Boundary element methods for axisymmetric heat conduction and thermoelastic deformations in nonhomogeneous solids

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Boundary Element Methods for Axisymmetric Heat Conduction and Thermoelastic Deformations in Nonhomogeneous Solids

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by

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Nanyang Technological University in partial fulfilment of the
requirement for the degree of Doctor of Philosophy in
School of Mechanical and Aerospace Engineering

Abstract

This thesis is concerned with the development of boundary element techniques for the numerical solution of several important classes of axisymmetric heat conduction and thermoelastic problems involving nonhomogeneous solids with material properties that vary continuously in space. The problems under consideration have applications in the analyses of functionally graded materials which play an important role in engineering.

The classes of axisymmetric problems considered in this thesis may be categorized as follows: (a) nonsteady heat conduction in a nonhomogeneous solid with temperature dependent material properties, (b) nonclassical heat conduction, based on the dual-phase-lag heat conduction model, in a nonhomogeneous solid, and (c) thermoelastostatic and thermoelastodynamic deformations in nonhomogeneous solids.

The problems are formulated in terms of boundary-domain integral equations. The dual-reciprocity method is applied to transform approximately the domain integrals in the boundary-domain integral equations into boundary integrals. New axisymmetric interpolating functions, which are bounded in the axisymmetric coordinate plane but are expressed in terms of relatively simple elementary functions, are proposed for use in the dual-reciprocity method.

In the last part of the thesis, an alternative boundary element approach, based on the theory of complex variables, is proposed for solving an axisym-

metric problem involving steady heat conduction in a nonhomogeneous solid. With the aid of the axisymmetric interpolating functions, the problem is recast into one requiring the construction of a complex function which is analytic in the axisymmetric solution domain and which is such that the relevant boundary conditions are satisfied. Cauchy integral formulae are discretized to obtain a boundary element procedure for constructing the required complex function numerically. Unlike the real boundary element approach for axisymmetric problems, the complex variable approach does not require the evaluation of complicated fundamental solutions involving elliptic integrals.

To check the validity of all the boundary element procedures, they are applied to solve specific problems with known analytical solutions. The numerical results obtained agree well with the analytical solutions. New results are also obtained for some problems which may be of some practical interest, such as axisymmetric laser heating of particular functionally graded solids.

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Chapter 1

General Introduction

1.1 Motivation

Functionally graded materials have nonhomogeneous properties that may be modeled as varying continuously from point to point in space. Such materials are designed for specific functions and applications in modern engineering (Miyamoto, Kaysser, Rabin et al. [61]). Examples include orthopaedic implants for shoulder and knee joint replacements designed by functionally grading the properties of various biopolymers (Pompe, Worch, Epple et al. [76]), ceramic metal functionally graded materials used in green automobiles (Li, Jian and Min [54]), and graded-index polymer optical fibers for high-speed transmission of data (Ishigure, Nihei and Koike [47]).

Due to the ever growing importance of functionally graded materials in engineering, many researchers have focused their attention on the analysis of problems involving nonhomogeneous materials whose properties are given by continuously varying spatial functions (see, for example, Zhou, Li and Lai [103], Gao, Zhang and Guo [38], Wang, Qin and Kang [94] and Wang, Mai and Noda [93]). Mathematically, such problems are inherently difficult to solve analytically as they are governed by partial differential equations with variable

coefficients. Analytical solutions have usually been obtained only for material properties that vary according to specific elementary functions of relatively simple forms. For example, Delale and Erdogan [33] and Dineva, Rangelov and Manolis [34] considered exponentially varying elastic moduli of solids in their analytical works in order to reduce the governing partial differential equations into more tractable forms.

Axisymmetric structures are common in engineering applications. It is not surprising then that during the last ten years or so there has been considerable interest in the analysis of axisymmetric functionally graded materials, that is, axisymmetric materials whose properties are graded continuously in the axial and radial directions. Recently, Keles and Tutuncu [51] computed the dynamic displacement and stress fields in hollow cylinders and spheres with material properties graded in the radial direction according to a simple power law; Clements and Kusuma [27] examined the axisymmetric loading of an elastic half-space whose elastic moduli vary quadratically in the axial direction; and Matysiak, Kulchytsky-Zhyhailo and Perkowski [59] considered the Reissner-Sagoci problem for a homogeneous coating on a functionally graded half-space with shear modulus that varies axially according to a simple power law.

1.2 Boundary Element Approaches for Non-homogeneous Materials

In general, numerical methods are needed for analyses of nonhomogeneous materials with general variable properties. The boundary element method, which first appeared in the works of Jaswon [48], Rizzo [79] and Symm [87] in the 1960s, is now one of the few established numerical methods in engineering analysis. An advantage of the boundary element method, as it is applied

to time independent problems governed by homogeneous linear elliptic partial differential equations with constant coefficients, is that the numerical solution procedure requires only the boundary of the solution domain to be discretized into elements (Wrobel and Aliabadi [98]). As one of the earliest mesh reduction techniques for the numerical solution of boundary value problems, the boundary element method is an interesting and attractive alternative to the more established finite element method which requires the entire solution domain to be meshed into elements.

Nevertheless, for analyses of nonhomogeneous materials, a suitable fundamental solution for the governing partial differential equation with general variable coefficients, which is needed to obtain a strictly boundary integral equation for the boundary element method, is mathematically difficult to derive. If the fundamental solution for the corresponding homogeneous media is used instead to transform the governing differential equation into an integral equation, the resulting integral formulation contains a boundary integral as well as a domain integral. If the solution domain is meshed into elements (like in the finite element method) to deal with the domain integral, the advantage of having to discretize only the boundary is lost. To treat the domain integral effectively or to derive integral formulations without the domain integrals, various approaches may be found in the literature. A brief review on the different boundary element approaches is given below.

In examining two-dimensional flows in porous media with permeability given by particular elementary functions, Cheng [22] used suitable substitution of variables to reduce the governing partial differential equation with variable coefficients to the Laplace's equation which could be easily solved by standard boundary element procedures. Later on, Rangogni [78] applied the boundary element method together with the perturbation technique to study Darcy's

flow in a porous medium with permeability slightly perturbed by a general spatial function, reducing the problem under consideration a pair of subproblems governed by the Laplace's equation and the Poisson's equation.

Martin, Richardson, Gray et al. [57] derived a fundamental solution for elastostatic deformations of exponentially graded materials.

For antiplane elastostatic problems or steady-state two-dimensional heat conduction, Clements [25] derived special fundamental solutions for the case in which shear modulus or thermal conductivity is a continuous function of a single Cartesian coordinate. Ang, Kusuma and Clements [10] extended the work in [25] to include the case where the shear modulus or thermal conductivity varies according to the functions of the product form $X(x)Y(y)$, where x and y are Cartesian coordinates.

Kassab and Divo [50] derived a generalized fundamental solution for heat conduction in solids with general variable thermal conductivity and applied the fundamental solution to obtain a strictly boundary integral formulation for the heat conduction problem.

Park and Ang [72] devised a complex variable boundary element method for general shear modulus or thermal conductivity which varies with only one Cartesian coordinates. The complex variable boundary element approach in [72] was extended by Park, Ang and Kang [73] to the case where the shear modulus is given by $X(x)Y(y)$.

Ang, Clements and Vahdati [9] applied the dual-reciprocity method pioneered by Brebbia and Nardini [18] to treat the domain integral which appears in the boundary-domain integral formulation of a two-dimensional heat conduction problem with variable thermal conductivity. Gao, Zhang and Guo [38] and Gao, Zhang, Sladek et al. [39] had formulated some two- and three-dimensional elastic problems involving functionally graded materials in

terms of boundary-domain integral equations and applied the radial integration method to treat the domain integrals. Recently, in solving two-dimensional heat conduction problem with variable thermal conductivity, Al-Jawary and Wrobel [3] had also used the radial integration method to reduce the domain integral to a line integral.

The boundary element analyses cited in the last few paragraphs are for non-homogeneous materials with continuously varying properties in two-dimensional space and general three-dimensional space. Axisymmetric boundary element methods are well developed for homogeneous materials (see Cruse, Snow and Wilson [29], Wrobel and Brebbia [99], Owatsiriwong, Phansri, Kong et al. [68], Chopra and Dargush [23], Long, Kuai, Chen et al. [56] and Savruk and Danilovich [83]). Nevertheless, it seems that very few axisymmetric boundary element solutions for functionally graded materials may be found in the literature. In Perrey-Debain, Gervais and Guilbaud [75], a wave propagation problem (Helmholtz equations) in nonhomogeneous and axisymmetric solid was investigated using dual-reciprocity boundary element method. Ochiai, Sladek and Sladek [65] proposed a triple-reciprocity boundary element approach for analyzing an axisymmetric elastic problems for nonhomogeneous materials.

More relevant references on boundary element methods are given in later chapters in which boundary element methods are developed for solving numerically several important classes of axisymmetric heat conduction and thermoelastic problems involving functionally graded materials.

1.3 The Present Thesis

1.3.1 Research Objective and Scope

The main objective of the thesis is to develop boundary element techniques for the numerical solution of several important classes of axisymmetric heat conduction and thermoelastic problems involving nonhomogeneous isotropic solids. The material properties of the axisymmetric solids vary in the axial and radial directions according to sufficiently smooth general functions of space.

Both static and dynamic heat conduction and thermoelastic problems are considered in this thesis. For heat conduction, boundary element solutions are derived for nonlinear problems in which the thermal conductivity is temperature dependent and also for nonclassical problems, in addition to classical linear problems. Problems involving thermomechanical deformations of solids are examined in the context of the classical linear theory of thermoelasticity.

1.3.2 Overview of Remaining Chapters

The remaining part of the thesis consists of seven chapters as outlined below.

Chapter 2 contains the mathematical preliminaries for the research works. The governing partial differential equations and the corresponding boundary-domain integral equations for relevant partial differential equations are given.

In Chapter 3, a nonsteady axisymmetric classical heat conduction problem involving a nonhomogeneous solid with temperature dependent thermal conductivity is considered. With the aid of Kirchhoff's transformation, suitable substitution of variables and the axisymmetric heat conduction fundamental solution for homogeneous materials, the nonlinear governing partial differential equation is transformed into a boundary-domain integral equation suitable for boundary element analysis. The first order time derivative of the temperature

is approximated by a central finite difference formula. The boundary-domain integral equation contains an integral over the boundary of the solution domain and a double integral over the solution domain. New axisymmetric interpolating functions, which are bounded on the axis of revolution of the axisymmetric solid but are in relatively simple elementary forms, are proposed for use in the dual-reciprocity boundary element method for converting the domain integral into a line integral. The heat conduction problem is eventually reduced to nonlinear algebraic equations which are solved by a corrective-predictor (iterative) procedure.

Axisymmetric heat conduction in nonhomogeneous solids, based on the dual-phase-lag heat flux model, is considered in Chapter 4. The governing equation for the nonclassical heat conduction is a third order partial differential equation with variable coefficients, containing first and second order time derivatives of the temperature. As in Chapter 3, the partial differential equation is transformed into an boundary-domain integral equation. The time-derivatives of the temperature and flux in the boundary-domain integral equation are suppressed by using the Laplace transform. The problem is finally reduced to a system of linear algebraic equations in the Laplace transform domain by using the dual-reciprocity method together with the axisymmetric interpolating functions proposed in Chapter 3. Once the linear algebraic equations are solved, the temperature in the physical domain can be approximately recovered by a numerical method for inverting Laplace transforms.

Chapter 5 deals with the problem of determining the time independent axisymmetric temperature and thermoelastic displacement and stress fields in a nonhomogeneous solid. The thermal part of the problem, which is based on classical theory of heat conduction, can be easily solved using the boundary element procedures in Chapter 3 and 4. For the thermoelastic part, the funda-

mental solution for elastostatic deformations of homogeneous solids is used to reduce the thermoelastostatic equations into boundary-domain integral equations. New axisymmetric interpolating functions, which are bounded but are in relatively simple elementary forms for easy computation, are constructed for treating the domain integrals in the thermoelastic analysis.

In Chapter 6, the work in Chapter 5 is extended to include axisymmetric thermoelastodynamic analysis of nonhomogeneous solids. For the dynamic problem, the heat equation and the momentum equation cannot be decoupled. The boundary element procedure in the earlier chapter can, however, be easily adapted to solve the problem in the Laplace transform domain.

An alternative boundary element method, based on the theory of complex variables, is proposed in Chapter 7 for solving numerically the classical steady axisymmetric heat equation for nonhomogeneous solids. The approach is to reduce the heat conduction problem to constructing a complex function which is analytic in the solution domain. Cauchy integral formulae are applied to construct the required complex function. Unlike the real axisymmetric boundary element method in the earlier chapters, the complex variable boundary element approach does not require the computation of rather complicated fundamental solutions which are expressed in terms of complete elliptic integrals of the first and second kind.

Lastly, Chapter 8 summarizes the main contributions of the thesis and gives some suggestions for research extensions.

1.4 Publications in Journals

Yun BI and Ang WT, “A dual-reciprocity boundary element approach for axisymmetric nonlinear time-dependent heat conduction in a nonhomogeneous

solid”, *Engineering Analysis with Boundary Elements*, vol. 34, pp. 697-706, 2010.

Yun BI and Ang WT, “A dual-reciprocity boundary element simulation of axisymmetric dual-phase-lag heat conduction in nonhomogeneous media”, *Computer Modeling in Engineering & Sciences*, vol. 65, pp. 217-244, 2010.

Ang WT and Yun BI, “A complex variable boundary element method for axisymmetric heat conduction in a nonhomogeneous solid”, *Applied Mathematics and Computation*, vol. 218, pp. 2225-2236, 2011.

Yun BI and Ang WT, “A dual-reciprocity boundary element method for axisymmetric thermoelastostatic analysis of nonhomogeneous materials”, *Engineering Analysis with Boundary Elements*, vol. 36, pp. 1776-1786, 2012.

Chapter 2

Mathematical Preliminaries

2.1 Basic Equations of Heat Conduction

The law of conservation of thermal energy in a solid is mathematically expressed by (see, for example, Wang, Zhou and Wei [96])

$$\underline{\nabla} \bullet \underline{\mathbf{q}} = -\rho c \frac{\partial T}{\partial t} + Q, \quad (2.1)$$

where T is the temperature field which is a scalar function of spatial coordinates and time t , $\underline{\mathbf{q}}$ is the heat flux vector field, Q is the internal heat generation term and ρ and c are respectively the density and the specific heat capacity of the material occupying the solid.

According to the Fourier classical heat flux model for heat conduction in solids, $\underline{\mathbf{q}}$ is related to the temperature by the relation

$$\underline{\mathbf{q}} = -\kappa \underline{\nabla} T, \quad (2.2)$$

where κ is the thermal conductivity coefficient.

Substitution of (2.2) into (2.1) leads to the classical heat equation

$$\underline{\nabla} \bullet (\kappa \underline{\nabla} T) + Q = \rho c \frac{\partial T}{\partial t}. \quad (2.3)$$

For the classical heat conduction, equation (2.3) is the partial differential equation to solve for the temperature field. In Chapter 3, for axisymmetric heat conduction, (2.3) is solved for the case where the thermal conductivity κ is dependent on spatial coordinates and temperature T . For such a case, the governing partial differential equation is nonlinear.

In the nonclassical heat flux model proposed by Tzou [89, 90], modifications are made to (2.3) by introducing time phase lags into the heat flux vector $\underline{\mathbf{q}}$ and the temperature gradient $\underline{\nabla}T$ in (2.2). Such a model gives rise to a more complicated heat equation in the form of a third order partial differential equations. More details on the nonclassical governing equation are given in Chapter 4 where the dual-phase-lag model is used to examine axisymmetric heat conduction in a functionally graded solid.

2.2 Basic Equations of Thermoelasticity

The momentum equation in a solid is given by (see, Sadd [82])

$$\underline{\nabla} \bullet \underline{\underline{\sigma}} + \underline{\mathbf{F}} = \rho \frac{\partial^2 \underline{\mathbf{u}}}{\partial t^2}, \quad (2.4)$$

where $\underline{\underline{\sigma}}$ is the stress tensor which is function of spatial coordinates and time t , $\underline{\mathbf{F}}$ is body force vector and $\underline{\mathbf{u}}$ is the displacement vector in the thermoelastic field.

For thermoelastic problems, the stress is related to the displacement and temperature by the Duhamel-Neumann constitutive relation [82], that is

$$\underline{\underline{\sigma}} = 2\mu \underline{\underline{\epsilon}} + \left\{ \frac{2\mu\nu}{1-2\nu} [\underline{\nabla} \bullet \underline{\mathbf{u}}] - \beta(T - T_0) \right\} \underline{\underline{\mathbf{I}}}, \quad (2.5)$$

where T_0 is the reference temperature at which the solid does not experience any thermally induced stress, $\underline{\underline{\epsilon}}$ is strain tensor, $\underline{\underline{\mathbf{I}}}$ is the identity tensor of

order two, μ is shear modulus, ν is Poisson's ratio and β is stress-temperature coefficient.

The strain-displacement relations are given by

$$\underline{\underline{\epsilon}} = \frac{1}{2}[\underline{\nabla} \underline{\mathbf{u}} + (\underline{\nabla} \underline{\mathbf{u}})^T], \quad (2.6)$$

where $(\underline{\nabla} \underline{\mathbf{u}})^T$ denotes the transpose of $\underline{\nabla} \underline{\mathbf{u}}$.

For the nonhomogeneous materials considered in this thesis, μ and β are taken to be sufficiently smooth spatial functions and the Poisson's ratio ν is assumed to be a constant. For such nonhomogeneous materials, substituting (2.5) and (2.6) into (2.4), we find that the thermoelastic equation is given by

$$\begin{aligned} & \mu \nabla^2 \underline{\mathbf{u}} + \frac{\mu}{1-2\nu} \underline{\nabla}(\underline{\nabla} \bullet \underline{\mathbf{u}}) + \underline{\mathbf{F}} \\ = & \rho \frac{\partial^2 \underline{\mathbf{u}}}{\partial t^2} + \beta \underline{\nabla} T + (T - T_0) \underline{\nabla} \beta \\ & - \underline{\nabla} \mu \{ \underline{\nabla} \underline{\mathbf{u}} + (\underline{\nabla} \underline{\mathbf{u}})^T + \frac{2\nu}{1-2\nu} [\underline{\nabla} \bullet \underline{\mathbf{u}}] \underline{\mathbf{I}} \}. \end{aligned} \quad (2.7)$$

In general, for thermoelasticity, the thermal energy equation in thermoelasticity is given by [63]

$$\underline{\nabla} \bullet \underline{\mathbf{q}} = -T \frac{\partial S}{\partial t} + Q, \quad (2.8)$$

where S is the entropy given by

$$S = \beta(\underline{\nabla} \bullet \underline{\mathbf{u}}) + \frac{\rho c}{T_0}(T - T_0). \quad (2.9)$$

Therefore, substituting (2.2) and (2.9) into (2.8), we obtain the heat conduction equation for thermoelasticity as given by

$$\underline{\nabla} \bullet (\kappa \underline{\nabla} T) + Q = \beta T_0 \frac{\partial}{\partial t} [\underline{\nabla} \bullet \underline{\mathbf{u}}] + \rho c \frac{\partial T}{\partial t}. \quad (2.10)$$

For the thermoelastodynamic problem considered in Chapter 6, equations (2.7) and (2.10) are the governing equations for solving the temperature and

the elastic deformations of the nonhomogeneous material. Both temperature T and displacement $\underline{\mathbf{u}}$ are present together in both (2.7) and (2.10). This allows for the coupling effects between thermal and elastic behaviors.

A thermoelastostatic problem is considered in Chapter 5. For time-independent problems, the governing equations (2.7) and (2.10) can be reduced to

$$\begin{aligned}
 & \mu \nabla^2 \underline{\mathbf{u}} + \frac{\mu}{1-2\nu} \nabla(\nabla \bullet \underline{\mathbf{u}}) + \underline{\mathbf{F}} \\
 & = \beta \nabla T + (T - T_0) \nabla \beta \\
 & - \nabla \mu \{ \nabla \underline{\mathbf{u}} + (\nabla \underline{\mathbf{u}})^T + \frac{2\nu}{1-2\nu} [\nabla \bullet \underline{\mathbf{u}}] \underline{\mathbf{I}} \}, \\
 & \nabla \bullet (\kappa \nabla T) + Q = 0.
 \end{aligned} \tag{2.11}$$

The thermal equation, which is the second equation in (2.11), does not involve the displacements. Thus, the temperature field in thermal equation can be determined independent of the displacement field. Once the temperature is determined, the displacements can be determined from the first equation in (2.11) by lumping the known temperature terms together with the body force $\underline{\mathbf{F}}$.

2.3 Axisymmetric Boundary-domain Integral Equations

With reference to an $Oxyz$ Cartesian coordinate frame, consider a solid occupying the three-dimensional region R bounded by a closed surface S . The body is axisymmetrical about the z axis, that is, the region R and its boundary S can be obtained by rotating respectively a two-dimensional region and a curve by an angle of 360° about the z -axis. On the rz (axisymmetric coordinate) plane the two-dimensional region and the curve are denoted by Ω and Γ re-

spectively. Figure 2.1 gives a sketch of Ω (shaded region) and Γ . Here the axisymmetrical coordinate r is defined by $r = \sqrt{x^2 + y^2 + z^2}$.

Figure 2.1: A sketch of the axisymmetric body on the rz plane.

2.3.1 Heat Equation

Let us assume that the partially differential equation governing heat conduction in the region R can be written in the form

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = D, \quad (2.12)$$

where D denotes an expression which may contain x , y , z and T and its partial derivatives.

From (2.12), we may derive the boundary-domain integral equation (as

shown in Ang [4])

$$\begin{aligned} \gamma(\xi, \eta, \zeta)T(\xi, \eta, \zeta) &= \iint_S (T(x, y, z) \frac{\partial}{\partial n} [\Phi_{3D}(x, y, z; \xi, \eta, \zeta)] \\ &\quad - \Phi_{3D}(x, y, z; \xi, \eta, \zeta) \frac{\partial}{\partial n} [T(x, y, z)]) d\ell(x, y, z) \\ &\quad + \iiint_R (\Phi_{3D}(x, y, z; \xi, \eta, \zeta) D) dx dy dz, \end{aligned} \quad (2.13)$$

where (ξ, η, ζ) is a source point, $\partial T / \partial n = n_x \partial T / \partial x + n_y \partial T / \partial y + n_z \partial T / \partial z$, $[n_x, n_y, n_z]$ is the outward unit normal vector to the surface,

$$\gamma(\xi, \eta, \zeta) = \begin{cases} 0 & \text{if } (\xi, \eta, \zeta) \notin R \cup S, \\ 1/2 & \text{if } (\xi, \eta, \zeta) \text{ lies on smooth path of } S, \\ 1 & \text{if } (\xi, \eta, \zeta) \in R, \end{cases} \quad (2.14)$$

and $\Phi_{3D}(x, y, z; \xi, \eta, \zeta)$ is the fundamental solution of three-dimensional Laplace's equation, that is

$$\Phi_{3D}(x, y, z; \xi, \eta, \zeta) = -\frac{1}{4\pi \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}}. \quad (2.15)$$

As mentioned earlier on, R is axisymmetric about the z axis. If T and D are independent of the polar coordinate θ (defined by $x = r \cos \theta$ and $y = r \sin \theta$), the integration with respect to θ in the boundary and domain integrals in (2.13) can be carried out analytically. Thus, the three-dimensional boundary-domain integral equation in (2.13) can be reduced to a boundary-domain integral equation over the curve Γ and the domain Ω on the rz plane (more details can be found in Brebbia, Telles and Wrobel [19]), that is,

$$\begin{aligned} \gamma(r_0, z_0)T(r_0, z_0) &= \int_{\Gamma} (T(r, z) G_1(r, z; r_0, z_0; n_r, n_z) \\ &\quad - G_0(r, z; r_0, z_0) \frac{\partial}{\partial n} [T(r, z)]) ds(r, z) \\ &\quad + \iint_{\Omega} (G_0(r, z; r_0, z_0) D) r dr dz, \end{aligned} \quad (2.16)$$

where $ds(r, z)$ denotes the length of an infinitesimal part of the boundary curve Γ and the axisymmetric fundamental solutions G_0 and G_1 are given by

$$\begin{aligned}
 & G_0(r, z; r_0, z_0) \\
 &= \int_0^{2\pi} \Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0) d\theta \\
 &= -\frac{K(m(r, z; r_0, z_0))}{\pi \sqrt{a(r, z; r_0, z_0) + b(r; r_0)}}, \tag{2.17}
 \end{aligned}$$

$$\begin{aligned}
 & G_1(r, z; r_0, z_0; n_r, n_z) \\
 &= \int_0^{2\pi} \frac{\partial}{\partial n} [\Phi_{3D}(r \cos \theta, r \sin \theta, z; r_0, 0, z_0)] d\theta \\
 &= -\frac{1}{\pi \sqrt{a(r, z; r_0, z_0) + b(r; r_0)}} \\
 &\quad \times \left\{ \frac{n_r(r, z)}{2r} \left[\frac{r_0^2 - r^2 + (z_0 - z)^2}{a(r, z; r_0, z_0) - b(r; r_0)} E(m(r, z; r_0, z_0)) \right. \right. \\
 &\quad \left. \left. - K(m(r, z; r_0, z_0)) \right] \right. \\
 &\quad \left. + n_z(r, z) \frac{z_0 - z}{a(r, z; r_0, z_0) - b(r; r_0)} E(m(r, z; r_0, z_0)) \right\}, \tag{2.18}
 \end{aligned}$$

and

$$\begin{aligned}
 m(r, z; r_0, z_0) &= \frac{2b(r; r_0)}{a(r, z; r_0, z_0) + b(r; r_0)} \\
 a(r, z; r_0, z_0) &= r_0^2 + r^2 + (z_0 - z)^2, b(r; r_0) = 2rr_0, \tag{2.19}
 \end{aligned}$$

where K and E denotes the complete elliptic integral of the first and second kind respectively. For $0 \leq m(r, z; r_0, z_0) \leq 1$, K and E are defined in Abramowitz and Stegun [1], that is

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}, \quad E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} d\theta. \tag{2.20}$$

Note that $G_0(r, z; r_0, z_0)$ is the fundamental solution for steady axisymmetric heat conduction in a homogeneous solid.

The axisymmetric boundary-domain integral equation in (2.16) will be used in Chapter 3, 4, 5 and 6 for developing boundary element methods for solving heat conduction and thermoelastic problems. In axisymmetric boundary element procedures, only the curve Γ has to be discretized into elements.

2.3.2 Thermoelastic Equations

In thermoelasticity, the momentum equations may be written in Cartesian form as

$$\nabla^2 u_i + \frac{1}{1-2\nu} u_{j,j} + D_i = 0 \text{ for } i, j = x, y, z, \quad (2.21)$$

where D_i denotes an expression which may contain x, y, z, u_i and its partial derivatives and other functions such as those associated with the temperature field. Note that lowercase Latin subscripts such as i and j here are given the values x, y and z and the usual Einsteinian convention of summing over a repeated subscripts over x, y and z is assumed.

The Betti's theorem and Somigliana identity given in [37] can be applied to transform the momentum equations in (2.21) into boundary-domain integral equations given by

$$\begin{aligned} \gamma(\xi, \eta, \zeta) u_i(\xi, \eta, \zeta) &= \iint_S (U_{ij}(x, y, z; \xi, \eta, \zeta) p_j(x, y, z; n_x, n_y, n_z) \\ &\quad - T_{ij}(x, y, z; \xi, \eta, \zeta; n_x, n_y, n_z) u_j(x, y, z)) d\ell(x, y, z) \\ &\quad + \iiint_R (U_{ij}(x, y, z; \xi, \eta, \zeta) D_j) dx dy dz, \end{aligned} \quad (2.22)$$

where $p_i = n_j(u_{i,j} + u_{j,i} + \frac{2\nu}{1-2\nu} u_{kk} \delta_{ij})$ and $U_{ij}(x, y, z; \xi, \eta, \zeta)$ is three-dimensional fundamental solutions for homogeneous elastostatic problems and $T_{ij}(x, y, z;$

$\xi, \eta, \zeta; n_x, n_y, n_z$) is the corresponding stress function as given by

$$U_{ij}(x, y, z; \xi, \eta, \zeta) = \frac{1}{16\pi(1-\nu)} \frac{1}{\vartheta(x, y, z; \xi, \eta, \zeta)} \times ((3-4\nu)\delta_{ij} + \vartheta_{,i}\vartheta_{,j}), \quad (2.23)$$

$$T_{ij}(x, y, z; \xi, \eta, \zeta; n_x, n_y, n_z) = -\frac{1}{8\pi(1-\nu)} \frac{1}{\vartheta(x, y, z; \xi, \eta, \zeta)^2} \times \left\{ \frac{\partial \vartheta(x, y, z; \xi, \eta, \zeta)}{\partial n} [(1-2\nu)\delta_{ij} + 3\vartheta_{,i}\vartheta_{,j}] - (1-2\nu)(n_j\vartheta_{,i} - n_i\vartheta_{,j}) \right\}, \quad (2.24)$$

where

$$\vartheta(x, y, z; \xi, \eta, \zeta) = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}, \quad (2.25)$$

$\vartheta_{,i} = \partial \vartheta / \partial x_i$ and δ_{ij} is Kronecker delta.

If the displacement and all relevant quantities are independent of θ , the axisymmetric boundary-domain integral equation can be obtained by performing analytically the integration with respect to θ in the three-dimensional boundary-domain integral equation (2.22) (see Bakr [16]). The axisymmetric boundary-domain integral equations are given by

$$\begin{aligned} \gamma(r_0, z_0)u_J(r_0, z_0) &= \int_{\Gamma} \{ \Phi_{KJ}(r, z; r_0, z_0)p_K(r, z; n_r, n_z) \\ &\quad - \Psi_{KJ}(r, z; r_0, z_0; n_r, n_z)u_K(r, z) \} r ds(r, z) \\ &\quad + \iint_{\Omega} (\Phi_{KJ}(r, z; r_0, z_0)D_K) r dr dz, \end{aligned} \quad (2.26)$$

where the uppercase Latin subscripts (such as J and K) are given the values r and z , u_K , p_K and D_K are the axisymmetric components of quantities whose Cartesian components are u_i , p_i and D_i respectively, $\Phi_{KJ}(r, z; r_0, z_0)$ and $\Psi_{KJ}(r, z; r_0, z_0; n_r, n_z)$ are the fundamental solutions of the elastic equations

for homogeneous solids which are given by

$$\begin{aligned}
 \Phi_{rr}(r, z; r_0, z_0) &= \frac{1}{8\pi(1-\nu)r_0C(r, z; r_0, z_0)} \{((3-4\nu)(r_0^2 + r^2) \\
 &\quad + 4(1-\nu)(z_0 - z)^2)K(m(r, z; r_0, z_0)) \\
 &\quad + (-[C(r, z; r_0, z_0)]^2(3-4\nu) \\
 &\quad - \frac{(z_0 - z)^2}{D(r, z; r_0, z_0)}A(r, z; r_0, z_0))E(m(r, z; r_0, z_0))\}, \\
 \Phi_{rz}(r, z; r_0, z_0) &= \frac{(z_0 - z)}{8\pi(1-\nu)C(r, z; r_0, z_0)} \{-K(m(r, z; r_0, z_0)) \\
 &\quad + \frac{B(r, z; r_0, z_0)}{D(r, z; r_0, z_0)}E(m(r, z; r_0, z_0))\}, \\
 \Phi_{zr}(r, z; r_0, z_0) &= \frac{r(z_0 - z)}{8\pi(1-\nu)r_0C(r, z; r_0, z_0)} \{K(m(r, z; r_0, z_0)) \\
 &\quad - \frac{A(r, z; r_0, z_0) - 2r_0^2}{D(r, z; r_0, z_0)}E(m(r, z; r_0, z_0))\}, \\
 \Phi_{zz}(r, z; r_0, z_0) &= \frac{r}{4\pi(1-\nu)C(r, z; r_0, z_0)} \{(3-4\nu)K(m(r, z; r_0, z_0)) \\
 &\quad + \frac{(z_0 - z)^2}{D(r, z; r_0, z_0)}E(m(r, z; r_0, z_0))\}, \tag{2.27}
 \end{aligned}$$

$$\begin{aligned}
 \Psi_{rr}(r, z; r_0, z_0; n_r, n_z) &= -\frac{r}{2\pi(1-\nu)}(\Lambda_1(r, z; r_0, z_0)n_r(r, z) \\
 &\quad + \Lambda_2(r, z; r_0, z_0)n_z(r, z)), \\
 \Psi_{rz}(r, z; r_0, z_0; n_r, n_z) &= -\frac{r}{2\pi(1-\nu)}(\Lambda_3(r, z; r_0, z_0)n_r(r, z) \\
 &\quad + \Lambda_4(r, z; r_0, z_0)n_z(r, z)), \\
 \Psi_{zr}(r, z; r_0, z_0; n_r, n_z) &= -\frac{r}{2\pi(1-\nu)}(\Lambda_5(r, z; r_0, z_0)n_r(r, z) \\
 &\quad + \Lambda_6(r, z; r_0, z_0)n_z(r, z)), \\
 \Psi_{zz}(r, z; r_0, z_0; n_r, n_z) &= -\frac{r}{2\pi(1-\nu)}(\Lambda_7(r, z; r_0, z_0)n_r(r, z) \\
 &\quad + \Lambda_8(r, z; r_0, z_0)n_z(r, z)), \tag{2.28}
 \end{aligned}$$

$$\begin{aligned}
 & \Lambda_1(r, z; r_0, z_0) \\
 = & \frac{1}{2r_0r^2C(r, z; r_0, z_0)} \{ (1 - 2\nu)(A(r, z; r_0, z_0) + H(r, z; r_0, z_0)) \\
 & - \frac{1}{[C(r, z; r_0, z_0)]^2 D(r, z; r_0, z_0)} (-2(z_0 - z)^6 + (-5r_0^2 - 4r^2)(z_0 - z)^4 \\
 & + (5r_0^2r^2 - 4r_0^4 - r^4)(z_0 - z)^2 + (r^2 - r_0^2)^3) \} K(m(r, z; r_0, z_0)) \\
 & + \frac{1}{2r_0r^2C(r, z; r_0, z_0)D(r, z; r_0, z_0)} \{ -(1 - 2\nu)(2A(r, z; r_0, z_0)B(r, z; r_0, z_0) \\
 & + 3r^2(A(r, z; r_0, z_0) - 2r_0^2)) \\
 & + \frac{1}{[C(r, z; r_0, z_0)]^2 D(r, z; r_0, z_0)} (-2(z_0 - z)^8 + (-6r^2 - 7r_0^2)(z_0 - z)^6 \\
 & + (-9r_0^4 + 2r_0^2r^2 - 5r^4)(z_0 - z)^4 \\
 & + (-5r_0^6 + 10r_0^4r^2 - 5r_0^2r^4)(z_0 - z)^2 \\
 & + (-r_0^8 + 2r_0^6r^2 - 2r_0^2r^4 + r^8)) \} E(m(r, z; r_0, z_0)), \tag{2.29}
 \end{aligned}$$

$$\begin{aligned}
 & \Lambda_2(r, z; r_0, z_0) = \Lambda_5(r, z; r_0, z_0) \\
 = & \frac{z_0 - z}{2r_0rC(r, z; r_0, z_0)} \{ (1 - 2\nu) \\
 & + \frac{1}{[C(r, z; r_0, z_0)]^2 D(r, z; r_0, z_0)} ((z_0 - z)^2(3A(r, z; r_0, z_0) - 2(z_0 - z)^2) \\
 & + 2(r_0^2 - r^2)^2) \} K(m(r, z; r_0, z_0)) \\
 & + \frac{z_0 - z}{2r_0rC(r, z; r_0, z_0)D(r, z; r_0, z_0)} \{ -(1 - 2\nu)A(r, z; r_0, z_0) \\
 & - \frac{1}{[C(r, z; r_0, z_0)]^2 D(r, z; r_0, z_0)} ((z_0 - z)^4(4A(r, z; r_0, z_0) - 3(z_0 - z)^2) \\
 & + (r_0^2 - r^2)^2(2A(r, z; r_0, z_0) + 3(z_0 - z)^2)) \} E(m(r, z; r_0, z_0)), \tag{2.30}
 \end{aligned}$$

$$\begin{aligned}
 & \Lambda_3(r, z; r_0, z_0) \\
 = & -\frac{z_0 - z}{2r^2[C(r, z; r_0, z_0)]^3 D(r, z; r_0, z_0)} (2r^2(r^2 - r_0^2 + 2(z_0 - z)^2) \\
 & + A(r, z; r_0, z_0)B(r, z; r_0, z_0))K(m(r, z; r_0, z_0)) \\
 & + \frac{z_0 - z}{C(r, z; r_0, z_0)D(r, z; r_0, z_0)} \left\{ (1 - 2\nu) - \frac{1}{2r^2[C(r, z; r_0, z_0)]^2 D(r, z; r_0, z_0)} \right. \\
 & \times (-[H(r, z; r_0, z_0)]^3 + r^2(z_0 - z)^2(2r_0^2 + r^2 - 5(z_0 - z)^2) \\
 & \left. + r^2(7r_0^4 - 11r_0^2 r^2 + 5r^4) \right\} E(m(r, z; r_0, z_0)), \tag{2.31}
 \end{aligned}$$

$$\begin{aligned}
 & \Lambda_4(r, z; r_0, z_0) = \Lambda_7(r, z; r_0, z_0) \\
 = & \frac{1}{2rC(r, z; r_0, z_0)} \{ (1 - 2\nu) \\
 & + \frac{(z_0 - z)^2}{[C(r, z; r_0, z_0)]^2 D(r, z; r_0, z_0)} B(r, z; r_0, z_0) \} K(m(r, z; r_0, z_0)) \\
 & + \frac{1}{2rC(r, z; r_0, z_0)D(r, z; r_0, z_0)} \{ -(1 - 2\nu)B(r, z; r_0, z_0) \\
 & + \frac{(z_0 - z)^2}{[C(r, z; r_0, z_0)]^2 D(r, z; r_0, z_0)} (-A(r, z; r_0, z_0)B(r, z; r_0, z_0) \\
 & + 6r^2(A(r, z; r_0, z_0) - 2r_0^2)) \} E(m(r, z; r_0, z_0)), \tag{2.32}
 \end{aligned}$$

$$\begin{aligned}
 & \Lambda_6(r, z; r_0, z_0) \\
 = & \frac{1}{2r_0C(r, z; r_0, z_0)} \{ (1 - 2\nu) \\
 & - \frac{(z_0 - z)^2}{[C(r, z; r_0, z_0)]^2 D(r, z; r_0, z_0)} (A(r, z; r_0, z_0) - 2r_0^2) \} K(m(r, z; r_0, z_0)) \\
 & + \frac{1}{2r_0C(r, z; r_0, z_0)D(r, z; r_0, z_0)} \{ -(1 - 2\nu)(A(r, z; r_0, z_0) - 2r_0^2) \\
 & + \frac{(z_0 - z)^2}{[C(r, z; r_0, z_0)]^2 D(r, z; r_0, z_0)} (A(r, z; r_0, z_0)(A(r, z; r_0, z_0) - 2r_0^2) \\
 & - 6r_0^2 B(r, z; r_0, z_0)) \} E(m(r, z; r_0, z_0)), \tag{2.33}
 \end{aligned}$$

$$\begin{aligned}
& \Lambda_8(r, z; r_0, z_0) \\
= & \frac{(z_0 - z)^3}{[C(r, z; r_0, z_0)]^3 D(r, z; r_0, z_0)} K(m(r, z; r_0, z_0)) \\
& + \frac{(z_0 - z)}{C(r, z; r_0, z_0) D(r, z; r_0, z_0)} \\
& \times \left\{ -(1 - 2\nu) - \frac{4(z_0 - z)^2}{[C(r, z; r_0, z_0)]^2 D(r, z; r_0, z_0)} A(r, z; r_0, z_0) \right\} \\
& \times E(m(r, z; r_0, z_0)), \tag{2.34}
\end{aligned}$$

where

$$\begin{aligned}
A(r, z; r_0, z_0) &= r_0^2 + r^2 + (z_0 - z)^2, \\
B(r, z; r_0, z_0) &= r_0^2 - r^2 + (z_0 - z)^2, \\
C(r, z; r_0, z_0) &= \sqrt{(r_0 + r)^2 + (z_0 - z)^2}, \\
D(r, z; r_0, z_0) &= (r_0 - r)^2 + (z_0 - z)^2, \\
H(r, z; r_0, z_0) &= r_0^2 + (z_0 - z)^2. \tag{2.35}
\end{aligned}$$

The axisymmetric boundary-domain integral equations in (2.26) will be used in Chapter 5 and 6 to solve thermoelastic problems involving nonhomogeneous materials.

2.4 Numerical Inversion of Laplace Transforms

In Chapter 4 and 6, the Laplace transformation is used to suppress the time derivatives of unknown functions in the governing partial differential equations. The problems under consideration are solved in the Laplace transform domain by boundary element methods. The solutions in the physical domain can be recovered by using a numerical technique for inverting Laplace transforms. Davies and Martin [32] had given a survey and comparison of some numerical techniques for inverting Laplace transforms.

According to the Stehfest's algorithm (Stehfest [85]), if $G(p)$ is the Laplace transform of $g(t)$, where p is the Laplace transform parameter, then

$$g(t) \simeq \frac{\ln(2)}{t} \sum_{n=1}^{2M} V_n G\left(\frac{n \ln(2)}{t}\right), \quad (2.36)$$

where M is a positive integer and

$$V_n = (-1)^{n+M} \sum_{m=\lceil (n+1)/2 \rceil}^{\min(n,M)} \frac{m^M (2m)!}{(M-m)! m! (m-1)! (n-m)! (2m-n)!}, \quad (2.37)$$

where $[b]$ denotes the integer part of the real number b .

Note that the inversion formula in (2.36) requires $G(p)$ to be evaluated for only real Laplace transform parameter p .

2.5 Complex Functions and Cauchy Integral Formulae

In Chapter 7, a complex variable boundary element method is proposed for solving an axisymmetric heat conduction problem. The method makes use of following results from the theory of complex variables [20].

If f is a well defined complex function of the variable $z = x + iy$ ($i = \sqrt{-1}$) in the region R bounded by a simple closed curve C on the Oxy plane, such that f is analytic on $C \cup R$, then $\phi(x, y) = \text{Re}\{f(z)\}$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ throughout } C \cup R,$$

and the Cauchy integral formulae

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{z - z_0} \text{ and } f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^2}$$

hold for any point z_0 lying in the interior of R .

Chapter 3

Axisymmetric Nonlinear Heat Conduction in Nonhomogeneous Solids

3.1 Introduction

Nonlinearities may occur in heat conduction problems due to temperature dependent material properties, nonlinear boundary conditions (such as the Stefan-Boltzmann radiation condition) and nonlinear heat sources. Experimental observations suggest that the thermal conductivity and specific heat capacity of metallic solids are strongly dependent on temperature during processes such as metal quenching. Thus, there is considerable interest among many researchers in developing boundary element techniques for the numerical solution of nonlinear heat conduction problems involving materials with temperature dependent thermal conductivity.

