relative dynamics. In the following, we briefly discuss the linearized models describing the relative motion of a member satellite w.r.t. the FRP in a close formation along a circular or an elliptical reference orbit.

In Fig. 7.2, $\mathbf{R}_r$ is the position vector of the FRP w.r.t. to the center of the Earth and $\mathbf{r}$ denotes the relative position of a member satellite w.r.t. the FRP. The motion of the member satellite, expressed in the Earth-centered inertial (ECI) coordinate frame X - Y - Z , under the influence of the ideal gravitational attraction of the Earth is given by

$$\ddot{\mathbf{r}}^I = \mathbf{g}(\mathbf{R}_r + \mathbf{r}) - \mathbf{g}(\mathbf{R}_r)$$

where, for any vector $\boldsymbol{\rho}$, $\mathbf{g}(\boldsymbol{\rho})$ is defined as $\mathbf{g}(\boldsymbol{\rho}) = -\frac{\mu}{|\boldsymbol{\rho}|^3}\boldsymbol{\rho}$ and μ is the gravitational parameter of the Earth. Here, the superscript 'I' in $\ddot{\mathbf{r}}^I$ indicates that the quantity is expressed in the ECI coordinate frame. For a close formation, $|\mathbf{r}| \ll |\mathbf{R}_r|$, and $\mathbf{g}(\mathbf{R}_r + \mathbf{r})$ can be linearized about $\mathbf{R}_r$ to give the motion equation [187]

$$\ddot{\mathbf{r}}^I = \frac{\mu}{R_r^5}(3\mathbf{R}_r\mathbf{R}_r^T - R_r^2\mathbf{I})\mathbf{r}$$

where $R_r = |\mathbf{R}_r|$. We can write this equation in terms of the relative acceleration expressed in a rotating coordinate frame centered at the FRP as

$$\ddot{\mathbf{r}} = \frac{\mu}{R_r^5}(3\mathbf{R}_r\mathbf{R}_r^T - R_r^2\mathbf{I})\mathbf{r} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r} \quad (7.1)$$

Figure 7.2: Schematic of a satellite formation in an elliptical reference orbit around the Earth. In the rotating coordinate frame, x , y and z represent the radial, along-track and cross-track (out-of-plane) directions respectively.

where $\boldsymbol{\omega}$ is the angular velocity of the rotating coordinate frame. Next, let us consider $\mathbf{r} = [r_x \ r_y \ r_z]^T$ where r_x, r_y, r_z are the components of the relative position vector $\mathbf{r}$ in the rotating coordinate frame along the x -, y - and z -directions, which respectively represent the radial (along the radius vector $\mathbf{R}_r$), along-track (along the direction of motion of the reference point) and cross-track (normal to the reference orbital plane) directions. Then, $\mathbf{R}_r = [R_r \ 0 \ 0]^T$ and $\boldsymbol{\omega} = [0 \ 0 \ \omega]^T = [0 \ 0 \ \dot{\theta}]^T$ where $\theta = \theta_{ref}$. So, (7.1) can be written as

$$\begin{bmatrix} \ddot{r}_x \\ \ddot{r}_y \\ \ddot{r}_z \end{bmatrix} = \frac{\mu}{R_r^3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 2\omega \dot{r}_y + \dot{\omega} r_y + \omega^2 r_x \\ -2\omega \dot{r}_x - \dot{\omega} r_x + \omega^2 r_y \\ 0 \end{bmatrix}. \quad (7.2)$$

Using the fundamental orbital mechanics describing planetary motion [187], R_r , ω and $\dot{\omega}$ can be written as

$$R_r = \frac{a(1 - e^2)}{1 + e \cos \theta}, \quad \omega = \sqrt{\frac{\mu}{a^3}} \frac{(1 + e \cos \theta)^2}{(1 - e^2)^{3/2}}, \quad \dot{\omega} = \frac{-2e \sin \theta}{1 + e \cos \theta} \omega^2 \quad (7.3)$$

where $a = a_{ref}$ and $e = e_{ref}$.

Therefore, the equations of relative motion based on the ideal gravitational attraction

only can be written as

$$\begin{aligned}
\ddot{r}_x &= (3 + e \cos \theta) k_0 \omega^{\frac{3}{2}} r_x - 2e \sin \theta k_0 \omega^{\frac{3}{2}} r_y + 2\omega \dot{r}_y \\
\ddot{r}_y &= 2e \sin \theta k_0 \omega^{\frac{3}{2}} r_x + e \cos \theta k_0 \omega^{\frac{3}{2}} r_y - 2\omega \dot{r}_x \\
\ddot{r}_z &= -k_0 \omega^{\frac{3}{2}} r_z
\end{aligned} \tag{7.4}$$

where $k_0 = \sqrt{\omega_0}$ with $\omega_0 = \sqrt{\frac{\mu}{a^3(1-e^2)^3}}$.

For an elliptical reference orbit, the angular speed ω is a function of the true anomaly θ . So the equations of relative motion in (7.4) are time-varying. The true anomaly θ is a monotonically increasing function of the mean anomaly M given by $M = E - e \sin E$, where $E = 2 \tan^{-1} \left(\sqrt{(1-e)/(1+e)} \tan(\theta/2) \right)$ is called the eccentric anomaly. M varies linearly with time t and is given by $M = \bar{\omega}(t - t_p)$, where $\bar{\omega} = \sqrt{\mu/a^3}$ is the mean motion of the reference orbit and t_p is the time of periapsis passage.

For a circular reference orbit, $e = 0$ and $\omega = \bar{\omega}$. So the equations of relative motion in (7.4) reduce to time-invariant equations

$$\begin{aligned}
\ddot{r}_x &= 3\bar{\omega}^2 r_x + 2\bar{\omega} \dot{r}_y \\
\ddot{r}_y &= -2\bar{\omega} \dot{r}_x \\
\ddot{r}_z &= -\bar{\omega}^2 r_z
\end{aligned} \tag{7.5}$$

which are popularly known as the Hill-Clohessy-Wiltshire (HCW) equations [188]. The unforced equations of relative motion in (7.4) or (7.5) are linearized equations based on the Keplerian motion of the satellites and they neglect the effects of the higher order terms, differential perturbations caused by the gravity harmonics including J_2 and differential effects of other disturbances such as aerodynamic drag, solar radiation pressure and the influences of other bodies including the sun and the moon. For formations in a low-Earth orbit, the effect of the second order gravity harmonic, J_2 is particularly significant. Including the linearized version of the J_2 effect in the equations of relative motion for a circular reference orbit results in modified HCW equations which can be written in the homogeneous form as [189]

$$\begin{aligned}
\ddot{r}_x &= (5c_1^2 - 2) \bar{\omega}^2 r_x + 2\bar{\omega} c_1 \dot{r}_y \\
\ddot{r}_y &= -2\bar{\omega} c_1 \dot{r}_x \\
\ddot{r}_z &= -(3c_1^2 - 2) \bar{\omega}^2 r_z
\end{aligned} \tag{7.6}$$

where $c_1 = \sqrt{c_2 + 1}$ and $c_2 = \frac{3}{8} J_2 \frac{R_{Earth}^2}{a^2} (1 + 3 \cos 2i_{ref})$. Here, R_{Earth} is the radius of the Earth.

With the unforced approximate equations of relative motion defined as in (7.4), (7.5) or (7.6), the linear relative dynamical model representing the actual relative motion of a

member satellite w.r.t. the FRP can be expressed as

$$\dot{x} = \check{A}x + \check{B}_u u + \check{B}_d d \quad (7.7)$$

where $x = [r_x \ r_y \ r_z \ \dot{r}_x \ \dot{r}_y \ \dot{r}_z]^T$ represents the state, $u = [u_x \ u_y \ u_z]^T$ represents the control input and $d = [d_x \ d_y \ d_z]^T$ represents the ‘disturbance’ accelerations that account for modeling errors including the effects of nonlinearities, perturbations and external disturbances. The system matrix $\check{A}$ is given by

$$\check{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ (3 + e \cos \theta) k_0 \omega^{\frac{3}{2}} & -2e \sin \theta k_0 \omega^{\frac{3}{2}} & 0 & 0 & 2\omega & 0 \\ 2e \sin \theta k_0 \omega^{\frac{3}{2}} & e \cos \theta k_0 \omega^{\frac{3}{2}} & 0 & -2\omega & 0 & 0 \\ 0 & 0 & -k_0 \omega^{\frac{3}{2}} & 0 & 0 & 0 \end{bmatrix} \quad (7.8)$$

in the case of the motion represented by (7.4), whereas, when representing the motion equations (7.5) or (7.6), $\check{A}$ is time-invariant and is given by

$$\check{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ (5c_0^2 - 2)\bar{\omega}^2 & 0 & 0 & 0 & 2c_0\bar{\omega} & 0 \\ 0 & 0 & 0 & -2c_0\bar{\omega} & 0 & 0 \\ 0 & 0 & -(3c_0^2 - 2)\bar{\omega}^2 & 0 & 0 & 0 \end{bmatrix} \quad (7.9)$$

where $c_0 = 1$ when representing the motion equations in (7.5) and $c_0 = c_1$ when the equations in (7.6) are considered. The matrices $\check{B}_u$ and $\check{B}_d$ are defined as $\check{B}_u = \check{B}_d = [\mathbf{0} \ I_3]^T$.

Note that in the linear model (7.7) with the system matrices defined above, the dynamics in the z -direction (cross-track) are decoupled with those in the x - y plane and can be treated separately.

7.2.2 Solution of the linearized model

We first consider the solution to (7.7) when the system matrix $\check{A}$ is time-varying as in (7.8). The solution to this model with the time t as the independent variable is not straight-forward. Since θ varies monotonically with the independent variable t , it is natural to express the time-varying relative dynamical model with θ as the independent variable [175]. Using the relationships $\dot{X} = \frac{dX}{d\theta}\dot{\theta} = X'\dot{\theta}$ and $\ddot{X} = X''\dot{\theta}^2 + X'\ddot{\theta}$, and defining new dependent variables $\tilde{x} = \rho x$, $\tilde{y} = \rho y$, $\tilde{z} = \rho z$, where $\rho = 1 + e \cos \theta = \sqrt{\omega}/k_0$, the time-varying model (7.7) with $\check{A}$ given by (7.8) can be expressed in a relatively simpler

form as

$$\tilde{x}' = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3/\rho & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tilde{x} + \frac{1}{k_0^4 \rho^3} (\ddot{B}_u u + \ddot{B}_d d) \quad (7.10)$$

where the state is defined by $\tilde{x} = [\tilde{r}_x \ \tilde{r}_y \ \tilde{r}_z \ \tilde{r}'_x \ \tilde{r}'_y \ \tilde{r}'_z]^\top$.

The homogeneous solution to this system at any final true anomaly θ starting from an initial state $\tilde{x}_0$ at θ_0 is given by

$$\tilde{x}(\theta) = \Phi_{\theta_0}^\theta \tilde{x}_0 \quad (7.11)$$

where the state transition matrix $\Phi_{\theta_0}^\theta = \Phi_\theta \Phi_{\theta_0}^{-1}$ with

$$\Phi_\theta = \begin{bmatrix} \rho s & 0 & 0 & 2 - 3e\rho s\vartheta & -\rho c & 0 \\ \rho c + c & 1 & 0 & -3\rho^2\vartheta & \rho s + s & 0 \\ 0 & 0 & s & 0 & 0 & c \\ s_d & 0 & 0 & -3es_d\vartheta - 3es/\rho & -c_d & 0 \\ -2\rho s & 0 & 0 & -3 + 6e\rho s\vartheta & 2\rho c - e & 0 \\ 0 & 0 & c & 0 & 0 & -s \end{bmatrix} \quad (7.12)$$

where $s = \sin \theta$, $c = \cos \theta$, $s_d = \cos \theta + e \cos 2\theta$, $c_d = -\sin \theta - e \sin 2\theta$ and $\vartheta = \omega_0(t - t_0)$. Here, the times t_0 and t correspond to the true anomalies θ_0 and θ respectively. The matrix Φ_{θ_0} is obtained by evaluating (7.12) at $\theta = \theta_0$ for which $\vartheta = 0$. The inverse of the matrix Φ_{θ_0} exists for all true anomalies and all eccentricities $e \in [0, 1]$ [190]. The solution to the homogeneous equations of relative motion with θ as the independent variable has been derived in the literature in various forms (see, e.g., [190, 191]). Solution (7.11-7.12) considered here is based on the results in [190].

The solution in (7.11) is in the θ domain. Since θ varies monotonically with t as mentioned in the last subsection, the solution at any time t can be easily obtained from the solution at the corresponding true anomaly $\theta(t)$.

In the case of a circular reference orbit, the relative dynamics are time-invariant and the solution can be directly obtained in the time-domain. For the standard HCW equations, given an initial condition x_0 at time $t_0 = 0$, the homogeneous solution at any time t is given by

$$x = \begin{pmatrix} 4 - 3 \cos \bar{\omega} t & 0 & 0 & \frac{\sin \bar{\omega} t}{\bar{\omega}} & 2 \frac{1 - \cos \bar{\omega} t}{\bar{\omega}} & 0 \\ -6(\bar{\omega} t - \sin \bar{\omega} t) & 1 & 0 & -2 \frac{1 - \cos \bar{\omega} t}{\bar{\omega}} & -3t + 4 \frac{\sin \bar{\omega} t}{\bar{\omega}} & 0 \\ 0 & 0 & \cos \bar{\omega} t & 0 & 0 & \frac{\sin \bar{\omega} t}{\bar{\omega}} \\ 3\bar{\omega} \sin \bar{\omega} t & 0 & 0 & \cos \bar{\omega} t & 2 \sin \bar{\omega} t & 0 \\ -6\bar{\omega}(1 - \cos \bar{\omega} t) & 0 & 0 & -2 \sin \bar{\omega} t & -3 + 4 \cos \bar{\omega} t & 0 \\ 0 & 0 & -\bar{\omega} \sin \bar{\omega} t & 0 & 0 & \cos \bar{\omega} t \end{pmatrix} x_0. \quad (7.13)$$

The solution to the modified HCW equations can also be obtained in a straightforward manner.

7.2.3 Passive relative orbits and initialization conditions

Possible geometrical patterns for a satellite formation are those that can be maintained for long periods of time without the use of much fuel. For this, the member satellites are required to be in passive (periodic) relative orbits (orbits that do not drift apart monotonically with time) with respect to the FRP. Such passive relative orbits can be obtained with particular choices of initialization conditions.

We first consider the solution in (7.13) for an insight on the nature of the possible periodic solutions that give periodic relative orbits. It is obvious from (7.13) that all components of x except r_y vary periodically with time. The component r_y is also periodic if the initial condition x_0 is chosen such that

$$\dot{r}_{y_0} = -2\bar{\omega} r_{x_0}. \quad (7.14)$$

With this condition satisfied, the natural periodic relative motion will be of the form

$$\begin{aligned} r_x &= \beta_x \sin(\bar{\omega} t + \phi_x) \\ r_y &= 2\beta_x \cos(\bar{\omega} t + \phi_x) + \delta_y \\ r_z &= \beta_z \sin(\bar{\omega} t + \phi_z) \end{aligned} \quad (7.15)$$

where

$$\begin{aligned} \beta_x &= \sqrt{r_{x_0}^2 + \left(\frac{\dot{r}_{x_0}}{\bar{\omega}}\right)^2}, \quad \phi_x = \tan^{-1} \left(\frac{r_{x_0}}{\dot{r}_{x_0}/\bar{\omega}} \right) \\ \beta_z &= \sqrt{r_{z_0}^2 + \left(\frac{\dot{r}_{z_0}}{\bar{\omega}}\right)^2}, \quad \phi_z = \tan^{-1} \left(\frac{r_{z_0}}{\dot{r}_{z_0}/\bar{\omega}} \right) \end{aligned}$$

The first two equations in (7.15) describe a motion in an ellipse centered at $(0, \delta_y)$ on the radial/along-track (x - y) plane. The motion in the cross-track (z -) direction given by the third equation in (7.15), is a simple harmonic oscillation and is decoupled from the

Table 7.1: Common maintainable satellite formation configurations

Configuration	Design specifications	Initialization conditions ¹
In-plane along-track	$\delta_y = \delta \neq 0, \beta_x = 0, \beta_z = 0$	$r_{y_0} = \delta$
In-ground-track	$\delta_y = \delta \neq 0, \beta_x = 0,$ $\beta_z = \beta = \frac{\omega_{earth}}{\bar{\omega}} \delta \sin i_{ref}$ $\phi_z = -\frac{\pi}{2}$	$r_{y_0} = \delta,$ $r_{z_0} = -\beta$
Cross-track pendulum	$\delta_y = \delta \neq 0, \beta_x = 0,$ $\beta_z = \beta > 0, \phi_z = \phi \in [0, 2\pi)$	$r_{y_0} = \delta,$ $r_{z_0} = \beta \sin \phi, \dot{r}_{z_0} = \beta \bar{\omega} \cos \phi$
Cartwheel	$\delta_y = 0, \beta_z = 0,$ $\beta_x = \beta > 0, \phi_x = \phi \in [0, 2\pi)$	$r_{x_0} = \beta \sin \phi, \dot{r}_{x_0} = \beta \bar{\omega} \cos \phi,$ $r_{y_0} = 2\dot{r}_{x_0}/\bar{\omega}, \dot{r}_{y_0} = -2\bar{\omega} r_{x_0}$
Circular (Radius $\beta > 0$)	$\delta_y = 0, \beta_x = \frac{1}{2}\beta, \beta_z = \frac{\sqrt{3}}{2}\beta$ $\phi_x = \phi_z = \phi \in [0, 2\pi)$	$r_{x_0} = \frac{1}{2}\beta \sin \phi, \dot{r}_{x_0} = \frac{1}{2}\beta \bar{\omega} \cos \phi,$ $r_{y_0} = 2\dot{r}_{x_0}/\bar{\omega}, \dot{r}_{y_0} = -2\bar{\omega} r_{x_0},$ $r_{z_0} = \pm\sqrt{3}r_{x_0}, \dot{r}_{z_0} = \pm\sqrt{3}\dot{r}_{x_0}$
Projected circular (Radius $\beta > 0$)	$\delta_y = 0, \beta_x = \frac{1}{2}\beta, \beta_z = \beta$ $\phi_x = \phi_z = \phi \in [0, 2\pi)$	$r_{x_0} = \frac{1}{2}\beta \sin \phi, \dot{r}_{x_0} = \frac{1}{2}\beta \bar{\omega} \cos \phi,$ $r_{y_0} = 2\dot{r}_x/\bar{\omega}, \dot{r}_{y_0} = -2\bar{\omega} r_{x_0},$ $r_{z_0} = \pm 2r_{x_0}, \dot{r}_{z_0} = \pm 2\dot{r}_{x_0}$

motion in the x - y plane. Combining the simple harmonic motion in the z -plane with the elliptical motion in the x - y plane will result in a family of ellipses that represent formation flying patterns that are maintainable. These patterns are thus determined by the five design parameters that define the periodic relative motion between the satellites, viz., the along-track separation from the FRP (δ_y), the size of the ellipse in the radial/along-track plane (β_x), the initial location within that ellipse (ϕ_x), the amplitude of oscillation on the cross-track plane (β_z) and the initial phase in that oscillation (ϕ_z). Several maintainable formation patterns that can be useful from the application point have been explored in the literature (see, e.g., [163]). Table 7.1 summarizes the design specifications and the initialization conditions for the common maintainable configurations.

