Representations of tolerance graphs

Eisermann, Birk

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TECHNOLOGICAL
UNIVERSITY

Representations of Tolerance Graphs

BIRK EISERMANN

Division of Mathematical Sciences

School of Physical and Mathematical Sciences

2012

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BIRK EISERMANN

Division of Mathematical Sciences
School of Physical and Mathematical Sciences

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— Birk Eisermann

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Abstract

This thesis presents research about tolerance graphs. First some of the background of the class of tolerance graphs and closely related graph classes is provided. Important properties like 'weakly chordal' and representation models (by intervals on the real line, parallelograms in the real plane or parallelepipeds in the 3-dimensional space) are introduced. An overview of standard algorithms on tolerance graphs and related graph classes is given.

We contribute an improved version of a criterion of Golumbic, Monma, Trenk about the distinction between bounded and unbounded tolerance graphs. The new criterion can recognize more tolerance graphs as bounded. Furthermore, an explicit construction of tolerance representations of bipartite tolerance graphs is presented. It can serve as a certificate of an already existing recognition algorithm of bipartite tolerance graphs.

These representations as well as representations of unit tolerance graphs are shown to be of small size. In other words, there are non-negative integer tolerance representations for graphs of these two classes with a maximal integer in $O(n^2)$. For general tolerance graphs, a new bound of $n^2 2^n$ is established, which improves the previously known bound of $1 + 5n2^{5n}$. The new bound is proved to be correct by integer linear programming techniques. These techniques also yield a moderately fast recognition of tolerance graphs with a small number of vertices, although the recognition is NP-complete.

Chapter 1

Introduction

1.1 Basic Terms

This thesis deals with a certain class of graphs which admit geometrical representations – tolerance graphs. The connection between such a graph and its representation is given by a mapping of the nodes of the graph to geometrical objects. Most of the time we consider intervals on the real line or certain parallelograms in the real plane as geometrical objects. These objects need to have a suitable placement such that pairwise intersections of them yield the edges of the graph.

1.2 Motivation and History

Consider the following kind of scheduling problem: There are some resources, e.g. rooms, which are limited in number and there are users of these resources. Each user requires access to some of the resources over certain time intervals. These intervals can be drawn as intervals on a real line (time line). By mapping each interval to a

vertex of a graph and each intersection of two intervals to an edge between the corresponding vertices, we create a graph known as interval graph. Then questions about the satisfiability of the requests can be solved by graph theoretic methods. But sometimes there are too few resources to admit a solution. Therefore constraints have to be relaxed to yield a solution. This leads to considering the class of tolerance graphs. They allow having certain amounts of overlap of the intervals, which translates to a certain overlap in usage of the resources. That way more solutions can be found.

Scheduling problems are one kind of applications of tolerance graphs. Other kinds of application are in temporal reasoning (for example, reconstructing a lost or unknown chronology of events) or in molecular biology (for example, reconstructing a DNA sequence from fragments).

Tolerance graphs have been studied for nearly three decades (see [3, 4, 6, 13, 15, 17, 20, 44]). They were introduced by Golumbic and Monma in 1982 in [15]. Then several works focused on finding a characterization of them and fast standard algorithms on them. The earlier works investigated subclasses of tolerance graphs. For example, in [17] and [2] and [13], characterizations were given for tolerance graphs that are trees. In the special case of trees, the classes of bounded tolerance graphs, interval graphs, permutation graphs are equivalent. Furthermore, in [29] and [3] it was discovered that the class of bounded tolerance graphs is equivalent to the class of parallelogram graphs.

Then in 2004 a monograph on tolerance graphs was published by Golumbic and Trenk [19], which contains many results about tolerance graphs.

More recent results can be found in two articles of Busch and Isaak (see [7, 8]). A characterization of tolerance graphs that are bipartite is presented in [7] involving the

existence of a partition into 'consecutive stars'. This structural property is further developed in [8] into a linear time algorithm to recognize bipartite tolerance graphs. The algorithm checks whether the graph does not contain a certain tree as subgraph and whether each block of the graph has a certain property. But there is no linear time algorithm for the recognition of tolerance graphs.

In fact, Mertzios, Sau, and Zaks showed in their articles [37, 38] that the general recognition problem for tolerance graphs and bounded tolerance graphs is NP-complete. Furthermore, they introduced a new representation model of tolerance graphs in [36], which led to faster standard algorithms on this graph class. They have also investigated the distinction between bounded and unbounded tolerance graphs in [34, 39].

1.3 Problems and Contributions

We briefly list the problems discussed in this thesis and what our contributions to them are.

Problem I is the decision problem whether a tolerance graph is bounded. 'Boundedness' is basically a property of the representation. A tolerance representation is bounded if for each interval the tolerated size of overlap is not larger than its length. We improve a criterion on boundedness of tolerance graphs of Golumbic, Monma, and Trotter.

Problem II is about recognition of tolerance graphs that are bipartite. Previously, a linear recognition algorithm had been known, but it did not yield a certificate. We construct a tolerance representation from the data calculated by the algorithm. This

representation can serve as a certificate.

Problem III is about the (general) tolerance graph recognition problem. Since it was proved to be NP-complete, there is no efficient recognition algorithm unless $P = NP$. We present a reasonable fast way to decide whether a small graph (up to about 15 vertices) is a tolerance graph.

Problem IV is about the size of tolerance representations. It was only known that there is an exponential bound on the largest integer in a non-negative integer tolerance representation in the number n of intervals. In general we reduce $1 + 5n2^{5n}$ to n^22^n and in certain cases (unit tolerance graphs, bipartite tolerance graphs) we can give a polynomial bound on the integers occurring in a suitable integer tolerance representation. In terms of storage space for the integers constituting the representation, in general we improve from $5n$ bits to n bits and in the two cases (just mentioned) even to $\log(n)$ bits.

1.4 Overview of Chapters

In Chapter 2 we recall several definitions and results which are used throughout the thesis. We also give some background knowledge that is related to the topics of this thesis. For example, super- and sub-classes and an overview of standard algorithms on these graph classes.

In Chapter 3 we consider the boundedness of tolerance graphs (Problem I). A criterion of Golumbic, Monma, and Trotter about recognition of bounded tolerance graphs can be improved to recognize more tolerance graphs as bounded.

In Chapter 4 we explicitly construct a non-negative integer tolerance represen-

tation of an arbitrary bipartite tolerance graph (Problem II and Problem IV). Our representation has an upper bound of $4n^2$ for each integer occurring in the representation. It is based on the algorithm of Busch and Isaak in [8] for recognizing graphs as bipartite tolerance graphs. But this algorithm does not provide an explicit certificate. The authors only mentioned one algorithm for constructing a representation whose size is not polynomially bounded. Hence, our representation can also serve as a certificate.

For the remaining two chapters, the focus is on constructing non-negative integer tolerance representations with small upper bound on the integers, called width of the representation. Hayward and Shamir showed in [25] that every tolerance graph on n vertices has a non-negative integer tolerance representation whose width is exponentially bounded. The upper bound on the width is huge: $1 + 5n2^{5n}$. They conjectured that every tolerance graph has a representation in which the bit size of every integer is poly-logarithmic bounded.

In Chapter 5 beside showing how to apply linear programming techniques to construct tolerance representations, we also present some computations (Problem III). We show that this technique yields a reasonably fast way to recognize bounded and not bounded tolerance graphs, although both recognition problems are NP-complete.

In Chapter 6, we investigate the width of a representation of a general tolerance graph more deeply (Problem IV). We reduce the known exponential bound of $1 + 5n2^{5n}$ for the width of a representation to n^22^n . For the class of unit tolerance graphs, we prove that these graphs have an integer tolerance representation with width bounded by n^2 . The techniques used for the proofs come from integer linear programming and are introduced in Chapter 5.

Chapter 2

Preliminaries

2.1 Basic Definitions and Examples

Throughout this thesis we assume that graphs are finite, undirected, connected, and have no loops and no multiple edges. We use the following graph theoretical notations similar to those used in [17, 19]. For a graph G , we denote its vertex set by $V(G)$ and its edge set by $E(G)$. If two vertices v and w are adjacent we write $v \sim w$ or $(v, w) \in E(G)$. The set of all neighbors of a vertex v is denoted by $Adj(v)$. The *complement* of a graph G is the graph $\overline{G}$ which has the same vertex set as G but the edges and non-edges are interchanged. That is $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{(v, w) \in V(G) \times V(G) : (v, w) \notin E(G), v \neq w\}$. A *subgraph* of a graph G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G) \cap (V(H) \times V(H))$. An *induced subgraph* is a subgraph H with $V(H) \subseteq V(G)$ and $E(H) = E(G) \cap (V(H) \times V(H))$. A sequence of adjacent vertices in G without any repeated vertex is called a *path*. An induced path in G is also called *chordless*.

In this work the class of intersection graphs and especially its subclasses are considered.

Definition 1. A graph is called *intersection graph* if it has a representation by a collection of sets in the following sense: There is a mapping of the vertices to sets such that two sets have a non-empty intersection if and only if there is an edge between the corresponding vertices.

We list some (common) examples for the kinds of collections of sets that can occur:

1. a collection of intervals on the real line;
2. a collection of parallelograms $P(v)$ (or trapezoids $Z(v)$) in the real plane with vertices $A(v), B(v), C(v), D(v)$ (counter-clockwise) such that for all $v \in V(G)$, the line segments $[B(v), A(v)]$ lie on a line l_1 and the line segments $[C(v), D(v)]$ lie on a line l_2 parallel to l_1 ; a degeneration by $B(v) = A(v)$ or $C(v) = D(v)$ is allowed;
3. a collection of real valued functions over an interval $[a, b]$;

The corresponding graphs are called

1. *interval graphs*,
2. *parallelogram graphs* (*trapezoid graphs*),
3. *function graphs*, respectively.

A graph theoretical property is called *hereditary* if, for every graph G which has this property, all induced subgraphs of G also have this property. For example, being

an intersection graph is hereditary. The reason is that the removal of a set does not destroy the property of being an intersection graph and furthermore, removing a set is equivalent to taking an induced subgraph.

Now we come to a class of graphs which does not trivially fulfill the definition of intersection graph, but later on, after some transformation we can see the link.

Let $V = \{1, 2, \dots, n\}$. For each $i \in V$, let $I(i) = [L(i), R(i)] \subset \mathbb{R}$ be a closed interval on the real line with left endpoint $L(i)$ and right endpoint $R(i)$, and let $T(i) \in \mathbb{R}^+ \cup \{\infty\}$ be a positive value or infinity, called *tolerance*. We write $|I(i)|$ for the length of the interval, that is $R(i) - L(i)$. Let $\mathcal{I} = \{I(i) : i \in V\}$ and $\mathcal{T} = \{T(i) : i \in V\}$. The pair $\Gamma = (\mathcal{I}, \mathcal{T})$ is called a *tolerance representation*. If $T(i) \leq |I(i)|$ then the tolerance $T(i)$ is called *bounded* otherwise *unbounded*. If all tolerances in a tolerance representation are bounded then the tolerance representation is called *bounded* otherwise *unbounded*.

A tolerance representation is called *integer* if all interval endpoints and tolerances are integers, analogously for *non-negative integer*. The *width* of a non-negative integer tolerance representation is defined as the largest value among all interval endpoints and tolerances.

A tolerance representation Γ *generates* the graph $G(\Gamma)$ with the vertex set $V(\Gamma) = \{1, \dots, n\}$, where n is the number of intervals in Γ , and edge set

$$E(\Gamma) = \{(i, j) \in V(\Gamma) \times V(\Gamma) : |I(i) \cap I(j)| \geq \min \{T(i), T(j)\}\},$$

where $|I(i) \cap I(j)|$ denotes the length of the intersection of $I(i)$ and $I(j)$. A graph G is called *tolerance graph*, if there is a tolerance representation Γ generating G , in other words, $G = G(\Gamma)$. In this case, we say that G *has a tolerance representation* Γ or Γ

Figure 2.1: examples of tolerance graphs with representations

is a tolerance representation of G . If G has a bounded tolerance representation Γ , then G is called a *bounded* tolerance graph, otherwise an *unbounded* tolerance graph.

A first example of a tolerance graph is the cycle C_4 of length 4 as shown in Figure 2.1. It has Γ_1 as tolerance representation. The label $(I(v), T(v))$ on each interval states the name of the interval $I(v)$ and tolerance $T(v)$ of vertex v . For simplicity, most of the representations in this thesis will be integer tolerance representations. Since we have $T(v) \leq |I(v)|$ for all $v \in \{1, 2, 3, 4\}$, Γ_1 is a bounded tolerance representation and C_4 is a bounded tolerance graph.

The second graph in Figure 2.1 is the tolerance graph which is commonly denoted

as T_2 with the tolerance representation Γ_2 , which is unbounded. Later by Result 32 we will see that T_2 has no bounded tolerance representation.

The points $L(v) + T(v)$ and $R(v) - T(v)$ are called the *left* and *right tolerance point* of $I(v)$, respectively. Each of these points is also called a *critical point* of the tolerance representation.

As one can see, the definition of tolerance graph does not satisfy the definition of intersection graph. The existence of an edge is not simply derived from whether there is an intersection of the intervals. It depends on the size of intersection between the intervals. This kind of graph is also called *measured intersection graph*. Later with Result 24 we explain a result by Mertzios and Sau and Zaks which shows that a tolerance graph has an intersection model by parallelepipeds.

There are several variations of tolerance graphs. Here we want to highlight two of them. One variation is what is refereed to as proper tolerance graph. A tolerance representation of G is called *proper* if no interval in the representation is a subinterval of another. A tolerance graph G is called *proper* if it has a proper tolerance representation. For an example of a proper tolerance graph with one of its proper tolerance representations, see Figure 2.2.

Proposition 2. *Let Γ be a proper tolerance representation, let π be a permutation of $\{1, \dots, n\}$ such that $L(\pi(1)) < \dots < L(\pi(n))$. Then we have $R(\pi(1)) < \dots < R(\pi(n))$.*

Proof. Let i and j be two arbitrary indices. If we had $L(i) = L(j)$ then for $R(i) \leq R(j)$, we would get $I(i) \subseteq I(j)$, and for $R(i) > R(j)$, we would get $I(j) \subseteq I(i)$. But both contradict the fact that Γ is a proper tolerance representation. Hence, we can assume without loss of generality $L(i) < L(j)$. If $R(i) \geq R(j)$ then we would

Figure 2.2: proper tolerance graph with proper tolerance representation

have $I(j) \subset I(i)$ — again a contradiction. Thus, we must have $R(i) < R(j)$ and this implies the proposition. $\square$

Result 3 ([19, Proposition 2.29]). *Proper tolerance graphs are bounded tolerance graphs.*

Proof. Consider a proper tolerance graph with a proper tolerance representation Γ . Let $T(v)$ be a tolerance that is unbounded in Γ . We show that reducing $T(v)$ to $|I(v)|$ creates no new edge. Let w be a vertex that is not adjacent to v , in other words, $|I(v) \cap I(w)| < \min\{T(v), T(w)\}$. Since Γ is proper, we have $I(v) \not\subset I(w)$. This implies $|I(v) \cap I(w)| < |I(v)|$. Hence, $|I(v) \cap I(w)| < \min\{|I(v)|, T(w)\}$ holds and replacing $T(v)$ by $|I(v)|$ does not create a new edge. $\square$

Another variation of tolerance graphs considered in this thesis are unit tolerance graphs. A graph is called *unit tolerance graph* if it has a tolerance representation where all intervals have the same length. An example is shown in Figure 2.3.

In [3] the relationship to the class of 50% tolerance graphs was investigated.

Figure 2.3: unit tolerance graph C_4 with unit tolerance representation

Definition 4. For $k \in [0, 100]$, a $k\%$ *tolerance graph* is a tolerance graph G that has a tolerance representation where $T(v) = \frac{k}{100} \cdot |I(v)|$ for all $v \in V(G)$.

Result 5 ([3, Theorem 1]). *A graph G is a unit tolerance graph if and only if G is 50% tolerance graph.*

Finally, unit tolerance graphs are related to proper tolerance graphs in the following way:

Result 6 ([3, Corollary 10]). *The class of unit tolerance graphs is strictly contained in the class of proper tolerance graphs.*

2.2 Basic Tools

In some cases it is helpful to write down the length of the intersection of two intervals in a more explicit form. Following proposition is well known.

Proposition 7. *For $I(i) = [L(i), R(i)]$ and $I(j) = [L(j), R(j)]$, we have*

$$|I(i) \cap I(j)| = \max\{0, \min\{R(i), R(j)\} - \max\{L(i), L(j)\}\} \quad (2.1)$$

Proof. There are six possible orderings of the endpoints $L(i), R(i), L(j), R(j)$ of the intervals $I(i)$ and $I(j)$. Without loss of generality, we can assume that $L(i) \leq L(j)$. (Otherwise we interchange i and j .) Hence, there are only three cases to check.

The first case is that $L(i) \leq R(i) \leq L(j) \leq R(j)$. Thus, the intersection of $I(i)$ and $I(j)$ is empty or just one point, and $\max\{0, \min\{R(i), R(j)\} - \max\{L(i), L(j)\}\} = \max\{0, R(i) - L(j)\} = 0 = |I(i) \cap I(j)|$. The second case is that $L(i) \leq L(j) \leq R(i) \leq R(j)$. Thus, $I(i)$ and $I(j)$ intersect partially, and $\max\{0, \min\{R(i), R(j)\} - \max\{L(i), L(j)\}\} = \max\{0, R(i) - L(j)\} = R(i) - L(j) = |I(i) \cap I(j)|$. The last case is that $L(i) \leq L(j) \leq R(j) \leq R(i)$. Thus, $I(j)$ is subinterval of $I(i)$ and we have $\max\{0, \min\{R(i), R(j)\} - \max\{L(i), L(j)\}\} = \max\{0, R(j) - L(j)\} = |I(j)| = |I(i) \cap I(j)|$. $\square$

2.3 Basic properties of tolerance graphs

In this subsection we recall and present possible modifications of tolerance representations under which tolerance graphs are invariant. In other words, the tolerance graph will remain the same if such a change is applied to its tolerance representation.

The next proposition shows that a small perturbation to an interval or a tolerance is possible without changing the generated tolerance graph. This result is well-known. For example, similar statements are used in the proof of Lemma 2.3 in [19] (that is Result 10 in this thesis). Nevertheless our version is not explicitly stated in any source known to us.

Proposition 8. *Let Γ be a tolerance representation and $\epsilon = \min\{|a - t| : a \in A, t \in \mathcal{T}, a \neq t\}$ where $A = \{R(i) - L(j) : i, j \in V(\Gamma)\}$ and $\mathcal{T} = \{T(v) : v \in V(\Gamma)\}$. (This*

guarantees $\epsilon > 0$.)

Furthermore, let Γ' be a tolerance representation that is the same as Γ except that for one vertex x

- the interval $I(x)$ is replaced by $I'(x)$ such that $I(x) \subset I'(x)$ and $|I'(x)| = |I(x)| + \epsilon_1$ for some $0 \leq \epsilon_1 < \epsilon$ and
- the tolerance $T(x)$ is replaced by $T'(x) := T(x) - \epsilon_2$ for some $0 \leq \epsilon_2 < \epsilon - \epsilon_1$.

Then Γ and Γ' generate the same tolerance graph, i.e., $G(\Gamma) = G(\Gamma')$.

Proof. The definition of ϵ guarantees that it is positive because $a \neq t$.

We only need to show that $G(\Gamma)$ and $G(\Gamma')$ have the same edges. The graph obtained from $G(\Gamma)$ by removing the vertex x and the graph obtained from $G(\Gamma')$ by removing vertex x are the same, because the corresponding intervals and tolerances are not changed. Hence, consider x and an arbitrary vertex $w \neq x$. Note that the interval $I(w)$ with tolerance $T(w)$ of Γ is the same in Γ' .

In the case that (x, w) is an edge of $G(\Gamma)$ we have $|I(x) \cap I(w)| \geq \min\{T(x), T(w)\}$. Hence, $|I'(x) \cap I(w)| \geq |I(x) \cap I(w)| \geq \min\{T(x), T(w)\} \geq \min\{T'(x), T(w)\}$. Thus, (x, w) is also an edge of $G(\Gamma')$.

If x and w are not adjacent in $G(\Gamma)$ we have $|I(x) \cap I(w)| < \min\{T(x), T(w)\}$. By Proposition 7, $|I(x) \cap I(w)|$ is equal to one of the values 0 , $R(x) - L(x)$, $R(x) - L(w)$, $R(w) - L(x)$, $R(w) - L(w)$. Hence, by definition of ϵ we have $\epsilon \leq T(x) - |I(x) \cap I(w)|$ and $\epsilon \leq T(w) - |I(x) \cap I(w)|$. Thus, $\epsilon \leq \min\{T(x), T(w)\} - |I(x) \cap I(w)|$ and for $\epsilon_1 + \epsilon_2 < \epsilon$, $|I(x) \cap I(w)| + \epsilon_1 + \epsilon_2 < \min\{T(x), T(w)\}$. Now we have $|I'(x) \cap I(w)| + \epsilon_2 \leq |I(x) \cap I(w)| + \epsilon_1 + \epsilon_2 < \min\{T(x), T(w)\} \leq \min\{T'(x), T(w)\} + \epsilon_2$. Thus $|I'(x) \cap I(w)| < \min\{T'(x), T(w)\}$ and there is no edge between x and w in $G(\Gamma')$. $\square$

A similar method for changing several intervals simultaneously is used in the proof of our Theorem 40.

Since a tolerance graph has many tolerance representations one can require additional properties from certain representations. To prove that these conditions can be satisfied, one can show that changing the tolerance representation in a certain way does not generate a different graph.

There are two different kinds of such requirements on tolerance representations: 'regular' and 'expansive'.

Definition 9. A *regular* tolerance representation is a tolerance representation that satisfies following properties:

1. any two end points of intervals are distinct,
2. any two bounded tolerances are different, and
3. each unbounded tolerance is equal to ∞ .

To prove the following result, we need to introduce some notation. With respect to some tolerance representation Γ , let V_B be the set of vertices $v \in V(G)$ that have bounded tolerances in Γ , i.e., $T(v) \leq |I(v)|$, and let V_U be the set of vertices $v \in V(G)$ that have unbounded tolerances in Γ , i.e., $T(v) > |I(v)|$. Hence, $V(G)$ can be partitioned into V_B and V_U . We define two multisets, the multiset of critical points $P = \{L(v), L(v) + T(v), R(v) - T(v), R(v) : v \in V(G)\}$ and the multiset of bounded tolerances and lengths of intervals with unbounded tolerances $Q = \{T(x) : x \in V_B\} \cup \{|I(x)| : x \in V_U\}$.

Now we can show

Result 10 ([19, Lemma 2.3]). *Each tolerance graph G has a regular tolerance representation.*

Proof. Let Γ be some tolerance representation of G . Let ϵ_P be the smallest non-zero difference between any two points of P and let ϵ_Q be the smallest non-zero difference between any two points of Q and let $\epsilon_0 = \min\{\epsilon_P, \epsilon_Q\}$. Note that $\epsilon_0 \leq \epsilon$ for ϵ as in Proposition 8.

1. If there is a repeated left or right endpoint, then we apply Proposition 8 by decreasing or increasing, respectively, one of the repeated left or right endpoints by $\epsilon_1 = \frac{\epsilon}{2}$. The new representation still represents G .
2. If there is a repeated tolerance, $T(v) = T(w)$, then we apply Proposition 8 and decrease $T(v)$ by $\epsilon_2 = \frac{\epsilon}{2}$. The new representation still represents G .
3. If there is an unbounded tolerance then it is larger than the length of its interval and, thus, it will never be less or equal to the length of the intersection of its interval with another interval. Hence, we can set the tolerance to ∞ without changing the graph generated.

□

Result 11 (Variation of Result 10, [19, Lemma 2.3]). *Every tolerance graph G has a regular (non-negative) integer tolerance representation.*

Proof. We start with a regular tolerance representation as in proof of Result 10. For each irrational value of P or Q , we replace it with a rational value by applying Proposition 8. This yields a 'rational' regular tolerance representation. Then we can

multiply all numbers in P and Q by the least common multiple of all denominators. That results in a regular integer tolerance representation of G . By shifting all intervals by the same amount to the right such that no interval point is negative we get a regular non-negative integer tolerance representation. $\square$

Another way to modify a tolerance representation without changing the corresponding tolerance graph is described in [19]:

Result 12 ([17, p. 159]). *For any positive number M , we can increase each interval in length by M symmetrically about its center and add M to its tolerance and the corresponding tolerance graph remains invariant under this change.*

Hence, each interval $[L(v), R(v)]$ with tolerance $T(v)$ is replaced by a new interval $[L(v) - \frac{M}{2}, R(v) + \frac{M}{2}]$ with tolerance $T(v) + M$. This new representation generates the same tolerance graph. In such a tolerance representation, let us consider an interval with a bounded tolerance. Then its tolerance points, which are $L(v) + T(v)$ and $R(v) - T(v)$, remain the same but the left and right interval endpoints are decreased and increased, respectively.