For problems involving temperature dependent material properties or nonlinear heat sources, the governing partial differential equation contains nonlinear terms. A commonly used technique for converting nonlinear terms that arise from temperature dependent thermal conductivity into linear ones is to

employ the Kirchhoff's transformation.

For a steady-state heat conduction problem in which the thermal conductivity is dependent on temperature only, the Kirchhoff's transformation is used in Azevedo and Wrobel [14] to reduce the nonlinear governing heat equation to the Laplace's equation. The problem under consideration is then formulated in terms of strictly boundary integral equations.

The Kirchhoff's transformation is also used by Goto and Suzuki [42] and Kikuta, Togoh and Tanaka [52] to solve two- and three-dimensional nonsteady heat conduction problems involving temperature dependent thermal conductivity. Nevertheless, the problems are formulated in terms of boundary-domain integral equations instead of strictly boundary integral equations. The domain integrals are mainly because of the presence of time derivative of temperature in the governing equations. In [42] and [52], domains with very simple geometries are discretized into many small cells in order to treat the domain integrals.

Recently, the work in [14], [42] and [52], which is for thermally isotropic solids with thermal conductivity that depends only on temperature, is extended by Clements and Budhi [26], Azis and Clements [15] and Ang and Clements [7] to thermally anisotropic solids with thermal conductivity that varies with temperature and spatial coordinates. In Ang and Clements [7], the domain integral in the formulation of the problem is treated by the dual-reciprocity method. The dual-reciprocity method for treating the domain integral is also employed in a very recent paper by Mohammadi, Hematiyan and Marin [62].

This chapter ¹ considers a nonlinear time-dependent axisymmetric heat conduction problem in a nonhomogeneous and isotropic material with tem-

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perature dependent properties. After Kirchhoff's transformation is applied on the governing equation, the problem is formulated in terms of a boundary-domain integral equation. To treat the domain integral without discretizing the solution domain, dual-reciprocity method is used. The dual-reciprocity method is used to approximate the domain integral in terms of a line integral by using suitable interpolating functions. New interpolating functions, which are bounded in the whole axisymmetric plane but are expressed in relatively forms in terms of elementary functions, are proposed here for treating the domain integral. The time derivative of temperature present in the integrand of the domain integral is approximated using a finite difference formula. At any given time level, if the temperature is assumed known at the previous time level, the problem under consideration can be formulated in terms of a system of nonlinear algebraic equations to be solved using a predictor-corrector (iterative) procedure. The numerical procedure presented here is applied to solve some specific problems including one that involves the laser heating of a cylindrical solid. For problems which have known exact solutions, the accuracy of the numerical solutions obtained is assessed.

3.2 Statement of Problem

Consider a thermally isotropic solid occupying the three-dimensional region R . If T is the temperature inside the solid, then the conservation of energy and the classical Fourier's law of heat conduction require the temperature to satisfy the partial differential equation

$$\underline{\nabla} \cdot (\kappa \underline{\nabla} T) + Q = \rho c \frac{\partial T}{\partial t} \text{ in } R \text{ for } t \geq 0, \quad (3.1)$$

where κ , ρ , c are respectively thermal conductivity, density and specific heat capacity which vary with spatial coordinates and temperature. Note that the

above equation is (2.3) in Chapter 2.

The three-dimensional region R is assumed to be axisymmetric. Specifically, it is obtained by rotating the two-dimensional region Ω on the axisymmetric rz plane in Figure 2.1 by an angle of 360° about the z axis. The rotation of the curve Γ on the rz plane gives the surface boundary of R . For the axisymmetric heat conduction problem considered here, it is assumed that the material properties, the temperature field and the internal heat source are independent of the polar angle θ defined by $x = r \cos \theta$ and $y = r \sin \theta$, that is, the temperature and internal heat source are given by $T(r, z, t)$ and $Q(r, z, t)$ respectively.

The thermal conductivity is functionally graded in the radial and axial directions of the solid of revolution. Specifically, it is taken to be temperature dependent, such that

$$\kappa = g(r, z)h(T), \quad (3.2)$$

where $g(r, z)$ is a suitably given function which is positive in Ω and $h(T)$ is a function which is integrable with respect to T . The density ρ and the specific heat capacity c are also dependent on r, z and T .

Mathematically, the problem of interest in this chapter is to solve (3.1) (in axisymmetric coordinates) together with (3.2) subject to the initial-boundary conditions

$$\begin{aligned} T(r, z, 0) &= f_0(r, z) \text{ in } \Omega, \\ T(r, z, t) &= f_1(r, z, t) \text{ on } \Gamma_1 \text{ for } t > 0, \\ g(r, z)h(T)\frac{\partial T}{\partial n} &= f_2(r, z, t) \text{ on } \Gamma_2 \text{ for } t > 0, \end{aligned} \quad (3.3)$$

where Γ_1 and Γ_2 are nonintersecting curves such that $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\partial T/\partial n$ denotes the outward normal derivative of T on Γ and $f_0(r, z)$, $f_1(r, z, t)$ and $f_2(r, z, t)$ are suitably given functions.

3.3 Transformed Equations

In order to treat the nonlinear partial differential equation given in (3.1), Kirchhoff's transformation [13, 69] is used. In Kirchhoff's transformation, a new variable Θ is introduced as defined by

$$\Theta(r, z, t) = \int h(T) dT \equiv \mathcal{K}(T). \quad (3.4)$$

Using (3.4), the nonlinear governing partial differential equation defined by (3.1) and (3.2) can be rewritten as

$$g \nabla_{\text{axis}}^2 \Theta = -Q - \underline{\nabla} g \bullet (\underline{\nabla} \Theta) + S(r, z, \Theta) \frac{\partial \Theta}{\partial t}, \quad (3.5)$$

where ∇_{axis}^2 which is the axisymmetric Laplacian differential operator defined by

$$\nabla_{\text{axis}}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad (3.6)$$

and

$$S(r, z, \Theta) = \frac{\rho(r, z, \mathcal{M}(\Theta)) c(r, z, \mathcal{M}(\Theta))}{h(\mathcal{M}(\Theta))}, \quad (3.7)$$

if one assumes that the Kirchhoff's transformation in (3.4) can be inverted to give the temperature as $T = \mathcal{K}^{-1}(\Theta) = \mathcal{M}(\Theta)$.

Note that partial derivatives of Θ with respect to spatial coordinates, given by the components of $\underline{\nabla} \Theta$, are present in (3.5). To avoid having such spatial derivatives in the boundary-domain integral equation (to be derived later on), the substitution

$$\Theta = \frac{1}{\sqrt{g}} \psi \quad (3.8)$$

is applied in (3.5) to obtain the governing partial differential equation

$$\nabla_{\text{axis}}^2 \psi = -\frac{Q}{\sqrt{g}} + B(r, z) \psi + D(r, z, \psi) \frac{\partial \psi}{\partial t}, \quad (3.9)$$

where

$$B(r, z) = \frac{1}{\sqrt{g(r, z)}} \nabla_{\text{axis}}^2 (\sqrt{g(r, z)}), \quad D(r, z, \psi) = \frac{1}{g} S(r, z, \frac{1}{\sqrt{g}} \psi). \quad (3.10)$$

The function g is assumed to be such that $\nabla_{\text{axis}}^2(\sqrt{g})$ exists in the solution domain R .

As ψ is a function of r , z and t , equation (3.9) can be written out more explicitly as

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{Q(r, z, t)}{\sqrt{g(r, z)}} + B(r, z) \psi + D(r, z, \psi) \frac{\partial \psi}{\partial t}, \quad (3.11)$$

The initial-boundary conditions in (3.3) can be rewritten as

$$\begin{aligned} \psi(r, z, 0) &= \sqrt{g(r, z)} \mathcal{K}(f_0(r, z)) \text{ in } \Omega, \\ \psi(r, z, t) &= \sqrt{g(r, z)} \mathcal{K}(f_1(r, z, t)) \text{ on } \Gamma_1 \text{ for } t > 0, \\ \frac{\partial}{\partial n} [\psi(r, z, t)] &= \frac{\psi(r, z, t)}{2g(r, z)} \frac{\partial}{\partial n} [g(r, z)] \\ &\quad + \frac{1}{\sqrt{g(r, z)}} f_2(r, z, t) \text{ on } \Gamma_2 \text{ for } t > 0, \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} \frac{\partial}{\partial n} [\psi(r, z, t)] &= n_r(r, z) \frac{\partial}{\partial r} [\psi(r, z, t)] + n_z(r, z) \frac{\partial}{\partial z} [\psi(r, z, t)], \\ \frac{\partial}{\partial n} [g(r, z)] &= n_r(r, z) \frac{\partial}{\partial r} [g(r, z)] + n_z(r, z) \frac{\partial}{\partial z} [g(r, z)], \end{aligned} \quad (3.13)$$

where $n_r(r, z)$ and $n_z(r, z)$ are the components of the outward unit normal vector on Γ at the point (r, z) in the r and z direction respectively.

Once $\psi(r, z, t)$ (hence $\Theta(r, z, t)$) is obtained by solving (3.11) in Ω subject to the initial-boundary conditions in (3.12), the temperature $T(r, z, t)$ may be obtained by inverting the Kirchhoff's transformation in (3.4).

3.4 Boundary-domain Integral Formulation

From the partial differential equation given in (3.11), a boundary-domain integral equation can be obtained by the use of reciprocal theorem together with suitably fundamental solutions as shown in Chapter 2. The boundary-domain integral equation in terms of integrals over Γ and Ω is

$$\begin{aligned}
 & \gamma(r_0, z_0)\psi(r_0, z_0, t) \\
 = & \iint_{\Omega} G_0(r, z; r_0, z_0) \left\{ -\frac{Q(r, z, t)}{\sqrt{g(r, z)}} + B(r, z)\psi \right. \\
 & + D(r, z, \psi) \frac{\partial}{\partial t} [\psi(r, z, t)] \} r dr dz \\
 & + \int_{\Gamma} (\psi(r, z, t) G_1(r, z; r_0, z_0) - G_0(r, z; r_0, z_0) \frac{\partial}{\partial n} [\psi(r, z, t)]) r ds(r, z) \\
 & \text{for } (r_0, z_0) \in \Omega \cup \Gamma,
 \end{aligned} \tag{3.14}$$

where $\gamma(r_0, z_0) = 1$ if (r_0, z_0) lies in the interior of Ω , $\gamma(r_0, z_0) = 1/2$ if (r_0, z_0) lies on a smooth part of Γ , $ds(r, z)$ denotes the length of an infinitesimal part of the curve Γ , and the fundamental solutions for axisymmetric heat conduction $G_0(r, z; r_0, z_0)$ and $G_1(r, z; r_0, z_0; n_r, n_z)$ are given as (2.17) and (2.18) in Chapter 2.

Note that the boundary-domain integral equation (3.14) does not contain any partial derivative of the unknown function with respect to the spatial coordinates r and z . If the boundary-domain integral equation is derived directly by using (3.5) (instead of (3.11)) with Θ as an unknown function, the integrand of the domain integral will contain first order partial derivatives of Θ with respect to the spatial coordinates. The presence of those partial derivatives may be regarded as a disadvantage in the numerical solution of the boundary-domain integral equation, as they have to be approximated in some way. Thus, the boundary-domain integral equation in (3.14) is preferred over the one derived

directly from (3.5) with unknown Θ .

3.5 Dual-reciprocity Boundary Element Procedures

A dual-reciprocity boundary element method based on the boundary-domain integral equation in (3.14) is described here for the approximate solution of the initial-boundary value problem defined by (3.11) and (3.12).

3.5.1 Boundary Approximations

The curve Γ of the solution domain is discretized into N straight line elements denoted by $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(N-1)}$ and $\Gamma^{(N)}$. The starting and ending points of a typical element $\Gamma^{(k)}$ are given by $(r^{(k)}, z^{(k)})$ and $(r^{(k+1)}, z^{(k+1)})$ respectively. Two points on the element $\Gamma^{(k)}$, denoted by $(r_0^{(k)}, z_0^{(k)})$ and $(r_0^{(N+k)}, z_0^{(N+k)})$, are chosen as

$$\begin{aligned} (r_0^{(k)}, z_0^{(k)}) &= (r^{(k)}, z^{(k)}) + \tau(r^{(k+1)} - r^{(k)}, z^{(k+1)} - z^{(k)}), \\ (r_0^{(N+k)}, z_0^{(N+k)}) &= (r^{(k)}, z^{(k)}) \\ &\quad + (1 - \tau)(r^{(k+1)} - r^{(k)}, z^{(k+1)} - z^{(k)}), \end{aligned} \quad (3.15)$$

where τ is a chosen number such that $0 < \tau < 1/2$.

If the function ψ at $(r_0^{(k)}, z_0^{(k)})$ and $(r_0^{(N+k)}, z_0^{(N+k)})$ is denoted by $\psi^{(k)}(t)$ and $\psi^{(N+k)}(t)$ respectively, then the boundary temperature is approximated using

$$\begin{aligned} \psi(r, z, t) \simeq & \frac{[s^{(k)}(r, z) - (1 - \tau)\ell^{(k)}]\psi^{(k)}(t) - [s^{(k)}(r, z) - \tau\ell^{(k)}]\psi^{(N+k)}(t)}{(2\tau - 1)\ell^{(k)}} \\ & \text{for } (r, z) \in \Gamma^{(k)}, \end{aligned} \quad (3.16)$$

where $\ell^{(k)} = s^{(k)}(r^{(k+1)}, z^{(k+1)})$ and $s^{(k)}(r, z)$ is the arc length along the element

$\Gamma^{(k)}$ as defined by

$$s^{(k)}(r, z) = \sqrt{(r - r^{(k)})^2 + (z - z^{(k)})^2}. \quad (3.17)$$

Similarly, $q(r, z, t) = \partial\psi/\partial n$ is approximated using

$$q(r, z, t) \simeq \frac{[s^{(k)}(r, z) - (1 - \tau)\ell^{(k)}]q^{(k)}(t) - [s^{(k)}(r, z) - \tau\ell^{(k)}]q^{(N+k)}(t)}{(2\tau - 1)\ell^{(k)}} \quad \text{for } (r, z) \in \Gamma^{(k)}, \quad (3.18)$$

if $q^{(k)}(t) = q(r_0^{(k)}, z_0^{(k)}, t)$ and $q^{(N+k)}(t) = q(r_0^{(N+k)}, z_0^{(N+k)}, t)$.

Note that the approximations in (3.16) and (3.18) which are also used in Ang and Ooi [11] do not guarantee the continuity of $\psi(r, z, t)$ and $q(r, z, t)$ from one element to the next. They give rise to what are known as discontinuous linear elements in the literature (París and Cañas [71]).

With (3.16) and (3.18), the boundary-domain integral equation (3.14) can be approximately written as

$$\begin{aligned} & \gamma(r_0, z_0)\psi(r_0, z_0, t) \\ = & \iint_{\Omega} G_0(r, z; r_0, z_0) \left\{ -\frac{Q(r, z, t)}{\sqrt{g(r, z)}} + B(r, z)\psi \right. \\ & + D(r, z, \psi) \frac{\partial}{\partial t} [\psi(r, z, t)] \} r dr dz \\ & + \sum_{k=1}^N \frac{1}{(2\tau - 1)\ell^{(k)}} \{ [-(1 - \tau)\ell^{(k)}\mathcal{F}_2^{(k)}(r_0, z_0) + \mathcal{F}_4^{(k)}(r_0, z_0)]\psi^{(k)}(t) \\ & + [\tau\ell^{(k)}\mathcal{F}_2^{(k)}(r_0, z_0) - \mathcal{F}_4^{(k)}(r_0, z_0)]\psi^{(N+k)}(t) \\ & - [-(1 - \tau)\ell^{(k)}\mathcal{F}_1^{(k)}(r_0, z_0) + \mathcal{F}_3^{(k)}(r_0, z_0)]q^{(k)}(t) \\ & - [\tau\ell^{(k)}\mathcal{F}_1^{(k)}(r_0, z_0) - \mathcal{F}_3^{(k)}(r_0, z_0)]q^{(N+k)}(t) \}, \quad (3.19) \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{F}_1^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} G_0(r, z; r_0, z_0) r ds(r, z), \\
 \mathcal{F}_2^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} G_1(r, z; r_0, z_0) r ds(r, z), \\
 \mathcal{F}_3^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} s(r, z) G_0(r, z; r_0, z_0) r ds(r, z), \\
 \mathcal{F}_4^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} s(r, z) G_1(r, z; r_0, z_0) r ds(r, z).
 \end{aligned} \tag{3.20}$$

The integrals over $\Gamma^{(k)}$ in (3.20) can be evaluated using numerical method such as a highly accurate Gaussian quadrature.

3.5.2 Axisymmetric Interpolating Functions and Treatment of Domain Integral

To treat the integral over the domain Ω in (3.19) using the dual-reciprocity method, L well-spaced out collocation points are chosen in the interior of the domain Ω . These points are denoted by $(r_0^{(2N+1)}, z_0^{(2N+1)})$, $(r_0^{(2N+2)}, z_0^{(2N+2)})$, $\dots$, $(r_0^{(2N+L-1)}, z_0^{(2N+L-1)})$ and $(r_0^{(2N+L)}, z_0^{(2N+L)})$. The points $(r_0^{(k)}, z_0^{(k)})$ and $(r_0^{(N+k)}, z_0^{(N+k)})$ on the element $\Gamma^{(k)}$ ($k = 1, 2, \dots, N$), as defined in (3.15), are also used as collocation points. The collocation points will be used to approximate the domain integral to linear integral with the use of suitably interpolating function.

The functions appear in the domain integral in (3.19) are approximated as

$$-\frac{Q(r, z, t)}{\sqrt{g(r, z)}} + B(r, z)\psi + D(r, z, \psi)\frac{\partial}{\partial t}[\psi(r, z, t)]$$

$$\begin{aligned}
 &\simeq \sum_{j=1}^{2N+L} \varphi^{(j)}(r, z) \sum_{k=1}^{2N+L} W^{(kj)} \left\{ -\frac{Q(r_0^{(k)}, z_0^{(k)}, t)}{\sqrt{g(r_0^{(k)}, z_0^{(k)})}} + B(r_0^{(k)}, z_0^{(k)}) \psi^{(k)}(t) \right. \\
 &\quad \left. + D(r_0^{(k)}, z_0^{(k)}, \psi^{(k)}(t)) \frac{d}{dt} [\psi^{(k)}(t)] \right\}. \tag{3.21}
 \end{aligned}$$

and the domain integral may be approximately given by

$$\begin{aligned}
 &\iint_{\Omega} G_0(r, z; r_0, z_0) \left\{ -\frac{Q(r, z, t)}{\sqrt{g(r, z)}} + B(r, z) \psi \right. \\
 &\quad \left. + D(r, z, \psi) \frac{\partial}{\partial t} [\psi(r, z, t)] \right\} r dr dz \\
 &\simeq \sum_{k=1}^{2N+L} \left\{ -\frac{Q(r_0^{(k)}, z_0^{(k)}, t)}{\sqrt{g(r_0^{(k)}, z_0^{(k)})}} + B(r_0^{(k)}, z_0^{(k)}) \psi^{(k)}(t) \right. \\
 &\quad \left. + D(r_0^{(k)}, z_0^{(k)}, \psi^{(k)}(t)) \frac{d}{dt} [\psi^{(k)}(t)] \right\} \sum_{j=1}^{2N+L} W^{(kj)} \Psi^{(j)}(r_0, z_0), \tag{3.22}
 \end{aligned}$$

where $\psi^{(k)}(t) = \psi(r_0^{(k)}, z_0^{(k)}, t)$ for $k = 1, 2, \dots, 2N + L$, the coefficients $W^{(kj)}$ are defined implicitly by

$$\sum_{j=1}^{2N+L} W^{(kj)} \varphi^{(p)}(r_0^{(j)}, z_0^{(j)}) = \begin{cases} 0 & \text{if } p \neq k \\ 1 & \text{if } p = k \end{cases} \quad \text{for } p, k = 1, 2, \dots, 2N + L, \tag{3.23}$$

the functions $\Psi^{(j)}(r_0, z_0)$ are expressed in terms of line integrals over Γ as

$$\begin{aligned}
 \Psi^{(j)}(r_0, z_0) &= \gamma(r_0, z_0) \chi^{(j)}(r_0, z_0) + \int_{\Gamma} r G_0(r, z; r_0, z_0) \frac{\partial}{\partial n} [\chi^{(j)}(r, z)] ds(r, z) \\
 &\quad - \int_{\Gamma} r \chi^{(j)}(r, z) G_1(r, z; r_0, z_0) ds(r, z) \\
 &\quad \text{for } j = 1, 2, \dots, 2N + L, \tag{3.24}
 \end{aligned}$$

and the functions $\varphi^{(p)}$ and $\chi^{(p)}$ have to satisfy

$$\frac{\partial^2 \chi^{(p)}}{\partial r^2} + \frac{1}{r} \frac{\partial \chi^{(p)}}{\partial r} + \frac{\partial^2 \chi^{(p)}}{\partial z^2} = \varphi^{(p)}. \tag{3.25}$$

The interpolating functions for approximating the domain integral are not unique. In axisymmetric case, for example in the work of Wang, Mattheij and ter Morsche [95], the interpolating functions $\varphi^{(p)}$ and $\chi^{(p)}$ are obtained based on interpolating functions that are used in three-dimensional case. The three-dimensional interpolating functions are $\nu(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)}))$ and $\omega(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)}))$, which have the relationship given by

$$\nabla_{\text{axis}}^2 \omega(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)})) = \nu(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)})), \quad (3.26)$$

where the function $\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)})$ is defined by

$$\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)}) = \sqrt{(r \cos \theta - r_0^{(p)})^2 + r^2 \sin^2 \theta + (z - z_0^{(p)})^2}. \quad (3.27)$$

In axisymmetric case, the functions $\varphi^{(p)}$ and $\chi^{(p)}$ can be obtained by integrating the functions $\nu(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)}))$ and $\omega(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)}))$ with respect to θ from 0 to 2π , that is

$$\begin{aligned} \varphi^{(p)}(r, z) &= \int_0^{2\pi} \nu(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)})) d\theta, \\ \chi^{(p)}(r, z) &= \int_0^{2\pi} \omega(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)})) d\theta. \end{aligned} \quad (3.28)$$

In Wang, Mattheij and ter Morsche [95], the functions $\nu(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)}))$ and $\omega(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)}))$ are chosen to be

$$\begin{aligned} \nu(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)})) &= \sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)}), \\ \omega(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)})) &= \frac{1}{12} [\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)})]^3. \end{aligned} \quad (3.29)$$

From (3.28) together with complete integral K and E (as defined in (2.20) in Chapter 2)

$$\int_0^{2\pi} (1 - m \sin^2(t))^{3/2} dt = \frac{1}{3} \begin{cases} 4(-1 + m)K(m) + 8(2 - m)E(m) & \text{if } 0 \leq m < 1, \\ 8 & \text{if } m = 1, \end{cases} \quad (3.30)$$

the choice of $\nu(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)}))$ and $\omega(\sigma(r, \theta, z; r_0^{(p)}, z_0^{(p)}))$ in (3.29) gives rise to

$$\begin{aligned}\varphi^{(p)}(r, z) &= 4\sqrt{a(r, z; r_0^{(p)}, z_0^{(p)}) + b(r; r_0^{(p)})E(m(r, z; r_0^{(p)}, z_0^{(p)}))}, \\ \chi^{(p)}(r, z) &= \frac{1}{9}[a(r, z; r_0^{(p)}, z_0^{(p)}) + b(r; r_0^{(p)})]^{3/2} \\ &\quad \times \begin{cases} (m(r, z; r_0^{(p)}, z_0^{(p)}) - 1)K(m(r, z; r_0^{(p)}, z_0^{(p)})) \\ + [4 - 2m(r, z; r_0^{(p)}, z_0^{(p)})]E(m(r, z; r_0^{(p)}, z_0^{(p)})) \\ \text{if } 0 \leq m(r, z; r_0^{(p)}, z_0^{(p)}) < 1, \\ 2 \text{ if } m(r, z; r_0^{(p)}, z_0^{(p)}) = 1. \end{cases} \quad (3.31)\end{aligned}$$

The first order partial derivatives of the function $\chi^{(p)}(r, z)$ in (3.31), as required in the evaluation of the first line integral on the right hand side of (3.24), can be obtained from (3.31) together with the results (see, for example, Whittaker and Watson [97])

$$\frac{d}{dm}(K(m)) = \frac{1}{2m}\left(\frac{E(m)}{1-m} - K(m)\right), \quad \frac{d}{dm}(E(m)) = \frac{1}{2m}(E(m) - K(m)). \quad (3.32)$$

As constructed by using (3.28) and (3.26) together with (3.29), the functions $\varphi^{(p)}(r, z)$ and $\chi^{(p)}(r, z)$ as given in (3.31) are quite complicated in form and they contain elliptic integrals. As mentioned earlier that the choice of interpolating functions are not unique. Interpolating functions that are in simpler form may be more desirable.

In the earlier works on the axisymmetric dual-reciprocity boundary element method, such as Partridge, Brebbia and Wrobel [74], the approach is to choose $\chi^{(p)}(r, z)$ as a simple function of the distance between the field point (r, z) and the collocation point $(r_0^{(p)}, z_0^{(p)})$ in the axisymmetric coordinate plane and determine the function $\varphi^{(p)}(r, z)$ from (3.25). For example, if $\chi^{(p)}(r, z)$ is chosen as

$$\chi^{(p)}(r, z) = \frac{1}{9}[\sigma(r, 0, z; r_0^{(p)}, z_0^{(p)})]^3, \quad (3.33)$$

where $\sigma(r, 0, z; r_0^{(p)}, z_0^{(p)})$ is the distance between the points (r, z) and $(r_0^{(p)}, z_0^{(p)})$ on the rz plane, that is,

$$\sigma(r, 0, z; r_0^{(p)}, z_0^{(p)}) = \sqrt{(r - r_0^{(p)})^2 + (z - z_0^{(p)})^2}, \quad (3.34)$$

then $\varphi^{(p)}(r, z)$ is given by

$$\varphi^{(p)}(r, z) = \left[\frac{4}{3} - \frac{r_0^{(p)}}{3r}\right]\sigma(r, 0, z; r_0^{(p)}, z_0^{(p)}), \quad (3.35)$$

Although the interpolating function $\varphi^{(p)}(r, z)$ constructed in this manner is simple in form (compared to (3.31)), its magnitude tends to infinity as $r \rightarrow 0$. Thus, if a certain part of the boundary of the domain Ω lies on the z axis (where $r = 0$), the use of (3.35) may compromise the accuracy of the approximation in (3.21).

Now if the function $\chi^{(p)}(r, z)$ can be chosen in such a way that the second term on the left hand side of (3.25) (that is, $r^{-1}\partial\chi^{(p)}/\partial r$) tends to a finite number as r tends to zero then $\varphi^{(p)}(r, z)$ which is bounded in Ω can be constructed. To do this, a method which take into consideration the virtual mirror image of the collocation point $(r_0^{(p)}, z_0^{(p)})$ about the z -axis (that is, the point $(-r_0^{(p)}, z_0^{(p)})$) is proposed here.

Specifically, we propose here to modify $\chi^{(p)}(r, z)$ in (3.33) to take the form

$$\chi^{(p)}(r, z) = \frac{1}{9}\{[\sigma(r, 0, z; r_0^{(p)}, z_0^{(p)})]^3 + [\sigma(r, 0, z; -r_0^{(p)}, z_0^{(p)})]^3\}. \quad (3.36)$$

The new interpolating function $\varphi^{(p)}(r, z)$ corresponding to $\chi^{(p)}(r, z)$ in (3.36) is then given by

$$\varphi^{(p)}(r, z) = \left[\frac{4}{3} - \frac{r_0^{(p)}}{3r}\right]\sigma(r, 0, z; r_0^{(p)}, z_0^{(p)}) + \left[\frac{4}{3} + \frac{r_0^{(p)}}{3r}\right]\sigma(r, 0, z; -r_0^{(p)}, z_0^{(p)}). \quad (3.37)$$

One may easily check that the function $\varphi^{(p)}(r, z)$ in (3.37) tends to a finite number as $r \rightarrow 0$.

Like (3.31), (3.37) gives $\varphi^{(p)}(r, z)$ which is not partially differentiable with respect to r or z at $(r_0^{(p)}, z_0^{(p)})$. The use of $\varphi^{(p)}(r, z)$ given by either (3.31) or (3.37) however does not pose any difficulty for the problem here. This is because the left hand side of (3.21) does not contain any partial derivative of the unknown function ψ with respect to r or z (hence the approximation of the domain integral as outlined in (3.22) does not involve any partial derivative of $\varphi^{(p)}(r, z)$). If spatial derivatives of the unknown function are present in the domain integral then interpolating functions which are partially differentiable in Ω are needed in the dual-reciprocity method. It should be obvious now why it is advantageous to use the substitution in (3.8) in formulating the problem.

3.5.3 Time-stepping and Iterative Procedure

Approximating $\psi^{(k)}(t)$ and its first order derivative by

$$\begin{aligned}\psi^{(k)}(t) &\simeq \frac{1}{2}[\psi^{(k)}(t + \frac{1}{2}\Delta t) + \psi^{(k)}(t - \frac{1}{2}\Delta t)], \\ \frac{d}{dt}[\psi^{(k)}(t)] &\simeq \frac{\psi^{(k)}(t + \frac{1}{2}\Delta t) - \psi^{(k)}(t - \frac{1}{2}\Delta t)}{\Delta t},\end{aligned}\tag{3.38}$$

and letting (r_0, z_0) in (3.19) be given in turn by $(r_0^{(1)}, z_0^{(1)})$, $(r_0^{(2)}, z_0^{(2)})$, $\dots$, $(r_0^{(2N+L-1)}, z_0^{(2N+L-1)})$ and $(r_0^{(2N+L)}, z_0^{(2N+L)})$, one finds that (3.19) and (3.22) give

$$\begin{aligned}&\frac{1}{2}\gamma(r_0^{(m)}, z_0^{(m)})[\psi^{(m)}(t + \frac{1}{2}\Delta t) + \psi^{(m)}(t - \frac{1}{2}\Delta t)] \\ &= \sum_{k=1}^{2N+L} \left\{ -\frac{Q(r_0^{(k)}, z_0^{(k)}, t)}{\sqrt{g(r_0^{(k)}, z_0^{(k)})}} + B(r_0^{(k)}, z_0^{(k)})\psi^{(k)}(t) \right. \\ &\quad \left. + F^{(k)}(t)\frac{\psi^{(k)}(t + \frac{1}{2}\Delta t) - \psi^{(k)}(t - \frac{1}{2}\Delta t)}{\Delta t} \right\} \mu^{(km)}\end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^N \frac{1}{(2\tau - 1)\ell^{(k)}} \{ [-(1 - \tau)\ell^{(k)}\mathcal{F}_2^{(k)}(r_0^{(m)}, z_0^{(m)}) \\
 & \quad + \mathcal{F}_4^{(k)}(r_0^{(m)}, z_0^{(m)})] \psi^{(k)}(t) \\
 & \quad + [\tau\ell^{(k)}\mathcal{F}_2^{(k)}(r_0^{(m)}, z_0^{(m)}) - \mathcal{F}_4^{(k)}(r_0^{(m)}, z_0^{(m)})] \psi^{(N+k)}(t) \\
 & \quad - [-(1 - \tau)\ell^{(k)}\mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)}) + \mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)})] q^{(k)}(t) \\
 & \quad - [\tau\ell^{(k)}\mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)}) - \mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)})] q^{(N+k)}(t) \} \\
 & \quad \text{for } m = 1, 2, \dots, 2N + L, \tag{3.39}
 \end{aligned}$$

where

$$F^{(k)}(t) = D(r_0^{(k)}, z_0^{(k)}, \psi^{(k)}(t)), \quad \mu^{(km)} = \sum_{j=1}^{2N+L} W^{(kj)} \Psi^{(j)}(r_0^{(m)}, z_0^{(m)}). \tag{3.40}$$

Application of the boundary conditions on the second and third lines of (3.12) into (3.39) yields

$$\begin{aligned}
 & \gamma(r_0^{(m)}, z_0^{(m)}) \{ \frac{\alpha^{(m)}}{2} [\psi^{(m)}(t + \frac{1}{2}\Delta t) + \psi^{(m)}(t - \frac{1}{2}\Delta t)] + \beta^{(m)} R^{(m)}(t) \} \\
 = & \sum_{k=1}^{2N+L} \{ B(r_0^{(k)}, z_0^{(k)}) (\frac{\alpha^{(k)}}{2} [\psi^{(k)}(t + \frac{1}{2}\Delta t) + \psi^{(k)}(t - \frac{1}{2}\Delta t)] + \beta^{(k)} R^{(k)}(t)) \\
 & + F^{(k)}(t) (\alpha^{(k)} \frac{\psi^{(k)}(t + \frac{1}{2}\Delta t) - \psi^{(k)}(t - \frac{1}{2}\Delta t)}{\Delta t} \\
 & \quad + \beta^{(k)} \frac{d}{dt} [R^{(k)}(t)]) - \frac{Q(r_0^{(k)}, z_0^{(k)}, t)}{\sqrt{g(r_0^{(k)}, z_0^{(k)})}} \} \mu^{(km)} \\
 & + \sum_{k=1}^N \frac{1}{(2\tau - 1)\ell^{(k)}} \{ [-(1 - \tau)\ell^{(k)}\mathcal{F}_2^{(k)}(r_0^{(m)}, z_0^{(m)}) + \mathcal{F}_4^{(k)}(r_0^{(m)}, z_0^{(m)}) \\
 & \quad - \alpha^{(k)} Z^{(k)}(t) (-(1 - \tau)\ell^{(k)}\mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)}) + \mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)}))] \\
 & \quad \times (\frac{\alpha^{(k)}}{2} [\psi^{(k)}(t + \frac{1}{2}\Delta t) + \psi^{(k)}(t - \frac{1}{2}\Delta t)] + \beta^{(k)} R^{(k)}(t)) \\
 & \quad + [\tau\ell^{(k)}\mathcal{F}_2^{(k)}(r_0^{(m)}, z_0^{(m)}) - \mathcal{F}_4^{(k)}(r_0^{(m)}, z_0^{(m)})]
 \end{aligned}$$

$$\begin{aligned}
 & -\alpha^{(N+k)} Z^{(N+k)}(t) (\tau \ell^{(k)} \mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)}) - \mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)})) \\
 & \times \left(\frac{\alpha^{(N+k)}}{2} [\psi^{(N+k)}(t + \frac{1}{2}\Delta t) + \psi^{(N+k)}(t - \frac{1}{2}\Delta t)] + \beta^{(N+k)} R^{(N+k)}(t) \right) \\
 & - [-(1 - \tau) \ell^{(k)} \mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)}) + \mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)})] \\
 & \times [\alpha^{(k)} Y^{(k)}(t) + \beta^{(k)} q^{(k)}(t)] \\
 & - [\tau \ell^{(k)} \mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)}) - \mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)})] \\
 & \times [\alpha^{(N+k)} Y^{(N+k)}(t) + \beta^{(N+k)} q^{(N+k)}(t)] \} \\
 & \text{for } m = 1, 2, \dots, 2N + L,
 \end{aligned} \tag{3.41}$$

where $R^{(m)}(t)$, $\alpha^{(m)}$, $\beta^{(m)}$, $Y^{(p)}(t)$ and $Z^{(p)}(t)$ for $m = 1, 2, \dots, 2N + L$ and $p = 1, 2, \dots, N, N + 1, \dots, 2N - 1, 2N$ are defined by

$$\begin{aligned}
 R^{(m)}(t) &= \sqrt{g(r_0^{(m)}, z_0^{(m)})} \mathcal{K}(f_1(r_0^{(m)}, z_0^{(m)}, t)), \\
 \alpha^{(m)} &= \begin{cases} 0 & \text{if } (r_0^{(m)}, z_0^{(m)}) \text{ lies on a boundary element} \\ & \text{where } \psi \text{ is specified,} \\ 1 & \text{otherwise,} \end{cases} \\
 \beta^{(m)} &= 1 - \alpha^{(m)}, \\
 Y^{(p)}(t) &= \frac{1}{\sqrt{g(r_0^{(p)}, z_0^{(p)})}} f_2(r_0^{(p)}, z_0^{(p)}, t), \\
 Z^{(p)}(t) &= \frac{1}{2g(r_0^{(p)}, z_0^{(p)})} (n_r(r, z) \frac{\partial}{\partial r} [g(r, z)] \\
 & \quad + n_z(r, z) \frac{\partial}{\partial z} [g(r, z)]) \Big|_{(r, z) = (r_0^{(p)}, z_0^{(p)})}.
 \end{aligned} \tag{3.42}$$

Note that $[n_r(r_0^{(p)}, z_0^{(p)}), n_z(r_0^{(p)}, z_0^{(p)})]$ gives the outward unit normal vector to the boundary element which contains the collocation point $(r_0^{(p)}, z_0^{(p)})$. Thus, $[n_r(r_0^{(k)}, z_0^{(k)}), n_z(r_0^{(k)}, z_0^{(k)})]$ and $[n_r(r_0^{(N+k)}, z_0^{(N+k)}), n_z(r_0^{(N+k)}, z_0^{(N+k)})]$ both refer to the outward unit normal vector to the boundary element denoted by $\Gamma^{(k)}$. Also, note that the function $f_2(r_0^{(p)}, z_0^{(p)}, t)$ (hence $Y^{(p)}(t)$) is defined only if $(r_0^{(p)}, z_0^{(p)})$ is a collocation point on a boundary element where ψ is not specified. (In (3.41), $Y^{(p)}(t)$ is always multiplied to $\alpha^{(p)}$. Hence, the calculation of

$Y^{(p)}(t)$ is needed only if $(r_0^{(p)}, z_0^{(p)})$ is a collocation point on a boundary element where ψ is not specified.)

In case of the nonlinear coefficient $F^{(n)}(t)$, iterative procedure is required. If $F^{(n)}(t)$ and $\psi^{(n)}(t - \frac{1}{2}\Delta t)$ are assumed known for $n = 1, 2, \dots, 2N + L$ then (3.41) constitutes a system of $2N + L$ linear algebraic equations containing $2N + L$ unknowns. There are L unknowns at the interior collocation points as given by $\psi^{(p)}(t + \frac{1}{2}\Delta t)$ for $p = 2N + 1, 2N + 2, \dots, 2N + L$. The remaining $2N$ unknowns are given either by $\psi^{(k)}(t + \frac{1}{2}\Delta t)$ and $\psi^{(N+k)}(t + \frac{1}{2}\Delta t)$ (if ψ is not specified on $\Gamma^{(k)}$) or $q^{(k)}(t)$ and $q^{(N+k)}(t)$ (if ψ is specified on $\Gamma^{(k)}$). The unknowns can be determined numerically by repeating the steps below until the numerical values of ψ at the selected points are obtained at the desired time level.

1. From the initial condition given in (3.12), compute the values of $\psi^{(n)}(0)$ for $n = 1, 2, \dots, 2N + L$. Choose a small positive time-step Δt . Set the integer $J = 0$. Go to Step 2.
2. Estimate the values of $F^{(n)}((J + \frac{1}{2})\Delta t)$ using the latest known values of $\psi^{(n)}(J\Delta t)$, that is, $F^{(n)}((J + \frac{1}{2})\Delta t) \simeq D(r_0^{(n)}, z_0^{(n)}, \psi^{(n)}(J\Delta t))$. Go to Step 3.
3. Using the latest known values of $F^{(n)}((J + \frac{1}{2})\Delta t)$ and $\psi^{(n)}(J\Delta t)$, let $t = (J + \frac{1}{2})\Delta t$ in (3.41) to set up a system of linear algebraic equations and solve for the unknowns. The unknowns are given by $\psi^{(m)}((J + 1)\Delta t)$ for $m = 2N + 1, 2N + 2, \dots, 2N + L$, and either by $\psi^{(k)}((J + 1)\Delta t)$ and $\psi^{(N+k)}((J + 1)\Delta t)$ (if ψ is not specified on $\Gamma^{(k)}$) or by $q^{(k)}((J + \frac{1}{2})\Delta t)$ and $q^{(N+k)}((J + \frac{1}{2})\Delta t)$ (if ψ is specified on $\Gamma^{(k)}$) for $k = 1, 2, \dots, N$. Go to Step 4.

4. Use the latest known values of $\psi^{(n)}((J+1)\Delta t)$ obtained in Step 3 above to compute $\psi^{(n)}((J+\frac{1}{2})\Delta t) = \frac{1}{2}[\psi^{(n)}((J+1)\Delta t) + \psi^{(n)}(J\Delta t)]$ for $n = 1, 2, \dots, 2N+L$. Re-calculate $F^{(n)}((J+\frac{1}{2})\Delta t)$ using $F^{(n)}((J+\frac{1}{2})\Delta t) \simeq D(r_0^{(n)}, z_0^{(n)}, \psi^{(n)}((J+\frac{1}{2})\Delta t))$. Check whether the newly obtained values of $F^{(n)}((J+\frac{1}{2})\Delta t)$ agree with the previous values to within a specified number of significant figures. If the required convergence is not achieved, go to Step 3. Otherwise, increase the current value of J by 1 and go to Step 2.

3.6 Specific Problems

The dual-reciprocity boundary element procedure described in previous section is applied here to solve some specific problems including one that involves the laser heating of a cylindrical solid.

The relatively simple interpolating functions $\varphi^{(p)}(r, z)$ proposed in (3.37) are used in all the problems below to treat the domain integral. For the first three problems which have known exact solutions, we have also repeated all the calculations by using the more complicated interpolating functions $\varphi^{(p)}(r, z)$ in (3.31) and found that they and the new interpolating functions in (3.37) deliver numerical solutions that are of comparable accuracy.

Problem 1

Take the solution domain $1 < r < 2$, $0 < z < 1$ (a hollow cylinder). The coefficients κ , ρ and c are given by $\kappa = 1 + r^2$ and $\rho c = 1$. The function Q is given by

$$Q = -\left\{\frac{21}{2}r^2 + \frac{1}{2}z^2 + 6\right\} \exp\left(-\frac{1}{2}t\right) - \frac{4}{r^4}.$$

The initial-boundary conditions are taken to be

$$\begin{aligned}
 T(r, z, 0) &= \frac{1}{r^2} + r^2 + z^2 \text{ for } 1 < r < 2, 0 < z < 1, \\
 T(r, 0, t) &= \frac{1}{r^2} + r^2 \exp(-\frac{1}{2}t) \text{ for } 1 < r < 2 \text{ and } t > 0, \\
 T(r, 1, t) &= \frac{1}{r^2} + (r^2 + 1) \exp(-\frac{1}{2}t) \text{ for } 1 < r < 2 \text{ and } t > 0, \\
 (1 + r^2) \frac{\partial T}{\partial r} \Big|_{r=1} &= -4 + 4 \exp(-\frac{1}{2}t) \text{ for } 0 < z < 1 \text{ and } t > 0, \\
 (1 + r^2) \frac{\partial T}{\partial r} \Big|_{r=2} &= -\frac{5}{4} + 20 \exp(-\frac{1}{2}t) \text{ for } 0 < z < 1 \text{ and } t > 0.
 \end{aligned}$$

For the problem here, the curve Γ in the boundary-domain integral formulation is a closed one which comprises the four sides of the square region $1 < r < 2$, $0 < z < 1$, on the rz plane. To apply the dual-reciprocity boundary element method to solve the problem numerically, each of the sides is discretized into N_0 elements of equal length (so that $N = 4N_0$) and the M^2 interior collocation points are chosen to be given by $(r, z) = (1 + i/(M + 1), j/(M + 1))$ for $i = 1, 2, \dots, M$ and $j = 1, 2, \dots, M$.

