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**ANALYSIS OF STRUCTURES WITH
UNCERTAIN PARAMETERS USING INTERVAL
METHOD**

CAI ZHENHAN

SCHOOL OF CIVIL & ENVIRONMENTAL ENGINEERING

NANYANG TECHNOLOGICAL UNIVERSITY

SINGAPORE

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Cai Zhenhan

School of Civil & Environmental Engineering

Nanyang Technological University

**A Thesis Submitted to the Nanyang Technological University in
Fulfillment of the Requirements for the Degree of Doctor of
Philosophy**

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LIST OF SYMBOLS

<i>Symbol</i>	<i>Description</i>
Δ	<i>Uncertainty part</i>
x^I	<i>Real interval number</i>
$\underline{x}$	<i>Lower bound of interval number x^I</i>
$\bar{x}$	<i>Upper bound of interval number x^I</i>
x^c	<i>Midpoint of interval number x^I</i>
r	<i>Radius of x</i>
α	<i>0-midpoint interval</i>
$I(R)$	<i>Set of all intervals over real number</i>
$r(x^I)$	<i>Continuous unary operation on R</i>
$q(A^I, B^I)$	<i>Distance between two intervals</i>
K^I	<i>Stiffness matrix</i>
u^I	<i>Displacement vector</i>
p^I	<i>Load vector</i>
Q	<i>Constraint matrix</i>
λ^I	<i>Vector of m Lagrange multipliers</i>
M^I	<i>Mass matrix</i>
λ	<i>Eigenvalue</i>
C^I	<i>Damping matrix</i>
$\text{int}(\mathbf{y}^I)$	<i>Interior of $\mathbf{y}^I$</i>
F^I	<i>Load vector</i>
E	<i>Young's modulus</i>
L	<i>Length of the element</i>
T	<i>Transformation matrix</i>
Π	<i>Potential energy</i>
v^I	<i>Eigenvector</i>
D^I	<i>Inverse of product of mass and stiffness matrices</i>
I	<i>Identity matrix</i>
d^I	<i>Relative uncertainty</i>

ω	<i>Nature frequency</i>
ρ	<i>Mass density</i>
β	<i>Uncertainty factor</i>
$\mathbf{X}^I(t)$	<i>Displacement response</i>
$\dot{\mathbf{X}}^I(t)$	<i>Velocity response</i>
$\ddot{\mathbf{X}}^I(t)$	<i>Acceleration response</i>
J	<i>Continuous Fourier transform</i>
$\tilde{\mathbf{F}}^I$	<i>Fourier transform of load vector</i>
$\tilde{\mathbf{X}}^I$	<i>Fourier transform of response</i>
$\mathbf{H}(\omega)$	<i>Frequency response function</i>
$\delta(x)$	<i>Dirac Delta function</i>
$Re(a)$	<i>Real part of complex number</i>
$\mathcal{L}\{f(t)\}$	<i>Laplace transform of a function</i>
$\mathbf{L}\mathbf{F}^I(s)$	<i>Laplace transform of load vector</i>
$\mathcal{L}^{-1}\{F(s)\}$	<i>Inverse Laplace transform of a function</i>

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SUMMARY

The uncertainties of structural parameters may lead to large and unexpected excursion of responses that may result in drastic reduction in accuracy and precision of the operation. Therefore, the concept of uncertainty plays an important role in the investigation of various engineering problems. In this thesis, the effect of uncertainty in structural parameters and applied forces on the linear structural systems is studied. Uncertain parameters are introduced in the form of intervals. The structural stiffness matrix, mass matrix and loading vectors are described as the sum of two parts corresponding to the deterministic matrix and the uncertainty matrix of the interval parameters. Interval analysis is conducted with the finite element (IFE) method to analyze the system response. The static analysis, the eigenvalue analysis, the frequency response function (FRF) analysis and the dynamic response analysis of structures are carried out. Some general interval techniques are used to solve equations with intervals, such as the element-by-element (EBE) technique, a special matrix treatment, the perturbation technique and so on.

In the static analysis, the element-by-element (EBE) technique is implemented to avoid the element coupling. A special matrix treatment to handle the interval matrix multiplication is used to reduce the overestimation due to multi-occurrence of interval variables. Then, the Brouwer's fixed point theorem and Krawczyk's operator are applied to obtain the displacements of the structure. Illustrative examples show that the proposed IFE method can achieve the sharpest possible results.

In the eigenvalue analysis, the perturbation technique is applied to remove the higher order terms of uncertainties in the eigenvalue equation, and a special

treatment is used to reduce the overestimation of interval calculation. In the proposed method, it is only required to solve a system of linear equations for each eigenvalue and its corresponding eigenvector. Neither iteration nor calculation of inverse of interval matrices is needed, which saves a lot of computational efforts. The effectiveness of the proposed method is demonstrated by numerical results which are compared with the vertex solutions and other researchers' work.

In the FRF analysis, the EBE technique is extended to dynamic system to formulate the IFE equation which is called EBE-IFE method. The structural FRF is obtained by solving the linear equations using the Brouwer's fixed point theorem. The transient responses of proposed method are compared with the vertex solutions and the direct IFE solutions to demonstrate the effectiveness of the proposed method.

In the dynamic response analysis, the Laplace transform is used to convert the dynamic equations into a linear system of equations. The matrix perturbation technique is then utilized, and the inverse Laplace transform is employed to obtain the structural responses. The proposed IFE method for dynamic systems can reduce the computational cost compared with the regular perturbation methods. Its accuracy is validated by the vertex solution and other researchers' work. An interesting finding is the divergent response of the undamped system, which is verified by the vertex solution.

It is worth mentioning that the proposed IFE methods based on the perturbation technique are suitable for small uncertainties (up to 10% in the thesis) of system parameters since the formulation is limited to the first order terms, which results in an inner enclosure of the system response.

CHAPTER 1 INTRODUCTION

1.1 Background

In engineering design, it is important to calculate structural response quantities such as the displacement, stress and vibration responses against a given set of design parameters. However, in most practical situations, the structural parameters are uncertain, such as the inaccuracy of the measurement, errors in the manufacturing and assembly process, invalidity of some components and uncertainty in boundary conditions etc. The uncertainties of structural parameters may lead to large and unexpected excursion of responses that may result in drastic reduction in accuracy and precision of the operation. Therefore, the concept of uncertainty plays an important role in the investigation of various engineering problems. Depending on the nature of imprecision, the analysis/design of the system can be conducted using interval analysis, fuzzy theory or probabilistic approaches. In some analysis of engineering problems, it is more realistic to model problems as uncertainties, that is, undetermined. It is because, for example, some loading, like wind load, often varies over certain range instead of one determined value. Even in the computational analysis, error often exists in the whole progress which bounds the results in some range rather than certain value. The performance characteristics of most engineering structures change during their lifetime because of aging, creep, wear and their operating conditions. The design of such systems should take care of the intervals of deviation of variables from their nominal values.

In some uncertainty cases, the variety in the system parameters is neglected for the convenience of analysis, and it is fortunate to obtain the response still valid and reasonably in some extent. However, we have to acknowledge the variety existence

in the response due to uncertainty in the system parameters, that is, the propagation of system uncertainty. Furthermore, sometimes the uncertainty analysis of system cannot be neglected because it does exaggerate in the final performance output or when high accuracy results are demanded.

Therefore various research and investigation have been conducted on this topic. According to different type of uncertainties, several distinct methods have been introduced to handle the analysis. For problems with distribution description of the variety in the system parameters, probability theory is the traditional approach to handle uncertainty. One of the classical probabilistic methods (Ang and Tang, 2006) is the Monte Carlo simulation or the stochastic finite element method. This approach requires sufficient statistical data to justify the assumed statistical distributions. It has been validated that, given sufficient statistical data, the probability theory describes the stochastic uncertainty well. However, probabilistic modeling cannot handle situations with incomplete or little information on which to evaluate a probability, or when that information is nonspecific, ambiguous, or conflicting.

For less information on the parameters of the system, the fuzzy set theory was introduced by Zadeh (1965 and 1978). The application of fuzzy sets concept into structural analysis has been studied. This application was considered by Ayyub (1998, 2005 and 2006) systematically. A mono and multi-objective optimization approach for structural uncertainty, using fuzzy set and random set theories was developed by Tonon and Bernardini (1999). Uncertainty was expressed by the fuzzy numbers, and the meaning was clarified by random set and possibilities theories. The fuzzy sets were always applied to the problem stating with presumption or preference via α -cut and the imprecision was interpreted as user's choice of a particular value for the uncertain parameter. It was found that, using such

presumption, the uncertain problems can be handled well. For example, some mechanical systems can be analyzed through this method (Qiu and Rao, 2005). If the problems are not suitable for the α -cut assumption or no information on the parameters is available, the fuzzy theory cannot be used.

For more general case that less information is given and only the range of each uncertainty quantity can be specified, the interval method has been widely used. Originally it was introduced to deal with computational errors. Rounding off error in computation progress can be bounded as interval. In this sense, interval arithmetic serves as the calculation tool for the generalized models of uncertainty.

A lot in mathematic aspect for interval calculation was contributed by Hargreaves (2002). He developed a general code called Intlab in Matlab to deal with interval quantity. It can be used as a platform to conduct the interval analysis conveniently. The Intlab is adopted in this work as a basic tool to deal with the interval quantity. It should be noticed that the overestimate effect which is often called “dependency problem” is essential and often exists in the interval based analysis. It occurs due to the neglect of parameter dependency and duplicate appearance of parameter in interval formulation (Wu and Rao, 2005).

Moore (1966), and Alefeld and Herzberger (1983) have done the pioneering work of interval analysis in mathematics. However, it is still difficult to apply these results to practical engineering problems due to the lack of available methods that are able to handle the overestimation caused by dependency and wide intervals. Recently, various generalized models of uncertainty have been applied within the context of the finite element analysis (FEA) to solve partial differential equations with uncertain parameters. The FEA with interval valued parameters results in the Interval Finite Element Analysis (IFEA). Normally IFEA solution techniques can

be utilized to model the uncertain structure systems. Some improvement of this modeling was investigated by Muhanna and Mullen (2001 and 2007), Gao et al. (2010) and Chen et al. (2002). In Muhanna and collaborators' work, they introduced a new formulation for interval-based uncertainty treatment that resulted in a sharp solution enclosure. It was called element-by-element (EBE) technique to reduce the overestimation.

Although plenty of effort has been made and improvement has been achieved in the interval method to solve the uncertainty problems, there are still few investigations on dynamic analysis of undetermined structural systems. The dynamic problems are receiving more and more attention. The natural frequency was considered by Chen et al. (1994), the interval eigenvalue analysis was developed by Deif (1991), the frequency response analysis was studied by Moens and Vandepitte (2007), the analysis of vibration control of structures was investigated by Chakraborty and Roy (2011), the response of structures was studied by Qiu and Wang (2006) and the nonlinear dynamic analysis was considered by Qiu et al. (2009). However, the overestimation problem still existed in their studies. This thesis will focus on not only the static analysis, but also the dynamic analysis of structures with uncertain parameters, aiming at improving both accuracy and efficiency of the conventional interval analysis. The eigensolutions, the magnitude of Frequency Response Function (FRF), and the dynamic response of structural systems under varied parameters will be obtained using the interval method. The overestimation problem will be handled using the EBE-IFEA method and a special treatment of matrix multiplication. In view of the lack of general IFEA code for structural problems, a general IFEA code will be developed in Matlab.

1.2 Research Objective and Scope

The objective of this research is:

To develop effective and efficient IFE methods to calculate both the static and dynamic response of the linear structures, such as the displacement, eigenvalue, mode shape, FRF and dynamic response of structures with interval parameters;

The scopes of this research are:

- (1) To improve the accuracy of interval arithmetic by eliminating the overestimation in interval calculations, and illustrate that the improved interval arithmetic is effective and efficient;

Some general interval techniques will be applied in structural analysis with interval parameters, such as the EBE technique, the special matrix treatment, the perturbation theory and the Brouwer's fixed point theorem.

- (2) To analyze the static performance of structure with interval parameters;
The displacements of the structures will be studied.

- (3) To analyze the dynamic performance of structure with interval parameters;

The eigenvalue, the eigenvector, the FRF and the dynamic response of the structures will be studied.

- (4) To develop a general IFEA code for both static and dynamic analysis of structures using interval analysis method;

Upon the development of IFE methods, a general-purpose software for uncertainty analysis will be developed and coded in MATLAB.

1.3 Organization of Thesis

This thesis consists of 7 chapters. The rest of the research is organized as follows:

In Chapter 2, a detailed review of three methods to model uncertainty which are interval analysis, fuzzy theory, and probabilistic approaches is given. The theoretical background of the interval arithmetic is also presented, including the research development on the uncertainty theory. More importantly, the development of IFEA methods for structural analysis is presented, including static analysis, eigenvalue analysis, natural frequency analysis, FRF analysis, dynamic response analysis, nonlinear system analysis and etc.

Chapter 3 presents the direct interval analysis and proposes an IFEA method using EBE technique for static analysis of structure systems. The comparison of these two methods is also presented. The results show that the developed IFEA method has significantly improved the accuracy of analysis by reducing the overestimation of the interval calculation.

In Chapter 4, based on the perturbation method, an IFE method for eigenvalue problem is presented. The computation of the proposed method focuses on solving a system of linear equations after removing the second order terms of uncertainties, and there is no need to calculate the inverse of interval matrix. Two numerical examples are studied to demonstrate the efficiency of the proposed method.

Chapter 5 proposes an IFE method for the FRF analysis of structures with uncertain parameters. The EBE technique is extended to dynamic systems to formulate the equation. The special treatment of matrix multiplication and the Brouwer's fixed point theorem are also applied to handle the interval calculation. The proposed method is validated by comparing with the vertex solution.

In Chapter 6, an IFE method for dynamic systems with uncertain parameters using Laplace transform is proposed to alleviate computational cost of solving equations compared with the conventional perturbation methods. The effectiveness of the proposed method is demonstrated by numerical examples.

Chapter 7 closes the thesis by concluding the important findings and making recommendations for future research.

CHAPTER 2 REVIEW OF THEORY AND PREVIOUS WORK

2.1 Uncertainties

Today with the development of research, precision is more and more emphasized. How to deal with the uncertain values should be considered in real life engineering practice. All engineering design problems involve imprecision, approximation or uncertainty to varying degrees. Uncertainties in engineering design may arise from the incomplete knowledge about the system behavior, model parameter, measurement and environmental loading conditions. Depending on the nature of the imprecision, the analysis/design of the system can be conducted using interval analysis, fuzzy theory or probabilistic approaches. In interval analysis, the uncertain parameter is denoted by a simple range. In addition to the range, if a preference function is used to describe the desirability of using different values within the range, fuzzy theory can be used. In fuzzy theory, the imprecision is interpreted as the designer's choice to use a particular value for the uncertain parameter. On the other hand, if the uncertain parameter is described as a random variable following a specified probability distribution, the probabilistic approach can be used.

2.1.1 Probabilistic Approach

The probabilistic concept has been well established for the extension of deterministic finite element method towards uncertainty assessment. This has led to a number of probabilistic finite element procedures that are often used in conjunction with Monte Carlo Simulation. Probabilistic methods are especially suitable in the case of model variabilities for which both the range and the

probability density function are available.

Probability theory is the traditional approach to handle uncertainty. This approach requires sufficient statistical data to justify the assumed statistical distributions. Analysts agree that, given sufficient statistical data, the probability theory describes the stochastic uncertainty well. However, probabilistic modeling cannot handle situations with incomplete or little information on which to evaluate a probability, or when that information is nonspecific, ambiguous or conflicting (Ferson and Ginzburg, 1996). Many generalized models of uncertainty have been developed to treat such situations, including fuzzy sets and possibility theory (Zadeh, 1978), Dempster-Shafer (1967) theory of evidence, random set (Kendall, 1974), probability bounds (Berleant, 1993), imprecise probabilities (Walley, 1991), convex model (Ben-Haim and Elishakoff, 1990), and so on.

A new approach for probabilistic characterization of linear elastic redundant trusses with uncertainty on the various members subjected to deterministic loads acting on the structure nodes has been presented by Mario Di Paola (2004). The method is based on the simple observation that variations of structural parameters are equivalent to superimposed strains on a reference structure depending on the axial forces on the elastic modulus of the original structure as well as on the uncertainty (virtual distortion method approach).

Cai and Lin (1988) proposed a method to obtain an approximate solution of the joint probability density function (PDF) for a non-linear single degree of freedom (SDOF) system, and presented an example for the case of Coulomb damping. Spencer et al. (1994) presented a probabilistic controller optimization method for a SDOF structure considering the uncertainty of structural parameters. In their study, the probability of peak response exceeding a predefined value was used as an

objective function and the root mean square of the control force was used as a constraint. Hong and Wang (2002) presented a method to assess the probability of peak response exceeding a critical value assuming that the uncertainty in seismic excitation of all potential earthquakes can be expressed by the uncertainty in the parameters of the power spectral density function. Spanos et al. (2003) presented a solution scheme for the random vibration of non-linear systems under Gaussian excitations. In their study, responses were estimated by expanding in a trigonometric Fourier series based on concepts of spectral theory for random process. Dimentberg et al. (2000) and Ying and Zhu (2003) applied the stochastic dynamic programming principle to a variety of non-linearly controlled system simplified by the stochastic averaging method. Especially, in the minimization of the first-passage failure probability, bang–bang type controllers with limited control force were obtained by Dimentberg et al. (2000). But this type of controllers may exhibit aforementioned problems in the actual implementation. In addition, the stochastic responses were predicted only for the stochastically averaged system. Park et al. (2005) presented the probabilistic analysis in performing on the bounded non-linear control of SDOF systems. Numerical examples were given to verify the solution given in their study comparing analytically obtained PDFs with statistically obtained ones.

Even though the non-linear control algorithms with limited control force have been widely explored and have shown promising results, the probabilistic analysis of those algorithms has been made restrictively. This is due to the fact that it is hard to obtain the analytical solution of the joint PDF for the non-linear system, which results from the non-linear controller even for the linear structural behavior.

A recent benchmark study by Schueller and co-workers (Schueller and Pradlwarter, 2007) indicated that advanced Monte Carlo methods have shown promise for

tackling complex systems. Subset Simulation, line sampling, and collectively methods that make use of Markov Chain Monte Carlo have demonstrated their variance reduction capability while retaining certain robustness in applicability. An importance sampling method was presented by Au (2009) for computing the first passage probability of elasto-plastic structures under stochastic excitations. The importance sampling distribution corresponds to shifting the mean of the excitation to an ‘adapted’ stochastic process whose future is determined based on information only up to the present. Numerical results for a single-degree-of freedom elasto-plastic structure showed that the proposed method was significant improvement in variance reduction over importance sampling using design points reported recently.

The uncertainty quantification and uncertainty propagation in structural mechanics using stochastic analysis were addressed by Schueller (2009). Monte Carlo methods were proposed for studying the variability in the structural properties and for their propagation to the response. The numerical example demonstrates the applicability of Line Sampling to general linear uncertain FE systems under Gaussian distributed excitation. Importance Sampling using the estimated structural design point and an adapted Line Sampling procedures involves only approximately one hundred conditional reliability evaluations for a robust estimation of the total unconditional reliability.

The probabilistic concept is so far the most popular method for numerical uncertainty modeling. Its popularity has led to a number of probabilistic finite element procedures. However, probabilistic modeling is not the only way to describe the uncertainty, and uncertainty is not equal to randomness. Indeed, the probabilistic approaches are not able to deliver reliable results at the required precision without sufficient experimental data to validate regarding the joint

probability densities of the random variables or functions involved. Therefore, one may recognize that uncertainties in parameters can be modeled on the basis of alternative, non-probabilistic conceptual frameworks.

Uncertainty has so far been tackled by the probability notion. However, this does not solve the problem generated by imprecise or vague knowledge. It forms the approved framework to treat concepts of non-probabilistic uncertainty, and to exploit, in a same formalism, imprecision and uncertainty. This can be taken into consideration with the notion of fuzzy sets.

2.1.2 Fuzzy Theory

Fuzzy sets have arisen from the idea of partial membership of a class, a category with ill-defined boundaries, the gradual move in the passage from one situation to another, a generalization of the classical theory of a set allowing the intermediate situation between everything and nothing. The developments of this concept supply the means to represent and handle inaccurately described vague or imprecise knowledge. They also establish an interface between the data described symbolically and numerically. The fuzzy logic makes it possible to work with such knowledge.

The concept of fuzzy sets, introduced by Zadeh (1965), has gained an increasing popularity during the last few decades. Its most important property is that it is capable of describing linguistic and therefore incomplete information in a non-probabilistic manner. Whereas a classical set clearly distinguishes between members and non-members a fuzzy set introduces a degree of membership, represented by the membership function. For a fuzzy set $\tilde{x}$, the membership function $\mu_{\tilde{x}}(x)$ describes the grade of membership to the fuzzy set for each

element x in the domain X :

$$\tilde{x} = \{(x, \mu_{\tilde{x}}(x)) | (x \in X)(\mu_{\tilde{x}}(x) \in [0,1])\} \quad (2.1)$$

If $\mu_{\tilde{x}}(x) = 1$, x is definitely a member of the set $\tilde{x}$, whereas if $\mu_{\tilde{x}}(x) = 0$, x is definitely not a member of the set $\tilde{x}$. For all x with $0 < \mu_{\tilde{x}}(x) < 1$, the membership is not certain. The most frequently applied membership function shapes are the triangular and Gaussian shape.

If F is a fuzzy number with a membership function μ_F , then the α -cut is defined as the set

$$F_\alpha = \{z \in Z, \mu_F(z) \geq \alpha\} \quad (2.2)$$

For the i th α -cut, the lower and upper bounds $(\underline{f}, \bar{f})$ are given by

$$\underline{f} = \min\{f: f \in F_\alpha\} \quad \bar{f} = \max\{f: f \in F_\alpha\} \quad (2.3)$$

If a set of n α -levels is constructed for all fuzzy quantities in a given problem, each of these quantities may be represented by a 2 by n array as follows (see Fig. 2.1):

$$F = \{(\underline{f}, \bar{f})_{\alpha 1}, (\underline{f}, \bar{f})_{\alpha 2}, \dots, (\underline{f}, \bar{f})_{\alpha n}\} \quad (2.4)$$

Figure 2.1 Triangular fuzzy number (Chen and Rao, 1997)

Fuzzy arithmetic is based on the extension principle (Tonon and Bernardini, 1999). The computational features of the extension principle can be achieved by using the α -cut representation of fuzzy numbers. Since any fuzzy number can be represented in interval form, as indicated in Eq. (2.4), fuzzy arithmetic operations can be implemented using interval (operation) analysis at each of the n α -levels independently. The solution of fuzzy arithmetic equations will be the result of interval operations carried out by a nested family of intervals for each fuzzy number. All the rules used for interval arithmetic are equally applicable to fuzzy arithmetic at each α -level.

Many papers (Brown, 1980; Brown and Yao, 1983) have discussed the application of the fuzzy set theory to structural design and in particular in structural optimization (Rao, 1987), in random vibration with application to seismic structures (Wang and Ou, 1990) and in finite element analysis of engineering systems containing vague information, such as boundary conditions, implying prescribed displacements in static analysis.

The fuzzy finite element methodology is a new area of finite-element analysis (FEA)

that began in the early 1990s. Rao and Sawyer (1995) presented a fuzzy finite-element approach for static analysis of engineering systems. Valliappan and Pham (1993) considered the finite-element-based static analysis of a foundation by employing the theory of fuzzy sets. The automatic generation of finite-element mesh using fuzzy sets was achieved by Shimuzu and Hiroaki (1993). Their method used the fuzzy set theory to represent the human thought process in the modeling process. The random vibration in the presence of fuzzy excitation inputs, with application to aseismic structures, was investigated by Wang and Ou (1990) using a SDOF system with deterministic (crisp) system parameters, in which an approach different from the finite-element method was considered for the solution. Muhanna and Mullen (1995, 1996 and 1999) developed a FEA procedure that utilized the concept of fuzzy sets through interval calculations. They also computed the response of different structural systems due to geometric and loading uncertainties. Other researchers such as Chen and Rao (1997), Chen and Lee (2007), Lee and Chen (2010) have contributed in the area of fuzzy finite element based dynamic analysis.

So far, the fuzzy theory has been widely used in dealing with the uncertainty. A lot of works have been done to verify that the fuzzy theory has good performance in uncertainty analysis. But it has the limitation of obtaining the numerical solutions. The interval method can be used to overcome this shortcoming.

2.1.3 Interval Analysis

Measurements are never completely accurate, so in reality, the actual value x_i of i -th measured quantity can differ from the measurement result $\tilde{x}_i$. Because of these measurement errors $\Delta x_i = \tilde{x}_i - x_i$, the result $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$ of data processing is, in general, different from the actual value $y = f(x_1, \dots, x_n)$ of the desired quantity

y.

It is desirable to describe the error $\Delta y := \tilde{y} - y$ of the result of data processing. To do so, it must have some information about the errors of direct measurements.

The errors Δ of direct measurements may come from several aspects. Firstly, the manufacturer of the measuring instrument must supply with an estimate of the upper bound Δ_i on the measurement error. (If no such upper bound is supplied, this means that no accuracy is guaranteed, and the corresponding “measuring instrument” is practically useless.) In this case, once a measurement is performed and the result is $\tilde{x}_i$, the actual (unknown) value of the measured quantity x_i belongs to the interval $x_i^I = [\underline{x}_i, \overline{x}_i]$, where $\underline{x}_i = \tilde{x}_i - \Delta_i$ and $\overline{x}_i = \tilde{x}_i + \Delta_i$.

In many practical situations, it is difficult to know the interval $[-\Delta_i, \Delta_i]$ of possible values of the measurement error; also it is hard to know the probability of different values Δx_i within this interval. This knowledge underlies the traditional engineering approach to estimate the error of indirect measurement, in which it can be assumed that probability distributions is known for measurement errors Δx_i .

In practice, the desired probabilities of different values of Δx_i can be determined by comparing the measured results obtained by this instrument with the same quantity by a standard (much more accurate) measuring instrument. Since the standard measuring instrument is much more accurate than the current one, the difference between these two measurement results is practically equal to the measurement error; thus, the empirical distribution of this difference is close to the desired probability distribution for measurement error.

However, in some cases, there is no information about the probabilities of Δx_i ; the

only information is an estimate on the upper bound on the measurement error. In this case, the only information about the actual value x_i of the measured quantity is that it belongs to the interval $x_i^I = [\tilde{x}_i - \Delta i, \tilde{x}_i + \Delta i]$. In such situations, the only information about the (unknown) actual value of $y = f(x_1, \dots, x_n)$ is that y belongs to the range $y^I = [\underline{y}, \bar{y}]$ of the function f over the box $x_1 \times \dots \times x_n$:

$$y^I = [\underline{y}, \bar{y}] = \{f(x_1, \dots, x_n) | x_1 \in X_1^I, \dots, x_n \in X_n^I\} \quad (2.5)$$

The process of computing this interval range based on the input intervals x_i is called interval computations.

Over the past decades, a number of methods have been developed that include uncertain model properties in the FEA and the quantification of the uncertainty on the analysis result. One of such approaches, based on a set theoretic formulation, is an unknown-but-bounded model (convex models). Such set models of uncertainty have been applied by Dief (1986) in linear programming and system theory. Recently, such set models of uncertainties in parameters have drawn interest for both the robust of system control and the structural failure measures. For example, the convex model was introduced by Ben-Haim and Elishakoff (1990), discussed later by Lindberg (1991), for the study of dynamic response and failure of structures under pulsed parametric loading; the convex model has been applied in determining the upper and lower bounds of static response for structures by Liu et al. (1994). The convex model (Ganzerli and Pantelides, 2000) has also been applied to the optimal design of structures with uncertain parameters.

Since the mid-1960s, a new method called the interval analysis has appeared. Moore (1966), Alefeld and Herzberger (1983) have done the pioneering work. Mathematically, linear interval equations, nonlinear interval equations, and interval

eigenvalue problems have been resolved partly. But because of the complexity of the algorithm, it is difficult to apply these results to practical engineering problems. Therefore, the interval methods that use interval arithmetic in engineering applications started in the 1990s. Recently, the interval finite element method (IFEM) was presented by Chen et al. (2002b) which makes the method easier to deal with the dynamic response analysis for the complex structures with interval parameters. The comparisons of the static and the dynamic responses of structures with uncertain parameters using non-probabilistic interval analysis and the random perturbation were conducted by Qiu et al. (2003 and 2004). It is seen that the range of the dynamical or static response of structures with uncertain-but-bounded parameters obtained by the interval analysis method contains that obtained by the probabilistic approach.

It should be pointed out that the previous interval analysis methods for computing the upper and lower bounds of response of structures with interval parameters are based on the first-order perturbation and the first-order Taylor series. Although the procedure is easy to implement and incorporate into the finite element code, the applications of the previous methods are limited to the case when the interval uncertainties of parameters are small. If the parameters of uncertainties are fairly large or are the combination of large number of uncertain parameters, the accuracy of the computational results will become unacceptable. Thus, it is highly desirable to present a more accurate method for computing the upper and lower bounds of responses of structures with fairly large uncertainties of interval parameters.

2.2 Theoretical Basis of Interval Analysis

In this thesis, matrix will be introduced in bold face and interval quantities (interval number, interval vector, interval matrix) will be denoted by attaching a superscript

“I” (Hansen, 1992), with the bracket “[.]” specifying the lower bound and upper bound.