The periodic relative motion described by the equations in (7.15) and the details of formation configurations mentioned in Table 7.1 are based on the HCW model. A slightly modified set of equations will represent similar configurations when the modified HCW equations are considered. When the reference orbit is elliptical, periodic relative orbits still exist but have more complicated configurations. It can be seen from the solution in (7.11-7.12) that the states will vary periodically if the terms containing ϑ (which is the only quantity monotonically growing with time) in the expression for $\tilde{x}(\theta)$ in (7.11) are canceled out. This can be achieved if

$$(3\rho_{\theta_0} + e^2 - 1)\tilde{r}_{x_0} + e s_{\theta_0} \tilde{r}'_{x_0} + \rho_{\theta_0}^2 \tilde{r}'_{y_0} = 0.$$

The above condition is not time-invariant and depends on the true anomaly of the reference orbit. If t_0 corresponds to the true anomaly $\theta_0 = 0$, this condition can be

¹Initial values of positions and velocities not mentioned are zero.

expressed with time t as the independent variable as follows:

$$\dot{r}_{y_0} = -\bar{w} \frac{2+e}{\sqrt{(1+e)(1-e)^3}} r_{x_0}. \quad (7.16)$$

This is the condition for periodic relative motion on the radial/along-track (x - y) plane. The motion in the cross-track (z -) direction, which is a simple harmonic oscillation, is decoupled from that on the x - y plane as in the case of the circular reference orbit. Formation configurations similar to those in the case of the circular reference orbit can be obtained for an elliptical reference orbit of small eccentricity by a proper selection of initial positions and velocities.

7.3 Relative Position Control with Dynamic-Policy-Based RHC

In the previous section, we discussed the linearized dynamical model representing the relative translational motion of member satellites with respect to the FRP, which can either be a real satellite or a virtual reference point on the reference orbit. The linearized dynamics also give an insight into the nature of the relative motion and the possible passive formation configurations that are maintainable in the long term. In SFF applications, member satellites need to be initialized into the desired configurations which form passive ‘apertures’. Moreover, since the relative orbits are based on linearized dynamics ignoring the effects of perturbations and other disturbances, satellites initialized into passive configurations will require some mechanism to ensure that they do not drift away from the desired configurations under the influence of perturbations and disturbances. A proper control mechanism is necessary during the initialization of the formation and also during the long-term operation (keeping) of the formation. In this section, we explore the use of the RHC algorithms developed in the earlier chapters in the control of the relative state of a member satellite in a formation orbiting the Earth.

We need a linear discrete-time dynamical model representing the relative motion in order to use the RHC algorithms developed in the earlier chapters. The discrete-time counterpart of the the continuous-time model of (7.7) will be of the form

$$x(t+1) = A(t)x(t) + B_u(t)u(t) + B_d(t)d(t) \quad (7.17)$$

where $t \in \mathbb{Z}_+$ represents the discrete time steps. The discrete-time model is usually based on the assumption that the control input remains constant over the time step. In the case of satellites, thrusters are commonly used to effect velocity changes. So a constant control input may not be available throughout the time step. While considering the discrete-time model of the relative motion, we, therefore, assume that control input will be applied only

during a fraction of the sampling time. In the following, we discuss the details in the case of both circular and elliptical reference orbits.

7.3.1 Control in formations around a circular reference orbit

7.3.1.1 Determination of the discrete-time dynamical model

When the reference orbit is circular, the linearized dynamical model is time-invariant. Considering a sampling time of T_s seconds and assuming that the control input is applied for a fraction σ of the sampling time (i.e., for σT_s seconds), the discrete-time system matrices $A(t)$ and $B_u(t)$ can be defined as

$$A(t) = A = e^{\check{A}T_s}, \quad B_u(t) = B_u = \int_0^{\sigma T_s} e^{\check{A}(T_s-\tau)} d\tau$$

where the matrix $\check{A}$ is defined in (7.9). The choice of the matrix $B_d(t) = B_d$ depends on the way the disturbance $d(t)$, which accounts for modeling errors resulting from nonlinearities and perturbations as well as the effects of external disturbances including the differential aerodynamic drag, is modeled. One way is to consider $d(t)$ as the disturbance acceleration and determine the bounds of its components using the analytic expressions of the effects of nonlinearities, J_2 , drag and other disturbances. Assuming that the variation of the disturbance acceleration during each time step is negligible, B_d , in this case, can be defined as $B_d(t) = B_d = \int_0^{T_s} e^{\check{A}(T_s-\tau)} d\tau$. Another way is to consider the disturbance directly in the discrete-time model for each state component and estimate the bounds of the disturbance components by comparing the transition of state made by the linear model in a single time step with the one obtained for the same time interval through a realistic nonlinear numerical simulation that includes all perturbations and disturbances. In this chapter, we consider the second approach, and, assuming that the expected relative ‘error’ states to be handled by the control input are bounded in a set $\mathcal{S}_{dx}$, estimate the disturbance bounds by comparing state transitions with the linear and nonlinear models for several initial relative ‘error’ states lying in $\mathcal{S}_{dx}$ and at several reference points along the reference orbit. Once the proper bounds of the disturbance components are estimated from the simulation observations, the disturbance term $B_d d(t)$ can be replaced by $B_w w(t)$ where B_w is suitably chosen so that the model conforms with the LTV models used in the earlier chapters where the disturbance $w(t)$ is assumed to be componentwise bounded by 1.

7.3.1.2 Computation of the dynamic control policy

Once the disturbance input is properly modeled, controller matrices defining the dynamic policy can be easily determined. The dynamic control policy can be used either as the main policy, or as the terminal policy appended to a finite-horizon RHC law. If the

Table 7.2: Bounds of disturbance components in Illustration 7.1

Linear Model	$r_x(\text{cm})$	$r_y(\text{cm})$	$r_z(\text{cm})$	$v_x(\text{mm/s})$	$v_y(\text{mm/s})$	$v_z(\text{mm/s})$
HCW	18.3	17.8	32.7	6.1	2.2	2.1
Modified HCW	16.1	17.3	31.8	5.4	2.0	1.8

dynamic policy involves the disturbance feedforward term, its use as the terminal policy requires a disturbance-based finite-horizon input parameterization. Using the dynamic policy as the terminal policy and optimizing the finite-horizon free variables on-line usually improves performance but it naturally increases the on-line computational burden. We consider an alternative in which the feedforward gains $E_j(t+i|t)$'s to be used in the finite horizon are 'extracted' from the off-line computed dynamic policy and only the free perturbation components $\bar{v}(t+i|t)$'s are optimized on-line. Essentially, this allows us to increase the size of the feasibility domain and to improve the control performance without much increase in the computational complexity.

Illustration 7.1. As an illustration, we consider the case of a reference orbit of semi-major axis $a_{ref} = 7500$ km, eccentricity $e_{ref} = 0.00005$ and inclination $i_{ref} = 25^\circ$. Assuming the constraints $-2.5 \text{ km} \leq r_x(t), r_z(t) \leq 2.5 \text{ km}$, $-5 \text{ km} \leq r_y(t) \leq 5 \text{ km}$, $-2.5 \text{ m/s} \leq v_x(t), v_z(t) \leq 2.5 \text{ m/s}$ and $-5 \text{ m/s} \leq v_y(t) \leq 5 \text{ m/s}$, estimates of the componentwise upper bounds of the total disturbance are obtained through simulations. In the simulations, each corner of the box defined by the position and velocity constraints is propagated both with the linearized version of the relative dynamical model and with a high-fidelity nonlinear propagator that considers the effects of the second order zonal and tesseral harmonics, sun and moon effects and the atmospheric drag². The reference point is also assumed to be a satellite and is propagated in the same way as the member satellite but without considering the atmospheric drag. Comparisons are made for several reference point locations around the reference orbit. Table 7.2 lists the largest absolute values of the disturbance components observed from these comparisons with both the HCW and the modified HCW models. A sampling time of $T_s = 60\text{s}$ and $\sigma = 1/6$ are considered for both of the discrete time models.

Next, we determine the controller matrices and feasible invariant sets for the dynamical model based on the modified HCW equations. Fig. 7.3 shows the projections of the feasible invariant sets on r_x - r_y , v_x - v_y and r_z - v_z subspaces. The sets $\bar{\mathcal{X}}_{r_x r_y}$, $\bar{\mathcal{X}}_{v_x v_y}$ and $\bar{\mathcal{X}}_{r_z v_z}$ correspond to the case when no disturbance feedforward is considered in the control law whereas the sets $\mathcal{X}_{r_x r_y}^f$, $\mathcal{X}_{v_x v_y}^f$ and $\mathcal{X}_{r_z v_z}^f$ correspond to the case when full disturbance feedforward is considered. The sets $\mathcal{X}_{r_x r_y}$, $\mathcal{X}_{v_x v_y}$ and $\mathcal{X}_{r_z v_z}$ correspond to the controller matrices that satisfy the H_∞ bound condition with $\gamma = 0.5$. Here, we consider $Q = \text{diag}(10^{-6}I_2, 0.1I_2)$ and $R = 10^4 I_2$, and choose K to be the unconstrained LQ optimal gain. An input constraint $\|u(t)\|_\infty \leq 1\text{mm/s}^2$ on the control acceleration is also considered.

Fig. 7.4 shows the performances of the dynamic-policy-based RHC laws when used for an initial 'error' state on the x - y plane. The plots show the time variation of the r_x and r_y components when the nominal and min-max RHC schemes employing the dynamic policy as the main policy are used for the regulation of the error state to the origin. Here, we assume that the 'error' state in the Hill's frame is available at each time step for the computation of the control input components. For a realistic simulation, the satellite 'error'

²For nonlinear propagation, we use Spacecraft Control Toolbox V6.1 from Princeton Satellite Systems.

Figure 7.3: Projections, on r_x - r_y , v_x - v_y and r_z - v_z subspaces, of feasible invariant sets for a member satellite with a circular reference orbit.

state is transformed to the ECI coordinate frame and propagated using the high-fidelity nonlinear numerical propagation routine. The computed control input components, after an appropriate transformation to the ECI coordinate frame, are applied for the first $\sigma T_s = 10$ s of the sampling interval of $T_s = 60$ s. The corresponding successor ‘error’ state in the Hill’s frame is computed with respect to the new reference point which is propagated in a similar way but without the control input. We can observe from the plots that even though the performances of the various RHC schemes are not much different, the nominal scheme using the dynamic policy without disturbance feedforward performs in a slightly more conservative way than the other two schemes which allow larger feasible invariant sets with the inclusion of the feedforward term.

Fig. 7.5 shows the control performances when the dynamic policy is used as the terminal control policy appended to a finite-horizon RHC law, and compares it with a scheme using a static terminal control law. The plots show the time variation of the r_x and r_y components for an initial ‘error’ state on the x - y plane with RHC schemes employing dynamic and static terminal policies. While a horizon length of $N = 5$ is considered for the nominal and the min-max schemes using the dynamic terminal policy, the nominal scheme using the optimal static terminal law requires a horizon length of $N = 60$ for the initial ‘error’ state considered. Disturbance feedforward is considered only in the min-max scheme and for this scheme the feedforward gains for the finite horizon are not optimized on-line but are obtained from the off-line-optimized terminal controller matrices. It can be observed from the plots that the schemes that use the dynamic terminal policy have control performances similar to that of the scheme using the terminal optimal static policy which requires a far larger horizon length and hence has a significantly higher computational complexity.

7.3.2 Control in formations around an elliptical reference orbit

7.3.2.1 Determination of the discrete-time dynamical model

When the reference orbit is elliptical, the linearized dynamical model of the relative satellite motion is no longer time-invariant; it varies with the true anomaly of the reference orbit (θ). The continuous-time system matrix $\tilde{A}$ is a function of θ and, since θ varies

Figure 7.4: Performances of nominal and min-max RHC schemes in relative satellite position control for an initial ‘error’ state $[-0.60 \ 1.20 \ 0.00 \ 0.45 \ 0.95 \ 0.00]^T$. (a) Time variation of the r_x component, and (b) Time variation of the r_y component.

Figure 7.5: Performances of RHC schemes with static and dynamic terminal policies in satellite relative state control for an initial ‘error’ state $[-0.30 \ 0.60 \ 0.00 \ 0.22 \ 0.48 \ 0.00]^T$. (a) Time variation of the r_x component, and (b) Time variation of the r_y component.

monotonically with time t , the discrete-time counterparts of the continuous-time system matrices can be obtained for each time step t by employing the closed form solution in (7.11) and/or numerical integration. Obtaining a polytopic model for the time-varying discrete-time system matrices from a series of system matrices numerically computed for all representative values of θ is not impossible, but it will likely result in a very complicated model. So, we look for an approximate discrete-time model in which the system matrices are expressed in a closed form in terms of some time-varying parameters. Indeed, since the time variation of the system matrices is rather slow, we can consider a rather small sampling time and assume θ to remain constant during the time interval so that the discrete-time system matrices can be approximated in terms of θ or any parameter that depends on θ .

From (7.8), we note that all the terms in the matrix $\check{A}$ can be expressed in terms of the time-varying orbital velocity ω . Since $1 + e \cos \theta = \omega/k_0$, we express $e \cos \theta$ as $\sqrt{\omega}/k_0 - 1$ and $e \sin \theta$ as either $\sqrt{1 - (\sqrt{\omega}/k_0 - 1)^2}$ or $-\sqrt{1 - (\sqrt{\omega}/k_0 - 1)^2}$. Considering a sampling

Figure 7.6: Schematic of a 3-vertex polytope that bounds a point (ω, ω^2) where ω lies between ω_1 and ω_2 . Based on this, system matrices with elements which are second order polynomials of ω can be bounded within a polytope defined by three vertex matrices.

time of T_s seconds, and assuming that the orbital velocity $\omega(t)$ remains constant during the time step t of T_s seconds, the system matrices can be defined as

$$A(t) = e^{\check{A}(\omega(t)) T_s}, \quad B_u(t) = \int_0^{\sigma T_s} e^{\check{A}(\omega(t)) (T_s - \tau)} d\tau$$

where we assume that the control input is applied for a fraction σ of the sampling time T_s . We express the matrices $A(t)$ and $B_u(t)$ approximately in terms of the variable $\omega(t)$ by expanding $e^{\check{A}(\omega(t)) T_s}$ to a finite number of terms. Further, since $\omega(t)$ is small (of the order 10^{-3} or smaller), for a small T_s (up to a few tens of seconds), we can approximate each term in the resulting matrices $A(t)$ and $B_u(t)$ by its Taylor series expansion, up to a small order, about the mean value of the variable $\omega(t)$, i.e., $\omega_0 = k_0^2$. Assuming small values of the eccentricity e (up to, say, 0.1 or 0.2), we consider approximate discrete-time models employing up to the first or the second order approximation.

- i. **Approximate discrete-time model 1 (ADTM-1):** We first discuss the approximate discrete-time model in which we expand each term of $A(t)$ and $B_u(t)$ about ω_0 up to the second power of $\omega(t)$. We call this model ADTM-1. As the true anomaly increases with time t , $\omega(t)$ varies between $\omega_1 = (1-e)\bar{\omega}$ and $\omega_2 = (1+e)\bar{\omega}$ and $(\omega(t))^2$ varies between ω_1^2 and ω_2^2 . Since $\omega(t)$ is always positive, we can observe from Fig. 7.6 that any $(\omega(t), (\omega(t))^2)$ with $\omega(t)$ lying in the interval $[\omega_1, \omega_2]$ can be bounded in a polytope described by three vertices, viz., (ω_1, ω_1^2) , (ω_2, ω_2^2) and $(\frac{\omega_1 + \omega_2}{2}, \omega_1 \omega_2)$. Therefore, clearly, the system matrices $A(t)$ and $B_u(t)$ can be assumed to lie within a polytope defined by three vertices during each half orbit when the sign of $\sin \theta$ remains unchanged. Together, we can form the polytopic model of the system matrices for all times with six vertices.
- ii. **Approximate discrete-time model 2 (ADTM-2)** Further approximation can

be done for a small e by neglecting the terms that contain the product of $e \sin \theta$ or $e \cos \theta$ with $k_0 \omega^{\frac{3}{2}}$ in the expression for $\check{A}$ in (7.8), and expanding the terms in $A(t)$ and $B_u(t)$ up to the first power of $\omega(t)$. We refer to the resulting approximate model as ADTM-2. Clearly, the polytopic model in this case is much simpler with only two vertices that correspond to the values ω_1 and ω_2 of $\omega(t)$.

Finally, for the modeling of the disturbance, we follow the simulation-based approach mentioned in the last subsection. For each state component, we determine the bounds of the disturbance component by comparing the per-step state transitions resulting from the LTV model with the corresponding transitions obtained with a high-fidelity nonlinear numerical integration. Accordingly, we express the disturbance term as $B_w w(t)$ where the disturbance $w(t)$ is assumed to be componentwise bounded by 1.

7.3.2.2 Computation of the dynamic policy

We can compute the controller matrices for the polytopic system model in the context of both the nominal and the min-max RHC schemes. Together with the controller matrices, we also need to compute the feasible invariant sets. Unlike in the case of time-invariant system matrices, in the polytopic case, computing the actual polyhedral invariant set can be difficult because of the higher dimension of the system. So we compute appropriate inner approximations of the polyhedral sets by employing the approximation technique discussed in Appendix A. Furthermore, in the case of ADTM-1, depending on the size of the ‘error’ states envisaged, we can consider the two half orbits of the reference point ($\theta \in [0, \pi]$ and $\theta \in (\pi, 2\pi)$) separately. This allows us to reduce the conservativeness associated with the polytopic bounding of the system matrices and also to simplify the computation of the polyhedral feasible invariant sets as it limits the number of vertices of the polytopic system to 3. However, care should be taken while dealing with the regulation of larger ‘error’ states so that the problem remains feasible when the reference point moves from one half-orbit to the other.