The governing partial differential equation is linear and T is related to ψ by $\psi = \sqrt{1 + r^2}T$. Hence, for the problem here, at a given time level $t = (J + \frac{1}{2})\Delta t$, it is not necessary to iterate between Steps 3 and 4 (in the procedure outlined in Section 3.5).

Two sets of numerical values are obtained for T . The first set (Set A) is obtained by using $N_0 = 10$, $M = 3$ and $\Delta t = 0.30$, while the second set (Set B) by $N_0 = 20$, $M = 15$ and $\Delta t = 0.10$. In both sets, the parameter τ in the discontinuous linear elements is chosen to be 0.25. In Table 3.1, at selected interior points and time $t = 0.45$, the numerical values of T in Sets A and B

Table 3.1: Numerical and exact values of T at selected interior points and time $t = 0.45$.

Point (r, z)	Set A	Set B	Exact
(1.25, 0.25)	1.93380	1.93744	1.93759
(1.50, 0.25)	2.28949	2.29095	2.29101
(1.75, 0.25)	2.82081	2.82186	2.82189
(1.25, 0.50)	2.08189	2.08711	2.08731
(1.50, 0.50)	2.43835	2.44064	2.44073
(1.75, 0.50)	2.96997	2.97156	2.97162
(1.25, 0.75)	2.33285	2.33671	2.33685
(1.50, 0.75)	2.68863	2.69022	2.69027
(1.75, 0.75)	3.21999	3.22113	3.22115

are compared with the exact solution given by

$$T(r, z, t) = \frac{1}{r^2} + (r^2 + z^2) \exp\left(-\frac{1}{2}t\right).$$

Both sets of numerical values of T are reasonably accurate. The percentage errors of the numerical values in Sets A and B are less than 0.25% and 0.09% respectively. It is obvious that the numerical values of T converge to the exact solution (that is, there is a significant improvement in the accuracy of the numerical values) when the calculation is refined by reducing the sizes of the boundary elements used, increasing the number of interior collocation points and decreasing the time-step.

The theoretical analysis of the stability of the time-stepping scheme may be mathematically difficult. Nevertheless, from numerical experiments, the scheme appears to be stable for a reasonably wide range of Δt tested. Furthermore, since the approximations in (3.38) are $O([\Delta t]^2)$ accurate, it seems not necessary to use extremely small Δt to obtain accurate numerical solutions.

To analyze the accuracy of the numerical results, the L_2 norm error defined

by

$$L_2 \text{ norm error} = \sqrt{\frac{\sum_{i=1}^{2N+L} (T_{\text{exact}}(i) - T_{\text{BEM}}(i))^2}{\sum_{i=1}^{2N+L} [T_{\text{exact}}(i)]^2}},$$

where $T_{\text{exact}}(i)$ and $T_{\text{BEM}}(i)$ denote the temperature values of the exact and the boundary element solutions respectively at the i -th collocation point (in the numerical procedure), is calculated using the numerical solutions of Set A and Set B and compared in Figure 3.1. It is clearly seen that in Figure 3.1, the numerical results for Set B is more accurate than the results obtained in Set A as norm of Set B is much smaller than the norm of Set A.

Figure 3.1: L_2 norm error for Set A and Set B against t .

Problem 2

The solution domain is taken to be $0 < r < 1$, $0 < z < 1$ (a solid cylinder). The coefficients κ , ρ and c are given by $\kappa = (1+z)(1+T)$ and $\rho c = 1$ and the function Q is given by

$$Q(r, z, t) = \frac{(1+z)^2 - 2t - 4t^2 - 2zt - 2zt^2}{(1+z)^3(1+t)^2}.$$

The initial-boundary conditions are taken to be

$$\begin{aligned} T(r, z, 0) &= 1 \text{ for } 0 < r < 1, \ 0 < z < 1, \\ T(r, 0, t) &= \frac{1+2t}{1+t} \text{ for } 0 < r < 1, \ t > 0, \\ T(r, 1, t) &= \frac{2+3t}{2(1+t)} \text{ for } 0 < r < 1, \ t > 0, \\ (1+z)(1+T) \left. \frac{\partial T}{\partial r} \right|_{r=1} &= 0 \text{ for } 0 < z < 1, \ t > 0. \end{aligned}$$

For the problem here, the curve Γ on the rz plane consists of three straight line segments of unit length. To apply the dual-reciprocity boundary element method to solve the problem numerically, each of these line segments is discretized into N_0 equal length boundary elements (so that $N = 3N_0$) and the M^2 interior collocation points are chosen to be given by $(r, z) = (i/(M+1), j/(M+1))$ for $i = 1, 2, \dots, M$ and $j = 1, 2, \dots, M$.

According to the Kirchhoff's transformation, the function ψ may be taken to be related to T by $\psi = \sqrt{1+z}(T + \frac{1}{2}T^2)$. Here the governing partial differential equation in ψ is nonlinear with the coefficient $D(r, z, \psi)$ of $\partial\psi/\partial t$ being given by

$$D(r, z, \psi) = \frac{1}{(1+z)\sqrt{1+2\psi/\sqrt{1+z}}}.$$

At a given time level $t = (J + \frac{1}{2})\Delta t$, Steps 3 and 4 (in the numerical procedure outlined in Section 3.5) are iterated until the value of $F^{(n)}((J + \frac{1}{2})\Delta t)$ averaged

Table 3.2: Numerical and exact values of T at $(r, z) = (0.50, 0.50)$ and at selected time levels.

Time t	Set A	Set B	Exact
0.15	1.08398	1.08675	1.08696
0.45	1.20811	1.20683	1.20690
0.75	1.28491	1.28567	1.28571
1.05	1.34189	1.34145	1.34146
1.35	1.38269	1.38297	1.38298

over all the collocation points does not change by more than 0.001% from one iteration to the next. In the numerical results presented below, not more than 10 iterations are required in the calculation.

As in Problem 1 above, two sets of numerical results are obtained for T . Sets A and B are obtained by using $(N_0, M, \Delta t) = (10, 7, 0.30)$ and $(N_0, M, \Delta t) = (20, 15, 0.10)$ respectively. In both sets, the parameter τ in the discontinuous linear elements is set to 0.25. In Table 3.2, the numerical values of T at the point $(r, z) = (0.50, 0.50)$ are compared with the exact solution of the problem at various time levels. The exact solution is given by

$$T(r, z, t) = 1 + \frac{t}{(1+z)(1+t)}.$$

As may be expected, the numerical values of T in Set B are found to be more accurate than those in Set A. The percentage errors in the numerical values in Sets A and B are less than 0.3% and 0.02% respectively.

Problem 3

Take the solution domain Ω on the rz plane to be the region bounded by the lines $r = z$, $r = 0$ and $z = 1$. (Rotating Ω by an angle of 360° about the z axis gives rise to R which is a conical region.) The coefficients κ , ρ and c are given

by $\kappa = T$ and $\rho c = T$. The function Q is given by

$$Q = \left(\frac{1}{2} - 2(r^2 + z^2)\right) \exp(-(t + r^2 - z^2)).$$

The initial-boundary conditions are taken to be

$$\begin{aligned} T(r, z, 0) &= \exp\left(-\frac{1}{2}(r^2 - z^2)\right) \text{ for } (r, z) \text{ in } R, \\ T(r, z, t) &= \exp\left(-\frac{t}{2}\right) \text{ on } r = z \text{ for } 0 < z < 1, \\ T \frac{\partial T}{\partial n} &= T - \exp\left(-\frac{1}{2}(t + r^2 - 1)\right) + \exp(-(t + r^2 - 1)) \\ &\text{on } z = 1 \text{ for } 0 < r < 1 \text{ and } t > 0. \end{aligned}$$

For the problem under consideration here, with the transformation $\psi = \frac{1}{2}T^2$, the boundary condition involving the flux $\partial T/\partial n$ and temperature T (that is, the so called Robin condition) can be rewritten as

$$\begin{aligned} \frac{\partial \psi}{\partial n} &= \sqrt{2\psi} - \exp\left(-\frac{1}{2}(t + r^2 - 1)\right) + \exp(-(t + r^2 - 1)) \\ &\text{on } z = 1 \text{ for } 0 < r < 1 \text{ and } t > 0. \end{aligned}$$

Unlike in (3.12), the above transformed boundary condition for the problem here contains a nonlinear term. The nonlinear term in the boundary condition can be treated as explained below.

At a fixed time level $t = (J + \frac{1}{2})\Delta t$ ($J = 0, 1, 2, \dots$), the nonlinear term in the transformed boundary condition is first approximated using the solution ψ at $t = J\Delta t$. The solution ψ at $t = (J + \frac{1}{2})\Delta t$ can then be obtained approximately by solving the linear algebraic equations in the boundary element procedure as explained in Section 3.5. The approximation of the nonlinear term in the boundary condition can be updated by using the just obtained solution ψ at $t = (J + \frac{1}{2})\Delta t$ and the boundary element procedure can be used again to solve for ψ at $t = (J + \frac{1}{2})\Delta t$. The process can be iterated until the

temperature at the collocation points converges to within a specified number of significant figures.

Figure 3.2: A graphical comparison of numerical and exact temperature on $z = 1$, $0 < r < 1$, at time $t = 0.95$.

To obtain some numerical results, the boundary Γ is discretized into 100 elements, the parameter τ in the discontinuous elements is taken to be 0.25, 36 well spaced points are selected as interior collocation points and the time-step Δt is chosen to be 0.10. Iteration on the nonlinear boundary condition is stopped when the temperature does not change by more than 0.001% at all the collocation points. Typically, not more than 10 iterations are required in the calculation. As the temperature is not known a priori on the boundary $z = 1$ ($0 < r < 1$), the numerical values of the boundary temperature there

Figure 3.3: A graphical comparison of numerical and exact temperature at point $(r, z) = (0.40, 0.70)$ over the period $0 < t < 2$.

at $t = 0.95$ are compared graphically with the exact temperature $T(r, z, t) = \exp(-\frac{1}{2}(t + r^2 - z^2))$ in Figure 3.2. In Figure 3.3, the numerical and the exact values of T at the point $(r, z) = (0.40, 0.70)$ are plotted against time t for $0 < t < 2$. The numerical and exact temperature agree well with each other. The percentage errors of the numerical values in Figures 3.2 and 3.3 are less than 0.12% and 0.07% respectively.

Problem 4

Consider a cylindrical solid of radius a and height b . Specifically, the cylindrical solid occupies the region $0 < r < a$, $0 < z < b$. It is subject to laser heating

on the surface $z = 0$. The laser heating may be modeled by taking

$$Q(r, z, t) = \phi(t)\mu(1 - R)I(r) \exp(-\mu z),$$

where μ is the laser absorption coefficient, R is the Fresnel surface reflectance, $\phi(t)$ is a given function controlling the laser heating and $I(r)$ is the incident irradiance at the center $(0, 0)$ on the surface $z = 0$. Specifically, $I(r)$ is chosen here to take the Gaussian form

$$I(r) = I_0 \exp\left(-\frac{2r^2}{w^2}\right),$$

where I_0 is the peak irradiance and w is the radius of the laser beam. For some details of problems involving laser heating of solids, refer to, for example, Gutierrez and Jen [43] and Ooi, Ang and Ng [66].

The initial-boundary conditions are given by

$$\begin{aligned} T(r, z, 0) &= T_0 \text{ for } (r, z) \text{ inside the solid,} \\ \kappa \frac{\partial T}{\partial n} &= 0 \text{ on } r = a \text{ for } t > 0, \\ \kappa \frac{\partial T}{\partial n} &= h_{\text{amb}}(T_{\text{amb}} - T) \text{ on } z = 0, \ 0 < r < a \text{ for } t > 0, \\ \kappa \frac{\partial T}{\partial n} &= 0 \text{ on } z = b, \ 0 < r < a \text{ for } t > 0, \end{aligned}$$

where T_0 is a given constant, h_{amb} is the ambient convection coefficient and T_{amb} is the ambient temperature.

As pointed out in [43], the high temperature gradient generated by the laser heating may result in significant changes in the thermal properties of the solid. The thermal conductivity κ and specific heat capacity c are modeled to vary with temperature in accordance with

$$\begin{aligned} \kappa &= \kappa_0 + \kappa_1(T - T_0), \\ c &= c_0 + c_1(T - T_0) + c_2(T - T_0)^2, \end{aligned}$$

where κ_0 , κ_1 , c_0 , c_1 and c_2 are given constants.

For the purpose of obtaining some results, the material constants of polymethyl methacrylate (PMMA) are used here. They are given by $\rho = 1180$ kg/m³, $\kappa_0 = 0.19$ W/(m K), $\kappa_1 = -0.19 \times 10^{-3}$ W/(m K²), $c_0 = 1500$ J/(kg K), $c_1 = 4.5$ J/(kg K²) and $c_2 = 0$ J/(kg K³) (Gutierrez and Jen [43]). The laser absorption coefficient μ is 5.0×10^4 m⁻¹. The other laser parameters required in the heat generation term are taken to be $(1 - R)I_0 = 1.0 \times 10^5$ W/m², $w = 0.0003$ m and $\phi(t) = 1$ for $t \geq 0$. The initial temperature T_0 , the ambient temperature T_{amb} and the ambient convection coefficient h_{amb} are 310 K, 300 K and 10 W/(m² K) respectively. The dimensions of the PMMA cylinder are given by $a = b = 1$ mm. The calculation of the temperature is carried using 60 elements on the curve Γ and 441 collocation points in the interior of the solution domain. The time-step used is $\Delta t = 0.093$ s.

The spatial temperature profiles over the solution domain $0 < r < 1$ mm, $0 < z < 1$ mm, at selected time instants $t = 9.18$ s, $t = 27.81$ s and $t = 46.44$ s are depicted by the plots in Figure 3.4 (under “nonlinear heat conduction”). Figure 3.4 also shows the corresponding temperature profiles for linear heat conduction (obtained by using $\kappa_1 = 0$, $c_1 = 0$ and $c_2 = 0$). Due to continuous laser heating, the temperature increases with time, as expected. According to the linear theory of heat conduction, at time $t = 46.44$ s, the temperature in the solid ranges from 420 K (in most part) to a maximum of over 500 K (at the center of heating). At the same time instant, the temperature given by the nonlinear theory is only around 440 K very close to the center of heating and is below 400 K in most part of the solid. Thus, it appears that linear theory predicts a quicker heating up of the solid than the nonlinear one.

To investigate further the effects of the temperature-dependent terms in the thermal conductivity and the specific heat capacity, we consider the following

Figure 3.4: Spatial temperature profiles for nonlinear and linear heat conduction at selected time instants.

cases:

Case I: $c_1 = 0$, $c_2 = 0$, $\kappa_1 = 0$ (linear heat conduction).

Case II: $c_1 = 0$, $c_2 = 0$, $\kappa_1 = -0.19 \times 10^{-3} \text{ W/(m K}^2\text{)}$.

Case III: $c_1 = 4.5 \text{ J/(kg K}^2\text{)}$, $c_2 = 0$, $\kappa_1 = 0$.

Case IV: $c_1 = 4.5 \text{ J/(kg K}^2\text{)}$, $c_2 = 0$, $\kappa_1 = -0.19 \times 10^{-3} \text{ W/(m K}^2\text{)}$.

All other parameters (such as ρ , c_0 , κ_0 , μ and T_0) are as given before for the PMMA solid. For all the four cases, the temperature at $r = 0.045 \text{ mm}$ and selected time instants ($t = 1.72 \text{ s}$, $t = 9.17 \text{ s}$ and $t = 46.44 \text{ s}$) is plotted against z ($0 < z < 1 \text{ mm}$) in Figures 3.5, 3.6 and 3.7. At earlier time, such as $t = 1.72 \text{ s}$, the temperature in each of the cases is quite close to one another. In Figures 3.5 and 3.6, the temperature in Case I is almost visually indistinguishable from that in Case II. Similarly, at earlier time, there is only a very small difference between the temperature in Case III and that in Case IV. This is perhaps not surprising as the value of κ_1 in Cases II and IV is relatively small, close to zero. Nevertheless, as time evolves, the difference in the temperature in each of the cases becomes more pronounced, as is obvious in Figure 3.7 where the temperature plots for $t = 46.44 \text{ s}$ are given. Each of the temperature-dependent terms in the thermal conductivity and the specific heat capacity in the PMMA cylindrical solid apparently has the effect of slowing down the heating of the solid.

3.7 Summary

The numerical solution of an axisymmetric heat conduction problem involving a nonhomogeneous solid with temperature-dependent properties is considered. Through the use of Kirchhoff's transformation and an appropriate substitution

Figure 3.5: Plots of temperature at $r = 0.045$ mm and time $t = 1.72$ s against z .

Figure 3.6: Plots of temperature at $r = 0.045$ mm and time $t = 9.17$ s against z .

Figure 3.7: Plots of temperature at $r = 0.045$ mm and time $t = 46.44$ s against z .

of variables, the problem is formulated in terms of a nonlinear boundary-domain integral equation. The boundary-domain integral equation contains an integral over a curve and a domain integral. The first order time derivative of the temperature is present in the integrand of the domain integral. With the time derivative of the temperature approximated by a finite-difference formula, the line and domain integrals are treated using a dual-reciprocity boundary element procedure. We have proposed relatively simple interpolating functions for use in the dual-reciprocity approximation of the domain integral. It appears that such interpolating functions have not been used before in the literature. The boundary-domain integral equation is eventually reduced to nonlinear algebraic equations which are solved iteratively at consecutive time levels.

To assess the validity and accuracy of the proposed numerical procedure, some problems which have known solutions are solved. The numerical results

agree favorably with the known solutions, indicating that the dual-reciprocity boundary element method together with the proposed interpolating function can be used to provide reliable and accurate numerical solutions for the non-linear axisymmetric heat equation. Lastly, the method is applied to study the effects of temperature-dependent material properties on the laser heating of a cylindrical solid.

Chapter 4

Axisymmetric Nonclassical Heat Conduction in Nonhomogeneous Solids

4.1 Introduction

The classical Fourier heat flux model assumes that the heat flux is instantaneously proportional to temperature gradient. This gives rise to the physically undesirable conclusion that thermal wave travels at infinite speed. A justification often given for using the classical model is that thermal waves are significant only over a very short length of time or only under extreme thermal conditions such as very low temperature or very high heat flow rate. Nevertheless, Kaminski [49] reported observing thermal waves at room temperature and Yuen and Lee [100] showed that thermal waves may possibly last over a longer period of time.

To take into consideration thermal waves, Cattaneo [21] and Vernotte [92] modified the Fourier constitutive relation in (2.2) by introducing a thermal relaxation time in the heat flux. Heat conduction problems solved using the Cattaneo-Vernotte model include Banerjee, Ogale, Das, et al. [17] (axisym-

metric heat conduction problem with continuous wave and short pulse laser heating), Liu, Chen and Xu [55] (bioheat transfer in skin), Chu, Lin and Lin [24] (conductive and radiative heat transfer in an isotropic scattering medium) and Hu, Fang and Du [46] (thermal wave scattering effect in semi-infinite functionally graded materials).

The Cattaneo-Vernotte model was generalized by Tzou [89, 90] in 1995 to take into account the influences of microscopic phenomena such as phonon-electron interaction and phonon scattering on the heating of solids. In the generalized model, which is widely known as the dual-phase-lag model, the Fourier constitutive relation in (2.2) is modified by introducing time phase lags in both the heat flux and the temperature gradient. The dual-phase-lag model has been successfully employed in the solutions of some thermal problems which include Antaki [12] (semi-infinite slab with surface heat flux), Tang and Araki [88] (slabs with laser-pulses heating), Ghazanfarian and Abbassi [40] (thin slab with the phonon scattering boundary condition), Ramadan, Tyfour and Al-Nimr [77] (metal with short-pulse laser heating) and Tzou and Chiu [91] (gold film with ultrafast laser heating).

The numerical solution of heat conduction problems based on nonclassical heat conduction models has been a subject of considerable interest in recent years. For example, Manzari and Manzari [60] had applied the finite element method to heat conduction based on Cattaneo-Vernotte model, Dai and Nassar [30], Zhang and Zhao [101] and Pan, Tang and Zhou [70] had outlined finite difference schemes for the dual-phase-lag heat conduction in one- and two-dimensional solution domains, and Ang [5] had proposed a boundary element method for two-dimensional dual-phase-lag heat conduction in a homogeneous thermally isotropic solid.

A dual-reciprocity boundary element approach is presented in this chapter

¹ for the numerical simulation of axisymmetric dual-phase-lag heat conduction in a thermally isotropic solid with properties that vary continuously in space. In the numerical approach here, the governing partial differential equation is first converted into a suitable boundary-domain integral equation. As in Chapter 3, the domain integral is transformed to boundary integral by using the newly proposed interpolating functions. The time derivatives of the temperature and the heat flux in the boundary-domain integral formulation are suppressed by using the Laplace transformation. The problem is eventually reduced to solving a system of linear algebraic equations. In Chapter 3, the time-stepping scheme is used to approximate the time derivatives of temperature from the integral equation. However, the presence of second order time derivative of temperature in the governing partial differential equation of the dual-phase-lag model requires at least three or more time levels in any finite difference approximation, hence resulting in linear algebraic equations with a larger number of unknowns. For a smaller number of unknowns in the linear algebraic equations, the Laplace transformation is used here instead to treat the time derivatives of the temperature. The linear algebraic equations are solved in the Laplace transform domain. The temperature in the physical domain is approximately recovered by a numerical method for inverting Laplace transforms. To check the validity of the boundary element procedure here, some specific axisymmetric nonclassical heat conduction problems involving nonhomogeneous materials are solved.

¹The work reported in this chapter has been published as: Yun BI and Ang WT, “A dual-reciprocity boundary element simulation of axisymmetric dual-phase-lag heat conduction in nonhomogeneous media”, *Computer Modeling in Engineering & Sciences*, vol. 65, pp. 217-244, 2010.

4.2 Basic Equations for the Dual-Phase-Lag Heat Conduction Model

According to the dual-phase-lag heat conduction model, the classical heat flux in (2.2) (given in Chapter 2) is modified to

$$\underline{\mathbf{q}}(r, z, t + \tau_q) = -\kappa \underline{\nabla} T(r, z, t + \tau_T), \quad (4.1)$$

where τ_q and τ_T are small magnitude time parameters giving the phase lags of the heat flux components and the temperature gradient respectively. Note that if $\tau_q = 0$ and $\tau_T = 0$ the Fourier classical model in (2.2) is recovered. A somewhat detailed discussion of the dual-phase-lag model can be found in the book by Zhang [102].

If the left and right hand sides of (4.1) are expanded as Taylor-Maclaurin series about $\tau_q = 0$ and $\tau_T = 0$ respectively, then ignoring second and higher order terms in τ_q and τ_T (that is, assuming that the phase lags are sufficiently small) gives

$$\underline{\mathbf{q}}(r, z, t) + \tau_q \frac{\partial}{\partial t} [\underline{\mathbf{q}}(r, z, t)] = -\kappa \underline{\nabla} (T(r, z, t) + \tau_T \frac{\partial}{\partial t} [T(r, z, t)]). \quad (4.2)$$

The use of (4.2) together with the law of conservation of thermal energy as shown in (2.1)

$$-\underline{\nabla} \bullet (\underline{\mathbf{q}}(r, z, t)) + Q = \rho c \frac{\partial}{\partial t} [T(r, z, t)]$$

yields

$$\begin{aligned} & \underline{\nabla} \bullet (\kappa \underline{\nabla} (T(r, z, t) + \tau_T \frac{\partial}{\partial t} [T(r, z, t)])) \\ &= \rho c (\frac{\partial}{\partial t} [T(r, z, t)] + \tau_q \frac{\partial^2}{\partial t^2} [T(r, z, t)]) - (Q + \tau_q \frac{\partial Q}{\partial t}). \end{aligned} \quad (4.3)$$

If the internal heat generation rate Q is of the form

$$Q = A(r, z)T(r, z, t) + B(r, z, t), \quad (4.4)$$

where $A(r, z)$ and $B(r, z, t)$ are given functions, (4.3) can be written as

$$\begin{aligned}
 & \underline{\nabla} \bullet (\kappa \underline{\nabla}(T(r, z, t) + \tau_T \frac{\partial}{\partial t}[T(r, z, t)])) \\
 = & \rho c \tau_q \frac{\partial^2}{\partial t^2}[T(r, z, t)] + (\rho c - \tau_q A(r, z)) \frac{\partial}{\partial t}[T(r, z, t)] \\
 & - (A(r, z)T(r, z, t) + B(r, z, t) + \tau_q \frac{\partial}{\partial t}[B(r, z, t)]). \quad (4.5)
 \end{aligned}$$

4.3 Initial-boundary Value Problem

The solution domain Ω and its boundary Γ are as sketched in Figure 2.1 (Chapter 2).

The temperature $T(r, z, t)$ and the heat flux vector $\underline{\mathbf{q}}(r, z, t)$ are determined by solving (4.5) together with (4.2) for $(r, z) \in \Omega$ and $t > 0$ subject to the initial-boundary conditions

$$\begin{aligned}
 T(r, z, 0) &= f_0(r, z) \text{ and } \frac{\partial}{\partial t}[T(r, z, t)] \Big|_{t=0} = f_1(r, z) \text{ for } (r, z) \in \Omega \cup \Gamma, \\
 \underline{\mathbf{q}}(r, z, 0) &= \underline{\mathbf{q}}_0(r, z) \text{ for } (r, z) \in \Omega \cup \Gamma, \\
 T(r, z, t) &= g_0(r, z, t) \text{ for } (r, z) \in \Gamma_1 \text{ for } t > 0, \\
 q(r, z, t) &= g_1(r, z, t) + g_2(r, z)T(r, z, t) \text{ for } (r, z) \in \Gamma_2 \text{ for } t > 0, \quad (4.6)
 \end{aligned}$$

where Γ_1 and Γ_2 are the non-intersecting curves such that $\Gamma_1 \cup \Gamma_2 = \Gamma$, the function $q(r, z, t) = \underline{\mathbf{n}}(r, z) \bullet \underline{\mathbf{q}}(r, z, t)$ is the component of the flux $\underline{\mathbf{q}}$ in the direction of the vector $\underline{\mathbf{n}}$, $\underline{\mathbf{n}}$ denotes the outward unit normal vector on the boundary Γ and $f_0(r, z)$, $f_1(r, z)$, $\underline{\mathbf{q}}_0(r, z)$, $g_0(r, z, t)$, $g_1(r, z, t)$ and $g_2(r, z)$ are suitably given functions. Note that if $g_2(r, z) = 0$ for $(r, z) \in \Gamma_2$ then the boundary condition on the last line gives the special case in which the heat flux is known on Γ_2 .

4.4 Boundary-domain Integral Formulation

To derive a suitable boundary-domain integral equation for the governing partial differential equation in (4.5), let

$$T(r, z, t) = \frac{1}{\sqrt{\kappa(r, z)}} \psi(r, z, t). \quad (4.7)$$

Substitution of (4.7) into (4.5) yields

$$\begin{aligned} & \nabla_{\text{axis}}^2 (\psi(r, z, t) + \tau_T \frac{\partial}{\partial t} [\psi(r, z, t)]) \\ = & C_2(r, z) \frac{\partial^2}{\partial t^2} [\psi(r, z, t)] + C_1(r, z) \frac{\partial}{\partial t} [\psi(r, z, t)] + C_0(r, z) \psi(r, z, t) \\ & - \frac{1}{\sqrt{\kappa(r, z)}} \{B(r, z, t) + \tau_q \frac{\partial}{\partial t} [B(r, z, t)]\}, \end{aligned} \quad (4.8)$$

where

$$\begin{aligned} C_0(r, z) &= \frac{\sqrt{\kappa(r, z)} \nabla_{\text{axis}}^2 (\sqrt{\kappa(r, z)}) - A(r, z)}{\kappa(r, z)}, \\ C_1(r, z) &= \frac{\rho c - \tau_q A(r, z)}{\kappa(r, z)} + \frac{\tau_T}{\sqrt{\kappa(r, z)}} \nabla_{\text{axis}}^2 (\sqrt{\kappa(r, z)}), \\ C_2(r, z) &= \frac{\rho c \tau_q}{\kappa(r, z)}. \end{aligned} \quad (4.9)$$

It is assumed here that $\nabla_{\text{axis}}^2 (\sqrt{\kappa(r, z)})$ exists in the solution domain Ω .

From (4.8), following the approach given in Chapter 2, one may derive the boundary-domain integral equation

$$\begin{aligned} & \gamma(r_0, z_0) (\psi(r_0, z_0, t) + \tau_T \frac{\partial}{\partial t} [\psi(r_0, z_0, t)]) \\ = & \iint_{\Omega} G_0(r, z; r_0, z_0) (C_2(r, z) \frac{\partial^2}{\partial t^2} [\psi(r, z, t)] + C_1(r, z) \frac{\partial}{\partial t} [\psi(r, z, t)] \end{aligned}$$

$$\begin{aligned}
 & +C_0(r, z)\psi(r, z, t) - \frac{1}{\sqrt{\kappa(r, z)}}\{B(r, z, t) + \tau_q \frac{\partial}{\partial t}[B(r, z, t)]\}rdrdz \\
 & + \int_{\Gamma} (\{\psi(r, z, t) + \tau_T \frac{\partial}{\partial t}[\psi(r, z, t)]\}G_1(r, z; r_0, z_0; n_r, n_z) \\
 & - G_0(r, z; r_0, z_0) \frac{\partial}{\partial n} \{\psi(r, z, t) + \tau_T \frac{\partial}{\partial t}[\psi(r, z, t)]\})rds(r, z) \\
 & \text{for } (r_0, z_0) \in \Omega \cup \Gamma \text{ and } t \geq 0,
 \end{aligned} \tag{4.10}$$

where $\partial[\psi(r, z, t)]/\partial n$ denotes the outward normal derivative of the function $\psi(r, z, t)$ on the boundary Γ , $\gamma(r_0, z_0) = 1$ if (r_0, z_0) lies in the interior of Ω , $\gamma(r_0, z_0) = 1/2$ if (r_0, z_0) lies on a smooth part of Γ , $ds(r, z)$ denotes the length of an infinitesimal part of the curve Γ , $G_0(r, z; r_0, z_0)$ and $G_1(r, z; r_0, z_0; n_r, n_z)$ are fundamental solutions given in (2.17) and (2.18) in Chapter 2.

4.5 Dual-reciprocity Boundary Element Procedures

In the boundary element procedure here, the boundary-domain integral equation in (4.10) is first to be transformed into Laplace transform domain and the domain integral is treated using dual-reciprocity method as described below.

4.5.1 Integral Equation in Laplace Transform Domain

Applying the Laplace transform with respect to time t ($0 \leq t < \infty$) on both sides of (4.10) and using the initial conditions in (4.6), we obtain the boundary-domain integral equation

$$\gamma(r_0, z_0)\phi(r_0, z_0, p) = \iint_{\Omega} G_0(r, z; r_0, z_0)[D(r, z, p)\phi(r, z, p) + F(r, z, p)]rdrdz$$

$$\begin{aligned}
 & + \int_{\Gamma} (\phi(r, z, p) G_1(r, z; r_0, z_0; n_r, n_z) \\
 & - G_0(r, z; r_0, z_0) \frac{\partial}{\partial n} [\phi(r, z, p)]) r ds(r, z) \\
 & \text{for } (r_0, z_0) \in \Omega \cup \Gamma \text{ and } p > 0, \quad (4.11)
 \end{aligned}$$

where p is the Laplace transform parameter (assumed to be real here), $\phi(r, z, p)$, $D(r, z, p)$ and $F(r, z, p)$ are given by

$$\begin{aligned}
 \phi(r, z, p) &= (1 + p\tau_T) \tilde{\psi}(r, z, p) - \tau_T \sqrt{\kappa(r, z)} f_0(r, z), \\
 D(r, z, p) &= \frac{1}{1 + p\tau_T} (p^2 C_2(r, z) + p C_1(r, z) + C_0(r, z)), \\
 F(r, z, p) &= \frac{\tau_T \sqrt{\kappa(r, z)} f_0(r, z)}{1 + p\tau_T} [p^2 C_2(r, z) + p C_1(r, z) + C_0(r, z)] \\
 &\quad - \sqrt{\kappa(r, z)} (C_2(r, z) [p f_0(r, z) + f_1(r, z)] + C_1(r, z) f_0(r, z)) \\
 &\quad - \frac{1}{\sqrt{\kappa(r, z)}} \{ (1 + p\tau_q) \tilde{B}(r, z, p) - \tau_q B(r, z, 0) \}, \quad (4.12)
 \end{aligned}$$

and $\tilde{\psi}(r, z, p)$ and $\tilde{B}(r, z, p)$ are the Laplace transform of $\psi(r, z, t)$ and $B(r, z, t)$ respectively, that is,

$$\begin{aligned}
 \tilde{\psi}(r, z, p) &= \int_0^{\infty} \psi(r, z, t) \exp(-pt) dt, \\
 \tilde{B}(r, z, p) &= \int_0^{\infty} B(r, z, t) \exp(-pt) dt. \quad (4.13)
 \end{aligned}$$

If we apply the Laplace transform on (4.2) as well as the boundary conditions in (4.6), we obtain

$$\begin{aligned}
 \phi(r, z, p) &= P_0(r, z, p) \text{ for } (r, z) \in \Gamma_1, \\
 \frac{\partial}{\partial n} [\phi(r, z, p)] &= P_1(r, z, p) + P_2(r, z, p) \phi(r, z, p) \text{ for } (r, z) \in \Gamma_2, \quad (4.14)
 \end{aligned}$$

where $P_0(r, z, p)$, $P_1(r, z, p)$ and $P_2(r, z, p)$ are given by

$$\begin{aligned}
 P_0(r, z, p) &= (1 + p\tau_T)\sqrt{\kappa(r, z)}\tilde{g}_0(r, z, p) - \tau_T\sqrt{\kappa(r, z)}f_0(r, z), \\
 P_1(r, z, p) &= \frac{1}{\sqrt{\kappa(r, z)}}\left[-\frac{1 + p\tau_q}{1 + p\tau_T}\tau_T g_2(r, z)f_0(r, z) \right. \\
 &\quad \left. - (1 + p\tau_q)\tilde{g}_1(r, z, p) + \tau_q q_0(r, z)\right], \\
 P_2(r, z, p) &= \frac{1}{2\kappa(r, z)}\frac{\partial}{\partial n}[\kappa(r, z)] - \left(\frac{1 + p\tau_q}{1 + p\tau_T}\right)\frac{g_2(r, z)}{\kappa(r, z)}, \tag{4.15}
 \end{aligned}$$

with $\tilde{g}_0(r, z, p)$ being the Laplace transform of $g_0(r, z, t)$ and $q_0(r, z)$ is given by $q_0(r, z) = \underline{n} \cdot \underline{q}_0(r, z)$.

In the Laplace transform domain, the problem is then to solve the integral equation in (4.11) for the unknown function $\phi(r, z, p)$ subject to the boundary conditions in (4.14).

4.5.2 Boundary Approximations

Similar to the procedure in Section 3.5.1, the boundary is discretized into N straight line elements $\Gamma^{(k)}$ ($k = 1, 2, \dots, N$). Two points on the element $\Gamma^{(k)}$ from boundary denoted by $(r_0^{(k)}, z_0^{(k)})$ and $(r_0^{(N+k)}, z_0^{(N+k)})$, are chosen as

$$\begin{aligned}
 (r_0^{(k)}, z_0^{(k)}) &= (r^{(k)}, z^{(k)}) + \tau(r^{(k+1)} - r^{(k)}, z^{(k+1)} - z^{(k)}), \\
 (r_0^{(N+k)}, z_0^{(N+k)}) &= (r^{(k)}, z^{(k)}) + (1 - \tau)(r^{(k+1)} - r^{(k)}, z^{(k+1)} - z^{(k)}) \tag{4.16}
 \end{aligned}$$

where τ is $0 < \tau < 1/2$.

If the function ϕ at $(r_0^{(k)}, z_0^{(k)})$ and $(r_0^{(N+k)}, z_0^{(N+k)})$ is denoted by $\phi^{(k)}(p)$ and $\phi^{(N+k)}(p)$ respectively, then it can be approximated as

$$\begin{aligned}
 \phi(r, z, p) &= \frac{[s^{(k)}(r, z) - (1 - \tau)\ell^{(k)}]\phi^{(k)}(p) - [s^{(k)}(r, z) - \tau\ell^{(k)}]\phi^{(N+k)}(p)}{(2\tau - 1)\ell^{(k)}} \\
 &\quad \text{for } (r, z) \in \Gamma^{(k)}, \tag{4.17}
 \end{aligned}$$

Similarly, if $h(r, z, p) = \partial[\phi(r, z, p)]/\partial n$ is given by $h^{(k)}(p)$ and $h^{(N+k)}(p)$ at $(r_0^{(k)}, z_0^{(k)})$ and $(r_0^{(N+k)}, z_0^{(N+k)})$ respectively, it can be approximately as

$$h(r, z, p) = \frac{[s^{(k)}(r, z) - (1 - \tau)\ell^{(k)}]h^{(k)}(p) - [s^{(k)}(r, z) - \tau\ell^{(k)}]h^{(N+k)}(p)}{(2\tau - 1)\ell^{(k)}} \quad \text{for } (r, z) \in \Gamma^{(k)}. \quad (4.18)$$

where $\ell^{(k)} = s^{(k)}(r^{(k+1)}, z^{(k+1)})$ and $s^{(k)}(r, z)$ is the arc length along the element $\Gamma^{(k)}$ as defined by

$$s^{(k)}(r, z) = \sqrt{(r - r^{(k)})^2 + (z - z^{(k)})^2}. \quad (4.19)$$

With (4.17) and (4.18), the line integrals in the integral equation in (4.11) can be approximated as

$$\begin{aligned} & \int_{\Gamma} \phi(r, z, p) G_1(r, z; r_0, z_0; n_r, n_z) r ds(r, z) \\ & \simeq \sum_{k=1}^N \frac{1}{(2\tau - 1)\ell^{(k)}} \{ [-(1 - \tau)\ell^{(k)} \mathcal{F}_2^{(k)}(r_0, z_0) + \mathcal{F}_4^{(k)}(r_0, z_0)] \phi^{(k)}(p) \\ & \quad + [\tau\ell^{(k)} \mathcal{F}_2^{(k)}(r_0, z_0) - \mathcal{F}_4^{(k)}(r_0, z_0)] \phi^{(N+k)}(p) \}, \end{aligned} \quad (4.20)$$

and

$$\begin{aligned} & \int_{\Gamma} h(r, z, p) G_0(r, z; r_0, z_0) r ds(r, z) \\ & \simeq \sum_{k=1}^N \frac{1}{(2\tau - 1)\ell^{(k)}} \{ [-(1 - \tau)\ell^{(k)} \mathcal{F}_1^{(k)}(r_0, z_0) + \mathcal{F}_3^{(k)}(r_0, z_0)] h^{(k)}(p) \\ & \quad + [\tau\ell^{(k)} \mathcal{F}_1^{(k)}(r_0, z_0) - \mathcal{F}_3^{(k)}(r_0, z_0)] h^{(N+k)}(p) \}, \end{aligned} \quad (4.21)$$

where

$$\mathcal{F}_1^{(k)}(r_0, z_0) = \int_{\Gamma^{(k)}} G_0(r, z; r_0, z_0) r ds(r, z),$$

$$\begin{aligned}
 \mathcal{F}_2^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} G_1(r, z; r_0, z_0; n_r, n_z) r ds(r, z), \\
 \mathcal{F}_3^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} s(r, z) G_0(r, z; r_0, z_0) r ds(r, z), \\
 \mathcal{F}_4^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} s(r, z) G_1(r, z; r_0, z_0; n_r, n_z) r ds(r, z),
 \end{aligned}$$

as defined in (3.20) in Chapter 3.

4.5.3 Dual-reciprocity Approximation of Domain Integral

The dual-reciprocity method can be used to approximate the domain integral over the domain Ω in (4.11). As described in Section 3.5.2, the dual-reciprocity method requires L well-spaced out collocation points to be chosen in the interior of the domain Ω . These points are denoted by $(r_0^{(2N+1)}, z_0^{(2N+1)})$, $(r_0^{(2N+2)}, z_0^{(2N+2)})$, $\dots$, $(r_0^{(2N+L-1)}, z_0^{(2N+L-1)})$ and $(r_0^{(2N+L)}, z_0^{(2N+L)})$. Besides the L interior points, $(r_0^{(k)}, z_0^{(k)})$ and $(r_0^{(N+k)}, z_0^{(N+k)})$ on the element $\Gamma^{(k)}$ ($k = 1, 2, \dots, N$) are also used as collocation points. Thus, there are $2N + L$ collocation points.