2.2.1 Definition of Interval Arithmetic

Basically a real interval number is a closed set in $\mathbb{R}$ that includes the possible range of an unknown real number, where $\mathbb{R}$ denotes the set of real number. So, a real interval number x^I is a nonempty set of real numbers

$$x^I = [\underline{x}, \bar{x}], \{x \in \mathbb{R}: \underline{x} \leq x \leq \bar{x}\} \quad (2.6)$$

where $\underline{x}$ and $\bar{x}$ are the lower and upper bounds of the interval number x respectively.

The midpoint x^c of x is,

$$x^c = \text{mid}(x) = \frac{1}{2}(\underline{x} + \bar{x}) \quad (2.7)$$

The radius of x is,

$$r = \text{rad}(x) = \frac{1}{2}(\bar{x} - \underline{x}) \quad (2.8)$$

Therefore, x can be also expressed as midpoint and radius that is,

$$x^I = \text{midrad}(x^c, r) \quad (2.9)$$

$$\text{or } x^I = x^c(1 + \alpha) \quad (2.10)$$

in which α is a 0-midpoint interval. For example, when x has 2% uncertainty, it

means $\alpha = [-0.01, 0.01]$, and $x^I = x^c(1 + [-0.01, 0.01])$.

Intervals of complex numbers may be of two types. A rectangular complex interval is

$$z^I = x^I + iy^I, \{x+iy: x \in x^I, y \in y^I\} \quad (2.11)$$

It produces a rectangle of complex numbers in the complex plane. It can cause overestimation of result when $z_1^I \times z_2^I$ is calculated. An alternative interval is defined as circular complex interval that is,

$$z^I = \text{midrad}(a, r) \quad \{z \in \mathbb{C}: |z - a| \leq r\} \quad (2.12)$$

It can result in an overestimation that is less than rectangular complex interval used.

2.2.2 Interval Arithmetic Operations

Interval arithmetic operations on real numbers can be explicitly expressed as follows,

$$x^I + y^I = [\underline{x} + \underline{y}, \bar{x} + \bar{y}] \quad (2.13)$$

$$x^I - y^I = [\underline{x} - \bar{y}, \bar{x} - \underline{y}] \quad (2.14)$$

$$x^I \times y^I = \left[\min \{ \underline{x}\underline{y}, \bar{x}\bar{y}, \underline{x}\bar{y}, \bar{x}\underline{y} \}, \max \{ \underline{x}\underline{y}, \bar{x}\bar{y}, \underline{x}\bar{y}, \bar{x}\underline{y} \} \right] \quad (2.15)$$

$$1/x^I = [1/\bar{x}, 1/\underline{x}] \text{ if } \underline{x} > 0 \text{ or } \bar{x} < 0 \quad (2.16)$$

$$\frac{x^I}{y^I} = x^I \times (1/y^I) \quad (2.17)$$

For the elementary interval operations, division by an interval containing zero is not defined. Note that any real number a is denoted by a degenerate interval $[a,a]$.

From above definition of interval operations, it suggests that interval addition and interval multiplication are both associative and commutative. Although the distribution law does not hold true always, the sub-distributive law and the inclusion monotonicity law hold true (Neumaier, 1990).

2.2.3 Interval Algebraic Properties

Table 2.1 Interval Algebraic Properties

1.	$A^I + B^I = B^I + A^I, \quad A^I \times B^I = B^I \times A^I,$
2.	$(A^I + B^I) + C^I = A^I + (B^I + C^I), \quad (A^I \times B^I) \times C^I = A^I \times (B^I \times C^I),$
3.	<p>$X^I = [0,0]$ and $Y^I = [1,1]$ are the unique neutral elements with respect to addition and multiplication; that is,</p> <p>$A^I = X^I + A^I = A^I + X^I$ for all $A^I \in \mathbf{I}(R) \Leftrightarrow X^I = [0,0],$</p> <p>$A^I = Y^I \times A^I = A^I \times Y^I$ for all $A^I \in \mathbf{I}(R) \Leftrightarrow Y^I = [1,1],$</p> <p>The set of all intervals over R is denoted by $\mathbf{I}(R)$</p>
4.	$\mathbf{I}(R)$ has no zero divisors,
5.	An arbitrary element $A^I = [a_1, a_2] \in \mathbf{I}(R)$ with $a_1 \neq a_2$ has no inverse element with respect to $+$ and $\times$. Nevertheless it follows that

	$0 \in A^I - A^I \text{ and } 1 \in A^I \div A^I,$ $A^I(B^I + C^I) \subseteq A^I B^I + A^I C^I$ $\subseteq$ is defined as ‘contained in’.
6.	$a(B^I + C^I) = aB^I + aC^I, a \in R$ $A^I(B^I + C^I) = A^I B^I + A^I C^I$ if $bc \geq 0$ for all $b \in B^I$ and $c \in C^I$
7.	Let $A^{I(k)}, B^{I(k)} \in I(R), k = 1, 2$, and assume that $A^{I(k)} \subseteq B^{I(k)}, k = 1, 2.$ Then for the operations $* \in \{+, -, \cdot, /\}$ it follows that $A^{I(1)} * A^{I(2)} \subseteq B^{I(1)} * B^{I(2)}$
8.	$X^I \subseteq Y^I \Rightarrow r(X^I) \subseteq r(Y^I),$ $x \in X^I \Rightarrow r(x) \subseteq r(X^I)$ where $r(X^I)$ is a continuous unary operation on R , and $r(X^I) = [\min_{x \in X} r(x), \max_{x \in X} r(x)]$, such as, $X^{I^k} (k \in R), e^{X^I}, \ln X^I, \sin X^I, \cos X^I,$

The distance between two intervals $A^I = [a_1, a_2], B^I = [b_1, b_2] \in I(R)$ is defined as, $q(A^I, B^I) = \max\{|a_1 - b_1|, |a_2 - b_2|\}.$

The absolute value of an interval $A^I = [a_1, a_2] \in I(R)$ is defined as $|A^I| = q(A^I, [0, 0]) = \max\{|a_1|, |a_2|\}.$

Let A^I, B^I, C^I and D^I be the interval, then

Table 2.2 Further Properties of Interval Arithmetic

1.	$q(A^I + B^I, A^I + C^I) = q(B^I, C^I)$
2.	$q(A^I + B^I, C^I + D^I) \leq q(A^I, C^I) + q(B^I, D^I)$
3.	$q(aB^I, aC^I) = a q(B^I, C^I), \quad a \in R$
4.	$q(A^I B^I, A^I C^I) \leq A^I q(B^I, C^I)$
5.	$ A^I \geq 0 \quad \text{and} \quad A^I = 0 \Leftrightarrow A^I = [0, 0]$ $ A^I + B^I \leq A^I + B^I $ $ xA^I = x A^I \quad \text{for} \quad x \in R$ $ A^I B^I = A^I B^I $
The width of an interval $A^I = [a_1, a_2]$ is defined to be $d(A^I) = a_2 - a_1 \geq 0$.	
6.	$A^I \subseteq B^I \Rightarrow d(A^I) \leq d(B^I)$
7.	$d(A^I \pm B^I) = d(A^I) \pm d(B^I)$
8.	$d(A^I B^I) \leq d(A^I) B^I + A^I d(B^I)$
9.	$d(A^I B^I) \geq \max\{ A^I d(B^I), d(A^I) B^I \}$
10.	$d(aB^I) = a d(B^I) \quad a \in R$
11.	$d(A^{In}) \leq n A^I ^{-1}d(A^I), n = 1, 2, \dots$
12.	$d((X^I - x)^n) \leq 2(d(X^I))^n \quad \text{for} \quad x \in X, n = 1, 2, \dots$
13.	$d(A^I) = A^I - A^I $
14.	$A^I \subseteq B^I \Rightarrow \frac{1}{2}(d(B^I) - d(A^I)) \leq q(A^I, B^I) \leq d(B^I) - d(A^I)$
For an interval $C^I \in \mathbf{I}(R)$ with $0 \in C^I$ it furthermore follows that	

15.	$ C^I \leq d(C^I) \leq 2 C^I $
Let A^I and B^I be intervals, and assume that A^I is a symmetric interval.	
16.	$A^I B^I = B^I A^I$
17.	$d(A^I B^I) = B^I d(A^I)$

2.2.4 Interval Overestimation Property (Dependency)

In the interval calculation, sub-distributive law $X^I(Y^I + Z^I) \subseteq X^I Y^I + X^I Z^I$ holds, which is different from real number case. It can be concluded that more than once occurrence of some interval variables can result in the overestimation of calculation. Another cause in the overestimation in the interval method is that each interval quantity is considered as an independent variable even when they represent the same physical quantity. Therefore the interval expression $x^I - x^I$ is not equal to zero in the interval calculation. It calculates all the possible results of difference lying within x^I instead of the difference between two identical numbers. Ignorance of dependency does lead to the overestimation of results.

So reducing the overestimation is a crucial issue to a successful interval analysis. In general, sharp results are obtained with proper understanding of the physical nature of the problem and reduction of the dependence. In the above example $x^I - x^I$, the exact solution could be achieved by evaluating $x^I - x^I$ as $x^I(1 - 1) = 0$.

2.2.5 Interval Vectors and Matrices

An interval vector is a vector whose components are interval numbers. An interval matrix is a matrix whose elements are interval numbers. The set of $m \times n$ interval matrices is denoted by $I(R)^{m \times n}$. An interval matrix $\mathbf{A}^I = (A_{ij})$ is interpreted as a set of real $m \times n$ matrices by the convention,

$$\mathbf{A}^I = \{A \in R^{m \times n} \mid A_{ij} \in A^I_{ij} \text{ for } i = 1, \dots, m; j = 1, \dots, n\} \quad (2.18)$$

An interval matrix is one that contains all real matrices, whose elements are obtained from all possible values between the lower and upper bound of its interval elements.

The interval arithmetic operations can also be extended to matrix computations. Here, some interval matrix operations and some properties of interval matrix are introduced in Table 2.3.

Table 2.3 Interval Matrix Operations and Properties

1.	<p>Let $\mathbf{A}^I = (A^I_{ij})$ and $\mathbf{B}^I = (B^I_{ij})$ be two $m \times n$ interval matrices. Then,</p> $\mathbf{A}^I + \mathbf{B}^I = (A^I_{ij} + B^I_{ij})$ $\mathbf{A}^I \mathbf{B}^I = (\sum_{v=1}^r A^I_{iv} B^I_{vj})$ $\mathbf{A}^I \mathbf{u} = (\sum_{v=1}^r A^I_{iv} U_v)$ $\mathbf{X} \mathbf{A}^I = \mathbf{A}^I \mathbf{X} = (\mathbf{X} \mathbf{A}^I_{ij})$
2.	Useful properties for the above operations.

	$\mathbf{A}^I + \mathbf{B}^I = \mathbf{B}^I + \mathbf{A}^I$ $\mathbf{A}^I + (\mathbf{B}^I + \mathbf{C}^I) = (\mathbf{A}^I + \mathbf{B}^I) + \mathbf{C}^I$ $\mathbf{A}^I + 0 = 0 + \mathbf{A}^I = \mathbf{A}^I,$ $\mathbf{A}^I I = I \mathbf{A}^I = \mathbf{A}^I,$ $\begin{cases} (\mathbf{A}^I + \mathbf{B}^I) \mathbf{C}^I \subseteq \mathbf{A}^I \mathbf{C}^I + \mathbf{B}^I \mathbf{C}^I \\ \mathbf{C}^I (\mathbf{A}^I + \mathbf{B}^I) \subseteq \mathbf{C}^I \mathbf{A}^I + \mathbf{C}^I \mathbf{B}^I \end{cases}, \quad (\text{subdistributivity})$ $(\mathbf{A}^I + \mathbf{B}^I) I = \mathbf{A}^I I + \mathbf{B}^I I$ $I(\mathbf{A}^I + \mathbf{B}^I) = I \mathbf{A}^I + I \mathbf{B}^I$ $\begin{cases} \mathbf{A}^I (\mathbf{B}^I \mathbf{C}^I) = (\mathbf{A}^I \mathbf{B}^I) \mathbf{C}^I \\ (\mathbf{A}^I \mathbf{B}^I) \mathbf{C}^I \subseteq \mathbf{A}^I (\mathbf{B}^I \mathbf{C}^I) \\ \mathbf{A}^I (\mathbf{B}^I \mathbf{C}^I) = (\mathbf{A}^I \mathbf{B}^I) \mathbf{C}^I \\ \mathbf{A}^I (\mathbf{B}^I \mathbf{C}^I) = (\mathbf{A}^I \mathbf{B}^I) \mathbf{C}^I \end{cases}$ <p>The associative law is in general not valid for interval matrices.</p>
3.	<p>If $\mathbf{A}^I = (\mathbf{A}_{ij}^I)$ is an interval matrix, then</p> <p>The real nonnegative matrix</p> $d(\mathbf{A}^I) := (d(\mathbf{A}_{ij}^I))$ <p>The real, nonnegative matrix</p> $ \mathbf{A}^I := (\mathbf{A}_{ij}^I)$
4.	<p>Let $\mathbf{A}^I = (\mathbf{A}_{ij}^I)$ and $\mathbf{B}^I = (\mathbf{B}_{ij}^I)$ be two interval matrices. The real nonnegative matrix</p> $q(\mathbf{A}^I, \mathbf{B}^I) := (q(\mathbf{A}_{ij}^I, \mathbf{B}_{ij}^I))$
5.	<p>Let $\mathbf{A}^I, \mathbf{B}^I, \mathbf{C}^I, \mathbf{D}^I \in M_{mn}(R(C))$. Then it can be got</p> $\mathbf{A}^I \subseteq \mathbf{C}^I, \mathbf{B}^I \subseteq \mathbf{D}^I \Rightarrow \mathbf{A}^I \cap \mathbf{B}^I \subseteq \mathbf{C}^I \cap \mathbf{D}^I$

2.2.6 Combinatorial Approach (Vertex Method)

If $f(x_1^I, x_2^I, \dots, x_n^I)$ is an arbitrary monotonic function of imprecise parameters, $x_1^I, x_2^I, \dots, x_n^I$, the range of f^I can be determined by considering all possible combination of extreme value of imprecise parameters. It can be defined,

$$x_i^I = [\underline{x}_i, \bar{x}_i] = [x_i^1, x_i^2] \quad (2.19)$$

The possible value of f^I can be determined as

$$f_r = f(x_1^i, x_2^j, \dots, x_n^k) \quad (2.20)$$

where $i, j, \dots, k=1$ or 2 . Therefore there is 2^n possible f value for such interval parameters.

$$f^I = [\underline{f}, \bar{f}] = [\min(f_r), \max(f_r)] \quad \text{for } r=1, 2, \dots, 2^n \quad (2.21)$$

Since linear structural analysis often constructs interval model with monotonic property, Eq. (2.21) always gives the correct extreme values of the response parameters such as displacement and corresponding stresses. Therefore, the vertex solution can be used to validate the results for an interval analysis (Rao and Berke, 1997). This is the reason why many previous studies (Muhanna and Mullen, 1995; Rao and Berke, 1997) on interval analysis were compared with the vertex solutions for validation. However, the computational cost of vertex method is huge if the number of uncertain parameters n becomes large. For example, if there are 10 uncertain parameters in a structural system, we need to perform $2^{10} = 1024$ times structural analysis to obtain the vertex solution.

2.3 Interval Finite Element Method (IFEM)

Recently, various generalized models of uncertainty have been applied within the context of the finite element method to solve partial differential equations with uncertain parameters. Regardless of the model adopted, the proper interval solution represents the first requirement for any further rigorous formulation. The use of IFEM solution techniques can be broadly classified into two groups, namely the optimization approach and the non-optimization approach. In the optimization approach (Koyluoglu et al., 1995), optimizations are performed to compute the minimal and maximal structural responses when the uncertain parameters are constrained as intervals.

This approach often encounters practical difficulties. Firstly, it requires an efficient and robust optimization algorithm. In most structural engineering problems, the interval finite element objective function is nonlinear and complicated, thus usually only an approximate solution is achievable. Secondly, this approach is computationally expensive. For each response quantity, two optimization problems must be solved to find the lower and the upper bounds.

More recently, non-optimization approaches for IFEM have been developed in a number of papers (Muhanna and Mullen, 2001; Rao and Berke, 1997). For linear elastic problems, this approach leads to a system of linear interval equations, then the solution is sought using various methods developed for this purpose. The major difficulty associated with this approach is the “dependency problem” (Hansen, 1992; Moore, 1979). The dependency in interval arithmetic leads to an overestimation of the solution.

A straightforward replacement of the system parameters (Muhanna and Mullen, 2001) with interval ones without taking care of the dependency problem is known as a direct application of interval arithmetic in the finite element method. Usually this method may lead to meaninglessly wide and even catastrophic results.

In the non-optimization category, a combinatorial approach (based on an exhaustive combination of the extreme values of the interval parameters) was used in the work done by Rao and Berke (1997). This approach gives exact solution in linear elastic problems. However, it is computationally tedious and expensive, and is limited to the solutions of small-scale problems. A convex modeling and superposition approach was proposed to analyze load uncertainty in the work done by Pantelides and Ganzerli (2001), and exact solution was obtained. However, the superposition is only applicable to load uncertainty. The combinatorial approach was used in the work done by Pantelides and Ganzerli (1999) to treat interval modulus of elasticity.

2.3.1 Static Interval Analysis

In this section, several typical methods of static analysis of structures are introduced.

The utility of interval methods in predicting the extreme values of the response parameters of structures was discussed by Rao and Berke (1997). Their work considered the modeling of uncertain structural systems using interval analysis. By representing each uncertain input parameter as an interval number, a static structural analysis problem could be expressed in the form of a system of linear interval equations. In addition to the direct and Gaussian elimination-based solution approaches, a combinatorial approach and an inequality-based method were presented to find the solution of interval equations. The range or interval of the

solution vector (response parameters) was found to increase with increasing size of the problem in all of the methods.

In Rao and Berke's work (1997), it had been observed that the widths (intervals) of the response parameters predicted become wider than the true widths with an increase in the size of the problem. To avoid the unnecessary growth of the intervals of the response parameters, a computationally efficient approximation procedure, termed the truncation method, was presented. The truncation method could be introduced as below.

Suppose c^I is the normal interval analysis results.

$$c^I = [\underline{c}, \bar{c}] \quad (2.22)$$

Then calculate a new interval value associated with c

$$d^I = [\underline{d}, \bar{d}] = [c^c + t(\underline{c} - c^c), c^c + t(\bar{c} - c^c)] \quad (2.23)$$

where c^c is the crisp case result, and t is the parameter specified for truncation. The crisp case refers to the case without uncertainty. The new interval analysis result e^I with truncation will be the inner and narrow range combined with c^I and d^I interval.

$$e^I = [\max\{\underline{c}, \underline{d}\}, \min\{\bar{c}, \bar{d}\}] \quad (2.24)$$

It was observed that the direct, the Gaussian elimination, and the inequality-based approaches tend to overestimate the extreme values of the response quantities, especially with increasing widths (ranges) and number of the uncertain input

parameters. However, the procedure can predict the extreme values of response accurately when the widths and number of uncertain parameters are small. When the widths or ranges of the uncertain parameters are large, the truncation procedure can be used to obtain reasonable estimates of the largest and smallest values of the response quantity. It was found that a value of the truncation parameter t equal to the maximum relative width of the interval numbers appearing in the problem data gives reasonably accurate results (which are comparable to the exact intervals given by the combinatorial approach).

Perturbation of structural system response is able to be estimated using the technique of stochastic finite element method (Nakagiri and Kubota, 1995), when the probability distributions on the uncertainties of the parameter are known. However, the evaluation of the structural response as probability distribution is unknown and as only the width of the perturbation of system parameter is known is realistically important (Koyluoglu et al. 1995). Matsumoto and Iwaya (2000) presented an interval finite element analysis (IFEA) method which deals with the structural system parameters as interval numbers with a certain width. To do this, the solution was approached by two kinds of formulation. In the first, an interval version of FEA was obtained from a standard one using ordinary real arithmetic by simply replacing each step by the corresponding interval arithmetic step. Then, the way of the construction of the element stiffness matrix was modified in order to obtain sharper bounds of response quantities. In the second approach, the interval version of FEA was formulated as an optimization problem in which all components of displacement vector were variables as well as the parameters of the structural system with uncertainties and a specific component of displacement vector was minimized/maximized as an objective function.

Muhanna and Mullen (2001) presented a non-traditional uncertainty treatment for

mechanics problems. Uncertainties were introduced as bounded possible values (intervals). To account for different types of uncertainties in linear static problems, an interval linear system of equations was developed, and this formulation was based on the element-by-element (EBE) technique. The equations are introduced as below.

In steady-state analysis, with the total potential energy method and Lagrange multiplier method, the structural model can be given,

$$\begin{bmatrix} \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^I \\ \boldsymbol{\lambda}^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}^I \\ \mathbf{t} \end{Bmatrix} \quad (2.25)$$

where $\mathbf{K}^I$ is the stiffness matrix, $\mathbf{u}^I$ is displacement vector, $\mathbf{p}^I$ is load vector, $\mathbf{Q}$ is a matrix of order $m \times n$ and $\boldsymbol{\lambda}^I$ is a vector of m Lagrange multipliers. The proposed interval formulation is based on the EBE technique. Each element has its own set of nodes, but the set of elements is disassembled, so that a node belongs to a single element. A set of additional constraints is introduced to force unknowns, associated with coincident nodes, to have identical values. The constraint equation $\mathbf{Q}\mathbf{u}^I = \mathbf{t}$ takes the form

$$\mathbf{Q}\mathbf{u}^I = \mathbf{0} \quad (2.26)$$

Then, Eq. (2.25) becomes,

$$\begin{bmatrix} \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^I \\ \boldsymbol{\lambda}^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}^I \\ \mathbf{0} \end{Bmatrix} \quad (2.27)$$

Eq. (2.27) represents the EBE interval-based formulation. A guaranteed enclosure for the solution of interval linear system of equations can be achieved and usually is not sharp and very conservative. However, an exact enclosure is not known to be

obtained in the general case of such systems. In their work, a very sharp enclosure for the solution set, due to loading, material and geometric uncertainty, was obtained.

Using the EBE technique, Muhanna et al. (2007) presented an interval approach for the treatment of parameter uncertainty for linear static structural mechanics problems. Interval analysis was applied to FEM to analyze the system response due to uncertain stiffness and loading. To avoid overestimation in interval arithmetic, the formulation was based on the EBE technique. Element matrices were formulated, based on the physics of materials, and the Lagrange multiplier method was applied to impose the necessary constraints for compatibility and equilibrium. Most sources of overestimation were eliminated, and a very sharp solution was obtained. Based on the developed formulation, the bounds on the system's displacements and element forces were obtained simultaneously and had the same level of accuracy.

With a series of IFEA of linear truss structures with interval parameters being investigated, a new method called Interval Factor Method (IFM) was proposed by Gao (2007a). Truss structures were used to illustrate examples of this method, in which structural physical parameters (Young's modulus), geometry (length and cross-sectional area of bar) and applied forces were considered as interval variables. The procedure of the IFM was as follows. Firstly, an interval variable (structural parameter or load) was expressed as an interval factor multiplied by the mean value of this interval variable. Secondly, the structural stiffness matrix was expressed as its deterministic value multiplied by the interval factors of the structural parameters. Finally, the structural static responses were expressed as the functions of these interval factors, and the computational expressions for interval structural displacement and stress responses could be obtained by means of the interval

operations. Therefore, the effect of the change of structural parameters and loads on the structural displacement and stress responses could be easily identified. This method allowed for uncertainty in the material parameters, geometric dimensions and applied forces. In addition, the lower and upper bounds of structural displacement and stress response could be obtained by using the computational expressions once the deterministic structural static responses obtained by the traditional FEA.

Gao et al. (2010) presented a hybrid probabilistic and interval method for engineering problems described by a mixture of random and interval variables. Random interval arithmetic, random interval moment method and random interval perturbation method are proposed to deal with the problems or systems having both random and interval variables. The random interval moment method for calculating the expectation and variance of random interval variables is developed by combining a random variable functional method and interval analysis. Two simple applications involving random interval variables are analyzed by using direct probabilistic and interval analyses. The accuracy of the random interval perturbation method, based on the first-order perturbation technique, is demonstrated. It should be noted that the accuracy of the random interval perturbation method can be further improved by a second-order perturbation, but this will require a greater computational effort, especially for large systems. Extensive applications to the static and dynamic analysis of structures with random and interval parameters/inputs form a worthwhile future study. The random interval perturbation method can be combined with finite element method to analyze many types of structures such as trusses, frames, plates, and other complex structures. However, it is very hard to use the current form of this method to predict the random interval response of structures with non-linear non-deterministic parameters. Methods based on perturbation theory cannot guarantee conservative results as the

higher order terms are neglected in computations. Sometimes, results obtained by interval analysis are not sharp.

An interval Monte Carlo method is developed by Zhang et al. (2010) for reliability assessment under parameter uncertainties represented by confidence intervals. The interval information of unknown parameters and the inherent uncertainties are propagated separately through reliability analysis. The IFEM is utilized to model the ranges of structural responses accurately and efficiently. Interval estimates of failure probability are computed which can provide a statement of confidence in the results of the reliability estimate. A wide interval p_f implies that epistemic uncertainties are large, thus additional data should be collected. The developed method can also be used to study the sensitivities of failure probability with respect to the changes in distribution parameters. The proposed interval Monte Carlo method and the Bayesian approach can obtain interval estimates on the limit state probability and provide decision-makers and designers useful information about the variability in reliability estimates. The interval p_f obtained from the proposed method tends to be wider than that from the Bayesian approach. This is to be expected since the interval approach only uses the information of the confidence bounds on the unknown parameters, while in the Bayesian approach prior information is assumed and incorporated in analysis.

2.3.2 Dynamic Interval Analysis

In many dynamic problems of engineering, one often encounters the following problem: given the stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$ of order n , determine the value of a scalar $\lambda = \omega^2$ and a nonzero vector $\mathbf{u}$ which satisfy the equation

$$\mathbf{K}\mathbf{u} = \lambda\mathbf{M}\mathbf{u} \quad (2.28)$$

Eq. (2.28) is called the generalized eigenvalue problem. $\lambda = \omega^2$ is called the eigenvalue of $\mathbf{K}$ and $\mathbf{M}$, and $\mathbf{u}$ is the associated eigenvector. The eigenvalues are generally assumed to be constant for identical structural systems. However, elements of matrices $\mathbf{K}$ and $\mathbf{M}$ can be neither measured exactly nor calculated exactly in experiments, which results in the variation of eigenvalues. The incomplete information about the elements of matrices $\mathbf{K}$ and $\mathbf{M}$ is a result of errors in observation, measurement errors, change in operating conditions, ageing, maintenance- induced errors, etc. In such cases we do not know the exact values of the elements of matrices $\mathbf{K}$ and $\mathbf{M}$. We know only the ends of intervals in which the elements of the matrices $\mathbf{K}$ and $\mathbf{M}$ are confined. Thus, the eigenvalues are uncertain variables whose uncertain properties are determined by the uncertain elements of the matrices $\mathbf{K}$ and $\mathbf{M}$.

The research in the interval eigenvalue problem has begun to emerge in recent years. Extensive research (Juang et al., 1987; Commercon, 1994; Kolev and Petrakieva, 2005) has been carried out by control engineers from the view point of stability and robustness analysis. Some works on the numerical methods for the interval eigenvalue problem have been reported in literature (Rohn, 1987; Hudak, 1984; Deif, 1991). Rohn (1987) studied the symmetric interval matrix and ended up with the formulae when the width of interval matrix has rank one. Hudak (1984) verified that the spectrum of eigenvalues of an interval matrix family was found to depend on finding a constant matrix under the constraint with the given interval matrix. Hallot and Bartlett (1987) discovered that the spectrum of the eigenvalues of an interval matrix family depends on the spectrum of its extremes set. Based on the invariance properties of the characteristic vectors' entries, Deif (1991) obtained a numerical method for interval eigenvalues. Qiu et al. (1993) extended Deif's method to the generalized interval eigenvalue problem. In structural analysis, Qiu et

al. (1995) used interval set models to evaluate the eigenvalue problems of structures with bounded uncertain parameters. Later, Qiu et al. (1996) presented a method based on an interval perturbation approximating formula. This method moved away certain limitations for matrices required by the Deif's method. Leng and He (2007) presented an interval perturbation method for the interval eigenvalue problem. The reliable upper and lower eigenvalue bounds for the standard interval eigenvalue problem can be obtained by their method. It is unconditional and very easy to be applied. If the interval bounding matrices $\underline{A}$ and $\overline{A}$ are very close, that is, the uncertain parameters vary within a small range, the Leng and He's method can produce very good bounds for eigenvalue intervals.

In 2003, a new method to solve the uncertain eigenvalue problem of structures with interval parameters was presented by Chen et al. (2003). The expressions of the interval stiffness matrix and interval mass matrix were developed directly from the interval parameters. The computations were based on the element level thus simplifying the calculations greatly, while the sharp bounds were guaranteed by the interval operations.

The eigenvalue problems of structures with multi-parameter were solved by Chen et al. (2009) using interval theory and the second-order Taylor series. In their method, the epsilon-algorithm was used to accelerate the convergence of the Neumann series to obtain the bounds of eigenvalues of structures with single interval parameter, thus increasing the computing accuracy and reducing the computational effort. By means of the stationary condition of Rayleigh Quotient and the minimax theorem of eigenvalues, the generalized eigenvalue problem of structures with bounded uncertain parameters was transformed into two different generalized eigenvalue problems by Chen et al. (1994). Leng and He (2010) presented a new procedure with two algorithms for solving the problem which can yield precise eigenvalue

bounds. This method can also be applied to the interval eigenvalue problems with parameter dependence.