As mentioned in the last subsection, the dynamic policy can be used either as the main policy or as the terminal policy. Furthermore, when the dynamic policy with the disturbance feedforward term is used as the terminal policy, the feedforward gains in the finite horizon can be obtained from the off-line-optimized controller matrices in order to simplify the on-line computations. However, since the controller matrices are polytopic, an appropriate convex combination of the vertices of the controller matrices $G(t)$ and $F(t)$ has to be selected for each time step t . Indeed, the convex combination should be the same as that of the vertices of system matrices. In particular, in the case of ADTM-1, if V_1 , V_2 and V_3 are the vertex matrices corresponding to the vertices (w_1, w_1^2) , (w_2, w_2^2) , $(\frac{w_1+w_2}{2}, w_1 w_2)$ in Fig.7.6, the proper combination at time t is given by $\lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3$ where $\lambda_1 = (w_2 - w(t))^2 / (w_2 - w_1)^2$, $\lambda_2 = (w(t) - w_1)^2 / (w_2 - w_1)^2$ and $\lambda_3 = 2(w_2 - w(t))(w(t) - w_1) / (w_2 - w_1)^2$. Similarly, in the case of ADTM-2, if V_1 and V_2 are the vertex

Figure 7.7: Upper bounds of disturbance components with increasing eccentricity.

matrices corresponding to ω_1 and ω_2 , the combination at time t is given by $\lambda_1 V_1 + \lambda_2 V_2$, where $\lambda_1 = (w_2 - w(t))/(w_2 - w_1)$ and $\lambda_2 = (w(t) - w_1)/(w_2 - w_1)$.

Illustration 7.2. We now consider an example with an elliptical reference orbit of semi-major axis $a_{ref} = 7500$ km, and inclination $i_{ref} = 25^\circ$. Assuming the constraints $-2 \text{ km} \leq r_x(t), r_z(t) \leq 2 \text{ km}$, $-4 \text{ km} \leq r_y(t) \leq 4 \text{ km}$, $-2 \text{ m/s} \leq v_x(t), v_z(t) \leq 2 \text{ m/s}$ and $-4 \text{ m/s} \leq v_y(t) \leq 4 \text{ m/s}$, componentwise upper bounds of the disturbance components are estimated through simulations as mentioned in Illustration 7.1. Fig. 7.7 shows the upper bounds of the disturbance components obtained for three different approximate discrete-time linear models obtained for $T_s = 30\text{s}$ and $\sigma = 1/3$ when the reference orbit eccentricity is increased from 0 to 0.1. The disturbance bounds for each state component obtained for ADTM-1 (solid lines) and ADTM-2 (dot-dashed lines) are compared with the bounds that will have to be considered if the time-invariant HCW model is used. The plots show that the approximate models ADTM-1 and ADTM-2 both are closer to the actual dynamics than the HCW model, and that ADTM-1 is more accurate than ADTM-2 for the dynamics in the x - y plane.

Next, we consider the case of $e = 0.05$ and, using the ADTM-1 model, determine the controller matrices and the invariant inner approximations of the corresponding feasible invariant sets. Here, we consider $Q = \text{diag}(10^{-6}I_2, 0.1I_2)$ and $R = 10^3I_2$, and choose K to be the unconstrained LQ optimal gain. We also consider an input constraint $\|u(t)\|_\infty \leq 2\text{mm/s}^2$ on the control acceleration.

Fig. 7.8 shows the projections of the inner approximations of the polyhedral feasible invariant sets $\bar{\mathcal{X}}$ and $\mathcal{X}^f$ on r_x - r_y , v_x - v_y and r_z - v_z subspaces. The sets $\bar{\mathcal{X}}$ and $\mathcal{X}^f$ correspond to the cases of no disturbance feedforward and full disturbance feedforward respectively. Fig. 7.9 shows the control of a relative ‘error’ state of a satellite with a nominal RHC scheme that uses the dynamic policy as the terminal policy appended to a standard finite-horizon control law of horizon $N = 5$. The curves represent the time variations of $r_x(t)$, $r_y(t)$ and $r_z(t)$ when the computed control is implemented at each time step in a realistic nonlinear propagation setting that considers the disturbance effects.

Figure 7.8: Projections, on r_x - r_y , v_x - v_y and r_z - v_z subspaces, of feasible invariant sets for a member satellite with an elliptical reference orbit ($e = 0.05$).

Figure 7.9: Control of the relative state of a satellite w.r.t. the FRP in an elliptical reference orbit: time variation of the position components.

7.4 SFF with Dynamic-Policy-Based RHC

An SFF application requires an active control of the relative states of the member satellites during several phases of the flying process. In a typical multi-mission SFF application, a group of satellites are to be configured into a definite stable formation pattern for one mission and then reconfigured to another after its completion. Active control is required for the initialization or configuration of the satellites into a definite pattern from a random initial pattern or some other stable formation pattern. Furthermore, even though the formation pattern attained for each mission is supported by the nominal linearized relative dynamics, control efforts in some small scale may be required during the mission for keeping the desired formation pattern intact. In this section, we explore the use of the dynamic-policy-based RHC schemes in the control of relative states of member satellites in the context of formation initialization / reconfiguration and formation keeping.

The study in this section is limited to the issue of control computations only. In our simulation studies, we make certain assumptions about some other aspects that need to be considered while developing the overall control strategy.

First, we assume that the relative state of each satellite w.r.t. the assumed reference point is available to the controller in the satellite in each time step t . In an actual application, the reference point may be a real satellite or may even be a mathematically propagated point which can be considered as a virtual satellite. In the latter case, the relative state of any satellite may not be directly measured. However, we assume that each satellite is equipped with the necessary measurement, navigation and/or communication devices so that it can determine its relative state w.r.t. the reference point. In a typical setup, neighbouring member satellites may use the onboard Global Positioning System (GPS) receivers together to form carrier-phase differential GPS measurements and employ appropriate filtering algorithms (see, e.g., [192]) to determine accurate estimates of the satellite states relative to the reference point in real-time. Even though sensing errors do exist even with the use of the most accurate navigation devices and filters, we do not consider sensing errors explicitly in control computations in the simulation studies of this section.

Second, we assume that the application of the computed control action in each satellite is perfect. In a real setup, however, control action is usually implemented with thrusters effecting velocity changes (ΔV 's) and errors in the implementation of ΔV 's can be a real concern. In practice, the ΔV implementation may be improved by the use of feedback from an accelerometer. Furthermore, the control algorithm can also be made robust to actuator uncertainties. However, we do not explicitly consider these issues in the simulation studies in this section.

7.4.1 Formation initialization / reconfiguration with dynamic-policy-based RHC

While setting up a formation flying mission, member satellites should be properly configured to a particular stable formation pattern. The control objective is to ultimately achieve the desired formation pattern about a point in the reference orbit. Since the reference point may not be fixed *a-priori* but can rather be chosen by the member satellites at the time of configuration, the issue of overall optimality naturally arises in the control problem.

We explore the problem in the context of slightly eccentric to circular reference orbits. The discrete-time relative dynamics of each member satellite in the case of circular and slightly eccentric reference orbits (HCW and ADTM-2 models) satisfy assumption A6.5 of Chapter 6 with $S_j = S = [0 \ 1 \ 0 \ 0 \ 0 \ 0]^T$. Therefore, we consider the problem as a consensus-related coordinated control problem and apply the schemes discussed in Chapter 6 for the solution of the initialization /reconfiguration problem. In the following illustrations, we discuss the details of two formation initialization/reconfiguration problems.

Illustration 7.3. In this example, we consider the initialization of satellites to an in-orbit formation in a slightly eccentric reference orbit. We consider $\mathcal{N} = 6$ identical satellites

Figure 7.10: Communication structures considered for a 6 member satellite formation.

randomly spread around a virtual reference point in a reference orbit of semi-major axis $a_{ref} = 7500$ km, inclination $i_{ref} = 25^\circ$ eccentricity $e = 0.02$. The control objective is to align the satellites along the reference orbit at a spacing of 100 meters in such a way that a suitably chosen overall cost function is minimized. Clearly, the member satellites need to negotiate for the point about which they should align.

We consider a quadratic cost with $Q = \text{diag}(10^{-6}I_2, 0.1I_2)$ and $R = 10^3I_2$, and choose K to be the unconstrained LQ optimal gain. Assuming the constraints $-2 \text{ km} \leq r_x(t), r_z(t) \leq 2 \text{ km}$, $-4 \text{ km} \leq r_y(t) \leq 4 \text{ km}$, $-2 \text{ m/s} \leq v_x(t), v_z(t) \leq 2 \text{ m/s}$, $-4 \text{ m/s} \leq v_y(t) \leq 4 \text{ m/s}$ and $\|u(t)\|_\infty \leq 2 \text{ mm/s}^2$, the controller matrices as well as the feasible invariant set are determined for a min-max RHC scheme with $\gamma = 0.5$ using the dynamical model ADTM-2 for $T_s = 30\text{s}$ and $\sigma = 1/3$.

We apply the control scheme discussed in Section 6.6.2 with a control horizon $N = 5$ for an H_∞ -based near-consensus. The initial satellite relative states considered for the simulation are shown in Table 7.3, and the inter-satellite communication structure is shown in Fig. 7.10 (left). The final desired positions of the satellites relative to the optimal reference point to be computed in a distributed way are fixed *a-priori*. The consensus point is determined at time $t = 0$ and updated at intervals of 10 time steps. We consider $\eta = 5$ for the averaging step employed in the distributed optimization of the consensus point.

In the simulation, we employ model ADTM-2 for each satellite in the computation of the control inputs. We transform the control input components computed at each time step to the ECI coordinate frame and employ high-fidelity nonlinear integration including the perturbations and disturbances for the propagation of each satellite. Fig. 7.11a shows the successive consensus points obtained during the control of the member satellites as per the distributed RHC scheme employed. The actual trajectories of the member satellites are shown in Fig. 7.11b. The satellites get aligned along the reference orbit on both sides of the point at which $r_y = \varsigma_{fin} = 0.48\text{km}$. From the simulation results, we note that the overall quadratic cost of initialization and the total ΔV 's required for the alignment about $r_y = \varsigma_{fin}$

Table 7.3: Initial subsystem states in Illustration 7.3

Sat.	r_x	r_y	r_z	v_x	v_y	v_z
Sat. 1	-0.39	-1.18	-0.70	0.64	0.40	-0.12
Sat. 2	0.14	-0.97	-0.80	-0.96	-0.36	0.64
Sat. 3	-0.78	0.39	0.46	0.08	1.20	0.40
Sat. 4	0.05	0.75	0.38	0.24	0.08	-0.40
Sat. 5	0.06	1.47	0.90	0.35	0.11	-0.40
Sat. 6	0.16	0.37	-0.38	-0.40	-0.28	0.40

Figure 7.11: Initialization of 6 member satellites to a linear in-orbit formation along an elliptical reference orbit with a distributed RHC scheme: (a) Successive values of the optimal reference point obtained every 10 time steps; (b) Actual trajectories that the satellites follow during initialization.

are about 9% and 12% lower than the corresponding cost and ΔV 's that would be required for an alignment about the original reference point.

Illustration 7.4. In this illustration, we consider a satellite formation reconfiguration problem in which a group of six satellites are required to move from one stable formation to another. The reference orbit considered is near-circular with semi-major axis $a_{ref} = 7500\text{km}$, eccentricity $e_{ref} = 0.00005$ and inclination, $i_{ref} = 25^\circ$. Originally, four of the satellites are assumed to be flying in a circular formation of radius 200m (cf. Table 7.1) and the remaining two moving along the reference orbit at distances -300m and -500m from the center of the circular formation (which is the existing formation reference point). The control problem is to configure the satellites in a projected circular formation of radius $\beta = 1\text{km}$. We assume that the satellites in the circular formation are initially at phases $j\pi - \pi/4$, $j = 1, \dots, 4$ (cf. Table 7.1).

For the desired projected circular formation, the desired state components of the member satellites, labeled $j = 1, 2, \dots, \mathcal{N}$, $\mathcal{N} = 6$, will be given by³

$$\begin{aligned} (r_x)_j(t) &= \frac{1}{2}\beta \sin(\bar{\omega}tT_s + 2\pi j/\mathcal{N} + \phi), & (v_x)_j(t) &= \frac{1}{2}\bar{\omega}\beta \cos(\bar{\omega}tT_s + 2\pi j/\mathcal{N} + \phi) \\ (r_y)_j(t) &= \delta + \beta \cos(\bar{\omega}tT_s + 2\pi j/\mathcal{N} + \phi), & (v_y)_j(t) &= -\bar{\omega}\beta \sin(\bar{\omega}tT_s + 2\pi j/\mathcal{N} + \phi) \\ (r_z)_j(t) &= \beta \cos(\bar{\omega}tT_s + 2\pi j/\mathcal{N} + \phi), & (v_z)_j(t) &= \bar{\omega}\beta \sin(\bar{\omega}tT_s + 2\pi j/\mathcal{N} + \phi) \end{aligned} \quad (7.18)$$

where δ is the offset of the center of the desired formation from the existing formation reference point and $\phi \in [0, 2\pi)$ defines the initial phase or orientation of the formation pattern. Unlike the problem in Illustration 7.3, the final desired states of the member satellites in this problem are not equilibrium points and they are time-varying. However, the dynamics of the ‘error’ states remain the same, and, when the initial phase ϕ is chosen *a-priori*, the problem is similar to the one in Illustration 7.3 with δ as the consensus variable. Therefore, determining δ as the consensus variable for a fixed ϕ is straightforward. However, from the optimality point of view, it is desirable to determine ϕ in real time along with δ .

Allowing ϕ to be an optimization variable turns the optimization problem into a nonlinear one. Employing dual decomposition, the problem can be transformed into a distributed

³Note that the variable t is used here, as elsewhere in the thesis except in Section 7.2, in the discrete-time sense, and tT_s , where $t \in \mathbb{Z}_+$, represents the actual time in seconds.

Figure 7.12: Reconfiguration of satellites to a 6 member projected circular formation: satellite trajectories and the final relative orbit.

problem with nonlinear subproblems and a concave master-problem. However, instead of solving the nonlinear problem, we consider a scheme in which the member satellites solve the consensus problem with only δ as the coupled variable for a set of fixed values of $\phi \in [0, 2\pi)$. Since the subproblems are simple convex problems in this case, the subproblems for various values of ϕ are solved in parallel in each member and, once the consensus variables converge, costs for various ϕ 's are computed in each member by employing the distributed averaging of the member costs. The value of ϕ and the corresponding δ leading to the minimum cost are then selected by the members. Since the scheme employs the selection of ϕ from among a number of fixed values only, the resulting solution is suboptimal. The solution will be close to optimal if the values of ϕ are considered at fairly close intervals.

As before, we consider a quadratic cost with $Q = \text{diag}(10^{-6}I_2, 0.1I_2)$ and $R = 10^3I_2$, and choose K to be the unconstrained LQ optimal gain. While modeling the disturbance bounds and determining the controller matrices, we consider that, for each member satellite, the components of the 'error' states satisfy $-1 \text{ km} \leq r_x(t) \leq 1 \text{ km}$, $-2 \text{ km} \leq r_y(t), r_z(t) \leq 2 \text{ km}$, $-1 \text{ m/s} \leq v_x(t) \leq 1 \text{ m/s}$, $-2 \text{ m/s} \leq v_y(t), v_z(t) \leq 2 \text{ m/s}$. In the control problem, these constraints are interpreted as being relative to the desired positions and velocities. We also consider absolute constraints on the input components: $-1 \text{ mm/s}^2 \leq u_x(t), u_y(t) \leq 1 \text{ mm/s}^2$ and $-2 \text{ mm/s}^2 \leq u_z(t) \leq 2 \text{ mm/s}^2$. The controller matrices as well as the feasible invariant set are determined for a nominal RHC scheme without disturbance feedforward employing the modified HCW model for $T_s = 60 \text{ s}$ and $\sigma = 1/6$.

We apply the set-based consensus algorithm with a nominal control scheme of horizon $N = 5$ for $\phi = 15^\circ, 45^\circ, \dots, 345^\circ$ at 30° intervals. The lowest cost is obtained for $\phi = 135^\circ$ and $\delta = 113 \text{ m}$. In the distributed computation of δ , for each value of ϕ , the inter-satellite communication structure of Fig. 7.10 (right) is considered and $\eta = 5$ is used. The coupling quantities ϕ and δ are computed only once in the beginning.

The simulation of the actual control phase is carried out in a realistic way as mentioned in the last illustration. Fig. 7.12 shows the actual trajectories of the member satellites as

they get reconfigured to the projected circular formation pattern. The overall quadratic cost and the total ΔV 's associated with the reconfiguration of the satellites to the optimized projected circular pattern were compared with the corresponding total cost and ΔV 's that would be required for a reconfiguration to a similar pattern centered at the original reference point and with a phase ϕ that is 180° apart from the optimal one. The costs were found to be about 80% and 24% lower in the former case when compared with the latter one.

7.4.2 Formation keeping with dynamic-policy-based RHC

SFF missions usually consider a stable formation pattern that is supported for a long period of time by the natural dynamics of the satellites. However, once the member satellites are initialized into a desired formation pattern, because of unmodeled disturbances, including differential J_2 and possible imperfections in initialization, formation members gradually drift away from their desired states, thus necessitating the use of control for formation keeping. Typically, the basic objective of the control mechanism designed for formation keeping is to minimize the overall fuel consumption.

For formation keeping control purposes, we consider a non-hierarchical control architecture based on the VS approach. This approach offers performance advantages (optimality etc.) of MIMO along with the robustness of L/F architectures [166]. We treat the formation pattern as a VS in which its members are assigned periodically varying positions with respect to a reference point. At each time step, every satellite obtains its relative state by some measurements and then tracks its desired state based on its state in the VS.

