Fast numerical methods for image restoration

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FAST NUMERICAL METHODS FOR IMAGE RESTORATION

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Summary

In the computer vision field, most problems can be described as energy functionals. The optimums of these energy functionals are the solutions of the computer vision problems. The fast numerical methods seeking the solutions are fundamentally important and highly demanded.

We mainly solve three different essential computer vision problems: image denoising problem, image segmentation problem and surface reconstruction problem. We will review the critical models such as the Rudin, Osher and Fatemi (ROF) model, TV-L1 model and Euler's elastica model for denoising and related problems. The Mumford-Shah model and the Chan-Vese model are also included for solving segmentation problem. In surface reconstruction problem, the weighted minimal surface model is introduced as background.

In this thesis, we use two types of fast numerical methods for solving these energy minimization problems. The first one is multiplier based method to the augmented Lagrangian function of TV-L1 model, for image denoising and image fusion problems. The other one is graph cuts technique for fast solving higher order curvature based models. It has been applied to solve the image denoising, segmentation and surface reconstruction problems.

The main part of this thesis consists of eight chapters, each of which is made

as self-contained as possible. In Chapter 1, short introductions of the important image processing problems and surface reconstruction problem are given. The corresponding fundamental energy functionals are introduced. Later outline of the thesis is listed. In Chapter 2, we briefly review the numerical methods and relevant knowledge for solving the problems in the following chapters.

In Chapter 3, we study the TV-L1 image approximation model under the primal and dual perspective, propose equivalent convex formulations and fast multiplier based algorithm. More specifically, we apply a convex approach to globally solve the corresponding discrete constrained optimization problem. In Chapter 4, we extend the study to TV-L1 and TV-L2 image fusion models. We investigate a general convex form for TV-L1 and TV-L2 models, provide fast multiplier based algorithm and compare the results of TV-L1 and TV-L2 models.

In Chapter 5, we present an efficient minimization algorithm based on graph cuts for minimizing the energy of the Euler's elastica model. Curvature based variational models reveal more local information which help us to preserve the thin and elongated structures of the objects. Our approaches obtain better results than ROF model in image denoising and inpainting, also keep fast computation speed. In Chapter 6, we propose a model based on the Mumford-Shah model with curvature information for segmentation which can be solved efficiently by graph cuts.

In Chapter 7, we extend our study in 2-D image processing problems to 3-D surface reconstruction problem. We modify the weighted minimal surface model with curvature term for better feature preservation compared with the conventional works. A fast and effective two-stage reconstruction algorithm is provided to reconstruct surfaces from sparse point data sets.

At last, in Chapter 8, we conclude our works and discuss about some possible research topics for investigation in the future.

Chapter 1

Introduction

In this thesis, we introduce several important image processing problems and surface reconstruction problem arising in computer vision. These problems have importance to be solved and the results can be applied in real applications. In mathematical scenario, these problems are described as energy minimization models; the optimal solution of the problem is the global minimizer of the corresponding energy functional. To minimize the proposed energy functional is not a trivial task. Most energy functionals concerning image problems are difficult to be optimized. Various optimization tools were presented in the last two decades, the fast numerical methods to obtain the global minimizer are still highly demanded.

In this chapter, we mainly provide the descriptions of three essential image processing and computer vision problems: image denoising, image segmentation and surface reconstruction. We also introduce the famous and widely used variational models for solving these problems.

Figure 1.1: Image corrupted by noise

1.1 Image Denoising and Related Problems

Image denoising is an essential task in image processing since noise usually happens in the transmission process. Due to its common existence and the requirement of clear image for applications, denoising problem has drawn many attentions for years. Various energy minimization models are proposed for removing different types and levels of noise.

Firstly, we assume that the unknown clear image $u(\mathbf{x})$ is perturbed by the additive noise $\eta(\mathbf{x})$, which can be the gaussian white noise or impulsive noise. The image degradation process can be written as:

$$u^0(\mathbf{x}) = u(\mathbf{x}) + \eta(\mathbf{x}),$$

and as shown in Figure 1.1. The goal of image denoising is to recover an image which is as close as $u(\mathbf{x})$ from the observed image $u^0(\mathbf{x})$. The result of this process can be obtained by ordinary least squares which is a maximum likelihood method, let $\mathbf{x} = (x, y) \in \mathbb{R}^2$:

$$\int_{\Omega} (u - u^0)^2 d\mathbf{x}. \quad (1.1)$$

This denotes the L2-norm of error between the true image u and the input u^0 .

1.1.1 The Tikhonov Model

By introducing the Tikhonov regularization [121] in the continuous domain for solving image restoration problems, the Tikhonov model was firstly proposed:

$$\min_u \left\{ E(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{\alpha}{2} \int_{\Omega} (u - u^0)^2 d\mathbf{x} \right\}, \quad (1.2)$$

where

- $\Omega \subset \mathbb{R}^N$ denotes the image domain, which is usually a rectangle in $\mathbb{R}^2$ in this thesis;
- $|\nabla u| = \sqrt{u_x^2 + u_y^2}$;
- α is a relative parameter measuring the balance of the regularization term and the data fitting term.

The first term of (1.2) is the regularization term which is helpful to solve problem with the information: L2-norm of the gradient. The second term is called data fidelity term which makes sure that the outcome approximates the original data closely. The Tikhonov model was proved to remove the noise but cannot keep sharp discontinuities on the boundaries, due to the over penalty of the L2-norm of the gradient as the regularizer.

1.1.2 The Rudin-Osher-Fatemi (ROF) Model

Unlike the Tikhonov model, the authors proposed the ROF (Rudin-Osher-Fatemi) model which chose L1-norm of the gradient as the regularization term [103]. This regularization term which is called the total variation was firstly introduced for image denoising. Image with extreme details has high total variation value, i.e. the integral of the absolute value of the image gradient is high. We first introduce the total variation regularization and its properties here.

Total Variation Regularization

Total variation regularization is an important concept and plays fundamental role for solving image processing problems, many studies on total variation and TV regularization models are established. Here is the strict definition of total variation:

Definition 1.1. *In a bounded open domain $\Omega \subset \mathbb{R}^2$, the total variation of a function $u \in L^1(\Omega)$ is defined as:*

$$TV(u) = \sup_p \left\{ \int_{\Omega} u \operatorname{div} p d\mathbf{x} : p \in C_c^1(\Omega, \mathbb{R}^2), |p(\mathbf{x})| \leq 1, \forall \mathbf{x} \in \Omega \right\}. \quad (1.3)$$

By this definition $TV(u)$ is finite if and only if the distribution derivative Du is a finite Radon measure in domain Ω , so that $TV(u) = |Du|(\Omega)$. If the gradient of u which is $\nabla u \in L^1(\Omega; \mathbb{R}^2)$, then the total variation term of a function $u : \Omega \mapsto \mathbb{R}$ can be defined as:

$$TV(u) = \int_{\Omega} |\nabla u| d\mathbf{x}. \quad (1.4)$$

The function whose total variation semi-norm has a finite value is said to have bounded variations. The space $BV(\Omega)$ identifies the space of functions of globally bounded variations, which is defined as

$$BV(\Omega) = \{u \in L^1(\Omega) : TV(u) < \infty\}.$$

In $BV(\Omega)$ space, the functions are allowed to keep sharp discontinuities.

The Coarea Formula

BV functions hold an important property which is studied and applied extensively, called the coarea formula [50].

Theorem 1.1. *Let function $u \in BV(\Omega)$, and U^γ is denoted as the γ -upper level set of u ,*

$$U^\gamma = \{\mathbf{x} \in \Omega \subset \mathbb{R}^2 : u(\mathbf{x}) \geq \gamma\},$$

and let u^γ denote its characteristic function:

$$u^\gamma = \begin{cases} 1 & \text{if } u(\mathbf{x}) \geq \gamma, \\ 0 & \text{else.} \end{cases}$$

Then

$$TV(u) = \int_{-\infty}^{\infty} |\partial U^\gamma| d\gamma = \int_{\Omega} |\nabla u^\gamma| d\mathbf{x}, \quad (1.5)$$

which is called the coarea formula.

Here $|\partial U^\gamma|$ is the curve length of ∂U^γ , it illustrates that the total variation term is equivalent to the sum of the length of all level sets of $u(\mathbf{x})$.

The ROF model which is proved to remove gaussian noise and keep the edges in the mean time, is defined as the energy minimization problem with constraint:

$$\min_u \int_{\Omega} |\nabla u| d\mathbf{x}, \quad \text{s.t.} \quad \int_{\Omega} (u - u^0)^2 d\mathbf{x} = \sigma^2, \quad (1.6)$$

where σ^2 denotes a bound on the noise variance. In [29], Chambolle replaced the constraint as $\int_{\Omega} (u - u^0)^2 d\Omega \leq \sigma^2 |\Omega|$, so that we have the unconstrained ROF model:

$$\min_u \left\{ E(u) = \int_{\Omega} |\nabla u| + \frac{\alpha}{2} \int_{\Omega} (u - u^0)^2 d\mathbf{x} \right\}, \quad (1.7)$$

where α is the relative parameter. The ROF model holds important property that it is convex, so that the exact optimal solution can be found. The shortage is in the uniform background regions, the unwanted block-like artifacts which are so called staircasing effects exist.

Nowadays, the researchers mostly focus on solving the corresponding but simpler unconstrained model (1.7) instead of the original constrained model (1.6). A large number of researches on the ROF model and follow-ups have been proposed and drawn lots of attentions [103, 37, 26, 129, 119, 33, 127, 59].

1.1.3 Total Variation with L1 Data Fidelity Term (TV-L1) Model

Recalling that the data fidelity of the ROF model is L2-norm. If the data fidelity term is the absolute function, we call it TV-L1 model. First introduced for solving the impulsive noise removal problem, the TV-L1 model is defined as:

$$\min_u \left\{ \int_{\Omega} |\nabla u| \, d\mathbf{x} + \alpha \int_{\Omega} |u - u^0| \, d\mathbf{x} \right\}, \quad (1.8)$$

where α is a parameter.

TV-L1 model holds some valuable properties: It has good performance in removing impulsive noise e.g. salt and pepper noise; it is also contrast invariant, i.e. u is the solution for one noisy image u^0 , then cu is also the solution for input cu^0 . However, TV-L1 model is not strictly convex, meaning that there are possibly multiple global minimizers and difficult to compute numerically.

1.1.4 Euler's Elastica Denoising Model

Higher order models have been developed for yielding more natural results and preventing staircasing effect of second order models. Higher order models which minimize the energy functionals based on the curvature of the image are fundamentally important [34, 104, 120, 143]. The Euler's elastica model is of central importance, which minimizes the curvature of all level lines in the image.

The Euler's elastica of all level curves of an image u can be written as:

$$\int_{\ell=0}^L \int_{\gamma_{\ell}: u=\ell} (a + b \cdot |\kappa|^{\beta}(s)) ds d\ell,$$

where s is the arc length, the level curve γ_{ℓ} is $u = \ell$. The power β can be set to $\beta = 1$ as in [85], or $\beta = 2$ as in [34].

Note that the curvature of the level curve is:

$$\kappa = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right).$$

Figure 1.2: Inpainting process

Using this fact, the Euler's elastica model can be expressed more simply as:

$$\int_{\Omega} \left(a + b \left| \nabla \cdot \frac{\nabla u}{|\nabla u|} \right|^{\beta} \right) |\nabla u| d\mathbf{x}. \quad (1.9)$$

For image denoising, the Euler's elastic energy (1.9) is used as regularization term, with L2-norm data fidelity term the complete energy functional can be written as:

$$\min_u \left\{ E(u) = \int_{\Omega} |u - u^0|^2 d\mathbf{x} + \int_{\Omega} \left(a + b \left| \nabla \cdot \frac{\nabla u}{|\nabla u|} \right|^{\beta} \right) |\nabla u| d\mathbf{x} \right\}. \quad (1.10)$$

1.1.5 Related Problem I: Image Inpainting

Image inpainting is filling in missing or ruined parts of incomplete images based on information collected from vicinity. The incomplete part can be disturbed by noise as shown in Figure 1.2. The inpainting problem is closely related to denoising problem and one can be translated from the other. The main difference is in inpainting problems, the missing parts are bigger.

Total Variation Inpainting Model

In [36], the author proposed total variation inpainting model which is very similar as ROF model. We assume $\tilde{\Omega} \subset \Omega$ is the domain of missing information. The TV inpainting model is then formulated as:

$$\min_u \left\{ E(u) = \int_{\tilde{\Omega}} |\nabla u| d\mathbf{x} \quad \text{s.t.} \quad u = u^0 \quad \text{on } \partial\tilde{\Omega} \right\},$$

where $\partial\tilde{\Omega}$ denotes the piecewise smooth boundary of $\tilde{\Omega}$. The TV inpainting model works well if the domain of missing data $\tilde{\Omega}$ is not large. Since the regularization term is a penalty on the length of boundaries, if $\tilde{\Omega}$ is large the disconnected results will be reached. In Chapter 5 we will provide the numerical results.

Euler's Elastica Inpainting Model

In [34], the authors proposed the Euler's elastica inpainting model to overcome the shortage of TV inpainting model, where the energy functional is:

$$\min_u \left\{ E(u) = \int_{\tilde{\Omega}} \left(a + b \left| \nabla \cdot \frac{\nabla u}{|\nabla u|} \right|^\beta \right) |\nabla u| d\mathbf{x} \quad \text{s.t.} \quad u = u^0 \quad \text{on } \partial\tilde{\Omega} \right\},$$

where $\partial\tilde{\Omega}$ denotes the boundary of $\tilde{\Omega}$. This model is very similar to the Euler's elastica denoising model, it has better results than TV inpainting model for recovering the missing parts by involving curvature information.

1.1.6 Related Problem II: Image Fusion

Image fusion is a topic to visualize information from multiple images in one single image as shown in Figure 1.3. The images can be obtained from different focuses or even from different image sources. The problem has drawn many attentions because of wide applications such as medical imaging. The image fused from MRI and CT converts more information than single image and illustrates significant details for doctor diagnose.

Image fusion model can be easily extended from denoising models, the difference is that there are two or even more observed images. Given two input images f_1 and f_2 , in [126] the authors proposed a TV based method for image fusion:

$$\min_u \left\{ E(u) = \frac{1}{2} \int_{\Omega} w_1 (u - f_1)^2 d\mathbf{x} + \frac{1}{2} \int_{\Omega} w_2 (u - f_2)^2 d\mathbf{x} + \alpha \int_{\Omega} |\nabla u| d\mathbf{x} \right\}, \quad (1.11)$$

where the functions $w_1(\mathbf{x})$ and $w_2(\mathbf{x})$ are the pixelwise weight functions such that

$$w_1(\mathbf{x}) + w_2(\mathbf{x}) = 1, \quad w_{1,2}(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in \Omega. \quad (1.12)$$

Figure 1.3: Image fusion process

The convex optimization model with the L1-norm data fidelity term will be studied in Chapter 4:

$$\min_u \left\{ E(u) = \int_{\Omega} w_1 |u - f_1| \, d\mathbf{x} + \int_{\Omega} w_2 |u - f_2| \, d\mathbf{x} + \alpha \int_{\Omega} |\nabla u| \, d\mathbf{x} \right\}. \quad (1.13)$$

While in [70] the authors also studied the similar formulation with the constant weight functions.

1.2 Image Segmentation

Image segmentation problem is separating the image domain Ω into several distinct subregions $\{\Omega_i\}_{i=1}^n$, based on image intensity u^0 . These distinct subregions cover the whole image domain without overlapping as shown in Figure 1.4.

$$\Omega = \cup_{i=1}^n \Omega_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad \forall i \neq j.$$

The segmentation results are usually used to simplify the images for further analysis and track objects or edges of images. The fast numerical methods which can distinguish the segmentation of homogeneous intensity regions are highly demanded.

Figure 1.4: Fourphases segmentation example

1.2.1 The Mumford-Shah Model

The general Mumford-Shah model [89] is based on piecewise smooth approximation of the intensity function, the energy functional is:

$$\min_{u, \Gamma} \left\{ E(u, \Gamma) = \int_{\Omega} (u - u^0)^2 d\mathbf{x} + \mu \int_{\Omega \setminus \Gamma} |\nabla u|^2 d\mathbf{x} + \nu \int_{\Gamma} d\xi \right\}, \quad (1.14)$$

where Γ is a discontinuity set containing edges, μ and ν are the tuning parameters. The first term is the fidelity term ensuring that the solution u approximates the observed image u^0 . The second term makes sure that u is smooth except on edges. The last term penalizes the length of all the edges to ensure the perimeters should be short.

1.2.2 The Chan-Vese Model

The level set method was first introduced by Osher and Sethian in [94] for following fronts propagation. The most attractive feature of this method is the ability to handle unknown topology, which is common in real applications. Given an open subset $\Omega_{in} \subset \Omega \subset \mathbb{R}^N$, where Ω is a bounded domain. $\partial\Omega_{in}$ denotes the boundary of Ω_{in} , which can be expressed as the zero level set of a level set function ϕ as follows:

$$\begin{cases} \phi(\mathbf{x}) > 0, & \text{for } \mathbf{x} \in \Omega_{in}, \\ \phi(\mathbf{x}) < 0, & \text{for } \mathbf{x} \in \Omega \setminus \Omega_{in} \setminus \partial\Omega_{in}, \\ \phi(\mathbf{x}) = 0, & \text{for } \mathbf{x} \in \partial\Omega_{in}. \end{cases} \quad (1.15)$$

If $N = 2$, the boundary $\partial\Omega_{in}$ is a curve; if N is chosen as 3, $\partial\Omega_{in}$ represents the two dimensional surface.

Based on the level set method, in [32] Chan-Vese proposed to approximate the perimeter term $\int_{\Gamma} d\xi$ of (1.14) by the zero level set function of a Lipschitz continuous function $\phi : \Omega \mapsto \mathbb{R}$.

Recalling the definitions of the characteristic function χ^- and χ^+ of the interior and the exterior regions:

$$\chi^-(\mathbf{x}) = \begin{cases} 1, & \text{if } \phi(\mathbf{x}) \leq 0, \\ 0, & \text{if } \phi(\mathbf{x}) > 0. \end{cases} \quad \chi^+(\mathbf{x}) = \begin{cases} 0, & \text{if } \phi(\mathbf{x}) \leq 0, \\ 1, & \text{if } \phi(\mathbf{x}) > 0. \end{cases}$$

And one dimensional Heaviside function:

$$H(\phi) = \begin{cases} 0, & \text{if } \phi(\mathbf{x}) \leq 0, \\ 1, & \text{if } \phi(\mathbf{x}) > 0. \end{cases} \quad (1.16)$$

There is a relationship between the characteristic function and the Heaviside function: $\chi^-(\mathbf{x}) = 1 - H(\phi(\mathbf{x}))$, $\chi^+(\mathbf{x}) = H(\phi(\mathbf{x}))$.

Hence, the two-phase Chan-Vese model is:

$$\min_{\phi, c_1, c_2} \left\{ E(\phi, c_1, c_2) = \int_{\Omega} (H(\phi)(u^0 - c_1)^2 + (1 - H(\phi))(u^0 - c_2)^2 + \nu |\nabla H(\phi)|) d\mathbf{x} \right\}.$$

It has the best approximation in the L2 sense of the input u^0 as two phases.

1.3 Surface Reconstruction

Reconstructing a surface from unorganized point data set is a challenging task since the problem is ill-posed. Besides, noises, outliers, undersampling and non-uniformity are usual reconstruction difficulties and frequently observed in data acquisition procedure. Moreover, there is always no priori knowledge about the ordering of data points or the complicated surface topology. These difficulties make the surface reconstruction even more complicated. The connection between discrete data points and continuous surface is need to be recovered. This is one

Figure 1.5: Surface reconstruction example

of most important and complicated problems in computer vision. One example is given in Figure 1.5.

1.3.1 The Weighted Minimal Surface Model

The weighted minimal surface model was proposed in [140] and widely used for surface reconstruction:

$$\min_{\mathcal{S}} \left\{ E(\mathcal{S}) = \left(\int_{\mathcal{S}} d^p(x) ds \right)^{\frac{1}{p}}, 1 \leq p \leq \infty \right\},$$

where $\mathcal{S}$ is an arbitrary surface and ds is the surface area. $d(x) = d(x, P)$ here is the distance function from the point x to the nearest point of data set P . This model consists only of a fidelity term, suggesting that result surface would be as close to the input data as possible. This model can be approached by gradient descent method with good initial guess [140].

1.4 Outline of the Thesis

In this thesis, we first review some important models in image processing and computer vision as background. We introduce the fast continuous multiplier based method to approximate TV-L1 for denoising and fusion exactly. Then we focus on graph cuts for fast solving complicated higher order Euler's elastica

model for image denoising, inpainting and segmentation. Later, we extend the success to the surface reconstruction problem.

In Chapter 2, we mainly review the numerical methods for energy functionals of general form. We introduce the method of solving the associated Euler-Lagrange equation of variational model, the multiplier based methods and the graph cuts methods for the investigations in the following chapters.

In Chapter 3, we study the TV-L1 image approximation model under the primal and dual perspective and propose equivalent convex formulations. We also apply a convex approach to globally solve the discrete constrained optimization problem of image approximation. The essential contributions of this part can be summarized as:

1. We propose equivalent formulations under primal and dual perspective, and build up an analytical framework which results in a new variational variant of the original TV-L1 model.
2. The proposed equivalent TV-L1 formulation can be used as the convex relaxed model of its relevant discrete constrained image processing task.
3. We introduce a fast multiplier based algorithm which explores the equivalent primal-dual formulation through two simple projection substeps.

In Chapter 4, we extend the study of TV-L1 model and propose two variational approaches to image fusion which are closely related to the standard TV-L1 and TV-L2 image fusion models. We investigate their convex optimization formulations under the perspective of primal and dual, and provide fast multiplier based algorithms. Our main contributions in this part are:

1. We propose the equivalent convex formulations for TV-L1 and TV-L2 fusion models then provide fast multiplier based algorithms.
2. We show the studied image fusion models actually result in two new image decompositions of the weighted input image, which extend the convex relaxation models of TV-L1 approximation.

To overcome limitations, such as staircasing effects of the total variation (TV) regularized models, variational models involving higher order derivatives have been proposed and studied. In Chapter 5, we take curvature into consideration and present an efficient minimization algorithm based on graph cuts for minimizing the energy in the Euler’s elastica denoising model. We simplify the problem as a sum of easy graph representable problems. Our approach achieves better results than total variation (TV) in image denoising and inpainting, and keeps fast computation speed. The main contributions can be summarized as:

1. We propose an approach to show that the curvature based variational model can be simplified to that of solving a sequence of second order submodular markov random field models (MRFs).
2. Our approach is more efficient than previous gradient descent methods, thanks to the underlying graph cuts minimization technique.
3. This approach also theoretically eliminates metrication errors in the solutions.

Inspired by the performance of curvature based models in image denoising and inpainting problems. In Chapter 6, we propose a model based on the Mumford-Shah model with curvature information for image segmentation and solve it efficiently. We construct a graph so that the discrete variational model can be minimized exactly by finding the minimum cut via graph cuts. In this part, our dedications can be summarized as:

1. Curvature based variational model for image segmentation is proposed and solved by graph cuts efficiently.
2. Our method involves curvature information which is helpful to keep thin and elongated structures even the input image is noisy.

Above all are our works on 2-D image processing problems, we then extend the novel combination of curvature based variational model and efficient graph

cuts technique to 3-D surface reconstruction problem. In Chapter 7, we modify the weighted minimal surface model with curvature term for better feature preservation and introduce a fast hybrid approach. The advantages of our approach are:

1. The proposed surface reconstruction model, which involves curvature, allows varying flexibility and preserves feature details.
2. We provide a fast and effective two-stage reconstruction algorithm to recover surfaces from sparse point data sets. The merit is that we combine the effective and fast global reconstruction via graph cuts, with the accuracy of local swap later.
3. The energy discretization is conducted on the tetrahedral mesh rather than regular mesh, which helps to keep sharp features intrinsically.

In Chapter 8, we conclude our works which can be shown better than others numerically. Some possible research directions are discussed and will be solved in the future.

Numerical Methods

In this chapter, we briefly review three different numerical methods which can be used to minimize the energy functionals. The numerical methods will be explained based on general forms of energy functionals. In the following chapters, our energy functionals satisfy these general forms and the details will be given then.

2.1 Solving Euler-Lagrange Equations of Energy Functionals

The straightforward and traditional method of solving the energy minimization problem is solving its associated Euler-Lagrange equation. Thus, we first introduce the *Gâteaux* derivative: if the given energy functional $E(u)$ defined on a Banach space $\mathcal{X}$,

Definition 2.1. *The Gâteaux derivative of $E(u)$ at u in the direction of v is defined by:*

$$\frac{\partial E}{\partial u}v = \lim_{\varepsilon \rightarrow 0} \frac{E(u + \varepsilon v) - E(u)}{\varepsilon}. \quad (2.1)$$

If there exists $\tilde{u} \in \mathcal{X}^$ (the dual space of $\mathcal{X}$) s.t.*

$$\frac{\partial E}{\partial u}v = \tilde{u}(v) := \langle \tilde{u}, v \rangle, \forall v \in \mathcal{X}, \quad (2.2)$$

then $E(u)$ is Gâteaux differentiable at u , and the derivative value at u is $\tilde{u}$.

If

$$\frac{\partial E}{\partial u} = 0,$$

it is called the Euler-Lagrange equation for $E(u)$ and the point $u \in \mathcal{X}$ is a critical point of $E(u)$. This equation is a necessary condition for u being a local minimizer of the energy functional $E(u)$.

The local minima of energy functional correspond to the steady state of evolutions. The corresponding Euler-Lagrange equation of energy functional $E(u)$ can be obtained by:

$$\left. \frac{d}{d\varepsilon} E(u + \varepsilon v) \right|_{\varepsilon=0} = \frac{\partial E}{\partial u} v = 0.$$

By introducing an artificial time variable t and then applying the time marching/gradient descent method, the solution is obtained until the steady state is reached by iterations:

$$u^{n+1} = u^n - dt \frac{\partial E}{\partial u}.$$

This method is standard and easy, however it relies on the choices of parameters, the initial guess of the boundary and the change of topology.