In [17, p. 159], the property that the intersection of all intervals is a non-empty interval is also required for regular tolerance graphs. For this, we choose $M = \max \{ \max_{v \in V} L(v) - \min_{v \in V} R(v) + 2, 0 \}$ in Result 12. That will result in a common intersection of all intervals of length at least 1. In this thesis we do not use or require this additional property.

In general, the notion of a regular tolerance representation (with the three properties as described above) is used for simplifying proofs.

Still, it is possible to demand more.

Definition 13. A tolerance representation is called *expansive* if

1. all critical points are distinct and
2. all bounded tolerances and lengths of intervals with unbounded tolerances are distinct.

Hence, an expansive tolerance representation is regular if all its unbounded tolerances are equal to ∞ .

Result 14 (part of [25, Lemma 2]). *Each tolerance graph G has an expansive tolerance representation.*

Proof. First we construct a regular representation Γ' by Result 10. Hence, all interval endpoints and all bounded tolerances are distinct. To make all critical points distinct, we have to modify repeated tolerance points. For this, we slightly change the corresponding endpoints (by repeatedly applying Proposition 8) but not their tolerances. This results in changes of the tolerance points of course. Hence, we get a tolerance representation of the same graph where all critical points are distinct.

Now we need to modify repeated values in the multiset Q . Let ϵ be the smallest difference between any two critical points, that are points of the multiset P , and also any two points from Q . Let s_1 and s_2 be two equal elements in Q . We construct the new tolerance representation Γ'' from Γ' in following way. If s_2 is a tolerance $T(w)$, we replace $T(w)$ with $T(w) - \frac{\epsilon}{2}$, and otherwise if s_2 is the length $|I(w)|$ of the interval $I(w)$, we replace the right endpoint $R(w)$ of this interval with $R(w) + \frac{\epsilon}{2}$. By Proposition 8, Γ'' still represents the graph G , there are no (newly created) repeated

Figure 2.4: regular and expansive tolerance representation of C_4

elements in P but the number of repeated elements in Q is reduced by one. We repeat this step finitely many time until there is no repeated value in Q . The result is an expansive tolerance representation of G . $\square$

The notion expansive is used in [25] for a proof of a result on the width of a tolerance representation.

Result 15 ([25, Lemma 2]). *Every tolerance graph has an expansive integer tolerance representation.*

Proof. We start with an expansive tolerance representation in proof of Result 14. Each irrational value of P or Q is replaced by a rational value by applying Proposition 8. This yields a 'rational' expansive tolerance representation. Then we can multiply all numbers in P and Q by the least common multiple of all denominators. That results in an integer expansive tolerance representation of G . $\square$

For illustration, we show a regular and and expansive tolerance representation of C_4 in Figure 2.4. In the expansive representation the left tolerance point $L(v) + T(v)$ is marked with a closing parenthesis ')' and the right tolerance point $R(v) - T(v)$ is marked with an opening parenthesis '('.

2.4 Models of representation of tolerance graphs

The original (first published) model for representing tolerance graphs was given by Golumbic and Monma in [15]. Their model is the one described in the last subsection. It consists of a set of intervals on the real line and their corresponding tolerances. But this model does not match the definition of an intersection graph (compare Definition 1). The reason is that two intervals may intersect but the corresponding vertices may not be adjacent because the intersection is less than the minimum of the corresponding tolerances. For completeness, we introduce two other important models of representation.

In [29] a new model for a subclass of the tolerance graphs was found. They showed that each bounded tolerance graph has a certain representation as an intersection graph of parallelograms. That is, each vertex is assigned a certain parallelogram and two vertices are adjacent if and only if the corresponding parallelograms intersect.

A parallelogram representation is defined in [29] as follows: Consider two parallel lines l_1 and l_2 in the real plane with line l_1 above l_2 and, for simplicity, at distance 1. Each parallelogram needs to have one side on l_1 and the opposite side on l_2 . Hence, a vertex of a parallelogram $P(i)$ can be described by the type of line on which the vertex lies and the position on that line, that is a single number. The whole parallelogram $P(i)$ can be described by the 4-tuple $(L(i), l(i), r(i), R(i))$ where $L(i)$ is the position of the left vertex on line l_2 , $l(i)$ is the position of the left vertex on line l_1 , $r(i)$ is the position of the right vertex on line l_2 , and $R(i)$ is the position of the right vertex on the line l_1 . See also Figure 2.5. Furthermore, let the *slope* $\sigma(i)$ of the parallelogram $P(i)$ be defined by $\frac{1}{l(i)-L(i)} = \frac{1}{R(i)-r(i)}$. Note that in [36] the angle $\arctan(\sigma(i))$ is

Figure 2.5: parallelogram representation of C_4

Figure 2.6: two different intersection types of parallelograms

denoted by $\phi(i)$ and unfortunately called slope, too.

Our following formulation is useful for some proofs:

Proposition 16. *Two parallelograms $P(v)$ and $P(w)$ with $L(v) \leq L(w)$ have a non-empty intersection if and only if $L(w) \leq r(v)$ or $l(w) \leq R(v)$.*

Proof. Let two parallelograms $P(v)$ and $P(w)$ with $P(v) \cap P(w) \neq \emptyset$ and $L(v) \leq L(w)$ be given. We consider two cases, which are illustrated in Figure 2.6. If $L(w)$ lies between $L(v)$ and $r(v)$ then we have $L(w) \leq r(v)$. Otherwise $r(v) < L(w)$. The left side of $P(w)$, $[L(w), l(w)]$, intersects with the diagonal of $P(v)$, $[L(v), R(v)]$, and we have $l(w) \leq R(v)$. If we had $l(w) > R(v)$ then $P(v)$ would lie completely on the left of $P(w)$ — a contradiction.

For the converse, it is obvious that $L(w)$ lies on the side $[L(v), r(v)]$ of $P(v)$, or the diagonal $[L(w), R(w)]$ of parallelogram $P(w)$ and the left side $[L(v), l(v)]$ of the parallelogram $P(v)$ intersect. Hence, we have $P(v) \cap P(w) \neq \emptyset$. $\square$

Remark 17. In other words, the last proposition shows that the relative orderings of the critical points on each line are necessary and sufficient to determine whether two parallelograms intersect.

Result 18 (well known). *In a parallelogram representation R we can move all points of one line (l_1 or l_2) by a fixed amount k without changing the corresponding parallelogram graph.*

Proof. Since we do not change the relative orderings of critical points within each line, the corresponding graph will not change. $\square$

Now we show that a bounded tolerance graph is a parallelogram graph.

Result 19 ([29], [19, Theorem 2.9]). *Let $\Gamma = (\mathcal{I}, \mathcal{T})$ be a bounded tolerance representation of the bounded tolerance graph G where $\mathcal{I} = \{[L(v), R(v)] : v \in V(G)\}$ and $\mathcal{T} = \{T(v) : v \in V(G)\}$. Let $\mathcal{P} = \{P(v) : v \in V(G)\}$ be a parallelogram representation constructed from $\mathcal{R}$ by defining each parallelogram $P(v)$ by a 4-tupel $(L(v), L(v) + T(v), R(v) - T(v), R(v))$ describing the vertices. Then G is equal to the intersection graph $G(\mathcal{P})$ of $\mathcal{P}$.*

Proof. We show that for any two vertices $v, w \in V(G)$, $P(v) \cap P(w) \neq \emptyset$ if and only if there is an edge in G , in other words, $|I(v) \cap I(w)| \geq \min\{T(v), T(w)\}$. Without loss of generality, let $L(v) \leq L(w)$.

First consider the case that v is adjacent to w . Then we have

$$|I(v) \cap I(w)| \geq \min\{T(v), T(w)\}. \quad (2.2)$$

We show that the conditions of Proposition 16 hold. If $T(v) \leq T(w)$, that is $\min\{T(v), T(w)\} = T(v)$, then by calculating the intersection length by Proposi-

tion 7, (2.2) reduces to $\min\{R(v), R(w)\} - L(w) \geq T(v)$. This implies that $R(v) - L(w) \geq T(v)$ or equivalent, $L(w) \leq R(v) - T(v) = r(v)$. Otherwise if $T(v) > T(w)$ then (2.2) reduces to $\min\{R(v), R(w)\} - L(w) \geq T(w)$. This implies that $l(w) = L(w) + T(w) \leq R(v)$. Hence, by Proposition 16 the parallelograms intersect and we have $P(v) \cap P(w) \neq \emptyset$.

Second consider the case that v is not adjacent to w . Hence, we have

$$|I(v) \cap I(w)| < \min\{T(v), T(w)\}. \quad (2.3)$$

If $R(v) > R(w)$ then (2.3) reduces to $R(w) - L(w) < \min\{T(v), T(w)\}$. That implies $R(w) - L(w) < T(w)$ — a contradiction to the boundedness of $T(w)$. If $R(v) \leq R(w)$ then (2.3) reduces to $R(v) - L(w) < \min\{T(v), T(w)\}$. That implies $r(v) = R(v) - T(v) < L(w)$ as well as $R(v) < L(w) + T(w) = l(w)$. Thus, $P(v)$ lies completely on the left side of $P(w)$ and $P(v) \cap P(w) = \emptyset$. $\square$

Now we show that every parallelogram graph G is a bounded tolerance graph.

Result 20 ([29], [19, Theorem 2.9]). *Let G be a parallelogram graph with a parallelogram representation $\mathcal{P} = \{P(v) : v \in V(G)\}$ where each parallelogram $P(v)$ is described by the vertices $(L(v), l(v), r(v), R(v))$. Then $(\{I(v) : v \in V(G)\}, \{T(v) : v \in V(G)\})$ with $I(v) := [L(v), R(v)]$ and $T(v) := l(v) - L(v) = R(v) - r(v)$ is a bounded tolerance representation of G . That is, G is a bounded tolerance graph.*

Proof. Every parallelogram graph has a parallelogram representation where all parallelograms have positive slopes. Otherwise we can move line l_1 in the representation by an appropriate amount to the right (see Result 18). So we assume this and we have following inequalities: $L(v) < l(v)$, $r(v) < R(v)$, $l(v) \leq R(v)$, $L(v) \leq r(v)$ for each vertex $v \in V(G)$.

To show that the parallelogram graph G is a bounded tolerance graph, we check that two parallelograms $P(v)$ and $P(w)$ intersect if and only if we have $|I(v) \cap I(w)| \geq \min\{T(v), T(w)\}$. We always assume without loss of generality that $L(v) \leq L(w)$.

By Proposition 16 (intersection of two parallelograms), if $P(v)$ and $P(w)$ intersect then we have $L(w) \leq r(v)$ or $l(w) \leq R(v)$. Using Proposition 7 (calculation of the interval length) and the above definition of $T(v)$, we get following inequalities. In the case $L(w) \leq r(v)$ we have $|I(v) \cap I(w)| = \min\{R(v), R(w)\} - L(w) = \min\{r(v) + T(v), r(w) + T(w)\} - L(w) = \min\{r(v) - L(w) + T(v), r(w) - L(w) + T(w)\} \geq \min\{T(v), T(w)\}$ (because $r(v) - L(w) \geq 0$ and $r(w) - L(w) \geq 0$). In the case $l(w) \leq R(v)$ we have $|I(v) \cap I(w)| = \min\{R(v), R(w)\} - L(w) \geq \min\{l(w), r(w) + T(w)\} - L(w) = \min\{l(w) - L(w), r(w) - L(w) + T(w)\} \geq T(w) \geq \min\{T(v), T(w)\}$ (because $l(w) - L(w) = T(w)$ and $r(w) - L(w) \geq 0$).

If $P(v)$ and $P(w)$ do not intersect then by Proposition 16 we have $L(w) > r(v)$ and $l(w) > R(v)$. Hence, $|I(v) \cap I(w)| = \min\{R(v), R(w)\} - L(w) = R(v) - L(w)$ because $R(v) < l(w) \leq R(w)$. Furthermore, $R(v) - L(w) < R(v) - r(v) = T(v)$ and $R(v) - L(w) < l(w) - L(w) = T(w)$. Thus, we have $|I(v) \cap I(w)| < \min\{T(v), T(w)\}$. $\square$

The previous two results yield

Result 21 ([29], [19, Theorem 2.9]). *The class of bounded tolerance graphs is equal to the class parallelogram graphs.*

So far this model does not deal with unbounded tolerances. Mertzios, Sau, Zaks generalized the model in [36]. The new model also incorporates unbounded tolerance and thus, represents general tolerance graphs. It consists of a collection of

parallelepipeds in the three dimensional space. But first, we give the definition of a projection representation as in [34].

Definition 22. For a tolerance graph G with tolerance representation $\Gamma = (\{I(i) : i = 1, \dots, n\}, \{T(i) : i = 1, \dots, n\})$, the *projection representation* $\bar{\Gamma}$ is defined in the following way. Similar to a parallelogram representation, it uses two parallel lines l_1 and l_2 at distance 1. For each bounded tolerance $T(v)$, $\bar{\Gamma}$ contains the parallelogram $\overline{P(v)}$ with the vertices $(L(v), L(v) + T(v), R(v) - T(v), R(v))$. For each unbounded tolerance $T(v)$, $\bar{\Gamma}$ contains the degenerated parallelogram $\overline{P(v)}$ with the vertices $(L(v), R(v), L(v), R(v))$, that is just a diagonal line.

This definition can be extended to an intersection representation of tolerance graphs as shown by Mertzios, Sau, Zaks in [36].

Definition 23. Let G be a tolerance graph with a tolerance representation $(\{I(i) : i = 1, \dots, n\}, \{T(i) : i = 1, \dots, n\})$ and a projection representation $\{\overline{P(v)} : v \in V(G)\}$. For every $i = 1, \dots, n$, a *parallelepiped* $P(i)$ in $\mathbb{R}^3$ is defined as follows:

if $T(i)$ bounded, $P(i) := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \overline{P(i)}, 0 \leq z \leq \phi(i)\}$;
 if $T(i)$ unbounded, $P(i) := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \overline{P(i)}, z = \phi(i)\}$.

The set of parallelepipeds $\{P(i) : i = 1, \dots, n\}$ is called a *parallelepiped representation* of G .

Result 24 ([36, Theorem 1]). *Every tolerance graph has a parallelepiped representation.*

In other words, a tolerance graph is an intersection graph of parallelepipeds. For showing results about optimal run times of certain algorithms on tolerance graphs,

parallelepiped representations were used instead of tolerance representations. In this aspect the parallelepiped representation has some advantage over the tolerance representation though both can be transformed into each other.

2.5 Related graph classes

2.5.1 Superclasses of Tolerance Graphs

In this section we introduce some graph classes that include the class of tolerance graphs. Hence, the properties of these classes are also properties of tolerance graphs.

As we have seen in last section (Result 24), tolerance graphs are intersection graphs of parallelepipeds. Hence, they have the hereditary property.

A *hole* of a graph G is an induced cycle of length at least 5. An *antihole* of G is an induced (subgraph which is isomorphic to the) complement of a cycle of length at least 5. In other words, an anti-hole is the complement of a hole. This notion is the same as in [43]. A graph G is called *weakly chordal* or *weakly triangulated* if G contains no induced subgraph which is a cycle of length greater than 4 or a complement of such a cycle. In other words, a graph G is weakly chordal if G contains no hole or antihole. This definition is due to Hayward [24]. For the characterizations of weakly chordal graphs, the notion of two-pair is used. A *two-pair* is a pair of non-adjacent vertices in a graph such that every chordless path between the two vertices in the pair has exactly two edges.

Result 25 ([22], [23]). *A graph G is weakly chordal if and only if every induced subgraph of G is either complete or has a two-pair.*

In [23], G being complete or having a two-pair was shown to be sufficient for G being a weakly chordal graph. The necessity is obvious [22, p. 341]: Consider the contrapositive that a non-weakly chordal graph is neither complete nor has a two-pair. A non-weakly chordal graph must have a hole C_k , $k \geq 5$, or an antihole $\overline{C_k}$, $k \geq 5$, as an induced subgraph. Both subgraphs are not complete, and no two non-adjacent vertices form a two-pair because they are also connected by a path of length at least 3.

The other characterization of weakly chordal graphs is following.

Result 26 ([43, Algorithm Recognition]). *A graph G is weakly chordal if and only if repeatedly adding edges between two-pairs in G yields a complete graph.*

The proof is done partially by showing the correctness of an algorithm and also by using the observation that adding an edge between the two vertices of an two-pair does not destroy the property of being a weakly chordal graph.

Result 27 ([17, Theorem 2, Theorem 3]). *Tolerance graphs are weakly chordal.*

This fact follows from following two results and the definition of weakly chordal: First, a tolerance graph cannot contain a cycle C_k of length $k \geq 5$ (see [17, Theorem 2]). And second, a tolerance graph cannot contain the complement $\overline{C_k}$ of a cycle of length $k \geq 5$ (see [17, Theorem 3]).

Another property of tolerance graphs is perfectness. The *chromatic number* $\chi(H)$ of a graph H is the minimal number of colors needed to color the vertices of the graph such that adjacent vertices have different colors. A *clique* in graph is a complete subgraph. The *clique number* $\omega(H)$ of a graph H is the number of vertices of the

clique in H with the most vertices. If the chromatic number $\chi(H)$ is equal to the clique number $\omega(H)$ for all induced subgraphs H of a graph G , then the graph G is called *perfect*.

Result 28 ([17, Theorem 9]). *Tolerance graphs are perfect.*

There are several ways to show this. For the original proof in [17], we need to introduce some new notion first. An *orientation* of a graph is an assignment of directions to the edges. The orientation is called *acyclic* if there is no directed cycle in the arising directing graph. In [11] an induced subgraph isomorphic to the graph on four vertices a, b, c, d with oriented edges $(a, b), (b, c), (d, c)$ is called an *obstruction*. Furthermore, a graph G is called *perfectly orderable* if G has an acyclic orientation that contains no obstruction.

Now the proof in [17] shows that complements of tolerance graphs are perfectly orderable. Chvátal showed in [11] that perfectly orderable graphs are perfect. Hence, the complements of tolerance graphs are perfect. Finally, also the tolerance graphs themselves are perfect by the Perfect Graph Theorem.

Result 29 (Weak Perfect Graph Theorem, [30]). *A graph is perfect if and only if its complement is perfect.*

There is also a different way to prove Result 28 by using the 'Strong Perfect Graph Theorem'. In 2002, Chudnovsky, Robertson, Seymour, Thomas proved that the 'Strong Perfect Graph Conjecture' of C. Berge is true. A graph G is called *Berge* if every induced cycle C_k of length $k \geq 4$ or induced complement $\overline{C_k}$ of a cycle of length $k \geq 4$ has even length k . In other words, a graph is Berge if it has no induced cycle C_k or induced complement $\overline{C_k}$ of a cycle of odd length $k \geq 5$.

Result 30 (Strong Perfect Graph Theorem, [10]). *A graph is perfect if and only if it is Berge.*

Since tolerance graphs are weakly chordal, it follows from the Strong Perfect Graph Theorem that they are perfect.

Now we come to some property about orientation. A graph G is called *alternately orientable* if there is an orientation of G which is transitive on every chordless cycle of length ≥ 4 . An alternately orientable graph can have only even length chordless cycles.

Result 31 ([19, Theorem 2.18]). *Tolerance graphs are alternately orientable.*

The proof idea is that tolerance graphs do not contain $C_n, n \geq 5$, as induced subgraph. And there is an orientation F that can be shown to be transitive on every induced 4-cycle. It is called *tolerance orientation* and defined for a regular tolerance representation Γ by $xy \in F$ if and only if $xy \in E$ and $T(x) < T(y)$ where $T(v)$ is the tolerance of vertex v in Γ .

2.5.2 Superclasses of Bounded Tolerance Graphs

As we have already seen in Result 21, the class of bounded tolerance graphs is equivalent to the class of parallelogram graphs. The latter can be naturally generalized. A parallelogram is a special case of a trapezoid. Thus, there is the notion of *trapezoid graph* which is an intersection graph of trapezoids between two parallel lines l_1 and l_2 such that each trapezoid has one side on l_1 and one on l_2 .

Figure 2.7: graph with asteroidal triple 1, 7, 8

Hence, bounded tolerance graphs have trapezoid graphs as a superclass. This relationship is proper, because not every trapezoid graph is a tolerance graph. For example, see [13, Figure 3].

Three vertices a, b, c of a graph are called an *asteroidal triple* if removing any of the three vertices together with its neighbors will leave the remaining two vertices connected by a path.

For an example, see Figure (2.7), where a graph with an asteroidal triple is shown together with the three subgraphs where one vertex of the triple with neighborhood is removed. A graph is called *asteroidal triple free* (*AT-free*) if it does not contain an asteroidal triple as induced subgraph.

A graph G is called a *comparability graph* or *transitively orientable graph* if there is an orientation F of its edges such that for all $v, w, x \in V(G)$, we have $(v, w) \in F$

Figure 2.8: complement of graph T_2

and $(w, x) \in F$ implies $(v, x) \in F$. A graph is called *cocomparability graph* if it is the complement of a comparability graph. Golumbic, Rotem, and Urrutia showed in [18] that cocomparability graphs are 'function graphs'. This can be used to show that cocomparability graphs are AT-free. Furthermore,

Result 32 ([15, Theorem 3]). *A bounded tolerance graph is a cocomparability graph.*

As an application, we show that T_2 (see Figure 3.2 on page 50) is not a bounded tolerance graph because it is not a cocomparability graph. Consider the complement of T_2 as shown in Figure 2.8. We observe that $\overline{T_2}$ has some symmetries. Using cycle notation, the permutation $(2, 4, 6)(1, 5, 7)$ is an automorphism. Assume that there is a transitive orientation. Due to the symmetries we can choose the orientation of the following two edges without loss of generality as: $(4, 6)$ and $(6, 2)$. These will force the oriented edges $(4, 7)$ and $(7, 2)$ because of $6 \not\sim 7$. Furthermore, because of $4 \not\sim 3$ we must have the oriented edge $(3, 7)$ and because of $3 \not\sim 2$ we must have the oriented edge $(7, 3)$ — a contradiction. Hence, $\overline{T_2}$ is not a cocomparability graph, and thus T_2 is not a bounded tolerance graph.

We have the following result:

Result 33 ([15, Theorem 3], [18]). *Bounded tolerance graphs are cocomparability graphs and hence AT-free.*

2.5.3 Subclasses of Tolerance Graphs

We list some graph classes which are proper subclasses of the class of tolerance graphs, and more precisely, also subclasses of the class of bounded tolerance graphs. Proper tolerance graphs and unit tolerance graphs are tolerance graphs by their definitions.

Definition 34. Let $\mathcal{I} = \{I(1), I(2), \dots, I(n)\}$ be a set of intervals on the real line. An *interval graph* of $\mathcal{I}$ is the graph with vertex set $V = \{1, 2, \dots, n\}$ and edge set $E = \{(i, j) \in V \times V : I(i) \cap I(j) \neq \emptyset\}$. The set $\mathcal{I}$ is also called *interval representation*.

The class of interval graphs is equivalent to a class of tolerance graphs with constant tolerances.

Definition 35. A graph G is a *permutation graph* if there is a permutation π of the vertex set $V(G)$ such that (i, j) is an edge if and only if j precedes i in the permutation π .

The class of permutation graphs is equivalent to a class of tolerance graphs where the tolerances are equal to the length of their intervals.

Finally, the class of threshold graphs is as a common subclass of interval and permutation a class of tolerance graphs. Here we define a *threshold graph* just by its characterization as a graph that can be constructed from the empty graph by repeatedly adding an isolated vertex or a dominating vertex.

2.5.4 Hierarchy Picture

We summarize all the relationships shown in the last subsections in Figure 2.9. Separating examples are also given.

2.6 Algorithms on Tolerance Graphs and Related Graph Classes

2.6.1 Recognition

The recognition of interval graphs can be achieved in $O(n + m)$ time as is shown in [5, 27]. The same holds for permutation graphs as shown in [32]. According to [43], the recognition of weakly chordal graphs is possible in $O(mn^2)$ time. The recognition of trapezoid graphs takes $O(n(n + m))$ and is shown in [35].

2.6.2 Standard Algorithms

We list the run time classes for the coloring / independent set problem, maximum clique problem, minimum weighted clique cover (or coloring) problem, max. weighted independent set (or clique) problem in following table.