We consider two variants of VS: a) normal VS, and b) iterated VS. In the former, the reference point is propagated considering both nonlinearity and major disturbances including J_2 whereas, in the latter, an optimal reference point that gives the best match of the current measured states with the desired states is calculated so as to counter the effects of common disturbances and hence to reduce the overall fuel consumption. Here, we consider a weighted least squares solution to determine the optimal point (see, e.g., [4]). For a formation of $\mathcal{N}$ satellites, defining the formation state error $\tilde{\mathbf{x}} = [\tilde{x}_1^T \ \tilde{x}_2^T \ \dots \ \tilde{x}_{\mathcal{N}}^T]^T$ where $\tilde{x}_j = x_j - x_{jD}$ with x_j as the actual and x_{jD} as the desired state of the j th satellite, the formation state error after reference point update will be $\tilde{\mathbf{x}} - T x_0$ where x_0 is the reference state to be determined and $T = [I_{n_x} \ I_{n_x} \ \dots \ I_{n_x}]^T$. Considering $\mathbf{Q}^o = \text{diag}(Q_1^o, Q_2^o, \dots, Q_{\mathcal{N}}^o)$, the least squares solution that will minimize $\|\mathbf{Q}^o(\tilde{\mathbf{x}} - T x_0)\|^2$ will be given by $x_0 = (T^T \mathbf{Q}^o T)^{-1} T^T \mathbf{Q}^o \tilde{\mathbf{x}}$. The solution can be easily obtained with distributed computations by employing distributed weighted averaging when the matrices Q_j^o 's are diagonal matrices.

The preciseness with which the satellites have to track their desired positions depends on the application considered, and in many cases, small deviations from the desired positions may be tolerable. Assuming that state errors within certain bounds are allowed,

Table 7.4: Initial Hill states of member satellites in Illustration 7.5

Sat.	r_x (m)	r_y (m)	r_z (m)	$\dot{x}$ (m/s)	$\dot{y}$ (m/s)	$\dot{z}$ (m/s)
Sat.1	0.00	977.50	0.00	0.53	0.00	0.00
Sat.2	433.01	-425.00	0.00	-0.24	-0.91	0.00
Sat.3	-433.01	-425.00	0.00	-0.24	0.91	0.00

the local controller on each satellite may operate only at certain times so that the state error is kept within the chosen bound. It is desirable to choose a bound on the minimum objective value J_{min} as a criterion to turn on and turn off the controller. Once J_{min} exceeds a suitably chosen upper bound on the objective function, J_u , the controller is turned on and the control action is applied. The control action is stopped once its value goes below a suitably chosen lower bound J_l .

Illustration 7.5. A formation of three satellites on a reference orbit of semi-major axis $a = 7500$ km, eccentricity $e = 0.05$ and inclination $i = 25^\circ$ is considered. Following condition (7.16), the three satellites are initialized into almost elliptical passive orbits relative to the formation reference point. The relative orbits are chosen on the x - y plane and the satellites are placed at 120° separation forming a cartwheel formation (cf. Table 7.1). The initial relative states of the satellites are shown in Table 7.4.

The member satellites are propagated in the inertial coordinate frame using the high-fidelity nonlinear orbit propagator as mentioned in the previous illustrations. A time step of $T_s = 30$ s is considered for all measurements and control calculations. At each time step, the relative state of each satellite expressed in the Hill's frame is compared with its desired relative state determined from the initial condition using (7.11). The relative state error is then used in the RHC algorithm that uses an off-line-optimized dynamic policy without disturbance feedforward as the terminal policy appended to a standard RHC law of horizon length $N = 5$. The approximate dynamical model ADTM-2 is considered for the terminal dynamic policy. While modeling the disturbance bounds for the computation of the dynamic policy, the 'error' states are assumed to satisfy $25 \text{ m} \leq r_x(t), r_z(t) \leq 25 \text{ m}$, $-40 \text{ m} \leq r_y(t), r_z(t) \leq 40 \text{ m}$, $-25 \text{ mm/s} \leq v_x(t), v_z(t) \leq 25 \text{ mm/s}$, $-40 \text{ mm/s} \leq v_y(t) \leq 40 \text{ mm/s}$. Moreover, a control actuation time equal to $\sigma = 1/3$ of the sampling time T_s is considered, and componentwise input constraints $|u_x|, |u_y|, |u_z| \leq 0.2 \text{ mm/s}^2$ on control acceleration are applied while computing the controller matrices and the feasible invariant set. The gain K is chosen to be unconstrained LQ optimal for a quadratic cost with $Q = \text{diag}(10^{-6}I_2, 0.1I_2)$ and $R = 10^3I_2$.

Simulations are carried out for 10 complete orbits in the two cases: (a) normal VS, and (b) iterated VS. The controller is switched on and off depending on the value of the minimum cost J_{min} , the upper and lower thresholds for which, J_u and J_l , have been chosen such that position error of each member in each direction is limited within about ± 25 m. Fig. 7.13 shows relative position errors of all three satellites for both the cases. The deviations are generally periodic because of the periodic nature of differential J_2 , which is the major disturbing force. The effect of reference point redetermination can be seen in the plots on the right where position deviations are reduced by the update of the reference point.

Fig. 7.14 presents the corresponding use of control input for keeping the state errors of member satellites within bounds in both cases. The plots show the equivalent impulsive

Figure 7.13: Position errors of member satellites during formation keeping. (a) Case 1: Normal VS (left) and (b) Case 2: Iterated VS (right).

velocity changes ΔV 's applied on each satellite for formation keeping. The fuel requirement is found to be about 21% lower in the second case in which optimal reference points based on measured relative states are used in control calculations.

7.5 Conclusion

In this chapter, we have studied the application of the dynamic-policy-based RHC schemes in control problems that arise in SFF. We have briefly introduced the SFF application and discussed the issues related to the control of relative translational motion of the member satellites flying in a close formation. The details of the relative dynamics of satellites have been presented and the discrete-time versions of the linearized dynamics have been discussed. In the case of an elliptical reference orbit where the relative dynamics are time-varying, approximate polytopic models of the discrete-time dynamics have been presented. These models have been used in computing the dynamic control policy in the context of both the nominal and the min-max RHC approaches. RHC algorithms using the dynamic policy as the main or the terminal control policy have been used in problems associated with formation initialization, keeping and reconfiguration. Realistic simulations demonstrate the effective performance of the control algorithms.

Figure 7.14: Control actions on member satellites during formation keeping. (a) Case 1: Normal VS (left) and (b) Case 2: Iterated VS (right). The total ΔV 's required per orbit for Sat. 1, Sat. 2 and Sat. 3 in mm/s are 189, 203 and 186 in Case 1, and 134, 190 and 132 in Case 2.

The simulation studies in this chapter are more focussed on algorithms for control computation, and assumptions have been made on other important aspects of the overall control strategy such as state estimation and control actuation. A practical solution to the overall formation flying control problem will likely need a holistic approach in which all the aspects are treated together. Certainly, the proposed control algorithms can be adapted to particular requirements arising from the specific choice of filtering and actuation mechanisms considered in an overall control scheme.

Chapter 8

Conclusions

In this thesis, we have explored a class of computationally efficient RHC algorithms that are aimed at meeting the control requirements in existing and emerging applications which demand a guaranteed control performance under uncertain and constrained conditions. RHC has been a popular control technique in the process industries for several years and is now being increasingly applied in diverse areas ranging from biomedical to aerospace applications. The technique owes its success to the fact that it employs a real-time optimization of the control inputs allowing regular updates and corrections based on the current condition of the controlled system. However, since the control computations are based on the predicted plant behaviour over a finite time horizon, uncertainties in the plant description complicate the on-line optimization problem, especially when the system is constrained, and often result in the actual plant behaviour deviating considerably from the predicted (desired) one. Keeping the deviations within reasonable bounds and ensuring a desired control performance usually require substantial increase in on-line computations, thus leading to a fundamental tradeoff issue – that between performance and applicability, and on-line computational complexity. The RHC approach developed in this thesis is an attempt to address this tradeoff issue in the context of a class of LTV systems operating under constraints in computational and control resources. In the following, we give a summary of the results presented in this thesis and also discuss some directions for further investigations in the area.

8.1 Summary of Contributions

In Chapter 2, we have presented the RHC approach in a general nonlinear setting for constrained systems with uncertainties and discussed the need to include feedback in control predictions in order to ensure that the resulting control action is realistic and that the actual behaviour of the controlled system matches reasonably with the predicted one. We have also broadly discussed the conditions that guarantee the robust stability of the controlled system and briefly noted various suggestions on the solution of the closed-loop

RHC problem for specific classes of systems. In the subsequent chapters, we have focussed on a class of LTV systems with a polytopic system description and with (possibly unmeasurable) bounded additive disturbances. Our solution to the RHC problem for this class of systems is based on the use of a dynamic policy which has been proposed and analyzed in Chapter 3. The proposed dynamic policy employs time-varying controller dynamics with controller matrices that need not be explicitly determined but only assumed to follow the same convex combination as the plant matrices, and with a disturbance feed-forward term that does not require the disturbance to be really measured. This policy essentially results in dynamically evolving perturbations to the constant optimal static feedback input, where the perturbations incorporate the ‘uncertain’ system information into their dynamics, allowing a less conservative prediction of feasible control inputs.

In Chapter 3, we have also discussed the procedure for the synthesis of the dynamic policy for a nominal RHC scheme. Since we are more interested in simplifying the complexity of on-line computations, we have presented an approach in which the overall control optimization problem is split into two parts – one to be solved off-line and the other on-line. While the off-line part involves the optimization of the dynamics of the control policy, essentially to maximize the size of the feasible invariant set, the on-line part involves the optimization of the initial state of the controller in order to obtain the optimal control inputs minimizing a suitable quadratic cost function. We have also presented results on the stability of the controlled system under the nominal RHC scheme based on the proposed policy.

We have discussed the synthesis of the dynamic policy for an H_∞ -like min-max RHC scheme in Chapter 4. In this case, the off-line optimization problem is a BMI problem and we have shown that the special structure of the problem allows us to solve it using the alternating SDP approach. While optimizing the controller matrices off-line, the gain parameter can be adjusted so as to obtain a tradeoff between the size of the feasible invariant set associated with the controller matrices and the steady-state control performance. The on-line optimization problem is greatly simplified as in the case of the nominal scheme. The overall min-max RHC scheme allows us to ensure ℓ_2 stability of the controlled system.

In the context of both the nominal and min-max RHC schemes considered in Chapters 3 and 4, we have presented additional results for a class of polytopic (but certain) systems with (possibly unmeasurable) external disturbances. It has been observed that the proposed dynamic policy, when combined with a finite-horizon disturbance-based input policy, forms an effective RHC scheme for this class of systems. Compared with the conventional fixed optimal static feedback, the proposed policy as a terminal control law offers an enlarged feasibility domain, thus allowing a significant reduction in the horizon length N and the on-line computations. In both the cases, the corresponding stability results have been established.

In Chapter 5, we have extended the results on the dynamic-policy-based RHC schemes to the output feedback case. The approach is based on the use of a Luenberger observer

to estimate the state of the system from a measured output. The dynamic control policy is based on the estimated state instead of the actual state which is unknown, and its dynamics include the feedforward of the output error information. The overall control optimization problem is treated in two parts in a way similar to that in the case of the standard state feedback RHC schemes. We have considered the problem in the context of both the nominal and min-max-cost-based approaches.

In Chapter 6, we have discussed the use of dynamic-policy-based RHC schemes for a class of coordinated control problems where computational efficiency of the control algorithms is highly desirable. The distributed solution of an aggregate control problem involving a number of subsystems coupled by a shared objective usually requires a repeated solution of the individual sub-problems in the subsystems. The computational advantages of the dynamic-policy-based algorithms lead to an efficient solution of the subproblems and hence simplify the solution of the overall control problem. In the chapter, we have presented overall control schemes that employ a combination of a dynamic-policy-based RHC technique and an approximate subgradient method to ensure that the subsystems reach a suitably defined set-based or H_∞ -based near-consensus condition.

Finally, in Chapter 7, we have discussed the application of the proposed RHC algorithms in control problems arising in satellite formation flying. Problems related to the control of relative translational motion of the member satellites in a formation have been studied and the performances of the proposed RHC algorithms in the context of objectives such as optimal formation reconfiguration or keeping have been assessed with realistic nonlinear simulations.

8.2 Directions for Further Research

As we have mentioned in Chapter 2, the development of an RHC algorithm relies on computational and performance considerations which determine the way various aspects of the algorithm such as the nature of the cost function and the horizon length are chosen. The dynamic-policy-based solution to the closed-loop RHC problem that we have presented in this thesis employs certain assumptions on the form of the policy so that the control optimization problem becomes computationally viable. Furthermore, we allow a part of the control optimization problem to be carried out off-line, essentially to simplify the on-line computations. Predictably, such assumptions and simplifications do have some consequences on some other aspects of the control technique.

One of the limitations of the solution based on an off-line procedure is the resulting inflexibility. If, in a dynamic situation, there arises a need to modify one or several parameters of the control problem, such as the constraint levels or the weighting matrices in the cost function, it may not be possible to redetermine the controller matrices and the feasibility set for the modified parameters in real-time. Such cases are usually rare and can be practically handled by maintaining a set of off-line solutions for various values of

the parameters. However, it must be noted that, in such situations, the off-line solutions must be determined with much care considering all likely scenarios of the actual operation of the system. Another possible limitation of the proposed procedure is the difficulty in handling asymmetric state and control constraints in a non-conservative way. This may complicate the use of the method in certain tracking problems where some asymmetric state constraints are active. Finally, another important aspect of the solution employing a ‘partial off-line’ optimization is the issue of optimality. Clearly, the resulting control is suboptimal, since the on-line optimization of the control action is based on a policy whose dynamics are determined off-line with the objective of maximizing the feasible invariant set. Therefore, conceptually, such a solution may raise a concern on whether the basic philosophy of the RHC approach is being undermined. However, computational issues are usually of serious concern. Without the off-line procedure, the overall on-line optimization problem will be a nonlinear problem with several LMI and BMI constraints and solving such a problem will be a challenge particularly for systems with faster dynamics. Furthermore, since an ‘all on-line’ scheme will need to use ellipsoidal invariant sets instead of the polyhedral sets, it may actually lead to little improvement in performance when compared with the ‘partial off-line’ scheme. The latter scheme allows us to use the polyhedral feasible sets – computed off-line, possibly using some conservative approximation technique especially in the case of higher-dimensional polytopic systems – during the on-line cost minimization, resulting in reduced conservativeness and simpler on-line computations. Moreover, in the case of time-varying but known systems, further improvement in the control performance can be achieved with the use of the off-line-optimized dynamic policy as the terminal control policy appended to a standard RHC law of a short horizon.

The above mentioned limitations of the RHC schemes explored in this thesis mostly result from computational considerations. We can expect that these limitations will be gradually addressed with the development of more and more efficient computational procedures accompanied by further research on the RHC technique in the years to come. Furthermore, the RHC approach presented in this thesis can be further explored in different directions. In the following, we discuss a few ideas for further investigation.

Our study in this thesis has been focussed on a class of LTV systems with a polytopic description and with componentwise bounded additive disturbances which we treat as unmeasurable. It would be interesting to explore similar dynamic policies for other classes of uncertain systems, particularly systems with feedback uncertainty. Although it might be possible to model such systems, albeit in a conservative way, as polytopic systems and apply the dynamic-policy-based RHC schemes developed in this thesis, defining the policy dynamics directly in terms of the original ‘uncertainty’ information may lead to a separate set of advantages and insights. In the case of systems with feedback uncertainty, the ideas underlying the gain-scheduling methods used for certain kinds of parameter-varying systems can be useful in defining and synthesizing appropriate dynamic policies (cf. Remark 3.1). Another interesting case for investigation is that of a more general form of time-varying systems with known and unknown parameters and measurable and un-

measurable disturbances. In our development of the dynamic controller for time-varying systems, we have treated the time-varying parameters to be uncertain and the disturbances to be unmeasurable, and have allowed only the future control inputs and not the current input to depend on the current values of the time-varying parameters and disturbances. While this is necessary in the case of uncertain parameters and unmeasurable disturbances, certain improvements can certainly be expected if the dynamic policy allowed separate treatment of the parameters and disturbances that would be known in real-time.

The approach to the min-max RHC problem that we have considered in Chapter 4 allows us to adjust the value of the ℓ_2 gain γ in order to achieve a trade-off between the size of the feasible invariant set and the steady state control performance. So the selection of the value of γ , from all allowable values, is dictated by the desired size of the feasible invariant set. Clearly, it would be interesting to develop a procedure that allows the controlled system to eventually satisfy the ℓ_2 stability condition with the smallest (practically) allowable value γ_m of γ (cf. Section 4.3.3). Such a scheme will likely require a set of off-line solutions for increasing values $\gamma_0 = \gamma_m, \gamma_1, \gamma_2, \dots$ of γ and the corresponding feasible invariant sets $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \dots$ of increasing sizes subsuming the previous ones. The on-line algorithm will simply require a successive switch to the controllers corresponding to smaller values of γ with a guarantee of feasibility after each switch.

The treatment of the output feedback case in Chapter 5 leaves some room for further investigation. Since we assume, in our algorithms, that the system matrices will be known in real-time, one obvious case for further research in this line is that of systems which remain uncertain throughout the control operation. Another consideration could be on the way the ‘uncertainty’ information is fed to both the finite-horizon part and the terminal part of the controller. Essentially, when we parameterize the future control inputs as functions of output errors at earlier time steps, the output errors are treated as independent uncertain or unmeasurable quantities bounded in appropriate sets, and this ensures that the control problem remains feasible in the future time instants for predetermined state estimation error sets. In reality, however, future output errors are based on the current state estimation error and the current and future disturbances and measurement noises, which are the actual sources of uncertainty. Parameterizing the future control inputs as functions of these quantities (i.e., the current state estimation error and the current and future disturbances and noises) instead of the output errors will likely reduce conservativeness of the resulting control policy. However, it will also create difficulties in guaranteeing recursive feasibility because, if we do not separately consider the state estimation error at each future time instant, it will not be possible to ensure the feasibility of the problem for some assumed state estimation set at a future time instant. Therefore, even though the predicted control inputs will be feasible for the actual state of the system, difficulties will arise since the actual state will continue to remain unknown to us. Any solution to this problem will likely require a scheme in which the state uncertainty set and the feasible inputs are jointly determined at all time steps except at the beginning when an assumption is made about the initial state uncertainty

set.

Our discussion in Chapter 6 on the RHC-based solution to consensus-related coordinated control problems is focussed on systems with dynamically decoupled subsystems which are connected by a shared control objective. A similar exploration of the RHC-based solution to consensus problems with mildly coupled subsystems might be of interest in some applications. The solution to the problem in this context will likely require the use of an overall feedback gain that ensures consensus in some optimal way when the system is unconstrained, and a separate dynamic policy attached to each subsystem for handling the applicable constraints. The overall control optimization problem will then require a distributed solution to ensure the satisfaction of constraints, much like the solution of some other distributed MPC problems, unless the feasibility set for each subsystem is determined separately, considering the worst-case closed-loop interactions from the neighbouring subsystems.

