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# Discrete-geometric functions associated to polyhedral cones and point sets 

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# Discrete-geometric functions associated to polyhedral cones and point sets 



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1. $\qquad$
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## Abstract

In this report, we use Fourier analysis and Diophantine analysis to study functions associated to polyhedral cones and finite point sets. We present a close relationship between the function associated to integral cones and the classical Dedekind sums. The theory of the polytope algebra-the universal group for translation-invariant valuations-was developed by many mathematicians (see [MS83], [Bri97]). In this report, we employed forms of evaluation, namely rational function valuation, which lead us to Dedekind sums.

The report is constructed as follows. The first chapter is served as an introduction to the whole thesis. In Chapter 2, we consider the decomposition of the first quadrant cone into integral cones and study the asymptotic behavior of an infinite sum:

$$
f(c, d)=\frac{1}{4 \pi^{2}} \lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)} .
$$

The motivation for us to study this function arises from the Fourier transform of indicator function of cones. Decomposition of the first quadrant cone leads to an identity: the indicator function of the first quadrant cone is equal to the sum of indicator functions of the two cones after decomposition. In order to apply harmonic analysis in our work, we first smooth out indicator function of cones by Gaussian function. By doing so we may apply certain identities such as the Poisson Summation Formula to the modified indicator function. These facts combined grant us to explore the limit of an infinite sum $f(c, d)$ where the integral vector $(c, d) \in \mathbb{Z}^{2}$ is the common edge shared by two cones after decomposition of the first quadrant cone. In the end of Chapter 2, we discovered a nice relationship between $f(c, d)$ and the classical Dedekind sum $s(c, d)$ which is the main research object of Chapter 4.

In Chapter 3, we continue to use a similar technique which appeared earlier in Chapter 2 to investigate an infinite sum defined over cones. The main difference is here we focus on real cones while earlier in Chapter 2, we are interested in
integral cones. We managed to generalize our argument from integral cones to real cones. We studied the convergent property of the sum:

$$
\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} \mathrm{f}(\epsilon, \alpha, \beta)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+n)(m+\beta n)}
$$

which is defined on real cones. Our conclusion is when $\alpha$ and $\beta$ are both quadratic irrationals, this infinite series will converge absolutely. Meanwhile we also gave a sufficient condition for the existence of the limit:

$$
\lim _{\epsilon \rightarrow 0^{+}} f(\epsilon, \alpha, \beta) .
$$

In the following two chapters, our main interest lies in Dedekind sums. In Chapter 4, we focus on the classical Dedekind sum:

$$
s(c, d)=\sum_{k=0}^{d-1}\left(\left(\frac{k c}{d}\right)\right)\left(\left(\frac{k}{d}\right)\right)
$$

and answer the question of when two Dedekind sums are equal to each other. We have found a necessary condition which is $b \mid\left(1-a_{1} a_{2}\right)\left(a_{1}-a_{2}\right)$ in order for $s\left(a_{1}, b\right)$ to be equal to $s\left(a_{2}, b\right)$. A parallel analysis for the Dedekind-Rademacher sum, namely

$$
r_{n}(a, b)=\sum_{k=0}^{b-1}\left(\left(\frac{k a+n}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right)
$$

is also given in Chapter 4. We include part of the content from this chapter in our paper [JRW11]. In Chapter 5, our focus is on Zagier-Dedekind sums, or higher dimensional Dedekind sums:

$$
d\left(p ; a_{1}, \cdots, a_{n}\right)=(-1)^{n / 2} \sum_{k=1}^{p-1} \cot \frac{\pi k a_{1}}{p} \cdots \cot \frac{\pi k a_{n}}{p}
$$

where $p$ is a positive integer, $a_{1}, \cdots, a_{n}$ are integers relatively prime to $p$ and $n$ even. The condition for two Zagier-Dedekind sums to be equal to each other is slightly more complicated than the one we gave for classical Dedekind sums. An interesting fact about Zagier-Dedekind sums is that there is a nice relation
between $d\left(p ; a_{1}, \cdots, a_{n}\right)$ and counting lattice points whose definition depends on $a_{1}, a_{2}, \cdots, a_{n}$ and $p$.

In Chapter 6, we study a curve which is defined as a set of generalized centers of a finite point set. We call it $\mu$-curve for short. This curve is infinitely smooth, and it captures the symmetrical properties of the original point set such as radial symmetry, reflectional symmetry, and rotational symmetry. We generalize Weiszfeld's algorithm to find $\mu(r)$ through an iteration process for $r \geq 1$. We prove that the $\mu$-curve is invariant under rigid motions, and we conjecture that the nondegenerate $\mu$-curve is uniquely determined by a point set. An example is given to support this conjecture. We also give plenty of examples of the $\mu$-curve for different point sets in the end of this chapter.

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to my family

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## cosex 1

## Introduction and Outline

We aim to establish a connection between the well studied geometrical object called polyhedral cones and the classical number theoretic object called Dedekind sums. This chapter is informal, and therefore we do not define all the terms mentioned here. We will give the formal definitions carefully in the main body of this thesis, and focus on the intuition instead. We set off the pursuit of this connection by studying the decomposition of cones.


Figure 1.1: Decomposition of the first quadrant into three cones

To avoid overlap between adjacent cones, we define all cones here to be halfopen. As shown in Figure 1.1, we decompose the first quadrant $\mathcal{K}_{1 s t}$ into a disjoint union of three half-open cones: $\mathcal{K}_{y}, \mathcal{K}$ and $\mathcal{K}_{x}$.

$$
\begin{equation*}
\mathcal{K}_{1 s t}=\mathcal{K}_{y} \bigcup \mathcal{K} \bigcup \mathcal{K}_{x} \tag{1.1}
\end{equation*}
$$

We will utilize the concept of indicator functions, seeking a nice mathematical interpretation of decompositions of cones. From (1.1), by using indicator functions of cones, we obtain:

$$
\begin{equation*}
1_{\mathcal{K}_{1 s t}}(\mathbf{x})=1_{\mathcal{K}_{y}}(\mathbf{x})+1_{\mathcal{K}}(\mathbf{x})+1_{\mathcal{K}_{x}}(\mathbf{x}) \tag{1.2}
\end{equation*}
$$

where $\mathbf{x} \in \mathbb{R}^{2}$.
It is tempting for us to apply Fourier transforms to indicator functions at this point. But after examining the definition of the Fourier transform, we find out that the integral defined by the Fourier transform of a cone is divergent. In order to guarantee convergence, we multiply the indicator functions by a complex factor $e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle}$ first, that is,

$$
\begin{equation*}
1_{\mathcal{K}_{1 s t}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle}=1_{\mathcal{K}_{y}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle}+1_{\mathcal{K}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle}+1_{\mathcal{K}_{x}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle} \tag{1.3}
\end{equation*}
$$

where $\mathbf{x}, \mathbf{s} \in \mathbb{R}^{2}$.
Next we will convolve these "modified" indicator functions with Gaussian functions. The main reason is because we would like to apply Poisson Summation later and convolution makes sure all functions involved will lie in the space of Schwartz functions. Let

$$
G_{\epsilon}(\mathbf{x})=\frac{1}{\epsilon} e^{-\frac{\pi}{\epsilon}\|\mathbf{x}\|^{2}},
$$

where $\epsilon>0, \mathbf{x} \in \mathbb{R}^{2}$, and $\|\cdot\|$ denotes the Euclidean norm. Its Fourier transform is:

$$
\widehat{G_{\epsilon}}(\mathbf{s})=e^{-\pi \epsilon\|\mathbf{s}\|^{2}}
$$

Then (1.3) becomes:

$$
\begin{equation*}
1_{\mathcal{K}_{1 s t}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})=1_{\mathcal{K}_{y}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})+1_{\mathcal{K}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})+1_{\mathcal{K}_{x}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x}) \tag{1.4}
\end{equation*}
$$

Now we can take Fourier transform of (1.4) on both sides:

$$
\begin{align*}
\mathcal{F}\left(1_{\mathcal{K}_{1 s t}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})\right) & =\mathcal{F}\left(1_{\mathcal{K}_{y}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})\right) \\
& +\mathcal{F}\left(1_{\mathcal{K}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})\right) \\
& +\mathcal{F}\left(1_{\mathcal{K}_{x}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})\right) \tag{1.5}
\end{align*}
$$

We use both $\mathcal{F}(f(\mathbf{x}))$ and $\widehat{f}(\mathbf{x})$ interchangeably to denote the Fourier transform of $f(\mathbf{x})$. If $h(x)=(f * g)(x)$, then $\widehat{h}(\xi)=\widehat{f}(\xi) \widehat{g}(\xi)$. As a result, (1.5) becomes:

$$
\begin{align*}
\mathcal{F}\left(1_{\mathcal{K}_{1 s t}}(\mathbf{x}) e^{2 \pi i(\mathbf{x}, \mathbf{s}\rangle}\right) \mathcal{F}\left(G_{\epsilon}(\mathbf{x})\right) & =\mathcal{F}\left(1_{\mathcal{K}_{y}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle}\right) \mathcal{F}\left(G_{\epsilon}(\mathbf{x})\right) \\
& +\mathcal{F}\left(1_{\mathcal{K}}(\mathbf{x}) e^{2 \pi i(\mathbf{x}, \mathbf{s}\rangle}\right) \mathcal{F}\left(G_{\epsilon}(\mathbf{x})\right) \\
& +\mathcal{F}\left(1_{\mathcal{K}_{x}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle}\right) \mathcal{F}\left(G_{\epsilon}(\mathbf{x})\right) \tag{1.6}
\end{align*}
$$

Or equivalently:

$$
\begin{align*}
1_{\mathcal{K}_{1 s t}}(\mathbf{x}) e^{2 \pi i(\mathbf{x}, \mathbf{s}\rangle} \widehat{G_{\epsilon}}(\mathbf{x}) & =1_{\mathcal{K}_{y}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle}} \widehat{G_{\epsilon}}(\mathbf{x})+\overline{1_{\mathcal{K}}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle} \widehat{G_{\epsilon}}(\mathbf{x})} \\
& +1_{\mathcal{K}_{x}(\mathbf{x}) e^{2 \pi i\langle\mathbf{x}, \mathbf{s}\rangle}} \widehat{G_{\epsilon}}(\mathbf{x}) \tag{1.7}
\end{align*}
$$

Now we can apply the Stretch Theorem of Fourier transforms: Theorem B. 1 mentioned in Appendix B , and (1.7) becomes:

$$
\begin{align*}
\widehat{\mathcal{K}_{1 s t}} & (\mathbf{x}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{x}\|^{2}}
\end{align*}=\widehat{1_{\mathcal{K}_{y}}}(\mathbf{x}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{x}\|^{2}}+\widehat{1_{\mathcal{K}}}(\mathbf{x}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{x}\|^{2}}
$$

We will denote $\mathbf{x}+i \mathbf{s}$ as $\mathbf{z}=\mathbf{m}+i$ s from now on. Then (1.8) is equivalent to:

$$
\begin{align*}
\widehat{1_{\mathcal{K}_{1 s t}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}} & =\widehat{1_{\mathcal{K}_{y}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}}+\widehat{1_{\mathcal{K}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}} \\
& +\widehat{{\mathcal{K}_{x}}_{x}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}} \tag{1.9}
\end{align*}
$$

We now consider the case of $\mathbb{Z}^{2}$. Our next step will be summing over every term in (1.9) over the whole lattice $\mathbb{Z}^{2}$ :

$$
\begin{align*}
\sum_{\mathbf{m} \in \mathbb{Z}^{2}} \widehat{1_{\mathcal{K}_{1 s t}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}} & =\sum_{\mathbf{m} \in \mathbb{Z}^{2}}\left(\widehat{\mathcal{K}_{\mathcal{K}_{y}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}}+\widehat{\mathcal{1 K}_{\mathcal{K}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}}\right. \\
& \left.+\widehat{\mathcal{1}_{\mathcal{K}_{x}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}}\right) \tag{1.10}
\end{align*}
$$

Denote $\mathbf{z}=\mathbf{m}+i \mathbf{s}=\left(m_{1}+i s_{1}, m_{2}+i s_{2}\right)=\left(z_{1}, z_{2}\right)$.
Applying results that we will prove in Chapter 2, (1.10) becomes:

$$
\begin{align*}
\sum_{\mathbf{m} \in \mathbb{Z}^{2}} \frac{1}{z_{1} z_{2}} e^{-\pi \epsilon\|\mathbf{m}\|^{2}} & =\sum_{\mathbf{m} \in \mathbb{Z}^{2}}\left(\frac{\alpha}{z_{2}\left(\alpha z_{1}+\beta z_{2}\right)} e^{-\pi \epsilon\|\mathbf{m}\|^{2}}+\frac{|\operatorname{det} M|}{\left\langle\omega_{1}, \mathbf{z}\right\rangle\left\langle\omega_{2}, \mathbf{z}\right\rangle} e^{-\pi \epsilon\|\mathbf{m}\|^{2}}\right. \\
& \left.+\frac{\gamma}{z_{1}\left(\delta z_{1}+\gamma z_{2}\right)} e^{-\pi \epsilon\|\mathbf{m}\|^{2}}\right) \tag{1.11}
\end{align*}
$$

where

$$
M=\left(\begin{array}{ll}
\omega_{1} & \omega_{2}
\end{array}\right)=\left(\begin{array}{ll}
\alpha & \delta \\
\beta & \gamma
\end{array}\right)
$$

The motivation of us looking at these infinite sums mainly comes from papers written by Bruce C. Berndt, Paul Gunnells and Robert Sczech. In [Ber76], the author investigated the infinite sum

$$
\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{1}{m(c m+d n)} .
$$

In [GS03], Gunnells and Sczech defined an infinite sum on a lattice which becomes one of the motivations of what we are going to do next besides [Ber76]. Let $\mathcal{L}$ be a rank $l$ sublattice of $\mathbb{Z}^{d}$ satisfying $\mathcal{L}=\operatorname{Sat} \mathcal{L}$ where Sat $\mathcal{L}=\left\{\mathbf{m} \in \mathbb{Z}^{n}: k \mathbf{m} \in\right.$ $\mathcal{L}$ for some $k \in \mathbb{Z}\}$ denotes the saturation of $\mathcal{L}$. In other words, any $\mathbb{Z}$-basis of $\mathcal{L}$ can be extended to a $\mathbb{Z}$-basis of $\mathbb{Z}^{n}$. Here

$$
\sigma=\left(\begin{array}{llll}
\sigma_{1} & \sigma_{2} & \cdots & \sigma_{\mathrm{d}}
\end{array}\right)
$$

denotes an integral $(d \times d)$-matrix with primitive columns $\sigma_{\mathbf{1}}, \cdots, \sigma_{\mathbf{d}} .{ }^{1}$ For $\mathbf{v} \in \mathbb{R}^{d}$, define the Dedekind sum:

$$
S(L, \sigma, \mathbf{v})=(2 \pi i)^{-d} \sum_{\mathbf{x} \in L}^{\prime} \frac{e^{2 \pi i\langle\mathbf{x}, \mathbf{v}\rangle}}{\left\langle\mathbf{x}, \sigma_{\mathbf{1}}\right\rangle \cdots\left\langle\mathbf{x}, \sigma_{\mathbf{d}}\right\rangle}
$$

This series only converges conditionally and Sczech discussed the convergence of it in detail in [Scz93]. The dimensional one version of $S(L, \sigma, \mathbf{v})$ is:

$$
\sum_{x \in \mathbb{Z}}^{\prime} \frac{e^{2 \pi i v x}}{x}=B_{1}(v)=v-[v]-\frac{1}{2}
$$

[^0]which is the Sawtooth function we will encounter in Chapter 2.
We will continue our argument by letting $\left(s_{1}, s_{2}\right)=(0,0)$ in (1.11), and we will get four infinite sums. For each of them, we will sum it over the whole lattice $\mathbb{Z}^{2}$ excluding those points that make the denominators of the summands vanish. Note that in our infinite sums defined over cones, there is an extra $e^{-\epsilon\|\mathbf{m}\|^{2}}$ in the numerator which is absent from Gunnells and Sczech's work. The existence of this $\epsilon$ relaxed the condition for the series to converge, but allows us to investigate the situation when $\epsilon \rightarrow 0^{+}$. We denote $\mathbf{m} \in \mathbb{Z}^{2}$ by $(m, n) \in \mathbb{Z}^{2}$ from this point on. In conclusion, we get the following four different infinite sums:
$\diamond$
$$
S_{1}(\epsilon)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m n \neq 0}} \frac{1}{m n} e^{-\pi \epsilon\left(m^{2}+n^{2}\right)} .
$$
$\diamond$
$$
S_{2}(\epsilon, \alpha, \beta)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ n(\alpha m+\beta n) \neq 0}} \frac{\alpha}{n(\alpha m+\beta n)} e^{-\pi \epsilon\left(m^{2}+n^{2}\right)} .
$$
$\diamond$
$$
S_{3}(\epsilon, \alpha, \beta, \delta, \gamma)=\sum_{\substack{(m, n) \epsilon \mathbb{Z}^{2} \\(\alpha m+\beta n)(\delta m+\gamma n) \neq 0}} \frac{|\operatorname{det} M|}{(\alpha m+\beta n)(\delta m+\gamma n)} e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}
$$
where
\[

M=\left(\omega_{\mathbf{1}}, \omega_{\mathbf{2}}\right)=\left($$
\begin{array}{ll}
\alpha & \delta \\
\beta & \gamma
\end{array}
$$\right)
\]

$\diamond$

$$
S_{4}(\epsilon, \delta, \gamma)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(\delta m+\gamma) \neq 0}} \frac{\gamma}{m(\delta m+\gamma n)} e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}
$$

Our research path splits into two different directions based on whether the infinite sums are defined over integral cones or real cones:

1. When the cones are integral, we mainly investigate the sum:

$$
f(\epsilon, c, d)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{\widehat{G_{\epsilon}}(m, n)}{m(c m+d n)}=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \in \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)},
$$

and the following limit:

$$
f(c, d)=\frac{1}{4 \pi^{2}} \lim _{\epsilon \rightarrow 0^{+}} f(\epsilon, c, d)=\frac{1}{4 \pi^{2}} \lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)}
$$

Note that, $f(\epsilon, c, d)$ is essentially $S_{4}(\epsilon, c, d)$. We proved a relationship between $f(c, d)$ and the classical Dedekind sum $s(c, d)$ in Chapter 2 :

$$
d f(c, d)+c f(d, c)=s(c, d)+s(d, c)-\frac{\arctan (d / c)}{2 \pi}-\frac{\arctan (c / d)}{2 \pi}+\frac{1}{2}
$$

Note that there are several different representations for the classical Dedekind sums. One of them involves the Sawtooth function:

$$
s(c, d)=\sum_{k=0}^{d-1}\left(\left(\frac{k c}{d}\right)\right)\left(\left(\frac{k}{d}\right)\right)
$$

Another representation uses cotangent functions. Let $(h, k)=1$. Then

$$
s(h, k)=\frac{1}{2 \pi} \sum_{\substack{n=1 \\ n \neq 0(\bmod k)}}^{\infty} \frac{\cot (\pi h n / k)}{n} .
$$

Historically, Dedekind sums are named after Richard Dedekind. He introduced them to express the functional equation of the Dedekind eta function in 1877 which we can find in [Ded53]. It has many other connections besides the one we mentioned here. In 1951, Mordell published a paper [Mor51] in J. Indian Math Soc. which first connected lattice points in a tetrahedron with Dedekind sums, and thereafter led discrete geometry into a new era. In [BR], Dedekind sums arise from the study of lattice point enumeration of rational polytopes. Another interesting connection is mentioned in [Pom93], the author found an expression for the codimension two part of
the Todd class of an arbitrary toric variety given in terms of Dedekind sums. In [GP00], Garoufalidis and Pommersheim used toric geometry to explain properties of the values of zeta functions and Dedekind sums. The authors also relate cones with Dedekind sums, but their approach is fundamentally different from what we are doing here. Dedekins sums are also connected to random number generating. In The Art of Computer Programming, volume 2 [Knu77], Donald Knuth pointed out that Dedekind sums happen to be exactly the standard deviation of pseudo random number generators.
2. When the cones are real, we focus on the following sum:

$$
\mathrm{f}(\epsilon, \alpha, \beta)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+n)(m+\beta n)},
$$

and the following limit:

$$
\lim _{\epsilon \rightarrow 0^{+}} f(\epsilon, \alpha, \beta) .
$$

Note that $\mathrm{f}(\epsilon, \alpha, \beta)$ is essentially $S_{3}(\epsilon, \alpha, 1,1, \beta)$. We proved that the infinite series

$$
\mathrm{f}(\epsilon, \alpha, \beta)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+n)(m+\beta n)}
$$

converges absolutely for each fixed $\epsilon>0$, where both $\alpha$ and $\beta$ are quadratic irrationals in Chapter 3.

We also look into another infinite sum:

$$
\mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)=\sum_{\substack{\left(m, n, \in \mathbb{Z}^{2} \\(\alpha m+\beta n)(\delta m+\gamma n) \neq 0\right.}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+\beta n)(\delta m+\gamma n)},
$$

which is $S_{3}(\epsilon, \alpha, \beta, \delta, \gamma)$ in essence. We prove that if $\alpha \delta+\beta \gamma=0$, then the infinite series $\lim _{\epsilon \rightarrow 0^{+}} \mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)$ exists.

In Chapter 4, we answered the question: when are two Dedekind sums equal to each other? Let $b$ be a positive integer, and $a_{1}, a_{2}$ any two integers that are
relatively prime to $b$. If the Dedekind sum $s\left(a_{1}, b\right)$ is equal to $s\left(a_{2}, b\right)$, then

$$
b \mid\left(1-a_{1} a_{2}\right)\left(a_{1}-a_{2}\right)
$$

In Chapter 5, we partially answered the following question: when are two Zagier-Dedekind sums equal? We adopted the same technique used in Chapter 4 in our proof.

In the last part of this report, Chapter 6, we define a curve of generalized centers for a finite point set. Given any finite point set $S \subseteq \mathbb{R}^{d}$, and given $r \geq 1$, we define

$$
\mu(r)=\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S}\|a-x\|^{r}
$$

so that $\mu:[1, \infty) \rightarrow \mathbb{R}^{d}$.
The motivation for us to define our $\mu$-curve goes way back to Fermat. The curve $\mu(r)$ is a generalization of the Fermat point. We generalize Weiszfeld's algorithm to find $\mu(r)$ by an iteration process. We prove that to a certain extent, this curve captures the symmetric property of the original point set. We also prove that the $\mu$-curve is invariant under rigid motions. We conjecture that nondegenerate $\mu$-curves are uniquely determined by the point sets. An example is given in support of this conjecture. In the end of this chapter, we give several examples for the $\mu$-curve both in two-dimension and three-dimension.


## Integral cones and classical Dedekind

 sums
### 2.1 Cone decompositions

Dedekind sums arise naturally from decompositions of cones. Our main goal in this chapter is to study a Dedekind-like function defined on two-dimensional integral cones, and to interpret the reciprocity law of Dedekind sums as the interplay of two adjacent integral cones.

Definition 2.1. $A$ pointed polyhedral cone $\mathcal{K}_{\mathbf{v}} \subseteq \mathbb{R}^{d}$ is a set of the form

$$
\mathcal{K}_{\mathbf{v}}=\left\{\mathbf{v}+\sum_{k=1}^{m} \lambda_{k} \omega_{k} \mid \mathbf{v} \in \mathbb{R}^{d}, \text { all } \omega_{k} \in \mathbb{R}^{d}, \text { all } \lambda_{k} \geq 0\right\}
$$

where $\mathbf{v}, \omega_{1}, \cdots, \omega_{m}$ are such that there exists a hyperplane $H$ for which $H \cap \mathcal{K}=$ $\{\mathbf{v}\}$. Or equivalently, $\mathcal{K}$ does not contain an infinite line. The vector $\mathbf{v}$ is called the apex of $\mathcal{K}$.

All cones mentioned in this dissertation are pointed polyhedral cones unless stated otherwise.

Definition 2.2. For a pointed polyhedral cone $\mathcal{K}_{\mathbf{v}} \subseteq \mathbb{R}^{d}$, we call a finite set $\left\{\omega_{1}, \omega_{2}, \cdots, \omega_{m}\right\}$ generators of $\mathcal{K}_{\mathbf{v}}$ if every element in $\mathcal{K}$ can be written as

$$
\left\{\mathbf{v}+\sum_{k=1}^{m} \lambda_{k} \omega_{k} \mid \mathbf{v} \in \mathbb{R}^{d} \text {, all } \omega_{k} \in \mathbb{R}^{d} \text {, all } \lambda_{k} \geq 0\right\}
$$

where $\mathbf{v}$ is the vertex of $\mathcal{K}$.

Definition 2.3. Let the finite set $\left\{\omega_{1}, \omega_{2}, \cdots, \omega_{m}\right\}$ be generators of a pointed polyhedral cone $\mathcal{K}_{\mathbf{v}}$. The collection of vectors $\left\{\mathbf{e}_{1}, \cdots, \mathbf{e}_{d}\right\}$ is a minimal set of generators for $\mathcal{K}_{\mathbf{v}}$ provided that:

1. $\left\{\mathbf{e}_{1}, \cdots, \mathbf{e}_{d}\right\}$ are also generators of $\mathcal{K}_{\mathbf{v}}$.
2. $m \geq d$.
3. If another collection of vectors $\left\{\mathbf{y}_{1}, \cdots, \mathbf{y}_{d}\right\}$ generates the pointed cone $\mathcal{K}_{\mathbf{v}}$, then up to reordering, it must be the case that $\mathbf{y}_{j}$ is a scalar multiple of $\mathbf{e}_{j}$ for all $j=1,2, \cdots, d$.

Any vector in a minimal set of generators for $\mathcal{K}_{\mathbf{v}}$ is called an edge.

Definition 2.4. Define the first orthant cone to be:

$$
\mathcal{K}_{1 s t}=\mathbb{R}_{\geq 0}^{d}=\left\{\left(x_{1}, x_{2}, \cdots, x_{d}\right) \in \mathbb{R}^{d} \mid x_{1} \geq 0, x_{2} \geq 0, \cdots, x_{d} \geq 0\right\}
$$

When the dimension is clear, we also use $\mathcal{K}_{1 s t}$ to denote the first quadrant cone:

$$
\mathcal{K}_{1 s t}=\mathbb{R}_{\geq 0}^{2}=\left\{(x, y) \in \mathbb{R}^{2} \mid x \geq 0, y \geq 0\right\}
$$

Let a pointed cone $\mathcal{K}$ be an integral cone in the first quadrant as shown in Figure 2.1.

Let the two generators of $\mathcal{K}$ be $\omega_{1}=(a, b) \in \mathbb{Z}_{>0}^{2}$ and $\omega_{\mathbf{2}}=(c, d) \in \mathbb{Z}_{>0}^{2}$. Denote the pointed cone with generators $\omega_{\mathbf{y}}=(0,1)$ and $\omega_{\mathbf{1}}$ by $\mathcal{K}_{y}$. Denote


Figure 2.1: Decomposition of the first quadrant into three integral cones
the pointed cone with generators $\omega_{2}$ and $\omega_{x}=(1,0)$ by $\mathcal{K}_{x}$. To avoid overlap, we let every cone be half-open. For example, the first quadrant cone becomes $\mathcal{K}_{1 s t}=\left\{(x, y) \in \mathbb{R}^{2} \mid x \geq 0, y>0\right\}$. From Figure 2.1, we can decompose the first quadrant cone $\mathcal{K}_{1 s t}$ into a disjoint union of three half-open cones:

$$
\mathcal{K}_{1 s t}=\mathcal{K}_{y} \cup \mathcal{K} \cup \mathcal{K}_{x} .
$$

We can place the two generators of the pointed cone $\mathcal{K}$ as column vectors of a matrix $M$ :

$$
M=\left(\begin{array}{ll}
\omega_{\mathbf{1}} & \omega_{\mathbf{2}}
\end{array}\right)=\left(\begin{array}{ll}
a & c \\
b & d
\end{array}\right)
$$

Note that the four entries of $M$ are all positive integers.
Definition 2.5. For any cone $\mathcal{K} \subseteq \mathbb{R}^{d}$, we can associate an indicator function $1_{\mathcal{K}}$ to $\mathcal{K}$ which indicates membership of an element in $\mathcal{K}$. Specifically,

$$
1_{\mathcal{K}}(x)= \begin{cases}1 & \text { if } x \in \mathcal{K} \\ 0 & \text { otherwise }\end{cases}
$$

From Figure 2.1 and the decomposition of the first quadrant cone $\mathcal{K}_{1 s t}$, we get a nice identity about the indicator functions of the four cones $\mathcal{K}_{1 s t}, \mathcal{K}_{y}, \mathcal{K}$ and $\mathcal{K}_{x}$ :

$$
\begin{equation*}
1_{K_{1 s t}}=1_{\mathcal{K}_{y}}+1_{\mathcal{K}}+1_{\mathcal{K}_{x}} . \tag{2.1}
\end{equation*}
$$

The presence of the indicator function of a cone allows us to apply Fourier transforms to cones and thus leads to many amazing properties.

Definition 2.6. We define the Fourier transform of any compact set $\mathcal{P} \in \mathbb{R}^{d}$ as follows:

$$
\widehat{1_{\mathcal{P}}}(\mathbf{y})=\int_{\mathbb{R}^{d}} e^{-2 \pi i\langle\mathbf{x}, \mathbf{y}\rangle} 1_{\mathcal{P}}(\mathbf{x}) d \mathbf{x}=\int_{\mathcal{P}} e^{-2 \pi i\langle\mathbf{x}, \mathbf{y}\rangle} d \mathbf{x}
$$

This integral converges for compact bodies $\mathcal{P}$, but if we replace $\mathcal{P}$ with an unbounded cone $\mathcal{K}$, the convergence of the integral will not be guaranteed anymore. In order to employ the methods of harmonic analysis, we need to consider functions of complex variables. For this reason, we will let the variable $\mathbf{y}$ in the above Fourier transform be a complex vector:

$$
\mathbf{y}=\left(z_{1}, z_{2}, \cdots, z_{d}\right)=\mathbf{m}+i \mathbf{s}=\left(m_{1}+i s_{1}, m_{2}+i s_{2}, \cdots, m_{d}+i s_{d}\right)
$$

Definition 2.7. Let $\mathcal{K} \in \mathbb{R}^{d}$ be a pointed cone with apex at the origin. The polar cone $\mathcal{K}^{*}$ is defined as

$$
\mathcal{K}^{*}=\left\{\mathbf{x} \in \mathbb{R}^{d} \mid\langle\mathbf{x}, \mathbf{y}\rangle<0 \text { for all } \mathbf{y} \in \mathcal{K}\right\}
$$

Example 2.1. We give two examples of polar cones here.