Figure 2.9: some graph class relations

algorithm	weakly chordal	trapezoid	interval	permutation
coloring/ independent set	$O((n+m)n^3)$	$O(n \log n)$	$O(n+m)$	$O(n \log n)$
maximum clique	$O((n+m)n^3)$	$O(n \log n)$	$O(n \log n)$	$O(n)$
minimum weighted clique cover (or coloring)	$O((n+m)n^4)$	$O(n \log n)$	$O(n \log n)$	$O(n)$
max. weighted independent set (or clique)	$O((n+m)n^4)$	$O(n \log n)$	$O(n \log n)$	$O(n)$
Referenz	[22, 23]	[14]	[16]	from compa- rability graphs [16]

2.6.3 Standard Algorithms On Tolerance Graphs

Remember that tolerance graphs are perfect graphs. Hence, by [21] of Grötschel, Lovász, and Schrijver there are standard algorithms which run in polynomial time with respect to the number of vertices and edges. But in practical use, the performance of these algorithms is not good. The run time cannot easily be bounded by a polynomial in the input size ([41, p. 7]). Under some conditions the run time is bounded by a quartic polynomial in the input size. But there are faster algorithms, which exploit the structure of tolerance graphs. The best known and also optimal

algorithms are the ones by Mertzios, Sau, and Zaks in [36]. For tolerance graphs, we have the following run times:

algorithm	run time	reference
coloring, indep. set	$O(n \log n)$, optimal	[36]
maximum clique	$O(n \log n)$, optimal	[36]
maximum weight independent set (or clique)	$O(n^2)$, optimal	[36]
minimum weighted clique cover (or coloring)	$O((n + m)n^4)$	from weakly chordal, [22, 23]

2.7 Further Literature and Generalizations

More information about graph classes can be found in the book [6] of Brandstädt, Le, and Spinrad. There is a even larger database about graph classes which is accessible at [12].

There are also some generalizations of tolerance graphs. For example, *bitolerance graphs*. In a bitolerance graph, compared to a tolerance graph, it is allowed that the distance between left interval endpoint and left tolerance point is different from the one between right interval endpoint and right tolerance point (cf. [4, p. 12]).

Chapter 3

Boundedness Criterion

3.1 Previous work and definitions

There are only a few known results on the distinction between bounded and unbounded tolerance graphs. The papers introducing tolerance graphs [15, 17] use the notion of assertive vertex. A vertex u of a tolerance graph G is called *assertive*, if, for each tolerance representation Γ of G , we have $G(\Gamma) = G(\Gamma')$ where Γ' is the tolerance representation obtained from Γ by replacing the tolerance $T(u)$ by $\min\{T(u), |I(u)|\}$. Otherwise the vertex is called *non-assertive*. If all vertices of a tolerance graph G are assertive then G is bounded ([17, Remark 1]). Consider the graph and its representation in Figure 3.1. The given representation shows that vertex 5 is non-assertive. Thus, [17, Remark 1] is insufficient to show that this graph is a bounded tolerance graph. Later we will see (Corollary 42) that there is also a bounded tolerance representation.

Furthermore, we recall that a bounded tolerance graph is a complement of a com-

Figure 3.1: tolerance graph and its representation

parability graph (see Result 33) and that, as stated in Result 21, the class of bounded tolerance graphs is exactly the class of parallelogram graphs. Since a parallelogram is a special case of a trapezoid, bounded tolerance graphs are also trapezoid graphs.

We give two examples of unbounded tolerance graphs: the graph T_2 and the extended triangle Δ_3 . Both are drawn in Figure 3.2. Result 32 shows that both

Figure 3.2: graph T_2 and graph Δ_3

graphs are unbounded.

Result 36 ([19, Proposition 3.3]). *For a tolerance graph, attaching new vertices as leaves to each non-assertive vertex results in a non-tolerance graph.*

For the graph T_2 , the non-assertive vertices are 1, 5, and 7 by using the criterion

Figure 3.3: graph T_3

introduced in Result 39. Adding a vertex to each of them creates the graph T_3 (see Figure 3.3). Hence, by applying the last result on graph T_2 , the new graph T_3 is not a tolerance graph.

Other recent work on the characterization of bounded tolerance graphs can be found in [34].

3.2 Properties of Vertex Neighborhoods of Tolerance Graphs

If a tolerance graph G has only assertive vertices then G is bounded (see [17, Remark 1]). The converse does not hold: the graph in Figure 3.1 has a non-assertive vertex (vertex 5), but is still a bounded tolerance graph as Corollary 42 will show. We generalize the notion of assertive vertices such that the converse is also true.

Definition 37. A vertex u in a tolerance graph G is called *quasi-assertive*, if, for each tolerance representation Γ of G , there is a tolerance representation Γ' of G such that the tolerance of u in Γ' is bounded, and all tolerances that are bounded in Γ are also bounded in Γ' . Otherwise u is called *non-quasi-assertive*.

Note that every assertive vertex is quasi-assertive. However, the converse is not true: vertex 5 of the graph in Figure 3.1 is quasi-assertive (as Theorem 40 will show) but not assertive. The following does not hold if “quasi-assertive” is replaced by “assertive”.

Proposition 38. *For a tolerance graph $G = (V, E)$, all vertices are quasi-assertive if and only if G is a bounded tolerance graph.*

Proof. First assume that all vertices $v_1, v_2, \dots, v_n$ of G are quasi-assertive. Let Γ be an arbitrary tolerance representation of G . Since v_1 is quasi-assertive Γ can be changed to a representation Γ_1 of G in which the interval $I(v_1)$ is bounded, and all vertices with bounded tolerances in Γ have bounded tolerances in Γ_1 . Next, since v_2 is quasi-assertive Γ_1 can be changed to a representation Γ_2 in which the interval $I(v_2)$ is bounded, too, and all vertices with bounded tolerances in Γ_1 have bounded tolerances in Γ_2 . Continuing in this way for the vertices $v_i, i = 3, \dots, n$, we get the representations $\Gamma_3, \dots, \Gamma_n$. In the last representation Γ_n all tolerances are bounded. Thus, G is bounded.

Conversely, assume that G is a bounded tolerance graph and has a bounded tolerance representation Γ' . Let u be an arbitrary vertex. Then for every representation Γ the representation Γ' is one in which the tolerance $T(u)$ of u is bounded and all vertices with bounded tolerance in Γ have bounded tolerance in Γ' . Thus, vertex u is quasi-assertive. Since u is an arbitrary vertex, all vertices of G are quasi-assertive. $\square$

The notion of quasi-assertive vertex allows us to obtain a variation of a condition on neighborhoods in tolerance graphs from [17]:

Result 39 ([17, Lemma 2]). *Let $G = (V, E)$ be a tolerance graph and $u \in V$. If $Adj(u) \not\subseteq Adj(v)$ for all $v \in V \setminus \{u\}$, then u is assertive.*

For quasi-assertive vertices, we get the following result, which is of broader applicability:

Theorem 40. *Let $G = (V, E)$ be a tolerance graph and $u \in V$. If we have*

$$Adj(u) \not\subseteq Adj(v) \text{ or } |Adj(u)| + 1 \geq |Adj(v)| \quad (3.1)$$

for all $v \in V \setminus \{u\}$, then u is quasi-assertive.

Proof. Let Γ be an arbitrary tolerance representation of the graph G . We will modify the intervals $I(v)$ and tolerances $T(v)$ for $v \in V_u \cup \{u\} \subseteq V \setminus Adj(u)$, where V_u is defined later, and arrange them in a staggered way. The modification will keep all bounded tolerances of Γ bounded and will make the tolerance of u become bounded, too.

If the tolerance of u is bounded, there is nothing to show. Thus, assume $T(u) = \infty$.

If v is a vertex adjacent to u we have $|I(u) \cap I(v)| \geq \min\{T(u), T(v)\} = T(v) \geq \min\{|I(u)|, T(v)\}$. If v is a vertex in $V \setminus \{u\}$ such that v is not adjacent to u and $I(u) \not\subseteq I(v)$ then $|I(u) \cap I(v)| < \min\{T(u), T(v)\} = T(v)$ and $|I(u) \cap I(v)| < |I(u)|$. Thus, $|I(u) \cap I(v)| < \min\{|I(u)|, T(v)\}$. In both cases, replacing $T(u)$ by $\min\{T(u), |I(u)|\}$ in Γ does not change the adjacency relation between u and v .

It remains to consider the set

$$V_u := \{v \in V \setminus \{u\} : I(u) \subseteq I(v), u \text{ not adjacent to } v\}.$$

Suppose that V_u is not empty (otherwise there is nothing to show). If u has a neighbor w which is not in $Adj(v)$ for some $v \in V_u$, that is, $(u, w) \in E$ and $(v, w) \notin E$, then

we get a contradiction: $T(w) = \min\{T(u), T(w)\} \leq |I(u) \cap I(w)| \leq |I(v) \cap I(w)| < \min\{T(v), T(w)\} \leq T(w)$. Hence $Adj(u) \subseteq Adj(v)$ for all $v \in V_u$.

Taking (3.1) into account, there are two kinds of vertices in V_u with respect to the number of their neighbors: we have either vertices $v \in V_u$ that satisfy $Adj(u) \subsetneq Adj(v)$ and $|Adj(u)| + 1 = |Adj(v)|$ or vertices v that satisfy $Adj(u) = Adj(v)$. For the first kind, a vertex v has exactly one neighbor $w(v)$ that is not a neighbor of u . If $w(v) \notin V_u$ then we have $I(u) \not\subseteq I(w(v))$ because of $w(v) \notin V_u$, and we have $I(w(v)) \not\subseteq I(u)$. Otherwise, if we had $I(w(v)) \subseteq I(u)$, then $I(w(v))$ would have an unbounded tolerance since u is not adjacent to $w(v)$. Thus, we would have $|I(v) \cap I(w(v))| = |I(w(v))| < T(w(v))$ and $|I(v) \cap I(w(v))| \leq |I(v) \cap I(u)| < T(v)$. But $|I(v) \cap I(w(v))| \geq \min\{T(v), T(w(v))\}$ — a contradiction. Hence, the endpoints of $I(w(v))$ and $I(u)$ satisfy either *case L* with $L(w(v)) < L(u)$ and $R(w(v)) < R(u)$ or *case R* with $L(u) < L(w(v))$ and $R(u) < R(w(v))$. Note that we have $L(v) < L(u)$ in case L and $R(u) < R(v)$ in case R. Now we partition V_u into

$$A := \{v \in V_u : Adj(v) = Adj(u)\},$$

$$B := \{v \in V_u : Adj(v) = Adj(u) \cup \{w(v)\}, w(v) \in V_u\},$$

$$C := \{v \in V_u : Adj(v) = Adj(u) \cup \{w(v)\}, w(v) \notin V_u, L(w(v)) < L(u)\},$$

$$D := \{v \in V_u : Adj(v) = Adj(u) \cup \{w(v)\}, w(v) \notin V_u, L(u) < L(w(v))\}.$$

Let a, b, c, d denote the cardinality of A, B, C, D respectively, and s the cardinality of the whole set V_u . Note that, for $v \in B$, the vertex $w(v)$ is contained in V_u . Thus, we have $Adj(u) \subseteq Adj(w(v))$. Since $w(v)$ is neighbor of v we have $Adj(w(v)) = Adj(u) \cup \{v\}$, that is $w(v) \in B$. Hence b is even.

Let $A = \{v_1, \dots, v_a\}$ be ordered arbitrarily. Note that if two vertices $x, y \in V_u$ are

adjacent then $x, y \in B$. As the last paragraph shows a vertex $x \in B$ is adjacent to exactly one other vertex $y \in B$. Hence, let $B = \{v_{a+1}, \dots, v_{a+b}\}$ be ordered such that adjacent vertices are consecutive. Let $C = \{v_{a+b+1}, \dots, v_{a+b+c}\}$ be ordered ascending by the left interval endpoint, that is, for $v_p, v_q \in C$ we have $v_p \leq v_q$ if and only if $L(v_p) \leq L(v_q)$. And let $D = \{v_{a+b+c+1}, \dots, v_s\}$ be ordered ascending by the right interval endpoint, that is, for $v_p, v_q \in D$ we have $v_p \leq v_q$ if and only if $R(v_p) \leq R(v_q)$.

Next, we define a modified representation Γ' that is derived from Γ by replacing $(I(v), T(v))$ with $(I'(v), T'(v))$ for all vertices $v \in V_u \cup \{u\}$. By $L'(v)$ and $R'(v)$ we denote the left and right endpoint of $I'(v)$, respectively. Let $L'(v) = L(v)$ for $v \in C$, and $R'(v) = R(v)$ for $v \in D$. Let ϵ be the smallest non-zero difference between any two critical points of Γ . The remaining left interval endpoints are chosen to satisfy following relations (explicit values can be found in the [Appendix](#) but they are not necessary for this proof):

$$\begin{aligned} L(u) - \frac{\epsilon}{2} &< L'(v_1) < L'(v_2) < \dots < L'(v_a) < L'(v_{a+1}) = L'(v_{a+2}), \\ L'(v_{a+2}) &< L'(v_{a+3}) = L'(v_{a+4}) < \dots < L'(v_{a+b-1}) = L'(v_{a+b}) < L'(u), \\ L'(u) &< L'(v_{a+b+c+1}) < L'(v_{a+b+c+2}) < \dots < L'(v_s) = L(u). \end{aligned} \quad (3.2)$$

Similarly, the remaining right interval endpoints are chosen to satisfy

$$\begin{aligned} R(u) &= R'(v_{a+b+1}) < R'(v_{a+b+2}) < \dots < R'(v_{a+b+c}) < R'(v_1), \\ R'(v_1) &< R'(v_2) < \dots < R'(v_a) < R'(v_{a+1}) = R'(v_{a+2}) < R'(v_{a+3}), \\ R'(v_{a+3}) &= R'(v_{a+4}) < \dots < R'(v_{a+b-1}) = R'(v_{a+b}) < R'(u) < R(u) + \frac{\epsilon}{2}. \end{aligned} \quad (3.3)$$

We define following tolerances

$$T'(v) := \begin{cases} \infty & \text{if } v \in C \cup D \text{ and } T(v) = \infty, \\ \min\{T(v), |I'(v)|\} & \text{otherwise.} \end{cases}$$

As shown there is no need to modify intervals and tolerances for vertices not contained in $V_u \cup \{u\}$. Figure 3.4 visualizes the modified part of the tolerance representation.

Figure 3.4: modified intervals and interval $I(u)$

We show that the modified representation Γ' still generates G . In other words, we prove that $|I'(v) \cap I'(w)| < \min\{T'(v), T'(w)\}$ if and only if $|I(v) \cap I(w)| < \min\{T(v), T(w)\}$ for all $v \in V_u \cup \{u\}$ and for all $w \in V, w \neq v$. Remember that $(I'(w), T'(w)) = (I(w), T(w))$ for $w \notin V_u \cup \{u\}$. First consider a vertex $v \in V_u$ and a neighbor w such that:

- $w \in \text{Adj}(u)$: The adjacency between v and w in $G(\Gamma)$ remains the same in $G(\Gamma')$

because we have $|I'(v) \cap I(w)| \geq |I(u) \cap I(w)| \geq \min\{T(u), T(w)\} = T(w) \geq \min\{T'(v), T(w)\}$.

- $w \notin V_u \cup \text{Adj}(u)$: In this case we have $v \in C \cup D$. The adjacency of v and w is reflected in Γ' : the interval $I(w)$ lies either strictly on the left side of $R(u)$ (if $v \in C$) or strictly on the right side of $L(u)$ (if $v \in D$), and in both cases we have $I'(v) \cap I(w) = I(v) \cap I(w)$. Thus, $|I'(v) \cap I(w)| = |I(v) \cap I(w)| \geq \min\{T(v), T(w)\} \geq \min\{T'(v), T(w)\}$.
- $w \in V_u$: We have $v \in B$ and $I'(v) = I'(w)$ since v is a neighbor of w . Thus, $|I'(v) \cap I'(w)| = |I'(v)| \geq \min\{T'(v), T'(w)\}$.

Second consider a vertex $v \in V_u \cup \{u\}$ and a vertex $w \neq v$ that is not a neighbor of v . Let $\delta = \max\{L(u) - L'(v_1), L(u) - L'(u), R'(u) - R(u)\}$. Note that $0 \leq \delta < \frac{\epsilon}{2}$. For $z \in V_u \cup \{u\}$, we have $I(u) \subseteq I(z)$, and $L(z) - \delta \leq L'(z)$ and $R'(z) \leq R(z) + \delta$ by the definitions of L' and R' in (3.2) and (3.3).

- $w \in V_u \cup \{u\}$: In this case, we have

$$\begin{aligned} |I'(v) \cap I'(w)| &= \min\{R'(v), R'(w)\} - \max\{L'(v), L'(w)\} \\ &\leq \min\{R(v) + \delta, R(w) + \delta\} - \max\{L(v) - \delta, L(w) - \delta\} \\ &= |I(v) \cap I(w)| + 2\delta < \min\{T(v), T(w)\}. \end{aligned}$$

Furthermore, $I'(v) \not\subseteq I'(w)$ and $I'(w) \not\subseteq I'(v)$ because v is not a neighbor of w . Hence, $|I'(v) \cap I'(w)| < |I'(v)|$ and $|I'(v) \cap I'(w)| < |I'(w)|$. By the definitions of $T'(v)$ and $T'(w)$, we have $|I'(v) \cap I'(w)| < \min\{T'(v), T'(w)\}$.

- $w \notin V_u \cup \{u\}$: The interval of v is enlarged by at most 2δ and $(I'(w), T'(w)) = (I(w), T(w))$. Hence, $|I'(v) \cap I(w)| \leq |I(v) \cap I(w)| + 2\delta < \min\{T(v), T(w)\}$.

Furthermore, $|I'(v) \cap I(w)| \leq |I'(v)|$. If we had $|I'(v) \cap I(w)| = |I'(v)|$, then $I'(v) \subseteq I(w)$ and because of $I(u) \subseteq I'(v)$ also $I(u) \subseteq I(w)$. Therefore $w \in V_u$ — a contradiction. Thus, $|I'(v) \cap I(w)| < |I'(v)|$, and finally $|I'(v) \cap I(w)| < \min\{|I'(v)|, |I(w)|\}$.

Finally, the correctness of Γ' regarding the adjacency relationship between vertex u and each vertex $w \in V_u$ has been already shown and the adjacency relationship between vertex u and each vertex $w \notin V_u$ does not change since the difference $|I'(u) \cap I(w)| - |I(u) \cap I(w)|$ is less than ϵ .

In summary, the modified tolerance representation has still the same tolerance graph, but the tolerance $T'(u)$ is bounded now. $\square$

We reformulate Lemma 3 of [17].

Result 41 ([17, Lemma 3]). *If G is a tolerance graph with $Adj(u) \not\subseteq Adj(v)$ for all $u, v \in V(G)$ then G is a bounded tolerance graph.*

Combining Proposition 38 and Theorem 40, we obtain the following strengthening of this lemma, that is, we can recognize more tolerance graphs as bounded:

Corollary 42. *If G is a tolerance graph such that for all $u, v \in V(G)$*

$$Adj(u) \not\subseteq Adj(v) \text{ or } |Adj(u)| + 1 \geq |Adj(v)|$$

then G is a bounded tolerance graph.

3.3 Conclusion and Open Problems

Unfortunately, our condition on quasi-assertive vertices in Theorem 40 is not necessary: Note that the graph in Figure 3.5 is a bounded tolerance graph. Although for

the vertex u the condition is not satisfied, u is quasi-assertive. But the improvement

Figure 3.5: bounded tolerance graph with representation

of using quasi-assertive vertices over assertive vertices is shown for the graph of Figure 3.1. Consider the vertices 2 and 5. They are non-assertive but quasi-assertive by Theorem 40. Thus, by Proposition 38 the graph is a bounded tolerance graph. The same applies to a cycle C_4 of length 4 – all vertices are non-assertive but quasi-assertive.

On the other hand, the condition is as good as possible with respect to the size of the neighborhood. If $|Adj(u)| + 2 \geq |Adj(v)|$ for some $v \in V \setminus \{u\}$ then the vertex u need not be quasi-assertive. An example is the graph T_2 shown in Figure 3.2.

Furthermore, the problem of finding a characterization of bounded tolerance graphs is not completely solved (see [34]).

Chapter 4

Efficient Representation of Bipartite Tolerance Graphs

4.1 Preliminaries

4.1.1 Blocks and Cutpoints

We recall some definitions which help to structure an arbitrary graph. A vertex of a graph G whose removal disconnects G is called *cutpoint*. A connected graph can be decomposed into its *blocks*, which are the maximal connected subgraphs having no cutpoint themselves. Such a block is either called a *bridge* if it is a single edge, or otherwise a *2-connected block*. The *block-cutpoint-graph* of a graph G is the graph $bc(G)$ whose vertex set contains one vertex for each block of G and one for each cutpoint of G . In $bc(G)$ two vertices are adjacent if and only if one is a block and the other one is a cutpoint and the cutpoint is part of the block. By this definition, in $bc(G)$ there is no edge between two vertices corresponding to cutpoints of G although these

Figure 4.1: bipartite tolerance graph G

cutpoints might be adjacent in G . Note that the block-cutpoint-graph must be a tree.

Furthermore, for a subgraph H and a vertex v , the integer $\epsilon_H(v)$ denotes the *eccentricity* of the vertex v in the subgraph H . That is the maximal distance between v and some other vertex of H .

As in the article of Busch and Isaak [8] we let P_G denote the set of *leaves* of G , that is the set of vertices which have exactly one neighbor. For a block B , the set of cutpoints of B can be described by $\mathcal{B}(B) = \{v \in V(B) : \text{Adj}(v) \not\subseteq V(B)\}$ which is called the *boundary* of B . We partition $\mathcal{B}(B)$ into $\mathcal{B}^1(B) = \{v \in \mathcal{B}(B) : \text{Adj}(v) \setminus V(B) \subseteq P_G\}$, which consists of cutpoints separating B from the leaves of G , and $\mathcal{B}^2(B) = \mathcal{B}(B) \setminus \mathcal{B}^1(B)$, the remaining cutpoints.

Example 43. In Figure 4.1 we present a bipartite tolerance graph G and in Figure 4.2 we show its blocks and also the extended blocks (described in Subsection 4.2) if they differ from the original blocks. For this graph G , we will demonstrate each step of the construction of a tolerance representation throughout this chapter. We start with the block-cutpoint-graph $bc(G)$ in Figure 4.3. Each vertex of $bc(G)$ corresponding to a block in G is denoted by the name of the block and each vertex corresponding to a

Figure 4.2: (extended) blocks of graph G

Figure 4.3: block-cutpoint graph $bc(G)$

cutpoint in G is denoted by the name of the cutpoint. A few selected eccentricities are: $\epsilon_{B_1}(5) = 2$, $\epsilon_{B_2}(5) = 1$, $\epsilon_{B_3}(9) = 3$.

4.1.2 Consecutively orderable collections of sets

For a bipartite graph G , the two partition classes of the vertex set are denoted by X and Y . We recall some definitions from [8]. Let $U_1, U_2, \dots, U_p$ be a collection of subsets of a set W . Then the collection is called *consecutively orderable* if there is a permutation π of $I := \{1, \dots, p\}$ such that $U_{\pi(j)}$ contains $U_{\pi(i)} \cap U_{\pi(k)}$ for all i, j, k with

$\pi(i) \leq \pi(j) \leq \pi(k)$. A collection is called *consecutively ordered* if no permutation is necessary, i.e., if π is the identity permutation. For example, the subsets $U_1, U_2, \dots, U_5$ of $W = \{1, 2, \dots, 6\}$ as shown in Table 4.1 are consecutively ordered.

I	$W = \{1, 2, 3, 4, 5, 6\}$
1	$U_1 = \{ \quad 2, \quad 4, \quad 6 \}$
2	$U_2 = \{ \quad \quad 3, 4, \quad 6 \}$
3	$U_3 = \{ \quad \quad 3, 4, \quad 6 \}$
4	$U_4 = \{ \quad \quad 3, 4, 5 \quad \}$
5	$U_5 = \{1, \quad \quad 4, 5 \quad \}$

Table 4.1: consecutively ordered sets

Let $G = (X, Y, E)$ be a bipartite graph. G is called *X-consecutive*, if the set of neighborhoods $\{Adj(x_i) : x_i \in X\}$ is consecutively orderable. In other words, if there is an ordering of $X = \{x_1, \dots, x_k\}$ such that the set of the neighborhoods $Adj(x_1), Adj(x_2), \dots, Adj(x_k)$ is consecutively ordered. Analogously, G is called *Y-consecutive* if $\{Adj(y_j) : y_j \in Y\}$ is consecutively orderable. For simplicity, we call a subgraph $H = (V_1, V_2, F)$ of G with $V_1 \subseteq X$ and $V_2 \subseteq Y$ *X-consecutive* if H is V_1 -consecutive, and analogously *Y-consecutive* if H is V_2 -consecutive.