2.2 Multiplier Based Methods

Multiplier based method was introduced in [61] for finding a minimum of a function subject to some constraints, later there are many relevant literature for minimizing a convex function subject to linear equality constraints in numerical analysis [56]. The multiplier based methods have been recognized as a class of essential methods for minimization problems subject to equality constraints and can also be applied to total variation minimization [69].

Firstly, a general form of such a problem is:

$$\min_u f(u) \quad \text{s.t.} \quad u \in \mathcal{X}, \quad h_1(u) = h_2(u) = \dots = h_m(u) = 0, \quad (2.3)$$

where $f, h_1, h_2, \dots, h_m$ are real-valued functions on $\mathbb{R}^N$, and $\mathcal{X} \subset \mathbb{R}^N$ is given.

One class for solving the problem (2.3) is minimizing a sequence of the Lagrangian function, by introducing Lagrangian multiplier $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$:

$$L(u, \lambda) = f(u) + \sum_{i=1}^m \langle \lambda_i, h_i \rangle. \quad (2.4)$$

The minimizer can be obtained by alternative iterations:

$$\begin{aligned} u^k &= \arg \min_u L(u, \lambda^k), \\ \lambda_i^{k+1} &= \lambda_i^k + h_i(u^k), \quad i = 1, \dots, m. \end{aligned} \quad (2.5)$$

The above iteration can be regarded as the steepest ascent iteration for finding an optimum of the corresponding dual problem. Hence, this algorithm is also a primal-dual method.

In multiplier based methods, the penalty function method is combined with primal-dual philosophy and the penalty term is added to the Lagrangian function (2.4) rather than objective function f . The corresponding augmented Lagrangian function is:

$$L_c(u, \lambda) = f(u) + \sum_{i=1}^m \langle \lambda_i, h_i \rangle + \frac{c}{2} \sum_{i=1}^m \|h_i\|^2, \quad (2.6)$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the standard inner product and Euclidean norm respectively and parameter $c > 0$. A sequence of minimizations of (2.6) is performed, the minimizer u^k and the multiplier sequence $\{\lambda^k\}$ are obtained by the iterations:

$$\begin{aligned} u^k &= \arg \min_u L_c(u, \lambda^k), \\ \lambda_i^{k+1} &= \lambda_i^k + c^k h_i(u^k), \quad i = 1, \dots, m. \end{aligned}$$

Convex Analysis

We also provide some standard definitions and results from convex analysis and duality theory in Banach spaces $\mathcal{X}$ for the later chapters.

Definition 2.2. A function $f : \mathcal{X} \rightarrow (-\infty, \infty]$ is called:

- **convex** if for all $x, y \in \mathcal{X}$ and $0 \leq t \leq 1$,

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y);$$

- **proper** if it is not identically ∞ ;
- **lower semicontinuous (l.s.c)** at $x \in \mathcal{X}$ if

$$f(x) \leq \liminf_{y \rightarrow x} f(y).$$

A function f is l.s.c. if it is l.s.c. at all $x \in \mathcal{X}$.

Definition 2.3. The dual space $\mathcal{X}^*$ of $\mathcal{X}$ is consisting of all linear functionals on $\mathcal{X}$, i.e.

$$\mathcal{X}^* = \{\varphi : \mathcal{X} \mapsto \mathbb{R}, \quad \text{s.t. } \varphi \text{ is linear.}\} \quad (2.7)$$

Definition 2.4. The Legendre-Fenchel conjugate f^* of f is defined for each $y \in \mathcal{X}^*$ as follows:

$$f^*(y) = \sup_{x \in \mathcal{X}} \langle x, y \rangle - f(x). \quad (2.8)$$

Theorem 2.1. If f is a convex, proper and l.s.c. function on $\mathcal{X}$, so that $f^{**} = f$.

Definition 2.5. Let $f : \mathcal{X} \rightarrow (-\infty, \infty]$, the subdifferential of f at x is defined as:

$$\partial f(x) = \{x^* \in \mathcal{X}^* : f(y) - f(x) \geq \langle x^*, y - x \rangle, \forall y \in \mathcal{X}\} \quad (2.9)$$

2.3 Graph Cuts Methods

Graph optimization algorithms are efficient in finding the minimizers of the discrete energy problems. Since its being first introduced for binary image problems in [60], a large number of researches on graph cuts methods have been proposed in [19, 17, 67, 40, 41, 27, 28].

The basic idea of graph cuts methods is: in order to solve a given computer vision problem, a discrete energy function is formulated; then a special graph with nodes and edges is designed and constructed; later cut to two or several partitions. The minimum cut (min-cut) on the graph also minimizes the energy function. In the following, a brief review of graph theory and graph representable form are provided as background.

Graph Theory

First of all, we introduce and define directed capacitated graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. $\mathcal{V}$ is defined as a set of nodes and $\mathcal{E}$ is a set of directed edges that connect all the nodes. There are two distinguished vertices in $\mathcal{V}$: s and t are called the source and the sink. For each edge $(a, b) \in \mathcal{E}$, there is a capacity $c(a, b)$ which means the maximum flow that can go through the edge (a, b) . A cut on the graph $\mathcal{G}$ partitions the vertices into two different groups source set S and sink set T while $s \in S$ and $t \in T$. A cut is a binary partition of a graph which can be regarded as a labeling and the cost of the cut equals to the sum of capacities of all edges which go from S to T :

$$c(S, T) = \sum_{a \in S, b \in T, (a, b) \in \mathcal{E}} c(a, b).$$

Finding the cut with the smallest cost is the min-cut problem, which is defined as:

Definition 2.6. *The min-cut problem is to find the cut which minimizes:*

$$\min_{(S, T)} \sum_{a \in S, b \in T, (a, b) \in \mathcal{E}} c(a, b) \quad s.t. \quad S \cap T = \emptyset, S \cup T = \mathcal{V}, s \in S, t \in T. \quad (2.10)$$

Besides the capacity, flow is a function $f(a, b)$ which assigns a real value to each edge $(a, b) \in \mathcal{E}$. The neighborhood system of $a \in \mathcal{V}$ is defined as follows:

$$\begin{cases} \mathcal{N}_a^+, & \text{if } (a, b) \in \mathcal{E}, \text{ for all } b \in \mathcal{V}, \\ \mathcal{N}_a^-, & \text{if } (b, a) \in \mathcal{E}, \text{ for all } b \in \mathcal{V}. \end{cases}$$

There are constraint conditions of the flow $f(a, b)$ on directed capacitated graph:

- The capacity of an edge is the upper bound of the flow through this edge:

$$f(a, b) \leq c(a, b), \quad \forall (a, b) \in \mathcal{E}. \quad (2.11)$$

- for each node $\forall a \in \mathcal{V} \setminus \{s, t\}$,

$$\sum_{b \in \mathcal{N}_a^+} f(a, b) - \sum_{b \in \mathcal{N}_a^-} f(b, a) = 0. \quad (2.12)$$

The max-flow problem should satisfy the constraint conditions (2.11) and (2.12), and be defined as:

Definition 2.7. *The max-flow problem is to find the flow which maximizes:*

$$\max_f \sum_{a \in \mathcal{V}} f(s, a).$$

The min-cut equals to the max-flow value due to the duality theorem of Ford and Fulkerson [51].

Theorem 2.2. *The maximum amount of an $s - t$ flow equals to the minimum capacity of an $s - t$ cut.*

Hence max-flow algorithms can be applied to find the minimizer of the corresponding energy functionals. In order to solve such max-flow/min-cut problem, there are mainly two groups of algorithms: Ford and Fulkerson style "augmenting paths" in [51]; Goldberg and Tarjan style "push-relabel" in [57].

Graphs Representation

For energy minimization problem, a general and standard form of energy function can be given as follows:

$$\min_u \left\{ E(u) = \sum_{p \in \Omega_d} D_p(u_p) + \sum_{p \in \Omega_d, q \in \mathcal{N}_p^k} V_{p,q}(u_p, u_q) \right\}. \quad (2.13)$$

For each $p = (i, j) \in \Omega_d$, $u = \{u_p | p \in \Omega_d\}$ is a labeling which minimizes the energy functionals, where

- $\Omega_d = \{(i, j) : i = 1, \dots, N, j = 1, \dots, M\}$ is the set of all pixels.
- $\mathcal{N}_p^k$ is the k neighborhood system of pixel p .
- $D_p(u_p)$: the data penalty function that measures the cost of appointing the label u_p to the pixel p .
- $V_{p,q}(u_p, u_q)$: the interaction capacity which measures the cost of assigning labels u_p, u_q to the pixel p and its neighboring pixels q .

In order to understand what kind of energy can be minimized by graph cuts, we first state an important theorem in combinatorial optimization.

Theorem 2.3. *Let $x_i \in \{0, 1\}$ and let $E(x_1, x_2, \dots, x_n) = \sum_{i,j} a_{ij}x_i x_j + L$, where $L = \sum_i a_i x_i + \text{const}$. Then E can be minimized by graph cut techniques if and only if $a_{ij} \leq 0$ for all i, j .*

Here for any configuration $x_1, \dots, x_n$, in which $\nu_i \in S$, if $x_i = 0$, and $\nu_i \in T$, if $x_i = 1$ for all $\nu_i \in \mathcal{V}$, $i = 1, \dots, n$.

In [73] the authors provided the necessary and sufficient conditions for the energy minimization via graph cuts. Both energy functions with interactions between pairwise pixels ($\mathcal{F}^2$ class) and those with interactions among triples of pixels ($\mathcal{F}^3$ class) were studied. The extension to $\mathcal{F}^k$ class was also introduced in [54]. In later chapters, we mainly focus on solving a sequence of binary graph representable problems so that the theorem for $\mathcal{F}^2$ class is provided here.

Theorem 2.4. *Let E be a function of n binary variables of the following form:*

$$E(x_1, \dots, x_n) = \sum_i E^i(x_i) + \sum_{i < j} E^{i,j}(x_i, x_j). \quad (2.14)$$

Then, E is graph-representable if and only if each term $E^{i,j}$ satisfies the inequality

$$E^{i,j}(0, 0) + E^{i,j}(1, 1) \leq E^{i,j}(0, 1) + E^{i,j}(1, 0). \quad (2.15)$$

Computational Complexity

In the research of computational complexity, if there is an algorithm which can compute the exact solution in polynomial time, the problem is called polynomial solvable. The group of polynomial solvable problems is denoted as P . Meanwhile, NP refers to the group of problems which are not polynomial solvable. NP is one of the most essential complexity classes. The complexity class P is contained in NP .

The time complexity of "augmenting paths" [51] is $O(\mathcal{V}\mathcal{E}^2)$ and "push-relabel" algorithm [57] is $O(\mathcal{V}^2\mathcal{E})$. Both are polynomial, which suggest P problems. Another type of graph cut problems, N -way cut problems, also draw a lot of attentions from potential applications such as multiphase segmentation [10]. However, this N -way cut problem has been shown to be NP -hard.

A Convex and Exact Approach to Discrete Constrained TV-L1 Image Approximation

3.1 Introduction

Many tasks of image processing can be formulated and solved successfully by convex optimization models, e.g. image denoising [103, 96], image segmentation [21], image labeling [99, 12] etc.

The reduced convex formulations can be studied in a mathematically sound way and usually tackled by a tractable scheme in numerics. Minimizing the total variation function for such convex image processing formulations is of great importance [103, 117, 87, 21, 91, 23, 66, 62], which preserves edges and sharp features.

In the pioneer works [30, 31], Chan et al. proposed the TV-L1 regularized image approximation model:

$$\min_{u(\mathbf{x}) \in BV(\Omega)} \left\{ E(u(\mathbf{x})) := \int_{\Omega} |\nabla u(\mathbf{x})| \, d\mathbf{x} + \alpha \int_{\Omega} |u^0(\mathbf{x}) - u(\mathbf{x})| \, d\mathbf{x} \right\}, \quad (3.1)$$

which was first introduced and studied by Alliney [4, 3] for discrete one-dimensional

signals' denoising. Chan et al. showed an interesting property of the TV-L1 model (3.1): for the input binary image $u^0(\mathbf{x}) \in \{0, 1\}$, there exists at least one global optimum $u(\mathbf{x}) \in \{0, 1\}$. It follows that the convex TV-L1 formulation (3.1) actually solves the non convex optimization problem:

$$\min_{u(\mathbf{x}) \in \{0,1\}} \left\{ \int_{\Omega} |\nabla u(\mathbf{x})| d\mathbf{x} + \alpha \int_{\Omega} |u^0(\mathbf{x}) - u(\mathbf{x})| d\mathbf{x} \right\}, \quad (3.2)$$

globally and exactly. Hence (3.1) provides an exact convex relaxation of the binary constrained optimization problem (3.2). We further proved that rounding the computed result of (3.1) may give a series of global optimums of the binary constrained optimization model (3.2).

3.1.1 Previous Works and Motivations

With the help of coarea formula, Chan et al. [30, 31] proved that the energy functional $E(u)$ of (3.1) can be equivalently represented in terms of the upper level set sequence of the image functions $u(\mathbf{x})$ and $u^0(\mathbf{x})$, i.e.

$$E(u(\mathbf{x})) = \int_{-\infty}^{+\infty} \{ |\partial U^\gamma| + \alpha |U^\gamma \triangle F^\gamma| \} d\gamma, \quad (3.3)$$

where for each γ , U^γ and F^γ give the γ -upper level set of the unknown $u(\mathbf{x})$ and the input $u^0(\mathbf{x})$ respectively, such that

$$U^\gamma(\mathbf{x}) = \begin{cases} 1, & \text{when } u(\mathbf{x}) > \gamma \\ 0, & \text{when } u(\mathbf{x}) \leq \gamma \end{cases}, \quad \mathbf{x} = (x, y) \in \Omega. \quad (3.4)$$

$|\partial U^\gamma|$ gives the perimeter of U^γ and $|U^\gamma \triangle F^\gamma|$ denotes the area of the symmetric difference of the two level sets. The symmetric difference of two sets is the set of elements which are in either of the sets and not in their intersection as shown in Figure 3.1 the grey parts.

Yin et al. [133] pointed out that minimizing such a layer-wise energy function (3.3) actually amounts to properly stacking the optimal U^γ s, which corresponds to solving (3.2) for each given binary indicator function of F^γ . In other words, solving (3.1) can be reduced to optimizing a sequence of binary constrained problems

Figure 3.1: Symmetric difference of two sets

as (3.2). Since $U^{\gamma_1} \subset U^{\gamma_2}$ when $\gamma_1 \geq \gamma_2$, the process recovers the optimum $u^*(\mathbf{x})$ of (3.1) by properly arranging all the associated level sets U^γ , $\gamma \in (-\infty, +\infty)$. The same result was also discovered by Darbon et al. [40, 41] in an image graph setting where the anisotropic total variation term is considered and a fast graph cut based algorithm was introduced. Goldfarb and Yin [133] also developed an efficient pre-flow based graph cut approach to such L1 image approximation regularized by discretized total variation.

However, in the spatial continuous setting as stated in [133], such approach means both bad and good news for processing the gray-scale image in practice: on the one hand, the total number of gray values is finite, i.e. $u(\mathbf{x}) \in \{0, \dots, 255\}$, hence only a finite number of optimization problems (3.2) should be considered; on the other hand, solving (3.2) for each layer F^γ is not trivial; and in order to globally tackle (3.1), one has to examine a large number of obtained level sets to restrict its search legally. This makes such computation by direct addressing multiple layered level sets impractical to a real image processing task with a large number of different gray values. A similar and interesting work in spatially continuous image labeling along linearly ordered labels, i.e. layered level sets, was recently addressed in [13].

In addition, the PDE-descent method is often taken to numerically approximate the global optimum of (3.1) [31, 30, 133, 45], which smooths the total variation term by $\sqrt{u_x^2 + u_y^2 + \varepsilon^2}$. Actually, even if ε takes a small enough value, the coarea formula is no longer satisfied. As a matter of fact, new gray levels appear and the indicator functions are blurred.

Motivated by the above observations, we introduce the primal and dual perspective of the TV-L1 model (3.1) and study the exactness of (3.1) as the convex relaxation of the discrete constrained optimization problem:

$$\min_{u(\mathbf{x}) \in \{f_1, \dots, f_n\}} \left\{ E(u(\mathbf{x})) = \int_{\Omega} |\nabla u(\mathbf{x})| d\mathbf{x} + \alpha \int_{\Omega} |u^0(\mathbf{x}) - u(\mathbf{x})| d\mathbf{x} \right\}, \quad (3.5)$$

given $u^0(\mathbf{x}) \in \{f_1, \dots, f_n\}$. In this chapter, we assume the gray values f_i , $i = 1, \dots, n$, are ordered by $f_1 < \dots < f_n$. Clearly, integers $0, \dots, 255$ may be taken as the option in most cases. We greatly extend the result for the binary constrained image denoising (3.2), obtained by Chan et al. [30, 31], to the more general gray-value constrained case.

3.1.2 Contributions

Our main contributions can be summarized as follows:

1. We propose equivalent formulations in terms of primal and dual, and build up a new analytical framework which results in a new variational perspective of (3.1).
2. By the proposed equivalent formulations, we show that the TV-L1 formulation (3.1) can be used as the convex relaxed model of its relevant discrete constrained image processing task. This greatly extends the results achieved by Chan et al. [30, 31] to the more general cases. To the best of our knowledge, this is new.
3. We introduce an elegant multiplier based algorithm which explores the equivalent primal-dual formulation through two simple projection substeps,

instead of tackling the highly nonsmooth TV-L1 energy functional directly. Its reliability and efficiency are verified by optimization theories and experiments.

In parallel to our multiplier based method, several other dual formulations and algorithmic schemes were proposed recently in the literature, see [141, 48, 130, 119, 139, 108, 109, 45]. In contrast to [48, 130, 119, 139], we apply the proposed equivalent primal-dual and dual formulations as a complete approach, including both variational analysis and algorithms to study topics concerning (3.1), not just derive the algorithmic scheme. In addition, the primal-dual algorithm we have proposed here is different from [48, 130, 119, 139]. In our algorithm, the solution u is treated as the multiplier novelly.

3.2 Equivalent Models

We call the TV-L1 image approximation (3.1) *primal model* in this section, as comparison to the equivalent models introduced in this section. In this section, we also introduce its equivalent formulations under the primal and dual perspective.

3.2.1 Equivalent Primal-Dual Model

With the help of conjugates [100], the data term of (3.1) can be equally expressed by

$$\alpha \int_{\Omega} |u^0 - u| = \max_{q \in S_{\alpha}} \langle q, u^0 - u \rangle, \quad S_{\alpha} := \{q \mid |q(\mathbf{x})| \leq \alpha, \forall \mathbf{x} \in \Omega\}. \quad (3.6)$$

Moreover, it is well known that the total variation term of (3.1) can also be formulated [55] as

$$\int_{\Omega} |\nabla u| d\mathbf{x} = \max_{p \in C_1} \langle \operatorname{div} p, u \rangle, \quad C_1 := \{p \mid p \in C_c^1(\Omega, \mathbb{R}^2), |p(\mathbf{x})| \leq 1, \forall \mathbf{x} \in \Omega\}. \quad (3.7)$$

In view of (3.6) and (3.7), after some rearrangements, the TV-L1 approximation formulation (3.1) can be rewritten as

$$\max_{q \in S_\alpha} \max_{p \in C_1} \min_u \{ E(u; q, p) := \langle q, u^0 \rangle + \langle \operatorname{div} p - q, u \rangle \}, \quad (3.8)$$

which is called the equivalent *primal-dual model* to the primal model (3.1).

3.2.2 Equivalent Dual Model

From (3.8), we have the linear equality $\operatorname{div} p = q$, and the constrained maximization problem

$$\max_{q \in S_\alpha} \max_{p \in C_1} \{ D(q, p) := \langle q, u^0 \rangle \}, \quad \text{s.t.} \quad \operatorname{div} p = q. \quad (3.9)$$

Likewise, we call (3.9) the equivalent *dual model* to (3.1).

Clearly, the dual model (3.9) can be further simplified as

$$\max_q \langle q, u^0 \rangle, \quad \text{s.t.} \quad q \in S_\alpha \cap \operatorname{div} C_1. \quad (3.10)$$

3.2.3 Optimization Facts

For the primal-dual formulation (3.8), the conditions of the minimax theorem are all satisfied see e.g., [46, 49]. That is: the constraints of dual variables p and q are convex and the energy function is linear to both u and (p, q) , hence convex lower semi continuous for fixed p and q and concave upper semi continuous for fixed u . This follows that there exists at least one saddle point, see [46, 49]. In consequence, the min and max operators of the primal-dual model (3.8) can be interchanged, i.e.

$$\max_{q \in S_\alpha} \max_{p \in C_1} \{ \min_u E(u; q, p) \} = \min_u \{ \max_{q \in S_\alpha} \max_{p \in C_1} E(u; q, p) \}. \quad (3.11)$$

It is easy to see that the optimization of the primal-dual model (3.8) over the dual variables q and p react on the primal formulation (3.1) of TV-L1 image approximation, i.e. the right hand side of (3.11):

$$P(u) = E(u; q^*, p^*) = \max_{q \in S_\alpha} \max_{p \in C_1} E(u; q, p).$$

Likewise, the dual model (3.9) can be achieved by optimizing the image function $u(\mathbf{x})$ in (3.8), i.e. the left hand side of (3.11):

$$D(q, p) = E(u^*, q, p) = \min_u E(u; q, p). \quad (3.12)$$

3.3 Global and Exact Optimums

Now we show that the equivalent formulations in the previous section provide us an interesting perspective and a powerful tool to study the TV-L1 model. In this section, we study the nonconvex optimization problem (3.5) and show the TV-L1 formulation (3.1) gives the convex relaxed model of (3.5), i.e. its optimum solves (3.5) globally and exactly through the proposed rounding scheme. We state our results and proof in several propositions.

Proposition 3.1 (Extremum Principle). *Given the image function $u^0(\mathbf{x}) \in \{f_1, \dots, f_n\}$, $\forall \mathbf{x} \in \Omega$, along with ordering $f_1 < \dots < f_n$, each optimum $u^*(\mathbf{x})$ of (3.1) suffices $f_1 \leq u^*(\mathbf{x}) \leq f_n$ almost everywhere.*

Proof. Let u^* be the minimum of (3.1). Due to the convexity of (3.1), u^* is simply accepted as the global minimum. We first prove that $u^*(\mathbf{x}) \leq f_n$ for $\forall \mathbf{x} \in \Omega$.

If $u^*(\mathbf{x}) > f_n$ at some area $\tilde{\Omega} \subset \Omega$, then we define the function u^* which just thresholds the value $u^*(\mathbf{x})$ to be not larger than f_n , i.e.

$$u^*(\mathbf{x}) = \begin{cases} f_n & \text{at } \mathbf{x} \in \tilde{\Omega} \\ u^*(\mathbf{x}) & \text{at } \mathbf{x} \in \Omega \setminus \tilde{\Omega} \end{cases}.$$

Obviously, in view of $u^0(\mathbf{x}) \leq f_n$ and $u^*(\mathbf{x}) > f_n$ for $\forall \mathbf{x} \in \tilde{\Omega}$, we have

$$\begin{aligned} \int_{\Omega} |u^* - u^0| d\mathbf{x} &= \int_{\Omega \setminus \tilde{\Omega}} |u^* - u^0| d\mathbf{x} + \left\{ \int_{\tilde{\Omega}} |f_n - u^0| d\mathbf{x} + \int_{\tilde{\Omega}} |u^* - f_n| d\mathbf{x} \right\} \\ &= \int_{\Omega} |u^* - u^0| d\mathbf{x} + \int_{\tilde{\Omega}} |u^* - f_n| d\mathbf{x}. \end{aligned}$$

It follows that

$$\int_{\Omega} |u^0 - u^*| d\mathbf{x} < \int_{\Omega} |u^0 - u^*| d\mathbf{x}. \quad (3.13)$$

By the coarea formula of the total variation term:

$$\mathrm{TV}(u) = \int_{-\infty}^{+\infty} L_\gamma(u) d\gamma,$$

where $L_\gamma(u)$ is the length of the γ -upper level set of u , it follows that

$$\mathrm{TV}(u^*) < \mathrm{TV}(u^*), \quad (3.14)$$

because the f_n -upper level set of u^* is thresholded to vanish.

Observe (3.13) and (3.14), we must have

$$\int_{\Omega} |u^0 - u^*| d\mathbf{x} + \alpha \mathrm{TV}(u^*) < \int_{\Omega} |u^0 - u^*| d\mathbf{x} + \alpha \mathrm{TV}(u^*).$$

This is in contradiction to the fact that u^* is the global minimum of (3.1).

Likewise, we can also prove $u^*(\mathbf{x}) \geq f_1$, $\mathbf{x} \in \Omega$ in the same way. In consequence, we prove that each minimum $u^*(\mathbf{x})$ of (3.1) suffices $u^*(\mathbf{x}) \in [f_1, f_n]$. $\square$

Proposition 3.2. *Given a bounded scalar function $f_1 \leq u(\mathbf{x}) \leq f_n$ for $\forall \mathbf{x} \in \Omega$, if an optimal vector field p^* maximizes the integral $\int_{\Omega} u \operatorname{div} p d\mathbf{x}$ over the convex set C_1 , i.e.*

$$\int_{\Omega} |\nabla u| d\mathbf{x} = \int_{\Omega} u \operatorname{div} p^* d\mathbf{x},$$

then in view of (3.4), for every γ -upper level set $U^\gamma(\mathbf{x})$ of $u(\mathbf{x})$ with $\gamma \in [f_1, f_m)$, p^ also maximizes the integral $\int_{\Omega} U^\gamma \operatorname{div} p d\mathbf{x}$ over the convex set C_1 and*

$$\int_{\Omega} U^\gamma \operatorname{div} p^* d\mathbf{x} = |\partial U^\gamma|,$$

which is the perimeter of the level set $U^\gamma(\mathbf{x})$.