Figure 2.2: Cones and their polar cones

Lemma 2.1. The Fourier transform of the first orthant $\mathcal{K}_{1 s t} \subseteq \mathbb{R}_{\geq 0}^{d}$ is

$$
\widehat{1_{\mathcal{K}_{1 s t}}}(\mathbf{m}+i \mathbf{s})=\left(\frac{1}{2 \pi i}\right)^{d} \prod_{j=1}^{d} \frac{1}{m_{j}+i s_{j}}
$$

Proof. By Definition 2.6, we have:

$$
\begin{align*}
\widehat{1_{1 s t}}(\mathbf{m}+i \mathbf{s}) & =\int_{\mathbb{R}_{\geq 0}^{d}} e^{-2 \pi i\langle\mathbf{x}, \mathbf{m}+i \mathbf{s}\rangle} d \mathbf{x},  \tag{2.2}\\
& =\int_{0}^{\infty} \cdots \int_{0}^{\infty} e^{-2 \pi i x_{1}\left(m_{1}+i s_{1}\right)} \cdots e^{-2 \pi i x_{d}\left(m_{d}+i s_{d}\right)} d x_{1} d x_{2} \cdots d x_{d},  \tag{2.3}\\
& =\left.\left.\frac{e^{-2 \pi i x_{1}\left(m_{1}+i s_{1}\right)}}{-2 \pi i\left(m_{1}+i s_{1}\right)}\right|_{0} ^{\infty} \cdots \frac{e^{-2 \pi i x_{d}\left(m_{d}+i s_{d}\right)}}{-2 \pi i\left(m_{d}+i s_{d}\right)}\right|_{0} ^{\infty},  \tag{2.4}\\
& =\frac{1}{2 \pi i\left(m_{1}+i s_{1}\right)} \cdots \frac{1}{2 \pi i\left(m_{d}+i s_{d}\right)},  \tag{2.5}\\
& =\left(\frac{1}{2 \pi i}\right)^{d} \prod_{j=1}^{d} \frac{1}{\left(m_{j}+i s_{j}\right)} . \tag{2.6}
\end{align*}
$$

In (2.3), as long as $s_{1}, s_{2}, \cdots, s_{d}<0$, or equivalently, when $\mathbf{s}=\left(s_{1}, s_{2}, \cdots, s_{d}\right) \in$ $\mathcal{K}_{1 s t}^{*}$, the integral will converge.

An example of Lemma 2.1 in $\mathbb{R}^{2}$ is as follows.

Example 2.2. The Fourier transform of the first quadrant $\mathcal{K}_{1 s t} \subseteq \mathbb{R}_{\geq 0}^{2}$ is

$$
\widehat{\mathcal{K}_{1 s t}}(\mathbf{m}+i \mathbf{s})=\left(\frac{1}{2 \pi i}\right)^{2} \frac{1}{\left(m_{1}+i s_{1}\right)\left(m_{2}+i s_{2}\right)}
$$

where $\mathbf{s}=\left(s_{1}, s_{2}\right) \in \mathcal{K}_{1 s t}^{*}$.

We will define simple cones and then give a general result of the Fourier transform of simple cones in $\mathbb{R}^{2}$. To set off, we need to introduce the definition of tangent cones.

Definition 2.8. Let $\mathcal{P} \in \mathbb{R}^{d}$ be a non-empty polyhedron and let $\mathbf{v} \in \mathcal{P}$ be a point. We define the tangent cone of $\mathcal{P}$ at $\mathbf{v}$ by:

$$
\operatorname{tcone}(\mathcal{P}, \mathbf{v})=\{\mathbf{v}+\mathbf{y}: \mathbf{v}+\epsilon \mathbf{y} \in \mathcal{P} \text { for some } \epsilon>0\}
$$

This definition is given in [Bar08]. Here is an illustration of a polyhedron $\mathcal{P} \in \mathbb{R}^{2}$ and its tangent cones at three different points in $\mathcal{P}$.


Figure 2.3: A polyhedron $\mathcal{P}$ and its tangent cones
Notice that the tangent cone $\operatorname{tcone}(\mathcal{P}, B)$ is a half plane, and tcone $(\mathcal{P}, C)$ is the whole plane $\mathbb{R}^{2}$.

Definition 2.9. For a tangent cone $\mathcal{K} \subseteq \mathbb{R}^{d}$, if the number of its generators is equal to the dimension of $\mathcal{K}$, then we call $\mathcal{K}$ a simple cone.

Lemma 2.2. For any simple tangent cone $\mathcal{K} \subseteq \mathbb{R}^{2}$ with generators $\omega_{1}=(a, b) \in$ $\mathbb{Z}_{>0}^{2}$ and $\omega_{2}=(c, d) \in \mathbb{Z}_{>0}^{2}$, the Fourier transform of $\mathcal{K}$ is:

$$
\widehat{1_{\mathcal{K}}}(\mathbf{m}+i \mathbf{s})=\left(\frac{1}{2 \pi i}\right)^{2} \frac{|\operatorname{det}(M)|}{\left\langle\omega_{\mathbf{1}}, \mathbf{m}+i \mathbf{s}\right\rangle\left\langle\omega_{\mathbf{2}}, \mathbf{m}+i \mathbf{s}\right\rangle},
$$

where $\left(s_{1}, s_{2}\right) \in \mathcal{K}^{*}$ and

$$
M=\left(\begin{array}{ll}
\omega_{1} & \omega_{2}
\end{array}\right)=\left(\begin{array}{ll}
a & c \\
b & d
\end{array}\right) .
$$

Proof. The main idea is to find a linear transformation which could map the simple tangent cone $\mathcal{K} \subseteq \mathbb{R}^{2}$ bijectively onto the positive quadrant $\mathcal{K}_{1 s t}$, then by applying Corollary 2.2, we can find the Fourier transform of any simple tangent cone without much difficulty. According to Definition 2.6, we have:

$$
\begin{align*}
\widehat{\mathcal{L}_{\mathcal{K}}}(\mathbf{m}+i \mathbf{s}) & =\int_{\mathcal{K}} e^{-2 \pi i\langle\mathbf{x}, \mathbf{m}+i \mathbf{s}\rangle} d \mathbf{x},  \tag{2.7}\\
& =\int_{\mathcal{K}_{1 s t}} e^{-2 \pi i\langle M \mathbf{y}, \mathbf{m}+i \mathbf{s}\rangle}|\operatorname{det}(M)| d \mathbf{y},  \tag{2.8}\\
& =\int_{\mathcal{K}_{1 s t}} e^{-2 \pi i\left\langle\mathbf{y}, M^{T}(\mathbf{m}+i \mathbf{s})\right\rangle}|\operatorname{det}(M)| d \mathbf{y},  \tag{2.9}\\
& =\int_{\mathcal{K}_{1 s t}} e^{-2 \pi i\left\langle\mathbf{y},\left(\left\langle\omega_{1}, \mathbf{m}+i \mathbf{s}\right\rangle,\left\langle\omega_{\mathbf{2}}, \mathbf{m}+i \mathbf{s}\right\rangle\right)\right\rangle}|\operatorname{det}(M)| d \mathbf{y},  \tag{2.10}\\
& =\left(\frac{1}{2 \pi i}\right)^{2} \frac{|\operatorname{det}(M)|}{\left\langle\omega_{\mathbf{1}}, \mathbf{m}+i \mathbf{s}\right\rangle\left\langle\omega_{\mathbf{2}}, \mathbf{m}+i \mathbf{s}\right\rangle} . \tag{2.11}
\end{align*}
$$

In (2.8), we mapped $\mathcal{K}$ to $\mathcal{K}_{1 s t}$ by letting $\mathcal{K}=M \mathcal{K}_{1 s t}$ where

$$
M=\left(\omega_{1}, \omega_{2}\right)=\left(\begin{array}{ll}
a & c \\
b & d
\end{array}\right)
$$

The reason for (2.10) to hold is because we have

$$
M^{T}=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)
$$

and

$$
\begin{aligned}
M^{T}(\mathbf{m}+i \mathbf{s}) & =\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)(\mathbf{m}+i \mathbf{s}) \\
& =\langle(a, b), \mathbf{m}+i \mathbf{s}\rangle\langle(c, d), \mathbf{m}+i \mathbf{s}\rangle \\
& =\left\langle\omega_{1}, \mathbf{m}+i \mathbf{s}\right\rangle\left\langle\omega_{2}, \mathbf{m}+i \mathbf{s}\right\rangle \\
& =\left(a m_{1}+b m_{2}+i\left(a s_{1}+b s_{2}\right)\right)\left(c m_{1}+d m_{2}+i\left(c s_{1}+d s_{2}\right)\right)
\end{aligned}
$$

Since $\left(s_{1}, s_{2}\right) \in \mathcal{K}^{*}$, we always have $a s_{1}+b s_{2}<0$ and $c s_{1}+d s_{2}<0$. Therefore the convergence of the Fourier transform of $\mathcal{K}$ is guaranteed.
$\widehat{1_{\mathcal{K}}}(\mathbf{m}+i \mathbf{s})$ is in fact a rational function of $s_{1}$ and $s_{2}$, so it has a meromorphic continuation of all vectors $\left(s_{1}, s_{2}\right) \in \mathbb{C}^{2}$.

Example 2.3. The Fourier transform of the cone $\mathcal{K}_{y}$ with generators $\omega_{\mathbf{y}}=(0,1)$ and $\omega_{\mathbf{1}}=(a, b)$ is:

$$
\widehat{\mathcal{K}_{y}}(\mathbf{m}+i \mathbf{s})=\left(\frac{1}{2 \pi i}\right)^{2} \frac{a}{\left(m_{2}+i s_{2}\right)\left(a\left(m_{1}+i s_{1}\right)+b\left(m_{2}+i s_{2}\right)\right)},
$$

where $\mathbf{s}=\left(s_{1}, s_{2}\right) \in \mathcal{K}_{y}^{*}$.

This is a direct application of Lemma 2.2 by letting

$$
M=\left(\omega_{y}, \omega_{1}\right)=\left(\begin{array}{ll}
0 & a \\
1 & b
\end{array}\right) .
$$

Example 2.4. The Fourier transform of the cone $\mathcal{K}_{x}$ with generators $\omega_{\mathbf{2}}=$ $(c, d) \in \mathbb{Z}_{\geq 0}^{2}$ and $\omega_{\mathbf{x}}=(1,0)$ is:

$$
\widehat{\mathcal{K}_{x}}(\mathbf{m}+i \mathbf{s})=\left(\frac{1}{2 \pi i}\right)^{2} \frac{d}{\left(m_{1}+i s_{1}\right)\left(c\left(m_{1}+i s_{1}\right)+d\left(m_{2}+i s_{2}\right)\right)},
$$

where $\mathbf{s}=\left(s_{1}, s_{2}\right) \in \mathcal{K}_{x}^{*}$.

This is also a direct application of Lemma 2.2, with

$$
M=\left(\omega_{2}, \omega_{x}\right)=\left(\begin{array}{ll}
c & 1 \\
d & 0
\end{array}\right) .
$$

Now let us go back to equation (2.1):

$$
1_{K_{1 s t}}=1_{\mathcal{K}_{y}}+1_{\mathcal{K}}+1_{\mathcal{K}_{x}} .
$$

If we apply the Fourier transform to both sides of this equation, we will get

$$
\begin{align*}
\frac{1}{2 \pi i z_{1}} \frac{1}{2 \pi i z_{2}} & =\left(\frac{1}{2 \pi i}\right)^{2} \frac{a}{z_{2}\left(a z_{1}+b z_{2}\right)} \\
& +\left(\frac{1}{2 \pi i}\right)^{2} \frac{|\operatorname{det}(M)|}{\left(a z_{1}+b z_{2}\right)\left(c z_{1}+d z_{2}\right)} \\
& +\left(\frac{1}{2 \pi i}\right)^{2} \frac{d}{z_{1}\left(c z_{1}+d z_{2}\right)} \tag{2.12}
\end{align*}
$$

Further simplification of (2.12) gives us:

$$
\frac{1}{z_{1} z_{2}}=\frac{a}{z_{2}\left(a z_{1}+b z_{2}\right)}+\frac{|\operatorname{det}(M)|}{\left(a z_{1}+b z_{2}\right)\left(c z_{1}+d z_{2}\right)}+\frac{d}{z_{1}\left(c z_{1}+d z_{2}\right)}
$$

where $\mathbf{z}=\mathbf{m}+\mathbf{i s}=\left(m_{1}+i s_{1}, m_{2}+i s_{2}\right)=\left(z_{1}, z_{2}\right)$. We can view it as taking the transforms on both sides of (2.1), or a "valuation" at any $\mathbf{z}=\left(m_{1}+i s_{1}, m_{2}+i s_{2}\right) \in$ $\mathbb{C}^{2}$, therefore sending cones to rational functions[DR10].

Next we will introduce a class of rapidly decreasing functions: Schwartz functions.

Definition 2.10. Let $\mathcal{S}$ be the collection of rapidly decreasing continuous functions:

$$
\mathcal{S}\left(\mathbb{R}^{d}\right)=\left\{f \in \mathbb{C}^{\infty}\left(\mathbb{R}^{d}\right) \mid\|f\|_{\alpha, \beta}<\infty, \forall \alpha, \beta\right\}
$$

where $\alpha, \beta \in \mathbb{N}_{0}^{d}$ are multi-indices, $\mathbb{C}^{\infty}\left(\mathbb{R}^{d}\right)$ is the set of smooth functions from $\mathbb{R}^{d} \rightarrow \mathbb{C}$, and

$$
\|f\|_{\alpha, \beta}=\sup _{\mathbf{x} \in \mathbb{R}^{d} \mid}\left|\mathbf{x}^{\alpha} D^{\beta} f(\mathbf{x})\right|
$$

When the dimension $d$ is clear, we can also write $\mathcal{S}=\mathcal{S}\left(\mathbb{R}^{d}\right)$. In words, these are the infinitely differentiable functions whose derivatives decrease faster than any power of $\mathbf{x}$ at infinity. These functions have the properties that:

1. If $f(\mathbf{x})$ is in $\mathcal{S}$, then the Fourier transform $\mathcal{F} f(\mathbf{s})$ is in $\mathcal{S}$.
2. If $f(\mathbf{x})$ is in $\mathcal{S}$, then $\mathcal{F}^{-1} \mathcal{F} f=f$.

We also refer to the functions in $\mathcal{S}$ simply as Schwartz functions. Here $\mathcal{F}$ denotes the Fourier transform of a function:

$$
\mathcal{F} f(\mathbf{s})=\int_{\mathbb{R}^{d}} e^{-2 \pi i\langle\mathbf{s}, \mathbf{t}\rangle} f(\mathbf{t}) d \mathbf{t}
$$

and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform of a function:

$$
\mathcal{F}^{-1} g(\mathbf{t})=\int_{\mathbb{R}^{d}} e^{2 \pi i\langle\mathbf{s}, \mathbf{t}\rangle} g(\mathbf{s}) d \mathbf{s}
$$

Definition 2.11. Let $\phi$ be a Schwartz function in $\mathbb{R}^{d}$. Then

$$
\sum_{\mathbf{k} \in \mathbb{Z}^{d}} \mathcal{F} \phi(\mathbf{k})=\sum_{\mathbf{k} \in \mathbb{Z}^{d}} \phi(\mathbf{k})
$$

## This is called the Poisson Summation Formula.

This result also holds for some more subtle classes of functions (see Theorem 3.1.7 in [Gra09] for a more detailed description of the Poisson Summation Formula). An example is the function $f(x)=e^{-2 \pi\|\mathbf{x}\|}$ where $\mathbf{x} \in \mathbb{R}^{n}$, and $\|\cdot\|$ denotes the Euclidean norm. This function does not belong to $\mathcal{S}$, but the Poisson Summation Formula is true for $f$ and its Fourier transform.

The Poisson Summation Formula is an idea certainly at the heart of Fourier analysis (the interested reader may consult the book [Osg07]). Indicator functions of cones do not belong to $\mathcal{S}$ because they are not continuous in $\mathbb{R}^{d}$, let alone infinitely differentiable. If we want to apply the Poisson Summation Formula in our analysis of cones, we need to modify the indicator function of a cone by smoothing it out.

Definition 2.12. Let $(S, \sigma, \mu)$ be a measure space. Consider the set of all measurable functions from $S$ to $\mathbb{C}$ (or $\mathbb{R}$ ) whose absolute value has finite integral, or equivalently, that

$$
\|f\|_{1}:=\int_{S}|f| d \mu<\infty
$$

The set of such functions form a vector space, which is denoted by $L_{1}(\mathbb{R})$.

Definition 2.13 ([Sta05]). If $g$ is in $L_{1}(\mathbb{R})$ and has mass 1 and $g_{[\epsilon]}$ satisfies

$$
g_{[\epsilon]}(y)=\epsilon^{-1} g\left(\epsilon^{-1} y\right),
$$

then the collection

$$
\left\{g_{[\epsilon]}: \epsilon>0\right\}
$$

of functions is called the approximate identity for convolution -or simply the approximate identity-generated by $g$.

In light of this, we can smooth the indicator function of cones by convolving it with Gaussian functions. The main reason for us to choose them as the candidates of smoothing functions is because their Fourier transform is essentially themselves.

There are several different ways to define Gaussian functions. The reason why we choose the following definition for a Gaussian function is because its Fourier transform is relatively simpler than those of other Gaussians.

Definition 2.14. Define a Gaussian function $G_{\epsilon}(\mathbf{x})$ on $\mathbb{R}^{d}$ as follows:

$$
G_{\epsilon}(\mathbf{x})=\frac{1}{\epsilon^{d / 2}} e^{-\frac{\pi}{\epsilon}\|\mathbf{x}\|^{2}}
$$

where $\epsilon>0, \mathbf{x} \in \mathbb{R}^{d}$, and $\|\cdot\|$ denotes the Euclidean norm.

Example 2.5. For example, a Gaussian function in $\mathbb{R}^{2}$ would be:

$$
G_{\epsilon}(m, n)=\frac{1}{\epsilon} e^{-\frac{\pi}{\epsilon}\left(m^{2}+n^{2}\right)},
$$

where $\epsilon \in \mathbb{R}_{>0}$. It has Fourier transform:

$$
\widehat{G_{\epsilon}}(m, n)=e^{-\pi \epsilon\left(m^{2}+n^{2}\right)} .
$$

In Lemma 2.2, for any simple cone $\mathcal{K} \subset \mathbb{R}^{2}$ with generators $\omega_{1}=(a, b)$ and $\omega_{2}=(c, d)$, we can interpret the Fourier transform of $\mathcal{K}$ in a different way.

Let $\mathbf{z}=\left(z_{1}, z_{2}\right)=\left(m+i s_{1}, n+i s_{2}\right) \in \mathbb{C}^{2}$. We can write $\widehat{1_{\mathcal{K}}}(\mathbf{z})$ as:

$$
\begin{align*}
\widehat{\mathcal{L}_{\mathcal{K}}}(\mathbf{z}) & =\widehat{1_{\mathcal{K}}}(m, n),  \tag{2.13}\\
& =\left(\frac{1}{2 \pi i}\right)^{2} \frac{|\operatorname{det} M|}{\left\langle\omega_{1}, \mathbf{z}\right\rangle\left\langle\omega_{2}, \mathbf{z}\right\rangle},  \tag{2.14}\\
& =\left(\frac{1}{2 \pi i}\right)^{2} \frac{|\operatorname{det} M|}{\left\langle\omega_{1},\left(m+i s_{1}, n+i s_{2}\right)\right\rangle\left\langle\omega_{2},\left(m+i s_{1}, n+i s_{2}\right)\right\rangle},  \tag{2.15}\\
& =\left(\frac{1}{2 \pi i}\right)^{2} \frac{|\operatorname{det} M|}{\left(a m+b n+i\left(a s_{1}+b s_{2}\right)\right)\left(c m+d n+i\left(c s_{1}+d s_{2}\right)\right)} . \tag{2.16}
\end{align*}
$$

Following this interpretation of the Fourier transform of a cone, the convolution of the Gaussian function $G_{\epsilon}(m, n)$ and the indicator functions in (2.1) will give us the modified indicator functions:

$$
\begin{equation*}
\left(1_{\mathcal{K}_{1 s t}} * G_{\epsilon}\right)(m, n)=\left(1_{\mathcal{K}_{y}} * G_{\epsilon}\right)(m, n)+\left(1_{\mathcal{K}} * G_{\epsilon}\right)(m, n)+\left(1_{\mathcal{K}_{x}} * G_{\epsilon}\right)(m, n) \tag{2.17}
\end{equation*}
$$

where $(m, n) \in \mathbb{R}^{2}$. The modified functions $1_{\mathcal{K}} * G_{\epsilon}$ will fall in the category of "nice" functions for any cone $\mathcal{K}$. In other words, $\left(1_{\mathcal{K}} * G_{\epsilon}\right) \in \mathcal{S}$.

Now take the Fourier transform of (2.17) on both sides:

$$
\begin{equation*}
\left.\widehat{\left(1_{\mathcal{K}_{1 s t} * G_{\epsilon}}\right.}\right)(m, n)=\left(\widehat{\left.1_{\mathcal{K}_{y} * G_{\epsilon}}\right)}(m, n)+\left(\widehat{1_{\mathcal{K}} * G_{\epsilon}}\right)(m, n)+\left(\widehat{1_{\mathcal{K}_{x} * G_{\epsilon}}}\right)(m, n) .\right. \tag{2.18}
\end{equation*}
$$

We know that if $h(x)=(f * g)(x)$, then $\widehat{h}(\xi)=\widehat{f}(\xi) \widehat{g}(\xi)$. As a result, (2.18) becomes:

$$
\begin{equation*}
\widehat{1_{\mathcal{K}_{1 s t}}}(m, n) \widehat{G_{\epsilon}}(m, n)=\widehat{\widehat{\mathcal{K}_{y}}}(m, n) \widehat{G_{\epsilon}}(m, n)+\widehat{\mathcal{K}_{\mathcal{K}}}(m, n) \widehat{G_{\epsilon}}(m, n)+\widehat{\mathcal{K}_{x}}(m, n) \widehat{G_{\epsilon}}(m, n) . \tag{2.19}
\end{equation*}
$$

We can sum both sides of (2.19) over the whole lattice $\mathbb{Z}^{2}$ :

$$
\begin{align*}
& \sum_{(m, n) \in \mathbb{Z}^{2}} \frac{\widehat{G_{\epsilon}}(m, n)}{\left(m+i s_{1}\right)\left(n+i s_{2}\right)}=a \sum_{(m, n) \in \mathbb{Z}^{2}} \frac{\widehat{G_{\epsilon}}(m, n)}{\left(n+i s_{2}\right)\left(a\left(m+i s_{1}\right)+b\left(n+i s_{2}\right)\right)} \\
& \quad+d \sum_{(m, n) \in \mathbb{Z}^{2}} \frac{\widehat{G_{\epsilon}}(m, n)}{\left(m+i s_{1}\right)\left(c\left(m+i s_{1}\right)+d\left(n+i s_{2}\right)\right)} \\
& \quad+|\operatorname{det}(M)| \sum_{(m, n) \in \mathbb{Z}^{2}} \frac{\widehat{G_{\epsilon}}(m, n)}{\left(a\left(m+i s_{1}\right)+b\left(n+i s_{2}\right)\right)\left(c\left(m+i s_{1}\right)+d\left(n+i s_{2}\right)\right)} . \tag{2.20}
\end{align*}
$$

Here we applied the results from Example 2.2, Lemma 2.2 ,Example 2.3, and Example 2.4 which give us the Fourier transforms of indicator functions.

### 2.2 Fourier transform of integral cones in $\mathbb{R}^{2}$ and classical Dedekind sums

To simplify our argument later, instead of analyzing the decomposition of the first quadrant into three half-open cones as shown in Figure 2.1, we decompose the first quadrant into a disjoint union of two half-open cones where the edge belonging to $\mathcal{K}_{1}$ and adjacent to $\mathcal{K}_{2}$ is $\omega=(c, d) \in \mathbb{Z}_{>0}^{2}$.


Figure 2.4: Decomposition of the first quadrant into two integral cones

Imagine that we squeeze the cone $\mathcal{K}$ in Figure 2.1 until it becomes a line ray. We will then get a new decomposition of the first quadrant cone as is shown in Figure 2.4. The decomposition of $\mathcal{K}_{1 s t}$ becomes:

$$
\mathcal{K}_{1 s t}=\mathcal{K}_{1} \bigcup \mathcal{K}_{2} .
$$

The relationship between the indicator functions of the three cones in Figure 2.4 is:

$$
1_{\mathcal{K}_{1 s t}}=1_{\mathcal{K}_{1}}+1_{\mathcal{K}_{2}} .
$$

By applying Lemma 2.2, Example 2.3 and Example 2.4, we have:

$$
\begin{equation*}
\frac{1}{z_{1} z_{2}}=\frac{c}{z_{2}\left(c z_{1}+d z_{2}\right)}+\frac{d}{z_{1}\left(c z_{1}+d z_{2}\right)} \tag{2.21}
\end{equation*}
$$

where $\mathbf{z}=\left(z_{1}, z_{2}\right)=\left(m+i s_{1}, n+i s_{2}\right) \in \mathbb{C}^{2}$. Now we fix $\left(s_{1}, s_{2}\right) \in \mathbb{R}^{2}$. Following the argument used previously in Section 2.1, we will get:

$$
\begin{align*}
\sum_{(m, n) \in \mathbb{Z}^{2}} \frac{\widehat{G_{\epsilon}}(m, n)}{\left(m+i s_{1}\right)\left(n+i s_{2}\right)} & =c \sum_{(m, n) \in \mathbb{Z}^{2}} \frac{\widehat{G_{\epsilon}}(m, n)}{\left(n+i s_{2}\right)\left(c\left(m+i s_{1}\right)+d\left(n+i s_{2}\right)\right)} \\
& +d \sum_{(m, n) \in \mathbb{Z}^{2}} \frac{\widehat{G_{\epsilon}}(m, n)}{\left(m+i s_{1}\right)\left(c\left(m+i s_{1}\right)+d\left(n+i s_{2}\right)\right)} . \tag{2.22}
\end{align*}
$$

We now turn our attention to the second sum on the right hand side of (2.22):

$$
\begin{equation*}
\sum_{(m, n) \in \mathbb{Z}^{2}} \frac{\widehat{G_{\epsilon}}(m, n)}{\left(m+i s_{1}\right)\left(c\left(m+i s_{1}\right)+d\left(n+i s_{2}\right)\right)} \tag{2.23}
\end{equation*}
$$

In [Ber76], the author investigated the infinite sum

$$
\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{1}{m(c m+d n)}
$$

which is one of the motivations for what we are going to do next. Another motivation for us to choose the direction of our work comes from [GS03]. The authors defined an infinite sum on a lattice which looks very similar to what we have here. Let $\mathcal{L}$ be a rank $l$ sublattice of $\mathbb{Z}^{d}$ satisfying $\mathcal{L}=$ Sat $\mathcal{L}$ where Sat $\mathcal{L}=\left\{\mathbf{m} \in \mathbb{Z}^{n}: d \mathbf{m} \in \mathcal{L}\right.$ for some $\left.d \in \mathbb{Z}\right\}$ denotes the saturation of $\mathcal{L}$. In other words, any $\mathbb{Z}$-basis of $\mathcal{L}$ can be extended to a $\mathbb{Z}$-basis of $\mathbb{Z}^{n}$. Note that $\sigma$ denotes an integral $(d \times d)$-matrix with primitive columns $\sigma_{\mathbf{1}}, \cdots, \sigma_{\mathbf{d}} \cdot{ }^{1}$ For $\mathbf{v} \in \mathbb{R}^{d}$, define

$$
S(L, \sigma, \mathbf{v})=(2 \pi i)^{-d} \sum_{\mathbf{x} \in L} \frac{e^{\langle\mathbf{x}, \mathbf{v}\rangle}}{\left\langle\mathbf{x}, \sigma_{\mathbf{1}}\right\rangle \cdots\left\langle\mathbf{x}, \sigma_{\mathbf{d}}\right\rangle}
$$

[^1]Due to the above reasons, we define the following infinite sum based on (2.23).

Definition 2.15. Define an infinite series $f(\epsilon, c, d)$ by letting $\left(s_{1}, s_{2}\right)=(0,0)$ in (2.23). We will remove two lines $m=0$ and $c m+d n=0$ from the whole lattice $\mathbb{Z}^{2}$ :

$$
f(\epsilon, c, d)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{\widehat{G_{\epsilon}}(m, n)}{m(c m+d n)}=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \neq \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)} .
$$

We wish to exclude the two straight lines $m=0$, and $c m+d n=0$ in the definition of $f(\epsilon, c, d)$ because the denominator of all summands vanish at points on those two lines. We are interested in the asymptotic behavior of $f(\epsilon, c, d)$ when $\epsilon \rightarrow 0^{+}$.

Definition 2.16. Define

$$
f(c, d)=\frac{1}{4 \pi^{2}} \lim _{\epsilon \rightarrow 0^{+}} f(\epsilon, c, d)=\frac{1}{4 \pi^{2}} \lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)} .
$$

We will find out later that $f(c, d)$ defined here is closely related to the classical Dedekind sums $s(c, d)$. A nice relationship between these two functions is given later and we would like to remind the reader that the technique introduced by Q. N. Le and S. Robins [LR11] plays a pivotal rule.