4.2 Representation of an (Extended) Block

Let $G = (X, Y, E)$ be an arbitrary bipartite tolerance graph with n vertices and let B be a block of G . The *extended block* of B that is the induced subgraph of B

together with the leaves adjacent to $\mathcal{B}^1(B)$ is denoted by B' . The subgraph which is isomorphic to B' with an attached path of length 2 to each vertex $v \in \mathcal{B}^2(B)$ is denoted by B'' . Note that a vertex $v \in \mathcal{B}^2(B)$ is a cutpoint between the block B and a subgraph of G which is not just a bridge. Hence, an induced path of length 2 as mentioned above exists in G and B'' is a subgraph of G . Our construction of a tolerance representation of a block is based on the following result:

Result 44 ([8, Lemma 6]). *For a block B of a bipartite tolerance graph, the subgraph B'' is X - or Y -consecutive.*

We assume without loss of generality that B'' is Y -consecutive. Otherwise, if the subgraph $B'' = (V_1, V_2, F)$, $V_1 \subseteq X$, $V_2 \subseteq Y$, is V_1 -consecutive (X -consecutive) then we can swap the labels of the partition classes V_1 and V_2 such that we have $V_2 \subseteq X$ and $V_1 \subseteq Y$. This is possible, since in our construction the blocks will be considered separately. Thus, B'' becomes V_2 -consecutive (Y -consecutive).

Now we transform the graph-theoretical property of being Y -consecutive into a property of the adjacency matrix. We follow [8] and [45]. For a bipartite graph $G = (X, Y, E)$ (not necessary tolerance), the *reduced adjacency matrix* $A(G)$ is defined as the submatrix of the adjacency matrix of G whose rows correspond to the vertices of X and whose columns correspond to the vertices of Y . A matrix M is said to have the *consecutive 1's property for columns* (rows, respectively) if the rows (columns, respectively) of M can be permuted such that the 1's in every column (row, respectively) occur consecutively. The matrix is said *to be in the consecutive 1's form for columns* (rows) if the 1's in the columns (rows) occur consecutive. Tucker [45, p. 154] established the connection between the graph-theoretical property and the matrix

property: G is X -consecutive if and only if its reduced adjacency matrix $A(G)$ has the consecutive 1's property for columns. Analogously, G is Y -consecutive if and only if $A(G)$ has the consecutive 1's property for rows. Using these facts, we assume:

- Assumption 1: The reduced adjacency matrix $A(B'')$ is in the consecutive 1's form for rows.

Notation 45 (used for preparation of Proposition 48). For the subgraph B'' of a block B of a bipartite tolerance graph, we denote the vertices of $X(B'')$ by $x_1, \dots, x_p$ and the vertices of $Y(B'')$ by $y_1, \dots, y_q$. Hence, the rows of $A(B'')$ are enumerated from 1 to p and the columns from 1 to q . For a row i of $A(B'')$, we define $a(i)$ to be the index of the first column whose entry is one and $b(i)$ to be the index of the last such column, in short, $a : \{1, \dots, p\} \rightarrow \{1, \dots, q\}, i \mapsto \min\{j \in \{1, \dots, q\} : (A(B''))_{i,j} = 1\}$ and $b : \{1, \dots, p\} \rightarrow \{1, \dots, q\}, i \mapsto \max\{j \in \{1, \dots, q\} : (A(B''))_{i,j} = 1\}$.

Permuting the rows of $A(B'')$ is allowed since it will only change the labeling of the vertices in $X(B'')$. Thus, by a permutation of the rows it is possible to order the pairs $(a(i), b(i))$ lexicographically. Hence, we have

- Assumption 2: The rows i of $A(B'')$ are ordered such that the pairs $(a(i), b(i))$ are ordered lexicographically increasing.

We assume that for each $j_0 = 1, \dots, q$, the columns that are equal to column j_0 are placed consecutive in the matrix $A(B'')$. Otherwise, let $j_1, \dots, j_t$ be the columns equal to column j_1 . Let $j_1, \dots, j_s$ be consecutive but $j_s + 1 < j_{s+1}$. Then we can move and insert column j_{s+1} directly after column j_s without destroying the consecutive 1's form or the lexicographical ordering. By repeatedly applying this argument, we

can reorder the columns of $A(B'')$ such that the columns equal to column j_1 are consecutive. Hence, our third assumption:

- Assumption 3: The columns of $A(B'')$ that are equal are grouped together.

Now we investigate the bounded case more deeply. Theorem 3.9 of [19] states that for bipartite graphs, the graph classes of bounded tolerance graphs and AT-free graphs are equal. We use this to prove:

Proposition 46. *Let B be a 2-connected block of a bipartite bounded tolerance graph and let M be the reduced adjacency matrix $A(B'')$ of the graph B'' . Without loss of generality let M satisfy Assumptions 1, 2, 3:*

1. *M is in the consecutive 1's form for rows,*
2. *the rows of M are ordered such that the pairs $(a(i), b(i))$ are lexicographically increasing,*
3. *the columns of M that are equal are grouped together.*

Then we have $b(i) \leq b(j)$ for $i < j$.

Proof. It has been shown in the previous paragraphs that every graph B'' of a block B has a reduced adjacency matrix M satisfying the Assumptions 1, 2, 3. By the lexicographical ordering of the rows we have $a(i) \leq a(j)$ for $i < j$. If $a(i) = a(j)$ then $b(i) \leq b(j)$. But if $a(i) < a(j)$ we have to show that $b(i) \leq b(j)$.

For a contradiction, assume that there are two rows $i < j$ such that $a(i) < a(j)$ and $b(i) > b(j)$. This implies

$$a(i) < a(j) \leq b(j) < b(i). \quad (4.1)$$

Without loss of generality we assume that j is as small as possible and $i = j - 1$. So far we have following information about the matrix M presented as a submatrix in the following. Note that separate consecutive columns may be only one single column in reality.

submatrix	$a(i)$	$a(j) - 1$	$a(j)$	$b(j)$	$b(j) + 1$	$b(i)$
i	1	1	1	1	1	1
j	0	0	1	1	0	0

If M had only rows g with $M_{g,a(j)-1} = M_{g,b(j)+1}$ then the columns $a(j) - 1$ and $b(j) + 1$ had the same type. The fact that $a(j)$ has different type but $a(j) - 1 < a(j) < b(j) + 1$ contradicts Assumption 3. Hence, there must be a row h with

$$M_{h,a(j)-1} \neq M_{h,b(j)+1}. \quad (4.2)$$

Case 1: $M_{h,a(j)-1} = 1$. Because of $j = i + 1$ and Assumption 2, we must have

$h < i$ and $M_{h,s} = 1$ for all $s = a(i), \dots, a(j) - 1$, as shown in the next matrix:

submatrix	$a(i)$	$a(j) - 1$	$a(j)$	$b(j)$	$b(j) + 1$
h	1	1	?	?	0
i	1	1	1	1	1
j	0	0	1	1	0

Case 1.1: $b(h) < a(j)$ for every row h satisfying condition (4.2). Note that we have $b(l) \leq b(h)$ for each row $l < h$ by the assumption (4.1). If $y_s \in P_G$ for all $s = b(j) + 1, \dots, b(i)$ then we could move those columns between columns $a(j) - 1$ and $a(j)$ without destroying Assumptions 1, 2, 3 and the assumption (4.1) were false

— a contradiction. If there is a $y_s \notin P_G$ then we have a submatrix as shown in Figure 4.4 and the corresponding subgraph contains an asteroidal tripel x_h, x_j, x_k .

Case 1.2: $b(h) = a(j) < b(j)$. Hence, $M_{h,a(j)} = 1$ and $M_{h,b(j)} = 0$. The subgraph induced by $\{h, i, j, a(i), a(j), b(j), b(j) + 1\}$ is drawn in Figure 4.5. It contains the asteroidal triple $x_h, x_j, y_{b(j)+1}$.

Case 1.3: $b(h) = b(j)$. Hence, $M_{h,a(j)} = M_{h,b(j)} = 1$.

submatrix	$a(i) - 1$	$a(i)$	$b(j)$	$b(j) + 1$
h	?	1	1	0
i	0	1	1	1
j	0	0	1	0

Case 1.3.1: There is a row g such that $a(g) < a(i) \leq b(g) \leq b(i)$. It is possible that g is identical to h . As one can see in Figure 4.6, the vertices $y_{a(i)-1}, x_j, y_{b(j)+1}$ form an asteroidal triple independent of whether '?' is replaced by 0 or 1.

Case 1.3.2: If there is a row k such that $a(j) \leq a(k) \leq b(j) < b(k)$ then $M_{k,b(j)} = M_{k,b(j)+1} = 1$. Consider the subgraph induced by $\{x_h, x_i, x_j, x_k, y_{a(i)}, y_{b(j)}, y_{b(j)+1}\}$. As one can see in Figure 4.7, there is an asteroidal triple $y_{a(i)}, x_j, y_{b(j)+1}$.

Case 1.3.3: For a contradiction, we assume there is no row g such that $a(g) < a(i) \leq b(g) \leq b(i)$ and no row k such that $a(j) \leq a(k) \leq b(j) < b(k)$. If there is a row d with $M_{d,a(i)} = 1$ and $M_{d,b(j)} = 0$ then we can find the asteroidal triple $x_d, x_j, y_{b(j)+1}$ in the graph as shown in Figure 4.8. If there is a row f with $M_{f,a(i)-1} = M_{f,a(i)} = M_{f,b(j)} = 1$ (it is possible that h is identical with f .) there will be an asteroidal triple $y_{a(i)-1}, x_j, y_{b(j)+1}$ as shown in Figure 4.9. If there is no row d and no row f then we can move the columns $a(i), \dots, a(j) - 1$ behind the column $b(j)$ and

Figure 4.4: submatrix and induced subgraph with AT in case 1.1

Figure 4.5: submatrix and induced subgraph with AT in case 1.2

Figure 4.6: submatrix and induced subgraph with AT in case 1.3.1

submatrix	$a(i)$	$b(j)$	$b(j) + 1$
h	1	1	0
i	1	1	1
j	0	1	0
k	0	1	1

Figure 4.7: submatrix and induced subgraph with AT in case 1.3.2

submatrix	$a(i)$	$b(j)$	$b(j) + 1$
d	1	0	0
h	1	1	0
i	1	1	1
j	0	1	0

Figure 4.8: submatrix and induced subgraph with AT in case 1.3.3

submatrix	$a(i) - 1$	$a(i)$	$b(j)$	$b(j) + 1$
f	1	1	1	0
i	0	1	1	1
j	0	0	1	0

Figure 4.9: submatrix and induced subgraph with AT in case 1.3.3

move the row j in front of row h . Now assumption (4.1) is false — a contradiction.

Case 2: $M_{h,a(j)-1} = 0$. Since the rows are lexicographically ordered and $i = j - 1$, we have $j < h$.

submatrix	$a(i)$	$a(j) - 1$	$a(j)$	$b(j)$	$b(j) + 1$
i	1	1	1	1	1
j	0	0	1	1	0
h	0	0	?	?	1

Case 2.1: $b(j) < a(h)$ for each row h satisfying condition (4.2). If there is a row d with $a(d) \leq a(i) \leq b(d) < b(j)$. Then we have an asteroidal triple x_d, x_j, x_h in the subgraph induced by $\{x_d, x_i, x_j, x_h, y_{a(j)-1}, y_{b(j)}, y_{b(j)+1}\}$ as shown in Figure 4.10.

If there is no such row d then no row has an 1-entry between $a(j)$ and $b(j)$. Hence, we can move the columns $a(j), \dots, b(j)$ in front of the column $a(i)$ and sort row h above row i . The Assumptions 1, 2, 3 are not destroyed and the assumption (4.1) is false — a contradiction.

Case 2.2: $a(j) < a(h) \leq b(j)$. Hence, $M_{h,a(j)} = 0$ and $M_{h,b(j)} = 1$. Then the vertices $y_{a(j)-1}, x_j, x_h$ form an asteroidal triple as shown in Figure 4.11.

Case 2.3: $a(j) = a(h)$. Hence, $M_{h,a(j)} = M_{h,b(j)} = 1$.

submatrix	$a(j) - 1$	$a(j)$	$b(j)$	$b(j) + 1$
i	1	1	1	1
j	0	1	1	0
h	0	1	1	1

If there is a row d with $a(d) \leq a(i) \leq a(j) \leq b(d) \leq b(j)$ then the subgraph induced by $\{x_d, x_i, x_j, x_h, y_{a(j)-1}, y_{a(j)}, y_{b(j)+1}\}$ contains an asteroidal triple $y_{a(i)}, x_j, y_{b(j)+1}$ as

Figure 4.10: submatrix and induced subgraph with AT in case 2.1

Figure 4.11: submatrix and induced subgraph with AT in case 2.2

shown in Figure 4.12.

If there is no such row d then no row has an 1-entry between $a(j)$ and $b(j)$. If all columns $b(j) + 1, \dots, b(i)$ have non-zero entries in the rows i and h only then we can move these columns between the columns $a(j) - 1$ and $a(j)$ and interchange the rows j and h . The Assumptions 1, 2, 3 are not destroyed and the assumption (4.1) is false for rows $1, 2, \dots, i + 1$ — a contradiction. Hence, one of the columns $b(j) + 1, \dots, b(i)$ has an entry in some row $l > h$. The vertices x_d, x_j, x_l form an asteroidal triple as shown in Figure 4.13.

In every case there is either a contradiction or an asteroidal triple, which is not allowed for a bounded tolerance graph (because of Result 33), thus, a contradiction, too. Hence, the assumption was wrong and $a(i) \leq a(j)$ implies $b(i) \leq b(j)$. $\square$

Finally, with a few more definitions we can construct the tolerance representation of an extended block. We assume that the matrix $A(B'')$ has the form as in Proposition 46. Let $A(B')$ be the matrix derived from $A(B'')$ by deleting the rows or columns corresponding to the vertices of $V(B'') \setminus V(B')$. From here on we only consider this matrix $A(B')$.

Notation 47 (used for preparation of Proposition 48). Let $A(B')$ be the matrix derived from $A(B'')$ by deleting the rows or columns corresponding to the vertices of $V(B'') \setminus V(B')$. For $A(B')$, we denote the number of rows with their first non-zero entry in column j by $c_a(j)$. That is $c_a(j) := |\{i : a(i) = j\}|$. Similarly, we denote the number of rows with their last non-zero entry in column j by $c_b(j)$. That is $c_b(j) := |\{i : b(i) = j\}|$. To sum up these numbers over the columns from 1 to $j - 1$

Figure 4.12: submatrix and induced subgraph with AT in case 2.3

Figure 4.13: submatrix and induced subgraph with AT in case 2.3

(excluding j) we define the function $\Delta : \{1, \dots, q+1\} \rightarrow \{1, \dots, 2p\}$, $\Delta(1) := 0$ and

$$\Delta(j) := \sum_{k=1}^{j-1} (c_a(k) + c_b(k)) \text{ for } j \geq 2.$$

Furthermore, let α be a mapping from $\{1, \dots, p\}$ to the non-negative integers such that for $i_0 \in \{1, \dots, p\}$, the restriction of α to the set $S(i_0) = \{i : a(i_0) = a(i)\}$ is a bijection to $\{0, 1, \dots, |S(i_0)| - 1\}$. For example, the difference between the row i_0 and the lowest row with same a -value serves the purpose: $\alpha(i_0) = i_0 - \min\{i : a(i) = a(i_0)\}$. Similarly, let β be a mapping from $\{1, \dots, p\}$ to the non-negative integers such that the restriction of β to $Q(i_0) = \{i : b(i_0) = b(i)\}$ is a bijection to $\{0, 1, \dots, |Q(i_0)| - 1\}$.

Note that $\{c_a(j) : j = 1, \dots, p\}$ and $\{c_b(j) : j = 1, \dots, p\}$ are integer partitions of p . Hence, $\Delta(j)$ cannot exceed $2p$. Furthermore, $0 \leq \alpha(i) \leq c_a(a(i))$ and $0 \leq \beta(i) \leq c_b(b(i))$.

From here onwards, we assume that the matrices $A(B')$ and $A(B'')$ satisfy Assumptions 1, 2, 3. In Figure 4.14 we show the calculated values for the block B_3 of Example 43. Note that $B_3 = B'_3$.

Finally, we present the construction the tolerance representation of an extended block. For a subgraph S of bipartite graph $G = (X, Y, E)$, we define $X(S) := V(S) \cap X$ and $Y(S) := V(S) \cap Y$.

Proposition 48. *Let $B' = (X(B'), Y(B'), F)$ be an extended block of a bipartite tolerance graph with a reduced adjacency matrix $A(B')$ satisfying Assumptions 1, 2, 3.*

		j						
		1	2	$3 = q$				
i	$A(B'_3)$	12	11	15	$a(i)$	$b(i)$	$\alpha(i)$	$\beta(i)$
1	9	1	1	0	1	2	0	0
2	14	1	1	1	1	3	1	0
$3 = p$	13	0	1	1	2	3	0	1
$c_a(j)$		2	1	0				
$c_b(j)$		0	1	2				
$\Delta(j)$		0	2	4				

Figure 4.14: subgraph B'_3 , reduced adjacency matrix together with values needed to calculate its tolerance representation

Then the following is a non-negative integer tolerance representation Γ of B' :

$$I(y_j) := [j - 1 + \Delta(j) + c_a(j), j + \Delta(j) + c_a(j)],$$

$$T(y_j) := 1 \quad \text{for all } j = 1, \dots, q;$$

$$I(x_i) := [a(i) - 1 + \Delta(a(i)) + \alpha(i), b(i) + \Delta(b(i)) + c_a(b(i)) + \beta(i)],$$

$$T(x_i) := \begin{cases} |I(x_i)| & \text{if } B'' \text{ is bounded} \\ |I(x_i)| + 1 & \text{if } B'' \text{ is unbounded} \end{cases} \quad \text{for all } i = 1, \dots, p.$$

The representation Γ is bounded if B'' is a bounded tolerance graph. The width of Γ is less than $2n$.

Proof. First, we show that Γ generates a bipartite graph $G(\Gamma)$. For $y_i, y_j \in Y(B')$, $i < j$, we consider the intersection between $I(y_i)$ and $I(y_j)$. Since $\Delta(i) + c_a(i) \leq \Delta(j) + c_a(j)$, we have $R(y_i) = i + \Delta(i) + c_a(i) < j + \Delta(j) + c_a(j) = L(y_j)$. Hence, $|I(y_i) \cap I(y_j)|$ is always 0, which is less than $\min\{T(y_i), T(y_j)\} = 1$. Thus, there is no edge between y_i and y_j .

Now let $x_i, x_j \in X(B')$, $i < j$. The intersection between $I(x_i)$ and $I(x_j)$ has finite length. If B'' is unbounded then $T(x_i) > |I(x_i)|$ and $T(x_j) > |I(x_j)|$, and $|I(x_i) \cap I(x_j)| < \min\{T(x_i), T(x_j)\}$. Thus, there is no edge between x_i and x_j . If B'' is bounded, we show that

$$L(x_i) = a(i) - 1 + \Delta(a(i)) + \alpha(i) < a(j) - 1 + \Delta(a(j)) + \alpha(j) = L(x_j) \quad (4.3)$$

and

$$R(x_i) = b(i) + \Delta(b(i)) + c_a(b(i)) + \beta(i) < b(j) + \Delta(b(j)) + c_a(b(j)) + \beta(j) = R(x_j). \quad (4.4)$$

The inequality (4.3) is true because we have either $a(i) = a(j)$ and thus $\Delta(a(i)) = \Delta(a(j))$ but the rows are still different and $\alpha(i) < \alpha(j)$. Or we have $a(i) < a(j)$ and $\Delta(a(i)) + \alpha(i) \leq \Delta(a(i)) + c_a(a(i)) \leq \Delta(a(j))$. This yields $L(x_i) < L(x_j)$. To show the inequality (4.4) boundedness comes into play, i.e., we have $b(i) \leq b(j)$ for $i < j$ by Proposition 46. We have either $b(i) = b(j)$ and thus $\Delta(b(i)) + c_a(b(i)) = \Delta(b(j)) + c_a(b(j))$ but the rows are still different and $\beta(i) < \beta(j)$. Or we have $b(i) < b(j)$ and thus, $\Delta(b(i)) + c_a(b(i)) + \beta(i) \leq \Delta(b(i)) + c_a(b(i)) + c_b(b(i)) = \Delta(b(i) + 1) \leq \Delta(b(j)) \leq \Delta(b(j)) + c_a(b(j)) + \beta(j)$. This yields $R(x_i) < R(x_j)$.

The inequalities (4.3) and (4.4) together show that the interval $I(x_i)$ is not a proper subinterval of interval $I(x_j)$ and $I(x_j)$ is not a proper subinterval of $I(x_i)$. Hence, $|I(x_i) \cap I(x_j)| < \min\{|I(x_i)|, |I(x_j)|\} \leq \min\{T(x_i), T(x_j)\}$ and there is no edge between x_i and x_j . That makes $G(\Gamma)$ a bipartite graph.

Second, we consider the relationship between vertices of $X(B')$ and $Y(B')$. Let $x_i \in X(B')$ and $y_j \in Y(B')$. We have that x_i is adjacent to y_j if and only if $a(i) \leq j \leq b(i)$ in the reduced adjacency matrix $A(B')$. Now we show that $a(i) \leq j \leq b(i)$ if and only if $|I(x_i) \cap I(y_j)| \geq \min\{T(x_i), T(y_j)\}$ in the constructed representation Γ .

(1) Assume that $a(i) \leq j \leq b(i)$. Then $L(x_i) = a(i) - 1 + \Delta(a(i)) + \alpha(i) \leq a(i) - 1 + \Delta(a(i)) + c_a(a(i)) \leq L(y_j) < R(y_j) \leq b(i) + \Delta(b(i)) + c_a(b(i)) \leq b(i) + \Delta(b(i)) + c_a(b(i)) + \beta(i) = R(x_i)$. Hence, we have $I(y_j) \subseteq I(x_i)$ and $|I(x_i) \cap I(y_j)| = |I(y_j)| \geq 1 = \min\{T(x_i), T(y_j)\}$.

(2) Assume that $j < a(i)$ or $b(i) < j$. If $j < a(i)$ we have $R(y_j) \leq a(i) - 1 + \Delta(a(i) - 1) + c_a(a(i) - 1) \leq a(i) - 1 + \Delta(a(i)) + \alpha(i) = L(x_i)$. If $b(i) < j$ we have $R(x_i) = b(i) + \Delta(b(i)) + c_a(b(i)) + \beta(i) \leq b(i) + \Delta(b(i)) + c_a(b(i)) + c_b(b(i)) \leq (b(i) + 1) - 1 + \Delta(b(i) + 1) + c_a(b(i) + 1) = L(y_j)$.

Figure 4.15: tolerance representation of block B'_3

In sum, we have either $R(y_j) \leq L(x_i)$ or $R(x_i) \leq L(y_j)$ and thus, $|I(x_i) \cap I(y_j)| = 0 < \min\{T(x_i), T(y_j)\}$.

If B'' is a bounded tolerance graph then $T(x_i) = |I(x_i)|$ for $i = 1, \dots, p$ and $T(y_j) = 1 \leq |I(y_j)|$ for $j = 1, \dots, q$. Hence, for each $v \in V(B')$, we have $T(v) \leq |I(v)|$, and Γ is bounded.

The width of the constructed tolerance representation is at most

$$\begin{aligned} \max_{v \in V(B')} R(v) &= \max\{R(x_p), R(y_q)\} = \\ &= \max\{b(p) + \Delta(b(p)) + c_a(b(p)) + \beta(p), q + \Delta(q) + c_a(q)\} \leq \\ &= \max\{b(p) + \Delta(b(p) + 1), q + \Delta(q + 1)\} \leq q + 2p = n + p. \end{aligned}$$

Thus, this representation has a width of less than $2n$. □

Example 49. For the extended block B'_3 in Figure 4.14, we show the tolerance representation constructed by Proposition 48 in Figure 4.15.

Remark 50. In general, the constructed tolerance representation is not regular but it is possible to change the construction to yield a regular tolerance representation.

block type	description
L or B0	bridge with one cutpoint in $\mathcal{B}^1(B)$ and one leave (in P_G)
L	if cutpoint not on spine of $bc_\omega(G)$ (defined later)
B0	if cutpoint on spine of $bc_\omega(G)$ (defined later)
B1	bridge with two cutpoints, i.e., cutpoints in $\mathcal{B}^2(B)$
B2	block B with $\epsilon_B(c) = 2$ for each of its cutpoints c
B3	block B with $\epsilon_B(c) > 2$ for at least one cutpoint c

Table 4.2: block types

4.3 Arranging Blocks

Again, let G be a bipartite tolerance graph and $bc(G)$ the block-cutpoint-graph. For constructing a tolerance representation of G , the idea is to arrange the representations of the blocks of G by using $bc(G)$. Let $\epsilon_B(v)$ denote the eccentricity as mentioned before, that is the maximal distance between the vertex $v \in V(B)$ and any other vertex of B .

Note that a block B of a bipartite graph must be a single edge if we have $\epsilon_B(c) = 1$ for every cutpoint c of B . We distinguish different kinds of blocks as shown in Table 4.2.