Proof. Denote the interval $\Gamma = [f_1, f_n]$. The coarea formula gives

$$\int_{\Omega} |\nabla u| d\mathbf{x} = \int_{\Gamma} \int_{\Omega} |\nabla U^\gamma| d\mathbf{x} d\gamma. \quad (3.15)$$

By applying this formula we can deduce

$$\int_{\Omega} u \operatorname{div} p^* d\mathbf{x} = \int_{\Omega} |\nabla u| d\mathbf{x} = \int_{\Gamma} \int_{\Omega} |\nabla U^\gamma| d\mathbf{x} d\gamma = \int_{\Gamma} \left(\max_{p \in C_1} \int_{\Omega} U^\gamma \operatorname{div} p d\mathbf{x} \right) d\gamma. \quad (3.16)$$

By the fact that $u(\mathbf{x}) - f_1 = \int_{\Gamma} U^{\gamma}(\mathbf{x}) d\gamma$ for any $\mathbf{x} \in \Omega$, we have

$$\int_{\Omega} u \operatorname{div} p^* d\mathbf{x} = \int_{\Omega} \left(\int_{\Gamma} U^{\gamma}(\mathbf{x}) d\gamma \right) \operatorname{div} p^*(\mathbf{x}) d\mathbf{x} = \int_{\Gamma} \int_{\Omega} U^{\gamma} \operatorname{div} p^* d\mathbf{x} d\gamma. \quad (3.17)$$

Therefore, combining (3.16) and (3.17):

$$\int_{\Gamma} \int_{\Omega} U^{\gamma} \operatorname{div} p^* d\mathbf{x} d\gamma = \int_{\Gamma} \left(\max_{p \in C_1} \int_{\Omega} U^{\gamma} \operatorname{div} p d\mathbf{x} \right) d\gamma. \quad (3.18)$$

This equality (3.18) together with the fact that for any $\gamma \in [f_1, f_n]$

$$\int_{\Omega} U^{\gamma} \operatorname{div} p^* d\mathbf{x} \leq \max_{p \in C_1} \int_{\Omega} U^{\gamma} \operatorname{div} p d\mathbf{x}. \quad (3.19)$$

Then it follows that

$$\int_{\Omega} U^{\gamma} \operatorname{div} p^* d\mathbf{x} = \max_{p \in C_1} \int_{\Omega} U^{\gamma} \operatorname{div} p d\mathbf{x}$$

for almost every $\gamma \in [f_1, f_n]$. Clearly, the perimeter of the level set U^{γ} is given by

$$|\partial U^{\gamma}| = \int_{\Omega} |\nabla U^{\gamma}| d\mathbf{x} = \max_{p \in C_1} \int_{\Omega} U^{\gamma} \operatorname{div} p d\mathbf{x}.$$

□

Corollary 3.1. *Given a bounded scalar function $f_1 \leq u(\mathbf{x}) \leq f_n$ for $\forall \mathbf{x} \in \Omega$ and $n - 1$ different values γ_i , $i = 1, \dots, n - 1$, such that $f_1 \leq \gamma_1 < \dots < \gamma_{n-1} \leq f_n$, if an optimal vector field p^* maximizes the integral $\int_{\Omega} u \operatorname{div} p d\mathbf{x}$ over the convex set C_1 , then for the image function*

$$u^{\gamma}(\mathbf{x}) = \sum_{i=1}^{n-1} (f_{i+1} - f_i) U^{\gamma_i}(\mathbf{x}),$$

p^* also maximizes the integral $\int_{\Omega} u^{\gamma} \operatorname{div} p d\mathbf{x}$ over the convex set C_1 , i.e. we have

$$\int_{\Omega} |\nabla u^{\gamma}| d\mathbf{x} = \int_{\Omega} u^{\gamma} \operatorname{div} p^* d\mathbf{x}.$$

Proof. By virtue of Prop. 3.2, p^* also maximize the integral

$$\int_{\Omega} U^{\gamma_i} \operatorname{div} p d\mathbf{x},$$

over the convex set C_1 for each $\gamma_i, i = 1, \dots, n-1$.

Then it follows that for the piecewise constant image function

$$u^\gamma(\mathbf{x}) = \sum_{i=1}^{n-1} (f_{i+1} - f_i) U^{\gamma_i}(\mathbf{x}),$$

p^* also maximizes the integral

$$\int_{\Omega} u^\gamma \operatorname{div} p \, d\mathbf{x} = \sum_{i=1}^{n-1} \{ (f_{i+1} - f_i) \int_{\Omega} U^{\gamma_i} \operatorname{div} p \, d\mathbf{x} \},$$

over the convex set $p \in C_1$, because $f_1 < \dots < f_n$ is ordered such that

$$f_{i+1} - f_i > 0, \quad i = 1, \dots, n-1.$$

Therefore, we have

$$\int_{\Omega} |\nabla u^\gamma| \, d\mathbf{x} = \int_{\Omega} u^\gamma \operatorname{div} p^* \, d\mathbf{x}.$$

□

With helps of the above facts, we can prove the following proposition:

Proposition 3.3. *Given the image function $u^0(\mathbf{x}) \in \{f_1, \dots, f_n\}$, where $f_1 < \dots < f_n$ and the boundary of each concerning upper level set $F^{f_i}(\mathbf{x}), i = 1, \dots, n$, is regular, then for any given $n-1$ values $\gamma_i, i = 1, \dots, n-1$ such that*

$$f_1 < \gamma_1 < f_2 < \dots < \gamma_{n-1} < f_n, \quad (3.20)$$

we define the image function $u^\gamma(\mathbf{x})$ by the $n-1$ upper level sets (3.4) of the computed optimum $u^(\mathbf{x})$ of (3.1):*

$$u^\gamma(\mathbf{x}) = f_1 + \sum_{i=1}^{n-1} (f_{i+1} - f_i) U^{\gamma_i}(\mathbf{x}). \quad (3.21)$$

Then $u^\gamma(\mathbf{x}) \in \{f_1, \dots, f_n\}$ and $u^\gamma(\mathbf{x})$ gives an exact global optimum of (3.5).

Proof. Let (u^*, q^*, p^*) be the optimal primal-dual pair of (3.8). Hence the optimal dual variables q^* and p^* suffice that q^* maximizes the integral $\int_{\Omega} q(u^0 - u) d\mathbf{x}$ over the convex set S_{α} and p^* maximizes the integral $\int_{\Omega} u \operatorname{div} p d\mathbf{x}$ over the convex set C_1 .

In the mean time, $u^{\gamma}(\mathbf{x}) \in \{f_1, \dots, f_n\}$ as (3.21) can be rearranged as

$$u^{\gamma}(\mathbf{x}) = f_1 (1 - U^{\gamma_1}(\mathbf{x})) + \sum_{i=2}^{n-1} f_i (U^{\gamma_{i-1}}(\mathbf{x}) - U^{\gamma_i}(\mathbf{x})) + f_n U^{\gamma_{n-1}}(\mathbf{x}).$$

u^{γ} is also a global optimum of (3.1), which can be obtained by considering the following facts: By Coro. 3.1, p^* also maximizes the integral $\int_{\Omega} u^{\gamma} \operatorname{div} p d\mathbf{x}$ over the convex set C_1 and

$$\int_{\Omega} |\nabla u^{\gamma}| d\mathbf{x} = \langle u^{\gamma}, \operatorname{div} p^* \rangle. \quad (3.22)$$

At the next step, we can prove

$$\alpha \int_{\Omega} |u^0 - u^{\gamma}| d\mathbf{x} = \langle q^*, u^0 - u^{\gamma} \rangle. \quad (3.23)$$

The optimal dual variable $q^*(\mathbf{x})$ actually gives the sign of $u^0(\mathbf{x}) - u^*(\mathbf{x})$ at each $\mathbf{x} \in \Omega$, when $u^0(\mathbf{x}) \neq u^*(\mathbf{x})$; when $u^0(\mathbf{x}) = u^*(\mathbf{x})$, $q^*(\mathbf{x})$ can take any value in $[-\alpha, \alpha]$. Now we assume $u^*(\mathbf{x}) \in [f_k, f_{k+1}]$ for the position $\mathbf{x} \in \Omega$, then in view of (3.4) and (3.21), we have

$$u^*(\mathbf{x}) \in [f_k, \gamma_k] \implies u^{\gamma}(\mathbf{x}) = f_k,$$

and

$$u^*(\mathbf{x}) \in (\gamma_k, f_{k+1}] \implies u^{\gamma}(\mathbf{x}) = f_{k+1}.$$

Since $u^0(\mathbf{x}) \in \{f_1, \dots, f_n\}$, we can analyze $q^*(\mathbf{x})$ in two cases: $u^0(\mathbf{x}) \leq f_k$ and $u^0(\mathbf{x}) \geq f_{k+1}$.

- When $u^0(\mathbf{x}) \leq f_k$, in view of $u^*(\mathbf{x}) \geq f_k$, we have $q^*(\mathbf{x}) = -\alpha$ for $u^*(\mathbf{x}) > f_k$ or $q^*(\mathbf{x}) \geq -\alpha$ for $u^*(\mathbf{x}) = u^0(\mathbf{x})$ in order to maximize $q(\mathbf{x}) \cdot (u^0(\mathbf{x}) - u^*(\mathbf{x}))$ over $q(\mathbf{x}) \in [-\alpha, \alpha]$. Then in both cases, $q^*(\mathbf{x})$ also maximizes the product $q(\mathbf{x}) \cdot (u^0(\mathbf{x}) - f_k)$, or $q(\mathbf{x}) \cdot (u^0(\mathbf{x}) - f_{k+1})$, over $q(\mathbf{x}) \in [-\alpha, \alpha]$. Hence $q^*(\mathbf{x})$ maximizes $q(\mathbf{x}) \cdot (u^0(\mathbf{x}) - u^{\gamma}(\mathbf{x}))$ over $q(\mathbf{x}) \in [-\alpha, \alpha]$.

- When $u^0(\mathbf{x}) \geq f_{k+1}$, in view of $u^*(\mathbf{x}) \leq f_{k+1}$, we have $q^*(\mathbf{x}) = \alpha$ for $u^*(\mathbf{x}) < f_{k+1}$ or $q^*(\mathbf{x}) \leq \alpha$ for $u^*(\mathbf{x}) = u^0(\mathbf{x})$ in order to maximize $q(\mathbf{x}) \cdot (u^0(\mathbf{x}) - u^*(\mathbf{x}))$. In both cases, $q^*(\mathbf{x})$ also maximizes the product $q(\mathbf{x}) \cdot (u^0(\mathbf{x}) - f_k)$ or $q(\mathbf{x}) \cdot (u^0(\mathbf{x}) - f_{k+1})$, over $q(\mathbf{x}) \in [-\alpha, \alpha]$. Hence $q^*(\mathbf{x})$ maximizes $q(\mathbf{x}) \cdot (u^0(\mathbf{x}) - u^\gamma(\mathbf{x}))$ over $q(\mathbf{x}) \in [-\alpha, \alpha]$.

Therefore, we have q^* maximizes the integral $\langle q, u^0 - u^\gamma \rangle$ over the convex set S_α . Then the fact (3.23) is proved.

By virtue of (3.22), (3.23) and the dual model (3.9), we have

$$P(u^\gamma) = E(u^\gamma, p^*, q^*) = \langle q^*, u^0 \rangle + \langle u^\gamma, \operatorname{div} p^* - q^* \rangle = \langle q^*, u^0 \rangle = P(u^*).$$

Then it follows that u^γ is also a global minimum of (3.1) as u^* is global minimum of (3.1). Since (3.1) is just the relaxed version of (3.5), $u^\gamma(\mathbf{x}) \in \{f_1, \dots, f_n\}$ solves (3.5) exactly and globally. $\square$

The proposed rounding scheme (3.21) actually gives

$$u^\gamma(\mathbf{x}) = \begin{cases} f_1, & \text{when } u^*(\mathbf{x}) < \gamma_1 \\ f_i, & \text{when } \gamma_{i-1} \leq u^*(\mathbf{x}) < \gamma_i, i = 2, \dots, n-1 \\ f_n, & \text{when } u^*(\mathbf{x}) \geq \gamma_{n-1} \end{cases}.$$

In the experiments of this chapter, we adopt the above scheme to obtain rounding results.

3.4 Multiplier Based Algorithm

In this thesis, we build up the algorithm upon the equivalent primal-dual model (3.8). Clearly, the primal variable u works as the multiplier in (3.8) for the linear equality $\operatorname{div} p - q = 0$. The energy function of (3.8) gives the corresponding Lagrangian function. By these observations, we define its augmented Lagrangian function as

$$L_c(q, p, u) = \langle q, u^0 \rangle + \langle \operatorname{div} p - q, u \rangle - \frac{c}{2} \|\operatorname{div} p - q\|^2$$

where $c > 0$. Thereafter, the classical augmented Lagrangian algorithm can be applied, which gives a splitting optimization framework over each dual variables q and p respectively, by exploring projections to corresponding convex sets. To this end, we call Algorithm 1 the *multiplier based algorithm*.

Algorithm 1: Multiplier Based Algorithm

- Set the starting values: q^0 , p^0 and u^0 , and let $k = 1$;
- Start the k -th iteration which includes two successive sub-steps:
 1. Optimize q^{k+1} by fixing p^k and u^k :

$$\begin{aligned} q^{k+1} &:= \arg \max_{\|q\|_\infty \leq 1} L_c(q, p^k, u^k) \\ &= \arg \max_{\|q\|_\infty \leq 1} \langle q, u^0 \rangle - \frac{c}{2} \|q - (\operatorname{div} p^k - u^k/c)\|^2, \end{aligned}$$

which is approximated by the projection

$$q^{k+1} = \operatorname{Proj}_{\|q\|_\infty \leq 1}(u^0/c + (\operatorname{div} p^k - u^k/c)); \quad (3.24)$$

2. Optimize p^{k+1} by fixing q^{k+1} and u^k :

$$p^{k+1} := \arg \min_{p \in C_\lambda} \frac{1}{2} \|\operatorname{div} p - (q^{k+1} + u^k/c)\|^2, \quad (3.25)$$

which is the projection of $(q^{k+1} + u^k/c)$ to the convex set $\operatorname{div} C_\lambda$ and can be implemented by Chambolle's algorithm [26];

- Update u^{k+1} by

$$u^{k+1} = u^k + c(q^{k+1} - \operatorname{div} p^{k+1}); \quad (3.26)$$

and let $k = k + 1$, repeat until convergence.

Clearly, Algorithm 1 explores two simple projection sub-steps at each iteration, which properly avoids tackling the nonsmooth terms in (3.1) in a direct way.

3.5 Numerical Experiments

In all experiments, convergence is achieved by evaluating the following error:

$$\text{err} = c \|\text{div } p - q\| / \|u\| ,$$

which is the ratio of the primal-dual gap to the image approximation $u(\mathbf{x})$, see (3.26).

To evaluate the performance of rounded results in the following experiments, we take the energy difference associated to the computed optimum u^* and the rounded result u^γ which is evaluated by the ratio:

$$\text{ratio} = |P(u^*) - P(u^\gamma)| / P(u^*) .$$

3.5.1 Synthetic Image

A synthetic image $u^0(\mathbf{x}) \in \{0, 0.5, 1\}$ is taken for this experiment, see figure (c) of Fig. 3.2, which is colorized by red: 0, green: 0.5, blue: 1. We set the penalty parameter $\alpha = 1$ and the augmented parameter $c = 6$. By this, the algorithm runs for 600 iterations and converges with the error $\text{err} = 3.23 \times 10^{-7}$. Figure (a) of Fig. 3.2 shows the plot of convergence rate.

In this experiment, two rounding schemes are taken: $\{\gamma_1 = 0.25, \gamma_2 = 0.75\}$; $\{\gamma_1 = 0.35, \gamma_2 = 0.65\}$. For the computed result u^* , it gives the energy $P(u^*) = 2938.7$. The two corresponding rounded results produce the energy $P(u^\gamma) = 2937.3$, 2937.3, i.e. both rounding schemes give the same energy. The energy ratios are 4.76×10^{-4} .

3.5.2 Gray Value Images

For the given gray-value images $u^0(\mathbf{x})$ of the experiments, 256 gray levels are, naturally, encoded, i.e. $u^0(\mathbf{x}) \in \{0, \dots, 255\}$.

The experiment results given in Fig. 3.3 show the denoising of the penguin image (see figure (a) of Fig. 3.3), which is downloaded from the middlebury data

Figure 3.2: (a) convergence rate (600 iterations); (b) the ground truth image colored by red: 0, green: 0.5, blue: 1; figure (c) the input image $u^0(\mathbf{x})$; (d) the computed image $u^*(\mathbf{x})$ where $\alpha = 1$; (e) the image u^γ rounded by $\{\gamma_1 = 0.25, \gamma_2 = 0.75\}$; (f) the image u^γ rounded by $\{\gamma_1 = 0.35, \gamma_2 = 0.65\}$; (g) the difference between two rounded results.

set: <http://vision.middlebury.edu/MRF>. The rounding scheme is simply taken by $\gamma = \{0.5, 1.5, \dots, 255\}$, i.e. it just gives the nearest integer. In all experiments where $\alpha = 1.3, 1, 0.5$, the Algorithm 1 converges to an error below 2×10^{-6} within 300 iterations, see the graph (b) of Fig. 3.3. Experiment results of Fig. 3.3, with respect to $\alpha = 1.3, 1, 0.5$, show the ratios of energy differences are nearly zeros!

The images processed in the experiments, shown in Fig. 3.4, are downloaded from the Berkeley segmentation dataset and benchmark.

The comparisons of medical image reconstructions in Fig. 3.5, are recovered by our TV-L1 method and Yin's discrete graph cuts method with four neighbors in [133]. These medical images have large size, but we still keep most details which is clearer in zoomed electronic version. In all experiments, we set $\alpha = 0.5$.

Figure 3.3: (a) the input image $u^0(\mathbf{x})$; (b) plot of convergence curves (300 iterations): red circle line : $\alpha = 1.3$, blue solid line: $\alpha = 1$ and green star line: $\alpha = 0.5$. Figures of 2nd row show the computation results when $\alpha = 1.3, 1, 0.5$ respectively. Figures of 3rd row show the rounding results when $\alpha = 1.3, 1, 0.5$ respectively.

All experiment results show the ratios of energy differences are nearly zeros.

Figure 3.4: Four input images are shown in the 1st row; the computed images $u^*(\mathbf{x})$ are given in the 2nd row respectively; the rounded images $u^*(\mathbf{x})$ are shown in the 3rd row respectively. In all experiments, we set $\alpha = 0.5$.

Figure 3.5: the 1st row: medical image of ankle reconstruction by TV-L1 and Yin's discrete method; the 2nd row: medical image of spine reconstruction by TV-L1 and Yin's discrete method.

A Study on Convex Optimization

Approaches to Image Fusion

4.1 Introduction

Imaging fusion technologies have been developed to be an effective way to show different imagery information from various sources in one single image, which is especially interesting in many areas, e.g. remote sensing [128, 42], medical imaging [110, 131] and synthesis of multi-focused images [80, 126]. More specifically, given two or more information data which are from different sources and properly aligned, image fusion integrates all such data into one visualized image, mostly with higher spatial or spectral resolution. For example, given two images, which may capture the same scene but with different focuses (see the left two images of Fig. 4.1), fusing these two images clearly gives a better visual result (see the right two fused images of Fig. 4.1). In remote sensing and satellite imaging, the fused image, which is merged by multispectral data, effectively conveys more information [128, 107]. In medical imaging, while both the Magnetic Resonance (MR) and Computed Tomography (CT) imagery provide standard diagnostic tools other than fluoroscope or ultrasound techniques, it is well-known that a CT scan will adequately highlight the bone structure details while soft tissue

information is not clearly visible; on the other hand, a T2 weighted MR scan produces significantly better details for images of these tissues. In this respect, it is highly desirable to have a combined view of CT and MR images, which illustrates significant details both from both CT and MR inputs and assists clinical diagnoses.

Figure 4.1: Multi-focus image fusion (a) and (b) give two images exposed with different focuses; (c) and (d) are the fused image computed by the proposed methods (4.1) and (4.3) in this chapter.

Parallel to recent developments of image processing, many pixelwise image fusion methods have been proposed to tackle such problem of combining multiple images or informative data, e.g. the wavelet or contourlet based approaches [95, 90, 131], high-pass filtering method [2, 107] etc. In this section, we concentrate on the variational approaches to image fusion, which were explored in [98, 126, 70]. Energy minimization and variational methods have been developed to be a standard way to effectively and reliably handle many practical topics of image processing and computer vision both in mathematics and numerics. Successful applications include image denoising [103, 31, 138], image decomposition [8, 87] image segmentation [32, 31, 135, 136] and image deblurring [77] etc. In this regard, variational image fusion methods [126, 70, 98] provide such an elegant approach for the trade-off between redundant imagery information and image priors.

4.1.1 Contributions

In this chapter, we study the variational models to the integration of images with gray-scales, which were proposed or partially investigated by [126, 70]. We propose the convex optimization approach to the studied variational problems under the novel duality based perspective, and consider the exactness of the reduced convex relaxation model to the nonconvex TV-L1 based image fusion with the pixelwise constraint of discrete values. Our contributions can be summarized as follows:

1. We consider the two convex optimization models of image fusion based on standard techniques of TV-L2 and TV-L1 image approximation.
2. We propose their equivalent convex formulations under the new perspective of primal and dual, which also leads to the equivalent convex formulations.
3. We show the studied image fusion models actually result in two new image decompositions of the weighted input image, with helps of the proposed dual formations.

In addition, we focus on the TV-L1 based image fusion method and prove it gives an exact convex relaxation model to the corresponding image fusion problem constrained by a linearly-ordered discrete-value set to each pixel, i.e. it solves such nonconvex image fusion problem globally and exactly. This properly extends the convex relaxation models of TV-L1 image approximation, proposed by Chan et al. [30] and Yuan et al. [138], to TV-L1 based image fusion applications. Clearly, direct and global solvers to such discrete-constrained image fusion, especially over a large number of linearly-ordered discrete values for instance medical imaging, definitely result in a high load of memory and computation, e.g. graph cuts method [19, 67] and the continuous min-cut method [13]. To this end, the convex relaxation approach proposed in this chapter leads to a much easier and more efficient approach to the given discrete-constrained optimization problem. We also derive fast and reliable algorithms to the studied two image fusion

methods through their proposed dual formulations, which properly avoids non-smoothness of the energy functions and leads to simple and efficient numerical implementations.

4.2 Convex Optimization Models

Given two input images $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$, a total variation based method for image fusion was proposed by Wang et al. [126] such that

$$\min_u \left\{ E(u) = \frac{1}{2} \int_{\Omega} w_1 (u - f_1)^2 d\mathbf{x} + \frac{1}{2} \int_{\Omega} w_2 (u - f_2)^2 d\mathbf{x} + \alpha \int_{\Omega} |\nabla u| d\mathbf{x} \right\}, \quad (4.1)$$

where the functions $w_1(\mathbf{x})$ and $w_2(\mathbf{x})$ are the pixelwise weight functions such that

$$w_1(\mathbf{x}) + w_2(\mathbf{x}) = 1, \quad w_{1,2}(\mathbf{x}) \geq 0, \quad \forall x \in \Omega. \quad (4.2)$$

In this chapter, we extend (4.1) to the convex optimization model with the L1-norm data fidelity term:

$$\min_u \left\{ E(u) = \int_{\Omega} w_1 |u - f_1| d\mathbf{x} + \int_{\Omega} w_2 |u - f_2| d\mathbf{x} + \alpha \int_{\Omega} |\nabla u| d\mathbf{x} \right\}. \quad (4.3)$$

Similar formulation as (4.3) was also studied in [70] where the weight functions are given constant.

Clearly, both models (4.1) and (4.3) formulate the integration of two input images as the problem of convex optimization which can be generalized as

$$\min_u \left\{ E(u) = \int_{\Omega} w_1 D(f_1 - u) d\mathbf{x} + \int_{\Omega} w_2 D(f_2 - u) d\mathbf{x} + \alpha \int_{\Omega} |\nabla u| d\mathbf{x} \right\}, \quad (4.4)$$

where $D(\cdot)$ is a positive convex function. In this chapter, we call (4.4), along with (4.1) and (4.3), the *primal model*.

In the following parts, we investigate (4.4) under the perspective of primal and dual and build up its connections to variational image decomposition.

4.2.1 Equivalent Convex Formulations

Let $D^*(q)$ be the respective conjugate of the convex function $D(v)$ such that

$$D(v) = \max_q \{vq - D^*(q)\} . \quad (4.5)$$

For the model (4.1) where the function D is in quadratic form, i.e. $D(v) = \frac{1}{2}v^2$, we have

$$D^*(q) = \frac{1}{2}q^2 . \quad (4.6)$$

For the problem (4.3) where D is an absolute function, i.e. $D(v) = |v|$, we have

$$D^*(q) = I_\delta(q \in [-1, 1]) \quad (4.7)$$

where $I_\delta(q \in [-1, 1])$ is the characteristic function of the convex set $q \in [-1, 1]$.

We also recall that the dual formulation of the total variation function [55]

$$\alpha \int_\Omega |\nabla u| \, d\mathbf{x} = \max_{p \in C_\alpha} \int_\Omega u \operatorname{div} p \, d\mathbf{x} \quad (4.8)$$

where C_α is a convex set defined by

$$C_\alpha := \{p \mid p \in C_c^1(\Omega, \mathbb{R}^2), |p(\mathbf{x})| \leq \alpha, \forall \mathbf{x} \in \Omega\} . \quad (4.9)$$

By simple computation, in view of (4.5) and (4.8), the primal formulation (4.4) can be rewritten as

$$\begin{aligned} \min_u \max_{q_1, q_2} \max_{p \in C_\alpha} & \int_\Omega w_1 (q_1 f_1 - D^*(q_1)) \, d\mathbf{x} + \int_\Omega w_2 (q_2 f_2 - D^*(q_2)) \, d\mathbf{x} \\ & + \langle \operatorname{div} p - (w_1 q_1 + w_2 q_2), u \rangle . \end{aligned} \quad (4.10)$$

In this chapter, we call (4.10) the equivalent *primal-dual model* of (4.4).