Appendix A

Computation of Robust Positively Invariant Sets for Linear Polytopic Systems

In this appendix, we briefly discuss the computational procedures used to determine robust positively invariant (RPI) sets, particularly maximal admissible sets (MASs) or their inner invariant approximations and minimal RPI sets or their outer invariant approximations, for linear polytopic systems. Such sets have been extensively used in the RHC schemes explored in this thesis.

The focus of our discussion in the sequel is the class of linear dynamical systems described by

$$x(t+1) = \Phi(t)x(t) + B_w(t)w(t) \quad (\text{A.1})$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state and $w(t) \in \mathbb{R}^{n_w}$ is an additive disturbance. The time-varying system matrices $\Phi(t)$ and $B_w(t)$ are known to satisfy $[\Phi(t) \ B_w(t)] \in \bar{\Omega} = \text{Co} \left\{ [\Phi^{(j)} \ B_w^{(j)}], j = 1, \dots, n_{\bar{\Omega}} \right\}$ and the disturbance $w(t)$ is assumed to be bounded in a convex and compact set $\mathcal{S}_w \in \mathbb{R}^{n_w}$.

Definition A.1. A set $\mathcal{S}_x$ is RPI for system (A.1) if

$$\text{Rea}_{(\bar{\Omega}, \mathcal{S}_w)}(\mathcal{S}_x) \subseteq \mathcal{S}_x \quad (\text{A.2})$$

where $\text{Rea}_{(\bar{\Omega}, \mathcal{S}_w)}(\mathcal{S}_x)$ is the one-step forward reachable set from $\mathcal{S}_x$, defined as

$$\text{Rea}_{(\bar{\Omega}, \mathcal{S}_w)}(\mathcal{S}_x) = \left\{ \Phi x + B_w w \mid x \in \mathcal{S}_x, [\Phi \ B_w] \in \bar{\Omega}, w \in \mathcal{S}_w \right\}. \quad (\text{A.3})$$

Recalling Definition 3.1, a convex set $\mathcal{S}_x$ is RPI for the polytopic system in (A.1) if

$$\Phi^{(j)}x + B_w^{(j)}w \in \mathcal{S}_x, \quad j = 1, \dots, n_{\bar{\Omega}}, \quad \forall x \in \mathcal{S}_x, \forall w \in \mathcal{S}_w. \quad (\text{A.4})$$

A sufficient condition for the existence of an RPI set for system (A.1) is the existence of a matrix $P \succ 0$ such that

$$P - \Phi^{(j)\top} P \Phi^{(j)} \succ 0, \quad j = 1, \dots, n_{\bar{\Omega}} \quad (\text{A.5})$$

which implies that the polytopic system (A.1) is quadratically stable. We are interested in the computation of ellipsoidal and polyhedral sets that are RPI for system (A.1). Since our basic objective is to compute RPI sets for system (A.1) in the context of the determination of either the MAS (which we also refer to as the largest feasible invariant set) or the minimal RPI set, we must also make sure that all system constraints are satisfied for all states within such sets. Further, since we deal with control strategies which are state-feedback-based, we assume that all system constraints (constraints on the state and the input) can be expressed in terms of the state constraint $x(t) \in \mathbb{X}$ where $\mathbb{X}$ is a compact polyhedral set defined as $\mathbb{X} = \{x \mid M_x x \leq \mathbf{1}\}$, where $M_x \in \mathbb{R}^{m_x \times n_x}$.

A.1 Ellipsoidal RPI Set

We first consider the computation of an ellipsoidal RPI set, which is represented as $\mathcal{E}_x = \{x \mid x^\top W^{-1} x \leq 1\}$ where W is a positive definite matrix. A sufficient condition for the invariance of an ellipsoidal set $\mathcal{E}_x$ for system (A.1) when $\mathcal{S}_w$ is a hypercube is given by Lemma 3.1, according to which, the set $\mathcal{E}_x$ is RPI in the presence of disturbances $w(t) \in \mathbb{R}^{n_w}$ satisfying $\|w(t)\|_\infty \leq 1$ if there exist positive constants $\alpha^{(j)}$ and non-negative matrices $R_w^{(j)} \in \mathbb{D}_+^{n_w}$, satisfying

$$\begin{bmatrix} W & \Phi^{(j)} W & B_w^{(j)} \\ * & \alpha^{(j)} W & 0 \\ * & * & R_w^{(j)} \end{bmatrix} \succeq 0, \quad \text{Tr}(R_w^{(j)}) \leq 1 - \alpha^{(j)}, \quad j = 1, \dots, n_{\bar{\Omega}}. \quad (\text{A.6})$$

In the following, we present the proof of this lemma.

Proof of Lemma 3.1

We proceed by proving the statement for a time-invariant system. An ellipsoidal set $\mathcal{E}_x = \{x \mid x^\top W^{-1} x \leq 1\}$ is positively invariant for a dynamical system $x(t+1) = \Phi x(t) + B_w w(t)$ with a stable Φ in the presence of disturbances $w(t) \in \mathbb{R}^{n_w}$ satisfying $\|w(t)\|_\infty \leq 1$ if and only if $(\Phi x(t) + B_w w(t))^\top W^{-1} (\Phi x(t) + B_w w(t)) \leq 1 \quad \forall (x(t), w(t))$ such that $x(t)^\top W^{-1} x(t) \leq 1$ and $w(t)_{[i]}^2 \leq 1, i = 1, \dots, n_w$. Applying the S -procedure [111, Sec. 2.6.3], this condition is satisfied if there exist $\alpha \geq 0$ and a non-negative diagonal matrix $R_w \in \mathbb{D}_+^{n_w}$ such that

$$\begin{aligned} 1 - (\Phi x(t) + B_w w(t))^\top W^{-1} (\Phi x(t) + B_w w(t)) - \alpha(1 - x(t)^\top W^{-1} x(t)) \\ - (\text{Tr}(R_w) - w(t)^\top R_w w(t)) \geq 0 \end{aligned} \quad (\text{A.7})$$

for all $(x(t), w(t))$. Clearly, condition (A.7) is satisfied for all $(x(t), w(t))$ if and only if the following matrix inequality holds

$$\begin{bmatrix} \alpha W^{-1} & 0 \\ 0 & R_w \end{bmatrix} - \begin{bmatrix} \Phi^T \\ B_w^T \end{bmatrix} W^{-1} [\Phi \ B_w] \succeq 0 \quad (\text{A.8})$$

for some $\alpha \geq 0$ and $R_w \in \mathbb{D}_+^{n_w}$ such that $1 - \alpha - \text{Tr}(R_w) \geq 0$. This condition, along with the fact that $W \succeq 0$, can be equivalently expressed as

$$\begin{bmatrix} \alpha W^{-1} & 0 & \Phi^T \\ * & R_w & B_w^T \\ * & * & W \end{bmatrix} \succeq 0, \quad R_w \in \mathbb{D}_+^{n_w}, \text{Tr}(R_w) \leq 1 - \alpha. \quad (\text{A.9})$$

Pre- and post-multiplying the matrix inequality in (A.9) by $\text{diag}([W, I_{n_w}, I_{n_x}])$ and rearranging its row-column pairs, we get the equivalent condition

$$\begin{bmatrix} W & \Phi W & B_w \\ * & \alpha W & 0 \\ * & * & R_w \end{bmatrix} \succeq 0, \quad R_w \in \mathbb{D}_+^{n_w}, \text{Tr}(R_w) \leq 1 - \alpha. \quad (\text{A.10})$$

Finally, since (A.10) is affine in α, Φ, B_w and R_w for a fixed W , the set $\mathcal{E}_x$ will be invariant for the system (A.1) if (A.10) is satisfied for all the vertices of the polytope $\bar{\Omega}$, hence the condition (A.6). $\blacksquare$

Note that the lines of reasoning used in the above proof can be used to derive the robust positive invariance condition when the disturbance set S_w has slightly different forms. In particular, when S_w is a unit ball, i.e., $S_w = \{w \mid \|w\| \leq 1\}$, the set $\mathcal{E}_x$ is RPI for (A.1) if there exist positive constants $\alpha^{(j)}, j \in 1, \dots, n_{\bar{\Omega}}$ satisfying

$$\begin{bmatrix} W & \Phi^{(j)} W & B_w^{(j)} \\ * & \alpha^{(j)} W & 0 \\ * & * & (1 - \alpha^{(j)}) I \end{bmatrix} \succeq 0, \quad j = 1, \dots, n_{\bar{\Omega}}. \quad (\text{A.11})$$

Moreover, when S_w is a more general polyhedral C -set¹ described as $S_w = \{w \mid M_w w \leq \mathbf{1}\} = \text{Co}\{w^{(1)}, w^{(2)}, \dots, w^{(\nu)}\}$, where $w^{(1)}, \dots, w^{(\nu)}$ are the vertices defining the polytopic set, the set $\mathcal{E}_x$ is RPI for (A.1) if there exist positive constants $\alpha^{(j)}, j \in 1, \dots, n_{\bar{\Omega}}$ and a positive definite matrix R_w satisfying

$$\left. \begin{aligned} & \begin{bmatrix} W & \Phi^{(j)} W & B_w^{(j)} \\ * & \alpha^{(j)} W & 0 \\ * & * & (1 - \alpha^{(j)}) R_w \end{bmatrix} \succeq 0, \quad j = 1, \dots, n_{\bar{\Omega}}, \\ & 1 - w^{(j)T} R_w w^{(j)} \geq 0, \quad j = 1, \dots, \nu. \end{aligned} \right\} \quad (\text{A.12})$$

---

¹A C -set is a convex and compact set with the origin in its interior.

Conditions (A.6), (A.11) and (A.12) are matrix inequality conditions which are LMIs when α^j 's are constant. As already mentioned in Remark 3.2, they are not necessary but only sufficient conditions for the invariance of the set $\mathcal{E}_x$ in the presence of disturbances of respective forms and therefore the resulting $\mathcal{E}_x$ is only a conservative approximation. However, they provide tractable conditions which can be practically employed in computing controller matrices and feasible invariant sets for RHC schemes as has been discussed in Chapters 3 and 4. Indeed, while computing the maximal (or the minimal) ellipsoidal RPI set, we can maximize the determinant of the matrix W (or W^{-1}) under the invariance constraint (A.6), (A.11) or (A.12) and a feasibility constraint

$$\begin{bmatrix} Z & M_x W \\ * & W \end{bmatrix} \succeq 0, \quad Z_{[j,j]} \leq 1, \quad j = 1, \dots, m_x \quad (\text{A.13})$$

which ensures that the states within the set $\mathcal{E}_x$ satisfy all the system constraints [111]. The optimization problem can be solved using any algorithm applicable for solving SDP problems (see, e.g., [124, 125]).

A.2 Polyhedral RPI Set

Ellipsoidal RPI sets are usually easy to compute since the set representation is simple and the computation can be carried out efficiently using LMI solvers. However, for the class of systems we consider, the maximal and the minimal invariant sets are more closely represented by polyhedral sets when the set $\mathcal{S}_w$ is a polyhedral C -set, and ellipsoidal invariant approximations of these sets are usually conservative. Moreover, the use of ellipsoidal invariant sets in the on-line control schemes often leads to an SDP problem rather than a QP or an LP problem which can be solved more easily. In the following, we discuss the computation of (an inner approximation of) the maximal and (an outer approximation of) the minimal polyhedral RPI sets for system (A.1) assuming that the sets $\mathbb{X}$ and $\mathcal{S}_w$ are polyhedral C -sets.

A.2.1 Maximal admissible set

For the construction of the MAS, which is the maximal-volume RPI set in which all constraints are satisfied, we use an iterative procedure based on the following geometric invariance condition.

Lemma A.1. A set $\mathcal{S}_x$ is RPI for system (A.1) if and only if

$$\mathcal{S}_x \subseteq \text{Pre}_{(\bar{\Omega}, \mathcal{S}_w)}(\mathcal{S}_x) \quad (\text{A.14})$$

where $\text{Pre}_{(\bar{\Omega}, \mathcal{S}_w)}(\mathcal{S}_x)$ is the pre-image set of $\mathcal{S}_x$ defined as

$$\text{Pre}_{(\bar{\Omega}, \mathcal{S}_w)}(\mathcal{S}_x) = \{x \mid \Phi x + B_w w \subseteq \mathcal{S}_x, \quad \forall [\Phi \ B_w] \in \bar{\Omega}, \forall w \in \mathcal{S}_w\}. \quad (\text{A.15})$$

Since the system is polytopic, the pre-image set can be computed using the definition

$$\text{Pre}_{(\bar{\Omega}, \mathcal{S}_w)}(\mathcal{S}_x) = \{x \mid \Phi^{(j)} x + B_w^{(j)} w \subseteq \mathcal{S}_x, j = 1, \dots, n_{\bar{\Omega}} \forall w \in \mathcal{S}_w\}. \quad (\text{A.16})$$

In order to determine the largest constraint-admissible set $\mathcal{X}$ satisfying the invariance condition mentioned in (A.14), we start from the set $\mathcal{X}^{(0)} = \mathbb{X}$ and repeatedly compute $\mathcal{X}^{(k+1)} = \mathcal{X}_{-1}^{(k)} \cap \mathbb{X}$ where $\mathcal{X}_{-1}^{(k)} = \text{Pre}_{(\bar{\Omega}, \mathcal{S}_w)}(\mathcal{X}^{(k)})$ for increasing values of $k = 0, 1, 2, \dots$ until we have $\mathcal{X}^{(k)} \subseteq \mathcal{X}^{(k+1)}$. The latter condition is the ideal stopping condition for the iterative procedure mentioned and if it is attained at a finite $k = k^*$, we set $\bar{\mathcal{X}} = \mathcal{X}^{(\infty)} = \mathcal{X}^{(k^*)}$, where $\bar{\mathcal{X}} = \mathcal{X}^{(\infty)} = \bigcap_{k=0}^{\infty} \mathcal{X}^{(k)}$ is the MAS. However, in general, this condition may not be satisfied in a finite number of steps k and a practical procedure will need to use a stopping criterion based on numerical tolerances. The following algorithm details the above mentioned procedure including a practical stopping criterion. We assume that $\mathbb{X} = \{x \mid M_x x \leq \mathbf{1}\}$ and $\mathcal{S}_w = \{w \mid M_w w \leq \mathbf{1}\}$ are polyhedral C -sets.

Algorithm A.1: Computation of the MAS for a linear polytopic system

1. Initialization:
  - (a) Choose the tolerance $\epsilon > 0$ and the maximum number of iterations k_{max} .
  - (b) Set $k = 0$ and set $\mathcal{X}^{(k)} = \{x \mid \mathcal{P}^{(k)} x \leq \mathbf{1}\} = \mathbb{X}$, i.e., set $\mathcal{P}^{(k)} = M_x$.
2. Compute the pre-image set $\mathcal{X}_{-1}^{(k)} = \{x \mid \mathcal{P}_{-1}^{(k)} x \leq \mathbf{1}\}$ of the set $\mathcal{X}^{(k)}$:
$$\mathcal{P}_{-1}^{(k)} = \begin{bmatrix} (\text{diag}(\mathbf{b}^{(1)}))^{-1} \mathcal{P}^{(k)} \Phi^{(1)} \\ (\text{diag}(\mathbf{b}^{(2)}))^{-1} \mathcal{P}^{(k)} \Phi^{(2)} \\ \vdots \\ (\text{diag}(\mathbf{b}^{(n_{\bar{\Omega}})}))^{-1} \mathcal{P}^{(k)} \Phi^{(n_{\bar{\Omega}})} \end{bmatrix}, \quad \mathbf{b}^{(j)} = \mathbf{1} - \max_{w \in \mathcal{S}_w} \mathcal{P}^{(k)} B_w^{(j)} w, \quad j = 1, \dots, n_{\bar{\Omega}}$$
3. Set $\mathcal{X}^{(k+1)} = \mathcal{X}_{-1}^{(k)} \cap \mathbb{X} = \mathcal{X}_{-1}^{(k)} \cap \mathcal{X}^{(k)}$, i.e., set $\mathcal{P}^{(k+1)} = \begin{bmatrix} \mathcal{P}^{(k)} \\ \mathcal{P}_{-1}^{(k)} \end{bmatrix}$.
4. If $\mathcal{X}^{(k)} \in (1 + \epsilon)\mathcal{X}^{(k+1)}$, set $\mathcal{X} = \mathcal{X}^{(k)}$. Exit.
5. If $\mathcal{X}^{(k+1)} = \emptyset$, set $\mathcal{X} = \emptyset$. Exit.
6. If $k \leq k_{max}$, set $k = k + 1$. Go to Step 2. ◇

In the above algorithm, we have used a numerical tolerance in Step 4 in order to have a practical stopping condition. With an $\epsilon > 0$, this condition is a relaxed form of the ideal invariance condition (A.14) which, if used, would result in the actual MAS, possibly with an infinitely large number of steps. Choosing an arbitrarily small positive ϵ and a correspondingly large k_{max} , we can obtain a set $\mathcal{X}$ that is arbitrarily close to the MAS. We note the following result from [113, Chap. 5] in this regard.

Lemma A.2. Let $k_{max} = \infty$. If $\bar{\mathcal{X}} = \mathcal{X}^{(\infty)} = \bigcap_{k=0}^{\infty} \mathcal{X}^{(k)}$ is a C -set, then for any tolerance $\epsilon > 0$, Algorithm A.1 stops in a finite number of steps successfully. Further, if $\bar{\mathcal{X}} = \mathcal{X}^{(\infty)} = \emptyset$, the algorithm stops in a finite number of steps unsuccessfully. □

The procedure mentioned in Algorithm A.1 is based on the basic iterative steps applicable with any forms of convex sets. Consequently, it requires us to compute the successive pre-image sets whose complexity, in terms of the number of linear constraints used to represent them, grows exponentially with the number of steps when the system is polytopic. Clearly, the procedure can be improved by directly computing the set $\mathcal{X}_{-1}^{(k)} \cap \mathcal{X}^{(k)}$ without explicitly determining the set $\mathcal{X}_{-1}^{(k)}$. This can be done by simply adding to the constraint set $\mathcal{P}^{(k)} \leq \mathbf{1}$ those constraints from $\mathcal{P}_{-1}^{(k)} \leq \mathbf{1}$ that are not redundant with each other and with those in the former. Moreover, with repeated additions of new constraints to the constraint set $\mathcal{P}^{(k)} \leq \mathbf{1}$, it is usually likely that some of the constraints added in the earlier steps will become redundant. So the redundancy of the constraints can be evaluated and the redundant constraints removed at regular intervals in order to reduce the complexity of the set being computed. The following procedure, based on one presented in [128] is a computationally efficient version of Algorithm A.1 applicable when the relevant sets are all polyhedral.