Definition 2.17. Define the first Bernoulli polynomial

$$
B_{1}(x)= \begin{cases}x-\frac{1}{2} & \text { if } x \in[0,1] \\ 0 & \text { otherwise }\end{cases}
$$

The Fourier transform of $B_{1}$ is:

$$
\begin{aligned}
\widehat{B_{1}}(\xi) & =\int_{0}^{1}\left(x-\frac{1}{2}\right) e^{-2 \pi i x \xi} d x \\
& =\frac{e^{2 \pi i \xi}-1}{4 \pi^{2} \xi^{2}}+\frac{e^{2 \pi i \xi}+1}{-4 \pi i \xi}
\end{aligned}
$$

When $\xi \in \mathbb{Z}_{\neq 0}$, we have

$$
\begin{equation*}
\widehat{B_{1}}(\xi)=-\frac{1}{2 \pi i \xi} \tag{2.24}
\end{equation*}
$$

When $\xi=0$, it is easy to see that $\widehat{B_{1}}(\xi)=0$. If $B_{1}^{\prime}(x)=B_{1}\left(x-x_{0}\right)$ for some $x_{0} \in \mathbb{R}$, then $B_{1}^{\prime}$ lives on the interval $\left[x_{0}, x_{0}+1\right]$. The translation property of the Fourier transform says:

$$
\widehat{B_{1}^{\prime}}(\xi)=e^{-2 \pi i x_{0} \xi} \widehat{B_{1}}(\xi)
$$

Definition 2.18. Define a two-dimension function $F$ as follows:

$$
\begin{aligned}
F\left(x_{1}, x_{2}\right) & = \begin{cases}B_{1}\left(x_{1}\right) B_{1}\left(x_{2}\right) & \text { if } x_{1} \in[0,1] \text { and } x_{2} \in[0,1] \\
0 & \text { otherwise } .\end{cases} \\
& = \begin{cases}\left(x_{1}-\frac{1}{2}\right)\left(x_{2}-\frac{1}{2}\right) & \text { if } x_{1} \in[0,1] \text { and } x_{2} \in[0,1], \\
0 & \text { otherwise. }\end{cases}
\end{aligned}
$$

$F$ is a product of two Bernoulli polynomials of independent variables. The Fourier transform of $F$ is:

$$
\begin{aligned}
\widehat{F}\left(\xi_{1}, \xi_{2}\right) & =\int_{0}^{1} \int_{0}^{1}\left(x_{1}-\frac{1}{2}\right)\left(x_{2}-\frac{1}{2}\right) e^{-2 \pi i x_{1} \xi_{1}} e^{-2 \pi i x_{2} \xi_{2}} d x_{1} d x_{2}, \\
& =\int_{0}^{1}\left(x_{1}-\frac{1}{2}\right) e^{-2 \pi i x_{1} \xi_{1}} d x_{1} \int_{0}^{1}\left(x_{2}-\frac{1}{2}\right) e^{-2 \pi i x_{2} \xi_{2}} d x_{2}, \\
& =\left(\frac{e^{2 \pi i \xi_{1}-1}}{4 \pi^{2} \xi_{1}^{2}}+\frac{e^{2 \pi i \xi_{1}+1}}{-4 \pi i \xi_{1}}\right)\left(\frac{e^{2 \pi i \xi_{2}-1}}{4 \pi^{2} \xi_{2}^{2}}+\frac{e^{2 \pi i \xi_{2}+1}}{-4 \pi i \xi_{2}}\right) .
\end{aligned}
$$

When $\xi_{1} \in \mathbb{Z}_{\neq 0}$ and $\xi_{2} \in \mathbb{Z}_{\neq 0}$, we have:

$$
\widehat{F}\left(\xi_{1}, \xi_{2}\right)=-\frac{1}{4 \pi^{2} \xi_{1} \xi_{2}} .
$$

When $\xi_{1}=0$ or $\xi_{2}=0$, we have:

$$
\widehat{F}\left(\xi_{1}, \xi_{2}\right)=0
$$

Lemma 2.3. Let $\omega_{1}=(\alpha, \beta) \in \mathbb{Z}^{2}$, and $\omega_{2}=(\delta, \gamma) \in \mathbb{Z}^{2}$. Let

$$
K=\binom{\omega_{1}^{T}}{\omega_{2}^{T}}=\left(\begin{array}{ll}
\alpha & \beta \\
\delta & \gamma
\end{array}\right) \in G L_{2}(\mathbb{Z})
$$

Denote the inverse transpose of $K$ by $K^{-T}$. Let $\xi \in \mathbb{Z}_{\neq 0}^{2}$. Denote $F \circ K^{-T}(\xi)$ by $\mathfrak{F}(\xi)$. Then we have:

$$
\widehat{\mathfrak{F}}(\xi)=\left(\widehat{F \circ K^{-T}}\right)(\xi)= \begin{cases}-\frac{|\operatorname{det} K|}{4 \pi^{2}\left\langle\omega_{1}^{T}, \xi\right\rangle\left\langle\omega_{2}^{T}, \xi\right\rangle} & \text { if }\left\langle\omega_{\mathbf{1}}{ }^{T}, \xi\right\rangle\left\langle\omega_{\mathbf{2}}{ }^{T}, \xi\right\rangle \neq 0 \\ 0 & \text { otherwise }\end{cases}
$$

The proof of this lemma follows from Theorem B. 1 in Appendix B.
Our goal is to study

$$
f(c, d)=\frac{1}{4 \pi^{2}} \lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)} .
$$

The denominator in the summand of $f(c, d)$ is a product of two inner products:

$$
m(c m+d n)=\langle(1,0),(m, n)\rangle \times\langle(c, d),(m, n)\rangle
$$

As shown in Figure 2.4, we denote $\omega=(c, d) \in \mathbb{Z}_{>0}^{2}$ and $\omega_{x}=(1,0)$. Let

$$
J=\binom{\omega^{T}}{\omega_{x}^{T}}=\left(\begin{array}{ll}
c & d \\
1 & 0
\end{array}\right)
$$

We have $\operatorname{det} J=-d$, and $J^{-T}=\left(\begin{array}{cc}0 & 1 / d \\ 1 & -c / d\end{array}\right)$. From the definition of $\mathfrak{F}$ in
Lemma 2.3, we have

$$
\begin{equation*}
\mathfrak{F}\left(\xi_{1}, \xi_{2}\right)=F\left(J^{-T}\left(\xi_{1}, \xi_{2}\right)\right)=F\left(\frac{\xi_{2}}{d}, \xi_{1}-\frac{c \xi_{2}}{d}\right) \tag{2.25}
\end{equation*}
$$

where $\xi=\left(\xi_{1}, \xi_{2}\right) \in \mathbb{Z}_{\neq 0}^{2}$.
Lemma 2.3 also tells us that

$$
\widehat{\mathfrak{F}}\left(\xi_{1}, \xi_{2}\right)= \begin{cases}-\frac{|\operatorname{det} J|}{4 \pi^{2} \xi_{1}\left(c \xi_{1}+d \xi_{2}\right)}=-\frac{d}{4 \pi^{2} \xi_{1}\left(c \xi_{1}+d \xi_{2}\right)} & \text { if } \xi_{1}\left(c \xi_{1}+d \xi_{2}\right) \neq 0 \\ 0 & \text { otherwise }\end{cases}
$$

On the one hand,

$$
\begin{align*}
& \lim _{\epsilon \rightarrow 0^{+}} \sum_{(m, n) \in \mathbb{Z}^{2}} \widehat{\mathfrak{F}}(m, n) \widehat{G_{\epsilon}}(m, n) \\
&=\lim _{\epsilon \rightarrow 0^{+}} \sum_{(m, n) \in \mathbb{Z}^{2}} \widehat{\mathfrak{F}}(m, n) e^{-\pi \epsilon\left(m^{2}+n^{2}\right)},  \tag{2.26}\\
&=\lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
m(c m+d n) \neq 0}}\left(-\frac{d}{4 \pi^{2}}\right) \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)},  \tag{2.27}\\
& \quad=-d f(c, d) . \tag{2.28}
\end{align*}
$$

By Lemma 2.3, when $m(c m+d n)=0$, we have $\widehat{\mathfrak{F}}(m, n)=0$. Therefore the equality in (2.27) holds.

On the other hand, the Poisson summation formula guarantees that:

$$
\sum_{(m, n) \in \mathbb{Z}^{2}} \widehat{\mathfrak{F}}(m, n) \widehat{G_{\epsilon}}(m, n)=\sum_{(m, n) \in \mathbb{Z}^{2}} \widehat{\mathfrak{F} * G_{\epsilon}}(m, n)=\sum_{(m, n) \in \mathbb{Z}^{2}}\left(\mathfrak{F} * G_{\epsilon}\right)(m, n)
$$

So we hope to give another identity of $\lim _{\epsilon \rightarrow 0^{+}} \sum_{(m, n) \in \mathbb{Z}^{2}} \widehat{\mathfrak{F}}(m, n) \widehat{G_{\epsilon}}(m, n)$ by analyzing $\sum_{(m, n) \in \mathbb{Z}^{2}}\left(\mathfrak{F} * G_{\epsilon}\right)(m, n)$ more carefully.

The function $\mathfrak{F}(m, n)=F \circ J^{-T}(m, n)$ lives on a region where $\mathfrak{F}(m, n) \neq$ 0 . The definition of $F$ implies that $J^{-T}(m, n) \in[0,1] \times[0,1]$ (or equivalently $\left.(m, n) \in J^{T}([0,1] \times[0,1])\right)$ if and only if $\mathfrak{F}(m, n) \neq 0$. Thus the support of $\mathfrak{F}$ is the parallelogram $H$ with vertices $(0,0),(1,0),(c+1, d)$, and $(c, d)$. The reason is as follows:

$$
J=\left(\begin{array}{ll}
c & d \\
1 & 0
\end{array}\right)
$$

then we have

$$
J^{T}\binom{1}{0}=\left(\begin{array}{ll}
c & 1 \\
d & 0
\end{array}\right)\binom{1}{0}=\binom{c}{d}
$$

and similarly, $J^{T}\binom{0}{1}=\binom{1}{0}, J^{T}\binom{1}{1}=\binom{c+1}{d}$.
Hence,


Figure 2.5: The parallelogram where $\mathfrak{F}$ lives

$$
\begin{align*}
\operatorname{supp}(F)=H & =\left\{(m, n) \in \mathbb{R}^{2} \mid(m, n) \in J^{T}([0,1] \times[0,1])\right\}  \tag{2.29}\\
& =\left\{(c m+n, d m) \in \mathbb{R}^{2} \mid(m, n) \in[0,1] \times[0,1]\right\}  \tag{2.30}\\
& =\left\{\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2} \left\lvert\, x_{1} \in\left[\frac{c x_{2}}{d}, \frac{c x_{2}}{d}+1\right]\right., x_{2} \in[0, d]\right\} . \tag{2.31}
\end{align*}
$$

Our aim here is to analyze $\sum_{(m, n) \in \mathbb{Z}^{2}}\left(\mathfrak{F} * G_{\epsilon}\right)(m, n)$. We will introduce a lemma [LR11] which incorporates a class of functions defined on a polytope $\mathcal{P}$ and the solid angle defined on $\mathcal{P}$.

Lemma 2.4. If $f$ is a continuous function on the polytope $\mathcal{P}$ in $\mathbb{R}^{d}$, and is zero outside $\mathcal{P}$, then for all $\mathbf{x} \in \mathbb{R}^{d}$,

$$
\lim _{\epsilon \rightarrow 0^{+}}\left(f * G_{\epsilon}\right)(\mathbf{x})=f(\mathbf{x}) \omega_{\mathcal{P}}(\mathbf{x}) .
$$

Here $\omega_{\mathcal{P}}(\mathbf{x})$ denotes the solid angle of $\mathcal{P}$ at $\mathbf{x}$.

Proof. We can compute $\left(f * G_{\epsilon}\right)$ according to the definition of convolution in Fourier analysis. We remind our reader of the definition of Gaussian functions given by Definition 2.14:

$$
\begin{gather*}
G_{\epsilon}(\mathbf{x})=\frac{1}{\epsilon^{d / 2}} e^{-\frac{\pi}{\epsilon}\|\mathbf{x}\|^{2}} \\
\left(f * G_{\epsilon}\right)(\mathbf{x})=\int_{\mathbb{R}^{d}} f(\mathbf{y}) G_{\epsilon}(\mathbf{x}-\mathbf{y}) d \mathbf{y}  \tag{2.32}\\
=\int_{\mathcal{P}} f(\mathbf{y}) G_{\epsilon}(\mathbf{x}-\mathbf{y}) d \mathbf{y}  \tag{2.33}\\
=\int_{\mathcal{P}} f(\mathbf{y}) G_{\epsilon}(\mathbf{y}-\mathbf{x}) d \mathbf{y}  \tag{2.34}\\
\operatorname{let} \mathbf{u}=\mathbf{y}-\mathbf{x} \\
=\int_{T_{-\mathbf{x}}(\mathcal{P})} f(\mathbf{u}+\mathbf{x}) G_{\epsilon}(\mathbf{u}) d \mathbf{u}  \tag{2.35}\\
=  \tag{2.36}\\
\int_{\frac{1}{\epsilon} T_{-\mathbf{x}}(\mathcal{P})} f(\mathbf{x}+\mathbf{v} \sqrt{\epsilon}) G_{1}(\mathbf{v}) d \mathbf{v}
\end{gather*}
$$

Here $T_{-\mathbf{x}}(\mathcal{P})$ denotes the translation of $\mathcal{P}$ by the vector $-\mathbf{x}$. Since the polytope $\mathcal{P}$ is closed and bounded, the function $f$ is uniformly continuous. Thus, when $\epsilon$ approaches 0 , the limit of $f * G_{\epsilon}(\mathbf{x})$ is:

$$
\begin{aligned}
\lim _{\epsilon \rightarrow 0^{+}}\left(f * G_{\epsilon}\right)(\mathbf{x}) & =f(\mathbf{x}) \int_{\mathcal{K}} G_{1}(\mathbf{v}) d \mathbf{v} \\
& =f(\mathbf{x}) \omega_{\mathcal{K}_{\mathbf{x}}}(0) \\
& =f(\mathbf{x}) \omega_{\mathcal{P}}(\mathbf{x})
\end{aligned}
$$

where $\mathcal{K}_{\mathbf{x}}$ is the tangent cone of $\mathcal{P}$ at the vertex $\mathbf{x}$.

We can interpret $\lim _{\epsilon \rightarrow 0^{+}} \sum_{(m, n) \in \mathbb{Z}^{2}} \widehat{\mathfrak{F}}(m, n) \widehat{G_{\epsilon}}(m, n)$ from this point of view.

$$
\begin{align*}
\lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
m(c m+d n) \neq 0}} \widehat{\mathfrak{F}}(m, n) \widehat{G_{\epsilon}}(m, n) & =\lim _{\epsilon \rightarrow 0^{+}} \sum_{(m, n) \in \mathbb{Z}^{2}} \widehat{\mathfrak{F}}(m, n) \widehat{G_{\epsilon}}(m, n),  \tag{2.37}\\
& =\lim _{\epsilon \rightarrow 0^{+}} \sum_{(m, n) \in \mathbb{Z}^{2}} \widehat{\mathfrak{F} * G_{\epsilon}}(m, n),  \tag{2.38}\\
& =\lim _{\epsilon \rightarrow 0^{+}} \sum_{(m, n) \in \mathbb{Z}^{2}}\left(\mathfrak{F} * G_{\epsilon}\right)(m, n),  \tag{2.39}\\
& =\sum_{\left(x_{1}, x_{2}\right) \in \mathbb{Z}^{2} \cap H} \mathfrak{F}\left(x_{1}, x_{2}\right) \omega_{H}\left(x_{1}, x_{2}\right),  \tag{2.40}\\
& =\sum_{x_{2}=0}^{d}\left(\sum_{x_{1} \in \mathbb{Z} \cap\left[\frac{c x_{2}}{d}, \frac{c x_{2}}{d}+1\right]} \mathfrak{F}\left(x_{1}, x_{2}\right) \omega_{H}\left(x_{1}, x_{2}\right)\right) \tag{2.41}
\end{align*}
$$

Here (2.39) is true due to the Poisson Summation Formula, and (2.40) holds because of Lemma 2.4.

Let the inner sum in (2.41) be:

$$
f\left(x_{2}\right)=\sum_{x_{1} \in \mathbb{Z} \cap\left[\frac{c x_{2}}{d}, \frac{c x_{2}}{d}+1\right]} \mathfrak{F}\left(x_{1}, x_{2}\right) \omega_{H}\left(x_{1}, x_{2}\right) .
$$

$\diamond$ We consider the case when $x_{2} \neq 0$ and $x_{2} \neq d$ first.

1. If $\frac{c x_{2}}{d}$ is an integer, then

$$
\begin{aligned}
f\left(x_{2}\right) & =\frac{1}{2}\left(\mathfrak{F}\left(\frac{c x_{2}}{d}, x_{2}\right)+\mathfrak{F}\left(\frac{c x_{2}}{d}+1, x_{2}\right)\right), \\
& =\frac{1}{2}\left(F\left(\frac{x_{2}}{d}, 0\right)+F\left(\frac{x_{2}}{d}, 1\right)\right), \\
& =\frac{1}{2}\left(B_{1}\left(\frac{x_{2}}{d}\right)\left(-\frac{1}{2}\right)+B_{1}\left(\frac{x_{2}}{d}\right)\left(\frac{1}{2}\right)\right), \\
& =0=\left(\left(\frac{c x_{2}}{d}\right)\right)=-\left(\left(\frac{x_{2}}{d}\right)\right)\left(\left(\frac{c x_{2}}{d}\right)\right) .
\end{aligned}
$$

2. If $\frac{c x_{2}}{d}$ is not an integer, then

$$
\begin{aligned}
f\left(x_{2}\right) & =1 \cdot \mathfrak{F}\left(\left\lfloor\frac{c x_{2}}{d}\right\rfloor+1, x_{2}\right) \\
& =F\left(\frac{x_{2}}{d},\left\lfloor\frac{c x_{2}}{d}\right\rfloor+1-\frac{c x_{2}}{d}\right) \\
& =\left(\left(\frac{x_{2}}{d}\right)\right)\left(\left(1-\left\{\frac{c x_{2}}{d}\right\}\right)\right) \\
& =-\left(\left(\frac{x_{2}}{d}\right)\right)\left(\left(\frac{c x_{2}}{d}\right)\right) .
\end{aligned}
$$

Here we use the following identities of Sawtooth functions :

$$
((-x))=((1-x))=-((x))
$$

and

$$
((\{x\}))=((x))
$$

for all $x \notin \mathbb{Z}$.


Figure 2.6: Sawtooth function

In conclusion, when $x_{2} \neq 0$ and $x_{2} \neq d$, we have:

$$
\sum_{x_{1} \in \mathbb{Z} \cap\left[\frac{c x_{2}}{d}, \frac{c x_{2}}{d}+1\right]} \mathfrak{F}\left(x_{1}, x_{2}\right) \omega_{H}\left(x_{1}, x_{2}\right)=-\left(\left(\frac{x_{2}}{d}\right)\right)\left(\left(\frac{c x_{2}}{d}\right)\right) .
$$

$\diamond$ When $x_{2}=0$. Let $\theta=\omega_{H}(0,0)$, then from Figure 2.5 we have $\omega_{H}(0,0)=$ $\omega_{H}(c+1, d)=\frac{\arctan (d / c)}{2 \pi}$. And we have $\omega_{H}(1,0)=\omega_{H}(c, d)=\frac{1}{2}-\theta$.

When $x_{2}=0$, we have:

$$
\begin{align*}
f\left(x_{2}\right)=f(0) & =\mathfrak{F}(0,0) \cdot \omega_{H}(0,0)+\mathfrak{F}(1,0) \cdot \omega_{H}(1,0),  \tag{2.42}\\
& =F(0,0) \theta+F(0,1)\left(\frac{1}{2}-\theta\right),  \tag{2.43}\\
& =\frac{1}{4} \theta+\left(-\frac{1}{4}\right)\left(\frac{1}{2}-\theta\right),  \tag{2.44}\\
& =\frac{1}{2} \theta-\frac{1}{8} \tag{2.45}
\end{align*}
$$

Here (2.43) is due to the definition of $\mathfrak{F}$ in (2.25).
$\diamond$ When $x_{2}=d$, we have:

$$
\begin{align*}
f\left(x_{2}\right)=f(d) & =\mathfrak{F}(c, d) \cdot \omega_{H}(c, d)+\mathfrak{F}(c+1, d) \cdot \omega_{H}(c+1, d),  \tag{2.46}\\
& =F(1,0)\left(\frac{1}{2}-\theta\right)+F(1,1) \theta,  \tag{2.47}\\
& =\left(-\frac{1}{4}\right)\left(\frac{1}{2}-\theta\right)+\frac{1}{4} \theta,  \tag{2.48}\\
& =\frac{1}{2} \theta-\frac{1}{8} . \tag{2.49}
\end{align*}
$$

Here (2.46) is due to the definition of $\mathfrak{F}$ in (2.25) as well.
Therefore (2.41) becomes:

$$
\begin{align*}
\lim _{\epsilon \rightarrow 0^{+}} & \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
m(c m+m) \neq 0}} \widehat{\mathfrak{F}}(m, n) \widehat{G_{\epsilon}}(m, n)  \tag{2.50}\\
& =-\sum_{x_{2}=1}^{d-1}\left(\left(\frac{x_{2}}{d}\right)\right)\left(\left(\frac{c x_{2}}{d}\right)\right)+\left(\frac{1}{2} \theta-\frac{1}{8}\right)+\left(\frac{1}{2} \theta-\frac{1}{8}\right)  \tag{2.51}\\
& =-\sum_{x_{2}=1}^{d-1}\left(\left(\frac{x_{2}}{d}\right)\right)\left(\left(\frac{c x_{2}}{d}\right)\right)+\theta-\frac{1}{4}  \tag{2.52}\\
& =-\sum_{x_{2}=1}^{d}\left(\left(\frac{x_{2}}{d}\right)\right)\left(\left(\frac{c x_{2}}{d}\right)\right)+\frac{\arctan (d / c)}{2 \pi}-\frac{1}{4}  \tag{2.53}\\
& =-s(c, d)+\frac{\arctan (d / c)}{2 \pi}-\frac{1}{4} \tag{2.54}
\end{align*}
$$

So (2.28) becomes:

$$
\begin{equation*}
-s(c, d)+\frac{\arctan (d / c)}{2 \pi}-\frac{1}{4}=-d f(c, d) \tag{2.55}
\end{equation*}
$$

or equivalently:

$$
\begin{equation*}
d f(c, d)=s(c, d)-\frac{\arctan (d / c)}{2 \pi}+\frac{1}{4} \tag{2.56}
\end{equation*}
$$

This result gives us the main theorem in this chapter.
Theorem 2.1. (Wang) Let $c$ and $d$ be positive integers. Define

$$
f(c, d)=\frac{1}{4 \pi^{2}} \lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)},
$$

then we have:

$$
d f(c, d)=s(c, d)-\frac{\arctan (d / c)}{2 \pi}+\frac{1}{4}
$$

Corollary 2.1. (Wang)

$$
\begin{equation*}
c f(d, c)=s(d, c)-\frac{\arctan (c / d)}{2 \pi}+\frac{1}{4} . \tag{2.57}
\end{equation*}
$$

This is a direct corollary of Theorem 2.1 by exchanging $c$ and $d$.
Adding up the two equations (2.56) and (2.57) gives us:

$$
\begin{equation*}
d f(c, d)+c f(d, c)=s(c, d)+s(d, c)-\frac{\arctan (d / c)}{2 \pi}-\frac{\arctan (c / d)}{2 \pi}+\frac{1}{2} \tag{2.58}
\end{equation*}
$$

Now let's take a look at a right triangle with leg lengths equal to $c$ and $d$.
From Figure 2.7, we have

$$
\begin{aligned}
& \theta_{1}=\frac{\arctan (c / d)}{2 \pi} \\
& \theta_{2}=\frac{\arctan (d / c)}{2 \pi}
\end{aligned}
$$

and

$$
\begin{equation*}
\theta_{1}+\theta_{2}=\frac{\arctan (c / d)}{2 \pi}+\frac{\arctan (d / c)}{2 \pi}=\frac{\pi}{2}=\frac{1}{4} \tag{2.59}
\end{equation*}
$$



Figure 2.7: A right triangle

The equality (2.59) together with (2.58) lead us to a very nice relationship between $f(c, d)$ and $s(c, d)$.

Corollary 2.2. (Wang) Let

$$
f(c, d)=\frac{1}{4 \pi^{2}} \lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)},
$$

for $(c, d) \in \mathbb{Z}_{>0}^{2}$. Then the following identity holds:

$$
d f(c, d)+c f(d, c)=s(c, d)+s(d, c)+\frac{1}{4}
$$

where $s(c, d)$ is the classical Dedekind sum.

This result follows directly from (2.58) and (2.59).
Corollary 2.3. (Wang) Let

$$
f(c, d)=\frac{1}{4 \pi^{2}} \lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)},
$$

for $(c, d) \in \mathbb{Z}_{>0}^{2}$. Then the following identity is true:

$$
d f(c, d)+c f(d, c)=\frac{1}{12}\left(\frac{c}{d}+\frac{d}{c}+\frac{1}{c d}\right) .
$$

Before we proceed to our next chapter, we would like to mention another representation of classical Dedekind sums [Ber76]. Let $(h, k)=1$. Then

$$
s(h, k)=\frac{1}{2 \pi} \sum_{\substack{n=1 \\ n \neq 0(\bmod k)}}^{\infty} \frac{\cot (\pi h n / k)}{n} .
$$

What is worth mentioning here is in [Ber76], the sum

$$
f(\theta)=\lim _{N \rightarrow \infty} \frac{1}{2 \pi}\left\{\sum_{\substack{0<m / h<R_{N} \\ m \neq 0(\bmod h)}} \frac{\cot (\pi m / \theta)}{m}+\sum_{\substack{0<n / k<R_{N} \\ n \neq 0(\bmod k)}} \frac{\cot (\pi n \theta)}{n}\right\}
$$

converges for any irrational $\theta$, but the individual two sums diverge. When $\theta$ is irrational, then

$$
f(\theta)=\frac{1}{12}\left(\theta+\frac{1}{\theta}\right)-\frac{1}{4}
$$

Recall the infinite sum defined in Definition 2.15:

$$
f(\epsilon, c, d)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)} .
$$

and its limit defined by Definition 2.16:

$$
f(c, d)=\frac{1}{4 \pi^{2}} \lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\ m(c m+d n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{m(c m+d n)} .
$$

We will extend the infinite sum $f(\epsilon, c, d)$ from integer parameters to real parameters in Chapter 3. We will look at a more general sum:

$$
\mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+\beta n)(\delta m+\gamma n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+\beta n)(\delta m+\gamma n)}
$$

and prove this extended infinite sum converges when both $\frac{\alpha}{\beta}$ and $\frac{\gamma}{\delta}$ are quadratic irrational numbers. Furthermore, we will prove that the $\operatorname{limit}^{\lim } \lim ^{+} \mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)$ exists when $\frac{\beta \gamma}{\alpha \delta}=-1$.


## Real cones and a certain infinite series

### 3.1 Fourier transform of real cones

Our main goal in this chapter is to discuss the convergence of an infinite series defined over real cones. In Chapter 2, we investigated an infinite sum defined over integral cones which is closely related to the classical Dedekind sums. Here we will extend the sum and proceed over real cones using the same argument, developed earlier in Chapter 2.


Figure 3.1: Decomposition of the first quadrant into three real cones

As in Chapter 2, we decompose the first quadrant $\mathcal{K}_{1 s t}$ into a disjoint union of three half-open real cones as shown in Figure 3.1: $\mathcal{K}_{y}, \mathcal{K}$, and $\mathcal{K}_{x}$. We have the following decomposition:

$$
\begin{equation*}
\mathcal{K}_{1 s t}=\mathcal{K}_{y} \bigcup \mathcal{K} \bigcup \mathcal{K}_{x} \tag{3.1}
\end{equation*}
$$

Denote the generators of $\mathcal{K}$ by $\omega_{1}=(\alpha, \beta) \in \mathbb{R}_{>0}^{2}$ and $\omega_{2}=(\gamma, \delta) \in \mathbb{R}_{>0}^{2}$. Then we have the generators of $K_{y}$ are $\omega_{\mathbf{y}}=(0,1)$ and $\omega_{\mathbf{1}}$. The generators of $\mathcal{K}_{x}$ are $\omega_{2}$ and $\omega_{\mathrm{x}}=(1,0)$.

From (3.1), we find a relationship involving the indicator functions of three real cones:

$$
\begin{equation*}
1_{\mathcal{K}_{1 s t}}(\mathbf{x})=1_{\mathcal{K}_{y}}(\mathbf{x})+1_{\mathcal{K}}(\mathbf{x})+1_{\mathcal{K}_{x}}(\mathbf{x}) \tag{3.2}
\end{equation*}
$$

where $\mathbf{x} \in \mathbb{R}^{2}$, and $1_{\mathcal{K}}$ is the indicator function defined in Definition 2.5. Next we multiply both sides of (3.2) by a factor $e^{2 \pi\langle\mathbf{x}, \mathbf{s}\rangle}$ :

$$
\begin{equation*}
1_{\mathcal{K}_{1 s t}}(\mathbf{x}) e^{2 \pi<\mathbf{x}, \mathbf{s}\rangle}=1_{\mathcal{K}_{y}}(\mathbf{x}) e^{2 \pi\langle\mathbf{x}, \mathbf{s}\rangle}+1_{\mathcal{K}}(\mathbf{x}) e^{2 \pi<\mathbf{x}, \mathbf{s}\rangle}+1_{\mathcal{K}_{x}}(\mathbf{x}) e^{2 \pi<\mathbf{x}, \mathbf{s}\rangle}, \tag{3.3}
\end{equation*}
$$

where $\mathbf{s} \in \mathbb{R}^{2}$. The reason for us to do so is to guarantee the convergence of Fourier transform (see Definition 2.6) of indicator functions.