Observe that u is unconstrained and the convex formulation (4.10) suffices the minimax theorem [46, 49] for our cases (4.1) and (4.3) in this study, the min and max operators of (4.10) are interchangeable. The minimization of (4.10) over u , therefore, leads to the linear equality

$$w_1 q_1 + w_2 q_2 = \operatorname{div} p , \quad (4.11)$$

and the corresponding constrained maximization problem as follows

$$\begin{aligned} \max_{q_1, q_2} \max_{p \in C_\alpha} \int_{\Omega} w_1 (q_1 f_1 - D^*(q_1)) \, d\mathbf{x} + \int_{\Omega} w_2 (q_2 f_2 - D^*(q_2)) \, d\mathbf{x} \quad (4.12) \\ \text{s.t.} \quad w_1 q_1 + w_2 q_2 = \text{div } p. \end{aligned}$$

Similarly, we call (4.12) the equivalent *dual model* of (4.4).

4.2.2 Variational Image Decompositions

With helps of the conjugates (4.5), we will see that the optimum of the variational image fusion (4.4) actually proposes a decomposition of the weighted input image $u^0(\mathbf{x}) := w_1(\mathbf{x})f_1(\mathbf{x}) + w_2(\mathbf{x})f_2(\mathbf{x})$, $\mathbf{x} \in \Omega$. More specifically, we have

Proposition 4.4. *Given the optimum (q_1^*, q_2^*, p^*, u^*) of the equivalent primal-dual model (4.10), (q_1^*, q_2^*, p^*, u^*) just gives rise to the decomposition of the weighted input image $(w_1 f_1 + w_2 f_2)(\mathbf{x})$, $\mathbf{x} \in \Omega$, such that*

$$u^0 := w_1 f_1 + w_2 f_2 = u^* + v^* \quad (4.13)$$

where

$$v^* = w_1 v_1^* + w_2 v_2^*, \quad v_1^* \in \partial D(q_1^*), \quad v_2^* \in \partial D(q_2^*).$$

Proof. Observe the conjugate formulations (4.5), we have

$$f_1 - u^* = v_1^* \in \partial D(q_1^*), \quad f_2 - u^* = v_2^* \in \partial D(q_2^*).$$

Here, ∂D is the subdifferential of convex function D . Recall that $w_1(\mathbf{x}) + w_2(\mathbf{x}) = 1$ for $\forall \mathbf{x} \in \Omega$, then we have

$$w_1 v_1^* + w_2 v_2^* = w_1 (f_1 - u^*) + w_2 (f_2 - u^*) = (w_1 f_1 + w_2 f_2) - u^*.$$

Then (4.13) simply follows. $\square$

Consider the conjugates (4.6) and Prop. 4.4, the L2-norm based image fusion problem (4.1) results in the following image decomposition:

Corollary 4.2. *Given the optimum (q_1^*, q_2^*, p^*, u^*) of the equivalent primal-dual model (4.10) corresponding to (4.1), (q_1^*, q_2^*, p^*, u^*) just gives rise to the decomposition of the weighted input image $(w_1 f_1 + w_2 f_2)(\mathbf{x})$, $\mathbf{x} \in \Omega$, such that*

$$u^0 := w_1 f_1 + w_2 f_2 = u^* + \operatorname{div} p^*. \quad (4.14)$$

Proof. In view of (4.6), we have $f_1 - u^* = q_1^*$ and $f_2 - u^* = q_2^*$. It follows

$$u^0 := w_1 f_1 + w_2 f_2 = (w_1 q_1^* + w_2 q_2^*) + u^*.$$

In view of the linear equality constraint (4.11), i.e. $w_1 q_1^* + w_2 q_2^* = \operatorname{div} p^*$, then we have the following and prove Coro. 4.2.

$$u^0 := w_1 f_1 + w_2 f_2 = u^* + \operatorname{div} p^*.$$

□

Consider the conjugates (4.7) and Prop. 4.4, the L1-norm based image fusion problem (4.3) leads to the following image decomposition:

Corollary 4.3. *The image fusion problem (4.3) is equivalent to*

$$\min_{p \in C_\alpha} \|(w_1 f_1 + w_2 f_2) - \operatorname{div} p\|^2, \quad (4.15)$$

i.e. the projection of the weighted input image $(w_1 f_1 + w_2 f_2)(\mathbf{x})$, $\mathbf{x} \in \Omega$, to the convex set $\operatorname{div} C_\alpha$.

Proof directly follows from the image decomposition model of Coro. 4.2 and (4.6). Clearly, similar results of image decomposition and projections as Coro. 4.2 and Coro. 4.3 were proposed in [26], which showed that TV-L2 image denoising amounts to image decomposition (4.14) and projection (4.15) of the single input image $u^0(\mathbf{x})$ instead of the weighted image $w_1 f_1 + w_2 f_2$.

For the image fusion problem (4.3), it leads to image decomposition as follows:

Corollary 4.4. *Given the optimum (q_1^*, q_2^*, p^*, u^*) of the equivalent primal-dual model (4.10) which is equivalent to (4.3), (q_1^*, q_2^*, p^*, u^*) just gives rise to the decomposition of the weighted input image $(w_1 f_1 + w_2 f_2)(\mathbf{x})$, $x \in \Omega$, such that*

$$u^0 := w_1 f_1 + w_2 f_2 = u^* + v^* \quad (4.16)$$

where

$$v^* = w_1 v_1^* + w_2 v_2^*, \quad v_1^* \in \partial I_S(q_1^*), \quad v_2^* \in \partial I_S(q_2^*),$$

I_S is the characteristic function of the set $S = \{q \mid q(\mathbf{x}) \in [-1, 1], \forall \mathbf{x} \in \Omega\}$.

Its proof directly follows by the conjugates (4.7) and Prop. 4.4.

4.3 Global and Exact Optimization

Now we focus on the TV-L1 based approach (4.3) and consider the respective discrete-valued optimization problem

$$\min_{u(\mathbf{x}) \in \mathcal{L}} \left\{ E(u) = \int_{\Omega} w_1 |u - f_1| d\mathbf{x} + \int_{\Omega} w_2 |u - f_2| d\mathbf{x} + \alpha \int_{\Omega} |\nabla u| d\mathbf{x} \right\}, \quad (4.17)$$

where we assume the two input images $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ take discrete values which are linearly ordered such that

$$f_i(\mathbf{x}) \in (\mathcal{L}_i := \{l_1^i, \dots, l_{n_i}^i\}), \quad l_1^i < l_2^i < \dots < l_{n_i}^i; \quad i = 1, 2 \quad (4.18)$$

and $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ is the combination set of $\mathcal{L}_1$ and $\mathcal{L}_2$. We also assume

$$\mathcal{L} = \{l_1, \dots, l_n\}, \quad l_1 < l_2 < \dots < l_n, \quad (4.19)$$

includes n discrete values.

In this section, we show that the TV-L1 based image fusion problem (4.3) amounts to an exact convex relaxation model of the integer-constrained optimization problem (4.17), i.e. the optimum of the convex optimization problem (4.3) results in the global and exact integer-valued optimum of (4.17). A similar

result was recently proposed by [138], where the authors proved that the convex TV-L1 image approximation does give global and exact optima to the corresponding discrete-constrained TV-L1 approximation. In order to reach our results, we should refer to the Proposition 3.1, 3.2, 3.3 and Corollary 3.1 in Chapter 3, due to the limit space we omit the details. Its detailed proof can be derived by the same way as [138].

4.4 Multiplier Based Algorithms

In this section, we propose fast numerical algorithms to image fusion problems (4.1) and (4.3) through their respective dual formulations. By Coro. 4.3, we observe that the image fusion problem (4.1) corresponds to the projection of the image $w_1 f_1 + w_2 f_2$ to the convex set $\text{div } C_\alpha$. It directly leads to the same duality based algorithm as [26] proposed by Chambolle. We list its iterative projected-gradient descent steps for computing the dual variable p as follows:

$$p^{i+1} = \mathbf{Proj}_{C_\alpha} \left(p^i + \tau \nabla ((w_1 f_1 + w_2 f_2) - \text{div } p^i) \right),$$

where $\tau > 0$ gives the step-size at each iteration.

With helps of (4.7) and (4.10), the TV-L1 based image fusion problem (4.3) can be written as the following primal-dual formulation:

$$\min_u \max_{q_1, q_2} \max_{p \in C_\alpha} \int_{\Omega} q_1 f_1 d\mathbf{x} + \int_{\Omega} q_2 f_2 d\mathbf{x} + \langle \text{div } p - (q_1 + q_2), u \rangle \quad (4.20)$$

$$\text{s.t. } q_1(\mathbf{x}) \in [-w_1(\mathbf{x}), w_1(\mathbf{x})], \quad q_2(\mathbf{x}) \in [-w_2(\mathbf{x}), w_2(\mathbf{x})]. \quad (4.21)$$

Also in view of (4.12), its equivalent dual model can be formulated as

$$\max_{q_1, q_2} \max_{p \in C_\alpha} \int_{\Omega} q_1 f_1 d\mathbf{x} + \int_{\Omega} q_2 f_2 d\mathbf{x} \quad (4.22)$$

$$\text{s.t. } q_1(\mathbf{x}) \in [-w_1(\mathbf{x}), w_1(\mathbf{x})], \quad q_2(\mathbf{x}) \in [-w_2(\mathbf{x}), w_2(\mathbf{x})]$$

$$q_1 + q_2 = \text{div } p. \quad (4.23)$$

Algorithm 2: Multiplier based algorithm to TV-L1 image fusion

- Set the starting values: q_1^0, q_2^0, p^0 and u^0 , and let $k = 1$;
- Start the k -th iteration which includes three successive sub-steps:
 1. Optimize q_1^{k+1} by fixing q_2^k, p^k and u^k :

$$\begin{aligned} q_1^{k+1} &:= \arg \max_{|q_1(\mathbf{x})| \leq w_1(\mathbf{x})} L_c(q_1, q_2^k, p^k, u^k) \\ &= \arg \max_{|q_1(\mathbf{x})| \leq w_1(\mathbf{x})} \langle q_1, f_1 \rangle - \frac{c}{2} \|q_1 - (\operatorname{div} p^k - q_2^k - u^k/c)\|^2, \end{aligned}$$

which is approximated by

$$q_1^{k+1} = \operatorname{Proj}_{|q_1(\mathbf{x})| \leq w_1(\mathbf{x})} (f_1/c + (\operatorname{div} p^k - q_2^k(\mathbf{x}) - u^k/c)); \quad (4.24)$$

2. Optimize q_2^{k+1} by fixing q_1^{k+1}, p^k and u^k :

$$\begin{aligned} q_2^{k+1} &:= \arg \max_{|q_2(\mathbf{x})| \leq w_2(\mathbf{x})} L_c(q_1^{k+1}, q_2, p^k, u^k) \\ &= \arg \max_{|q_2(\mathbf{x})| \leq w_2(\mathbf{x})} \langle q_2, f_2 \rangle - \frac{c}{2} \|q_2 - (\operatorname{div} p^k - q_1^{k+1} - u^k/c)\|^2, \end{aligned}$$

which is approximated by

$$q_2^{k+1} = \operatorname{Proj}_{|q_2(\mathbf{x})| \leq w_2(\mathbf{x})} (f_2/c + (\operatorname{div} p^k - q_1^{k+1}(\mathbf{x}) - u^k/c)); \quad (4.25)$$

3. Optimize p^{k+1} by fixing q_1^{k+1}, q_2^{k+1} and u^k :

$$p^{k+1} := \arg \min_{p \in C_\alpha} \|\operatorname{div} p - (q_1^{k+1} + q_2^{k+1} + u^k/c)\|^2, \quad (4.26)$$

which is the projection of $(q_1^{k+1} + q_2^{k+1} + u^k/c)$ to the convex set $\operatorname{div} C_\alpha$.

- Update u^{k+1} by

$$u^{k+1} = u^k + c(q_1^{k+1} + q_2^{k+1} - \operatorname{div} p^{k+1}); \quad (4.27)$$

and let $k = k + 1$, repeat untill convergence.

We see that the image $u(\mathbf{x})$ in the primal-dual formulation (4.20), which is what we wish to obtain, just works as the multiplier function to the linear equality constraint (4.23) of the dual model (4.22). In addition, the energy function of (4.20) gives the corresponding Lagrangian function to the dual formulation (4.22). Through these observations, we define its augmented Lagrangian function as

$$L_c(q_1, q_2, p, u) = \langle q_1, f_1 \rangle + \langle q_2, f_2 \rangle + \langle \operatorname{div} p - (q_1 + q_2), u \rangle - \frac{c}{2} \|\operatorname{div} p - (q_1 + q_2)\|^2$$

where $c > 0$.

In this chapter, we apply the classical augmented Lagrangian algorithm [101, 15] through its augmented Lagrangian function $L_c(q_1, q_2, p, u)$, which includes the following steps at k -th iteration as Algorithm 2. The algorithm gives a splitting optimization framework over each dual variables q_1 , q_2 and p respectively, by exploring projection to their corresponding convex set. To this end, we call it the *multiplier based algorithm to TV-L1 image fusion*. It explores three simple sub-steps: (4.24), (4.25) and (4.26) at each iteration, which properly avoids tackling the nonsmooth terms in (4.3) in a direct way. The substeps of (4.24) and (4.25) are easy and cheap to compute. For the projection substep (4.26), we can use one or a few steps of the iterative projected-gradient decent algorithm to approximately solve (4.26) as follows:

$$p^{i+1} = \mathbf{Proj}_{C_\alpha} \left(p^i + \tau \nabla \{ \operatorname{div} p^i - ((q_1^{i+1} + q_2^{i+1}) + u^i/c) \} \right). \quad (4.28)$$

Interestingly, our experiments show that just one single step of the above iteration (4.28), with a proper step-size τ , is needed to make the algorithm converge. This implements the algorithm in a very fast way, mostly superlinear.

4.5 Numerical Experiments

In this section, we first fuse two binary images to show the fundamental differences between (4.1) and (4.3). Then experiments for both medical imaging and remote sensing are given for comparisons. Note experiment results may vary

Figure 4.2: Fusing binary images (a) and (b) give the two input binary image; (c) and (d) show the results computed by the TV-L2 and TV-L1 based methods respectively.

with different choices of weights $w_1(\mathbf{x})$ and $w_2(\mathbf{x})$. In our experiments, firstly we use the gradient based weighted method to obtain the weighted functions $w_1(\mathbf{x}) = \frac{|\nabla f_1|}{|\nabla f_1| + |\nabla f_2|}$ and $w_2(\mathbf{x}) = 1 - w_1(\mathbf{x})$.

4.5.1 Fusing Binary Images

Given two binary images (see the two images on the leftside of Fig. 4.2), i.e. $f_{1,2}(\mathbf{x}) \in \{0, 1\}$, we computed the fused image by both two approaches: (4.1) and (4.3). For the TV-L2 based method (4.1), we set $\alpha = 3$ and its fused result $u(\mathbf{x})$ is shown by the 3rd image of Fig. 4.2. For the TV-L1 based method (4.3), we set $\alpha = 1$ and its fused result $u(\mathbf{x})$ is shown by the last image of Fig. 4.2. The choices of α are tuned to obtain the best results. Clearly, the TV-L1 based method gives the binary optimum which takes the value either 0 or 1 nearly everywhere. This is in contrast to the result of the TV-L2 based approach.

4.5.2 Applications to Medical Imaging and Remote Sensing

Besides the fusion experiment of multi-focused images (shown in Fig. 4.1), we also made fusion experiments for medical imaging and remote-sensing images.

Both experiments are computed by a Ubuntu desktop with AMD Athalon 64 X2 5600. The TV-L1 based method (4.3) takes couple of seconds. Except one additional step of (4.24) and (4.25), its algorithmic scheme has the same complexities as the fast TV-L1 method proposed in [138]. All the images are adjusted into the same grayscale range for comparisons. Fig. 4.3 shows the fusion experiment of medical imaging, which integrates the images from CT and MRI (see Fig. 4.3). The TV-L1 based method performs visually better than the TV-L2 based method in preserving high-contrast and details (see the enlarged image patches for comparisons). Fig. 4.4 shows the image fusion experiment of remote sensing, where two images from different spectral channels are fused by the studied two methods respectively. Detailed comparison of the enlarged patches (see the images at 2nd row of Fig. 4.4) clearly indicates better visual result by the TV-L1 based method.

Figure 4.3: Fusing medical images 1st row: the left two images show two input images of spine discs by CT and MRI respectively; the right two images show the fused images computed by (4.1) and (4.3) respectively. 2nd row: the left two images show the zoomed image patches cropped by the red lines on the same position of CT and MRI images respectively; the right two images show the fused results at the patched area computed by (4.1) and (4.3) respectively. 3rd row: it shows another experiment of fusing images from brain CT (see the first image) and T2 (see the second image); the results by TV-L2 based method and TV-L1 based method are given by the third and fourth images respectively.

Figure 4.4: Fusing images from two spectral bands 1st row: the left two images show the input images of remote sensing images from two different spectral channels; the right two images show the fused images computed by (4.1) and (4.3) respectively. 2nd row: the left two images show the zoomed image patches cropped by the red lines on the same position of the input images respectively; the right two images show the fused results at the patched area computed by (4.1) and (4.3) respectively.

Graph Cuts for Curvature based Image Denoising

5.1 Introduction

Image denoising is the problem of recovering a true image u from an observed noisy image u^0 . In this chapter, we assume the original image u has been perturbed by additive gaussian noise η :

$$u^0 = u + \eta.$$

The variational approach is an important paradigm for solving image denoising problems when the image is defined on a continuous domain.

The Tikhonov model [121] was firstly proposed, it can remove the noise but cannot keep sharp discontinuities. Unlike the Tikhonov model with L2-norm of the gradient as the regularizer which has over smoothing and blurring on the boundaries, the ROF (Rudin-Osher-Fatemi) model [103] instead has L1-norm of the gradient as the regularization term, which is called the total variation. The ROF model was the first image denoising model for 2D-noisy images with additive gaussian white noise using total variation as a regularization term. Total variation is a powerful notion in such variational problems. The main advantage of total

variation is the ability to preserve discontinuities. In the mean time, the L2 data term works better for gaussian noise removal and the L1 non-smooth data fidelity term has better results for impulsive noise removal. Several numerical methods have been developed for minimizing the energy in such TV-based models, and most of them are based on solving the corresponding Euler-Lagrange equations in the continuous setting. Recently, an equivalence between the anisotropic total variation denoising model and a class of binary second order MRF (markov random field) models has been established in the discrete and quantized setting [40, 41, 27], which has resulted in some very efficient algorithms. In case of L gray values, an optimum can be obtained by solving $\log_2(L) - 1$ binary second order MRFs independently. Such binary MRFs have been studied extensively in computer vision and it is well known that they can be solved very efficiently by graph cuts. As a result, graph cuts based algorithms can be used to efficiently find exact global minima of TV models in the discrete setting.

In spite of the simplicity of TV-based models, they have some disadvantages, for instance the so called staircasing effect: piecewise constant images are favored over piecewise smooth images. To counteract these disadvantages, more sophisticated models have recently been developed, where the energy functionals depend on higher order derivatives; see [134, 82, 84, 88, 85, 34, 35, 143, 137], etc. Models that minimize curvature based functionals have been demonstrated to perform particularly well. Of central importance is the Euler's elastica model [88, 85, 34]. In contrast to the total variation model, which minimizes the length of each level line in the image, Euler's elastica minimizes the total curvature of each level line in the image. As a result, the reconstructed image will have a more natural appearance. The Euler's elastica model can also preserve discontinuities.

The main disadvantage of such higher order methods is the difficulty and complexity of computation. Traditionally, the Euler-Lagrange or gradient descent equations are derived. These are nonlinear partial differential equations (PDEs) of order four or even higher. Numerically solving these equations is a

time consuming process. Another disadvantage of these methods is the possibility of getting stuck in a local minimum, since the curvature based functionals are not convex.

In this chapter we will propose an algorithm based on graph cuts for efficiently minimizing the energy in the Euler's elastica model. In the discrete setting, this model has the form of a higher order MRF. In contrast to second order MRFs, these are *NP*-hard and can generally not be solved exactly. Not much work has been devoted to higher order MRFs so far, however in the last few years some work has appeared, much of which was developed simultaneously with ours. Some recent work can be found in [68, 102, 71]. Very recently a technique was also proposed in [75] based on dual decomposition and message passing [76]. This method will in general find an approximate solution by solving a simpler relaxed problem. However, the curvature based models we study do not fit directly into their framework. A linear programming relaxation approach for region based segmentation models with curvature regularity was proposed in [105]. This approach will also find an approximate solution by solving a simpler linear programming relaxation of an integer optimization formulation of the problem. This approach could potentially be adapted to denoising by increasing the size of the linear program (L -times as large in case of L gray levels). However, the computational complexity would be quite heavy compared to ours, due to the size of the linear program. The same authors also presented an interesting technique for globally optimizing functions of curvature of a curve in case there are no data term (e.g. for edge based segmentation models in 2D) by minimum ratio cycles [104]. However, this result is not applicable for energy functionals with a separate data fitting term. In [47], the authors introduced minimization of curvature of region boundaries by graph cut, inspired by the work of [105]. The higher order curvature term was rewritten as a non-submodular second order term. The resulting non-submodular energy minimization problem is still *NP*-hard, but by applying non-submodular minimization techniques, global or close to global

solutions could be obtained on many instances. Their method is applicable to binary labeling problems. In contrast we are interested in multilabeling problems, where each label represents a gray value.

The curvature based models we study do not directly fit into the framework of [75]. One reason being that in the discrete setting, the interaction term in the Euler’s elastica model consists of a multiplication of one higher order factor and one second order factor. Curvature is also very complex and nonlinear in the discrete setting. Our approach is different, we instead show that the problem can be simplified to that of solving a sequence of much easier problems, in the form of second order submodular MRFs. The sequence of solutions to these simpler problems has connections to the gradient flow of the energy function and will converge to a local minimum. We claim our approach to be more efficient than previous approaches like [75, 105] due to the underlying graph cuts minimization technique and logarithmic complexity in the number of gray values.

Like all other curvature based approaches (except the edge based model [104]), ours cannot be guaranteed to find a global minimum. The solution may to some extent depend on the initialization. However, for the denoising application, the noisy input image is available as a good initialization. Furthermore, numerical experiments indicate that our algorithm for the Euler’s elastica model is rather robust to initialization. Our numerical results also show that the minimization of functionals related to curvature prevents shrinking effects and staircasing effects while keeping sharp edges and smooth interior, compared to the total variation denoising model. Restoration results are generated with higher Peak Signal to Noise Ratio (PSNR) values than the ROF model.

The rest of this chapter is organized as follows: In section 5.2 we review total variation and graph representable energy functions. We show how graph cuts can be used to efficiently minimize these energy functions. In section 5.3 we propose an algorithm for minimizing the Euler’s elastica model. Based on our general formulation of energy functional, we can make use of the connection between

minimization problems and binary MRFs, and solve the problem via graph cuts. In section 5.4, we carry out numerical experiments with our method, and compare them to the results of TV in several aspects such as Peak Signal to Noise Ratio (PSNR) values, residual images and visual effects.

5.2 Total Variation and Graph Representable Energy Functionals

In this section we discuss algorithms for total variation and graph representable second order MRFs, that will be used as substeps in our algorithm for minimizing higher order MRFs. Assume the input image u^0 is defined on a continuous domain Ω . Here we focus on two dimensional problems where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, but our algorithms and results can easily be generalized to higher dimensions. To recover a denoised image u , can be formulated as a minimization problem with constraints such as $\int_{\Omega} (u(\mathbf{x}) - u^0(\mathbf{x}))^2 d\mathbf{x} \leq \sigma^2 |\Omega|$, where σ^2 denotes a bound on the noise variance and $|\Omega|$ is the area of image domain. The total variation functional of u , defined as $TV(u) = \int_{\Omega} |\nabla u|$, was first used as a regularization term for image denoising in the ROF model taking the form

$$\min_{u \in BV(\Omega)} \left\{ E^{TV}(u) = a \int_{\Omega} |\nabla u(\mathbf{x})| + \int_{\Omega} (u(\mathbf{x}) - u^0(\mathbf{x}))^2 d\mathbf{x} \right\}. \quad (5.1)$$

The parameter $a > 0$ balances the TV regularization term and the fitting term. The space BV of functions of bounded variation is defined as $BV(\Omega) = \{u \in L^1(\Omega) : TV(u) < \infty\}$. There are two variants of the total variation term, the anisotropic variant:

$$TV_1(u) = \int_{\Omega} |\nabla u|_1 = \int_{\Omega} |u_x| + |u_y| d\mathbf{x},$$

and the isotropic variant:

$$TV_2(u) = \int_{\Omega} |\nabla u|_2 = \int_{\Omega} \sqrt{u_x^2 + u_y^2} d\mathbf{x}.$$

The anisotropic variant is graph representable and can be efficiently minimized by graph cuts as explained later. However, $TV_1(u)$ has the disadvantage of penalizing stronger in horizontal and vertical directions, i.e. it is not rotationally invariant. This may lead to some blocky artifacts in the processed image, which are often called metrication errors in the discrete setting.

Both $TV_1(u)$ and $TV_2(u)$ satisfy coarea formulae, which will be used extensively in the following sections. For a positive function u of bounded variation, we define the functions of upper level sets:

$$\theta^\ell = \begin{cases} 1 & \text{if } u \geq \ell, \\ 0 & \text{else.} \end{cases} \quad (5.2)$$

The coarea formula is given by

$$\int_{\Omega} |\nabla u(\mathbf{x})| d\mathbf{x} = \int_{\ell=0}^L \int_{\Omega} |\nabla \theta^\ell(\mathbf{x})| d\mathbf{x} d\ell, \quad (5.3)$$

where the image $u(\mathbf{x})$ is a positive function and L is the maximum value of u .