Algorithm A.2: Efficient computation of the MAS for a linear polytopic system

1. Initialization:
  - (a) Choose the tolerance $\epsilon > 0$, and the redundancy check interval $\sigma > 0$.
  - (b) Set $\mathcal{P} = M_x$, $r = r_o = \text{No. of rows of } \mathcal{P}$, $k = 1$.
2. Repeat until $k \leq r$,
  - (a) Set $p = \mathcal{P}_{[k,:]}^T$.
  - (b) Repeat for $j = 1, \dots, n_{\bar{Q}}$
    - i. Compute $b = 1 - \max_{w \in \mathcal{S}_w} p^T B_w^{(j)} w$.
    - ii. Add the constraint $p^T \Phi^{(j)} x \leq b$ to $\mathcal{P} x \leq \mathbf{1}$ if it is not redundant w.r.t. the constraints in $\mathcal{P} x \leq \mathbf{1}$:
      - Compute $c = \max_{\mathcal{P} x \leq \mathbf{1}} b^{-1} p^T \Phi^{(j)} x$.
      - If $c > 1 + \epsilon$, set $\mathcal{P} = \begin{bmatrix} \mathcal{P} \\ b^{-1} p^T \Phi^{(j)} \end{bmatrix}$.
  - (c) Set $r_n = \text{No. of rows of } \mathcal{P}$.
  - (d) If $r_n - r_o > \sigma r_o$, remove the redundant constraints in $\mathcal{P} x \leq \mathbf{1}$:
    - i. Set $i = 1$.
    - ii. Repeat until $i \leq r_n$,
      - Set $p = \mathcal{P}_{[i,:]}^T$
      - Set $\mathcal{P}_i = \begin{bmatrix} \mathcal{P}_{[1:i-1,:]} \\ \mathcal{P}_{[i+1:r_n,:]} \end{bmatrix}$
      - Compute $c = \max_{\mathcal{P}_i x \leq \mathbf{1}} p^T x$.
      - If $c \leq 1$,
        - Set $\mathcal{P} = \mathcal{P}_i$
        - If $i < k$, set $k = k - 1$.

- Set $i = i - 1$, $r_n = r_n - 1$.
- Set $i = i + 1$.
- iii. Set $r_o = r_n$.
- (e) Set $r = \text{No. of rows of } \mathcal{P}$, $k = k + 1$. ◇

Algorithm A.2 improves upon Algorithm A.1 in terms of computational efficiency and can be efficiently used for low dimensional systems with a few vertices defining their polytopic model. For higher dimensional systems, however, the presence of polytopic uncertainty makes the polyhedral representation of the MAS complex with a very large number of constraints. In such a situation, we may need to employ some approximation technique to practically compute a feasible invariant set that is an inner approximation of the MAS. In the following, we discuss one such approximation technique that was explored in [128] and is known by the name *pruning*.

Essentially, the *pruning* technique aims to avoid the addition of more than one constraints to the constraint set $\mathcal{P}x \leq \mathbf{1}$ in each pass through Step 2b of Algorithm A.2. Clearly, without any approximation mechanism in place, there can be an addition of up to $n_{\bar{\Omega}}$ new constraints in each such pass. Since these additional constraints are likely to be somewhat similar in the sense that all of them arise from the same parent constraint $p^T x \leq 1$, we check whether a slight tightening of one of the additional constraints makes some or all of the other additional constraints redundant, and, if it indeed makes so, we add the slightly tightened constraint instead of the original additional constraints. The question then is about which additional constraint to tighten and how much.

The problem of tightening one additional constraint to effect the redundancy of another additional constraint with respect to a set of existing constraints can be seen in the following way. Suppose we have a polyhedral set represented by a set of existing constraints $Px \leq \mathbf{1}$ and we are required to approximate the set defined by $Px \leq \mathbf{1}$ plus two additional constraints $p_1^T x \leq 1$ and $p_2^T x \leq 1$ by a set represented by $Px \leq \mathbf{1}$ and a tightened form of any one of the additional constraints, say $p_1^T x \leq 1$. Writing the tightened form of $p_1^T x \leq 1$ as $\mu p_1^T x \leq 1$ ($\mu > 1$), the problem of finding the minimum tightening required to make the constraint $p_2^T x \leq 1$ redundant can be expressed as the problem of finding the smallest μ such that

$$\max_{Px \leq \mathbf{1}, \mu p_1^T x \leq 1} p_2^T x \leq 1. \quad (\text{A.17})$$

This problem can be written as [128]

$$\text{s. t. } \left. \begin{array}{l} \min_{x, \nu} \nu \\ \begin{bmatrix} P & 0 \\ p_1^T & -1 \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix}, \quad p_2^T x = 1. \end{array} \right\} \quad (\text{A.18})$$

If the optimal value ν^* of ν in (A.18) is positive, the required smallest value of μ satisfying (A.17) will be given by $\mu^* = 1/\nu^*$.

The above result presents a way to systematically decide the constraint tightening strategy for lowering the number of constraint additions in Step 2b of Algorithm A.2. If two or more among the $n_{\bar{\Omega}}$ additional constraints arising in each pass through this step are non-redundant, we can compute the extent of tightening required in each such constraint to make each of the remaining ones redundant, and accordingly decide if any of the constraints can be tightened to render some others redundant. Note that a suitable limit on the maximum tightening allowed should be chosen such that the algorithm leads to the converged computation of a non-empty invariant set. The following piece of algorithm presents a modified form of Step 2b of Algorithm A.2 that allows systematic constraint tightening in order to reduce the number of constraint additions in each pass through this step. Here, we assume that the level of constraint tightening, represented by the quantity μ , is limited by the maximum allowed value $\mu_m > 1$.

Algorithm A.3: Modified version of Step 2b of Algorithm A.2

- i. Define $\mathcal{N}_C = \{1, 2, \dots, n_{\bar{\Omega}}\}$. Set $\eta = n_{\bar{\Omega}}$.
- ii. For each $j \in \mathcal{N}_C$,
  - Compute $b_{[j]} = 1 - \max_{w \in \mathcal{S}_w} p^T B_w^{(j)} w$.
  - Set $h_j = b_{[j]}^{-1} \Phi^{(j)T} p$.
- iii. For each $j \in \mathcal{N}_C$, compute $c_{[j]} = \max_{\mathcal{P}x \leq \mathbf{1}} h_j^T x$.
- iv. If for all $j \in \mathcal{N}_C$, $c_{[j]} \leq 1 + \epsilon$, then go to Step 2c (of Algorithm A.2).
- v. If for any $j \in \mathcal{N}_C$, $c_{[j]} > 1 + \epsilon$ and, for all $j \in \mathcal{N}_C \setminus \{j\}$, $c_{[j]} \leq 1 + \epsilon$ then, set $\mathcal{P} = [\mathcal{P}^T \ h_j]^T$. Go to Step 2c (of Algorithm A.2).
- vi. If $\eta = 1$ and j is the only element of $\mathcal{N}_C$, set $\mathcal{P} = [\mathcal{P}^T \ h_j]^T$. Go to Step 2c (of Algorithm A.2).
- vii. Set d as the vector of all j 's such that $j \in \mathcal{N}_C$ and $c_{[j]} > 1 + \epsilon$.
- viii. Set $l = \text{length}(d)$, define $\mathcal{N}_d = \{1, 2, \dots, l\}$ and set $\Lambda = I_l$.
- ix. For each constraint $h_{d[\kappa]}^T x \leq 1$, $\kappa \in \mathcal{N}_d$, compute the level of tightening required to make each of the other constraints redundant:
  - For each $\iota \in \mathcal{N}_d, \iota \neq \kappa$,
    - Solve $\min_{x, \nu} \nu$ s. t. $\begin{bmatrix} \mathcal{P} & 0 \\ h_{d[\kappa]}^T & -1 \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix}$, $h_{d[\iota]}^T x = 1$
    - Set $\Lambda_{[\kappa, \iota]} = 1/\nu^*$.
- x. Find the constraint which makes the most of the other constraints redundant with the least allowable tightening of itself and add its tightened form to $\mathcal{P}x \leq \mathbf{1}$:
  - Define $\mathcal{N}_m$ as the set of indices of the rows of Λ which have the largest number of elements less than or equal to μ_m .
  - Find $(\kappa^*, \iota^*) = \arg \min_{\kappa \in \mathcal{N}_m} \max_{\iota \in \mathcal{N}_d, \Lambda_{[\kappa, \iota]} \leq \mu_m} \Lambda_{[\kappa, \iota]}$

- Set $\mathcal{P} = \begin{bmatrix} \mathcal{P} \\ \Lambda_{[\kappa^*, \iota^*]} h_{d_{[\kappa^*]}}^T \end{bmatrix}$
- xi. Define e as the vector of column indices of the elements of the κ^* th row of Λ which are greater than μ_m .
- xii. Set $\eta = \text{length}(e)$. Define $\mathcal{N}_C = \{d_{[e_{[1]}]}, d_{[e_{[2]}]}, \dots, d_{[e_{[\eta]}]}\}$.
- xiii. Go to Step iii. ◇

Here, in the second step of Step x of Algorithm A.3, the optimal values κ^* and ι^* may not always be unique. We assume that any one of the non-unique solutions is picked up in such a situation. Moreover, when no allowable tightening of any constraint leads to the redundancy of any other constraints, the optimal value of the objective function will be 1, and any κ in the set $\mathcal{N}_m$ can be a solution. A simple strategy in such a situation would be to add all of the additional constraints to the existing constraint set. However, the above procedure allows the recheck of the possibility of effecting redundancy among the remaining constraints after the addition of each additional constraint. The conditions in Steps iv-vi take care of the termination of the operations in Algorithm A.3.

The maximum level of constraint tightening μ_m used in Algorithm A.3 can be tuned to vary the extent to which the MAS is approximated. With the use of a larger μ_m , the resulting set will be less complex with lower number of constraints defining it. A smaller μ_m more closer to 1, on the other hand, will result in a more closer approximation to the actual MAS, albeit with a correspondingly more complex representation. Care should, however, be taken to avoid higher levels of constraint tightening so that the algorithm successfully computes a set more or less close to the actual MAS.

A.2.2 Minimal RPI set

In the last subsection, we discussed the construction of (an inner invariant approximation) of the MAS, which is the largest feasible RPI set, for the polytopic system in (A.1). For this, we employed an iterative procedure involving the computation of successive pre-image sets starting from the given feasible set $\mathbb{X}$. In this subsection, we discuss the computation of (an outer approximation of) the minimal RPI set for system (A.1).

The minimal RPI set for a system, if it exists, is actually the infinite-time reachability set for the state of the system with the disturbance $w(t)$ as the input. While we have employed an iterative procedure involving the backward computation of pre-image sets for the construction of the MAS, we will consider the computation of successive forward reachable sets starting from the set $\{0\}$ in order to determine (an approximation of) the minimal RPI set. We assume that the minimal RPI set exists for the system being considered and that it is contained well inside the constraint set $\mathbb{X}$ so that no feasibility issues need to be considered.

For the determination of the smallest RPI set $\mathcal{R}$ satisfying the standard condition (A.2), we wish to start from the set $\mathcal{R}^{(0)} = \{0\}$ and, for increasing values of $k = 0, 1, 2, \dots$,

iteratively compute the set $\mathcal{R}^{(k+1)} = \mathcal{R}_{+1}^{(k)}$ where $\mathcal{R}_{+1}^{(k)} = \text{Rea}_{(\bar{\Omega}, \mathcal{S}_w)}(\mathcal{R}^{(k)})$ is the one-step forward reachable set from $\mathcal{R}^{(k)}$. The set $\mathcal{R}^{(k)}$, therefore, is the k -step reachable set from $\mathcal{R}^{(0)} = \{0\}$. The above mentioned iterative procedure generates a sequence of forward reachable sets $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \mathcal{R}^{(3)} \dots$ satisfying the successive inclusion $\mathcal{R}^{(1)} \subseteq \mathcal{R}^{(2)} \subseteq \mathcal{R}^{(3)} \dots$, and, conceptually the set $\mathcal{R}^{(k)}$ should converge to the minimal RPI set $\bar{\mathcal{R}} = \mathcal{R}^{(\infty)}$ as $k \rightarrow \infty$ providing the set exists. However, unlike the case of pre-image sets, the forward reachable sets are not necessarily convex for a polytopic system even when $\mathcal{S}_w$ is a C -set [113, Sec 6.1.2] and therefore cannot be computed easily by only considering the vertices of the uncertainty set $\bar{\Omega}$. Fortunately, the convex hulls of the successive forward reachable sets can, however, be computed rather easily. We note the following result in this regard [127], [113, Chap. 6].

Lemma A.3. Let $\mathcal{R}^{(k)}$ be the k -step reachable set for system (A.1) from $\mathcal{R}^{(0)} = \{0\}$. Also, let the set sequence $\{\hat{\mathcal{R}}^{(k)}\}_{k=1,2,\dots}$ be defined recursively from $\hat{\mathcal{R}}^{(0)} = \{0\}$ as

$$\hat{\mathcal{R}}^{(k+1)} = \text{Co} \left\{ \bigcup_{j=1, \dots, n_{\bar{\Omega}}} (\Phi^{(j)} \hat{\mathcal{R}}^{(k)} \oplus B_w^{(j)} \mathcal{S}_w) \right\}, k = 1, 2, \dots \quad (\text{A.19})$$

Then, for any $k \in \{1, 2, \dots\}$, $\hat{\mathcal{R}}^{(k)} = \text{Co}\{\mathcal{R}^{(k)}\}$ and hence $\hat{\mathcal{R}}^{(\infty)} = \text{Co}\{\mathcal{R}^{(\infty)}\} = \text{Co}\{\bar{\mathcal{R}}\}$. $\square$

Lemma A.3 implies that the convex hull of the reachability sets can be computed by simply forward propagating the vertices. So, if $\{x^{(1)}, x^{(2)}, \dots, x^{(\eta)}\}$ are the vertices defining the set $\hat{\mathcal{R}}^{(k)}$, and if $\mathcal{S}_w = \text{Co}\{w^{(1)}, w^{(2)}, \dots, w^{(\nu)}\}$, the set $\hat{\mathcal{R}}^{(k+1)}$ can be computed using

$$\hat{\mathcal{R}}^{(k+1)} = \text{Co} \left\{ \Phi^{(j)} x^{(j)} + B_w^{(j)} w^{(l)}, j = 1, \dots, n_{\bar{\Omega}}, j = 1, \dots, \eta, l = 1, \dots, \nu \right\}. \quad (\text{A.20})$$

Although the repetitive procedure to compute the successive convex hulls of reachable sets using (A.20) is simple in form, its implementation may not be so simple as the number of vertices defining the set $\hat{\mathcal{R}}^{(k)}$ will grow exponentially as k increases. Indeed, the ideal stopping condition $\hat{\mathcal{R}}^{(k+1)} \subseteq \hat{\mathcal{R}}^{(k)}$ of the iterative procedure will rarely be attained without propagating the reachable sets for a considerably large number of steps. However, more practical stopping conditions based on numerical tolerances can be employed as in the case of the computation of the MAS so that a close inner approximation of the minimal RPI set can be obtained in a reasonable number of steps. Furthermore, an outer approximation of the minimal RPI set can be obtained either by slightly enlarging the inner approximation or by considering a slightly enlarged disturbance set during the construction of its inner approximation. We will not further discuss such approximation techniques here. The readers are referred to [127] for the details of some systematic methods to compute reasonable approximations of the minimal RPI set.

Appendix B

Expressions for Some Constant Matrices Used in the Thesis

B.1 Constant Matrices Used in Section 3.4.2

First we define the two functions:

$$\psi_1(\mathcal{A}(t), \mathcal{A}(t+1), \dots, \mathcal{A}(t+N-1)) = \begin{bmatrix} I \\ \mathcal{A}(t) \\ \mathcal{A}(t+1)\mathcal{A}(t) \\ \vdots \\ \mathcal{A}(t+N-1)..\mathcal{A}(t) \end{bmatrix} \quad (\text{B.1a})$$

$$\psi_2(\mathcal{A}(t), \mathcal{A}(t+1), \dots, \mathcal{A}(t+N-1)) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ \mathcal{A}(t+1) & I_n & \cdots & 0 \\ \mathcal{A}(t+2)\mathcal{A}(t+1) & \mathcal{A}(t+2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}(t+N-1)..\mathcal{A}(t+1) & \mathcal{A}(t+N-1)..\mathcal{A}(t+2) & \cdots & I_n \end{bmatrix}. \quad (\text{B.1b})$$

The constant matrices are then defined as

$$\begin{aligned}
\mathbf{A}(t) &= \psi_1(\mathcal{A}(t), \dots, \mathcal{A}(t+N-1)), \quad \mathcal{A}(t+i) = A(t+i) + B(t+i)K \\
\mathbf{B}_u(t) &= \psi_2(\mathcal{A}(t), \dots, \mathcal{A}(t+N-1)) \times \text{diag}(B_u(t), B_u(t+1), \dots, B_u(t+N-1)) \\
\mathbf{B}_w(t) &= \psi_2(\mathcal{A}(t), \dots, \mathcal{A}(t+N-1)) \times \text{diag}(B_w(t), B_w(t+1), \dots, B_w(t+N-1)) \\
\check{\mathbf{A}}(t) &= \mathbf{KJA}(t), \quad \mathbf{K} = I_N \otimes K, \quad \mathbf{J} = \begin{bmatrix} I_{Nn_x} & 0 \end{bmatrix} \\
\check{\mathbf{B}}_u(t) &= \mathbf{KJB}_u(t) + I_{Nn_u} \\
\check{\mathbf{B}}_w(t) &= \mathbf{KJB}_w(t).
\end{aligned}$$

Moreover, the the matrices in the constraint expression (3.34) are defined as

$$\begin{aligned}
\mathcal{M}_x &= \mathbf{M}_x \mathbf{A}(t) + \mathbf{M}_u [\check{\mathbf{A}}(t)^T \quad 0]^T \\
\mathcal{M}_u &= \mathbf{M}_x [\mathbf{B}_u(t) \quad 0] + \mathbf{M}_u \times \text{diag}(\check{\mathbf{B}}_u(t), I_{n_\xi}) \\
\mathcal{M}_w &= \mathbf{M}_x \mathbf{B}_w(t) + \mathbf{M}_u [\check{\mathbf{B}}_w(t)^T \quad 0]^T \\
\mathbf{M}_x &= \text{diag}(I_N \otimes M_x, M_{zx}), \quad \mathbf{M}_u = \text{diag}(I_N \otimes M_u, M_{z\xi})
\end{aligned}$$

where M_{zx} and $M_{z\xi}$ are the first n_x and the rest n_ξ columns of M_z respectively.

