Credit risk pricing in a general framework

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2012

Huang, H. (2012). Credit risk pricing in a general framework. Doctoral thesis, Nanyang Technological University, Singapore.

<https://hdl.handle.net/10356/50620>

<https://doi.org/10.32657/10356/50620>

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2012

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A thesis submitted to the Nanyang Technological University
in partial fulfilment of the requirement for the degree of
Doctor of Philosophy

2012

Acknowledgements

I must first express my deep gratitude towards my supervisor, Professor Yao Shuntian for his endless help and strong support throughout all the time of my research, the preparation of the dissertation. I have benefited much from this advisory guidance and research collaboration over the years. It is a great pleasure for me to conduct this thesis under his supervision.

I would like to express my sincere appreciation to the other members of my PhD committee: Professor Zhou Jie and Professor Yohanes Eko Riyanto for their support and good advice.

I would also like to thank Professor Tan Kim Heng for his kindness and valuable suggestion for my academic research and personal life.

Last but not least, my thanks go to the Nanyang Technological University for its generous financial support of my PhD study.

Abstract

In the literature, two principal approaches are widely used for credit risk modeling: structural models and reduced form models.

The evolution of firms' structural variables, such as firms' asset and debt values, are applied to determine the time of default in structural models. In these models, a default event occurs when the value of the assets falls below some threshold for the first time.

On the other hand, the reduced-form models avoid the link between the default arrival and firms' capital structures, and directly apply an exogenous counting process to the default arrival time. Therefore, the default time is usually a totally inaccessible stopping time.

The methodology of this thesis falls into the latter category. In the thesis, a (doubly stochastic) marked Poisson process $(T_n, Q_n)_{n \geq 1}$ is introduced to model a sequence of credit event arrivals. In the pair of $(T_n, Q_n)_{n \geq 1}$, the argument T_n indicates the default event arrival time, and upon each default arrival, a random mark Q_n is drawn from the mark space $(M, \mathcal{M})$ to identify the default event. By the theorem of thinning Poisson process, two independent marked Poisson processes can be separated to model two different kinds of arrivals: a default event with liquidation effect and a default event with non-liquidation effect. This characterization distinguishes our approach from traditional approaches, in which the default event is usually modeled as the first jump of a counting process. Under this setting, a general pricing

framework for defaultable claims is provided and it is applied to value defaultable bonds under different recovery schemes.

In the following chapter, we follow a similar methodology to Schönbucher (2000) to derive the drift restrictions in the interest rate analysis framework of Heath, Jarrow and Morton (1992). Different arbitrage free conditions are consequently examined and Girsanov's theorem is discussed under the marked Poisson framework.

At the end of the thesis, we provide a specification of the model: a Hull-White example. An Ornstein-Uhlenbeck process is applied to model the dynamics of the mean-reversed default free interest rate and the dynamics of the default intensity process. Consequently, analytical solutions for defaultable bonds are derived and other basic credit derivatives, such as credit default swap (CDS) and convertible bonds, are priced under the model specification.

Notation

$\mathbb{Q}$	Risk neutral probability measure
$\mathbb{P}$	Real world probability measure
$\mathcal{F}_t^W$	Time t background filtration
$\mathcal{F}_t^N$	Time t default information filtration
$W(t)$	Wiener Process
$N(t)$	Poisson Process
$(M, \mathcal{M})$	An independent mark space including all types of default events
$A \in \mathcal{M}$	A set of default events with liquidation effect
τ_i	Time of i -th jump of a Poisson process
τ_A	Time of the first default event with liquidation effect
$\lambda(t)$	Intensity process of a Poisson process
$h_1(t)$	Loss function of a default event with liquidation effect
$h_2(t, q)$	Loss function of a default event with non-liquidation effect
$DC(t, T)$	Market value of a time t contingent claim with maturity T
$V_T(t)$	Pre-default market price of a time t contingent claim with maturity T
$B(t, T)$	Market price of the time t default free bond with face value \$1
$B_d(t, T)$	Market price of the time t defaultable bond with face value \$1
$f(t, T)$	Default free instantaneous forward rate
$f_d(t, T)$	Defaultable instantaneous forward rate
■	End of proof

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Chapter 1

Introduction and Literature Review

Although default risk modeling can be traced back to the classic option valuation formula in Black and Scholes (1973), this area of research has been largely overlooked for a long period. Since the 1990's, however, default risk¹ modeling and default risk management has received intense focus from financial market practitioners, regulatory circles, and academic studies.

From financial market practitioners, since the markets for credit derivatives were created in the early 1990s in London and New York, they have experienced a strong growth. In 2007, the year before the sub-prime crisis, according to the mid-year survey of ISDA (International Swaps and Derivatives Association, INC), the notional amount of outstanding credit derivatives grew by 32% in the first six months of the year from \$34.42 trillion to \$45.46 trillion. The annual growth rate of credit derivatives was 75% at mid-year 2006. Moreover, according to the Securities Industry and Financial Markets Association, the issuance of collateralized debt obligations almost tripled between 2004 and 2007, amounting to \$489 billion in 2007. These results have made the market of credit derivatives the most innovative and fastest growing market segment of derivative securities before the subprime bond market's collapsing in 2007. The rapid development of the market has promoted an urgent need for more flexible and sophisticated default risk models to price and manage these

¹ The terms "default (events/risk)" and "credit (events/risk)" will be used as synonyms in this thesis.

credit-sensitive financial instruments, especially when securitization techniques are introduced to fund the trading of credit risks and make the trading system more complicated.

From the regulatory point of view, the central banks' monitoring developments in credit derivatives are important in terms of three interlinked perspectives²:

1. the conduct of monetary policy,
2. financial stability
3. and market standards.

Furthermore, the New Basel Accord (International Convergence of Capital Measurement and Capital Standard, Basel II, 26 January 2004) has promoted tighter standards for credit risk management, and obligated financial institutions to fulfill a variety of regulatory capital requirements. Credit derivatives and structured credit markets can therefore significantly enhance banks' capability to handle default risks and meet the standards of the New Basel Accord. For example, price discovery in the credit derivatives market can reduce the risk of mispricing loans. Moreover, securitization, which is another credit risk transfer instrument, offers the possibility of more effective management of the liquidity risk of traditionally illiquid loans in the balance sheet for the banks' market operations. To take significant advantage of the credit derivative markets, banks are required to move from the traditional market operation of "buy-and-hold" model to the "originate-and-distribute" model³.

² Quoted from the keynote address by Jean-Claude Trichet, President of the European Central Bank at the 22nd Annual General Meeting of the International Swaps and Derivatives Association (ISDA) Boston, 18 April 2007

³ The European Central Bank, Financial Stability Review, Box 12, December 2006

Therefore, effective credit models are required for banks to decide the distribution of portfolios of credit risks and assets to other market players.

Finally, the sub-prime mortgage crisis in 2008, which has spread worldwide, revealed that most of the existing models of credit risks have underestimated the default risks in the context of the global financial crisis. The failure of these models has made it difficult, if not impossible in some cases, for investors to manage the credit risks associated with countries, corporations, financial institutions, and even some financial instruments. It suggests that the existing credit risk models should be reviewed and new modeling techniques need to be developed.

From an academic point of view, the rapid increase of credit instrument markets gives academics new impetus to develop credit risk models, and the analysis of credit risks opens the door to new fields of research that interface with other areas like continuous-time corporate finance (e.g. Leland (1994), Leland and Tof (1996), etc).

The recent global financial crisis also reminds academics of the need to improve the existing models of credit risks to accommodate recent changes in economic and financial circumstances or the occurrence of extreme events.

The literature review shows that two principal approaches are widely used for credit risk modeling: structural models and reduced form models.

The evolution of firms' structural variables, such as asset and debt values, which follow a continuous diffusion process, is used in structural models to determine the time of default. In these models, a default event occurs when the value of the assets falls below some threshold for the first time. As a result, the models can be solved in

the Black and Scholes option valuation framework, which usually means that the default time is predictable.

On the other hand, reduced-form models avoid the inter-link between default arrival and firms' capital structure, and directly apply an exogenous counting process to model default arrival times. In reduced form models, the default times are usually a sequence of totally inaccessible stopping time.

1.1 Structural Models

Structural models are first introduced by Merton (1974), Black and Cox (1976), followed by Shimko et al. (1993) and Longstaff and Schwartz (1995). In these models, the default time is derived from the evolution of the firm's capital structures, and as a result, the default arrival time is generated endogenously.

1.1.1 Modeling Methodology

In Merton's (1974) model, unrealistic constraints are imposed to achieve an analytical solution under the Black-Scholes framework. However, the principal methodology of structural models can still be extracted from the simple specification.

Merton's assumptions are:

- The firm's value V is regarded as the value of the underlying asset, which follows a lognormal diffusion process (under the martingale measure):

$$\frac{dV_t}{V_t} = rdt + \sigma dW(t) \quad (1.1.1)$$

where r and σ are constants and $W(t)$ is a standard Wiener process under the martingale measure.

- The firm is only funded by issuing equities and bonds. Therefore firm's capital structure is comprised of its equity and a defaultable corporate bond with face value D and maturing time T . At any time t before the bond maturity, $0 \leq t \leq T$, the firm's value V_t is:

$$V_t = E_t + B_d(t, T) \quad (1.1.2)$$

where E_t is the equity price and $B_d(t, T)$ is the time t price of the defaultable bond which matures at T .

- The interest rate r is a constant and the term structure of interest rates is flat
- The firm pays no dividend over the lifespan of the defaultable corporate bond.
- The default only occurs exactly at the maturity time T of the corporate bond if the firm value is insufficient to pay the outstanding debt at T .

Because debt is senior to equity, at time T , the equity price is

$$E_T = \max(V_T - D, 0) \quad (1.1.3)$$

Respectively, the corporate bond price is

$$B(T, T) = \min(D, V_T) = D - \max(0, D - V_T)$$

Consequently, the firm's equity can be regarded as a European vanilla call option, and the corporate bond is equivalent to a portfolio comprised of a short position of a European vanilla put and a default-free loan with face value D maturing at T .

By the Black-Scholes formula, the price of the equity and corporate bond at time t , $0 \leq t < T$ can be derived as following:

$$E_t = e^{-r(T-t)} [e^{r(T-t)} V_t \Phi(d_1) - D \Phi(d_2)] \quad (1.1.4)$$

$$B(t, T) = V_t - e^{-r(T-t)} [e^{r(T-t)} V_t \Phi(d_1) - D \Phi(d_2)] \quad (1.1.5)$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function and d_1, d_2 are given as:

$$d_1 = \frac{\ln\left(\frac{e^{r(T-t)}V_t}{D}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The probability of default at time T is then given as:

$$\mathbb{Q}(V_T < D) = \Phi(-d_2) \quad (1.1.6)$$

Merton's model assumes that the default can only happen at maturity T of the defaultable corporate bond. If the firm value drops to some insufficient level upon which it cannot maintain its operation before the maturity of the debt, but it is able to recover and honor the debt at maturity, the default will not happen in Merton's model. To avoid this modeling flaw, Black and Cox (1976) alternatively propose a first passage model, in which a default is triggered whenever the firm's value V_t falls below a threshold $\bar{K}$ during the life span of the corporate bond.

1.1.2 Pros and Cons of the Structural Models

In most structural models, the default is triggered as the firm value falls below some threshold or barrier. Default events are directly linked to firms' capital structure, and the default time and default probability can be derived endogenously. Consequently, the main advantage of structural models is that, from an economic point of view, the reasons for a firm to default are well explained in these models. This intrinsic feature makes the structural models well-suited for some issues of corporate finance such as the relative powers of shareholders and creditors. When firms experience financial distress, equity holders may act strategically, forcing concessions from debt holders

and paying less than the originally contracted interest payments⁴. It thus opens a door to the analysis of the optimal capital structure and strategic debt servicing.

However, the drawbacks of structural models are as obvious as their advantages.

Firstly, to implement Merton's model, one needs to estimate the evolution of the firm value V_t , which in turn makes the model implementation very difficult if not impossible, since many of the firm's assets are typically not tradable and the firm value process is fundamentally unobservable.

Secondly, the arrival of the default event is predictable in the specification of structural models. Since the default time is endogenously derived from the capital structure, which is driven by a continuous diffusion process, the default does not come as a surprise, i.e. it can be announced in advance by the distance to default (distance between the firm's current value and its default threshold). The predictability of such models implies that the short term credit spreads are close to zero. This result contradicts market observation, as empirical studies show that short-term credit spreads are lower bounded⁵.

Another drawback of structural models is their bad performance in empirical studies. Eom, Helwege and Huang (2003) carry out an empirical analysis of five structural models (Merton; Geske; Leland and Toft; Longstaff and Schwartz; and Collin-Dufresne and Goldstein), and arrive at the unpleasant conclusion that "Using estimates from the implementations we consider most realistic, we agree that the five

⁴ See Mella Barral (1997)

⁵ See Jones et al. (1984), Franks and Torous (1989), Sarig and Warga (1989) and Fons (1994)

structural bond pricing models do not accurately price corporate bonds. However, the difficulties are not limited to the underprediction of spreads. ... they all share the same problem of inaccuracy, as each has a dramatic dispersion of predicted spreads.”

1.1.3 Advances in Structural Modeling

In order to apply the theory of European options pricing developed by Black and Scholes (1973), Merton simplifies his structural model to meet the requirements of the Black-Scholes framework. Following researchers have sought to remove these restrictions and make the models more realistic.

The assumption of a constant interest rate and a flat term structure is one of the major criticisms that Merton’s model has received. Jones et al. (1984) suggest that “there exists evidence that introducing stochastic interest rates, as well as taxes, would improve the model’s performance.” Stochastic interest rates have been considered, among others, by Ronn and Verma (1986), Kim, Ramaswamy and Sundaresan (1993), Nielsen et al. (1993), Longstaff and Schwartz (1995), and Briys and de Varenne (1997), which allow the introduction of the correlation between the firm’s asset value and the short rate.

Another problem of Merton’s model is the simple capital structure assumption, in which only an equity and a zero coupon bond are considered. Longstaff and Schwartz (1995) extend the model by considering both default risk and interest-rate risk simultaneously. Risky corporate bonds with periodic coupon payments are evaluated even within a sophisticated firm capital structure. Subsequent researchers demonstrate that the correlation of a firm’s assets with the risk free interest rate can affect the

value of defaultable corporate bonds significantly. In particular, the defaultable term structure is introduced by using the Vasicek (1977) approach.

Finally, to solve the problem of the predictability of the default arrival time and the zero short term spread within structural models, Zhou (1997, 2001a) and Schönbucher (1996) apply jump diffusion processes for the evolution of the firm's value in Merton's model. Fouque, Sircar and Solna (2005) consider the effect of stochastic volatility, finding that it indeed increases short-term spreads. These efforts make the models more realistic in generating the shape of a credit spread term structure, comparing to the classical structural models that seem to underestimate credit spread values with respect to market data.

1.2 Reduced Form Models

Reduced form models are first introduced by Jarrow and Turnbull (1995) to solve the intrinsic predictability problem within structural approaches. These reduced form models are characterized by a more flexible methodology for modeling credit risks, in which the default events arrive at a sequence of totally inaccessible stopping times⁶. As a result, the models become more implementable and the parameters governing the default hazard rate can be extracted from market data.

1.2.1 Modeling Methodology

In contrast to structural models, the time of default in reduced form models is not determined via the value of the firm, but it is the first jump of an exogenously given point process. In this way, the default time can be defined as:

$$\tau = \inf\{t \in \mathbb{R}^+ : N(t) > 0\} \quad (1.2.1)$$

where $N(t)$ is a point process with a compensator $\mu(t) = \int_0^t \lambda(u) du$, and the intensity process $\lambda(t)$ is usually assumed to be an adapted positive $\mathcal{F}_t^W$ -process.

In most cases the point process is specified as a doubly stochastic process, therefore, the survival probability can be derived as:

$$\mathbb{Q}(N(t) = 0 | \mathcal{F}_t^W) = \mathbb{Q}(\tau > t | \mathcal{F}_t^W) = e^{-\int_0^t \lambda(s) ds} \quad (1.2.2)$$

Consequently, given no default up to time t , the intensity can be derived as:

⁶ In most literature, when a default event occurs, the price process is terminated. Therefore only the first stopping time is relevant.

$$\lambda(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbb{Q}[\tau \in (t, t + \Delta t] | \tau > t]}{\Delta t} = \frac{f(t)}{1 - F(t)} \quad (1.2.3)$$

Conditioning on the background information filtration $\mathcal{F}_t^W$, $F(t) = \mathbb{Q}(\tau \leq t)$ is the default probability and $f(t)$ is the density function of $F(t)$.

With the specification of an exogenous recovery, the pricing building blocks can be derived and defaultable bonds and other credit derivatives can be priced by using these pricing building blocks.

1.2.2 Research Advances in Reduced Form Models

Jarrow and Turnbull (1995) is the first paper that studies default risks by introducing reduced form models, where a deterministic Poisson process with a constant intensity is applied. In contrast, Jarrow et al. (1997) consider the issuer's rating transition matrix as the fundamental variable driving the default process and the rating dynamics are modeled as a Markov chain. Lando (1998) generalizes the model of Jarrow and Turnbull (1995) by introducing a Cox process, which allows stochastic intensity. If the Cox process is conditional on a realization of the intensity, it becomes an inhomogeneous Poisson process. The default is described as the first jump of the underlying Cox process. Restrictions on the default time are imposed to obtain a remarkable degree of analytical tractability.

Das and Sundaram (2000) derive a more flexible model for defaultable financial claims. Within their model, a defaultable financial claim's price is equal to the expected value of future payoffs discounted by a defaultable interest rate. As a result, the existing term structure models in the literature, such as Cox et al. (1985) and Heath et al. (1992) can directly be applied to model the defaultable term structure.

Academics and practitioners have followed this pricing methodology to carry out credit spread term structure analysis. Duffie and Singleton (1999) provide a discrete-time reduced-form model to value defaultable bonds and other defaultable financial claims by arbitrage-free arguments. A forward spread process is added to the forward risk-free rate process to obtain the arbitrage-free drift restriction within the HJM Framework⁷. By using the "recovery of market value" condition⁸, they provide a recursive formula that is easy to implement.

Schönbucher (1998) also applies this defaultable HJM approach and explores the arbitrage free drift restrictions for the dynamics of the term structure of defaultable interest rates under different conditions. A similar result is obtained in Pugachevsky (1999), or Maksymiuk and Gatarek (1999). In these models, the HJM drift restrictions are extracted by applying the arbitrage free condition without assuming any jumps to default. They also show that the risky forward rate is in fact the sum of the risk-free forward rate and the credit spread.

Jeanblanc and Rutkowski (2002), Brigo and Morini (2005) and Jamshidian (2004) suggest a different approach to defaultable bond and credit derivative modeling: in a probability space equipped with a sub-filtration structure the default is modeled as a Cox process.

Gaspar and Slinko (2008) first construct a doubly stochastic marked Poisson process to model the influence of macroeconomic factors on both probability of default (PD) and loss given default (LGD). The model parameters are then calibrated to market data.

⁷ Refer to Heath et al. (1992)

⁸ Refer to Duffie and Singleton (1999)

1.3 Reconciliation of Structural Models and Reduced Form Models

Although beyond the scope of this thesis, it is necessary to note that some researchers have made attempts to fill the gap between the structural and reduced form approaches. They either introduce jump processes into the firm value evolution of structural models⁹ or develop hybrid models which incorporate the firm's capital fundamentals as state variables into the intensity processes of the reduced form models¹⁰. However, as argued by Duffie and Lando (2001), these efforts do not theoretically reconcile the structural and reduced form approaches in a consistent framework.

Structural models can be consistently transformed into reduced form models when introducing incomplete information. This result is shown by Kusuoka (1999), Duffie and Lando (2001), Çetin et al. (2004), Giesecke (2004, 2005), Giesecke and Goldberg (2004), and Guo, Jarrow and Zeng (2005), among others, through different modeling approaches.

In the model of Duffie and Lando (2001), the asset process V_t is assumed to follow a lognormal process and the default time is fixed by the firm's managers to maximize the value of equity, as described by Leland and Toft (1996). However, at time t , investors can not directly observe V_t , as it is obscured by a noise term A_t . In other words, the information set revealed to the investors at time t is $\mathcal{F}_t = \sigma(Z_{t_i}, 0 \leq t_i \leq t)$, where $Z_t = V_t + A_t$. Also, the investors can only observe this information periodically. In this particular specification, the default time τ becomes totally inaccessible. Duffie

⁹ For example, see Schönbucher(1996) or Zhou (1997).