Let us also mention a generalization of the above functional which is graph representable and will be used in later sections

$$E^{TVg}(u) = \int_{\Omega} f^1(\mathbf{x})|u_x| + f^2(\mathbf{x})|u_y| d\mathbf{x} + \int_{\Omega} g(u(\mathbf{x}), \mathbf{x}) d\mathbf{x}, \quad (5.4)$$

where $f^1, f^2 : \Omega \mapsto \mathbb{R}$ are any positive functions and $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a convex function in u . In case $f^1 = f^2 = a$ and $g(u(\mathbf{x}), \mathbf{x}) = |u(\mathbf{x}) - u^0(\mathbf{x})|^2$, (5.4) reduces to the anisotropic TV denoising model. The coarea formula for this general functional is given by

$$\int_{\Omega} f^1(\mathbf{x})|u_x| + f^2(\mathbf{x})|u_y| d\mathbf{x} = \int_{\ell=0}^L \int_{\Omega} f^1(\mathbf{x})|\theta_x^\ell| + f^2(\mathbf{x})|\theta_y^\ell| d\mathbf{x} d\ell, \quad (5.5)$$

where L is the maximum value of u .

5.2.1 Discretization of Energy Functional

We now consider the discrete and quantized versions of (5.1) and (5.4). To this end, define the uniform grid with the grid size equals to 1:

$$\Omega_d = \{(i, j) : i = 1, \dots, N, j = 1, \dots, M\}, \quad (5.6)$$

and let the discrete functions $u_{i,j} : \Omega_d \mapsto \{0, \dots, L-1\}$ and $u_{i,j}^0 : \Omega \mapsto \{0, \dots, L-1\}$ be defined on this grid. $\{0, \dots, L-1\}$ is the set of gray values. In Fig. 5.1, Ω_d is depicted as the set of filled circles.

The anisotropic total variation model can be written as

$$E_d^{TV}(u) = \sum_{(i,j) \in \Omega_d} |u_{i,j} - u_{i,j}^0|^2 + a \sum_{(i,j) \in \Omega_d} |u_{i+1,j} - u_{i,j}| + |u_{i,j+1} - u_{i,j}|. \quad (5.7)$$

Let us now focus on the generalized model (5.4). To express the discrete version of this model it will be useful to define two new sets of grid points

$$\Omega_d^x = \{(i + \frac{1}{2}, j) : i = 1, \dots, N, j = 1, \dots, M\},$$

and

$$\Omega_d^y = \{(i, j + \frac{1}{2}) : i = 1, \dots, N, j = 1, \dots, M\}.$$

In Fig. 5.1, Ω_d^x is depicted as the set of squares and Ω_d^y is depicted as the set of empty circles. In the discrete setting, the functions $f^1 : \Omega_d^x \mapsto \mathbb{R}$ and $f^2 : \Omega_d^y \mapsto \mathbb{R}$ are defined on Ω_d^x and Ω_d^y respectively and the function $g : \Omega_d \mapsto \mathbb{R}$ is defined on Ω_d . The discrete version of (5.4) can now be written

$$E_d^{TVg}(u) = \sum_{i,j} g_{i,j}(u_{i,j}) + \sum_{i,j} f_{i+\frac{1}{2},j}^1 |u_{i+1,j} - u_{i,j}| + f_{i,j+\frac{1}{2}}^2 |u_{i,j+1} - u_{i,j}|. \quad (5.8)$$

Both the models (5.7) and (5.8) are anisotropic and use a 4-neighborhood system for the regularization term. It is possible to create "more" isotropic versions by using a larger neighborhood systems, and hence reduce metrication errors in the solutions. However, for simplicity we stick to 4-neighborhood systems in this chapter.

5.2.2 Equivalent Representation as Set of Binary MRFs

In this section we show how (5.7) and (5.8) can be minimized by graph cuts using an algorithm which has a logarithmic complexity in the number of gray values L .

Figure 5.1: Grid definition

In order to simplify notation, we construct an equivalent representation of (5.8).

For each $p = (i, j) \in \Omega_d$, we define the neighborhood system

$$\mathcal{N}_p^4 = \{(i \pm 1, j), (i, j \pm 1)\} \cap \Omega_d.$$

For $p \in \Omega_d$ and $q \in \mathcal{N}_p^4$ the discrete total variation model (5.7) can be written more compactly using the 4 neighborhood system as

$$E_d^{TV}(u) = \sum_{p \in \Omega_d} |u_p - u_p^0|^2 + \sum_{p \in \Omega_d} \sum_{q \in \mathcal{N}_p^4} \frac{1}{2} w_{pq} |u_p - u_q|. \quad (5.9)$$

The weights w_{pq} are simply set to a for all neighbors p and q . Using larger neighborhood can generate results that are more isotropic.

For the more general form (5.8), the notations p^1, p^2 and q^1, q^2 represent the first and second component of p and q respectively. Then (5.8) can be rewritten as:

$$E_d^{TVg}(u) = \sum_{p \in \Omega_d} g(u_p) + \sum_{p \in \Omega_d} \sum_{q \in \mathcal{N}_p^k} w_{pq} |u_p - u_q|, \quad (5.10)$$

with the weights w_{pq} given as follows:

$$w_{pq} = \begin{cases} f_{\frac{p^1+q^1}{2}, \frac{p^2+q^2}{2}}^1 & \text{if } p^2 = q^2, \\ f_{\frac{p^1+q^1}{2}, \frac{p^2+q^2}{2}}^2 & \text{if } p^1 = q^1. \end{cases}$$

Darbon et. al. [40, 41] and Chambolle [27] showed independently that the discrete TV model (5.7) and (5.9) could be very efficiently minimized via graph

cuts, provided the input image is quantized. The same result can also be adapted to the generalized model (5.8) and (5.10). In the quantized and discretized case it is assumed that $u : \Omega_d \mapsto [0, 1, \dots, L - 1]$. In this case there are $L - 1$ distinct functions of upper level sets $\ell = 0, \dots, L - 1$, which are point-wise given by

$$\theta_p^\ell = \begin{cases} 1 & \text{if } u_p \geq \ell, \\ 0 & \text{else.} \end{cases} \quad (5.11)$$

In the discrete setting the coarea formula can be written

$$\sum_{p \in \Omega_d} \sum_{q \in \mathcal{N}_p^k} w_{pq} |u_p - u_q| = \sum_{\ell=0}^{L-1} \sum_{p \in \Omega_d} \sum_{q \in \mathcal{N}_p^k} w_{pq} |\theta_p^\ell - \theta_q^\ell|, \quad (5.12)$$

where the weights w_{pq} can be derived e.g. by the Cauchy Crofton formula [16]. In this way, the cut based length on the grid graph approximates to the Euclidean length of curve, i.e.

$$\int_{\Omega} |\nabla \theta^\ell(\mathbf{x})| d\mathbf{x} d\ell = \sum_{p \in \Omega_d} \sum_{q \in \mathcal{N}_p^k} w_{pq} |\theta_p^\ell - \theta_q^\ell|.$$

If u is a function, it must be single valued at each point $p \in \Omega_d$. In that case the family $\{\theta^\ell\}_{\ell=0}^{L-2}$ is monotonically decreasing: $\theta^{\ell+1} \leq \theta^\ell$, $\forall \ell = 0, \dots, L - 2$. For any such family of monotonically decreasing binary functions, the function u can be recovered by the formula $u_p = \max\{\ell : \theta_p^\ell = 1\}$.

Using these facts it was shown in [27], [40], [41] and [58] that (5.9) can be written

$$\sum_{\ell=0}^{L-2} E^\ell(\theta^\ell) + \sum_{p \in \Omega_d} |u_p^0|,$$

where

$$\begin{aligned} E^\ell(\theta^\ell) &= \sum_{p \in \Omega_d} \sum_{q \in \mathcal{N}_p^k} w_{pq} |\theta_p^\ell - \theta_q^\ell| + \sum_{p \in \Omega_d} (|\ell + 1 - u_p^0|^2 - |\ell - u_p^0|^2)(1 - \theta_p^\ell) \\ &= \sum_{p \in \Omega_d} \sum_{q \in \mathcal{N}_p^k} w_{pq} |\theta_p^\ell - \theta_q^\ell| + \sum_{p \in \Omega_d} w_p (1 - \theta_p^\ell). \end{aligned} \quad (5.13)$$

So that,

$$w_p = |\ell + 1 - u_p^0|^2 - |\ell - u_p^0|^2,$$

which will be used in the following sections.

The minimization problem can therefore be written in terms of θ^ℓ as

$$\min_{\{\theta^\ell\}_{\ell=0}^{L-2}} \sum_{\ell=0}^{L-2} E^\ell(\theta^\ell), \quad (5.14)$$

under the constraints $\theta^{\ell+1} \leq \theta^\ell, \forall \ell = 0, \dots, L-2$. It was shown in [27] that by minimizing each $E^\ell(\theta^\ell)$ independently, these constraints were automatically satisfied. Furthermore, it was observed that each $E^\ell(\theta^\ell)$ has the form of a binary MRF, and could be efficiently minimized via graph cuts as was shown by Greig et. al in 1989 [60]. This leads to several algorithms. Since there is a lot of redundancy between the problems $E^\ell(\theta^\ell)$, it is not necessary to solve the problems sequentially. That would yield an algorithm of linear complexity in the number of gray values. Instead, a dyadic algorithm was proposed in [27, 40], which has a logarithmic complexity in the number of gray values, i.e. at most $\log_2(L) - 1$ binary MRFs had to be solved.

5.2.3 Graph Cuts Procedure

Graph cuts has been used in vision problems for a long time. We will briefly review the concept and show how it can be used to efficiently solve binary problems of the form (5.13). It was first observed in [60] that binary energy functions of the form (5.10), can be represented on a graph and minimized by max-flow/min-cut algorithms. This construction was also later studied in [17] and [73].

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined as a set of nodes $\mathcal{V}$ and a set of directed edges $\mathcal{E}$ that connect all the nodes. For 2D grids of the form (5.6), the set of vertices $\mathcal{V}$ is defined as

$$\mathcal{V} = \{\nu_p \mid p \in \Omega_d\} \cup \{s\} \cup \{t\},$$

where the two distinguished vertices in $\mathcal{V}$, s and t are called the source and the sink. Then the "cut" $c(S, T)$ of the graph is a partition of $\mathcal{V}$ into two sets with $s \in S$ and $t \in T$. The set of edges $\mathcal{E}$ is defined by:

$$\mathcal{E} = \{(\nu_p, \nu_q) \mid p \in \Omega_d, q \in \mathcal{N}_p^k\} \cup \{(s, \nu_p) \mid \forall p \in \Omega_d\} \cup \{(\nu_p, t) \mid \forall p \in \Omega_d\}.$$

Let $(a, b) \in \mathcal{E}$ denote an edge in $\mathcal{E}$, and let $c(a, b)$ denote the "capacity". The cost of the cut is defined as the sum of costs of severed edges:

$$c(S, T) = \sum_{a \in S, b \in T, (a, b) \in \mathcal{E}} c(a, b).$$

Based on the energy functionals (5.10) and (5.13), the costs on the edges from source to every vertex are $c(s, \nu_p) = \max(w_p, 0)$ and the costs from every vertex to the sink are $c(\nu_p, t) = \max(-w_p, 0)$; the costs added on edges between vertices corresponding to neighboring grid points are $c(\nu_p, \nu_q) = w_{\frac{p^1+q^1}{2}, \frac{p^2+q^2}{2}}$. The minimum cut on $\mathcal{G}$ corresponds to a minimizer of the energy functional and equals the maximum flow value due to the duality theorem of Ford and Fulkerson [51]. Hence max-flow algorithms can be used to minimize energy functionals of the form (5.13). In image processing applications, such max-flow algorithms have been shown to be extremely efficient [17].

5.3 Minimization of Higher Order Energy Functionals involving Curvature

Recently, there have been developed several variational image processing models based on higher order derivatives. Some fourth order PDEs [35, 34, 82, 84, 137, 143] are especially important. The motivation is to overcome limitations of second order models like total variation. Typically, minimization methods for these models are based on solving their corresponding Euler-Lagrange equations. These equations get increasingly complicated, and numerical solutions are usually rather inefficient. In this chapter, we make use of graph cuts as a minimization technique.

Unfortunately, discretizing the higher order energy functional directly will yield a very difficult combinatorial optimization problem, which is an example of an NP -hard higher order MRF. There has been little work on tackling such higher order MRFs, see the introduction for a short survey. The recent work of [75] gives details on how to compute approximate solutions to such NP -hard higher order MRFs by using dual decomposition [76]. The idea is to decompose the difficult MRF problem into easier subproblems. Typically each subproblem itself is a higher order MRF over a much smaller graph, even just over one single clique. The subproblems are solved independently. Based on the solutions a master assigns new parameters to each subproblem and they are solved independently over again. This process is iterated until convergence. It can be shown that such a process solves a linear programming relaxation of the problem. Even though this relaxation is not exact, the solutions can be accepted as suboptimal.

We will instead introduce an artificial discrete time variable. The problem of advancing from one time step to the next will have the graph representable form (5.8) considered in section 5.2.1 and can therefore be efficiently minimized by graph cuts, as explained in section 5.2.2. The sequence of solutions at each time step will converge to a local minimum of the original problem. Similarly as [75], this approach also simplifies the difficult problem into easier subproblems which are solved iteratively until convergence. We claim our approach is more efficient, mainly because our subproblems can be solved by an algorithm which has a logarithmic complexity in the number of labels. Furthermore, the underlying graph cut optimization technique has proven to be extremely efficient for practical image processing problems. The complexity of solving the subproblems in [75] is at least linear in the number of labels as far as we can see. Our approach can handle curvature based energy functionals that do not fit easily into the framework of [75].

The higher order models that minimize some functionals of the curvature of the image [34, 104, 120, 143] are especially important. Euler's elastica is an

important such curvature based model which was first introduced in image processing in [88]. It was studied further in [85, 34], where PDE based optimization methods were developed. In image processing the model can be formulated as the minimization of Euler's elastica of all level curves of the image. The Euler's elastica of a curve Γ is given by the energy

$$\int_{\Gamma} (a + b \cdot |\kappa|^\beta(s)) ds, \quad (5.15)$$

where a and b are two parameters and κ is the curvature of Γ at position s . By setting $b = 0$, $E(\Gamma)$ measures the total length of the curve. If $a = 0$, $E(\Gamma)$ measures the total curvature of the curve. Therefore, the Euler's elastica of all level curves of an image u can be written as:

$$\int_{\ell=0}^L \int_{\gamma_\ell: u=\ell} (a + b \cdot |\kappa|^\beta(s)) ds d\ell. \quad (5.16)$$

The power β can be set to $\beta = 1$ as in [85], or $\beta = 2$ as in [34]. The choice of $\beta = 1$ allows cracks (sharp corners) of the level curves. By setting $\beta = 2$ differentiation is much easier, such that the Euler-Lagrange equations do not get too complicated. Our approach can handle any choice of β with equal difficulty. For the denoising application we found that $\beta = 1$ tends to give best results. Note that the curvature of the level curve can be expressed:

$$\kappa = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right). \quad (5.17)$$

Using this fact, (5.16) can be expressed more simply as

$$\int_{\Omega} \left(a + b \left| \nabla \cdot \frac{\nabla u}{|\nabla u|} \right|^\beta \right) |\nabla u| d\mathbf{x}. \quad (5.18)$$

Euler's elastica can be used as either an image inpainting model or an image denoising model. In the inpainting application, we assume $\tilde{\Omega} \subset \Omega$ is the domain of missing information. The model is then formulated as

$$\min_u \int_{\tilde{\Omega}} \left(a + b \left| \nabla \cdot \frac{\nabla u}{|\nabla u|} \right|^\beta \right) |\nabla u| d\mathbf{x} \quad (5.19)$$

$$\text{s.t. } u = u_0 \quad \text{on } \partial\tilde{\Omega},$$

where $\partial\tilde{\Omega}$ denotes the boundary of $\tilde{\Omega}$.

Here our emphasis is instead on image denoising, in which case the complete energy functional can be written as

$$\min_u \left\{ E^{\text{EL}}(u) = \int_{\Omega} |u - u^0|^2 d\mathbf{x} + \int_{\Omega} \left(a + b \left| \nabla \cdot \frac{\nabla u}{|\nabla u|} \right|^\beta \right) |\nabla u| d\mathbf{x} \right\}. \quad (5.20)$$

If the parameter b is set to $b = 0$, this model reduces to the usual total variation denoising model, which in view of the discussion above minimizes the total length of all the level lines in the image. If $a = 0$ (5.20) minimizes the total curvature of all the level lines in the image. Euler's elastica is therefore obviously more sophisticated than total variation, as the resulting level lines will have a more natural appearance. Theoretically, Euler's elastica may also suffer from the staircasing effect, just like total variation, however we found that this model generally gives smoother results. Just as in the TV model, there are anisotropic and isotropic variants of (5.20) depending on the choice on whether one uses the 1-norm or 2-norm to measure the magnitude of the gradient ∇u . Our algorithm allows to use a mixture with 1-norm in the nominator and 2-norm in the denominator

$$\min_u \left\{ E^{\text{EL}}(u) = \int_{\Omega} |u - u^0|^2 d\mathbf{x} + \int_{\Omega} \left(a + b \left| \nabla \cdot \frac{\nabla u}{|\nabla u|_2} \right|^\beta \right) |\nabla u|_1 d\mathbf{x} \right\}.$$

In this chapter we will develop graph cuts based methods for minimizing the energy in the models (5.19) and (5.20). Neither of these models have the graph representable form considered in Section 5.2. However, by introducing an artificial discrete time variable, we show that a minimum can be obtained by solving a sequence of much easier graph representable problems. Instead of deriving and discretizing the Euler-Lagrange equations or gradient descent equations, we start by discretizing the energy functional itself.

5.3.1 Discretization of Curvature based Energy Functionals

The energy functionals (5.19) and (5.20) are discretized on the domain Ω_d . We assume further that u and u^0 are quantized, i.e. their function values are constrained to a finite set $u, u^0 : \Omega_d \mapsto [0, 1, \dots, L - 1]$. The discrete gradient ∇u is given by the forward difference scheme:

$$\begin{aligned}\nabla u &= (u^x, u^y), \\ u_{i+\frac{1}{2},j}^x &= u_{i+1,j} - u_{i,j}, \\ u_{i,j+\frac{1}{2}}^y &= u_{i,j+1} - u_{i,j},\end{aligned}\tag{5.21}$$

where $u^x : \Omega_d^x \mapsto \mathbb{R}$ and $u^y : \Omega_d^y \mapsto \mathbb{R}$. Alternatively, the central difference scheme can be used as a rougher approximation on Ω_d

$$\begin{aligned}\nabla u &= (u^x, u^y), \\ u_{i,j}^x &= (u_{i+1,j} - u_{i-1,j})/2, \\ u_{i,j}^y &= (u_{i,j+1} - u_{i,j-1})/2.\end{aligned}\tag{5.22}$$

The second order derivatives are discretized as

$$\begin{aligned}u_{i,j}^{xx} &= u_{i+1,j} - 2u_{i,j} + u_{i-1,j}, \\ u_{i,j}^{yy} &= u_{i,j+1} - 2u_{i,j} + u_{i,j-1},\end{aligned}\tag{5.23}$$

where $u^{xx} : \Omega_d \mapsto \mathbb{R}$ and $u^{yy} : \Omega_d \mapsto \mathbb{R}$.

The curvature κ can be discretized with central differences by

$$\kappa = \frac{u^{xx}(u^y)^2 - 2u^x u^y u^{xy} + u^{yy}(u^x)^2}{((u^x)^2 + (u^y)^2)^{\frac{3}{2}}}.\tag{5.24}$$

See e.g. [20] page 40.

Hence $\kappa : \Omega_d \mapsto \mathbb{R}$ is defined on Ω_d . In order to define κ on Ω_d^x and Ω_d^y , we have used linear interpolation in connection with (5.24)

$$\begin{aligned}\kappa_{i+\frac{1}{2},j} &= \frac{\kappa_{i+1,j} + \kappa_{i,j}}{2} : \Omega_d^x \mapsto \mathbb{R}, \\ \kappa_{i,j+\frac{1}{2}} &= \frac{\kappa_{i,j+1} + \kappa_{i,j}}{2} : \Omega_d^y \mapsto \mathbb{R},\end{aligned}\tag{5.25}$$

In our experience (5.24) in combination with (5.25) yielded best results. It will be referred to as DI. We have also experimented with some other discretizations, such as the forward scheme

$$\kappa = \left(\frac{u^x}{\sqrt{(u^x)^2 + (u^y)^2}} \right)^x + \left(\frac{u^y}{\sqrt{(u^x)^2 + (u^y)^2}} \right)^y, \quad (5.26)$$

The averaging scheme (5.25) in combination with (5.26) will be referred to as DII, it results in some blurring around the edges. Another option, which is often used in numerical solutions of PDEs involving curvature is the min-mod [93] discretization

$$\begin{aligned} \kappa_{i+\frac{1}{2},j} &= \text{minmod}(\kappa_{i+1,j}, \kappa_{i,j}) : \Omega_d^x \mapsto \mathbb{R}, \\ \kappa_{i,j+\frac{1}{2}} &= \text{minmod}(\kappa_{i,j+1}, \kappa_{i,j}) : \Omega_d^y \mapsto \mathbb{R}. \\ \text{minmod}(A, B) &= \frac{\text{sign}(A) + \text{sign}(B)}{2} \min(|A|, |B|). \end{aligned} \quad (5.27)$$

For this discretization to be valid it is required that the time-step in the PDE is set quite low. We therefore experienced this discretization was not very suitable in our algorithm. We instead found the following modification of the min-mod to be more suitable (referred to as DIII)

$$\text{minmod}(A, B) = \begin{cases} |A + B|/2 & \text{if } \text{sign}(A) \neq \text{sign}(B), \\ \frac{\text{sign}(A) + \text{sign}(B)}{2} \min(|A|, |B|) & \text{otherwise.} \end{cases} \quad (5.28)$$

The discrete version of the Euler's elastica denoising model (5.20) can now be written with any of the discretizations of κ as

$$\begin{aligned} \min_u E_d^{\text{EL}}(u) &= \sum_{(i,j) \in \Omega_d} \left(a + b \cdot |\kappa_{i+\frac{1}{2},j}|^\beta \right) \left| u_{i+\frac{1}{2},j}^x \right| \\ &+ \left(a + b \cdot |\kappa_{i,j+\frac{1}{2}}|^\beta \right) \left| u_{i,j+\frac{1}{2}}^y \right| + \sum_{(i,j) \in \Omega_d} |u_{i,j} - u_{i,j}^0|^2. \end{aligned} \quad (5.29)$$

Note that the anisotropic gradient factor is discretized with a 4-neighborhood system. A more isotropic version of the elastica model could be derived by using a larger neighborhood system (e.g. 8 neighbors), but for simplicity we stick to 4 neighborhood systems in this chapter.

We will now define the discrete version of the Euler's elastica inpainting model. Let $\tilde{\Omega}_d \subset \Omega_d$ denote the discrete inpainting domain, the discrete version the inpainting model (5.19) can then be written

$$\sum_{(i,j) \in \tilde{\Omega}_d} \left(a + b \cdot |\kappa_{i+\frac{1}{2},j}|^\beta \right) \left| u_{i+\frac{1}{2},j}^x \right| + \left(a + b \cdot |\kappa_{i,j+\frac{1}{2}}|^\beta \right) \left| u_{i,j+\frac{1}{2}}^y \right| \quad (5.30)$$

$$\text{s.t. } u = u^0 \quad \text{on } \partial\tilde{\Omega}_d.$$

In order to fit the framework of our algorithm, the side constraint must be replaced by a convex data term. Since u is assumed quantized, it is easily seen that (5.30) can be equivalently rewritten as

$$\begin{aligned} \min_u E_d^{EL}(u) = & \sum_{(i,j) \in \tilde{\Omega}_d} \left(a + b \cdot |\kappa_{i+\frac{1}{2},j}|^\beta \right) \left| u_{i+\frac{1}{2},j}^x \right| \\ & + \left(a + b \cdot |\kappa_{i,j+\frac{1}{2}}|^\beta \right) \left| u_{i,j+\frac{1}{2}}^y \right| + \lambda \sum_{(i,j) \in \Omega_d \setminus \tilde{\Omega}_d} |u_{i,j} - u_{i,j}^0|^2, \end{aligned} \quad (5.31)$$

provided the parameter λ is sufficiently large compared to a and b .

The discrete models (5.29) and (5.31) are now combinatorial optimization problems.

5.3.2 Minimization based on Iterative Graph Cuts

It is no doubt the optimization problems (5.29) and (5.31) are very complex and difficult to solve exactly, since they do not have the form considered in (5.8). Furthermore, (5.29) and (5.31) do not fit easily into the framework of [75], because of the nonlinear mixture of pairwise and higher order factors. We will show that the problem can be simplified by introducing an artificial discrete time variable. This is related to the work of [18, 28], where gradient flows of interphases were studied in an integral framework, in case the original energy functional is graph representable in the discrete setting. Our work differs in two main directions: we study images instead of interphases, and we do not require the original energy functional to be graph representable, i.e. we allow

more complicated energy functionals involving curvature. Furthermore, we are interested in the final steady state solution, not the evolution sequence itself. We start by the first generalization: let E be a graph representable energy functional of the form (5.4), depending on images u . The gradient flow of E can be regarded as an infinitesimal sequence of images u^t such that

$$u^{t+dt} = \arg \min_{u: \|u - u^t\| < \varepsilon(dt)} E(u).$$

where $\varepsilon(dt) \rightarrow 0$ as $dt \rightarrow 0$. Choosing the L2 norm $\|u\|_2^2 = \int_{\Omega} |u|^2 d\mathbf{x}$ in the constraint set above will have nice properties, which will soon become apparent. By adding the constraint as a penalty term, this can equivalently be formulated as

$$u^{t+dt} = \arg \min_u E(u) + \gamma(dt) \|u - u^t\|,$$

where $\gamma(dt) \rightarrow \infty$ as $dt \rightarrow 0$, e.g. $\gamma(dt) = \frac{1}{dt}$.