The natural frequencies of a structural system are perhaps the most essential characteristics in determining the dynamic behavior of the structure. In deterministic analysis of structural systems, the closed-form functional or approximate numerical schemes are used to determine the natural frequencies and their corresponding mode shapes. The FEM is one of the schemes that accomplish this task by performing a generalized eigenvalue problem using the structure's stiffness and mass matrices. However, in the presence of uncertainty in the geometric and/or material characteristics of the system, the determination of bounds on natural frequencies requires two major considerations: first, modifications in the representation of the characteristics due to the existing uncertainty and second, development of schemes that are capable of considering the presence of uncertainty throughout the solution process. Those developed schemes must be consistent with the system's physical behavior and also be computationally feasible.

The method for computing the upper and lower bounds on natural frequencies of structures with bounded uncertain (or interval) parameters were investigated by Chen et al. (1994). The stiffness matrix and the mass matrix of structures, whose elements had the initial errors, were unknown except for the fact that they belonged to given bounded matrix sets. The set of possible matrices could be described by the interval matrix. By means of the stationary condition of Rayleigh Quotient and the min/max theorem of eigenvalues, the generalized eigenvalue problem of structures with bounded uncertain parameters could be transformed into two different generalized eigenvalue problems.

In the previous works only direct estimation methods for the interval of natural

frequency were proposed. The modal interval analysis method to estimate modal parameters, frequency response function (FRF), and mode shapes of structures with uncertain-but-bounded parameters was presented by Sim et al. (2007). By using modal analysis and interval calculus, the method of computing upper and lower bounds of parameters such as, natural frequencies, modal shapes, and FRFs were investigated. Theoretically, it was possible to analyze the uncertain-but-bounded parameters of a structure by using modal analysis and interval calculus. They could be estimated by modal interval analysis. On the basis of the estimated intervals, the engineering structure parameters could be applied into engineering design.

In the framework of dynamic structural analysis, an IFEM to calculate envelope FRF of uncertain structures was developed by Moens and Vandepitte (2004). In their work, the undamped systems were considered and the methodology was based on the modal superposition principle. The modal rectangle method (MR), the modal rectangle method with eigenfrequency interval correction (MRE) and the locally optimized modal rectangle method with eigenfrequency interval correction (OMRE) were used. However, in the MR method, modes with modal parameters that reach zero inside the input parameter space were omitted from the superposition. In the MRE method, the eigenfrequency ranges was added to improve the modal parameter range approximation. So the exact eigenfrequency bounds were important in the MRE. In the OMRE method, the reduction of the conservatism in the FRF was controlled by choosing the discrete frequencies.

Moens and Vandepitte (2007) introduced the concept of interval sensitivities, applicable in the context of general numerical interval analysis. This concept enables the quantification of the relative effect of individual uncertain input parameters on the output range obtained by the interval analysis. A high interval sensitivity of the output with respect to a specific uncertain input parameter

indicates a strong influence of this parameter on the obtained uncertainty on the result of the analysis. The response range sensitivities are obtained as analytic functions of end point gradients of the model parameters, which follow directly from the optimization part of this algorithm. Consequently, the calculation of the response range sensitivities is computationally not more expensive than the original interval FRF analysis. The interval sensitivity concept is also shown to be appropriate for the analysis of the effect of input interval uncertainties on the resulting uncertainty on the output side of the problem, as it helps to identify the most crucial uncertainties in the defined problem.

The interval dynamic characteristics analysis of interval structures was investigated (Gao, 2007b) using the IFM proposed by Gao (2007a). The computational expressions for the lower and upper bounds and interval changed ratio of the natural frequency and mode shape were derived from Rayleigh quotient by means of the interval operations. The influences of the change of the structural parameters on the natural frequency and mode shape were demonstrated by using truss structures.

Based on the matrix perturbation theory and interval extension of function, the upper and lower bounds of dynamic response were obtained by Chen et al. (2002a) using Taylor series expansion. In their method, the stiffness matrix, mass matrix and damping matrix are expressed as,

$$\begin{aligned}
\mathbf{K}_i^{le}(b^l) &= \mathbf{K}_i^{le}(b^c) + \sum_{j=1}^m \mathbf{K}_{i,j}^c (b_j^l - b_j^c) \\
\mathbf{M}_i^{le}(b^l) &= \mathbf{M}_i^{le}(b^c) + \sum_{j=1}^m \mathbf{M}_{i,j}^c (b_j^l - b_j^c) \\
\mathbf{C}_i^{le}(b^l) &= \mathbf{C}_i^{le}(b^c) + \sum_{j=1}^m \mathbf{C}_{i,j}^c (b_j^l - b_j^c)
\end{aligned} \tag{2.29}$$

Based on the above approach, the lower and upper bounds of dynamic responses of structures are obtained. However, the computational accuracy is limited if the number and width of the interval parameters are too large because the method is based on the 1st order perturbation.

Subsequently, Chen and Wu (2004) and Chakraborty and Roy (2011) extended the matrix perturbation theory and interval extension of function method to the vibration control of structures subjected to parametric uncertainties. Qiu and Wang (2003; 2005) also proposed two similar methods based on Taylor series expansion and parameter perturbation. The Taylor series expansion method was later extended to nonlinear dynamic analysis by Qiu et al. (2009) and Rama Rao et al. (2010). Gao et al. (2007a) presented the IFM to calculate the dynamic response of trusses. However, in the IFM method, it is required that the uncertainties are synchronized (i.e., all the uncertain parameters vary synchronically and achieve their minimum or maximum values at the same time.) according to the interval factor and the results are quite conservative. In summary, the perturbation theory and Taylor series expansion have been used by many researchers to obtain the dynamic response. However, the computational cost is high since it is required to solve a system of n -dimensional 2nd ordered linear ordinary differential equations (ODE) for m times, where n is the degree of freedom (DOF) of the problem and m is the number of independent intervals.

The uncertainty concept plays an important role in the control problems of the vibration structures. Many studies (Zhang et al., 2007) have been conducted on control problems of the vibration of structure subjected to parameter uncertainties, though upper and lower bounds of dynamic response of closed-loop systems are not investigated adequately. In the work of Zhang et al. (2007), the interval analysis was used to deal with the control problems with uncertain parameters. The

uncertain parameters were described by the interval model. The control problems of the uncertain systems were transformed into ones of the deterministic systems. By using the method of pole allocation, the state feedback gain matrix of the system with deterministic parameters was obtained, and then it was applied to the actual uncertain systems. Using the interval analysis and IFEM, a new method for estimating the upper and lower bounds of the dynamic response of closed-loop system with interval parameters was used. Another theoretical basis of the research was the application of the precise time integration (PTI) method, which was a time stepping method of ODEs proposed by Zhong and Williams (1994). The method had many numerical advantages such as unconditionally stable and highly precise numerical results, which approached the full computer precision.

In general interval analysis, all uncertain parameters are considered to act simultaneously. While this is often the most realistic representation of the physical condition of the actual product, for design purposes, it is of great value to know the contribution of the individual uncertainties to the response range obtained from the interval analysis. This enables a designer to distinct between the non-deterministic influences that have an important contribution to the fuzziness on the dynamic behavior of the design, and those that have little or no influence. This distinction is very valuable in the definition of e.g. tight design tolerances or realistic allowable working conditions, and as such, could lead to less conservative designs. In dynamic analysis with additive uncertainties, assuming ‘simultaneous action’ or ‘synchronized uncertainty’ will lead to very conservative scenario which is also unrealistic. For this context, the analysis with all uncertain parameters considered not to act simultaneously is studied in this thesis.

2.4 Summary

In the static IFEA, several methods, such as Gaussian elimination-based solution approaches, inequality-based method, interval-truncation approach, stochastic FEM, EBE technique and IFM, were used to overcome the overestimation in interval calculations. All these methods showed certain improvement in the accuracy of the solution. In this thesis, the EBE technique will be used due to its general and easy implementation in FEA.

In the dynamic IFEA, several problems, such as eigenvalue problem, natural frequency of structure, FRF, modal shape of structure, dynamic responses and control problems of the vibration structures, were studied. But in these problems, the overestimation problem was not well solved. Thus, in this thesis, the overestimation issue in interval dynamic analysis will be addressed by developing more accurate and efficient IFEA methods.

CHAPTER 3 STATIC ANALYSIS OF TRUSS STRUCTURES USING INTERVAL METHOD

3.1 Introduction

In this chapter, interval finite-element approach will be used for uncertainty treatment in linear truss structures. Stiffness and load parameters are considered to have unknown values that are included in defined possible ranges. Uncertainty is introduced by assigning interval values for parameters in both stiffness matrices and load terms in the finite element formulation.

As mentioned in Chapter 2, the main difficulty in interval computation is the overestimation due to variable dependency and failure of some algebraic laws that are valid only in real deterministic arithmetic. Such overestimation may produce extremely conservative and even meaningless results. To overcome this difficulty, the challenge is to eliminate overestimation sources to achieve the sharpest possible results.

Therefore, a formulation, based on the EBE technique will be developed and used with the IFEA. Based on this EBE-IFEA technique, a general IFEM code will be developed for static analysis of truss structures. A number of illustrative examples will be presented to show the effectiveness of the technique.

3.2 Solving System of Linear Interval Equations

3.2.1 Direct Method

It is common to construct a model in the engineering analysis,

$$Ax = b \quad \text{where} \quad (A \in A^I, b \in b^I) \quad (3.1)$$

The solution of Eq. (3.1) is understood to mean

$$x = \{x | Ax = b, A \in A^I, b \in b^I, x = bA^{-1}\} \quad (3.2)$$

Eq. (3.2) denotes a set of solutions in x space such that for every $A \in A^I, b \in b^I$ there existis a x that satisfies the equation.

The solution set and its hull of Eq. (3.2) have been investigated by some researchers (Rohn, 1995). A number of methods have been developed to find x for the general linear interval equations such as interval Gauss elimination, interval Gauss-Seidel iteration, Krawczyk's iteration, and fixed-point iteration (Gay, 1982; Jansson, 1991; Neumaier, 1987 and 1990; Rump, 1983 and 1992).

To directly solve these interval equations, a new method was proposed by Hargreaves (2002). It was an improved version based on the Krawczyk's method and Hansen-Bliek-Rohn-Ning-Kearfott-Neumaier's method (Neumaier, 1999; Ning and Kearfott, 1997). Hargreaves (2002) integrated this method into Matlab toolbox as Intlab and compared the calculation results with other methods. Hereafter, this method is referred to as the direct method.

However, the overestimation laid in the interval method is noticed, typically solving the linear interval equations. There are currently several techniques to reduce this effect. Among them, the direct approach tries to express the solution explicitly in order to reduce the multiple occurrence of interval variable and avoid the redundant

independency. The Brouwer's fixed point theorem has been proven to possess high accuracy in interval analysis (Muhanna et al., 2007). Therefore, in this work, the Brouwer's fixed point theorem will be used and presented in the next section.

3.2.2 Brouwer's Fixed Point Theorem

One typical approach to find the solution of a linear system $A^I y^I = p^I$ is to transform it into a fixed point equation $\varphi(y^I) = y^I$, in which

$$\varphi(y^I) = y^I - R(A^I y^I - p^I) = R p^I + (I - R A^I) y^I \quad (3.3)$$

where R is a nonsingular matrix, and all the parameters except R are the interval values. From the Brouwer's fixed point theorem, it follows,

$$R p^I + (I - R A^I) y^I \in y^I \quad (3.4)$$

If an interval vector y^I satisfying Eq. (3.4) can be found, then y^I contains the solution of $A^I y^I = p^I$.

Based on this result, it can be derived (Rump, 2001),

$$R p^I + (I - R A^I) y^I \subseteq \text{int}(y^I) \quad (3.5)$$

where $\text{int}(y^I)$ means the interior of y^I . Eq. (3.5) can also be given in the form (Neumaier, 1990),

$$R p^I - R A^I y_0 + (I - R A^I) y^{*I} \subseteq \text{int}(y^{*I}) \quad (3.6)$$

where $\mathbf{y}^I = \mathbf{y}_0 + \mathbf{y}^{*I}$ and $\mathbf{y}_0$ is a real vector. In particular, $\mathbf{A}^{c-1}$ is a good choice for $\mathbf{R}$ (Neumaier, 1990), and $\mathbf{y}_0 = \mathbf{R}\mathbf{p}^c$. Assigning $\mathbf{z}^I = \mathbf{R}\mathbf{p}^I - \mathbf{R}\mathbf{A}^I\mathbf{y}_0$, $\mathbf{H}^I = (\mathbf{I} - \mathbf{R}\mathbf{A}^I)$, iteration can be constructed in the following form,

$$\mathbf{y}^{*n+1I} = \mathbf{z}^I + \mathbf{H}^I\mathbf{y}^{*nI} \quad (\text{for } n=0,1,2,\dots) \quad (3.7)$$

The stopping criteria for Eq. (3.6) becomes

$$\mathbf{y}^{*n+1I} \subseteq \text{int}(\mathbf{y}^{*nI}) \quad (3.8)$$

The above algorithm is designed for non-parametric linear interval equations; i.e., the coefficients in the system are assumed to vary independently between their bounds. For many engineering problems, the coefficients have complex dependency relations. For example, the stiffness matrix in FEM is symmetric and positive definite. To account for the dependency effect, one approach is to adapt the solver for non-parametric interval equations. This approach usually involves reformulation of the coefficient matrix and the right hand side vector. The Brouwer's fixed point theorem has been shown to be efficient and accurate for solving linear interval equations (Muhanna et al., 2007). Thus it will be adopted in this thesis.

3.3 Static IFEA for Truss Structures

3.3.1 Overestimation in IFEA

We start with the example of two bar truss shown in Fig. 3.1 to show the overestimation in IFEA. The truss is subjected to a unit load at node 3. Based on the direct method of IFEA formulation, the equilibrium equation is given in the form,

$$[\mathbf{K}^I]\{\mathbf{U}^I\} = \{\mathbf{F}^I\} \quad (3.9)$$

where $[\mathbf{K}^I]$ is the stiffness matrix, $\{\mathbf{U}^I\}$ is the vector of displacement, and $\{\mathbf{F}^I\}$ is the vector of loads.

Figure 3.1. Two bar truss

Eq. (3.9) can be rewritten in the explicit form,

$$\begin{bmatrix} \mathbf{k}_1^I + \mathbf{k}_2^I & -\mathbf{k}_2^I \\ -\mathbf{k}_2^I & \mathbf{k}_2^I \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1^I \\ \mathbf{u}_2^I \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad (3.10)$$

where $\mathbf{k}_1^I$ and $\mathbf{k}_2^I$ are the interval stiffness parameters of the first and second finite element, respectively.

If the interval numbers of $[0.98, 1.02]$ and $[1.96, 2.04]$ are assigned for $\mathbf{k}_1^I$ and $\mathbf{k}_2^I$ respectively, the direct method of IFEM takes the following form,

$$\begin{bmatrix} [2.94, 3.06] & [-2.04, -1.96] \\ [-2.04, -1.96] & [1.96, 2.04] \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1^I \\ \mathbf{u}_2^I \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad (3.11)$$

The solution of $\begin{Bmatrix} \mathbf{u}_1^I \\ \mathbf{u}_2^I \end{Bmatrix}$ is obtained by Krawczyk operator. The details can be found

in Intlab (Hargreaves, 2002).

$$\begin{cases} \mathbf{u}_1^I = [0.723, 1.278] \\ \mathbf{u}_2^I = [1.160, 1.840] \end{cases} \quad (3.12)$$

On the other hand, the exact solution can be achieved by solving Eq. (3.10) symbolically,

$$\begin{aligned} \mathbf{u}_1^I &= \frac{1}{\mathbf{k}_1^I} = \frac{1}{[0.98, 1.02]} = [0.980, 1.021] \\ \mathbf{u}_2^I &= \frac{\mathbf{k}_1^I + \mathbf{k}_2^I}{\mathbf{k}_1^I \mathbf{k}_2^I} = \frac{1}{\mathbf{k}_1^I} + \frac{1}{\mathbf{k}_2^I} = \frac{1}{[0.98, 1.02]} + \frac{1}{[1.96 + 2.04]} \\ &= [1.471, 1.531] \end{aligned} \quad (3.13)$$

Comparing Eq. (3.12) and (3.13), the overestimation of Eq. (3.12) is apparent. As noted earlier, the amount of overestimation depends on the width of interval quantities, the problem size and the problem complexity. The two main factors that lead to overestimation are the element coupling and multiple occurrences of the interval variables. Overcoming the dependency problem in the two bar example is easy to achieve by the exact solution in Eq. (3.13). However, it is difficult to do so if the structure becomes more complicated since the exact formulation is not available.

3.3.2 Direct Method in IFEA

In the context of the finite element analysis, the governing equation for displacements is in the form of Eq. (3.9). The stiffness matrix $[\mathbf{K}^I]$ is the global stiffness matrix of the structure. The applied forces are considered as interval

variables and can be expressed as,

$$\{\mathbf{F}^I\} = \{f_1^I, \dots, f_i^I, \dots, f_n^I\}^T \quad (i=1, \dots, n) \quad (3.14)$$

where f_i^I is the force applied on the i^{th} node.

Directly solving Eq. (3.9), the result of the equation can be easily obtained as following,

$$\{\mathbf{U}^I\} = [\mathbf{K}^I]^{-1}\{\mathbf{F}^I\} \quad (3.15)$$

If the information of truss system property as $\mathbf{K}^I$ and $\mathbf{F}^I$ is given, the interval displacements of the uncertain structure will be obtained. But this method does not reduce any overestimation in the analysis, as shown in Eq. (3.12), because it does not reduce the occurrence of element stiffness in the interval calculation. Therefore, the direct IFEA method is not desirable.

3.3.3 EBE-IFEA Method

To reduce the overestimation in the interval finite element solutions, the issues of coupling and multiple occurrences of interval variables have to be handled properly. Therefore, two techniques are adopted here to solve these problems. One is the EBE method and the other is a special treatment of matrix formulation that eliminates the multiple occurrences of interval variables.

The EBE method can be shown using the stiffness matrix. Suppose that there are m elements in the analyzed truss structure. The stiffness matrix $\mathbf{K}$ of truss structure in global coordinate can be expressed as,

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_1^e & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_2^e & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_m^e \end{bmatrix} \quad (3.16)$$

where m is the number of elements.

$$\mathbf{K}_i^e = \mathbf{T}_i^{eT} \times \frac{E_i^e A_i^e}{L_i^e} \times \mathbf{G} \times \mathbf{T}_i^e \quad (3.17)$$

where $\mathbf{K}_i^e$, E_i^e , A_i^e and L_i^e are the i^{th} element's stiffness matrix, Young's modulus, cross sectional area and length, respectively. $\mathbf{G}$ is a 6×6 matrix, where $g_{11} = g_{44} = 1$, $g_{14} = g_{41} = -1$ and other elements are zero for three-dimensional truss elements. $\mathbf{T}_i^e$ is the 6×6 transformation matrix that translates the local coordinates of the i^{th} element to the global coordinates, and $\mathbf{T}_i^{eT}$ is its transpose.

The EBE approach increases the number of degrees of freedom (DOF) in the system but avoids the element coupling. This EBE method is also applicable for the interval variables. Therefore, the interval stiffness can be expressed as,

$$\mathbf{K}^I = \begin{bmatrix} \mathbf{K}_1^{Ie} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_2^{Ie} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_m^{Ie} \end{bmatrix} \quad (3.18)$$

The treatment of eliminating the multiple occurrences of interval variables can be illustrated by the two bar problem in Fig. 3.1. Each element stiffness matrix $\mathbf{K}^{\mathbf{e}^1}$ in the two bar truss is,

$$\mathbf{K}_1^I = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha_1^I & 0 \\ 0 & \alpha_1^I \end{bmatrix}$$

$$\mathbf{K}_2^I = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha_2^I & 0 \\ 0 & \alpha_2^I \end{bmatrix} \quad (3.19)$$

where α_i^I is the interval quantity. So the system stiffness matrix $\mathbf{K}^I$ can be written as,

$$\mathbf{K}^I = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha_1^I & 0 & 0 & 0 \\ 0 & \alpha_1^I & 0 & 0 \\ 0 & 0 & \alpha_2^I & 0 \\ 0 & 0 & 0 & \alpha_2^I \end{bmatrix} \quad (3.20)$$

When the stiffness matrix multiplies the displacement vector, the terms α_i^I and x_i^I can be treated as,

$$\begin{bmatrix} \alpha_1^I & 0 & 0 & 0 \\ 0 & \alpha_1^I & 0 & 0 \\ 0 & 0 & \alpha_2^I & 0 \\ 0 & 0 & 0 & \alpha_2^I \end{bmatrix} \begin{Bmatrix} x_1^I \\ x_2^I \\ x_3^I \\ x_4^I \end{Bmatrix} = \begin{bmatrix} x_1^I & 0 \\ x_2^I & 0 \\ 0 & x_3^I \\ 0 & x_4^I \end{bmatrix} \begin{Bmatrix} \alpha_1^I \\ \alpha_2^I \end{Bmatrix} = [\mathbf{X}^I][\Lambda^I] \quad (3.21)$$

The above procedure shows how generally $[\alpha^I][x^I]$ can be rewritten as a new formulation $[\mathbf{X}^I][\Lambda^I]$. This special treatment of matrix formulation can eliminate the multiple occurrences of interval variables. In the left side of Eq. (3.21), the occurrence of α_1^I appears twice. After the special treatment, the occurrence of α_1^I only appears once in the right side of Eq. (3.21).

The special treatment Eq. (3.21) will be applied in the structural analysis as illustrated in the below procedure. In static analysis, the FEM formula can be

derived from the minimizing of the following potential energy Π ,

$$\Pi = \frac{1}{2} \mathbf{u}^I T \mathbf{K}^I \mathbf{u}^I - \mathbf{u}^I T \mathbf{F}^I \quad (3.22)$$

with the conditions

$$\frac{\partial \Pi}{\partial \mathbf{u}_i^I} = 0 \quad \text{for all } i \quad (3.23)$$

where, $\mathbf{K}^I$, $\mathbf{u}^I$ and $\mathbf{F}^I$ are the stiffness matrix, displacement vector and load vector respectively. Assume that the r linearly independent discrete constraints $\mathbf{Q}\mathbf{u}^I - \mathbf{t} = 0$ can be imposed onto the solution, where $\mathbf{Q}$ and $\mathbf{t}$ contain constants. To impose constraints by Lagrange multipliers, it multiplies $\mathbf{Q}\mathbf{u}^I - \mathbf{t}$ by a row vector $\boldsymbol{\lambda}^I$ that contains as many Lagrange multipliers λ_i^I as the number of constraint equations and adds this to the potential energy Eq. (3.22) (Cook et al., 1989),

$$\Pi^* = \frac{1}{2} \mathbf{u}^I T \mathbf{K}^I \mathbf{u}^I - \mathbf{u}^I T \mathbf{F}^I + \boldsymbol{\lambda}^I T \mathbf{Q}\mathbf{u}^I - \mathbf{t} \quad (3.24)$$

Letting $\frac{\partial \Pi^*}{\partial \mathbf{u}^I} = 0$ and $\frac{\partial \Pi^*}{\partial \boldsymbol{\lambda}^I} = 0$, it can be obtained,

$$\begin{bmatrix} \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^I \\ \boldsymbol{\lambda}^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^I \\ \mathbf{t}^I \end{Bmatrix} \quad (3.25)$$

Considering that the compatibility conditions in the two bar problem takes the form $\mathbf{Q}\mathbf{u}^I - \mathbf{t} = 0$ and $\mathbf{t} = 0$, Eq. (3.25) reduces to

$$\begin{bmatrix} \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^I \\ \boldsymbol{\lambda}^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^I \\ \mathbf{0} \end{Bmatrix} \quad (3.26)$$

The linear interval Eq. (3.26) can be solved by the Brouwer's fixed point theorem mentioned in Section 3.2.2. However, this theorem carries the implicit assumption that the coefficients of $\mathbf{A}^I$ are independent among themselves and that the components of $\mathbf{p}^I$ vary independently. To eliminate the coupling and multiple occurrences of interval variables, the EBE-IFEA technique has to be applied to reduce the dependency effect.

For linear system problem,

$$\mathbf{A}^I \mathbf{y}^I = \mathbf{p}^I \quad (3.27)$$

with

$$\mathbf{A}^I = \left[\begin{bmatrix} \mathbf{K}^c & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{K}^c \mathbf{d}^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right], \quad \mathbf{y}^I = \begin{Bmatrix} \mathbf{u}^I \\ \boldsymbol{\lambda}^I \end{Bmatrix}, \quad \mathbf{p}^I = \begin{Bmatrix} \mathbf{F}^I \\ \mathbf{0} \end{Bmatrix} \quad (3.28)$$

$\mathbf{A}^I$ can be decomposed further

$$\mathbf{A}^I = \begin{bmatrix} \mathbf{K}^c & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{K}^c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d}^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{A}^c + \mathbf{S} \mathbf{D}^I \quad (3.29)$$

The final equation to calculate the displacements can be derived from Eq. (3.7) as shown below,

$$\mathbf{y}^{*n+1I} = \mathbf{R} \mathbf{p}^I - \mathbf{y}_0 - \mathbf{R} \mathbf{S} \mathbf{D}^I (\mathbf{y}_0 + \mathbf{y}^{*nI}) \quad (3.30)$$

where $\mathbf{R} = \mathbf{A}^{c-1}$ and $\mathbf{y}_0 = \mathbf{R} \mathbf{p}^c$.

The method shown in Eq. (3.21) can be used in the term $\mathbf{D}^I (\mathbf{y}_0 + \mathbf{y}^{*nI})$ in Eq.

(3.30) to eliminate the multiple occurrences of the interval variables. Eq. (3.8) is used for stopping the iteration of Eq. (3.30). After n iterations, the enclosure $\mathbf{y}^I$ is given by

$$\mathbf{y}^I = \mathbf{y}^{*n+1I} + \mathbf{y}_0 \quad (3.31)$$

The obtained interval vector $\mathbf{y}^I$ contains two parts, $\mathbf{u}^I$ and $\boldsymbol{\lambda}^I$.

The proposed methods are implemented in Matlab with the interval calculation tool box IntLab (Hargreaves, 2002). The developed computer programme for truss problem is given in Appendix A.

3.4 Numerical Examples for Static Analysis

In this section, three numerical examples are presented to show how the EBE-IFEA method reduces the overestimation in the interval arithmetic. Based on this method, a general interval FEM code (Appendix A) for static analysis of truss structures has been developed with uncertainties in modulus of elasticity, area of cross-section and load.

3.4.1 Two-bay Truss

The first example is a two-bay truss shown in Fig. 3.2. There is a load P applied at node 2 negative y direction. The load is uncertain and described by an interval $[10,20]$ kN. The cross-sectional area $A_i = 0.01\text{m}^2$, and the uncertain modulus of elasticity $E_i^I = [199,201]\text{GPa}$, $i=1,\dots,11$. The modulus of elasticity of each element is assumed to vary independently.

Figure 3.2. Two-bay truss

The force vector $\{\mathbf{F}\}$ of this structure is shown below.

For the direct method of IFEA,

$$\{\mathbf{F}^I\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\mathbf{f}^I \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad (i \text{ rows, for } i = 3 \times \text{number of nodes}) \quad (3.32)$$

For the EBE-IFEA method,

$$\{\mathbf{F}^I\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\mathbf{f}^I \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad (i \text{ rows, for } i = 3 \times 2 \times \text{number of elements}) \quad (3.33)$$

where $\mathbf{f}^I = [10,20]\text{kN}$. According to the current system parameter condition specified above, the selected displacements are calculated using both the direct IFEA and EBE-IFEA methods. The results are listed in Table 3.1,

Table 3.1. Selected vertical and horizontal displacements of two-bay truss

Nodal displacements(m)	v2 lower	v2 upper	u4 lower	u4 upper	v4 lower	v4 upper
Combinatorial approach (Muhanna et al., 2007) $\times 10^{-5}$	-20.0326	-9.9166	1.9489	4.0041	-0.9984	-0.4886
Crisp case $\times 10^{-5}$	-14.9490	-14.9490	2.9631	2.9631	-0.7408	-0.7408
Direct IFEA (Muhanna et al., 2007) $\times 10^{-5}$	-21.7050	-8.1936	1.4618	4.4644	-1.1089	-0.3726
EBE-IFEA $\times 10^{-5}$	-20.0683	-9.8303	1.9360	3.9903	-0.9962	-0.4854
Direct IFEA Width Ratio	0.4519	0.4519	0.5067	0.5067	0.4969	0.4970
EBE-IFEA Width Ratio	0.3425	0.3424	0.3466	0.3467	0.3448	0.3448

The crisp case refers to the case where all the parameters are deterministic in calculation. The results of such case are also termed as the middle values in the literature.

The width ratio in Table 3.1 is obtained by,

$$\text{Width Ratio} = \left| \frac{(\text{Direct IFEA value or EBE - IFEA value}) - \text{Crisp value}}{\text{Crisp value}} \right| \quad (3.34)$$

The results of EBE-IFEA are compared with those of combinatorial approach and

the direct IFEA. From Table 3.1, it can be noticed that the EBE-IFEA method narrows the range of response of the truss structure, thus reducing the overestimation of the direct method of IFEA. It can be also found that the EBE-IFEA width ratio is not significantly reduced compared with the direct IFEA width ratio. This is due to the large width of the loading variable.