¹⁰ See Crouhy et al (1998), Madan and Unal (1998, 2000) or Davydov et al. (1999).

and Lando derive the intensity of default in terms of the conditional asset distribution and the default threshold. Therefore, with noise and observation reduction, the structural model is transformed into a reduced form model. Kusuoka (1999) proposes a more abstract method to derive a similar result, where the relevant processes Z_t can be observed continuously.

Instead of introducing a noise term into the system, Giesecke and Goldberg (2004a) consider the case in which, although the firm's value process is observed continuously by the market investors, the default barrier is unknown to them and is taken as an independent random curve. Consequently, the investors have to work under a distribution function for the value of this default barrier and the default time τ is rendered totally inaccessible. Giesecke and Goldberg (2004a) argue that "this result is consistent with recent experiences at Enron, WorldCom, and Tyco, which surely show that investors cannot observe the default barrier. In each of these cases, the true level of liabilities was not disclosed to the public." Several variants of the above model are developed in later studies of Giesecke et al. Giesecke and Goldberg (2004b) develop a model in which investors do not have information about the firms' asset value processes or the information about the default threshold. This approach is then used to model the default correlation among different firms. Starting with a structural model in which the firm's asset value follows a continuous Wiener process and investors have been equipped with complete information about both the level of the firm's asset value and the default threshold, Giesecke (2005) discusses different cases of incomplete information: (i) investors can continuously observe the information of the asset value but only have access to incomplete information of default barrier. With this specification of incomplete information, there does not exist a default intensity process; (ii) investors can only observe the incomplete information for the asset value,

but have complete information of the default threshold, and in this case the default compensator is differentiable and therefore admits an intensity representation; (iii) both the information for the asset value process and the default threshold are incomplete to investors. In this case the default intensity representation can also be derived.

Çetin et al. (2004) extend the approach of Duffie and Lando (2001), assuming that investors receive only a subfiltration of the information set that is available to firm's managers. They claim that the default time is a predictable event for firm's managers, since they have enough information about the firm's fundamentals. But public investors do not have access to that information. Instead they observe a reduced version of this information. In their model, the firm's Cash Flow (CF) is the variable which triggers default, after reaching some default threshold during a given period of time. Firm's managers can see the CF levels, but investors only receive information about the sign of the CF, making the default time an unpredictable event for them. In this setting, they derive the default intensity as seen by the market.

In light of the different information assumptions between reduced and structural model, Jarrow and Protter (2004) argue the different usages of structural and reduced form models:

“Which model is preferred—structural or reduced form—depends on the purpose for which the model is being used. If one is using the model for risk management purposes—pricing and hedging—then the reduced form perspective is the correct one to take. Prices are determined by the market, and the market equilibrates based on the information that it has available to make its decisions. In marking-to-market, or judging market risk, reduced form models are the preferred modeling methodology.

Instead, if one represents the management within a firm, judging its own firm's default risk for capital considerations, then a structural model may be preferred.

However, this is not the approach one wants to take for pricing a firm's risky debt or related credit derivatives.”

Consistent with the argument in Jarrow and Protter (2004), the methodology in this thesis can be categorized as a reduced form approach, and the model in this thesis is constructed purposefully to price tradable default risk and thus the default derivatives in the market.

1.4 Structure of the Thesis

The thesis is organized as following:

In chapter 2, a probability space and a (doubly stochastic) marked Poisson process are constructed. Under this construction, this thesis distinguishes two types of default events: default with liquidation effect (*LE*-default) and default with non-liquidation effect (*NLE*-default). A General pricing formula for default risks under the marked Poisson Process is then proposed. Next, an *NL*-default loss is specified as proportional loss to the claim's par value and defaultable bond pricing formulae under different recovery schemes are also thoroughly discussed.

Chapter 3 discusses the arbitrage free dynamics of the defaultable zero coupon bond under the framework of HJM, where the Schönbucher's (1998) approach is extended to a marked Poisson process. Arbitrage free drift restrictions are derived and change of probability measure is discussed.

Chapter 4 proposes an implementation of the models. The Hull-White mean-reversion process is applied to the model due to its tractability, and closed form solutions are given under this model specification. The credit default swap (CDS) is priced under this setting and the analytical solution is derived.

The thesis is concluded with Chapter 5.

Chapter 2

General Pricing Framework

2.1 Introduction

In this paper, default risks refer to the risks that a legal entity fails to honor its payment liability on any significant financial transaction in a timely manner. For example, a bond issuer may default, by failing to repay principal, coupon or interest rate on a bond it issued on the due dates. Upon the credit risks, a related default event (or credit event) can be defined. In 1999, the ISDA (International Swaps and Derivatives Association) standardized the definition of a possible credit event as:

- Bankruptcy
- Failure to Pay
- Restructuring
- Repudiation
- Moratorium
- Obligation Acceleration
- Obligation Default

Schönbucher suggests¹¹ that the other two cases listed below may also be treated as credit events:

- A rating downgrade below given threshold

¹¹ Refer to Schönbucher (2003) for more details

- Changes in the credit spread

In a particular credit derivative contract, the definition of a credit event may differ from the above. It may depend on the particular terms of the agreement and the negotiating power of the counterparties of the contract.

In respect of default modeling, most reduced form models (also known as intensity based models) treat a default event as the first jump in a (doubly stochastic) Poisson process. When a default event arrives, the price process of the defaultable claim will be terminated. Therefore, only the first jump time τ_1 in the Poisson process is relevant.

However, in the real world, when a default of obligation occurs, most bankruptcy codes provide alternative procedures to deal with defaulted debt and the debtors. Defaulted firms in financial distress are not necessarily liquidated because they can always choose to be reorganized. In this case, debtors are paid in terms of newly issued debt or required to inject new funds. One of the obvious advantages of not being liquidated is to avoid inefficient liquidation sales. And in some cases, liquidation is even impossible to be executed, for example, liquidation that may cause massive unemployment, or other adverse macroeconomic and political impacts.

This argument is supported by Franks and Torous (1996) and Arturo, Ivo and Ning (2006). Franks and Torous (1996) in their empirical study find out that the majority of firms in financial distress are reorganized instead of being liquidated.

Also, Arturo, Ivo and Ning (2006) explores a comprehensive sample of small and large corporate bankruptcies in Arizona and New York from 1995 to 2001 and compares the bankruptcy cost of Chapter 7 liquidations and Chapter 11 reorganizations in United States Bankruptcy Code from four aspects: the change in

the estate's value during bankruptcy (a measure of indirect costs); the time spent in bankruptcy (another and more common measure of indirect costs); the expenses submitted to and approved by the bankruptcy court (a measure of direct costs); and the recovery rates for creditors and APR (absolute priority rule) violations.

They conclude that Chapter 7 liquidations appear to be no faster or cheaper than Chapter 11 reorganizations. Chapter 11 reorganizations seem to preserve assets better than Chapter 7 liquidations, thereby allowing creditors to recover relatively more.

In the following sections, we develop a model that can accommodate multiple defaults. The proposed model can capture not only the liquidation effect of default events (e.g. first jump effect in most intensity based models) but also the reorganization effect (e.g. multiple defaults effect in Schönbucher's models) by using marked Poisson processes. After studying a long list of papers devoted to intensity based models, we find that very few authors model the default events with a marked point process, except Schönbucher (1998), who first applies a marked point process to model the multiple defaults, and Gaspar & Slinko (2008), who first use the doubly stochastic marked Poisson process into their empirical study. However, Schönbucher (1998) assumes that a default always leads to a reorganization and issuers never be liquidated; and Gaspar & Slinko (2008) focuses more on the correlation between default probability (PD) and loss given default (LGD). Therefore, it is not surprising that the results in the both previous paper share the similarities with most of other reduced form models. The contribution of the thesis is that it further exploits the theoretical flexibility of (doubly) stochastic marked Poisson process and generalizes the model to accommodate the different types of default events according to their default consequences.

2.2 Mathematical Setup

Assumption 2.2.1: Probability Space

Unless stated otherwise, we assume that there always exists a frictionless financial market equipped with a risk neutral (equivalent martingale) probability space $(\Omega, \mathbb{Q}, \mathcal{G}, \{\mathcal{G}_t\}_{0 \leq t < \infty})$, where $\{\mathcal{G}_t\}_{0 \leq t < \infty}$ are filtrations to which all stochastic processes are adapted and $\mathcal{G} = \sigma(\bigcup_{t < \infty} \mathcal{G}_t)$. The probability measure $\mathbb{Q}$ is interpreted as a risk neutral probability measure (spot martingale measure) and usually different from the real world probability measure $\mathbb{P}$. The arbitrage free condition requires all discounted price processes of financial claims are martingale under the risk neutral probability measure $\mathbb{Q}$. The probability $\mathbb{Q}$ may not be unique, on the other hand, if the market is not complete. In that case, other constraints or investment preferences should be imposed in order to decide which martingale probability measure is proper.

Assumption 2.2.2: the Underlying Stochastic Processes

We assume that all default free background uncertainties are driven by an n -dimensional Wiener process $W(t) = (W_1(t), W_2(t), \dots, W_n(t))$, i.e., all the trading assets in the default free financial market, such as default free bonds, are driven by the multi-dimensional Wiener process $W(t)$.

In addition, we define a marked point process as a sequence of pairs $(T_n, Q_n)_{n \geq 1}$, in which $(T_n)_{n \geq 1}$ is a univariate point process and $(Q_n)_{n \geq 1}$ is a sequence of i.i.d. random variables on an independent mark space $(M, \mathcal{M})$.

To model the arrivals of default events, the marked point process in this thesis is specified as a marked Poisson process which will be defined in the following.

Therefore, in $(T_n, Q_n)_{n \geq 1}$, the argument T_n indicates the time dependence of the marked Poisson process while Q_n is a random mark representing what kind of loss that this default event may cause.

Upon the mark space, an independent distribution function $\phi(A) = \int_A \phi(dq)$ is defined for any measurable set $A \in \mathcal{M}$. $\phi(A)$ is a conditional probability given that a credit event will happen in the next instant.

A marked point process can also be associated to a counting process $N(t, A)$ for each $A \in \mathcal{M}$:

$$N(t, A) = \sum_{n \geq 1} 1_{\{T_n \leq t\}} \cdot 1_{\{Q_n \in A\}} \quad (2.2.1)$$

$1_{\{\tau_A > t\}}$ is an indicator function which is equal to 1 when $\tau_A > t$, and zero otherwise.

We can therefore define the associated σ -finite (random) counting measure

$$\Lambda((s, t], A) = N(t, A) - N(s, A), \quad t > s \geq 0, \quad A \in \mathcal{M}. \quad (2.2.2)$$

This measure allows us to obtain a more concise expression in the form of integral:

$$\int_s^t \int_A H(u, y) \Lambda(du, dq) = \sum_{n \geq 1} H(T_n, Q_n) \cdot 1_{\{s < T_n \leq t\}} \cdot 1_{\{Q_n \in A\}} = \sum_{n=N_s}^{N_t} H(T_n, Q_n) \cdot 1_{\{Q_n \in A\}} \quad (2.2.3)$$

where $H(t, q)$ is a predictable process.

Definition 2.2.3: Filtrations

A family of sigma-fields $\{\mathcal{G}_t : t \in \mathbb{R}^+\} = \{\mathcal{G}_t\}$ is called a filtration (or history), if

$$\mathcal{G}_s \subseteq \mathcal{G}_t, \text{ for all } s \leq t$$

The concept of filtration represents the information flow that has been revealed as time evolves.

Here, we distinguish several different filtrations:

- the background filtrations $\{\mathcal{F}_t^W\}_{0 \leq t < \infty}$ are generated by the multi-dimensional Wiener process, i.e.,

$$\mathcal{F}_t^W := \sigma\{W(s) : 0 \leq s \leq t\} \cup \mathcal{N}$$

where $\mathcal{N}$ contains all $\mathbb{Q}$ -null subsets of Ω .

In this sense, the filtrations $\{\mathcal{F}_t^W\}_{0 \leq t < \infty}$ reveal all the background economic information other than the uncertainties about the jump process (or information about credit event arrivals), and $\mathcal{F}^W = \bigvee_{t \geq 0} \mathcal{F}_t^W$ contains both the past and future background economic information.

- Since the marked Poisson process can also be represented as a counting measure $\Lambda(dt, dq)$ on the product space $(\mathbb{R}^+ \times M, \mathcal{B}^+ \otimes \mathcal{M})$, in which $\mathcal{B}^+$ is the Borel sigma field on $\mathbb{R}^+$, we can define the filtrations $(\mathcal{F}_t^N)_{0 \leq t < \infty}$ as:

$$\mathcal{F}_t^N := \sigma\{N(s, A); s \leq t, A \in \mathcal{M}\}$$

The filtrations $(\mathcal{F}_t^N)_{0 \leq t < \infty}$ are information about default events up to time t , which are generated by the marked Poisson process $(T_n, Q_n)_{n \geq 1}$.

- Also, we define

$$\mathcal{G}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^N,$$

The filtration $(\mathcal{G}_t)_{0 \leq t < \infty}$ can be considered as a flow of accumulated information available to a market trader at time t .¹²

Remarks

- Since $\mathcal{F}^W$ is generated by the multi-dimensional Wiener process $W(t)$ the realization paths of which are continuous almost surely, any process adapted to the filtration $\mathcal{F}_t^W$ will be predictable, i.e., for any stopping τ , there exists a sequence of stopping time which announce τ such that:

$$\tau_1 < \tau_2 < \dots < \tau, \text{ and } \lim_{n \rightarrow \infty} \tau_n = \tau$$

Otherwise, τ is considered as a totally inaccessible stopping time under probability measure $\mathbb{Q}$, if there does not exist any predictable stopping time which can give information about τ , i.e., $\mathbb{Q}(\tau = \tilde{\tau} < \infty) = 0$ for any predictable stopping time $\tilde{\tau}$.¹³ Intuitively, a predictable default arrival time means that although a default is considered as an uncertain event, it cannot technically provide “enough uncertainties” to the modelers, since it can be

¹² See Bielecki.T.R and Rutkowski.M (2002) for more details about the sigma field enlargement.

¹³ Refer to Capuano et al. (2009) for more details.

anticipated with almost certainty by watching the historical realizations of the stochastic process.

Definition 2.2.4-i: Marked Poisson Process

We call $\tilde{\Lambda}(dt, dq)$ an $\mathcal{F}_t^N$ -marked Poisson process if there exists a deterministic measure $\tilde{\mu}$ on the measurable space $(\mathbb{R}^+ \times M, \mathcal{B}^+ \otimes \mathcal{M})$, such that:

$$\begin{aligned} \mathbb{Q}(\tilde{\Lambda}((s, t] \times A) = k \mid \mathcal{F}_s^N) &= \frac{\tilde{\mu}((s, t] \times A)^k}{k!} \cdot e^{-\tilde{\mu}((s, t] \times A)} \\ &= \frac{(\int_s^t \int_A \phi(dq) \tilde{\lambda}(t) dt)^k}{k!} \cdot e^{-\int_s^t \int_A \phi(dq) \tilde{\lambda}(t) dt} \end{aligned} \quad (2.2.4)$$

almost surely for any measurable set $A \in \mathcal{M}$, where $\tilde{\lambda}(t)$ is a deterministic function.

Definition 2.2.4-ii: Doubly Stochastic Marked Poisson Process

We call $\Lambda(dt, dq)$ an $\mathcal{G}_t$ -doubly stochastic Poisson process if there exists an $\mathcal{F}^W$ -measurable random measure μ , on the measurable space $(\mathbb{R}^+ \times M, \mathcal{B}^+ \otimes \mathcal{M})$, such that:

$$\begin{aligned} \mathbb{Q}(\Lambda((s, t] \times A) = k \mid \mathcal{F}_s^N \vee \mathcal{F}_t^W) &= \frac{(\mu((s, t] \times A))^k}{k!} \cdot e^{-\mu((s, t] \times A)} \\ &= \frac{(\int_s^t \int_A \phi(dq) \lambda(t) dt)^k}{k!} \cdot e^{-\int_s^t \int_B \phi(dq) \lambda(t) dt} \end{aligned} \quad (2.2.5)$$

a.s. for any measurable set $A \in \mathcal{M}$, where $\lambda(t)$ is an $\mathcal{F}_t^W$ -adapted process.¹⁴

Assumption 2.2.5: the Intensity process for a (Doubly Stochastic) Marked Poisson Process

We assume that for each measurable set $A \in \mathcal{M}$, the (doubly stochastic) Poisson process $N_t(A) = N((0, t] \times A)$ admits an intensity process $\mu_t(A) = \mu((0, t] \times A)$, such that, $N((0, t] \times A) - \mu((0, t] \times A)$ is a $\mathcal{G}_t$ -martingale. This then leads to a measure-valued intensity $\mu_t(dq)$ such that:

$E[\int_0^t \int_M H(s, q)(\Lambda(ds, dq) - \mu_s(dq)ds)]$ is an $\mathcal{G}_t$ -martingale for all nonnegative $\mathcal{G}_t$ -predictable and M -marked processes H .

We may let $\mu_t(dq)$ depend on some background $\mathcal{F}_t^W$ random process, leading to a doubly stochastic marked Poisson process and if $\mu_t(dq)$ is a deterministic time-inhomogeneous function, it is a marked Poisson process.

In the following pricing models, we do not try to model the compensator $\mu_t(dq)$.

Instead we separate it into two quantities: the $\mathcal{F}_t^W$ -adapted intensity $\lambda(t)$ of the Poisson point process $N(t, M)$, and the conditional marked distribution function $\phi(dq)$ on the marked space M , i.e.,

$$\mu_t(dq) = \lambda(t) \cdot \phi(dq),$$

¹⁴ See Last and Brandt (1995) for a thorough treatment of marked point processes on the real line.

where $\phi(dq)$ is a deterministic function and the intensity $\lambda(t)$ can be stochastic in the case of doubly stochastic Poisson point process. Because $\mu_t(A)$ is a random product measure on $\mathbb{R}^+ \times M$, we are always allowed to do so.¹⁵

Theorem 2.2.6-i: Thinning Doubly Stochastic Marked Poisson Process

Suppose a doubly stochastic marked Poisson process $\Lambda(dt, dq)$ admits an $\mathcal{F}_t^W$ -adapted compensator process $\mu(dt, dq)$, i.e., for any measurable set $A \in \mathcal{M}$,

$$\mathbb{Q}(\Lambda((s, t] \times A) = k \mid \mathcal{F}_s^N \vee \mathcal{F}_t^W) = \frac{\mu((s, t] \times A)^k}{k!} \cdot e^{-\mu((s, t] \times A)} \text{ a.s.} \quad (2.2.6)$$

Suppose that each jump occurrence can be classified as being either a type-I or type-II event with respect to its mark identity. For generality, we allow the independent probability of a jump being classified as type-I or type-II event to depend on the time at which it occurs. That is, when a jump happens, a random mark q is independently drawn from the mark space M , and if $q \in A$, $A \in \mathcal{M}$, the jump event is classified as type-I event, otherwise it is classified as type-II event. Specifically, when the jump occurs at time t , the event is classified as being type-I event with probability

$$p = \int_A K(t, dq) \text{ and a type-II event with probability } 1 - p = 1 - \int_A K(t, dq), \quad A \in \mathcal{M},$$

where $K(t, dq)$ is the time dependent probability measure. If we let $\Lambda_x(t)$, $x = 1, 2$

represent the counting process of type- x events up to time t , then conditioning on $\mathcal{F}_t^W$,

$\Lambda_1(t)$ and $\Lambda_2(t)$ are two independent Poisson processes having respective $\mathcal{F}_t^W$ -

adapted intensity $\mu_t p$ and $\mu_t(1 - p)$

¹⁵ See Last and Brandt (1995) for further details. Also, the proof about the existence and the construction procedure of such a (doubly) stochastic marked Poisson process can be found in Gaspar and Slinko (2008)

where

$$p = \frac{1}{t} \int_0^t \int_A K(s, dq) ds,$$

And $\mu_t = \mu_t(M)$ is the compensator for the Poisson process $\Lambda_t(M)$

Proof.