The domain integral is approximated as

$$\begin{aligned}
 &\iint_{\Omega} G_0(r, z; r_0, z_0) [D(r, z, p) \phi(r, z, p) + F(r, z, p)] r dr dz \\
 &\simeq \sum_{k=1}^{2N+L} [D(r_0^{(k)}, z_0^{(k)}, p) \phi^{(k)}(p) + F(r_0^{(k)}, z_0^{(k)}, p)] \sum_{j=1}^{2N+L} W^{(kj)} \Psi^{(j)}(r, z),
 \end{aligned} \tag{4.22}$$

where $\phi^{(k)}(p) = \phi(r_0^{(k)}, z_0^{(k)}, p)$ for $k = 1, 2, \dots, 2N + L$, the coefficients $W^{(kj)}$

are defined implicitly by

$$\sum_{j=1}^{2N+L} W^{(kj)} \varphi^{(n)}(r_0^{(j)}, z_0^{(j)}) = \begin{cases} 0 & \text{if } n \neq k \\ 1 & \text{if } n = k \end{cases} \quad \text{for } n, k = 1, 2, \dots, 2N+L, \quad (4.23)$$

the function $\Psi^{(j)}(r, z)$ is expressed in terms of line integrals over Γ as

$$\begin{aligned} \Psi^{(j)}(r_0, z_0) &= \gamma(r_0, z_0) \chi^{(j)}(r_0, z_0) + \int_{\Gamma} r G_0(r, z; r_0, z_0) \frac{\partial}{\partial n} [\chi^{(j)}(r, z)] ds(r, z) \\ &\quad - \int_{\Gamma} r \chi^{(j)}(r, z) G_1(r, z; r_0, z_0; n_r, n_z) ds(r, z) \\ &\quad \text{for } j = 1, 2, \dots, 2N+L, \end{aligned} \quad (4.24)$$

the functions $\varphi^{(n)}$ and $\chi^{(n)}$ are local interpolating functions centered about the collocation point $(r_0^{(n)}, z_0^{(n)})$ and required to satisfy the partial differential equation

$$\frac{\partial^2}{\partial r^2} [\chi^{(n)}(r, z)] + \frac{1}{r} \frac{\partial}{\partial r} [\chi^{(n)}(r, z)] + \frac{\partial^2}{\partial z^2} [\chi^{(n)}(r, z)] = \varphi^{(n)}(r, z). \quad (4.25)$$

We take the interpolating functions $\chi^{(p)}(r, z)$ and $\varphi^{(n)}(r, z)$ to be those in Chapter 3 as given by

$$\chi^{(p)}(r, z) = \frac{1}{9} \{ [\sigma(r, 0, z; r_0^{(p)}, z_0^{(p)})]^3 + [\sigma(r, 0, z; -r_0^{(p)}, z_0^{(p)})]^3 \}. \quad (4.26)$$

$$\begin{aligned} \varphi^{(n)}(r, z) &= \left[\frac{4}{3} - \frac{r_0^{(n)}}{3r} \right] \sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) \\ &\quad + \left[\frac{4}{3} + \frac{r_0^{(n)}}{3r} \right] \sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)}). \end{aligned} \quad (4.27)$$

4.5.4 System of Linear Algebraic Equations

From (4.20), (4.21) and (4.22), the integral differential equation in (4.11) can be approximated as

$$\begin{aligned}
 & \gamma(r_0^{(m)}, z_0^{(m)})\phi^{(m)}(p). \\
 = & \sum_{k=1}^{2N+L} [D(r_0^{(k)}, z_0^{(k)}, p)\phi^{(k)}(p) + F(r_0^{(k)}, z_0^{(k)}, p)]\mu^{(km)} \\
 & + \sum_{k=1}^N \frac{1}{(2\tau-1)\ell^{(k)}} \{[-(1-\tau)\ell^{(k)}\mathcal{F}_2^{(k)}(r^{(m)}, z^{(m)}) \\
 & \quad + \mathcal{F}_4^{(k)}(r^{(m)}, z^{(m)})]\phi^{(k)}(p) \\
 & \quad + [\tau\ell^{(k)}\mathcal{F}_2^{(k)}(r^{(m)}, z^{(m)}) - \mathcal{F}_4^{(k)}(r^{(m)}, z^{(m)})]\phi^{(N+k)}(p) \\
 & \quad - [-(1-\tau)\ell^{(k)}\mathcal{F}_1^{(k)}(r^{(m)}, z^{(m)}) + \mathcal{F}_3^{(k)}(r^{(m)}, z^{(m)})]h^{(k)}(p) \\
 & \quad - [\tau\ell^{(k)}\mathcal{F}_1^{(k)}(r^{(m)}, z^{(m)}) - \mathcal{F}_3^{(k)}(r^{(m)}, z^{(m)})]h^{(N+k)}(p)\} \\
 & \text{for } m = 1, 2, \dots, 2N+L,
 \end{aligned} \tag{4.28}$$

where $\mu^{(km)}$ is given by

$$\mu^{(km)} = \sum_{j=1}^{2N+L} W^{(kj)}\Psi^{(j)}(r^{(m)}, z^{(m)}). \tag{4.29}$$

The boundary conditions in (4.14) can be applied in (4.28) to obtain

$$\begin{aligned}
 & \gamma(r_0^{(m)}, z_0^{(m)})\{\alpha^{(m)}\phi^{(m)}(p) + \beta^{(m)}R^{(m)}(p)\} \\
 = & \sum_{k=1}^{2N+L} [D(r_0^{(k)}, z_0^{(k)}, p)(\alpha^{(k)}\phi^{(k)}(p) + \beta^{(k)}R^{(k)}(p)) + F(r_0^{(k)}, z_0^{(k)}, p)]\mu^{(km)} \\
 & + \sum_{k=1}^N \frac{1}{(2\tau-1)\ell^{(k)}} \{[-(1-\tau)\ell^{(k)}\mathcal{F}_2^{(k)}(r_0^{(m)}, z_0^{(m)}) + \mathcal{F}_4^{(k)}(r_0^{(m)}, z_0^{(m)}) \\
 & \quad - \alpha^{(k)}P_2^{(k)}(p)(-(1-\tau)\ell^{(k)}\mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)}) + \mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)}))]\}
 \end{aligned}$$

$$\begin{aligned}
 & \times (\alpha^{(k)} \phi^{(k)}(p) + \beta^{(k)} R^{(k)}(p)) \\
 & + [\tau \ell^{(k)} \mathcal{F}_2^{(k)}(r_0^{(m)}, z_0^{(m)}) - \mathcal{F}_4^{(k)}(r_0^{(m)}, z_0^{(m)}) \\
 & - \alpha^{(N+k)} P_2^{(N+k)}(p) (\tau \ell^{(k)} \mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)}) - \mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)}))] \\
 & \times (\alpha^{(N+k)} \phi^{(N+k)}(p) + \beta^{(N+k)} R^{(N+k)}(p)) \\
 & - [-(1 - \tau) \ell^{(k)} \mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)}) + \mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)})] \\
 & \times [\alpha^{(k)} P_1^{(k)}(p) + \beta^{(k)} h^{(k)}(p)] \\
 & - [\tau \ell^{(k)} \mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)}) - \mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)})] \\
 & \times [\alpha^{(N+k)} P_1^{(N+k)}(p) + \beta^{(N+k)} h^{(N+k)}(p)] \} \\
 & \text{for } m = 1, 2, \dots, 2N + L,
 \end{aligned} \tag{4.30}$$

where $R^{(m)}(p)$, $\alpha^{(m)}$, $\beta^{(m)}$, $P_1^{(n)}$, and $P_2^{(n)}$ (for $m = 1, 2, \dots, 2N + L$ and $n = 1, 2, \dots, N, N + 1, \dots, 2N - 1, 2N$) are defined by

$$\begin{aligned}
 \alpha^{(m)} &= \begin{cases} 0 & \text{if } (r_0^{(m)}, z_0^{(m)}) \text{ lies on a boundary element} \\ & \text{where } \phi \text{ is known,} \\ 1 & \text{otherwise,} \end{cases} \\
 \beta^{(m)} &= 1 - \alpha^{(m)}, \quad R^{(m)}(p) = P_0(r_0^{(m)}, z_0^{(m)}, p), \\
 P_1^{(n)} &= P_1(r_0^{(n)}, z_0^{(n)}, p), \quad P_2^{(n)} = P_2(r_0^{(n)}, z_0^{(n)}, p).
 \end{aligned} \tag{4.31}$$

Note that the functions $P_1(r_0^{(n)}, z_0^{(n)}, p)$ and $P_2(r_0^{(n)}, z_0^{(n)}, p)$ are defined only for the collocation point $(r_0^{(n)}, z_0^{(n)})$ which lies on a boundary element where ϕ is not known. Thus, in (4.30), the coefficient $\alpha^{(n)}$ is always multiplied to $P_1^{(n)}$ and $P_2^{(n)}$. Likewise, $\beta^{(m)}$ is always multiplied to $R^{(m)}(p)$ since $P_0(r_0^{(m)}, z_0^{(m)}, p)$ is defined only for $(r_0^{(m)}, z_0^{(m)})$ which lies on a boundary element where ϕ is known.

The system (4.30) comprises $2N + L$ linear algebraic equations in $2N + L$ unknowns. The unknowns are $\phi^{(2N+n)}(p)$ for $n = 1, 2, \dots, L$, $\phi^{(k)}(p)$ if ϕ is not known on the boundary element $\Gamma^{(k)}$, and $h^{(k)}(p)$ if ϕ is known on the boundary element $\Gamma^{(k)}$.

4.5.5 Inversion of Laplace Transform Solution

Once $\phi(r, z, p)$ is determined, the Laplace transform of $\psi(r, z, t)$, that is, $\tilde{\psi}(r, z, p)$, can be evaluated from (4.12). The temperature $T(r, z, t)$ can then be recovered by inverting $\tilde{\psi}(r, z, p)/\sqrt{\kappa(r, z)}$ by using Stehfest's algorithm (2.36) and (2.37) as described in Section 2.4.

Furthermore, once $h(r, z, p)$ is known at all points on the boundary Γ , the Laplace transform of the normal flux across the boundary, that is, $\tilde{q}(r, z, p)$, can be determined from

$$h(r, z, p) = \frac{1}{\sqrt{\kappa(r, z)}} [-(1 + p\tau_q)\tilde{q}(r, z, p) + \tau_q q_0(r, z)] + \frac{1}{2\kappa(r, z)} \frac{\partial}{\partial n} [\kappa(r, z)], \quad (4.32)$$

and the Stehfest's formula can be used to invert $\tilde{q}(r, z, p)$ to obtain the normal flux $q(r, z, t)$.

4.6 Specific Problems

In this section, the Laplace transform dual-reciprocity boundary element method presented above is used to solve some specific problems. The first three problems are initial-boundary value problems which have known exact solutions. They are specially designed to test the validity of the method as well as to assess its accuracy. In the last problem, the method is applied to simulate the axisymmetric dual-phase-lag heat conduction in a particular exponentially graded cylindrical solid. The effects of the phase lags and the spatial variations of the thermal properties on the temperature distribution are examined.

Problem 1

For a test problem, take the solution domain as $0 < r < 1$, $0 < z < 1$ (a solid cylinder). The coefficient κ , ρ , c , τ_T and τ_q are given by $\kappa = 1 + z$, $\rho c = 1$,

$\tau_T = 1/10$ and $\tau_q = 1/5$. The internal heat generation term Q is given by

$$Q(r, z, t) = -T(r, z, t) - \frac{9}{4}(3 + 4z) \exp(-t)$$

The initial-boundary conditions are given by

$$\begin{aligned} T(r, z, 0) &= r^2 + z^2 \text{ for } 0 < r < 1, 0 < z < 1, \\ \frac{\partial}{\partial t}[T(r, z, t)] \Big|_{t=0} &= -(r^2 + z^2) \text{ for } 0 < r < 1, 0 < z < 1, \\ T(1, z, t) &= (1 + z^2) \exp(-t) \text{ for } 0 < z < 1, t > 0, \\ \underline{\mathbf{q}}(r, z, 0) &= -\frac{9}{4}r(1 + z)\underline{\mathbf{e}}_r - \frac{9}{4}(z^2 + z)\underline{\mathbf{e}}_z \text{ for } 0 < r < 1, 0 < z < 1, \\ q(r, 0, t) &= 0 \text{ for } 0 < r < 1, t > 0, \\ q(r, 1, t) &= -\frac{9}{2} \exp(-t) \text{ for } 0 < r < 1, t > 0, \end{aligned}$$

where $\underline{\mathbf{e}}_r$ and $\underline{\mathbf{e}}_z$ are unit magnitude vectors pointing in the positive direction of the r and z axes respectively.

From (4.14), in the Laplace transform domain, the problem is to determine $\phi(r, z, p)$ which satisfies the boundary conditions

$$\begin{aligned} \phi(1, z, p) &= \sqrt{1 + z}(1 + z^2) \left[\frac{1 + \frac{1}{10}p}{1 + p} - \frac{1}{10} \right] \text{ for } 0 < z < 1, p > 0, \\ \frac{\partial}{\partial n}[\phi(r, z, p)] \Big|_{z=0} &= -\frac{1}{10}r^2 \frac{1 + \frac{1}{5}p}{1 + \frac{1}{10}p} + r^2 \frac{1 + \frac{1}{5}p}{1 + p} \\ &\quad + \left(-\frac{1}{2} - \frac{1 + \frac{1}{5}p}{1 + \frac{1}{10}p} \right) \phi(r, 0, p) \\ &\text{for } 0 < r < 1, p > 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial n}[\phi(r, z, p)] \Big|_{z=1} &= \frac{1}{\sqrt{2}} \left[-\frac{1}{10}(r^2 + 1) \frac{1 + \frac{1}{5}p}{1 + \frac{1}{10}p} \right. \\ &\quad \left. + \frac{1 + \frac{1}{5}p}{1 + p} \left(r^2 + \frac{11}{2} \right) - \frac{9}{10} \right] + \left(\frac{1}{4} - \frac{1 + \frac{1}{5}p}{2(1 + \frac{1}{10}p)} \right) \phi(r, 1, p) \\ &\text{for } 0 < r < 1, p > 0. \end{aligned}$$

Table 4.1: Numerical and exact values of $T(r, z, 0.10)$.

(r, z)	Set A	Set B	Exact
(0.25,0.25)	0.11832	0.11312	0.11310
(0.50,0.25)	0.28338	0.28278	0.28276
(0.75,0.25)	0.56613	0.56554	0.56552
(0.25,0.50)	0.28312	0.28277	0.28276
(0.50,0.50)	0.45269	0.45243	0.45242
(0.75,0.50)	0.73547	0.73519	0.73518
(0.25,0.75)	0.56572	0.56553	0.56552
(0.50,0.75)	0.73535	0.73519	0.73518
(0.75,0.75)	1.01813	1.01795	1.01794

For the specific problem here, the boundary Γ comprises three equal length straight lines. Each of the straight lines is divided into N_0 equal length elements, so that the total number of elements is $3N_0$, that is, $N = 3N_0$. The interior collocation points are taken to be given by $(i/(L_0 + 1), j/(L_0 + 1))$ for $i, j = 1, 2, \dots, L_0$, that is, the total number of interior collocation points is given by L_0^2 . In the approximate formula (2.36) for inverting Laplace transformation, we take $M = 4$.

In Table 4.1, numerical values of the temperature at $t = 0.10$, obtained using $(N_0, L_0) = (10, 3)$ (Set A) and $(N_0, L_0) = (40, 15)$ (Set B), are compared with the exact solution of the test problem given by

$$T(r, z, t) = (r^2 + z^2) \exp(-t).$$

The numerical values in Set B are significantly more accurate than those in Set A. Thus, the numerical solution converges to the exact one when the calculation is refined by using more elements and collocation points.

The numerical temperature at $(r, z) = (0.50, 0.50)$, obtained using $N_0 = 10$

and $L_0 = 9$, is compared graphically with the exact temperature over the time period $0 < t < 2$ in Figure 4.1. The graphs for the numerical and the exact temperature are almost visually indistinguishable.

Figure 4.1: A graphical comparison of the numerical and the exact temperature at $(r, z) = (0.50, 0.50)$ over the time period $0 < t < 2$.

For the initial-boundary conditions here, the temperature is specified at $r = 1$ for $t > 0$. Thus, it may be of interest to find out if the normal flux across $r = 1$ can be accurately recovered by the dual-reciprocity boundary element procedure. In Figure 4.2, we plot the numerical and the exact flux $\underline{\mathbf{q}}(r, z, t) \bullet \underline{\mathbf{e}}_r$ at time $t = 0.10$ against $0 < z < 1$ on the boundary $r = 1$. There is a good agreement between the numerical and the exact flux. The numerical values of the normal flux in Figure 4.2 are computed using $N_0 = 10$ and $L_0 = 9$. The percentage errors in the numerical values are less than 0.20%.

The L_2 norm error is calculated for the numerical T obtained using boundary element method. The norm for Set A and Set B at time $t = 0.1$ are

Figure 4.2: A graphical comparison of the numerical and the exact normal flux at time $t = 0.10$ against $0 < z < 1$ on the boundary $r = 1$.

Figure 4.3: L_2 norm error for Set A and Set B against t .

respectively given by 0.00042264 and 0.00002015. Set B is much smaller and therefore more accurate than the results in Set A. In Figure 4.3, the norm is plotted against time t . Note that the results in Figure 4.3 are obtained using $M = 4$ in the Stehfest's algorithm.

Figure 4.4: L_2 norm error for Set A against t at $M = 4$, $M = 6$ and $M = 8$.

In Laplace transform method, the choice of inversion terms M used for Laplace inversion is important. In Figure 4.3, the norms are relatively small number when $t < 1.5$ for both results from Set A and Set B. This shows that the results obtained by using $M = 4$ give accurate results up to $t = 1.5$. In order to obtain results with good accuracy at higher time level, a greater value of M is needed. As shown in Figure 4.4, the L_2 norm error at different value of M are compared and plotted against t . From the figure, high accuracy (small value of norm error) can be obtained up to $t = 2$ and $t = 4$ respectively for $M = 4$ and $M = 6$. For $M = 8$, the results are in good accuracy up to $t = 7$.

Thus, the number of terms to be used in the Stehfest's algorithm depends on the range of time in the calculation of the solution. In general, more terms are needed if the solution has to be calculated at a higher time level.

Problem 2

Consider a hollow cylinder occupying the region $1 < r < 2$, $1 < z < 2$. The values of ρc , τ_T and τ_q are respectively taken to be 1, 1/10 and 1/5. The thermal conductivity of the hollow cylinder is set as $\kappa = r + z$ and the internal heat generation is given by

$$Q(r, z, t) = -\frac{51}{52}(5r + 6z) \sin(t) + \left(\frac{5}{52}(5r + 6z) + \left(\frac{1}{2}r^2 + z^2\right)\right) \cos(t).$$

The initial conditions are given by

$$\begin{aligned} T(r, z, 0) &= 0 \text{ for } 1 < r < 2, 1 < z < 2, \\ \frac{\partial}{\partial t}[T(r, z, t)] \Big|_{t=0} &= \frac{1}{2}r^2 + z^2 \text{ for } 1 < r < 2, 1 < z < 2, \\ \underline{q}(r, z, 0) &= \frac{5}{52}r(r + z)\underline{e}_r + \frac{5}{26}z(r + z)\underline{e}_z \text{ for } 1 < r < 2, 1 < z < 2, \end{aligned}$$

and the boundary conditions by

$$\begin{aligned} T(1, z, t) &= \left(\frac{1}{2} + z^2\right) \sin(t) \text{ for } 1 < z < 2, t > 0, \\ T(2, z, t) &= (2 + z^2) \sin(t) \text{ for } 1 < z < 2, t > 0, \\ q(r, 1, t) &= \frac{51}{26}(r + 1) \sin(t) - \frac{5}{26}(r + 1) \cos(t) \text{ for } 1 < r < 2, t > 0, \\ q(r, 2, t) &= -\frac{51}{13}(r + 2) \sin(t) + \frac{5}{13}(r + 2) \cos(t) \text{ for } 1 < r < 2, t > 0. \end{aligned}$$

In the Laplace transform domain, the initial-boundary conditions give

$$\begin{aligned} \phi(1, z, p) &= \sqrt{1 + z} \frac{1 + \frac{1}{10}p}{1 + p^2} \left(\frac{1}{2} + z^2\right) \text{ for } 1 < z < 2, p > 0, \\ \phi(2, z, p) &= \sqrt{2 + z} \frac{1 + \frac{1}{10}p}{1 + p^2} (2 + z^2) \text{ for } 1 < z < 2, p > 0, \end{aligned}$$

$$\begin{aligned}
 \left. \frac{\partial}{\partial n} [\phi(r, z, p)] \right|_{z=1} &= \frac{1}{\sqrt{r+1}} \left[-\frac{1+\frac{1}{5}p}{1+p^2} \left(\frac{51}{26}(r+1) \right. \right. \\
 &\quad \left. \left. - \frac{5}{26}(r+1)p - \left(\frac{1}{2}r^2 + 1 \right) \right) - \frac{1}{26}(r+1) \right] \\
 &\quad + \frac{1}{r+1} \left(-\frac{1}{2} - \frac{1+\frac{1}{5}p}{1+\frac{1}{10}p} \right) \phi(r, 1, p) \\
 &\text{for } 1 < r < 2, p > 0,
 \end{aligned}$$

and

$$\begin{aligned}
 \left. \frac{\partial}{\partial n} [\phi(r, z, p)] \right|_{z=2} &= \frac{1}{\sqrt{r+2}} \left[-\frac{1+\frac{1}{5}p}{1+p^2} \left(-\frac{51}{13}(r+2) + \frac{5}{13}(r+2)p - \left(\frac{1}{2}r^2 + 4 \right) \right) \right. \\
 &\quad \left. + \frac{1}{13}(r+2) \right] + \frac{1}{r+2} \left(\frac{1}{2} - \frac{1+\frac{1}{5}p}{1+\frac{1}{10}p} \right) \phi(r, 2, p) \\
 &\text{for } 1 < r < 2, p > 0.
 \end{aligned}$$

As this test problem involves the sinusoidal functions $\sin(t)$ and $\cos(t)$ in the heat generation term and the initial boundary conditions, it may be of interest to see if the numerical solution can be recovered properly within one period of the sinusoidal functions, that is, within the time interval $0 \leq t \leq 2\pi$. To obtain some numerical results, the boundary Γ is discretized into 160 elements, 225 well spaced interior points are selected to be interior collocation points, the parameter τ in the discontinuous elements is taken to be 0.25, and $M = 11$ is used in the numerical Laplace transform inversion formula. At the points (1.25, 1.25), (1.50, 1.50) and (1.75, 1.75), the numerical and the exact temperature are plotted against time t for $0 \leq t \leq 2\pi$ in Figure 4.5. The exact solution given by $T(r, z, t) = (\frac{1}{2}r^2 + z^2) \sin(t)$ is plotted using thick solid lines for all the three points. It appears that temperature can be properly recovered over the time period $0 \leq t \leq 2\pi$ by the boundary element procedure

in Section 4.5. On the whole, the numerical values of T seem to be more accurate at lower time t . There is a slight deterioration in the accuracy of the numerical temperature near $t = 2\pi$. More terms may be required in the numerical Laplace transform formula to achieve the same level of accuracy for larger t .

Figure 4.5: Plots of the numerical and the exact temperature over $0 \leq t \leq 2\pi$ at the points $(1.25, 1.25)$, $(1.50, 1.50)$ and $(1.75, 1.75)$. The exact solution at the points is plotted using thick solid lines.

Problem 3

Take the solution domain to be the hemispherical in shape given by $r^2 + z^2 < 1$, $0 < z < 1$. The values of phase lags τ_T and τ_q are taken to be $1/10$ and $1/5$ respectively. The thermal conductivity κ and heat capacity c vary spatially exponentially as $\kappa = \kappa_0 \exp(\sigma z)$ and $c = c_0 \exp(\sigma z)$ with $\kappa_0 = c_0 = \sigma = 1$. The density is taken to be $\rho = \rho_0 = 1$ and the internal heat generation by

$$Q(r, z, t) = \frac{5}{4} \left(\left(2 + \frac{14}{5} r^2 \right) \sin(2z) - \frac{9}{5} (2 + r^2) \cos(2z) \right) \exp(z - t).$$

The initial-boundary conditions are given by

$$\begin{aligned} T(r, z, 0) &= (2 + r^2) \sin(2z) \text{ for } 0 < r < 1, 0 < z < 1, \\ \frac{\partial}{\partial t} [T(r, z, t)] \Big|_{t=0} &= -(2 + r^2) \sin(2z) \text{ for } 0 < r < 1, 0 < z < 1, \\ \underline{\mathbf{q}}(r, z, 0) &= -\frac{9}{4} \exp(z) (r \sin(2z) \underline{\mathbf{e}}_r + (2 + r^2) \cos(2z) \underline{\mathbf{e}}_z) \\ &\text{for } 0 < r < 1, 0 < z < 1, \\ T(r, 0, t) &= 0 \text{ for } 0 < r < 1, t > 0, \\ T(r, z, t) &= (3 - z^2) \sin(2z) \exp(-t) \\ &\text{for } r^2 + z^2 = 1, 0 < r < 1, 0 < z < 1, t > 0. \end{aligned}$$

For the numerical solution, the boundary is discretized into 60 elements and 37 well spaced collocation points are selected in the interior of the solution domain. For the Stehfest's algorithm in (2.36), M is chosen to be 8. The normal flux of the temperature is not specified in the boundary conditions. Thus, the numerical normal flux on the circular part of the boundary ($r^2 + z^2 = 1$) is compared with exact normal flux given by

$$\begin{aligned} q(r, z, t) &= -\frac{9}{4} \exp(z - t) (r^2 \sin(2z) + (2 + r^2) z \cos(2z)) \\ &\text{for } r^2 + z^2 = 1, 0 < r < 1, 0 < z < 1. \end{aligned}$$

Note that the exact solution of the problem under consideration here is $T(r, z, t) = (2 + r^2) \sin(2z) \exp(-t)$.

Figure 4.6 gives a graphical comparison of the numerical and exact normal flux on $r^2 + z^2 = 1$, $0 < r < 1$, $0 < z < 1$ at time $t = 1.50$, 2.50 and 3.0 . The normal flux is plotted against γ which is defined by $\gamma = (180/\pi) \arctan(r/z)$. On the whole, there is a reasonably good agreement between the numerical and exact normal flux. Perhaps, as may be expected, the numerical values of the flux have been observed to be slightly less accurate at points very near the sharp corners $(r, z) = (0, 1)$ and $(r, z) = (1, 0)$ of the solution domain (where γ is given by 0 and 90 respectively).

Figure 4.6: A graphical comparison of the numerical and the exact normal flux across $r^2 + z^2 = 1$, $0 < r < 1$, $0 < z < 1$ at time $t = 1.5$, 2.5 and 3.0 , against γ .

Problem 4

Consider now the heating of a nonhomogeneous material which occupies the cylindrical region $0 < r < a$, $0 < z < a$, where a is a positive constant. The thermal conductivity κ , density ρ and specific heat c of the materials are assumed to vary exponentially in the z direction according to $\kappa = \kappa_0 \exp(\sigma z)$ and $\rho c = \rho_0 c_0 \exp(\sigma z)$, where σ is a constant. The center of the surface $z = 0$ of the cylinder is subject to a uniform heat flux q_0 over a small region of radius of $r_c < a$.

Mathematically, the problem is to solve (4.3) with $A = 0$ and $B = 0$ (that is, with no internal heat generation) inside the cylindrical region $0 < r < a$, $0 < z < a$, subject to the initial conditions

$$\begin{aligned} T(r, z, 0) &= T_0 \text{ for } 0 < r < a, \ 0 < z < a, \\ \left. \frac{\partial}{\partial t} [T(r, z, t)] \right|_{t=0} &= 0 \text{ for } 0 < r < a, \ 0 < z < a, \\ \underline{\mathbf{q}}(r, z, 0) &= 0 \text{ for } 0 < r < a, \ 0 < z < a, \end{aligned}$$

and the boundary conditions

$$\begin{aligned} q(r, a, t) &= 0 \text{ for } 0 < r < 1, \ t > 0, \\ q(a, z, t) &= 0 \text{ for } 0 < z < a, \ t > 0, \\ q(r, 0, t) &= 0 \text{ for } r_c < r < 1, \ t > 0, \\ q(r, 0, t) &= -(H(t) - H(t - \tau_0))q_0 \text{ for } 0 < r < r_c, \ t > 0, \end{aligned}$$

where q_0 is the magnitude of the specified heat flux, τ_0 is the duration of the heating on the surface $z = 0$ and H denotes the unit-step Heaviside function.

To solve this particular problem for the non-dimensionalized temperature $\kappa_0(T - T_0)/(aq_0)$, it is sufficient to input the five non-dimensionalized parameters which are listed below:

- τ_T/τ_q (non-dimensionalized phase in temperature gradient)
- τ_0/τ_q (non-dimensionalized duration of surface heating)
- $a\sigma$ (non-dimensionalized grading parameter of thermal properties)
- r_c/a (non-dimensionalized radius of the region of heating)
- $\kappa_0\tau_q/(a^2\rho_0c_0)$ (non-dimensionalized thermal diffusivity)

If the radius a of the solid cylinder is several centimeter long (for example, $a = 0.05\text{m}$) and κ_0 , ρ_0 and c_0 are respectively the values of the thermal conductivity, density and specific heat of a typical biological tissue with (for example) phase lag $\tau_q = 16\text{s}$ then $\kappa_0\tau_q/(a^2\rho_0c_0)$ may have a magnitude of the order 10^{-3} (see Zhou, Zhang and Chen [104]). Taking $\kappa_0\tau_q/(a^2\rho_0c_0) = 10^{-3}$ and $r_c/a = 0.2$, we examine the effects of varying the non-dimensionalized parameters τ_T/τ_q and $a\sigma$ on the non-dimensionalized temperature $\kappa_0(T - T_0)/(aq_0)$ for cases in which the material is subject to a short duration of surface heating (with $\tau_0/\tau_q = 0.1$) and continuous heating (with $\tau_0/\tau_q \rightarrow \infty$) respectively.

To carry out the numerical simulation with good accuracy, the boundary of the solution domain is discretized into elements and collocation points are placed in the interior of the solution domain. To model the rapid change in temperature distribution in the region near where the surface $z = 0$ is subject to heating, most of the interior collocation points are concentrated within a small region near $(r, z) = (0, 0)$. More specifically, collocation points are placed in the region $0 < r < 0.2a$, $0 < z < 0.2a$, and the other points are distributed evenly in the remaining part of the solution domain. For the numerical inversion of the Laplace transform solution, $M = 8$ is used in the Stehfest's formula in (2.36).

Figure 4.7: Plots of the non-dimensionalized temperature at the center of heating against t/τ_q for selected values of τ_T/τ_q (short duration heating).

For $a\sigma = 0$ (homogeneous solid), $\tau_0/\tau_q = 0.1$ (short duration heating on the surface $0 < r < r_c$, $z = 0$) and some selected values of τ_T/τ_q , the non-dimensionalized temperature $\kappa_0(T - T_0)/(aq_0)$ at $(r/a, z/a) = (0, 0)$ is plotted against t/τ_q in Figure 4.7. For each τ_T/τ_q , the non-dimensionalized temperature $\kappa_0(T(0, 0, t) - T_0)/(aq_0)$ reaches its just before the end of the heating on the surface $0 < r < r_c$, $z = 0$ (that is, at around $t = 0.095\tau_q < \tau_0$). From Figure 4.7, it is apparent that $\kappa_0(T(0, 0, t) - T_0)/(aq_0)$ has a higher maximum value, is more oscillatory in time and is slower in approaching the steady state ($T = T_0$) for τ_T/τ_q with a smaller magnitude. Thus, it appears

that a larger value of τ_T/τ_q has a greater damping effect on the thermal wave response in the solid, as may perhaps be expected.

Figure 4.8: Plots of the non-dimensionalized temperature at the center of heating against t/τ_q for $\tau_T/\tau_q = 0.01$ and selected values of $a\sigma$ (continuous heating).

To examine the effect of the non-dimensionalized parameter $a\sigma$ on the heating of the cylindrical solid, we now consider the case of continuous heating on the surface $0 < r < r_c$, $z = 0$ ($\tau_0/\tau_q \rightarrow \infty$) with $\tau_T/\tau_q = 0.01$. The non-dimensionalized temperature $\kappa_0(T - T_0)/(aq_0)$ at $(r/a, z/a) = (0, 0)$ is plotted against t/τ_q in Figure 4.8 for selected values of $a\sigma$. For time $t/\tau_q < 0.01$, the graphs of $\kappa_0(T(0, 0, t) - T_0)/(aq_0)$ appear to be almost indistinguishable for

Figure 4.9: Plots of non-dimensionalized temperature against z/a at for $\tau_T/\tau_q = 0.01$ and selected values of $a\sigma$ at various time instants (continuous heating).

$a\sigma = -3$, $a\sigma = 0$ and $a\sigma = 3$. The difference in the temperature $\kappa_0(T(0, 0, t) - T_0)/(aq_0)$ for the three selected values of $a\sigma$ only becomes more obvious as time evolves. To show this for the temperature at other points inside the solid, the spatial variation of $\kappa_0(T - T_0)/(aq_0)$ on $r/a = 0$ with z/a (for $0 < z/a < 0.1$) is examined in Figure 4.9 at time $t/\tau_q = 0.1$, $t/\tau_q = 0.5$, $t/\tau_q = 1$ $t/\tau_q = 2$. At small time $t/\tau_q = 0.5$, the difference in the non-dimensionalized temperature $\kappa_0(T(0, z, t) - T_0)/(aq_0)$ for each of the three cases $a\sigma = -3$, $a\sigma = 0$ and $a\sigma = 3$ is obvious only for a very small range z/a near the center of the surface heating.

The difference becomes more pronounced over an increasing larger range of z/a as time increases. From Figure 4.8 and 4.9, it is apparent that as time evolves the temperature for $a\sigma = 0$ (homogeneous solid) will become lower than that for $a\sigma = -3$ but higher than that for $a\sigma = 3$ over an increasingly larger region of the solid.

4.7 Summary

The numerical simulation of axisymmetric dual-phase-lag heat conduction in nonhomogeneous thermally isotropic solids is considered. The temperature field is obtained numerically by using a dual-reciprocity boundary element method to solve the governing partial differential equation in the Laplace transform domain. The task of determining the temperature field is eventually reduced to solving a system of linear algebraic equations. Once the linear algebraic equations are solved, the physical temperature can be recovered by using a numerical algorithm for inverting Laplace transform.

The numerical approach presented is applicable for nonhomogeneous solids with thermal conductivity, density and specific heat which are given by rather general functions of the axisymmetrical spatial coordinates. Through the use of appropriate interpolating functions, the domain integral which arises in the integral formulation of the problem is approximated in terms of integral over the boundary of the solution domain. Thus, in the numerical implementation, only the boundary of the solution domain has to be discretized into elements.

To assess the validity and accuracy of the proposed numerical procedure, some test problems which have known solutions are solved. The numerical solutions obtained indicate that the dual-reciprocity boundary element method presented can be used as a reliable and accurate computational tool for analyz-

ing axisymmetric dual-phase-lag heat conduction in nonhomogeneous solids. The method is also applied to simulate the heating of a particular exponentially graded cylindrical solid. The effects of the spatial variations of the thermal properties and the phase lags in the temperature gradient and flux fields on the thermal behaviors of the solid are examined. The numerical results obtained appear to be intuitively and qualitatively acceptable.

Chapter 5

Axisymmetric Thermoelastostatic Deformations in Nonhomogeneous Solids

5.1 Introduction

Boundary element methods for studying elastic deformations in solids have been well established since the seminal work of Rizzo [79] in 1967. In [79], the boundary integral equations for two-dimensional elastostatics were discretized into linear algebraic equations and successfully implemented on the computer to solve plane elastostatic problems. The approach was later extended by Cruse [28] to three-dimensional elastostatic problems and by Cruse, Snow and Wilson [29] to axisymmetric problems.

Boundary element techniques for thermoelasticity problems may be developed using the basic boundary integral equations and fundamental equations of elasticity. For thermoelastostatic problems involving homogeneous materials, the effects of the temperature fields on the deformations may be incorporated into the body force terms as in Rizzo and Shippy [80] and Cruse, Snow and

Wilson [29] (which had solved both elastic and thermoelastic problem). Such an approach leads to integral equations containing boundary and domain integrals. The body force terms give rise to domain integrals. Nevertheless, for thermoelastostatic deformations in homogeneous materials, since the temperature can be determined independent of the displacement and stress fields, the domain integrals do not contain any unknown functions and can be regarded as particular solutions. Boundary element analyses of thermoelastic deformations of nonhomogenous bodies or bodies with temperature dependent thermal properties in two- and three-dimensional Cartesian spaces may be found in Ghosh and Mukherjee [41] (two-dimensional static thermoelasticity), Sladek, Sladek and Markechova [84] (two-dimensional thermoelasticity in temperature dependent material) and Matsumoto, Guzik and Tanaka [58] (three-dimensional thermoelasticity with temperature dependent material). In these studies, the problems under consideration are formulated in terms of boundary-domain integral equations.

There are some works on boundary element methods for axisymmetric thermoelastic problems. For example, Ochiai and Sekiya [64] solved a thermoelastostatic problem with a general heat generation, Rudolphi [81] had combined the boundary and finite element for solving axisymmetric thermoelasticity and Chopra and Dargush [23] analyzed the thermal stress behavior in quasi-static thermoelastic solid. However, the axisymmetric works in [23], [64] and [81] are restricted to problem involving homogeneous materials. From the literature search conducted, it appears that there is either no work or very limited work on the boundary element methods for solving axisymmetric thermoelastic problems involving nonhomogeneous material. As in two- and three-dimensional problems in Cartesian spaces, the integral formulations of axisymmetric thermoelastic problems concerning nonhomogeneous materials

with material properties that vary continuously in space contain domain integrals, if the formulations are derived using the axisymmetric boundary integral equations and fundamental solutions for homogeneous materials.

In this chapter ¹, a dual-reciprocity boundary element method is proposed for determining the axisymmetric thermoelastostatic fields in a nonhomogeneous isotropic body. The thermal conductivity, shear modulus and stress-temperature coefficients are functionally graded along the axial and radial directions of the axisymmetric body, while the Poisson's ratio is constant. The governing equations of thermoelasticity problem comprise the heat and momentum equations. The boundary element procedure for the heat equation is similar to the procedures given in Chapter 3 and 4. For the momentum equations, the integral formulation of the problem under consideration is derived by using the fundamental solution for the partial differential equations governing the axisymmetric elastostatic deformation of a homogeneous body. In addition to the usual boundary integrals over a curve on the axisymmetric plane, the formulation contains domain integrals due to the continuously varying material properties and thermal effects. The domain integrals can be expressed in terms of boundary integrals using the dual-reciprocity method similar to the approach used in solving the heat equation. For the dual-reciprocity method, axisymmetric interpolating functions for momentum equations that are bounded in the solution domain but are in relatively simple elementary forms for easy computation are proposed here. In thermoelastostatic problems, the temperature field can be calculated first using the heat conduction equation as it is independent of the displacement and stress fields. Therefore,

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temperature on the boundary and in the solution domain can be treated as known when solving the momentum equations. The dual-reciprocity boundary element approach here is successfully applied to solve several axisymmetric thermoelastostatic problems for specific variations of the thermal conductivity, shear modulus and stress-temperature coefficients.

5.2 Statement of Problem

Consider an axisymmetric solid with domain Ω and boundary Γ in rz plane as sketched in Figure 2.1 in Chapter 2. The material occupying the solid is isotropic and nonhomogeneous with its thermal conductivity κ , shear modulus μ and stress-temperature coefficient β functionally graded, that is, κ , μ and β are positive smoothly varying functions of r and z in Ω . The Poisson's ratio ν of the material is a constant such that $-1 < \nu < 1/2$.

The thermoelastic fields in the solid are independent of time and vary with only the spatial coordinates r and z . Furthermore, in the cylindrical polar coordinates r , θ and z , the only non-zero components of the displacement are u_r and u_z , and the non-zero stress components are σ_{rr} , σ_{rz} , $\sigma_{\theta\theta}$ and σ_{zz} . The components of the axisymmetric thermoelastic traction are $t_r = \sigma_{rr}n_r + \sigma_{rz}n_z$ and $t_z = \sigma_{rz}n_r + \sigma_{zz}n_z$, where n_r and n_z are respectively the r and z components of the unit outward normal vector to Γ .

At each point on Γ , either the temperature or the normal heat flux which may be expressed as a linear function of the unknown boundary temperature and any two of the four components u_r , u_z , t_r and t_z are suitably specified. The problem is to determine the thermoelastostatic fields throughout the solid.

The governing equations for thermoelastostatic problems with temperature $T(r, z)$, displacements $u_r(r, z)$ and $u_z(r, z)$, as given in (2.11) in Chapter 2 ,

are

$$\kappa \nabla_{\text{axis}}^2 T + \frac{\partial \kappa}{\partial r} \frac{\partial T}{\partial r} + \frac{\partial \kappa}{\partial z} \frac{\partial T}{\partial z} = -Q, \quad (5.1)$$

where ∇_{axis}^2 is as defined in (3.6), Q is internal heat generation and

$$\begin{aligned} & \nabla_{\text{axis}}^2 u_r - \frac{u_r}{r^2} + \frac{1}{1-2\nu} \frac{\partial}{\partial r} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) \\ = & \frac{1}{\mu} \left\{ \beta \frac{\partial T}{\partial r} + \frac{\partial \beta}{\partial r} (T - T_0) - F_r - \frac{\partial \mu}{\partial z} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \right. \\ & \left. - 2 \frac{\partial \mu}{\partial r} \left[\frac{\partial u_r}{\partial r} + \frac{\nu}{1-2\nu} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) \right] \right\}, \\ & \nabla_{\text{axis}}^2 u_z + \frac{1}{1-2\nu} \frac{\partial}{\partial z} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) \\ = & \frac{1}{\mu} \left\{ \beta \frac{\partial T}{\partial z} + \frac{\partial \beta}{\partial z} (T - T_0) - F_z - \frac{\partial \mu}{\partial r} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \right. \\ & \left. - 2 \frac{\partial \mu}{\partial z} \left[\frac{\partial u_z}{\partial z} + \frac{\nu}{1-2\nu} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) \right] \right\}, \end{aligned} \quad (5.2)$$

where T_0 is a constant reference temperature at which the body does not experience any thermally induced stress, F_r and F_z are the body forces in r and z direction respectively.

The problem under consideration here is to solve (5.1) and (5.2) in Ω subject to the boundary conditions specified on Γ .