3.4.2 Three-bay Truss

The second example is a three-bay truss shown in Fig. 3.3. There are two loads P applied at the bottom joints. The loads are constant and described by the value 20 kN. Each element has a cross-sectional area $A_i=0.01\text{m}^2$, and an uncertain modulus of elasticity $E_i^I = [199,201]\text{GPa}$, $i=1,\dots,16$. The modulus of elasticity of each element is assumed to vary independently.

Figure 3.3. Three-bay truss

For the above system, the solution methods are similar to those of the first example. The displacements are compared in Table 3.2.

Table 3.2. Selected vertical and horizontal displacements of three-bay truss

Nodal displacements(m)	u2 lower	u2 upper	v2 lower	v2 upper
Direct IFEA x 10^{-5}	-5.6272	-0.6470	-80.6251	-35.7011
Crisp case x 10^{-5}	-3.1372	-3.1372	-58.1631	-58.1631
EBE-IFEA x 10^{-5}	-3.1540	-3.1202	-58.4603	-57.8658
Direct IFEA Width Ratio	0.7937	0.7938	0.3862	0.3862
EBE-IFEA Width Ratio	0.0054	0.0054	0.0051	0.0051
Nodal displacements(m)	u5 lower	u5 upper	v5 lower	v5 upper
Direct IFEA x 10^{-5}	8.3689	22.7158	-2.4806	-0.9354
Crisp case x 10^{-5}	15.5424	15.5424	-1.7080	-1.7080
EBE-IFEA x 10^{-5}	15.4619	15.6228	-1.7169	-1.6992
Direct IFEA Width Ratio	0.4615	0.4615	0.4523	0.4523
EBE-IFEA Width Ratio	0.0052	0.0052	0.0052	0.0052
Nodal displacements(m)	u6 lower	u6 upper	v6 lower	v6 upper
Direct IFEA x 10^{-5}	4.2787	13.1419	-76.0015	-32.4510
Crisp case x 10^{-5}	8.7103	8.7103	-54.2262	-54.2262
EBE-IFEA x 10^{-5}	8.6649	8.7557	-54.5028	-53.9496
Direct IFEA Width Ratio	0.5088	0.5088	0.4016	0.4016
EBE-IFEA Width Ratio	0.0052	0.0052	0.0051	0.0051

The results of EBE-IFEA are compared with those of the direct IFEA. From Table 3.2, it can be found that the width ratio for the direct IFEA ranges from 0.38 to 0.79 while that for the EBE-IFEA is only around 0.005. Thus, the EBE-IFEA method significantly reduces the overestimation of the direct IFEA though the problem size increases.

3.4.3 25-bar Space Truss

The third example is a 25-bar space truss structure shown in Fig. 3.4. The load P in the positive Y direction is acted on node 1 of the structure. The load is constant and described by the value 30 kN. Each element has a cross-sectional area $A_i = 4.0 \times 10^{-4} \text{ m}^2$, and an uncertain modulus of elasticity $E_i^I = [69.6, 72.4] \text{ GPa}$, $i=1, \dots, 25$. The modulus of elasticity of each element is assumed to vary independently. The lengths of bars are constant and determined by the joint coordinates as shown in Fig. 3.4.

Figure 3.4. 25-bar space truss structure (unit: m)

The selected displacements are given in Table 3.3.

Table 3.3. Selected vertical displacements of 25-bar space truss

Nodal displacements(m)	v2 lower	v2 upper
Direct IFEA x 10 ⁻³	-18.5696	29.2897
Crisp case x 10 ⁻³	5.3601	5.3601
EBE-IFEA x 10 ⁻³	5.2121	5.5080
Direct IFEA Width Ratio	4.4644	4.4644
EBE-IFEA Width Ratio	0.0276	0.0276

The results of EBE-IFEA are compared with those of the direct IFEA. The results show that the EBE-IFEA method is also applicable for the space truss structures and significantly narrows the range of the response of the structure. In the example, the direct IFEA method actually yields un-acceptable results with the width ratio over 4.

3.5 Summary

The developed formulations of EBE-IFEA used in this chapter are derived from the Brouwer's fixed point theorem. The EBE-IFEA method is applied on static interval analysis. Several numerical examples show that the EBE-IFEA method has significant improvement in reducing the overestimation of the interval calculation. With the increase of DOFs, the width ratio of EBE-IFEA becomes smaller compared with that of the direct method of IFEA. This is because with more DOFs, the occurrence of the interval variables in the calculation increases in the direct method, resulting in larger overestimation in the results. While the occurrence of the interval variables does not increase in the proposed method due to the special treatment and the EBE technique.

In this chapter, it is found the EBE method is efficient for solving the interval problems. Therefore, this method will be extended in dynamic analysis of truss structures in Chapter 5 and Chapter 6.

CHAPTER 4 INTERVAL ANALYSIS OF EIGENVALUE PROBLEM FOR STRUCTURES

4.1 Introduction

The research on how to determine the bounds of eigenvalues for interval eigenvalue problems has begun to emerge in recent decades. As reviewed in Chapter 2, it is difficult to solve the equations with uncertain parameters directly and some iteration methods to solve this problem are studied. In this chapter, based on the perturbation method, a direct interval analysis method for eigenvalue problem is presented. In proposed method, no iteration of interval calculation is needed. The structural physical parameters are considered as interval variables. The structural stiffness matrix, mass matrix, eigenvalues and eigenvectors are thus described as the sum of two parts corresponding to the deterministic matrix and the uncertainty of the interval parameters. Subsequently, the matrix perturbation technique is used to remove the higher order terms of the general eigenvalue equation to obtain the eigensolutions.

To overcome the overestimation in interval calculation, a special treatment Eq. (3.21) of matrix multiplication is used to reduce multi-occurrence of interval variables. In the proposed method, it is only required to solve a system of linear equations for each eigenvalue and its corresponding eigenvector. There is also no need to calculate the inverse of interval matrix, which saves a lot of computational efforts.

Finally, the effectiveness of the proposed method is demonstrated by numerical examples compared with the vertex solutions and other researchers' work.

4.2 Interval Eigenvalue Problem

Eigenvalue problems are important in structural dynamic analysis. In the general cases (i.e. the mass and stiffness matrices are real symmetric or Hermitian.), the eigenvalues are real numbers when the system is undamped. In this chapter, the undamped structures are considered, and the general interval eigenvalue problem with n degrees can be expressed as follows (Qiu et al., 1995),

$$\mathbf{K}^I \mathbf{v}_i^I = \lambda_i^I \mathbf{M}^I \mathbf{v}_i^I \quad (4.1)$$

where $\mathbf{K}^I = [k_{ij}^I]$ is the stiffness matrix, $\mathbf{M}^I = [m_{ij}^I]$ is the mass matrix, $\mathbf{v}_i^I = \{v_i^I\}$ is the eigenvector or the mode shape and λ_i^I is the i th eigenvalue.

Multiplying the inverse of $\mathbf{M}^I$ multiplies on both sides of Eq. (4.1), it becomes,

$$\mathbf{D}^I \mathbf{v}_i^I = \lambda_i^I \mathbf{v}_i^I \quad (4.2)$$

where $\mathbf{D}^I = \mathbf{M}^{I^{-1}} \mathbf{K}^I$.

If the interval terms of Eq. (4.2) are written as,

$$\mathbf{K}^I = \mathbf{K}^c + \Delta \mathbf{K}^I \quad (4.3)$$

$$\mathbf{M}^I = \mathbf{M}^c + \Delta \mathbf{M}^I \quad (4.4)$$

$$\lambda_i^I = \lambda_i^c + \Delta \lambda_i^I \quad (4.5)$$

$$\mathbf{v}_i^I = \mathbf{v}_i^c + \Delta \mathbf{v}_i^I \quad (4.6)$$

$$\mathbf{D}^I = \mathbf{D}^c + \Delta \mathbf{D}^I \quad (4.7)$$

where the superscript I indicates an interval quantity, the superscript c indicates the middle value of an interval quantity, which is deterministic, and Δ indicates the uncertainty part.

Eqs. (4.5), (4.6) and (4.7) are applied into Eq. (4.2), then

$$\begin{aligned} (\mathbf{D}^c + \Delta \mathbf{D}^I)(\mathbf{v}_i^c + \Delta \mathbf{v}_i^I) &= (\lambda_i^c + \Delta \lambda_i^I)(\mathbf{v}_i^c + \Delta \mathbf{v}_i^I) \\ \mathbf{D}^c \mathbf{v}_i^c + \mathbf{D}^c \Delta \mathbf{v}_i^I + \Delta \mathbf{D}^I \mathbf{v}_i^c + \Delta \mathbf{D}^I \Delta \mathbf{v}_i^I \\ &= \lambda_i^c \mathbf{v}_i^c + \lambda_i^c \Delta \mathbf{v}_i^I + \Delta \lambda_i^I \mathbf{v}_i^c + \Delta \lambda_i^I \Delta \mathbf{v}_i^I \end{aligned} \quad (4.8)$$

From the crisp case, the equation of eigenvalue problem can be obtained as,

$$\mathbf{D}^c \mathbf{v}_i^c = \lambda_i^c \mathbf{v}_i^c \quad (4.9)$$

and neglecting the second order terms of uncertainty in Eq. (4.8), Eq. (4.8) can be written as,

$$\Delta \mathbf{D}^I \mathbf{v}_i^c = \lambda_i^c \Delta \mathbf{v}_i^I + \Delta \lambda_i^I \mathbf{v}_i^c - \mathbf{D}^c \Delta \mathbf{v}_i^I \quad (4.10)$$

It can be also expressed as,

$$[\lambda_i^c I - \mathbf{D}^c, \mathbf{v}_i^c] \begin{Bmatrix} \Delta \mathbf{v}_i^I \\ \Delta \lambda_i^I \end{Bmatrix} = \Delta \mathbf{D}^I \mathbf{v}_i^c \quad (4.11)$$

For a n -DOF system, it is found that the number of equations in Eq. (4.11) is n and the number of unknowns in $\begin{Bmatrix} \Delta \mathbf{v}_i^I \\ \Delta \lambda_i^I \end{Bmatrix}$ is $(n + 1)$. This is reasonable because the values of eigenvectors can be determined as relative values which may be scaled or normalized to certain extent. Therefore, we can fix one component in the eigenvector $\mathbf{v}_i^I$, and then, in the uncertainty part of eigenvector $\Delta \mathbf{v}_i^I$, the corresponding component becomes zero. The number of unknowns in $\begin{Bmatrix} \Delta \mathbf{v}_i^I \\ \Delta \lambda_i^I \end{Bmatrix}$ of Eq. (4.11) is thus n , and Eq. (4.11) can be solved.

To solve Eq. (4.11), the right hand side term, $\Delta \mathbf{D}^I \mathbf{v}_i^c$ must be obtained first. From Eq. (4.4), it can be obtained,

$$\mathbf{M}^{I-1} = (\mathbf{M}^c + \Delta \mathbf{M}^I)^{-1} = (\mathbf{I} + \mathbf{M}^{c-1} \Delta \mathbf{M}^I)^{-1} \mathbf{M}^{c-1} \quad (4.12)$$

The Maclaurin series expansion can be applied in Eq. (4.12) and the higher order terms are neglected, then it becomes,

$$\mathbf{M}^{I-1} = (\mathbf{I} - \mathbf{M}^{c-1} \Delta \mathbf{M}^I) \mathbf{M}^{c-1} \quad (4.13)$$

So,

$$\begin{aligned} \mathbf{D}^I &= (\mathbf{I} - \mathbf{M}^{c-1} \Delta \mathbf{M}^I) \mathbf{M}^{c-1} (\mathbf{K}^c + \Delta \mathbf{K}^I) \\ &= (\mathbf{M}^{c-1} - \mathbf{M}^{c-1} \Delta \mathbf{M}^I \mathbf{M}^{c-1}) (\mathbf{K}^c + \Delta \mathbf{K}^I) \end{aligned}$$

$$= \mathbf{M}^{c-1} \mathbf{K}^c + (\mathbf{M}^{c-1} \Delta \mathbf{K}^I - \mathbf{M}^{c-1} \Delta \mathbf{M}^I \mathbf{M}^{c-1} \mathbf{K}^c - \mathbf{M}^{c-1} \Delta \mathbf{M}^I \mathbf{M}^{c-1} \Delta \mathbf{K}^I) \quad (4.14)$$

Again, the higher order of uncertain terms are neglected in Eq. (4.14) and Eq. (4.7) becomes

$$\begin{aligned} \mathbf{D}^I &= \mathbf{D}^c + \Delta \mathbf{D}^I \\ &= \mathbf{M}^{c-1} \mathbf{K}^c + (\mathbf{M}^{c-1} \Delta \mathbf{K}^I - \mathbf{M}^{c-1} \Delta \mathbf{M}^I \mathbf{M}^{c-1} \mathbf{K}^c) \end{aligned} \quad (4.15)$$

where $\mathbf{D}^c = \mathbf{M}^{c-1} \mathbf{K}^c$ and $\Delta \mathbf{D}^I = \mathbf{M}^{c-1} \Delta \mathbf{K}^I - \mathbf{M}^{c-1} \Delta \mathbf{M}^I \mathbf{M}^{c-1} \mathbf{K}^c$.

In order to reduce the overestimation of the interval calculation, a special treatment Eq. (3.21) of term $\Delta \mathbf{D}^I \mathbf{v}_i^c$ in Eq. (4.11) is introduced in the following,

$$\Delta \mathbf{D}^I \mathbf{v}_i^c = \mathbf{M}^{c-1} \Delta \mathbf{K}^I \mathbf{v}_i^c - \mathbf{M}^{c-1} \Delta \mathbf{M}^I \mathbf{M}^{c-1} \mathbf{K}^c \mathbf{v}_i^c \quad (4.16)$$

If the product $\Delta \mathbf{K}^I \mathbf{v}_i^c$ in the first term at the right-hand side of Eq. (4.16) is calculated directly, the overestimation could be very large. Therefore, the method to reduce the multiple occurrences of the interval variables in Eq. (4.16) is very important. In this chapter, only the truss systems are considered. Although the chapter focuses only on trusses, the developed method is applicable to other structures like beams and frames, as long as the finite element formulation part of truss is replaced by that of the beam or frame.

For example, for the j th element in a 2D truss structure, $\Delta \mathbf{K}_j^I$ in $\Delta \mathbf{K}^I$ can be expressed as,

$$\Delta \mathbf{K}_j^I = \mathbf{T}_j^T \begin{bmatrix} k_j^c & -k_j^c \\ -k_j^c & k_j^c \end{bmatrix} \mathbf{T}_j d_j^I$$

$$= k_j^c \begin{bmatrix} c_j^2 & c_j s_j & -c_j^2 & -c_j s_j \\ c_j s_j & s_j^2 & -c_j s_j & -s_j^2 \\ -c_j^2 & -c_j s_j & c_j^2 & c_j s_j \\ -c_j s_j & -s_j^2 & c_j s_j & s_j^2 \end{bmatrix} d_j^l \quad (4.17)$$

where d_j^l is the relative uncertainty of $\mathbf{K}_j^l$, $\mathbf{T}_j$ is the 2×4 transformation matrix that translates the local coordinates of the j^{th} element to the global coordinates, and $\mathbf{T}_j^T$ is its transpose. $c_j = \cos\theta$ and $s_j = \sin\theta$, where θ is the angle from the truss member to horizontal coordinate.

For example, since the first element consists of nodes 1 and 3, its global stiffness matrix can be expressed as $\Delta \mathbf{K}_1^l$, i.e.,

$$\Delta \mathbf{K}_1^l = k_1^c \begin{bmatrix} c_1^2 & c_1 s_1 & 0 & 0 & -c_1^2 & -c_1 s_1 & 0 & \dots & 0 \\ c_1 s_1 & s_1^2 & 0 & 0 & -c_1 s_1 & -s_1^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -c_1^2 & -c_1 s_1 & 0 & 0 & c_1^2 & c_1 s_1 & 0 & \dots & 0 \\ -c_1 s_1 & -s_1^2 & 0 & 0 & c_1 s_1 & s_1^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n \times n} d_1^l \quad (4.18)$$

The calculation of $\Delta \mathbf{K}_1^l \mathbf{v}_i^c$ can be written as,

$$\Delta \mathbf{K}_1^l \mathbf{v}_i^c = \mathcal{H} \mathbf{k}_1 \mathbb{Q} \mathbf{k}_1^l$$

$$= \begin{bmatrix} (c_1^2 v_1^c + c_1 s_1 v_2^c - c_1^2 v_5^c - c_1 s_1 v_6^c) k_1^c & 0 & \cdots & 0 \\ (c_1 s_1 v_1^c + s_1^2 v_2^c - c_1 s_1 v_5^c - s_1^2 v_6^c) k_1^c & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ (-c_1^2 v_1^c - c_1 s_1 v_2^c + c_1^2 v_5^c + c_1 s_1 v_6^c) k_1^c & 0 & \cdots & 0 \\ (-c_1 s_1 v_1^c - s_1^2 v_2^c + c_1 s_1 v_5^c + s_1^2 v_6^c) k_1^c & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n} \begin{Bmatrix} d_1^I \\ \vdots \\ 0 \\ \vdots \\ 0 \end{Bmatrix}_{n \times 1} \quad (4.19)$$

$$\text{where } \mathcal{H} \mathbf{k}_1 = \begin{bmatrix} (c_1^2 v_1^c + c_1 s_1 v_2^c - c_1^2 v_5^c - c_1 s_1 v_6^c) k_1^c & 0 & \cdots & 0 \\ (c_1 s_1 v_1^c + s_1^2 v_2^c - c_1 s_1 v_5^c - s_1^2 v_6^c) k_1^c & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ (-c_1^2 v_1^c - c_1 s_1 v_2^c + c_1^2 v_5^c + c_1 s_1 v_6^c) k_1^c & 0 & \cdots & 0 \\ (-c_1 s_1 v_1^c - s_1^2 v_2^c + c_1 s_1 v_5^c + s_1^2 v_6^c) k_1^c & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n} \quad \text{and}$$

$$\mathbb{Q} \mathbf{k}_1^I = \begin{Bmatrix} d_1^I \\ \vdots \\ 0 \\ \vdots \\ 0 \end{Bmatrix}_{n \times 1}.$$

For the multiplication of stiffness matrix of each element $\Delta \mathbf{K}_j^I$ with $\mathbf{v}_i^c$, the above calculation method can be applied. After all $\mathcal{H} \mathbf{k}_j$ and $\mathbb{Q} \mathbf{k}_j^I$ ($j=1 \dots n$) are obtained, $\Delta \mathbf{K}^I \mathbf{v}_i^c$ can be expressed as,

$$\Delta \mathbf{K}^I \mathbf{v}_i^c = \sum_{j=1}^n \mathcal{H} \mathbf{k}_j \sum_{j=1}^n \mathbb{Q} \mathbf{k}_j^I \quad (4.20)$$

Subsequently, $\mathbf{M}^{c-1} \Delta \mathbf{K}^I \mathbf{v}_i^c$ is obtained.

For the second term $\mathbf{M}^{c-1} \Delta \mathbf{M}^I \mathbf{M}^{c-1} \mathbf{K}^c \mathbf{v}_i^c$ at the right-hand side of Eq. (4.16), the method to deal with $\Delta \mathbf{M}^I \mathbf{M}^{c-1} \mathbf{K}^c \mathbf{v}_i^c$ is the same as the method for $\Delta \mathbf{K}^I \mathbf{v}_i^c$. Let the part $\mathbf{M}^{c-1} \mathbf{K}^c \mathbf{v}_i^c$ be calculated first, then $\Delta \mathbf{M}^I$ multiplying $\mathbf{M}^{c-1} \mathbf{K}^c \mathbf{v}_i^c$ takes the following form,

$$\Delta \mathbf{M}^I \mathbf{M}^{c-1} \mathbf{K}^c \mathbf{v}_i^c = \sum_{j=1}^n \mathcal{H} \mathbf{m}_j \sum_{j=1}^n \mathbb{Q} \mathbf{m}_j^I \quad (4.21)$$

The above procedure illustrates the special treatment Eq. (3.21) of matrix multiplication for the term $\Delta \mathbf{D}^I \mathbf{v}_i^c$, the $\begin{Bmatrix} \Delta \mathbf{v}_i^I \\ \Delta \lambda_i^I \end{Bmatrix}$ can then be readily solved from Eq. (4.11).

$$\begin{aligned} \begin{Bmatrix} \Delta \mathbf{v}_i^I \\ \Delta \lambda_i^I \end{Bmatrix} &= [\lambda_i^c I - \mathbf{D}^c, \mathbf{v}_i^c]^{-1} \Delta \mathbf{D}^I \mathbf{v}_i^c \\ &= \mathbf{G} (\mathbf{M}^{c-1} \sum_{j=1}^n \mathcal{H} \mathbf{k}_j \sum_{j=1}^n \mathbb{Q} \mathbf{k}_j^I - \mathbf{M}^{c-1} \sum_{j=1}^n \mathcal{H} \mathbf{m}_j \sum_{j=1}^n \mathbb{Q} \mathbf{m}_j^I) \end{aligned} \quad (4.22)$$

where $\mathbf{G} = [\lambda_i^c I - \mathbf{D}^c, \mathbf{v}_i^c]^{-1}$. It should be mentioned that the coefficient matrix $[\lambda_i^c I - \mathbf{D}^c, \mathbf{v}_i^c]$ in Eq. (4.22) is deterministic. To find its inverse matrix, $\mathbf{G}$, no interval calculation is needed. Finally, $\mathbf{v}_i^I$ and λ_i^I can be obtained from Eqs. (4.5)

and (4.6) after $\begin{Bmatrix} \Delta \mathbf{v}_i^I \\ \Delta \lambda_i^I \end{Bmatrix}$ is obtained.

The above proposed method is implemented in Matlab with the interval calculation tool box IntLab (Hargreaves, 2002). The developed computer programme is given

in Appendix B.

4.3 Numerical Examples

4.3.1 Five-Story Frame

In order to illustrate the effectiveness of the presented interval analysis method for the eigenvalue problem of the structures with uncertain parameters, a five-story frame structure with uncertain-but-bounded parameters is considered and shown in Fig. 4.1(a). Assume that the horizontal members are very rigid compared to the columns of the frame. This assumption reduces the system to only five DOFs shown in Fig. 4.1(b), indicated by coordinates $X_1, X_2, \dots, X_5$ in the figure. This structure can thus be modeled by the five mass-spring system shown in Fig. 4.1(b). The masses of the structure, which are lumped at the floor levels, have interval values as $M_1^I = [29, 31]$, $M_2^I = [26, 28]$, $M_3^I = [26, 28]$, $M_4^I = [24, 26]$ and $M_5^I = [17, 19]$ kg. The total stiffness of each story also has interval values as $k_1^I = [2000, 2020]$, $k_2^I = [1800, 1850]$, $k_3^I = [1600, 1630]$, $k_4^I = [1400, 1420]$ and $k_5^I = [1200, 1210]$ N/m. These interval physical parameters of the system are listed in Table 4.1. This example has been studied by Sim et al. (2007), and Qiu et al. (1993).

Figure 4.1. (a) Five-story frame and (b) 5-DOF modeled system

From the defined interval parameters, we can obtain the mid-value matrix of stiffness matrix $\mathbf{K}^c$, uncertainty of stiffness matrix $\Delta\mathbf{K}^I$, mid-matrix of mass matrix $\mathbf{M}^c$ and uncertainty of mass matrix $\Delta\mathbf{M}^I$.

$$\mathbf{K}^c = \begin{bmatrix} 3835 & -1825 & 0 & 0 & 0 \\ -1825 & 3440 & -1615 & 0 & 0 \\ 0 & -1615 & 3025 & -1410 & 0 \\ 0 & 0 & -1410 & 2615 & -1205 \\ 0 & 0 & 0 & -1205 & 1205 \end{bmatrix}$$

$$\Delta\mathbf{K}^I = \begin{bmatrix} (-35, 35) & (-25, 25) & 0 & 0 & 0 \\ (-25, 25) & (-40, 40) & (-15, 15) & 0 & 0 \\ 0 & (-15, 15) & (-25, 25) & (-10, 10) & 0 \\ 0 & 0 & -10 & (-15, 15) & (-5, 5) \\ 0 & 0 & 0 & (-5, 5) & (-5, 5) \end{bmatrix} \quad (4.23)$$

$$\mathbf{M}^c = \text{diag}\{30, 27, 27, 25, 18\}$$

$$\Delta \mathbf{M}^I = \text{diag}\{(-1, 1), (-1, 1), (-1, 1), (-1, 1), (-1, 1)\} \quad (4.24)$$

Table 4.1 Interval physical parameters of the system (Sim et al., 2007)

Interval for each stiffness parameters (N/m)		Interval for each mass parameters (kg)	
Bounded value	Uncertainty	Bounded value	Uncertainty
$K_1^I = [2000, 2020]$	$\Delta K_1 = 10$	$M_1^I = [29, 31]$	$\Delta M_1 = 1$
$K_2^I = [1800, 1850]$	$\Delta K_2 = 25$	$M_2^I = [26, 28]$	$\Delta M_2 = 1$
$K_3^I = [1600, 1630]$	$\Delta K_3 = 15$	$M_3^I = [26, 28]$	$\Delta M_3 = 1$
$K_4^I = [1400, 1420]$	$\Delta K_4 = 10$	$M_4^I = [24, 26]$	$\Delta M_4 = 1$
$K_5^I = [1200, 1210]$	$\Delta K_5 = 5$	$M_5^I = [17, 19]$	$\Delta M_5 = 1$

Hence, the eigensolutions for the deterministic system with mid-parameters $\mathbf{K}^c$ and $\mathbf{M}^c$ can be solved. The results are listed in Table 4.2.

Table 4.2 Eigensolutions for system with mid-point parameters

		Mode Number				
		1	2	3	4	5
Eigenvalues	λ^c	6.1662	44.078	103.567	165.5908	219.4201
Natural frequency (m/s)	ω^c	2.4831834	6.6391264	10.176787	12.868209	14.812836
Eigenvector	$\mathbf{v}^c$	$\mathbf{v}_1^c$	$\mathbf{v}_2^c$	$\mathbf{v}_3^c$	$\mathbf{v}_4^c$	$\mathbf{v}_5^c$
		0.2496324	-0.619347	0.8903048	-1.006789	3.1898827
		0.4992648	-0.852858	0.3550887	0.624802	-4.802786
		0.7299461	-0.488065	-0.864646	0.7388549	3.7829912
		0.9080216	0.3415515	-0.547098	-1.47341	-2.278592
		1	1	1	1	1

The comparison of interval eigenvalues obtained by the proposed method with those of Deif's method (Qiu et al., 1993), Sim et al. (2007) and the vertex solution is listed in Table 4.3. It can be observed from Table 4.3 that the results obtained by the proposed method are very close to the vertex solution which is accurate. It yields tight bounds in the eigenvalues, though the bounds are slightly larger than the vertex solution. These results demonstrate that the proposed method is superior to the Deif's method and the Sim's method.

Table 4.3 Comparison of interval eigenvalues of the proposed method with other methods

	Bound	Notation	Mode Number				
			1	2	3	4	5
Eigenvalues obtained by proposed method	Lower	$\underline{\lambda}$	5.8894	42.0982	97.9695	156.0711	208.4646
	Upper	$\bar{\lambda}$	6.4431	46.0578	109.1646	175.1106	230.3756
Eigenvalues obtained by Deif's method (Qiu et al., 1993)	Lower	$\underline{\lambda}$	4.6166	40.6428	98.1800	157.8433	209.5148
	Upper	$\bar{\lambda}$	7.8303	47.8200	109.3993	174.0029	230.0845
Eigenvalues obtained by Sim et al. (2007)	Lower	$\underline{\lambda}$	4.6166	40.7540	98.5720	158.8600	211.5100
	Upper	$\bar{\lambda}$	7.8303	47.7020	108.9900	172.8900	227.9500
Eigenvalues	Lower	$\underline{\lambda}$	5.8581	42.1293	98.8564	158.0514	209.5148

obtained by vertex method	Upper	$\bar{\lambda}$	6.5020	46.0088	108.6890	173.7776	230.0845
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Table 4.4 lists the comparison of interval eigenvectors obtained by the proposed method with the vertex solution and those by Sim et.al. (2007). As mentioned, the eigenvectors are relative values. Therefore, each eigenvector is normalized to the fifth component which is set at 1 in Table 4.4.