B.2 Constant Matrices Used in Section 5.3.3

Let us consider the functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ defined in the previous section. We define the constant matrices as:

$$\begin{aligned}
\mathbf{A}(t) &= \psi_1(\mathcal{A}(t), \dots, \mathcal{A}(t+N-1)) \\
\mathbf{B}_u(t) &= \psi_2(\mathcal{A}(t), \dots, \mathcal{A}(t+N-1)) \times \text{diag}(B_u(t), \dots, B_u(t+N-1)) \\
\mathbf{B}_\eta(t) &= \psi_2(\mathcal{A}(t), \dots, \mathcal{A}(t+N-1)) \times \text{diag}(L, \dots, L) \\
\check{\mathbf{A}}(t) &= \mathbf{KJA}(t), \quad \check{\mathbf{B}}_u(t) = \mathbf{KJB}_u(t) + I_{Nn_u} \\
\check{\mathbf{B}}_\eta(t) &= \mathbf{KJB}_\eta(t), \quad \mathbf{K} = I_N \otimes K, \quad \mathbf{J} = \begin{bmatrix} I_{Nn_x} & 0 \end{bmatrix} \\
\mathbf{C} &= \text{diag}(C, \dots, C) \\
\mathbf{D} &= \text{diag}(D, \dots, D) \\
\tilde{\mathbf{A}}(t) &= \psi_1(\tilde{\mathcal{A}}(t), \dots, \tilde{\mathcal{A}}(t+N-2)) \\
\tilde{\mathbf{B}}_w(t) &= \mathbf{J} \psi_2(\tilde{\mathcal{A}}(t), \dots, \tilde{\mathcal{A}}(t+N-1)) \times \text{diag}(B_w(t), \dots, B_w(t+N-1)) \\
\tilde{\mathbf{B}}_e(t) &= \mathbf{J} \psi_2(\tilde{\mathcal{A}}(t), \dots, \tilde{\mathcal{A}}(t+N-1)) \times \text{diag}(-LD, \dots, -LD).
\end{aligned}$$

Moreover, the the matrices in the constraint expression (5.18) are defined as

$$\begin{aligned}
\mathcal{M}_{\hat{x}} &= \mathbf{M}_{\mathbf{x}} \mathbf{A}(t) + \mathbf{M}_{\mathbf{u}} [\check{\mathbf{A}}(t)^{\text{T}} \ 0]^{\text{T}} \\
\mathcal{M}_u &= \mathbf{M}_{\mathbf{x}} [\mathbf{B}_{\mathbf{u}}(t) \ 0] + \mathbf{M}_{\mathbf{u}} \times \text{diag}(\check{\mathbf{B}}_{\mathbf{u}}(t), I_{n_{\xi}}) \\
\mathcal{M}_{\eta} &= \mathbf{M}_{\mathbf{x}} \mathbf{B}_{\eta}(t) + \mathbf{M}_{\mathbf{u}} [\check{\mathbf{B}}_{\eta}(t)^{\text{T}} \ 0]^{\text{T}} \\
\mathcal{M}_{\tilde{x}} &= \begin{bmatrix} I_N \otimes M_x \\ 0 \end{bmatrix}
\end{aligned}$$

where $\mathbf{M}_{\mathbf{x}}$ and $\mathbf{M}_{\mathbf{u}}$ are defined as in the previous section.

Author's Publications

- [1] A. Gautam, Y. C. Soh, and Y. C. Chu. Robust predictive control of satellite formations. *In Proceedings of the ICARCV*, pages 844-849, 2008.
- [2] A. Gautam, Y. C. Chu, and Y. C. Soh. Robust MPC with optimized controller dynamics for linear polytopic systems with bounded additive disturbances. *In Proceedings of the 7th Asian Control Conference*, pages 1322-1327, 2009.
- [3] A. Gautam, Y. C. Chu, and Y. C. Soh. Model predictive control of linear time-varying systems with bounded disturbances using a dynamic terminal policy. *In Proceedings of the IEEE Conference on Decision and Control*, pages 3999-4004, 2009.
- [4] A. Gautam, Y. C. Chu, and Y. C. Soh. Optimized dynamic policy for receding horizon control of linear time-varying systems with bounded disturbances. *IEEE Transactions on Automatic Control*, 57(4):973-988, 2012
- [5] A. Gautam, Y. C. Chu, and Y. C. Soh. Robust H_∞ receding horizon control for a class of coordinated control problems involving dynamically decoupled subsystems. Submitted for publication.
- [6] A. Gautam, Y. C. Soh, and Y. C. Chu. Guaranteed disturbance attenuation versus feasibility in min-max model predictive control using optimized dynamic policy. Submitted for publication.
- [7] A. Gautam, Y. C. Chu, and Y. C. Soh. Dynamic-policy-based output-feedback receding horizon control for LTV systems with bounded disturbances. In preparation.

Bibliography

- [1] J. Baillieul and P. J. Antsaklis. Control and communication challenges in networked real-time systems. *Proceedings of the IEEE*, 94(1):9–28, 2007.
- [2] R. M. Murray. Recent research in cooperative control of multivehicle systems. *Journal of Dynamic Systems, Measurement, and Control*, 129(5):517–529, 2007.
- [3] R. W. Beard, J. R. Lawton, and F. Y. Hadaegh. A coordination architecture for spacecraft formation control. *IEEE Transactions on Control Systems Technology*, 9(6):777–790, 2001.
- [4] L. Breger, I. Gokhan, M. Tillerson, and J. P. How. Cooperative spacecraft formation flying: model predictive control with open- and closed-loop robustness. *Modern Astrodynamics 1, Elsevier Astrodynamics Series*, pages 237–277, 2006.
- [5] R. W. Beard, T. W. McLain, D. Nelson, D. Kingston, and D. Johanson. Decentralized cooperative aerial surveillance using fixed-wing miniature UAVs. *Proceedings of the IEEE*, 94(7):1306–1324, 2006.
- [6] D. J. Stilwell and B. E. Bishop. Platoons of underwater vehicles. *IEEE Control Systems Magazine*, 20(6):45–52, 2000.
- [7] C. Tomlin, G. J. Pappas, and S. Sastry. Conflict resolution for air traffic management: A study in multi-agent hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):509–521, 1998.
- [8] Ü Özgüner, C. Stiller, and K. Redmill. Systems for safety and autonomous behavior in cars: The DARPA Grand Challenge experience. *Proceedings of the IEEE*, 95(2):397–412, 2007.
- [9] S. A. Middlebrooks and J. B. Rawlings. Model predictive control of $\text{Si}_{1-x}\text{Ge}_x$ thin film chemical-vapor deposition. *IEEE Transactions on Semiconductor Manufacturing*, 20(2):114–125, 2007.
- [10] R. S. Parker and G. Clermont. Systems engineering medicine: engineering the inflammation response to infectious and traumatic challenges. *Journal of the Royal Society Interface*, 7:989–1013, 2010.
- [11] R. Hovorka, J. Kremen, J. Blaha, M. Matias, K. Anderlova, L. Bosanska, T. Roubicek, M. E. Wilinska, L. J. Chassin, S. Svacina, and M. Haluzik. Blood glucose control by a model predictive control algorithm with variable sampling rate

- versus a routine glucose management protocol in cardiac surgery patients: A randomized controlled trial. *The Journal of Clinical Endocrinology & Metabolism*, 92(8):2960–2964, 2007.
- [12] J. M. Maciejowski. *Predictive Control with Constraints*. Prentice Hall, 2000.
  - [13] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. M. O. Scokaert. Constrained model predictive control: stability and optimality. *Automatica*, 36:789–814, 2000.
  - [14] M. Morari and J. H. Lee. Model predictive control: past, present and future. *Computers and Chemical Engineering*, 23:667–682, 1999.
  - [15] A. A. Jalali and V. Nadimi. A survey on robust model predictive control from 1999-2006. In *Proceedings of the International Conference on CIMCA-IAWTIC*, pages 207–212, 2006.
  - [16] C. E. García, D. M. Prett, and M. Morari. Model predictive control: Theory and practice — A survey. *Automatica*, 25(3):335–348, 1989.
  - [17] J. B. Rawlings, E. S. Meadows, and K. R. Muske. Nonlinear model predictive control: A tutorial and survey. In *Proceedings ADCHEM'94*, Kyoto, Japan, 1994.
  - [18] A. Bemporad and M. Morari. Robust model predictive control: A survey. *Robustness in Identification and Control (Lecture Notes in Control and Information Sciences,)*, 245:207–226, 1999.
  - [19] S. J. Qin and T. A. Badgwell. A survey of industrial model predictive control technology. *Control Engineering Practice*, 11(7):733–764, 2003.
  - [20] A. Alessio and A. Bemporad. A survey on explicit model predictive control. *Nonlinear Model Predictive Control (Lecture Notes in Control and Information Sciences,)*, 384:345–369, 2009.
  - [21] R. R. Bitmead, M. Gevers, and V. Wertz. *Adaptive Optimal Control: The Thinking Man's GPC*. Prentice-Hall, NJ, 1990.
  - [22] R. Soeterboek. *Predictive control - A unified approach*. Prentice-Hall, NJ, 1992.
  - [23] E. Mosca. *Optimal, Predictive and Adaptive Control*. Prentice Hall, 1994.
  - [24] J. A. Rossiter. *Model-based predictive control - A practical approach*. CRC Press, 2003.
  - [25] E. F. Camacho and C. Bordens. *Model Predictive Control*. Springer, 2004.
  - [26] W. H. Kwon and S. Han. *Receding Horizon Control*. Springer, 2005.
  - [27] A. I. Propoi. Use of linear programming methods for synthesizing sampled-data automatic systems. *Automation and Remote Control*, 24(3):837–844, 1963.
  - [28] E. B. Lee and L. Markus. *Foundations of Optimal Control Theory*. Wiley, New York, 1967.

- [29] Y. A. Thomas. Linear quadratic optimal estimation and control with receding horizon. *Electronics Letters*, 11:19–21, 1975.
- [30] W. H Kwon and A. E. Pearson. A modified quadratic cost function problem and feedback stabilization of a linear system. *IEEE Transactions on Automatic Control*, 22(5):838–842, 1977.
- [31] J. Richalet, A. Rault, J. L. Testud, and J. Papon. Model predictive heuristic control: Applications to industrial processes. *Automatica*, 14:413–428, 1978.
- [32] C. R. Cutler and B. C. Ramaker. Dynamic matrix control - a computer control algorithm. In *Proceedings Joint Automatic Control Conference*, 1980.
- [33] C. E. García and A. M. Morshedi. Quadratic programming solution of dynamic matrix control (QDMC). *Chemical Engineering Communications*, 46:73–87, 1986.
- [34] V. Peterka. Predictor-based self-tuning control. *Automatica*, 20(1):39–50, 1984.
- [35] B. E. Ydstie. Extended horizon adaptive control. In *Proceedings 9th IFAC World Congress*, Budapest, Hungary.
- [36] D. W. Clarke, C. Mohtadi, and P. S. Tuffs. Generalized predictive control. Part I: The basic algorithm. *Automatica*, 23(2):137–148, 1987.
- [37] D. W. Clarke, C. Mohtadi, and P. S. Tuffs. Generalized predictive control. Part II: Extensions and interpretations. *Automatica*, 23(2):149–160, 1987.
- [38] B. L. Kleinman. An easy way to stabilize a linear constant system. *IEEE Transactions on Automatic Control*, 15(12):692–692, 1970.
- [39] W. H. Kwon, A. M. Bruckstein, and T. Kailath. Stabilizing state-feedback design via the moving horizon method. *International Journal of Control*, 37(3):631–643, 1983.
- [40] C. C. Chen and L. Shaw. On receding horizon feedback control. *Automatica*, 18:349–352, 1982.
- [41] S. S. Keerthi and E. G. Gilbert. Optimal, infinite horizon feedback laws for a general class of constrained discrete time systems: Stability and moving-horizon approximations. *Journal of Optimization Theory and Applications*, 57:265–293, 1988.
- [42] D. Q. Mayne and H. Michalska. Receding horizon control of non-linear systems. *IEEE Transactions on Automatic Control*, 35(5):814–824, 1990.
- [43] W. H. Kwon and D. G. Byun. Receding horizon tracking control as a predictive control and its stability properties. *International Journal of Control*, 50:1807–1824, 1989.
- [44] H. Demircioglu and D. W. Clarke. Generalized predictive control with end point weighing. *IEE Proceedings D: Control Theory & Applications*, 140(4):275–282, 1993.

- [45] J. B. Rawlings and K. R. Muske. Stability of constrained receding horizon control. *IEEE Transactions on Automatic Control*, 38(10):1512–1516, 1993.
- [46] H Michalska and D. Q. Mayne. Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 38:1623–1632, 1993.
- [47] L. Chisci, A. Lombardi, and E. Mosca. Dual receding horizon control of constrained discrete-time systems. *European Journal of Control*, 32(10):1361–1379, 1996.
- [48] M. Sznaier and M. J. Damborg. Suboptimal control of linear systems with state and control inequality constraints. In *Proceedings of the IEEE Conference on Decision and Control*, pages 761–762, Los Angeles, 1987.
- [49] E. G. Gilbert and K. T. Tan. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36(9):1008–1020, 1991.
- [50] D. Chmielewski and V. Manousiouthakis. On constrained infinite-time linear quadratic control. *Systems & Control Letters*, 29:121–129, 1996.
- [51] P. M. O. Scokaert and D. Q. Mayne. Min-max feedback model predictive control for constrained linear systems. *Automatica*, 34(8):1136–1142, 1998.
- [52] T Parisini and R. Zoppoli. A receding horizon regulator for nonlinear systems and a neural approximation. *Automatica*, 31(10):1443–1451, 1995.
- [53] H. Chen and F. Allgöwer. A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 34(10):1205–1217, 1998.
- [54] J. W. Lee, W. H. Kwon, and J. Choi. On stability of constrained receding horizon control with finite terminal weighing matrix. *Automatica*, 34(12):1607–1612, 1998.
- [55] G. De Nicolao, L. Magni, and R. Scattolini. On the robustness of receding horizon control with terminal constraints. *IEEE Transactions on Automatic Control*, 41:451–453, 1996.
- [56] M. V. Kothare, V. Balakrishnan, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 2:278–285, 1996.
- [57] B. Kouvaritakis, J. A. Rossiter, and J. Schuurmans. Efficient robust predictive control. *IEEE Transactions on Automatic Control*, 45(8):1545–1549, 2000.
- [58] Z. Wan and M. V. Kothare. An efficient offline formulation of robust model predictive control using linear matrix inequalities. *Automatica*, 39:837–846, 2003.
- [59] M. Bacic, M. Cannon, Y. I. Lee, and B. Kouvaritakis. General interpolation in MPC and its advantages. *IEEE Transactions on Automatic Control*, 48(6):1092–1096, 2003.
- [60] M. Cannon and B. Kouvaritakis. Optimizing prediction dynamics for robust MPC. *IEEE Transactions on Automatic Control*, 52(11):1892–1897, 2005.

- [61] B. Pluymers, L. Roobrouck, L. Buijs, J. A. K. Suykens, and B. D. Moor. Constrained linear MPC with time-varying terminal cost using convex combinations. *Automatica*, 41:831–837, 2005.
- [62] L. Imsland, J. A. Rossiter, B. Pluymers, and J. Suykens. Robust triple mode MPC. *International Journal of Control*, 81(4):679–689, 2008.
- [63] Y. I. Lee and B. Kouvaritakis. Constrained receding horizon predictive control for systems with disturbances. *International Journal of Control*, 72(11):1027 – 1032, 2001.
- [64] L. Chisci, J. A. Rossiter, and G. Zappa. Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37:1019–1028, 2001.
- [65] K. B. Kim. Design of receding horizon controls for constrained time-varying systems. *IEEE Transactions on Automatic Control*, 49(12):2248–2253, 2004.
- [66] E. C. Kerrigan and J. M. Maciejowski. Feedback min-max model predictive control using a single linear program: Robust stability and the explicit solution. *International Journal Robust and Nonlinear Control*, 14:395–413, 2004.
- [67] D. Q. Mayne, M. M. Seron, and S. V. Raković. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41:219–224, 2005.
- [68] S. V. Raković and Mayne D. Q. A simple tube controller for efficient robust model predictive control of constrained linear discrete time systems subject to bounded disturbances. In *Proceedings of the IFAC World Congress*, pages 241–246, 2005.
- [69] D. R. Ramirez and E. F. Camacho. Piecewise affinity of min-max MPC with bounded additive disturbances. *Automatica*, 42:295–302, 2006.
- [70] P. J. Goulart, E. C. Kerrigan, and J. M. Maciejowski. Optimization over state feedback policies for robust control with constraints. *Automatica*, 42:523–533, 2006.
- [71] T. Alamo, D. Muñoz de la Peña, and E. F. Camacho. Min-Max MPC based on network problem. *Systems and Control Letters*, 57:185–192, 2008.
- [72] D. Sui, L. Feng, M. Hovd, and C. J. Ong. Decomposition principle in model predictive control for linear systems with bounded disturbances. *Automatica*, 45:1917–1922, 2009.
- [73] J. H. Lee and Z. Yu. Worst-case formulations of model predictive control for systems with bounded parameters. *Automatica*, 33(5):763–781, 1997.
- [74] A. Bemporad, F. Borrelli, and M. Morari. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Transactions on Automatic Control*, 48(9):1600–1606, 2003.
- [75] W. Langson, I. Chrysoschoos, S. V. Raković, and D. Q. Mayne. Robust model predictive control using tubes. *Automatica*, 40:125–133, 2004.