5.3 Boundary-domain Integral Formulations

5.3.1 Heat Conduction

The governing partial differential equation (5.1) may be rewritten as

$$\nabla_{\text{axis}}^2 (\sqrt{\kappa} T) = -\frac{Q}{\sqrt{\kappa}} + T \cdot \nabla_{\text{axis}}^2 (\sqrt{\kappa}). \quad (5.3)$$

Following the analysis in Chapter 2, we can recast (5.3) into the boundary-domain integral equation

$$\begin{aligned}
 & \gamma(r_0, z_0) \sqrt{\kappa(r_0, z_0)} T(r_0, z_0) \\
 = & \int_{\Gamma} \left\{ T(r, z) [\sqrt{\kappa(r, z)} G_1(r, z; r_0, z_0; n_r, n_z) - \frac{\partial \sqrt{\kappa(r, z)}}{\partial n} G_0(r, z; r_0, z_0)] \right. \\
 & \left. - \sqrt{\kappa(r, z)} G_0(r, z; r_0, z_0) q(r, z; n_r, n_z) \right\} r ds(r, z) \\
 & + \iint_{\Omega} G_0(r, z; r_0, z_0) \left[-\frac{Q(r, z)}{\sqrt{\kappa(r, z)}} + T(r, z) \cdot \nabla_{\text{axis}}^2 (\sqrt{\kappa(r, z)}) \right] r dr dz \\
 & \text{for } (r_0, z_0) \in \Omega \cup \Gamma,
 \end{aligned} \tag{5.4}$$

where $\gamma(r_0, z_0) = 1$ if (r_0, z_0) lies in the interior of Ω , $\gamma(r_0, z_0) = 1/2$ if (r_0, z_0) lies on a smooth part of Γ , $ds(r, z)$ denotes the length of an infinitesimal part of the curve Γ , $\underline{n}(r, z) = [n_r(r, z), n_z(r, z)]$ (for $(r, z) \in \Gamma$) is the unit normal vector to Γ pointing out of Ω , $G_0(r, z; r_0, z_0)$ and $G_1(r, z; r_0, z_0; n_r, n_z)$ are fundamental solutions of heat conduction problem as given in (2.17) and (2.18) in Chapter 2 and $q(r, z; n_r, n_z)$ is defined by

$$q(r, z; n_r, n_z) = n_r(r, z) \frac{\partial}{\partial r} [T(r, z)] + n_z(r, z) \frac{\partial}{\partial z} [T(r, z)]. \tag{5.5}$$

5.3.2 Thermoelastostatics

For convenience, uppercase Latin subscript such as J or K to be the values of r and z and Einsteinian convention of summing over a repeated subscripts are adopted. The boundary-domain integral equation for (5.2) can be derived as explained in Chapter 2. It is given by (see Bakr [16])

$$\begin{aligned}
 \gamma(r_0, z_0) u_K(r_0, z_0) = & \int_{\Gamma} (\Phi_{JK}(r, z; r_0, z_0) p_J(r, z; n_r, n_z) \\
 & - \Psi_{JK}(r, z; r_0, z_0; n_r, n_z) u_J(r, z)) r ds(r, z)
 \end{aligned}$$

$$\begin{aligned}
 & + \iint_{\Omega} \frac{1}{\mu(r, z)} \Phi_{JK}(r, z; r_0, z_0) \left\{ -\beta(r, z) \frac{\partial}{\partial x_J} [T(r, z)] \right. \\
 & \quad - \frac{\partial}{\partial x_J} [\beta(r, z)] (T(r, z) - T_0) + F_J(r, z) \\
 & \quad + \frac{\partial}{\partial x_J} [\mu(r, z)] \frac{2\nu}{(1 - 2\nu)r} u_r(r, z) \\
 & \quad \left. + X_{JN}(r, z) \frac{\partial}{\partial z} [u_N(r, z)] + Y_{JN}(r, z) \frac{\partial}{\partial r} [u_N(r, z)] \right\} r dr dz \\
 & \quad \text{for } (r_0, z_0) \in \Omega \cup \Gamma, \tag{5.6}
 \end{aligned}$$

where $x_r = r$, $x_z = z$, $\Phi_{JK}(r, z; r_0, z_0)$ and $\Psi_{JK}(r, z; r_0, z_0; n_r, n_z)$ are fundamental solutions for elasticity problem as given in (2.27) and (2.28) in Chapter 2, $p_J(r, z; n_r, n_z)$ are defined by

$$\begin{aligned}
 p_r(r, z; n_r, n_z) &= 2 \left(\frac{\partial u_r}{\partial r} + \frac{\nu}{1 - 2\nu} \left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right] \right) n_r(r, z) \\
 &\quad + \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) n_z(r, z), \\
 p_z(r, z; n_r, n_z) &= \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) n_r(r, z) \\
 &\quad + 2 \left(\frac{\partial u_z}{\partial z} + \frac{\nu}{1 - 2\nu} \left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right] \right) n_z(r, z), \tag{5.7}
 \end{aligned}$$

and

$$\begin{aligned}
 X_{rr}(r, z) &= \frac{\partial \mu(r, z)}{\partial z}, \quad X_{rz}(r, z) = \frac{2\nu}{1 - 2\nu} \frac{\partial \mu(r, z)}{\partial r}, \\
 X_{zr}(r, z) &= \frac{\partial \mu(r, z)}{\partial r}, \quad X_{zz}(r, z) = \frac{\partial \mu(r, z)}{\partial z} \frac{2(1 - \nu)}{1 - 2\nu}, \\
 Y_{rr}(r, z) &= \frac{\partial \mu(r, z)}{\partial r} \frac{2(1 - \nu)}{1 - 2\nu}, \quad Y_{rz}(r, z) = \frac{\partial \mu(r, z)}{\partial z}, \\
 Y_{zr}(r, z) &= \frac{2\nu}{1 - 2\nu} \frac{\partial \mu(r, z)}{\partial z}, \quad Y_{zz}(r, z) = \frac{\partial \mu(r, z)}{\partial r}. \tag{5.8}
 \end{aligned}$$

Note that $p_J(r, z; n_r, n_z)$ are related to the axisymmetric components $t_J(r, z; n_r, n_z)$ of the thermoelastic tractions by

$$t_J(r, z; n_r, n_z) = \mu(r, z) p_J(r, z; n_r, n_z) - \beta(r, z) [T(r, z) - T_0] \delta_{JL} n_L(r, z), \tag{5.9}$$

where δ_{JN} is the Kronecker-delta defined by $\delta_{rr} = \delta_{zz} = 1$ and $\delta_{rz} = \delta_{zr} = 0$.

5.4 Dual-reciprocity Boundary Element Procedures

5.4.1 Boundary Approximations

The boundary curve Γ of solution domain is discretized into N straight line elements denoted by $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(N-1)}$ and $\Gamma^{(N)}$. The starting and ending points of an element $\Gamma^{(k)}$ are given by $(r^{(k)}, z^{(k)})$ and $(r^{(k+1)}, z^{(k+1)})$ respectively. Over $\Gamma^{(k)}$, the functions T, q, u_J and p_J are respectively approximated as constants $T^{(k)}, q^{(k)}, u_J^{(k)}$ and $p_J^{(k)}$. Furthermore, $(r_0^{(1)}, z_0^{(1)}), (r_0^{(2)}, z_0^{(2)}), \dots, (r_0^{(N-1)}, z_0^{(N-1)})$ and $(r_0^{(N)}, z_0^{(N)})$ are the midpoints of $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(N-1)}$ and $\Gamma^{(N)}$ respectively. Therefore, the boundary integral equation in (5.4) and (5.6) are given by

$$\begin{aligned}
 & \gamma(r_0, z_0) \sqrt{\kappa(r_0, z_0)} T(r_0, z_0) \\
 = & \sum_{k=1}^N T^{(k)} \int_{\Gamma^{(k)}} [\sqrt{\kappa(r, z)} G_1(r, z; r_0, z_0; n_r, n_z) \\
 & - \frac{\partial \kappa(r, z)}{\partial n} G_0(r, z; r_0, z_0)] r ds(r, z) \\
 & - \sum_{k=1}^N q^{(k)} \int_{\Gamma^{(k)}} \sqrt{\kappa(r, z)} G_0(r, z; r_0, z_0) r ds(r, z) \\
 & + \iint_{\Omega} G_0(r, z; r_0, z_0) \left[-\frac{Q(r, z)}{\sqrt{\kappa(r, z)}} \right. \\
 & \left. + T(r, z) \cdot \nabla_{\text{axis}}^2 (\sqrt{\kappa(r, z)}) \right] r dr dz, \tag{5.10}
 \end{aligned}$$

and

$$\begin{aligned}
 \gamma(r_0, z_0) u_K(r_0, z_0) = & \sum_{k=1}^N p_J^{(k)} \int_{\Gamma^{(k)}} \Phi_{JK}(r, z; r_0, z_0) r ds(r, z) \\
 & - \sum_{k=1}^N u_J^{(k)} \int_{\Gamma^{(k)}} \Psi_{JK}(r, z; r_0, z_0; n_r, n_z) r ds(r, z)
 \end{aligned}$$

$$\begin{aligned}
 & + \iint_{\Omega} \frac{1}{\mu(r, z)} \Phi_{JK}(r, z; r_0, z_0) \left\{ -\beta(r, z) \frac{\partial}{\partial x_J} [T(r, z)] \right. \\
 & \quad \left. - \frac{\partial}{\partial x_J} [\beta(r, z)] (T - T_0) + F_J(r, z) \right. \\
 & \quad + \frac{\partial}{\partial x_J} [\mu(r, z)] \frac{2\nu}{(1 - 2\nu)r} [u_r(r, z)] \\
 & \quad \left. + X_{JN}(r, z) \frac{\partial}{\partial z} [u_N(r, z)] + Y_{JN}(r, z) \frac{\partial}{\partial r} [u_N(r, z)] \right\} r dr dz \\
 & \quad \text{for } m = 1, 2, \dots, N + L,
 \end{aligned} \tag{5.11}$$

where the integrals over $\Gamma^{(k)}$ in (5.10) and (5.11) can be evaluated numerically using Gaussian quadrature. Noted that constant elements are used for boundary approximation in this chapter, while discontinuous linear elements are used in Chapter 3 and 4.

5.4.2 Axisymmetric Interpolating Functions and Dual-reciprocity Approximation of Domain Integrals

The axisymmetric interpolating functions, as given and used in Chapter 3 and 4, are again adopted here for treating the domain integral in (5.10). The interpolating functions for thermal equation will be modified to apply in the dual-reciprocity method for the thermoelastic domain integral in (5.11).

By using dual-reciprocity method, the double integral in (5.4) and (5.6) can be approximated to line integrals over Γ . Select L well-spaced out collocation points in $\Omega \cup \Gamma$, which are denoted by $(r_0^{(N+1)}, z_0^{(N+1)})$, $(r_0^{(N+2)}, z_0^{(N+2)})$, $\dots$, $(r_0^{(N+L-1)}, z_0^{(N+L-1)})$ and $(r_0^{(N+L)}, z_0^{(N+L)})$. Besides the L collocation points, the points $(r_0^{(k)}, z_0^{(k)})$ at element $\Gamma^{(k)}$ ($k = 1, 2, \dots, N$) are also used as collocation points. Therefore, there are total of $N + L$ collocation points. Note that none of the selected points lies on the z axis.

To treat the double integral in (5.4), let

$$-\frac{Q(r, z)}{\sqrt{\kappa(r, z)}} + T(r, z) \cdot \nabla_{\text{axis}}^2(\sqrt{\kappa(r, z)}) \simeq \sum_{n=1}^{N+L} \alpha^{(n)} \varphi^{(n)}(r, z) \text{ for } (r, z) \in \Omega, \quad (5.12)$$

where $\alpha^{(n)}$ are constant coefficients to be determined and $\varphi^{(n)}(r, z)$ is a local interpolating function which can be written in the form

$$\nabla_{\text{axis}}^2 \chi^{(n)}(r, z) = \varphi^{(n)}(r, z), \quad (5.13)$$

then the domain integral can be given by

$$\begin{aligned} & \iint_{\Omega} G_0(r, z; r_0, z_0) \left[-\frac{Q(r, z)}{\sqrt{\kappa(r, z)}} + T(r, z) \cdot \nabla_{\text{axis}}^2(\sqrt{\kappa(r, z)}) \right] r dr dz \\ & \simeq \sum_{n=1}^{N+L} \alpha^{(n)} W^{(n)}(r_0, z_0), \end{aligned} \quad (5.14)$$

where

$$\begin{aligned} W^{(n)}(r_0, z_0) &= \gamma(r_0, z_0) \chi(r_0, z_0; r_0^{(n)}, z_0^{(n)}) \\ &+ \int_{\Gamma} [G_0(r, z; r_0, z_0) \frac{\partial}{\partial n} [\chi(r, z; r_0^{(n)}, z_0^{(n)})] \\ &- G_1(r, z; r_0, z_0; n_r, n_z) \chi(r, z; r_0^{(n)}, z_0^{(n)})] r ds(r, z), \end{aligned} \quad (5.15)$$

and

$$\frac{\partial}{\partial n} [\chi^{(n)}(r, z)] = n_r(r, z) \frac{\partial}{\partial r} [\chi^{(n)}(r, z)] + n_z(r, z) \frac{\partial}{\partial z} [\chi^{(n)}(r, z)]. \quad (5.16)$$

We may let $(r, z) = (r_0^{(k)}, z_0^{(k)})$ ($k = 1, 2, \dots, N+L$) in (5.12) to obtain

$$\begin{aligned} \sum_{n=1}^{N+L} \alpha^{(n)} \varphi^{(n)}(r_0^{(k)}, z_0^{(k)}) &= -\frac{Q(r_0^{(k)}, z_0^{(k)})}{\sqrt{\kappa(r_0^{(k)}, z_0^{(k)})}} \\ &+ T(r_0^{(k)}, z_0^{(k)}) \cdot \nabla_{\text{axis}}^2(\sqrt{\kappa(r, z)}) \Big|_{(r, z) = (r_0^{(k)}, z_0^{(k)})} \\ \text{for } k &= 1, 2, \dots, N+L, \end{aligned} \quad (5.17)$$

which may be solved as a system of $N + L$ linear algebraic equations to find the unknown coefficients $\alpha^{(n)}$.

Using the same interpolating function as shown in Chapter 3 and 4, the local interpolating function $\varphi^{(n)}(r, z)$ is constructed from (5.13) by letting

$$\chi^{(n)}(r, z) = \frac{1}{9} \{ [\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^3 + [\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^3 \}, \quad (5.18)$$

where $\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) = \sqrt{(r - r_0^{(n)})^2 + (z - z_0^{(n)})^2}$.

The function $\varphi^{(n)}(r, z)$ corresponding to $\chi^{(n)}(r, z)$ in (5.18) is then given by

$$\varphi^{(n)}(r, z) = \left[\frac{4}{3} - \frac{r_0^{(n)}}{3r} \right] \sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) + \left[\frac{4}{3} + \frac{r_0^{(n)}}{3r} \right] \sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)}). \quad (5.19)$$

The interpolating function $\varphi^{(n)}(r, z)$ in (5.19) is bounded at all points (r, z) in the region $r > 0$.

Formula (5.14) together with (5.16), (5.17), (5.18) and (5.19) can be used to reduce the double integral over Ω in (5.4) to line integrals over Γ . Therefore, the boundary-domain integral equation is given by

$$\begin{aligned} & \gamma(r_0^{(m)}, z_0^{(m)}) \sqrt{\kappa(r_0^{(m)}, z_0^{(m)})} T^{(m)} \\ &= \sum_{k=1}^N T^{(k)} \int_{\Gamma^{(k)}} [\sqrt{\kappa(r, z)} G_1(r, z; r_0^{(m)}, z_0^{(m)}; n_r, n_z) \\ & \quad - \frac{\partial \kappa(r, z)}{\partial n} G_0(r, z; r_0^{(m)}, z_0^{(m)})] r ds(r, z) \\ & \quad - \sum_{k=1}^N q^{(k)} \int_{\Gamma^{(k)}} \sqrt{\kappa(r, z)} G_0(r, z; r_0^{(m)}, z_0^{(m)}) r ds(r, z) \\ & \quad + \sum_{n=1}^{N+L} \alpha^{(n)} W^{(n)}(r_0^{(m)}, z_0^{(m)}) \\ & \quad \text{for } m = 1, 2, \dots, N + L, \end{aligned} \quad (5.20)$$

and

$$\begin{aligned} \sum_{n=1}^{N+L} \alpha^{(n)} \varphi^{(n)}(r_0^{(k)}, z_0^{(k)}) &= -\frac{Q(r_0^{(k)}, z_0^{(k)})}{\sqrt{\kappa(r_0^{(k)}, z_0^{(k)})}} \\ &\quad + T^{(k)} \cdot \nabla_{\text{axis}}^2 (\sqrt{\kappa(r, z)}) \Big|_{(r, z) = (r_0^{(k)}, z_0^{(k)})} \\ &\quad \text{for } k = 1, 2, \dots, N+L, \end{aligned} \quad (5.21)$$

The double integral in (5.6) will be treated using similar approach.

We use the local interpolating functions $\varphi_{IJ}^{(n)}(r, z)$ given by

$$\begin{aligned} \varphi_{rJ}^{(n)}(r, z) &= \nabla_{\text{axis}}^2 \chi_{rJ}^{(n)}(r, z) - \frac{\chi_{rJ}^{(n)}(r, z)}{r^2} \\ &\quad + \frac{1}{1-2\nu} \frac{\partial}{\partial r} \left(\frac{\partial}{\partial r} [\chi_{rJ}^{(n)}(r, z)] \right) + \frac{\chi_{rJ}^{(n)}(r, z)}{r} + \frac{\partial}{\partial z} [\chi_{zJ}^{(n)}(r, z)], \\ \varphi_{zJ}^{(n)}(r, z) &= \nabla_{\text{axis}}^2 \chi_{zJ}^{(n)}(r, z) \\ &\quad + \frac{1}{1-2\nu} \frac{\partial}{\partial z} \left(\frac{\partial}{\partial r} [\chi_{rJ}^{(n)}(r, z)] \right) + \frac{\chi_{rJ}^{(n)}(r, z)}{r} + \frac{\partial}{\partial z} [\chi_{zJ}^{(n)}(r, z)]. \end{aligned} \quad (5.22)$$

If we let

$$\begin{aligned} f_J(r, z) &= \frac{1}{\mu} \left\{ -\beta(r, z) \frac{\partial}{\partial x_J} [T(r, z)] - \frac{\partial}{\partial x_J} [\beta(r, z)] (T - T_0) \right. \\ &\quad + F_J(r, z) + \frac{\partial}{\partial x_J} [\mu(r, z)] \frac{2\nu}{(1-2\nu)r} [u_r(r, z)] \\ &\quad \left. + X_{JN}(r, z) \frac{\partial}{\partial z} [u_N(r, z)] + Y_{JN}(r, z) \frac{\partial}{\partial r} [u_N(r, z)] \right\}, \end{aligned} \quad (5.23)$$

then the function $f_J(r, z)$ can be approximated using

$$f_J(r, z) \simeq \sum_{n=1}^{N+L} \varphi_{JN}^{(n)}(r, z) \alpha_N^{(n)} \text{ for } (r, z) \in \Omega, \quad (5.24)$$

where $\alpha_N^{(n)}$ are constant coefficients to be determined, then the domain integrals are given by

$$\iint_{\Omega} \Phi_{JK}(r, z; r_0, z_0) f_J(r, z) r dr dz \simeq \sum_{n=1}^{N+L} \alpha_N^{(n)} W_{KN}^{(n)}(r_0, z_0), \quad (5.25)$$

where

$$\begin{aligned} W_{KN}^{(n)}(r_0, z_0) &= -\gamma(r_0, z_0) \chi_{KN}^{(n)}(r_0, z_0) \\ &\quad + \int_{\Gamma} (\Phi_{JK}(r, z; r_0, z_0) \tau_{JN}^{(n)}(r, z; n_r, n_z) \\ &\quad - \Psi_{JK}(r, z; r_0, z_0; n_r, n_z) \chi_{JN}^{(n)}(r, z)) r ds(r, z), \end{aligned} \quad (5.26)$$

and

$$\begin{aligned} \tau_{rN}^{(n)}(r, z; n_r, n_z) &= 2n_r(r, z) \left\{ \frac{\partial}{\partial r} [\chi_{rN}^{(n)}(r, z)] + \frac{\nu}{1-2\nu} \left(\frac{\partial}{\partial r} [\chi_{rN}^{(n)}(r, z)] \right. \right. \\ &\quad \left. \left. + \frac{\chi_{rN}^{(n)}(r, z)}{r} + \frac{\partial}{\partial z} [\chi_{zN}^{(n)}(r, z)] \right) \right\} \\ &\quad + n_z(r, z) \left\{ \frac{\partial}{\partial z} [\chi_{rN}^{(n)}(r, z)] + \frac{\partial}{\partial r} [\chi_{zN}^{(n)}(r, z)] \right\}, \\ \tau_{zN}^{(n)}(r, z; n_r, n_z) &= n_r(r, z) \left\{ \frac{\partial}{\partial z} [\chi_{rN}^{(n)}(r, z)] + \frac{\partial}{\partial r} [\chi_{zN}^{(n)}(r, z)] \right\} \\ &\quad + 2n_z(r, z) \left\{ \frac{\partial}{\partial z} [\chi_{zN}^{(n)}(r, z)] + \frac{\nu}{1-2\nu} \left(\frac{\partial}{\partial r} [\chi_{rN}^{(n)}(r, z)] \right. \right. \\ &\quad \left. \left. + \frac{\chi_{rN}^{(n)}(r, z)}{r} + \frac{\partial}{\partial z} [\chi_{zN}^{(n)}(r, z)] \right) \right\}. \end{aligned} \quad (5.27)$$

To construct functions $\varphi_{IJ}^{(n)}(r, z)$ that are bounded at all points (r, z) for $r > 0$, we take

$$\begin{aligned} \chi_{rr}^{(n)}(r, z) &= \chi^{(n)}(r, z) - \frac{2}{9} [\sigma(0, 0, z; r_0^{(n)}, z_0^{(n)})]^3, \\ \chi_{zr}^{(n)}(r, z) &= \chi_{rz}^{(n)}(r, z) = 0, \\ \chi_{zz}^{(n)}(r, z) &= \chi^{(n)}(r, z), \end{aligned} \quad (5.28)$$

where $\chi^{(n)}(r, z)$ is as defined in (5.18).

The functions $\varphi_{IJ}^{(n)}(r, z)$ and $\tau_{IJ}^{(n)}(r, z; n_r, n_z)$ constructed using (5.22), (5.27) and (5.28) are

$$\begin{aligned}
 & \varphi_{rr}^{(n)}(r, z) \\
 = & -\left\{ \left(\frac{4}{3} - \frac{r_0^{(n)}}{3r} \right) \sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) + \left(\frac{4}{3} + \frac{r_0^{(n)}}{3r} \right) \sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)}) \right. \\
 & - \frac{2}{3} (z - z_0^{(n)})^2 [\sigma(0, 0, z; r_0^{(n)}, z_0^{(n)})]^{-1} - \frac{2}{3} \sigma(0, 0, z; r_0^{(n)}, z_0^{(n)}) \\
 & - \frac{1}{9r^2} [[\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^3 + [\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^3 \\
 & - 2[\sigma(0, 0, z; r_0^{(n)}, z_0^{(n)})]^3] + \frac{1}{1-2\nu} \left[\left(\frac{2}{3} - \frac{r_0^{(n)}}{3r} \right) \sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) \right. \\
 & + \left(\frac{2}{3} + \frac{r_0^{(n)}}{3r} \right) \sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)}) + \frac{1}{3} (r - r_0^{(n)})^2 [\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^{-1} \\
 & + \frac{1}{3} (r + r_0^{(n)})^2 [\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^{-1} - \frac{1}{9r^2} [[\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^3 \\
 & \left. \left. + [\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^3 - 2[\sigma(0, 0, z; r_0^{(n)}, z_0^{(n)})]^3] \right\},
 \end{aligned}$$

$$\begin{aligned}
 & \varphi_{zr}^{(n)}(r, z) \\
 = & -\frac{(z - z_0^{(n)})}{3(1-2\nu)} \{ (r - r_0^{(n)}) [\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^{-1} \\
 & + (r + r_0^{(n)}) [\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^{-1} \\
 & + \frac{1}{r} [\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) + \sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)}) - 2\sigma(0, 0, z; r_0^{(n)}, z_0^{(n)})] \},
 \end{aligned}$$

$$\begin{aligned}
 & \varphi_{rz}^{(n)}(r, z) \\
 = & -\frac{(z - z_0^{(n)})}{3(1-2\nu)} \{ (r - r_0^{(n)}) [\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^{-1} \\
 & + (r + r_0^{(n)}) [\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^{-1} \},
 \end{aligned}$$

$$\begin{aligned}
 & \varphi_{zz}^{(n)}(r, z) \\
 = & -\left\{\left(\frac{4}{3} - \frac{r_0^{(n)}}{3r}\right)\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) + \left(\frac{4}{3} + \frac{r_0^{(n)}}{3r}\right)\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})\right. \\
 & + \frac{1}{3(1-2\nu)}[(z - z_0^{(n)})^2[\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^{-1} \\
 & + (z - z_0^{(n)})^2[\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^{-1} \\
 & \left. + \sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) + \sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]\right\}, \tag{5.29}
 \end{aligned}$$

and $\tau_{IJ}^{(n)}(r, z; n_r, n_z)$ in (5.27) are given in explicit form as

$$\begin{aligned}
 & \tau_{rr}^{(n)}(r, z; n_r, n_z) \\
 = & 2n_r(r, z)\left\{\frac{1-\nu}{3(1-2\nu)}[(r - r_0^{(n)})\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})\right. \\
 & + (r + r_0^{(n)})\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})] \\
 & + \frac{\nu}{9(1-2\nu)r}[[\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^3 + [\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^3 \\
 & - 2[\sigma(0, 0, z; r_0^{(n)}, z_0^{(n)})]^3]\} \\
 & + n_z(r, z)\left\{\frac{(z - z_0^{(n)})}{3}[\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) + \sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})\right. \\
 & \left. - 2\sigma(0, 0, z; r_0^{(n)}, z_0^{(n)})]\right\},
 \end{aligned}$$

$$\begin{aligned}
 & \tau_{zz}^{(n)}(r, z; n_r, n_z) \\
 = & n_r(r, z)\left\{\frac{(z - z_0^{(n)})}{3}[\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})\right. \\
 & + \sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)}) - 2\sigma(0, 0, z; r_0^{(n)}, z_0^{(n)})]\} \\
 & + 2n_z(r, z)\left\{\frac{\nu}{3(1-2\nu)}[(r - r_0^{(n)})\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})\right. \\
 & + (r + r_0^{(n)})\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})] + \frac{\nu}{9(1-2\nu)r}[[\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^3 \\
 & + [\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^3 - 2[\sigma(0, 0, z; r_0^{(n)}, z_0^{(n)})]^3]\},
 \end{aligned}$$

$$\begin{aligned}
 & \tau_{rz}^{(n)}(r, z; n_r, n_z) \\
 = & \frac{2n_r(r, z)\nu(z - z_0^{(n)})}{3(1 - 2\nu)}[\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) + \sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})] \\
 & + \frac{n_z(r, z)}{3}[(r - r_0^{(n)})\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) + (r + r_0^{(n)})\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})], \\
 \\
 & \tau_{zz}^{(n)}(r, z; n_r, n_z) \\
 = & \frac{n_r(r, z)}{3}[(r - r_0^{(n)})\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) + (r + r_0^{(n)})\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})] \\
 & + \frac{2n_z(r, z)(1 - \nu)(z - z_0^{(n)})}{3(1 - 2\nu)}[\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) \\
 & + \sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]. \tag{5.30}
 \end{aligned}$$

With $\chi_{KN}^{(n)}(r, z)$ as given in (5.28), $\varphi_{IJ}^{(n)}(r, z)$ can be shown to be bounded at all points (r, z) in the region $r > 0$. In Agnantiaris, Polyzos and Beskos [2], bounded interpolating functions are constructed by integrating axially the corresponding interpolating functions for the three-dimensional dual-reciprocity boundary element method. Explicit expressions for the interpolating functions constructed from the axial integration are, however, very complicated compared to those given in (5.29). In [2], it appears that the integrals from the axial integration are computed numerically. Thus, (5.29) and (5.30) give the axisymmetric interpolating functions that are simpler to evaluate for use in the dual-reciprocity method.

We collocate (5.24) by taking $(r, z) = (r_0^{(k)}, z_0^{(k)})$ ($k = 1, 2, \dots, M$) to obtain

$$\sum_{n=1}^{N+L} \varphi_{JN}^{(n)}(r_0^{(k)}, z_0^{(k)}) \alpha_N^{(n)} = f_J(r_0^{(k)}, z_0^{(k)}) \text{ for } k = 1, 2, \dots, N + L. \tag{5.31}$$

The coefficients $\alpha_N^{(n)}$ can then be obtained in terms of $f_J(r_0^{(k)}, z_0^{(k)})$ by inverting (5.31).

Therefore, the integral equations in (5.6) can be approximately reduced to the linear algebraic equations

$$\begin{aligned}
 \gamma(r_0^{(m)}, z_0^{(m)})u_K^{(m)} &= \sum_{n=1}^{N+L} \alpha_N^{(n)} W_{KN}^{(n)}(r_0^{(m)}, z_0^{(m)}) \\
 &+ \sum_{k=1}^N p_J^{(k)} \int_{\Gamma^{(k)}} \Phi_{JK}(r, z; r_0^{(m)}, z_0^{(m)}) r ds(r, z) \\
 &- \sum_{k=1}^N u_J^{(k)} \int_{\Gamma^{(k)}} \Psi_{JK}(r, z; r_0^{(m)}, z_0^{(m)}; n_r, n_z) r ds(r, z) \\
 &\text{for } m = 1, 2, \dots, N+L,
 \end{aligned} \tag{5.32}$$

and

$$\begin{aligned}
 &\sum_{n=1}^{N+L} \varphi_{JN}^{(n)}(r_0^{(k)}, z_0^{(k)}) \alpha_N^{(n)} \\
 &= \frac{1}{\mu(r_0^{(k)}, z_0^{(k)})} \left\{ -\beta(r_0^{(k)}, z_0^{(k)}) \frac{\partial}{\partial x_J} [T(r, z)] \Big|_{(r,z)=(r_0^{(k)}, z_0^{(k)})} \right. \\
 &\quad - \frac{\partial}{\partial x_J} [\beta(r, z)] \Big|_{(r,z)=(r_0^{(k)}, z_0^{(k)})} (T^{(k)} - T_0) \\
 &\quad + F_J(r_0^{(k)}, z_0^{(k)}) + \frac{\partial}{\partial x_J} [\mu(r, z)] \Big|_{(r,z)=(r_0^{(k)}, z_0^{(k)})} \frac{2\nu}{(1-2\nu)r^{(k)}} u_r^{(k)} \\
 &\quad + X_{JN}(r_0^{(k)}, z_0^{(k)}) \frac{\partial}{\partial z} [u_N(r, z)] \Big|_{(r,z)=(r_0^{(k)}, z_0^{(k)})} \\
 &\quad \left. + Y_{JN}(r_0^{(k)}, z_0^{(k)}) \frac{\partial}{\partial r} [u_N(r, z)] \Big|_{(r,z)=(r_0^{(k)}, z_0^{(k)})} \right\} \\
 &\text{for } k = 1, 2, \dots, N+L.
 \end{aligned} \tag{5.33}$$

Unlike (5.21), the right hand side of (5.33) contains values of the first order partial derivatives of the unknown functions. To approximate those values, let

$$T(r, z) \simeq \sum_{m=1}^{N+L} t^{(m)} \chi^{(m)}(r, z),$$

$$\begin{aligned}
 u_r(r, z) &\simeq \sum_{m=1}^{N+L} v_r^{(m)} \bar{\chi}^{(m)}(r, z), \\
 u_z(r, z) &\simeq \sum_{m=1}^{N+L} v_z^{(m)} \chi^{(m)}(r, z),
 \end{aligned} \tag{5.34}$$

where $\chi^{(m)}(r, z)$ is as defined in (5.18) and $\bar{\chi}^{(m)}(r, z)$ by

$$\bar{\chi}^{(m)}(r, z) = \frac{1}{9} \{ [\sigma(r, 0, z; r_0^{(m)}, z_0^{(m)})]^3 - [\sigma(r, 0, z; -r_0^{(m)}, z_0^{(m)})]^3 \}. \tag{5.35}$$

If we collocate (5.34) by letting $(r, z) = (r_0^{(k)}, z_0^{(k)})$ for $k = 1, 2, \dots, N+L$ and invert the resulting equations to determine the constant coefficients $t^{(m)}$ and $v_N^{(m)}$, we obtain

$$\begin{aligned}
 \frac{\partial}{\partial x_J} [T(r, z)] &= \sum_{p=1}^{N+L} T^{(p)} \phi_J^{(p)}(r, z), \\
 \frac{\partial}{\partial x_J} [u_r(r, z)] &= \sum_{p=1}^{N+L} u_r^{(p)} \bar{\phi}_J^{(p)}(r, z), \\
 \frac{\partial}{\partial x_J} [u_z(r, z)] &= \sum_{p=1}^{N+L} u_{zJ}^{(p)} \phi_J^{(p)}(r, z),
 \end{aligned} \tag{5.36}$$

where

$$\begin{aligned}
 \phi_J^{(p)}(r, z) &= \sum_{m=1}^{N+L} \omega^{(mp)} \frac{\partial}{\partial x_J} [\chi^{(m)}(r, z)], \\
 \bar{\phi}_J^{(p)}(r, z) &= \sum_{m=1}^{N+L} \bar{\omega}^{(mp)} \frac{\partial}{\partial x_J} [\bar{\chi}^{(m)}(r, z)],
 \end{aligned} \tag{5.37}$$

and

$$\begin{aligned}
 \sum_{m=1}^{N+L} \chi^{(m)}(r_0^{(k)}, z_0^{(k)}) \omega^{(mp)} &= \begin{cases} 1 & \text{if } k = p, \\ 0 & \text{if } k \neq p, \end{cases} \\
 \sum_{m=1}^{N+L} \bar{\chi}^{(m)}(r_0^{(k)}, z_0^{(k)}) \bar{\omega}^{(mp)} &= \begin{cases} 1 & \text{if } k = p, \\ 0 & \text{if } k \neq p. \end{cases}
 \end{aligned} \tag{5.38}$$

It follows that (5.33) becomes

$$\begin{aligned}
 & \sum_{n=1}^{N+L} \varphi_{JN}^{(n)}(r_0^{(k)}, z_0^{(k)}) \alpha_N^{(n)} \\
 = & \frac{1}{\mu(r_0^{(k)}, z_0^{(k)})} \left\{ -\beta(r_0^{(k)}, z_0^{(k)}) \sum_{p=1}^{N+L} T^{(p)} \phi_J^{(p)}(r_0^{(k)}, z_0^{(k)}) \right. \\
 & - \left. \frac{\partial}{\partial x_J} [\beta(r, z)] \right|_{(r,z)=(r_0^{(k)}, z_0^{(k)})} (T^{(k)} - T_0) \\
 & + F_J(r_0^{(k)}, z_0^{(k)}) + \left. \frac{\partial}{\partial x_J} [\mu(r, z)] \right|_{(r,z)=(r_0^{(k)}, z_0^{(k)})} \frac{2\nu}{(1-2\nu)r} u_r^{(k)} \\
 & + \sum_{p=1}^{N+L} [X_{Jr}(r_0^{(k)}, z_0^{(k)}) u_r^{(p)} \bar{\phi}_z^{(p)}(r_0^{(k)}, z_0^{(k)}) + X_{Jz}(r_0^{(k)}, z_0^{(k)}) u_z^{(p)} \phi_z^{(p)}(r_0^{(k)}, z_0^{(k)})] \\
 & + \sum_{p=1}^{N+L} [Y_{Jr}(r_0^{(k)}, z_0^{(k)}) u_r^{(p)} \bar{\phi}_r^{(p)}(r_0^{(k)}, z_0^{(k)}) + Y_{Jz}(r_0^{(k)}, z_0^{(k)}) u_z^{(p)} \phi_r^{(p)}(r_0^{(k)}, z_0^{(k)})] \} \\
 & \text{for } k = 1, 2, \dots, N+L.
 \end{aligned} \tag{5.39}$$

In (5.18) and (5.35), $u_r(r, z)$ and the first order partial derivatives of $T(r, z)$ and $u_z(r, z)$ with respect to r behave as $O(r)$ for small r . Such behaviors are expected of the temperature $T(r, z)$ and the displacement components $u_r(r, z)$ and $u_z(r, z)$ if the solution domain contains points (r, z) where r can be zero.

With $T^{(p)}$ determined from (5.20) and (5.21), we may now solve (5.32) and (5.39) as a system of $4(N+L)$ linear algebraic equations in $4(N+L)$ unknowns. The unknowns are $\alpha_J^{(n)}$ ($n = 1, 2, \dots, N+L$), two unknowns from the four boundary components $u_r^{(k)}$, $u_z^{(k)}$, $p_r^{(k)}$ and $p_z^{(k)}$ ($k = 1, 2, \dots, N$) and the unknown displacement components $u_r^{(i)}$ and $u_z^{(i)}$ ($i = N+1, N+2, \dots, N+L$) at interior collocation points.

5.5 Specific Problems

Boundary element method is applied here to solve several test problems. The first two problems are boundary value problems with known exact solutions. They are used to access the validity and accuracy of the proposed method. In the last problem, the boundary element method is applied to solve a thermoelastostatic problem in exponentially graded cylindrical solid where the material properties vary exponentially in the axial direction. The effect of the stress-temperature coefficient β is examined.

Problem 1

For a particular test problem, take the solution domain as $0 < r < 1, 0 < z < 1$. The thermal conductivity κ , the stress-temperature coefficient β , the shear modulus μ , the Poisson's ratio ν and the reference temperature T_0 are taken to be given by $\kappa = (r^2 + z + 1)^2$, $\beta = z^2 + 1$, $\mu = r^2 + 1$, $\nu = 0.3$ and $T_0 = 0$ respectively. The internal heat generation Q and body force terms F_r and F_z are given by

$$\begin{aligned} Q(r, z) &= (r^2 + z + 1)(12z^2 + 6r^2z - 6z - 36r^2 - 12), \\ F_r(r, z) &= -2r^3 - 4rz^2 - 38r, \\ F_z(r, z) &= -5z^4 - 8r^2z - 3z^2 - 10z + 16r^2 + 8. \end{aligned}$$

The boundary conditions are taken to be given by

$$\left. \begin{aligned} T(r, 0) &= 3r^2 \\ u_r(r, 0) &= 3r \\ u_z(r, 0) &= -2r^2 \end{aligned} \right\} \text{ for } 0 < r < 1,$$

$$\left. \begin{aligned} \frac{\partial T}{\partial z} \Big|_{z=1} &= -3 \\ u_r(r, 1) &= 4r \\ u_z(r, 1) &= 4 - 2r^2 \end{aligned} \right\} \text{ for } 0 < r < 1,$$

$$\left. \begin{aligned} T(1, z) &= 3 - z^3 \\ u_r(1, z) &= 3 + z^2 \\ u_z(1, z) &= 4z - 2 \end{aligned} \right\} \text{ for } 0 < z < 1.$$

To obtain some numerical results, the boundary is discretized into N equal length elements and L evenly distributed collocation points are chosen inside the domain. The numerical results are obtained with three different sets of N and L : $(N, L) = (30, 9)$ (Set A), $(N, L) = (90, 121)$ (Set B) and $(N, L) = (270, 361)$ (Set C).

Table 5.1: A comparison of the numerical and exact values of T at selected interior points.

Point (r, z)	Set A	Set B	Set C	Exact
(0.25, 0.25)	0.193720	0.173006	0.172138	0.171875
(0.50, 0.25)	0.745310	0.735069	0.734542	0.734375
(0.75, 0.25)	1.672804	1.671976	1.671919	1.671875
(0.25, 0.50)	0.090651	0.064348	0.062965	0.062500
(0.50, 0.50)	0.640126	0.626256	0.625321	0.625000
(0.75, 0.50)	1.565350	1.562901	1.562625	1.562500
(0.25, 0.75)	-0.195026	-0.231369	-0.233598	-0.234375
(0.50, 0.75)	0.354515	0.330494	0.328720	0.328125
(0.75, 0.75)	1.278146	1.267007	1.265953	1.265625

Numerical values of T , u_r and u_z at selected interior points are compared with the exact solution of the problem in Tables 5.1, 5.2 and 5.3 respectively. The exact solution is given by

$$\begin{aligned} T(r, z) &= 3r^2 - z^3, \\ u_r(r, z) &= 3r + rz^2, \\ u_z(r, z) &= 4z - 2r^2. \end{aligned}$$

From the tables, it is obvious that the accuracy of the numerical values improves when more boundary elements and interior collocation points are employed in the boundary element calculation.

In the particular problem under consideration here, the tractions are not known a priori on the boundary. In Figures 5.1 and 5.2, the numerically computed traction components t_r and t_z on the boundary $z = 1$ are plotted against r (for $0 < r < 1$) and compared with the values of t_r and t_z calculated from the exact solution of the problem. The numerical values of t_r and t_z from Set A (that is, from the boundary element computation using $(N, L) = (30, 9)$) are rather inaccurate at boundary points very close to the sharp corner point $(r, z) = (1, 1)$. As shown in the plots for Sets B and C, the numerical values of t_r and t_z at boundary points near $(r, z) = (1, 1)$, however, converge to the exact values when more elements and interior collocation points are used in the numerical calculation.

Problem 2

Consider a hollow cylindrical solid whose thermoelastic properties are radially graded. More specifically, the solid occupies the region $r_1 < r < r_2, 0 < z < z_1$, and thermal conductivity κ , the stress-temperature coefficient β , the shear modulus μ , the Poisson's ratio ν are chosen as $\kappa = \kappa_0 r$, $\beta = \beta_0 r$, $\mu = \mu_0 r$ and $\nu = 1/10$ respectively. The reference temperature is taken to be given by $T_0 = 0$ and the internal heat generation and body force terms by $Q(r) = 0$, $F_r(r) = 0$ and $F_z(r) = 0$.

Table 5.2: A comparison of the numerical and exact values of u_r at selected interior points.

Point (r, z)	Set A	Set B	Set C	Exact
(0.25, 0.25)	0.766878	0.765905	0.765715	0.765625
(0.50, 0.25)	1.533480	1.531916	1.531472	1.53125
(0.75, 0.25)	2.300714	2.298296	2.297356	2.296875
(0.25, 0.50)	0.815198	0.813377	0.812794	0.812500
(0.50, 0.50)	1.629552	1.626573	1.625527	1.625000
(0.75, 0.50)	2.443177	2.439522	2.438174	2.437500
(0.25, 0.75)	0.893627	0.891704	0.891000	0.890625
(0.50, 0.75)	1.786779	1.783241	1.781937	1.781250
(0.75, 0.75)	2.678461	2.674354	2.672723	2.671875

Table 5.3: A comparison of the numerical and exact values of u_z at selected interior points.

Point (r, z)	Set A	Set B	Set C	Exact
(0.25, 0.25)	0.871572	0.874477	0.874852	0.875000
(0.50, 0.25)	0.496347	0.499169	0.499743	0.500000
(0.75, 0.25)	-0.129133	-0.126418	-0.125473	-0.125000
(0.25, 0.50)	1.876029	1.876178	1.875435	1.875000
(0.50, 0.50)	1.500802	1.500796	1.500297	1.500000
(0.75, 0.50)	0.874228	0.874962	0.875001	0.875000
(0.25, 0.75)	2.880382	2.877597	2.875904	2.875000
(0.50, 0.75)	2.505703	2.502466	2.500857	2.500000
(0.75, 0.75)	1.878916	1.876782	1.875625	1.875000

Figure 5.1: A graphical comparison of the numerical and exact traction component t_r on the boundary $0 < r < 1$, $z = 1$.

Figure 5.2: A graphical comparison of the numerical and exact traction component t_z on the boundary $0 < r < 1$, $z = 1$.

The boundary conditions are given by

$$\left. \begin{aligned} T(r_1, z) &= T_1 \\ T(r_2, z) &= T_2 \\ t_r(r_1, z) &= 0 \\ t_r(r_2, z) &= 0 \\ t_z(r_1, z) &= 0 \\ t_z(r_2, z) &= 0 \end{aligned} \right\} \text{ for } 0 < z < z_1$$

and

$$\left. \begin{aligned} \left. \frac{\partial T}{\partial z} \right|_{z=0} &= 0 \\ \left. \frac{\partial T}{\partial z} \right|_{z=1} &= 0 \\ t_r(r, z_1) &= 0 \\ t_r(r, z_2) &= 0 \\ u_z(r, z_1) &= 0 \\ u_z(r, z_2) &= 0 \end{aligned} \right\} \text{ for } r_1 < r < r_2,$$

where T_1 and T_2 are given constants.