First, we prove that at time t , $\Lambda_1(t)$ and $\Lambda_2(t)$ are two independent Poisson variables,

conditioning on $\mathcal{F}_t^W$:

$$\begin{aligned} & \mathbb{Q}(\Lambda_1(t) = k_1, \Lambda_2(t) = k_2 \mid \mathcal{F}_t^W) \\ &= \sum_{k=0}^{\infty} \mathbb{Q}(\Lambda_1(t) = k_1, \Lambda_2(t) = k_2 \mid \Lambda_t(M) = k, \mathcal{F}_t^W) \cdot \mathbb{Q}(\Lambda_t(M) = k \mid \mathcal{F}_t^W) \end{aligned}$$

Since for any $k \neq k_1 + k_2$,

$$\mathbb{Q}(\Lambda_1(t) = k_1, \Lambda_2(t) = k_2 \mid \Lambda_t(M) = k, \mathcal{F}_t^W) = 0$$

therefore,

$$\begin{aligned} & \mathbb{Q}(\Lambda_1(t) = k_1, \Lambda_2(t) = k_2 \mid \mathcal{F}_t^W) \\ &= \mathbb{Q}(\Lambda_1(t) = k_1, \Lambda_2(t) = k_2 \mid \Lambda_t(M) = k_1 + k_2, \mathcal{F}_t^W) \cdot \mathbb{Q}(\Lambda_t(M) = k_1 + k_2 \mid \mathcal{F}_t^W) \end{aligned}$$

In a Poisson Process, given the information of the number of jumps, the arrival times are i.i.d. uniformly distributed. That is, conditioning on $\Lambda_t(M) = k$ and $\mathcal{F}_t^W$, the k

arrival times $S_1, \dots, S_k$ have the same distribution as the order statistics corresponding to k i.i.d. uniformly distributed on the time interval $(0, t)$ ¹⁶.

$$\text{Then, } \mathbb{Q}(\Lambda_1(t) = k_1, \Lambda_2(t) = k_2 \mid \Lambda_t(M) = k_1 + k_2, \mathcal{F}_t^W) = \binom{k_1 + k_2}{k_1} \cdot p^{k_1} \cdot (1-p)^{k_1+k_2}$$

$$\text{where } p = \frac{1}{t} \int_0^t \int_A K(s, dq) ds$$

Consequently,

$$\begin{aligned} \mathbb{Q}(\Lambda_1(t) = k_1, \Lambda_2(t) = k_2 \mid \mathcal{F}_t^W) &= \binom{k_1 + k_2}{k_1} \cdot p^{k_1} \cdot (1-p)^{k_1+k_2} \cdot \mathbb{Q}(\Lambda_t(M) = k_1 + k_2 \mid \mathcal{F}_t^W) \\ &= \binom{k_1 + k_2}{k_1} \cdot p^{k_1} \cdot (1-p)^{k_1+k_2} \cdot \frac{(\mu_t)^{k_1+k_2}}{(k_1 + k_2)!} \cdot e^{-\mu_t} = e^{-tp\mu_t} \cdot \frac{(tp\mu_t)^{k_1}}{k_1!} \cdot e^{-t(1-p)\mu_t} \cdot \frac{(t(1-p)\mu_t)^{k_2}}{k_2!} \end{aligned}$$

That is, conditioning on $\mathcal{F}_t^W$, $\Lambda_1(t)$ and $\Lambda_2(t)$ are two independent Poisson variables for any fixed time t .

And since all the augmented sigma-algebras generated by $\Lambda_x(t)$, $x = 1, 2$,

$$\mathcal{F}_\infty^{\Lambda_x(t)} = \sigma\{\Lambda_x(t) : t \in \mathbb{R}^+\}, \quad x = 1, 2$$

are independent, $\Lambda_1(t)$ and $\Lambda_2(t)$ are two independent Poisson processes, given the background information $\mathcal{F}_t^W$.

This completes the proof. ■

¹⁶ See Ross (1982) for more details.

The theorem can be generalized to multiple cases as follows:

Theorem 2.2.6-ii: Thinning Doubly Stochastic Marked Poisson Process

(Multi-Factor Case)

Let $\Lambda(dt, dq, \omega)$ be a marked Poisson process and $B_1, \dots, B_m \in \mathcal{M}$, $m \in \mathbb{N}$, are mutually disjoint sets. Then $\Lambda(\cdot \times B_1), \dots, \Lambda(\cdot \times B_m)$ are independent Poisson processes with intensity measures $\mu(\cdot \times B_1), \dots, \mu(\cdot \times B_m)$.

Proof. See Last and Brandt (1995)

2.3 General Pricing Framework

Assumption 2.3.1: Arbitrage Free Pricing Condition

The arbitrage free condition of financial markets is taken as given in our thesis and in these markets, all the financial claims can be continuously traded without cost. As pointed out by Harrison & Kreps (1979), in order to rule out arbitrage opportunities, there should always exist at least one spot martingale measure $\mathbb{Q}$. Under this probability measure $\mathbb{Q}$, any price process of tradable securities, which pays no coupons or dividends, will follow a G_t martingale at time t , when discounted by the risk free interest rate. The martingale measure is not necessarily unique, i.e., the market may not be complete. In that case, additional constraints should be imposed to decide which martingale probability measure is the proper one to use.

Definition 2.3.2: Risk Free Account $S(t)$

We assume there exists a saving account where at time t the cash flow $S(t)$ will be accumulated at the instantaneous default free short rate $r(t)$:

$$S(t) = e^{\int_0^t r(u) du} \quad (2.3.1)$$

Definition 2.3.3: Default Events

As stated in the introduction of this chapter, default events happen when a legal entity fails to honor its payment obligations on time.

Mathematically, we assume that:

- Default events arrive at a sequence of the stopping times $\tau_1 < \tau_2 < \dots$, where τ_i is the time of the i -th jump of a (doubly stochastic) Poisson process

$$N(t) = \Lambda(t, M).$$
- At each default arrival time, a mark q is drawn from the mark space $(M, \mathcal{M})$ to indicate a negative impact of the default event on the issuer of the defaultable claim and such negative impact is measured by the loss function $h(t, q)$.
- We distinguish between two types of default events: a default with liquidation effect (*LE*-default) and a default with non-liquidation effect (*NLE*-default).

Mathematically, we define a measurable set $A \in \mathcal{M}$ upon the underlying (doubly stochastic) marked Poisson process $\Lambda(dt, dq)$, whenever the mark is from this set, i.e., $q_n \in A$, the default event is identified as an *LE*-default and it leads to a liquidation process. In this case, the price process of the related defaultable claim is terminated and the claim holders will receive a remnant of their defaultable claim. That is, the measurable set A represents absorbing (ruined) states of the price process, and the independent probability

$\phi(A) = \int_A \phi(dq)$ is defined as the liquidation probability conditioned on a default event happening. Regarding the different effects that a default may cause, we define two (doubly stochastic) Poisson processes on the underlying (doubly stochastic) marked Poisson process:

$$N_1(t) = N(t, A) \text{ with intensity } \lambda(t)\phi(A), \text{ and}$$

$$N_2(t) = N(t, M \setminus A) \text{ with intensity } \lambda(t)\phi(A)$$

Definition 2.3.4: Default time τ

The default time τ is modeled by a sequence of random arrival times of the underlying (doubly stochastic) marked Poisson process. τ is assumed to be a stopping time with respect to the filtration $\{\mathcal{G}_t\}_{t \geq 0}$. Under this setting, τ is totally inaccessible and therefore not a stopping time (neither adapted) with respect to filtration $\mathcal{F}_t^W$ which is generated by the continuous Wiener process. This essential difference distinguishes reduced form models (or intensity based models) from structural models.

Specifically, we define τ_A as the first stopping time when a jump happens and a mark q falls in the measurable set A , i.e., $\tau_A = \inf \{t : N(t, A, \omega) \geq 1\}$. This means a default event occurs and for the first time it triggers the liquidation process prior to or at the maturity date T of the defaultable claim. The underlying stochastic process then enters the absorbing state, and the price process of the defaultable claim is terminated.

Definition 2.3.5: Final Payoff X_T

The final payoff of the defaultable claim X_T is the promised contingent payoff given no default events happen before the maturity T of the claim, i.e., the obligor's obligation to be redeemed at the maturity T . We assume that X_T is an independent $\mathcal{F}_T^W$ -adapted nonnegative random variable.

Definition 2.3.6: Loss Given Default

We distinguish between two types of loss given default here:

At each default arrival time, if the random mark $q \in A$, i.e., the default event leads to a liquidation process and terminates the price process of the defaultable claim, the creditors suffer a loss and the defaulted obligor's remaining assets are distributed to the creditors. The time dependent function $h_1(t)$ ¹⁷ is then applied to measure the loss magnitude. If the random mark $q \notin A$, we assume that the holders of the defaultable claim suffer a loss of $h_2(t, q)$ in this default event.

Both functions of $h_1(t)$ and $h_2(t, q)$ are assumed to be continuous and predictable, therefore adapted to the filtration $\mathcal{F}_t^W$.

In order to approximate reality, more regularities are imposed on the jump functions: $h_1(t)$ and $h_2(t, q)$ are assumed to be non-positive and lower bounded by some negative real number almost surely. And if the $h_i(t, q)$, $i = 1, 2$ represent quota losses to the current market value of the defaultable claim at each arrival time of a default event, we usually require $-1 < h_i(t, q) \leq 0$, $i = 1, 2$ ¹⁸. The condition $h_2(t, q) \leq 0$ can be relaxed in some cases, since our model can also be generalized to capture the arrivals of “good news” to the obligors, such as improvement of the issuer's financial performance and position, upgrading of the issuer's credit rating, etc.

Within our framework, we leave the economic meaning of the mark unspecified. It can be used to represent the firms' value loss, deterioration of credit rating,

¹⁷ When the defaultable claim issuer goes bankrupt it triggers a legal liquidation process, which is solely regulated by the bankruptcy codes and is less related to the default event itself. Thus, it is reasonable to assume that this loss is a mark independent function $h_1(t)$ rather than a mark dependent function $h_1(t, q)$.

¹⁸ $h_2(t, q) = 0$ means that a default event happens but does not affect the market price of the defaultable claim.

restructuring, etc. If it is assumed that the mark represents asset value of a corporate, and the measurable absorbing set A is some lower boundary of the firm's value. Then we can get a hybrid model;¹⁹ If we define the mark as a quota loss to the pre-default value of a defaultable claim, then we get a structure similar to multiple default models²⁰, except in this case we consider the reorganization effect of a default as well as a liquidation effect of a bankruptcy; If we define the mark space M as a union of finite states, then we can get a credit rating transition model.

Definition 2.3.7: Defaultable Contingent Claims

A defaultable claim is a financial contract that promises a series of future payments which will be contingent on future default risks.

By the definition above, at time t , a defaultable contingent claim with maturity T ,

$DC(t, T)$, can therefore be well defined as a quintuple²¹:

$$DC(t, T) = (X_T, \tau, h_1, h_2, \phi) \quad (2.3.2)$$

in which X_T is the time T final payoff; τ is the default time; $h_1(t)$ and $h_2(t, q)$ constitute the loss magnitudes; and ϕ is the distribution function for the random mark.

¹⁹ Hybrid models are introduced by Crouhy et al (1998), Madan and Unal (1998, 2000) or Davydov et al. (1999). In those models the default intensities are directly linked to the current value of the firm's assets. Hybrid models try to incorporate some advantages of firm value models and reduced form models. Technically speaking, hybrid models belong to the category of reduced form models since the default time in the set-up is still a totally inaccessible stopping time.

²⁰ Refer to Schönbucher (1998, 2000)

²¹ Without ambiguity, $DC(t, T)$ is also used for the price process the defaultable claim in the following context.

Definition 2.3.8: Pre-default Price Process

We define an auxiliary stochastic process: the pre-default price process $V_T(t)$ of $DC(t, T)$ ²²:

Therefore, given there has been no arrival of default event by time t , $V_T(t)$ is the market value of the defaultable claim $DC(t, T)$.

Theorem 2.3.9

Under the Spot Martingale Measure, we have the following non-arbitrage defaultable contingent claims price formula:

$$\begin{aligned}
 DC(t, T) &= 1_{\{\tau_A > t\}} \tilde{S}(t) E^{\mathbb{Q}}[\tilde{S}(T)^{-1} X_T + \int_t^T \phi(A) \lambda(u) \tilde{S}(u)^{-1} (1 + h_1(u)) d\mathcal{U}_{\mathcal{F}_t^W}] \\
 &+ 1_{\{\tau_A > t\}} \tilde{S}(t) E^{\mathbb{Q}}[\int_t^T \int_{M \setminus A} h_2(u, q) \phi(dq) \tilde{S}(u)^{-1} \lambda(u) du | \mathcal{F}_t^W] \\
 &+ \int_0^t \frac{S(t)}{S(u)} 1_{\{\tau_A > u\}} \int_{M \setminus A} h_2(u, q) \Lambda(du, dq) \quad (2.3.3)
 \end{aligned}$$

And

$$\begin{aligned}
 V_T(t) &= \tilde{S}(t) E^{\mathbb{Q}}[\tilde{S}(T)^{-1} X_T + \int_t^T \phi(A) \lambda(u) \tilde{S}(u)^{-1} (1 + h_1(u)) d\mathcal{U}_{\mathcal{F}_t^W}] \\
 &+ \tilde{S}(t) E^{\mathbb{Q}}[\int_t^T \int_{M \setminus A} h_2(u, q) \phi(dq) \tilde{S}(u)^{-1} \lambda(u) du | \mathcal{F}_t^W] \quad (2.3.4)
 \end{aligned}$$

where $\tilde{S}(t) = e^{\int_0^t (\phi(A) \lambda(s) + r(s)) ds}$ is the defaultable discount rate.

²² The idea of this auxiliary stochastic process stems from Duffie & Singleton (1999).

Remarks of Theorem 2.3.9

- $E^{\mathbb{Q}}[\cdot | \mathcal{F}_t^W]$ is an expectation under the spot martingale measure $\mathbb{Q}$, conditional on the background information up to time t . We take it as given that a spot martingale measure $\mathbb{Q}$ exists in the first place, i.e., the market is arbitrage free, and in the following section, we will later derive such a spot martingale $\mathbb{Q}$ from the real world probability measure $\mathbb{P}$ by Girsanov's theorem.
- The first argument $\tilde{S}(T)^{-1} X_T$ inside the expectation refers to the discounted (under defaultable discount rate) final payment.
- The second term of the expectation $\int_t^T \phi(A) \lambda(u) \tilde{S}(u)^{-1} (1 + h_1(u)) du$ is the discounted recovery payoff to the claim holders immediately after the occurrence of liquidation prior to the maturity date T . It models the expected settlement value at the time of liquidation.
- The third term of the expectation $\int_t^T \int_{M \setminus A} h_2(u, q) \phi(dq) \tilde{S}(u)^{-1} \lambda(u) du$ is the discounted future loss if the issuer has not yet been liquidated during the time between t and T . Noting that our setting here is different from the popular definitions in most of the literature. In some extreme cases (e.g. the issuer has already been liquidated), the gain process can be negative with positive probability. This is because it implies that when an NLD event happens, the creditors are required to inject funds. The loss has been withdrawn from the price process and does not disappear even when the price process is terminated. (Therefore, it can be considered as a series of negative coupons paid at random

dates up to the time $\tau_A \wedge T$). Other regularities will be introduced to remove this possibility in next section.

- The last term $\int_0^t \frac{S(t)}{S(u)} 1_{\{\tau_A > u\}} \int_{M \setminus A} h_2(u, q) \Lambda(du, dq)$ counts the cumulative losses up to current time t .
- The pre-default price process V_t is adapted to the filtration $\mathcal{F}_t^W$. So it is predictable. However the price process of $DC(t, T)$ is adapted to the filtration $\mathcal{G}_t$.
- When a default event occurs and the price process of this defaultable claim is not terminated, the market price of this claim will jump with magnitude of $h_2(u, q)$ at the default time u . Furthermore, if $h_2(t, q)$ is a deterministic function of t and q , and continuous on the first variable t , due to the independence of the mark distribution, we have:

$$\begin{aligned}
& S(t) E^{\mathbb{Q}} \left[\int_t^T S(u)^{-1} 1_{\{\tau_A > u\}} \int_{M \setminus A} h_2(u, q) \Lambda(du, dq) \mid \mathcal{G}_t \right] \\
&= E^{\mathbb{Q}} \left[\int_t^T e^{-\int_t^u (\phi(A)\lambda(s) + r(s)) ds} \int_{M \setminus A} h_2(u, q) \phi(dq) \lambda(u) du \mid \mathcal{F}_t^W \right] \\
&= E^{\mathbb{Q}} \left[\int_t^T e^{-\int_t^u (\phi(A)\lambda(s) + r(s)) ds} \bar{h}_2^{M \setminus A}(u) \lambda(u) du \mid \mathcal{F}_t^W \right]
\end{aligned}$$

where $\bar{h}_2^{M \setminus A}(u)$ is a truncated expectation in the mark space, and,

$$\bar{h}_2^{M \setminus A}(t) = \int_{M \setminus A} h_2(t, q) \phi(dq)$$

- By Theorem 2.2.4, $N_1(t) = N(t, A)$ and $N_2(t, M \setminus A) = N(t, M \setminus A)$ are two independent Poisson processes. Therefore, conditional on $\mathcal{F}_T^W$, $1_{\{\tau_A > T\}}$ and

$$\int_0^T \int_{M \setminus A} h_2(u, q) \Lambda(du, dq) = \int_0^T h_2(u, Q) N_2(du, Q) \text{ are independent.}$$

The following lemma are needed to prove Theorem 2.3.3:

Lemma 2.3.10

For any $\mathcal{G}_T$ - measurable random variable Y and $t \in \mathbb{R}^+$, we have

$$E^{\mathbb{Q}}[1_{\{\tau > t\}} Y \mid \mathcal{G}_t] = 1_{\{\tau > t\}} \frac{E^{\mathbb{Q}}[1_{\{\tau > t\}} Y \mid \mathcal{F}_t^W]}{\mathbb{Q}\{\tau > t \mid \mathcal{F}_t^W\}} \quad (2.3.5)$$

where τ is a stopping time with respect to the filtration $(\mathcal{G}_t)_{0 \leq t < \infty}$ and $\mathcal{F}_t^W \subseteq \mathcal{G}_t$.

Proof of Lemma 2.3.10

Refer to Bielecki and Rutkowski (2002).

Proof of Theorem 2.3.9

If we take as given the risk neutral probability measure $\mathbb{Q}$, then in the sense of Harrison & Kreps (1979), the gain processes generated by every asset at time t , when discounted by the saving account $S(t) = e^{\int_0^t r(u) du}$, should be a $\mathcal{G}_t$ martingale under $\mathbb{Q}$.

Suppose at the beginning, an individual holds a defaultable contingent claim

$DC(0, T)$, and at time t , the discounted gain process of this investment $G_t S(t)^{-1}$ is:

$$G_t S(t)^{-1} = S(t)^{-1} [DC(t, T) + 1_{\{\tau_A \leq t\}} \frac{S(t)}{S(\tau_A)} (1 + h_1(\tau_A))] \quad ^{23}$$

At the maturity T , the final payoff of this gain process G_T will be

$$G_T = 1_{\{\tau_A > T\}} X_T + \int_0^T 1_{\{\tau_A > u\}} \frac{S(T)}{S(u)} \int_M h_2(u, q) \Lambda(du, dq) + 1_{\{\tau_A \leq T\}} \frac{S(T)}{S(\tau_A)} (1 - h_1(\tau_A))$$

Therefore, the discounted final payoff $G_T S(T)^{-1}$ will be:

$$\begin{aligned} G_T S(T)^{-1} &= 1_{\{\tau_A > T\}} S(T)^{-1} X_T + 1_{\{\tau_A \leq T\}} S(\tau_A)^{-1} (1 + h_1(\tau_A)) \\ &+ \int_0^T S(u)^{-1} 1_{\{\tau_A > u\}} \int_{M \setminus A} h_2(u, q) \Lambda(du, dq) \end{aligned}$$

By the arbitrage free argument, the discounted gain process $G_t S(t)^{-1}$ should be a $\mathcal{G}_t$ martingale under the risk neutral probability measure $\mathbb{Q}$:

$$G_t S(t)^{-1} = E^{\mathbb{Q}}[G_T S(T)^{-1} | \mathcal{G}_t]$$

Substitute the expression of $G_t S(t)^{-1}$ and $G_T S(T)^{-1}$, we have:

$$\begin{aligned} DC(t, T) &= S(t) E^{\mathbb{Q}}[S(T)^{-1} 1_{\{\tau_A > T\}} X_T + 1_{\{t < \tau_A \leq T\}} (1 + h_1(\tau_A)) S(\tau_A)^{-1} \\ &+ \int_0^T S(u)^{-1} 1_{\{\tau_A > u\}} \int_{M \setminus A} h_2(u, q) \Lambda(du, dq) | \mathcal{G}_t] \end{aligned}$$

²³We have a trivial result, if the contingent claim defaulted before time t , i.e., $\tau_A \leq t$. The value of the defaultable claim is zero, $DC(t, T) = 1_{\{\tau_A > t\}} V_t = 0$, since we get $1_{\{\tau_A > t\}} = 0$ in this case. The discounted recovery payoff reduces to $h_1(\tau_A) S(\tau_A) S(t)^{-1}$. That is, after default, the gain process of this investment becomes a saving account with initial capital $(1 + h_1(\tau_A))$ invested at time τ_A .