Table 4.4 Comparison of interval eigenvectors of proposed method with other methods

Eigenvectors obtained by proposed method						
		Mode Number				
Bound	Notation	1	2	3	4	5
Lower	$\underline{v}$	$\underline{v}_1$	$\underline{v}_2$	$\underline{v}_3$	$\underline{v}_4$	$\underline{v}_5$
		0.2304	-0.6898	0.7637	-1.2345	1.4006
		0.4735	-0.9167	0.2789	0.4126	-7.2298
		0.7087	-0.5393	-0.9897	0.4637	2.4337
		0.8982	0.2854	-0.6428	-1.6369	-2.6170
		1	1	1	1	1
Upper	$\bar{v}$	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$
		0.2689	-0.5489	1.0171	-0.7794	4.9780
		0.5251	-0.7888	0.4315	0.8374	-2.3734
		0.7512	-0.4369	-0.7396	1.0140	5.1308
		0.9176	0.3978	-0.4513	-1.3102	-1.9383
		1	1	1	1	1
Eigenvectors obtained by Sim et al. (2007)						
Lower	$\underline{v}$	$\underline{v}_1$	$\underline{v}_2$	$\underline{v}_3$	$\underline{v}_4$	$\underline{v}_5$

Upper	$\bar{v}$	0.2383	-0.6976	0.694	-1.415	2.2305
		0.4758	-0.9602	0.2771	0.4443	-6.8745
		0.6957	-0.5496	-1.1081	0.5256	2.6428
		0.8651	0.3032	-0.7011	-2.0708	-3.2605
		1	1	1	1	1
		$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$
		0.2621	-0.5501	1.1412	-0.7163	4.5675
		0.5234	-0.7574	0.4557	0.8776	-3.3571
		0.7653	-0.4332	-0.6742	1.0381	5.414
		0.9518	0.3846	-0.4267	-1.0484	-1.5909
<hr/>						
Eigenvectors obtained by vertex method						
Lower	$\underline{v}$	$\underline{v}_1$	$\underline{v}_2$	$\underline{v}_3$	$\underline{v}_4$	$\underline{v}_5$
		0.2421	-0.6632	0.768	-1.307	1.7902
		0.4864	-0.8985	0.2482	0.4106	-7.8135
		0.7179	-0.5485	-0.9502	0.4672	2.6008
		0.9005	0.2932	-0.6697	-1.7045	-2.6375
		1	1	1	1	1
		$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$
		0.2576	-0.5779	1.0259	-0.7695	5.5024
		0.5126	-0.808	0.4783	0.8891	-2.9202
		0.7422	-0.4285	-0.7816	1.0615	5.347
0.9151	0.3885	-0.4325	-1.2656	-1.9494		
1	1	1	1	1		

By comparing the results, the proposed method also yields tight bounds for the most part of eigenvectors. To clearly illustrate the differences among the results, Fig. 4.2 plots the five modal shapes obtained by the proposed method, the vertex solution

and by Sim et al. (2007). Overall, the accuracy of the mode shapes obtained by the proposed method is satisfactory.

(a) Deformation of first-mode shape

(b) Deformation of second-mode shape

(c) Deformation of third-mode shape

(d) Deformation of fourth-mode shape

(e) Deformation of fifth-mode shape

Figure 4.2 Deformation of mode shapes for bounded modal frequencies

4.3.2 Eight Bar Truss

The eight-bar truss considered in Qiu et al. (2005) as shown in Fig. 4.3 is analyzed. The cross-sectional areas of members 1, 2, 3, 4 and 6 are considered to be interval variables. They are $A_i^l = [A^c - \beta A^c, A^c + \beta A^c]$, $i=1, 2, 3, 4, 6$, where $A^c = 2.0 \times 10^{-4} m^2$. β is the uncertainty factor that will be varied to check the eigenvalue bounds with increasing uncertainty. The cross-sectional areas of members 5, 7 and 8 are deterministic and taken as $A_5 = A_7 = A_8 = 1.0 \times 10^{-4} m^2$. The Young's modulus of the material is $E = 200 GN/m^2$. The mass density is $\rho = 7800 kg/m^3$. In this example, the uncertainties come from area A , thus the mass matrix and the stiffness matrix will vary synchronically.

Figure 4.3 The eight bar truss

Table 4.5 shows the comparison of the eigenvalue bounds obtained by the proposed method with the parameter vertex solution as β equals to 0.02 i.e., 2% variation in cross sectional area. The results of the vertex method are the accurate solutions, however, its computational cost is much higher than the proposed method. In order to obtain n eigenvalues (where n is number of degrees of freedom and $n = 8$ in

this example) and their corresponding eigenvectors, the vertex method needs to solve the whole system for $n \times 2^m$ times where m is the number of independent uncertainties and $m = 5$ in this example. The conventional perturbation methods need to solve the system for $n \times m$ times. However, the proposed method needs only to solve the system n times.

Table 4.5 Interval eigenvalues by the proposed method and vertex method
($\beta = 0.02$)

	Eigenvalues obtained by vertex method		Eigenvalues obtained by proposed method	
	$\underline{\lambda}$	$\bar{\lambda}$	$\underline{\lambda}$	$\bar{\lambda}$
1	6.99E+05	7.25E+05	6.94E+05	7.30E+05
2	3.87E+06	3.97E+06	3.83E+06	4.00E+06
3	8.09E+06	8.42E+06	8.03E+06	8.48E+06
4	1.50E+07	1.53E+07	1.49E+07	1.54E+07
5	2.95E+07	2.98E+07	2.92E+07	3.01E+07
6	3.16E+07	3.23E+07	3.14E+07	3.25E+07
7	3.59E+07	3.63E+07	3.58E+07	3.64E+07
8	4.58E+07	4.60E+07	4.54E+07	4.64E+07

In Figs. 4.4 shows the comparison of the bounds of the first four eigenvalues obtained using the proposed method and the vertex method as β varies from 0 to 0.02. The eigenvalue bounds increase monotonically with the increase of β (Qiu et al., 2005) in the figure, but the ratio of the range of eigenvalues should be nearly constant if $\beta\%$ is small. The results obtained by the proposed method are very close to those of vertex solution, which demonstrates the effectiveness of the proposed method.

(a) The first eigenvalue

(b) The second eigenvalue

(c) The third eigenvalue

(d) The fourth eigenvalue

Figure 4.4. The bounds on the first four eigenvalues of the eight bar truss

4.4 Summary

In this chapter, an interval analysis method with perturbation technique has been presented to solve the eigenvalues and eigenvectors of dynamic systems with

interval parameters. The computation of the proposed method focuses on solving a linear system of equations after removing the second order terms of uncertainties. Unlike the methods proposed by other researchers, this method does not require the calculation of inverse of interval matrix. A special treatment of matrix operation is adopted to avoid multiple occurrences of the interval variables, thus reducing the overestimation during the interval calculation. For n eigenvalues and corresponding eigenvectors, the linear equation system needs to be solved for n times. The effectiveness of the proposed method has been demonstrated by comparing the numerical results obtained by the proposed method with those obtained by Deif's method (Qiu et al., 1993), the method of Sim et al. (2007) and the vertex solution.

It is worth mentioning that the proposed method is suitable for small uncertainties of system parameters since the formulation is limited to the first order terms, which results in an inner enclosure of the system response.

CHAPTER 5 INTERVAL ANALYSIS OF FREQUENCY RESPONSE OF STRUCTURES

5.1 Introduction

As reviewed in Chapter 2, Moens and Vandepitte (2004) commented that it is not advisable to directly solve the interval FRF. To overcome the overestimation problem, several methods are proposed in this chapter. Therefore, an IFE method for the FRF of structures with uncertain parameters is proposed. Similar to Chapter 4, the structural stiffness matrix and mass matrix are described as the sum of two parts corresponding to the deterministic matrix and the uncertainty of the interval parameters. Subsequently, the Fourier transform is applied on the interval dynamic equation to convert it into a system of linear equations. Then, the structural FRF is obtained by solving the linear equations using the Brouwer's fixed point theorem. Finally, the historical response of dynamic system under harmonic loading can be obtained by the inverse Fourier transform.

To overcome the overestimation in interval calculation, the EBE method is extended to dynamic system to formulate the IFE equation. During the interval calculation, a special treatment Eq. (3.21) of matrix multiplication is employed to further reduce the overestimation.

This method of this IFE analysis is developed in MATLAB. Two numerical examples are presented to validate the EBE-IFE method and the results are compared with the vertex solution to demonstrate the effectiveness of the proposed method.

5.2 FRF Analysis

5.2.1 Direct Method with IFE

For the systems under dynamic loading, the general interval equation with damping can be expressed as follows (Meirovotch, 1980; Weaver and Johnston, 1987),

$$\mathbf{M}^I \ddot{\mathbf{X}}^I(t) + \mathbf{C}^I \dot{\mathbf{X}}^I(t) + \mathbf{K}^I \mathbf{X}^I(t) = \mathbf{F}^I(t) \quad (5.1)$$

where $\mathbf{M}^I = [m_{ij}^I]$, $\mathbf{C}^I = [c_{ij}^I]$ and $\mathbf{K}^I = [k_{ij}^I]$ are the interval mass, damping and stiffness matrix respectively; $\mathbf{F}^I(t) = \{f_i^I(t)\}$ is the external load vector; $\mathbf{X}^I(t) = \{x_i^I(t)\}$, $\dot{\mathbf{X}}^I(t) = \{\dot{x}_i^I(t)\}$ and $\ddot{\mathbf{X}}^I(t) = \{\ddot{x}_i^I(t)\}$ are the displacement, velocity and acceleration vectors of the finite element assemblage, respectively. In the rest of thesis, the damping is only considered as Rayleigh damping which is suitable for all the proposed methods. The damping matrix $\mathbf{C}^I$ can be written as,

$$\mathbf{C}^I = a\mathbf{K}^I + b\mathbf{M}^I \quad (5.2)$$

where a and b are the coefficients of Rayleigh damping. The Fourier transform can be used to obtain the FRF from Eq. (5.1). The continuous Fourier transform pair of a function $F(t)$, defined on the time domain from $-\infty$ to $+\infty$, is given as

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} J(\omega) e^{+i\omega t} d\omega \quad (5.3)$$

$$F^{-1}[F(t)] = J(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(t) e^{-i\omega t} dt \quad (5.4)$$

where $J(\omega)$ is the continuous Fourier transform (CFT) and ω is the angular frequency. The first form, Eq. (5.3), is the inverse transform while the second form, Eq. (5.4), is the forward transform. This arbitrary convention arises because the signal to be transformed usually originates in the time domain.

Using the Fourier transform of $\mathbf{X}^I(t)$ and $\mathbf{F}^I(t)$, Eq. (5.1) can be written as,

$$F^{-1}[\mathbf{M}^I \ddot{\mathbf{X}}^I(t) + \mathbf{C}^I \dot{\mathbf{X}}^I(t) + \mathbf{K}^I \mathbf{X}^I(t)] = F^{-1}[\mathbf{F}^I(t)] \quad (5.5)$$

It is easy to get,

$$(-\omega^2 \mathbf{M}^I + i\omega \mathbf{C}^I + \mathbf{K}^I) F^{-1}[\mathbf{X}^I(t)] = F^{-1}[\mathbf{F}^I(t)] \quad (5.6)$$

Let $\tilde{\mathbf{X}}^I = F^{-1}[\mathbf{X}^I(t)]$ and $\tilde{\mathbf{F}}^I = F^{-1}[\mathbf{F}^I(t)]$, Eq. (5.6) can be written as,

$$(-\omega^2 \mathbf{M}^I + i\omega \mathbf{C}^I + \mathbf{K}^I) \tilde{\mathbf{X}}^I = \tilde{\mathbf{F}}^I \quad (5.7)$$

and

$$\text{magnitue of } \tilde{\mathbf{X}}^I = |\tilde{\mathbf{X}}^I| = |\tilde{\mathbf{Z}}^{I-1} \tilde{\mathbf{F}}^I| \quad (5.8)$$

where $\tilde{\mathbf{Z}}^I = -\omega^2 \mathbf{M}^I + i\omega \mathbf{C}^I + \mathbf{K}^I$ and let $\mathbf{H}(\omega) = \tilde{\mathbf{Z}}^{I-1}$. Here, the magnitude means the magnitude of each element of the matrix, not the magnitude of the matrix.

Letting $\tilde{\mathbf{F}} = \{1 \ 0 \ \dots \ 0\}^T$, the first column of $\tilde{\mathbf{Z}}^{I-1}$ can be solved from

$\tilde{\mathbf{Z}}^I \tilde{\mathbf{X}}^I = \tilde{\mathbf{F}}$. Then, $\tilde{\mathbf{X}}^I = \tilde{\mathbf{Z}}^{I^{-1}} \tilde{\mathbf{F}} = \{\tilde{\mathbf{Z}}_{11}^{I^{-1}} \quad \tilde{\mathbf{Z}}_{21}^{I^{-1}} \quad \dots \quad \tilde{\mathbf{Z}}_{m1}^{I^{-1}}\}^T$. And the rest columns of $\tilde{\mathbf{Z}}^{I^{-1}}$ can be solved similarly by letting $\tilde{\mathbf{F}} = \{0 \quad 1 \quad 0 \quad \dots \quad 0\}^T$, $\tilde{\mathbf{F}} = \{0 \quad 0 \quad 1 \quad \dots \quad 0\}^T$ and so on. So all the elements of $\tilde{\mathbf{Z}}^{I^{-1}}$ can be obtained, then the magnitude of each element of the matrix $\tilde{\mathbf{Z}}^{I^{-1}}$ can be derived. Eventually, plotting each magnitude of FRF under each frequency, a transform function can be produced.

For a harmonic load $\mathbf{F}(t)$, the response of structure can be obtained by using the Fourier transform. This procedure can be illustrated using an example shown in Fig. 5.1.

Figure 5.1 Four-story tower structure

It can be assumed that the force vector is,

$$\mathbf{F}(t) = \{\cos(at) \quad 0 \quad 0 \quad 0\}^T \quad (5.9)$$

Using the CFT, Eq. (5.9) can be taken as,

$$F^{-1}[\mathbf{F}(t)] = \tilde{\mathbf{F}}(\omega) = \left\{ \frac{\sqrt{2\pi}}{2} [\delta(\omega - a) + \delta(\omega + a)] \quad 0 \quad 0 \quad 0 \right\}^T \quad (5.10)$$

where $\delta(x)$ is the Dirac Delta function. Considering Eqs. (5.7) and (5.10), and

$$\tilde{\mathbf{Z}}^{\mathbf{I}^{-1}}(\omega) = (-\omega^2 \mathbf{M}^{\mathbf{I}} + i\omega \mathbf{C}^{\mathbf{I}} + \mathbf{K}^{\mathbf{I}})^{-1} \quad (5.11)$$

Solution of Eq. (5.7) can be calculated as,

$$\begin{aligned} \tilde{\mathbf{X}}^{\mathbf{I}}(\omega) &= \tilde{\mathbf{Z}}^{\mathbf{I}^{-1}}(\omega) \left\{ \frac{\sqrt{2\pi}}{2} \delta(\omega - a) \quad 0 \quad 0 \quad 0 \right\}^T \\ &\quad + \tilde{\mathbf{Z}}^{\mathbf{I}^{-1}}(\omega) \left\{ \frac{\sqrt{2\pi}}{2} \delta(\omega + a) \quad 0 \quad 0 \quad 0 \right\}^T \end{aligned} \quad (5.12)$$

After the inverse Fourier transform, Eq. (5.12) is transformed to,

$$\begin{aligned} \mathbf{X}^{\mathbf{I}}(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\mathbf{X}}^{\mathbf{I}}(\omega) e^{i\omega t} d\omega \\ &= \frac{1}{2} [\tilde{\mathbf{Z}}^{\mathbf{I}^{-1}}(a) \{e^{-iat} \quad 0 \quad 0 \quad 0\}^T + \tilde{\mathbf{Z}}^{\mathbf{I}^{-1}}(-a) \{e^{iat} \quad 0 \quad 0 \quad 0\}^T] \end{aligned} \quad (5.13)$$

From above Eq. (5.13), it can be noticed that $\tilde{\mathbf{Z}}^{\mathbf{I}^{-1}}(a) \{e^{-iat} \quad 0 \quad 0 \quad 0\}^T$ and $\tilde{\mathbf{Z}}^{\mathbf{I}^{-1}}(-a) \{e^{iat} \quad 0 \quad 0 \quad 0\}^T$ are the complex conjugate pairs. The sum of a complex number and its conjugate is twice as its real part, so

$$\mathbf{X}^{\mathbf{I}}(t) = \text{Re}[\tilde{\mathbf{Z}}^{\mathbf{I}^{-1}}(a) \{e^{-iat} \quad 0 \quad 0 \quad 0\}^T] \quad (5.14)$$

where $\text{Re}(a)$ means the real part of a complex number. Thus, using Eqs. (5.8) and (5.14) directly, the interval magnitude of FRF and the interval responses of structure can be obtained.

To find the sharp enclosures of the FRF and the structural responses, it is crucial to solve Eq. (5.7) accurately. Moens and Vandepitte (2004) commented that it is not advisable to directly solve the linear interval equations Eq. (5.7) since it may introduce large overestimation as shown in the results of direct IFE analysis in Section 5.3. In the next two subsections, several methods are proposed to overcome the above overestimation problem.

5.2.2 FRF Analysis with EBE-IFE Method

To reduce the overestimation in the interval finite element solutions, the issues of coupling and multiple occurrences of interval variables have to be handled properly. The EBE technique was adopted to solve these problems (Muhanna and Mullen, 2001; Muhanna et al., 2007) in the static analysis. The EBE-IFE method extends the EBE technique to the dynamic analysis to reduce the overestimation.

The EBE method has been introduced in Chapter 3. The interval stiffness matrix $\mathbf{K}^I$ of truss structure in global coordinate can be expressed as,

$$\mathbf{K}^I = \begin{bmatrix} \mathbf{K}_1^{I^e} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_2^{I^e} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_m^{I^e} \end{bmatrix} \quad (5.15)$$

The interval mass matrix $\mathbf{M}^I$ can also be expressed in the similar form as the

stiffness matrix as,

$$\mathbf{M}^I = \begin{bmatrix} \mathbf{M}_1^{I^e} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2^{I^e} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_m^{I^e} \end{bmatrix} \quad (5.16)$$

$$[\mathbf{M}^{I^e}] = \frac{1}{2} \rho^{I^e} \mathbf{A}^{I^e} \mathbf{L}^{I^e} [\mathbf{I}] \quad (5.17)$$

where $[\mathbf{M}^{I^e}]$ is the e^{th} element's mass matrix, ρ is the density and $[\mathbf{I}]$ is a six-order identity matrix.

The system stiffness matrix $\mathbf{K}^I$ in the EBE approach is singular, so the Lagrange multiplier method is used to ensure the compatibility conditions and to eliminate the singularity of $\mathbf{K}^I$. Therefore, the dynamic EBE approach of system equation uses the same theorem as the static analysis does. Using the Lagrange equations of the first kind, the Lagrange multipliers and the constraints are considered,

$$\lambda^T (\mathbf{Q}\mathbf{u} - \mathbf{t}) \quad (5.18)$$

If the compatibility conditions are considered, it can be obtained,

$$\lambda^T \mathbf{Q} = \mathbf{0} \quad (5.19)$$

$$\mathbf{Q}\mathbf{u} = \mathbf{0} \quad (5.20)$$

So, from the EBE technique, the general vibration equation with the interval parameters can be written as,

$$\begin{bmatrix} \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{x}^I(t) \\ \lambda^I \end{Bmatrix} + \begin{bmatrix} \mathbf{C}^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{x}}^I(t) \\ \lambda^I \end{Bmatrix} + \begin{bmatrix} \mathbf{M}^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}}^I(t) \\ \lambda^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^I(t) \\ \mathbf{0} \end{Bmatrix} \quad (5.21)$$

where $\mathbf{Q}$ is the constraint matrix of the structure. $\mathbf{Q}$ can be illustrated by the example in Fig.5.1,

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \quad (5.22)$$

After Eq. (5.21) is applied with Fourier transform, the following equation can be obtained,

$$\left(\begin{bmatrix} \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} + i\omega \begin{bmatrix} \mathbf{C}^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M}^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{x}^I(\omega) \\ \lambda^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^I(\omega) \\ \mathbf{0} \end{Bmatrix} \quad (5.23)$$

It can be rewritten as,

$$\begin{bmatrix} \mathbf{K}^I + i\omega \mathbf{C}^I - \omega^2 \mathbf{M}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{x}^I(\omega) \\ \lambda^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^I(\omega) \\ \mathbf{0} \end{Bmatrix} \quad (5.24)$$

Substituting Eq. (5.2) into Eq. (5.24), we obtain,

$$\begin{bmatrix} (-\omega^2 + i\omega b) \mathbf{M}^I + (i\omega a + 1) \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{x}^I(\omega) \\ \lambda^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^I(\omega) \\ \mathbf{0} \end{Bmatrix} \quad (5.25)$$

Eq. (5.25) can be expressed as,

$$\left(\begin{bmatrix} (-\omega^2 + i\omega b) \mathbf{M}^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} (i\omega a + 1) \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{x}^I(\omega) \\ \lambda^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^I(\omega) \\ \mathbf{0} \end{Bmatrix} \quad (5.26)$$

For an element with interval parameters, the interval parameter could be factored

out from the element stiffness matrix and mass matrix. Considering the i th finite element in the structure, the element stiffness matrix $\mathbf{K}_i^I$ and mass matrix $\mathbf{M}_i^I$ can be expressed in the form,

$$\mathbf{K}_i^I = \mathbf{K}_i^c(\mathbf{I} + \mathbf{d}_i^I) \quad (5.27)$$

$$\mathbf{M}_i^I = \mathbf{M}_i^c(\mathbf{I} + \mathbf{d}_i^I) \quad (5.28)$$

where each element of $\mathbf{K}_i^c$ and $\mathbf{M}_i^c$ are the midpoint of each element of $\mathbf{K}_i^I$ and $\mathbf{M}_i^I$ respectively, $\mathbf{I}$ is the identity matrix and $\mathbf{d}_i^I$ is an interval diagonal matrix. Following the same procedure for each element, the system stiffness matrix $\mathbf{K}^I$ and mass matrix $\mathbf{M}^I$ constructed by the EBE technique can be expressed as,

$$\mathbf{K}^I = \mathbf{K}^c(\mathbf{I} + \mathbf{d}^I) \quad (5.29)$$

$$\mathbf{M}^I = \mathbf{M}^c(\mathbf{I} + \mathbf{d}^I) \quad (5.30)$$

Applying this factorization, Eq. (5.26) can be written as,

$$\begin{aligned} & \left(\begin{bmatrix} (-\omega^2 + i\omega b)\mathbf{M}^c + (i\omega a + 1)\mathbf{K}^c & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} + (i\omega a + 1) \begin{bmatrix} \mathbf{K}^c \mathbf{d}_1^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right. \\ & \quad \left. + (-\omega^2 + i\omega b) \begin{bmatrix} \mathbf{M}^c \mathbf{d}_2^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{x}^I(\omega) \\ \boldsymbol{\lambda}^I \end{Bmatrix} \\ & = \begin{Bmatrix} \mathbf{F}^I(\omega) \\ \mathbf{0} \end{Bmatrix} \end{aligned} \quad (5.31)$$

To use the Brouwer's fixed point theorem which has been introduced in Chapter 3 in the formulation above, Eq. (5.31) is rewritten in the linear system $\mathbf{A}^I \mathbf{y}^I = \mathbf{p}^I$, with

$$\begin{aligned}
\mathbf{A}^I &= \begin{bmatrix} (-\omega^2 + i\omega b)\mathbf{M}^c + (i\omega a + 1)\mathbf{K}^c & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} + (i\omega a + 1) \begin{bmatrix} \mathbf{K}^c \mathbf{d}_1^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
&\quad + (-\omega^2 + i\omega b) \begin{bmatrix} \mathbf{M}^c \mathbf{d}_2^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
\mathbf{y}^I &= \begin{Bmatrix} \mathbf{x}^I(\omega) \\ \lambda^I \end{Bmatrix}, \quad \mathbf{p}^I = \begin{Bmatrix} \mathbf{F}^I(\omega) \\ \mathbf{0} \end{Bmatrix}
\end{aligned} \tag{5.32}$$

$\mathbf{A}^I$ can be decomposed further,

$$\begin{aligned}
\mathbf{A}^I &= \begin{bmatrix} (-\omega^2 + i\omega b)\mathbf{M}^c + (i\omega a + 1)\mathbf{K}^c & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} + (i\omega a + 1) \begin{bmatrix} \mathbf{K}^c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d}_1^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
&\quad + (-\omega^2 + i\omega b) \begin{bmatrix} \mathbf{M}^c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d}_2^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
&= \mathbf{A}^c + \mathbf{S}_1 \mathbf{D}_1^I + \mathbf{S}_2 \mathbf{D}_2^I
\end{aligned} \tag{5.33}$$

Using the fixed point iteration form, the $\mathbf{y}^{*n+1^I}$ can be expressed as,

$$\mathbf{y}^{*n+1^I} = \mathbf{R} \mathbf{p}^I - \mathbf{y}_0 - \mathbf{R}(\mathbf{S}_1 \mathbf{D}_1^I + \mathbf{S}_2 \mathbf{D}_2^I)(\mathbf{y}_0 + \mathbf{y}^{*n^I}) \quad \text{for } n=0,1,2,\dots \tag{5.34}$$

where $\mathbf{R} = \mathbf{A}^{c-1}$ and $\mathbf{y}_0 = \mathbf{R} \mathbf{p}^c$.

The special treatment Eq. (3.21) of the right hand side part $(\mathbf{S}_1 \mathbf{D}_1^I + \mathbf{S}_2 \mathbf{D}_2^I)(\mathbf{y}_0 + \mathbf{y}^{*n^I})$ is similar to that in Chapter 3 Section 3.3. If the termination condition of Eq. (5.34) is satisfied after n iterations, the enclosure $\mathbf{y}^I$ is given by

$$\mathbf{y}^I = \mathbf{y}^{*n+1^I} + \mathbf{y}_0 \tag{5.35}$$

Thus, the magnitude of FRF of the structure can be obtained. The transient response

can be found using the inverse Fourier transform similar to that used in the direct IFE method shown in Eq. (5.14). The convergence of iterations of Eq. (5.34) has been proven by Muhanna et al. (2007).

The EBE-IFE method is implemented in Matlab with the interval calculation tool box IntLab (Hargreaves, 2002). The developed computer programme for truss problem is given in Appendix C.

5.3 Numerical Examples

5.3.1 Four-story Tower

In this example, a four-story tower structure shown in Fig. 5.1 is considered. Each story has the same mass $M_1 = M_2 = M_3 = M_4 = 100$ kg with 0.5% uncertainty. The system stiffness is $K_1 = K_2 = K_3 = K_4 = 1000$ KN/m with 1% uncertainty. The stiffness and the mass density of each element are assumed to vary independently. The damping is assumed to be $\mathbf{C}^I = 0.001 \times \mathbf{K}^I + 1 \times \mathbf{M}^I$. Note that the response of a dynamic system is always sensitive to the choice of $\mathbf{C}^I$.

To solve this four-story system, the direct method, the EBE-IFE method and the vertex method are used. The magnitudes of FRF from $X1$ to $X4$ which are FRFs $H_{11}(\omega)$, $H_{22}(\omega)$, $H_{33}(\omega)$ and $H_{44}(\omega)$ are shown in Fig. 5.2, where $H_{ij}(\omega)$ is the FRF of output X_i with respect to input X_j .

(a) Magnitude of $H_{11}(\omega)$

(b) Magnitude of $H_{22}(\omega)$

(c) Magnitude of $H_{33}(\omega)$

(d) Magnitude of $H_{44}(\omega)$

Figure 5.2. Magnitude of FRF of four-story tower under different frequency

From Fig. 5.2, it is evident that the EBE-IFE method holds the high precision of the interval arithmetic by comparing with the vertex solution. The IFE method has significantly reduced the overestimation by comparing with the direct method, where “direct” indicates the results of direct IFE method and “present” means the results of the EBE-IFE method. It is also noticed that the four peaks in the figures represent the four resonance frequencies of the structure which are about 0.731, 3.022, 4.832 and 5.911 Hz.

When a harmonic load is applied on the first story of this structure, the historical response of the structure can be obtained. The force is assigned to be,

$$\mathbf{F}(t) = \mathbf{F}_1(t) + \mathbf{F}_2(t) + \mathbf{F}_3(t)$$

$$= \begin{Bmatrix} \cos(0.731t) \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \cos(3.022t) \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \cos(4.832t) \\ 0 \\ 0 \\ 0 \end{Bmatrix} \mathbf{N} \quad (5.36)$$

The system can be solved by using Eq. (5.14). So the transient response of the structure is obtained as,

$$\mathbf{x}(t) = \text{Re} \left(\tilde{\mathbf{Z}}^{-1}(0.731) \begin{Bmatrix} e^{-i0.731t} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \right) + \text{Re} \left(\tilde{\mathbf{Z}}^{-1}(3.022) \begin{Bmatrix} e^{-i3.022t} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \right) + \text{Re} \left(\tilde{\mathbf{Z}}^{-1}(4.832) \begin{Bmatrix} e^{-i4.832t} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \right) \quad (5.37)$$

The caused transient responses of the structure are shown in Fig. 5.3.

(a) Response history of X1

(b) Response history of X2

(c) Response history of X3

(d) Response history of X4

Figure 5.3. Response history of four-story tower

The comparison of the EBE-IFE method with the direct method and the vertex solution is shown in Fig. 5.3. The EBE-IFE method shows much higher accuracy than the direct method. The improvement of the results is significant.

5.3.2 Two-bay Truss

The structure in this example is the two-bay truss shown in Fig. 3.2. Each element has a cross-sectional area $A_i = 0.01\text{m}^2$. Each element's mass density is $\rho = 7800\text{ kg/m}^3$. The modulus of elasticity of each element is assumed to vary independently. The damping is assumed to be $\mathbf{C}^I = 0.001 \times \mathbf{K}^I + 1 \times \mathbf{M}^I$. Two cases of uncertain modulus of elasticity are considered, (1) $E_i^I = [199,201]\text{GPa}$, i.e., 0.5% uncertainty; and (2) $E_i^I = [190,210]\text{GPa}$, i.e., 5% uncertainty; $i=1,\dots,11$.