Definition 51. The *weighted block-cutpoint-graph* $bc_\omega(G)$ is the block-cutpoint-graph together with a vertex labeling called weight ω such that $\omega(\text{cutpoint}) = 0$, $\omega(L) = \omega(B0) = 0$, $\omega(B1) = 1$, $\omega(B2) = 2$, and $\omega(B3) = 3$. A path in $bc_\omega(G)$ starting and ending in a vertex of type B1 or B2 or B3 and of highest weight is called a *spine* of $bc_\omega(G)$. The blocks on the spine are enumerated $B_1, \dots, B_s$.

Figure 4.16: weighted block-cutpoint-graph $bc_w(G)$ of Example 43 (spine marked bold; weights above or below vertices)

Note that the spine does not including any vertex of type B0 or L because such a vertex has degree one and is a leave. Hence, the calculation of the spine is independent of vertices of type B0 and L. Thus, after fixing the spine we can decide whether a 'leave vertex' is of type L or B0. Blocks of type L correspond to leaves of blocks and are automatically included in the tolerance representation of the extended block constructed by Proposition 48.

In Figure 4.16 we show the weighted block-cutpoint-graph of the graph in our Example 43.

4.4 Representation of the entire graph

Now we construct the tolerance representation of the whole graph G . We consider the blocks on the spine of $bc_w(G)$ first and the remaining non-spine blocks later. Assume that $s \geq 2$ otherwise we already know the representation of the whole graph. Let c_i be the cutpoint between block B_i and B_{i+1} . Note that $c_i \neq c_{i+1}$ for $s \geq 3$ and $i = 1, \dots, s-2$. Let Γ_i be the tolerance representation of the extended block of B_i as constructed in Proposition 48. We denote by $G_{1,i}$ the subgraph consisting of the

extended block $B'_1, \dots, B'_i$ without the cutpoint c_i for $i = 1, \dots, s-1$, or the subgraph of $B'_1, \dots, B'_i$ for $i = s$. The representation of $G_{1,i}$ is denoted by $\Gamma_{1,i}$. In an iterative way, we expand $\Gamma_{1,1}$ to a tolerance representation $\Gamma_{1,s}$. The cutpoint c_i does not need to be considered separately since it is included in the block B'_i .

In this section we have to deal with different representations simultaneously. One is the newly constructed representation and the others are the old representations of individual blocks. To distinguish them, we will use capital letters for the new one, i.e., $I(i) = [L(i), R(i)], T(i)$, and small letters for the old ones, i.e. $J(i) = [r(i), l(i)], t(i)$.

For the construction, it is necessary that the intervals corresponding to the vertices of $\mathcal{B}^2(B_i)$, which are the cutpoints c_{i-1} and c_i of block B_i , can be located at the sides of the tolerance representation of Γ_i . We have:

Result 52 ([8, Lemma 7]). *For a 2-connected block B of a bipartite tolerance graph G with $|\mathcal{B}^2(B)| = 2$, say $\mathcal{B}^2(B) = \{c, d\}$, the extended block B' has a partition into stars that are consecutively orderable (that is, a consecutive star partition) such that its first star contains c and its last star contains d .*

Remark 53. This result can be related to the reduced adjacency matrix. Again, without loss of generality, we assume that $A(B')$ is the consecutive 1's form for rows. Recall that columns are enumerated from 1 to q . The vertex c is either identically with vertex y_1 or is adjacent to y_1 . Thus, c corresponds to the first column or to a row i with $a(i) = 1$, i.e., which has a 1 in the first column. Analogously, d corresponds to either the last column or a row i with $b(i) = q$, i.e., which has a 1 in the last column.

Considering the tolerance representation Γ_i of the extended block B'_i by using Proposition 48, the interval $I(c_{i-1})$ is either located left most or it can be extended

to the left without changing the generated tolerance graph. That is true, because if $t(c_{i-1}) = 1$ or $t(c_{i-1}) = \infty$ then changing $l(c_{i-1})$ to $l(c_{i-1}) - k$, for some positive integer k , does not create a new edge. Note that by the construction of Γ_i there is no interval $J(v)$ with $r(v) \leq l(c_{i-1})$. If $1 < t(c_{i-1}) < \infty$ then B_i'' is bounded and $c_{i-1} \in X(B_i')$.

submatrix of $A(B')$	y_k	y_l
w	1	0
c_{i-1}	1	1

By permutations of rows and columns preserving Assumptions 1, 2, 3 this matrix can be transformed into one for that $\alpha(\text{row of } c_{i-1}) = 0$. Hence, assume that no such transformation exists. Then there must be another vertex x .

submatrix of $A(B')$	y_k	y_l
w	1	0
c_{i-1}	1	1
x	0	1

Together with the induced path (c_{i-1}, p_1, p_2) of length 2 which is attached to cut-point c_{i-1} , the vertices w, x, p_2 form an asteroidal triple in B'' — a contradiction to B'' is bounded.

Analogously, the interval $I(c_i)$ is located right most or it can be extended to the right without changing the generated tolerance graph.

In the next theorem, we show how the representation Γ_{i+1} can be modified and then attached to $\Gamma_{1,i}$ in order to obtain $\Gamma_{1,i+1}$. For a vertex $v \in V(\Gamma_{1,i})$, we denote the corresponding interval and tolerance in the representation $\Gamma_{1,i}$ by $I(v) = [L(v), R(v)]$ and $T(v)$. For a vertex $v \in V(\Gamma_{i+1}) \setminus \{c_i, c_{i+1}\}$, $i = 1, \dots, s-2$, $s \geq 2$, or $v \in$

$V(\Gamma_{i+1}) \setminus \{c_i\}$, $i = s - 1$, we denote the corresponding interval and tolerance in Γ_{i+1} by $J(v) = [l(v), r(v)]$ and $t(v)$. Note that vertex c_i has two corresponding intervals and tolerances, one is denoted by $J^-(c_i) = [l^-(c_i), r^-(c_i)]$ with $t^-(c_i)$ in Γ_i and the other is denoted by $J^+(c_i) = [l^+(c_i), r^+(c_i)]$ with $t^+(c_i)$ in Γ_{i+1} . These two intervals will be merged into one new interval $I(c_i) = [L(c_i), R(c_i)]$ with tolerance $T(c_i)$. Let

$$m_i := \max \left\{ |J^-(c_i)|, |J^+(c_i)| \right\}.$$

Let $M_0 := 0$ and

$$M_i := M_{i-1} + \max_{v \in V(\Gamma_i)} r(v) + m_i.$$

Since it is important for our construction to which partition class a cutpoint c_i belongs we introduce a short hand for each case.

condition	type of cutpoint c_i
$c_i \in X(B_i)$ and $c_i \in X(B_{i+1})$	XX-cutpoint
$c_i \in X(B_i)$ and $c_i \in Y(B_{i+1})$	XY-cutpoint
$c_i \in Y(B_i)$ and $c_i \in X(B_{i+1})$	YX-cutpoint
$c_i \in Y(B_i)$ and $c_i \in Y(B_{i+1})$	YY-cutpoint

Theorem 54 (construction step of representation of spine). *Let $\Gamma_{1,1}$ be the representation Γ_1 without $J(c_1)$ and in case that c_1 is a YX-cutpoint with the following modifications: for all $v \in \text{Adj}(c_1) \cap V(\Gamma_1)$, $r(v)$ is increased to $R(v) := r(v) + m_1$ and $t(v)$ is increased to $T(v) := t(v) + m_1 = |I(v)|$.*

For $s \geq 2$, the representation $\Gamma_{1,i+1}$, $i = 1, \dots, s - 1$, is constructed from $\Gamma_{1,i}$ and Γ_{i+1} in the following way. Add the interval $I(c_i) := [M_{i-1} + l^-(c_i), M_i + r^+(c_i)]$ and

if c_i is a YY -cutpoint then let $T(c_i) := 1$ otherwise let $T(c_i) := m_i + 1$. Let

$$W := \begin{cases} V(\Gamma_{i+1}) \setminus \{c_i, c_{i+1}\} & \text{if } i < s - 1, \\ V(\Gamma_{i+1}) \setminus \{c_i\} & \text{if } i = s - 1. \end{cases}$$

For each vertex $w \in W$, add the following interval $I(w)$ with

$$\begin{aligned} \bullet \quad L(w) &:= \begin{cases} M_i + l(w) - m_i & \text{if } c_i \text{ is } XY\text{-cutpoint and } w \in \text{Adj}(c_i) \\ M_i + l(w) & \text{otherwise} \end{cases} \\ \bullet \quad R(w) &:= \begin{cases} M_i + r(w) + m_{i+1} & \text{if } c_{i+1} \text{ is } YX\text{-cutpoint and } w \in \text{Adj}(c_{i+1}) \\ M_i + r(w) & \text{otherwise} \end{cases} \\ \bullet \quad T(w) &:= \begin{cases} |I(w)| & \text{if } t(w) \text{ is bounded} \\ \infty & \text{if } t(w) \text{ is not bounded} \end{cases} \end{aligned}$$

Proof. One step is to proof that $\Gamma_{1,1}$ is a representation of block B'_1 without c_1 . But this will easily follow from a later step.

We assume that $\Gamma_{1,i}$, $i \in \{1, \dots, s-1\}$, is a representation of the subgraph consisting of $B'_1, \dots, B'_i$ without c_i . By our construction, the representation $\Gamma_{1,i}$ becomes part of $\Gamma_{1,i+1}$ without any modification. Hence, to proof that $\Gamma_{1,i+1}$ represents the subgraph $G_{1,i+1}$ it suffices to show:

1. In $\Gamma_{1,i+1}$ there is no intersection between an interval of a vertex $v \in V(\Gamma_{1,i})$ and an interval of a vertex $w \in W$. In detail, we have $R(v) \leq L(w)$.
2. The cutpoint c_i is adjacent to a vertex $v \in V(\Gamma_i) \setminus \{c_i\}$ if and only if $|I(c_i) \cap I(v)| \geq \min\{T(c_i), T(v)\}$ in $\Gamma_{1,i+1}$.

3. The cutpoint c_i is adjacent to a vertex $w \in W$ if and only if $|I(c_i) \cap I(w)| \geq \min\{T(c_i), T(w)\}$ in $\Gamma_{1,i+1}$.
4. All the intervals $I(w)$ with tolerances $T(w)$ of $\Gamma_{1,i+1}$, $w \in W$, generate the same graph as Γ_{i+1} without $J^+(c_i)$ and, if $i < s - 1$, without $J^-(c_{i+1})$. In other words, they generate the extended block B'_{i+1} without the cutpoints c_i and, if $i < s - 1$, without c_{i+1} .

Figure 4.17 visualizes how Γ_{i+1} is added to $\Gamma_{1,i}$ for $i < s$.

Figure 4.17: construction of $\Gamma_{1,i+1}$

1. Let $v \in V(\Gamma_{1,i})$ and $w \in W$. There are two possible upper bounds for $R(v)$: If c_i is YX-cutpoint and v is adjacent to c_i then we have $R(v) \leq M_{i-1} + \max_{x \in V(\Gamma_i)} r(x) + m_i$ and otherwise we have $R(v) \leq M_{i-1} + \max_{x \in V(\Gamma_i)} r(x)$. And there are two possible lower bounds for $L(w)$: If c_i is XY-cutpoint and w is adjacent to c_i , then we have $M_i - m_i \leq L(w)$ and otherwise we have $M_i \leq L(w)$. By the definition of M_i we have $M_{i-1} + \max_{x \in V(\Gamma_i)} r(x) + m_i = M_i$. Note that c_i can either be a YX-cutpoint or a XY-cutpoint, but not both. Hence, the

conditions that $R(v) \leq M_{i-1} + \max_{x \in V(\Gamma_i)} r(x) + m_i$ and $M_i - m_i \leq L(w)$ do not occur simultaneously, and we always have $R(v) \leq L(w)$.

2. For c_i is adjacent to $v \in V(\Gamma_i) \setminus \{c_i\}$, we show $|I(c_i) \cap I(v)| \geq \min\{T(c_i), T(v)\}$.

And we show the contrapositive of the inversion: If c_i is not adjacent to $v \in V(\Gamma_i) \setminus \{c_i\}$, then $|I(c_i) \cap I(v)| < \min\{T(c_i), T(v)\}$.

Assume v is adjacent to c_i . If $c_i \in Y(\Gamma_{1,i})$ then $v \in X(\Gamma_{1,i})$ because $G(\Gamma_{1,i})$ is bipartite. By the construction of Γ_i in Proposition 48, we have $|J^-(c_i)| = 1 = t(c_i)$ and $J^-(c_i) \subseteq J(v)$. That implies $l(v) \leq l^-(c_i) < r^-(c_i) \leq r(v)$. Adding M_{i-1} to each term, we get $M_{i-1} + l(v) \leq M_{i-1} + l^-(c_i) < M_{i-1} + r^-(c_i) \leq M_{i-1} + r(v)$. With some additional estimations, we get $L(v) = M_{i-1} + l(v) \leq L(c_i) = M_{i-1} + l^-(c_i) < M_{i-1} + r^-(c_i) \leq M_{i-1} + r(v) \leq R(v) \leq M_{i-1} + m_i + r(v) \leq M_i + r^+(c_i) = R(c_i)$. That is $L(v) \leq L(c_i) < R(v) \leq R(c_i)$.

- If c_i is a YY-cutpoint then $T(c_i) = 1$. Thus, $|I(v) \cap I(c_i)| = R(v) - L(c_i) \geq 1 = T(c_i) = \min\{T(v), T(c_i)\}$.
- If c_i is a YX-cutpoint then $T(c_i) = m_i + 1$ and $R(v) = M_{i-1} + r(v) + m_i$. Thus, $|I(v) \cap I(c_i)| = R(v) - L(c_i) = r(v) + m_i - l^-(c_i) \geq 1 + m_i = T(c_i) \geq \min\{T(v), T(c_i)\}$.

If $c_i \in X(\Gamma_i)$ then $v \in Y(\Gamma_i)$. By the construction of Γ_i in Proposition 48, we have $|J(v)| = 1 = t(v)$ and $J(v) \subseteq J^-(c_i)$. Since c_i cannot be an YX-cutpoint we have $R(v) = M_{i-1} + r(v)$. If $c_{i-1} \in Y(\Gamma_i)$ then v is not adjacent to c_{i-1} because $v \in Y(\Gamma_i)$. Otherwise $c_{i-1} \in X(\Gamma_i)$ and c_{i-1} is not a XY-cutpoint. The last two facts imply that we have $L(v) = M_{i-1} + l(v)$ independently of to

Figure 4.18: ordering of endpoints of $J(v)$ and $J^-(c_i)$

which partition c_{i-1} belongs. Since the interval $I(v)$ is just the shift of $J(v)$ by M_{i-1} to the right and $J(v) \subseteq J^-(c_i)$, the length of $I(v) \cap I(c_i)$ is the same as $J(v) \cap J^-(c_i)$. Furthermore, because of $t(v) = 1$ and $|J(v)| = 1$ we have $T(v) = |I(v)| = 1$. Finally, we have $|I(v) \cap I(c_i)| = |J(v) \cap J^-(c_i)| = 1 = T(v) = \min\{T(v), T(c_i)\}$.

If v is not adjacent to c_i , we have $|J(v) \cap J^-(c_i)| < \min\{t(v), t^-(c_i)\}$. For the new endpoints, we have $M_{i-1} + l(v) - m_{i-1} \leq L(v) \leq M_{i-1} + l(v)$, $R(v) = M_{i-1} + r(v)$, $L(c_i) = M_{i-1} + l^-(c_i)$, $R(c_i) = M_i + r^+(c_i)$. Hence,

$$\begin{aligned}
 & |I(v) \cap I(c_i)| \\
 &= \max\{0, \min\{R(v), R(c_i)\} - \max\{L(v), L(c_i)\}\} \\
 &\leq \max\{0, \min\{r(v), M_i + r^+(c_i) - M_{i-1}\} - \max\{l(v) - m_{i-1}, l^-(c_i)\}\} \\
 &= \max\{0, r(v) - \max\{l(v) - m_{i-1}, l^-(c_i)\}\} = (*)
 \end{aligned} \tag{4.5}$$

In detail, we consider all different orderings of the endpoints of $J(v)$ and $J^-(c_i)$, also illustrated in Figure 4.18.

Case 1+2: $l(v) < r(v) \leq l^-(c_i) < r^-(c_i)$ or $l(v) < l^-(c_i) \leq r(v) < r^-(c_i)$.

Hence, $\min\{r(v), r^-(c_i)\} = r(v)$ and $\max\{l(v), l^-(c_i)\} = l^-(c_i)$.

$$\begin{aligned} (*) &= \max\{0, r(v) - l^-(c_i)\} = |J(v) \cap J^-(c_i)| \\ &< \min\{t(v), t^-(c_i)\} \leq \min\{T(v), T(c_i)\}. \end{aligned}$$

Case 3: $l(v) \leq l^-(c_i) < r^-(c_i) \leq r(v)$. Since v is not adjacent to c_i , $t^-(c_i) = \infty$ and $t(v) > |J^-(c_i)| \geq 1$, i.e., $t(v) = |J(v)|$. Hence, $\min\{r(v), r^-(c_i)\} = r^-(c_i)$ and $\max\{l(v), l^-(c_i)\} = l^-(c_i)$. Thus,

$$\begin{aligned} (*) &= \max\{0, r(v) - l^-(c_i)\} < \begin{cases} |J(v)| = t(v) \\ \infty = t^-(c_i) \end{cases} \\ &= \min\{t(v), t^-(c_i)\} \leq \min\{T(v), T(c_i)\} \end{aligned}$$

Case 4: $l^-(c_i) \leq l(v) < r(v) \leq r^-(c_i)$. Since v is not adjacent to c_i , $t(v) = \infty$ and $t^-(c_i) > |J(v)| \geq 1$, i.e., $t^-(c_i) = |J^-(c_i)|$. Hence, $\min\{r(v), r^-(c_i)\} = r(v)$ and $\max\{l(v), l^-(c_i)\} = l(v)$. Thus,

$$\begin{aligned} (*) &\leq \max\{0, r(v) - l(v)\} = |J(v)| < \begin{cases} t^-(c_i) \\ \infty = t(v) \end{cases} \\ &= \min\{t(v), t^-(c_i)\} \leq \min\{T(v), T(c_i)\} \end{aligned}$$

Case 5+6: $l^-(c_i) < l(v) < r^-(c_i) < r(v)$ or $l^-(c_i) < r^-(c_i) \leq l(v) < r(v)$. In other words, the interval $J(v)$ is located right of $l^-(c_i)$. But by the construction by Proposition 48 and by Remark 53 to have $l^-(c_i) < l(v)$ is impossible.

3. This part is similar to part 2. For the sake of completeness we present the details, too. First we show that if c_i is adjacent to $w \in W$, then $|I(c_i) \cap I(w)| \geq$

$\min\{T(c_i), T(w)\}$. Second, we show the contrapositive of the inversion: For c_i is not adjacent to $w \in W$, we have $|I(c_i) \cap I(w)| < \min\{T(c_i), T(w)\}$.

Let w be adjacent to c_i . If $c_i \in Y(\Gamma_{i+1})$ then $w \in X(\Gamma_{i+1})$ because $G(\Gamma_{i+1})$ is bipartite. By the construction of Γ_{i+1} in Proposition 48, we have $|J^+(c_i)| = 1 = t(c_i)$ and $J^+(c_i) \subseteq J(w)$. That implies $l(w) \leq l^+(c_i) < r^+(c_i) \leq r(w)$. Adding M_i to each term, we get $M_i + l(w) \leq M_i + l^+(c_i) < M_i + r^+(c_i) \leq M_i + r(w)$. With some additional estimations, we get $L(c_i) = M_{i-1} - l^-(c_i) < M_i - m_i + l(w) \leq L(w) \leq M_i + l(w) \leq M_i + l^+(c_i) < R(c_i) = M_i + r^+(c_i) \leq M_i + r(w) \leq R(w)$. That is $L(c_i) < L(w) < R(c_i) \leq R(w)$.

- If c_i is a YY-cutpoint then $T(c_i) = 1$. Thus, $|I(w) \cap I(c_i)| = R(c_i) - L(w) \geq 1 = T(c_i) = \min\{T(w), T(c_i)\}$.
- If c_i is a XY-cutpoint then $T(c_i) = m_i + 1$ and $L(w) = M_i + l(w) - m_i$. Thus, $|I(w) \cap I(c_i)| = R(c_i) - L(w) = r^+(c_i) - l(w) + m_i \geq 1 + m_i = T(c_i) \geq \min\{T(w), T(c_i)\}$.

If $c_i \in X(\Gamma_{i+1})$ then $w \in Y(\Gamma_{i+1})$. By the construction of Γ_{i+1} in Proposition 48, we have $|J(w)| = 1 = t(w)$ and $J(w) \subseteq J^+(c_i)$. Since c_i cannot be an XY-cutpoint we have $L(w) = M_i + l(w)$. If $c_{i+1} \in Y(\Gamma_{i+1})$ then w is not adjacent to c_{i+1} because $w \in Y(\Gamma_{i+1})$. Otherwise $c_{i+1} \in X(\Gamma_{i+1})$ and c_{i+1} is not a YX-cutpoint. The last two fact imply that independent of which partition c_{i+1} belongs to, we have $R(w) = M_i + r(w)$. Since the interval $I(w)$ is just the shift of $J(w)$ by M_i to the right and $J(w) \subseteq J^+(c_i)$, the length of $I(w) \cap I(c_i)$ is the same as $J(w) \cap J^+(c_i)$. Furthermore, because of $t(w) = 1$ and $|J(w)| = 1$ we have $T(w) = |I(w)| = 1$. Finally, we have $|I(w) \cap I(c_i)| = |J(w) \cap J^+(c_i)| =$

Figure 4.19: ordering of endpoints of $J(w)$ and $J^+(c_i)$

$$1 = T(w) = \min\{T(w), T(c_i)\}.$$

If w is not adjacent to c_i , we have $|J(w) \cap J^+(c_i)| < \min\{t(w), t^+(c_i)\}$. For the new endpoints, we have $L(w) = M_i + l(w)$, $M_i + r(w) \leq R(w) \leq M_i + m_{i+1} + r(w)$, $L(c_i) = M_{i-1} + l^-(c_i)$, $R(c_i) = M_i + r^+(c_i)$. Hence,

$$\begin{aligned} & |I(w) \cap I(c_i)| \\ &= \max\{0, \min\{R(w), R(c_i)\} - \max\{L(w), L(c_i)\}\} \\ &\leq \max\{0, \min\{r(w) + m_{i+1}, r^+(c_i)\} - \max\{l(w), M_{i-1} + l^-(c_i) - M_i\}\} \\ &= \max\{0, \min\{r(w) + m_{i+1}, r^+(c_i)\} - l(w)\} = (*) \end{aligned} \tag{4.6}$$

In detail, we consider all different ordering of the endpoints of $J(w)$ and $J^+(c_i)$, as also illustrated in Figure 4.19.

Case 1+2: $l(w) < r(w) \leq l^+(c_i) < r^+(c_i)$ or $l(w) < l^+(c_i) \leq r(w) < r^+(c_i)$. In other words, the interval $J(w)$ is located left of $r^+(c_i)$. But by the construction by Proposition 48 and by Remark 53 to have $r(w) < r^+(c_i)$ is impossible.

Case 3: $l(w) \leq l^+(c_i) < r^+(c_i) \leq r(w)$. Since w is not adjacent to c_i , $t^+(c_i) = \infty$ and $t(w) > |J^+(c_i)| \geq 1$, i.e., $t(w) = |J(w)|$. Hence, $\min\{r(w), r^+(c_i)\} = r^+(c_i)$

and $\max\{l(w), l^+(c_i)\} = l^+(c_i)$. Thus,

$$\begin{aligned} (*) &= \max\{0, r^+(c_i) - l(w)\} < \begin{cases} |J(w)| = t(w) \\ \infty = t^+(c_i) \end{cases} \\ &= \min\{t(w), t^+(c_i)\} \leq \min\{T(w), T(c_i)\} \end{aligned}$$

Case 4: $l^+(c_i) \leq l(w) < r(w) \leq r^+(c_i)$. Since w is not adjacent to c_i , $t(w) = \infty$ and $t^+(c_i) > |J(w)| \geq 1$, i.e., $t^+(c_i) = |J^+(c_i)|$. Hence, $\min\{r(w), r^+(c_i)\} = r(w)$ and $\max\{l(w), l^+(c_i)\} = l(w)$. Thus,

$$\begin{aligned} (*) &\leq \max\{0, r^+(c_i) - l(w)\} < \begin{cases} |J^+(c_i)| = t^+(c_i) \\ \infty = t(w) \end{cases} \\ &= \min\{t(w), t^+(c_i)\} \leq \min\{T(w), T(c_i)\} \end{aligned}$$

Case 5+6: $l^+(c_i) < r(w) < r^+(c_i) < r(w)$ or $l^+(c_i) < r^+(c_i) \leq l(w) < r(w)$.