By applying lemma 2.3.10, we obtain

$$\begin{aligned}
E^{\mathbb{Q}}\left[\frac{S(t)}{S(T)}1_{\{\tau_A > T\}}X_T \mid \mathcal{G}_t\right] &= E^{\mathbb{Q}}\left[\frac{S(t)}{S(T)}1_{\{\tau_A > t\}}1_{\{\tau_A > T\}}X_T \mid \mathcal{G}_t\right] \\
&= 1_{\{\tau_A > t\}} \frac{E^{\mathbb{Q}}\left[\frac{S(t)}{S(T)}1_{\{\tau_A > T\}}X_T \mid \mathcal{F}_t^W\right]}{\mathbb{Q}\{\tau_A > t \mid \mathcal{F}_t^W\}}
\end{aligned}$$

Since the final payoff X_T is an independent $\mathcal{F}_T^W$ -measurable variable, we have

$$E^{\mathbb{Q}}\left[\frac{S(t)}{S(T)}1_{\{\tau_A > T\}}X_T \mid \mathcal{G}_t\right] = 1_{\{\tau_A > t\}} \frac{E^{\mathbb{Q}}[E^{\mathbb{Q}}[1_{\{\tau_A > T\}} \mid \mathcal{F}_T^W] \frac{S(t)}{S(T)}X_T \mid \mathcal{F}_t^W]}{\mathbb{Q}\{\tau_A > t \mid \mathcal{F}_t^W\}}$$

Given the background information up to time t (conditional on $\mathcal{F}_t^W$), the Poisson process $N_1(t)$ is like a deterministic Poisson process with time-inhomogeneous intensity $\phi(A)\lambda(t)$, then

$$\mathbb{Q}\{\tau_A > t \mid \mathcal{F}_t^W\} = e^{-\int_t^T \phi(A)\lambda(s)ds}$$

Therefore,

$$\begin{aligned}
E^{\mathbb{Q}}\left[\frac{S(t)}{S(T)}1_{\{\tau_A > T\}}X_T \mid \mathcal{G}_t\right] &= 1_{\{\tau_A > t\}} E^{\mathbb{Q}}\left[e^{-\int_t^T (\phi(A)\lambda(s) + r(s))ds} X_T \mid \mathcal{F}_t^W\right] \\
&= 1_{\{\tau_A > t\}} \tilde{S}(t) E^{\mathbb{Q}}[\tilde{S}(T)^{-1} X_T \mid \mathcal{F}_t^W]
\end{aligned}$$

Similarly, we have

$$E^{\mathbb{Q}}[1_{\{t < \tau_A \leq T\}} h_1(\tau_A) \frac{S(t)}{S(\tau_A)} | \mathcal{G}_t] = 1_{\{\tau_A > t\}} \frac{E^{\mathbb{Q}}[1_{\{t < \tau_A \leq T\}} (1 + h_1(\tau_A)) \frac{S(t)}{S(\tau_A)} | \mathcal{F}_t^W]}{\mathbb{Q}\{\tau_A > t | \mathcal{F}_t^W\}}$$

$$= 1_{\{\tau_A > t\}} E^{\mathbb{Q}}[\int_t^T \phi(A) \lambda(u) e^{-\int_t^u (\phi(A) \lambda(s) + r(s)) ds} (1 + h_1(u)) du | \mathcal{F}_t^W]$$

$$= 1_{\{\tau_A > t\}} \tilde{S}(t) E^{\mathbb{Q}}[\int_t^T \phi(A) \lambda(u) \tilde{S}(u)^{-1} (1 + h_1(u)) du | \mathcal{F}_t^W]$$

And since $\int_t^T \frac{S(t)}{S(u)} 1_{\{\tau_A > u\}} \int_{M \setminus A} h_2(u, q) \Lambda(du, dq)$ is a $\mathcal{G}_T$ -measurable variable, again,

lemma 2.3.10 implies

$$E^{\mathbb{Q}}[\int_t^T \frac{S(t)}{S(u)} 1_{\{\tau_A > u\}} \int_{M \setminus A} h_2(u, q) \Lambda(du, dq) | \mathcal{G}_t]$$

$$= 1_{\{\tau_A > t\}} \frac{E^{\mathbb{Q}}[\int_t^T \int_{M \setminus A} \frac{S(t)}{S(u)} 1_{\{\tau_A > u\}} h_2(u, q) \Lambda(du, dq) | \mathcal{F}_t^W]}{\mathbb{Q}\{\tau_A > t | \mathcal{F}_t^W\}}$$

$$= 1_{\{\tau_A > t\}} \frac{E^{\mathbb{Q}}[E^{\mathbb{Q}}[\int_t^T \int_{M \setminus A} \frac{S(t)}{S(u)} 1_{\{\tau_A > u\}} h_2(u, q) \Lambda(du, dq) | \mathcal{F}_T^W] | \mathcal{F}_t^W]}{\mathbb{Q}\{\tau_A > t | \mathcal{F}_t^W\}}$$

$$= 1_{\{\tau_A > t\}} E^{\mathbb{Q}}[\int_t^T \int_{M \setminus A} h_2(u, q) \phi(dq) e^{-\int_t^u (\phi(A) \lambda(s) + r(s)) ds} \lambda(u) du | \mathcal{F}_t^W]$$

$$= 1_{\{\tau_A > t\}} \tilde{S}(t) E^{\mathbb{Q}}[\int_t^T \int_{M \setminus A} h_2(u, q) \phi(dq) \tilde{S}(u)^{-1} \lambda(u) du | \mathcal{F}_t^W]$$

Combining all the results above, we obtain

$$DC(t, T) = 1_{\{\tau_A > t\}} \tilde{S}(t) E^{\mathbb{Q}}[\tilde{S}(T)^{-1} X + \int_t^T \phi(A) \lambda(u) \tilde{S}(u)^{-1} (1 + h_1(u)) du | \mathcal{F}_t^W]$$

$$+1_{\{\tau_A > t\}} \tilde{S}(t) E^{\mathbb{Q}} \left[\int_t^T \int_{M \setminus A} h_2(u, q) \phi(dq) \tilde{S}(u)^{-1} \lambda(u) du \mid \mathcal{F}_t^W \right]$$

$$+ \int_0^t \frac{S(u)}{S(t)} 1_{\{\tau_A > u\}} \int_{M \setminus A} h_2(u, q) \Lambda(du, dq)$$

And by the definition of V_t , given that the defaultable claim has not been defaulted

by time t , we get ²⁴

$$V_T(t) = \tilde{S}(t) E^{\mathbb{Q}} \left[\tilde{S}(T)^{-1} X + \int_t^T \phi(A) \lambda(u) \tilde{S}(u)^{-1} (1 + h_1(u)) du \mid \mathcal{F}_t^W \right]$$

$$+ \tilde{S}(t) E^{\mathbb{Q}} \left[\int_t^T \int_{M \setminus A} h_2(u, q) \phi(dq) \tilde{S}(u)^{-1} \lambda(u) du \mid \mathcal{F}_t^W \right]$$

■

²⁴ Refer to Bielecki and Rutkowski (2002) for detailed arguments supporting the replacement of $\mathcal{G}_t$ with a strictly smaller filtration $\mathcal{F}_t^W$ in various kinds of situations. A similar method is provided by Lando (1998) for pricing defaultable claim under a Cox process setting.

2.4 Pricing under Different Recovery Schemes

In this section, prices of the defaultable zero coupon bonds under different recovery schemes are discussed. The defaultable zero coupon bond is a basic defaultable claim in financial market and hence, a keystone for pricing other complicated credit derivatives.

In the literature, a variety of models are proposed for the recovery of defaultable claims. In Duffie and Singleton (1997, 1999), the recovery of a market value model (also known as fractional recovery model) is developed. When the bond defaults, the holders of the claim will receive compensation in terms of an equivalent defaultable bond. Therefore the recovery is expressed as a fraction of the bond's pre-default market value. The model of recovery of par is used in Duffie (1998), where the holders of the defaultable claims are compensated with a fraction of the claims' face value when the default occurs. In Jarrow and Turnbull (1995), an alternative model of recovery of treasury is offered, in which the compensation is made in terms of the value of default-free bonds on default. Schönbucher (1998) introduces the concept of multiple defaults and proves that it is equivalent to the recovery of market value model.

Under marked Poisson processes things become complicated, since now we have to consider two types of recoveries: $h_1(t)$ for the one-time jump process and $h_2(t, q)$ for the multiple jumps process. To avoid the negative gain problem in the previous subsection, we specify the non-liquidation default loss process $h_2(t, q)$ as following:

Definition 2.4.1: Remaining Valuing process

At each non-liquidation default event arrival, we specify the default loss as a fractional loss $h_2(u, q)$ of the face value of the defaultable bond, i.e. define a remaining value process $L(t)$ with the dynamics:

$$\frac{dL(t)}{L(t-)} = \int_{M \setminus A} h_2(t, q) \Lambda(dt, dq) \quad (2.4.1)$$

with $-1 < h_2(u, q) \leq 0$ almost surely and initial condition $L(0) = 1$.

At time t , the face value of the defaultable bond is $L(t)$ given that the defaultable bond price process is not terminated, and thus the final payoff of the defaultable bond at maturity T will be: $1_{\{\tau_A > T\}} L(T)$.

Definition 2.4.2: Defaultable Zero Coupon Bonds

We assume that there always exists a market that allows continuous trading of defaultable zero coupon bonds with any maturity. For simplicity, the promised payoff of a defaultable zero coupon bond at the maturity T is assumed to be $1_{\{T < \tau_A\}}$, where $1_{\{T < \tau_A\}}$ is an indicator function and τ_A is the first time that the debtor goes bankrupt and has to liquidate his assets. At time t , the market price of the defaultable bond with maturity T is denoted by $B_d(t, T)$.

We assume that $B_d(t, T)$ is continuously differentiable with respect to t and T , $\mathbb{Q}$ -a.s.

Definition 2.4.3 Defaultable Bond Price under Zero Recovery $B_d^0(t, T)$

This is the simplest case when an obligor gets into bankruptcy and enters the liquidation process. The defaultable claims holders are assumed to get no compensation, i.e., $h_1(\tau_A) = -1$ at liquidation time τ_A .

Proposition 2.4.4 Pricing $B_d^0(t, T)$

We then have the following pricing formula for the time t defaultable bond $B_d^0(t, T)$ with zero recovery:

$$B_d^0(t, T) = 1_{\{\tau_A > t\}} L(t) E^{\mathbb{Q}} \left[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W \right] \quad (2.4.2)$$

and the pre-default price process $V_T^0(t)$ is:

$$V_T^0(t) = E^{\mathbb{Q}} \left[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W \right] \quad (2.4.3)$$

We need the following Lemma to prove Proposition 2.4.4:

Lemma 2.4.5

Consider a defaultable claim with final payoff X_T at maturity T . For the purpose of computing its price at any time t prior to maturity,

$E^{\mathbb{Q}} \left[e^{-\int_t^T r(s) ds} L(T) X_T \mid \mathcal{F}_t^W \right]$, it is equivalent to use the following two dynamics

for the remaining value process $L(t)$:

$$\frac{dL(t)}{L(t-)} = \int_{M \setminus A} h_2(t, q) \Lambda(dt, dq)$$

and

$$\frac{dL(t)}{L(t-)} = \bar{h}_2^{M \setminus A}(t) dN_2(t)$$

Proof of Lemma 2.4.5

By respectively calculating $E^{\mathbb{Q}}[e^{-\int_t^T r(s)ds} L(T) X_T | \mathcal{F}_t^W]$ with two dynamics of $L(t)$ mentioned above and then comparing the results.

See also Raquel M Gaspar & Irina Slinko (2008)

Proof of Proposition 2.4.4

By lemma 2.4.5, instead of

$$\frac{dL(t)}{L(t-)} = \int_{M \setminus A} h_2(t, q) \Lambda(dt, dq),$$

we can now use the following dynamics for computations:

$$\frac{dL(t)}{L(t-)} = \bar{h}_2^{M \setminus A}(u) dN_2(u)$$

For every fixed t , we define

$$Z(u) = e^{-\int_t^u \bar{h}_2^{M \setminus A}(s) \lambda(s) ds} L(u), \quad u \geq t,$$

where $\bar{h}_2^{M \setminus A}(t) = \int_{M \setminus A} h(q, t) \phi(dq)$ is a truncated expectation on the mark space at time t .

Then by applying Ito formula to $Z(u)$, we obtain:

$$\frac{dZ(u)}{Z(u-)} = \bar{h}_2^{M \setminus A}(u)[dN_2(u) - \lambda(u)\phi(M \setminus A)du]$$

Therefore, conditioned on the filtration $\mathcal{F}_u^W$, $Z(u)$ is an $\mathcal{G}_u$ -martingale under the risk neutral measure $\mathbb{Q}$, i.e., we have

$$Z(t) = E^{\mathbb{Q}}[Z(T) | \mathcal{G}_t]$$

By theorem 2.2.6, conditional on $\mathcal{F}_T^W$, $1_{\{\tau_A > T\}}$ and $Z(T)$ are two independent (doubly stochastic) Poisson Process), we have,

$$\begin{aligned} B_d^0(t, T) &= E^{\mathbb{Q}}\left[\frac{S(t)}{S(T)} 1_{\{\tau_A > T\}} L(T) | \mathcal{G}_t\right] \\ &= E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left[\frac{S(t)}{S(T)} 1_{\{\tau_A > T\}} e^{\int_t^T \bar{h}_2^{M \setminus A}(s) \lambda(s) ds} Z(T) | \mathcal{F}_T^W \vee \mathcal{F}_t^N\right] | \mathcal{G}_t\right] \\ &= E^{\mathbb{Q}}\left[\frac{S(t)}{S(T)} 1_{\{\tau_A > T\}} e^{\int_t^T \bar{h}_2^{M \setminus A}(s) \lambda(s) ds} E^{\mathbb{Q}}[Z(T) | \mathcal{F}_T^W \vee \mathcal{F}_t^N] | \mathcal{G}_t\right] \\ &= E^{\mathbb{Q}}\left[1_{\{\tau_A > T\}} e^{\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + r(s)] ds} E^{\mathbb{Q}}[Z(T) | \mathcal{F}_T^W \vee \mathcal{F}_t^N] | \mathcal{G}_t\right] \\ &= E^{\mathbb{Q}}\left[1_{\{\tau_A > T\}} e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + r(s)] ds} Z(t) | \mathcal{G}_t\right] \\ &= Z(t) E^{\mathbb{Q}}\left[1_{\{\tau_A > T\}} e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + r(s)] ds} | \mathcal{G}_t\right] \end{aligned}$$

By applying lemma 2.3.10, we obtain

$$\begin{aligned}
& E^{\mathbb{Q}}[1_{\{\tau_A > T\}} e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + r(s)] ds} \mid \mathcal{G}_t] \\
&= E^{\mathbb{Q}}[1_{\{\tau_A > t\}} 1_{\{\tau_A > T\}} e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + r(s)] ds} \mid \mathcal{G}_t] \\
&= 1_{\{\tau_A > t\}} \frac{E^{\mathbb{Q}}[1_{\{\tau_A > T\}} e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W]}{\mathbb{Q}\{\tau_A > t \mid \mathcal{F}_t^W\}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
B_d^0(t, T) &= E^{\mathbb{Q}}\left[\frac{S(t)}{S(T)} 1_{\{\tau_A > T\}} L(T) \mid \mathcal{G}_t\right] \\
&= 1_{\{\tau_A > t\}} Z(t) \frac{E^{\mathbb{Q}}[1_{\{\tau_A > T\}} e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W]}{\mathbb{Q}\{\tau_A > t \mid \mathcal{F}_t^W\}} \\
&= 1_{\{\tau_A > t\}} L(t) E^{\mathbb{Q}}[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W]
\end{aligned}$$

■

Remarks

- If there are no liquidation risks, that is, $\tau_A = \infty$, or $A = \emptyset$, the original pricing model is reduced to

$$B_d^0(t, T) = L(t) E^{\mathbb{Q}}[e^{-\int_t^T [-\bar{h}_2(s) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W],$$

And respectively, the pre-default price $V_T^0(t)$ is given as:

$$V_T^0(t) = E^{\mathbb{Q}}[e^{-\int_t^T [-\bar{h}_2(s) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W]$$

This is exactly the same results as Schönbucher's multiple defaults model²⁵.

In this case, $\bar{h}_2(t) = \int_M h_2(t, q) \phi(dq)$ is the expected jump magnitude at each default time, and regarding this result, in fact we also prove that the multiple defaults model coincides with the recovery of market value model in Duffie and Singleton (1997, 1999). Both recovery models share a similar structure of the modified time t discount rate:

$$\hat{r}(t) = -\bar{h}_2(t)\lambda(t) + r(t)$$

- Taking into account the liquidation risk, the modified discount rate now should include two parts in the original default free interest rate, i.e. the loss rate caused by the default event with liquidation effect $\phi(A)\lambda(t)$, and the loss rate caused by the default with non-liquidation effect $-\bar{h}_2^{M \setminus A}(t)\lambda(t)$. As a result, the discount rate is modified to

$$\tilde{r}^0(t) = -\bar{h}_2^{M \setminus A}(t)\lambda(t) + \phi(A)\lambda(t) + r(t)$$

- This setting implies that the defaultable forward rate is continuous at each default time.

Definition 2.4.6 Defaultable Bond Price under Recovery of Par Value

$$B_d^{PV}(t, T)$$

Under the recovery scheme, when the obligor goes bankrupt and enters the liquidation process, bond holders are compensated with an immediate bullet payment at the

²⁵ See Schönbucher (1998) for the details of the multiple defaults models.

liquidation time τ_A , that is, a fixed quota of the face value of the bond, or

$1 + h_1(\tau_A) = \delta(\tau_A)$, where $\delta(t)$ is a function of time with positive value.