The temperature and the other thermoelastic fields vary with r only. The problem may be solved analytically. The exact solution is given by

$$\begin{aligned} T(r, z) &= T_1 + (T_2 - T_1) \frac{(r_1 - r)r_2}{(r_1 - r_2)r}, \\ u_r(r, z) &= C_2 r^{-\frac{1}{2}(1 - \frac{\sqrt{5-14\nu+9\nu^2}}{-1+\nu})} + C_1 r^{-\frac{1}{2}(1 + \frac{\sqrt{5-14\nu+9\nu^2}}{-1+\nu})} \\ &\quad + \frac{r\beta_0(\nu - \frac{1}{2})}{\mu_0} \left(\frac{r_2(T_2 - T_1)}{r_1 - r_2} - T_1 \right), \\ u_z(r, z) &= 0, \end{aligned}$$

where C_1 and C_2 are constants determining by using the boundary conditions $\sigma_{rr}(r_1, z) = 0$ and $\sigma_{rr}(r_2, z) = 0$.

The boundary element solution in Section 5.4 will be checked here against the exact solution above for the particular case in which $r_1 = 1$, $r_2 = 2$, $z_1 = 1$, $T_1 = 1$, $T_2 = 2$, $\kappa_0 = 1$, $\beta_0 = 1$ and $\mu_0 = 1$. For this particular case, the constants C_1 and C_2 are given by $C_1 = -0.758832$ and $C_2 = 0.256974$.

Table 5.4: A comparison of the numerical and exact values of T , u_r and u_z at selected interior points.

Point (r, z)	Numerical (T, u_r, u_z)	Exact (T, u_r, u_z)
(1.25, 0.25)	(1.39990, 0.82146, 0.00009)	(1.40000, 0.81992, 0)
(1.50, 0.25)	(1.66664, 0.98261, 0.00006)	(1.66667, 0.98108, 0)
(1.75, 0.25)	(1.85717, 1.16617, 0.00004)	(1.85714, 1.16460, 0)
(1.25, 0.50)	(1.39995, 0.82148, 0.00000)	(1.40000, 0.81992, 0)
(1.50, 0.50)	(1.66664, 0.98262, 0.00000)	(1.66667, 0.98108, 0)
(1.75, 0.50)	(1.85715, 1.16618, 0.00000)	(1.85714, 1.16460, 0)
(1.25, 0.75)	(1.39990, 0.82146, -0.00009)	(1.40000, 0.81992, 0)
(1.50, 0.75)	(1.66664, 0.98261, -0.00006)	(1.66667, 0.98108, 0)
(1.75, 0.75)	(1.85717, 1.16617, -0.00004)	(1.85714, 1.16460, 0)

The boundary Γ consists of four straight lines. For the boundary element solution, each straight line is discretized into N_0 elements, so the total number of element is $N = 4N_0$. The interior collocation points are chosen at point $(1 + i/(L_0 + 1), j/(L_0 + 1))$ for $i, j = 1, 2, \dots, L_0$, therefore, the total number of interior collocation points is given by $L = L_0^2$. To obtain some numerical results, $N_0 = 40$ and $L_0 = 15$ are used.

In Table 5.4, the numerical values of the temperature T and the displacement components u_r and u_z are obtained and compared with the values computed from the exact solution. There is a good agreement between the numerical and exact values.

Figure 5.3 shows plots of the numerical and the exact temperature against r for $1 < r < 2$ at $z = 0.50$. (Note that the temperature is independent of z .) The two plots for the numerical and the exact temperature are visually almost indistinguishable.

The interior thermoelastic stresses $\sigma_{rr}, \sigma_{rz}, \sigma_{zz}$ and $\sigma_{\theta\theta}$ can be computed numerically using (2.5) as shown in Chapter 2, explicitly for axisymmetric

Figure 5.3: Plots of the numerical and the exact temperature against r for $1 < r < 2$ at $z = 0.50$.

coordinate

$$\begin{aligned}
 \sigma_{rr} &= 2\mu\left(\frac{\partial u_r}{\partial r} + \frac{\nu}{1-2\nu}\left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right]\right) - \beta(T - T_0), \\
 \sigma_{zz} &= 2\mu\left(\frac{\partial u_z}{\partial z} + \frac{\nu}{1-2\nu}\left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right]\right) - \beta(T - T_0), \\
 \sigma_{\theta\theta} &= 2\mu\left(\frac{u_r}{r} + \frac{\nu}{1-2\nu}\left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right]\right) - \beta(T - T_0), \\
 \sigma_{rz} &= \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right),
 \end{aligned}$$

and (5.36). For the particular problem here, σ_{rz} vanishes throughout the solution domain since the displacement components u_r and u_z are functions of

Figure 5.4: Plots of the numerical and the exact thermoelastic stresses against r for $1 < r < 2$ at $z = 0.50$.

r alone. Plots of the numerically computed and the exact σ_{rr}, σ_{zz} and $\sigma_{\theta\theta}$ against r for $1 < r < 2$ at $z = 0.50$ are given in Figure 5.4. The numerical and the exact values of the thermoelastic stresses are in good agreement with each other.

Problem 3

Consider a solid cylinder which occupies the region $0 < r < H, 0 < z < H$, where H is a positive constant. The thermal conductivity, shear modulus and stress-temperature coefficient of the cylinder vary exponentially along the z

axis as given respectively by $\kappa = \kappa_0 \exp(-hz)$, $\mu = \mu_0 \exp(-hz)$ and $\beta = \beta_0 \exp(-hz)$, where κ_0, μ_0, β_0 and h are positive constants.

There is no heat generation and body forces in the cylinder, that is, $Q = 0$ and $F_r = F_z = 0$. A uniform heat flux q_0 enters the solid through the part of the boundary where $0 < r < H/2$, $z = 0$. The end of the cylinder at $z = H$ is maintained at uniform temperature T_0 (that is, the constant reference temperature at which the body does not experience any thermally induced stress) and is attached to a rigid wall (so that $u_r = u_z = 0$ on $0 < r < H$, $z = H$). The parts of the boundary given by $H/2 < r < H$, $z = 0$ and $0 < z < H$, $r = H$ are thermally insulated. Apart from the fixed end at $z = H$, the boundary of the cylinder is free of traction.

For this problem, if $\beta_0 = 0$, the cylinder is undeformed, that is, the deformation of the cylinder is purely due to thermal effects. Taking $\nu = 0.3$, $hH = 0.50$ and $(q_0 H)/(\kappa_0 T_0) = 0.30$, we examine the effects of varying $\beta_0 T_0/\mu_0$ on the displacements and stresses on different parts of the boundary of the cylinder. For the dual-reciprocity boundary element method, the boundary of the cylinder is discretized into 270 equal length element and 361 well distributed interior collocation points are chosen.

The displacement components u_r and u_z are not known a priori at the end of the cylinder at $z = 0$. In Figures 5.5 and 5.6, u_r/H and u_z/H at $z = 0$ are plotted against r/H ($0 < r/H < 1$) for selected values of $\beta_0 T_0/\mu_0$. From the figures, it is obvious that increasing $\beta_0 T_0/\mu_0$ has the effect of increasing the magnitudes of u_r/H and u_z/H . The effect of $\beta_0 T_0/\mu_0$ on u_r appears to be more pronounced at larger distance from the center $(0, 0)$ of the cylinder while the effect on u_z seems to be greater nearer to $(0, 0)$.

Plots of u_r and u_z on the traction free surface $r = H$ against z/H ($0 < z/H < 1$) are given in Figures 5.7 and 5.8 for selected values of $\beta_0 T_0/\mu_0$. As

Figure 5.5: Plots of u_r/H at $z = 0$ against r/H for selected values of $\beta_0 T_0 / \mu_0$.

Figure 5.6: Plots of u_z/H at $z = 0$ against r/H for selected values of $\beta_0 T_0 / \mu_0$.

Figure 5.7: Plots of u_r/H at $r = H$ against z/H for selected values of $\beta_0 T_0 / \mu_0$.

Figure 5.8: Plots of u_z/H at $r = H$ against z/H for selected values of $\beta_0 T_0 / \mu_0$.

before, the magnitudes of u_r/H and u_z/H increase with increasing $\beta_0 T_0/\mu_0$. The displacement component u_r decreases in magnitude as z/H increases and the component u_z changes in sign near $z/H = 0.40$. As expected, both components u_r and u_z are extremely small near the fixed end of the cylinder at $z = H$.

Figure 5.9: Plots of σ_{rr}/μ_0 at $z = 0$ against r/H for selected values of $\beta_0 T_0/\mu_0$.

For selected values of $\beta_0 T_0/\mu_0$, the radial stress σ_{rr}/μ_0 and the hoop stress $\sigma_{\theta\theta}/\mu_0$ on $z = 0$ are plotted against r/H ($0 < r/H < 1$) in Figures 5.9 and 5.10 respectively, and the longitudinal stress σ_{zz}/μ_0 and the hoop stress $\sigma_{\theta\theta}/\mu_0$ on $r = H$ are plotted against z/H ($0 < z/H < 1$) in Figures 5.11 and 5.12 respectively. Increasing $\beta_0 T_0/\mu_0$ appears to increase the magnitudes of the stresses. At the end $z = 0$ of the cylinder, the radial stress σ_{rr}/μ_0 has larger magnitude at points nearer to the center where heating occurs. The magnitude of the longitudinal stress σ_{zz}/μ_0 on the cylindrical surface $r = H$ is larger as

Figure 5.10: Plots of $\sigma_{\theta\theta}/\mu_0$ at $z = 0$ against r/H for selected values of $\beta_0 T_0/\mu_0$.

Figure 5.11: Plots of σ_{zz}/μ_0 at $r = H$ against z/H for selected values of $\beta_0 T_0/\mu_0$.

Figure 5.12: Plots of $\sigma_{\theta\theta}/\mu_0$ at $r = H$ against z/H for selected values of $\beta_0 T_0/\mu_0$.

z/H gets closer to 1, that is, the magnitude of the stress is larger at points nearer to the fixed end of the cylinder.

In order to examine the effects of the grading parameter h on the state of stress in the solid, stresses along boundary $0 < r < H$, $z = 0$ are plotted in Figure 5.13 and 5.14 for three selected values of h ($hH = 0$, $hH = 0.5$ and $hH = 1$). Note that $h = 0$ corresponds to the case of a homogeneous solid whose material properties do not change from point to point in space. From Figure 5.14, it appears that the magnitude of the stress $\sigma_{\theta\theta}$ at a fixed point on the boundary $0 < r < H$, $z = 0$ decreases as h increases. The same effect is also observed in Figure 5.13 for the stress σ_{rr} on $0 < r < r_0$, $z = 0$, where r_0/H is approximately found to be 0.77. The stress σ_{rr} at a fixed point on $r_0 < r < H$, $z = 0$, however, increases as h increases.

Figure 5.13: Plots of σ_{rr}/μ_0 at $z = 0$ against r/H at $\beta_0 T_0/\mu_0 = 0.02$ for selected values of hH .

Figure 5.14: Plots of $\sigma_{\theta\theta}/\mu_0$ at $z = 0$ against r/H at $\beta_0 T_0/\mu_0 = 0.02$ for selected values of hH .

5.6 Summary

A dual-reciprocity boundary element procedure is implemented for calculating axisymmetric thermoelastostatic fields in nonhomogeneous solids with material properties that vary continuously from point to point in space. New interpolating functions that are bounded but in relatively simple elementary forms are used in the dual-reciprocity method for treating domain integrals which appear in the integral formulation of the thermoelastic problem under consideration.

Numerical results for specific test problems with known analytical closed-form solutions indicate that the temperature, displacement and stress fields can be accurately computed by the proposed dual-reciprocity boundary element approach.

Chapter 6

Axisymmetric Thermoelastodynamic Deformations in Nonhomogeneous Solids

6.1 Introduction

In the thermoelastostatic problems considered in Chapter 5, the temperature field can be determined independent of the displacement and stress fields in the solid. Thus, in solving the momentum equations for the displacement and stress fields, the temperature is regarded as known and incorporated into the body force terms. Nevertheless, for time-dependent problems, it may not be desirable to ignore the interdependence between the temperature and the displacement and stress fields. A fully coupled thermoelastic problem should be considered.

In a thermoelastodynamic problem, Dargush and Banerjee [31] (thermoelasticity in homogeneous material) used time-dependent fundamental solution to derive boundary-only integral equation. On the other hand, using fundamental solution of time-independent and homogeneous momentum equation, the time-

dependent parameters are present in the governing integral equation. The time derivatives in the integral equation can be treated using time-stepping scheme or Laplace transformation for boundary element analysis (such as Kögl and Gaul [53] used time-stepping scheme and Suh and Tosaka [86] used Laplace transformation).

The current problem deals with axisymmetric coupled thermoelastodynamic problem involving nonhomogeneous material. The governing partial differential equations are first transformed into integral equations in Laplace transform domain. Boundary integral and domain integrals are involved. The domain integrals can be approximated to line integral using dual-reciprocity method together with specific interpolating functions. After the unknowns are obtained, the solutions can be converted back to the results in physical domain numerically using Stehfest's algorithm.

6.2 Statement of Problem

In the coupled thermoelastodynamic problem, the temperature T , displacements u_r, u_z satisfy the partial differential equations given by (2.10) as shown in Chapter 2. Writing the equations in axisymmetric coordinates (see Figure 2.1) in rz plane, the governing equations for thermoelastodynamic problems are

$$\underline{\nabla} \cdot (\kappa \underline{\nabla} T) + Q = \beta T_0 \frac{\partial}{\partial t} \left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right] + \rho c \frac{\partial T}{\partial t}, \quad (6.1)$$

and

$$\begin{aligned} & \nabla_{\text{axis}}^2 u_r - \frac{u_r}{r^2} + \frac{1}{1-2\nu} \frac{\partial}{\partial r} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) \\ &= \frac{1}{\mu} \left\{ \beta \frac{\partial T}{\partial r} + \frac{\partial \beta}{\partial r} (T - T_0) - F_r - \frac{\partial \mu}{\partial z} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \right\} \end{aligned}$$

$$\begin{aligned}
 & -2\frac{\partial\mu}{\partial r}\left[\frac{\partial u_r}{\partial r} + \frac{\nu}{1-2\nu}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right)\right] \\
 & + \rho\frac{\partial^2 u_r}{\partial t^2}\}, \\
 & \nabla_{\text{axis}}^2 u_z + \frac{1}{1-2\nu}\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) \\
 = & \frac{1}{\mu}\left\{\beta\frac{\partial T}{\partial z} + \frac{\partial\beta}{\partial z}(T - T_0) - F_z - \frac{\partial\mu}{\partial r}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\right. \\
 & \left.- 2\frac{\partial\mu}{\partial z}\left[\frac{\partial u_z}{\partial z} + \frac{\nu}{1-2\nu}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right)\right]\right. \\
 & \left.+ \rho\frac{\partial^2 u_z}{\partial t^2}\right\}, \tag{6.2}
 \end{aligned}$$

where T_0 is a constant reference temperature at which the body does not experience any thermally induced stress, the κ , β , ρ , c , ν , μ are thermal conductivity, stress-temperature coefficient, density, specific heat capacity, Poisson's ratio and shear moduli respectively, F_r , F_z are body forces in r and z direction and Q is internal heat generation. Material properties κ , β , ρ , c , μ are functionally graded in spatial coordinates r and z and the Poisson's ratio is a constant such that $-1 < \nu < 1/2$. In axisymmetric coordinate, the temperature, internal heat generation, displacements and body forces are dependent on time and vary with spatial coordinates, they are respectively given by $T(r, z, t)$, $Q(r, z, t)$, $u_r(r, z, t)$, $u_z(r, z, t)$, $F_r(r, z, t)$ and $F_z(r, z, t)$.

The problem here is to solve (6.1) and (6.2) subject to the initial conditions as given by

$$\begin{aligned}
 T(r, z, 0) &= f_0(r, z) \quad \text{in } \Omega, \\
 u_J(r, z, 0) &= g_{J0}(r, z) \quad \text{in } \Omega, \\
 \left.\frac{\partial u_J(r, z, t)}{\partial t}\right|_{t=0} &= g_{J1}(r, z) \quad \text{in } \Omega, \tag{6.3}
 \end{aligned}$$

and the boundary conditions that either temperature T or heat flux q and any two of the four components u_r , u_z , t_r , t_z are suitably specified at each point of

boundary Γ .

6.3 Boundary-domain Integral Formulations

6.3.1 Heat Equation

The governing heat equation in (6.1) may be rewritten as

$$\nabla_{\text{axis}}^2(\sqrt{\kappa}T) = -\frac{Q}{\sqrt{\kappa}} + T \cdot \nabla_{\text{axis}}^2\sqrt{\kappa} + \frac{\rho c}{\sqrt{\kappa}} \frac{\partial T}{\partial t} + \frac{\beta T_0}{\sqrt{\kappa}} \frac{\partial}{\partial t} \left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right]. \quad (6.4)$$

Following the procedures in Chapter 2, the boundary-domain integral equation for (6.4) is given by

$$\begin{aligned} & \gamma(r_0, z_0) \sqrt{\kappa(r_0, z_0)} T(r_0, z_0, t) \\ = & \int_{\Gamma} \left\{ T(r, z, t) [\sqrt{\kappa(r, z)} G_1(r, z; r_0, z_0; n_r, n_z) - \frac{\partial \sqrt{\kappa(r, z)}}{\partial n} G_0(r, z; r_0, z_0)] \right. \\ & - \sqrt{\kappa(r, z)} G_0(r, z; r_0, z_0) q(r, z, t; n_r, n_z) \left. \right\} r ds(r, z) \\ & + \iint_{\Omega} G_0(r, z; r_0, z_0) \left[-\frac{Q(r, z, t)}{\sqrt{\kappa(r, z)}} + T(r, z, t) \cdot \nabla_{\text{axis}}^2(\sqrt{\kappa(r, z)}) \right. \\ & + \frac{\rho(r, z)c(r, z)}{\sqrt{\kappa(r, z)}} \frac{\partial T(r, z, t)}{\partial t} \\ & + \frac{\beta(r, z)T_0}{\sqrt{\kappa(r, z)}} \frac{\partial}{\partial t} \left[\frac{\partial u_r(r, z, t)}{\partial r} + \frac{u_r(r, z, t)}{r} + \frac{\partial u_z(r, z, t)}{\partial z} \right] \left. \right] r dr dz \\ & \text{for } (r_0, z_0) \in \Omega \cup \Gamma, \end{aligned} \quad (6.5)$$

where $G_0(r, z; r_0, z_0)$, $G_1(r, z; r_0, z_0; n_r, n_z)$ are fundamental solutions as shown in (2.17) and (2.18) in Chapter 2.

6.3.2 Thermoelasticity

As explained in Chapter 2, the partial differential equations in (6.2) can be used to derive the boundary-domain integral equations

$$\begin{aligned}
 \gamma(r_0, z_0)u_K(r_0, z_0, t) = & \int_{\Gamma} (\Phi_{JK}(r, z; r_0, z_0)p_J(r, z, t; n_r, n_z) \\
 & - \Psi_{JK}(r, z; r_0, z_0; n_r, n_z)u_J(r, z, t))rds(r, z) \\
 & + \iint_{\Omega} \frac{1}{\mu(r, z)}\Phi_{JK}(r, z; r_0, z_0)\{-\beta(r, z)\frac{\partial}{\partial x_J}[T(r, z, t)] \\
 & - \frac{\partial}{\partial x_J}[\beta(r, z)](T(r, z, t) - T_0) + F_J(r, z, t) \\
 & + \frac{\partial}{\partial x_J}[\mu(r, z, t)]\frac{2\nu}{(1-2\nu)r}[u_r(r, z, t)] \\
 & + X_{JN}(r, z)\frac{\partial}{\partial z}[u_N(r, z, t)] + Y_{JN}(r, z)\frac{\partial}{\partial r}[u_N(r, z, t)] \\
 & - \rho(r, z)\frac{\partial^2 u_K(r, z, t)}{\partial t^2}\}rdrdz \\
 & \text{for } (r_0, z_0) \in \Omega \cup \Gamma,
 \end{aligned} \tag{6.6}$$

where $x_r = r, x_z = z, \Phi_{JK}(r, z; r_0, z_0)$ and $\Psi_{JK}(r, z; r_0, z_0; n_r, n_z)$ are the fundamental solutions as given in (2.27) and (2.28) in Chapter 2, $p_J(r, z, t; n_r, n_z)$ are given by

$$\begin{aligned}
 p_r(r, z, t; n_r, n_z) = & 2\left(\frac{\partial u_r}{\partial r} + \frac{\nu}{1-2\nu}\left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right]\right)n_r(r, z) \\
 & + \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)n_z(r, z), \\
 p_z(r, z, t; n_r, n_z) = & \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)n_r(r, z) \\
 & + 2\left(\frac{\partial u_z}{\partial z} + \frac{\nu}{1-2\nu}\left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right]\right)n_z(r, z), \tag{6.7}
 \end{aligned}$$

and

$$\begin{aligned}
 X_{rr}(r, z) &= \frac{\partial \mu(r, z)}{\partial z}, \quad X_{rz}(r, z) = \frac{2\nu}{1-2\nu} \frac{\partial \mu(r, z)}{\partial r}, \\
 X_{zr}(r, z) &= \frac{\partial \mu(r, z)}{\partial r}, \quad X_{zz}(r, z) = \frac{\partial \mu(r, z)}{\partial z} \frac{2(1-\nu)}{1-2\nu}, \\
 Y_{rr}(r, z) &= \frac{\partial \mu(r, z)}{\partial r} \frac{2(1-\nu)}{1-2\nu}, \quad Y_{rz}(r, z) = \frac{\partial \mu(r, z)}{\partial z}, \\
 Y_{zr}(r, z) &= \frac{2\nu}{1-2\nu} \frac{\partial \mu(r, z)}{\partial z}, \quad Y_{zz}(r, z) = \frac{\partial \mu(r, z)}{\partial r}.
 \end{aligned} \tag{6.8}$$

The traction $t_J(r, z, t; n_r, n_z)$ is related to $p_J(r, z, t; n_r, n_z)$ by

$$\begin{aligned}
 t_J(r, z, t; n_r, n_z) &= \mu(r, z) p_J(r, z, t; n_r, n_z) \\
 &\quad - \beta(r, z) [T(r, z, t) - T_0] \delta_{JL} n_L(r, z),
 \end{aligned} \tag{6.9}$$

where δ_{JN} is Kronecker-delta which gives the value 1 when $J = N$ and else is 0.

6.4 Dual-reciprocity Boundary Element Procedures

6.4.1 Integral Equations in Laplace Transform Domain

Together with the initial conditions in (6.3), the Laplace transformation is applied on the integral equations (6.5) and (6.6) to obtain

$$\begin{aligned}
 &\gamma(r_0, z_0) \sqrt{\kappa(r_0, z_0)} \tilde{T}(r_0, z_0, p) \\
 &= \int_{\Gamma} \left\{ \tilde{T}(r, z, p) [\sqrt{\kappa(r, z)} G_1(r, z; r_0, z_0; n_r, n_z) - \frac{\partial \sqrt{\kappa(r, z)}}{\partial n} G_0(r, z; r_0, z_0)] \right. \\
 &\quad \left. - \sqrt{\kappa(r, z)} G_0(r, z; r_0, z_0) \tilde{q}(r, z, p; n_r, n_z) \right\} r ds(r, z)
 \end{aligned}$$

$$\begin{aligned}
 & + \iint_{\Omega} G_0(r, z; r_0, z_0) \left[-\frac{\tilde{Q}(r, z, p)}{\sqrt{\kappa(r, z)}} + \tilde{T}(r, z, p) \cdot \nabla_{\text{axis}}^2(\sqrt{\kappa(r, z)}) \right. \\
 & + \frac{\rho(r, z)c(r, z)}{\sqrt{\kappa(r, z)}} [p\tilde{T}(r, z, p) - f_0(r, z)] \\
 & + \frac{\beta(r, z)T_0}{\sqrt{\kappa(r, z)}} \left[\frac{\partial}{\partial r}(p\tilde{u}_r(r, z, p) - g_{r0}(r, z)) + \frac{1}{r}(p\tilde{u}_r(r, z, p) - g_{r0}(r, z)) \right. \\
 & \left. + \frac{\partial}{\partial z}((p\tilde{u}_z(r, z, p) - g_{z0}(r, z))) \right] r dr dz \\
 & \text{for } (r_0, z_0) \in \Omega \cup \Gamma,
 \end{aligned} \tag{6.10}$$

and

$$\begin{aligned}
 & \gamma(r_0, z_0)\tilde{u}_K(r_0, z_0, p) \\
 = & \int_{\Gamma} (\Phi_{JK}(r, z; r_0, z_0)\tilde{p}_J(r, z, p; n_r, n_z) \\
 & - \Psi_{JK}(r, z; r_0, z_0; n_r, n_z)\tilde{u}_J(r, z, p)) r ds(r, z) \\
 & + \iint_{\Omega} \frac{1}{\mu(r, z)} \Phi_{JK}(r, z; r_0, z_0) \left\{ -\beta(r, z) \frac{\partial}{\partial x_J} [\tilde{T}(r, z, p)] \right. \\
 & - \frac{\partial}{\partial x_J} [\beta(r, z)] (\tilde{T}(r, z, p) - \frac{T_0}{p}) + \tilde{F}_J(r, z, p) \\
 & + \frac{\partial}{\partial x_J} [\mu(r, z)] \frac{2\nu}{(1-2\nu)r} \tilde{u}_r(r, z, p) \\
 & + X_{JN}(r, z) \frac{\partial}{\partial z} [\tilde{u}_N(r, z, p)] + Y_{JN}(r, z) \frac{\partial}{\partial r} [\tilde{u}_N(r, z, p)] \\
 & \left. - \rho(r, z) [p^2 \tilde{u}_K(r, z, p) - pg_{K0}(r, z) - g_{K1}(r, z)] \right\} r dr dz \\
 & \text{for } (r_0, z_0) \in \Omega \cup \Gamma,
 \end{aligned} \tag{6.11}$$

where p is the Laplace transform parameter and $\tilde{T}(r, z, p)$, $\tilde{q}(r, z, p; n_r, n_z)$, $\tilde{Q}(r, z, p)$, $\tilde{u}_J(r, z, p)$, $\tilde{p}_J(r, z, p; n_r, n_z)$ and $\tilde{F}_J(r, z, p)$ are the Laplace transform functions defined as

$$\tilde{T}(r, z, p) = \int_0^{\infty} T(r, z, t) \exp(-pt) dt,$$

$$\begin{aligned}
 \tilde{q}(r, z, p; n_r, n_z) &= \int_0^\infty q(r, z, t; n_r, n_z) \exp(-pt) dt, \\
 \tilde{Q}(r, z, p) &= \int_0^\infty Q(r, z, t) \exp(-pt) dt, \\
 \tilde{u}_J(r, z, p) &= \int_0^\infty u_J(r, z, t) \exp(-pt) dt, \\
 \tilde{p}_J(r, z, p; n_r, n_z) &= \int_0^\infty p_J(r, z, t; n_r, n_z) \exp(-pt) dt, \\
 \tilde{F}_J(r, z, p) &= \int_0^\infty F_J(r, z, t) \exp(-pt) dt.
 \end{aligned} \tag{6.12}$$

6.4.2 Boundary Approximations

Similar to the approach in Chapter 3 and 4, discontinuous linear elements are used here. The boundary curve Γ is first discretized into N straight line elements denoted by $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(N-1)}$ and $\Gamma^{(N)}$, and two points on the element $\Gamma^{(k)}$, are chosen as

$$\begin{aligned}
 (r_0^{(k)}, z_0^{(k)}) &= (r^{(k)}, z^{(k)}) + \tau(r^{(k+1)} - r^{(k)}, z^{(k+1)} - z^{(k)}), \\
 (r_0^{(N+k)}, z_0^{(N+k)}) &= (r^{(k)}, z^{(k)}) \\
 &\quad + (1 - \tau)(r^{(k+1)} - r^{(k)}, z^{(k+1)} - z^{(k)}),
 \end{aligned} \tag{6.13}$$

where τ has value $0 < \tau < 1/2$.

The functions $\tilde{T}(r, z, p)$ at $(r_0^{(k)}, z_0^{(k)})$ and $(r_0^{(N+k)}, z_0^{(N+k)})$ is denoted by $T^{(k)}(p)$ and $T^{(k+N)}(p)$ respectively, and $\tilde{T}(r, z, p)$ is approximated as

$$\begin{aligned}
 \tilde{T}(r, z, p) &= \frac{[s^{(k)}(r, z) - (1 - \tau)\ell^{(k)}]T^{(k)}(p) - [s^{(k)}(r, z) - \tau\ell^{(k)}]T^{(N+k)}(p)}{(2\tau - 1)\ell^{(k)}} \\
 &\quad \text{for } (r, z) \in \Gamma^{(k)},
 \end{aligned} \tag{6.14}$$

where $s^{(k)}(r, z) = \sqrt{(r - r^{(k)})^2 + (z - z^{(k)})^2}$ and $\ell^{(k)} = s^{(k)}(r^{(k+1)}, z^{(k+1)})$. The approximation in (6.14) is also applied to functions $\tilde{q}(r, z, p; n_r, n_z)$, $\tilde{u}_J(r, z, p)$ and $\tilde{p}_J(r, z, p; n_r, n_z)$. The functions at points $(r_0^{(k)}, z_0^{(k)})$ and $(r_0^{(k+N)}, z_0^{(k+N)})$ are respectively given by $q^{(k)}(p)$ and $q^{(k+N)}(p)$, $u_J^{(k)}(p)$ and $u_J^{(k+N)}(p)$ and $p_J^{(k)}(p)$ and $p_J^{(k+N)}(p)$.

Therefore, the boundary integral in (6.10) becomes

$$\begin{aligned}
 & \int_{\Gamma} \left\{ \tilde{T}(r, z, p) [\sqrt{\kappa(r, z)} G_1(r, z; r_0, z_0; n_r, n_z) - \frac{\partial \sqrt{\kappa(r, z)}}{\partial n} G_0(r, z; r_0, z_0)] \right. \\
 & \quad \left. - \sqrt{\kappa(r, z)} G_0(r, z; r_0, z_0) \tilde{q}(r, z, p; n_r, n_z) \right\} r ds(r, z) \\
 = & \sum_{k=1}^N \frac{1}{(2\tau - 1)\ell^{(k)}} \{ (\mathcal{F}_4^{(k)}(r_0, z_0) \\
 & \quad - (1 - \tau)\ell^{(k)}\mathcal{F}_2^{(k)}(r_0, z_0)) \sqrt{\kappa(r_0^{(k)}, z_0^{(k)})} T^{(k)}(p) \\
 & \quad - (\mathcal{F}_4^{(k)}(r_0, z_0) - \tau\ell^{(k)}\mathcal{F}_2^{(k)}(r_0, z_0)) \sqrt{\kappa(r_0^{(k+N)}, z_0^{(k+N)})} T^{(k+N)}(p) \\
 & \quad - (\mathcal{F}_3^{(k)}(r_0, z_0) - (1 - \tau)\ell^{(k)}\mathcal{F}_1^{(k)}(r_0, z_0)) \\
 & \quad \times \left[\frac{\partial \sqrt{\kappa(r, z)}}{\partial n} \right]_{(r, z) = (r_0^{(k)}, z_0^{(k)})} T^{(k)}(p) + \sqrt{\kappa(r_0^{(k)}, z_0^{(k)})} q^{(k)}(p) \\
 & \quad + (\mathcal{F}_3^{(k)}(r_0, z_0) - \tau\ell^{(k)}\mathcal{F}_1^{(k)}(r_0, z_0)) \\
 & \quad \times \left[\frac{\partial \sqrt{\kappa(r, z)}}{\partial n} \right]_{(r, z) = (r_0^{(k+N)}, z_0^{(k+N)})} T^{(k+N)}(p) \\
 & \quad + \sqrt{\kappa(r_0^{(k+N)}, z_0^{(k+N)})} q^{(k+N)}(p) \} \}, \tag{6.15}
 \end{aligned}$$

where $\mathcal{F}_1^{(k)}, \mathcal{F}_2^{(k)}, \mathcal{F}_3^{(k)}, \mathcal{F}_4^{(k)}$ are as defined in (3.20) in Chapter 3, which are

$$\begin{aligned}
 \mathcal{F}_1^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} G_0(r, z; r_0, z_0) r ds(r, z), \\
 \mathcal{F}_2^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} G_1(r, z; r_0, z_0) r ds(r, z),
 \end{aligned}$$

$$\begin{aligned}\mathcal{F}_3^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} s(r, z) G_0(r, z; r_0, z_0) r ds(r, z), \\ \mathcal{F}_4^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} s(r, z) G_1(r, z; r_0, z_0) r ds(r, z).\end{aligned}$$

and the boundary integrals in (6.11) are

$$\begin{aligned}& \int_{\Gamma} (\Phi_{JK}(r, z; r_0, z_0) \tilde{p}_J(r, z, p; n_r, n_z) \\ & - \Psi_{JK}(r, z; r_0, z_0; n_r, n_z) \tilde{u}_J(r, z, p)) r ds(r, z) \\ = & \sum_{k=1}^N \frac{1}{(2\tau - 1)\ell^{(k)}} \{ (\bar{\mathcal{G}}_{JK}^{(k)}(r_0, z_0) - (1 - \tau)\ell^{(k)}\mathcal{G}_{JK}^{(k)}(r_0, z_0)) p_J^{(k)}(p) \\ & - (\bar{\mathcal{G}}_{JK}^{(k)}(r_0, z_0) - \tau\ell^{(k)}\mathcal{G}_{JK}^{(k)}(r_0, z_0)) p_J^{(k+N)}(p) \\ & - (\bar{\mathcal{F}}_{JK}^{(k)}(r_0, z_0) - (1 - \tau)\ell^{(k)}\mathcal{F}_{JK}^{(k)}(r_0, z_0)) u_J^{(k)}(p) \\ & + (\bar{\mathcal{F}}_{JK}^{(k)}(r_0, z_0) - \tau\ell^{(k)}\mathcal{F}_{JK}^{(k)}(r_0, z_0)) u_J^{(k+N)}(p) \},\end{aligned}\tag{6.16}$$

where

$$\begin{aligned}\mathcal{G}_{JK}(r_0, z_0) &= \int_{\Gamma^{(k)}} \Phi_{JK}(r, z; r_0, z_0) r ds(r, z), \\ \bar{\mathcal{G}}_{JK}^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} s(r, z) \Phi_{JK}(r, z; r_0, z_0) r ds(r, z), \\ \mathcal{F}_{JK}^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} \Psi_{JK}(r, z; r_0, z_0; n_r, n_z) r ds(r, z), \\ \bar{\mathcal{F}}_{JK}^{(k)}(r_0, z_0) &= \int_{\Gamma^{(k)}} s(r, z) \Psi_{JK}(r, z; r_0, z_0; n_r, n_z) r ds(r, z).\end{aligned}\tag{6.17}$$

6.4.3 Dual-reciprocity Approximation of Domain Integrals

The domain integrals in (6.10) and (6.11) can be treated using the dual-reciprocity method. By using selected collocation points in $\Omega \cup \Gamma$, the do-

main integral can be approximated into line integral. Select L well-spaced out collocation points in the interior of solution domain Ω , which are denoted by $(r_0^{(2N+1)}, z_0^{(2N+1)}), (r_0^{(2N+2)}, z_0^{(2N+2)}), \dots, (r_0^{(2N+L-1)}, z_0^{(2N+L-1)}), (r_0^{(2N+L)}, z_0^{(2N+L)})$. The boundary points $(r_0^{(k)}, z_0^{(k)})$ and $(r_0^{(k+N)}, z_0^{(k+N)})$ at element $\Gamma^{(k)}$ (for $k = 1, 2, \dots, N$) are also used as collocation points together with those selected interior points. Therefore, there are total of $2N + L$ collocation points.

In heat equation, let

$$\begin{aligned}
 f(r, z, p) = & -\frac{\tilde{Q}(r, z, p)}{\sqrt{\kappa(r, z)}} + \tilde{T}(r, z, p) \cdot \nabla_{\text{axis}}^2(\sqrt{\kappa(r, z)}) \\
 & + \frac{\rho(r, z)c(r, z)}{\sqrt{\kappa(r, z)}} [p\tilde{T}(r, z, p) - f_0(r, z)] \\
 & + \frac{\beta(r, z)T_0}{\sqrt{\kappa(r, z)}} \left[\frac{\partial}{\partial r}(p\tilde{u}_r(r, z, p) - g_{r0}(r, z)) \right. \\
 & + \frac{1}{r}(p\tilde{u}_r(r, z, p) - g_{r0}(r, z)) \\
 & \left. + \frac{\partial}{\partial z}(p\tilde{u}_z(r, z, p) - g_{z0}(r, z)) \right], \tag{6.18}
 \end{aligned}$$

and then the function $f(r, z, p)$ can be approximated as

$$f(r, z, p) \simeq \sum_{n=1}^{2N+L} \alpha^{(n)}(p) \varphi^{(n)}(r, z) \text{ for } (r, z) \in \Omega, \tag{6.19}$$

where $\alpha^{(n)}(p)$ are constant coefficients and $\varphi^{(n)}(r, z)$ is interpolating function. The interpolating function needs to satisfy

$$\nabla_{\text{axis}}^2 \chi^{(n)}(r, z) = \varphi^{(n)}(r, z) \tag{6.20}$$

and the same interpolating functions which are used in Chapter 3, 4 and 5,

that is

$$\begin{aligned}\chi^{(n)}(r, z) &= \frac{1}{9}\{[\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^3 + [\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^3\}. \\ \varphi^{(n)}(r, z) &= [\frac{4}{3} - \frac{r_0^{(n)}}{3r}]\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)}) \\ &\quad + [\frac{4}{3} + \frac{r_0^{(n)}}{3r}]\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)}),\end{aligned}\quad (6.21)$$

are again to be adopted here.

Therefore, the domain integral in (6.10) becomes

$$\iint_{\Omega} G_0(r, z; r_0, z_0) f(r, z, p) \simeq \sum_{n=1}^{2N+L} \alpha^{(n)}(p) W^{(n)}(r_0, z_0), \quad (6.22)$$

where

$$\begin{aligned}W^{(n)}(r_0, z_0) &= \gamma(r_0, z_0) \chi^{(n)}(r_0, z_0) + \int_{\Gamma} [G_0(r, z; r_0, z_0) \frac{\partial}{\partial n} [\chi^{(n)}(r, z)] \\ &\quad - G_1(r, z; r_0, z_0; n_r, n_z) \chi^{(n)}(r, z)] r ds(r, z),\end{aligned}\quad (6.23)$$

and

$$\frac{\partial}{\partial n} [\chi^{(n)}(r, z)] = n_r(r, z) \frac{\partial}{\partial r} [\chi^{(n)}(r, z)] + n_z(r, z) \frac{\partial}{\partial z} [\chi^{(n)}(r, z)]. \quad (6.24)$$

Let $(r, z) = (r_0^{(k)}, z_0^{(k)})$ ($k = 1, 2, \dots, 2N + L$) in (6.19), the coefficient $\alpha^{(n)}(p)$ can be obtained by solving the system of $2N + L$ algebraic equations given by

$$\sum_{n=1}^{2N+L} \alpha^{(n)}(p) \varphi^{(n)}(r_0^{(k)}, z_0^{(k)}) = f(r_0^{(k)}, z_0^{(k)}, p) \text{ for } k = 1, 2, \dots, 2N + L. \quad (6.25)$$

Similarly for thermoelastic equations, let

$$\begin{aligned}
 h_J(r, z, p) = & \frac{1}{\mu(r, z)} \left\{ -\beta(r, z) \frac{\partial}{\partial x_J} [\tilde{T}(r, z, p)] \right. \\
 & - \frac{\partial}{\partial x_J} [\beta(r, z)] (\tilde{T}(r, z, p) - \frac{T_0}{p}) + \tilde{F}_J(r, z, p) \\
 & + \frac{\partial}{\partial x_J} [\mu(r, z)] \frac{2\nu}{(1-2\nu)r} \tilde{u}_r(r, z, p) \\
 & + X_{JN}(r, z) \frac{\partial}{\partial z} [\tilde{u}_N(r, z, p)] + Y_{JN}(r, z) \frac{\partial}{\partial r} [\tilde{u}_N(r, z, p)] \\
 & \left. - \rho(r, z) [p^2 \tilde{u}_K(r, z, p) - pg_{K0}(r, z) - g_{K1}(r, z)] \right\}, \quad (6.26)
 \end{aligned}$$

and the function $h_J(r, z, p)$ are approximated as

$$h_J(r, z, p) \simeq \sum_{n=1}^{2N+L} \alpha_N^{(n)}(p) \varphi_{JN}^{(n)}(r, z) \text{ for } (r, z) \in \Omega, \quad (6.27)$$

where $\alpha_N^{(n)}(p)$ are constant coefficients and $\varphi_{JN}^{(n)}(r, z)$, as shown in (5.29) in Chapter 5, are interpolating functions for treating the domain integral in momentum equations. The interpolating functions need to satisfy (5.22) as given in Chapter 5.