- [76] H. Fukushima and R. R. Bitmead. Robust constrained predictive control using comparison model. *Automatica*, 41:97–106, 2005.
- [77] S. V. Raković, A. R. Teel, and Mayne D. Q. Simple robust control invariant tubes for some classes of nonlinear discrete time systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 6397–6402, 2006.
- [78] A. B. Açıkmeşe and J. M. Carson. A nonlinear model predictive algorithm with proven robustness and resolvability. In *Proceedings of the American Control Conference*, pages 887–893, 2006.
- [79] L. Imsland, N. Bar, and B. A. Foss. More efficient predictive control. *Automatica*, 41:1395–1403, 2005.
- [80] R. S. Smith. Robust model predictive control of constrained linear systems. In *Proceedings of the American Control Conference*, pages 245–250, 2004.
- [81] J. Löfberg. Approximations of closed-loop minimax MPC. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1438–1442, 2003.
- [82] C. Wang, C. J. Ong, and M. Sim. Model predictive control using affine disturbance feedback for constrained linear system. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1275–1280, 2007.
- [83] P. J. Goulart, E. C. Kerrigan, and T. Alamo. Control of constrained discrete-time systems with bounded ℓ_2 gain. *IEEE Transactions on Automatic Control*, 54(5):1105–1111, 2009.
- [84] R. Findeisen, L. Imsland, F. Allgöwer, and B. A. Foss. State and output feedback nonlinear model predictive control: An overview. *European Journal of Control*, 9(2-3):190–206, 2003.
- [85] A. Bemporad and A. Garulli. Output-feedback predictive control of constrained linear systems via set-membership state estimation. *International Journal of Control*, 73(8):655–665, 2000.
- [86] Y. I. Lee and B. Kouvaritakis. Receding horizon output feedback control for linear systems with input saturation. *IEE Control Theory and Applications*, 148(2):109–115, 2001.
- [87] L. Chisci and G. Zappa. Feasibility in predictive control of constrained linear systems: the output feedback case. *International Journal of Robust and Nonlinear Control*, 12:465–487, 2002.
- [88] J. Löfberg. Towards joint state estimation and control in minimax mpc. In *Proceedings 15th IFAC World Congress*, 2002.
- [89] D. Q. Mayne, S. V. Raković, R. Findeisen, and F. Allgöwer. Robust output feedback model predictive control of constrained linear systems. *Automatica*, 42:1217–1222, 2006.

- [90] P. J. Goulart and E. C. Kerrigan. Output feedback receding horizon control of constrained systems. *International Journal of Control*, 80(1):9–20, 2007.
- [91] Z. Wan and M. V. Kothare. Robust output feedback model predictive control using off-line linear matrix inequalities. *Journal of Process Control*, 12:763–774, 2002.
- [92] C. Løvvaas, M. M. Seron, and G. C. Goodwin. Robust output-feedback model predictive control for systems with unstructured uncertainty. *Automatica*, 44:1933–1943, 2008.
- [93] B. Ding and L. Xie. Dynamic output feedback robust model predictive control with guaranteed quadratic boundedness. In *Proceedings of the IEEE Conference on Decision and Control*, pages 8034–8039, 2009.
- [94] A. Richards and J. How. Decentralized model predictive control of cooperating UAVs. In *Proceedings of IEEE Conference on Decision and Control*, pages 4286–4291, 2004.
- [95] J. Balderud, L. Giovanini, and M. R. Katebi. Distributed control of underwater vehicles. *Proceedings of the IME, Part M: Journal of Engineering for the Maritime Environment*, 222(2):95–107, 2008.
- [96] G. Chaloulos, P. Hokayem, and J. Lygeros. Distributed hierarchical MPC for conflict resolution in air traffic control. In *Proceedings of the American Control Conference*, pages 3945–3950, 2010.
- [97] W. Dunbar. Distributed receding horizon control of cost coupled systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 2510–2515, 2007.
- [98] T. Keviczky, F. Borrelli, and G. J. Balas. Decentralized receding horizon control for large scale dynamically decoupled systems. *Automatica*, 42:2105–2115, 2006.
- [99] A. Richards and J. P. How. Robust distributed model predictive control. *International Journal of Control*, 80(9):1517–1531, 2007.
- [100] B. Johansson, A. Speranzon, M. Johansson, and K. H. Johansson. On decentralized negotiation of optimal consensus. *Automatica*, 44:1175–1179, 2008.
- [101] H.T. Zhang, M.Z.Q. Chen, G.B. Stan, T. Zhou, and J.M. Maciejowski. Collective behavior coordination with predictive mechanisms. *IEEE Circuits and Systems Magazine*, 8(3):67–85, 2008.
- [102] G. Ferrari-Trecate, L. Galbusera, M.P.E. Marciandi, and R. Scattolini. Model predictive control schemes for consensus in multi-agent systems with single- and double-integrator dynamics. *IEEE Transactions on Automatic Control*, 54(11):2560–2572, 2009.
- [103] A. Gautam, Y. C. Chu, and Y. C. Soh. Optimized dynamic policy for receding horizon control of linear time-varying systems with bounded disturbances. *IEEE Transactions on Automatic Control*, 57(4):973–988, 2012.

- [104] A. Gautam, Y. C. Chu, and Y. C. Soh. Model predictive control of linear time-varying systems with bounded disturbances using a dynamic terminal policy. In *Proceedings of the IEEE Conference on Decision and Control*, pages 3999–4004, 2009.
- [105] A. Gautam, Y. C. Chu, and Y. C. Soh. Robust MPC with optimized controller dynamics for linear polytopic systems with bounded additive disturbances. In *Proceedings of the 7th Asian Control Conference*, pages 1322–1327, 2009.
- [106] A. Gautam, Y. C. Chu, and Y. C. Soh. Guaranteed disturbance attenuation versus feasibility in min-max model predictive control using optimized dynamic policy. Submitted for publication.
- [107] A. Gautam, Y. C. Chu, and Y. C. Soh. Dynamic-policy-based output-feedback receding horizon control for LTV systems with bounded disturbances. In preparation.
- [108] A. Gautam, Y. C. Chu, and Y. C. Soh. Robust H_∞ receding horizon control for a class of coordinated control problems involving dynamically decoupled subsystems. Submitted for publication.
- [109] A. Gautam, Y. C. Soh, and Y. C. Chu. Robust predictive control of satellite formations. In *Proceedings of the ICARCV*, pages 844–849, 2008.
- [110] A. Gautam, Y. C. Chu, and Y. C. Soh. Satellite formation reconfiguration with dynamic-policy-based distributed receding horizon control. In preparation.
- [111] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*. SIAM, 1994.
- [112] A. Packard and J. Doyle. The complex structured singular value. *Automatica*, 29:71–109, 1993.
- [113] F. Blanchini and S. Miani. *Set-Theoretic Methods in Control*. Springer, 2008.
- [114] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.
- [115] Z. P. Jiang and Wang Y. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37:857–869, 2001.
- [116] B. Pluymers, M. V. Kothare, J. A. K. Suykens, and B. D. Moor. Robust synthesis of constrained linear state feedback using LMIs and polyhedral sets. In *Proceedings of the American Control Conference*, pages 881–886, 2006.
- [117] J. M. Carson. *Robust Model Predictive Control with a Reactive Safety Mode*. PhD Thesis, California Institute of Technology, 2008.
- [118] A. Ben-Tal, S. Boyd, and A. Nemirovski. Extending scope of robust optimization: Comprehensive robust counterparts of uncertain problems. *Mathematical Programming, Series B*, 107:63–89, 2006.

- [119] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, 2004.
- [120] O. Boni and A. Ben-Tal. Adjustable robust counterpart of conic quadratic problems. *Mathematical Methods of Operations Research*, 68(2):1432–2994, 2008.
- [121] P. J. Goulart, E. C. Kerrigan, and D. Ralph. Efficient robust optimization for robust control with constraints. *Mathematical Programming*, 114(1):115–147, 2008.
- [122] A. Richards. Robust model predictive control for time-varying systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 3747–3752, 2005.
- [123] C. Scherer, P. Gahinet, and M. Chilali. Multi-objective output feedback control via LMI optimization. *IEEE Transactions on Automatic Control*, 42(7):896–911, 1997.
- [124] Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. SIAM, 1994.
- [125] L. Vandenberghe, S. Boyd, and S. Wu. Determinant maximization with linear matrix inequality constraints. *SIAM Journal on Matrix Analysis and Applications*, 19:499–533, 1998.
- [126] I. Kolmanovsky and E. G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4(4):317–367, 1998.
- [127] S. V. Raković, K. I. Kouramas, E. C. Kerrigan, J. Allwright, and D. Q. Mayne. The minimal robust positively invariant set for linear difference inclusions and its robust positively invariant approximations. Technical Report EEE/C&P/SVR/9-d/2005, Imperial College London, 2005.
- [128] B. Pluymers. *Robust Model Based Predictive Control - An Invariant Set Approach*. PhD Thesis, K. U. Leuven (Belgium), 2006.
- [129] Cs. Mészáros. Fast cholesky factorization for interior point methods of linear programming. *Computers & Mathematics with Applications*, 31(4/5):49–51, 1996.
- [130] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali. The LMI control toolbox. In *Proceedings of IEEE Conference on Decision and Control*, pages 2038 – 2041, 1994.
- [131] S. Lall and K. Glover. A game theoretic approach to moving horizon control. In D. W. Clarke, editor, *Advances in model-based predictive control*, pages 131–144. Oxford University Press, 1994.
- [132] L. Magni, H. Nijmeijer, and A. J. van der Schaft. A receding-horizon approach to the nonlinear H_∞ control problem. *Automatica*, 37(3):429–435, 2001.
- [133] H. Chen and C. W. Scherer. Moving horizon H_∞ control with performance adaptation for constrained linear systems. *Automatica*, 42(6):1033–1040, 2006.

- [134] M. Kočvara and M. Stingl. PENNON- a code for convex nonlinear and semidefinite programming. *Optimization Methods and Software*, 8(3):317–333, 2003.
- [135] L. E. Ghaoui, F. Oustry, and M. AitRami. A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control*, 42(8):1171–1176, 1997.
- [136] P. Apkarian and H. D. Tuan. Robust control via concave minimization local and global algorithms. *IEEE Transactions on Automatic Control*, 45(2):299–305, 2000.
- [137] J. F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 1(11):625–653, 1999.
- [138] F. C. Schweppe. Recursive state estimation: Unknown but bounded errors and system inputs. *IEEE Transactions on Automatic Control*, 13(1):22–28, 1968.
- [139] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.
- [140] F. Giulietti, L. Pollini, and M. Innocenti. Autonomous formation flight. *IEEE Control Systems Magazine*, 20(6):34–44, 2000.
- [141] J. Cortes, X. Martinez, and F. Bullo. Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. *IEEE Transactions on Automatic Control*, 51(8):1289–1298, 2006.
- [142] A. Jadbabaie, J. Lin, and Morse A. S. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.
- [143] R. Olfati-saber. Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Transactions on Automatic Control*, 51(3):401–420, 2006.
- [144] W. Ren and R. Beard. *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*. Springer, 2008.
- [145] R. Olfati-saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.
- [146] W. Ren and R. W. Beard. Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.
- [147] L. Moreau. Stability of multi-agent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, 2005.
- [148] S. Martinez. A convergence result for multi-agent systems subject to noise. In *Proceedings of the American Control Conference*, pages 4537–4542, 2009.
- [149] G. Xie and L. Wang. Consensus control for a class of networks of dynamic agents. *Int. J. Robust Nonlinear Contr.*, 17(10-11):941–959, 2007.

- [150] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9):1465–1476, 2004.
- [151] Z. Qu, J. Wang, and R. A. Hull. Cooperative control of dynamical systems with application to autonomous vehicles. *IEEE Transactions on Automatic Control*, 53(4):894–911, 2008.
- [152] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. *Systems and Control Letters*, 53:65–78, 2004.
- [153] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. *IEEE Transactions on Information Theory*, 52(6):2508–2530, 2006.
- [154] G. A. de Castro and F. Paganini. Convex synthesis of controllers for consensus. In *Proceedings of the American Control Conference*, pages 4933–4938, 2004.
- [155] N. Z. Shor. *Nondifferential Optimization and Polynomial Problems*. Kluwer Academic Publishers, Netherlands, 1998.
- [156] D. P. Bertsekas. *Nonlinear Programming*. Athena Scientific, Belmont, MA, 1999.
- [157] B. T. Polyak. *Introduction to Optimization*. Optimization Software Inc., New York, 1987.
- [158] Y. Nesterov. *Introductory Lectures on Convex Optimization: A Basic Course*. Kluwer Academic Publishers, MA, 2004.
- [159] K. C. Kiwiel. Convergence of approximate and incremental subgradient methods for convex optimization. *SIAM J. Optim.*, 14(3):807–840, 2004.
- [160] P. R. Lawson. The terrestrial planet finder. In *Proceedings of the IEEE Aerospace Conference*, pages 2005–2011, 2001.
- [161] C. V. M. Fridlund. Darwin- the infrared space interferometry mission. *ESA Bulletin*, 103:20–63, 2000.
- [162] G. Krieger, A. Moreira, H. Fiedler, I. Hajnsek, M. Werner, M. Younis, and M. Zink. TanDEM-X: A satellite formation for high-resolution SAR interferometry. *IEEE Transactions on Geoscience and Remote Sensing*, 45(11):3317–3341, 2007.
- [163] C. Sabol, R. Burns, and C. A. McLaughlin. Satellite formation flying design and evolution. *Journal of Spacecraft and Rockets*, 38(2):270–278, 2001.
- [164] D. Massonnet and C. Elachi. High-resolution land topography. *Comptes Rendus Geoscience*, 338(14-15):1029–1041, 2006.
- [165] M. Martin, P. Klupar, S. Kilberg, and J. Winter. TechSat 21 and revolutionizing space missions using microsatellites. In *Proceedings of the AIAA Conference on Small Satellites*, 2001.
- [166] D.P. Scharf, F.Y. Hadaegh, and S. R. Ploen. A survey of spacecraft formation flying guidance and control (Part II): Control. In *Proceedings of the American Control Conference*, pages 2976–2985, 2004.

- [167] G. Sholomitsky, O. Prilutsky, and V. Rodin. Infra-red space interferometer. In *28th International Astronautical Federation Congress*, 1977.
- [168] P. K. C. Wang and F. Y. Hadaegh. Coordination and control of multiple microspacecraft moving in formation. *Journal of Astronautical Sciences*, 44(3):315–355, 1996.
- [169] D. P. Scharf, F. Y. Hadaegh, and S. R. Ploen. A survey of spacecraft formation flying guidance and control (Part I): Guidance. In *Proceedings of the American Control Conference*, pages 1733–1739, 2003.
- [170] K. T. Alfriend, S. R. Vadali, P. Gurfil, J. P. How, and L. S. Breger. *Spacecraft Formation Flying: Dynamics, control and navigation*. Elsevier Ltd., 2010.
- [171] V. Kapila, A. G. Sparks, J. M. Buffington, and Q. Yan. Spacecraft formation flying: Dynamics and control. *Journal of Guidance, Control, and Dynamics*, 23(3):561–564, 2000.
- [172] J. R. Carpenter. A preliminary investigation of decentralized control for satellite formations. In *Proceedings of the American Control Conference*, pages 63–74, 2000.
- [173] S.R. Starin, R.K. Yedavalli, and A.G. Sparks. Design of a lqr controller of reduced inputs for multiple spacecraft formation flying. In *Proceedings of the American Control Conference*, pages 1327–1332, 2001.
- [174] Q. Yan, V. Kapila, and M.S. de Queiroz. Nonlinear dynamics, trajectory generation, and output feedback control of multiple spacecraft in elliptical orbits. In *Proceedings of the American Control Conference*, pages 839–843, 2000.
- [175] M. Tillerson, G. Inalhan, and J. P. How. Co-ordination and control of distributed spacecraft systems using convex optimization techniques. *International Journal of Robust and Nonlinear Control*, 12:207–242, 2002.
- [176] F. Bacconi, E. Mosca, and A. Casavola. Hybrid constrained formation flying control of micro-satellites. *IET Control Theory and Applications*, 1(2):513–521, 2007.
- [177] H. Wong, V. Kapila, and A.G. Sparks. Adaptive output feedback tracking control of spacecraft formation. *International Journal of Robust and Nonlinear Control*, 12(2-3):117–139, 2002.
- [178] R. Pongvithithum, S. M. Veres, S. B. Gabriel, and E. Rogers. Universal adaptive control of satellite formation flying. *International Journal of Robust and Nonlinear Control*, 78(1):45–52, 2005.
- [179] Y. Xu, N. Fitz-Coy, R. Lind, and A. Tatsch. μ -control for satellite formation flying. *Journal of Aerospace Engineering*, 21(1):10–21, 2007.
- [180] B. Wu, D. Wang, E. K. Poh, and G. Xu. Nonlinear optimization of low-thrust trajectory for satellite formation: Legendre pseudospectral approach. *Journal of Guidance, Control, and Dynamics*, 32(4):1371–1381, 2009.

- [181] P. Gurfil and D. Mishne. Cyclic spacecraft formation: Relative motion control using line-of-sight measurements only. *Journal of Guidance, Control, and Dynamics*, 30(1):214–226, 2007.
- [182] D.P. Scharf, F.Y. Hadaegh, and S. R. Ploen. Precision formation delta-v requirements for distributed platforms in Earth orbit. In *Proceedings of SPIE - Enabling Sensor and Platform Technologies for Spaceborne Remote Sensing*, pages 312–323, 2005.
- [183] W. Ren and W. B. Beard. A decentralized scheme for spacecraft formation flying via the virtual structure approach. In *Proceedings of the American Control Conference*, pages 1746–1751, 2003.
- [184] J. A. Fax. *Optimal and Cooperative Control of Vehicle Formations*. PhD Thesis, California Institute of Technology, 2002.
- [185] G. M. Belanger, S. Ananyev, J. L. Speyer, D. F. Chichka, and J. R. Carpenter. Decentralized control of satellite clusters under limited communication. *Journal of Guidance, Control, and Dynamics*, 29(1):134–145, 2006.
- [186] U. Ulybyshev. Long-term formation keeping of satellite constellation using linear-quadratic controller. *Journal of Guidance, Control, and Dynamics*, 21(1):109–115, 1998.
- [187] J. E. Prussing and B. A. Conway. *Orbital Mechanics*. Oxford University Press, 1993.
- [188] W. H. Clohessy and R. S. Wiltshire. Terminal guidance system for satellite rendezvous. *Journal of the Aerospace Sciences*, 27(9):653–658, 1960.
- [189] S. A. Schweighart and R. J. Sedwick. High-fidelity linearized j_2 model for satellite formation flight. *Journal of Guidance, Control, and Dynamics*, 25(6):207–242, 2002.
- [190] K. Yamanaka and F. Ankersen. New state transition matrix for relative motion on an arbitrary elliptical orbit. *Journal of Guidance, Control, and Dynamics*, 25(1):60–66, 2002.
- [191] G. Inalhan, M. Tillerson, and J. P. How. Relative dynamics and control of spacecraft formations in eccentric orbits. *Journal of Guidance, Control, and Dynamics*, 25(1):48–59, 2002.
- [192] S. Leung and O. Montenbruck. Real-time navigation of formation-flying spacecraft using Global-Positioning-System measurements. *Journal of Guidance, Control, and Dynamics*, 28(2):226–235, 2005.