Proposition 2.4.7 Pricing $B_d^{PV}(t, T)$

Under the recovery of Par value scheme, the time t defaultable bond price $B_d^{PV}(t, T)$

is:

$$B_d^{PV}(t, T) = 1_{\{\tau_A > t\}} L(t) E^{\mathbb{Q}} \left[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W \right] \\ + 1_{\{\tau_A \leq t\}} L(t) E^{\mathbb{Q}} \left[\delta \int_t^T \phi(A) \lambda(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)) ds} du \mid \mathcal{F}_t^W \right] \quad (2.4.4)$$

And the pre-default market price is given as:

$$V_T^{PV}(t) = E^{\mathbb{Q}} \left[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W \right] \\ + E^{\mathbb{Q}} \left[\delta \int_t^T \phi(A) \lambda(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)) ds} du \mid \mathcal{F}_t^W \right] \quad (2.4.5)$$

Proof of Proposition 2.4.7

Since at the liquidation time τ_A , the face value of the defaultable bond is $L(\tau_A)$, and

consequently the recovery is $\delta(\tau_A)L(\tau_A)$. By definition, the price formula is now

modified as:

$$B_d^{PV}(t, T) = E^{\mathbb{Q}} \left[\frac{S(t)}{S(\tau_A)} \delta(\tau_A) L(\tau_A) 1_{\{t < \tau_A < T\}} + \frac{S(t)}{S(T)} 1_{\{\tau_A > T\}} L(T) \mid \mathcal{G}_t \right]$$

From Lemma 2.3.10, we obtain,

$$E^{\mathbb{Q}}\left[\frac{S(t)}{S(T)}1_{\{\tau_A>T\}}L(T)|\mathcal{G}_t\right] = 1_{\{\tau_A>t\}}L(t)E^{\mathbb{Q}}\left[e^{-\int_t^T[-\bar{h}_2^{M\setminus A}(s)\lambda(s)+\phi(A)\lambda(s)+r(s)]ds}\middle|\mathcal{F}_t^W\right].$$

And,

$$E^{\mathbb{Q}}\left[\frac{S(t)}{S(\tau_A)}\delta(\tau_A)L(\tau_A)1_{\{t<\tau_A<T\}}|\mathcal{G}_t\right]$$

$$=1_{\{\tau_A>t\}}E^{\mathbb{Q}}\left[\int_t^T\phi(A)\lambda(u)e^{-\int_t^u(\phi(A)\lambda(s)+r(s))ds}\delta(u)L(u)du|\mathcal{F}_t^W\right]$$

$$=1_{\{\tau_A>t\}}E^{\mathbb{Q}}\left[\int_t^T\phi(A)\lambda(u)e^{-\int_t^u(-\bar{h}_2^{M\setminus A}(s)\lambda(s)+\phi(A)\lambda(s)+r(s))ds}\delta(u)Z(u)du|\mathcal{F}_t^W\right]$$

$$=1_{\{\tau_A>t\}}E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left[\int_t^T\phi(A)\lambda(u)e^{-\int_t^u(-\bar{h}_2^{M\setminus A}(s)\lambda(s)+\phi(A)\lambda(s)+r(s))ds}\delta(u)Z(u)du|\mathcal{F}_T^W\vee\mathcal{F}_t^N\right]|\mathcal{F}_t^W\right]$$

$$=1_{\{\tau_A>t\}}E^{\mathbb{Q}}\left[\int_t^T\phi(A)\lambda(u)e^{-\int_t^u(-\bar{h}_2^{M\setminus A}(s)\lambda(s)+\phi(A)\lambda(s)+r(s))ds}\delta(u)E^{\mathbb{Q}}[Z(u)|\mathcal{F}_T^W\vee\mathcal{F}_t^N]du|\mathcal{F}_t^W\right]$$

$$=1_{\{\tau_A>t\}}E^{\mathbb{Q}}\left[\int_t^T\phi(A)\lambda(u)e^{-\int_t^u(\bar{h}_2^{M\setminus A}(s)\lambda(s)+\phi(A)\lambda(s)+r(s))ds}\delta(u)Z(t)du|\mathcal{F}_t^W\right]$$

$$=1_{\{\tau_A>t\}}L(t)E^{\mathbb{Q}}\left[\int_t^T\phi(A)\lambda(u)\delta(u)e^{-\int_t^u(-\bar{h}_2^{M\setminus A}(s)\lambda(s)+\phi(A)\lambda(s)+r(s))ds}du|\mathcal{F}_t^W\right]$$

Therefore,

$$B_d^{PV}(t,T) = 1_{\{\tau_A>t\}}L(t)E^{\mathbb{Q}}\left[e^{-\int_t^T[-\bar{h}_2^{M\setminus A}(s)\lambda(s)+\phi(A)\lambda(s)+r(s)]ds}\middle|\mathcal{F}_t^W\right]$$

$$+1_{\{\tau_A>t\}}L(t)E^{\mathbb{Q}}\left[\int_t^T\phi(A)\lambda(u)\delta(u)e^{-\int_t^u(-\bar{h}_2^{M\setminus A}(s)\lambda(s)+\phi(A)\lambda(s)+r(s))ds}du|\mathcal{F}_t^W\right] \quad \blacksquare$$

Again, $\bar{h}_2^{M \setminus A}(t)$ is a truncated expectation on the mark space at time t .

Remarks

- In the pricing formula above, the second term

$E^{\mathbb{Q}}[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s)\lambda(s) + \phi(A)\lambda(s) + r(s)]ds} | \mathcal{F}_t^W]$ refers to the discounted final promised

payoff, if there is no liquidation occurring before the maturity. However, if we compare the second term with most results in other intensity models of the literature, we can find out that they are with similar structures.

- Roughly speaking, the expression $E^{\mathbb{Q}}[\int_t^T \phi(A)\lambda(u)du | \mathcal{F}_t^W]$ in first term can be interpreted as the risk neutral probability of default during time t to T , given the background information up to time t . And both terms are discounted by the modified defaultable rate:

$$\tilde{r}^0(t) = -\bar{h}_2^{M \setminus A}(t)\lambda(t) + \phi(A)\lambda(t) + r(t)$$

Definition 2.4.7 Defaultable Bond Price under Recovery of Treasury Value

$$B_d^{TV}(t, T)$$

In the fractional recovery of treasury value scheme, at time τ_A when the obligor goes bankrupt with liquidation, bond owners receive a recovery payment that is equal to a fixed fraction $\delta(\tau_A)$ of a market value of an equivalent default free bond (e.g. a government bond with the same future promised payment and maturity). This is not a standard market convention, but an instructive tool in the models of Duffie & Singleton (1999), Jarrow & Turnbull (1995)

Proposition 2.4.8 Pricing $B_d^{TV}(t, T)$

Under the recovery of treasury value scheme, the time t defaultable bond price

$B_d^{TV}(t, T)$ is given as:

$$B_d^{TV}(t, T) = 1_{\{\tau_A > t\}} e^{-\int_t^T r(s) ds} L(t) E^{\mathbb{Q}} \left[\int_t^T \phi(A) \lambda(u) \delta(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s)) ds} du \mid \mathcal{F}_t^W \right] \\ + 1_{\{\tau_A \leq t\}} L(t) E^{\mathbb{Q}} \left[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W \right]$$

And accordingly, the pre-default market price V_t^{TV} is given as

$$V_t^{TV}(t) = e^{-\int_t^T r(s) ds} E^{\mathbb{Q}} \left[\int_t^T \phi(A) \lambda(u) \delta(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s)) ds} du \mid \mathcal{F}_t^W \right] \\ + E^{\mathbb{Q}} \left[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W \right]$$

Proof of Proposition 2.4.8

By definition, $B_d^{TV}(t, T)$ can be deduced as following:

$$B_d^{TV}(t, T) = E^{\mathbb{Q}} [\delta(\tau_A) B(\tau_A, T) l(\tau_A) 1_{\{t < \tau_A < T\}} \frac{S(t)}{S(\tau_A)} + S(T)^{-1} L(T) 1_{\{\tau_A > T\}} \mid \mathcal{G}_t].$$

Applying a similar argument, we get:

$$E^{\mathbb{Q}} [\delta(\tau_A) B(\tau_A, T) l(\tau_A) 1_{\{t < \tau_A < T\}} \frac{S(t)}{S(\tau_A)} \mid \mathcal{G}_t] \\ = 1_{\{\tau_A > t\}} E^{\mathbb{Q}} \left[\int_t^T e^{-\int_t^u (\phi(A) \lambda(s) + r(s)) ds} B(u, T) \phi(A) \lambda(u) \delta(u) L(u) du \mid \mathcal{F}_t^W \right]$$

$$\begin{aligned}
&= 1_{\{\tau_A > t\}} E^{\mathbb{Q}} \left[\int_t^T e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)) ds} \phi(A) \lambda(u) e^{-\int_u^T r(s) ds} \delta(u) Z(u) du \mid \mathcal{F}_t^W \right] \\
&= 1_{\{\tau_A > t\}} e^{-\int_t^T r(s) ds} E^{\mathbb{Q}} [E^{\mathbb{Q}} \left[\int_t^T \phi(A) \lambda(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s)) ds} \delta(u) Z(u) du \mid \mathcal{F}_T^W \vee \mathcal{F}_t^N \right] \mid \mathcal{F}_t^W] \\
&= 1_{\{\tau_A > t\}} e^{-\int_t^T r(s) ds} E^{\mathbb{Q}} \left[\int_t^T \phi(A) \lambda(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s)) ds} \delta(u) E^{\mathbb{Q}} [Z(u) \mid \mathcal{F}_T^W \vee \mathcal{F}_t^N] du \mid \mathcal{F}_t^W \right] \\
&= 1_{\{\tau_A > t\}} e^{-\int_t^T r(s) ds} E^{\mathbb{Q}} \left[\int_t^T \phi(A) \lambda(u) e^{-\int_t^u (\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s)) ds} \delta(u) Z(u) du \mid \mathcal{F}_t^W \right] \\
&= 1_{\{\tau_A > t\}} e^{-\int_t^T r(s) ds} L(t) E^{\mathbb{Q}} \left[\int_t^T \phi(A) \lambda(u) \delta(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s)) ds} du \mid \mathcal{F}_t^W \right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
B_d^{TV}(t, T) &= 1_{\{\tau_A > t\}} e^{-\int_t^T r(s) ds} L(t) E^{\mathbb{Q}} \left[\int_t^T \phi(A) \lambda(u) \delta(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s)) ds} du \mid \mathcal{F}_t^W \right] \\
&+ 1_{\{\tau_A > t\}} L(t) E^{\mathbb{Q}} \left[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W \right]
\end{aligned}$$

■

Remark

If the default event is modeled as the first jump of a point process as in most cases in the literature, i.e. $A = M$, and the recovery rate δ is constant, the pricing formula for V_t^{TV} is reduced to:

$$\begin{aligned}
V_T^{TV}(t) &= E^{\mathbb{Q}} \left[e^{-\int_t^T r(s) ds} \int_t^T \lambda(u) \delta e^{-\int_t^u \lambda(s) ds} du + e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t^W \right] \\
&= E^{\mathbb{Q}} \left[\delta e^{-\int_t^T r(s) ds} (1 - e^{-\int_t^T \lambda(s) ds}) + e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t^W \right]
\end{aligned}$$

$$\begin{aligned}
&= E^{\mathbb{Q}} \left[\delta e^{-\int_t^T r(s)ds} + (1-\delta) e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t^W \right] \\
&= \delta V_t^0 + (1-\delta) B(t, T)
\end{aligned}$$

That is, the pre-default bond price $V_T^{TV}(t)$ can be express as combination of a pre-default bond price with zero recovery and an equivalent default free bond price.²⁶

This is well explained in economics. Under the recovery scheme, the proportion δ of the market value of this defaultable bond is always guaranteed no matter whether the bond defaulted or not, and then the other quota $(1-\delta)$ of the value is exactly $V_T^0(t)$, which is totally exposed to the default risk without any recovery.

However, it is not true in our case. In fact we can easily prove that under the setting of the marked Poisson framework,

$$V_T^{TV}(t) < (1-\delta)V_T^0(t) + \delta B(t, T) \quad \text{a.s.}$$

This is because in our setting, all default free bond prices follow continuous processes. Therefore, there are no “equivalent” default free counterparts to defaultable bonds.

Definition 2.4.9 Defaultable Bond Price under Recovery of Market Value

$$B_d^{MV}(t, T)$$

Recovery of market value is a most popular market practice convention. The idea of recovery of market value is inspired by the recovery rules of OTC derivatives. The ISDA master agreement for swap contracts specifies that, on the default of one

²⁶ Refer to Bielecki and Rutkowski (2002) for a rigorous mathematical proof or Schönbucher (2003) for a detailed explanation.

counterparty, the other counterparty's claim is the market value of a “non-defaulted, but otherwise equivalent security”²⁷. These kinds of recovery models are first proposed in Duffie and Singleton's models (1997, 1999). In their models, the recovery process $h_2(\tau_A)$ is endogenously specified and is expressed as some non-anticipating functional form of the defaultable bond value process, i.e.

$$h_2(t) = f(B^d(t-, T), t).$$

Then the price of this defaultable zero coupon bond will be:

$$B_d^{MV}(t, T) = S(t)E^{\mathbb{Q}}[1_{\{t < \tau_A < T\}}S(\tau_A)^{-1}f(B^d(\tau_A-, T), \tau_A) + S(T)^{-1}L(T)1_{\{\tau_A > T\}} | \mathcal{G}_t]$$

If the recovery payoff function $f(B^d(\tau_A-, T), \tau_A)$ is not a linear function of the market value of the defaultable bond, usually we cannot get an explicit solution to this pricing formula.

However, if we assume the recovery payoff can be expressed as a linear function of the market value of the defaultable bond, that is,

$$h_1(t) = l(t)B^d(t-, T), \quad -1 \leq l(t) \leq 0, \text{ where } l(t) \text{ is } \mathcal{F}_t^W \text{ adapted process that can be}$$

consider as an exogenous loss rate measurable function, a similar term structure of defaultable interest rate compared to the default free counterpart can be achieved, except in this case the discounted factor will be the default free rate plus the expected losses $R = r - l(t)\lambda(t)$ ²⁸.

Now we would like to apply this result to the marked Poisson case.

²⁷ Refre to Schönbucher (2003) for more details

²⁸ Refer to Duffie & Singleton (1999) for more details.

Proposition 2.4.10 Pricing $B_d^{MV}(t, T)$

Under the recovery of market value scheme, the time t defaultable bond price

$B_d^{MV}(t, T)$ is:

$$B_d^{MV}(t, T) = 1_{\{\tau_A > t\}} L(t) V_T^{MV}(t) = 1_{\{\tau_A > t\}} L(t) E^{\mathbb{Q}} \left[e^{\int_t^T [-\tilde{h}_2^{M \setminus A}(u) \lambda(u) - \phi(A) l(u) \lambda(u) + r(u)] du} \mid \mathcal{F}_t^W \right] \quad (2.4.6)$$

And the pre-default market price $V_T^{TV}(t)$ is:

$$V_T^{MV}(t) = E^{\mathbb{Q}} \left[e^{\int_t^T [-\tilde{h}_2^{M \setminus A}(u) \lambda(u) - \phi(A) l(u) \lambda(u) + r(u)] du} \mid \mathcal{F}_t^W \right] \quad (2.4.7)$$

Proof of Proposition 2.4.10

Define the recovery rate at time t as: $\delta(t) = 1 + l(t)$

By definition the defaultable bond price under the recovery of market value scheme is:

$$B_d^{MV}(t, T) = E^{\mathbb{Q}} \left[1_{\{t < \tau_A < T\}} \frac{S(t)}{S(\tau_A)} \delta(\tau_A) B_d^{MV}(\tau_A^-, T) + \frac{S(t)}{S(T)} L(T) 1_{\{\tau_A > T\}} \mid \mathcal{G}_t \right]$$

By definition of the pre-default market value V_t^{MV} , we get:

$$\begin{aligned} & E^{\mathbb{Q}} \left[1_{\{t < \tau_A < T\}} \frac{S(t)}{S(\tau_A)} \delta(\tau_A) B_d^{MV}(\tau_A^-, T) \mid \mathcal{G}_t \right] \\ &= E^{\mathbb{Q}} \left[1_{\{t < \tau_A < T\}} \frac{S(t)}{S(\tau_A)} \delta(\tau_A) 1_{\{\tau_A > t\}} L(\tau_A^-) V_T^{MV}(\tau_A^-) \mid \mathcal{G}_t \right] \end{aligned}$$

Notice that the pre-default market price V_t^{MV} is continuous at default times, i.e.

$$V_T^{MV}(t-) = V_T^{MV}(t).$$

Only countably many jumps happen in the process $L(t)$ for any finite time interval $[0, t]$, therefore we can replace $L(t-)$ with $L(t)$ in all the equations above to simplify the calculation²⁹.

Consequently,

$$\begin{aligned} & E^{\mathbb{Q}}[1_{\{t < \tau_A < T\}} \frac{S(t)}{S(\tau_A)} \delta(\tau_A) B_d^{MV}(\tau_A-, T) | \mathcal{G}_t] \\ &= E^{\mathbb{Q}}[1_{\{t < \tau_A < T\}} \frac{S(t)}{S(\tau_A)} \delta(\tau_A) 1_{\{\tau_A > t\}} L(\tau_A) V_T^{MV}(\tau_A) | \mathcal{G}_t] \\ &= 1_{\{\tau_A > t\}} E^{\mathbb{Q}}[\int_t^T \phi(A) \lambda(u) e^{-\int_t^u (\phi(A) \lambda(s) + r(s)) ds} \delta(u) V_u^{MV} L(u) du | \mathcal{F}_t^W] \\ &= 1_{\{\tau_A > t\}} E^{\mathbb{Q}}[E^{\mathbb{Q}}[\int_t^T \phi(A) \lambda(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)) ds} \delta(u) V_T^{MV}(u) Z(u) du | \mathcal{F}_t^W \vee \mathcal{F}_t^N] | \mathcal{F}_t^W] \\ &= 1_{\{\tau_A > t\}} E^{\mathbb{Q}}[\int_t^T \phi(A) \lambda(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)) ds} \delta(u) V_T^{MV}(u) E^{\mathbb{Q}}[Z(u) | \mathcal{F}_t^W \vee \mathcal{F}_t^N] du | \mathcal{F}_t^W] \\ &= 1_{\{\tau_A > t\}} E^{\mathbb{Q}}[\int_t^T \phi(A) \lambda(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)) ds} \delta(u) V_T^{MV}(u) Z(t) du | \mathcal{F}_t^W] \\ &= 1_{\{\tau_A > t\}} L(t) E^{\mathbb{Q}}[\int_t^T \phi(A) \lambda(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)) ds} \delta(u) V_T^{MV}(u) du | \mathcal{F}_t^W] \end{aligned}$$

We hence obtain:

²⁹ This is because the Lebesgue measure on the summation of those jump time points is zero.

$$B_d^{MV}(t, T) = 1_{\{\tau_A > t\}} L(t) E^{\mathbb{Q}} \left[\int_t^T \phi(A) \lambda(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)) ds} \delta(u) V_T^{MV}(u) du \mid \mathcal{F}_t^W \right]$$

$$+ 1_{\{\tau_A > t\}} L(t) E^{\mathbb{Q}} \left[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W \right]$$

and respectively,

$$V_T^{MV}(t) = E^{\mathbb{Q}} \left[\int_t^T \phi(A) \lambda(u) \delta(u) V_T^{MV}(u) e^{-\int_t^u (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)) ds} d \mid \mathcal{F}_t^W \right]$$

$$+ E^{\mathbb{Q}} \left[e^{-\int_t^T [-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \mid \mathcal{F}_t^W \right]$$

Define: $\tilde{S}(t) = e^{\int_0^t (-\bar{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)) ds}$,

and $M(t) = E^{\mathbb{Q}} \left[\int_0^T \phi(A) \lambda(u) \delta(u) V_T^{MV}(u) \tilde{S}(u)^{-1} du + \tilde{S}(T)^{-1} \mid \mathcal{F}_t^W \right]$.