To solve this system, three methods, the direct method, the EBE-IFE method and vertex solution, are used to compare with each other. For Case 1, magnitude of FRF of node 4 in the x direction $H_{33}(\omega)$ is shown in Fig. 5.4, magnitude of FRF of node 4 in the y direction $H_{44}(\omega)$ is shown in Fig. 5.5, and magnitude of FRF of node 2 in the y direction $H_{22}(\omega)$ is shown in Fig. 5.6.

Figure 5.4 Magnitude of $H_{33}(\omega)$ under different frequency

Figure 5.5 Magnitude of $H_{44}(\omega)$ under different frequency

Figure 5.6 Magnitude of $H_{22}(\omega)$ under different frequency

In Fig. 5.4, the vertical axis is the magnitude of $H_{33}(\omega)$, and the horizontal axis is the frequency ranging from 0 to 700 Hz. The peaks of the curve represent the natural frequencies of the structure. The first natural frequency of this structure is 170 Hz and the second one is 313 Hz, which are calculated based on the constant value of the parameters of the system. From the comparisons shown in Figs. 5.4 to 5.6, it can be concluded that the EBE-IFE method possesses the high accuracy in computation since it matches very well with the vertex solution. Furthermore, it is obvious that the EBE-IFE method significantly reduces the overestimation of the direct method.

Fig. 5.7 shows the results obtained from Case 2 with higher (5%) uncertainty in E . It is obvious that the results of direct method do not converge and show no meaning, but the results of the proposed EBE-IFE method are comparable with the vertex solution. Therefore, with the increased uncertainty of 5% in the structural properties, the EBE-IFE method is able to generate accurate results compared with the vertex

solution, while the direct method does not work. The practice of choosing the lower/upper bound of modulus of elasticity all structural members is the origin of the vertex method. Its result is one case in the vertex solutions, and hence, it is not considered as an independent method in the original thesis. Such case has been extracted from the vertex solution and compared with other results in Fig.5.8. It can be concluded that the accuracy of this simplest method is not enough, thus it is necessary to develop more sophisticated approach as the proposed EBE-IFE method.

Figure 5.7 Magnitude of $H_{22}(\omega)$ with 5% uncertainty

Figure 5.8 Comparison of $H_{22}(\omega)$ with 5% uncertainty

5.3.3 25-bar Space Truss

The structure in this example is the 25-bar space truss shown in Fig. 3.4. Each element has a cross-sectional area $A_i = 0.1\text{m}^2$, and an uncertain modulus of elasticity $E_i^I = [63,77]\text{GPa}$, $i=1,\dots,11$ (i.e. 10% uncertainty in E). Each element's mass density is $\rho = 7800\text{ kg/m}^3$. The modulus of elasticity of each element is assumed to vary independently. The damping is assumed to be $\mathbf{C}^I = 0.01 \times \mathbf{K}^I + 1 \times \mathbf{M}^I$.

The results of $H_{11}(\omega)$ which is magnitude of FRF of node 1 in the x direction are calculated in Fig. 5.9. Because the uncertainty of this example is high, the results of direct method do not converge. Fig. 5.9 shows the comparison between the results of the EBE-IFE method and the vertex solution. Again, the EBE-IFE method matches reasonably well with the vertex solution, but it is much more efficient than the vertex method as discussed in Chapter 4. Fig. 5.10 shows the magnitude of FRF of node 1 in the x direction with 2%, 5% and 10% uncertainty. With the small

uncertainties, the higher accuracy of the results can be obtained.

Figure 5.9 Magnitude of $H_{11}(\omega)$ with 10% uncertainty

Figure 5.10 Magnitude of $H_{11}(\omega)$ with different percentage uncertainty

The cases of 5% and 10% uncertainties in Sections 5.3.2 and 5.3.3 show that the direct method does not work for these high percentages of uncertainties. However, the proposed EBE-IFE method still generates accurate results as compared to the

vertex solution. Thus, the proposed method is applicable for 10% uncertainty. It is worth mentioning that the applicable range of uncertainty for an interval method is problem-dependant. Although this 10% range may vary for other problems, it can be concluded that the applicable range of uncertainty for the proposed method is much higher than that of the direct method.

5.4 Summary

In this chapter, the FRF of structures with uncertainty in structural parameters are studied by IFE method. The EBE technique used previously in static analysis is extended to dynamic system. The special treatment of matrix multiplication is also used to reduce the multiple occurrences of the interval variables. The Brouwer's fixed point theorem is then applied to obtain the FRF. In addition, the transient responses of the structures under harmonic external loading are obtained by inverse Fourier transform of the system equation. Finally, three numerical examples, a four story tower, a two-bay truss and 25-bar space truss, are studied. The results show the effectiveness of the EBE-IFE method by comparing with the vertex solution. Results also show that the EBE-IFE method is able to significantly reduce the overestimation of the direct IFE method.

Note that the EBE-IFEA approach will introduce additional DOFs to the problem solving. The computational cost is higher than the direct method, but the results obtained are much more accurate, as illustrated previously. Meanwhile, The computational cost of the proposed method is much less than the vertex method, yet the results are comparable.

It is also worth mentioning that the Rayleigh damping was employed due to its simplicity and common use in engineering practice. In the Rayleigh damping, both

mass and stiffness matrices are involved in calculation. In the proposed method, the multiple occurrence of uncertain mass and stiffness parameters in the Rayleigh damping is prevented by using the special treatment and EBE technique. If other damping like modal damping is used, the derivation of the IFEA formulation will be similar and the special treatment and EBE technique are still applicable.

CHAPTER 6 INTERVAL ANALYSIS OF DYNAMIC RESPONSE OF STRUCTURES USING LAPLACE TRANSFORM

6.1 Introduction

As reviewed in Chapter 2, the perturbation theory and Taylor series expansion have been used by many researchers to obtain the dynamic response with interval parameters. However, the computational cost is high since it is required to solve a system of n -dimensional 2^{nd} ordered linear ODE for m times, where n is the degree of freedom (DOF) of the problem and m is the number of independent intervals. Therefore, in this chapter, an interval analysis method for dynamic systems with uncertain parameters using Laplace transform is proposed to alleviate the times of solving equations compared with regular perturbation method. The Laplace transform is first used to convert the dynamic equations into a linear system of equations. The matrix perturbation technique is then utilized to remove the higher order terms in the equations, and the inverse Laplace transform is employed to obtain the structural dynamic responses. In addition, the EBE idea is used in dynamic analysis. A special matrix treatment is also considered for both the EBE and non-EBE cases to reduce the overestimation of interval analysis and to facilitate the inverse Laplace transform.

To prove the effectiveness of the proposed method, several numerical examples are presented by comparing with the vertex solutions and other researchers' work. The proposed method is suitable for small uncertainties of system parameters since the formulation is limited to the first order terms, which results in an inner enclosure of the system response.

6.2 Interval Dynamic Analysis of Structures

6.2.1 Laplace Transform based Dynamic Analysis Method

For a deterministic system under external loading, the Laplace transform based dynamic analysis method is presented by Douglas (2008). The Laplace transform will be used here to solve the general interval dynamic equation with n DOFs.

$$\mathbf{M}^I \ddot{\mathbf{X}}^I(t) + \mathbf{C}^I \dot{\mathbf{X}}^I(t) + \mathbf{K}^I \mathbf{X}^I(t) = \mathbf{F}^I(t) \quad (5.1)$$

The Laplace transform is defined as follows,

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t) dt \quad (6.1)$$

where the parameter s is a complex number, $s = \gamma + i\omega$, with real numbers γ and ω .

The Laplace transform of $\ddot{\mathbf{X}}^I(t)$ and $\dot{\mathbf{X}}^I(t)$ can be expressed as,

$$\mathcal{L}\{\ddot{\mathbf{X}}^I(t)\} = s^2 \mathcal{L}^I(s) - s\mathbf{X}^I(0) - \dot{\mathbf{X}}^I(0) \quad (6.2)$$

and

$$\mathcal{L}\{\dot{\mathbf{X}}^I(t)\} = s\mathcal{L}^I(s) - \dot{\mathbf{X}}^I(0) \quad (6.3)$$

After the Laplace transform, Eq. (5.1) becomes,

$$(\mathbf{M}^I s^2 + \mathbf{C}^I s + \mathbf{K}^I) \mathcal{L}^I(s) = \mathbf{L} \mathbf{F}^I(s) + (\mathbf{M}^I s + \mathbf{C}^I) \mathbf{X}^I(0) + \mathbf{M}^I \dot{\mathbf{X}}^I(0) \quad (6.4)$$

where $\mathcal{L}^I(s)$, $\mathbf{L} \mathbf{F}^I(s)$, $\mathbf{X}^I(0)$ and $\dot{\mathbf{X}}^I(0)$ are the Laplace transform of $\mathbf{X}^I(t)$, the Laplace transform of $\mathbf{F}^I(t)$, the system initial displacement and velocity, respectively. If all the initial conditions of the structure are zero, which means $\mathbf{X}^I(0) = \mathbf{0}$, $\dot{\mathbf{X}}^I(0) = \mathbf{0}$ and $\ddot{\mathbf{X}}^I(0) = \mathbf{0}$, Eq. (6.4) can be written as,

$$(\mathbf{M}^I s^2 + \mathbf{C}^I s + \mathbf{K}^I) \mathcal{L}^I(s) = \mathbf{L} \mathbf{F}^I(s) \quad (6.5)$$

With the Rayleigh damping Eq. (5.2), Eq. (6.5) becomes,

$$(\mathbf{M}^I(s^2 + bs) + \mathbf{K}^I(as + 1)) \mathcal{L}^I(s) = \mathbf{L} \mathbf{F}^I(s) \quad (6.6)$$

The interval matrices $\mathcal{L}^I$, $\mathbf{M}^I$, $\mathbf{K}^I$ and $\mathbf{L} \mathbf{F}^I$ can be expressed in terms of the mid-matrix and uncertainty as

$$\mathcal{L}^I = \mathcal{L}^c + \Delta \mathcal{L}^I \quad (6.7)$$

$$\mathbf{M}^I = \mathbf{M}^c + \Delta \mathbf{M}^I \quad (6.8)$$

$$\mathbf{K}^I = \mathbf{K}^c + \Delta \mathbf{K}^I \quad (6.9)$$

$$\mathbf{L} \mathbf{F}^I = \mathbf{L} \mathbf{F}^c + \Delta \mathbf{L} \mathbf{F}^I \quad (6.10)$$

Then, $\Delta \mathbf{M}^I = \mathbf{M}^I - \mathbf{M}^c$ and $\Delta \mathbf{K}^I = \mathbf{K}^I - \mathbf{K}^c$ can be obtained.

Substituting Eqs. (6.7), (6.8), (6.9) and (6.10) into Eq. (6.6) leads to,

$$\begin{aligned}
& [\mathbf{M}^c(s^2 + bs) + \mathbf{K}^c(as + 1)]\mathcal{L}^c(s) \\
& + [\Delta\mathbf{M}^I(s^2 + bs) + \Delta\mathbf{K}^I(as + 1)]\mathcal{L}^c(s) \\
& + [\mathbf{M}^c(s^2 + bs) + \mathbf{K}^c(as + 1)]\Delta\mathcal{L}^I(s) \\
& + [\Delta\mathbf{M}^I(s^2 + bs) + \Delta\mathbf{K}^I(as + 1)]\Delta\mathcal{L}^I(s) = \mathbf{L}\mathbf{F}^c(s) + \Delta\mathbf{L}\mathbf{F}^I(s) \quad (6.11)
\end{aligned}$$

With the mid-value of the parameters, it can be obtained,

$$[\mathbf{M}^c(s^2 + bs) + \mathbf{K}^c(as + 1)]\mathcal{L}^c(s) = \mathbf{L}\mathbf{F}^c(s) \quad (6.12)$$

If the second order uncertain terms in Eq. (6.11) are neglected and Eq. (6.12) is considered, Eq. (6.11) can be written as,

$$\mathbf{A} \times \Delta\mathcal{L}^I(s) = \Delta\mathbf{L}\mathbf{F}^I(s) - [\Delta\mathbf{M}^I(s^2 + bs) + \Delta\mathbf{K}^I(as + 1)]\mathcal{L}^c(s) \quad (6.13)$$

where

$$\mathbf{A} = \mathbf{M}^c(s^2 + bs) + \mathbf{K}^c(as + 1) \quad (6.14)$$

Therefore, from Eq. (6.13), the uncertainty of $\mathcal{L}^I(s)$ can be obtained,

$$\Delta\mathcal{L}^I(s) = \mathbf{A}^{-1}\Delta\mathbf{L}\mathbf{F}^I(s) - \mathbf{A}^{-1}\Delta\mathbf{M}^I(s^2 + bs)\mathcal{L}^c(s) - \mathbf{A}^{-1}\Delta\mathbf{K}^I(as + 1)\mathcal{L}^c(s) \quad (6.15)$$

After taking the inverse Laplace transform of each term in Eq. (6.15), the uncertainty part of the dynamic response of the structure can be obtained. And the inverse Laplace transform is defined as follows,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds \quad (6.16)$$

where γ is a real number so that the contour path of integration is in the region of convergence of $F(s)$ normally requiring $\gamma > \text{Re}(S_p)$ for every singularity S_p of $F(s)$ and $i^2 = -1$.

Hence, Eq. (6.15) becomes,

$$\begin{aligned} \Delta \mathbf{X}^I(t) = \mathcal{L}^{-1}\{\mathbf{A}^{-1}\Delta \mathbf{F}^I(s)\} \\ - \mathcal{L}^{-1}\{\mathbf{A}^{-1}\Delta \mathbf{M}^I(s^2 + bs)\mathcal{L}^c(s)\} - \mathcal{L}^{-1}\{\mathbf{A}^{-1}\Delta \mathbf{K}^I(as + 1)\mathcal{L}^c(s)\} \end{aligned} \quad (6.17)$$

It should be noted that it is difficult to integrate the terms with uncertainty. Thus, a special treatment Eq. (3.21) of matrix is used to extract the uncertainties outside of the integral. For example, letting

$$\Delta \mathbf{K}^I = \begin{bmatrix} \Delta k_1^I & \Delta k_1^I & 0 \\ \Delta k_1^I & \Delta k_1^I + \Delta k_2^I & \Delta k_2^I \\ 0 & \Delta k_2^I & \Delta k_2^I \end{bmatrix} \quad \mathcal{L}^c(s) = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \quad (6.18)$$

the last term in Eq. (6.17) can be rewritten as,

$$\begin{aligned} \mathcal{L}^{-1}\{\mathbf{A}^{-1}\Delta \mathbf{K}^I(as + 1)\mathcal{L}^c(s)\} &= \mathcal{L}^{-1}\left\{\mathbf{A}^{-1}(as + 1) \begin{bmatrix} \Delta k_1^I & \Delta k_1^I & 0 \\ \Delta k_1^I & \Delta k_1^I + \Delta k_2^I & \Delta k_2^I \\ 0 & \Delta k_2^I & \Delta k_2^I \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix}\right\} \\ &= \mathcal{L}^{-1}\left\{\mathbf{A}^{-1}(as + 1) \begin{bmatrix} L_1 + L_2 & 0 \\ L_1 + L_2 & L_2 + L_3 \\ 0 & L_2 + L_3 \end{bmatrix} \begin{Bmatrix} \Delta k_1^I \\ \Delta k_2^I \end{Bmatrix}\right\} \\ &= \mathcal{L}^{-1}\left\{\mathbf{A}^{-1}(as + 1) \begin{bmatrix} L_1 + L_2 & 0 \\ L_1 + L_2 & L_2 + L_3 \\ 0 & L_2 + L_3 \end{bmatrix} \begin{Bmatrix} \Delta k_1^I \\ \Delta k_2^I \end{Bmatrix}\right\} \end{aligned} \quad (6.19)$$

The same method can be used to deal with the second term at the right hand side of Eq. (6.17). By using this special treatment Eq. (3.21), not only the integration of

uncertain quantities can be calculated but also the overestimation of interval calculation can be reduced.

Using the inverse Laplace transform, $\mathbf{X}^c(t)$ can be solved from Eq. (6.12),

$$\begin{aligned}\mathbf{X}^c(t) &= \mathcal{L}^{-1}\{\mathcal{L}^c(s)\} \\ &= \mathcal{L}^{-1}\{[\mathbf{M}^c(s^2 + bs) + \mathbf{K}^c(as + 1)]^{-1}\mathbf{L}\mathbf{F}^c(s)\} \\ &= \mathcal{L}^{-1}\{\mathbf{A}^{-1}\mathbf{F}\mathbf{F}^c(s)\}\end{aligned}\quad (6.20)$$

Then, the displacement intervals equal to the mid-value of displacement plus the uncertainty part, as

$$\mathbf{X}^I(t) = \mathbf{X}^c(t) + \Delta\mathbf{X}^I(t) \quad (6.21)$$

6.2.2 Dynamic Analysis with EBE Technique

To reduce the overestimation in the interval finite element solutions, the issues of coupling and multiple occurrences of interval variables have to be handled properly. In this section, the EBE method is extended to the dynamic analysis to further reduce the overestimation.

The interval stiffness matrix $\mathbf{K}^I$ and the interval mass matrix $\mathbf{M}^I$ have been introduced in Chapter 5, Eqs. (5.15) and (5.16). The damping matrix is Rayleigh Damping. Using the Lagrange equations of the first kind, the general vibration equation with the interval parameters can be written as,

$$\begin{bmatrix} \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{X}^I(t) \\ \boldsymbol{\lambda}^I \end{Bmatrix} + \begin{bmatrix} \mathbf{C}^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{X}}^I(t) \\ \boldsymbol{\lambda}^I \end{Bmatrix} + \begin{bmatrix} \mathbf{M}^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{X}}^I(t) \\ \boldsymbol{\lambda}^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^I(t) \\ \mathbf{0} \end{Bmatrix} \quad (5.20)$$

where $\mathbf{Q}$ is the constraint matrix of the structure.

After using the Laplace transform, Eq. (5.20) becomes,

$$\begin{aligned} & \begin{bmatrix} \mathbf{M}^I s^2 + \mathbf{C}^I s + \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathcal{L}^I(s) \\ \mathcal{L}^I(\lambda) \end{Bmatrix} \\ &= \begin{Bmatrix} \mathbf{F} \mathbf{F}^I(s) \\ \mathbf{0} \end{Bmatrix} + \begin{bmatrix} \mathbf{C}^I + \mathbf{M}^I s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{X}^I(0) \\ \mathbf{0} \end{Bmatrix} + \begin{bmatrix} \mathbf{M}^I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{X}}^I(0) \\ \mathbf{0} \end{Bmatrix} \end{aligned} \quad (6.22)$$

where $\mathcal{L}^I(\lambda)$ is the Laplace transform of λ^I . If all the initial conditions of the structure are zero, which means $\mathbf{X}^I(0) = \mathbf{0}$, $\dot{\mathbf{X}}^I(0) = \mathbf{0}$ and $\ddot{\mathbf{X}}^I(0) = \mathbf{0}$, Eq. (6.22) can be simplified as,

$$\begin{bmatrix} \mathbf{M}^I s^2 + \mathbf{C}^I s + \mathbf{K}^I & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathcal{L}^I(s) \\ \mathcal{L}^I(\lambda) \end{Bmatrix} = \begin{Bmatrix} \mathbf{L} \mathbf{F}^I(s) \\ \mathbf{0} \end{Bmatrix} \quad (6.23)$$

Using the Rayleigh Damping, Eq. (6.23) becomes,

$$\begin{bmatrix} \mathbf{M}^I(s^2 + bs) + \mathbf{K}^I(1 + as) & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathcal{L}^I(s) \\ \mathcal{L}^I(\lambda) \end{Bmatrix} = \begin{Bmatrix} \mathbf{L} \mathbf{F}^I(s) \\ \mathbf{0} \end{Bmatrix} \quad (6.24)$$

Substituting Eqs. (6.7), (6.8), (6.9) and (6.10) into Eq. (6.24) leads to,

$$\begin{aligned} & \begin{bmatrix} \mathbf{A} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathcal{L}^c(s) \\ \mathcal{L}^c(\lambda) \end{Bmatrix} + \begin{bmatrix} \mathbf{A} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta \mathcal{L}^I(s) \\ \Delta \mathcal{L}^I(\lambda) \end{Bmatrix} \\ &+ \left\{ \begin{bmatrix} \Delta \mathbf{K}^I(as + 1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \Delta \mathbf{M}^I(s^2 + bs) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \begin{Bmatrix} \mathcal{L}^c(s) \\ \mathcal{L}^c(\lambda) \end{Bmatrix} \\ &+ \left\{ \begin{bmatrix} \Delta \mathbf{K}^I(as + 1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \Delta \mathbf{M}^I(s^2 + bs) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \begin{Bmatrix} \Delta \mathcal{L}^I(s) \\ \Delta \mathcal{L}^I(\lambda) \end{Bmatrix} \\ &= \begin{Bmatrix} \mathbf{L} \mathbf{F}^c(s) \\ \mathbf{0} \end{Bmatrix} + \begin{Bmatrix} \Delta \mathbf{L} \mathbf{F}^I(s) \\ \mathbf{0} \end{Bmatrix} \end{aligned} \quad (6.25)$$

With the mid-value of the parameters, it can be obtained,

$$\begin{bmatrix} \mathbf{A} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathcal{L}^c(s) \\ \mathcal{L}^c(\lambda) \end{Bmatrix} = \begin{Bmatrix} \mathbf{L}\mathbf{F}^c(s) \\ \mathbf{0} \end{Bmatrix} \quad (6.26)$$

and neglecting the second order terms of uncertainty in Eq. (6.25), Eq. (6.25) can be written as,

$$\begin{aligned} & \begin{bmatrix} \mathbf{A} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta\mathcal{L}^I(s) \\ \Delta\mathcal{L}^I(\lambda) \end{Bmatrix} \\ &= \begin{Bmatrix} \Delta\mathbf{F}^I(s) \\ \mathbf{0} \end{Bmatrix} - \left\{ \begin{bmatrix} \Delta\mathbf{K}^I(as + 1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{M}^I(s^2 + bs) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \begin{Bmatrix} \mathcal{L}^c(s) \\ \mathcal{L}^c(\lambda) \end{Bmatrix} \end{aligned} \quad (6.27)$$

where the term $\begin{Bmatrix} \mathcal{L}^c(s) \\ \mathcal{L}^c(\lambda) \end{Bmatrix}$ can be obtained from Eq. (6.26). Then, taking the inverse Laplace transform of Eq. (6.27), the uncertain part of the displacement of vibration system and the Lagrange multipliers can be obtained,

$$\begin{aligned} \begin{Bmatrix} \Delta\mathbf{X}^I(t) \\ \Delta\boldsymbol{\lambda}^I \end{Bmatrix} &= \mathcal{L}^{-1} \left(\begin{bmatrix} \mathbf{A} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix}^{-1} \begin{Bmatrix} \Delta\mathbf{F}^I(s) \\ \mathbf{0} \end{Bmatrix} \right) \\ &\quad - \mathcal{L}^{-1} \left(\begin{bmatrix} \mathbf{A} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \Delta\mathbf{K}^I(as + 1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathcal{L}^c(s) \\ \mathcal{L}^c(\lambda) \end{Bmatrix} \right) \\ &\quad - \mathcal{L}^{-1} \left(\begin{bmatrix} \mathbf{A} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \Delta\mathbf{M}^I(s^2 + bs) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathcal{L}^c(s) \\ \mathcal{L}^c(\lambda) \end{Bmatrix} \right) \end{aligned} \quad (6.28)$$

Similar to Eq. (6.17), it is difficult to integrate the terms with interval parameters in Eq. (6.28). Thus, the special matrix treatment for Eq. (6.17) needs to be used to facilitate the integration of uncertainty quantities in Eq. (6.28). Taking the inverse Laplace transform of Eq. (6.26), the deterministic part of the displacement and the Lagrange multipliers can be obtained,

$$\begin{Bmatrix} \mathbf{X}^c(t) \\ \boldsymbol{\lambda}^c \end{Bmatrix} = \mathcal{L}^{-1} \left(\begin{bmatrix} \mathbf{A} & \mathbf{Q}^T \\ \mathbf{Q} & \mathbf{0} \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{L}\mathbf{F}^c(s) \\ \mathbf{0} \end{Bmatrix} \right) \quad (6.29)$$

Figure 6.1 Flow chart of proposed method

After $\begin{Bmatrix} \Delta \mathbf{X}^I(t) \\ \Delta \boldsymbol{\lambda}^I \end{Bmatrix}$ and $\begin{Bmatrix} \mathbf{X}^c(t) \\ \boldsymbol{\lambda}^c \end{Bmatrix}$ are obtained, the displacement of vibration system and the Lagrange multipliers can be easily calculated as the sum of their mid-values and the corresponding uncertainty parts,

$$\begin{Bmatrix} \mathbf{X}^I(t) \\ \boldsymbol{\lambda}^I \end{Bmatrix} = \begin{Bmatrix} \mathbf{X}^c(t) \\ \boldsymbol{\lambda}^c \end{Bmatrix} + \begin{Bmatrix} \Delta \mathbf{X}^I(t) \\ \Delta \boldsymbol{\lambda}^I \end{Bmatrix} \quad (6.30)$$

The proposed method can be summarized in flow chart shown in Fig. 6.1. It is implemented in Matlab with the interval calculation tool box IntLab (Hargreaves, 2002). The developed computer programme is given in Appendix D.

6.3 Numerical Examples

6.3.1 Four-Story Frame

In order to illustrate the effectiveness of the presented interval based method for the dynamic analysis of the structures with uncertainty, a four-story frame tower structure shown in Chapter 5, Fig. 5.1 is analyzed. Each story has the same mass $M_1 = M_2 = M_3 = M_4 = 100 \text{ kg}$ with 5% uncertainty. The system stiffness is $K_1 = K_2 = K_3 = K_4 = 1000 \text{ kN/m}$ with 10% uncertainty. The stiffness and the mass density of each element are assumed to vary independently. A harmonic external loading $F = \cos(3t) \text{ kN}$ is applied on the first story in the horizontal direction with initial conditions $\mathbf{x}^I(0) = \mathbf{0}$, $\dot{\mathbf{x}}^I(0) = \mathbf{0}$. Two cases are considered for this structure, i.e., Case 1 undamped vibration, and Case 2 damped vibration by assuming damping matrix as $\mathbf{C}^I = 0.01 \times \mathbf{K}^I + 1 \times \mathbf{M}^I$. The first four natural frequencies for the nominal structure with mid-point value of parameters are calculated as 0.731, 3.022, 4.832 and 5.911 Hz.

Figure 6.2. Response of first story without damping in horizontal direction within 5 seconds

Figure 6.3. Response of first story without damping in horizontal direction within 40 seconds

For Case 1, the displacement responses of the first story in the horizontal direction obtained by the proposed method without using the EBE technique are compared

with the vertex solution in Fig. 6.2 and Fig. 6.3. In Fig. 6.2, during the first 5 seconds, the upper and lower bounds of the dynamic response are agreeable with the vertex solution. The width of interval response increases noticeably with time. Fig. 6.3 describes the response within the first 40 seconds. It can be found that the dynamic responses of the undamped structure diverged due to the uncertain system parameters. This divergent behavior is validated by the vertex solution, which indicates that it is due to the nature of the problem rather than the numerical error caused by the proposed method. This behavior of the un-damped system can be attributed to the free vibration of the system which does not decay with time due to zero damping. In addition, the response of damped system is dominated by the forced vibration in long-run and is always convergent, while the responses of the undamped system depend on both free and forced vibration. However, the divergence in response can only be observed in un-damped system. For the damped system, the response is always convergent, as will be illustrated in the next paragraph and the later example.

Figure 6.4 Response of first story with damping in horizontal direction within 12 seconds

For Case 2, the displacement responses of the first story during the first 12 seconds obtained by the proposed method with and without the EBE technique are shown in Fig. 6.4. It can be observed that the dynamic responses converge under damping because the responses of damped system are controlled by the forced vibration and the free vibration diminishes with time. By comparing the results with and without the EBE technique, it can be found that the dynamic response with the EBE technique is almost the same as that without the EBE technique and sharp interval responses are obtained in both cases. It indicates that the special treatment Eq. (3.21) of the matrix in Eq. (6.17) is more effective to reduce the overestimation than the EBE technique.

6.3.2 Eight-Bar Truss

Consider the plane truss subjected to a harmonic sinusoidal excitation $P(t) = -p^I \sin(20\pi t) N$ at node 3 in the vertical direction with the initial conditions $\mathbf{x}^I(0) = \mathbf{0}$ and $\dot{\mathbf{x}}^I(0) = \mathbf{0}$, as shown in Chapter 4, Fig. 4.3, which has been studied by Qiu and Wang (2003). The truss is modeled by 6 nodes and 8 elements. Assume that the cross-sectional area of elements 1, 2, 3 and 4 are $1.0 \times 10^{-4} m^2$, and those for elements 5, 6, 7 and 8 are $1.2 \times 10^{-4} m^2$. The Young's modulus, the mass density of the plane truss and the harmonic excitation amplitude are assumed to be interval parameters. These interval parameters are: $E^I = [190 \times 10^9, 210 \times 10^9] N/m^2$, $\rho^I = [7410, 8190] kg/m^3$ and $p^I = [95, 105] N$. Again, two cases are considered for this structure, i.e., Case 1 undamped vibration, and Case 2 damped vibration with damping matrix assumed to be $\mathbf{C}^I = 10^{-5} \times \mathbf{K}^I + 1 \times \mathbf{M}^I$. The natural frequencies for the nominal structure with mid-point value of parameters are calculated as 108.6, 296.9, 399.6, 544.6, 823.6, 987.6, 1024.5 and 1114.9 Hz.