Taking the convolution of $1_{\mathcal{K}}(\mathbf{x}) e^{2 \pi<\mathbf{x}, \mathbf{s}>}$ in (3.3) with Gaussian gives us:

$$
\begin{align*}
1_{\mathcal{K}_{1 s t}}(\mathbf{x}) e^{2 \pi\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x}) & =1_{\mathcal{K}_{y}}(\mathbf{x}) e^{2 \pi\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})+1_{\mathcal{K}}(\mathbf{x}) e^{2 \pi<\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x}) \\
& +1_{\mathcal{K}_{x}}(\mathbf{x}) e^{2 \pi\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x}) \tag{3.4}
\end{align*}
$$

where

$$
G_{\epsilon}(\mathbf{x})=\frac{1}{\epsilon} e^{-\frac{\pi}{\epsilon}\|\mathbf{x}\|^{2}}
$$

for $\mathbf{x} \in \mathbb{R}^{2}$, and its Fourier transform is:

$$
\widehat{G_{\epsilon}}(\mathbf{s})=e^{-\pi \epsilon\|\mathbf{s}\|^{2}}
$$

Then we take the Fourier transform of both sides in (3.4):

$$
\begin{aligned}
\mathcal{F}\left(1_{\mathcal{K}_{1 s t}}(\mathbf{x}) e^{2 \pi\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})\right) & =\mathcal{F}\left(1_{\mathcal{K}_{y}}(\mathbf{x}) e^{2 \pi<\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})\right)+\mathcal{F}\left(1_{\mathcal{K}}(\mathbf{x}) e^{2 \pi\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})\right) \\
& +\mathcal{F}\left(1_{\mathcal{K}_{x}}(\mathbf{x}) e^{2 \pi\langle\mathbf{x}, \mathbf{s}\rangle} * G_{\epsilon}(\mathbf{x})\right)
\end{aligned}
$$

We know that if $h(x)=(f * g)(x)$, then $\hat{h}(\xi)=\widehat{f}(\xi) \widehat{g}(\xi)$. Following this property of convolution in Fourier analysis, the above equation is equivalent to:

$$
\begin{align*}
1_{\mathcal{K}_{1 s t}} \widehat{(\mathbf{x}) e^{2 \pi<\mathbf{x}}, \mathbf{s}>} e^{-\pi \epsilon\|\mathbf{x}\|^{2}} & =1_{\mathcal{K}_{y}(\mathbf{x}) e^{2 \pi<\mathbf{x}, \mathbf{s}>}} \overline{e^{-\pi \epsilon\|\mathbf{x}\|^{2}}}+\overline{1_{\mathcal{K}}(\mathbf{x}) e^{2 \pi<\mathbf{x}, \mathbf{s}>}} e^{-\pi \epsilon\|\mathbf{x}\|^{2}} \\
& +1_{\mathcal{K}_{x}} \overline{(\mathbf{x}) e^{2 \pi<\mathbf{x}, \mathbf{s}>}} e^{-\pi \epsilon\|\mathbf{x}\|^{2}} \tag{3.5}
\end{align*}
$$

Theorem B. 3 in Appendix B allows us to simplify (3.5) to:

$$
\begin{align*}
\widehat{\mathcal{K}_{\mathcal{K}_{1 s t}}}(\mathbf{x}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{x}\|^{2}} & =\widehat{1_{\mathcal{K}_{y}}}(\mathbf{x}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{x}\|^{2}}+\widehat{\mathcal{1}_{\mathcal{K}}}(\mathbf{x}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{x}\|^{2}} \\
& +\widehat{1_{\mathcal{K}_{x}}}(\mathbf{x}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{x}\|^{2}} . \tag{3.6}
\end{align*}
$$

We will use $\mathbf{m}+i$ s instead of $\mathbf{x}+i$ from now on.
Now we sum both sides of (3.6) over the whole lattice $\mathbb{Z}^{2}$ :

$$
\begin{align*}
\sum_{\mathbf{m} \in \mathbb{Z}^{2}} \widehat{1_{\mathcal{K}_{1 s t}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}} & =\sum_{\mathbf{m} \in \mathbb{Z}^{2}} \widehat{1_{\mathcal{K}_{y}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}}+\sum_{\mathbf{m} \in \mathbb{Z}^{2}} \widehat{1_{\mathcal{K}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}} \\
& +\sum_{\mathbf{m} \in \mathbb{Z}^{2}} \widehat{1_{\mathcal{K}_{x}}}(\mathbf{m}+i \mathbf{s}) e^{-\pi \epsilon\|\mathbf{m}\|^{2}} \tag{3.7}
\end{align*}
$$

Let $\mathbf{z}=\left(z_{1}, z_{2}\right)=\mathbf{m}+i \mathbf{s}=\left(m+i s_{1}, n+i s_{2}\right) \in \mathbb{C}^{2}$. According to Example
2.2, Lemma 2.2, Example 2.3, and Example 2.4, we have:

$$
\begin{aligned}
\frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{z_{1} z_{2}} & =\frac{\alpha e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{\left\langle\omega_{y}, \mathbf{m}+i \mathbf{s}\right\rangle\left\langle\omega_{1}, \mathbf{m}+i \mathbf{s}\right\rangle}+\frac{\gamma e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{\left\langle\omega_{x}, \mathbf{m}+i \mathbf{s}\right\rangle\left\langle\omega_{2}, \mathbf{m}+i \mathbf{s}\right\rangle} \\
& +\frac{|\operatorname{det} M| e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{\left\langle\omega_{1}, \mathbf{m}+i \mathbf{s}\right\rangle\left\langle\omega_{2}, \mathbf{m}+i \mathbf{s}\right\rangle},
\end{aligned}
$$

which can also be written as:

$$
\begin{align*}
\sum_{(m, n) \in \mathbb{Z}^{2}} & \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{\left(m+i s_{1}\right)\left(n+i s_{2}\right)} \\
& =\alpha \sum_{(m, n) \in \mathbb{Z}^{2}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{\left(n+i s_{2}\right)\left(\alpha m+\beta n+i\left(\alpha s_{1}+\beta s_{2}\right)\right)} \\
& +\gamma \sum_{(m, n) \in \mathbb{Z}^{2}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{\left(m+i s_{1}\right)\left(\delta m+\gamma n+i\left(\delta s_{1}+\gamma s_{2}\right)\right)} \\
& +|\operatorname{det} M| \sum_{(m, n) \in \mathbb{Z}^{2}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{\left(\alpha m+\beta n+i\left(\alpha s_{1}+\beta s_{2}\right)\right)\left(\delta m+\gamma n+i\left(\delta s_{1}+\gamma s_{2}\right)\right)} . \tag{3.8}
\end{align*}
$$

Here the matrix $M$ is defined as :

$$
M=\binom{\omega_{1}^{T}}{\omega_{2}^{T}}=\left(\begin{array}{ll}
\alpha & \beta \\
\delta & \gamma
\end{array}\right)
$$

where $\omega_{1}=(\alpha, \beta)$ and $\omega_{2}=(\delta, \gamma)$ are the two generators of cone $\mathcal{K}$.
We will focus on an infinite sum by letting $\left(s_{1}, s_{2}\right)=(0,0)$ in the following sum appearing on the right hand side of (3.8):

$$
\sum_{(m, n) \in \mathbb{Z}^{2}} \frac{\widehat{G_{\epsilon}}(m, n)}{\left(\alpha m+\beta n+i\left(\alpha s_{1}+\beta s_{2}\right)\right)\left(\delta m+\gamma n+i\left(\delta s_{1}+\gamma s_{2}\right)\right)} .
$$

We will define a new infinite sum based on it excluding two lines: $\alpha m+\beta n=0$ and $\delta m+\gamma n=0$. As mentioned in Chapter 2, the motivation for us to focus on the following sum comes from [Ber76] and [GS03].

Definition 3.1. Define

$$
f(\epsilon, \alpha, \beta, \delta, \gamma)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+\beta n)(\delta m+\gamma n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+\beta n)(\delta m+\gamma n)},
$$

where $\epsilon \in \mathbb{R}_{>0}, \alpha, \beta, \delta, \gamma \in \mathbb{R}$.

We would like to mention the fact that $\mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)$ in Definition 3.1 is an extension of $f(\epsilon, c, d)$ in Definition 2.15. If we let $\alpha=1, \beta=0, \delta=c$, and $\gamma=d$, then $\mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)=\mathrm{f}(\epsilon, 1,0, c, d)=f(\epsilon, c, d)$.

We will answer the following two questions with respect to $\mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)$ :

1. Is the infinite sum $\mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)$ convergent?
2. What is the asymptotic behavior of $\mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)$ ? In other words, does the limit $\lim _{\epsilon \rightarrow 0^{+}} \mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)$ exist?

We first introduce some formal definitions [Grü03] related to polytopes and tangent cones.

Definition 3.2. Let $K$ be a convex subset of $\mathbb{R}^{d}$. A point $x \in K$ is an extreme point of $K$ provided given any $y, z \in K$, and any $0<\lambda<1$, such that $x=$ $\lambda y+(1-\lambda) z$, we must have $x=y=z$.

In other words, $x$ is an extreme point of $K$ if it does not belong to the relative interior of any segment contained in $K$.

Definition 3.3. Let $K$ be a convex subset of $\mathbb{R}^{d}$. A set $F \in K$ is a face of $K$ if either $F=\phi$ or $F=K$, or if there exists a supporting hyperplane $H$ of $K$ such that $F=K \cap H$.

Definition 3.4. A compact convex set $K \in \mathbb{R}^{d}$ is a polytope provided $K$ has finitely many extreme points.

Definition 3.5. We can attach a cone to each face $F$ of a polytope $\mathcal{P}$, namely its tangent cone, defined by

$$
\mathcal{K}_{F}=\{\mathbf{x}+\lambda(\mathbf{y}-\mathbf{x}): \mathbf{x} \in F, \mathbf{y} \in \mathcal{P}, \lambda \geq 0\} .
$$

The tangent cone of $F$ is the union of all rays that have a base point in F and point "towards" $\mathcal{P}$.

Recall that in Chapter 2, we defined a tangent cone of a polyhedron at some point $\mathbf{v}$ in Definition 2.8. Let $\mathcal{P} \in \mathbb{R}^{d}$ be a non-empty polyhedron and let $\mathbf{v} \in \mathcal{P}$ be a point. We define the tangent cone of $\mathcal{P}$ at $\mathbf{v}$ by:

$$
\operatorname{tcone}(\mathcal{P}, \mathbf{v})=\{\mathbf{v}+\mathbf{y}: \mathbf{v}+\epsilon \mathbf{y} \in \mathcal{P} \text { for some } \epsilon>0\}
$$

In other words, the tangent cone of $\mathcal{P}$ at $\mathbf{v}$ is the union of all rays that share the base point $\mathbf{v}$ and point "towards" $\mathcal{P}$, which is consistent with Definition 3.5.

Definition 3.6 ([DR10],[BR07]). Suppose $\mathcal{P} \in \mathbb{R}^{d}$ is a convex d-polytope. The solid angle $\omega_{\mathcal{P}}(\mathbf{x})$ of a point $\mathbf{x}$ with respect to $\mathcal{P}$ equals the proportion of a small ball centered at $\mathbf{x}$ that is contained in $\mathcal{P}$.

Consider the solid angle $\omega_{\mathcal{P}}(\mathbf{n})$ at a lattice point $\mathbf{n} \in \mathbb{Z}^{d}$ in a polytope $\mathcal{P}$ :

$$
\omega_{\mathcal{P}}(\mathbf{n})=\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{\epsilon^{d / 2}} \int_{\mathcal{P}} e^{-\frac{\pi}{\epsilon}\|\mathbf{t}-\mathbf{n}\|^{2}} d \mathbf{t}
$$

This definition arises from a Gaussian function centering at $\mathbf{n}$ with respect to $l^{2}$-norm that is normalized to have a total mass of 1 and then integrating to calculate the proportion of mass contained in $\mathcal{P}$.

For $\epsilon>0$, and $\mathbf{t} \in \mathbb{R}^{d}$, we define

$$
\phi_{\epsilon}(\mathbf{t})=\frac{1}{\epsilon^{d / 2}} e^{-\frac{\pi}{\epsilon}\|\mathbf{t}\|^{2}} .
$$

Therefore, $\omega_{\mathcal{P}}(\mathbf{n})$ becomes the convolution of the indicator function $1_{\mathcal{P}}$ and $\phi_{\epsilon}$ :

$$
\begin{align*}
\omega_{\mathcal{P}}(\mathbf{n}) & =\lim _{\epsilon \rightarrow 0^{+}} \int_{\mathcal{P}} \phi_{\epsilon}(\mathbf{t}-\mathbf{n}) d \mathbf{t}  \tag{3.9}\\
& =\lim _{\epsilon \rightarrow 0^{+}} \int_{\mathcal{P}} \phi_{\epsilon}(\mathbf{n}-\mathbf{t}) d \mathbf{t}  \tag{3.10}\\
& =\lim _{\epsilon \rightarrow 0^{+}} \int_{\mathbf{t} \in \mathbb{R}^{d}} 1_{\mathcal{P}}(\mathbf{t}) \phi_{\epsilon}(\mathbf{n}-\mathbf{t}) d \mathbf{t}  \tag{3.11}\\
& =\lim _{\epsilon \rightarrow 0^{+}}\left(1_{\mathcal{P}} * \phi_{\epsilon}\right)(\mathbf{n}) \tag{3.12}
\end{align*}
$$

Now we introduce a discrete volume $A_{\mathcal{P}}=\sum_{\mathbf{n} \in \mathbb{Z}^{d}} \omega_{\mathcal{P}}(\mathbf{n})$, which is an approximation of the volume of $\mathcal{P}$ [DeS].

Definition 3.7. For any pointed polyhedral cone $\mathcal{K}_{\mathbf{v}} \in \mathbb{R}^{d}$, we define a determinant of the cone as:

$$
\left|\operatorname{det} \mathcal{K}_{\mathbf{v}}\right|=\left|\operatorname{det}\left(\mathbf{w}_{1}(\mathbf{v}), \mathbf{w}_{2}(\mathbf{v}), \cdots, \mathbf{w}_{d}\right)\right|
$$

where $\mathbf{w}_{k}(\mathbf{v})(k=1,2, \cdots, d)$ denote the generators of $\mathcal{K}_{\mathbf{v}}$.

Suppose a polytope $\mathcal{P} \in \mathbb{R}^{d}$ has $N$ many vertices $\mathbf{v}_{1}, \mathbf{v}_{2}, \cdots, \mathbf{v}_{N}$. Denote the
tangent cone at each vertex $\mathbf{v}_{j}$ by $\mathcal{K}_{\mathbf{v}_{j}}(j=1,2, \cdots, N)$. Then we have:

$$
\begin{align*}
A_{\mathcal{P}}= & \sum_{\mathbf{n} \in \mathbb{Z}^{d}} \omega_{\mathcal{P}}(\mathbf{n}),  \tag{3.13}\\
= & \sum_{\mathbf{n} \in \mathbb{Z}^{d}} \lim _{\epsilon \rightarrow 0^{+}}\left(1_{\mathcal{P}} * \phi_{\epsilon}\right)(\mathbf{n}),  \tag{3.14}\\
= & \lim _{\epsilon \rightarrow 0^{+}} \sum_{\mathbf{n} \in \mathbb{Z}^{d}}\left(1_{\mathcal{P}} * \phi_{\epsilon}\right)(\mathbf{n}),  \tag{3.15}\\
= & \lim _{\epsilon \rightarrow 0^{+}} \sum_{\mathbf{n} \in \mathbb{Z}^{d}} \widehat{1_{\mathcal{P}} * \phi_{\epsilon}}(\mathbf{n}),  \tag{3.16}\\
= & \lim _{\epsilon \rightarrow 0^{+}} \sum_{\mathbf{n} \in \mathbb{Z}^{d}} \widehat{1_{\mathcal{P}}}(\mathbf{n}) \widehat{\phi_{\epsilon}}(\mathbf{n}),  \tag{3.17}\\
= & \lim _{\epsilon \rightarrow 0^{+}} \sum_{\mathbf{n} \in \mathbb{Z}^{d} \backslash V}\left(\frac{\left|\operatorname{det} \mathcal{K}_{\mathbf{v}_{\mathbf{1}}}\right| e^{\left\langle\mathbf{v}_{\mathbf{1}}, \mathbf{n}\right\rangle}}{\prod_{k=1}^{d}\left\langle\mathbf{w}_{\mathbf{k}}\left(v_{1}\right), \mathbf{n}\right\rangle}+\cdots\right. \\
& \quad+\lim _{\epsilon \rightarrow 0^{+}} \sum_{\mathbf{n} \in V} \hat{1}_{\mathcal{P}}(\mathbf{n}) e^{-\epsilon \pi\|\mathbf{n}\|^{2}} . \tag{3.18}
\end{align*}
$$

where $\mathbf{w}_{\mathbf{k}}\left(\mathbf{v}_{j}\right)(k=1,2, \cdots, d)$ denote the generators of the tangent cone $\mathcal{K}_{v_{j}}$, and $V$ denotes the set of vectors which are orthogonal to any $\mathbf{w}_{\mathbf{k}}\left(\mathbf{v}_{j}\right), j=1,2, \cdots, N$. In (3.15) we exchanged summation and limit. The reason is as follows: we know that $1_{\mathcal{P}} * \phi_{\epsilon}(\mathbf{n}) \in L_{1}(\mathbb{R})$, therefore dominated convergence theorem guarantees us that we can interchange the sum and limit ${ }^{1} ;(3.16)$ is due to Poisson-summation formula [Osg07]; (3.17) is due to a well-known identity: the Fourier transform of a convolution of two functions is equal to the product of the Fourier transform of both functions; the last equality (3.18) is due to Brion's theorem (the proof of Brion's theorem can be found in [BR07] or Theorem A. 3 in Attachment A). Note that when $\mathbf{v}_{j}$ is the origin, then the term $e^{\left\langle\mathbf{v}_{j}, \mathbf{n}\right\rangle}$ in (3.18) is equal to 1.

Example 3.1. To illustrate $A_{\mathcal{P}}$, we look at an example where $\mathcal{P}$ is a triangle in $\mathbb{R}^{2}$ with vertices $v_{1}=(0,0), v_{2}=(0,1)$ and $v_{3}=(\sqrt{2}, 0)$.

[^2]

Figure 3.2: The triangle $\mathcal{P}$ and its real tangent cones

We have

$$
\begin{gathered}
\left|\operatorname{det} \mathcal{K}_{v_{1}}\right|=\left|\operatorname{det}\left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right)\right|=1, \\
\left|\operatorname{det} \mathcal{K}_{v_{2}}\right|=\left|\operatorname{det}\left(\begin{array}{cc}
0 & \sqrt{2} \\
-1 & -1
\end{array}\right)\right|=\sqrt{2}, \\
\left|\operatorname{det} \mathcal{K}_{v_{3}}\right|=\left|\operatorname{det}\left(\begin{array}{cc}
-\sqrt{2} & -1 \\
1 & 0
\end{array}\right)\right|=1 .
\end{gathered}
$$

For the cones $\mathcal{K}_{v_{1}}, \mathcal{K}_{v_{2}}$ and $\mathcal{K}_{v_{3}}$, the denominator in (3.18) becomes:

$$
\begin{gathered}
\prod_{k=1}^{d}\left\langle\mathbf{w}_{\mathbf{k}}\left(v_{1}\right), \mathbf{n}\right\rangle=n_{1} n_{2} \\
\prod_{k=1}^{d}\left\langle\mathbf{w}_{\mathbf{k}}\left(v_{2}\right), \mathbf{n}\right\rangle=\left(-n_{2}\right)\left(\sqrt{2} n_{1}-n_{2}\right)
\end{gathered}
$$

and

$$
\prod_{k=1}^{d}\left\langle\mathbf{w}_{\mathbf{k}}\left(v_{3}\right), \mathbf{n}\right\rangle=\left(-n_{1}\right)\left(-\sqrt{2} n_{1}+n_{2}\right)
$$

Remark 3.1. Note that for any cone in $\mathbb{R}^{2}$ with generators $\omega_{1}=(a, b)$ and $\omega_{2}=(c, d)$, as long as $b, c \neq 0$, we can always write $\prod_{k=1}^{2}\left\langle\mathbf{w}_{k}\left(v_{i}\right), \mathbf{n}\right\rangle$ in the form of $C(\alpha m+n)(m+\beta n)$ where $C \in \mathbb{R}$ is some constant, and $\alpha, \beta \in \mathbb{R}$.

There is an infinite series arising from (3.18):

$$
\mathrm{f}(\epsilon, \alpha, \beta)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+n)(m+\beta n)},
$$

assuming the vertex of the cone is identical with the origin. This infinite series is a simplified version of the infinite series defined in Definition 3.1. From now on we focus on the convergence of this series and the existance of the limit:

$$
\lim _{\epsilon \rightarrow 0^{+}} f(\epsilon, \alpha, \beta) .
$$

Theorem 3.2. (Wang)
The infinite series

$$
f(\epsilon, \alpha, \beta)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+n)(m+\beta n)}
$$

converges absolutely for each fixed $\epsilon>0$, where both $\alpha$ and $\beta$ are quadratic irrationals.

### 3.2 Proof of the convergence property of an infinite sum defined on real cones, and Diophantine analysis

Before we proceed to the proof of Theorem 3.2, we recall the definition of quadratic irrationals first, and then give another definition describing how near a point and a straight line is. We will also introduce Liouville's theorem as it will become the main tool in the proof of Theorem 3.2.

Definition 3.8. Suppose $\alpha \in \mathbb{R}$ is an irrational number, it is called a quadratic irrational if it satisfies a quadratic equation

$$
a \alpha^{2}+b \alpha+c=0
$$

with $a, b, c \in \mathbb{Z}$ and $a \neq 0$.

Definition 3.9. Given two straight lines: $l_{1}: \alpha x+y=0 \in \mathbb{R}^{2}$ and $l_{2}: x+\beta y=$ $0 \in \mathbb{R}^{2}$, a point $P=(m, n) \in \mathbb{Z}^{2}$ is said to be near to one of them if the distance between $P$ and $l_{i}(i=1,2)$ is less than $\epsilon^{*}$ with

$$
0<\epsilon^{*}<\min \left\{\left|\frac{1-\alpha \beta}{\beta \sqrt{1+\alpha^{2}}}\right|,\left|\frac{1-\alpha \beta}{\alpha \sqrt{1+\beta^{2}}}\right|\right\} .
$$

If the two given lines are of more general form, say $l_{1}: \alpha m+\beta n=0$, and $l_{2}: \gamma m+\delta n=0$, we can always use elementary computation to transform them into the form mentioned above as long as $\beta, \gamma \neq 0$.

Lemma 3.1. If $\alpha$ is an irrational number which is the root of a polynomial $f$ of degree $n>0$, with integer coefficients, then there exists a real number $A>0$ such that, for all integers $p$ and $q$ with $q>0$,

$$
\left|\alpha-\frac{p}{q}\right|>\frac{A}{q^{n}} .
$$

This lemma is also known as Liouville's theorem (on diophantine approximation). The proof of this lemma can be found in [Sch80] (also see [Oxt80]).

Now we come to prove our main result Theorem 3.2. Without loss of generality, we assume $\alpha, \beta>0$.

Proof. In order to prove the convergence of this series, we classify the points in $\mathbb{Z}^{2}$ into 3 classes:

1. Points near to the straight line $l_{1}: \alpha m+n=0$;
2. Points near to the straight line $l_{2}: m+\beta n=0$;
3. All other lattice points except the origin.

We can prove that

$$
\mathrm{f}(\epsilon, \alpha, \beta)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+n)(m+\beta n)}
$$

converges for points "away" from $l_{1}$ and $l_{2}$ without too much trouble. Then we only need to take care of those points near to $l_{1}$ and $l_{2}$.
$\diamond$ Case 1. When points $P=(m, n)$ are near to $l_{1}: \alpha m+n=0$, we have

$$
\operatorname{dist}\left(P, l_{1}\right)=\frac{|\alpha m+n|}{\sqrt{\alpha^{2}+1}}<\epsilon^{*}
$$

i.e. $|\alpha m+n|<\sqrt{1+\alpha^{2}} \epsilon^{*}$, which is equivalent to $-\sqrt{1+\alpha^{2}} \epsilon^{*}<\alpha m+n<$ $\sqrt{1+\alpha^{2}} \epsilon^{*}$. Then we'll have the following inequalities:

$$
\begin{align*}
-\sqrt{1+\alpha^{2}} \epsilon^{*}-\alpha m<n & <\sqrt{1+\alpha^{2}} \epsilon^{*}-\alpha m \\
-\beta \sqrt{1+\alpha^{2}} \epsilon^{*}-\alpha \beta m<\beta n & <\beta \sqrt{1+\alpha^{2}} \epsilon^{*}-\alpha \beta m \\
-\beta \sqrt{1+\alpha^{2}} \epsilon^{*}-\alpha \beta m+m<\beta n+m & <\beta \sqrt{1+\alpha^{2}} \epsilon^{*}-\alpha \beta m+m \\
\text { i.e. }|\beta n+m-(1-\alpha \beta) m| & <\beta \sqrt{1+\alpha^{2}} \epsilon^{*} \tag{3.19}
\end{align*}
$$

Meanwhile we have

$$
\begin{equation*}
||\beta n+m|-|(1-\alpha \beta) m|| \leq|\beta n+m-(1-\alpha \beta) m| \tag{3.20}
\end{equation*}
$$

(3.19) and (3.20) give us:

$$
\begin{align*}
& \quad||\beta n+m|-|(1-\alpha \beta) m||<\beta \sqrt{1+\alpha^{2}} \epsilon^{*} \\
& -\beta \sqrt{1+\alpha^{2}} \epsilon^{*}<|\beta n+m|-|(1-\alpha \beta) m|<\beta \sqrt{1+\alpha^{2}} \epsilon^{*} \\
& -\beta \sqrt{1+\alpha^{2}} \epsilon^{*}+|(1-\alpha \beta) m|<|\beta n+m|<\beta \sqrt{1+\alpha^{2}} \epsilon^{*}+|(1-\alpha \beta) m| . \tag{3.21}
\end{align*}
$$

Since $\beta \epsilon^{*}>0$, we have $\beta \epsilon^{*}<\beta \epsilon^{*}|m|$ for all $m \in \mathbb{Z}_{\neq 0}$, then from (3.21) we get:

$$
\begin{aligned}
|\beta n+m| & >-\beta \sqrt{1+\alpha^{2}} \epsilon^{*}+|(1-\alpha \beta) m| \\
& >-\beta \sqrt{1+\alpha^{2}} \epsilon^{*}|m|+|1-\alpha \beta||m|=\left(|1-\alpha \beta|-\beta \sqrt{1+\alpha^{2}} \epsilon^{*}\right)|m| .
\end{aligned}
$$

Therefore,

$$
\begin{align*}
|(\alpha m+n)(m+\beta n)| & =|\alpha m+n||m+\beta n| \\
& =|m|\left|\alpha+\frac{n}{m}\right||m+\beta n| \\
& \geq|m| \frac{A}{m^{2}}\left(|1-\alpha \beta|-\beta \sqrt{1+\alpha^{2}} \epsilon^{*}\right)|m|  \tag{3.22}\\
& =A\left(|1-\alpha \beta|-\beta \sqrt{1+\alpha^{2}} \epsilon^{*}\right) \tag{3.23}
\end{align*}
$$

Note: Since $0<\epsilon^{*}<\frac{|1-\alpha \beta|}{\beta \sqrt{1+\alpha^{2}}}$, we can always make sure that $|1-\alpha \beta|-$ $\beta \sqrt{1+\alpha^{2}} \epsilon^{*}$ is positive. In (3.22), $A$ is some positive constant. The reason why (3.22) holds is due to the fact that $\alpha$ is a quadratic irrational. It follows from Definition 3.8 that $\alpha$ is the root of some polynomial $f(x)=$ $a x^{2}+b x+c$ where $a, b, c \in \mathbb{Z}$ and $a \neq 0$. By Lemma 3.1, or Liouville's theorem (on Diophantine approximation), we know that

$$
\left|\alpha-\frac{n}{m}\right|>\frac{A}{m^{2}},
$$

where $A$ is some positive constant. Thus when $\epsilon>0$, (3.23) implies

$$
\frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{|(\alpha m+n)(m+\beta n)|} \leq \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{A\left(|1-\alpha \beta|-\beta \sqrt{1+\alpha^{2}} \epsilon^{*}\right)}
$$

where $\epsilon^{*}$ is defined in Definition 3.9.

As a result,

$$
\begin{aligned}
|\mathrm{f}(\epsilon, \alpha, \beta)|= & \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{|(\alpha m+n)(m+\beta n)|}, \\
\leq & \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{A\left(|1-\alpha \beta|-\beta \sqrt{1+\alpha^{2}} \epsilon^{*}\right)} \\
= & \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
(\alpha m+n)(m+\beta n) \neq 0}} g(m, n, \epsilon, A, \alpha, \beta),
\end{aligned}
$$

where we define $g(m, n, \epsilon, A, \alpha, \beta)=\frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{A\left(|1-\alpha \beta|-\beta \sqrt{1+\alpha^{2}} \epsilon^{*}\right)}$. Without much effort we can see that $\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} g(m, n, \epsilon, A, \alpha, \beta)$ converges for fixed $\epsilon>0, A>0$ and $\alpha, \beta$ both quadratic irrationals. We conclude from the comparison test that for any $\epsilon>0$, the infinite series

$$
\mathrm{f}(\epsilon, \alpha, \beta)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+n)(m+\beta n)}
$$

converges absolutely when points $P=(m, n)$ are near to $l_{1}$.
$\diamond$ Case 2. When points $P=(m, n)$ are near to $l_{2}: m+\beta n=0$, we have

$$
\operatorname{dist}\left(P, l_{2}\right)=\frac{|m+\beta n|}{\sqrt{\beta^{2}+1}}<\epsilon^{*}
$$

i.e. $|m+\beta n|<\sqrt{\beta^{2}+1} \epsilon^{*}$, which is equivalent to $-\sqrt{\beta^{2}+1} \epsilon^{*}<m+\beta n<$ $\sqrt{\beta^{2}+1} \epsilon^{*}$. Therefore we have the following inequalities:

$$
\begin{align*}
&-\sqrt{\beta^{2}+1} \epsilon^{*}-\beta n<m<\sqrt{\beta^{2}+1} \epsilon^{*}-\beta n \\
&-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}-\alpha \beta n<\alpha m<\alpha \sqrt{\beta^{2}+1} \epsilon^{*}-\alpha \beta n \\
&-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}+n-\alpha \beta n<\alpha m+n<\alpha \sqrt{\beta^{2}+1} \epsilon^{*}+n-\alpha \beta n \\
& \text { i.e. }|\alpha m+n-(1-\alpha \beta) n|<\alpha \sqrt{\beta^{2}+1} \epsilon^{*} . \tag{3.24}
\end{align*}
$$

Meanwhile we have:

$$
\begin{equation*}
||\alpha m+n|-|(1-\alpha \beta) n|| \leq|\alpha m+n-(1-\alpha \beta) n| \tag{3.25}
\end{equation*}
$$

It follows from (3.24) and (3.25) that

$$
\begin{array}{r}
||\alpha m+n|-|(1-\alpha \beta) n||<\alpha \sqrt{\beta^{2}+1} \epsilon^{*}, \\
-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}<|\alpha m+n|-|(1-\alpha \beta) n|<\alpha \sqrt{\beta^{2}+1} \epsilon^{*}, \\
-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}+|(1-\alpha \beta) n|<|\alpha m+n|<\alpha \sqrt{\beta^{2}+1} \epsilon^{*}+|(1-\alpha \beta) n| . \tag{3.26}
\end{array}
$$

Since $\alpha \epsilon^{*}>0$, we have $\alpha \epsilon^{*}<\alpha \epsilon^{*}|n|$ for all $n \in \mathbb{Z}_{\neq 0}$, then from (3.26) we get

$$
\begin{align*}
|\alpha m+n| & >-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}+|1-\alpha \beta||n|, \\
& >-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}|n|+|1-\alpha \beta||n|, \\
& =\left(|1-\alpha \beta|-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}\right)|n| . \tag{3.27}
\end{align*}
$$

Therefore,

$$
\begin{align*}
|(\alpha m+n)(m+\beta n)| & =|\alpha m+n||m+\beta n|, \\
& =|\alpha m+n||n|\left|\beta+\frac{m}{n}\right|, \\
& \geq\left(|1-\alpha \beta|-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}\right)|n||n| \frac{B}{n^{2}},  \tag{3.28}\\
& =B\left(|1-\alpha \beta|-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}\right), \tag{3.29}
\end{align*}
$$

where $B$ is some positive constant.
Since $0<\epsilon^{*}<\frac{|1-\alpha \beta|}{\alpha \sqrt{1+\beta^{2}}}$, we always have $|1-\alpha \beta|-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}>0$. The reason why (3.28) holds is due to the fact that $\beta$ is a quadratic irrational. It follows from Definition 3.8 that $\beta$ is the root of some polynomial $g(x)=$ $a^{\prime} x^{2}+b^{\prime} x+c^{\prime}$ where $a^{\prime}, b^{\prime}, c^{\prime} \in \mathbb{Z}$ and $a^{\prime} \neq 0$. Lemma 3.1 or Liouville's theorem (on diophantine approximation) gives us that

$$
\left|\beta-\frac{m}{n}\right|>\frac{B}{n^{2}},
$$

where $B>0$.