Then $M(t)$ is a $\mathbb{Q}$ -martingale with respect to the filtration $\mathcal{F}^W$, since it is the expectation of an $\mathcal{F}_T^W$ -measurable random variable conditioned on the background information up to time t . The expression of V_t^{MV} can then be transformed to:

$$V_T^{MV}(t) = \tilde{S}(t)(M(t) - \int_0^t \phi(A) \lambda(u) \tilde{S}(u)^{-1} \delta(u) V_T^{MV}(u) du)$$

Because all the processes in the equation above are adapted to the continuous filtration $\mathcal{F}_t^W$ generated by the multi-dimensional Wiener process, Ito's formula implies³⁰:

³⁰ Refer to Øksendal (2005) or Protter (1990) for a version of Ito's formula that applies in general semi-martingale cases.

$$dV_T^{MV}(t) = (-\bar{h}_2^{M \setminus A}(t)\lambda(t) + \phi(A)\lambda(t) + r(t))\tilde{S}(t)(M(t) - \int_0^t \phi(A)\lambda(u)\tilde{S}(u)^{-1}\delta(u)V_T^{MV}(u)du)dt \\ + \tilde{S}(t)(dM(t) - \phi(A)\lambda(t)\tilde{S}(t)^{-1}\delta(t)V_T^{MV}(t)dt)$$

A simplification of the above equation yields:

$$dV_T^{MV}(t) = (-\bar{h}_2^{M \setminus A}(t)\lambda(t) + \phi(A)\lambda(t) + r(t))V_T^{MV}(t)dt + \tilde{S}(t)dM(t) - \phi(A)\lambda(t)\delta(t)V_T^{MV}(t)dt \\ = [-\bar{h}_2^{M \setminus A}(t)\lambda(t) + \phi(A)(1 - \delta(t))\lambda(t) + r(t)]V_T^{MV}(t)dt + \tilde{S}(t)dM(t) \\ = [-\bar{h}_2^{M \setminus A}(t)\lambda(t) - \phi(A)\lambda(t)l(t) + r(t)]V_T^{MV}(t)dt + \tilde{S}(t)dM(t)$$

Define the defaultable time t savings account under market value recovery as:

$$\tilde{S}^{MV}(t) = e^{\int_0^t [-\bar{h}_2^{M \setminus A}(u)\lambda(u) + (1 - \delta(u))\phi(A)\lambda(u) + r(u)]du}$$

Accordingly the modified defaultable discount rate is:

$$\tilde{r}^{MV}(t) = -\bar{h}_2^{M \setminus A}(t)\lambda(t) - \phi(A)\lambda(t)l(t) + r(t)$$

We have:

$$dV_T^{MV}(t) = \tilde{r}^{MV}(t)V_T^{MV}(t)dt + \tilde{S}(t)dM(t)$$

$$dV_T^{MV}(t) - \tilde{r}^{MV}(t)V_T^{MV}(t)dt = \tilde{S}(t)dM(t)$$

Multiply both side by $\tilde{S}^{MV}(t)^{-1}$,

$$\tilde{S}^{MV}(t)^{-1}dV_T^{MV}(t) + V_T^{MV}(t)d\tilde{S}^{MV}(t)^{-1} = \tilde{S}^{MV}(t)^{-1}\tilde{S}(t)dM(t)$$

$$d(\tilde{S}^{MV}(t)^{-1}V_T^{MV}(t)) = \tilde{S}^{MV}(t)^{-1}\tilde{S}(t)dM(t)$$

By the martingale representation theorem, the process $\tilde{S}^{MV}(t)^{-1}V_T^{MV}(t)$ is a $\mathbb{Q}$ -

martingale with respect to the filtration $\mathcal{F}_t^W$, and we obtain:

$$\tilde{S}^{MV}(t)^{-1}V_T^{MV}(t) = E^{\mathbb{Q}}[\tilde{S}^{MV}(t)^{-1}V_T^{MV}(T) | \mathcal{F}_t^W]$$

$$V_T^{MV}(t) = \tilde{S}^{MV}(t)E^{\mathbb{Q}}[\tilde{S}^{MV}(T)^{-1}V_T^{MV}(T) | \mathcal{F}_t^W]$$

With the boundary condition $V_T^{MV}(T) = 1$, that is, given no default at maturity T , the payoff for the defaultable bond is 1.

We have

$$V_T^{MV}(t) = E^{\mathbb{Q}}[e^{-\int_t^T [(-\tilde{h}_2^{M \setminus A}(u)\lambda(u) - \phi(A)\lambda(u)l(u) + r(t))]du} | \mathcal{F}_t^W]$$

Therefore by definition:

$$B_d^{MV}(t, T) = 1_{\{\tau_A > t\}} L(t) V_T^{MV}(t) = 1_{\{\tau_A > t\}} L(t) E^{\mathbb{Q}}[e^{-\int_t^T [(-\tilde{h}_2^{M \setminus A}(u)\lambda(u) - \phi(A)\lambda(u)l(u) + r(t))]du} | \mathcal{F}_t^W] \blacksquare$$

Remark

- Mathematically speaking, the recovery models mentioned above just differ in the measure of the remaining value of the defaulted claims: defaultable bonds holders are compensated with defaultable bonds, cash, and default-free bonds respectively after the default event lead to a liquidation process. Within limits, these models can be transformed into each other theoretically, if different

recovery rates are applied. In this sense, these recovery models are mathematically equivalent.

- the results above can easily be extended to the defaultable coupon bonds.

Suppose a defaultable bond $B_d^c(t, T)$ has coupon payments c_i , $i = 1, 2, \dots, n$ at a sequence of future dates $0 < T_1, T_2, \dots, T_n \leq T$, which are known in advance.

Since by convention, the coupons receive no recovery on default, the price of this defaultable coupon bond can be modeled as:

$$B_d^c(t, T) = \sum_{i=1}^n c_i B_d^0(t, T_i) + B_d(t, T),$$

in which, $\sum_{i=1}^n c_i B_d^0(t, T_i)$ can be considered as a summation of a sequence of

zero coupon bonds with zero recovery and face value c_i , $i = 1, 2, \dots, n$. And

$B_d(t, T)$ is the price of zero coupon with a specified recovery.

- We find that the results of RMV share a similar structure in expression with the results of zero recovery setting. This similarity is justified in the arguments of Lando (1998). Receiving a fraction $1 + l(t)$ of pre-default value in the event of default of a contract is equivalent, from a pricing perspective, to receiving the outcome of a lottery in which the full pre-default value is received with probability $1 + l(t)$ and 0 is received with probability $-l(t)$, i.e. the event of default has been retained with probability $-l(t)$. This in turn may be viewed as a default process in which there is 0 recovery but where the default intensity has been thinned using the process $-l(t)$, producing a new default intensity of $-l(t)\lambda(t)$.

Chapter 3

The Dynamics of the Defaultable Rates

3.1 Introduction

In literature and market practice about fixed income, there are two popular approaches that can proceed to study interest rate dynamics and price interest-sensitive securities. The first one is the so called “classical approach”, such as models of Ho and Lee (1986), Vasicek (1977) and Cox, Ingersoll and Ross (1985). In those models, by using the arbitrage free arguments between default free bond prices with different maturities and the spot interest rate under the risk neutral measure $\mathbb{Q}$, i.e.,

$$B(t, T_i) = E^{\mathbb{Q}}[e^{\int_t^{T_i} r(s)ds} | \mathcal{F}_t^W] \quad (3.1.1)$$

a risk adjusted model can be extracted for the spot rate. This process involves modeling the drift term of the spot rate dynamics, as well as calibrating observed volatilities. In this approach, the Markovian property of the spot rate is always assumed.³¹

On the other hand, in the Heath, Jarrow and Morton (1992) framework (HJM framework), another arbitrage free argument between instantaneous forward rate and market prices of default free bonds is used, i.e., $B(t, T_i) = e^{\int_t^{T_i} f(t,s)ds}$, and then arbitrage free dynamics of k -dimensional forward rates $f(t, T_i)$ are obtained. The drift term of

³¹ Refer to Neftci (2000) for a detailed introduction.

the process automatically fits to market data once the volatility functions and market prices of risk are calibrated. It is similar to the Black-Scholes' environment where there is no need to model the expected rate on the underlying stock but only modeling or calibrating the volatility is needed. The HJM framework is considered as more flexible and general than the classical approach, because it is automatically calibrated to the currently observed yield curves.

Various finite dimensional Markovian HJM models under different forward rate volatility restrictions are studied in Ritchken and Sankarasubramanian (1995), Bhar and Chiarella (1997) and Inui and Kijima (1998), which are developed under diffusion processes. Chiarella and Nikitopoulos (2003) extend the HJM framework by including a jump-diffusion process.

The HJM framework was extended to include the analysis of credit risk in Duffie & Singleton (1999) and Schönbucher (2000)

Duffie & Singleton (1999) first introduce the HJM methodology to the analysis of defaultable interest rates. They provide a discrete-time reduced-form model to evaluate defaultable bonds and other defaultable financial claims by arbitrage-free arguments. A forward spread process is added to the forward risk-free rate process to obtain the arbitrage-free drift restriction within the HJM Framework. By using the "recovery of market value" condition, they provide a recursive formula that is easy to implement.

Schönbucher (1998) systematically studies different forms of non-arbitrage condition between default free and defaultable term structures. In addition, he showed that the forward rate credit spread offers the link between the defaultable and default free term

structures. A similar result is obtained in Pugachevsky (1999), or Maksymiuk and Gatarek (1999). In these models, HJM drift restriction is extracted by applying the arbitrage free condition obtained and without assuming any jumps to default. They also show that the risky forward rate is in fact the sum of the risk-free forward rate and the credit spread.

While Chiarella, Maina and Nikitopolous (2007) incorporate stochastic volatility structure into the generalized framework of Schönbucher (2000), Chiarella, Nikitopoulos and Schlögl (2007) generalize the results of Schönbucher (2000, 2003), and under multiple default assumption, their study specifies a level and time dependent volatility structure of forward rate, which may result in a Markovian dynamics of the spot rate.

3.2 Default Free Heath-Jarrow-Morton Framework

In this section, we briefly review the Heath, Jarrow and Morton (HJM) Framework. The section may be regarded as a simple introduction to the modeling philosophy and our main goal is to emphasize the crucial steps which will recur in the later discussion of defaultable models. The HJM approach is considered as a very general framework within which we can develop more specific arbitrage free models, where the current forward rate curves are parts of the input. Particularly, if we are given sufficient forward rates with different maturities, usually we can specify the dynamics of the forward rate structure for all maturities T , if they are driven by countable randomness.

Definition 3.2.1: Default Free Bond Price and Default Free Forward Rate

- We assume that there always exists a market that allows continuous trading of default free zero coupon bonds with any maturities. For simplicity, the promised final payoff of default free bond at the maturity T is \$1. At time t , the market price of the default free bond with maturity T is denoted by $B(t, T)$. Also we assume that $B(t, T)$ is continuously differentiable with respect to t and T , $\mathbb{Q}$ -a.s. And since we postulate that $B(t, T)$ is adapted to the background information filtration $\mathcal{F}_t^W$, which is generated by multi-dimension Wiener process, $B(t, T)$ is also a continuous process almost surely.
- We define the continuously compounded instantaneous default free forward rate at time t for the date $T > t$ as

$$f(t, T) = -\frac{\partial}{\partial T} \ln B(t, T) \quad (3.2.1)$$

- At the same time, the instantaneous default free short rate $r(t)$ is defined as:

$$r(t) = f(t, t) \quad (3.2.2)$$

Assumption 3.2.2: Technical Conditions for the HJM Framework

- The model is assumed to be set up on the filtered probability space $(\Omega, \mathbb{Q}, \mathcal{F}^W, \{\mathcal{F}_t^W\}_{0 \leq t < \infty})$ that is defined in the previous chapter³²
- We take the initial default free forward rate structure $f(0, T)$ as given for any maturity T . And we assume given a fixed maturity T , the dynamics of $f(t, T)$ are diffusion processes satisfying:

$$df(t, T) = \alpha(t, T)dt + \beta^{Tr}(t, T)dW(t) = \alpha(t, T)dt + \sum_{i=1}^n \beta_i(t, T)dW_i(t) \quad (3.2.3)$$

or in integral form:

$$\begin{aligned} f(t, T) &= f(0, T) + \int_0^t \alpha(u, T)du + \int_0^t \beta^{Tr}(u, T)dW(u) \\ &= f(0, T) + \int_0^t \alpha(u, T)du + \sum_{i=1}^n \int_0^t \beta_i(u, T)dW_i(u) \end{aligned}$$

where $\beta^{Tr}(u, T)$ is a transpose vector of $\beta(u, T)$ and $\beta_i(u, T)$ is the i -th component of the column vector $\beta(u, T)$.

- For all maturities T , $\alpha(t, T)$ and $\beta(t, T)$ are $\mathcal{F}_t^W$ - adapted processes

³² The related filtration $\mathcal{G}$ is reduced to $\mathcal{F}^W$, since we assume that no jump process is involved in the default free case.

- $\int_0^T |\alpha(t, T)| dt < \infty$ and $\int_0^T \beta^{Tr}(t, T) \beta(t, T) dt < \infty$ almost surely.

- $\int_0^T \int_0^u |\alpha(t, u)| dt du < \infty$ almost surely.

- The initial forward rate structure is deterministic and satisfies

$$\int_0^T |f(0, u)| du < \infty.$$

- Also we assume that the sufficient condition of Fubini's Theorem is satisfied, i.e.,

$$E^{\mathbb{Q}}[\int_0^T \int_0^u |\beta(s, u)| dW(s) du] < \infty$$

Theorem 3.2.3: Arbitrage Free Drift Restriction under the HJM Framework

If there are no arbitrage opportunities and if the dynamics of the forward rates under the spot martingale measure $\mathbb{Q}$ are given as in assumption 2.3.1, then

$$\alpha(t, T) = \beta^{Tr}(t, T) \int_t^T \beta(t, u) du = \sum_{i=1}^n \beta_i(t, T) \int_t^T \beta_i(t, u) du$$

Or equivalently, under the real world probability measure $\mathbb{P}$, the arbitrage free drift

$$\text{restriction is: } \alpha(t, s) = \sum_{i=1}^n \beta(t, T)_i \int_t^s \beta_i(t, u) du = \sum_{i=1}^n \beta(t, T)_i \eta_i$$

Where η_i is the market price of risk in the sense of Black-Scholes's framework.

Proof

By assumption 3.2.2, and definition 3.2.1, we obtain:

$$r(t) = f(t, t) = f(0, t) + \int_0^t \alpha(u, t) du + \int_0^t \beta^{Tr}(u, t) dW(u)$$

Define $I(t) = -\int_t^T f(t, u) du$ and by applying Ito's formula, we obtain

$$dI(t) = f(t, t)dt - \int_t^T df(t, u)du = r(t)dt - \int_t^T [\alpha(t, u)dt + \sum_{i=1}^n \beta_i(t, T)dW_i(t)]du$$

Now we define: $\bar{\alpha}(t, T) = \int_t^T \alpha(t, u)du$,

$$\text{and } \bar{\beta}^{Tr}(t, u) = \int_t^T \beta^{Tr}(t, u)du = \sum_{i=1}^n \int_t^T \beta_i(t, u)du$$

Then under Assumption 3.2.3, the interchange of the order of stochastic differential and Riemann integration is justified by Stochastic Fubini's Theorem. Hence we obtain

$$\begin{aligned} dI(t) &= r(t)dt - \int_t^T [\alpha(t, u)dt + \sum_{i=1}^n \beta_i(t, T)dW_i(t)]du \\ &= r(t)dt - (\int_t^T \alpha(t, u)du)dt + \sum_{i=1}^n (\int_t^T \beta_i(t, T)du)dW_i(t) \\ &= r(t)dt - \bar{\alpha}(t, T)dt - \bar{\beta}^{Tr}(t, u)dW(t) \end{aligned}$$

By definition 3.2.1, we have

$$f(t, T) = -\frac{\partial}{\partial T} \ln B(t, T), \text{ or } B(t, T) = e^{I(t)}$$

By applying Ito's formula, we obtain

$$\begin{aligned} dB(t, T) &= e^{I(t)}dI(t) + \frac{1}{2}e^{I(t)}(dI(t))^2 \\ dB(t, T) &= B(t, T)[r(t)dt - \bar{\alpha}(t, T)dt - \bar{\beta}^{Tr}(t, T)dW(t)] + \frac{1}{2}B(t, T)\bar{\beta}^{Tr}(t, T)\bar{\beta}(t, T)dt \end{aligned}$$

$$\frac{dB(t,T)}{B(t,T)} = [r(t) - \bar{\alpha}(t,T) + \frac{1}{2} \bar{\beta}^{Tr}(t,T) \bar{\beta}(t,T)] dt + \bar{\beta}^{Tr}(t,T) dW(t)$$

Under the spot martingale probability measure $\mathbb{Q}$, the arbitrage free condition

requires that the drift term of the dynamics $\frac{dB(t,T)}{B(t,T)}$ should be $r(t)$. Therefore we

have

$$r(t) - \bar{\alpha}(t,T) + \frac{1}{2} \bar{\beta}^{Tr}(t,T) \bar{\beta}(t,T) = r(t),$$

$$\text{or, } \bar{\alpha}(t,T) = \frac{1}{2} \bar{\beta}^{Tr}(t,T) \bar{\beta}(t,T).$$

By differentiating both sides with respect to T , we get the no-arbitrage drift restriction under HJM framework

$$\alpha(t,T) = \beta^{Tr}(t,T) \int_t^T \beta(t,u) du = \sum_{i=1}^n \beta_i(t,T) \int_t^T \beta_i(t,u) du$$

And by applying Girsanov's theorem, we also get the equivalent drift restriction under the real world probability measure $\mathbb{P}$ ³³

$$\alpha(t,s) = \sum_{i=1}^n \beta(t,T)_i \int_t^s \beta_i(t,u) du = \sum_{i=1}^n \beta_i(t,T) \eta_i$$

■

³³ What we provide here is a less rigorous proof for the HJM theorem. Refer to the original paper of Heath et al. (1992) for a complete mathematical proof.

3.3 Defaultable Heath-Jarrow-Morton Framework

Definition 3.3.1: Defaultable Bond Price and Defaultable Forward Rate

- The defaultable bond is defined as in definition 2.4.1
- We define the continuously compounded instantaneous defaultable forward rate at time t for the date $T > t$ as

$$f_d(t, T) = -\frac{\partial}{\partial T} \ln V_T(t) \quad (3.3.1)$$

where $V_T(t)$ is the pre-default bond price defined in definition 2.3.8

- As a result, the instantaneous defaultable short rate $r_d(t)$ can be define as following: $r_d(t) = f_d(t, t)$ (3.3.2)

Assumption 3.3.2:

- Definition 3.3.1 implies that the defaultable forward rate $f_d(t, T)$ is continuous at each default time³⁴. Therefore the defaultable forward rate dynamics are assumed to be driven only by a diffusion process:

$$f_d(t, T) = f_d(0, T) + \int_0^t \alpha_d(u, T) du + \int_0^t \beta_d^{Tr}(u, T) dW(u)$$

or in differential form,

$$df_d(t, T) = \alpha_d(t, T)dt + \beta_d^{Tr}(t, T)dW(t) = \alpha_d(t, T)dt + \sum_{i=1}^n \beta_{d,i}(t, T)_i dW_i(t) \quad (3.3.3)$$

³⁴ By this definition the forward rate $f_d(t, T)$ is unique up to liquidation time τ_A and it becomes arbitrary after τ_A , i.e. after the defaultable price process has been terminated.

where $\beta_d^{Tr}(u, T)$ is a transpose matrix of $\beta_d(u, T)$ and $\beta_{d,i}(u, T)$ is the i -th component of $\beta_d(u, T)$.

Again similar regularities are imposed as in the previous sub-section:

- For all maturities T , $\alpha(t, T)$ and $\beta_d(t, T)$ are $\mathcal{F}_t^W$ - adapted processes.
- $\int_0^T |\alpha(t, T)| dt < \infty$ and $\int_0^T \beta_d^{Tr}(t, T) \beta_d(t, T) dt < \infty$ almost surely.
- $\int_0^T \int_0^u |\alpha(t, u)| dt du < \infty$ almost surely.
- The initial defaultable forward rate structure is deterministic and satisfies

$$\int_0^T |f_d(0, u)| du < \infty.$$

- Also we assume that the sufficient condition of Fubini's Theorem is satisfied,

$$\text{i.e., } E^{\mathbb{Q}}[\int_0^T \int_0^u |\beta_d(s, u)| dW(s) du] < \infty.$$

Proposition 3.3.3 The Dynamics of the Defaultable Bond Price within the HJM Framework

Under the zero recovery scheme, the price of a defaultable zero coupon bond that is specified in Definition 2.4, $B_d^0(t, T)$ will follow the dynamics:

$$\begin{aligned} \frac{dB_d^0(t, T)}{B_d^0(t-, T)} &= (r_d(t) - \int_t^T \alpha_d(t, u) du + \frac{1}{2} \sum_{i=1}^n (\int_t^T \beta_i(t, u) du)^2) dt \\ &+ \sum_{i=1}^n (\int_t^T \beta_{d,i}(t, T) du) dW_i(t) - dN_1(t) + \int_{M \setminus A} h_2(t, q) \Lambda(dt, dq) \end{aligned} \quad (3.3.4)$$

We need the following generalized Ito formula to prove Proposition 3.3.3:

Lemma 3.3.4 Ito Product rule

Let X_t and Y_t be two $\{\mathcal{G}_t\}$ -Semi-martingales, assume that X_t is a finite variation process, and then we have: $d(X_t Y_t) = X_{t-} dY_t + Y_t dX_t$.

Proof of lemma 3.3.4

See Protter 1990 or Øksendal et al. (2005) .