Therefore, the domain integrals in (6.11) can be approximated as

$$\iint \Phi_{JK}(r, z; r_0, z_0) h_J(r, z, p) \simeq \sum_{n=1}^{2N+L} \alpha_N^{(n)}(p) W_{KN}^{(n)}(r_0, z_0), \quad (6.28)$$

where

$$\begin{aligned}
 W_{KN}^{(n)}(r_0, z_0) = & -\gamma(r_0, z_0) \chi_{KN}^{(n)}(r, z) \\
 & + \int_{\Gamma} (\Phi_{JK}(r, z; r_0, z_0) \tau_{JN}^{(n)}(r, z) \\
 & - \Psi_{JK}(r, z; r_0, z_0; n_r, n_z) \chi_{JN}^{(n)}(r, z)) r ds(r, z), \quad (6.29)
 \end{aligned}$$

and

$$\begin{aligned}
 \tau_{rN}^{(n)}(r, z; n_r, n_z) &= 2n_r(r, z) \left\{ \frac{\partial}{\partial r} [\chi_{rN}^{(n)}(r, z)] + \frac{\nu}{1-2\nu} \left(\frac{\partial}{\partial r} [\chi_{rN}^{(n)}(r, z)] \right. \right. \\
 &\quad \left. \left. + \frac{\chi_{rN}^{(n)}(r, z)}{r} + \frac{\partial}{\partial z} [\chi_{zN}^{(n)}(r, z)] \right) \right\} \\
 &\quad + n_z(r, z) \left\{ \frac{\partial}{\partial z} [\chi_{rN}^{(n)}(r, z)] + \frac{\partial}{\partial r} [\chi_{zN}^{(n)}(r, z)] \right\}, \\
 \\
 \tau_{zN}^{(n)}(r, z; n_r, n_z) &= n_r(r, z) \left\{ \frac{\partial}{\partial z} [\chi_{rN}^{(n)}(r, z)] + \frac{\partial}{\partial r} [\chi_{zN}^{(n)}(r, z)] \right\} \\
 &\quad + 2n_z(r, z) \left\{ \frac{\partial}{\partial z} [\chi_{zN}^{(n)}(r, z)] + \frac{\nu}{1-2\nu} \left(\frac{\partial}{\partial r} [\chi_{rN}^{(n)}(r, z)] \right. \right. \\
 &\quad \left. \left. + \frac{\chi_{rN}^{(n)}(r, z)}{r} + \frac{\partial}{\partial z} [\chi_{zN}^{(n)}(r, z)] \right) \right\}. \tag{6.30}
 \end{aligned}$$

The interpolating functions $\varphi_{IJ}^{(n)}(r, z)$ that are bounded at all points (r, z) for $r > 0$ are obtained from (6.30) using

$$\begin{aligned}
 \chi_{rr}^{(n)}(r, z) &= \chi^{(n)}(r, z) - \frac{2}{9} [\sigma(0, 0, z; r_0^{(n)}, z_0^{(n)})]^3, \\
 \chi_{zr}^{(n)}(r, z) &= \chi_{rz}^{(n)}(r, z) = 0, \\
 \chi_{zz}^{(n)}(r, z) &= \chi^{(n)}(r, z), \tag{6.31}
 \end{aligned}$$

and $\chi^{(p)}(r, z)$ is given by (6.21). The functions $\varphi_{IJ}^{(n)}(r, z)$ and $\tau_{IJ}^{(n)}(r, z; n_r, n_z)$ are given in (5.29) and (5.30) in Chapter 5.

Taking $(r, z) = (r_0^{(k)}, z_0^{(k)})$ ($k = 1, 2, \dots, 2N + L$) in (6.27) to give

$$\sum_{n=1}^{2N+L} \alpha_N^{(n)}(p) \varphi_{JN}^{(n)}(r_0^{(k)}, z_0^{(k)}) \simeq h_J(r_0^{(k)}, z_0^{(k)}, p) \text{ for } k = 1, 2, \dots, 2N + L, \tag{6.32}$$

the coefficients $\alpha_N^{(n)}$ can be obtained by solving linear algebraic equations in (6.32).

6.4.4 Approximations of First Order Spatial Derivatives of Temperature and Displacements

The functions $f(r_0^{(k)}, z_0^{(k)}, p)$ and $h_J(r_0^{(k)}, z_0^{(k)}, p)$ contain first order partial derivatives of unknown functions. To treat them, first let the unknown functions to be approximated using specific functions $\chi^{(n)}(r, z)$ and $\bar{\chi}^{(n)}(r, z)$, then

$$\begin{aligned}\tilde{T}(r, z, p) &\simeq \sum_{m=1}^{2N+L} t^{(m)}(p) \chi^{(m)}(r, z), \\ \tilde{u}_r(r, z, p) &\simeq \sum_{m=1}^{2N+L} v_r^{(m)}(p) \bar{\chi}^{(m)}(r, z), \\ \tilde{u}_z(r, z, p) &\simeq \sum_{m=1}^{2N+L} v_z^{(m)}(p) \chi^{(m)}(r, z),\end{aligned}\tag{6.33}$$

where $\chi^{(p)}(r, z)$ is given by (6.21) and $\bar{\chi}^{(p)}(r, z)$ is

$$\bar{\chi}^{(n)}(r, z) = \frac{1}{9} \{ [\sigma(r, 0, z; r_0^{(n)}, z_0^{(n)})]^3 - [\sigma(r, 0, z; -r_0^{(n)}, z_0^{(n)})]^3 \}.$$

The coefficients $t^{(m)}(p)$, $v_r^{(m)}(p)$, $v_z^{(m)}(p)$ can be obtained by solving a system of algebraic equations after letting $(r, z) = (r_0^{(k)}, z_0^{(k)})$ to collocate (6.33). Therefore, the first order partial derivatives of unknown functions can be obtained as

$$\begin{aligned}\frac{\partial}{\partial x_J} [\tilde{T}(r, z, p)] &= \sum_{d=1}^{N+L} T^{(d)}(p) \phi_J^{(d)}(r, z), \\ \frac{\partial}{\partial x_J} [\tilde{u}_r(r, z, p)] &= \sum_{d=1}^{N+L} u_r^{(d)}(p) \bar{\phi}_J^{(d)}(r, z), \\ \frac{\partial}{\partial x_J} [\tilde{u}_z(r, z, p)] &= \sum_{d=1}^{N+L} u_{zJ}^{(d)}(p) \phi_J^{(d)}(r, z),\end{aligned}\tag{6.34}$$

where

$$\begin{aligned}\phi_J^{(d)}(r, z) &= \sum_{m=1}^{N+L} \omega^{(md)} \frac{\partial}{\partial x_J} [\chi^{(m)}(r, z)], \\ \bar{\phi}_J^{(d)}(r, z) &= \sum_{m=1}^{N+L} \bar{\omega}^{(md)} \frac{\partial}{\partial x_J} [\bar{\chi}^{(m)}(r, z)],\end{aligned}$$

$$\begin{aligned}\sum_{m=1}^{N+L} \chi^{(m)}(r_0^{(k)}, z_0^{(k)}) \omega^{(md)} &= \begin{cases} 1 & \text{if } k = d, \\ 0 & \text{if } k \neq d, \end{cases} \\ \sum_{m=1}^{N+L} \bar{\chi}^{(m)}(r_0^{(k)}, z_0^{(k)}) \bar{\omega}^{(md)} &= \begin{cases} 1 & \text{if } k = d, \\ 0 & \text{if } k \neq d. \end{cases}\end{aligned}\quad (6.35)$$

Therefore, from (6.15), (6.22) and (6.33), the integral differential equation in (6.10) can be approximated as

$$\begin{aligned}& \gamma(r_0^{(m)}, z_0^{(m)}) \sqrt{\kappa(r_0^{(m)}, z_0^{(m)})} T^{(m)}(p) \\ &= \sum_{k=1}^N \frac{1}{(2\tau - 1)\ell^{(k)}} \{ (\mathcal{F}_4^{(k)}(r_0^{(m)}, z_0^{(m)}) - (1 - \tau)\ell^{(k)} \mathcal{F}_2^{(k)}(r_0^{(m)}, z_0^{(m)})) \\ & \quad \times \sqrt{\kappa(r_0^{(k)}, z_0^{(k)})} T^{(k)}(p) \\ & \quad - (\mathcal{F}_4^{(k)}(r_0^{(m)}, z_0^{(m)}) - \tau\ell^{(k)} \mathcal{F}_2^{(k)}(r_0^{(m)}, z_0^{(m)})) \sqrt{\kappa(r_0^{(k+N)}, z_0^{(k+N)})} T^{(k+N)}(p) \\ & \quad - (\mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)}) - (1 - \tau)\ell^{(k)} \mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)})) \\ & \quad \times \left[\frac{\partial \sqrt{\kappa(r, z)}}{\partial n} \right]_{(r, z) = (r_0^{(k)}, z_0^{(k)})} T^{(k)}(p) + \sqrt{\kappa(r_0^{(k)}, z_0^{(k)})} q^{(k)}(p) \\ & \quad + (\mathcal{F}_3^{(k)}(r_0^{(m)}, z_0^{(m)}) - \tau\ell^{(k)} \mathcal{F}_1^{(k)}(r_0^{(m)}, z_0^{(m)})) \\ & \quad \times \left[\frac{\partial \sqrt{\kappa(r, z)}}{\partial n} \right]_{(r, z) = (r_0^{(k+N)}, z_0^{(k+N)})} T^{(k+N)}(p) \end{aligned}$$

$$\begin{aligned}
 & + \sqrt{\kappa(r_0^{(k+N)}, z_0^{(k+N)})} q^{(k+N)}(p) \} \} \\
 & + \sum_{n=1}^{2N+L} \alpha^{(n)}(p) W^{(n)}(r_0^{(m)}, z_0^{(m)}) \\
 & \text{for } m = 1, 2, \dots, 2N + L,
 \end{aligned} \tag{6.36}$$

together with

$$\begin{aligned}
 & \sum_{n=1}^{2N+L} \alpha^{(n)}(p) \varphi^{(n)}(r_0^{(k)}, z_0^{(k)}) \\
 = & - \frac{Q(r_0^{(k)}, z_0^{(k)}, p)}{\sqrt{\kappa(r_0^{(k)}, z_0^{(k)})}} + T^{(k)}(p) \cdot \nabla_{\text{axis}}^2 (\sqrt{\kappa(r, z)}) \Big|_{(r,z)=(r_0^{(k)}, z_0^{(k)})} \\
 & + \frac{\rho(r_0^{(k)}, z_0^{(k)}) c(r_0^{(k)}, z_0^{(k)})}{\sqrt{\kappa(r_0^{(k)}, z_0^{(k)})}} [p T^{(k)}(p) - f_0(r_0^{(k)}, z_0^{(k)})] \\
 & + \frac{\beta(r_0^{(k)}, z_0^{(k)}) T_0}{\sqrt{\kappa(r_0^{(k)}, z_0^{(k)})}} \left[\frac{1}{r} (p u_r^{(k)}(p) - g_{r0}(r_0^{(k)}, z_0^{(k)})) + A_1^{(k)}(p) \right. \\
 & \left. - g_{r0}(r_0^{(d)}, z_0^{(d)}) \bar{\phi}_r^{(d)}(r_0^{(k)}, z_0^{(k)}) - g_{z0}(r_0^{(d)}, z_0^{(d)}) \phi_z^{(d)}(r_0^{(k)}, z_0^{(k)}) \right], \\
 & \text{for } k = 1, 2, \dots, 2N + L,
 \end{aligned} \tag{6.37}$$

where

$$\begin{aligned}
 A_1^{(k)}(p) = & \frac{\beta(r_0^{(k)}, z_0^{(k)}) T_0}{\sqrt{\kappa(r_0^{(k)}, z_0^{(k)})}} \sum_{d=1}^{2N+L} \{ p [u_r^{(d)}(p) \bar{\phi}_r^{(d)}(r_0^{(k)}, z_0^{(k)}) + u_z^{(d)}(p) \phi_z^{(d)}(r_0^{(k)}, z_0^{(k)})] \\
 & - g_{r0}(r_0^{(d)}, z_0^{(d)}) \bar{\phi}_r^{(d)}(r_0^{(k)}, z_0^{(k)}) - g_{z0}(r_0^{(d)}, z_0^{(d)}) \phi_z^{(d)}(r_0^{(k)}, z_0^{(k)}) \}, \tag{6.38}
 \end{aligned}$$

and from (6.16), (6.28) and (6.33), the integral equations in (6.11) are approximated as

$$\begin{aligned}
 & \gamma(r_0^{(m)}, z_0^{(m)}) u_K^{(m)}(p) \\
 = & \sum_{k=1}^N \frac{1}{(2\tau - 1)\ell^{(k)}} \{ (\bar{\mathcal{G}}_{JK}^{(k)}(r_0^{(m)}, z_0^{(m)}) - (1 - \tau)\ell^{(k)} \mathcal{G}_{JK}^{(k)}(r_0^{(m)}, z_0^{(m)})) p_J^{(k)}(p)
 \end{aligned}$$

$$\begin{aligned}
 & - (\bar{\mathcal{G}}_{JK}^{(k)}(r_0^{(m)}, z_0^{(m)}) - \tau \ell^{(k)} \mathcal{G}_{JK}^{(k)}(r_0^{(m)}, z_0^{(m)})) p_J^{(k+N)}(p) \\
 & - (\bar{\mathcal{F}}_{JK}^{(k)}(r_0^{(m)}, z_0^{(m)}) - (1 - \tau) \ell^{(k)} \mathcal{F}_{JK}^{(k)}(r_0^{(m)}, z_0^{(m)})) u_J^{(k)}(p) \\
 & + (\bar{\mathcal{F}}_{JK}^{(k)}(r_0^{(m)}, z_0^{(m)}) - \tau \ell^{(k)} \mathcal{F}_{JK}^{(k)}(r_0^{(m)}, z_0^{(m)})) u_J^{(k+N)}(p) \} \\
 & + \sum_{n=1}^{2N+L} \alpha_N^{(n)}(p) W_{KN}^{(n)}(r_0^{(m)}, z_0^{(m)}) \\
 & \text{for } m = 1, 2, \dots, 2N + L,
 \end{aligned} \tag{6.39}$$

together with

$$\begin{aligned}
 & \sum_{n=1}^{2N+L} \alpha_N^{(n)}(p) \varphi_{JN}^{(n)}(r_0^{(k)}, z_0^{(k)}) \\
 = & \frac{1}{\mu(r_0^{(k)}, z_0^{(k)})} \left\{ -\beta(r_0^{(k)}, z_0^{(k)}) \sum_{d=1}^{N+L} T^{(d)}(p) \phi_J^{(d)}(r_0^{(k)}, z_0^{(k)}) \right. \\
 & - \left. \frac{\partial}{\partial x_J} [\beta(r, z)] \Big|_{(r,z)=(r_0^{(k)}, z_0^{(k)})} (T^{(k)}(p) - \frac{T_0}{p}) \right. \\
 & + F_J(r_0^{(k)}, z_0^{(k)}, p) + \left. \frac{\partial}{\partial x_J} [\mu(r, z)] \Big|_{(r,z)=(r_0^{(k)}, z_0^{(k)})} \frac{2\nu}{(1-2\nu)r} u_r^{(k)}(p) \right. \\
 & - \rho(r_0^{(k)}, z_0^{(k)}) [p^2 u_J^{(k)}(p) - p g_{J0}(r_0^{(k)}, z_0^{(k)}) - g_{J1}(r_0^{(k)}, z_0^{(k)})] \\
 & + \sum_{d=1}^{N+L} [X_{Jr}(r_0^{(k)}, z_0^{(k)}) u_r^{(d)}(p) \bar{\phi}_z^{(d)}(r_0^{(k)}, z_0^{(k)}) \\
 & + X_{Jz}(r_0^{(k)}, z_0^{(k)}) u_z^{(d)}(p) \phi_z^{(d)}(r_0^{(k)}, z_0^{(k)})] \\
 & + \sum_{d=1}^{N+L} [Y_{Jr}(r_0^{(k)}, z_0^{(k)}) u_r^{(d)}(p) \bar{\phi}_r^{(d)}(r_0^{(k)}, z_0^{(k)}) \\
 & + Y_{Jz}(r_0^{(k)}, z_0^{(k)}) u_z^{(d)}(p) \phi_r^{(d)}(r_0^{(k)}, z_0^{(k)})] \} \\
 & \text{for } k = 1, 2, \dots, 2N + L.
 \end{aligned} \tag{6.40}$$

The problem of interest here is to solve (6.36) and (6.39) as a system of $3(2N + L)$ linear algebraic equations with $3(2N + L)$ unknowns after the coefficients $\alpha_N^{(n)}(p)$ and $\alpha_N^{(n)}(p)$ are calculated from (6.37) and (6.40). The unknowns are $T^{(k)}(p)$ or $q^{(k)}(p)$ and two of the four components $u_r^{(k)}(p)$, $u_z^{(k)}(p)$,

$p_r^{(k)}(p)$, $p_z^{(k)}(p)$ ($k = 1, 2, \dots, N$) on boundary $\Gamma^{(k)}$ depend on the boundary conditions, and $T^{(2N+n)}(p)$, $u_r^{(2N+n)}(p)$ and $u_z^{(2N+n)}(p)$ for $n = 1, 2, \dots, L$ in the solution domain.

6.4.5 Inversion of Laplace Transform Solution

The solutions are determined in the Laplace transform domain. Once the unknowns such as $T^{(k)}(p)$, $q^{(k)}(p)$, $u_J^{(k)}(p)$ or $p_J^{(k)}(p)$ are calculated from the boundary element procedure explained in previous section, those results can be transformed back to results in physical domain using numerical inversion method. The Stehfest's algorithm as shown in (2.36) in Chapter 2 is again used here.

6.5 Specific Problems

Problem 1

For a test problem, let the solution domain be given by $0 < r < 1$, $0 < z < 1$. The materials properties are

$$\begin{aligned}\rho &= 2, c = 1, \\ \beta &= 1 + r^2, \mu = 1 + z^2, \\ \kappa &= (1 + z^2)^2, \nu = 0.3.\end{aligned}$$

The reference temperature is given by $T_0 = 1$. The function Q , F_r and F_z are respectively

$$\begin{aligned}Q(r, z, t) &= -2(5 + 6r + 2r^2 + 6r^3 + 15z^2 + 9z^4) \exp(-t), \\ F_r(r, z, t) &= -2r + (-42 - 42z^2 + 4r + 8r^2 + 8r^3 + 2rz^2 + 2rz) \exp(-t), \\ F_z(r, z, t) &= (2z + 2zr^2 - 6z^2 - 36rz - r^2) \exp(-t).\end{aligned}$$

The initial-boundary conditions are given by

$$\begin{aligned}
 T(r, z, 0) &= 2r^2 + z^2, \\
 u_r(r, z, 0) &= (4r - z)r, \\
 u_z(r, z, 0) &= z^2 - 0.5r^2, \\
 \left. \frac{\partial}{\partial t} [u_r(r, z, t)] \right|_{t=0} &= -(4r - z)r, \\
 \left. \frac{\partial}{\partial t} [u_z(r, z, t)] \right|_{t=0} &= -(z^2 - 0.5r^2), \\
 \left. \begin{aligned}
 \frac{\partial}{\partial n} [T(r, z, t)]|_{z=0} &= 0 \\
 t_r(r, 0, t) &= 2r \exp(-t) \\
 t_z(r, 0, t) &= -18r \exp(-t) \\
 &\quad + (1 + r^2)(2r^2 \exp(-t) - 1)
 \end{aligned} \right\} \text{ for } \begin{cases} 0 < r < 1, \\ t > 0, \end{cases} \\
 \left. \begin{aligned}
 T(r, 1, t) &= (2r^2 + 1) \exp(-t) \\
 u_r(r, 1, t) &= (4r - 1)r \exp(-t) \\
 u_z(r, 1, t) &= (1 - 0.5r^2) \exp(-t)
 \end{aligned} \right\} \text{ for } \begin{cases} 0 < r < 1, \\ t > 0, \end{cases} \\
 \left. \begin{aligned}
 T(1, z, t) &= (2 + z^2) \exp(-t) \\
 t_r(1, z, t) &= 2(1 + z^2)(1 - z) \exp(-t) \\
 &\quad - 2(2 + z^2) \exp(-t) + 2 \\
 t_z(1, z, t) &= -2(2 + z^2) \exp(-t)
 \end{aligned} \right\} \text{ for } \begin{cases} 0 < z < 1, \\ t > 0. \end{cases}
 \end{aligned}$$

To obtain some numerical results, the boundary of the solution domain prescribed here is discretized into N elements with equal length and L well-spaced out collocation points are chosen inside the solution domain. The numerical solutions are obtained with different sets of N and L , which are given by $(N, L) = (15, 9)$ (Set A), $(N, L) = (30, 49)$ (Set B) and $(N, L) = (60, 121)$ (Set C). Numerical results of T , u_r and u_z at selected interior points are computed and compared with the exact solutions given by

$$\begin{aligned}
 T(r, z, t) &= (2r^2 + z^2) \exp(-t), \\
 u_r(r, z, t) &= (4r - z)r \exp(-t), \\
 u_z(r, z, t) &= (z^2 - 0.5r^2) \exp(-t),
 \end{aligned}$$

at $t = 0.1$ in Table 6.1, 6.2 and 6.3 respectively.

Table 6.1: A numerical comparison of the numerical and exact values of T at $t = 0.1$ at selected interior points.

Point (r, z)	Set A	Set B	Set C	Exact
(0.25, 0.25)	0.163829	0.168984	0.169443	0.169657
(0.50, 0.25)	0.505899	0.508498	0.508819	0.508971
(0.75, 0.25)	1.072466	1.074005	1.074322	1.074494
(0.25, 0.50)	0.331487	0.338554	0.339090	0.339314
(0.50, 0.50)	0.674333	0.678065	0.678455	0.678628
(0.75, 0.50)	1.240809	1.243613	1.243974	1.244151
(0.25, 0.75)	0.615238	0.621311	0.621840	0.622076
(0.50, 0.75)	0.956934	0.960777	0.961192	0.961390
(0.75, 0.75)	1.523033	1.526280	1.526700	1.526913

Table 6.2: A numerical comparison of the numerical and exact values of u_r at $t = 0.1$ at selected interior points.

Point (r, z)	Set A	Set B	Set C	Exact
(0.25, 0.25)	0.169939	0.169665	0.169664	0.169657
(0.50, 0.25)	0.791902	0.791751	0.791741	0.791733
(0.75, 0.25)	1.866750	1.866229	1.866232	1.866227
(0.25, 0.50)	0.113629	0.113127	0.113100	0.113105
(0.50, 0.50)	0.678718	0.678664	0.678625	0.678628
(0.75, 0.50)	1.697012	1.696598	1.696561	1.696570
(0.25, 0.75)	0.057326	0.056634	0.056574	0.056552
(0.50, 0.75)	0.566046	0.565613	0.565549	0.565523
(0.75, 0.75)	1.527951	1.526972	1.526927	1.526913

Note that the parameter τ in the discontinuous linear elements in the calculation is chosen to be 0.25, and the terms for Laplace inversion is chosen to be $M = 5$. From Table 6.1, 6.2 and 6.3, the results are more accurate when

Table 6.3: A numerical comparison of the numerical and exact values of u_z at $t = 0.1$ at selected interior points.

Point (r, z)	Set A	Set B	Set C	Exact
(0.25, 0.25)	0.028039	0.028314	0.028277	0.028276
(0.50, 0.25)	-0.056532	-0.056543	-0.056553	-0.056552
(0.75, 0.25)	-0.198005	-0.197925	-0.197929	-0.197933
(0.25, 0.50)	0.197513	0.198001	0.197941	0.197933
(0.50, 0.50)	0.113332	0.113117	0.113107	0.113105
(0.75, 0.50)	-0.028257	-0.028255	-0.028272	-0.028276
(0.25, 0.75)	0.480290	0.480796	0.480718	0.480695
(0.50, 0.75)	0.396152	0.395886	0.395875	0.395866
(0.75, 0.75)	0.254502	0.254533	0.254493	0.254486

the results are obtained by finer calculation. It is obvious that the accuracy is improving from Set A to Set B and then to Set C. In other words, the accuracy of the numerical results improves when more boundary element and interior collocation points are used in the calculation.

Along the boundary $0 < r < 1, z = 1$, the tractions are not known a priori. The tractions along the boundary $0 < r < 1, z = 1$ can be calculated from boundary element procedure. Select $(N, L) = (60, 121)$, the tractions are plotted in Figure 6.1 and 6.2 and they are compared with the exact tractions at $t = 0.1$. As shown in Figure 6.1 and 6.2, the numerical and exact values are in good agreement with each other.

Figure 6.1: Comparison of the numerical and exact traction component t_r along the boundary $0 < r < 1$, $z = 1$ at $t = 0.1$.

Figure 6.2: Comparison of the numerical and exact traction component t_z along the boundary $0 < r < 1$, $z = 1$ at $t = 0.1$.

Problem 2

Consider a problem with hollow cylinder, which has solution domain of $1 < r < 2$, $0 < z < 1$. The parameters are respectively given by

$$\begin{aligned}\rho &= r^2 + z^2, c = 2, \\ \beta &= r^2 + z^2, \mu = r + z, \\ \kappa &= (r^2 + z^2)^2, \nu = 0.3.\end{aligned}$$

The function Q , F_r and F_z are respectively given by

$$\begin{aligned}Q(r, z, t) &= -\frac{r^2 + z^2}{r} \{ \sin(t) [16r^3 z - r^2 + 4r + 4rz^3 + z^2] \\ &\quad - 2r^3 z \cos(t) + 16r^3 z + 4rz^3 \}, \\ F_r(r, z, t) &= -\frac{1}{2r^2} \{ \cos(t) [4r^5 + 2r^4 z^2 + 4r^3 z^2 - 4r^3 \\ &\quad + 2r^2 z^2 + 20r^2 + r^2 z - 4rz^2 - 7z^3] \\ &\quad + (1 + \sin(t)) [-8r^5 z - 4r^3 z^3] + 4r^3 \}, \\ F_z(r, z, t) &= \frac{1}{2r^2} \{ \cos(t) [2r^4 z - 4r^3 + 7r^2 + 2r^2 z^3 \\ &\quad - 10rz - 4rz^2 - 12r - 11z^2] \\ &\quad + (1 + \sin(t)) [2r^5 + 6r^3 z^2] - 4rz \}.\end{aligned}$$

The problem here is to solve (6.1) and (6.2) subject to initial-boundary conditions

$$\begin{aligned}T(r, z, 0) &= r^2 z, \\ u_r(r, z, 0) &= z^2 + 2r, \\ u_z(r, z, 0) &= 2 - rz, \\ \left. \frac{\partial u_r}{\partial t} \right|_{t=0} &= 0, \left. \frac{\partial u_z}{\partial t} \right|_{t=0} = 0, \\ \left. \begin{aligned} T(r, 0, t) &= 0 \\ u_r(r, 0, t) &= 2r \cos(t) \\ u_z(r, 0, t) &= 2 \cos(t) \end{aligned} \right\} \text{ for } \begin{aligned} 1 &< r < 2, \\ t &> 0, \end{aligned}\end{aligned}$$

$$\begin{aligned}
 \left. \begin{aligned} T(r, 1, t) &= r^2(\sin(t) + 1) \\ u_r(r, 1, t) &= (1 + 2r) \cos(t) \\ u_z(r, 1, t) &= (2 - r) \cos(t) \end{aligned} \right\} \text{ for } \begin{aligned} 1 < r < 2, \\ t > 0, \end{aligned} \\
 \left. \begin{aligned} \frac{\partial}{\partial n}[T(r, z, t)]|_{r=1} &= -2z(\sin(t) + 1) \\ u_r(1, z, t) &= (z^2 + 2) \cos(t) \\ u_z(1, z, t) &= (2 - z) \cos(t) \end{aligned} \right\} \text{ for } \begin{aligned} 0 < z < 1, \\ t > 0, \end{aligned} \\
 \left. \begin{aligned} \frac{\partial}{\partial n}[T(r, z, t)]|_{r=2} &= 4z(\sin(t) + 1) \\ u_r(2, z, t) &= (z^2 + 4) \cos(t) \\ u_z(2, z, t) &= 2(1 - z) \cos(t) \end{aligned} \right\} \text{ for } \begin{aligned} 0 < z < 1, \\ t > 0. \end{aligned}
 \end{aligned}$$

For the hollow cylinder problem here, the boundary curve Γ comprises four straight lines of the region $1 < r < 2$, $0 < z < 1$. To apply boundary element method, the boundary Γ is discretized into N elements with equal length. The dual-reciprocity method is applied for treating the domain integrals in the boundary-domain integral formulation. Therefore, L well-spaced out collocation points are chosen to implement the dual-reciprocity method.

To obtain some numerical results, $N = 80$ and $L = 121$ are used. In the calculation, the parameter τ for discontinuous linear elements are chosen to be 0.25. The results are first obtained in Laplace domain, the results in physical domain are calculated by using Stehfest's algorithm with $M = 10$. In Figure 6.3, 6.4 and 6.5, the numerical results of T , u_r and u_z are plotted against t at $(r, z) = (1.5, 0.5)$ and the results are compared with the exact solutions which are given by

$$\begin{aligned}
 T(r, z, t) &= r^2 z (\sin(t) + 1), \\
 u_r(r, z, t) &= (z^2 + 2r) \cos(t), \\
 u_z(r, z, t) &= (2 - rz) \cos(t).
 \end{aligned}$$

All results in Figure 6.3, 6.4 and 6.5 agree well with the exact solutions. For the results at $t > 5$, the accuracy can be improved by using more terms for numerical inversion of Laplace transform formula.

Figure 6.3: Comparisons of numerical and exact T at $(r, z) = (1.5, 0.5)$ over $0 < t < 6$.

Figure 6.4: Comparison of numerical and exact u_r at $(r, z) = (1.5, 0.5)$ over $0 < t < 6$.

Figure 6.5: Comparison of numerical and exact u_z at $(r, z) = (1.5, 0.5)$ over $0 < t < 6$.

Problem 3

Consider a solid cylinder which occupies the region $0 < r < H, 0 < z < H$, where H is a positive constant. The thermal conductivity, density, stress-temperature coefficient and shear modulus of the cylinder, which are functionally graded exponentially along the z -axis, are respectively given by

$$\begin{aligned}\kappa &= \kappa_0 \exp(-hz), \rho = \rho_0 \exp(-hz), \\ \beta &= \beta_0 \exp(-hz), \mu = \mu_0 \exp(-hz),\end{aligned}$$

and the specific heat $c = c_0$ where $\kappa_0, \rho_0, \beta_0, \mu_0, c_0$ and h are positive constants.

No internal heat generation and no body forces apply to the cylinder, which is, $Q = 0, F_r = F_z = 0$. The problem is to solve (6.1) and (6.2) subject to initial-boundary conditions. Initially, the temperature of the solid remains at T_0 (same as the reference temperature where the solid does not experience any

thermally induced stress) and there is no deformation. Therefore, the initial conditions are given by

$$\begin{aligned} T(r, z, 0) &= T_0, \\ u_r(r, z, 0) &= 0, u_z(r, z, 0) = 0, \\ \frac{\partial}{\partial n}[u_r(r, z, t)] \Big|_{t=0} &= 0, \frac{\partial}{\partial n}[u_z(r, z, t)] \Big|_{t=0} = 0, \\ \text{for } 0 < r < H, 0 < z < H. \end{aligned}$$

On the surface $0 < r < H, z = H$, the temperature is maintained at T_0 and the cylinder surface is attached to rigid wall. A uniform heat flux q_0 is applied at the center surface of cylinder $0 < r < H/2, z = 0$. The remaining boundary along $0 < z < H, r = H$ and $H/2 < r < H, z = 0$ are thermally insulated. Besides the part which is attached to rigid wall, the remaining boundary of cylinder is free of traction. Therefore, the boundary conditions can be given explicitly by

$$\begin{aligned} \left. \begin{aligned} q(r, H, t) &= 0 \\ u_r(r, H, t) &= 0 \\ u_z(r, H, t) &= 0 \end{aligned} \right\} & \text{for } 0 < r < H, t > 0, \\ \left. \begin{aligned} q(H, z, t) &= 0 \\ t_r(H, z, t) &= 0 \\ t_z(H, z, t) &= 0 \end{aligned} \right\} & \text{for } 0 < z < H, t > 0, \\ \left. \begin{aligned} q(r, 0, t) &= 0 \\ t_r(r, 0, t) &= 0 \\ t_z(r, 0, t) &= 0 \end{aligned} \right\} & \text{for } H/2 < r < H, t > 0, \\ \left. \begin{aligned} q(r, 0, t) &= q_0 \\ t_r(r, 0, t) &= 0 \\ t_z(r, 0, t) &= 0 \end{aligned} \right\} & \text{for } 0 < r < H/2, t > 0. \end{aligned}$$

In this coupled time-dependent thermoelastic problem, the deformations of the cylinder are induced by the thermal effect when the stress-temperature coefficient β is not zero. Takes $\nu = 0.3$, $hH = 0.5$ and $q_0H/\kappa_0T_0 = 0.02$, the

effect of $\beta_0/\rho_0 c_0$ are examined along different parts of boundary. In boundary element analysis, the boundary of the solution domain is discretized to 120 equal length elements and 225 well distributed interior collocation points are selected. The parameter τ in the discontinuous linear elements in the calculation is taken to be 0.25, and the terms for Laplace inversion is chosen to be $M = 8$.

Along $0 < r < H, z = H$, the stresses are obtained and compared with different values of $\beta_0/\rho_0 c_0$. The stresses are plotted in Figure 6.6, 6.7, 6.8 and 6.9 against r/H at $\kappa_0 t/\rho_0 c_0 H^2 = 1.6$. With larger value of $\beta_0/\rho_0 c_0$, the stresses are higher. From the figures, the stresses are higher when r/H gets to 1.

The displacements are not known along the boundary $0 < z < H, r = H$ at a priori. In Figure 6.10 and 6.11, the displacements u_r/H and u_z/H at $r = H$ are plotted against z/H at $\kappa_0 t/\rho_0 c_0 H^2 = 1.6$ for selected values of $\beta_0/\rho_0 c_0$. From the figures, it is obvious that increasing the values of $\beta_0/\rho_0 c_0$ affects the displacements by increasing the magnitudes of u_r/H and u_z/H . At points near $z/H = 0$, the effect of $\beta_0/\rho_0 c_0$ on both u_r and u_z is greater and the effect getting smaller at larger z/H .

The displacements u_r and u_z along boundary $0 < r < H, z = 0$ which is subject to traction free condition are shown in Figure 6.12 and 6.13 at $\kappa_0 t/\rho_0 c_0 H^2 = 1.6$ for selected values of $\beta_0/\rho_0 c_0$. Similar to the displacements along boundary $0 < z < H, r = H$, the displacements u_r and u_z increase when the value of $\beta_0/\rho_0 c_0$ is higher. From the figures, the effect of $\beta_0/\rho_0 c_0$ is greater at further distance from the center of cylinder ($r = 0$) for u_r but the effect of $\beta_0/\rho_0 c_0$ is greater at points nearer to the center $r = 0$ for u_z .

In Figure 6.14, 6.15, 6.16 and 6.17, the stresses along $0 < r < H, z = H$ at $\beta_0/\rho_0 c_0 = 2$ are plotted against r/H at selected time $\kappa_0 t/\rho_0 c_0 H^2$. From the figures, all the stresses σ_{rr} , σ_{rz} , σ_{zz} and $\sigma_{\theta\theta}$ appear to be larger at higher time

Figure 6.6: Plots of σ_{rr}/μ_0 on $z/H = 1$ against r/H at $\kappa_0 t/\rho_0 c_0 H^2 = 1.6$ for selected values of $\beta_0/\rho_0 c_0$.

Figure 6.7: Plots of σ_{rz}/μ_0 on $z/H = 1$ against r/H at $\kappa_0 t/\rho_0 c_0 H^2 = 1.6$ for selected values of $\beta_0/\rho_0 c_0$.

Figure 6.8: Plots of σ_{zz}/μ_0 on $z/H = 1$ against r/H at $\kappa_0 t/\rho_0 c_0 H^2 = 1.6$ for selected values of $\beta_0/\rho_0 c_0$.

Figure 6.9: Plots of $\sigma_{\theta\theta}/\mu_0$ on $z/H = 1$ against r/H at $\kappa_0 t/\rho_0 c_0 H^2 = 1.6$ for selected values of $\beta_0/\rho_0 c_0$.

Figure 6.10: Plots of u_r/H at $r = H$ against z/H at $\kappa_0 t / \rho_0 c_0 H^2 = 1.6$ for selected values of $\beta_0 / \rho_0 c_0$.

Figure 6.11: Plots of u_z/H at $r = H$ against z/H at $\kappa_0 t / \rho_0 c_0 H^2 = 1.6$ for selected values of $\beta_0 / \rho_0 c_0$.

Figure 6.12: Plots of u_r/H at $z = 0$ against r/H at $\kappa_0 t / \rho_0 c_0 H^2 = 1.6$ for selected values of $\beta_0 / \rho_0 c_0$.

Figure 6.13: Plots of u_z/H at $z = 0$ against r/H at $\kappa_0 t / \rho_0 c_0 H^2 = 1.6$ for selected values of $\beta_0 / \rho_0 c_0$.

Figure 6.14: Plots of σ_{rr}/μ_0 at $z = H$ against r/H at $\beta_0/\rho_0 c_0 = 2$ for selected values of $\kappa_0 t / \rho_0 c_0 H^2$.

Figure 6.15: Plots of σ_{rz}/μ_0 at $z = H$ against r/H at $\beta_0/\rho_0 c_0 = 2$ for selected values of $\kappa_0 t / \rho_0 c_0 H^2$.

Figure 6.16: Plots of σ_{zz}/μ_0 at $z = H$ against r/H at $\beta_0/\rho_0 c_0 = 2$ for selected values of $\kappa_0 t / \rho_0 c_0 H^2$.

Figure 6.17: Plots of $\sigma_{\theta\theta}/\mu_0$ at $z = H$ against r/H at $\beta_0/\rho_0 c_0 = 2$ for selected values of $\kappa_0 t / \rho_0 c_0 H^2$.

level.

To examine the effects of the grading parameter h on the state of stress in the solid, stresses at the point $(r/H, z/H) = (0.2, 1)$ are plotted against time $\kappa_0 t / \rho_0 c_0 H^2$ at $\beta_0 / \rho_0 c_0 = 2$ for selected values of hH in Figures 6.18, 6.19, 6.20 and 6.21. The results obtained indicate that the stresses at the fixed point increase in magnitude as hH decreases. It appears that the difference between the stresses for the homogeneous solid and nonhomogeneous ($hH = 0.1, hH = 0.2$) is more pronounced as time increases.

Figure 6.18: Plots of σ_{rr}/μ_0 at $(r/H, z/H) = (0.2, 1)$ against $\kappa_0 t / \rho_0 c_0 H^2$ at $\beta_0 / \rho_0 c_0 = 2$ for selected values of hH .

6.6 Summary

Dual-reciprocity boundary element method has solved an axisymmetric thermoelastodynamic problem involving nonhomogeneous material properties with functionally graded in spatial coordinates. The boundary-domain integral

Figure 6.19: Plots of σ_{rz}/μ_0 at $(r/H, z/H) = (0.2, 1)$ against $\kappa_0 t / \rho_0 c_0 H^2$ at $\beta_0 / \rho_0 c_0 = 2$ for selected values of hH .

Figure 6.20: Plots of σ_{zz}/μ_0 at $(r/H, z/H) = (0.2, 1)$ against $\kappa_0 t / \rho_0 c_0 H^2$ at $\beta_0 / \rho_0 c_0 = 2$ for selected values of hH .

Figure 6.21: Plots of $\sigma_{\theta\theta}/\mu_0$ at $(r/H, z/H) = (0.2, 1)$ against $\kappa_0 t / \rho_0 c_0 H^2$ at $\beta_0 / \rho_0 c_0 = 2$ for selected values of hH .

equations are first obtained and they are transformed and solved in Laplace domain. The domain integrals involved in the integral equation are solved by dual-reciprocity method together with specified interpolating functions. After the unknowns are solved in Laplace domain, the Stehfest's algorithm is used to invert the results to physical domain. The dual-reciprocity boundary element method is applied to solve several test problems. Using several test problems with exact solutions, the proposed method has proven that it can solve the problems with good accuracy. Therefore, the proposed numerical procedure provides accurate and reliable numerical solutions for thermoelastodynamic problems.

Chapter 7

Complex Variable Boundary Element Method for Axisymmetric Heat Conduction

7.1 Introduction

An alternative boundary element method based on the theory of complex variable was proposed by Hromadka and Lai [44] for solving the Laplace's equation in the two-dimensional Cartesian space. Since then, many authors have applied such a complex variable boundary element approach to various problems (see, for examples, Ang, Clements and Cooke [8], Dumir and Kumar [36], Ostanin, Mogilevskaya, Labuz et al. [67] and Hromadka and Zillmer [45]). Park and Ang [72] and Park, Ang and Kang [73] have applied the method to analyze an elliptic partial differential equation governing the behaviors of a two-dimensional solid with material properties that vary continuously in space. The complex variable boundary element method used in [72] and [73] differs from that in [44] in the manner which the flux boundary conditions are treated.

Recently, Dobroskok and Linkov [35] extended the complex variable boundary element method to analyze a two-dimensional transient diffusion problem.

Radial basis functions are used to approximately re-formulate the problem under consideration as one governed by the two-dimensional Laplace's equation which may be solved by the complex variable boundary element method. The approach in [35] is used by Ang [6] to solve a two-dimensional heat conduction problem involving a nonhomogeneous anisotropic solid.

In this chapter ¹, guided by the analysis in [6] and [35], we use axisymmetric interpolating functions similar to those constructed in Chapter 3 to reduce the axisymmetric steady heat conduction equation for a nonhomogeneous solid to a form amenable to complex variable boundary element analysis. The complex variable boundary element method presented here for the axisymmetric heat conduction should provide a useful and interesting alternative to the real boundary element methods in Chapter 3 and 4. Unlike the real boundary element methods, the complex variable approach does not require the computation of a rather complicated fundamental solution which is expressed in terms of complete elliptic integrals of the first and second kind. To assess the validity and accuracy of the complex variable boundary element procedure presented here, several specific axisymmetric heat conduction problems involving functionally graded materials are solved.

7.2 A Steady-state Axisymmetric Heat Conduction Problem

Consider an axisymmetric thermally isotropic solid occupying the region Ω with boundary Γ as sketched in Figure 2.1. The axisymmetric temperature in the solid is assumed to be independent of time and given by $T(r, z)$. The

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temperature satisfies the partial differential equation

$$\underline{\nabla} \bullet (\kappa \underline{\nabla} T) + Q = 0 \text{ in } \Omega, \quad (7.1)$$

where κ is the thermal conductivity and Q is the internal heat source generation rate which depends on only r and z .

The thermal conductivity is functionally graded in the radial and axial directions of the solid of revolution, that is,

$$\kappa = g(r, z), \quad (7.2)$$

where g is a suitably given function that is positive in Ω .

Of interest here is the numerical solution of (7.1) together with (7.2) subject to the boundary conditions

$$\begin{aligned} T(r, z) &= f_1(r, z) \text{ on } \Gamma_1, \\ g(r, z) \frac{\partial T}{\partial n} &= f_2(r, z) + f_3(r, z)T(r, z) \text{ on } \Gamma_2, \end{aligned} \quad (7.3)$$

where Γ_1 and Γ_2 are non-intersecting surfaces such that $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\partial T / \partial n$ denotes the outward normal derivative of T on Γ and $f_1(r, z)$, $f_2(r, z)$ and $f_3(r, z)$ are suitably given functions of r and z .

If we let

$$T(r, z) = \frac{1}{\sqrt{g(r, z)}} w(r, z), \quad (7.4)$$

then (7.1) can be written as

$$\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial z^2} = -\frac{Q(r, z)}{\sqrt{g(r, z)}} + B(r, z)w - \frac{1}{r} \frac{\partial w}{\partial r}, \quad (7.5)$$

where

$$B(r, z) = \frac{1}{\sqrt{g(r, z)}} \nabla_{\text{axis}}^2 (\sqrt{g(r, z)}), \quad (7.6)$$

the function g is assumed to be such that $B(r, z)$ is bounded in the solution domain Ω .