We can now state and prove the following theorem, which is the image counterpart of Theorem 1 given in [18] for interphases.

Theorem 5.1. *Given time t_0 , let u^t be the solution of*

$$u^t = \arg \min_u E(u) + \frac{1}{2(t - t_0)} \|u - u^{t_0}\|_2^2, \quad (5.32)$$

where $\|u - u^{t_0}\|_2^2 = \int_{\Omega} |u - u^{t_0}|^2 d\mathbf{x}$. Then, as $t \rightarrow t_0$

$$u^t = u^{t_0} - \frac{\partial E}{\partial u}(u^t)(t - t_0) + o(\Delta t), \quad (5.33)$$

i.e.

$$\frac{\partial u^t}{\partial t} = -\frac{\partial E}{\partial u}(u^t), \quad (5.34)$$

where $\frac{dE}{du}$ is the gradient of E with respect to the L2 inner product.

Proof. Since u^t minimizes (5.32)

$$0 = \frac{\partial}{\partial u} \left\{ E(u) + \frac{1}{2(t - t_0)} \|u - u^{t_0}\|_2^2 \right\} (u^t)$$

$$= \frac{\partial E}{\partial u}(u^t) + \frac{1}{(t - t_0)}(u^t - u^{t_0}).$$

By rearranging we get

$$\frac{u^t - u^{t_0}}{t - t_0} = -\frac{\partial E}{\partial u}(u^t),$$

or, as $t \rightarrow t_0$

$$\frac{\partial u^t}{\partial t} = -\frac{\partial E}{\partial u}(u^t)$$

□

In the time discrete setting we let Δt denote the step size and $u^n \approx u^{t_0+n\Delta t}$ denote the function u at time step n . Then u^{n+1} is given by

$$u^{n+1} = \arg \min_u E(u) + \frac{1}{2\Delta t} \|u^{n+1} - u^n\|^2 \quad (5.35)$$

and the discrete gradient flow can then be written

$$u^{n+1} = u^n + \frac{1}{2\Delta t} \frac{\partial E}{\partial u}(u^{n+1}),$$

This can therefore be seen as a fully implicit time discretization, where all the factors in $\frac{\partial E}{\partial u}$ are evaluated at the unknown u^{n+1} .

We will now consider problems of the form (5.29) where the energy functional E itself is not graph representable. By instead making an appropriate semi-implicit time discretization (some factors are handled implicitly and some explicitly), the gradient flow can be approximated by efficiently solving a problem of the form (5.10) to get from one time step to the next. We also directly move to the spatially discretized models from now on.

We start by considering (5.18), which is simplest. Letting u^1 be some initial guess and u^0 the input image, we propose the following scheme for minimizing (5.29): for $n = 1, \dots$, solve

$$\begin{aligned} u^{n+1} = \arg \min_{u^{n+1}} \sum_{(i,j) \in \Omega_d} & \left(a + b \cdot |\kappa_{i+\frac{1}{2},j}^n|^\beta \right) \left| (u_{i+\frac{1}{2},j}^x)^{n+1} \right| + \left(a + b \cdot |\kappa_{i,j+\frac{1}{2}}^n|^\beta \right) \left| (u_{i,j+\frac{1}{2}}^y)^{n+1} \right| \\ & + \sum_{(i,j) \in \Omega_d} |u_{i,j}^{n+1} - u_{i,j}^0|^2 + \frac{1}{2\Delta t} \sum_{(i,j) \in \Omega_d} |u_{i,j}^{n+1} - u_{i,j}^n|^2, \end{aligned} \quad (5.36)$$

The same scheme also applies to the inpainting model (5.31) by replacing the data term with $\lambda \sum_{(i,j) \in \Omega_d \setminus \tilde{\Omega}_d} |u_{i,j} - u_{i,j}^0|^2$.

Observe that if u^n and k^n at time step n are known, this is an optimization problem of the same form as (5.8) for the unknown u^{n+1} , with f^1, f^2 and g chosen as

$$\begin{aligned} f_{i+\frac{1}{2},j}^1 &= a + b \cdot |\kappa_{i+\frac{1}{2},j}^n|^\beta : \Omega_d^x \rightarrow \mathbb{R}, \\ f_{i,j+\frac{1}{2}}^2 &= a + b \cdot |\kappa_{i,j+\frac{1}{2}}^n|^\beta : \Omega_d^y \rightarrow \mathbb{R}, \\ g_{i,j} &= |u_{i,j}^{n+1} - u_{i,j}^0|^2 + \frac{1}{2\Delta t} |u_{i,j}^{n+1} - u_{i,j}^n|^2 : \Omega_d \rightarrow \mathbb{R}. \end{aligned} \tag{5.37}$$

The function $g_{i,j}$ is convex in $u_{i,j}^{n+1}$ as required, being a sum of two convex functions. Therefore, in the spatially discrete setting one can get from one time step to the next by efficiently minimizing $\log_2(L) - 1$ binary MRFs by graph cuts as described in section 5.2.2.

The penalty term of (5.36) $\frac{1}{2\Delta t} \sum_{i,j} |u_{i,j}^{n+1} - u_{i,j}^n|^2$ is essential for ensuring convergence of the algorithm. Without this term, the algorithm would alternate. As we see, it is incorporated in the algorithm directly in the data term of the second order MRF subproblem. If Δt is low, the penalty term is large, therefore the solution u is not allowed make a large displacement from each time step to the next. The relation to the gradient flow holds if the time step Δt is chosen sufficiently small. However, we experienced that the algorithm still converges to the same solutions even if the time step is set quite large. On the other hand, a large choice of Δt may result in oscillations as the sequence u^{n+1} reaches a stationary point. As a compromise, we found it useful to choose a large Δt in the beginning (e.g. first 10-15 iterations) and set Δt lower in the final iterations. It is difficult to find a Δt which completely prevents small oscillations around the stationary point. A useful technique is to add artificial intermediate gray values in the computation, i.e. to compute on a finer label scale. For instance, if an output of 256 gray values is required (i.e. label set: $\{0, 1, \dots, 255\}$), 1024 gray values can be used in the computation (label set in computation: $\{0, 0.25, 0.5, \dots, 254.75, 255\}$). In this case there are 3 artificial gray levels between each true gray value. This strategy

Figure 5.2: Inpainting example: The region with random noise denotes the inpainting domain. By minimizing the total curvature of the level lines, our method can fill in the missing regions appropriately. The data term is set to zero in the inpainting domain (the noisy regions). Figure (c) and (g) shows the result of total variation, which minimizes the total length of the level lines. Figure (d) and (h) shows our result of minimizing Euler's elastica with $a = 0$ and using the noisy image as initialization. Some artifacts can be observed due to the simple 4-neighborhood implementation.

Figure 5.3: Denoising results for Dragonfly(480×320) with $\sigma^2 = 0.01$. Thin elongated structures are better preserved by the Euler's elastica model. (a) input noisy image, (b) recovered by TV, (c) recovered by Euler's elastica (DI). Set of pixels with nonzero gradient (level lines): (d) TV, (e) Euler's elastica. The reconstructed image is constant in the white regions, observe that TV favors piecewise constant images to a larger degree than Euler's elastica.

Figure 5.4: Energy as a function of the iteration count for Algorithm 3 (Euler's elastica) applied to the Lena image (a) and the Pepper image (b). Red: noisy image $u^1 = u^0$ used as initialization. Blue: constant image $u^1 = 155$ used as initialization. Convergence is reached in around 30-40 iteration for noisy initialization.

aims to force most of the oscillations to occur at the fine scale (adding 3 such artificial labels seems to be sufficient). In the end, the solution is converted to 256 gray values by rounding. This approach will not add much to the cost of the algorithm, since computation with 1024 gray values instead of 256 requires to solve $9 = \log_2(1024) - 1$ binary problems instead of $7 = \log_2(256) - 1$ in each iteration.

Algorithm 3: Algorithm for Euler's Elastica Model

- Step1: Input image u^0 , set $u^1 = u^0$;
 - Step2: For $n = 1, 2, 3, \dots$ until converge,
find u^{n+1} by solving (5.36) with u^0 and u^n as fixed inputs. This subproblem is a second order MRF with parameters from (5.37), and is solved via graph cuts by as described in section 5.2.2 and 5.2.3;
 - Step3: Output u^{n+1} .
-

Figure 5.5: Denoising results for Lena(512×512) with $\sigma^2 = 0.01$. (a) original image, (b) input noisy image, (c) recovered image by TV, (d) recovered image by Euler's elastica DIII, (e) recovered image by Euler's elastica average DII, (f) recovered image by Euler's elastica DI.

Figure 5.6: Denoising results for part of Lena with $\sigma^2 = 0.01$. (a) original image, (b) input noisy image, (c) recovered image by TV, (d) recovered image by Euler's elastica DIII, (e) recovered image by Euler's elastica DII, (f) recovered image by Euler's elastica DI.

Figure 5.7: Difference between reconstructed image and noisy image for part of Lena. (a) TV, (b) Euler's elastica DIII, (c) Euler's elastica DII, (d) Euler's elastica DI.

Figure 5.8: Denoising results for Pepper(512×512) with $\sigma^2 = 0.01$. (a) original image, (b) input noisy image, (c) recovered image by TV, (d) recovered image by Euler's elastica DIII, (e) recovered image by Euler's elastica DII, (f) recovered image by Euler's elastica DI.

Figure 5.9: Denoising results for part of Pepper with $\sigma^2 = 0.01$. a) original image, (b) input noisy image, (c) recovered image by TV, (d) recovered image by Euler's elastica DIII, (e) recovered image by Euler's elastica DII, (f) recovered image by Euler's elastica DI.

Figure 5.10: Set of pixels with nonzero gradient (level lines). (a) Lena reconstructed by TV (from Fig. 5.5 c), (b) Lena reconstructed by Euler's elastica DI (from Fig. 5.5 f), (c) Pepper reconstructed by TV (from Fig. 5.8 c), (d) Pepper reconstructed by Euler's elastica DI (from Fig. 5.8 f).

To summarize, Algorithm 3 iteratively solves a sequence of second order MRF problems until convergence. Each of these MRF problems are converted to $\log_2(L) - 1$ binary MRF problems, where L is the number of gray values, each of which are solved efficiently by graph cuts. Therefore the complexity is logarithmic in the number of gray values. The approach of [75] is also based on iteratively solving a sequence of simpler problems until convergence to an approximate solution of the original problem. However, the complexity of solving the simpler problems increases at least linearly with L as far as we can see.

5.4 Numerical Experiments

In this section, the algorithm is tested numerically and the results are compared to anisotropic total variation generated by graph cuts. The codes are implemented in C++. For max-flow computation, we have used the implementation of Boykov et. al. reported in [17]. The numerical experiments were performed on an HP laptop with an Intel(R) Core(TM) 2 Duo CPU T7500 @ 2.19GHz, and 2.00GB of RAM. The results are compared visually and by measuring the Peak Signal to Noise Ratio (PSNR) between the ground truth and the output of the different methods. The PSNR is defined as:

$$\text{PSNR}(u, v) = 10 \log_{10} \frac{255^2}{\frac{1}{MN} \sum_{i,j} (u_{i,j} - v_{i,j})^2},$$

where $u_{i,j}$ and $v_{i,j}$ denote the pixel values of initial ground truth images and restored images respectively. In addition, the residual image $r = u^0 - v$ where u^0 is the noisy input image and v is the restored output image is shown for some of the experiments.

We start by giving illustrative inpainting examples in Figure 5.2 which clearly show the advantage of using curvature information over total variation. The inpainting domain $\tilde{\Omega}$ consists of the noisy regions. Random noise has been added in the inpainting domain such that an initialization is available for our method. The same result was also obtained by initializing $u^1 = 255$ (all white) in the

inpainting domain. As we see, our algorithm for minimization of Euler's elastica perfectly connects the level lines such that the missing region is inpainted in the visually most logical manner. Total variation on the other hand, just minimizes the total length of the level lines, which leads to much less reasonable results. The sudden turns in the level lines are not penalized by TV.

Figure 5.3 also shows the same effect on a more realistic image denoising example. The thin elongated structures are better preserved by the Euler's elastica model. Furthermore, the level lines have a smoother, more natural appearance. Figure 5.3 (d) and (e) show the set of pixels where the gradient of the image is nonzero, i.e the set of all level lines. The reconstructed image is constant in each white region. As we see, total variation favors solutions that are piecewise constant to a larger degree than Euler's elastica, the so called staircasing effect. In all the examples so far we used discretization (DI). The power in the Euler's elastica model is set to $\beta = 1$.

For Lena (512×512) and Pepper (512×512) presented in Figure 5.5-5.9, gaussian noise with a variance of $\sigma^2 = 0.01$ was added. In these experiments we have selected the regularization parameters in TV and the curvature based model to optimize results visually. In the Euler's elastica model (Algorithm 3) we set $a = 0$, and $\beta = 1$, so that the curvature term does all the regularization. The case that $a > 0$ is not so interesting, since that would be a balancing of TV and curvature regularization. Results with different discretizations of the curvature are shown: (d) DIII, (e) DII and (f) DI.

In all these experiments we observe that the algorithm for Euler's elastica yields smoother results than total variation. In all the TV results one can clearly notice the staircasing effect: piecewise constant image are favored. The Euler's elastica model may also suffer from the staircasing effect, but to a less degree than TV.

The results are also compared by measuring the Peak Signal to Noise Ratio (PSNR) in Table1, which shows that the curvature based methods clearly perform

best for all the experiments. The residual images for the curvature based methods also include less signal information than total variation.

We found that Algorithm 3 for Euler's elastica converged in around 30-40 iterations. This can also be seen from the graphs in Figure 5.4 where we have plotted the energy $E^{\text{EL}}(u^n)$ as a function of the iteration count n for the two images Lena and Pepper. The graphs also show that results generated by Algorithm 3 are quite insensitive to initialization. The red curve depicts energies after initialization with the noisy image $u^1 = u^0$, while the blue curve depicts energies after initialization with the constant image $u^1 = 155$. They both converge to solutions of almost the same energy. For initializations with the noisy image u^0 , most of the change happen during the first 5-10 iterations. The cpu time for our implementation is around 6 seconds per iteration for the lena and pepper images (size 512×512) with 1024 labels, if the max-flow problems are solved from scratch in each iteration. However, since the edge weights only change a little from each iteration to the next, one should in practice reuse flow from the previous iteration to get a much faster algorithm. For instance, the work of Kohli et. al. [72] presented such a max-flow algorithm for dynamically changing MRF models. When the changes in the weights are small this algorithm is shown to be significantly faster than traditional max-flow algorithms. However, a careful implementation of such methods is out of the scope of this chapter.

Img. No.	Table1: PSNR results comparison ($\sigma^2 = 0.01$)			
	Name	TV	Euler's Elastica, DII	Euler's Elastica, DI
Fig. 5.5	Lena	21.86dB	23.46dB	26.59dB
Fig. 5.8	Pepper	24.86dB	26.80dB	28.07dB
Fig. 5.3	Dragonfly	15.71dB	-	18.87dB

Graph Cuts for Curvature based Image Segmentation

6.1 Introduction

Image segmentation includes determining the important parts in images, then partitioning the image domain into several distinct homogeneous subregions. It is an important component in computer vision and useful in many real applications such as medical diagnosis.

We classify the different types of approaches to solve segmentation problems in the discrete and continuous environments. In the discrete setting, some image segmentation models for instance the watershed segmentation methods and region merging algorithms have been proposed. This kind of models is easy to solve by fast algorithms. However, the solutions always rely on the choices of parameters and are not strictly mathematical. In this scenario, some mathematically well studied continuous models are proposed. The Mumford-Shah segmentation model was first introduced in [89] and has been investigated intensively in the past few years. Later, the two-phase Chan-Vese model was proposed in [32]. It represents the partitions by several overlapping level set functions. By introducing new level set functions into framework, this method can be extended to more regions using

intersections between inside and outside of the level sets. In [10], the multiphase segmentation problems are solved by graph cuts, the same results are obtained and computation costs are saved a lot.

The level set method was firstly introduced by Osher and Sethian [94] for tracking moving fronts. Nowadays, it is a well applied instrument for numerical realization [78]. The level set method has an implicit and flexible representation of interfaces of different subregions. The original idea is to represent the interface as the zero level set of a function, and the level set evolution is computed on a fixed coordinate system since it is a parameter free formulation. Later, the piecewise constant level set method was proposed and used discontinuities to represent the interfaces [81]. Comparatively, this method has the advantages as follows: refraining the re-initialization of the level set function when evolution and making use of only one level set function instead of several ones.

The Mumford-Shah model is not convex, the solution obtained by traditional gradient descent method might be far from globally optimal solution. Recently, the relationship between graph cuts and continuous variational models has been investigated. Graph cuts technique is a useful optimization technique which can solve image segmentation problems by minimizing discrete energy functionals. It has higher efficiency than traditional methods and is able to find global minimum. Thus, instead of solving the corresponding Euler-Lagrange equations, we propose to minimize the energy functional by graph cuts. By discretizing the energy functional and finding the minimum cut on a designed graph, the solution for the segmentation problem will be obtained.

We are inspired by the details preservation results of higher order models in denoising problems in Chapter 5. The Euler's elastica model is one of the most important higher order models, which minimizes the total curvature of each level line in the image. In this chapter, we propose a model based on the Mumford-Shah model with curvature information and solve it globally and efficiently. We will construct a graph so that the discrete variational model can be minimized

exactly by finding the minimum cut on this graph. A linear programming relaxation approach for region based segmentation models with curvature regularity was also proposed in [105]. This approach tries to find the approximation by solving linear programming relaxation of an integer optimization formulation of the problem. However, the computation complexity is heavier than ours due to the huge size of the linear program. Our approach is more efficient and has logarithmic complexity, thanks to the underlying graph cuts minimization technique.

The rest of this chapter is organized as follows. In Section 6.2, we review the previous works on image segmentation including the Mumford-Shah model and the piecewise constant level set model. In Section 6.3, we formulate the curvature based segmentation variational model. In Section 6.4, we describe the computation procedure by graph cuts and provide the fast numerical algorithm. Section 6.5 is devoted to the numerical experiments for multiple segmentation applications, the comparisons of visual effects and PSNR values with previous works are provided.

6.2 Previous Works on Image Segmentation

We give a short review on image segmentation works. Beginning from 2-D cases, input image u^0 is defined on a bounded open set $\Omega \subset \mathbb{R}^2$ and in our investigation u^0 can either contain noise or not. The segmentation task can be considered as obtaining a set of subdomains $\{\Omega_i\}_{i=1}^n$ which cover the whole image domain Ω without overlapping as shown in Figure 6.1:

$$\Omega = \cup_{i=1}^n \Omega_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad \forall i \neq j.$$

The number of n can be defined before segmentation or unknown.

Some of the most important and powerful image segmentation approach is region based segmentation [89, 32, 81]. Many of such method adopted the length

Figure 6.1: Partition of the image domain into several subdomains

regularization. For length-regularized energy functional, the traditional continuous partial differential equation (PDE) approaches and later the max-flow/min-cut graph cuts approach are widely applied.

6.2.1 Mumford-Shah Model

The Mumford-Shah functional [89] is a well-known and widely applied model for separating the image into distinct homogeneous subregions $\{\Omega_i\}_{i=1}^n$. We assume the original image is u^0 in the domain Ω and u is regarded as the approximation image. We should minimize the following energy functional:

$$\min_{u, \Gamma} \left\{ E(u, \Gamma) = \int_{\Omega} (u - u^0)^2 d\mathbf{x} + \mu \int_{\Omega \setminus \Gamma} |\nabla u|^2 d\mathbf{x} + \nu \int_{\Gamma} d\xi \right\}, \quad (6.1)$$

where Γ is a discontinuity set containing edges, μ and ν are the tuning parameters. The meanings of each term in this model are: the first term asks that u approximates u^0 closely; the second term is the smoothness item and asks that u and hence u^0 do not change a lot; the third term asks that the boundaries should be as short as possible.

In real applications, it is satisfied in most cases when assuming u is the piecewise constant function. Thus, the smoothness item $\mu \int_{\Omega \setminus \Gamma} |\nabla u|^2 d\mathbf{x}$ can be removed from the original Mumford-Shah model. Also, $\{\Gamma_i\}_{i=1}^n$ denotes the interfaces between the different subregions $\{\Omega_i\}_{i=1}^n$. Therefore, the reduced form of model (6.1) is simplified as:

$$\min_{u, \Gamma_i} \left\{ E(u, \Gamma_i) = \int_{\Omega} (u - u^0)^2 d\mathbf{x} + \nu \sum_{i=1}^n \int_{\Gamma_i} d\xi \right\}. \quad (6.2)$$

The numerical approximation of the Mumford-Shah functional is usually based on the solution of corresponding Euler-Lagrange equations.

6.2.2 Piecewise Constant Level Set Model

Finding the optimizers of the energy minimization problem such as the Mumford-Shah model, is far from trivial. A breakthrough is approaching the solution by the level set function which was proposed by Chan and Vese in [32]. The Chan-Vese model is able to find objects whose edges are not necessarily defined by gradient as original active contour models, so it is called active contour without edges. They also found an easy way to solve the perimeter part of the Mumford-Shah model. They approximated the term $\sum_{i=1}^n \int_{\Gamma_i} d\xi$ by the zero level set function of a Lipschitz continuous function $\phi(\mathbf{x}) : \Omega \mapsto \mathbb{R}$.

Later on, in [81] the authors firstly introduced the method of minimizing the Mumford-Shah model by piecewise constant level set function. The energy functional can be minimized on the n non overlapping subdomains $\{\Omega_i\}_{i=1}^n$, and n is arbitrary. Here, the n disjoint subdomains $\{\Omega_i\}_{i=1}^n$ are described with a piecewise constant level set function ϕ :

$$\phi = i \quad \text{in } \Omega_i \quad \text{for } i = 1, 2, \dots, n. \quad (6.3)$$

The interfaces between different phases are the discontinuities of the piecewise constant level set function ϕ obviously.

The characteristic function χ_i of each non overlapping subdomain Ω_i can be obtained:

$$\chi_i = \frac{\prod_{j=1, j \neq i} (\phi - j)}{\prod_{k=1, k \neq i} (i - k)}. \quad (6.4)$$

The important term in the Mumford-Shah model—the length of the boundary of interfaces Γ_i is easily calculated by the characteristic functions:

$$\int_{\Gamma_i} d\xi = \int_{\Omega} |\nabla \chi_i| d\mathbf{x}. \quad (6.5)$$

The piecewise constant function u is assumed to be constant in each region Ω_i . Based on the derivation above, the piecewise constant function u which approximates u^0 can be expressed as the linear combination of characteristic function:

$$u = c_i \quad \text{in} \quad \Omega_i, \quad u = \sum_{i=1}^n c_i \chi_i. \quad (6.6)$$

Also, the piecewise constant level set function ϕ can be expressed by its corresponding characteristic function χ :

$$\phi = \sum_{i=1}^n i \chi_i, \quad \nabla \chi_i = \chi'_i \nabla \phi.$$

Therefore, the length term $\sum_{i=1}^n \int_{\Gamma_i} d\xi$ can be approximated by the total variation of the piecewise constant level set function ϕ , we have the alternative minimization problem of u and ϕ :

$$\min_{\{c_i\}, \phi} \left\{ E(\{c_i\}, \phi) = \sum_{i=1}^n \int_{\Omega} (u - u^0)^2 d\mathbf{x} + \frac{\nu}{2} \sum_{i=1}^n \int_{\Omega} |\nabla \phi| d\mathbf{x} \right\}. \quad (6.7)$$

6.3 Curvature based Image Segmentation Model

The most well-known total variation is widely applied in image segmentation as a regularization term. Due to the limitations such as staircasing effect of second order models, several higher order derivatives [35, 34, 83, 84] have been introduced in image processing. The higher order models that minimize curvature based

functionals of the image are important in applications. The Euler's elastic model is one of the most essential models. In human perception, the recovery relied on curvature is more natural than length regularization based construction. Also, curvature is important when keeping the detailed information of image, although the computation challenges are the complexity and nonlinear expression in the discrete setting.

There are many ways to solve the curvature based variational model, typically, the solving of their Euler-Lagrange equations is adopted. Unfortunately, the equations are complicated and the numerical solutions are difficult to reach. In this chapter, we make use of graph cuts to achieve the global approximation iteratively.

6.3.1 Curvature based Variational Model

Besides the introduction of previous image segmentation works we have discussed, in this subsection we will introduce the curvature based variational model. The Euler's elastica model is one of the most important curvature based models in image processing problems, which was first introduced in [88]. It was studied and broadened in [34, 142] for inpainting problems, and corresponding PDE based solutions were proposed.

In the mean time, curvature regularity for region based segmentation approach has drawn lots of attentions [104, 106, 105, 47]. A linear programming relaxation is proposed and extended in [106], they are solving linear programming relaxation of an integer optimization formula of the problem. Due to the size of the linear program, the computation complexity is very heavy compared to our graph cuts approach.

Also, we should emphasize that segmentation for noisy image is a challenging task. Curvature based approaches reveal more detail information from noisy input, keep some important but thin and weak structures.

In the previous work, Euler's elastica can be used as both image inpainting

and image denoising models, see [34, 105, 104, 118, 9]. In this chapter, we focus on the extension to image segmentation problem. It is a natural way to combine curvature information into the Mumford-Shah model, in which case the complete energy functional can be obtained as adding curvature information on (6.7):

$$\min_{\{c_i\}, \phi} \left\{ E(\{c_i\}, \phi) = \sum_{i=1}^n \int_{\Omega} (u - u^0)^2 d\mathbf{x} + \frac{\nu}{2} \sum_{i=1}^n \int_{\Omega} (a + b \left(\nabla \cdot \frac{\nabla u}{|\nabla u|} \right)^2) |\nabla \phi| d\mathbf{x} \right\}. \quad (6.8)$$

Note that the curvature of the level curve can be expressed:

$$\kappa = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right).$$

This model has tight connection with previous energy functional. By setting $a = 1, b = 0$, $E(\{c_i\}, \phi)$ is as same as original model we derived from.