Proof of proposition 3.3.3

By proposition 2.4.4, we have the following pricing formula for the defaultable bond with zero recovery $B_d^0(t, T)$:

$$\begin{aligned} B_d^0(t, T) &= 1_{\{\tau_A > t\}} L(t) V_T^0(t) \\ &= 1_{\{\tau_A > t\}} L(t) E^{\mathbb{Q}} \left[e^{\int_t^T [(-\tilde{h}_2^{M \setminus A}(u) + \phi(A) + \phi(A)\delta)\lambda(u) + r(t)] du} \mid \mathcal{F}_t^W \right] \end{aligned}$$

Notice that the liquidation time τ_A is in fact the first arrival time of the (doubly stochastic) Poisson process $N_1(t)$ with intensity process $\phi(A)\lambda(t)$, and we define the liquidation default indicator process $I(t)$ generated by τ_A as:

$$I(t) = 1_{\{\tau_A \leq t\}} = 1_{\{N_1(t) > 0\}} .$$

By differentiating both sides, we obtain: $dI(t) = dN_1(t)$.

Applying lemma 2.4.5 to the $B_d^0(t, T)$ pricing formula and noticing that all Poisson processes are with finite variation, and $L(t)$ follows the dynamic:

$$\frac{dL(t)}{L(t-)} = \int_{M \setminus A} h_2(t, q) \Lambda(dt, dq)$$

we have:

$$B_d^0(t, T) = (-I(t)L(t)V_T^0(t))$$

$$dB_d^0(t, T) = d(-I(t)L(t)V_T^0(t))$$

$$= -I(t-)d(L(t)V_T^0(t)) - L(t-)V_T^0(t)d(\mathbb{L})$$

From the Ito product rule

$$d(L(t)V_T^0(t)) = V_T^0(t)d(\mathbb{L}) + L(t-)d(V_T^0(t))$$

Combining the above results, we have:

$$dB_d^0(t, T) = -L(t-)V_T^0(t)d(\mathbb{L}) - I(t-)V_T^0(t)d(\mathbb{L}) - I(t-)L(t-)d(V_T^0(t))$$

Rearranging the expression above, we have:

$$\frac{dB_d^0(t, T)}{B_d^0(t-, T)} = -dN_1(t) + \frac{dL(t)}{L(t-)} + \frac{dV_T^0(t)}{V_T^0(t)}$$

$$\frac{dB_d^0(t, T)}{B_d^0(t-, T)} = -dN_1(t) + \int_{M \setminus A} h_2(t, q) \Lambda(dt, dq) + \frac{dV_T^0(t)}{V_T^0(t)}$$

Since $V_T^0(t)$ is $\mathcal{F}_t^W$ -adapted, and from Definition 3.3.1, we have

$$V_T^0(t) = e^{-\int_t^T f_d(t, u) du}.$$

Again, by applying Ito formula and Fubini's theorem, and with arguments similar to the default free HJM framework in the previous sub-section, we have

$$\begin{aligned} d\left(-\int_t^T f_d(t,u)du\right) &= r_d(t)dt - \int_t^T [\alpha_d(t,u)dt + \sum_{i=1}^n \beta_{d,i}(t,u)dW_i(t)]du \\ &= r_d(t)dt - \left(\int_t^T \alpha_d(t,u)du\right)dt + \sum_{i=1}^n \left(\int_t^T \beta_{d,i}(t,u)du\right)dW_i(t) \end{aligned}$$

Hence,

$$\frac{dV_T^0(t)}{V_T^0(t)} = (r_d(t) - \int_t^T \alpha_d(t,u)du + \frac{1}{2} \sum_{i=1}^n (\int_t^T \beta_i(t,u)du)^2)dt + \sum_{i=1}^n (\int_t^T \beta_{d,i}(t,u)du)dW_i(t)$$

Therefore we obtain the dynamic representation for the defaultable bond price

$$B_d^0(t,T):$$

$$\begin{aligned} \frac{dB_d^0(t,T)}{B_d^0(t-,T)} &= (r_d(t) - \int_t^T \alpha_d(t,u)du + \frac{1}{2} \sum_{i=1}^n (\int_t^T \beta_i(t,u)du)^2)dt \\ &+ \sum_{i=1}^n (\int_t^T \beta_{d,i}(t,T)du)dW_i(t) - dN_1(t) + \int_{M \setminus A} h_2(t,q)\Lambda(dt,dq), \end{aligned}$$

which completes the proof. ■

Remark

- The zero recovery scheme is used here to simplify the calculation. This simplification in fact sacrifices little generality, because two models with different recovery rates can be made compatible by adjusting the intensity function (Lando, 1998).
- For the consideration of unifying the expression, we may use the symbolic notation for the marked term $\int_{M \setminus A} h_2(t, q) \Lambda(dt, dq)$:

$$\int_{M \setminus A} h_2(t, q) \Lambda(dt, dq) \stackrel{sym}{=} h_2(t, Q) dN_2(t).$$

However, this notation is only used as a short hand for the marked Poisson process, and cannot be directly used for calculations³⁵. Then we have:

$$\begin{aligned} \frac{dB_d^0(t, T)}{B_d^0(t-, T)} &\stackrel{sym}{=} (r_d(t) - \int_t^T \alpha_d(t, u) du + \frac{1}{2} \sum_{i=1}^n (\int_t^T \beta_i(t, u) du)^2) dt \\ &+ \sum_{i=1}^n (\int_t^T \beta_{d,i}(t, T) du) dW_i(t) - dN_1(t) + h_2(t, Q) dN_2(t) \end{aligned}$$

Theorem 3.3.5 Arbitrage Free Drift Restrictions in the Defaultable HJM

Framework

As pointed out in assumption 2.3.1, the arbitrage free condition is equivalent to the existence of a spot martingale measure $\mathbb{Q}$, in which all discounted prices of tradable securities are martingales.

³⁵ Refer to Hanson (2007)

Consequently, under the dynamics in proposition 3.3.3, the defaultable bond price

$B_d^0(t, T)$ is a martingale when discounted by the risk free rate $r(t)$.

Since

$$E^{\mathbb{Q}}[dN_1(t)] = E^{\mathbb{Q}}[1_{\{\tau_A > dt\}}] = \mathbb{Q}(dN_1(t) = 0) = \phi(A)\lambda(t)dt$$

$$\text{and } E^{\mathbb{Q}}[\int_{M \setminus A} h_2(t, q) \Lambda(dt, dq)] = \bar{h}_2^{M \setminus A}(t)\lambda(t)dt,$$

we have:

$$\begin{aligned} \frac{dB_d^0(t, T)}{B_d^0(t-, T)} &= (r_d(t) - \int_t^T \alpha_d(t, u)du + \frac{1}{2} \sum_{i=1}^n (\int_t^T \beta_{d,i}(t, u)du)^2 - \phi(A)\lambda(t) + \bar{h}_2^{M \setminus A}(t)\lambda(t))dt \\ &+ \sum_{i=1}^n (\int_t^T \beta_{d,i}(t, T)du) dW_i(t) - (dN_1(t) - \phi(A)\delta(t)\lambda(t)dt) \\ &+ (h_2(t, Q)dN_2(t) - \bar{h}_2^{M \setminus A}(t)\lambda(t)dt) \end{aligned}$$

In other words, the arbitrage free condition requires that the drift term of the above dynamics of defaultable bond price is $r(t)$, i.e.

$$r(t) = r_d(t) - \int_t^T \alpha_d(t, u)du + \frac{1}{2} \sum_{i=1}^n (\int_t^T \beta_{d,i}(t, u)du)^2 - \phi(A)\delta(t)\lambda(t) + \bar{h}_2^{M \setminus A}(t)\lambda(t) \quad (3.3.5)$$

By Proposition 2.4.4, the defaultable discount rate under zero recovery is derived as following:

$$r_d(t) = r(t) + \phi(A)\delta(t)\lambda(t) - \bar{h}_2^{M \setminus A}(t)\lambda(t)$$

Combined the above two equations, we obtain the arbitrage free drift restriction for defaultable bonds in HJM framework:

$$\int_t^T \alpha_d(t, u) du = \frac{1}{2} \sum_{i=1}^n \left(\int_t^T \beta_{d,i}(t, u) du \right)^2$$

■

This result yields the following proposition.

Proposition 3.3.6

The following statements are equivalent:

- The dynamics of defaultable bond and default free bond are arbitrage free.
- The defaultable instantaneous short rate $r_d(t)$ and default free short rate $r(t)$ satisfy: $r_d(t) = r(t) + \phi(A)\delta(t)\lambda(t) - \bar{h}_2^{M \setminus A}(t)\lambda(t)$.
- The drift and variation coefficients of the default free forward rate in assumption 3.2.2 satisfy: $\int_t^T \alpha(t, u) du = \frac{1}{2} \sum_{i=1}^n \left(\int_t^T \beta_i(t, u) du \right)^2$ for all $t < T$.

And the drift and variation coefficients of the defaultable forward rate in assumption 3.3.2 satisfy:

$$\int_t^T \alpha_d(t, u) du = \frac{1}{2} \sum_{i=1}^n \left(\int_t^T \beta_{d,i}(t, u) du \right)^2 \text{ for all } t < T.$$

- The defaultable instantaneous short rate $r_d(t)$ and default free short rate $r(t)$ satisfy: $r_d(t) = r(t) + \phi(A)\delta(t)\lambda(t) - \bar{h}_2^{M \setminus A}(t)\lambda(t)$.

The dynamics of the default free bond price $B(t, T)$ is given by:

$$\frac{dB(t, T)}{B(t, T)} = r(t)dt + \sum_{i=1}^n \left(\int_t^T \beta_i(t, u) du \right) dW_i(t).$$

The dynamics of the pre-default bond price V_t^0 is given by:

$$\frac{dV_T^0(t)}{V_T^0(t)} = r_d(t)dt + \sum_{i=1}^n \left(\int_t^T \beta_{d,i}(t, u) du \right) dW_i(t).$$

Proof of Proposition 3.3.6

The results can be directly derived from theorem 3.3.5.

3.4 Change of measure

In previous chapters, we take as given the existence of risk neutral measure. Under this measure, all the calculations are undertaken and the connections between the dynamics of defaultable forward rates and defaultable bonds are clarified. In this section we demonstrate how to change the real world probability measure to the risk neutral probability measure, which is an extension of the standard techniques of change of measure for continuous finance problems.

Theorem 3.4.1

Given a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}^W, \{\mathcal{F}_t^W\}_{0 \leq t < \infty})$ defined in 2.2.1. Let $t \in [0, T]$ with T given and assume that there exists an n -dimensional predictable process $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))$ such that: $\int_0^t \|\varphi(s)\|^2 ds < \infty$, for all finite t .

Define the Radon-Nikodym derivative $X_1(t)$ by

$$\frac{dX_1(t)}{X_1(t)} = -\varphi(t)dW(t) = -\sum_{i=1}^n \varphi_i(t)dW_i(t), \text{ with } X_1(0) = 1$$

and, suppose that, for all finite t , $E^{\mathbb{P}}[X_1(t)] = 1$. Then there exists a probability

measure $\mathbb{Q}$ on $\mathcal{F}^W$, equivalent to $\mathbb{P}$ with $d\mathbb{Q} = X_1(T)d\mathbb{P}$ such tha

$$dW(t) = d\bar{W}(t) + \varphi(t)dt$$

Where $W(t)$ is a $\mathbb{Q}$ -Wiener process and $\bar{W}(t)$ is a $\mathbb{P}$ -Wiener process.

Proof of Theorem 3.4.1

See Øksendal 2003

Theorem 3.4.2 (Girsanov Theorem for Jump-Diffusion Processes)

Given a filtered probability space $(\Omega, \mathbb{P}, \mathcal{G}, \{\mathcal{G}_t\}_{0 \leq t < \infty})$, On the finite time interval $[0, T]$, define two strictly positive predictable processes $\theta_1(t)$, $\theta_2(t, q)$ such that

$$\int_0^t \int_M |\theta_i(s, q)| \phi(dq) ds < \infty, \quad i = 1, 2, \quad t \leq T.$$

Assume an M -marked Poisson process presented by a counting measure $\Lambda_2(dt, dq)$ and a Poisson process $N_1(t)$, both of which are independent and defined in Definition 2.2.3.

Define $X_2(t)$ with the dynamics:

$$\frac{dX_2(t)}{X_2(t)} = (\theta_1(t) - 1)(N_1(t) - \phi(A)\lambda(t)dt)$$

and $X_3(t)$ with the dynamics:

$$\frac{dX_3(t)}{X_3(t)} = \int_{M \setminus A} (\theta_2(t, q) - 1)(\Lambda(dq, dt) - \phi(dq)\lambda(t)dt)$$

Consequently, if $X(t)$ is given as: $X(t) = X_1(t)X_2(t)X_3(t)$

and it is assumed $E^{\mathbb{P}}[X(t)] = 1$ the Radon-Nikodym derivative, $X(t)$ defines an equivalent measure $\mathbb{Q}$ with $d\mathbb{Q} = X(T)d\mathbb{P}$, such that all the defaultable bond price processes specified previously are martingale.

In this case, the dynamics of the Radon-Nikodym derivative $X(t)$ process is given as following:

$$\begin{aligned} \frac{dX(t)}{X(t)} &= \frac{dX_1(t)}{X_1(t)} + \frac{dX_2(t)}{X_2(t)} + \frac{dX_3(t)}{X_3(t)} \\ \frac{dX(t)}{X(t)} &= -\sum_{i=1}^n \varphi_i(t) dW_i(t) + (\theta_1(t) - 1)(N_1(t) - \phi(A)\lambda(t)dt) \\ &\quad + \int_{M \setminus A} (\theta_2(t, q) - 1)(\Lambda(dq, dt) - \phi(dq)\lambda(t)dt) \end{aligned}$$

The solution to the above stochastic differential equation is

$$\begin{aligned} X(t) &= \exp \left\{ -\sum_{i=1}^n \int_0^t \varphi_i(s) dW_i(s) - \frac{1}{2} \sum_{i=1}^n \int_0^t \varphi_i^2(s) ds \right\} \\ &\quad \cdot \exp \left(\int_0^t (1 - \theta_1(s)) \phi(A) \lambda(s) ds \right) N_1(t) \\ &\quad \cdot \exp \left(\int_0^t \int_{M \setminus A} (1 - \theta_2(s, q)) \phi(dq) \lambda(s) ds + \int_0^t \int_{M \setminus A} \ln(1 - \theta_2(s, q)) \Lambda(ds, dq) \right) \end{aligned}$$

Proof of Theorem 3.4.2

See Øksendal 2005 or Runggaldier 2002³⁶

Remarks

- In financial context, $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))$ is the premium for the market risk which is generated by default free uncertainties. The component $\theta_1(t)$ is the

³⁶ W.J. Runggaldier, "Jump Diffusion Models". In : Handbook of Heavy Tailed Distributions in Finance (S.T. Rachev, ed.)

market price on the non-liquidation default risk and $\int_{M \setminus A} \theta_2(t, q) \phi(dq)$ is the market premium for the liquidation risk.

- The change of measure from $\mathbb{P}$ to $\mathbb{Q}$ only change the drift term in the dynamics of defaultable bonds and leaves the volatilities of the defaultable bond prices unchanged.

Chapter 4

An Example of Model Specification and Credit Derivatives Pricing

4.1 Introduction

In this section we present the Hull-White specification of the stochastic processes for interest rates and default intensities. Vasicek (1977) assumes that the risk free short rate $r(t)$ evolves as an Ornstein-Uhlenbeck process with constant coefficients. And Hull-White (1993) extends the model to accommodate time varying interest rate and volatility. In this chapter, we will adopt the extended Hull-White model with time t dependent coefficients.

The specification has the disadvantage of introducing negative interest rates and intensities. However, this troubling problem can be alleviated: by judiciously choosing the parameters, the probability of interest rates and default intensities becomes very small. Moreover, this drawback is greatly compensated by the model's analytical tractability. Closed form solutions for the defaultable bond pricing can be derived, which in turn, helps us to understand the influence of model parameters well and makes it possible to calibrate the model to market data.

4.2 Specification of the Model: the Hull-White Example

Assumption 4.2.1

We assume the filtration of information flow is defined in chapter 2, and the default free short rate $r(t)$ follows the dynamics³⁷:

$$dr(t) = (\kappa_r(t) - \alpha_r(t)r(t))dt + \sigma_r(t)\sqrt{\frac{1+\rho}{2}}dW_1(t) + \sigma_r(t)\sqrt{\frac{1-\rho}{2}}dW_2(t) \quad (4.2.1)$$

where, $W_1(t)$ and $W_2(t)$ are two independent standard Wiener processes.

Similarly, the intensity process of the doubly stochastic Poisson process

$N(t) = N(t, M)$ that is associated to the doubly stochastic marked Poisson process $\Lambda(t, q)$ is given as:

$$d\lambda(t) = (\kappa_\lambda(t) - \alpha_\lambda(t)\lambda(t))dt + \sigma_\lambda(t)\sqrt{\frac{1+\rho}{2}}dW_1(t) - \sigma_\lambda(t)\sqrt{\frac{1-\rho}{2}}dW_2(t) \quad (4.2.2)$$

The coefficients $\kappa_r(t)$, $\alpha_r(t)$, $\sigma_r(t)$, $\kappa_\lambda(t)$, $\alpha_\lambda(t)$ and $\sigma_\lambda(t)$ are assumed to be deterministic time t function, and $\kappa_r(t) > 0$, $\kappa_\lambda(t) > 0$.

Remark

- The most important feature of the model is the property of mean reversion. For example, in the process of default free interest rates, the ratio $\bar{r}(t) = \frac{\kappa_r(t)}{\alpha_r(t)}$ is defined as the long run equilibrium rate. If the current interest rate level is greater than $\bar{r}$, the negative drift term will pull down the interest rate in the direction of $\bar{r}$, and vice versa. In this case $\alpha_r(t)$ is regarded as the speed of adjustment in the mean reversion process. There are also compelling economic arguments to justify the mean reversion process. When the interest rate is far greater than equilibrium rate, the economy tends to slow down and borrowers require fewer funds because of higher borrowing costs. Therefore, the rate tends to decline to its equilibrium. On the contrary when the rate is low, there

³⁷ To ease the computation burden, a two dimension Wiener process is presented here to introduce the correlation between intensity process and default free interest rate process, and it can be easily extended to the k -dimensional case.

tends to be high demand for funds on the part of the borrowers and the interest rate tends to increase. This feature is particularly attractive because without it, interest rates could drift permanently upward the way stock prices do and this is simply not observed in practice.

- $r(t)$ and $\lambda(t)$ are two correlated process with correlation:

$$dr(t)d\lambda(t) = \rho \cdot \sigma_\lambda(t) \cdot \sigma_r(t) \cdot dt$$

We assumed that the correlation is negative with $-1 < \rho < 0$, because of the well understood relationship of default probability and the economic level: in periods of recession (indexed by low interest rates) also tend to be periods of high default probability reflecting some sort of economic collapse, while periods of economic boom are perceived as safe periods with low default probability.

Assumption 4.2.2

The independent mark Q is uniformly distributed on the interval $[0,1]$ and the absorbing set is $A = \{q : q \in [0, c]\}$, i.e. whenever the realization of random mark, q is lower than the liquidation boundary c , or $q \in A$, the default event will lead to a liquidation process and terminate the defaultable claim. And the non-liquidation loss quota function $h_2(t, q)$ we shall use is given as: $h_2(t, q) = e^{-(T-t)q} - 1$ (4.2.3)

Proposition 4.2.3

Under the assumption 4.2.1, the default free bond price will follow the dynamics:

$$\frac{dB(t, T)}{B(t, T)} = r(t)dt + \sigma^r(t)A(t)\sqrt{\frac{1+\rho}{2}}dW_1(t) + \sigma^r(t)A(t)\sqrt{\frac{1-\rho}{2}}dW_2(t) \quad (4.2.4)$$

where $A(t) = -\int_t^T e^{-\int_t^s \alpha^r(u)du} ds$.