Using axisymmetric model, the boundary value problem defined by (7.1) together with (7.2) and (7.3) can be reformulated as one that requires solving (7.5) subject to

$$\begin{aligned} w(r, z) &= \sqrt{g(r, z)} f_1(r, z) \text{ on } \Gamma_1, \\ \frac{\partial}{\partial n}[w(r, z)] &= \frac{1}{g(r, z)} \left\{ \frac{1}{2} \frac{\partial}{\partial n}[g(r, z)] + f_3(r, z) \right\} w(r, z) \\ &\quad + \frac{1}{\sqrt{g(r, z)}} f_2(r, z) \text{ on } \Gamma_2, \end{aligned} \quad (7.7)$$

and

$$\begin{aligned} \frac{\partial}{\partial n}[w(r, z)] &= n_r(r, z) \frac{\partial}{\partial r}[w(r, z)] + n_z(r, z) \frac{\partial}{\partial z}[w(r, z)], \\ \frac{\partial}{\partial n}[g(r, z)] &= n_r(r, z) \frac{\partial}{\partial r}[g(r, z)] + n_z(r, z) \frac{\partial}{\partial z}[g(r, z)], \end{aligned} \quad (7.8)$$

with $n_r(r, z)$ and $n_z(r, z)$ being the components of the outward unit normal vector on Γ at the point (r, z) in the r and z direction respectively.

7.3 Interpolating Function Approximation and Complex Variable Formulation

To approximate the right hand side of (7.5) using a meshfree approximation, we choose P well spaced out collocation points in $\Omega \cup \Gamma$. (The collocation points may lie on $r = 0$ if the solution domain R contains Cartesian points (x, y, z) such that $x^2 + y^2 = 0$.) The collocation points in the axisymmetric coordinates (r, z) are denoted by $(r_0^{(1)}, z_0^{(1)})$, $(r_0^{(2)}, z_0^{(2)})$, $\dots$, $(r_0^{(P-1)}, z_0^{(P-1)})$ and $(r_0^{(P)}, z_0^{(P)})$.

We make the approximation

$$-\frac{Q(r, z)}{\sqrt{g(r, z)}} + B(r, z)w - \frac{1}{r} \frac{\partial w}{\partial r} \simeq \sum_{p=1}^P \alpha^{(p)} \varphi_1^{(p)}(r, z), \quad (7.9)$$

where $\alpha^{(p)}$ is a constant coefficient and the interpolating function $\varphi_1^{(p)}(r, z)$ centered about $(r_0^{(p)}, z_0^{(p)})$ is taken here to be of the form

$$\varphi_1^{(p)}(r, z) = \sigma(r, z; r_0^{(p)}, z_0^{(p)}) + \sigma(r, z; -r_0^{(p)}, z_0^{(p)}), \quad (7.10)$$

where

$$\sigma(r, z; r_0^{(p)}, z_0^{(p)}) = \sqrt{(r - r_0^{(p)})^2 + (z - z_0^{(p)})^2}. \quad (7.11)$$

We can let (r, z) in (7.9) be given by $(r_0^{(m)}, z_0^{(m)})$ for $m = 1, 2, \dots, P$, to set up a system of linear algebraic equations in $\alpha^{(p)}$. The system can be inverted to obtain

$$\alpha^{(p)} = \sum_{m=1}^P \left\{ -\frac{Q^{(m)}}{\sqrt{g^{(m)}}} + B^{(m)}w^{(m)} - \left(\frac{1}{r} \frac{\partial w}{\partial r} \right) \Big|_{(r,z)=(r_0^{(m)}, z_0^{(m)})} \right\} \chi^{(mp)}, \quad (7.12)$$

where $w^{(m)} = w(r_0^{(m)}, z_0^{(m)})$, $B^{(m)} = B(r_0^{(m)}, z_0^{(m)})$, $Q^{(m)} = Q(r_0^{(m)}, z_0^{(m)})$, $g^{(m)} = g(r_0^{(m)}, z_0^{(m)})$ and $\chi^{(mp)}$ are constants defined by

$$\sum_{m=1}^P \varphi_1^{(p)}(r_0^{(m)}, z_0^{(m)}) \chi^{(mr)} = \begin{cases} 1 & \text{if } p = r, \\ 0 & \text{if } p \neq r. \end{cases} \quad (7.13)$$

Note that the value of $(1/r)\partial w/\partial r$ at $(r, z) = (r_0^{(m)}, z_0^{(m)})$ must be interpreted in a limiting sense if $r_0^{(m)} = 0$, that is,

$$\left(\frac{1}{r} \frac{\partial w}{\partial r} \right) \Big|_{(r,z)=(r_0^{(m)}, z_0^{(m)})} = \lim_{r \rightarrow 0^+} \left(\frac{1}{r} \frac{\partial w}{\partial r} \right) \Big|_{z=z_0^{(m)}} \quad \text{if } r_0^{(m)} = 0. \quad (7.14)$$

Guided by the analysis in Ang [6] and Dobroskok and Linkov [35], for the solution of (7.5), we write

$$w(r, z) = w_0(r, z) + w_1(r, z) \quad (7.15)$$

and choose $w_0(r, z)$ to satisfy

$$\frac{\partial^2 w_0}{\partial r^2} + \frac{\partial^2 w_0}{\partial z^2} = -\frac{Q(r, z)}{\sqrt{g(r, z)}} + B(r, z)w - \frac{1}{r} \frac{\partial w}{\partial r}, \quad (7.16)$$

so that $w_1(r, z)$ is to be obtained by solving

$$\frac{\partial^2 w_1}{\partial r^2} + \frac{\partial^2 w_1}{\partial z^2} = 0. \quad (7.17)$$

Noted that the axisymmetric interpolating functions in (7.10), which is similar to the interpolating functions proposed in Chapter 3.

From (7.9), (7.10) and (7.12), an approximate solution of (7.16) may be given by

$$\begin{aligned} w_0(r, z) \simeq & \sum_{m=1}^P D^{(m)}(r, z) \left\{ -\frac{Q^{(m)}}{\sqrt{g^{(m)}}} + B^{(m)} w^{(m)} \right. \\ & \left. - \left(\frac{1}{r} \frac{\partial w}{\partial r} \right) \Big|_{(r, z) = (r_0^{(m)}, z_0^{(m)})} \right\}, \end{aligned} \quad (7.18)$$

where

$$D^{(m)}(r, z) = \sum_{p=1}^P \chi^{(mp)} \tau^{(p)}(r, z), \quad (7.19)$$

and

$$\begin{aligned} \tau^{(p)}(r, z) = & \frac{1}{9} \sigma(r, z; r_0^{(p)}, z_0^{(p)})^3 \\ & + \frac{1}{9} \sigma(r, z; -r_0^{(p)}, z_0^{(p)})^3. \end{aligned} \quad (7.20)$$

To work out a suitable formula for approximating $\partial w / \partial r$ at $(r, z) = (r_0^{(m)}, z_0^{(m)})$, take

$$w(r, z) \simeq \sum_{p=1}^P \beta^{(p)} \varphi_2^{(p)}(r, z), \quad (7.21)$$

where

$$\begin{aligned} \varphi_2^{(p)}(r, z) = & \sigma(r, z; r_0^{(p)}, z_0^{(p)})^5 \\ & + \sigma(r, z; -r_0^{(p)}, z_0^{(p)})^5 \end{aligned} \quad (7.22)$$

If we let (r, z) in (7.21) be given by $(r_0^{(m)}, z_0^{(m)})$ for $m = 1, 2, \dots, P$ to set up a system of linear algebraic equations in $\beta^{(p)}$, the system can be inverted to obtain

$$\beta^{(p)} = \sum_{m=1}^P \eta^{(mp)} w^{(m)}, \quad (7.23)$$

where

$$\sum_{m=1}^P \varphi_2^{(p)}(r_0^{(m)}, z_0^{(m)}) \eta^{(mr)} = \begin{cases} 1 & \text{if } p = r, \\ 0 & \text{if } p \neq r. \end{cases} \quad (7.24)$$

From (7.21) and (7.23), we obtain the approximation

$$\frac{1}{r} \frac{\partial w}{\partial r} \simeq \sum_{n=1}^P L^{(n)}(r, z) w^{(n)}, \quad (7.25)$$

where

$$L^{(n)}(r, z) = \sum_{p=1}^P \eta^{(np)} \frac{1}{r} \frac{\partial}{\partial r} (\varphi_2^{(p)}(r, z)) \text{ for } r > 0. \quad (7.26)$$

Because of the choice of $\varphi_2^{(p)}(r, z)$ in (7.22), the function $L^{(n)}(r, z)$ is bounded in the entire region $r > 0$ and it tends to a finite value as $r \rightarrow 0^+$. The complex variable boundary element method which will be presented later requires the evaluation of $L^{(n)}(r, z)$ at $r = 0$, if the solution domain R contains points (x, y, z) such that $x^2 + y^2 = 0$. We define $L^{(n)}(0, z)$ as the limit of $L^{(n)}(r, z)$ as $r \rightarrow 0^+$. More specifically,

$$\begin{aligned} L^{(n)}(0, z) = & \sum_{p=1}^P 10\eta^{(np)} \{ 3[r_0^{(p)}]^2 ([r_0^{(p)}]^2 + [z - z_0^{(p)}]^2)^{1/2} \\ & + ([r_0^{(p)}]^2 + [z - z_0^{(p)}]^2)^{3/2} \}. \end{aligned} \quad (7.27)$$

Also, note that the choice of $\varphi_2^{(p)}(r, z)$ in (7.22) ensures that $w(r, z)$ as approximated in (7.21) is sufficiently smooth to give a good approximation of $(1/r)\partial w/\partial r$ at or near $r = 0$. For $w(r, z)$ given by particular functions, numerical experiments indicate that the approximation of $(1/r)\partial w/\partial r$ in (7.25) near

$r = 0$ is quite poor in accuracy if we replace $\varphi_2^{(p)}(r, z)$ in (7.22) by less smooth interpolating functions such as $\sigma(r, z; r_0^{(p)}, z_0^{(p)})^3 + \sigma(r, z; -r_0^{(p)}, z_0^{(p)})^3$.

With (7.25), we find that (7.18) can be rewritten as

$$\begin{aligned} w_0(r, z) \simeq & \sum_{m=1}^P D^{(m)}(r, z) \left\{ -\frac{Q^{(m)}}{\sqrt{g^{(m)}}} + B^{(m)} w^{(m)} \right. \\ & \left. - \sum_{n=1}^P L^{(n)}(r_0^{(m)}, z_0^{(m)}) w^{(n)} \right\}. \end{aligned} \quad (7.28)$$

The general solution of (7.17) is given by

$$w_1(r, z) = \operatorname{Re}\{F(z + ir)\}, \quad (7.29)$$

where $i = \sqrt{-1}$ and $F(z + ir)$ is an arbitrary complex function that is analytic in the region Ω on the rz plane, this is as explained in Section 2.5.

The boundary condition in (7.7) can be rewritten as

$$\begin{aligned} w(r, z) &= p_1(r, z) \text{ for } (r, z) \in \Gamma_1, \\ &\sum_{m=1}^P E^{(m)}(r, z) w^{(m)} + p_3(r, z) \operatorname{Re}\{F(z + ir)\} \\ &+ \operatorname{Re}\{[n_z(r, z) + in_r(r, z)]F'(z + ir)\} \\ &= p_2(r, z) \text{ for } (r, z) \in \Gamma_2, \end{aligned} \quad (7.30)$$

where the prime denotes differentiation with respect to the relevant argument and

$$\begin{aligned} p_1(r, z) &= \sqrt{g(r, z)} f_1(r, z), \\ p_2(r, z) &= \frac{f_2(r, z)}{\sqrt{g(r, z)}} + \sum_{m=1}^P \{G^{(m)}(r, z) + D^{(m)}(r, z) p_3(r, z)\} \frac{Q^{(m)}}{\sqrt{g^{(m)}}}, \end{aligned}$$

$$\begin{aligned}
 p_3(r, z) &= -\frac{1}{g(r, z)} \left\{ \frac{1}{2} \frac{\partial}{\partial n} [g(r, z)] + f_3(r, z) \right\}, \\
 E^{(m)}(r, z) &= B^{(m)} [G^{(m)}(r, z) + D^{(m)}(r, z) p_3(r, z)] \\
 &\quad - \sum_{n=1}^P [G^{(n)}(r, z) + D^{(n)}(r, z) p_3(r, z)] L^{(m)}(r_0^{(n)}, z_0^{(n)}) \\
 G^{(m)}(r, z) &= n_r(r, z) \frac{\partial}{\partial r} [D^{(m)}(r, z)] + n_z(r, z) \frac{\partial}{\partial z} [D^{(m)}(r, z)]. \quad (7.31)
 \end{aligned}$$

If we can construct $F(z + ir)$ that is analytic in Ω and find the constants $w^{(1)}, w^{(2)}, \dots, w^{(P-1)}$ and $w^{(P)}$ such that (7.30) is satisfied, then we have approximately solved the boundary value problem stated in Section 7.2. The required solution of the boundary value problem is then approximately given by

$$\begin{aligned}
 T(r, z) \simeq & \frac{1}{\sqrt{g(r, z)}} \left(\sum_{m=1}^P D^{(m)}(r, z) \left\{ -\frac{Q^{(m)}}{\sqrt{g^{(m)}}} + B^{(m)} w^{(m)} \right. \right. \\
 & \left. \left. - \sum_{n=1}^P L^{(n)}(r_0^{(n)}, z_0^{(n)}) w^{(n)} \right\} + \operatorname{Re}\{F(z + ir)\} \right). \quad (7.32)
 \end{aligned}$$

7.4 Complex Variable Boundary Element Procedures

According to the Cauchy integral formula (as shown in Section 2.5), for $(r_0, z_0) \in \Omega$, we can write

$$2\pi i F(z_0 + ir_0) = \oint_{(z,r) \in \Xi} \frac{F(z + ir) d(z + ir)}{(z - z_0 + i[r - r_0])}, \quad (7.33)$$

$$2\pi i F'(z_0 + ir_0) = \oint_{(z,r) \in \Xi} \frac{F(z + ir) d(z + ir)}{(z - z_0 + i[r - r_0])^2}, \quad (7.34)$$

where Ξ the curve enclosing the region Ω is assigned the anticlockwise direction.

For the region Ω as sketched in Figure 2.1 (where the curve Γ is not a closed curve), the curve Ξ in (7.33) and (7.34) comprises Γ and C , where C is the

portion on the z axis (Γ and C form a closed curve). As $\partial T/\partial r$ is expected to be 0 on C , we may impose the additional condition:

$$\begin{aligned} & \sum_{m=1}^P E^{(m)}(r, z) w^{(m)} + p_3(r, z) \operatorname{Re}\{F(z + ir)\} \\ & + \operatorname{Re}\{[n_z(r, z) + in_r(r, z)]F'(z + ir)\} \\ = & p_4(r, z) \text{ for } (r, z) \text{ on } C, \end{aligned} \quad (7.35)$$

where

$$p_4(r, z) = \sum_{m=1}^P \{G^{(m)}(r, z) + D^{(m)}(r, z)p_3(r, z)\} \frac{Q^{(m)}}{\sqrt{g^{(m)}}}. \quad (7.36)$$

Note that $n_z(r, z) = 0$ and $n_r(r, z) = -1$ for $(r, z) \in C$. Also, the definition of function $f_3(r, z)$ in $p_3(r, z)$ is extended to include $f_3(r, z) = 0$ for $(r, z) \in C$.

If Γ is a closed curve then $\Xi = \Gamma$ and (7.35) is not applicable.

We shall now apply (7.33) and (7.34) to devise a procedure for constructing numerically $F(z + ir)$ and finding $w^{(1)}, w^{(2)}, \dots, w^{(P-1)}$ and $w^{(P)}$ to satisfy the boundary conditions on Ξ . The boundary conditions are given by (7.30) and (7.35) (if the latter is applicable).

Put N closely packed points $(r^{(1)}, z^{(1)}), (r^{(2)}, z^{(2)}), \dots, (r^{(N-1)}, z^{(N-1)})$ and $(r^{(N)}, z^{(N)})$ on the curve Ξ following the anticlockwise direction. For $m = 1, 2, \dots, N$, define $\Xi^{(m)}$ to be the straight line segment from $(r^{(m)}, z^{(m)})$ to $(r^{(m+1)}, z^{(m+1)})$ (with $(r^{(N+1)}, z^{(N+1)}) = (r^{(1)}, z^{(1)})$). The first N collocation points in (7.9) and (7.10) are chosen to be midpoints of $\Xi^{(1)}, \Xi^{(2)}, \dots, \Xi^{(N-1)}$ and $\Xi^{(N)}$, that is,

$$(r_0^{(m)}, z_0^{(m)}) = \frac{1}{2}(r^{(m)} + r^{(m+1)}, z^{(m)} + z^{(m+1)}) \text{ for } m = 1, 2, \dots, N. \quad (7.37)$$

Another M collocation points are given by $(r_0^{(N+1)}, z_0^{(N+1)}), (r_0^{(N+2)}, z_0^{(N+2)}), \dots, (r_0^{(N+M-1)}, z_0^{(N+M-1)})$ and $(r_0^{(N+M)}, z_0^{(N+M)})$ are chosen to lie in the interior of Ω . (Thus, the total number of collocation points is given by $P = N + M$.)

Following Park and Ang [72], we make the approximation $\Xi \simeq \Xi^{(1)} \cup \Xi^{(2)} \cup \dots \cup \Xi^{(N)}$ and discretize the Cauchy integral formula in (7.33) as

$$2\pi i F(Z) = \sum_{k=1}^N (u^{(k)} + iv^{(k)}) [\lambda(Z^{(k)}, Z^{(k+1)}, Z) + i\theta(Z^{(k)}, Z^{(k+1)}, Z)]$$

for $Z \in \Omega$, (7.38)

where $Z = z + ir$, $Z^{(m)} = z^{(m)} + ir^{(m)}$, $u^{(k)}$ and $v^{(k)}$ are real constants given by $u^{(k)} + iv^{(k)} = F(z_0^{(k)} + ir_0^{(k)})$ and λ and θ are real parameters defined by

$$\begin{aligned} \lambda(a, b, c) &= \ln |b - c| - \ln |a - c| \\ \theta(a, b, c) &= \begin{cases} \Theta(a, b, c) & \text{if } \Theta(a, b, c) \in (-\pi, \pi] \\ \Theta(a, b, c) + 2\pi & \text{if } \Theta(a, b, c) \in [-2\pi, -\pi] \\ \Theta(a, b, c) - 2\pi & \text{if } \Theta(a, b, c) \in (\pi, 2\pi] \end{cases} \\ \Theta(a, b, c) &= \text{Arg}(b - c) - \text{Arg}(a - c). \end{aligned} \quad (7.39)$$

Note that $\text{Arg}(z)$ denotes the principal argument of the complex number z . If the solution domain Ω is convex in shape, $\theta(a, b, c)$ can be calculated directly from

$$\theta(a, b, c) = \cos^{-1} \left(\frac{|b - c|^2 + |a - c|^2 - |b - a|^2}{2 |b - c| |a - c|} \right). \quad (7.40)$$

If we let $Z \rightarrow \widehat{Z}^{(k)} = z_0^{(k)} + ir_0^{(k)}$ (for each of the collocation points), the imaginary part of (7.38) gives

$$u^{(k)} = \frac{1}{2\pi} \sum_{m=1}^N \{u^{(m)} \theta(Z^{(m)}, Z^{(m+1)}, \widehat{Z}^{(k)}) + v^{(m)} \lambda(Z^{(m)}, Z^{(m+1)}, \widehat{Z}^{(k)})\}$$

for $k = 1, 2, \dots, N + M$. (7.41)

From (7.15), (7.28) and (7.29), we find that

$$u^{(k)} = \sum_{p=1}^{N+M} c^{(kp)} w^{(p)} + h^{(k)}, \quad (7.42)$$

where

$$\begin{aligned}
 c^{(kp)} &= -D^{(p)}(r_0^{(k)}, z_0^{(k)})B^{(p)} + \sum_{n=1}^{N+M} D^{(n)}(r_0^{(k)}, z_0^{(k)})L^{(p)}(r_0^{(n)}, z_0^{(n)}) \\
 &\quad + \begin{cases} 1 & \text{if } k = p \\ 0 & \text{if } k \neq p \end{cases}, \\
 h^{(k)} &= \sum_{m=1}^{N+M} D^{(m)}(r_0^{(k)}, z_0^{(k)}) \frac{Q^{(m)}}{\sqrt{g^{(m)}}}.
 \end{aligned} \tag{7.43}$$

Hence, (7.41) can be written as

$$\begin{aligned}
 \sum_{p=1}^{N+M} d^{(kp)} w^{(p)} + e^{(k)} &= \frac{1}{2\pi} \sum_{m=1}^N v^{(m)} \lambda(Z^{(m)}, Z^{(m+1)}, \widehat{Z}^{(k)}) \\
 &\text{for } k = 1, 2, \dots, N + M,
 \end{aligned} \tag{7.44}$$

where

$$\begin{aligned}
 e^{(k)} &= h^{(k)} - \frac{1}{2\pi} \sum_{m=1}^N h^{(m)} \theta(Z^{(m)}, Z^{(m+1)}, \widehat{Z}^{(k)}), \\
 d^{(kp)} &= c^{(kp)} - \frac{1}{2\pi} \sum_{m=1}^N c^{(mp)} \theta(Z^{(m)}, Z^{(m+1)}, \widehat{Z}^{(k)}).
 \end{aligned} \tag{7.45}$$

The first boundary condition in (7.30) can be written as

$$w^{(k)} = p_1(r_0^{(k)}, z_0^{(k)}) \text{ if } T \text{ is specified on } \Xi^{(k)}. \tag{7.46}$$

The formula in (7.34) can be used to derive

$$\begin{aligned}
 \pi i F'(\widehat{Z}^{(k)}) &= \sum_{m=1}^N (u^{(m)} + i v^{(m)}) \\
 &\quad \times [q(Z^{(m)}, Z^{(m+1)}, \widehat{Z}^{(k)}) + i r(Z^{(m)}, Z^{(m+1)}, \widehat{Z}^{(k)})] \\
 &\text{for } k = 1, 2, \dots, N,
 \end{aligned} \tag{7.47}$$

where q and r are real parameters defined by

$$q(a, b, c) + i r(a, b, c) = -\frac{1}{b - c} + \frac{1}{a - c}. \tag{7.48}$$

For further details, one may refer to Park and Ang [72].

With (7.47), the boundary condition on the second line of (7.30) and the one in (7.35) can be written as

$$\sum_{p=1}^{N+M} T^{(kp)} w^{(p)} - \sum_{m=1}^N Y^{(km)} v^{(m)} = X^{(k)} \text{ if } \frac{\partial T}{\partial n} \text{ is specified on } \Xi^{(k)}, \quad (7.49)$$

where $T^{(kp)}$ are given by

$$T^{(kp)} = E^{(p)}(r_0^{(k)}, z_0^{(k)}) + p_3(r_0^{(k)}, z_0^{(k)}) c^{(kp)} + \sum_{m=1}^N c^{(mp)} R^{(km)}, \quad (7.50)$$

the coefficients $X^{(k)}$ are defined by

$$\begin{aligned} X^{(k)} = & -p_3(r_0^{(k)}, z_0^{(k)}) h^{(k)} - \sum_{m=1}^N h^{(m)} R^{(km)} \\ & + \begin{cases} p_2(r_0^{(k)}, z_0^{(k)}) & \text{if } \Xi^{(k)} \text{ does not lie on } r = 0 \\ p_4(r_0^{(k)}, z_0^{(k)}) & \text{if } \Xi^{(k)} \text{ lies on } r = 0 \end{cases}, \end{aligned} \quad (7.51)$$

the real parameters $R^{(km)}$ and $Y^{(km)}$ are defined by

$$\begin{aligned} R^{(km)} + iY^{(km)} = & \frac{1}{\pi} (r(Z^{(m)}, Z^{(m+1)}, \widehat{Z}^{(k)}) - iq(Z^{(m)}, Z^{(m+1)}, \widehat{Z}^{(k)})) \\ & \times [n_z^{(k)} + in_r^{(k)}], \end{aligned} \quad (7.52)$$

and $[n_r^{(k)}, n_z^{(k)}]$ is the outward unit normal vector to $\Gamma^{(k)}$.

Equations (7.46) and (7.49) require the functions p_1 , p_2 and p_3 to be evaluated at the midpoint $(r_0^{(k)}, z_0^{(k)})$ of the boundary element $\Xi^{(k)}$. According to (7.31), p_1 , p_2 and p_3 are expressed in terms of f_1 , f_2 and f_3 given by the boundary conditions in (7.7). Now, depending on the geometry of Γ , the midpoint $(r_0^{(k)}, z_0^{(k)})$ may or may not lie on the actual physical boundary Γ . If $(r_0^{(k)}, z_0^{(k)})$ does not lie on Γ , then the value of f_i ($i = 1, 2, 3$) needed in the calculation of p_i at $(r_0^{(k)}, z_0^{(k)})$ may be taken to be given by the average value of f_i at the

endpoints of $\Xi^{(k)}$, as the endpoints of every boundary element are chosen to lie on Γ .

We may solve (7.44) for $k = 1, 2, 3, \dots, N + M$, together with (7.46), (7.49) and $v^{(N)} = 0$, as a system of $2N + M$ linear algebraic equations for the constants $w^{(p)}$ ($p = 1, 2, \dots, N + M$) and $v^{(m)}$ ($m = 1, 2, \dots, N - 1$). Note that $v^{(N)}$ is set to zero to ensure that the imaginary part of the complex function $F(z + ir)$ is uniquely determined by the linear algebraic equations. It is assumed that T is specified on some part of Γ so that the temperature (hence the real part of $F(z + ir)$) is unique. The over-determined system of linear algebraic equations may be solved by using the method of least squares. Once the constants $w^{(p)}$ are found, $u^{(k)}$ can be computed from (7.42) and the required complex function $F(Z)$ is given by (7.38). Note that the value of the solution T at the collocation point $(r_0^{(k)}, z_0^{(k)})$ is given $w^{(k)} / \sqrt{g(r_0^{(k)}, z_0^{(k)})}$. The value of T at any other point in the solution domain can be approximately calculated from (7.32) and (7.38).

7.5 Specific Problems

To assess the validity and accuracy of the complex variable boundary element method here, it is applied to solve the following specific problems.

Problem 1

Here the governing partial differential equation given by (7.1) and (7.2) with

$$\begin{aligned} g(r, z) &= (r^2 + 1)^2, \\ Q(r, z) &= \frac{\pi^2}{16}(r^2 + 1)^2 \cos\left(\frac{\pi}{4}z\right) - 12r^4 - 16r^2 - 4, \end{aligned}$$

is to be solved in the domain as $0 < r < 1$, $0 < z < 1$, subject to

$$\begin{aligned}\left.\frac{\partial T}{\partial n}\right|_{z=0} &= 0 \text{ for } 0 < r < 1, \\ \left.\frac{\partial T}{\partial n}\right|_{z=1} &= -\frac{\pi\sqrt{2}}{8} \text{ for } 0 < r < 1, \\ T(1, z) &= 1 + \cos\left(\frac{\pi}{4}z\right) \text{ for } 0 < z < 1.\end{aligned}$$

As the solution domain contains points on $r = 0$, the complex variable boundary element procedure requires the additional condition

$$\left.\frac{\partial T}{\partial n}\right|_{r=0} = 0 \text{ for } 0 < z < 1.$$

In order to obtain some numerical results, the open boundary Γ and the line segment $r = 0$, $0 < z < 1$, are discretized into N equal length elements and M well spaced out collocation points are chosen inside the domain. In Table 7.1, three sets of numerical values of T are obtained by and compared with the exact solution

$$T(r, z) = r^2 + \cos\left(\frac{\pi}{4}z\right)$$

at 9 selected interior collocation points. Sets A, B and C are calculated using $(N, M) = (20, 9)$, $(N, M) = (40, 49)$ and $(N, M) = (80, 361)$ respectively. The average absolute errors of the numerical values of T at the selected points are given in the last row of Table 7.1. It is obvious that there is a significant improvement in the accuracy of the numerical values of T as N and M increases. The convergence rate is as may be expected since only constant elements are employed in the calculation here.

Table 7.1: Numerical and exact values of T at selected interior points.

Point (r, z)	Set A	Set B	Set C	Exact
(0.25, 0.25)	1.11764	1.05853	1.05241	1.04329
(0.50, 0.25)	1.25478	1.25205	1.24301	1.23079
(0.75, 0.25)	1.55035	1.57106	1.55906	1.54329
(0.25, 0.50)	1.01564	1.00502	0.99734	0.98638
(0.50, 0.50)	1.20628	1.19799	1.18763	1.17388
(0.75, 0.50)	1.52317	1.51689	1.50359	1.48638
(0.25, 0.75)	0.91183	0.90878	0.90275	0.89397
(0.50, 0.75)	1.11442	1.10213	1.09327	1.08147
(0.75, 0.75)	1.43680	1.42092	1.40921	1.39397
Average absolute error	0.03306	0.02100	0.01191	-

Problem 2

The governing partial differential equation is given by (7.1) and (7.2) with

$$\begin{aligned} g(r, z) &= z + 1, \\ Q(r, z) &= -4z - 6. \end{aligned}$$

It is to be solved in a concave solution domain, specifically the one sketched in Figure 7.1. Note that the curved part of the boundary of the solution domain is defined by $(r - 2)^2 + (z - 2)^2 = 1$, $1 < r < 2$, $1 < z < 2$.

The boundary conditions of the problem are

$$\begin{aligned} \frac{\partial}{\partial n}[T(r, z)] \Big|_{z=2} &= 2 \text{ for } 0 < r < 1, \\ \frac{\partial}{\partial n}[T(r, z)] &= 2r(r - 2) + 2(z - 2) \\ &\text{for } (r - 2)^2 + (z - 2)^2 = 1, 1 < r < 2, 1 < z < 2, \\ T(2, z) &= 4 + 2z \text{ for } 0 < z < 1, \\ T(r, 0) &= r^2 \text{ for } 0 < r < 2. \end{aligned}$$

For the purpose of obtaining some numerical results, the curved part of

Figure 7.1: Solution domain for Problem 2.

the boundary is discretized into $2N_0$ elements, each of the straight parts given by $r = 0$, $0 < z < 2$ and $z = 0$, $0 < r < 2$ into $2N_0$ elements, and each of the remaining parts given by $r = 2$, $0 < z < 1$ and $z = 2$, $0 < r < 1$ into N_0 elements. Thus, the total number of elements is given by $N = 8N_0$. The interior collocation points are chosen to be evenly distributed inside the solution domain.

The normal derivative of the temperature (that is, the heat flux) is specified on the curve part of the boundary. We compare the numerically computed temperature on the semi-circle with the exact temperature given by

$$T(r, z) = r^2 + 2z.$$

Figure 7.2 gives plots of the numerical and exact temperature against the angle $\theta = \arctan(r/z)$ for $(r - 2)^2 + (z - 2)^2 = 1$, $1 < r < 2$, $1 < z < 2$. The two plots obtained by using 192 boundary elements ($N_0 = 24$) and 419 interior

collocation points are in good agreement with each other.

Figure 7.2: Plots of numerical and exact temperature on the curved part of the boundary.

The normal heat flux $g\partial T/\partial n$ on $r = 2, 0 < z < 1$ is not known a priori (from the boundary conditions of the problem). It can be calculated directly from the complex variable boundary element solution. In Figure 7.3, the numerically calculated normal heat flux on $r = 2, 0 < z < 1$ is plotted against z and compared with the one computed from the exact solution. The numerical and the exact values of the flux show good agreement with each other, except at points near the sharp corners $(2, 0)$ and $(2, 1)$ where there is a loss in the accuracy of the numerical calculation. Nevertheless, at any fixed point near a sharp corner point, further calculations show that the accuracy of the numerical flux can be improved significantly by employing more elements near the corner.

Figure 7.3: Plots of numerical and exact flux on $r = 2$, $0 < z < 1$.

Problem 3

Take the solution domain to be $1 < r < 2, 1 < z < 2$ (a hollow cylinder) and the function g and Q in (7.1) and (7.2) to be given by

$$\begin{aligned} g(r, z) &= r + z, \\ Q(r, z) &= -\frac{1}{r} - 1. \end{aligned}$$

The governing partial differential equation is to be solved in the solution domain subject to the boundary conditions

$$\begin{aligned} \left. \frac{\partial T}{\partial n} \right|_{r=1} &= -1 \text{ for } 1 < z < 2, \\ \left. \frac{\partial T}{\partial n} \right|_{r=2} &= \frac{1}{2} \text{ for } 1 < z < 2, \\ T(r, 1) &= 1 + \ln r \text{ for } 1 < r < 2, \\ T(r, 2) &= 2 + \ln r \text{ for } 1 < r < 2. \end{aligned}$$

To obtain some numerical results, each side of the square solution domain in the rz plane is discretized into N_0 equal length elements. The interior collocation points are chosen as $(1 + j/N_0, 1 + k/N_0)$ for $j = 1, 2, \dots, N_0 - 1$ and $k = 1, 2, \dots, N_0 - 1$, so the total interior collocation points are $M = (N_0 - 1)^2$. In Table 7.2, numerical values of T at selected interior points, which are obtained using $N_0 = 4, 12$ and 24 , are compared with the exact solution

$$T(r, z) = z + \ln r.$$

There is an obvious reduction in the average absolute error of the numerical values at the selected interior points when the calculation is refined using larger N_0 . The average absolute error for $N_0 = 4$ is two and the half times larger than that for $N_0 = 12$, and the average absolute error for $N_0 = 12$ is twice as large as that for $N_0 = 24$.

Table 7.2: Numerical and exact values of T at selected interior points.

(r, z)	$N_0 = 4$	$N_0 = 12$	$N_0 = 24$	Exact
(1.25, 1.25)	1.43249	1.46465	1.46904	1.47314
(1.25, 1.50)	1.63930	1.64889	1.65221	1.65547
(1.25, 1.75)	1.79015	1.80422	1.80694	1.80962
(1.50, 1.25)	1.71717	1.72041	1.72183	1.72314
(1.50, 1.50)	1.90420	1.90527	1.90535	1.90547
(1.50, 1.75)	2.05984	2.06070	2.06013	2.059616
(1.75, 1.25)	1.98831	1.97740	1.97520	1.973144
(1.75, 1.50)	2.16944	2.16182	2.15856	2.15547
(1.75, 1.75)	2.34167	2.31680	2.31312	2.30962
Average absolute error	1.61×10^{-2}	0.61×10^{-2}	0.31×10^{-2}	-

Problem 4

Consider a solid cylinder that occupies the region $0 < r < L$, $0 < z < L$, where L is a positive constant. The thermal conductivity κ of the solid varies in the z direction as $\kappa = \kappa_0(1 + \alpha z)^2$, where κ_0 and α are positive constants.

It is assumed that there is no internal heat generation in the cylindrical solid, that is, $Q = 0$. A portion of the cylindrical surface at $z = 0$ is subject to heating with uniform flux q_0 , while the cylindrical surface at $z = L$ has heat removed by the convection process. The remaining cylindrical surface is thermally insulated. More precisely, the boundary conditions are given by

$$\begin{aligned}\kappa \frac{\partial T}{\partial n} &= h_{\text{amb}}(T_{\text{amb}} - T) \text{ for } 0 < r < L, z = L, \\ \kappa \frac{\partial T}{\partial n} &= 0 \text{ for } 0 < z < L, r = L, \\ \kappa \frac{\partial T}{\partial n} &= 0 \text{ for } L/2 < r < L, z = 0, \\ \kappa \frac{\partial T}{\partial n} &= q_0 \text{ for } 0 < r < L/2, z = 0,\end{aligned}$$

where h_{amb} and T_{amb} are the ambient heat convection coefficient and ambient temperature respectively.

To compute the non-dimensionalized temperature $\kappa_0(T - T_{\text{amb}})/(q_0 L)$ using the complex variable boundary element method (CVBEM) here, the exterior boundary of the cylindrical solid on the axisymmetric coordinate plane is discretized into $3N_0$, and $(N_0 - 1)^2$ well distributed interior collocation points are chosen. As the solution domain contains points on the z axis, the complex variable boundary element method procedure requires the line segment $0 < z < L$, $r = 0$, to be discretized into elements. It (the line segment) is divided up into N_0 elements. For comparison, numerical values of $\kappa_0(T - T_{\text{amb}})/(q_0 L)$ are also obtained by using the axisymmetric boundary element method (A-BEM) with discontinuous linear elements as described in Chapter 3 and 4.

Table 7.3: Numerical values of $\kappa_0(T - T_{\text{amb}})/(q_0L)$.

	$N_0 = 8$		$N_0 = 16$		$N_0 = 20$	
$(r/L, z/L)$	CVBEM	A-BEM	CVBEM	A-BEM	CVBEM	A-BEM
(0.25, 0.25)	0.25290	0.24883	0.25087	0.24880	0.25044	0.24880
(0.25, 0.50)	0.19919	0.19993	0.20007	0.19993	0.20009	0.19993
(0.25, 0.75)	0.15342	0.15686	0.15572	0.15688	0.15601	0.15688
(0.50, 0.25)	0.14558	0.14515	0.14555	0.14515	0.14549	0.14515
(0.50, 0.50)	0.12652	0.12776	0.12748	0.12776	0.12758	0.12776
(0.50, 0.75)	0.10942	0.11190	0.11105	0.11191	0.11127	0.11191
(0.75, 0.25)	0.07174	0.07260	0.07236	0.07260	0.07244	0.07260
(0.75, 0.50)	0.06535	0.06674	0.06631	0.06674	0.06643	0.06674
(0.75, 0.75)	0.05919	0.06129	0.06054	0.06129	0.06074	0.06129

For $\alpha L = 1/10$, the numerical values of $\kappa_0(T - T_{\text{amb}})/(q_0L)$ at selected interior points, obtained using the CVBEM and the A-BEM with $N_0 = 8$, 16 and 20, are given in Table 7.3. It is obvious that the CVBEM and A-BEM solutions approach each other as N_0 increases. Note that only constant elements are used in the CVBEM. Thus, as expected, the convergence rate of the CVBEM solution is slower than that of the linear element A-BEM.

On the plane $0 < r/L < 1$, $z/L = 0$, where the surface heating occurs, the temperature is not known a priori. We plot the non-dimensionalized temperature $\kappa_0(T - T_{\text{amb}})/(q_0L)$ against r/L on the surface $0 < r/L < 1$, $z/L = 0$, for selected values of the non-dimensionalized parameter αL in Figure 7.4. The plots in Figure 7.4 are obtained using the CVBEM and the A-BEM with $N_0 = 16$. The numerical values of the surface temperature calculated using the CVBEM are in close agreement with those obtained using the A-BEM. The temperature is lower if αL has a larger value. This is to be expected, as the solid conducts heat away better from where the surface heating occurs if

the thermal conductivity $\kappa = \kappa_0(1 + \alpha z)^2$ is larger.

Figure 7.4: Plots of $\kappa_0(T - T_{\text{amb}})/(q_0 L)$ against r/L on the surface $0 < r/L < 1$, $z/L = 0$.

Different mathematical approaches are used in the CVBEM and the A-BEM to reduce the problem under consideration to linear algebraic equations. In the A-BEM, the coefficients of the linear algebraic equations are given by integrals with rather complicated integrands which are expressed in terms of elliptic integrals. The integrands are singular if the collocation point (in the boundary element procedure) lies on the element where the integration is performed. The integrals in the coefficients of the linear algebraic equations in the A-BEM have to be evaluated numerically. Thus, the computational speed

of the A-BEM, even for constant elements, is highly dependent on the numerical integration quadratures chosen to evaluate the coefficients. On the other hand, the coefficients of the linear algebraic equations in the CVBEM here are expressed explicitly in terms of simple elementary functions and can be evaluated analytically. However, a few parts of those coefficients are given by expressions involving multiple sum (specifically those involving the function $w_0(r, z)$ in (7.28)), which may become computationally less efficient to evaluate for a very high number of collocation points. Thus, it may be difficult to compare the CVBEM and the A-BEM directly in a fair manner.

Nevertheless, we have made a straightforward comparison of the computational time needed by the CVBEM and the A-BEM codes to set up the linear algebraic equations for the particular problem here, with N_0 (the integer that determines the number of elements and collocation points) given value up to 16. In general, it appears that the computational time needed by both codes is of the same order of magnitude. If N_0 is equal to 12 or less, the CVBEM seems to perform better than the A-BEM. For example, the time needed by the A-BEM to set up the linear algebraic equations is around 3 times bigger than that needed by the CVBEM if $N_0 \leq 8$. It appears that the efficiency of the CVBEM and A-BEM codes are roughly the same for N_0 between 12 and 16. For $N_0 = 16$, we find that the time needed by the CVBEM is about 1.8 times larger than that needed by the A-BEM.

7.6 Summary

The problem of axisymmetric steady-state heat conduction problem in a non-homogeneous isotropic solid is considered here. The problem is reformulated approximately as one governed by the two-dimensional Laplace's equation to

be solved by constructing a suitable analytic complex function. The Cauchy integral formula and its differentiated form are used to reduce the numerical construction of the analytic function for solving a system of linear algebraic equations. The numerical procedure does not require the solution domain to be divided into small elements. Only the boundary is discretized into straight line elements.

To assess its validity and accuracy, the proposed complex variable boundary element procedure is applied to solve some specific cases of the axisymmetric heat conduction problem. In all the cases, the numerical solutions obtained agree favourably with known solutions and convergence in the numerical values obtained is observed when the number of boundary elements and interior collocation points is increased. This suggests that the complex variable boundary element formulation presented here is correctly derived and it can be used as an accurate and reliable tool for the analysis of the axisymmetric heat conduction problem.

Chapter 8

Research Contributions and Extensions

8.1 Summary of Contributions

The contributions of the thesis may be summarized as follows.

- Boundary element solutions based on boundary-domain integral formulations are derived for several important classes of axisymmetric heat conduction and thermoelastic problems involving functionally graded solids. The solutions are successfully implemented on the computer.
- New axisymmetric interpolating functions, which are bounded in the axisymmetric coordinate plane but are expressed in terms of relatively simple elementary functions, are constructed for use in the dual-reciprocity method for treating the domain integrals in the axisymmetric boundary-domain integral formulations of heat conduction and thermoelasticity.
- A complex variable boundary element method is developed for solving an axisymmetric problem involving steady heat conduction in a nonhomogeneous solid. Earlier works on the complex variable boundary element methods have been mainly confined to two-dimensional problems in

Cartesian space. Thus, the extension of the complex variable boundary element approach to axisymmetric problems appears to be new.

8.2 Research Extensions

Some possible extensions for the works in this thesis are suggested below.

- All the boundary element analyses presented here are for isotropic solids. It may be worthwhile to consider extending the works here to particular anisotropic solids. The governing partial differential equations for non-homogeneous anisotropic solids are mathematically more complicated.
- The works in Chapter 3, 5 and 6 are considered in the context of the classical theory of heat conduction. Extension to the nonclassical dual-phase-lag heat flux model may be considered.
- The complex variable boundary element method in Chapter 7 may be further developed to include axisymmetric thermoelastic analysis of non-homogeneous solids.

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