Figure 6.5 Response of node 3 within 0.1 seconds

Table 6.1 Comparison of numerical results without EBE method and with EBE method

Time(s)	Lower bound w/o EBE (m)	Upper bound w/o EBE (m)	Lower bound with EBE (m)	Upper bound with EBE (m)
0	0	0	0	0
0.01	-1.54E-05	-1.14E-05	-1.57E-05	-1.14E-05
0.02	-2.45E-05	-1.87E-05	-2.42E-05	-1.96E-05
0.03	-2.39E-05	-1.91E-05	-2.42E-05	-1.88E-05
0.04	-1.45E-05	-1.14E-05	-1.48E-05	-9.89E-06
0.05	-1.63E-06	2.75E-06	-1.88E-06	4.46E-06
0.06	9.45E-06	1.84E-05	1.03E-05	1.95E-05
0.07	1.68E-05	2.70E-05	1.64E-05	2.79E-05
0.08	1.79E-05	2.48E-05	1.79E-05	2.69E-05
0.09	1.09E-05	1.39E-05	1.10E-05	1.64E-05
0.1	-4.74E-06	1.94E-06	-3.88E-06	1.87E-06

For Case 1, the structural responses are computed by the proposed method and the displacement response of node 3 in the vertical direction is plotted in Fig. 6.5. It is observed from Fig. 6.5 and Table 6.1 that the results obtained with and without the EBE technique are almost the same. It shows that the EBE technique cannot produce better results, which agrees with the conclusion drawn from the example in Section 6.3.1. The results obtained by the proposed method match well with the work of Qiu and Wang (2003), as shown in Fig. 6.6. To further study the divergent behavior of response for undamped system, Fig.6.7 presents the response range obtained from 0 to 0.4 seconds. Again, the divergent phenomenon in this undamped system is obvious.

Figure 6.6 Comparison of response with Qiu and Wang's work (2003)

Figure 6.7 Response of node 3 of undamped system

For Case 2, the displacement response of node 3 in the vertical direction is shown in Fig. 6.8, illustrating again the convergence of dynamic response in the damped system.

Figure 6.8 Response of node 3 of damped system

6.4 Summary

An interval based dynamic analysis method using Laplace transform has been presented to determine the vibration responses of structures with uncertain parameters. The EBE technique is extended for the dynamic interval analysis. The computation of the proposed method focuses on the Laplace and inverse Laplace transform. There is no need to calculate the structural response for each parameter perturbation for obtaining the interval change ranges of structural response. The computational cost is thus significantly reduced. The effectiveness of the proposed method has been demonstrated by numerical examples.

The results of the proposed method are comparable with those obtained by other researchers and the vertex solutions. Also, it is found that the EBE technique is unable to generate better solutions compared with the usual non-EBE interval method in presented examples. More importantly, an observation from the present study is the divergent phenomenon of response in the undamped examples, which is verified by the vertex solution. With the first order perturbation, the proposed equations can be simplified. However, this is not an issue in terms of computational requirements. It is worth mentioning that all the interval parameters need to vary independent and the dynamic load is taken as deterministic in nature in the proposed method. The proposed method is suitable for small uncertainties of system parameters since the formulation is limited to the first order terms, which results in an inner enclosure of the system response. The applicability of the proposed method with the regard to maximum level of uncertainty can be studied for the future work. Furthermore, the higher order terms can reserve in proposed method in the future work. The iteration method can be employed to solve the equations. Therefore, the improvement of proposed method can be achieved.

CHAPTER 7 CONCLUSIONS AND FUTURE WORK

7.1 Conclusions and Contributions

In engineering design, it is important to calculate structural response quantities such as the displacement, stress and vibration responses against a given set of design parameters. However, in most practical situations, the structural parameters are uncertain, such as the inaccuracy of the measurement, errors in the manufacturing and assembly process, invalidity of some components and uncertainty in boundary conditions etc. The uncertainties of structural parameters may lead to large and unexpected excursion of responses that may result in drastic reduction in accuracy and precision of the operation. Therefore, the concept of uncertainty plays an important role in the investigation of various engineering problems. Many methods have been developed to solve the uncertainty problems, such as the probability theory, fuzzy theory and interval theory. The interval methods that use interval arithmetic in engineering applications started in the 1990s. In the interval analysis method, the uncertain structural parameters are described as interval variables and only their lower and upper bounds are required.

In this thesis, the effect of uncertainty in individual structural parameters and applied forces on the linear truss systems is studied. Uncertain parameters are introduced in the form of intervals. Interval analysis is applied to calculate the system responses, such as static displacements, eigensolutions, FRF and vibration response. The Matlab programming codes are developed to solve these responses.

7.1.1 Static Analysis

This part proposed an EBE-IFE method in static interval analysis. The Brouwer's

fixed point theorem and Krawczyk's operator are applied on the EBE-IFE method to reduce the overestimation in interval calculation. Stiffness matrices are formulated based on the physics of materials, and the Lagrange multiplier method is applied to impose the necessary constraints for compatibility and equilibrium. Several numerical examples show that the EBE-IFE method has significant improvement in reducing the overestimation of the interval calculation by comparing with the direct IFE method.

7.1.2 Eigenvalue Analysis

This part presented an interval analysis method with perturbation technique to solve the eigenvalues and eigenvectors of structures with interval parameters. The perturbation method was used to remove the second order terms of uncertainties. Unlike the methods proposed by other researchers, this method does not require the calculation of inverse of interval matrix. Special treatment of matrix operation is adopted to avoid multiple occurrences of the interval variables. For n eigenvalues and corresponding eigenvectors, the eigenvalue equation needs to be solved for n times. The effectiveness of the proposed method has been demonstrated by comparing the numerical results obtained by the proposed method with those obtained by other researchers' methods and the vertex solution.

7.1.3 FRF Analysis

This part provided the FRF analysis with uncertainty in structural parameters solving by the proposed IFE method. The EBE technique was extended to dynamic system to formulate the dynamic equation. The Fourier transform was applied to convert the equation into a system of linear equations. The special treatment of matrix multiplication is used to reduce the multiple occurrences of the interval

variables. Eventually, the Brouwer's fixed point theorem is applied in the proposed IFE method to obtain the FRF. In addition, the historical responses of the structures under harmonic external loading are obtained by the inverse Fourier transform. Applying all the techniques proposed for FRF analysis, the FRF can be solved directly. The results of the numerical examples showed the effectiveness and high accuracy of the proposed IFE method by comparing with the vertex solution.

7.1.4 Dynamic Response

This part proposed an interval based dynamic analysis method using Laplace transform to determine the vibration responses of structures with uncertain parameters. The EBE technique was extended for the dynamic interval analysis. The computation of the proposed method focused on the Laplace and inverse Laplace transform. The perturbation method was applied. However, there was no need to calculate the structural response for each parameter perturbation. The computational cost was thus significantly reduced. The results of the proposed method were comparable with those obtained by other researchers and the vertex solutions. Also, the EBE technique was unable to generate better solutions compared with the usual non-EBE interval method in the presented examples. More importantly, a new observation from the present study was the divergent phenomenon of response in the undamped systems. It is worth mentioning that the proposed method is suitable for small uncertainties of system parameters since the formulation is limited to the first order terms.

7.2 Future Work

Based on the proposed IFE methods, further development work can be considered in the following aspects.

1. To obtain the system responses of the frame structure. In this thesis, the developed Matlab programming codes are only applicable for truss system. Therefore, the programming codes for frame structures will be developed for analyzing the responses using interval method.
2. To conduct non-linear analysis of the structure. Structures with non-linear behavior will be studied using the IFE method. With the help of the perturbation method and Taylor series expansion, the estimated response of the nonlinear dynamic system can be obtained by the IFE method. The results of IFE method should be acceptable because the similar work has been done by Qiu et. al. (2009). However, the partial derivatives of the nonlinear dynamic equations need to be determined numerically, which will lead to high computational cost, especially for high DOFs systems.
3. To apply the proposed IFEA method on the large scale structures, such as reinforced concrete and steel buildings, bridges and so on.
4. To optimize structural configuration under uncertainty. Together with certain optimization/search algorithms, the developed method will be employed to optimize some structures whose performances are dependent on uncertain variables.

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APPENDIX A

```
clc
clear

% NE number of elements
% NP number of points
% NF number of freedom
% NM number of materials
% NR number of restraint
% Co coordinate of nodes
% NAE element type
% AE material property

format short e
format long

% read data
NP=xlsread('twobay truss.xls',1,'B3');
NE=xlsread('twobay truss.xls',1,'C3');
NF=xlsread('twobay truss.xls',1,'D3');
NM=xlsread('twobay truss.xls',1,'E3');
NR=xlsread('twobay truss.xls',1,'F3');
aaa=sprintf('B6:E%d',NP+5);
Co=xlsread('twobay truss.xls',1,aaa);
bbb=sprintf('B3:E%d',NE+2);
ME=xlsread('twobay truss.xls',2,bbb);
ooo=sprintf('E3:E%d',NE+2);
NAE=xlsread('twobay truss.xls',2,ooo);
ccc=sprintf('C3:C%d',NM+2);
M1=xlsread('twobay truss.xls',3,ccc);
ddd=sprintf('G3:G%d',NM+2);
M2=xlsread('twobay truss.xls',3,ddd);
AE1=[M1,M2];
eee=sprintf('D3:D%d',NM+2);
M3=xlsread('twobay truss.xls',3,eee);
fff=sprintf('H3:H%d',NM+2);
M4=xlsread('twobay truss.xls',3,fff);
AEu=[M3,M4];
AE=infsup(AE1,AEu);
ggg=sprintf('B3:E%d',NR+2);
```

```

NRR=xlsread('twobay truss.xls',4,ggg);
hhh=sprintf('C3:C%d',2*NE+2);
P1=xlsread('twobay truss.xls',5,hhh);
iii=sprintf('G3:G%d',2*NE+2);
P2=xlsread('twobay truss.xls',5,iii);
jjj=sprintf('K3:K%d',2*NE+2);
P3=xlsread('twobay truss.xls',5,jjj);
kkk=sprintf('O3:O%d',2*NE+2);
P4=xlsread('twobay truss.xls',5,kkk);
P1=[P1,P2,P3,P4];
l11=sprintf('H3:H%d',2*NE+2);
P5=xlsread('twobay truss.xls',5,l11);
mmm=sprintf('L3:L%d',2*NE+2);
P6=xlsread('twobay truss.xls',5,mmm);
nnn=sprintf('P3:P%d',2*NE+2);
P7=xlsread('twobay truss.xls',5,nnn);
Pu=[P1,P5,P6,P7];
P=infsup(P1,Pu);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%

% Form structure stiffness matrix
A=(2*NE);
for i=1:A
    for j=1:A
        K(i,j)=infsup(0,0);
    end
end
for ne=1:NE
    AKE=FKE(NP,ne,NM,ME,AE,NAE,Co);
    T=FT(NP,ne,ME,Co);
    AKET=AKE*T;
    TT=T';
    AKEE=TT*AKET;
    K=FAKK(ne,AKEE,K);
end

K0=K;

% Form matrix C
C(1,1:(6*NE))=0;

```



```

r=1;
for i=1:(NE-1);
    for j=1:2;
        a=ME(i,j+1);
        for k=(i+1):NE;
            if ME(k,2)==a;
                C(r,6*(i-1)+3*(j-1)+1)=1;
                C(r,6*(k-1)+1)=-1;
                C(r+1,6*(i-1)+3*(j-1)+2)=1;
                C(r+1,6*(k-1)+2)=-1;
                C(r+2,6*(i-1)+3*(j-1)+3)=1;
                C(r+2,6*(k-1)+3)=-1;
                r=r+3;
                break
            elseif ME(k,3)==a;
                C(r,6*(i-1)+3*(j-1)+1)=1;
                C(r,6*(k-1)+4)=-1;
                C(r+1,6*(i-1)+3*(j-1)+2)=1;
                C(r+1,6*(k-1)+5)=-1;
                C(r+2,6*(i-1)+3*(j-1)+3)=1;
                C(r+2,6*(k-1)+6)=-1;
                r=r+3;
                break
            end
        end
    end
end

% Deal with constraint
for i=1:NR
    NI=NRR(i,2);
    K(NI,NI)=1e25;
end

% Form matrix A

KK=[K,C';C,zeros((r-1),(r-1))];

% Form load matrix
for i=1:(2*NE)
    for j=1:NF
        F((i-1)*NF+j,1)=P(i,j+1);
    end
end

```

```

end
FF=[F;zeros((r-1),1)];

% Solve equation with iteration method

X=ITSOL(NF,NE,K,r,C,FF,K0)

function [AKE]=FKE(NP,NE,NM,ME,AE,NAE,Co);
N1=ME(NE,2);
N2=ME(NE,3);
X1=Co(N1,2);
Y1=Co(N1,3);
Z1=Co(N1,4);
X2=Co(N2,2);
Y2=Co(N2,3);
Z2=Co(N2,4);
XX=X2-X1;
YY=Y2-Y1;
ZZ=Z2-Z1;
S=sqrt(XX*XX+YY*YY+ZZ*ZZ);
NM1=NAE(NE);
E0=AE(NM1,1);
A0=AE(NM1,2);
C=E0*A0/S;
AKE(1,1)=C;
AKE(2,2)=C;
AKE(1,2)=-C;
AKE(2,1)=-C;

function [T]=FT(NP,NE,ME,Co);
N1=ME(NE,2);
N2=ME(NE,3);
X1=Co(N1,2);
Y1=Co(N1,3);
Z1=Co(N1,4);
X2=Co(N2,2);
Y2=Co(N2,3);
Z2=Co(N2,4);
XX=X2-X1;
YY=Y2-Y1;
ZZ=Z2-Z1;
S=sqrt(XX*XX+YY*YY+ZZ*ZZ);

```

```

C1=XX/S;
C2=YY/S;
C3=ZZ/S;
T(1,1)=C1;
T(1,2)=C2;
T(1,3)=C3;
T(2,4)=C1;
T(2,5)=C2;
T(2,6)=C3;

function [K]=FAKK(NE,AKEE,K);
for j=1:6;
    for k=1:6;
        K(6*(NE-1)+j,6*(NE-1)+k)=AKEE(j,k);
    end;
end;

function [Calph]=FAL(D,m);

alph=D(1,1);
Calph=[alph];
n=1;
for i=1:m
    door=0;
    for j=1:n
        if D(i,i)==Calph(j);
            door=1;
            break
        end
    end
    if door==0
        Calph=[Calph;D(i,i)];
        n=n+1;
    end
end

function [M]=FM(Calph,m,n,Xstar,D,X0);

clear M
% E=infsup(0.5,0.5);
M(1,1)=Xstar(1,1)+X0(1,1);

```

```

for i=1:n;
    alph=Calph(i,1);
    for j=1:m
        if D(j,j)==alph;
            M(j,i)=Xstar(j,1)+X0(j,1);
        end
    end
end

function [X]=ITSOL(NF,NE,K,r,C,FF,K0);
format short e
format long
for i=1:(2*NE*Nf);
    for j=1:(2*NE*Nf);
        Khat(i,j)=0.5*(inf(K(i,j))+sup(K(i,j)));
    end
end
for i=1:(2*NE*Nf);
    for j=1:(2*NE*Nf);
        K0hat(i,j)=0.5*(inf(K0(i,j))+sup(K0(i,j)));
    end
end
for i=1:(2*NE*Nf);
    for j=1:(2*NE*Nf);
        d(i,j)=infsup(0,0);
    end
end
for i=1:NE;
    if K0(6*(i-1)+1,6*(i-1)+1)~=0;

d(6*(i-1)+1,6*(i-1)+1)=(K0(6*(i-1)+1,6*(i-1)+1)-K0hat(6*(i-1)+1,6*(
i-1)+1))/K0hat(6*(i-1)+1,6*(i-1)+1);
        for j=2:6;
            d(6*(i-1)+j,6*(i-1)+j)=d(6*(i-1)+1,6*(i-1)+1);
        end
        elseif K0(6*(i-1)+2,6*(i-1)+2)~=0;

d(6*(i-1)+2,6*(i-1)+2)=(K0(6*(i-1)+2,6*(i-1)+2)-K0hat(6*(i-1)+2,6*(
i-1)+2))/K0hat(6*(i-1)+2,6*(i-1)+2);
        for j=1:6;
            d(6*(i-1)+j,6*(i-1)+j)=d(6*(i-1)+2,6*(i-1)+2);
        end
    end
end

```

```

elseif K0(6*(i-1)+3,6*(i-1)+3)~=0;

d(6*(i-1)+3,6*(i-1)+3)=(K0(6*(i-1)+3,6*(i-1)+3)-K0hat(6*(i-1)+3,6*(
i-1)+3))/K0hat(6*(i-1)+3,6*(i-1)+3);
    for j=1:6;
        d(6*(i-1)+j,6*(i-1)+j)=d(6*(i-1)+3,6*(i-1)+3);
    end
end
end
D=[d,zeros((2*NE*NF),(r-1));zeros((r-1),(2*NE*NF)),zeros((r-1),(r-1
))];
Ahat=[Khat,C';C,zeros((r-1),(r-1))];
R=inv(Ahat);
S=[Khat,zeros((2*NE*NF),(r-1));zeros((r-1),(2*NE*NF)),zeros((r-1),(
r-1))];
b=FF;
for i=1:(2*NE*NF+r-1);
    bhat(i,1)=0.5*(inf(b(i,1))+sup(b(i,1)));
end
X0=R*bhat;

% Form matrix M and Calpha (handle the dependency problem)
% Slove X

m=2*NE*NF+r-1;
for j=1:m
    Xstar(j,1)=infsup(0,0);
end
Calph=FAL(D,m);
n=size(Calph,1);
p=0;
q=0;
while p<m && q<200,
    q=q+1;
    M=FM(Calph,m,n,Xstar,D,X0);
    U=Xstar;
    Xstar=R*b-X0-R*S*M*Calph;
    V=Xstar;
    p=0;
    for i=1:m
        if inf(U(i))<=inf(V(i)) && sup(U(i))>=sup(V(i));
            p=p+1;
        end
    end
end

```

```
        end  
    end
```

```
X=Xstar+X0
```

APPENDIX B

```
clc
clear

% NE number of elements
% NP number of points
% NF number of freedom
% NM number of materials
% NR number of restraint
% Co coordinate of nodes
% NAE element type
% AE material property

format short e
format long

syms s

%Input data
%control data

NP=xlsread('twobay truss.xls',1,'B3');
NE=xlsread('twobay truss.xls',1,'C3');
NF=xlsread('twobay truss.xls',1,'D3');
NM=xlsread('twobay truss.xls',1,'E3');
NR=xlsread('twobay truss.xls',1,'F3');
aaa=sprintf('B6:E%d',NP+5);
Co=xlsread('twobay truss.xls',1,aaa);
bbb=sprintf('B3:E%d',NE+2);
ME=xlsread('twobay truss.xls',2,bbb);
ooo=sprintf('E3:E%d',NE+2);
NAE=xlsread('twobay truss.xls',2,ooo);
ccc=sprintf('C3:C%d',NM+2);
M1=xlsread('twobay truss.xls',3,ccc);
ddd=sprintf('G3:G%d',NM+2);
M2=xlsread('twobay truss.xls',3,ddd);
AE1=[M1,M2];
eee=sprintf('D3:D%d',NM+2);
M3=xlsread('twobay truss.xls',3,eee);
fff=sprintf('H3:H%d',NM+2);
```

```

M4=xlsread('twobay truss.xls',3,fff);
AEu=[M3,M4];
AE=infsup(AEl,AEu);
ggg=sprintf('B3:E%d',NR+2);
NRR=xlsread('twobay truss.xls',4,ggg);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%

% Form structure stiffness matrix
A=NP*NF;
for i=1:A
    for j=1:A
        K(i,j)=infsup(0,0);
    end
end
for ne=1:NE
    AKE=FKE(NP,ne,NM,ME,AE,NAE,Co);
    T=FT(NP,ne,ME,Co);
    AKET=AKE*T;
    TT=T';
    AKEE=TT*AKET;
    IS=FIS(NF,ne,ME);
    K=FAKK(NF,AKEE,IS,K);
end

K(1,:)=[];
K(1,:)=[];
K(1,:)=[];
K(:,1)=[];
K(:,1)=[];
K(:,1)=[];

K(1,:)=[];
K(1,:)=[];
K(1,:)=[];
K(:,1)=[];
K(:,1)=[];
K(:,1)=[];

K(3,:)=[];
K(:,3)=[];

```



```

K(5,:)=[];
K(:,5)=[];
K(7,:)=[];
K(:,7)=[];
K(9,:)=[];
K(:,9)=[];

% Form mass matrix

wei=7800;
mm1=wei*2e-4*2^0.5;
mm2=wei*2e-4;
mm4=wei*2e-4;
mm3=wei*2e-4;
mm6=wei*2e-4*2^0.5;

m1=wei*infsup(0.000196,0.000204)+0.5*wei*2^0.5*infsup(0.000196,0.00
0204)+0.5*wei*1e-4;
m3=wei*1e-4;
m4=wei*1e-4+0.5*wei*2^0.5*infsup(0.000196,0.000204);
for i=1:8
    for j=1:8
        M(i,j)=infsup(0,0);
    end
end
for i=1:2
    M(i,i)=m1;
    M(i+2,i+2)=m1;
    M(i+4,i+4)=m3;
    M(i+6,i+6)=m4;
end

% Force

Kc=mid(K);
Mc=mid(M);

dM=M-Mc;

D=inv(M)*K;
% D=inv(K)*M;
Dc=inv(Mc)*Kc;

```

dD=D-Dc ;

ne=1 ;

AKE=FKE (NP ,ne ,NM ,ME ,AE ,NAE ,Co) ;

T=FT (NP ,ne ,ME ,Co) ;

AKET=AKE*T ;

TT=T' ;

kk1=TT*AKET ;

k1=mid (kk1) ;

k1 (3 , :)=[] ;

k1 (5 , :)=[] ;

k1 (: ,3)=[] ;

k1 (: ,5)=[] ;

ne=2 ;

AKE=FKE (NP ,ne ,NM ,ME ,AE ,NAE ,Co) ;

T=FT (NP ,ne ,ME ,Co) ;

AKET=AKE*T ;

TT=T' ;

kk2=TT*AKET ;

k2=mid (kk2) ;

k2 (3 , :)=[] ;

k2 (5 , :)=[] ;

k2 (: ,3)=[] ;

k2 (: ,5)=[] ;

ne=3 ;

AKE=FKE (NP ,ne ,NM ,ME ,AE ,NAE ,Co) ;

T=FT (NP ,ne ,ME ,Co) ;

AKET=AKE*T ;

TT=T' ;

kk3=TT*AKET ;

k3=mid (kk3) ;

k3 (3 , :)=[] ;

k3 (5 , :)=[] ;

k3 (: ,3)=[] ;

k3 (: ,5)=[] ;

ne=4 ;

AKE=FKE (NP ,ne ,NM ,ME ,AE ,NAE ,Co) ;

T=FT (NP ,ne ,ME ,Co) ;

AKET=AKE*T ;

TT=T' ;

```

kk4=TT*AKET;
k4=mid(kk4);
k4(3,:)=[];
k4(5,:)=[];
k4(:,3)=[];
k4(:,5)=[];

ne=5;
AKE=FKE(NP,ne,NM,ME,AE,NAE,Co);
T=FT(NP,ne,ME,Co);
AKET=AKE*T;
TT=T';
kk5=TT*AKET;
k5=mid(kk5);
k5(3,:)=[];
k5(5,:)=[];
k5(:,3)=[];
k5(:,5)=[];

ne=6;
AKE=FKE(NP,ne,NM,ME,AE,NAE,Co);
T=FT(NP,ne,ME,Co);
AKET=AKE*T;
TT=T';
kk6=TT*AKET;
k6=mid(kk6);
k6(3,:)=[];
k6(5,:)=[];
k6(:,3)=[];
k6(:,5)=[];

ne=7;
AKE=FKE(NP,ne,NM,ME,AE,NAE,Co);
T=FT(NP,ne,ME,Co);
AKET=AKE*T;
TT=T';
kk7=TT*AKET;
k7=mid(kk7);
k7(3,:)=[];
k7(5,:)=[];
k7(:,3)=[];
k7(:,5)=[];

```

```

ne=8;
AKE=FKE(NP,ne,NM,ME,AE,NAE,Co);
T=FT(NP,ne,ME,Co);
AKET=AKE*T;
TT=T';
kk8=TT*AKET;
k8=mid(kk8);
k8(3,:)=[];
k8(5,:)=[];
k8(:,3)=[];
k8(:,5)=[];

[V,A]=eig(Dc);
for i=1:8
    Y1(i,1)=V(i,2);
end

w=A(2,2);

d1=[0;0;k1(3,3)*Y1(3)+k1(3,4)*Y1(4);k1(4,3)*Y1(3)+k1(4,4)*Y1(4);0;0;0;0];
d2=[0;0;k2(3,3)*Y1(3)+k2(3,4)*Y1(4);k2(4,3)*Y1(3)+k2(4,4)*Y1(4);0;0;0;0];
d3=[k3(3,3)*Y1(1)+k3(3,4)*Y1(2);k3(4,3)*Y1(1)+k3(4,4)*Y1(2);0;0;0;0;0;0];
d4=[k4(1,1)*Y1(1)+k4(1,2)*Y1(2)+k4(1,3)*Y1(3)+k4(1,4)*Y1(4);k4(2,1)*Y1(1)+k4(2,2)*Y1(2)+k4(2,3)*Y1(3)+k4(2,4)*Y1(4);k4(3,1)*Y1(1)+k4(3,2)*Y1(2)+k4(3,3)*Y1(3)+k4(3,4)*Y1(4);k4(4,1)*Y1(1)+k4(4,2)*Y1(2)+k4(4,3)*Y1(3)+k4(4,4)*Y1(4);0;0;0;0];
d5=[k6(1,1)*Y1(1)+k6(1,2)*Y1(2)+k6(1,3)*Y1(7)+k6(1,4)*Y1(8);k6(2,1)*Y1(1)+k6(2,2)*Y1(2)+k6(2,3)*Y1(7)+k6(2,4)*Y1(8);0;0;0;0;k6(3,1)*Y1(1)+k6(3,2)*Y1(2)+k6(3,3)*Y1(7)+k6(3,4)*Y1(8);k6(4,1)*Y1(1)+k6(4,2)*Y1(2)+k6(4,3)*Y1(7)+k6(4,4)*Y1(8)];

dky=[d1,d2,d3,d4,d5];

KY=Kc*Y1;

u1=[0;0;mm1*KY(3);mm1*KY(4);0;0;0;0];
u2=[0;0;mm2*KY(3);mm2*KY(4);0;0;0;0];
u3=[mm3*KY(1);mm3*KY(2);0;0;0;0;0;0];
u4=[mm4*KY(1);mm4*KY(2);mm4*KY(3);mm4*KY(4);0;0;0;0];
u5=[mm6*KY(1);mm6*KY(2);0;0;0;0;mm6*KY(7);mm6*KY(8)];

```

```

dmu=[u1,u2,u3,u4,u5];

% dD=inf(dD);
B=[(w*eye(8)-Dc),Y1];
B(:,8)=[ ];

y=[infsup(-0.02,0.02);infsup(-0.02,0.02);infsup(-0.02,0.02);infsup(
-0.02,0.02);infsup(-0.02,0.02)];

% x=inv(B)*(inv(Mc)*dky*y-dM*(inv(Mc)*inv(Mc)*Kc*Y1));

x=inv(B)*(inv(Mc)*dky-inv(Mc)*inv(Mc)*0.5*dmu)*y;

lenda=x(8)+w

function [K]=FAKK(NF,AKEE,IS,K);
for i=1:6
    for j=1:6
        K(IS(i),IS(j))=K(IS(i),IS(j))+AKEE(i,j);
    end
end

function [AKE]=FKE(NP,NE,NM,ME,AE,NAE,Co);
N1=ME(NE,2);
N2=ME(NE,3);
X1=Co(N1,2);
Y1=Co(N1,3);
Z1=Co(N1,4);
X2=Co(N2,2);
Y2=Co(N2,3);
Z2=Co(N2,4);
XX=X2-X1;
YY=Y2-Y1;
ZZ=Z2-Z1;
S=sqrt(XX*XX+YY*YY+ZZ*ZZ);
NM1=NAE(NE);
E0=AE(NM1,1);
A0=AE(NM1,2);
C=E0*A0/S;

```

```

AKE(1,1)=C;
AKE(2,2)=C;
AKE(1,2)=-C;
AKE(2,1)=-C;

```

```

function [T]=FT(NP,NE,ME,Co);
N1=ME(NE,2);
N2=ME(NE,3);
X1=Co(N1,2);
Y1=Co(N1,3);
Z1=Co(N1,4);
X2=Co(N2,2);
Y2=Co(N2,3);
Z2=Co(N2,4);
XX=X2-X1;
YY=Y2-Y1;
ZZ=Z2-Z1;
S=sqrt(XX*XX+YY*YY+ZZ*ZZ);
C1=XX/S;
C2=YY/S;
C3=ZZ/S;
T(1,1)=C1;
T(1,2)=C2;
T(1,3)=C3;
T(2,4)=C1;
T(2,5)=C2;
T(2,6)=C3;