In this chapter, we will develop graph cuts method rather than gradient descent method for minimizing the energy in the model (6.8). Although initially it is not a graph representable model, we will still get the approximation by iteratively graph cuts. Instead of discretizing the Euler-Lagrange equations, we begin by discretizing the energy functional itself.

6.3.2 Discretization of Variational Model

The typical and traditional minimization for higher order variational model is solving its associated Euler-Lagrange equation. The procedure is complicated due to the existence of high order derivatives. Our approach begins to discretize the variational model itself and solves it via graph cuts.

The energy functional is discretized on the discrete domain Ω_d . The central difference scheme can be applied as an approximation of derivatives on Ω_d . We assume u and u^0 are quantized and defined on the uniform grid:

$$\Omega_d = \{(i, j) : i = 1, \dots, N, j = 1, \dots, M.\}$$

We let the discrete functions $u_{i,j} : \Omega_d \mapsto \{0, \dots, L-1\}$ and $u_{i,j}^0 : \Omega \mapsto \{0, \dots, L-1\}$ be defined on this grid, where $\{0, \dots, L-1\}$ is the set of gray values. The discrete

gradient ∇u can be given as:

$$\nabla u = (u^x, u^y),$$

$$u_{i,j}^x = \frac{u_{i+1,j} - u_{i-1,j}}{2}, \quad u_{i,j}^y = \frac{u_{i,j+1} - u_{i,j-1}}{2},$$

where $u^x : \Omega_d \rightarrow \mathbb{R}$ and $u^y : \Omega_d \rightarrow \mathbb{R}$. In the same way, the second order derivatives are discretized as:

$$\begin{aligned} u_{i,j}^{xx} &= u_{i+1,j} - 2u_{i,j} + u_{i-1,j}, \\ u_{i,j}^{yy} &= u_{i,j+1} - 2u_{i,j} + u_{i,j-1}, \end{aligned} \tag{6.9}$$

where $u^{xx} : \Omega_d \rightarrow \mathbb{R}$ and $u^{yy} : \Omega_d \rightarrow \mathbb{R}$.

The curvature κ which will be used extensively can be discretized with central differences by:

$$\kappa = \frac{u^{xx}(u^y)^2 - 2u^x u^y u^{xy} + u^{yy}(u^x)^2}{((u^x)^2 + (u^y)^2)^{\frac{3}{2}}}. \tag{6.10}$$

In order to simplify computation, the anisotropic version of total variation term is adopted as:

$$\int_{\Omega} |\nabla \phi| d\mathbf{x} = \int_{\Omega} |\nabla \phi|_1 d\mathbf{x} = \int_{\Omega} |\phi^x| + |\phi^y| d\mathbf{x} = \sum_{i,j} |\phi_{i+1,j} - \phi_{i,j}| + |\phi_{i,j+1} - \phi_{i,j}|. \tag{6.11}$$

We also construct an equivalent representation of each node. Taking the four neighborhood system as example. For each $p = (i, j) \in \Omega_d$, its four neighbors are:

$$\mathcal{N}_p^4 = \{(i \pm 1, j), (i, j \pm 1)\} \cap \Omega_d.$$

The discrete version of the energy functional is:

$$E_d(\{c_i\}, \phi) = \sum_{p \in \Omega_d} (u_p - u_p^0)^2 + \sum_{p \in \Omega_d} \sum_{q \in \mathcal{N}_p^4} \left(a + b \cdot \frac{\kappa_p^2 + \kappa_q^2}{2} \right) w_{pq} |\phi_p - \phi_q|, \tag{6.12}$$

where $w_{pq} = \frac{1}{\|p-q\|}$.

Figure 6.2: Example of a directed capacitated graph and the cut for segmentation.

6.4 Graph Cuts for solving Our Model

Graph cuts method is relatively new, but it is widely applied to solve many computer vision problems efficiently. The theory of graph cuts was first introduced as a useful optimization instrument in computer vision and graphics by Greig et. al [60]. Later it was studied extensively and connected to the continuous methods [18, 17, 16, 73, 40, 41, 19].

6.4.1 Graph Construction

In order to solve our discrete energy functional (6.12) by graph cuts, the minimizer of (6.12) should be found by finding the minimum cut C over a suitable constructed graph $\mathcal{G}$ with fixed $\{c_i\}$:

$$\min_{C: \text{cut on } \mathcal{G}} |C| = \min_{\phi} E_d(\{c_i\}, \phi). \quad (6.13)$$

We first provide the simplest example of directed capacitated graph in Figure 6.2. A one level graph for this two phase segmentation problem is constructed. After cuts, each grid point can be classified to source s or sink t . For solving the more complicated and general multiphase segmentation problems via graph cuts algorithm, we have to introduce one more dimension, i.e. for a 2-D image of size

$N \times M$, we should construct a 3-D graph consisting of $N \times M \times (\mathcal{L} - 1)$ vertices. Here, $\mathcal{L}$ is the number of phases and hence $\mathcal{L} - 1$ is the number of graph level.

So that, a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined as a set of nodes $\mathcal{V}$ and a set of directed edges $\mathcal{E}$ that connect all the nodes. We let the vertex $\nu_{p,l}$ correspond to a grid point (p, l) on a 3-D grids:

$$\{(p, l) \in \mathbb{R}^2 \times \mathbb{R} | p \in \Omega_d, l \in \{1, 2, \dots, \mathcal{L} - 1\}\}.$$

The set of vertices $\mathcal{V}$ is defined as

$$\mathcal{V} = \{\nu_{p,l} | p \in \Omega_d, l \in \{1, 2, \dots, \mathcal{L} - 1\}\} \cup \{s\} \cup \{t\}.$$

According to the original energy functional, the set of edges $\mathcal{E}$ consists of a data fitting term and a regularization term respectively:

$$\mathcal{E} = \mathcal{E}_D \cup \mathcal{E}_R = \cup_{p \in \Omega_d} \mathcal{E}_p \cup \mathcal{E}_R,$$

where

$$\mathcal{E}_p = \cup_{p \in \Omega_d} ((s, \nu_{p,1}) \cup_{l=1}^{\mathcal{L}-2} (\nu_{p,l}, \nu_{p,l+1}) \cup (\nu_{p,\mathcal{L}-1}, t)),$$

and

$$\mathcal{E}_R = \{(\nu_{p,l}, \nu_{q,l}) | p \in \Omega_d, q \in \mathcal{N}_p^4, l \in \{1, 2, \dots, \mathcal{L} - 1\}\}.$$

The minimum cut on $\mathcal{G}$ corresponds to a minimizer of the discrete energy functional and equals the maximum flow value on the graph due to the duality theorem of Ford and Fulkerson [51].

For such multiphase image segmentation problem, the minimal cut on graph is constructed with the piecewise constant level set functions. For each grid point $p \in \Omega_d$ and severed edges $\mathcal{C}$, the level set function is as follows:

$$\begin{cases} \phi_p = 1, & \text{if } (s, \nu_{p,1}) \in \mathcal{C}, \\ \phi_p = l + 1, & \text{if } (\nu_{p,l}, \nu_{p,l+1}) \in \mathcal{C}, \\ \phi_p = \mathcal{L}, & \text{if } (\nu_{p,\mathcal{L}-1}, t) \in \mathcal{C}. \end{cases} \quad (6.14)$$

The corresponding edge costs (weights) on the data edges can be defined with constant σ as:

$$\begin{cases} c(s, \nu_{p,1}) = |u_p^0 - c_1|^2 + \sigma, \\ c(\nu_{p,l}, \nu_{p,l+1}) = |u_p^0 - c_l|^2 + \sigma, \\ c(\nu_{p,\mathcal{L}}, t) = |u_p^0 - c_{\mathcal{L}}|^2 + \sigma. \end{cases} \quad (6.15)$$

The costs (weights) set on the edges for regularization terms are:

$$c(\nu_{p,l}, \nu_{q,l}) = \left(a + b \cdot \frac{\kappa_p^2 + \kappa_q^2}{2} \right) w_{pq}, \quad (6.16)$$

and the calculation of curvature κ_p and κ_q are based on (6.10). Finally, the cut can be found on the graph as (6.13).

6.4.2 Minimization based on Iterative Graph Cuts

We approximate the model itself directly and find the cut on the given graph. Our work is related to [18, 28] but different since we study images instead of interphases, so that the original energy functional to be graph representable is not compulsory.

Furthermore, we are interested in the final steady state rather than the evolution sequence. We let E_d be a graph representable energy functional, depending on level set function. So that:

$$\phi^{n+1} = \arg \min_{\tilde{\phi}} E_d(\{c_i\}^n, \tilde{\phi}).$$

6.4.3 Algorithm

In this subsection, we provide detailed algorithm for iterative graph cuts procedure. The initial values $\{c_i\}^0$ can be obtained by the isodata algorithm which is introduced in [123]. For our segmentation problems, the Algorithm 4 normally converges in 10 iterations.

Algorithm 4: Algorithm for Curvature based Segmentation Model

1 Estimate initial values $\{c_i\}^0$, for iteration $n = 1, 2, 3, \dots$, while

$(\|\{c_i\}^n - \{c_i\}^{n-1}\| < tol)$:

1. Calculate curvature for each iteration as (6.10) for the next step graph cuts procedure.
2. Use graph cuts to estimate ϕ^{n+1} from (6.13):

$$\phi^{n+1} = \arg \min_{\tilde{\phi}} E_d(\{c_i\}^n, \tilde{\phi}).$$

3. Compute the characteristic functions $\{\chi_i\}^{n+1}$, $i = 1, 2, 3, \dots, n$ from ϕ^{n+1} as in (6.4):

$$\chi_i = \frac{\prod_{j=1, j \neq i} (\phi - j)}{\prod_{k=1, k \neq i} (i - k)}.$$

4. Update $\{c_i\}^{n+1}$ from continuous form:

$$\{c_i\}^{n+1} = \frac{\int_{\Omega} u^0 \chi_i^n d\mathbf{x}}{\int_{\Omega} \chi_i^n d\mathbf{x}}, \quad i = 1, 2, \dots, \mathcal{L},$$

either from discrete form:

$$\{c_i\}^{n+1} = \frac{\sum_{p \in P} u^0 \chi_{i,p}^n}{\sum_{p \in P} \chi_{i,p}^n}, \quad i = 1, 2, \dots, \mathcal{L}.$$

5. Update $n \leftarrow n + 1$.
-

6.5 Numerical Experiments

In this section, various examples are presented to illustrate the effectiveness, efficiency and better results of the proposed method compared to original Mumford-Shah model (M-S). The codes are implemented in C++. For max-flow computation, we have made use of the implementation provided by Boykov et al. in [17]. All experiments were conducted on a PC with Intel Pentium 4 CPU of 3.2GHz and 4GB memory. Some test images are from the Berkeley Segmentation Dataset and Benchmark (<http://www.eecs.berkeley.edu/Research/Projects/CS/vision/bsds/>) and SIPI Image Database (<http://sipi.usc.edu/database/>).

We conduct experiments of clear image and noisy images, including synthetic and real ones. In the first subsection, we focus on clear image segmentation. The input image u^0 does not contain any noise. Compared with the original Mumford-Shah model, we can find that the curvature based model in image segmentation allows more details. In the second subsection, we conduct segmentation for noisy images. In contrast to traditional length regularization, our approach can preserve relevant thin and elongated structures and more details as well.

6.5.1 Clear Image Segmentation

We start by giving two classic and real images for four phases segmentation experiment. In Figure 6.3, our input is clear image "clock". Compared with the segmentation result of Mumford-Shah model, we find that the details on the left bottom are clear since original model favors piecewise constant result. Our computation time is 11.735s for 256×256 "clock". In Figure 6.4, our curvature segmentation result keeps more details such as the texture on lena's hat and hair. Our computation time is 9.344s for 256×256 "lena".

Figure 6.3: Segmentation comparison of clear image "clock"

Figure 6.4: Segmentation comparison of clear image "lena"

6.5.2 Noisy Image Segmentation

Noisy image segmentation is an important issue in segmentation problems. Curvature based image denoising results have been demonstrated more natural than total variational recovery [9]. We found that based on our numerical tests, curvature based segmentation can reach clear results without noise and keep small and thin structures as well.

In Figure 6.5, we evaluate segmentation results on a synthetic image of four regions interrupted by gaussian noise. We have segmented sharp corners and holes well. Our result of phase 2 is quite clear and the left edge of phase 4 is even better than the Mumford-Shah result. Also, we only need one level set function and the graph cuts computation speed is fast.

Then we investigate the real image segmentation problems. In Figure 6.6, we apply our method on the gaussian noisy image "cameraman". In the background part, we obtain nearly the same results as previous method. If zoom in and focus on the tripod part of this image, our result keeps the essential details of tripod clearly.

The thin and elongated structures are very challenging in segmentation of Figure 6.7. Comparably, the dragonfly's thin and elongated front legs are kept in our results. This example is always adopted as curvature segmentation illustration, but noisy input is rarely used.

We also show our method works well in real application such as medical image processing. In Figure 6.8, the input is noisy brain image. We use four phases segmentation to extract the background and three different tissue classes: cerebrospinal fluid, gray matter and white matter. In order to show clearly, we list the gray matter phase and show the better result. Also need to mention the computation time of graph cuts approach is dramatically improved rather than traditional ones.

Finally, we compared the PSNR values of our curvature based method with original Mumford-Shah model. Providing the results of the Peak Signal to Noise

Figure 6.5: Synthetic noisy example comparisons.

Figure 6.6: Segmentation comparison of noisy image "cameraman".

Figure 6.7: Segmentation comparison of noisy image "dragonfly".

(a) input noisy brain image

(b) M-S

(c) curvature segmentation

(d) M-S result of grey matter

(e) curvature segmentation of grey matter

Figure 6.8: real noisy image, segmentation results by M-S and curvature, comparison about grey matter phase.

Img. No.	Table2: PSNR results comparison		
	Name	M-S	Curvature
Fig. 6.5	synthetic	$29.13dB$	$29.92dB$
Fig. 6.6	Cameraman	$23.09dB$	$23.41dB$

Ratio (PSNR) in Table1, which shows that the curvature based methods clearly perform better than Mumford-Shah model.

Curvature based Energy Minimizers for Feature Sensitive Surface Reconstruction

7.1 Introduction

In computer graphics community, surface reconstruction from unorganized point data sets has drawn many attentions. To establish a connection between discrete scanned data points and continuous surface model becomes one of the most challenging problems in this area, since that neither the relationship between the data points nor the topology of the final surface is known as a priori. The laser scanner technology has been developed significantly recently and there are various applications such as medical imaging, visualization, CAD designs, 3D shape processing. Despite the development of scanner techniques, there is still undesirable interruption in the capture procedure such as noise and data missing. Extensive research has been conducted and tremendous advances have been made to handle these difficulties. Therefore, our study focuses on finding a robust and effective reconstruction method which could recover the surface with the sharp features from scanned data sets.

In previous surface reconstruction works, most of them can be classified as

explicit methods and implicit methods. Explicit methods include geometric approaches concerning parameters, usually based on Delaunay triangulation and dual Voronoi diagram [6, 5, 7, 43]. This popular technique is using the Voronoi diagram to recover a polyhedral surface from points data sets. The primary advantage is the theoretical guarantee that there exists a sub-complex of Delaunay triangulation of the input, which is homeomorphic to the real surface given sufficient point data sets.

There are also explicit approaches utilizing variational models and solving corresponding partial differential equations (PDEs). For this kind of methods, appropriate global parametrization is required but actually difficult to obtain in real sparse data sets. Also, most of the explicit algorithms require the input data sets should be without noise. In fact, some of these algorithms are even more sensitive to noise. Sharp features are high frequency portion in the signal processing language, which means the normal data acquisition resolution could not fulfill the sufficient sampling requirement.

In order to gain flexibility of topological representation and mathematical facilities, some researchers turned to the non-parametric approaches which are using implicit surfaces and volumetric methods. Such as [63, 140, 132, 52, 113, 114, 11, 22, 38, 14, 111, 115, 112, 92, 53]. The success of the weighted minimal surface model in [140] and its variants prove the effectiveness of this methodology by using zero level set function to represent the surface. In this approach, a scalar field function is constructed on regular grids and the output surface is a designed level set of the field function. In such variational models, the most popular regularization term added in geometric modeling is based on the area, which is designed for noise removal but not for feature preservation.

In the mean time, the application of Euler's elastica model in image processing inspires the graphics community [88, 34, 86, 9, 118] and some surface works oriented to curvature have been proposed, see [52]. However, most of implicit reconstruction methods utilize the regular grids to discretize the energy functional.

The consequence of this framework is the staircasing observed in the reconstructed surface. Some smoothing post-processing is needed more or less. The smoothing procedure counteracts most of feature preserving effects. See [52], an advanced work on feature preservation with curvature information is proposed, the result surface is still far away from ground truth especially on the sharp edges.

In computer vision problems, graph cuts techniques have been widely used to find the global minimum of energy functional [16, 19, 17]. Due to its fast computation and the good approximation to solution compared with the iterative methods such as gradient descent, graph cuts method is also adopted in many works such as surface reconstruction problem [64, 65, 74, 79, 97, 116, 124]. The work about how to apply graph cuts to solve non graph representable high order problems is also discussed in [68, 9]. Based on the previous research, graph cuts can be applied to minimize complicated energy functional on implicit surface.

In our study, a novel method for surface reconstruction especially with sharp features is developed. The weighted minimal surface model [140] has been added with a curvature term for better feature preservation. Rather than traditional discretization on regular mesh, in our approach the variational model is discretized on the tetrahedral mesh. A graph is constructed dual to the obtained mesh then the global minimum is obtained by graph cuts. The curvature term as well as the closeness term are assigned as the graph edge weights, in this way we can get good approximation to this non graph representable energy functional. In the algorithm, the high order curvature term is calculated based on the last iteration result and graph cuts are performed iteratively. Then the local swap is applied to the obtained mesh partitioning while the criterion is the energy decreasing. In this stage, the energy functional involved curvature term is calculated based on the explicit triangulated two dimensional manifolds acquired via graph cuts.

The novelty is our method integrates Delaunay-based tetrahedral mesh and curvature based variational model. Although both are complicated in computation, our method takes the advantages of both successfully. The curvature based

model helps to preserve the features better than previous methods. The Delaunay triangulation guarantees the existence of reliable recovered surface to the ground truth given sufficient sampling. More important is that the tetrahedral mesh guarantees the better capability of representing piecewise linear surfaces with sharp corners and cusps. In the comparisons and the known earlier works, we found that the representation based on grid intrinsically corrupt the sharp features. The precise information is coarsened in the beginning. Consequently the ground truth are difficult to be reconstructed exactly. In our method, ground truth could be reconstructed accurately, which will be illustrated in our examples.

The rest of this chapter is organized as follows. In Section 7.2, we review some related works and give an overview of our proposed two stage method. In Section 7.3, we propose a graph cuts based method as the first stage, global minimization. In Section 7.4, the local swap method is proposed to recover the remaining features. Most of the examples we have tested are recovered as ground truth. In Section 6.5, various numerical experiments including real and synthetic examples are conducted; the results are shown and compared with other method to indicate the advantage of our method.

7.1.1 Contributions

Important features of our proposed approach are as follows:

1. The proposed surface reconstruction model, which involves curvature, allows varying flexibility and feature preservation. The constructed shape is more precise than any piecewise linear approximation especially on edges.
2. We have fast and effective two-stage reconstruction algorithms to recover surfaces from sparse point data sets. The combination of two stages is straightforward and beneficial. We combine the effective and fast global reconstruction via graph cuts, with the accuracy of local swap later.

3. The energy discretization is conducted on the tetrahedral mesh, which is helpful for keep sharp features intrinsically.

7.2 Curvature Based Surface Reconstruction Model

We propose a new curvature based surface reconstruction model. Inspired by the minimal-surface-like model introduced by Zhao et al. in [140], which is solved by a gradient descent way. This weighted minimal surface model can handle complicated topological cases easily, although it has drawbacks on recovering sharp edge features. In the mean time, models that minimize curvature based functionals have been proved to perform well in keeping features. The extensive use of curvature information in surface reconstruction models would be ideal that combines the merits of both.

Regarding the minimization problem, the graph cuts technique is used to solve the curvature based energy functional in our approach. Compared with the iterative gradient descent approach [140, 52], graph cuts method provides a faster solver. Another difference is that the discretization of energy functional is calculated on tetrahedral mesh instead of regular grids to avoid staircasing effect.

7.2.1 Related Works and Motivations

A brief introduction of the weighted minimal surface model proposed by Zhao et al. in [140] will be given in this subsection:

$$E(\mathcal{S}) = \left(\int_{\mathcal{S}} d^p(x) ds \right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty, \quad (7.1)$$

where $\mathcal{S}$ is an arbitrary surface and ds is the surface area. $d(x) = d(x, P)$ here is the distance function from the point x to the nearest point of data set P . If $p < \infty$, $E(\mathcal{S})$ is based on surface area; when $p = \infty$, $E(\mathcal{S})$ becomes the Hausdroff distance between $\mathcal{S}$ and P . To find a local minimum of $E(\mathcal{S})$, the gradient flow

of energy functional is:

$$\frac{d\mathcal{S}(t)}{dt} = - \left(\int_{\mathcal{S}} d^p(x) ds \right)^{\frac{1}{p}-1} d^{p-1}(x) \left(\nabla d(x) \cdot \mathbf{n} + \frac{1}{p} d(x) \kappa \right) \mathbf{n}, \quad (7.2)$$

and the steady state should satisfy Euler-Lagrange equation:

$$d^{p-1}(x) \left(\nabla d(x) \cdot \mathbf{n} + \frac{1}{p} d(x) \kappa \right) \mathbf{n} = 0,$$

where $\mathbf{n}$ is the unit outward normal and κ is the mean curvature. The surface is $\mathcal{S}(t) = \{x : \phi(x, t) = 0\}$. After differentiating the energy functional with respect to t , the energy functional can be rewritten as:

$$\frac{d\phi(\mathcal{S}(t), t)}{dt} = \frac{\partial \phi}{\partial t} + \frac{d\mathcal{S}(t)}{dt} \cdot \nabla \phi = 0. \quad (7.3)$$

We have got $\frac{d\mathcal{S}(t)}{dt}$ from (7.2), substituting it into (7.3), it follows that:

$$\frac{\partial \phi}{\partial t} = -\frac{1}{p} |\nabla \phi| \left(\int d^p(x) \delta(\phi) |\nabla \phi| dx \right)^{\frac{1}{p}-1} \nabla \cdot \left(d^p(x) \frac{\nabla \phi}{|\nabla \phi|} \right).$$

Here the level set function re-expresses the partial differential equation with respect to $\mathcal{S}$ to that of ϕ , which is a scalar function and easier to handle. The surface is captured iteratively, by locating set $\mathcal{S}(t)$ for which ϕ vanishes.

Besides the gradient descent method, in [52] the authors reconstructed the surface on an unorganized data set by local radial basis functions. They solved Zhao's model ($p = 1$) by the evolution which is based on distance function $d(x) = d(x, P)$ and the curvature $\kappa(x)$ defined as below:

$$\kappa(x) = \nabla \cdot \left(\frac{\nabla d(x)}{|\nabla d(x)|} \right).$$

In this approach, both the $d(x)$ and $\kappa(x)$ are discretized and computed analytically on the regular mesh grids as:

$$\begin{aligned} \kappa = \frac{1}{||\nabla d||^3} & (d_x^2 d_{yy} - 2d_x d_y d_{xy} + d_y^2 d_{xx} + d_x^2 d_{zz} - 2d_x d_z d_{xz} \\ & + d_z^2 d_{xx} + d_y^2 d_{zz} - 2d_y d_z d_{yz} + d_z^2 d_{yy}). \end{aligned}$$

In the mean time in imaging field, curvature based minimization models can better preserve natural appearance when reconstruction. Euler's elastica model is an important such curvature based model, which was first introduced in image processing in [88]. For solving this kind of curvature based model, traditionally, the Euler-Lagrange or gradient descent equations are derived. The new approach is proposed in [9], based on the general formulation of energy functional, we can solve the high order problem faster via graph cuts.

Inspired by the minimal surface model and the accuracy of curvature information, we would like to combine these two parts and propose a new curvature based surface reconstruction model. Instead of the gradient descent method, we solve the minimization problem via graph cuts due to its fast performance on graphs dual to Delaunay tetrahedral meshes too [19].

7.2.2 Overview of the Proposed Method

Our new energy functional is an improvement of the weighted minimal surface model, which is also related to the geodesic active contours approaches [24, 25, 39]. We firstly introduce our model which can recover not only the smooth parts but also the features such as sharp edges and corners as follows:

$$E(\mathcal{S}) = \int_{\mathcal{S}} (d(x) + \lambda|\kappa(x)|) ds. \quad (7.4)$$

Here $\lambda|\kappa(x)|$ is the regularization term which replace the area term in the Zhao et al.'s model. The distance function $d(x)$ for each point x is the fidelity term. Then we proposed an efficient two-stage strategy:

1. Based on the primal mesh and dual graph, we construct the graph and assign the graph weight according to the energy functional. Apply graph cuts to minimize the energy functional and get the surface initialization for next step;
2. Input is the explicit surface obtained by the first stage, we then adopt local swap to recover the remaining features without oscillation.