Hence, $\min\{r(w), r^+(c_i)\} = r^+(c_i)$ and $\max\{l(w), l^+(c_i)\} = l(w)$.

$$\begin{aligned} (*) &= \max\{0, r^+(c_i) - l(w)\} = |J(w) \cap J^+(c_i)| \\ &< \min\{t(w), t^+(c_i)\} \leq \min\{T(w), T(c_i)\}. \end{aligned}$$

4. For $w_1, w_2 \in W$, we show that $|I(w_1) \cap I(w_2)| \geq \min\{T(w_1), T(w_2)\}$ if and only if w_1 is adjacent to w_2 .

If w_1 is adjacent to w_2 , then without loss of generality $w_1 \in Y(\Gamma_{i+1})$ and $w_2 \in X(\Gamma_{i+1})$. We show that the interval $I(w_1)$ is as long as $J(w_1)$. If $c_i \in Y(\Gamma_{i+1})$ then c_i is not adjacent to w because of bipartiteness, and if $c_i \in X(\Gamma_{i+1})$ then c_i is not a XY-cutpoint. Hence, $L(w_1) = l(w_1) + M_i$. The vertex $w_1 \in Y(\Gamma_{i+1})$

cannot be adjacent to another vertex in $Y(\Gamma_{i+1})$. Hence, w_1 is never a neighbor of a YX-cutpoint c_{i+1} and $R(w_1) = r(w_1) + M_i$. Thus, $I(w)$ is just a shift of $J(w)$ by M_i to the right and $|I(w)| = |J(w)| = 1$ as well as $T(w_1) = 1$. Hence, $|I(w_1) \cap I(w_2)| \geq 1 = \min\{T(w_1), T(w_2)\}$.

If w_1 is not adjacent to w_2 , then $|J(w_1) \cap J(w_2)| < \min\{t(w_1), t(w_2)\}$. If B'' is not bounded and $w_1, w_2 \in X(\Gamma_{i+1})$ then $t(w_1) = t(w_2) = \infty$ and $T(w_1) = T(w_2) = \infty$. Hence, $|I(w_1) \cap I(w_2)| < \infty = \min\{T(w_1), T(w_2)\}$.

Otherwise without loss of generality $t(w_1) \leq |J(w_1)|$. Then $J(w_1)$ cannot be subinterval of $J(w_2)$ else w_1 would be adjacent to w_2 .

$$|I(w_1) \cap I(w_2)| = \max\{0, \min\{R(w_1), R(w_2)\} - \max\{L(w_1), L(w_2)\}\} = (*)$$

Now we distinguish two cases.

Case 4.1: $l(w_1) < l(w_2)$ (and $r(w_1) < r(w_2)$). If the endpoints $L(w_2)$ and $R(w_1)$ are just shifted by M_i then independent of $L(w_1)$ and $R(w_2)$ we get

$$\begin{aligned} (*) &= \max\{0, \min\{r(w_1), r(w_2)\} - \max\{l(w_1), l(w_2)\}\} \\ &= |J(w_1) \cap J(w_2)| \\ &< \min\{t(w_1), t(w_2)\} \\ &= \min\{T(w_1), T(w_2)\} \end{aligned}$$

If $L(w_2)$ is extended to $l(w_2) + M_i - m_i$ then w_2 is adjacent to c_i and c_i is a XY-cutpoint. By Remark 53 there is no interval with $r(v) < r(c_i)$ that does not intersect $J(c_i)$. Hence, since $r(w_2) < r(c_i)$, w_1 is also adjacent to c_i and $L(w_2)$ is extended to $l(w_2) + M_i - m_i$. Hence, both tolerances $T(w_1)$ and $T(w_2)$ must be increased by m_i .

If $R(w_1)$ is extended to $r(w_1) + M_i + m_{i+1}$ then w_1 is adjacent to c_{i+1} and c_{i+1} is a YX-cutpoint. By Remark 53 there is no interval with $l(c_{i+1}) < l(v)$ that does not intersect $J(c_{i+1})$. Hence, since $l(c_{i+1}) < l(w_2)$, w_2 is also adjacent to c_{i+1} and $R(w_2)$ is extended to $r(w_2) + M_i + m_{i+1}$. Hence, both tolerances $T(w_1)$ and $T(w_2)$ must be increased by m_{i+1} .

If $R(w_1)$ is extended or $L(w_2)$ is extended, $(*)$ is less than $\min\{T(w_1), T(w_2)\}$.

We show the proof for when both interval endpoints are extended. The other proofs are similar.

$$\begin{aligned}
 (*) &= \max\{0, \min\{r(w_1) + m_{i+1}, r(w_2) + m_{i+1}\} - \max\{l(w_1) - m_i, l(w_2) - m_i\}\} \\
 &= \max\{0, r(w_1) + m_{i+1} - l(w_2) + m_i\} \\
 &= |J(w_1) \cap J(w_2)| + m_{i+1} + m_i \\
 &< \min\{t(w_1), t(w_2)\} + m_{i+1} + m_i \\
 &= \min\{T(w_1), T(w_2)\}
 \end{aligned}$$

Case 4.2: $l(w_2) < l(w_1)$ (and $r(w_2) < r(w_1)$) is symmetric to Case 4.1. If the endpoints $L(w_1)$ and $R(w_2)$ are just shifted by M_i then independent of $L(w_2)$ and $R(w_1)$ we get

$$\begin{aligned}
 (*) &= \max\{0, \min\{r(w_1), r(w_2)\} - \max\{l(w_1), l(w_2)\}\} \\
 &= |J(w_1) \cap J(w_2)| \\
 &< \min\{t(w_1), t(w_2)\} \\
 &= \min\{T(w_1), T(w_2)\}
 \end{aligned}$$

If $L(w_1)$ is extended to $l(w_1) + M_i - m_i$ then $L(w_2)$ is extended to $l(w_2) + M_i - m_i$ and both tolerances $T(w_1)$ and $T(w_2)$ must be increased by m_i . If $R(w_2)$ is

extended to $r(w_2) + M_i + m_{i+1}$ then $R(w_1)$ is extended to $r(w_1) + M_i + m_{i+1}$ and both tolerances $T(w_1)$ and $T(w_2)$ must be increased by m_{i+1} .

If $L(w_1)$ is extended or $R(w_2)$ is extended, $(*)$ is less than $\min\{T(w_1), T(w_2)\}$.

This time we show the proof for when $L(w_1)$ and thus also $L(w_2)$ are extended.

The other proofs are similar

$$\begin{aligned}
 (*) &= \max\{0, \min\{r(w_1), r(w_2)\} - \max\{l(w_1) - m_i, l(w_2) - m_i\}\} \\
 &= \max\{0, r(w_1) - l(w_2) + m_i\} \\
 &= |J(w_1) \cap J(w_2)| + m_i \\
 &< \min\{t(w_1), t(w_2)\} + m_i \\
 &= \min\{T(w_1), T(w_2)\}
 \end{aligned}$$

Now the proof that $\Gamma_{1,1}$ represents block B'_1 without c_1 follows from Point 4. $\square$

Remark 55. In some special cases, if c_i is not a XY-cutpoint and not a YX-cutpoint, in $\Gamma_{1,i+1}$ there will be 'gap', i.e., the interval $I(c_i)$ has no intersection with other intervals between $M_i - m_i$ and M_i . In this case the constant M_i can be decreased by m_i . In detail, $M_i = M_{i-1} + \max_{v \in V(\Gamma_i)} r(v)$.

Now we define the points P_i of the representation $\Gamma_{1,s}$ where we have $L(j) < P_i < R(j)$ only for $j = c_i$. Let

$$P_i := \begin{cases} M_i - m_i & \text{if } c_i \text{ is XY-cutpoint,} \\ M_i & \text{otherwise.} \end{cases}$$

Proposition 56. For $j \in V(\Gamma_{1,s})$ we have $L(j) < P_i < R(j)$ if and only if $j = c_i$.

Proof. This follows from last Theorem 54 and especially from Point 1 of its proof: There is no intersection between an interval of $\Gamma_{1,i}$ and an interval of Γ_{i+1} in $\Gamma_{1,i+1}$, except $I(c_i)$. $\square$

Finally, we show how to add the non-spine vertices of the weighted block-cutpoint-graph $bc(G)$, which are not yet included in the tolerance representation $\Gamma_{1,s}$. First we exclude some cases.

Theorem 57. *Let G be a bipartite tolerance graph with weighted block-cutpoint-graph $bc_\omega(G)$. Consider an arbitrary component K in $bc_\omega(G)$ that arises from removing a cutpoint C in $bc_\omega(G)$ and that only consists of non-spine vertices. Let H be the subgraph of G that corresponds to the vertices of K and let $c \in V(G)$ be the cutpoint between H and the corresponding vertex/vertices of C . Then the eccentricity $\epsilon_H(c)$ is less than 3, i.e., there is no vertex $v \in V(H)$ such that the distance to c is larger than or equal 3.*

Proof. For a contradiction, we assume $\epsilon_H(c) \geq 3$, i.e., that a vertex $v \in V(H)$ at distance at least 3 from c exists. This implies that there is an induced path of length 3 in H . Since no path of weight 2 in C can give rise to a subgraph H' of G with $\epsilon_{H'}(c) \geq 3$, the component C must contain a path p_1 of weight 3. Since the vertices of C are not on the spine of $bc_\omega(G)$, there must be two paths p_2 and p_3 on the spine starting in c with weight at least 3 (otherwise the path p_1 in C were on the spine). Both paths p_2 and p_3 will induce paths in G of length at least 3. Hence, G contains an induced T_3 — a contradiction. $\square$

Corollary 58. *Using the notation of Theorem 57, the component C of $bc_\omega(G)$ does*

Figure 4.20: forbidden and allowed subgraphs of $bc_\omega(G)$

not contain one of the following paths: $c-B3$ or $c-B2-c-B2$ or $c-B1-c-B2$ or $c-B1-c-B1-c-B1$.

By Corollary 58 the allowed non-spine blocks are of type L, B1, or B2. Only cutpoints at distance 1 from a cutpoint on the spine may have leaves attached. For illustration we show one forbidden and one allowed subgraph of $bc_\omega(G)$ in Figure 4.20 (vertex names denote the type only).

Note that a 2-connected block is an induced subgraph. For a block B of type B1 or B2 and for a cutpoint $c_i \in V(B)$ of G , the neighborhood $Adj(c_i)$ forms a complete partition class of the bipartite subgraph B . Hence, the expression $Adj(c_i)$ -consecutive is well defined.

Theorem 59 (adding non-spine blocks). *Let B be a block not on the spine. Let Γ be a representation of $B_1, \dots, B_s$ and possibly some non-spine blocks excluding B . Let $\Gamma'_B = (\{[l(k), r(k)] : k \in V(B')\}, \{t(k) : k \in V(B')\})$ be the tolerance representation of the block B' constructed using Proposition 48, and let $W(\Gamma')$ be the width of Γ'_B .*

Let

$$d = \begin{cases} 2 \cdot W(\Gamma'_B) - 2 & \text{Case a: if } T(c_i) = 1 \text{ and } B \text{ is } \text{Adj}(c_i)\text{-consecutive,} \\ T(c_i)W(\Gamma'_B) & \text{Case b: if } T(c_i) > 1 \text{ and } B \text{ is not } \text{Adj}(c_i)\text{-consecutive,} \\ W(\Gamma'_B) & \text{otherwise;} \end{cases}$$

and let $\tilde{P}_i$ be a point such that $L(j) < \tilde{P}_i < R(j)$ if and only if $j = c_i$. Let $V_1 = \{j \in V(\Gamma) \setminus \{c_i\} : R(j) \leq \tilde{P}_i\}$ and $V_2 = \{j \in V(\Gamma) \setminus \{c_i\} : \tilde{P}_i \leq L(j)\}$. The new representation Γ' consists of following intervals and tolerances.

- If not Case a, let $I'(j) = I(j)$ for $j \in V_1$. Let $I'(c_i) = [L(c_i), R(c_i) + d]$ and let $I'(j) = [L(j) + d, R(j) + d]$ for $j \in V_2$. For all $j \in V(\Gamma)$, let $T'(j) = T(j)$.
- Otherwise if Case a then for $j \in V_1 \setminus \text{Adj}(c_i)$, let $I'(j) = I(j)$ and $T'(j) = T(j)$, and for $j \in V_1 \cap \text{Adj}(c_i)$, $I'(j) = [L(j), R(j) + W(\Gamma'_B) - 1]$ and

$$T'(j) = \begin{cases} T(j) + W(\Gamma'_B) - 1 & \text{if } T(j) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

For vertex c_i , $I'(c_i) = [L(c_i), R(c_i) + d]$ and $T'(c_i) = W(\Gamma'_B)$. For $j \in V_2 \setminus \text{Adj}(c_i)$, $I'(j) = [L(j) + d, R(j) + d]$ and $T'(j) = T(j)$, and for $j \in V_2 \cap \text{Adj}(c_i)$, $I'(j) = [L(j) - (W(\Gamma'_B) - 1) + d, R(j) + d]$ and

$$T'(j) = \begin{cases} T(j) + W(\Gamma'_B) - 1 & \text{if } T(j) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

- For $k \in V(B')$, we add

$$I'(k) = [d \cdot l(k) + \tilde{P}_i, d \cdot r(k) + \tilde{P}_i], \quad T'(k) = d \cdot t(k) \text{ if Case b,}$$

$$I'(k) = [l(k) + \tilde{P}_i, r(k) + \tilde{P}_i], \quad T'(k) = t(k) \text{ otherwise.}$$

Proof. By Proposition 56 there exists $\tilde{P}_i$ for $\Gamma_{1,s}$. If non-spine blocks are added by Theorem 59, the existence of $\tilde{P}_i$ is still guaranteed for Γ . We show that the new representation Γ' generates the same graph as $G(\Gamma)$ united with block B' .

The graph $G(\Gamma) \setminus \{c_i\}$ has two subgraphs $G[V_1]$ and $G[V_2]$ which are connected only by the cutpoint c_i . Note that $G[V_1]$ or $G[V_2]$ might consist of more than one component.

1. In Case a, Γ' restricted to vertices of V_1 is identical to Γ restricted to V_1 except for the intervals of vertices $j \in V_1 \cap \text{Adj}(c_i)$. But since $T(c_i) = 1$, we still have $|I'(j) \cap I'(c_i)| = |I(j) \cap I(c_i)| + W(\Gamma'_B) - 1 \geq \min\{T(j), T(c_i)\} + W(\Gamma'_B) - 1 = W(\Gamma'_B) = T'(c_i) \geq \min\{T'(j), T'(c_i)\}$, thus $j \sim c_i$. Otherwise (not Case a) Γ' restricted to vertices of V_1 is identical to Γ restricted to V_1 .

2. If all numbers of the representation Γ_B are scaled by the same factor d we still have a representation for B . And the same is true if we shift the representation by $\tilde{P}_i$. Hence, $G(\Gamma'_B) = B$.

3. In Case a, Γ' restricted to vertices of V_2 is identical to Γ restricted to V_2 and shifted by d and except the intervals of vertices $j \in V_2 \cap \text{Adj}(c_i)$. But since $T(c_i) = 1$, we still have $|I'(j) \cap I'(c_i)| = |I(j) \cap I(c_i)| + W(\Gamma'_B) - 1 \geq \min\{T(j), T(c_i)\} + W(\Gamma'_B) - 1 = W(\Gamma'_B) = T'(c_i) \geq \min\{T'(j), T'(c_i)\}$, thus $j \sim c_i$. Otherwise (not Case a) Γ' restricted to vertices of V_2 is identical to Γ restricted to V_2 and shifted by d .

4. Because of the new created 'gap' from $\tilde{P}_i$ to $\tilde{P}_i + d$, we have $|I'(v) \cap I'(w)| = 0$ for $v \in V(\Gamma) \setminus \{c_i\}$ and $w \in V(B') \setminus \{c_i\}$. And finally, $I(c_i)$ connects all three subgraphs.

That implies $G(\Gamma')$ is equal to $G(\Gamma)$ united with B' . $\square$

We conclude:

Corollary 60. *For a bipartite tolerance graph G and its weighted block-cutpoint graph $bc_\omega(G)$, applying the Theorem 54 on all spine blocks of $bc_\omega(G)$ and applying Theorem 59 on each non-spine block of $bc_\omega(G)$ yield a tolerance representation Γ of G . The representation Γ is bounded if and only if G is bounded.*

Note that for a bounded bipartite tolerance graph, all extended blocks are bounded and the construction steps do not produce any unbounded tolerances. Thus, this construction results in a bounded or unbounded tolerance representation depending on whether the tolerance graph is bounded or unbounded.

Theorem 61. *A bipartite tolerance graph G has an integer tolerance representation Γ where the largest occurring integer is bounded by $4n^2$. Thus, the storage size of the representation in bits is $O(n \log n)$.*

Proof. We constructed Γ by Corollary 60. Let $bc_\omega(G)$ be the block-cutpoint-graph of G and let the spine of $bc_\omega(G)$ consist of the blocks $B_1, \dots, B_s$ with $k_1, \dots, k_s$ vertices. We have $k := k_1 + k_2 + \dots + k_s \leq n$. The largest integer in the tolerance representation of block B_i is at most $2k_i$ by Proposition 48. In above construction (Theorem 54) the blocks $B_1, \dots, B_s$ are placed beside each other with certain gaps in between. The sizes of the gaps are determined by m_i which is bounded by the longest interval of both neighboring blocks. That is in the worst case a block can be enlarged to each side by $2k_i$. Hence, the width can be at most $6k_i$. Summing up over in the worst case, the width of all blocks together is at most $6k$.

Then there are non-spine blocks $\tilde{B}_1, \dots, \tilde{B}_p$ with width $W(\tilde{B}_1), \dots, W(\tilde{B}_p)$. In the worst case a non-spine block $\tilde{B}_j$ with width $W(\tilde{B}_j)$ can be enlarged to width $T(c_i)W(\tilde{B}_j)$ where $T(c_i)$ is some bounded tolerance value (see Case b of Theorem

59). An upper bound of $T(c_i)$ is the length of the longest interval in a representation of a block increased by 1, i.e., $\max_{i=1,\dots,s} 2k_i + 1$. A simpler upper bound of $T(c_i)$ is $2n$. The width of all non-spine blocks $\tilde{B}_1, \dots, \tilde{B}_p$ together is $W(\tilde{B}_1) + W(\tilde{B}_2) + \dots + W(\tilde{B}_p) \leq 2(n - k)$. After possibly scaling these blocks, the width of them is at most $2n2(n - k) = 4n^2 - 4kn$. Altogether we construct a representation where the biggest integer is at most $6k + 4n^2 - 4kn = 4n^2 + 2k(3 - 2n) < 4n^2$. $\square$

Theorem 62. *The tolerance representation of Corollary 60 can be constructed in $O(n)$ time.*

Proof. The first step of the construction is to find the components of the bipartite tolerance graph. This can be done in linear time $O(n)$. The second step is to transform all the reduced adjacency matrices to be in the consecutive 1's form. It is possible to construct the consecutive 1's form efficiently, for example, see [31] for a linear time algorithm. Finally, the construction of the representation of the spine blocks and adding the non-spine blocks costs $O(n)$ if we combine Theorem 54 and Theorem 59. That is we add spine blocks and non-spine blocks in sequence as they appear in the block-cutpoint-graph $bc_\omega(G)$. In sum, the run time is in $O(n)$. $\square$

Remark 63. The above tolerance representation can be used as a certificate for the recognition algorithm of bipartite tolerance graphs described in [8, p. 17].

For a bipartite bounded tolerance graph, our construction will give a bounded representation. But checking more carefully, for an interval of length k , the tolerance will be k . Thus, our construction yields a direct way to show the well known result:

Corollary 64 (cf. [19, Theorem 3.9]). *A bipartite bounded tolerance graph is a permutation graph.*

Chapter 5

Recognition of Tolerance Graphs and Integer Linear Programming

5.1 Complexity of Recognition

The complexity of the problem to decide whether a given graph is a tolerance graph, was an open question for over two decades. Just a few years ago it was identified. One step towards the determination of the complexity was archived by Hayward and Shamir. In their article [25], they proved that the problem belongs to the class of NP problems. There was some evidence that the problem might be polynomial time solvable. Especially the recognition of the super classes (weakly chordal graphs) and the subclasses (interval graphs, permutation graphs) is in polynomial time (cf. Chapter 2.6.1). But Mertzios and Sau and Zaks showed in [38] that the recognition problem is NP-hard. That is we cannot construct any polynomial run time algorithm for it, if the classes P and NP differ.

Here we present the main ideas of both articles. We start with showing that the problem is in the class NP. Afterwards we continue with showing the NP-hardness.

Result 65 ([25, Corollary 4]). *The recognition problem of tolerance graphs is in the class NP.*

We give a sketch of the proof based on [25]. If a problem belongs to the class NP then every solution can be verified in polynomial time with respect to the input size. In this case a solution to the recognition problem is a tolerance representation and the input size is determined by the number n of vertices or the number m of edges. One can verify that a given tolerance representation corresponds to the given tolerance graph by following steps. For each pair of vertices v and w of the graph, one checks that there is an edge between the vertices v and w if and only if the corresponding intervals $I(v)$ and $I(w)$ have an intersection that is at least $\min\{T(v), T(w)\}$. If all interval endpoints $L(v)$, $R(v)$ and the values of the tolerances $T(v)$ are in the range $[0, M]$ where $\log(M) \in O(n + m)$, then the calculation of $|I(v) \cap I(w)|$ and also $\min\{I(v), I(w)\}$ will take polynomial time. But a problem will arise if the interval endpoints or tolerance values are from an arbitrarily large range, in other words, if the number of bits to represent those values is not bounded. Then the calculation of $|I(v) \cap I(w)|$ and $\min\{I(v), I(w)\}$ will take arbitrarily much time.

Hence, the main work is to find the upper bound M on the values (of a non-negative integer tolerance representation) with $\log(M) \in O(n + m)$. Then the verification can be done in polynomial time. Let Γ be a solution, that is a tolerance representation of the graph. We extract the order of the critical points (that are $L(v)$, $L(v) + T(v)$, $R(v) - T(v)$, $R(v)$) and the order of the tolerances from Γ . Both

orders together determine the tolerance graph uniquely (as shown in Lemma 1 of [25]). These orders are total / linear and can be transformed into a linear inequality system. In general for some variables c_i , $i = 1, \dots, n$, the ordering $c_1 < c_2 < \dots < c_n$ can be written as $c_{i+1} - c_i > 0$ for $i \in \{1, \dots, n-1\}$. That is a linear inequality system. The system of the orders of critical points and tolerances has a feasible basic solution because Γ is one. Furthermore, the linear system can be transformed into standard form and there is a solution by the Cramer's Rule, where each entry can be expressed by determinants. Thus, an upper bound for each entry can be established, which is exponential in the number of vertices of the graph. Since the entries correspond to the critical points and tolerances of the tolerance representation, one can choose this value for M .

Remark 66. The recognition problem of bounded tolerance graphs is as well in NP. For checking whether the representation corresponds to the graph, we have to check additionally that each tolerance is less or equal to the length of its interval.

Result 67 ([38, Theorem 4.1]). *The recognition problem of bounded tolerance graphs and tolerance graphs is NP-hard.*

We can only give a sketch of the proofs in [37] (detailed version) and [38] (shortened version).

First, the recognition of bounded tolerance graphs is shown to be NP-hard, by providing a reduction from the monotone-Not-All-Equal-3-SAT (monotone-NAE-3-SAT) problem. Let φ be a boolean formula in conjunctive normal form with three literals in every clause (3-CNF) and let φ be also *monotone*, that is, no variable is negated. The formula φ is called *NAE-satisfiable* if there exists a truth assignment of

the variables of φ such that every clause has at least one true variable and one false variable. Given a monotone 3-CNF formula φ , a trapezoid graph H_φ is constructed, which is a parallelogram graph if and only if φ is NAE-satisfiable. Note that being a parallelogram graph is equivalent to being a bounded tolerance graph (Result 21).

The main tool is an algorithm (which uses an approach similar as in [9]) that transforms a given trapezoid graph into a permutation graph, while preserving a certain acyclic orientation property. For that, the notion of an acyclic orientation of a permutation graph and a trapezoid graph is extended. Furthermore, several properties of permutation and trapezoid graphs are used. The constructed permutation graph does not depend on any particular trapezoid representation of the input graph.

Second, this reduction can be extended to the recognition of general tolerance graphs. The constructed graph H_φ is a tolerance graph if and only if it is a bounded tolerance graph. Hence, both problems are NP-hard.

The last two results together show:

Result 68 ([38, Theorem 3.3,4.1]). *The recognition problem of tolerance graphs as well as the recognition problem of bounded tolerance graphs is NP-complete.*

5.2 Recognition Algorithm

By the last result, the recognition of tolerance graphs will need exponential run time in general if the classes P and NP differ. Keeping this in mind we present some reasonable fast way to solve the problem.