```

```

function [IS]=FIS(NF,NE,ME);
for i=1:2
    for j=1:NF
        II=(i-1)*NF+j;
        IS(II)=(ME(NE,i+1)-1)*NF+j;
    end
end

```

```

function [AKE]=FKE(NP,NE,NM,ME,AE,NAE,Co);
N1=ME(NE,2);
N2=ME(NE,3);

```

```

X1=Co(N1,2);
Y1=Co(N1,3);
Z1=Co(N1,4);
X2=Co(N2,2);
Y2=Co(N2,3);
Z2=Co(N2,4);
XX=X2-X1;
YY=Y2-Y1;
ZZ=Z2-Z1;
S=sqrt(XX*XX+YY*YY+ZZ*ZZ);
NM1=NAE(NE);
E0=AE(NM1,1);
A0=AE(NM1,2);
C=E0*A0/S;
AKE(1,1)=C;
AKE(2,2)=C;
AKE(1,2)=-C;
AKE(2,1)=-C;

```

APPENDIX C

```
clc;%it is a 4 dof system like a vibration absorbers with damping.
clear;
format short e
format long

% input m and k
m1=midrad(100,0.5);
m2=midrad(100,0.5);
m3=midrad(100,0.5);
m4=midrad(100,0.5);
k1=midrad(1000,10);
k2=midrad(1000,10);
k3=midrad(1000,10);
k4=midrad(1000,10);
F1=midrad(1,0);

%form K
for i=1:24
    for j=1:24
        K(i,j)=infsup(0,0);
    end
end
K(1,1)=k1;
K(4,4)=k1;
K(1,4)=-k1;
K(4,1)=-k1;
K(7,7)=k2;
K(10,10)=k2;
K(7,10)=-k2;
K(10,7)=-k2;
K(13,13)=k3;
K(16,16)=k3;
K(13,16)=-k3;
K(16,13)=-k3;
K(19,19)=k4;
K(22,22)=k4;
K(19,22)=-k4;
K(22,19)=-k4;

K0=K;
```



```

for i=1:24
    K(i,i)=1e25;
end
K(4,4)=k1;
K(7,7)=k2;
K(10,10)=k2;
K(13,13)=k3;
K(16,16)=k3;
K(19,19)=k4;
K(22,22)=k4;

%form C
for i=1:9
    for j=1:24
        C(i,j)=0;
    end
end
C(1,4)=1;
C(1,7)=-1;
C(2,5)=1;
C(2,8)=-1;
C(3,6)=1;
C(3,9)=-1;
C(4,10)=1;
C(4,13)=-1;
C(5,11)=1;
C(5,14)=-1;
C(6,12)=1;
C(6,15)=-1;
C(7,16)=1;
C(7,19)=-1;
C(8,17)=1;
C(8,20)=-1;
C(9,18)=1;
C(9,21)=-1;

%form M
for i=1:24
    for j=1:24
        M(i,j)=infsup(0,0);
    end
end
end

```

```

for i=1:24
    M(i,i)=0.5*m1;
end
M(22,22)=m4;
M(23,23)=m4;
M(24,24)=m4;

%form load
for i=1:33
    FF(i,1)=infsup(0,0);
end
FF(22,1)=F1;
Ke=K; % Ke is the constraint stiffness matrix
Kf=K0; % Kf is the unconstraint stiffness matrix

%solve equation
NF=3;
NE=4;
r=10;
v=0;
b=0;
while v<80
    w=v*0.1;
    % Z=-w^2*M+w*C*1i+K;
    K=(-w^2+1i*w*1)*M+(1i*w*0.001+1)*Ke;
    K0=(-w^2+1i*w*1)*M+(1i*w*0.001+1)*Kf;
    X=ITSOL(NF,NE,K,r,C,FF,K0);
    b=b+1;
    X1(b,1)=X(22,1);
    v=v+1;
end
u=linspace(0,8,80);
X1l=inf(abs(X1));
X1u=sup(abs(X1));
X1m=mid(abs(X1));
plot(u,X1l,u,X1u,u,X1m);% upper lower crisp case of X1 response
grid on;

function [X]=ITSOL(NF,NE,K,r,C,FF,K0);
format short e
format long
for i=1:(2*NE*Nf);

```

```

    for j=1:(2*NE*NF);
        Khat(i,j)=0.5*(inf(K(i,j))+sup(K(i,j)));
    end
end
for i=1:(2*NE*NF);
    for j=1:(2*NE*NF);
        K0hat(i,j)=0.5*(inf(K0(i,j))+sup(K0(i,j)));
    end
end
for i=1:(2*NE*NF);
    for j=1:(2*NE*NF);
        d(i,j)=infsup(0,0);
    end
end
for i=1:NE;
    if K0(6*(i-1)+1,6*(i-1)+1)~=0;

d(6*(i-1)+1,6*(i-1)+1)=(K0(6*(i-1)+1,6*(i-1)+1)-K0hat(6*(i-1)+1,6*(
i-1)+1))/K0hat(6*(i-1)+1,6*(i-1)+1);
        for j=2:6;
            d(6*(i-1)+j,6*(i-1)+j)=d(6*(i-1)+1,6*(i-1)+1);
        end
        elseif K0(6*(i-1)+2,6*(i-1)+2)~=0;

d(6*(i-1)+2,6*(i-1)+2)=(K0(6*(i-1)+2,6*(i-1)+2)-K0hat(6*(i-1)+2,6*(
i-1)+2))/K0hat(6*(i-1)+2,6*(i-1)+2);
        for j=1:6;
            d(6*(i-1)+j,6*(i-1)+j)=d(6*(i-1)+2,6*(i-1)+2);
        end
        elseif K0(6*(i-1)+3,6*(i-1)+3)~=0;

d(6*(i-1)+3,6*(i-1)+3)=(K0(6*(i-1)+3,6*(i-1)+3)-K0hat(6*(i-1)+3,6*(
i-1)+3))/K0hat(6*(i-1)+3,6*(i-1)+3);
        for j=1:6;
            d(6*(i-1)+j,6*(i-1)+j)=d(6*(i-1)+3,6*(i-1)+3);
        end
    end
end
D=[d,zeros((2*NE*NF),(r-1));zeros((r-1),(2*NE*NF)),zeros((r-1),(r-1
))];
Ahat=[Khat,C';C,zeros((r-1),(r-1))];
R=inv(Ahat);
S=[Khat,zeros((2*NE*NF),(r-1));zeros((r-1),(2*NE*NF)),zeros((r-1),(

```

```

r-1))];
b=FF;
for i=1:(2*NE*Nf+r-1);
    bhat(i,1)=0.5*(inf(b(i,1))+sup(b(i,1)));
end
X0=R*bhat;

% Form matrix M and Calpha (handle the dependency problem)
% Solve X

m=2*NE*Nf+r-1;
for j=1:m
    Xstar(j,1)=infsup(0,0);
end
Calph=FAL(D,m);
n=size(Calph,1);
p=0;
q=0;
while p<m && q<200,
    q=q+1;
    M=FM(Calph,m,n,Xstar,D,X0);
    U=Xstar;
    Xstar=R*b-X0-R*S*M*Calph;
    V=Xstar;
    p=0;
    for i=1:m
        if inf(U(i))<=inf(V(i)) && sup(U(i))>=sup(V(i));
            p=p+1;
        end
    end
end
X=Xstar+X0;

clc;%it is a 4 dof system like a vibration absorbers with damping. m1
and m2 is the lower ,and upper mass respectively.
clear;
format short e
format long
m1=midrad(100,0.5);
m2=midrad(100,0.5);
m3=midrad(100,0.5);

```

```

m4=midrad(100,0.5);
k1=midrad(1000,10);
k2=midrad(1000,10);
k3=midrad(1000,10);
k4=midrad(1000,10);

for i=1:24
    for j=1:24
        K(i,j)=infsup(0,0);
    end
end
K(1,1)=k1;
K(4,4)=k1;
K(1,4)=-k1;
K(4,1)=-k1;
K(7,7)=k2;
K(10,10)=k2;
K(7,10)=-k2;
K(10,7)=-k2;
K(13,13)=k3;
K(16,16)=k3;
K(13,16)=-k3;
K(16,13)=-k3;
K(19,19)=k4;
K(22,22)=k4;
K(19,22)=-k4;
K(22,19)=-k4;

K0=K;

for i=1:24
    K(i,i)=1e25;
end
K(4,4)=k1;
K(7,7)=k2;
K(10,10)=k2;
K(13,13)=k3;
K(16,16)=k3;
K(19,19)=k4;
K(22,22)=k4;

for i=1:9
    for j=1:24

```

```

        C(i,j)=0;
    end
end
C(1,4)=1;
C(1,7)=-1;
C(2,5)=1;
C(2,8)=-1;
C(3,6)=1;
C(3,9)=-1;
C(4,10)=1;
C(4,13)=-1;
C(5,11)=1;
C(5,14)=-1;
C(6,12)=1;
C(6,15)=-1;
C(7,16)=1;
C(7,19)=-1;
C(8,17)=1;
C(8,20)=-1;
C(9,18)=1;
C(9,21)=-1;

for i=1:24
    for j=1:24
        M(i,j)=infsup(0,0);
    end
end
for i=1:24
    M(i,i)=0.5*m1;
end
M(22,22)=m4;
M(23,23)=m4;
M(24,24)=m4;

% CC=1*M+0.001*K;
for i=1:33
    FF(i,1)=infsup(0,0);
end

Ke=K; % Ke is the constraint stiffness matrix
Kf=K0; % Kf is the unconstraint stiffness matrix
NF=3;
NE=4;

```

```

r=10;
v=0;
b=0;
w1=0.731;
w2=3.022;
w3=4.832;
while v<200
    t=v*0.1;
    % Z=-w^2*M+w*C*1i+K;
    K=(-w1^2+1i*w1*1)*M+(1i*w1*0.001+1)*Ke;
    K0=(-w1^2+1i*w1*1)*M+(1i*w1*0.001+1)*Kf;
    FF(4,1)=exp(-1i*w1*t);
    XX1=ITSOL(NF,NE,K,r,C,FF,K0);% use the harmonic assumed solution
and transfer matrix to get the response. inverse matrix of impedance
is used.
    K=(-w2^2+1i*w2*1)*M+(1i*w2*0.001+1)*Ke;
    K0=(-w2^2+1i*w2*1)*M+(1i*w2*0.001+1)*Kf;
    FF(4,1)=exp(-1i*w2*t);
    XX2=ITSOL(NF,NE,K,r,C,FF,K0);
    K=(-w3^2+1i*w3*1)*M+(1i*w3*0.001+1)*Ke;
    K0=(-w3^2+1i*w3*1)*M+(1i*w3*0.001+1)*Kf;
    FF(4,1)=exp(-1i*w3*t);
    XX3=ITSOL(NF,NE,K,r,C,FF,K0);
    X=real(XX1)+real(XX2)+real(XX3);
    b=b+1;
    X1(b,1)=X(4,1);
    X2(b,1)=X(10,1);
    X3(b,1)=X(16,1);
    X4(b,1)=X(22,1);
    v=v+1;
end
u=linspace(0,20,200);
X1l=inf(X1);
X1u=sup(X1);
X2l=inf(X2);
X2u=sup(X2);
X3l=inf(X3);
X3u=sup(X3);
X4l=inf(X4);
X4u=sup(X4);
X1m=mid(X1);
plot(u,X1l,u,X1u,u,X1m);% upper lower crisp case of X1 response
grid on;

```

```

function [X]=ITSOL(NF,NE,K,r,C,FF,K0);
format short e
format long
for i=1:(2*NE*Nf);
    for j=1:(2*NE*Nf);
        Khat(i,j)=0.5*(inf(K(i,j))+sup(K(i,j)));
    end
end
for i=1:(2*NE*Nf);
    for j=1:(2*NE*Nf);
        K0hat(i,j)=0.5*(inf(K0(i,j))+sup(K0(i,j)));
    end
end
for i=1:(2*NE*Nf);
    for j=1:(2*NE*Nf);
        d(i,j)=infsup(0,0);
    end
end
for i=1:NE;
    if K0(6*(i-1)+1,6*(i-1)+1)~=0;

d(6*(i-1)+1,6*(i-1)+1)=(K0(6*(i-1)+1,6*(i-1)+1)-K0hat(6*(i-1)+1,6*(
i-1)+1))/K0hat(6*(i-1)+1,6*(i-1)+1);
        for j=2:6;
            d(6*(i-1)+j,6*(i-1)+j)=d(6*(i-1)+1,6*(i-1)+1);
        end
        elseif K0(6*(i-1)+2,6*(i-1)+2)~=0;

d(6*(i-1)+2,6*(i-1)+2)=(K0(6*(i-1)+2,6*(i-1)+2)-K0hat(6*(i-1)+2,6*(
i-1)+2))/K0hat(6*(i-1)+2,6*(i-1)+2);
        for j=1:6;
            d(6*(i-1)+j,6*(i-1)+j)=d(6*(i-1)+2,6*(i-1)+2);
        end
        elseif K0(6*(i-1)+3,6*(i-1)+3)~=0;

d(6*(i-1)+3,6*(i-1)+3)=(K0(6*(i-1)+3,6*(i-1)+3)-K0hat(6*(i-1)+3,6*(
i-1)+3))/K0hat(6*(i-1)+3,6*(i-1)+3);
        for j=1:6;
            d(6*(i-1)+j,6*(i-1)+j)=d(6*(i-1)+3,6*(i-1)+3);
        end
    end
end

```



```

end
D=[d,zeros((2*NE*NF),(r-1));zeros((r-1),(2*NE*NF)),zeros((r-1),(r-1))];
Ahat=[Khat,C';C,zeros((r-1),(r-1))];
R=inv(Ahat);
S=[Khat,zeros((2*NE*NF),(r-1));zeros((r-1),(2*NE*NF)),zeros((r-1),(r-1))];
b=FF;
for i=1:(2*NE*NF+r-1);
    bhat(i,1)=0.5*(inf(b(i,1))+sup(b(i,1)));
end
X0=R*bhat;

% Form matrix M and Calpha (handle the dependency problem)
% Slove X

m=2*NE*NF+r-1;
for j=1:m
    Xstar(j,1)=infsup(0,0);
end
Calph=FAL(D,m);
n=size(Calph,1);
p=0;
q=0;
while p<m && q<200,
    q=q+1;
    M=FM(Calph,m,n,Xstar,D,X0);
    U=Xstar;
    Xstar=R*b-X0-R*S*M*Calph;
    V=Xstar;
    p=0;
    for i=1:m
        if inf(U(i))<=inf(V(i)) && sup(U(i))>=sup(V(i));
            p=p+1;
        end
    end
end
end

X=Xstar+X0;

clc
clear

```

```

% NE number of elements
% NP number of points
% NF number of freedom
% NM number of materials
% NR number of restraint
% Co coordinate of nodes
% NAE element type
% AE material property

format short e
format long

%Input data
%control data
NP=xlsread('twobay truss.xls',1,'B3');
NE=xlsread('twobay truss.xls',1,'C3');
NF=xlsread('twobay truss.xls',1,'D3');
NM=xlsread('twobay truss.xls',1,'E3');
NR=xlsread('twobay truss.xls',1,'F3');
aaa=sprintf('B6:E%d',NP+5);
Co=xlsread('twobay truss.xls',1,aaa);
bbb=sprintf('B3:E%d',NE+2);
ME=xlsread('twobay truss.xls',2,bbb);
ooo=sprintf('E3:E%d',NE+2);
NAE=xlsread('twobay truss.xls',2,ooo);
ccc=sprintf('C3:C%d',NM+2);
M1=xlsread('twobay truss.xls',3,ccc);
ddd=sprintf('G3:G%d',NM+2);
M2=xlsread('twobay truss.xls',3,ddd);
AE1=[M1,M2];
eee=sprintf('D3:D%d',NM+2);
M3=xlsread('twobay truss.xls',3,eee);
fff=sprintf('H3:H%d',NM+2);
M4=xlsread('twobay truss.xls',3,fff);
AEu=[M3,M4];
AE=insup(AE1,AEu);
ggg=sprintf('B3:E%d',NR+2);
NRR=xlsread('twobay truss.xls',4,ggg);
hhh=sprintf('C3:C%d',2*NE+2);
P1=xlsread('twobay truss.xls',5,hhh);
iii=sprintf('G3:G%d',2*NE+2);
P2=xlsread('twobay truss.xls',5,iii);

```

```

jjj=sprintf('K3:K%d',2*NE+2);
P3=xlsread('twobay truss.xls',5,jjj);
kkk=sprintf('O3:O%d',2*NE+2);
P4=xlsread('twobay truss.xls',5,kkk);
P1=[P1,P2,P3,P4];
l1l=sprintf('H3:H%d',2*NE+2);
P5=xlsread('twobay truss.xls',5,l1l);
mmm=sprintf('L3:L%d',2*NE+2);
P6=xlsread('twobay truss.xls',5,mmm);
nnn=sprintf('P3:P%d',2*NE+2);
P7=xlsread('twobay truss.xls',5,nnn);
Pu=[P1,P5,P6,P7];
P=infsup(P1,Pu);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%

% Form structure stiffness matrix
A=(2*NE);
for i=1:A
    for j=1:A
        K(i,j)=infsup(0,0);
    end
end
for ne=1:NE
    AKE=FKE(NP,ne,NM,ME,AE,NAE,Co);
    T=FT(NP,ne,ME,Co);
    AKET=AKE*T;
    TT=T';
    AKEE=TT*AKET;
    K=FAKK(ne,AKEE,K);
end

K0=K;

% Form matrix C
C(1,1:(6*NE))=0;
r=1;
for i=1:(NE-1);
    for j=1:2;
        a=ME(i,j+1);
        for k=(i+1):NE;

```

```

        if ME(k,2)==a;
            C(r,6*(i-1)+3*(j-1)+1)=1;
            C(r,6*(k-1)+1)=-1;
            C(r+1,6*(i-1)+3*(j-1)+2)=1;
            C(r+1,6*(k-1)+2)=-1;
            C(r+2,6*(i-1)+3*(j-1)+3)=1;
            C(r+2,6*(k-1)+3)=-1;
            r=r+3;
            break
        elseif ME(k,3)==a;
            C(r,6*(i-1)+3*(j-1)+1)=1;
            C(r,6*(k-1)+4)=-1;
            C(r+1,6*(i-1)+3*(j-1)+2)=1;
            C(r+1,6*(k-1)+5)=-1;
            C(r+2,6*(i-1)+3*(j-1)+3)=1;
            C(r+2,6*(k-1)+6)=-1;
            r=r+3;
            break
        end
    end
end
end

% Deal with constraint
for i=1:NR
    NI=NRR(i,2);
    K(NI,NI)=1e25;
end

% Form matrix A

KK=[K,C';C,zeros((r-1),(r-1))];

% Solve equation with iteration method

for i=1:114
    FF(i,1)=infsup(0,0);
end
FF(5,1)=infsup(1,1);

for i=1:66
    for j=1:66
        M(i,j)=infsup(0,0);
    end
end

```

```

        end
    end
    for i=1:24
        M(i,i)=360;
    end
    for i=25:42
        M(i,i)=195;
    end
    for i=43:66
        M(i,i)=436;
    end
    MM=[M,zeros((2*NE*NF),(r-1));zeros((r-1),(2*NE*NF)),zeros((r-1),(r-1))];
    Kd=[K0,C';C,zeros((r-1),(r-1))];
    % CC=1*MM+0.001*Kd;
    Ke=K; % Ke is the constraint stiffness matrix
    Kf=K0; % Kf is the unconstraint stiffness matrix
    v=0;
    b=0;
    while v<80
        w=v*10;
        % Z=-w^2*MM+1i*w*CC+KK;
        K=(-w^2+1i*w*1)*M+(1i*w*0.001+1)*Ke;
        K0=(-w^2+1i*w*1)*M+(1i*w*0.001+1)*Kf;
        X=ITSOL(NF,NE,K,r,C,FF,K0); % use the harmonic assumed solution and
        transfer matrix to get the response. inverse matrix of impedance is used.
        b=b+1;
        X1(b,1)=X(5,1);
        v=v+1;
    end
    u=linspace(0,800,80);
    X1l=inf(abs(X1));
    X1u=sup(abs(X1));
    X1m=mid(abs(X1));
    plot(u,X1l,u,X1u,u,X1m); % upper lower crisp case of X1 response

```

APPENDIX D

```
clc
clear

% NE number of elements
% NP number of points
% NF number of freedom
% NM number of materials
% NR number of restraint
% Co coordinate of nodes
% NAE element type
% AE material property

format short e
format long
syms s

%input data
NP=xlsread('twobay truss.xls',1,'B3');
NE=xlsread('twobay truss.xls',1,'C3');
NF=xlsread('twobay truss.xls',1,'D3');
NM=xlsread('twobay truss.xls',1,'E3');
NR=xlsread('twobay truss.xls',1,'F3');
aaa=sprintf('B6:E%d',NP+5);
Co=xlsread('twobay truss.xls',1,aaa);
bbb=sprintf('B3:E%d',NE+2);
ME=xlsread('twobay truss.xls',2,bbb);
ooo=sprintf('E3:E%d',NE+2);
NAE=xlsread('twobay truss.xls',2,ooo);
ccc=sprintf('C3:C%d',NM+2);
M1=xlsread('twobay truss.xls',3,ccc);
ddd=sprintf('G3:G%d',NM+2);
M2=xlsread('twobay truss.xls',3,ddd);
AE1=[M1,M2];
eee=sprintf('D3:D%d',NM+2);
M3=xlsread('twobay truss.xls',3,eee);
fff=sprintf('H3:H%d',NM+2);
M4=xlsread('twobay truss.xls',3,fff);
AEu=[M3,M4];
AE=infsup(AE1,AEu);
ggg=sprintf('B3:E%d',NR+2);
```

```

NRR=xlsread('twobay truss.xls',4,ggg);
hhh=sprintf('C3:C%d',2*NE+2);
P1=xlsread('twobay truss.xls',5,hhh);
iii=sprintf('G3:G%d',2*NE+2);
P2=xlsread('twobay truss.xls',5,iii);
jjj=sprintf('K3:K%d',2*NE+2);
P3=xlsread('twobay truss.xls',5,jjj);
kkk=sprintf('O3:O%d',2*NE+2);
P4=xlsread('twobay truss.xls',5,kkk);
P1=[P1,P2,P3,P4];
l11=sprintf('H3:H%d',2*NE+2);
P5=xlsread('twobay truss.xls',5,l11);
mmm=sprintf('L3:L%d',2*NE+2);
P6=xlsread('twobay truss.xls',5,mmm);
nnn=sprintf('P3:P%d',2*NE+2);
P7=xlsread('twobay truss.xls',5,nnn);
Pu=[P1,P5,P6,P7];
P=infsup(P1,Pu);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%

% Form structure stiffness matrix
A=(2*NE);
for i=1:A
    for j=1:A
        K(i,j)=infsup(0,0);
    end
end
for ne=1:NE
    AKE=FKE(NP,ne,NM,ME,AE,NAE,Co);
    T=FT(NP,ne,ME,Co);
    AKET=AKE*T;
    TT=T';
    AKEE=TT*AKET;
    K=FAKK(ne,AKEE,K);
end

K0=K;

% Deal with constraint

```

```

K(1,:)=[];
K(1,:)=[];
K(1,:)=[];
K(:,1)=[];
K(:,1)=[];
K(:,1)=[];

K(3,:)=[];
K(:,3)=[];

K(3,:)=[];
K(3,:)=[];
K(3,:)=[];
K(:,3)=[];
K(:,3)=[];
K(:,3)=[];

K(5,:)=[];
K(:,5)=[];

K(5,:)=[];
K(5,:)=[];
K(5,:)=[];
K(:,5)=[];
K(:,5)=[];
K(:,5)=[];

K(7,:)=[];
K(:,7)=[];
K(9,:)=[];
K(:,9)=[];
K(11,:)=[];
K(:,11)=[];
K(13,:)=[];
K(:,13)=[];
K(15,:)=[];
K(:,15)=[];
K(17,:)=[];
K(:,17)=[];
K(19,:)=[];
K(:,19)=[];
K(21,:)=[];
K(:,21)=[];

```



```

K(23,:)=[];
K(:,23)=[];
K(25,:)=[];
K(:,25)=[];
K(27,:)=[];
K(:,27)=[];

K0=K;
% Form Matrix M
wei=infsup(7410,8190);
m1=wei*1e-4*2^0.5; % element 1
m2=wei*1e-4; % element 2,3,4
m3=wei*1.2e-4*2^0.5; % element 6
m4=wei*1.2e-4; % element 5,7,8

for i=1:6
    M(i,i)=m1/2; % 1
    M(i+6,i+6)=m2/2; % 2
    M(i+12,i+12)=m2/2; % 3
    M(i+18,i+18)=m2/2; % 4
    M(i+30,i+30)=m3/2; % 6
    M(i+24,i+24)=m4/2; % 5
    M(i+36,i+36)=m4/2; % 7
    M(i+42,i+42)=m4/2; % 8
end

M0=M;

M(1,:)=[];
M(1,:)=[];
M(1,:)=[];
M(:,1)=[];
M(:,1)=[];
M(:,1)=[];

M(3,:)=[];
M(:,3)=[];

M(3,:)=[];
M(3,:)=[];
M(3,:)=[];
M(:,3)=[];
M(:,3)=[];

```

```

M(:,3)=[ ];

M(5,:)=[ ];
M(:,5)=[ ];

M(5,:)=[ ];
M(5,:)=[ ];
M(5,:)=[ ];
M(:,5)=[ ];
M(:,5)=[ ];
M(:,5)=[ ];

M(7,:)=[ ];
M(:,7)=[ ];
M(9,:)=[ ];
M(:,9)=[ ];
M(11,:)=[ ];
M(:,11)=[ ];
M(13,:)=[ ];
M(:,13)=[ ];
M(15,:)=[ ];
M(:,15)=[ ];
M(17,:)=[ ];
M(:,17)=[ ];
M(19,:)=[ ];
M(:,19)=[ ];
M(21,:)=[ ];
M(:,21)=[ ];
M(23,:)=[ ];
M(:,23)=[ ];
M(25,:)=[ ];
M(:,25)=[ ];
M(27,:)=[ ];
M(:,27)=[ ];

M0=M;

% Form matrix Q

for i=1:18
    for j=1:26
        Q(i,j)=0;
    end
end

```

```

end
Q(1,1)=1;
Q(1,3)=-1;
Q(2,2)=1;
Q(2,4)=-1;
Q(3,1)=1;
Q(3,9)=-1;
Q(4,2)=1;
Q(4,10)=-1;
Q(5,1)=1;
Q(5,11)=-1;
Q(6,2)=1;
Q(6,12)=-1;
Q(7,5)=1;
Q(7,7)=-1;
Q(8,6)=1;
Q(8,8)=-1;
Q(9,5)=1;
Q(9,15)=-1;
Q(10,6)=1;
Q(10,16)=-1;
Q(11,5)=1;
Q(11,19)=-1;
Q(12,6)=1;
Q(12,20)=-1;
Q(13,13)=1;
Q(13,17)=-1;
Q(14,14)=1;
Q(14,18)=-1;
Q(15,13)=1;
Q(15,25)=-1;
Q(16,14)=1;
Q(16,26)=-1;
Q(17,21)=1;
Q(17,23)=-1;
Q(18,22)=1;
Q(18,24)=-1;

% Calculation

load LL
L=ilaplace(LL(6,1));

```

```

syms DL
for i=1:26
    for j=1:8
        DL(i,j)=0;
    end
end
DL(1,1)=LL(1);
DL(2,1)=LL(2);
DL(3,2)=LL(3);
DL(4,2)=LL(4);
DL(5,3)=LL(5);
DL(6,3)=LL(6);
DL(7,4)=LL(7);
DL(8,4)=LL(8);
DL(9,4)=LL(9);
DL(10,4)=LL(10);
DL(11,5)=LL(11);
DL(12,5)=LL(12);
DL(13,5)=LL(13);
DL(14,5)=LL(14);
DL(15,6)=LL(15);
DL(16,6)=LL(16);
DL(17,6)=LL(17);
DL(18,6)=LL(18);
DL(19,7)=LL(19);
DL(20,7)=LL(20);
DL(21,7)=LL(21);
DL(22,7)=LL(22);
DL(23,8)=LL(23);
DL(24,8)=LL(24);
DL(25,8)=LL(25);
DL(26,8)=LL(26);

for i=1:8
    alpha(i,1)=infsup(-0.05,0.05);
end

for i=1:26
    trLL(1,i)=LL(i,1);

```

```

end

DDLK=trLL*(mid(K0)*DL)/(-20*pi/(s^2+400*pi^2));
DDLm=trLL*(mid(M0)*DL)*s^2/(-20*pi/(s^2+400*pi^2));
for i=1:8
    L1(1,i)=ilaplace(DDLK(1,i));
    L2(1,i)=ilaplace(DDLm(1,i));
end

for i=1:200
    t=i*0.002;

    y0=double(vpa(subs(L,t)))*infsup(95,105);

    y1=double(vpa(subs(L1,t)))*alpha*100;

    y2=double(vpa(subs(L2,t)))*alpha*100;

    y=y0-y1-y2;

    z1(i,1)=y;

end

u=linspace(0,0.4,200);
z1l=inf(z1);
z1u=sup(z1);
z1m=mid(z1);
plot(u,z1l,u,z1u);

```