Then when $\epsilon>0,(3.29)$ implies:

$$
\begin{align*}
|f(\epsilon, \alpha, \beta)|= & \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{|(\alpha m+n)(m+\beta n)|}, \\
\leq & \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{B\left(|1-\alpha \beta|-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}\right)},  \tag{3.30}\\
= & g_{\substack{(m, n) \in \mathbb{Z}^{2} \\
(\alpha m+n)(m+\beta n) \neq 0}} g^{\prime}(m, n, \epsilon, B, \alpha, \beta), \tag{3.31}
\end{align*}
$$

where $\epsilon^{*}$ is defined as in Definition 3.9, and we define $g^{\prime}(m, n, \epsilon, B, \alpha, \beta)=$ $\frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{B\left(|1-\alpha \beta|-\alpha \sqrt{\beta^{2}+1} \epsilon^{*}\right)}$. Since

$$
\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} g^{\prime}(m, n, \epsilon, B, \alpha, \beta)
$$

converges, we conclude from the comparison test that for any $\epsilon>0$, and $\alpha, \beta$ both quadratic irrationals, $\mathrm{f}(\epsilon, \alpha, \beta)$ converges absolutely when points $P=(m, n)$ are near to $l_{2}$.
$\diamond$ Case 3. When lattice points $(m, n)$ are not near to $l_{1}$ and $l_{2}$, according to
Definition 3.9, we have $\left\{\begin{array}{l}|\alpha m+n| \geq \epsilon^{*}, \\ |m+\beta n| \geq \epsilon^{*} .\end{array}\right.$
As a result,

$$
\begin{align*}
|f(\epsilon, \alpha, \beta)|= & \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{|(\alpha m+n)(m+\beta n)|},  \tag{3.32}\\
\leq & \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{\epsilon^{* 2}},  \tag{3.33}\\
= & \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\
(\alpha m+n)(m+\beta n) \neq 0}} h(m, n, \epsilon), \tag{3.34}
\end{align*}
$$

where $\epsilon>0, \epsilon^{*}$ is defined in Definition 3.9, and $h(m, n, \epsilon)$ is defined as $\frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{\epsilon^{* 2}}$. It is a fact that $\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} h(m, n, \epsilon)$ converges, hence by the comparison test we have for any $\epsilon>0$,

$$
\mathrm{f}(\epsilon, \alpha, \beta)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+n)(m+\beta n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+n)(m+\beta n)}
$$

is absolutely convergent under this situation. This completes our proof.

Corollary 3.1. (Wang) Suppose

$$
M=\left(\begin{array}{cc}
\alpha & \beta \\
\delta & \gamma
\end{array}\right) \in G L_{2}(\mathbb{R})
$$

Then the infinite series

$$
f(\epsilon, \alpha, \beta, \delta, \gamma)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+\beta n)(\delta m+\gamma n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+\beta n)(\delta m+\gamma n)}
$$

converges absolutely for each fixed $\epsilon>0$, where both $\alpha / \beta$ and $\gamma / \delta$ are quadratic irrationals $(\beta, \delta \neq 0)$.

The proof of Corollary 3.1 follows from the proof of Theorem 3.2, but we need to define the range of $\epsilon$ more carefully. We can rewrite the denominator in the summand of $\mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)$ as:

$$
(\alpha m+\beta n)(\delta m+\gamma n)=\beta \delta\left(\frac{\alpha}{\beta} m+n\right)\left(m+\frac{\gamma}{\delta} n\right)
$$

Therefore, in this case, we need to define $0<\epsilon<\left|1-\frac{\alpha \gamma}{\beta \delta}\right|=\frac{|\operatorname{det}(M)|}{\beta \delta}$ in the proof.

### 3.3 Asymptotic analysis of the infinite series defined over real cones

We will prove the limit of the infinite sum:

$$
\mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)=\sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+\beta n)(\delta m+\gamma n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+\beta n)(\delta m+\gamma n)},
$$

where $\epsilon \in \mathbb{R}_{>0}$ exists under a certain condition .
Theorem 3.3. (Wang) If $\alpha \delta+\beta \gamma=0$, and $\alpha, \gamma \neq 0$, then the limit of the infinite series $\lim _{\epsilon \rightarrow 0^{+}} f(\epsilon, \alpha, \beta, \delta, \gamma)$ exists.

Define $s(\alpha, \beta, \delta, \gamma, m, n)=\frac{1}{(\alpha m+\beta n)(\delta m+\gamma n)}$. We will find the partial fraction of $s$ first.

$$
\begin{aligned}
s(\alpha, \beta, \delta, \gamma, m, n) & =\frac{1}{(\alpha m+\beta n)(\delta m+\gamma n)} \\
& =\frac{1}{\alpha \delta\left(m+\frac{\beta}{\alpha} n\right)\left(m+\frac{\gamma}{\delta} n\right)} \\
& =\frac{1}{\alpha \delta}\left(\frac{1}{m+\frac{\beta}{\alpha} n}-\frac{1}{m+\frac{\gamma}{\delta} n}\right) \frac{1}{\left(\frac{\gamma}{\delta}-\frac{\beta}{\alpha}\right) n}, \\
& =\left(\frac{1}{m+\frac{\beta}{\alpha} n}-\frac{1}{m+\frac{\gamma}{\delta} n}\right) \frac{1}{(\alpha \gamma-\beta \delta) n}, \\
& =\left(\frac{1}{m+\frac{\beta}{\alpha} n}-\frac{1}{m+\frac{\gamma}{\delta} n}\right) \frac{1}{\operatorname{det}(M) n},
\end{aligned}
$$

where

$$
M=\left(\begin{array}{cc}
\alpha & \beta \\
\delta & \gamma
\end{array}\right) \in G L_{2}(R)
$$

This result will assist us in our future proof of the existence of $\lim _{\epsilon \rightarrow 0^{+}} f$. Next we introduce a lemma before proceeding to the main proof.

Lemma 3.5. Let $C_{N}$ be a positively oriented circle of radius $R_{N}, 1 \leq N<\infty$, centered at the origin. Assume the sequence of radii $R_{N}$ is increasing to $\infty$,for irrational $\theta=\frac{h}{k}$,

$$
f(\theta)=\lim _{N \rightarrow \infty} \frac{1}{2 \pi}\left\{\sum_{0<m / h<R_{N}} \frac{\cot (\pi m / \theta)}{m}+\sum_{0<n / k<R_{N}} \frac{\cot (\pi n \theta)}{n}\right\}
$$

converges.
The proof of this lemma can be found in [Ber76].
Lemma 3.6. Let $\epsilon>0$. For $\forall z \in \mathbb{C}$,

$$
\begin{equation*}
\pi \cot (\pi z)=\lim _{\epsilon \rightarrow 0} \sum_{m \in \mathbb{Z}} \frac{e^{-\epsilon m^{2}}}{m+z} \tag{3.35}
\end{equation*}
$$

There is a series representation as a partial fraction expansion for the trigonometric function $\cot (z)$, where just translated reciprocal functions are summed up, such that the poles of the cotangent function and the reciprocal functions match [Rem91] [NP69]. It's a standard fact that

$$
\begin{equation*}
\pi \cot (\pi z)=\lim _{n \rightarrow \infty} \sum_{v=-n}^{v=n} \frac{1}{z+a}, \tag{3.36}
\end{equation*}
$$

where $z \in \mathbb{C}$.
Proof of Lemma 3.6. For any fixed $\epsilon>0$, we have:

$$
\begin{align*}
\lim _{\epsilon \rightarrow 0^{+}}\left(\sum_{m \in \mathbb{Z}} \frac{e^{-\epsilon m^{2}}}{m+z}\right) & =\lim _{\epsilon \rightarrow 0^{+}} \lim _{n \rightarrow \infty} \sum_{v=-n}^{n} \frac{e^{-\epsilon v^{2}}}{v+z}  \tag{3.37}\\
& =\lim _{n \rightarrow \infty} \lim _{\epsilon \rightarrow 0^{+}} \sum_{v=-n}^{n} \frac{e^{-\epsilon v^{2}}}{v+z}  \tag{3.38}\\
& =\lim _{n \rightarrow \infty} \sum_{v=-n}^{n} \frac{1}{v+z}  \tag{3.39}\\
& =\pi \cot (\pi z) \tag{3.40}
\end{align*}
$$

The reason why in (3.38) we can exchange the two limits is due to Lebesgue's dominated convergence theorem. Define $f_{n}(\epsilon, z)=\sum_{v=-n}^{n} \frac{e^{-\epsilon v^{2}}}{v+z}$. As long as we can
prove that $f_{n}(\epsilon, z)$ is uniform convergent, then it is safe for us to exchange the two limits.

$$
\begin{aligned}
f_{n}(\epsilon, z) & =\sum_{v=-n}^{n} \frac{e^{-\epsilon v^{2}}}{v+z} \\
& =\frac{e^{-\epsilon(-n)^{2}}}{-n+z}+\frac{e^{-\epsilon(-n+1)^{2}}}{-(n-1)+z}+\cdots+\frac{1}{z}+\cdots+\frac{e^{-\epsilon n^{2}}}{n+z},
\end{aligned}
$$

pair the $j$-th term and the $(2 n+2-j)$-th term:

$$
\begin{aligned}
& =e^{-\epsilon n^{2}}\left(\frac{1}{-n+z}+\frac{1}{n+z}\right)+e^{-\epsilon(1-n)^{2}}\left(\frac{1}{-(n-1)+z}+\frac{1}{n-1+z}\right)+ \\
& \cdots+e^{-\epsilon}\left(\frac{1}{-1+z}+\frac{1}{1+z}\right)+\frac{1}{z}, \\
& =e^{-\epsilon n^{2}} \frac{2 z}{z^{2}-n^{2}}+e^{-\epsilon(n-1)^{2}} \frac{2 z}{z^{2}-(n-1)^{2}}+\cdots+e^{-\epsilon} \frac{2 z}{z^{2}-1}+\frac{1}{z} \\
& =2 z \sum_{k=1}^{n} \frac{e^{-\epsilon k^{2}}}{z^{2}-k^{2}}+\frac{1}{z} .
\end{aligned}
$$

Since $\epsilon>0$, we have

$$
\left|\frac{e^{-\epsilon k^{2}}}{z^{2}-k^{2}}\right| \leq\left|\frac{1}{z^{2}-k^{2}}\right|
$$

And we know that $\sum_{k=1}^{n} \frac{1}{\left|z^{2}-k^{2}\right|}$ converges absolutely for any fixed $z \in \mathbb{C}$. Therefore, we proved the uniform convergence of $f_{n}(\epsilon, z)$.

Proof of Theorem 3.3. We recall that by the definition of f ,

$$
\lim _{\epsilon \rightarrow 0^{+}} f(\epsilon, \alpha, \beta, \delta, \gamma)=\lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+\beta n)(\delta m+\gamma n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+\beta n)(\delta m+\gamma n)} .
$$

In order to find $\lim _{\epsilon \rightarrow 0^{+}} \mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma)$, we will sum over $m$ first, and then sum
over $n$.

$$
\begin{array}{rl}
\lim _{\epsilon \rightarrow 0^{+}} f & f(\epsilon, \alpha, \beta, \delta, \gamma)=\lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{m \in \mathbb{Z} \\
(\alpha m+\beta n)(\delta m+\gamma n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+\beta n)(\delta m+\gamma n)}, \\
& =\lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{m \in \mathbb{Z} \\
(\alpha m+\beta n)(\delta m+\gamma n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{\operatorname{det}(M) n}\left(\frac{1}{m+\frac{\beta}{\alpha} n}-\frac{1}{m+\frac{\gamma}{\delta} n}\right), \\
& =\lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{m \in \mathbb{Z} \\
(\alpha m+\beta n)(\delta m+\gamma n) \neq 0}} \frac{e^{-\pi \epsilon n^{2}}}{\operatorname{det}(M) n}\left(\frac{e^{-\pi \epsilon m^{2}}}{m+\frac{\beta}{\alpha} n}-\frac{e^{-\pi \epsilon m^{2}}}{m+\frac{\gamma}{\delta} n}\right), \\
& =\lim _{\epsilon \rightarrow 0^{+}} \sum_{n \in Z^{*}}\left(\frac{e^{-\pi \epsilon n^{2}} \pi}{\operatorname{det}(M) n} \cot \left(\pi n \frac{\beta}{\alpha}\right)-\frac{e^{-\pi \epsilon n^{2}} \pi}{\operatorname{det}(M) n} \cot \left(\pi n \frac{\gamma}{\delta}\right)\right), \\
& =\lim _{\epsilon \rightarrow 0^{+}} \sum_{n \in Z^{*}} \frac{e^{-\pi \epsilon n^{2}} \pi}{\operatorname{det}(M)}\left(\frac{\cot \left(\pi n \frac{\beta}{\alpha}\right)}{n}-\frac{\cot \left(\pi n \frac{\gamma}{\delta}\right)}{n}\right) . \tag{3.45}
\end{array}
$$

Here $\mathbb{Z}^{*}$ signifies $\mathbb{Z} \backslash\{0\}$.
Define

$$
\mathrm{c}_{n}(\epsilon, \alpha, \beta, \delta, \gamma, n)=\frac{e^{-\pi \epsilon n^{2}} \pi}{\operatorname{det}(M)}\left(\frac{\cot \left(\pi n \frac{\beta}{\alpha}\right)}{n}-\frac{\cot \left(\pi n \frac{\gamma}{\delta}\right)}{n}\right)
$$

Then we have:

$$
\begin{align*}
\lim _{\epsilon \rightarrow 0^{+}} \mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma) & =\lim _{\epsilon \rightarrow 0^{+}} \sum_{n \in \mathbb{Z}^{*}} \mathrm{c}_{n}(\epsilon, \alpha, \beta, \delta, \gamma, n),  \tag{3.46}\\
& =\lim _{\epsilon \rightarrow 0^{+}} \sum_{n \in \mathbb{Z}^{*}} \frac{e^{-\pi \epsilon n^{2}} \pi}{\operatorname{det}(M)}\left(\frac{\cot \left(\pi n \frac{\beta}{\alpha}\right)}{n}-\frac{\cot \left(\pi n \frac{\gamma}{\delta}\right)}{n}\right) . \tag{3.47}
\end{align*}
$$

When $\frac{\beta}{\alpha} \frac{\gamma}{\delta}=-1$, and $\frac{\beta}{\alpha}, \frac{\gamma}{\delta}$ both irrationals, Lemma 3.5 tells us that

$$
\begin{aligned}
\lim _{\epsilon \rightarrow 0^{+}} \mathrm{f}(\epsilon, \alpha, \beta, \delta, \gamma) & =\frac{1}{\operatorname{det}(M)} \sum_{n \in \mathbb{Z}^{*}}\left(\frac{\cot \left(\pi n \frac{\beta}{\alpha}\right)}{n}-\frac{\cot \left(\pi n \frac{\gamma}{\delta}\right)}{n}\right), \\
& =\frac{1}{\operatorname{det}(M)} \sum_{n \in \mathbb{Z}^{*}}\left(\frac{\cot (\pi n \theta)}{n}+\frac{\cot \left(\pi n \frac{1}{\theta}\right)}{n}\right),
\end{aligned}
$$

exists, where $\theta=\frac{\beta}{\alpha}=-\frac{\delta}{\gamma}$.

When $\frac{\beta}{\alpha} \frac{\gamma}{\delta}=-1$ and $\frac{\beta}{\alpha}, \frac{\gamma}{\delta}$ are both rational numbers, we can prove the limit

$$
\lim _{\epsilon \rightarrow 0^{+}} f(\epsilon, \alpha, \beta, \delta, \gamma)=\lim _{\epsilon \rightarrow 0^{+}} \sum_{\substack{(m, n) \in \mathbb{Z}^{2} \\(\alpha m+\beta n)(\delta m+\gamma n) \neq 0}} \frac{e^{-\pi \epsilon\left(m^{2}+n^{2}\right)}}{(\alpha m+\beta n)(\delta m+\gamma n)},
$$

exists without much difficulty since we only need to take care of few points which are near to the two straight lines: $\alpha m+\beta n=0$ and $\delta m+\gamma n=0$.

In conclusion, $\lim _{\epsilon \rightarrow 0^{+}} \mathrm{f}$ exists when $\frac{\beta}{\alpha} \frac{\gamma}{\delta}=-1$.

We remind the reader that in general, for a polytope $\mathcal{P} \in \mathbb{R}^{d}$, the discrete volume of $\mathcal{P}$ is:

$$
\begin{aligned}
A_{\mathcal{P}}= & \sum_{\mathbf{n} \in \mathbb{Z}^{d}} \omega_{\mathcal{P}}(\mathbf{n}), \\
= & \lim _{\epsilon \rightarrow 0^{+}} \sum_{\mathbf{n} \in \mathbb{Z}^{d} \backslash V}\left(\frac{\left|\operatorname{det} \mathcal{K}_{\mathbf{v}_{1}}\right| e^{\left\langle\mathbf{v}_{1}, \mathbf{n}\right\rangle}}{\prod_{k=1}^{d}\left\langle\mathbf{w}_{k}\left(v_{1}\right), \mathbf{n}\right\rangle}+\cdots \quad+\frac{\left|\operatorname{det} \mathcal{K}_{\mathbf{v}_{N}}\right| e^{\left(\mathbf{v}_{N}, \mathbf{n}\right\rangle}}{\prod_{k=1}^{d}\left\langle\mathbf{w}_{k}\left(v_{N}\right), \mathbf{n}\right\rangle}\right) e^{-\epsilon \pi \|\left.\mathbf{n}\right|^{2}} \\
& +\lim _{\epsilon \rightarrow 0^{+}} \sum_{\mathbf{n} \in V} \hat{1}_{\mathcal{P}}(\mathbf{n}) e^{-\epsilon \pi\|\mathbf{n}\|^{2}} .
\end{aligned}
$$

Here $V$ denotes the set of vectors which are orthogonal to any $\mathbf{w}_{k}\left(v_{i}\right)$, for $1 \leq i \leq N$. These infinite lattice sums extend the f functions as defined in Definition 3.1. When $\mathbf{v}_{\mathbf{i}}$ is not the origin, the former part of $A_{\mathcal{P}}$ will be different from $f$, and it will be a very interesting research problem in our future work.


## When are two Dedekind sums equal?

### 4.1 An observation about Dedekind sums and a question arising from it

Dedekind sums arise naturally in many fields, most prominently in combinatorial geometry [BR07] and in the theory of modular forms [RG72]. We also see the classical Dedekind sums in Chapter 2 when we analyze an infinite sum defined on integral cones. Recall that the classical Dedekind sum is defined by:

$$
s(a, b)=\sum_{k=0}^{b-1}\left(\left(\frac{k a}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right),
$$

where $a$ and $b$ are any two relatively prime integers, and where the Sawtooth function is defined by

$$
((x))= \begin{cases}\{x\}-\frac{1}{2} & \text { if } x \notin \mathbb{Z} \\ 0 & \text { if } x \in \mathbb{Z}\end{cases}
$$

The Dedekind sum enjoys two important properties. The first of these properties is the periodicity of the Dedekind sums in the first variable, namely $s(a+k b, b)=s(a, b)$ for all $k \in \mathbb{Z}$. The second, and deeper, of these properties is
the famous reciprocity law for Dedekind sums :

$$
s(a, b)+s(b, a)=-\frac{1}{4}+\frac{1}{12}\left(\frac{a}{b}+\frac{1}{a b}+\frac{b}{a}\right),
$$

valid for any two relatively prime integers $a$ and $b$.
We made a quite interesting observation with respect to the values of Dedekind sums $s(a, b)$ for certain $b$. Here are some examples illustrating what we discovered. Note that we used the same color to indicate the same value of Dedekind sum $s(a, b)$ when $\operatorname{gcd}(a, b)=1$.

Example 4.1. Here we give a table of values of Dedekind sums $s(a, 11)$.

| $a$ | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $s(a, 11)$ | $\frac{15}{22}$ | $\frac{5}{11}$ | $\frac{3}{22}$ | $\frac{3}{22}$ | $-\frac{5}{22}$ |
| $a$ | 6 | 7 | 8 | 9 | 10 |
| $s(a, 11)$ | $\frac{5}{22}$ | $-\frac{3}{22}$ | $-\frac{3}{22}$ | $-\frac{5}{22}$ | $-\frac{15}{22}$ |

Table 4.1: Dedekind sum $s(a, 11)$

Example 4.2. Here we give a table of values of Dedekind sums $s(a, 15)$.

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $s(a, 15)$ | $\frac{91}{90}$ | $\frac{7}{18}$ | $\frac{1}{5}$ | $\frac{19}{90}$ | $-\frac{1}{18}$ | 0 | $-\frac{7}{18}$ |
| $a$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| $s(a, 15)$ | $\frac{7}{18}$ | 0 | $-\frac{1}{18}$ | $-\frac{19}{90}$ | $-\frac{1}{5}$ | $-\frac{7}{18}$ | $-\frac{91}{90}$ |

Table 4.2: Dedekind sum $s(a, 15)$

Example 4.3. Here we give a table of values of Dedekind sums $s(a, 19)$.

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $s(a, 19)$ | $\frac{51}{38}$ | $\frac{21}{38}$ | $\frac{9}{38}$ | $\frac{11}{38}$ | $\frac{11}{38}$ | $-\frac{9}{38}$ | $\frac{3}{38}$ | $-\frac{3}{38}$ | $-\frac{21}{38}$ |
| $a$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $s(a, 19)$ | $\frac{21}{38}$ | $\frac{3}{38}$ | $-\frac{3}{38}$ | $\frac{9}{38}$ | $-\frac{11}{38}$ | $-\frac{11}{38}$ | $-\frac{9}{38}$ | $-\frac{21}{38}$ | $-\frac{51}{38}$ |

Table 4.3: Dedekind sum $s(a, 19)$

It is very natural to ask under what conditions on the integers $a_{1}, a_{2}$, and $b$ is it true that

$$
s\left(a_{1}, b\right)=s\left(a_{2}, b\right) ?
$$

We answer this question with the following results.
Theorem 4.4. (Wang) Let $b$ be a positive integer, and $a_{1}, a_{2}$ any two integers that are relatively prime to $b$. If $s\left(a_{1}, b\right)=s\left(a_{2}, b\right)$, then

$$
b \mid\left(1-a_{1} a_{2}\right)\left(a_{1}-a_{2}\right)
$$

An immediate corollary of this theorem is the following result:
Corollary 4.1. (Wang) Let $p$ be a prime. Then $s\left(a_{1}, p\right)=s\left(a_{2}, p\right)$ if and only if $a_{1} \equiv a_{2} \bmod p$, or $a_{1} a_{2} \equiv 1 \bmod p$.

We note that the converse of Theorem 4.4 is false in general. Consider, for example, $b=40$, and $a_{1}=37, a_{2}=33$. Then $b \mid(1-37 \cdot 33)(37-33)=20 \cdot 4$, and yet $s(37,40)=-0.8125$ and $s(33,40)=-0.3125$, so that $s\left(a_{1}, b\right) \neq s\left(a_{2}, b\right)$ in this case.

We also study an analogous question for the Dedekind-Rademacher sums, which arise in Donald Knuth's work [Knu77] on pseudo-random number generators. Given any non-negative integer $n$, and any two relatively prime integers $a$ and $b$, we define the Dedekind-Rademacher sum by:

$$
r_{n}(a, b)=\sum_{k=0}^{b-1}\left(\left(\frac{k a+n}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right) .
$$

In order to state the corresponding reciprocity law for the Dedekind-Rademacher sums, we define

$$
\chi_{a}(n)= \begin{cases}1 & \text { if } a \mid n \\ 0 & \text { otherwise }\end{cases}
$$

Lemma 4.1 (Reciprocity law for Dedekind-Rademacher sums). Let $a$ and $b$ be relatively prime positive integers. Then for $n=1,2, \cdots, a+b$,

$$
\begin{aligned}
r_{n}(a, b)+r_{n}(b, a) & =\frac{n^{2}}{2 a b}-\frac{n}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{a b}\right)+\frac{1}{12}\left(\frac{b}{a}+\frac{a}{b}+\frac{1}{a b}\right) \\
& +\frac{1}{2}\left(\left(\left(\frac{a^{-1} n}{b}\right)\right)+\left(\left(\frac{b^{-1} n}{a}\right)\right)+\left(\left(\frac{n}{a}\right)\right)+\left(\left(\frac{n}{b}\right)\right)\right) \\
& +\frac{1}{4}\left(1+\chi_{a}(n)+\chi_{b}(n)\right),
\end{aligned}
$$

where $a a^{-1} \equiv 1 \bmod b$ and $b b^{-1} \equiv 1 \bmod a$.

The proof of Lemma 4.1 can be found, for example, in [BR07]. For the Dedekind-Rademacher sums, we have the following two results.

Here are some examples of Dedekind-Rademacher sums $r_{n}(a, b)$ for certain values of $n$ and $b$. By observing the values of $r_{n}(a, b)$ in the examples, we ask the question what is the relationship between $a, b$, and $n$ given two DedekindRademacher sums are equal.

Example 4.5. Here we give a table of values of Dedekind-Rademacher sums $r_{4}(a, 11)$.

| $a$ | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $r_{4}(a, 11)$ | $-\frac{15}{44}$ | $-\frac{7}{44}$ | $-\frac{9}{44}$ | $\frac{3}{44}$ | $\frac{3}{44}$ |
| $a$ | 6 | 7 | 8 | 9 | 10 |
| $r_{4}(a, 11)$ | $-\frac{3}{44}$ | $-\frac{3}{44}$ | $\frac{9}{44}$ | $\frac{7}{44}$ | $\frac{15}{44}$ |

Table 4.4: Dedekind-Rademacher sum $r_{4}(a, 11)$

Example 4.6. Here we give a table of values of Dedekind-Rademacher sums $r_{4}(a, 15)$.

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $r_{4}(a, 15)$ | $-\frac{37}{180}$ | $-\frac{17}{180}$ | $-\frac{1}{10}$ | $\frac{17}{180}$ | $\frac{1}{18}$ | $\frac{1}{10}$ | $\frac{53}{180}$ |
| $a$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| $r_{6}(a, 17)$ | $-\frac{53}{180}$ | $-\frac{1}{10}$ | $-\frac{1}{18}$ | $-\frac{17}{180}$ | $\frac{1}{10}$ | $\frac{17}{180}$ | $\frac{37}{180}$ |

Table 4.5: Dedekind-Rademacher sum $r_{4}(a, 15)$

Example 4.7. Here we give a table of values of Dedekind-Rademacher sums $r_{6}(a, 17)$.

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $r_{6}(a, 17)$ | $-\frac{35}{68}$ | $-\frac{1}{4}$ | $-\frac{7}{68}$ | $\frac{1}{68}$ | $-\frac{19}{68}$ | $\frac{15}{68}$ | $\frac{7}{68}$ | $\frac{11}{68}$ |
| $a$ | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $r_{6}(a, 17)$ | $-\frac{11}{68}$ | $-\frac{7}{68}$ | $-\frac{15}{68}$ | $\frac{19}{68}$ | $-\frac{1}{68}$ | $\frac{7}{68}$ | $\frac{1}{4}$ | $\frac{35}{68}$ |

Table 4.6: Dedekind-Rademacher sum $r_{6}(a, 17)$

The examples here give us a clue of what the relationship between $a_{1}, a_{2}, b$, and $n$ is given $r_{n}\left(a_{1}, b\right)=r_{n}\left(a_{2}, b\right)$.

Theorem 4.8. (Wang) Fix a non-negative integer $n$ and a positive integer b. Let $a_{1}$ and $a_{2}$ be any two integers that are relatively prime to $b$.

If $r_{n}\left(a_{1}, b\right)=r_{n}\left(a_{2}, b\right)$, then

$$
b \mid\left(6 n^{2}+1-a_{1} a_{2}\right)\left(a_{2}-a_{1}\right) .
$$

An immediate corollary for prime moduli follows from Theorem 4.8.