Proof of Proposition 4.2.3

From Assumption 4.2.1,

$$dr(t) = (\kappa_r(t) - \alpha_r(t)r(t))dt + \sigma_r(t)\left(\sqrt{\frac{1+\rho}{2}}dW_1(t) + \sqrt{\frac{1-\rho}{2}}dW_2(t)\right)$$

$$e^{\int_0^t \alpha_r(s) ds} dr(t) + e^{\int_0^t \alpha_r(s) ds} \alpha_r(t) r(t) dt = e^{\int_0^t \alpha_r(s) ds} (\kappa_r(t) dt + \sigma_r(t) (\sqrt{\frac{1+\rho}{2}} dW_1(t) + \sqrt{\frac{1-\rho}{2}} dW_2(t)))$$

$$d(e^{\int_0^t \alpha_r(s) ds} r(t)) = e^{\int_0^t \alpha_r(s) ds} (\kappa_r(t) dt + \sigma_r(t) (\sqrt{\frac{1+\rho}{2}} dW_1(t) + \sqrt{\frac{1-\rho}{2}} dW_2(t)))$$

The solution to the above stochastic differential equation of $r(t)$ is therefore:

$$r(t) = e^{-\int_0^t \alpha_r(s) ds} r(0) + \int_0^t e^{-\int_s^t \alpha_r(u) du} \kappa_r(s) ds + \int_0^t e^{-\int_s^t \alpha_r(u) du} \sigma_r(s) (\sqrt{\frac{1+\rho}{2}} dW_1(s) + \sqrt{\frac{1-\rho}{2}} dW_2(s))$$

And for any $T > t$, we then get

$$r(T) = e^{-\int_t^T \alpha_r(s) ds} r(t) + \int_t^T e^{-\int_s^T \alpha_r(u) du} \kappa_r(s) ds + \int_t^T e^{-\int_s^T \alpha_r(u) du} \sigma_r(s) (\sqrt{\frac{1+\rho}{2}} dW_1(s) + \sqrt{\frac{1-\rho}{2}} dW_2(s))$$

Under spot martingale measure $\mathbb{Q}$, the default free bond price is:

$$\begin{aligned} B(t, T) &= E^{\mathbb{Q}}[\exp(-\int_t^T r(s) ds) | \mathcal{F}_t^W] \\ &= E^{\mathbb{Q}}[\exp(-\int_t^T [e^{-\int_t^s \alpha_r(u) du} r(t) + \int_t^s e^{-\int_u^s \alpha_r(v) dv} \kappa_r(u) du] ds) | \mathcal{F}_t^W] \\ &\quad \cdot E^{\mathbb{Q}}[\exp(-\int_t^T \int_t^s e^{-\int_u^s \alpha_r(v) dv} \sigma_r(u) (\sqrt{\frac{1+\rho}{2}} dW_1(u) + \sqrt{\frac{1-\rho}{2}} dW_2(u)) ds) | \mathcal{F}_t^W] \end{aligned}$$

By Fubini's theorem,

$$\begin{aligned} &\int_t^T \int_t^s e^{-\int_u^s \alpha_r(v) dv} \cdot \sigma_r(u) (\sqrt{\frac{1+\rho}{2}} dW_1(u) + \sqrt{\frac{1-\rho}{2}} dW_2(u)) ds \\ &= \int_t^T \int_s^T e^{-\int_s^u \alpha_r(v) dv} \cdot \sigma_r(s) du (\sqrt{\frac{1+\rho}{2}} dW_1(s) + \sqrt{\frac{1-\rho}{2}} dW_2(s)) \end{aligned}$$

which is a normally distributed random variable independent of $\mathcal{F}_t^W$ with zero drift

and variance $\int_t^T (\int_s^T e^{-\int_s^u \alpha_r(v) dv} \sigma_r(s) du)^2 ds$.

If $x \sim N(\mu, \sigma^2)$ is a normal distribution variable then we have $E(e^x) = e^{\mu + \frac{1}{2}\sigma^2}$.

By applying this result, we obtain: $B(t, T) = e^{A(t)r(t)+C(t)}$

where $A(t) = -\int_t^T e^{-\int_t^s \alpha_r(u)du} ds$

and $C(t) = -\int_t^T \int_t^s e^{-\int_u^s \alpha_r(v)dv} \kappa_r(u) du ds + \frac{1}{2} \int_t^T (\int_s^T e^{-\int_s^u \alpha_r(v)dv} \sigma_r(s) du)^2 ds$.

By applying Ito formula, we obtain the dynamics of the default free bond price as:

$$\frac{dB(t, T)}{B(t, T)} = r(t)dt + \sigma_r(t)A(t)\sqrt{\frac{1+\rho}{2}}dW_1(t) + \sigma_r(t)A(t)\sqrt{\frac{1-\rho}{2}}dW_2(t)$$

■

Proposition 4.2.4

Given the assumption 4.2.1 and Assumption 4.2.2, and under zero recovery, the pre-default bond price will be of the form:

$$V_T^0(t) = e^{A(t)r(t)+C(t)} \cdot e^{A'(t)w(t)\lambda(t)+C'(t)} = B(t, T) \cdot e^{A'(t)w(t)\lambda(t)+C'(t)} \quad (4.2.5)$$

Proof of Proposition 4.2.4

By similar argument, the solution to the stochastic differential equation of $\lambda(t)$ is

$$\lambda(t) = e^{-\int_0^t \alpha^\lambda(u)du} \lambda(0) + \int_0^t e^{-\int_u^t \alpha^\lambda(v)dv} \kappa^\lambda(u) du + \int_0^t e^{-\int_u^t \alpha^\lambda(v)dv} \sigma^\lambda(u) \left(\sqrt{\frac{1+\rho}{2}} dW_1(u) - \sqrt{\frac{1-\rho}{2}} dW_2(u) \right)$$

and the pre-default price $V_T^0(t)$ with zero recovery in Definition 2.4.2 is given as:

$$V_T^0(t) = E^{\mathbb{Q}}[e^{-\int_t^T [-\frac{1}{2}M^A(s)\lambda(s) + \phi(A)\lambda(s) + r(s)]ds} | \mathcal{F}_t^W]$$

By Assumption 4.2.2, we have $\phi(A) = c$, and the truncated expectation of $h_2(t, q)$ is given as:

$$\begin{aligned}\bar{h}_2^{M \setminus A}(t) &= \int_c^1 h_2(t, q) \phi(dq) = \int_c^1 (e^{-(T-t)q} - 1) dq \\ &= \frac{1}{T-t} (e^{-c(T-t)} - e^{-(T-t)}) - (1-c)\end{aligned}$$

To ensure $\bar{h}_2^{M \setminus A}(t) \leq 0$, we require that $c < 1$. This is always guaranteed by the definition of the distribution function $\phi(\cdot)$.

Substitute the expression of $r(t)$ and $\lambda(t)$ into the price formula of $V_T^0(t)$, we have obtained the analytical solution for $V_T^0(t)$:

$$V_T^0(t) = e^{A(t)r(t)+C(t)} \cdot e^{A'(t)w(t)\lambda(t)+C'(t)} = B(t, T) \cdot e^{A'(t)w(t)\lambda(t)+C'(t)}$$

$$\text{where } A'(t) = -\int_t^T e^{-\int_t^s \alpha_\lambda(u) du} ds,$$

and

$$\begin{aligned}C'(t) &= -\int_t^T \left[\int_t^s e^{-\int_t^u \alpha_\lambda(v) dv} \kappa_\lambda(u) du ds + \frac{1}{2} \int_t^T \left(\int_s^T e^{-\int_s^u \alpha_\lambda(v) dv} \sigma_\lambda(s) du \right)^2 ds \right. \\ &\quad \left. + \int_t^T \rho \left(\int_s^T e^{-\int_s^u \alpha_r(v) dv} \sigma_r(s) du \right) \left(\int_s^T e^{-\int_s^u \alpha_\lambda(v) dv} \sigma_\lambda(s) du \right) ds \right]\end{aligned}$$

$$\text{and } w(t) = -\bar{h}_2^{M \setminus A}(t) + \phi(A) = 1 - \frac{1}{T-t} (e^{-c(T-t)} - e^{-(T-t)}).$$

■

4.3 Pricing Credit Risk Derivatives

4.3.1 Credit Default Swap (CDS)

A credit default swap (CDS) is a bilateral swap contract between the protection buyer and seller. In the contract, the protection seller pays the buyer P_{CDS} the difference between the face value of a stipulated loan or defaultable bond and its recovery value, if the default occurs at time τ_A before a stated expiration date T . In exchange, the buyer of protection makes periodic fee payments of some amount U to the seller, up to time $\tilde{T}$, where $\tilde{T} = \min(T, \tau_A)$.

4.3.1.1 Calculating S_{CDS}

Assuming the existence of the martingale measure $\mathbb{Q}$, the market price of the default protection S_{CDS} is valued as:

$$S_{CDS} = E^{\mathbb{Q}}[1_{\{\tau_A < T\}} e^{-\int_0^{\tau_A} r(s)ds} L(\tau_A)(1 - \delta(\tau_A)) | \mathcal{G}_0]$$

And based on the same arguments in the proof of proposition 2.4.7, we get

$$\begin{aligned} S_{CDS} &= E^{\mathbb{Q}}[1_{\{\tau_A < T\}} e^{-\int_0^{\tau_A} r(t)dt} L(\tau_A)(1 - \delta(\tau_A)) | \mathcal{G}_0] \\ &= E^{\mathbb{Q}}[\int_0^T \phi(A) \lambda(t)(1 - \delta(t)) e^{-\int_0^t (-\tilde{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s))ds} dt] \\ &= \phi(A) \int_0^T E^{\mathbb{Q}}[\lambda(t)(1 - \delta(t)) e^{-\int_0^t (-\tilde{h}_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s))ds}] dt \end{aligned}$$

Given the specifications of the default free interest rate process and the intensity process in the model of the previous sub-section, we have,

$$S_{CDS} = \phi(A) \int_0^T e^{b_0(t)} E^{\mathbb{Q}}[(a_0(t) + \int_0^t a_1(s) dW_1(s) + \int_0^t a_2(s) dW_2(s)) \cdot e^{\int_0^t b_1(s) dW_1(s) + \int_0^t b_2(s) dW_2(s)}] dt \quad (4.3.1)$$

Where

$$a_0(t) = (1 - \delta(t)) e^{-\int_0^t \alpha_r(s)ds} r(0) + (1 - \delta(t)) \int_0^t e^{-\int_s^t \alpha_r(u)du} \kappa_r(s) ds$$

$$a_1(s) = e^{-\int_s^T \alpha_r(u)du} \sigma_r(s) \sqrt{\frac{1 + \rho}{2}}$$

$$a_2(s) = e^{-\int_s^T \alpha_r(u) du} \sigma_r(s) \sqrt{\frac{1-\rho}{2}}$$

And,

$$\begin{aligned} b_0(t) &= -\int_0^t e^{-\int_0^s \alpha_r(u) du} r(0) + \frac{(T-s) - e^{-c(T-s)} - e^{-(T-s)}}{T-t} e^{-\int_0^s \alpha_\lambda(u) du} \lambda(0) ds \\ &\quad - \int_0^t \int_0^s (e^{-\int_u^s \alpha_r(v) dv} \kappa_r(u) + \frac{(T-s) - e^{-c(T-s)} - e^{-(T-s)}}{T-s} e^{-\int_u^s \alpha_\lambda(v) dv} \kappa_\lambda(u)) du ds \\ b_1(s) &= -\sqrt{\frac{1+\rho}{2}} \int_s^T e^{-\int_s^u \alpha_r(v) dv} \sigma_r(s) + \frac{(T-s) - e^{-c(T-s)} - e^{-(T-s)}}{T-s} e^{-\int_s^u \alpha_\lambda(v) dv} \sigma_\lambda(s) du \\ b_2(s) &= -\sqrt{\frac{1-\rho}{2}} \int_s^T e^{-\int_s^u \alpha_r(v) dv} \sigma_r(s) - \frac{(T-s) - e^{-c(T-s)} - e^{-(T-s)}}{T-s} e^{-\int_s^u \alpha_\lambda(v) dv} \sigma_\lambda(s) du \end{aligned}$$

Since $\int_0^t b_1(s) dW_1(s)$ and $\int_0^t b_2(s) dW_2(s)$ are independent normally distributed random variables, we have,

$$\begin{aligned} &E^{\mathbb{Q}}[a_0(t) e^{\int_0^t b_1(s) dW_1(s) + \int_0^t b_2(s) dW_2(s)}] \\ &= a_0(t) E^{\mathbb{Q}}[e^{\int_0^t b_1(s) dW_1(s)}] \cdot E^{\mathbb{Q}}[e^{\int_0^t b_2(s) dW_2(s)}] \\ &= a_0(t) e^{\frac{1}{2} \int_0^t (b_1(s)^2 + b_2(s)^2) ds} \end{aligned}$$

And

$$\begin{aligned} &E^{\mathbb{Q}}[\int_0^t a_1(s) dW_1(s) \cdot e^{\int_0^t b_1(s) dW_1(s) + \int_0^t b_2(s) dW_2(s)}] \\ &= E^{\mathbb{Q}}[\int_0^t a_1(s) dW_1(s) \cdot e^{\int_0^t b_1(s) dW_1(s)}] \cdot E^{\mathbb{Q}}[e^{\int_0^t b_2(s) dW_2(s)}] \\ &= e^{\int_0^t b_2(s)^2 ds} \cdot \int_0^t a_1(s) \cdot b_1(s) ds \end{aligned}$$

Similarly,

$$E^{\mathbb{Q}}[\int_0^t a_2(s) dW_2(s) \cdot e^{\int_0^t b_1(s) dW_1(s) + \int_0^t b_2(s) dW_2(s)}]$$

$$\begin{aligned}
&= E^{\mathbb{Q}} \left[\int_0^t a_2(s) dW_2(s) \cdot e^{\int_0^t b_2(s) dW_2(s)} \right] \cdot E^{\mathbb{Q}} \left[e^{\int_0^t b_1(s) dW_1(s)} \right] \\
&= e^{\int_0^t b_1(s)^2 ds} \cdot \int_0^t a_2(s) \cdot b_2(s) ds
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_{CDS} &= \phi(A) \cdot \int_0^T e^{b_0(t)} \cdot (a_0(t) e^{\frac{1}{2} \int_0^t b_1(s)^2 + b_2(s)^2 ds}) dt \\
&+ \phi(A) \cdot \int_0^T e^{b_0(t)} \cdot [e^{\int_0^t b_2(s)^2 ds} \cdot \int_0^t a_1(s) b_1(s) ds + e^{\int_0^t b_1(s)^2 ds} \cdot \int_0^t a_2(s) b_2(s) ds] dt \quad (4.3.2)
\end{aligned}$$

4.3.1.2 Calculating B_{CDS}

From the buyer's point of view, the buyer is required to make a series of payments in the due dates $T_1, T_2, \dots, T_n$ until time $\tilde{T}$, where $\tilde{T} = \min(T, \tau_A)$, and the total payment

of the buyer B_{CDS} is: $B_{CDS} = U \sum_{i=1}^n V_{T_i}^0(0)$

in which $V_{T_i}^0(0)$ is the time 0 market price of the pre-default zero coupon bond with zero recovery and maturity T_i . And $V_{T_i}^0(0)$ is given as:

$$\begin{aligned}
V_{T_i}^0(0) &= E^{\mathbb{Q}} \left[e^{-\int_0^{T_i} [h_2^{M \setminus A}(s) \lambda(s) + \phi(A) \lambda(s) + r(s)] ds} \right] \\
&= E^{\mathbb{Q}} \left[e^{b_0(T_i) + \int_0^{T_i} b_1(s) dW_1(s) + \int_0^{T_i} b_2(s) dW_2(s)} \right] \\
&= e^{b_0(T_i) + \int_0^{T_i} (b_1(s)^2 + b_2(s)^2) ds} \quad (4.3.3)
\end{aligned}$$

where the functions $b_i(\cdot)$, $i = 0, 1, 2$ are defined in expression (4.3.1)

$$\text{Consequently, } B_{CDS} = U \sum_{i=1}^n V_{T_i}^0(0) = U \sum_{i=1}^n e^{b_0(T_i) + \int_0^{T_i} (b_1(s)^2 + b_2(s)^2) ds} \quad (4.3.4)$$

where we preserve the standardization of the face value \$1 of the defaultable bond.

To calculate the fair rate U , called par CDS spread, we set the market value of the CDS equal to zero: $S_{CDS} = B_{CDS}$. And therefore the fair rate U can be solved analytically.

Chapter 5

Conclusion

5.1 Summary

In this thesis, a new default risk modeling approach is developed by introducing a (doubly stochastic) marked Poisson process. The literature shows that the default event is usually described as the first jump of the underlying point process in the previous study. This modeling approach is inconsistent with most bankruptcy codes. In reality, the consequence of a default event may be uncertain: it may lead to liquidation, or it may lead to reorganization. To cope with this fact, a sophisticated model with a marked Poisson process is necessary.

A marked Poisson process can be represented by a random product measure $\Lambda(dt, dq)$ or a sequence of the pair $(T_n, Q_n)_{n \geq 1}$, in which T_n indicates the occurrence time of a default event and Q_n is a random mark representing what kind of consequence that this default event may cause.

Accordingly, this thesis distinguishes between two types of default events with different implications for defaultable claim pricing: a default with liquidation effect (*LE*-default) and a default with non-liquidation effect (*NL*-default) by defining an absorbing subset A in the mark space $(M, \mathcal{M})$. Upon an occurrence of a default, if a random mark q is drawn from A , the default event will lead to a liquidation process

and thus terminate the pricing process of the defaultable claim; otherwise the defaultable claim will suffer a value loss and survive.

The thinning Poisson theorem presented in the thesis makes it possible to separate the original (doubly stochastic) Poisson into two independent (doubly stochastic) Poisson processes which represent different default events as mentioned above. As a result, we can derive a general pricing formula from a time t gain process under the arbitrage free arguments.

Since the loss given default is also one of the most important components in a credit event, it is one of the advantages for using a (doubly stochastic) marked Poisson process to model the default probability and loss given default (or, default recovery) simultaneously, which is followed by a non-arbitrage defaultable contingent claims pricing formula.

In academic studies and market practice, three recovery schemes are widely used, i.e., recovery of par value, recovery of treasury value, and recovery of market value.

Under these schemes, the recovery of defaultable bond is measured by the unit of face value of the bond, default free bond with same term structure, and market value of the defaultable bond, respectively. We prove that if the recovery payoff is well defined, the solutions to defaultable bond pricing problems under these different recovery schemes can be obtained as analytic expression conditional on the background information up to time t , i.e., $\mathcal{F}_t^W$.

Under this new marked Poisson process that underlies the price process of defaultable claims, this study follows the methodology of Schönbucher's work (1998), and extends his study to a marked Poisson process. The arbitrage free condition for the

dynamics of default free bond and defaultable bond can be equivalently express as an equation that regulates the relationship between the defaultable instantaneous short rate $r_d(t)$ and default free short rate $r(t)$:

$$r_d(t) = r(t) + \phi(A)\delta(t)\lambda(t) - \overline{h}_2^{M \setminus A}(t)\lambda(t)$$

Finally, a model implementation exercise is given under the assumption that dynamics for interest rates and default intensities follow an Ornstein-Uhlenbeck process. In this setting, the tradeoff of the model's analytical tractability is the possibility of introducing negative interest rate and negative default intensity. A closed form solution for zero-coupon defaultable bond pricing can be obtained, which constitutes the building blocks for pricing more complicated credit derivatives, such as CDS, convertible bond, etc.

5.2 Limitation of the Thesis and Future Research

In the thesis, the mark space attached to the credit events is defined as an independent and time-invariant probability space. Consequently, the information generated by the random marks is not included into the background filtration up to time t , i.e., $\mathcal{F}_t^W$. It can be argued that credit events may also have considerable repercussion effects, i.e., the attitude of a credit event may be decisive to a company's default intensity process $\lambda(t)$. For example, a credit rating downgrade may increase the company's debt collateral requirements, make the debt financing more costly, which in turn exacerbates its future credit risk. However, these repercussion effects cannot be measured due to the limitation of the underlying Poisson framework.

To remove these negativities and model more complicated payoff credit derivatives, a numerical algorithm to simulate the dynamics of the defaultable claims may need to be developed. More important, it is usually difficult to Markovian representation under the HJM system. With these respects, the volatility function specification in Chiarella et al. (2005) and the numerical scheme in Chiarella et al. (2008) may be adopted. If the mark space is separated into mutually independent sets, similarities

between the model developed in this project and the credit rating transition models may be observed.

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