5.2.1 Enumeration

First of all, a very simple approach to check whether a graph with n vertices is a tolerance graph is an enumeration. We could enumerate the non-negative integer tolerance representations that yield all tolerance graphs on n vertices. As shown by [25, Theorem 3] (Result 69) each tolerance graph has a non-negative integer tolerance representation. Such a tolerance representation consists of n left and n right interval endpoints and n tolerances. Each of these values would have to range over the integers from 0 to a certain upper bound, the width of the representation. The best-known width was $1 + 5n2^{5n}$ in [25, Theorem 3]. (Actually, this can be reduced as shown in Chapter 6.) However, using this approach we have to check much more than 2^{15n^2} combinations in average to decide whether a graph has a tolerance representation — that is actually impossible. But there are several symmetries which can be broken by adding some constraints. For example, the left endpoint must be strictly smaller than the right endpoint, and at least one interval must start at 0, and unbounded tolerances can be set to a value large than the width of the tolerance representation. But even if we reduced the width for the enumeration to n^2 , we still would need to check $O((n^2)^{3n})$ cases. Hence, this method is not suitable.

5.2.2 Setup of a Mixed Integer Linear Program

A more practical approach to recognize tolerance graphs is to apply linear programming techniques. These techniques were also used in [25] to prove a theorem about the width of an integer tolerance representation. That is the Result 65 in our thesis. Here we will set up a more general linear program.

Let $G = (V, E)$ be a graph. G is a tolerance graph if and only if it has a (non-negative) tolerance representation $\Gamma = (\{I(i)\}, \{T(i)\})$, $I(i) = [L(i), R(i)]$. We can change a tolerance representation having negative numbers into one without negative numbers by adding a positive constant to all interval endpoints such that they become non-negative. The values $L(i), R(i), T(i)$, $i \in V$, form a tolerance representation of G if and only if the following system (5.1) has a solution.

$$\begin{aligned}
 |I(i) \cap I(j)| &\geq \min\{T(i), T(j)\} && \text{for each } (i, j) \in E \\
 |I(i) \cap I(j)| &< \min\{T(i), T(j)\} && \text{for each } (i, j) \notin E \\
 L(i) &< R(i) && \text{for each } i \in V \\
 L(i), R(i) &\in \mathbb{R}_0^+ && \text{for each } i \in V \\
 T(i) &\in \mathbb{R}^+ && \text{for each } i \in V
 \end{aligned} \tag{5.1}$$

The inequality $L(i) < R(i)$ incorporates that the left and right interval endpoint of one interval are distinct, in other words, $|I(i)| > 0$. Furthermore, *all* non-negative tolerance representations $\Gamma = (\{I(i)\}, \{T(i)\})$, $I(i) = [L(i), R(i)]$ that generate G are described by the solution set of system (5.1).

The system has some non-linear expressions like minima and the length of interval intersections. To linearize them we need some modifications. The first one is to write down explicitly the length of the intersection of two intervals. For this, we use Proposition 7, i.e., $|I(i) \cap I(j)| = \max\{0, \min\{R(i), R(j)\} - \max\{L(i), L(j)\}\}$. Hence, the first line of system (5.1) becomes

$$\max\{0, \min\{R(i), R(j)\} - \max\{L(i), L(j)\}\} \geq \min\{T(i), T(j)\}. \tag{5.2}$$

Note that the minimum of any two tolerances $T(i)$ and $T(j)$ is always positive. Hence,

the left side of (5.2) cannot be negative and we can omit taking the outer maximum.

The second line of system (5.1) becomes

$$\max\{0, \min\{R(i), R(j)\} - \max\{L(i), L(j)\}\} < \min\{T(i), T(j)\}. \quad (5.3)$$

Again, $\min\{T(i), T(j)\}$ is positive by default and it is not necessary to raise the left side to 0 if $\min\{R(i), R(j)\} - \max\{L(i), L(j)\}$ is negative.

The rewritten system is equivalent to

$$\begin{aligned} \min\{R(i), R(j)\} - \max\{L(i), L(j)\} &\geq \min\{T(i), T(j)\} && \text{for each } (i, j) \in E \\ \min\{R(i), R(j)\} - \max\{L(i), L(j)\} &< \min\{T(i), T(j)\} && \text{for each } (i, j) \notin E \\ L(i) &< R(i) && \text{for each } i \in V \\ L(i), R(i) &\in \mathbb{R}_0^+ && \text{for each } i \in V \\ T(i) &\in \mathbb{R}^+ && \text{for each } i \in V \end{aligned} \quad (5.4)$$

In the next step we replace each minimum and maximum by expressions using absolute values. We use the formulas

$$\min\{a, b\} = \frac{1}{2}(a + b - |a - b|) \quad (5.5)$$

and

$$\max\{a, b\} = \frac{1}{2}(a + b + |a - b|) \quad (5.6)$$

where a and b are non-negative real numbers.

Eventually, only the absolute values are non-linear. Let d be an expression whose absolute value, say c , needs to be determined. We can calculate the absolute values using a few inequalities and some additional variables. We use two non-negative variables p and n . If d is non-negative then p shall contain the absolute value of d

and n shall be 0. Otherwise if d is negative then n shall contain the absolute value of d and p shall be 0. The requirement that only one of the variables p and n is non-zero must be satisfied. This will be achieved by using an auxiliary variable λ that takes only the values 0 or 1, i.e., a boolean variable. Let D_- be a lower bound for the expression d and D_+ be an upper bound for d . For example, let $a, b, \in [0, 50]$. Then the lower bound of a and b is 0 and the upper one is +50. The lower bound D_- of $a - b$ is -50 and the upper bound D_+ is +50. Now we list all necessary inequalities such that the sum $p + n$ yields the result $c = |d|$:

$$\begin{aligned}
 d - p + n &= 0 \\
 D_+ \lambda &\geq p \\
 -D_- (1 - \lambda) &\geq n \\
 p, n &\geq 0 \\
 \lambda &\in \{0, 1\}
 \end{aligned} \tag{5.7}$$

If we resolve all minimas and maximas in system (5.4) by using the formulas (5.5) and (5.6) then we are left with calculating the absolute values $|L(i) - L(j)|$, $|R(i) - R(j)|$, $|T(i) - T(j)|$. As upper bound D_+ and negative lower bound $-D_-$ we can use a general upper bound on all numbers of an integer tolerance representation, just depending on the number n of vertices of the graph. For instance, such an upper bound is $1 + 5n2^{5n}$ as shown in [25] or tighter $n^2 2^n$ as shown in Chapter 6, Theorem 75, in this thesis. As described above, we introduce the necessary auxiliary variables corresponding to p, n, λ . For $|L(i) - L(j)|$, $|R(i) - R(j)|$, $|T(i) - T(j)|$, respectively, the variables having the role of p are $L_{i,j}^+$, $R_{i,j}^+$, $T_{i,j}^+$, respectively. The ones having the role of n are $L_{i,j}^-$, $R_{i,j}^-$, $T_{i,j}^-$, respectively. The ones having the role of λ are $\lambda_{i,j}$,

$\rho_{i,j}$, $\tau_{i,j}$, respectively. For clearness, we also introduce the auxiliary variables $L_{i,j}$ for $\max\{L(i), L(j)\}$, $R_{i,j}$ for $\min\{R(i), R(j)\}$, and $T_{i,j}$ for $\min\{T(i), T(j)\}$. Finally, we arrived at a mixed integer linear system which is given here.

For each $(i, j) \in E$:

$$\begin{aligned}
 R(i) - L(i) &\geq T_{i,j} \\
 R(i) - L(j) &\geq T_{i,j} \\
 R(j) - L(i) &\geq T_{i,j} \\
 R(j) - L(j) &\geq T_{i,j} \\
 \frac{1}{2} (T(i) + T(j) - (T_{i,j}^+ + T_{i,j}^-)) &= T_{i,j} \\
 T(i) - T(j) - T_{i,j}^+ + T_{i,j}^- &= 0 \\
 D_+ \tau_{i,j} &\geq T_{i,j}^+ \\
 -D_- (1 - \tau_{i,j}) &\geq T_{i,j}^- \\
 T_{i,j}^+, T_{i,j}^- &\in \mathbb{R}_0^+ \\
 \tau_{i,j} &\in \{0, 1\}
 \end{aligned} \tag{5.8}$$

For each $(i, j) \notin E$:

$$\begin{aligned}
 R_{i,j} - L_{i,j} &< T(i) \\
 R_{i,j} - L_{i,j} &< T(j) \\
 \frac{1}{2} (L(i) + L(j) + (L_{i,j}^+ + L_{i,j}^-)) &= L_{i,j} \\
 L(i) - L(j) - L_{i,j}^+ + L_{i,j}^- &= 0 \\
 D_+ \lambda_{i,j} &\geq L_{i,j}^+ \\
 -D_- (1 - \lambda_{i,j}) &\geq L_{i,j}^- \\
 \frac{1}{2} (R(i) + R(j) - (R_{i,j}^+ + R_{i,j}^-)) &= R_{i,j} \\
 R(i) - R(j) - R_{i,j}^+ + R_{i,j}^- &= 0 \\
 D_+ \rho_{i,j} &\geq R_{i,j}^+ \\
 -D_- (1 - \rho_{i,j}) &\geq R_{i,j}^- \\
 L_{i,j}^+, L_{i,j}^-, R_{i,j}^+, R_{i,j}^- &\in \mathbb{R}_0^+ \\
 \lambda_{i,j}, \rho_{i,j} &\in \{0, 1\}
 \end{aligned} \tag{5.9}$$

For each $i \in V$:

$$\begin{aligned}
 L(i) &< R(i) \\
 L(i), R(i) &\in \mathbb{R}_0^+ \\
 T(i) &\in \mathbb{R}^+
 \end{aligned} \tag{5.10}$$

5.2.3 Computational Results

In this section we apply the above described techniques to calculate the number of tolerance graphs and bounded tolerance graphs for $n = 1$ to 9. For some $n \geq 10$, checking all graphs (more than 12.000.000) takes much time. But checking single graphs is still possible.

We consider all undirected graphs which may or may not be connected. We use a library and a collection of programs, called *nauty*, to generate all graphs on n vertices (cf. [33]). Each graph gives rise to an integer linear program (ILP) according to the system consisting of (5.8), (5.9), (5.10) and the ILP is solved by an ILP-solver like CPLEX or SoPlex. If there is no solution then the graph is not a tolerance graph, otherwise it is. We try to make the tolerance representation Γ (=solution) as small as possible and also bounded. For this, we utilize the objective function $f(\Gamma)$ of the ILP which is to be minimized. We use $\max\{R(i) : i \in V\}$ as penalty for the width of the representation and add the value $T(i) - |I(i)|$ for each unbounded tolerance (and 0 for bounded tolerances). That yields the expression $\max\{R(i) : i \in V\} + \sum_{i \in V} \max\{T(i) - |I(i)|, 0\}$, which needs to be linearized. Let W_1 be a number larger than any number in Γ . For each $i \in V$, we have

$$\begin{aligned} \max\{T(i) - |I(i)|, 0\} &= \\ \max\{T(i) - |I(i)| + W_1, W_1\} - W_1 &= \\ \frac{1}{2}(T(i) - (R(i) - L(i)) + 2W_1 + |T(i) - (R(i) - L(i))|) - W_1 &= \\ \frac{1}{2}(T(i) - R(i) + L(i) + |T(i) - R(i) + L(i)|). \end{aligned}$$

Since we are not interested in the actual value of the objective function, we can omit

the factor $\frac{1}{2}$. Hence, we get

$$f(\Gamma) = \max\{R(i) : i \in V\} + \sum_{i \in V} (T(i) - R(i) + L(i) + |T(i) - R(i) + L(i)|).$$

With a few additional conditions we can state $f(\Gamma)$ as a linearized function:

$$\tilde{f}(\Gamma) = W_{\max} + \sum_{i \in V} T(i) - R(i) + L(i) + p_i + n_i$$

subject to

$$R(i) \leq W_{\max} \quad \text{for all } i \in V$$

$$-T(i) + R(i) - L(i) - p_i + n_i = 0$$

$$D_+ \lambda_i \geq p_i$$

$$-D_- (1 - \lambda_i) \geq n_i$$

$$p_i, n_i \geq 0$$

$$\lambda_i \in \{0, 1\}$$

D_- and D_+ are chosen as previously.

Hence, in case the graph is a bounded tolerance graph, we get a bounded tolerance representation as solution, otherwise an unbounded one. To speed up the process for $n \geq 7$ we may first check whether a graph is weakly chordal. If a graph G is not weakly chordal, then G is not tolerance by Result 27. For the recognition of weakly chordal graphs, we use the algorithm of [40]. Basically it considers all induced path of length 4 and checks whether one can be 'extended' to an induced cycle of length at least 5.

The algorithm is following:

Algorithm 1: counting number of tolerance graphs on n vertices

```

1   $L :=$  set of graphs with  $n$  vertices;
2   $C := 0$ ;
3  for  $G$  in  $L$  do
4      for each pair  $v, w \in V(G) = \{1, 2, \dots, n\}, v < w$  do
5          if  $(v, w)$  is edge then
6              set up inequalities (5.8) for an edge
7          else
8              set up inequalities (5.9) for a non-edge;
9          fi
10     od
11     add inequalities (5.10);
12     solve ILP;
13     if no solution then
14          $G$  is not tolerance graph;
15     else
16          $G$  is tolerance graph;  $C := C + 1$ ;
17     fi
18 od
19 return  $C$ ;

```

One other advantage of using linear programming is that the upper bound on the values can be chosen large without affecting the computation time significantly.

vertices n	number of graphs			
	unlabeled	weakly chordal	tolerance	bounded tolerance
1	1	1	1	1
2	2	2	2	2
3	4	4	4	4
4	11	11	11	11
5	34	33	33	33
6	156	146	146	143
7	1044	886	881	815
8	12346	8483	8231	6627
9	274668	126029	113401	71739

Table 5.1: number of certain graphs with up to 9 vertices

The results of the computation are shown in Table 5.1. We list all numbers for graphs of up to 9 vertices. For verification, the total numbers of (unlabeled) graphs on n vertices are the same as the numbers which are listed in “The On-Line Encyclopedia of Integer Sequences” ([1]).

The remaining numbers were not listed in the On-Line Encyclopedia. The numbers of weakly chordal graphs do not seem to be published and were derived by using the recognition algorithm in [40] (already mentioned above). The numbers for tolerance graphs and bounded tolerance graphs are new and were calculated by using Algorithm 1. As a rough estimate of the time, for $n = 9$, the computation took about 20 days on a 4-core processor.

5.3 Open problems

It is not known whether unit tolerance graphs can be recognized in polynomial time.

A system for unit tolerance graphs is 'easier' to solve than the system consisting of (5.8), (5.9), (5.10), but is that enough?

Chapter 6

Width of Tolerance Representations

6.1 Preliminaries

Hayward and Shamir showed in [25] that every tolerance graph has a tolerance representation where each number can be represented by a polynomial number of bits. Here, 'polynomial' is meant with respect to the number of intervals. Their approach is following: For any given tolerance representation of a graph with n vertices, they extract the orderings of the critical points as well as the ordering of the bounded tolerances and lengths of intervals with unbounded tolerance. Both orderings together uniquely determine the tolerance graph. Each ordering can be transformed into a linear inequality system. A solution to this system is a tolerance representation of the graph. And each entry of the solution can be bounded by an exponential expression in the number n of vertices and the number m of edges. In other words, each entry needs a polynomial number of bits.

Recall that a tolerance representation is *integer* if all interval endpoints and tol-

erances are integer, and that it is *expansive* if all critical points are distinct and also all bounded tolerances and lengths of intervals with unbounded tolerances are distinct. The *width* of a non-negative integer tolerance representation is defined as the largest integer value among all interval endpoints and tolerances. In [25], it was shown (cf. Result 14) that every tolerance graph has an expansive tolerance representation. But the main result of their paper is:

Result 69 ([25, Theorem 3]). *Every tolerance graph G with n vertices has a non-negative integer tolerance representation with width at most $1 + 5n2^{5n}$.*

In that paper it is described details how to calculate such a tolerance representation from any tolerance representation by using a system of linear inequalities. A solution to that system yields a tolerance representation whose entries are bounded by $1 + 5n2^{5n}$. Furthermore, they gave a

Conjecture 70 ([25]). *Every tolerance graph G with n vertices has a tolerance representation in which each interval endpoint and tolerance can be represented by only a poly-logarithmic number of bits.*

For a subclass of tolerance graphs, the class of interval graphs, the conjecture is true. Every interval graph has an interval representation whose width is at most $2n$. That representation can be derived from any interval representation by repeatedly applying following well known proposition.

Proposition 71. *For an interval graph $G = (V, E)$, $V = \{1, 2, \dots, n\}$, let $\mathcal{I} = \{[L(i), R(i)] : i \in V; L(i), R(i) \in \mathbb{Z}_0^+\}$ be a non-negative integer interval representation of G . Let $W(\mathcal{I}) = \max\{L(i), R(i) : i \in V\}$, in other words, the width of the*

representation. If there is some integer $k \in [0, W(\mathcal{I})]$ such that $k \neq L(i)$ and $k \neq R(i)$ for all $i \in V$, then a new representation $\mathcal{I}'$ with $W(\mathcal{I}') = W(\mathcal{I}) - 1$ can be obtained from $\mathcal{I}$ by decreasing all $L(i)$ with $L(i) > k$ and all $R(i)$ with $R(i) > k$ by 1.

Proof. Note that intervals are only shortened. Hence, we only need to check that intersecting intervals remain intersecting. Since the relative ordering of the endpoints is not changed a pair of intersecting intervals of $\mathcal{I}$ will remain intersecting in $\mathcal{I}'$. $\square$

Since there are only $2n$ different endpoints, the width is at most $2n$.

Here we prove a slightly modified version of the conjecture in two cases and reduce the bound in the general case. The modification is in the requirement that we weaken the condition of expansive to regular. The definitions are given in Chapter 2 on page 26 and the following. That means, we do not require the tolerance points to be distinct from other points in the representation.

In the case of a unit tolerance graph with n vertices and m edges, we prove that it has a regular non-negative integer tolerance representation with width less than $\binom{n}{2} - m + 3n - 1$. That is less than n^2 . In the remaining cases, we can reduce the bound to $n^2 2^n$.

To calculate an upper bound for the width of a tolerance representation, we use similar linear programming techniques as in [25]. But in contrast to their proof, we start with a different system of linear inequalities and we find a tighter bound in all cases.

6.2 Setting up a New Linear System

Let a tolerance graph $G = (V, E)$, $V = \{1, 2, \dots, n\}$, and one of its regular non-negative integer tolerance representation $\Gamma_0 = (\{I_0(i) : i \in V\}, \{T_0(i) : i \in V\})$, $I_0(i) = [L_0(i), R_0(i)]$, be given. There is such a representation by Result 11. We consider the system (5.4) of Subsection 5.2.2 for the graph G . The tolerance representation Γ_0 is a solution of this system.

We can add a few requirements to this system without destroying the feasibility. The additional inequalities will just restrict the solution set. The first requirement is that all variables are integers as possible by Result 11. The second requirement is that the tolerance representation is regular. That is, all interval endpoints are distinct and the bounded tolerances are distinct. For the latter, we partition the vertex set V into the part V_B with bounded tolerances in Γ_0 and the part V_U with unbounded tolerances in Γ_0 . The additional restrictions to the system are

$$\begin{aligned} L(i) &\neq R(j) && \text{for each } i, j \in V \\ T(i) &\neq T(j) && \text{for each } i, j \in V_B \\ T(i) &= \infty && \text{for each } i \in V_U \end{aligned} \tag{6.1}$$

So far the new system is the following:

$$\begin{aligned}
 \min\{R(i), R(j)\} - \max\{L(i), L(j)\} &\geq \min\{T(i), T(j)\} \quad \text{for each } (i, j) \in E \\
 \min\{R(i), R(j)\} - \max\{L(i), L(j)\} &< \min\{T(i), T(j)\} \quad \text{for each } (i, j) \notin E \\
 L(i) &< R(i) \quad \text{for each } i \in V \\
 L(i) &\neq R(j) \quad \text{for each } i, j \in V \\
 T(i) &\neq T(j) \quad \text{for each } i, j \in V_B \\
 T(i) &= \infty \quad \text{for each } i, j \in V_U \\
 L(i), R(i) &\in \mathbb{Z}_0^+ \quad \text{for each } i \in V \\
 T(i) &\in \mathbb{Z}^+ \quad \text{for each } i \in V
 \end{aligned} \tag{6.2}$$

Our goal is to estimate the size of the smallest solution. But since some inequalities of the system (6.2) are not linear, it might be difficult to find a solution and estimate the size of the solution. Hence, we will simplify the system further. We extract information from the structure of the given tolerance representation Γ_0 , which is a solution to system (6.2). With that we create new inequalities. They will restrict the solution set additionally, but will not destroy the feasibility as Γ_0 remains a solution.

The left and right endpoints of intervals in Γ_0 are $L_0(i), i \in V$, and $R_0(i), i \in V$. Let them be denoted by $Q_0(j), j = 1, 2, \dots, 2n$, in such a way that they are ordered ascendingly $Q_0(1) < Q_0(2) < \dots < Q_0(2n)$. Furthermore, let the bounded tolerances $T_0(i), i \in V_B$, be ordered like $1 \leq T_0(t(1)) < T_0(t(2)) < \dots < T_0(t(|V_B|))$ where t is a mapping of $\{1, 2, \dots, |V_B|\}$ to V .

Additionally we require each solution of system (6.2), i.e., each tolerance representation Γ , to have the same two orderings of endpoints and tolerances as Γ_0 . Let $Q(i), i = 1, 2, \dots, 2n$, be the variables $L(i), R(i), i = 1, 2, \dots, n$. We shall have

$Q(1) < Q(2) < \dots < Q(2n)$ and

$$1 \leq T(t(1)) < T(t(2)) < \dots < T(t(|V_B|)). \quad (6.3)$$

Recall that $Q(i)$ is a place holder for some $L(i)$ or $R(j)$. This implies an ordering $l : \{1, 2, \dots, n\} \rightarrow V$ of the new left endpoints like

$$L(l(1)) < L(l(2)) < \dots < L(l(n)), \quad (6.4)$$

and an ordering $r : \{1, 2, \dots, n\} \rightarrow V$ of the new right endpoints like

$$R(r(1)) < R(r(2)) < \dots < R(r(n)). \quad (6.5)$$

These restrictions lead to several eliminations:

1. the minimas and maximas, because they can be determined from the above orderings (6.3), (6.4), and (6.5),
2. the inequalities $L(i) < R(i)$, for $i \in V$, Γ has the same order of endpoints as Γ_0 and $L_0(i) < R_0(i)$,
3. the inequalities, $L(i) \neq R(j)$, $T(i) \neq T(j)$, because the orderings are strict.

With the additional restrictions on the ordering of the variables, we arrive at a *linear* system of constraints:

$$\begin{aligned}
 R(a(i, j)) - L(b(i, j)) &\geq T(c(i, j)) \quad \text{for each } (i, j) \in E \\
 R(a(i, j)) - L(b(i, j)) &< T(c(i, j)) \quad \text{for each } (i, j) \notin E \\
 Q(i) &< Q(i+1) \quad \text{for } i \in \{1, 2, \dots, 2n-1\} \\
 T(t(i)) &< T(t(i+1)) \quad \text{for } i \in \{1, 2, \dots, |V_B| - 1\} \\
 1 &\leq T(t(1)) \\
 T(i) &= \infty \quad \text{for each } i \in V_U \\
 L(i), R(i) &\in \mathbb{Z}_0^+ \quad \text{for each } i \in V \\
 T(i) &\in \mathbb{Z}_0^+ \quad \text{for each } i \in V_B
 \end{aligned} \tag{6.6}$$

where $a(i, j)$ such that $R(a(i, j)) = \min\{R_0(i), R_0(j)\}$, $b(i, j)$ such that $L(b(i, j)) = \max\{L_0(i), L_0(j)\}$, $c(i, j)$ such that $T(c(i, j)) = \min\{T_0(i), T_0(j)\}$.

For the subsequent processing, we transform the system (6.6) into standard form. That is the form $Ax = b, x \geq 0$. We write the final integer linear system (6.7)

$$\begin{aligned}
 -R(a(i, j)) + L(b(i, j)) + T(c(i, j)) &\leq 0 \quad \text{for each } (i, j) \in E \\
 R(a(i, j)) - L(b(i, j)) - T(c(i, j)) &\leq -1 \quad \text{for each } (i, j) \notin E \\
 Q(i) - Q(i+1) &\leq -1 \quad \text{for } i \in \{1, 2, \dots, 2n-1\} \\
 T(t(i)) - T(t(i+1)) &\leq -1 \quad \text{for } i \in \{1, 2, \dots, |V_B| - 1\} \\
 -T(t(1)) &\leq -1 \\
 T(i) &= \infty \quad \text{for each } i \in V_U \\
 L(i), R(i) &\in \mathbb{Z}_0^+ \quad \text{for each } i \in V \\
 T(i) &\in \mathbb{Z}_0^+ \quad \text{for each } i \in V_B
 \end{aligned} \tag{6.7}$$

Since we only added requirements, the solution set of system (6.6) is a subset of

the solution set of system (5.4). Furthermore, the solution set is non-empty since Γ_0 is a solution to both systems.