Corollary 4.2. (Wang) Let $p$ be a prime. If $r_{n}\left(a_{1}, p\right)=r_{n}\left(a_{2}, p\right)$, then it follows that $a_{1} \equiv a_{2} \bmod p$, or

$$
a_{1} a_{2} \equiv 1+6 n^{2} \bmod p
$$

We note that here the results are a bit different than the corresponding results for the classical Dedekind sums. Namely, not only is the converse of Theorem 4.8 false in the case of the Dedekind-Rademacher sums, but even the converse of Corollary 4.2 is false in this case. The best way to elucidate this is by an example.

Example 4.9. Consider the Dedekind-Rademacher sum $r_{n}(a, b)$ when $n=6$, $b=23, a_{1}=3$, and $a_{2}=11$. We have

$$
r_{6}(3,23)=-\frac{3}{92}
$$

and

$$
r_{6}(11,23)=\frac{43}{92}
$$

Although $3 \cdot 11 \equiv 1+6 \cdot 6^{2} \bmod 23$, we don't have $r_{6}(3,23)=r_{6}(11,23)$.

One direction for our further research will be to investigate the number of solutions to the equation $s(x, b)=c$ and what the solutions are. Note that when $b$ is composite and $c$ is rational, the equation $s(x, b)=c$ might have more than 2 solutions in $x \in \mathbb{Z}$. In fact, Corollary 4.1 shows that if $b$ has $r$ distinct prime divisors, then the number of solutions to $s(x, b)=c$ is greater than or equal to $2^{r}$, by the usual elementary modular arithmetic arguments. It would be quite interesting to study how many integer solutions in $x \in \mathbb{Z}$ the equation $s(x, b)=c$ has in general. The same question also arises from topological considerations, and the correction terms of the Heegaard Floer Homology [BL90] is closely related to it.

### 4.2 How Reciprocity Law of Dedekind sums leads to the proof of the identity $s\left(a_{1}, b\right)=s\left(a_{2}, b\right)$, and more

We first introduce some lesser-known but useful properties of Dedekind sums. It is proved in [Gri74] that

$$
\begin{equation*}
6 b s(a, b) \in \mathbb{Z} \tag{4.1}
\end{equation*}
$$

for any two relatively prime integer $a$ and $b$. This property of Dedekind sums gives us a nice upper bound on the denominators that any Dedekind sum $s(a, b)$ may have, and it plays an interesting role in the proof of Theorem 4.4.

Now we come to the proofs of our results in the previous section.
Proof of Theorem 4.4. For any integers $a_{1}$ relatively prime to $b$, and $a_{2}$ relatively prime to $b$, Dedekind's Reciprocity law implies that we have the following two identities:

$$
\begin{align*}
& 12 a_{1} b\left(s\left(a_{1}, b\right)+s\left(b, a_{1}\right)\right)=-3 a_{1} b+a_{1}^{2}+b^{2}+1,  \tag{4.2}\\
& 12 a_{2} b\left(s\left(a_{2}, b\right)+s\left(b, a_{2}\right)\right)=-3 a_{2} b+a_{2}^{2}+b^{2}+1 . \tag{4.3}
\end{align*}
$$

Multiplying (4.2) with $a_{2}$, and multiplying (4.3) with $a_{1}$, we get

$$
\begin{align*}
& 12 a_{1} a_{2} b\left(s\left(a_{1}, b\right)+s\left(b, a_{1}\right)\right)=a_{2}\left(-3 a_{1} b+a_{1}^{2}+b^{2}+1\right)  \tag{4.4}\\
& 12 a_{1} a_{2} b\left(s\left(a_{2}, b\right)+s\left(b, a_{2}\right)\right)=a_{1}\left(-3 a_{2} b+a_{2}^{2}+b^{2}+1\right) \tag{4.5}
\end{align*}
$$

Subtracting (4.5) from (4.4) gives us

$$
\begin{align*}
& 12 a_{1} a_{2} b\left(s\left(a_{1}, b\right)+s\left(b, a_{1}\right)\right)-12 a_{1} a_{2} b\left(s\left(a_{2}, b\right)+s\left(b, a_{2}\right)\right)  \tag{4.6}\\
& =a_{2}\left(-3 a_{1} b+a_{1}^{2}+b^{2}+1\right)-a_{1}\left(-3 a_{2} b+a_{2}^{2}+b^{2}+1\right) .
\end{align*}
$$

We know, by assumption, that $s\left(a_{1}, b\right)=s\left(a_{2}, b\right)$, and therefore

$$
\begin{equation*}
12 a_{1} a_{2} b s\left(b, a_{1}\right)-12 a_{1} a_{2} b s\left(b, a_{2}\right)=a_{1}^{2} a_{2}+b^{2} a_{2}-b^{2} a_{1}+a_{2}-a_{2}^{2} a_{1}-a_{1} \tag{4.7}
\end{equation*}
$$

Using the fact (4.1) that $\left(6 a_{1}\right) s\left(b, a_{1}\right)$ and $\left(6 a_{2}\right) s\left(b, a_{2}\right)$ are both integers, we may reduce (4.7) mod $b$ to obtain the result

$$
\begin{equation*}
\left(a_{2}-a_{1}\right)\left(1-a_{1} a_{2}\right) \equiv 0 \bmod b . \tag{4.8}
\end{equation*}
$$

Lemma 4.2. For any relatively prime integers $a$ and $b$, we have

$$
12 b r_{n}(a, b) \in \mathbb{Z}
$$

Proof. Note that for relatively prime numbers $a$ and $b$, there's always a solution to the equation $k a+n \equiv 0(\bmod b)$, while $k \in\{0,1, \cdots, b-1\}$. We consider two different situations.
$\diamond$ When $k=0$, or equivalently if $n \equiv 0(\bmod b)$, then $r_{n}(a, b)=r_{0}(a, b)=$ $s(a, b)$ and it was pointed out in $[R G 72]$ that $6 b s(a, b) \in \mathbb{Z}$.
$\diamond$ When $k=k_{0} a+n \equiv 0(\bmod b)$ where $k_{0} \in\{1, \cdots, b-1\}$.

$$
\begin{aligned}
12 b r_{n}(a, b) & =12 b \sum_{k=0}^{b-1}\left(\left(\frac{k a+n}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right), \\
& =12 b \sum_{k=1, k \neq k_{0}}^{b-1}\left(\frac{k a+n}{b}-\left[\frac{k a+n}{b}\right]-\frac{1}{2}\right)\left(\frac{k}{b}-\frac{1}{2}\right), \\
& =12 b \sum_{k=1, k \neq k_{0}}^{b-1}\left(\frac{k(k a+n)}{b^{2}}-\frac{A}{2 b}+\frac{1}{4}\right), \\
& =12 b\left(\frac{a(b-1)(2 b-1)}{6 b}+\frac{n(b-1)}{2 b}-\frac{A(b-2)}{2 b}+\frac{b-2}{4}-\frac{C k_{0}}{b}\right), \\
& =2 a(b-1)(2 b-1)+6 n(b-1)-6 A(b-2)+3 b(b-2)-12 C k_{0} .
\end{aligned}
$$

where $A, C \in \mathbb{Z}$, and immediately we have $12 b r_{n}(a, b) \in \mathbb{Z}$.

Proof of Theorem 4.8. From the reciprocity law for the Dedekind-Rademacher sums, we know that when $a_{1}, a_{2}$ are relatively prime to $b$, we have:

$$
\begin{align*}
12 a_{1} b\left(r_{n}\left(a_{1}, b\right)+r_{n}\left(b, a_{1}\right)\right) & =6 n^{2}+a_{1}^{2}+b^{2}+1-9 a_{1} b \\
& -6 a_{1} b\left(\left[\frac{a_{1}^{-1} n}{b}\right]+\left[\frac{b^{-1} n}{a_{1}}\right]+\left[\frac{n}{b}\right]+\left[\frac{n}{a_{1}}\right]\right) \\
& +3 a_{1} b\left(\chi_{a_{1}}(n)+\chi_{b}(n)\right), \tag{4.9}
\end{align*}
$$

$$
\begin{align*}
12 a_{2} b\left(r_{n}\left(a_{2}, b\right)+r_{n}\left(b, a_{2}\right)\right) & =6 n^{2}+a_{2}^{2}+b^{2}+1-9 a_{2} b \\
& -6 a_{2} b\left(\left[\frac{a_{2}^{-1} n}{b}\right]+\left[\frac{b^{-1} n}{a_{2}}\right]+\left[\frac{n}{b}\right]+\left[\frac{n}{a_{2}}\right]\right) \\
& +3 a_{2} b\left(\chi_{a_{2}}(n)+\chi_{b}(n)\right) . \tag{4.10}
\end{align*}
$$

To simplify the ensuing algebra, we let

$$
\begin{gathered}
S_{a_{1}}=\left[\frac{a_{1}^{-1} n}{b}\right]+\left[\frac{b^{-1} n}{a_{1}}\right]+\left[\frac{n}{b}\right]+\left[\frac{n}{a_{1}}\right] \in \mathbb{Z} \\
S_{a_{2}}=\left[\frac{a_{2}^{-1} n}{b}\right]+\left[\frac{b^{-1} n}{a_{2}}\right]+\left[\frac{n}{b}\right]+\left[\frac{n}{a_{2}}\right] \in \mathbb{Z} \\
T_{a_{1}}=\chi_{a_{1}}(n)+\chi_{b}(n) \in \mathbb{Z} \\
T_{a_{2}}=\chi_{a_{2}}(n)+\chi_{b}(n) \in \mathbb{Z}
\end{gathered}
$$

we can rewrite (4.9) and (4.10) as follows:

$$
\begin{equation*}
12 a_{1} b\left(r_{n}\left(a_{1}, b\right)+r_{n}\left(b, a_{1}\right)\right)=6 n^{2}+a_{1}^{2}+b^{2}+1-9 a_{1} b-6 a_{1} b S_{a_{1}}+3 a_{1} b T_{a_{1}} \tag{4.11}
\end{equation*}
$$

$$
\begin{equation*}
12 a_{2} b\left(r_{n}\left(a_{2}, b\right)+r_{n}\left(b, a_{2}\right)\right)=6 n^{2}+a_{2}^{2}+b^{2}+1-9 a_{2} b-6 a_{2} b S_{a_{2}}+3 a_{2} b T_{a_{2}} . \tag{4.12}
\end{equation*}
$$

Multiplying (4.11) with $a_{2}$ gives us

$$
\begin{align*}
& 12 a_{1} a_{2} b\left(r_{n}\left(a_{1}, b\right)+r_{n}\left(b, a_{1}\right)\right)  \tag{4.13}\\
& =a_{2}\left(6 n^{2}+a_{1}^{2}+b^{2}+1-9 a_{1} b-6 a_{1} b S_{a_{1}}+3 a_{1} b T_{a_{1}}\right) .
\end{align*}
$$

Multiplying (4.12) with $a_{1}$ gives us

$$
\begin{align*}
& 12 a_{1} a_{2} b\left(r_{n}\left(a_{2}, b\right)+r_{n}\left(b, a_{2}\right)\right)  \tag{4.14}\\
& =a_{1}\left(6 n^{2}+a_{2}^{2}+b^{2}+1-9 a_{2} b-6 a_{2} b S_{a_{2}}+3 a_{2} b T_{a_{2}}\right) .
\end{align*}
$$

Since $r_{n}\left(a_{1}, b\right)=r_{n}\left(a_{2}, b\right)$, subtracting (4.14) from (4.13) we get

$$
\begin{align*}
& 12 a_{1} a_{2} b\left(r_{n}\left(b, a_{1}\right)-r_{n}\left(b, a_{2}\right)\right)  \tag{4.15}\\
& =\left(a_{2}-a_{1}\right)\left(6 n^{2}+1-a_{1} a_{2}\right)+b^{2}\left(a_{2}-a_{1}\right) \\
& -6 a_{1} a_{2} b\left(S_{a_{1}}+S_{a_{2}}\right)+3 a_{1} a_{2} b\left(T_{a_{1}}+T_{a_{2}}\right) .
\end{align*}
$$

We notice that, by Lemma (4.1), we have $12 a_{1} r_{n}\left(b, a_{1}\right) \in \mathbb{Z}$ and $12 a_{2} r_{n}\left(b, a_{2}\right) \in$ $\mathbb{Z}$. We may therefore reduce both sides of (4.15) modulo $b$ to obtain the result:

$$
0 \equiv\left(6 n^{2}+1-a_{1} a_{2}\right)\left(a_{2}-a_{1}\right) \bmod b
$$



## When are two Zagier-Dedekind sums equal?

### 5.1 Properties of Zagier-Dedekind sums and a condition for two such sums to be equal

In Don Zagier's 1973 paper [Zag73], the author introduced the following Zagier-Dedekind sum:

$$
d\left(p ; a_{1}, \cdots, a_{n}\right)=(-1)^{n / 2} \sum_{k=1}^{p-1} \cot \frac{\pi k a_{1}}{p} \cdots \cot \frac{\pi k a_{n}}{p}
$$

where $p$ is a positive integer, $n$ is an even integer, and $a_{1}, \cdots, a_{n}$ are integers relatively prime to $p$.

We will partially answer the question of when two Zagier-Dedekind sums are equal in this chapter. These Zagier-Dedekind sums enjoy very nice properties which are essential for us to prove our results later.

Theorem 5.1 (Zagier). Let $a_{0}, a_{1}, \cdots, a_{n}$ ( $n$ even) be pairwise coprime positive integers. Then

$$
\sum_{j=0}^{n} \frac{1}{a_{j}} d\left(a_{j} ; a_{0}, \cdots, \widehat{a_{j}}, \cdots, a_{n}\right)=1-\frac{l_{n}\left(a_{0}, \cdots, a_{n}\right)}{a_{0} \cdots a_{n}},
$$

where $l_{n}\left(a_{0}, \cdots, a_{n}\right)$ is the polynomial defined as the coefficient of $t^{n}$ in the power series expansion of

$$
\prod_{j=0}^{n} \frac{a_{j} t}{\tanh \left(a_{j} t\right)}=\prod_{j=0}^{n}\left(1+\frac{1}{3} a_{j}{ }^{2} t^{2}-\frac{1}{45} a_{j}{ }^{4} t^{4}+\frac{2}{945} a_{j}{ }^{6} t^{6}-\cdots\right)
$$

The proof of this theorem can be found in [Zag73]. These polynomials are recognized as Hirzebruch $L$-polynomials [Hir66] by topologists. The first values of $l_{n}\left(a_{0}, \cdots, a_{n}\right)$ are:

$$
\begin{align*}
l_{0}(a) & =1  \tag{5.1}\\
l_{2}(a, b, c) & =\left(a^{2}+b^{2}+c^{2}\right) / 3  \tag{5.2}\\
l_{4}(a, b, c, d, e) & =\left(5\left(a^{2}+b^{2}+c^{2}+d^{2}+e^{2}\right)^{2}\right. \\
& \left.-7\left(a^{4}+b^{4}+c^{4}+d^{4}+e^{4}\right)\right) / 90 . \tag{5.3}
\end{align*}
$$

Each $l_{n}$ is even in every variable, symmetric under interchange of the variables, and homogeneous of total degree $n . l_{n}$ can also be written as:

$$
l_{n}\left(a_{0}, \cdots, a_{n}\right)=L_{k}\left(p_{1}, \cdots, p_{k}\right)
$$

where $k=n / 2$ and $p_{i}(i=1, \cdots, k)$ is the $i^{\text {th }}$ elementary symmetric polynomial in $a_{0}{ }^{2}, \cdots, a_{n}{ }^{2}$. Then the first few polynomials $L_{k}$ are:

$$
\begin{align*}
L_{0} & =1  \tag{5.4}\\
L_{1}\left(p_{1}\right) & =p_{1} / 3  \tag{5.5}\\
L_{2}\left(p_{1}, p_{2}\right) & =\left(-p_{1}^{2}+7 p_{2}\right) / 45,  \tag{5.6}\\
L_{3}\left(p_{1}, p_{2}, p_{3}\right) & =\left(2 p_{1}^{3}-13 p_{1} p_{2}+62 p_{3}\right) / 945 . \tag{5.7}
\end{align*}
$$

Let

$$
\mu_{k}=\text { the denominator of } L_{k}\left(p_{1}, \cdots, p_{k}\right) .
$$

From (5.4) to (5.7), we have:

$$
\begin{aligned}
& \mu_{0}=1 \\
& \mu_{1}=3 \\
& \mu_{2}=45 \\
& \mu_{3}=945 .
\end{aligned}
$$

The general result of $\mu_{k}$ can be found in [Ati61]:

$$
\mu_{k}=\prod_{\substack{l \text { prime } \\ l \text { odd }}} l^{\left[\frac{n}{l-1}\right]} .
$$

Theorem 5.2 (Zagier). Let $p$ be a positive integer and $a_{1}, \cdots, a_{n}$ (neven) be integers prime to $p$. Then $d\left(p ; a_{1}, \cdots, a_{n}\right)$ is a rational number whose denominator divides

$$
\prod_{\substack{l \text { prime } \\ l>2 \\ l \mid p}} l^{\left[\frac{n}{l-1}\right]}
$$

Theorem 5.3 (Zagier). Let $p$ be a positive integer and $a_{1}, a_{2}, \cdots, a_{n}$ odd integers prime to $p$. Then

$$
d\left(2 p ; a_{1}, \cdots, a_{n}\right)-d\left(p ; a_{1}, \cdots, a_{n}\right)=p t_{p}\left(a_{1}, \cdots, a_{n}\right)
$$

where $t_{p}\left(a_{1}, \cdots, a_{n}\right)$ is the integer

$$
\begin{aligned}
& \operatorname{Card}\left\{k_{1}, \cdots, k_{n} \mid 0<k_{1}, \cdots, k_{n}<p \text { and } \frac{a_{1} k_{1}+\cdots+a_{n} k_{n}}{p} \text { an even integer }\right\} \\
& -\operatorname{Card}\left\{k_{1}, \cdots, k_{n} \mid 0<k_{1}, \cdots, k_{n}<p \text { and } \frac{a_{1} k_{1}+\cdots+a_{n} k_{n}}{p} \text { an odd integer }\right\}
\end{aligned}
$$

The proof of Theorem 5.2 and Theorem 5.3 can be found in [Zag73].
If we have two higher dimensional Dedekind sums with the same value, what can we say about the variables? Here we find a relationship between $a_{i}$ and $b_{j}$ provided $d\left(a_{0} ; a_{1}, \cdots, a_{n}\right)=d\left(b_{0} ; b_{1}, \cdots, b_{n}\right)$ and $a_{0}=b_{0}$.

We now state the main result of this chapter.

Theorem 5.4. (Wang) Let $a_{0}, \cdots, a_{n}$ be pairwise coprime positive integers, and let $b_{0}, \cdots, b_{n}$ be pairwise coprime positive integers. If we have $p=a_{0}=b_{0}$ and $d\left(a_{0} ; a_{1}, \cdots, a_{n}\right)=d\left(b_{0} ; b_{1}, \cdots, b_{n}\right)$, then the following equality holds:

$$
\mu\left(\prod_{i=0}^{n} a_{i} l_{n}\left(p, \cdots, b_{n}\right)-\prod_{i=0}^{n} b_{i} l_{n}\left(p, \cdots, a_{n}\right)\right) \equiv 0 \bmod p
$$

where $\mu=\operatorname{lcm}\left(\mu_{1}, \mu_{2}\right)$, $\mu_{1}$ and $\mu_{2}$ are defined as follows.
Denote the denominator of the Dedekind sum $d\left(a_{i} ; a_{0}, \cdots, \widehat{a_{i}}, \cdots, a_{n}\right)$ by $\mu_{n, a_{i}}$, and denote the denominator of the Dedekind sum $d\left(b_{i} ; b_{0}, \cdots, \widehat{b_{i}}, \cdots, b_{n}\right)$ by $\mu_{n, b_{i}}$. Then $\mu_{1} \stackrel{\text { def }}{=} \operatorname{lcm}\left(\mu_{n, a_{0}}, \mu_{n, a_{1}}, \cdots, \mu_{n, a_{n}}\right), \mu_{2} \stackrel{\text { def }}{=} \operatorname{lcm}\left(\mu_{n, b_{0}}, \mu_{n, b_{1}}, \cdots, \mu_{n, b_{n}}\right)$.

We can state the result of Theorem 5.4 in a slightly different way. Let $A=\mu\left(\prod_{i=0}^{n} a_{i}\right)$, and $B=\mu\left(\prod_{i=0}^{n} b_{i}\right)$. Then we have:

$$
A l_{n}\left(p, b_{1}, \cdots, b_{n}\right) \equiv B l_{n}\left(p, a_{1}, \cdots, a_{n}\right) \bmod p
$$

Corollary 5.1. (Wang) If $d\left(a_{0} ; a_{1}, a_{2}, a_{3}, a_{4}\right)=d\left(b_{0} ; b_{1}, b_{2}, b_{3}, b_{4}\right)$, where $p=a_{0}=$ $b_{0}$ is a positive integer, $p, a_{1}, \cdots, a_{4}$ are pairwise coprime, and $p, b_{1}, \cdots, b_{4}$ are also pairwise coprime to each other, then we have

$$
\mu\left(\prod_{i=1}^{4} a_{i} l_{4}\left(p, b_{1}, b_{2}, b_{3}, b_{4}\right)-\prod_{i=1}^{4} b_{i} l_{4}\left(p, a_{1}, a_{2}, a_{3}, a_{4}\right)\right) \equiv 0 \bmod p
$$

or

$$
\begin{aligned}
& \mu\left(\prod_{i=1}^{4} a_{i}\left(5\left(p^{2}+{b_{1}}^{2}+{b_{2}}^{2}+{b_{3}}^{2}+{b_{4}}^{2}\right)^{2}-7\left(p^{4}+b_{1}^{4}+b_{2}^{4}+b_{3}^{4}+b_{4}^{4}\right)\right) / 90\right. \\
& \left.\quad-\prod_{i=1}^{4} b_{i}\left(5\left(p^{2}+{a_{1}}^{2}+{a_{2}}^{2}+{a_{3}}^{2}+{a_{4}}^{2}\right)^{2}-7\left(p^{4}+{a_{1}}^{4}+{a_{2}}^{4}+{a_{3}}^{4}+a_{4}{ }^{4}\right)\right) / 90\right) \\
& \quad \equiv 0 \bmod p
\end{aligned}
$$

where $\mu=l$ cm $\left(\mu_{1}, \mu_{2}\right)$. We define $\mu_{1}$ and $\mu_{2}$ in the following way: denote the denominator of the Dedekind sum $d\left(a_{i} ; a_{0}, \cdots, \widehat{a_{i}}, \cdots, a_{4}\right)$ by $\mu_{4, a_{i}}$; similarly, denote the denominator of the Dedekind sum $d\left(b_{i} ; b_{0}, \cdots, \widehat{b}_{i}, \cdots, b_{4}\right)$ by $\mu_{4, b_{i}}$, then

$$
\mu_{1}=\operatorname{lcm}\left(\mu_{4, a_{0}}, \mu_{4, a_{1}}, \cdots, \mu_{4, a_{4}}\right), \mu_{2}=\operatorname{lcm}\left(\mu_{4, b_{0}}, \mu_{4, b_{1}}, \cdots, \mu_{4, b_{4}}\right) .
$$

### 5.2 Proofs of properties of Zagier-Dedekind sums and some experimental data of their values

Proof of Theorem 5.4. The reciprocity law for higher dimensional Zagier-Dedekind sums gives us:

$$
\begin{gather*}
\sum_{i=0}^{n} \frac{1}{a_{i}} d\left(a_{i} ; a_{0}, \cdots, \widehat{a_{i}}, a_{n}\right)=1-\frac{l_{n}\left(a_{0}, \cdots, a_{n}\right)}{a_{0} \cdots a_{n}},  \tag{5.8}\\
\sum_{i=0}^{n} \frac{1}{b_{i}} d\left(b_{i} ; b_{0}, \cdots, \widehat{b_{i}}, b_{n}\right)=1-\frac{l_{n}\left(b_{0}, \cdots, b_{n}\right)}{b_{0} \cdots b_{n}} . \tag{5.9}
\end{gather*}
$$

Denote the denominator of the Dedekind sum $d\left(a_{i} ; a_{0}, \cdots, \widehat{a_{i}}, \cdots, a_{n}\right)$ by $\mu_{n, a_{i}}$; denote the denominator of the Dedekind sum $d\left(b_{i} ; b_{0}, \cdots, \widehat{b}_{i}, \cdots, b_{n}\right)$ by $\mu_{n, b_{i}}$. Let $\mu_{1}=\operatorname{lcm}\left(\mu_{n, a_{0}}, \mu_{n, a_{1}}, \cdots, \mu_{n, a_{n}}\right), \mu_{2}=\operatorname{lcm}\left(\mu_{n, b_{0}}, \mu_{n, b_{1}}, \cdots, \mu_{n, b_{n}}\right)$. Let $\mu=$ $\operatorname{lcm}\left(\mu_{1}, \mu_{2}\right)$.

Multiplying (5.8) with $\mu \prod_{i=0}^{n} a_{i} \prod_{j=1}^{n} b_{j}$ :

$$
\begin{align*}
& \mu \prod_{j=1}^{n} b_{j} \sum_{k=0}^{n} a_{0} \cdots \widehat{a_{k}} \cdots a_{n} d\left(a_{k} ; a_{0}, \cdots, a_{n}\right)= \\
& \mu \prod_{i=0}^{n} a_{i} \prod_{j=1}^{n} b_{j}-\mu l_{n}\left(a_{0}, \cdots, a_{n}\right) \prod_{j=1}^{n} b_{j} . \tag{5.10}
\end{align*}
$$

Multiplying (5.9) with $\mu \prod_{i=0}^{n} b_{i} \prod_{j=1}^{n} a_{j}$ :

$$
\begin{align*}
& \mu \prod_{j=1}^{n} a_{j} \sum_{k=0}^{n} b_{0} \cdots \widehat{b_{k}} \cdots b_{n} d\left(b_{k} ; b_{0}, \cdots, b_{n}\right)= \\
& \mu \prod_{i=0}^{n} b_{i} \prod_{j=1}^{n} a_{j}-\mu l_{n}\left(b_{0}, \cdots, b_{n}\right) \prod_{j=1}^{n} a_{j} . \tag{5.11}
\end{align*}
$$

Note the first term in the LHS of (5.10) is $\mu \prod_{i=1}^{n} a_{i} \prod_{j=1}^{n} b_{j} d\left(a_{0} ; a_{1}, \cdots, a_{n}\right)$, and the first term in the LHS of (5.11) is $\mu \prod_{i=1}^{n} b_{i} \prod_{j=1}^{n} a_{j} d\left(b_{0} ; b_{1}, \cdots, b_{n}\right)$. Since we know that $d\left(a_{0} ; a_{1}, \cdots, a_{n}\right)=d\left(b_{0} ; b_{1}, \cdots, b_{n}\right)$, these two terms are, in fact, equal to each other. All the other terms in the LHS of (5.10) are of the form $p T_{1}=a_{0} T_{1}$ where $T_{1} \in \mathbb{Z}$, while all the other terms in the LHS of (5.11) are of the form $p T_{2}=b_{0} T_{2}$ where $T_{2} \in \mathbb{Z}$.

Subtract (5.11) from (5.10), and take modulus $p$ both sides, we have

$$
\mu\left(\prod_{i=1}^{n} a_{i} l_{n}\left(p, b_{1}, \cdots, b_{n}\right)-\prod_{i=1}^{n} b_{i} l_{n}\left(p, a_{1}, \cdots, a_{n}\right)\right) \equiv 0 \bmod p
$$

Proof of Corollary 5.1. The reciprocity law tells us that:

$$
\begin{gather*}
\sum_{i=0}^{4} \frac{1}{a_{i}} d\left(a_{i} ; a_{0}, \cdots, \widehat{a_{i}}, \cdots, a_{4}\right)=1-\frac{l_{4}\left(a_{0}, \cdots, a_{n}\right)}{a_{0} \cdots a_{n}},  \tag{5.12}\\
\sum_{i=0}^{4} \frac{1}{b_{i}} d\left(b_{i} ; b_{0}, \cdots, \widehat{b_{i}}, \cdots, b_{4}\right)=1-\frac{l_{4}\left(b_{0}, \cdots, b_{n}\right)}{b_{0} \cdots b_{n}}, \tag{5.13}
\end{gather*}
$$

where

$$
l_{4}(a, b, c, d, e)=\left(5\left(a^{2}+b^{2}+c^{2}+d^{2}+e^{2}\right)^{2}-7\left(a^{4}+b^{4}+c^{4}+d^{4}+e^{4}\right)\right) / 90
$$

Applying the same technique used in the proof of Theorem 5.4, we can finally get:

$$
\mu\left(\prod_{i=1}^{4} a_{i} l_{4}\left(p, b_{1}, b_{2}, b_{3}, b_{4}\right)-\prod_{i=1}^{4} b_{i} l_{4}\left(p, a_{1}, a_{2}, a_{3}, a_{4}\right)\right) \equiv 0 \bmod p
$$

where $\mu$ is defined as in Theorem 5.4.

### 5.3 Table of 4-dimensional Zagier-Dedekind Sums

We give a table of four-dimensional Dedekind sums for our reader's interest.