In our work the strategy contains two stages, where the first step reconstructs the surface from unorganized data sets via graph cuts, then the second step recovers the remaining features on the output surface. We combine the effective and fast reconstruction from the first stage, with the accuracy in the second stage of local refinement. The combination of two stages is straightforward and efficient. The result of first stage is not perfect but good enough as the input of the second stage, and the second stage recovers the sharp edges in a short time as mentioned in Section 7.4 since we only need to evaluate the energy change in a small region. Both stages are important and match each other very well. The flowchart of our method is as shown in Fig. 7.1, and we will describe the details in the following two sections. The methodology combines both the global and local methods and provides approaches to general surface reconstruction cases.

Figure 7.1: In left column, the flowchart of our method is shown. In the right column, brief illustrations of all stages are listed correspondingly.

7.3 Stage One: Surface Reconstruction based on Delaunay Triangulation and the Graph Cuts

In this section, our curvature based variational model is solved by graph cuts on Delaunay triangulation. The unstructured tetrahedral mesh $\mathcal{T}_h$ which provides more flexibility is used instead of structured grids. Now we only discuss two-phase problems, in which the ground truth surface $\mathcal{S}$ partitions the embedding domain $X \subset \mathbb{R}^3$ into two regions, inside and outside.

7.3.1 Tetrahedral Mesh and Delaunay Triangulation

Instead of the regular grid, the tetrahedral mesh is applied in the proposed method to discretize the domain of interest. Compared with the structured grids, the tetrahedral mesh is more flexible. More important is that tetrahedral mesh could intrinsically avoid the staircasing often encountered in the grid based methods. Basic notations of the tetrahedral mesh are briefly given as follows.

A tetrahedral mesh is defined as the division of a domain into a set of tetrahedra with the restriction that each triangle face is entirely shared by two adjacent tetrahedra. This division is also referred to as triangulation or tessellation. The tetrahedral mesh in our method is denoted by $\mathcal{T}_h$. The vertices are denoted by u, v, w, z , we define $\{v, u\}$ as the edge between v and u , $\{v, u, w\}$ is the triangle with the vertices u, v, w , and $\{u, v, w, z\}$ as the tetrahedron with vertices u, v, w, z . Also, $\{K_i\}_{i=1}^N$ is utilized to express all N tetrahedra in $\mathcal{T}_h$.

The explicit surface reconstruction methods are mainly based on a special kind of triangulation, the Delaunay triangulation, in which the circumsphere of every element does not contain any other vertices. According to [6], given enough sampling density, there exists $\mathcal{S}_{\mathcal{T}}$, a sub-complex of Delaunay triangulation of the data set, which is homeomorphic to the ground truth surface $\mathcal{S}$.

Given the input data set P , earlier works restrict the Delaunay triangulation of $P(DT(P))$ to some given subset D_{sub} of the domain. The main task in those works is to find such a subset containing the surface $\mathcal{S}$. After accomplishing that, to obtain the reconstructed surface is simply to extract the Delaunay triangulation $DT(P)$ restricted within the subset D_{sub} .

In the proposed method, some auxiliary points Q , which is referred by non-geometric points here, are added to P to generate Delaunay triangulation: $DT(P \cup Q)$. Due to the local property of Delaunay triangulation, if $\mathcal{S}_{\mathcal{T}} \subset DT(P)$ and Q is away enough from P , $\mathcal{S}_{\mathcal{T}} \subset DT(P \cup Q)$, namely the desired $\mathcal{S}_{\mathcal{T}}$ still exists in $DT(P \cup Q)$. Besides the introduction of non-geometric points changes the mesh structure and facilitates the application of graph techniques.

7.3.2 Discretization on Mesh

After the mesh $\mathcal{T}_h$ is obtained and the domain is be tessellated correspondingly, the energy functional could be discretized on $\mathcal{T}_h$.

$$\begin{aligned} E(\mathcal{S}) &= \int_{\mathcal{S}} (d(x) + \lambda|\kappa(x)|) ds, \\ E(\mathcal{S}_h) &= \int_{\mathcal{S}_h} (d(x) + \lambda|\kappa(x)|) ds. \end{aligned} \quad (7.5)$$

As mentioned, $\mathcal{S}_h$ is the approximation of surface $\mathcal{S}$, a two manifolds surface triangulation embedded in $\mathcal{T}_h$. The surface $\mathcal{S}$ separates the domain into two subdomains, while the surface triangulation $\mathcal{S}_h$ separates all tetrahedra into two subsets, namely “interior” and “exterior”. This labeling of all tetrahedra can be denoted as follows:

$$\phi_{\mathcal{S}_h}(K_i) = \begin{cases} c_1 & \text{if } K_i \text{ inside } \mathcal{S}_h, \\ c_2 & \text{if } K_i \text{ outside } \mathcal{S}_h. \end{cases}$$

This labeling is equivalent to a piecewise constant level set function choice. The surface triangulation $\mathcal{S}_h$ is the union of all triangle faces shared by two tetrahedra

from different subsets. This fact suggests the following relationship:

$$\mathcal{S}_h = \bigcup_{i,j} (K_i \cap K_j) \mathbf{1}_{\{\phi_{\mathcal{S}_h}(K_i) \neq \phi_{\mathcal{S}_h}(K_j)\}}, \quad (7.6)$$

and the indication function $\mathbf{1}_{\{\mathbf{T}\}}$ is also defined as:

$$\mathbf{1}_{\{\mathbf{T}\}} = \begin{cases} 1 & \text{if the statement } \mathbf{T} \text{ is true,} \\ 0 & \text{if the statement } \mathbf{T} \text{ is false.} \end{cases}$$

Based on which, the (7.5) could be discretized further.

$$\begin{aligned} E(\mathcal{S}_h) &= \int_{\mathcal{S}_h} (d(x) + \lambda|\kappa(x)|) ds \\ &= \int_{\sum \mathcal{S}_{i,j}} (d(x) + \lambda|\kappa(x)|) \mathbf{1}_{\{\phi_{\mathcal{S}_h}(K_i) \neq \phi_{\mathcal{S}_h}(K_j)\}} ds \\ &= \sum_{i,j} \int_{\mathcal{S}_{i,j}} (d(x) + \lambda|\kappa(x)|) \mathbf{1}_{\{\phi_{\mathcal{S}_h}(K_i) \neq \phi_{\mathcal{S}_h}(K_j)\}} ds \\ &= \sum_{i,j} (d_{i,j} + \lambda|\kappa_{i,j}|) \bar{S}_{i,j} \mathbf{1}_{\{\phi_{\mathcal{S}_h}(K_i) \neq \phi_{\mathcal{S}_h}(K_j)\}}, \end{aligned} \quad (7.7)$$

where

$$d_{i,j} = \frac{\int_{\mathcal{S}_{i,j}} d(x) ds}{\int_{\mathcal{S}_{i,j}} ds}, \quad \bar{S}_{i,j} = \int_{\mathcal{S}_{i,j}} ds, \quad \kappa_{i,j} = \frac{\int_{\mathcal{S}_{i,j}} \kappa(x) ds}{\int_{\mathcal{S}_{i,j}} ds}. \quad (7.8)$$

Calculation of Distance Function $d(x)$

The calculation of distance function $d(x)$ is a time-consuming task. As mentioned $d(x) = d(x, P)$, to computer all $d(x)$ for $x \in Q$ is of $O(|P||Q|)$ time complexity. In earlier study, quad-tree or oct-tree could also be used in similar search tasks. However, in our method, the property of Delaunay triangulation is utilized to computer $d(x)$ much faster since no auxiliary information is required to be prepared or stored in advance. The topological information of Delaunay triangulation is ready.

Since only the distance information of the vicinity of P is useful, the method starts with finding the n -ring neighborhood of each vertex in P . Instead of

calculating the distance from a non-geometric point to every data point, the distance from a data point to all non-geometric points around it is calculated. The algorithm is described in details in Algorithm 5.

Algorithm 5: Distance Computation

Input: Data Set P , and auxiliary points Q

Output: Distance for each $u \in Q$

```

1 foreach vertex  $v_i \in P$  do
2   Find the  $n$ -ring neighborhood of  $v_i$ ;
3   foreach vertex  $u \in n\text{-ring neighborhood}$  do
4     Calculate  $d(u, v_i)$ ;
5     Compare  $d(u, v_i)$  with previous  $d(u, v_j)$ .
6     if  $d(u, v_i) < d(u, v_j)$  then
7       update  $d(u) = d(u, v_i)$ .
8     end
9   end
10 end

```

In Algorithm 5, n is usually set to 1 or 2. Obviously for $n = 1$, $d(x)$ obtained from this algorithm is $d(x, P)$. For $n = 2$, this algorithm could also provide a reasonable approximation of $d(x, P)$. Since all non-geometric points within the n -ring neighborhood are to be calculated for distance to the data point, the time complexity is $O(m^n|P|)$, and m is the maximum vertex degree in $\mathcal{T}_h$.

Curvature Estimation

In this stage, to estimate curvature on the potential surface is the critical step. Since curvature could not be graph-representable as the area, the proposed method applies an iterative way in which curvature information of last step could be used.

Curvature is defined on the point of a manifold, namely the surface in this study. Such geometric characteristic on the point out of the surface is not defined. However, in the level set formulation, the curvature on the point out of surface

could be defined in the iso-surface meaning:

$$\kappa(x) = \nabla \cdot \left(\frac{\nabla d(x, \mathcal{S})}{|\nabla d(x, \mathcal{S})|} \right),$$

where $\mathcal{S}$ is the zero level set. To minimize the energy functional $E(\mathcal{S})$ where $\mathcal{S}$ is unknown, we adopt the result surface from the last iteration step. In the first iteration, $d(x, P)$ could approximate $d(x, \mathcal{S})$. To calculate the gradient and divergence on the tetrahedral mesh, some relevant methods are available in [122].

7.3.3 Graph Cuts and Iterations

A graph dual to the primal mesh is constructed. Such primal-dual relationship is shown in Fig. 7.2 and Table. 7.1. Therefore, we assign the fidelity and curvature terms as edge weights, the energy functional can be minimized efficiently by graph cuts, since it is graph representable. In Fig. 7.3, t_i, s_i are t -links in the graph for the sink and the source respectively; N_{ij} are n -links in the graph. The weight on the edge (i, j) now is set to $d_{i,j} + \lambda|\kappa_{i,j}^n|$. The detailed computation and algorithm are given as follows and Algorithm 6.

Figure 7.2: Primal mesh and dual graph

Figure 7.3: Graph edge weight assignment

The iteration is performed as follows:

Table 7.1: Primal-dual relationship

Primal Mesh $\mathcal{T}_h$	Dual Graph $\mathcal{G}$
Tetrahedron K_i	Node i
Triangle $\mathcal{S}_{i,j} = K_i \cap K_j$	Edge (i, j)
$\mathcal{S}_h = \bigcup \mathcal{S}_{i,j}$	$C = \bigcup (i, j)$

When $n = 0$,

$$E(\mathcal{S}_h^1) = \int_{\mathcal{S}_h^1} (d(x) + \lambda|\kappa^0(x)|) ds = \sum_{i,j} (d_{i,j} + \lambda|\kappa_{i,j}^0|) \bar{S}_{i,j} \mathbf{1}_{\{\phi_{\mathcal{S}_h}(K_i) \neq \phi_{\mathcal{S}_h}(K_j)\}},$$

and

$$\kappa_{i,j}^0 = \frac{\int_{\mathcal{S}_{i,j}} \kappa^0(x) ds}{\int_{\mathcal{S}_{i,j}} ds}, \kappa^0(x) = \nabla \cdot \left(\frac{\nabla d(x, P)}{|\nabla d(x, P)|} \right).$$

When $n > 1$,

$$E(\mathcal{S}_h^{n+1}) = \int_{\mathcal{S}_h^{n+1}} (d(x) + \lambda|\kappa^n(x)|) ds = \sum_{i,j} (d_{i,j} + \lambda|\kappa_{i,j}^n|) S_{i,j}^n \mathbf{1}_{\{\phi_{\mathcal{S}_h}(K_i) \neq \phi_{\mathcal{S}_h}(K_j)\}}.$$

and

$$\kappa_{i,j}^n = \frac{\int_{\mathcal{S}_{i,j}} \kappa^n(x) ds}{\int_{\mathcal{S}_{i,j}} ds}, \kappa^n(x) = \nabla \cdot \left(\frac{\nabla d(x, \mathcal{S}_h^n)}{|\nabla d(x, \mathcal{S}_h^n)|} \right).$$

The algorithm is described as follows.

Algorithm 6: Global Minimization in Each Iteration

Input: Data Set P

Output: A Surface Triangulation $\mathcal{S}_h$

- 1 Generate mesh and initialize $\kappa^0(x) = 0$;
 - 2 **foreach** point i **do**
 - 3 Compute distance to P : $d(x_i)$ and $\kappa(x_i)$;
 - 4 Construct graph to minimize energy functional (7.4);
 - 5 Extract the surface $\mathcal{S}_h$;
 - 6 **if** the energy difference $E(\mathcal{S}_h^{n+1}) - E(\mathcal{S}_h^n) \geq \epsilon$ **then**
 - 7 update κ , go to step 2
 - 8 **end**
 - 9 **end**
-

7.3.4 Discussion

By graph cuts, the proposed energy functional could be minimized globally. However, the iteration result is not perfect. The global minimization technique, i.e. graph cuts method, is not the main reason for the undesirable result. The reason is the inaccuracy of the curvature calculation. The tetrahedral mesh is a much sparser representation compared with the grid representation. The calculation based on such a sparse framework could not obtain a desirable result. From the results of this stage, we can observe that some elements have been recovered. However it is not the ground truth. Hence, the local swap based on more precise calculation would be applied sequentially.

7.4 Stage Two: Feature Recovery via Local Minimization

It is straightforward that we continue to refine the surface features based on the output surface of the first stage. Once the tetrahedral mesh is established, finding the embedded surface is equivalent to finding the labeling for all tetrahedra to partition the whole mesh. For each surface $\mathcal{S}$, there is one corresponding labeling L . The labeling L is a local minimum with respect to the energy functional in (7.4) if $E(L) \leq E(L')$ for any L' “near to” L . In the environment of discrete tetrahedral mesh, the labellings near to L are those within the move of a single tetrahedron. This move is usually referred to standard moves in computer vision. In this section, the object of the standard move operation is only changed from image pixels to volumetric tetrahedra.

7.4.1 Notions from Differential Geometry

When the explicit surface expression $\mathcal{S}_h$ is obtained from the first stage, we will adopt the method of [86]. For each point on the surface $\mathcal{S}_h$, we have the surface

approximation by its tangent plane. Local geometry of surface is measured by curvature. The mean curvature is defined as follows:

$$\kappa_H = \frac{1}{2\pi} \int_0^{2\pi} \kappa^N(\theta) d\theta, \quad (7.9)$$

which $\kappa^N(\theta)$ is the normal curvature and $\kappa^N(\theta) = \kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta)$, here κ_1, κ_2 are two principle curvature.

The operator $\mathbf{K}$ maps a point x_i on the surface to the vector:

$$\mathbf{K}(x_i) = 2\kappa(x_i)\mathbf{n}(x_i),$$

where $\mathbf{n}(x_i)$ is the normal vector. The mean curvature normal variable $\mathbf{K}$ is defined as a generality of Laplacian on manifolds [44]. By applying the Gauss' theorem, the integral of the mean curvature normal can be simplified as the following form:

$$\int \int_{\mathcal{A}_M} \mathbf{K}(x_i) dA = \frac{1}{2} \sum_{j \in N_1(i)} (\cot \alpha_{i,j} + \cot \beta_{i,j})(x_i - x_j), \quad (7.10)$$

where two triangles sharing the edge (x_i, x_j) , $\alpha_{i,j}$ and $\beta_{i,j}$ are the two angles facing to the shared edge as shown in Fig. 7.4, $N_1(i)$ is the set of 1-ring neighboring of vertex x_i and $\mathcal{A}_M$ is an infinitesimal area of the point x_i on this surface. Usually this region could be chosen as either the Voronoi cell around x_i or the Barycentric cell. Due to the optimal consideration, the Voronoi cell is applied in this study. Calculating the area of the Voronoi cell around a vertex on a triangulation is also provided in [86].

Figure 7.4: 1-ring neighbors and angles opposite to an edge

Figure 7.5: Change of energy for each standard move

Algorithm 7: Local Swap Procedure**Input:** The Old Surface $\mathcal{S}_h$ **Output:** A New Surface $\mathcal{S}'_h$

- 1 Start with the initial surface $\mathcal{S}_h$;
- 2 **foreach** *element* i **do**
 - 3 Move it to the other partition and obtain the new surface $\mathcal{S}'_h$;
 - 4 Re-calculate the after energy $E(\mathcal{S}'_h)$ from (7.12), and compare $E(\mathcal{S}'_h)$ with $E(\mathcal{S}_h)$;
 - 5 **if** $E(\mathcal{S}'_h) < E(\mathcal{S}_h)$ **then**
 - 6 confirm this move;
 - 7 otherwise undo.
 - 8 **end**
- 9 **end**

7.4.2 Discretization and Algorithm

We can get the mean curvature normal operator is:

$$\mathbf{K}(x_i) = \frac{1}{2\mathcal{A}_M} \sum_{j \in N_1(i)} (\cot \alpha_{i,j} + \cot \beta_{i,j})(x_i - x_j). \quad (7.11)$$

Therefore, the discretization of energy functional (7.4) can be written as follows and notation $\mathcal{S}_i$ is triangles constituting the whole surface $\mathcal{S}_h$:

$$E(\mathcal{S}_h) = \sum_i (d_i + \lambda |\kappa_i|) \mathcal{S}_i, \quad (7.12)$$

where

$$\mathcal{S}_h = \bigcup \mathcal{S}_i, d_i = \frac{1}{3} \sum_{j=1}^3 d(v_{ij}), \kappa_i = \frac{1}{3} \sum_{j=1}^3 \kappa(v_{ij}). \quad (7.13)$$

Here $v_{ij}, j = 1, 2, 3$ are three vertices of $\mathcal{S}_i$, the $d(v_{ij}) = d(v_{ij}, \mathcal{S}_h)$ which is the same as calculated in Section 7.3, and $\kappa(v_{ij})$ is calculated from (7.10) and (7.11).

We could perform the local swap algorithm on any element as follows.

This local swap algorithm could be illustrated in Fig. 7.5, actually we are comparing the energy in the rectangular since the other labellings are not changed in a single move. We do one element move and detect whether the energy decreases, in order to decide move this element to the new surface or not.

7.4.3 Discussion

If the surface is known or obtained already, the energy functional could be calculated explicitly by (7.12). Hence for the labeling swap of a single tetrahedral, the change of energy could also be calculated locally. This local swap of a single tetrahedral has the counterpart in image processing field. In our cases, the good initial surface makes the local stage work less likely to stuck in a local minimum far away from the global one. The combination of two stages is effective since the result of first stage is not perfect but good enough as the input of local swap, and then the sharp edges are recovered in a short time. Both stages are important and match each other very well.

7.5 Numerical Experiments

In this section, we have tested both synthetic and real examples to illustrate the effectiveness and robustness of the proposed method. We have compared our results with total variation model, especially on sharp edges and some zoomed parts of real surface examples. We can get more natural reconstruction surface and also keep sharp features. All experiments were conducted on a PC with

Figure 7.6: Comparison of TV result and our result

Intel Pentium 4 CPU of 3.2GHz and 4GB memory. The Armadillo, hand and Happy Budda examples are obtained from Stanford 3D Scanning Repository, the other examples were synthesized by ourselves. In the mesh generation stage, we adopted the incremental insert algorithm implemented by CGAL [1]. All surfaces are rendered by MeshLab. In the algorithms we only utilize the points location information.

7.5.1 Comparisons

At first, we use a simple reconstruction example in Fig. 7.6 to illustrate the advantage of using curvature information over total variation (TV). Our algorithm perfectly recovers the sharp edges and corners of cubes. Total variation just recovers the smooth faces. Some more identical examples were also approached in [125, 52], readers could compare the performance and find we have recovered the most features.

7.5.2 Real Surface Examples

The classic example Armadillo is shown in Fig. 7.7. We have compared our reconstruction of Armadillo with TV result and also zoomed the chest and back of

Figure 7.7: In the first row are: Point data sets of Armadillo, surface reconstruction of Armadillo by TV and by our method; the second row are the comparisons of TV and our method in zoomed part of Armadillo's chest and back.

the reconstructed surface. These parts are used to illustrate the good preservation of surface details in real example compared to TV. We also provide two more real examples, one is human hand skeleton. This example contains the complicated connecting part between fingers and palm. The zoomed part shows that the details are kept very well. The last real example is the famous example Happy Budda, we have shown the point data set, the surface reconstruction by our method and the zoomed part. The surface is smooth and the necklace part is recovered well.

Figure 7.8: In the first row are: Surface reconstruction of Hand by our method and the zoomed part of details; the second row are: Point data sets of Happy Budda, surface recovered by our method and zoomed part of Happy Budda's chest.

7.5.3 More Examples and the Point Clouds

In this subsection, we provide more examples to prove the robustness and faithfulness of our method. These examples which have sharp edges and corners were synthesized by ourselves. The following figures include points data sets in the first row and the reconstructed surfaces in the second row. In Fig. 7.9 from left to right, four geometries are shown: the atomic structure and several boolean operations: the unions of cubes and spheres. In Fig. 7.10, surface reconstruction results of four little complicated geometries are shown. In Fig. 7.11, two platonic solids, i.e. a dodecahedron and an icosahedron, a buckyball model and a brilliant cut diamond are shown. In Fig. 7.12, four interesting CAD models are shown, all of which have sharp edges or corners.

From all our reconstructed examples, most features are recovered especially the sharp edges. The reconstructed surfaces are natural and almost the ground truth surfaces. The accuracy of our feature-preserving operation will be improved and better performance could be expected in future works.

For all these experiments, we set the value of λ to 0.1. Table 7.2 gives the sizes of data sets of four surface examples and corresponding CPU time counted in seconds. The first column provides the examples' names. The second column gives the numbers of data points P . The third column is the mesh generation time. The fourth column is the number of generated tetrahedra, the fifth the graph cuts iteration cycles, the sixth the graph cuts time, the seventh the local swap iteration cycles, and the eighth the local swap time.

Figure 7.9: Unions of geometries

Figure 7.10: a perforated cube, two tangling ones, male and female symbol models

Figure 7.11: two platonic solids: dodecahedron and icosahedron, a buckyball and a brilliant cut diamond

Figure 7.12: four CAD models from left to right are: power wheel, tear drop, weight and bolt

Table 7.2: Statistics of four examples

Example	Data Set	Mesh Generation Time	Tetrahedra Number	Global Iteration Cycle	Total Graph Cut Time	Local Iteration Cycle	Local Swap Time
two cubes	2472	24.1	175851	3	0.33	5	5.57
two spheres	2653	34.2	218119	3	0.42	4	0.31
female symbol	4530	41.8	307968	3	0.63	7	11.57
bolt	2357	34.9	223346	3	0.41	6	2.57

Conclusion and Future Works

8.1 Conclusion

In this thesis, we have studied fast numerical methods include multiplier based methods and graph cuts methods, for image and surface restoration problems. We have reviewed several basic models incorporating total variation and curvature based regularization such as: the ROF model, the TV-L1 model, the Mumford-Shah model and the Euler's elastica model for image processing and surface reconstruction. We have discussed their properties and the most well used numerical approaches for these models.

Firstly, we have studied the TV-L1 image approximation model under the primal and dual perspective in a continuous way and proposed equivalent convex formulations. Later we extended the study and proposed two variational approaches to solve image fusion problems. We have investigated their convex optimization models and provided fast multiplier based algorithms. The experimental results showed that TV-L1 based fusion outperformed the TV-L2 method in image fusion problems.

Secondly, in order to overcome shortages of TV models such as stenciling effects, we studied higher order variational models based on curvature information.

We presented an efficient minimization algorithm based on graph cuts for minimizing the energy in the Euler's Elastica model for image denoising, inpainting and segmentation problems. Curvature based variational models helped us to keep thin and elongated structures and more details. Our approaches achieved better results than total variation regularized models and kept fast computation speed.

Finally, we have extended our works in imaging to surface reconstruction problem. We modified the weighted minimal surface model with curvature term for better feature preservation and more natural results. We have provided a fast and effective two-stage reconstruction algorithm to recover surfaces from sparse point data sets by graph cuts and local swap.

8.2 Future Works

In our previous works, there are some problems arising in the study which worth further research in the future.

- For surface reconstruction problems, most of previous works are recovering surface from 3-D point clouds or 2-D image information. There are many works on surface reconstruction from 3-D point data sets, but sufficient points are required. The 2-D image information of object are easily acquired nowadays, however the convergence and convexity of reconstruction from 2-D image information are not established in recent relevant works. Based on our experience of 2-D image processing and 3-D surface reconstruction, we would like to propose a new formulation combining 3-D scanned point data sets and the corresponding 2-D image information of the objects for better surface reconstruction which shall have the advantages of both; the fast algorithm can also be developed via graph cuts.
- We are considering to propose a new definition of curvature. In the previous works the curvature we have used is defined on a curve, which is in one

manifold. Curvature can also be defined in two manifolds as a matrix, the definition of curvature and the further solving procedure should be different. Thus, the denoising problem and segmentation problem on color images can be solved directly based on this new definition rather than solved in each channel.

- We have proposed iterative graph cuts algorithms for curvature based image denoising and segmentation as discussed. The relationship to gradient flow can be explored further. The efficiency of the algorithm can be improved even more by reusing flow from each iteration to the next, instead of solving the subproblems each iteration.

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List of Publications

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- ③ **Juan Shi** and Min Wan and Xue-Cheng Tai and Desheng Wang. Curvature Minimization for Surface Reconstruction with Features. *the Third International Conference on Scale Space and Variational Methods in Computer Vision (SSVM)*. 2011.
- ④ Jing Yuan and **Juan Shi** and Xue-Cheng Tai and Yuri Boykov. A Study on Convex Optimization Approaches to Image Fusion. *the Third International Conference on Scale Space and Variational Methods in Computer Vision (SSVM)*, 2011.
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- ⑥ **Juan Shi** and Xue-Cheng Tai. Curvature based Image Segmentation. In preparation, 2012.