In total, the system (6.7) has $k = \binom{n}{2} + 2n - 1 + |V_B|$ inequalities. Hence, we have $\binom{n}{2} + 2n - 1 \leq k \leq \binom{n}{2} + 3n - 1$. The first two lines describe $\binom{n}{2}$ inequalities. Line 3 describes $2n - 1$ inequalities corresponding to the linear ordering of the set of $L(i), i = 1, \dots, n$ together with $R(i), i = 1, \dots, n$. This can also be expressed by the linear inequalities $M_1 Q \leq -\mathbf{1}$ where $Q = (Q(1), Q(2), \dots, Q(2n))$ and M_1 is the following $(2n - 1) \times 2n$ matrix:

M_1	1	2	...	$2n - 1$	$2n$
1	+1	-1	0	...	0
2	0	+1	$\ddots$	$\ddots$	$\vdots$
$\vdots$		$\ddots$	$\ddots$	$\ddots$	
$2n - 2$	$\vdots$		$\ddots$	+1	-1
$2n - 1$	0	...	0	+1	-1

Similarly, the lines 4 and 5 of (6.7) describes the linear ordering of $T(t(i)), i \in \{1, 2, \dots, |V_B|\}$ by $|V_B|$ inequalities. This can be rewritten by the linear inequalities $M_2 T \leq -\mathbf{1}, -T(t(1)) \leq -1$, where T stands for $(T(t(1)), T(t(2)), \dots, T(t(|V_B|)))$ and M_2 is the following $(|V_B| - 1) \times |V_B|$ matrix:

M_2	1	2	...	$ V_B - 1$	$ V_B $
1	+1	-1	0	...	0
2	0	+1	$\ddots$	$\ddots$	$\vdots$
$\vdots$		$\ddots$	$\ddots$	$\ddots$	
$ V_B - 2$	$\vdots$		$\ddots$	+1	-1
$ V_B - 1$	0	...	0	+1	-1

Furthermore, let S be the vector of slack variables which has length k . For the vectors x and S , we denote the transposed vector by x^t and S^t . This should not be confused with the notation $T(t(i))$ where t denotes a permutation. We set up the following linear equation system by adding k slack variables S . I_k denotes the identity matrix of order k .

$$Ax = b, x \geq 0, x \in \mathbb{Z}^{2n+|V_B|+k} \quad (6.8)$$

where

x^T	$L(l(1))\dots$ $L(l(n))$	$R(r(1))\dots$ $R(r(n))$	$T(t(1))$	$\dots$ $T(t(V_B))$	S^T	b
edges	one +1 per row	one -1 per row	one +1 per row		I_k	0
non-edges	one -1 per row	one +1 per row	one -1 per row			-1
A linear order L and R	M_1		0			-1
linear order T	0	0	M_2			
	0	0	-1	0		-1

Finally, we recall some terms of linear programming as they can be found in the book of Schrijver [42] or in the book [26]. For a linear system $Ax = b$ where the

number of columns is greater than or equal to the number of rows, we call a subset B of the column indices of A a *basis*, if the columns whose indices are in B are linearly independent and the number of columns is maximal. For a basis B , the matrix A_B is defined to be the submatrix of A with all columns whose indices are in B . The matrix A_B is called *basis matrix* (with respect to B). That choice guarantees that A_B is invertible. A *basic solution* to $Ax = b$ is a vector $x = (x_B, x_N)$ such that there is a basis B and $x_B = A_B^{-1}b$ and $x_N = 0$.

Back to our problem; the matrix A has more columns than rows and the last k columns form an identity matrix. Hence, A has full row rank ($= k$). The system $Ax = b, x \geq 0$, has a basic feasible solution, say x_B , where B is a basis. We can write $x_B = A_B^{-1}b$. The inverse matrix A_B^{-1} can be expressed by the *adjugate matrix* $\text{adj}(A_B)$. By definition (e.g. as in [28, chapter 2]), the entry at position (i, j) of $\text{adj}(A_B)$ is equal to the *cofactor*

$$c_{i,j} := (-1)^{i+j} \det(\hat{M}_{j,i})$$

where $\hat{M}_{j,i}$ denotes the matrix which is obtained from A_B by deleting row j and column i . We have $A_B^{-1} = \frac{1}{\det(A_B)} \cdot \text{adj}(A_B)$ and thus, $x_B = \frac{1}{\det(A_B)} \cdot \text{adj}(A_B)b$.

The only reason that x_B might not be integral is the factor $\frac{1}{\det(A_B)}$. Note that if some square matrix C is integral then the value of a determinant of C is integer, because the determinant can be expressed as a sum of signed products of the entries of C . Thus, $\det(A_B)$ and $\text{adj}(A_B)$ must be integral. Since $\text{adj}(A_B)$ and b are integral, only the factor $\frac{1}{\det(A_B)}$ is not integer.

We can eliminate this non-integer factor by replacing b with $b' := b \cdot \det(A_B)$. We

define a new system

$$Ax = b', x \geq 0, x \in \mathbb{R}^{2n+|V_B|+k}. \quad (6.9)$$

System (6.8) is feasible if and only if system (6.9) is feasible. If x_0 is a solution to (6.8) then $x_0 \det(A_B)$ is one to (6.9). And if x'_0 is a solution to (6.9) then $x'_0 \cdot \frac{1}{\det(A_B)}$ is one to (6.8). Hence, system (6.9) has a basic solution (x'_B, x'_N) because system (6.8) is feasible. We have $x'_B = (A_B)^{-1} b' = \frac{1}{\det(A_B)} \cdot \text{adj}(A_B) b \cdot \det(A_B) = \text{adj}(A_B) b =$

$$\begin{bmatrix} c_{1,1} & \cdots & c_{1,k} \\ \vdots & \ddots & \vdots \\ c_{k,1} & \cdots & c_{k,k} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}.$$

Now x'_B is an integral vector and (x'_B, x'_N) corresponds to a non-negative integer tolerance representation.

6.3 Size of the Entries of a Solution

Our goal is to find an upper bound on the entries x_i of the integral vector $x'_B = (x_i)_{i=1,\dots,k}$. From above calculation we get

$$x_i = \sum_{j=1}^k c_{i,j} b_j = \sum_{j=1}^k (-1)^{i+j} \det(\hat{M}_{j,i}) b_j \quad (6.10)$$

This expression is similar to Cramer's Rule. We need to determine the values of the determinants $\det(\hat{M}_{j,i})$. For a submatrix C of A , we call a column j_0

L-column if column j_0 corresponds to $L(\gamma)$ for some $\gamma \in \{1, \dots, n\}$,

R-column if column j_0 corresponds to $R(\gamma)$ for some $\gamma \in \{1, \dots, n\}$,

T-column if column j_0 corresponds to $T(\gamma)$ for some $\gamma \in \{1, \dots, n\}$,

S-column if column j_0 corresponds to $S(\gamma)$ for some $\gamma \in \{1, \dots, k\}$.

We call a row of C an **LR -row** if it has no entries in T -columns. Similarly, **RT -row** if it has no entries in L -columns, and **LT -row** if it has no entries in R -columns. If there is an entry in the L -columns and one in the R -columns and one in the T -columns then we call it an **LRT -row**. The rows establishing the linear orders are of type **LL -rows**, **RR -rows**, **TT -rows**.

The matrix $\hat{M}_{j,i}$ is a submatrix of A . We express the determinant of the $(k-1) \times (k-1)$ matrix $\hat{M}_{j,i}$ by using the Laplace expansion along each S -column. An S -column has only one non-zero entry, which is $+1$. Let the number of S -columns be $k-d$ and d the number of the remaining columns. We have $d \leq 3n$ since there are at most $3n$ columns which are not S -columns. The remaining $d \times d$ matrix is called $\widetilde{M}_{j,i}$. Thus, we have $\det(\hat{M}_{j,i}) = \pm 1 \cdot \det(\widetilde{M}_{j,i})$. If we apply the absolute value on (6.10), we get

$$|x_i| = \left| \sum_{j=1}^k \det(\hat{M}_{j,i}) b_j \right| \quad \text{for } i \in \{1, \dots, k\}.$$

Recall that $b_j \in \{-1, 0\}$ and that $b_j \neq 0$ if and only if $j > m$.

$$\left| \sum_{j=1}^k \det(\hat{M}_{j,i}) b_j \right| \leq \left| \sum_{j=m+1}^k \det(\hat{M}_{j,i}) \right|$$

Using the triangle inequality together with the fact that $|\det(\hat{M}_{j,i})| = |\det(\widetilde{M}_{j,i})|$, it follows that

$$\left| \sum_{j=m+1}^k \det(\hat{M}_{j,i}) \right| \leq \sum_{j=m+1}^k |\det(\widetilde{M}_{j,i})| \quad (6.11)$$

Then we check whether $\widetilde{M}_{j,i}$ is totally unimodular. We use the criterion which was given by Ghoulia and Hourì, 1962¹.

¹cf. Padberg, 1976: Note on the total unimodularity of matrices

Result 72 ([42, chap 19.3(iv)]). *A matrix C is totally unimodular if and only if each collection J of columns of C can be split into two parts J^+ and J^- such that the sum of the columns of J^+ minus the sum of the columns of J^- is a vector with entries only 0, -1 , and $+1$.*

For a row i of C , we define $\Delta(i) := \sum_{j \in J^+} C_{i,j} - \sum_{j \in J^-} C_{i,j}$. Then the condition on the sum of vectors in Result 72 can be expressed componentwise:

Result 73 (Variation of Result 72). *A matrix C is totally unimodular if and only if each collection J of columns of C can be split into two parts J^+ and J^- such that $\Delta(i) \in \{-1, 0, +1\}$ for all $i \in \{1, \dots, k\}$.*

Let J be any subset of indices of columns of $\widetilde{M}_{j,i}$. We check whether Result 73 can be applied. We have only certain choices to assign the column indices θ, ι, κ to the parts J^+ and J^- as shown in Table 6.1.

We model the assignment of the column indices to the partition J^+ and J^- observing the condition $\Delta(i) \in \{-1, 0, +1\}$ for all $i \in \{1, \dots, k\}$ as a 2-colorability problem of some hypergraph H . A hypergraph is a generalization of the concept of graph where an edge can consist of more than two vertices. A 2-coloring of a hypergraph is an assignment of colors to all vertices such that each vertex is assigned one color and each edge has vertices of different colors and at most two colors are used. In this context we may denote one color with $+$ and the other one with $-$. Now we define the hypergraph H . Let the vertex set $V(H)$ of the hypergraph H be the set of columns $j_1, j_2, \dots$ of $\widetilde{M}_{j,i}$ together with auxiliary vertices

$$\begin{aligned} & \{x_{\theta, \iota} : \theta < \iota, \theta, \iota \in \{j_1, j_2, \dots\}, \theta, \iota \text{ are } L\text{- or } R\text{-columns}\} \cup \\ & \{y_{\iota, \kappa} : \iota, \kappa \in \{j_1, j_2, \dots\}, \iota \text{ is } R\text{- or } T\text{-column}, \kappa \text{ is } T\text{-column}\}. \end{aligned}$$

row type	non-zero entries			all possible assignments
	column θ cor- responds to	column ι cor- responds to	column κ cor- responds to	
LR	$L(\cdot) = 1$	$R(\cdot) = -1$	$-$	$\theta, \iota \in J^+$ or $\theta, \iota \in J^-$
	$L(\cdot) = -1$	$R(\cdot) = 1$	$-$	
RT	$-$	$R(\cdot) = 1$	$T(\cdot) = -1$	$\iota, \kappa \in J^+$ or $\iota, \kappa \in J^-$
	$-$	$R(\cdot) = -1$	$T(\cdot) = 1$	
LT	$L(\cdot) = 1$	$-$	$T(\cdot) = 1$	$\theta \in J^+, \kappa \in J^-$ or $\theta \in J^-, \kappa \in J^+$
	$L(\cdot) = -1$	$-$	$T(\cdot) = -1$	
LRT	$L(\cdot) = 1$	$R(\cdot) = -1$	$T(\cdot) = 1$	$\theta, \iota, \kappa \in J^+$ or $\theta, \iota \in J^+, \kappa \in J^-$ or $\theta \in J^+, \iota, \kappa \in J^-$ or $\theta, \iota, \kappa \in J^-$ or $\theta, \iota \in J^-, \kappa \in J^+$ or $\theta \in J^-, \iota, \kappa \in J^+$
	$L(\cdot) = -1$	$R(\cdot) = 1$	$T(\cdot) = -1$	
LL	$L(\cdot) = 1$	$L(\cdot) = -1$	$-$	$\theta, \iota \in J^+$ or $\theta, \iota \in J^-$
	$L(\cdot) = -1$	$L(\cdot) = 1$	$-$	
RR	$R(\cdot) = 1$	$R(\cdot) = -1$	$-$	$\theta, \iota \in J^+$ or $\theta, \iota \in J^-$
	$R(\cdot) = -1$	$R(\cdot) = 1$	$-$	
TT	$-$	$T(\cdot) = 1$	$T(\cdot) = -1$	$\iota, \kappa \in J^+$ or $\iota, \kappa \in J^-$
	$-$	$T(\cdot) = -1$	$T(\cdot) = 1$	

Table 6.1: choices for partition

The edge set $E(H)$ shall consist of the edges shown in Table 6.2 (only row types of the matrix with at least two non-zero entries need to be considered).

row type	edge(s) of hypergraph
LR	$\{\theta, x_{\theta, \iota}\}, \{x_{\theta, \iota}, \iota\}$
RT	$\{\iota, y_{\iota, \kappa}\}, \{y_{\iota, \kappa}, \kappa\}$
LT	$\{\theta, \kappa\}$
LRT	$\{\theta, x_{\theta, \iota}\}, \{x_{\theta, \iota}, \iota, y_{\theta, \iota}\}, \{y_{\theta, \iota}, \kappa\}$
LL	$\{\theta, x_{\theta, \iota}\}, \{x_{\theta, \iota}, \iota\}$
RR	$\{\theta, x_{\theta, \iota}\}, \{x_{\theta, \iota}, \iota\}$
TT	$\{\iota, y_{\iota, \kappa}\}, \{y_{\iota, \kappa}, \kappa\}$

Table 6.2: edges of the hypergraph

In some cases a 2-coloring is guaranteed. Let S_1 be the set of LR -rows of $\widetilde{M}_{i,j}$ and let S_2 be the set of RT -rows of $\widetilde{M}_{i,j}$ and let S_3 be the set of LT -rows of $\widetilde{M}_{i,j}$ and let S_4 be the set of LRT -rows of $\widetilde{M}_{i,j}$.

Corollary 74. *If one of the sets S_1, S_2, S_3 is empty for $\widetilde{M}_{i,j}$ then the hypergraph has a 2-coloring.*

Proof. If S_1 is empty, we let $J^+ = \{j \in J : \text{column } j \text{ is } R\text{- or } T\text{-column}\}$ and $J^- = J \setminus J^+ = \{j \in J : \text{column } j \text{ is } L\text{-column}\}$. If S_2 is empty, then we let $J^+ = \{j \in J : \text{column } j \text{ is } L\text{-column or } R\text{-column}\}$ and $J^- = J \setminus J^+ = \{j \in J : \text{column } j \text{ is } T\text{-column}\}$. If S_3 is empty, we let $J^+ = \{j \in J : \text{column } j \text{ is } L\text{-column or } R\text{-column or } T\text{-column}\}$ and $J^- = J \setminus J^+ = \emptyset$.

In all these cases we have $\Delta(i) = \begin{cases} 0 & \text{if row type is listed in Table 6.2} \\ \pm 1 & \text{otherwise, row } i \text{ has one non-zero entry.} \end{cases}$

That is $\Delta(i) \in \{-1, 0, +1\}$ for all $i = 1, \dots, d$. $\square$

Theorem 75. *The upper bound on each value of an integer tolerance representation is $n^2 2^n$. If each hypergraph H constructed of each submatrix of $\widetilde{M}_{i,j}$ has a 2-coloring then the upper bound reduces to n^2 .*

Proof. Let $x'_B = (x_i)_{i=1, \dots, k}$ be a solution to the system (6.9), in other words, an integer tolerance representation. We consider the absolute value of x_i for an arbitrary $i \in \{1, \dots, k\}$.

$$|x_i| \leq \sum_{j=m+1}^k \left| \det(\widetilde{M}_{j,i}) \right|$$

In system (6.8) $k - m$ is less than $\binom{n}{2} - m + 2n - 1 + n - 1$. Hence,

$$\begin{aligned} |x_i| &\leq \left(\binom{n}{2} - m + 3n - 1 \right) \max_{m < j \leq k} \left| \det(\hat{M}_{j,i}) \right| \\ &= \left(\binom{n}{2} - m + 3n - 1 \right) \max_{m < j \leq k} \left| \det(\widetilde{M}_{j,i}) \right| \\ &\leq n^2 \max_{m < j \leq k} \left| \det(\widetilde{M}_{j,i}) \right|, \end{aligned}$$

if $n > 4$. We compute $\det(\widetilde{M}_{j,i})$ further by using the Laplace expansion. Let S_x be the smallest set among S_1, S_2, S_3 . This set S_x must have cardinality less than or equal n (otherwise by the pigeonhole principle the sets S_1, S_2, S_3 were not disjoint). We expand along all rows that belong to S_x . A row of type LR, RT, LT has two non-zero entries which are ± 1 . Thus, expansion along one row contributes a factor of at most ± 2 , that is, $\left| \det(\widetilde{M}_{j,i}) \right| \leq 2 \cdot \left| \det(M_3) \right|$ where M_3 is the matrix with the maximal determinant among the two remaining matrices. Repeatedly applying until all rows of

S_x are expanded will contribute at most the factor 2^n . The remaining matrices are totally unimodular due to Corollary 74 since the set S_x is empty and thus, one of the sets S_1 or S_2 or S_3 . Thus, we have $|\det(\widetilde{M_{j,i}})| \leq 2^n \cdot |\pm 1| = 2^n$. Hence,

$$|x_i| \leq n^2 \max_{j, b_j \neq 0} |\det(\hat{M}_{j,i})| \leq n^2 2^n.$$

Given a 2-coloring, we have $\Delta(i) \in \{-1, 0, +1\}$, if we put the vertices with color $+$ in J^+ and the vertices with color $-$ in J^- . Considering each type of row (except rows with a single entry), the Table 6.1 yields $\Delta(i) \in \{-1, 0, +1\}$ for each i . Hence, for each $j \in \{1, \dots, d\}$, the matrix $\widetilde{M_{j,i}}$ is totally unimodular, and $\max_{j, b_j \neq 0} |\det(\widetilde{M_{j,i}})| = 1$ and $|x_i| \leq n^2$. $\square$

Corollary 76. *A unit tolerance graph has a regular integer tolerance representation where the maximum integer is bounded by n^2 .*

Proof. Let a be the length of every interval. Then we can express each right endpoint by $R(i) = L(i) + a$. Hence, we can omit the calculation of the variables $R(i)$. Hence, there are no rows with LR, RT, LRT type. By Corollary 74 each hypergraph has a 2-coloring and by last theorem the upper bound is n^2 . $\square$

6.4 Open problems

One open problem is to find a bound on the size of the solutions if there is a hypergraph which has no 2-coloring. Can we do better in this case? Or can we even show $|\det(D)| \leq 1$?

Another question is whether the simplification from expansive to regular has a significant impact on the width of the tolerance representation.

Appendix

Theorem 40 (Chapter 3) presents some conditions when a vertex of a tolerance graph is quasi-assertive. The property of a vertex of being quasi-assertive is purely determined by whether each of its tolerance representations can be modified to satisfy certain conditions. In our proof of Theorem 40 we do not explicitly show the modifications of the tolerance representations. Instead we only prove the existence of such representations.

Here we give explicit values for the intervals and tolerances in case someone wants to construct such a modified tolerance representation. We define $L'(x)$ and $R'(x)$ by $I'(x) = [L'(x), R'(x)]$. Let

$$I'(v_i) := \begin{cases} [L(v_i), R(u) + \frac{i-1}{s+1}\epsilon] & \text{for } v_i \in C, \\ [L(u) - \frac{s-(c+i-1)}{s+1}\epsilon, R(u) + \frac{c+i-1}{s+1}\epsilon] & \text{for } v_i \in A, \\ [L(u) - \frac{s-\delta(i)+c+a+1}{s+1}\epsilon, R(u) + \frac{\delta(i)+c+a-1}{s+1}\epsilon] & \text{for } v_i \in B, \\ [L(u) - \frac{s-(c+a+b+i)}{s+1}\epsilon, R(v_i)] & \text{for } v_i \in D, \end{cases}$$

and

$$\delta(2i) := \delta(2i+1) := 2i \quad \text{for } i \in \{1, \dots, b\}$$

and

$$T'(v_i) := \begin{cases} \infty & \text{if } v_i \in C \cup D \text{ and } t(v_i) = \infty, \\ \min \{T(v_i), |I'(v_i)|\} & \text{otherwise.} \end{cases}$$

Furthermore,

$$I'(u) := \left[L(u) - \frac{d}{s+1}\epsilon, R(u) + \frac{c+a+b}{s+1}\epsilon \right],$$

$$T'(u) := |I'(u)|.$$

The sets A, B, C, D and the constants a, b, c, d, s are as defined in the proof of Theorem 40. By the way, this approach was the original idea but it made the proof more difficult to understand.

List of Works

Below is the work published during my PhD studies in NTU.

B. Eisermann. A sufficient condition for boundedness of tolerance graphs. *Discrete Applied Mathematics*, 159:2165–2169, 2011.

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List of Symbols

$v \sim w$ vertex v is adjacent to vertex w

$Adj(v)$ neighborhood of vertex v

$A(G)$ reduced adjacency matrix of bipartite graph G ; *see* page 65

$a(i)$ index of first column of $A(G)$ whose entry in row i is 1; *see* page 66

$\alpha(\cdot)$ certain mapping from row indices to non-negative integers; *see* page 76

$b(i)$ index of last column of $A(G)$ whose entry in row i is 1; *see* page 66

$\beta(\cdot)$ certain mapping from column indices to non-negative integers; *see* page 76

B' extended block of B ; *see* page 65

B'' B' together with path of length 2 attached to each $v \in \mathcal{B}^2(B)$; *see* page 65

$\mathcal{B}(B)$ boundary of block B ; *see* page 62

$\mathcal{B}^1(B)$ subset of boundary (separating B from leaves of G); *see* page 62

$\mathcal{B}^2(B)$ subset of boundary, $\mathcal{B}(B) \setminus \mathcal{B}^1(B)$; *see* page 62

$bc(G)$ block-cutpoint-graph of graph G ; *see* page 61

$bc_\omega(G)$ weighted block-cutpoint-graph of graph G ; *see* page 81

$c_a(j)$ number of rows of $A(G)$ with their first non-zero entry in column j ; *see* page 74

$c_b(j)$ number of rows of $A(G)$ with their last non-zero entry in column j ; *see* page 74

$\overline{G}$ complement of graph G

$\Delta(j)$ *Chapter 4*: sum of c_a and c_b values from 1 to $j - 1$; *see* page 76

Chapter 6: sum of certain matrix entries; *see* page 131

$E(\cdot)$ edge set of a graph or subgraph

$\epsilon_H(v)$ eccentricity of vertex v in subgraph H ; *see* page 62

$G(\Gamma)$ graph generated by the representation Γ

Γ a tolerance representation

k number of inequalities of some system; *see* page 126

$L(\cdot), l(\cdot)$ left endpoint of an interval

$|I|$ length of interval I

n number of vertices of graph G ; *see* page 19

p number of rows of reduced adjacency matrix $A(\cdot)$; *see* page 66

P_G set of leaves of graph G ; *see* page 62

q number of columns of reduced adjacency matrix $A(\cdot)$; *see* page 66

$Q_0(\cdot)$ (left or right) interval endpoint; *see* page 123

$R(\cdot), r(\cdot)$ right endpoint of an interval

s Chapter 3: cardinality of V_u ; see page 54

Chapter 4: number of blocks on spine; see page 81

$T(\cdot), t(\cdot)$ tolerance

$T_0(\cdot)$ tolerance; see page 123

$V(\cdot)$ vertex set of a graph or subgraph

V_B set of vertices with bounded tolerances (with respect to some tolerance representation); see page 26

V_U set of vertices with unbounded tolerances (with respect to some tolerance representation); see page 26

V_u certain subset of vertices depending on vertex u ; see Chap. 3, page 53

$\omega(\cdot)$ weight of vertices of $bc_\omega(G)$; see page 81

$X(S)$ vertex subset of S belonging to X partition class; see page 76

$Y(S)$ vertex subset of S belonging to Y partition class; see page 76

$\mathbb{Z}^+$ positive integers

$\mathbb{Z}_0^+$ non-negative integers

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