Table 5.1: 4-dimensional Zagier-Dedekind Sums

| $d(n ; a, b, c, d)$ | $d(n ; a, b, c, d)$ |
| :--- | :--- |
| $\mathrm{d}(7 ; 1,1,1,3)=2$ | $\mathrm{~d}(7 ; 1,1,3,3)=6$ |
| $\mathrm{~d}(7 ; 1,1,2,3)=2$ | $\mathrm{~d}(7 ; 2,3,4,5)=6$ |
| $\mathrm{~d}(7 ; 1,2,3,4)=2$ | $\mathrm{~d}(7 ; 1,2,5,6)=6$ |
| Continued on next page |  |

Table 5.1 - continued from previous page

| $d(n ; a, b, c, d)$ | $d(n ; a, b, c, d)$ |
| :--- | :--- |
| $\mathrm{d}(7 ; 1,2,4,5)=2$ | $\mathrm{~d}(7 ; 4,4,5,5)=6$ |
| $\mathrm{~d}(7 ; 3,4,5,6)=2$ | $\mathrm{~d}(7 ; 5,5,6,6)=6$ |
| $\mathrm{~d}(7 ; 1,3,5,6)=2$ | $\mathrm{~d}(7 ; 1,3,4,6)=6$ |
| $\mathrm{~d}(13 ; 1,1,2,6)=-12$ | $\mathrm{~d}(13 ; 1,2,3,5)=12$ |
| $\mathrm{~d}(13 ; 2,3,4,6)=-12$ | $\mathrm{~d}(13 ; 2,3,4,7)=12$ |
| $\mathrm{~d}(13 ; 1,2,3,4)=16$ | $\mathrm{~d}(13 ; 1,1,1,6)=-40$ |
| $\mathrm{~d}(13 ; 2,4,5,7)=16$ | $\mathrm{~d}(13 ; 3,5,6,8)=-40$ |
| $\mathrm{~d}(13 ; 2,3,5,6)=16$ | $\mathrm{~d}(13 ; 3,5,5,7)=-40$ |
| $\mathrm{~d}(13 ; 3,5,6,7)=8$ | $\mathrm{~d}(16 ; 3,3,5,7)=2$ |
| $\mathrm{~d}(13 ; 4,5,6,8)=8$ | $\mathrm{~d}(16 ; 1,1,5,7)=2$ |
| $\mathrm{~d}(13 ; 1,1,3,5)=8$ | $\mathrm{~d}(16 ; 1,1,3,3)=142$ |
| $\mathrm{~d}(13 ; 1,1,3,6)=8$ | $\mathrm{~d}(16 ; 5,5,7,7)=142$ |
| $\mathrm{~d}(17 ; 1,3,4,5)=24$ | $\mathrm{~d}(17 ; 1,2,3,5)=32$ |
| $\mathrm{~d}(17 ; 2,3,4,5)=24$ | $\mathrm{~d}(17 ; 1,4,5,6)=32$ |
| $\mathrm{~d}(17 ; 1,2,3,4)=52$ | $\mathrm{~d}(17 ; 2,3,6,7)=-4$ |
| $\mathrm{~d}(17 ; 2,3,5,7)=52$ | $\mathrm{~d}(17 ; 2,5,6,7)=-4$ |
| $\mathrm{~d}(17 ; 3,4,6,7)=52$ | $\mathrm{~d}(17 ; 3,4,5,6)=-4$ |
| $\mathrm{~d}(17 ; 2,7,9,11)=-24$ | $\mathrm{~d}(17 ; 2,3,11,12)=28$ |
| $\mathrm{~d}(17 ; 4,7,9,11)=-24$ | $\mathrm{~d}(17 ; 3,5,7,9)=28$ |
| $\mathrm{~d}(17 ; 2,3,8,10)=-24$ | $\mathrm{~d}(17 ; 4,5,10,11)=28$ |
| $\mathrm{~d}(17 ; 1,1,2,3)=244$ | $\mathrm{~d}(17 ; 5,5,6,7)=64$ |
| $\mathrm{~d}(17 ; 2,5,5,7)=244$ | $\mathrm{~d}(17 ; 3,5,7,7)=64$ |
|  |  |

## $\square_{\text {Chapter }} 6$

## The curve of centers of a finite point set

### 6.1 Introduction to the $\mu$-curve: generalized centers of a finite point set

In the $17^{\prime}$ th century, Pierre De Fermat proposed the following problem, which has attracted the attention of the mathematical community ever since Fermat breathed fresh life into this problem. Given three points in the plane, find a fourth point $p$ in $\mathbb{R}^{2}$ such that the sum of the distances from $p$ to the three given points is minimized.

Pierre de Fermat received a letter from R. Descartes in August 1638, in which he was asked to investigate the following curves:

$$
\left\{\mathbf{x} \in \mathbb{R}^{2} \mid \sum_{i=1}^{4}\left\|p_{i}-\mathbf{x}\right\|=c\right\},
$$

for given points $p_{1}, p_{2}, p_{3}, p_{4} \in \mathbb{R}^{2}$, and where $c$ is a constant. He was probably inspired by it and asked another related question in 1643: "Given three points in the place, find the point having the minimal sum of distances to these three points." Many mathematicians have given answers to this very interesting question, and E. Torricelli gave the initial solution due to which this special point earns the name "Fermat-Torricelli point" [KM97]. This problem is also called the "Steiner problem" named after Jacob Steiner. In 1937, the 16 -year-old Endre

Weiszfeld (later also known as Emil Varshony) published a celebrated algorithm that has been successfully used to solve the Steiner problem. The original paper of the Hungarian Jewish Weiszfeld was written in French, and published in a Japanese journal (Tohoku). This problem was historically known by many names: the "facility location problem", the "Fermat-Weber problem", and the "Torricelli problem" (see [Cle88],[Dal00],[dF43],[Kuh73],[Wei37]).

Among one of the generalized problems of the Steiner problem was to find a point $p \in \mathbb{R}^{d}$ minimizing the sum of weighted Euclidean distances from $p$ to $N$ given points. Weiszfeld is the first one who discovered an iterative algorithm to tackle it. In this chapter, our focus is on a generalization of the Steiner problem.

Definition 6.1. Given any finite point set $S \subseteq \mathbb{R}^{d}$, and given $r \geq 1$, we define

$$
\mu(r)=\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S}\|a-x\|^{r} .
$$

We note that $\mu:[1, \infty) \rightarrow \mathbb{R}^{d}$.
In other words, we associate a certain continuous curve $\mu$ to any finite point set $S \subseteq \mathbb{R}^{d}$, and we will show that the $\mu$-curve embodies many symmetry properties of $S$. The reason we restrict our attention to $r \geq 1$ is that in this range of $r$, the function $\|a-x\|^{r}$ is a convex function of $r$. If $r<1$, we may not necessarily get a unique point for the minimization problem above.

We generalize the Fermat-Weber problem by finding a series of points $\mu(r) \in$ $\mathbb{R}^{d}(r \in[1, \infty))$, each of which minimizes the sum of the $r$-th power of the distances from $\mu(r)$ to $N$ many given points in $S$. It is an easy fact that $\mu(2)$ gives the traditional mean of the point set $S$; in other words, the center of mass of the point set $S$. Moreover, $\mu(1)$ gives us a nice definition for the multidimensional median of $S$. There are various competing definitions for multidimensional medians, but the definition given here is often used in the literature on facility location. It is also a standard fact that $\mu(\infty)$ is the center of the smallest sphere that contains all of the points of $S$ [DLMZ07].

The $\mu$-curve unifies all of these natural choices of centers of a finite point set $S \in \mathbb{R}^{d}$ into one curve, and gives us a new kind of "signature" for the point set $S$.

Definition 6.2. Call the curve comprising of the points $\mu(r)$ the curve of centers or simply the $\mu$-curve.

We will extend Weiszfeld's algorithm to find $\mu(r)$ for $1 \leq r<\infty$.
We first study geometric meanings of some special points on the $\mu$-curve and the relationship between it and hyperplane(s) of the point set. Next, we proceed to expand Weiszfeld's algorithm to find $\mu(r)$ and prove that the $\mu$-curve is $C^{\infty}$. Then we find an equivalent condition for $\mu$-curve to be degenerate, in other words, a single point. Then we conjecture an upper bound for the distance between any two points on the $\mu$-curve using moments of $S$. In the end, we illustrate the $\mu$ curve of several point sets and show that we can use $\mu$-curve to detect approximate reflective symmetry and radial symmetry of finite point sets.

Note that there's another way to interpret our $\mu$-curve.
Definition 6.3. We define the $t$-th moment of our given point set $S \subset \mathbb{R}^{d}$ as follows:

$$
M_{t}(x)=\sum_{a \in S}\|a-x\|^{t}
$$

for each real $t \geq 1$.

Then

$$
\mu(r)=\underset{x \in \mathbb{R}^{d}}{\arg \min } M_{r}(x) .
$$

In words, each point on our $\mu$-curve minimizes the $r$-th moment of our point set among all possible points $x \in \mathbb{R}^{d}$.

Throughout the paper, $S$ is any fixed, finite set of points in $\mathbb{R}^{d}$. We will sometimes treat the case of two-dimension $d=2$ separately. For the sake of clarity we first recall some standard definitions.

Definition 6.4. A set is called radially symmetric (about the origin) if whenever a point $p \in S$, then $-p \in S$.

We also recall the definition of symmetry about a hyperplane $H$ for any set $S$.

Definition 6.5. Given a point $p \in \mathbb{R}^{d}$, consider the perpendicular line $L$ to $H$ that passes through $x$, and call $p^{\prime}$ the reflected point about the hyperplane $H$ if $p^{\prime}$ is on the line $L$, but on the other side of $H$, and located the same distance from $H$ as the distance that $p$ is away from $H$. We say that $S$ is symmetric about the hyperplane $H$ if for every $p \in S$, we also have the reflected point $p^{\prime} \in S$.

Some natural intuitive questions immediately arise:

1. What does the length of $\mu$-curve say about the point set $S$ ? If the $\mu$-curve is short (but not zero) relative to the diameter of $S$, does this mean that $S$ is in some sense "random"?
2. Can we bound the length of $\mu$-curve in terms of certain moments of $S$ ?
3. To what extent does the $\mu$-curve uniquely capture information about the point set $S$ ?

Theorem 6.4. (Wang) If the point set $S$ has a hyperplane $H$ of symmetry, then the $\mu$-curve lies on the hyperplane $H$.

Proof. Suppose to the contrary that for a certain $r$ there exists a point $p=\mu(r)$ that lies off the hyperplane $H$. Construct the perpendicular line $L$ to $H$, from $p$, and reflect the point $p$ about the hyperplane $H$. In other words, we walk from $p$ along the perpendicular line $L$ to $H$ until we get to the other side of $H$, a distance equal to the distance that $p$ is away from $H$. Call the reflected point $q$. Now it's clear from the hypothesis concerning the symmetry that the set of distances $\left\|q-a_{i}\right\|$ is equal to the set of distances $\left\|p-a_{i}\right\|$. This means that both the point $p$ and the point $q$ lie on the $\mu$-curve for the same $r$, contradicting the uniqueness of the point $p=\mu(r)$ that is guaranteed by convexity of the $\mu$-curve, because $r \geq 1$.

Note that the $\mu$-curve is not a straight line segment in general. However, in $\mathbb{R}^{2}$, if $S$ does possess a line of symmetry, then it follows that the $\mu$-curve is indeed a line segment which lies on the line of symmetry of $S$.

Corollary 6.1. (Wang) If a finite point set $S \in \mathbb{R}^{d}$ has at least d different hyperplanes of symmetry whose normal vectors are linearly independent, then $\mu=\{0\}$.

Proof. From Theorem 6.4, we know that the $\mu$-curve lies simultaneously on the $d$ linearly independent hyperplanes $H_{1}, \ldots, H_{d}$. But these hyperplanes only intersect simultaneously at the origin.

### 6.2 Weiszfeld's algorithm

Weiszfeld's algorithm solves the Steiner problem, which corresponds to $\mu(1)$ on the $\mu$-curve. We now expand this algorithm to find $\mu(r)$ for any $1 \leq r<\infty$ in $\mathbb{R}^{2}$. Suppose we are given a set $S$ of $N$ points in $\mathbb{R}^{2}$, namely $S=\left\{\left(x_{1}, y_{1}\right), \ldots,\left(x_{N}, y_{N}\right)\right\}$, and we wish to find a point $(P, Q)$ that minimizes the sum of the Euclidean distances from $(P, Q)$ to each of the given points $\left(x_{i}, y_{i}\right)$. This magical point $(P, Q)$ is called the Steiner point of the point set $S$.

Pick an initial point in $\mathbb{R}^{2}$, call it $\left(P_{0}, Q_{0}\right)$, making sure that it does not coincide with any point of $S$. Weiszfeld's algorithm is then defined as follows. Let

$$
\left\{\begin{aligned}
P_{k+1}= & \frac{\sum_{i=1}^{N}\left(\frac{x_{i}}{\left\|\left(P_{k}, Q_{k}\right)-\left(x_{i}, y_{i}\right)\right\|}\right)}{\sum_{i=1}^{N}\left(\frac{1}{\left\|\left(P_{k}, Q_{k}\right)-\left(x_{i}, y_{i}\right)\right\|}\right)}, \\
Q_{k+1}= & \frac{\sum_{i=1}^{N}\left(\frac{y_{i}}{\left\|\left(P_{k}, Q_{k}\right)-\left(x_{i}, y_{i}\right)\right\|}\right)}{\sum_{i=1}^{N}\left(\frac{1}{\left\|\left(P_{k}, Q_{k}\right)-\left(x_{i}, y_{i}\right)\right\|}\right)} .
\end{aligned}\right.
$$

Here $\|\cdot\|$ denotes the Euclidean norm. The sequence ( $P_{k}, Q_{k}$ ) converging to a point $(P, Q)$ for any initial choice $\left(P_{0}, Q_{0}\right)$ is now a theorem of Weiszfeld [Wei37] that it does indeed converge to the Seiner point $(P, Q)$ of the point set $S$ on condition that during the evolution of the algorithm, none of the points $\left(P_{k}, Q_{k}\right)$ coincides with any point of $S$.

### 6.3 An extension of Weiszfeld's algorithm, and the smoothness of the $\mu$-curve

For our problem, we again suppose the points form a finite point set $S=$ $\left\{a_{1}, \ldots, a_{N}\right\}$. Our goal is to find a point $\mu(r) \in \mathbb{R}^{d}$, for each given $r \in[1, \infty)$, that minimizes the sum $\sum_{i=1}^{n}\left\|x-a_{i}\right\|^{r}$ over all $x \in \mathbb{R}$. Let

$$
F(x)=\sum_{i=1}^{n}\left\|x-a_{i}\right\|^{r}
$$

The gradient of $F$ can be calculated as

$$
\nabla F(x)=\sum_{i=1}^{n} r\left(x-a_{i}\right)\left\|x-a_{i}\right\|^{r-2}
$$

when $x$ does not coincide with any of the $a_{i}$ 's. Solving the equation $\nabla F(x)=0$ yields the optimality condition

$$
\begin{equation*}
\mu(r)=\frac{\sum_{i=1}^{n} r a_{i}\left\|\mu(r)-a_{i}\right\|^{r-2}}{\lambda(\mu(r))} \tag{6.1}
\end{equation*}
$$

### 6.3 An extension of Weiszfeld's algorithm, and the smoothness of the $\mu$-curve

where

$$
\lambda(\mu(r))=\sum_{i=1}^{n} r\left\|\mu(r)-a_{i}\right\|^{r-2}
$$

We assume that, unless otherwise specified, the $\mu$-curve does not meet any of the points in $S$. The reason for this assumption is because $\mu(1)$ is generally not uniquely defined.

Similar in spirit to Weiszfeld's algorithm, we define the following point sequence, which will yield the optimal point $\mu(r)$ for any $r$ in the range $r \in[1, \infty)$ :

$$
\mu(r)_{k+1}=\frac{\sum_{a_{i} \in S} a_{i} d_{i, k}^{r-2}}{\sum_{a_{i} \in S} d_{i, k}^{r-2}},
$$

where $d_{i, k}$ is the distance from $a_{i} \in S$ to $\mu(r)_{k}$.
Theorem 6.5. (Wang) Given any $\left(\mu(r)_{0}\right) \in \mathbb{R}^{d}$, define $\mu(r)_{k+1}=\frac{\sum_{a_{i} \in S} a_{i} d_{i, k}^{r-2}}{\sum_{a_{i} \in S} d_{i, k}^{r-2}}$ as above, $k=1,2, \ldots$.If no $\mu(r)_{k}$ is a point from the point set $S$, then $\lim _{k \rightarrow \infty} \mu(r)_{k}=$ $\mu(r)$.

The proof follows exactly from the one for Weiszfeld's algorithm. We now give a structure theorem that tells us the $\mu$-curve is infinitely smooth.

Theorem 6.6. (Wang) The unique solution to the minimization problem

$$
\underset{x \in \mathbb{R}}{\arg \min } \sum_{i=1}^{n}\left\|x-a_{i}\right\|^{r},
$$

as a function of the parameter $r \in(1, \infty)$, is $C^{\infty}$ at all $r$ for which $\mu(r)$ does not coincide with any of the $a_{i}$ 's.

Proof. Let $r$ be fixed, and let $\mu(r) \notin\left\{a_{1}, \ldots, a_{n}\right\}$. Since $\mu(r)$ solves the unconstrained minimization problem, then the gradient

$$
\sum_{i=1}^{n} r\left(x-a_{i}\right)\left\|x-a_{i}\right\|^{r-2}
$$

must vanish at $\mu(r)$. By viewing the above gradient as a $C^{\infty}$-function $g$ of the pair $(x, r)$, the Implicit Function Theorem will prove the theorem if the Jacobian
of $g$ with respect to $x$ is nonsingular at all $(\mu(r), r)$, with $r>1$, and $\mu(r) \neq a_{i}$. Indeed, the Jacobian is

$$
\begin{aligned}
J_{x} g(x, r): h \mapsto & r \sum_{i=1}^{n}\left((r-2)\left\|x-a_{i}\right\|^{r-4}\left(x-a_{i}\right)\left(x-a_{i}\right)^{T} h-\left\|x-a_{i}\right\|^{r-2} h\right) \\
& =r\left\|x-a_{i}\right\|^{r-4} \sum_{i=1}^{n}\left((r-2)\left(x-a_{i}\right)\left(x-a_{i}\right)^{T} h+\left\|x-a_{i}\right\|^{2} h\right),
\end{aligned}
$$

which is positive definite since

$$
\begin{aligned}
& h^{T} J_{x} g(x, r) h \\
& =r\left\|a_{i}-x\right\|^{r-4} \sum_{i=1}^{n}\left((r-2)\left(\left(a_{i}-x\right)^{T} h\right)^{2}+\left\|a_{i}-x\right\|^{2}\|h\|^{2}\right) \\
& \geq \begin{cases}r\left\|a_{i}-x\right\|^{r-4} \sum\left\|a_{i}-x\right\|^{2}\|h\|^{2} & \text { when } r \geq 2, \\
r\left\|a_{i}-x\right\|^{r-4} \sum\left((r-2)\left\|a_{i}-x\right\|^{2}\|h\|^{2}+\left\|a_{i}-x\right\|^{2}\|h\|^{2}\right) & \text { when } r \in(1,2)\end{cases} \\
& = \begin{cases}r\left\|a_{i}-x\right\|^{r-4} \sum\left\|a_{i}-x\right\|^{2}\|h\|^{2} & \text { when } r \geq 2, \\
r(r-1)\left\|a_{i}-x\right\|^{r-4} \sum\left\|a_{i}-x\right\|^{2}\|h\|^{2} & \text { when } r \in(1,2)\end{cases}
\end{aligned}
$$

$>0 \quad$ whenever $h \neq 0$.

Remark 6.1. When $\mu(r)$ coincides with one of the given points $a_{j}$, then the gradient

$$
g(x)=\sum_{i \in\{1, \ldots, n\} \backslash\{j\}} r\left(x-a_{i}\right)\left\|x-a_{i}\right\|^{r-2}+r \operatorname{sgn}\left(x-a_{j}\right)\left\|x-a_{j}\right\|^{r-1},
$$

where

$$
\operatorname{sgn}(x)= \begin{cases}x /\|x\| & \text { if } x \neq 0 \\ 0 & \text { otherwise }\end{cases}
$$

is no longer $C^{\infty}$ at $(\mu(r), r)$. In fact it should generally just be $C^{k}$, where $k$ is the least integer no less than $r-1$, since the term $r \operatorname{sgn}\left(x-a_{j}\right)\left\|x-a_{j}\right\|^{r-1}$ is only $C^{k}$ while the other summands are $C^{\infty}$ at $(\mu(r), r)$. In this case, the Implicit Function Theorem can only conclude that $\mu(r)$ is $C^{k}$-smooth at $r$. The exception is when $r$ is an even integer, in which case the gradient is still $C^{\infty}$.

### 6.4 Invariance of the $\mu$-curve under rigid motions and uniqueness of

 the $\mu$-curve
### 6.4 Invariance of the $\mu$-curve under rigid motions and uniqueness of the $\mu$-curve

A natural question that arises from the definition of the $\mu$-curve is what will happen to the curve after an orthogonal transformation to the point set. Here we prove that our $\mu$-curve bears very nice invariance property under rigid motions.

Theorem 6.1. (Wang) Let $S$ be a finite point set in $\mathbb{R}^{d}$, and $\mu(r)=\arg \min _{x \in \mathbb{R}^{d}} \sum_{a \in S} \| a-$ $x\left|\left.\right|^{r}\right.$.
(i) Fix any $\alpha \in \mathbb{R}_{\neq 0}$. Dilate $S$ by a factor $\alpha$, and denote the dilated point set by $\alpha S=\{\alpha a \mid a \in S\}$. Then $\mu_{\alpha S}(r)=\alpha \mu_{S}(r)$ for $\forall r \in[1, \infty)$.
(ii) Let $M \in \mathbb{O}(d)$ be a $d \times d$ orthogonal matrix, then $\mu_{M S}(r)=M\left(\mu_{S}(r)\right)$ for $\forall r \in[1, \infty)$.

Proof. (i) Define

$$
\mu_{S}(r)=\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S}\|a-x\|^{r},
$$

then we have

$$
\begin{aligned}
\mu_{\alpha S}(r) & =\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{b \in \alpha S}\|b-x\|^{r}, \\
& =\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S}\|\alpha a-x\|^{r}, \\
& =\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S} \alpha^{r}\|a-x / \alpha\|^{r}, \\
& =\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S}\|a-x / \alpha\|^{r}, \\
& =\alpha \mu_{S}(r) .
\end{aligned}
$$

(ii)

$$
\begin{align*}
\mu_{M S} & =\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{b \in M S}\|b-x\|^{r}, \\
& =\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S}\|M a-x\|^{r}, \\
& =\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S}\langle M a-x, M a-x\rangle^{\frac{r}{2}}, \\
& =\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S}\left\langle M^{t}(M a-x), M^{t}(M a-x)\right\rangle^{\frac{r}{2}},  \tag{6.2}\\
& =\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S}\left\langle a-M^{t} x, a-M^{t} x\right\rangle^{\frac{r}{2}}, \\
& =\underset{x \in \mathbb{R}^{d}}{\arg \min } \sum_{a \in S}\left\|a-M^{-1} x\right\|^{r}, \\
& =M \mu_{S}(r) .
\end{align*}
$$

Here equation (6.2) holds due to the fact that $\langle x, y\rangle=\left\langle M^{t} x, M^{t} y\right\rangle$ where $M$ therefor $M^{t}$ is an orthogonal matrix.

Another question that we can ask about our $\mu$-curve is whether it is uniquely determined by the point set or not. Asides from degenerate cases where $\mu$-curve is a single point, we conjecture that our $\mu$-curve is uniquely determined by the point set, in other words, given a $\mu$-curve, we can always find out what the original point set is assuming we know the number of points in $S$. The intuition comes from the Hilbert Basis Theorem[CLO97] which tells us that every ideal $I \subset k\left[x_{1}, \cdots, x_{n}\right]$ has a finite generating set. That is, $I=<g_{1}, \cdots, g_{s}>$ for some $g_{1}, \cdots, g_{s} \in I$, where $k$ is a field, and $k\left[x_{1}, \cdots, x_{n}\right]$ denotes the set of all polynomials in $x_{1}, \cdots, x_{n}$ with coefficients in $k$.

By using this result, we conjecture that the nondegenerate $\mu$-curve is uniquely determined by the point set.

Conjecture 6.1. The nondegenerate $\mu$-curve is uniquely determined by a point set $S$.

### 6.4 Invariance of the $\mu$-curve under rigid motions and uniqueness of the $\mu$-curve

Suppose we are give a $\mu$-curve, and assume the point set which the $\mu$-curve is defined on is $S=\left\{a_{1}, a_{2}, \cdots, a_{n}\right\}$, from our previous argument, we have the following formula:

$$
\mu(r)=\frac{\sum_{i=1}^{n} a_{i}\left\|\mu(r)-a_{i}\right\|^{r-2}}{\sum_{i=1}^{n}\left\|\mu(r)-a_{i}\right\|^{r-2}},
$$

which is essentially a set of infinitely many equations about the coordinates of $a_{i}$. According to Hilbert Basis Theorem, we know that every ideal in a polynomial ring over $\mathbb{R}$ is finitely generated, therefore, these equations are also finitely generated and there exists a solution. In other words, we can find what the original point set S is by solving these equations.

Example 6.2. In order to show how a $\mu$-curve is determined by a finite point set, we give an example here. Suppose we are given a $\mu$-curve of some point set which has 3 points in $\mathbb{R}^{2}$. We denote this point set by $S=\{(a, b),(c, d),(e, f)\} \in \mathbb{R}^{2}$. And suppose we know some of the points on the $\mu$-curve with respect to different r. For example, assume $\mu(2)=(3,3), \mu(4)=(3.1984,2.8016)$, and $\mu(6)=(3.2502,2.7498)$ (In fact, these points are on the $\mu$-curve of point set $\left.S_{0}=\{(1,3),(5,1),(3,5)\}\right)$. We will try to recover $S_{0}$ from three points on its $\mu$-curve. According to

$$
\mu(r)=\frac{\sum_{i=1}^{n} a_{i}\left\|\mu(r)-a_{i}\right\|^{r-2}}{\sum_{i=1}^{n}\left\|\mu(r)-a_{i}\right\|^{r-2}},
$$

we will be able to get 6 equations for $r=2,4,6$.
$\diamond$ When $r=2$, we have:

$$
\left\{\begin{array}{l}
3=\frac{a+c+e}{3}, \\
3=\frac{b+d+e}{3} .
\end{array}\right.
$$

$\diamond$ When $r=4$, we denote the distance between $(a, b)$ and $\mu(4)=(3.1984,2.8016)$ by $D_{1}$, the distance between $(c, d)$ and $\mu(4)=(3.1984,2.8016)$ by $D_{2}$, and
the distance between $(e, f)$ and $\mu(4)=(3.1984,2.8016)$ by $D_{3}$, then we have:

$$
\left\{\begin{array}{l}
3.1984=\frac{a \cdot D_{1}^{2}+c \cdot D_{2}^{2}+e \cdot D_{3}^{2}}{D_{1}^{2}+D_{2}^{2}+D_{3}^{2}}, \\
2.8016=\frac{b \cdot D_{1}^{2}+d \cdot D_{2}^{2}+f \cdot D_{3}^{2}}{D_{1}^{2}+D_{2}^{2}+D_{3}^{2}} .
\end{array}\right.
$$

Here $D_{1}=\sqrt{(3.1984-a)^{2}+(2.8016-b)^{2}}, D_{2}=\sqrt{(3.1984-c)^{2}+(2.8016-d)^{2}}$, and $D_{3}=\sqrt{(3.1984-e)^{2}+(2.8016-f)^{2}}$.
$\diamond$ When $r=6$, we denote the distance between $(a, b)$ and $\mu(6)=(3.2502,2.7498)$ by $D_{4}$, the distance between $(a, b)$ and $\mu(6)=(3.2502,2.7498)$ by $D_{5}$, and the distance between $(a, b)$ and $\mu(6)=(3.2502,2.7498)$ by $D_{6}$, then we have:

$$
\left\{\begin{array}{l}
3.2502=\frac{a \cdot D_{4}^{4}+c \cdot D_{5}^{4}+e \cdot D_{6}^{4}}{D_{4}^{4}+D_{5}^{4}+D_{6}^{4}}, \\
2.7498=\frac{b \cdot D_{4}^{4}+d \cdot D_{5}^{4}+f \cdot D_{6}^{4}}{D_{4}^{4}+D_{5}^{4}+D_{6}^{4}} .
\end{array}\right.
$$

Here $D_{4}=\sqrt{(3.2502-a)^{2}+(2.7498-b)^{2}}, D_{5}=\sqrt{(3.2502-c)^{2}+(2.7498-d)^{2}}$, and $D_{6}=\sqrt{(3.2502-e)^{2}+(2.7498-f)^{2}}$.

Put these 6 equations together:

$$
\left\{\begin{array}{l}
3=\frac{a+c+e}{3}, \\
3=\frac{b+d+e}{3}, \\
3.1984=\frac{a \cdot D_{1}^{2}+c \cdot D_{2}^{2}+e \cdot D_{3}^{2}}{D_{1}^{2}+D_{2}^{2}+D_{3}^{2}}, \\
2.8016=\frac{b \cdot D_{1}^{2}+d \cdot D_{2}^{2}+f \cdot D_{3}^{2}}{D_{1}^{2}+D_{2}^{2}+D_{3}^{2}}, \\
3.2502=\frac{a \cdot D_{4}^{4}+c \cdot D_{5}^{4}+e \cdot D_{6}^{4}}{D_{4}^{4}+D_{5}^{4}+D_{6}^{4}}, \\
2.7498=\frac{b \cdot D_{4}^{4}+d \cdot D_{5}^{4}+f \cdot D_{6}^{4}}{D_{4}^{4}+D_{5}^{4}+D_{6}^{4}}
\end{array}\right.
$$

The solution is ${ }^{1}$ :

$$
\left\{\begin{array}{l}
a=0.725 \approx 1 \\
b=2.859 \approx 3 \\
c=5.259 \approx 5 \\
d=0.893 \approx 1 \\
e=3.016 \approx 3 \\
f=5.249 \approx 5
\end{array}\right.
$$

From this example, we conjecture that nondegenerate $\mu$-curves are uniquely determined by point sets.

### 6.5 Degeneracy of the $\mu$-curve and symmetric properties of a point set

The following result gives a necessary and sufficient condition for the $\mu$-curve to consist only of the origin.

Theorem 6.7. (Wang) The $\mu$-curve is $\{0\}$ if and only if the point set $S$ lies on a finite union of concentric spheres centered at the origin, and satisfies the following property: those points lying on each sphere have the origin as their center of mass.

Proof. By the definition of $\mu$-curve, we know that

$$
\mu(r)=\frac{\sum_{i=1}^{n} r a_{i}\left\|\mu(r)-a_{i}\right\|^{r-2}}{\lambda(\mu(r))},
$$

where

$$
\lambda(\mu(r))=\sum_{i=1}^{n} r\left\|\mu(r)-a_{i}\right\|^{r-2} .
$$

[^3]When $\mu(r)=0$ for $\forall r \in(1, \infty)$, we have

$$
0=\frac{\sum_{i=1}^{n} a_{i}\left\|a_{i}\right\|^{r-2}}{\sum_{i=1}^{n}\left\|a_{i}\right\|^{r-2}},
$$

i.e.

$$
0=\sum_{i=1}^{n} \frac{a_{i}}{\left\|a_{i}\right\|}\left\|a_{i}\right\|^{r-1} .
$$

Let $S_{1}=\left\{a_{i} \in S: a_{i}\right.$ has the longest norm $L$ among all points in $\left.S\right\}$, the above equation can be written as

$$
0=\sum_{a_{i} \in S_{1}} \frac{a_{i}}{\left\|a_{i}\right\|} L^{r-1}+\sum_{a_{i} \in S-S_{1}} \frac{a_{i}}{\left\|a_{i}\right\|}\left\|a_{i}\right\|^{r-1},
$$

divide both sides by $L^{r-1}$, we get

$$
0=\sum_{a_{i} \in S_{1}} \frac{a_{i}}{\left\|a_{i}\right\|}+\sum_{a_{i} \in S-S_{1}} \frac{a_{i}}{\left\|a_{i}\right\|}\left(\frac{\left\|a_{i}\right\|}{L}\right)^{r-1} .
$$

Let $r$ approach $\infty$, notice that $\lim _{r \rightarrow \infty}\left(\frac{\left\|a_{i}\right\|}{L}\right)^{r-1}=0$ for $\forall r \in(1, \infty)$, we have

$$
0=\sum_{a_{i} \in S_{1}} \frac{a_{i}}{\left\|a_{i}\right\|}
$$

In other words, for those points with the longest norm $L$, their center of mass is the origin. The same argument can apply to all the points that are left. Thus we get our conclusion that if $\mu$-curve is $\{0\}$ for a point set $S$, then $S$ lies on a finite union of concentric spheres whose center is the origin and for those points lying on the same sphere, their center of mass is the origin.

Sufficiency follows by checking the (sufficient) optimality condition that defines our $\mu$-curve; in other words, checking the gradient:

$$
G(x)=-r \sum_{i=1}^{n}\left\|a_{i}-x\right\|^{r-2}\left(a_{i}-x\right)=0 .
$$

Notice that $G(x)$ can be written as:

$$
G(x)=-r \sum_{i=1}^{n} \frac{a_{i}-x}{\left\|a_{i}-x\right\|}\left\|a_{i}-x\right\|^{r-1} .
$$

If $S$ satisfies the property given in the theorem, we can classify all the points in $S$ according to their norms then $G(x)=0$ becomes obvious.

Another way to prove necessity is to use Vandermonde matrix defined by the coefficients of the equation

$$
0=\sum_{i=1}^{n}\left(\frac{a_{i}}{\left\|a_{i}\right\|}\right)\left\|a_{i}\right\|^{r-1}
$$

by letting $r$ be $1,2,3, \cdots, l$ where $l$ is the number of classes of $a_{i}$ due to the length of their norms.

Corollary 6.2. (Wang) If the point set $S \subset \mathbb{R}^{d}$ has radial symmetry about the origin, then the curve $\mu=\{0\}$.

Proof. For any point $p \in S$, by assumption we also have $-p \in S$. Thus, the sphere that passes through $p$ and $-p$ has the origin as the center of mass of the subcollection of two points $\{p,-p\}$. If we think of $S$ as a finite union of such antipodal pairs of points, each pair lying on its own sphere, then Theorem 6.7 shows that $\mu=\{0\}$.

To illustrate Corollary 6.7, the following figure shows a point set $S$ that is comprised of three subsets, where each of these subsets lies on a circle with the center of mass of that subset being the origin. We notice that the $\mu$-curve for this point set $S$ being exclusively $\{0\}$ which is fully explained by Theorem 6.7.


Figure 6.1: Example of the $\mu$-curve for a point set on concentric circles

If we wish to connect the ideas of radial symmetry to rotational symmetry, we can use Corollary 6.2 , but only for $\mathbb{R}^{2}$. In the case of $\mathbb{R}^{2}$, we have radial symmetry if and only if we have a rotation by angle equal to $\pi$ radians. However, if dimension $d \geq 3$, this equivalence is no longer valid. The two notions are now distinct, and further research is required to find more connections between rotationally symmetric point sets and the $\mu$-curve.

Corollary 6.3. (Wang) Let $S$ be any finite set of points in $\mathbb{R}^{d}$. If $S$ has d linearly independent hyperplanes of symmetry, then the points of $S$ must lie on spherical shells, all centered at the point of intersection of the hyperplanes, such that the center of mass of the points on each shell is this point of intersection.

Proof. If $S$ has $d$ linearly independent hyperplanes of symmetry, then by Corollary 6.4, our $\mu$-curve must consist of only the point of intersection. But by Theorem 6.7, the $\mu$-curve is this point if and only if $S$ has the desired property above, namely the "shell" property.

### 6.6 The moments of $S$ and their relation with the distance between any two points on the $\mu$-curve

We know for any point set in $\mathbb{R}^{1}$, the following inequality is true:

$$
\|\mu(2)-\mu(1)\| \leq \sigma,
$$

where $\sigma$ is the usual standard deviation of the point set $S$.
We will extend the inequality above in a very general way. As mentioned in the beginning of our paper, define the $t$-th moment of a given point set $S \subset \mathbb{R}^{d}$ as follows:

$$
M_{t}(x)=\sum_{a_{i} \in S}\left\|a_{i}-x\right\|^{t}
$$

for each real $t \geq 1$. With this definition, the $\mu$-curve can be interpreted as follows:

$$
\mu(r)=\underset{x \in \mathbb{R}^{d}}{\arg \min } M_{r}(x) .
$$

Or equivalently, each point on our curve minimizes the $r$-th moment of the point set $S$ among all possible points $x \in \mathbb{R}^{d}$.

We can also give the following interpretation to $\sigma$ in terms of the second moment of our point set: $\sigma^{2}=\frac{M_{2}(0)}{\|S\|}$, where $M_{2}(x)=\sum_{a_{i} \in S}\left\|a_{i}-x\right\|^{2}$ is the second moment of the point set $S$, and $\|S\|$ denotes the number of points in $S$. In [GL00], there is a whole chapter dedicated to moments related theory and we found a strong connection between our $\mu$-curve and the center of $P$ of order $r$ defined there. We give a brief idea of what the center of $P$ of order $r$ means.

Definition 6.6. Let $\mathbf{X}=\left(X_{1}, \cdots, X_{d}\right)$ be a $\mathbb{R}^{d}$-valued random variable with distribution $P$. Let $1 \leq r<\infty$ and assume that $E\left(\|\mathbf{X}\|^{r}\right)<\infty$, where $\|\cdot\|$
denotes any norm on $\mathbb{R}^{d}$. Define a center of $P$ of order $r$ by a point $\mathbf{a} \in \mathbb{R}^{d}$ such that

$$
E\|\mathbf{X}-\mathbf{a}\|^{r}=\inf _{\mathbf{b} \in \mathbb{R}^{d}} E\|\mathbf{X}-\mathbf{b}\|^{r}
$$

The $r$-th absolute moment of $P$ about the center is defined by

$$
V_{r}(P)=\inf _{\mathbf{a} \in \mathbb{R}^{d}} E\|\mathbf{X}-\mathbf{a}\|^{r} .
$$

If $S$ is a finite point set with $S=\left\{\mathbf{X}_{\mathbf{1}}, \mathbf{X}_{\mathbf{2}}, \cdots, \mathbf{X}_{\mathbf{N}}\right\}$ with distribution $P$, then $E(S)=\sum_{i=1}^{N} \mathbf{X}_{\mathbf{i}} P\left(\mathbf{X}_{\mathbf{i}}\right)$. If all points in $S$ are uniformly distributed, then $P\left(\mathbf{X}_{\mathbf{i}}\right)=\frac{1}{N}$, and $E(S)=\frac{1}{N} \sum_{i=1}^{N} \mathbf{X}_{\mathbf{i}}$. In this case, the above $V_{r}(P)$ can be written as

$$
V_{r}(P)=\inf _{\mathbf{a} \in \mathbb{R}^{d}} \frac{1}{N} \sum_{i=1}^{N}\left\|\mathbf{X}_{\mathbf{i}}-\mathbf{a}\right\|^{r}
$$

which is actually $\frac{1}{N} \inf M_{r}(x)$, while our focus is on $\arg \min M_{r}(x)$. They are closely related in the sense that once we know what $\mu(r)=\arg \min M_{r}(x)$ is, it will naturally leads us to $V_{r}(P)$.

In order to prove our main theorem, we need Conjecture 6.2. Experimental results show the correctness of it though at this point, we still have no idea of how to prove it.

Conjecture 6.2. $\|(\mu(r)-\mu(1))\| \leq \frac{\sum_{i=1}^{N}\left\|\left(\mu(r)-X_{i}\right)\right\|}{N}$.

If this conjecture is true, then we can find a nice bound for the $r$-th moment defined above.

## Theorem 6.8.

$$
\|\mu(r)-\mu(1)\| \leq\left(\frac{M_{k}(\mu(r))}{N}\right)^{\frac{1}{k}}
$$

where $k \in(1, \infty)$.

Proof.

$$
\begin{align*}
\|\mu(r)-\mu(1)\| & \leq E\left(\left\|\left(X_{i}-\mu(r)\right)\right\|\right.  \tag{6.3}\\
& =\frac{\sum_{i=1}^{N}\left\|X_{i}-\mu(r)\right\|}{N} \\
& =\frac{\sum_{i=1}^{N}\left(\left\|X_{i}-\mu(r)\right\|^{k}\right)^{\frac{1}{k}}}{N} \\
& \leq\left(\frac{\sum_{i=1}^{N}\left\|X_{i}-\mu(r)\right\|^{k}}{N}\right)^{\frac{1}{k}}  \tag{6.4}\\
& =\left(\frac{M_{k}(\mu(r))}{N}\right)^{\frac{1}{k}}
\end{align*}
$$

(6.3) holds due to Conjecture 6.2, since

$$
\sum_{i=1}^{N}\|(\mu(r)-\mu(1))\| \leq \sum_{i=1}^{N}\left\|\left(\mu(r)-X_{i}\right)\right\|
$$

therefore we have

$$
\begin{aligned}
\|\mu(r)-\mu(1)\| & =\frac{\sum_{i=1}^{N}\|(\mu(r)-\mu(1))\|}{N} \\
& \leq \frac{\sum_{i=1}^{N}\left\|\left(\mu(r)-X_{i}\right)\right\|}{N} \\
& =E\left(\left\|\left(X_{i}-\mu(r)\right)\right\|\right)
\end{aligned}
$$

(6.4) is true because the function $f(x)=x^{\frac{1}{k}}, k>1$ is concave, and Jensen's inequality tells us that when a function $f$ is concave, we'll have $f\left(\frac{\sum x_{i}}{N}\right) \geq$ $\frac{\sum f\left(x_{i}\right)}{N}$, and in our case $x_{i}=\left\|X_{i}-\mu(r)\right\|^{k}$. When $k=1$, this theorem is identical with Conjecture 6.2.

### 6.7 Examples of some $\mu$-curves

Here we obtain experimental information about the shape and other characteristics of some $\mu$-curves.

In the following example, the point set given by blue '*' consists of $\{(0,0),,(-$ $1,2),(3,0)\}$. The $\mu$-curve is plotted with step size 0.01 and $r \in[1,3]$. The first few
values of $r$ are plotted with green ' O ', while the last few values of $r$ are plotted with magenta ${ }^{\text {(*) }}$ to distinguish the beginning and the end of the curve. The same rules also apply to other examples followed.


Figure 6.2: the $\mu$-curve for $S=\{(0,0),(-1,2),(3,0)\}$

In this example, the way we plot out $\mu$-curve is to use iteration to approach to the best possible solution $\mu(r)$ in finite many steps. MATLAB also gives us another option to draw $\mu(r)$ directly for any given $r$. For any given $r$, in order to find $\mu(r)$, we can solve an equation system given by the extension of Weiszfeld's algorithm. In $\mathbb{R}^{2}$, we can suppose the coordinates of $\mu(r)$ are $x$ and $y$, and call the two equations given by (6.1) $F(x, y, r)$ and $G(x, y, r)$. We can plot the curves $F(x, y, r)$ and $G(x, y, r)$ in $\mathbb{R}^{2}$ when $r$ is fixed. An example when $r=6$ for the
point set $S=\{(0,0),,(-1,2),(3,0)\}$ is given below. Denote $\mu(6)$ by $(x, y)$. Then $F(x, y, 6)$ and $G(x, y, 6)$ is shown as follows:

$$
\left\{\begin{array}{l}
F(x, y, 6)=x-\frac{\sum_{i=1}^{3} a_{i}\left\|\mu(6)-a_{i}\right\|^{4}}{\sum_{i=1}^{3}\left\|\mu(6)-a_{i}\right\|^{4}}=0 \\
G(x, y, 6)=y-\frac{\sum_{i=1}^{3} a_{i}\left\|\mu(6)-a_{i}\right\|^{4}}{\sum_{i=1}^{3}\left\|\mu(6)-a_{i}\right\|^{4}}=0 .
\end{array}\right.
$$

We can write these two equations explicitly:

$$
\left\{\begin{array}{l}
F(x, y, 6)=x-\frac{-1 \cdot\left((x+1)^{2}+(y-2)^{2}\right)^{2}+3 \cdot\left((x-3)^{2}+y^{2}\right)^{2}}{\left(x^{2}+y^{2}\right)^{2}+\left((x+1)^{2}+(y-2)^{2}\right)^{2}+\left((x-3)^{2}+y^{2}\right)^{2}}=0 \\
G(x, y, 6)=y-\frac{2 \cdot\left((x+1)^{2}+(y-2)^{2}\right)^{2}}{\left(x^{2}+y^{2}\right)^{2}+\left((x+1)^{2}+(y-2)^{2}\right)^{2}+\left((x-3)^{2}+y^{2}\right)^{2}}=0
\end{array}\right.
$$

we can get $\mu(6)=(0.9595,0.9317)$ by iteration process. The numerical result is consistent with the graphic result shown below.


Figure 6.3: $\mu(6)$ as the solution to an equation system

In the following example, the point set given by blue '*' consists of $\{(-0.8,-$ $0.2),(0.7,-0.4),(0.3,0.5)\}$. The $\mu$-curve is plotted with step size 0.01 and $r \in[1,3]$.


Figure 6.4: the $\mu$-curve for $S=\{(-0.8,-0.2),(0.7,-0.4),(0.3,0.5)\}$

In the following example, the point set given by blue '*' consists of $\{(0,0),(3,3),(4,0)\}$. The $\mu$-curve is plotted with step size 0.05 and $r \in[1,15]$.


Figure 6.5: the $\mu$-curve for $S=\{(0,0),(3,3),(4,0)\}$

In the following example, the point set given by blue '*' consists of $\{(0,0),(3,0),(0,2)\}$. The $\mu$-curve is plotted with step size 0.05 and $r \in[1,15]$.


Figure 6.6: the $\mu$-curve for $S=\{(0,0),(3,0),(0,2)\}$

In the following example, the point set given by blue '*' consists of $\{(-0.6,-0.2),(0.4,0),(-0.3,0.5),(0.2,0.7)\}$. The $\mu$-curve is plotted with step size 0.01 and $r \in[1,3]$.


Figure 6.7: the $\mu$-curve for $S=\{(-0.6,-0.2),(0.4,0),(-0.3,0.5),(0.2,0.7)\}$

In the following example, the point set given by blue ${ }^{\text {'* }}$ ' consists of $\{(1,3),(2,0),(4,4),(4,1)\}$. The $\mu$-curve is plotted with step size 0.05 and $r \in[1,15]$.


Figure 6.8: the $\mu$-curve for $S=\{(1,3),(2,0),(4,4),(4,1)\}$

In the following example, the point set given by blue '*' consists of $\{(1,4),(2,0),(3,2),(5,4)\}$. The $\mu$-curve is plotted with step size 0.05 and $r \in[1,15]$.


Figure 6.9: the $\mu$-curve for $S=\{(1,4),(2,0),(3,2),(5,4)\}$

In the following example, the point set given by blue ${ }^{*}$ ' consists of $\{(1,1),(1,3),(4,4),(5,1),(2,0)\}$. The $\mu$-curve is plotted with step size 0.05 and $r \in[1,15]$.


Figure 6.10: the $\mu$-curve for $S=\{(1,1),(1,3),(4,4),(5,1),(2,0)\}$

In the following example, the point set $S=\{(1,1),(1,3),(4,4),(2,1),(2,0)\}$ is shown as blue ${ }^{(*)}$. The $\mu$-curve is plotted with step size 0.05 and $r \in[1,15]$.


Figure 6.11: the $\mu$-curve for $S=\{(1,1),(1,3),(4,4),(2,1),(2,0)\}$

In the following example, the point set $S=\{(2,3,7),(4,1,5),(1,6,4)\}$ is in three dimension, shown as blue ' $*$ '. The $\mu$-curve is plotted with step size 0.05 and $r \in[1,30]$.


Figure 6.12: the $\mu$-curve for $S=\{(2,3,7),(4,1,5),(1,6,4)\}$

In the following example, the point set is in three dimension, given by blue '*, consists of $\{(0,0,0),(3,2,6),(2,2,0),(6,2,5)\}$. The $\mu$-curve is plotted with step size 0.05 and $r \in[1,30]$.


Figure 6.13: the $\mu$-curve for $S=\{(0,0,0),(3,2,6),(2,2,0),(6,2,5)\}$

In the following example, the point set is in 3 dimension, given by blue ${ }^{\text {'* }}$, consists of $\{(1,3,2),(4,0,5),(2,6,7),(5,3,1)\}$. The $\mu$-curve is plotted with step size 0.05 and $r \in[1,30]$.


Figure 6.14: the $\mu$-curve for $S=\{(1,3,2),(4,0,5),(2,6,7),(5,3,1)\}$

In conclusion, $\mu$-curve tends to be degenerate if the point set has symmetric property to some extent; otherwise $\mu$-curve is just a nondegenerate curve lying inside the convex hull of the point set.


## Brion's theorem

The following theorems and their proofs can be found in [BR07]. A convex $d$ polytope with exactly $d+1$ vertices is called a $d$-simplex.

Theorem A. 1 (Brianchon-Gram identity for simplices). Let $\Delta$ be a d dimensional polytope. Then

$$
1_{\Delta}(\mathbf{x})=\sum_{\mathcal{F} \subseteq \Delta}(-1)^{\operatorname{dim} \mathcal{F}} 1_{\mathcal{K}_{\mathcal{F}}}(\mathbf{x})
$$

where the sum is taken over all nonempty faces $\mathcal{F}$ of $\Delta .1_{\Delta}(\mathbf{x})$ denotes the indicator function of simplex $\Delta$ as defined in the first chapter.

Let $S$ be any subset of $\mathbb{R}^{d}$, define

$$
\sigma_{S}(\mathbf{z})=\sigma_{S}\left(z_{1}, \cdots, z_{d}\right):=\sum_{\mathbf{m} \in S \cap \mathbb{Z}^{d}} \mathbf{z}^{\mathbf{m}} .
$$

We call $\sigma_{S}$ the integer-point transform of $S$.
Corollary A. 1 (Brion's theorem for simplices). Suppose $\Delta \subseteq \mathbb{R}^{d}$ is a rational polytope, and $\mathbf{z} \in \mathbb{C}^{d}$. Then we have the following identity of rational functions:

$$
\sigma_{\Delta}(\mathbf{z})=\sum_{\mathbf{v} \text { a vertex of } \Delta} \sigma_{\mathcal{K}_{\mathbf{v}}}(\mathbf{z}) .
$$

Theorem A. 2 (Brion's theorem: discrete form). Suppose $\mathcal{P}$ is a rational convex polytope in $\mathbb{R}^{d}$, and $\mathbf{z} \in \mathbb{C}^{d}$. Then we have the following identity of rational functions:

$$
\begin{equation*}
\sigma_{\mathcal{P}}(\mathbf{z})=\sum_{\mathrm{v}: a \text { vertex of } \mathcal{P}} \sigma_{\mathcal{K}_{\mathrm{v}}}(\mathbf{z}) . \tag{A.1}
\end{equation*}
$$

Theorem A. 3 (Brion's theorem: continuous form). Suppose $\mathcal{P}$ is a simple rational convex d-polytope. For a vertex cone $\mathcal{K}_{\mathbf{v}}$ of $\mathcal{P}$, fix a set of generators

$$
\mathbf{w}_{\mathbf{1}}(\mathbf{v}), \mathbf{w}_{\mathbf{2}}(\mathbf{v}), \cdots, \mathbf{w}_{\mathbf{d}}(\mathbf{v}) \in \mathbb{Z}^{d}
$$

Then

$$
\begin{align*}
\int_{\mathcal{P}} \exp (\mathbf{x} \cdot \mathbf{z}) d \mathbf{x} & =(-1)^{d} \sum_{\mathbf{v} \text { a vertex of } \mathcal{P}} \frac{\exp (\mathbf{v} \cdot \mathbf{z})\left|\operatorname{det}\left(\mathbf{w}_{\mathbf{1}}(\mathbf{v}), \mathbf{w}_{\mathbf{2}}(\mathbf{v}), \cdots, \mathbf{w}_{\mathbf{d}}(\mathbf{v})\right)\right|}{\prod_{k=1}^{d}\left(\mathbf{w}_{\mathbf{k}}(\mathbf{v}) \cdot \mathbf{z}\right)} \\
& =(-1)^{d} \sum_{\mathbf{v} \text { a vertex of } \mathcal{P}} \frac{\exp (\mathbf{v} \cdot \mathbf{z})\left|\operatorname{det} \mathcal{K}_{\mathbf{v}}\right|}{\prod_{k=1}^{d}\left(\mathbf{w}_{\mathbf{k}}(\mathbf{v}) \cdot \mathbf{z}\right)} \tag{A.2}
\end{align*}
$$

for all $\mathbf{z}$ such that the denominators on the right-hand side do not vanish.
Note that the left hand side of (A.2) is the Fourier transform of polytope $\mathcal{P}$, and it's equal to the sum of the Fourier transform of its tangent cones at their vertices $\mathbf{v}$.

## $\sqrt{4 m a x a t i x}$

## The Stretch Theorem and Shift Theorem for the Fourier transform on $\mathbb{R}^{d}$

Consider real- or complex-valued functions $f \in L_{1}(\mathbb{R})$ defined on $\mathbb{R}^{d}$. Let $\mathbf{x}=\left(x_{1}, \cdots, x_{d}\right) \in \mathbb{R}^{d}$. The Fourier transform of $f(\mathbf{x})$ for $\forall \xi=\left(\xi_{1}, \cdots, \xi_{d}\right) \in \mathbb{R}^{d}$ is defined by [Osg07]:

$$
\widehat{f}(\xi)=\int_{\mathbb{R}^{d}} f(\mathbf{x}) e^{-2 \pi i\langle\mathbf{x}, \xi\rangle} d \mathbf{x}
$$

We will introduce a few basic facts about higher dimensional Fourier transform here.

Theorem B. 1 (Stretch Theorem). Let $M \in G L_{n}(\mathbb{R})$, the general linear group over $\mathbb{R}$. For any function $f \in L_{1}(\mathbb{R})$, and $\forall \xi=\left(\xi_{1}, \cdots, \xi_{d}\right) \in \mathbb{R}^{d}$, the following identity holds:

$$
\widehat{f \circ M}(\xi)=\frac{1}{|\operatorname{det} M|} \widehat{f}\left(M^{-T} \xi\right)
$$

Theorem B. 2 (Shift Theorem). Suppose $\xi=\left(\xi_{1}, \cdots, \xi_{d}\right) \in \mathbb{R}^{d}$ and $\mathbf{b}=\left(b_{1}, b_{2}, \cdots, b_{d}\right) \in \mathbb{R}^{d}$ is a constant vector. For any function $f \in L_{1}(\mathbb{R})$, if $f(\mathbf{x}) \rightleftharpoons F(\xi)$, or in other words, the Fourier transform of $f(\mathbf{x})$ is denoted by $F(\xi)$, then the following identity holds:

$$
f(\mathbf{x}+\mathbf{b}) \rightleftharpoons e^{2 \pi i \mathbf{b} \cdot \xi} F(\xi)
$$

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Theorem B. 3 (Frequency Shift Theorem). Suppose $\xi=\left(\xi_{1}, \cdots, \xi_{d}\right) \in \mathbb{R}^{d}$ and $\xi^{0}=\left(\xi_{1}^{0}, \xi_{2}^{0}, \cdots, \xi_{d}^{0}\right) \in \mathbb{R}^{d}$ is a constant vector. For any function $f \in L_{1}(\mathbb{R})$, if $f(\mathbf{x}) \rightleftharpoons F(\xi)$, or in other words, the Fourier transform of $f(\mathbf{x})$ is denoted by $F(\xi)$, then the following identity holds:

$$
F\left(\xi-\xi^{\mathbf{0}}\right) \rightleftharpoons e^{2 \pi i \xi^{\mathbf{0}} \cdot \mathbf{x}} f(\mathbf{x})
$$

Proof of Theorem B.1.

$$
\begin{align*}
\widehat{f \circ M}(\xi) & =\int_{\mathbb{R}^{d}} f(M \mathbf{x}) e^{-2 \pi i(\mathbf{x}, \xi\rangle} d \mathbf{x},  \tag{B.1}\\
& =\frac{1}{|\operatorname{det} M|} \int_{\mathbb{R}^{d}} f(\mathbf{y}) e^{-2 \pi i\left\langle M^{-1} \mathbf{y}, \xi\right\rangle} d \mathbf{y},  \tag{B.2}\\
& =\frac{1}{|\operatorname{det} M|} \int_{\mathbb{R}^{d}} f(\mathbf{y}) e^{-2 \pi i\left\langle\mathbf{y}, M^{-T} \xi\right\rangle} d \mathbf{y},  \tag{B.3}\\
& =\frac{1}{|\operatorname{det} M|} \widehat{f}\left(M^{-T} \xi\right) . \tag{B.4}
\end{align*}
$$

In (B.2), we let $M \mathbf{x}=\mathbf{y}$, or equivalently, $\mathbf{x}=M^{-1} \mathbf{y}$, and the Jacobian determinant is $\frac{1}{|\operatorname{det} M|}$.

Proof of Theorem B.2.

$$
\begin{aligned}
\int_{\mathbb{R}^{d}} f(\mathbf{x}+\mathbf{b}) e^{-2 \pi i \xi \mathbf{x}} d \mathbf{x} & =\int_{\mathbb{R}^{d}} f(\mathbf{u}) e^{-2 \pi i \xi(\mathbf{u}-\mathbf{b})} d \mathbf{u} \\
& (\text { substituting } \mathbf{u}=\mathbf{x}+\mathbf{b}) \\
& =\int_{\mathbb{R}^{d}} f(\mathbf{u}) e^{-2 \pi i \xi \mathbf{u}} e^{-2 \pi i \xi(-\mathbf{b})} d \mathbf{u} \\
& =e^{-2 \pi i \xi(-\mathbf{b})} \int_{\mathbb{R}^{d}} f(\mathbf{u}) e^{-2 \pi i \xi \mathbf{u}} d \mathbf{u} \\
& =e^{-2 \pi i \xi(-\mathbf{b})} F(\xi) \\
& =e^{2 \pi i \xi \mathbf{b}} F(\xi)
\end{aligned}
$$

Proof of Theorem B.3.

$$
\begin{aligned}
F\left(\xi-\xi^{\mathbf{0}}\right) & =\int_{\mathbb{R}^{d}} f(\mathbf{x}) e^{-2 \pi i \xi-\xi^{\mathbf{0}} \cdot \mathbf{x}} d \mathbf{x}, \\
& =\int_{\mathbb{R}^{d}} f(\mathbf{x}) e^{-2 \pi i \xi \cdot \mathbf{x}} e^{-2 \pi i-\xi_{0} \cdot \mathbf{x}} d \mathbf{x}, \\
& =\int_{\mathbb{R}^{d}} f(\mathbf{x}) e^{-2 \pi i \xi \cdot \mathbf{x}} e^{2 \pi i \xi^{\mathbf{0}} \cdot \mathbf{x}} d \mathbf{x} .
\end{aligned}
$$

Therefore we have

$$
F\left(\xi-\xi^{\mathbf{0}}\right) \rightleftharpoons e^{2 \pi i \xi^{\mathbf{0}} \cdot \mathbf{x}} f(\mathbf{x})
$$

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[^0]:    ${ }^{1}$ the coordinates of $\sigma_{i}$ are relatively prime as a $d$-tuple.

[^1]:    ${ }^{1}$ the coordinates of $\sigma_{i}$ are relatively prime as a $d$-tuple.

[^2]:    ${ }^{1}$ we know that $1_{\mathcal{P}} * \phi_{\epsilon}(\mathbf{n})=\int_{\mathcal{P}} \frac{1}{\epsilon^{d / 2}} e^{-\frac{\pi}{\epsilon}\|\mathbf{n}-\mathbf{t}\| d \mathbf{t}}$. When $\epsilon$ is very small, we have $\lim _{\epsilon \rightarrow 0^{+}}\left(1_{\mathcal{P}} * \phi_{\epsilon}(\mathbf{n})\right)=\omega_{\mathcal{P}}(\mathbf{n})$, which is the solid angle at $\mathbf{n}$ in $\mathcal{P}$; when $\epsilon$ is big enough, we have $\int_{\mathcal{P}} \frac{1}{\epsilon^{d / 2}} e^{-\frac{\pi}{\epsilon}\|\mathbf{n}-\mathbf{t}\| d \mathbf{t}} \leq \int_{\mathcal{P}} \frac{1}{\epsilon^{d / 2}} e^{-\pi\|\mathbf{n}-\mathbf{t}\| d \mathbf{t}} \leq \int_{\mathcal{P}} e^{-\pi\|\mathbf{n}-\mathbf{t}\| d \mathbf{t}}$, and the latter integral is convergent.

[^3]:    ${ }^{1}$ We used Mathcad to solve the above nonlinear equation system. The method used is Newton-Raphson method, therefore the result is sensitive to initial guess. Here we set $\left(a_{0}, b_{0}, c_{0}, d_{0}, e_{0}, f_{0}\right)=(0,4,6,0,4,6)$.

